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Jan Awrejcewicz

Classical Mechanics

Kinematics and Statics

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Jan Awrejcewicz
Department of Automation
and Biomechanics
Łódź University of Technology
Łódź
Poland

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Preface

This book belongs to a series of three books written simultaneously (the remaining two are titled *Classical Mechanics: Dynamics* and *Classical Mechanics: Applied Mechanics and Mechatronics*). This book's *triad* attempts to cover different subjects in classical mechanics and creates a link between them by introducing them from the same root. The classical mechanics approach extended to the study of electro–magneto–mechanical systems is also emphasized. Another important objective in writing the series of three volumes with the repeated title *Classical Mechanics* was to include and unify sometimes different approaches to the subject in the English, Russian, Polish, and German literature. This explains why sometimes the list of references includes works written either in Russian or Polish (the English literature is now easily available using Google or Scopus). Although the list of references includes works written not necessarily in English, the way of the book material presentation does not require to read the reference sources.

This first volume contains problems of classical mechanics including kinematics and statics. It is recommended as a textbook for undergraduate and graduate students in mechanical and civil engineering and applied physics as well as for researchers and engineers dealing with mechanics. It could also be a main reference for other courses, but it is suited for a course in statics and dynamics.

In Chap. 1 the fundamental principles of mechanics are formulated, illustrated, and discussed. In the introduction, the general historical path of the development of mechanics and its pioneers is described with an emphasis on mechanical modeling of planetary motion and Newtonian mechanics. Three of Newton's laws are formulated and discussed. The definitions of force and mass are given, and then a classification of forces is introduced. Next, the principles of mechanics are given and briefly discussed. In addition, the impact of classical mechanics on electrodynamics (Maxwell's equations) and relativistic mechanics (Einstein's theories) is presented. Kinetic units, and in particular the principal SI units, are introduced and discussed. In Sect. 1.2, D'Alembert's principle is introduced, illustrated, and discussed. In Sect. 1.3, the principle of virtual work is derived and illustrated. In Sect. 1.4, the increment and the variation of a function are presented, and two examples supporting the introduced theoretical background are given.

Chapter 2 is devoted to statics. First, a concept of equilibrium is introduced, including the formulation of several theorems. The notations of moment of force about a point and about an axis are illustrated, statically determinate and indeterminate problems are defined, and the "freezing principle" is described. In Sect. 2.2, the geometrical equilibrium conditions of a plane force system are introduced and discussed. The force polygon and funicular polygon are introduced and illustrated through two examples. In Sect. 2.3, the geometrical conditions for a space system are formulated. In particular, Poinsot's method is described. Analytical equilibrium conditions are given in Sect. 2.4. The three-moments theorem is formulated and proved. Two reduction invariants as well as the fields of forces are defined. Two theorems regarding mechanical systems with parallel forces are formulated and proved, among others. In Sect. 2.5, mechanical interactions, constraints, and supports are described, and examples of supports carrying three-dimensional systems of forces are graphically illustrated. In addition, three computational examples are shown. Reductions of a space force system to a system of two skew forces (Sect. 2.6) and to a wrench (Sect. 2.7) are carried out. Theoretical considerations are supported by figures and examples, and two reduction invariants are also introduced. In Sect. 2.8, the phenomenon of friction is described including limiting and fully developed friction forces, three Coulomb laws, sliding friction, and rolling resistance. Three illustrative examples are also given. The paradoxical behavior of bodies coupled by friction is presented and discussed in Sect. 2.9, which includes the following examples: (a) supply of energy by means of friction; (b) Coulomb friction as a force exciting rigid body motion; (c) Coulomb friction as viscous damping. Two different approaches to the Euler formula derivation are shown in Sect. 2.10 while dealing with friction of strings wrapped around a cylinder. Friction models are presented and discussed in Sect. 2.11. First, some friction models used in research and frequently found in the literature are presented, and then particular attention is paid to the so-called CCZ (Coulomb–Contensou–Zhuravlev) three-dimensional friction model. Some computational examples putting emphasis on the coupling of sliding and rotation of a body moving on an inclined plane are included.

Chapter 3 is devoted to the geometry of masses. In Sect. 3.1, basic concepts are introduced including mass center, moment of inertia of a system of particles with respect to a plane (an axis), polar moment of inertia. Four illustrative computational examples are provided. Centroids of common shapes of areas, lines, and volumes are given in tables. Two Pappus–Guldinus rules are formulated and supplementary examples of their applications are provided. In Sect. 3.2, the moments of inertia (second moments) are discussed, and formulas for their values for common geometric figures and three-dimensional bodies are included in tables. The inertia matrix and its transformations are discussed in Sect. 3.3. Steiner theorems are formulated, and an illustrative example is provided. In Sect. 3.4, principal axes and principal moments on a plane are defined, and in particular the plane inertia circle (Mohr's circle) is discussed. The inertia tensor, principal axes of inertia, and ellipsoid of inertia are the theme of Sect. 3.5. In particular, an ellipsoid of inertia of a body, principal axes of inertia, invariants of an inertia tensor, and inertia triangle inequalities are illustrated and discussed. Section 3.6 presents the properties of principal and principal centroidal axes of inertia, whereas Sect. 3.7 addresses problems related to the determination of moments of inertia of a rigid body.

The kinematics of a particle, the curvilinear and normal coordinates, and kinematic pairs and chains constitute the focus of Chap. 4. In Sect. 4.1, the motion of a particle and trajectory (path) of motion are defined including its velocity and acceleration. Section 4.2 deals with selected problems of plane motion of a particle putting emphasis on circular, rectilinear and curvilinear motion and the vectorial approach. Section 4.3 focuses on introduction and application of rectangular and curvilinear coordinates in space. It includes the classification of a particle's motion with respect to acceleration. In Sect. 4.4, the concept of natural coordinates is illustrated and discussed. The motion of a radius vector and rectangular and curvilinear coordinates in space are introduced and their properties and applications described. Both vector and tensor notations are used, and in particular covariant and contravariant unit vectors are applied. Orthogonal and orthonormal bases are introduced, and then displacement, velocity, and acceleration components are defined via the kinetic energy of a particle. Unit vectors and their first derivatives are derived for rectangular, cylindrical, and spherical coordinate systems. The position, velocity, and acceleration of a particle in rectangular, cylindrical, spherical, and arbitrary curvilinear coordinates are reported. Two illustrative examples are given. Natural coordinates (velocities and accelerations, the Darboux vector, torsion of a curvature, Frenet–Serret formulas, and examples) are studied in Sect. 4.4. Kinematic pairs and chains, joint variables, and the Denavit–Hartenberg convention are considered in Sect. 4.5. Definitions of a class of kinematic pairs, a mechanism, and a group are introduced, and low-order kinematic pairs are presented in a table. Finally, Sect. 4.6 provides a classification of problems in kinematics.

In Chap. 5, the kinematics of a rigid body and the composite motion of a point (particle) are studied. In Sect. 5.1, translational and rotational motions are considered. The study includes the movement of a rigid body in a three-dimensional space and the definition of degree of freedom. In particular, angular velocities and angular accelerations as vectors and the notation of a vector of small rotation are illustrated. Planar motion is studied in Sect. 5.2. It is demonstrated that planar motion can be treated as a composition (geometric sum) of translational and rotational motions, and the first of Euler's theorem is formulated. Moving and fixed centrode concepts are illustrated and discussed. Two theorems regarding the instantaneous points of zero velocity (point C) and zero acceleration (point S) are formulated and then proved. Various vector methods of velocity and acceleration determination on the basis of a point's planar motion are presented and illustrated analytically and geometrically. Burmester's theorems are formulated and their applications are illustrated. A few illustrative examples are provided. In Sect. 5.3 (5.3), a composite point motion in three-dimensional (two-dimensional) space is illustrated and analyzed. The general motion of a rigid body in three-dimensional space is studied in Sect. 5.5. Following a brief introduction, the angular velocity and angular acceleration of a rigid body are defined using both vector and tensor calculus. Euler's proposal is introduced rigorously, and Euler's formula is derived. In particular, the Eulerian angles are illustratively introduced and explained step by step. A rotation matrix that is a product of three matrices of elementary rotations is derived, and their elements are explicitly given. The matrix's non-commutivity is discussed, among other topis. An important theorem is formulated and proved. In Sect. 5.5.4, Eulerian angles are introduced, whereas in Sect. 5.5.5, Euler's kinematic equations are formulated, and both Euler's angles and angular velocities are derived and graphically presented. The displacement of a rigid body with one point fixed is studied in a subsequent section. Several basic theorems regarding the displacement and rotation of a rigid body are formulated in Sect. 5.5.7. In the subsequent short subsections, the parallel translation and rotation of a rigid body and a homogenous transformation, kinematic states of a rigid body's velocity and acceleration in translational motion and in motion about a point are illustrated and studied. In particular, rotational, centripetal, and normal accelerations are analyzed. Then, in Sects. 5.5.12 and 5.5.13, the velocities and accelerations in the motion of a body about a fixed axis and in various coordinate systems are studied. The velocities of a point of a rigid body in various coordinate systems are considered in Sect. 5.5.14, the regular precessions of a rigid body are analyzed in Sect. 5.5.15 (two additional examples support the introduced theoretical considerations). Screw motion is illustrated and discussed in Sect. 5.5.16, whereas the geometrical interpretation of velocity and acceleration of a point of a rigid body in general motion is given in Sect. 5.5.17. The latter subsection also includes a few important theorems, sometimes rigorously proved. The composite motion of a rigid body is analyzed in Sect. 5.6. It consists of the composition of two instanteneous translational (rotational) motions, a couple of instanteneous rotations, and the composition of rotational motions of a rigid body about intersecting axes. Two theorems are formulated and proved, among others.

In Chap. 6, the kinematics of a deformable body is studied. In Sect. 6.1, the role of tensor notation in mechanics is described. In Sect. 6.2, particular attention is paid to the stress tensor. Symmetric and asymmetric tensors are introduced, and their actions are explained analytically and geometrically. Elastic body deformation is studied via introduction of a tensor governing the velocity of body deformation. Two simple examples of theoretical considerations are provided. The definition of Clapeyron systems is introduced, and then Betti's and Castigliano's theorems are formulated and proved.

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Łódź and Darmstadt Jan Awrejcewicz

Contents

Chapter 1
Fundamental Principles of Mechanics

1.1 Introduction

Mechanics is a branch of physics. In general, mechanics allows one to describe and predict the conditions of rest or movement of particles and bodies subjected to the action of forces. Aristotle[1] was among the first scholars to introduce the term mechanics. At first, the development of mechanics was related to that of knowledge about the modeling of the Universe. Plato,[2] Eudoxus,[3] and Aristotle are among the creators of the homocentric system, whereas Apollonius,[4] Hipparchus,[5] and Ptolemy[6] created the epicyclic system. The theory they developed, according to which the motionless Earth is the center of the Universe, is called the geocentric theory. As was mentioned previously, Ptolemy, an Alexandrian scholar, was the originator of this theory. He based his ideas on the works of Hipparchus, one of the greatest astronomers of antiquity who explained the complexity of the motion of planets while retaining the central location of the Earth and introducing the combination of circular motions. The geocentric theory is based on the assumptions that the immovable Earth is located at the center of the Universe and that other celestial bodies shaped like spheres revolve around Earth, moving uniformly in circular orbits.

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[1] Aristotle (384–322 BC), Greek philosopher (Plato's student).

[2] Plato (428–348 BC), Greek philosopher and mathematician.

[3] Eudoxus of Cnidus (408–355 BC), Greek philosopher, astronomer and mathematician.

[4] Apollonius of Perga (260–190 BC), Greek mathematician and astronomer (focused on conics and the movement of the Moon).

[5] Hipparchus of Nicaea (190–120 BC), Greek astronomer, mathematician, and geographer; considered a precursor of astronomy.

[6] Claudius Ptolemaeus (100–168), Greek mathematician, astronomer, and geographer; one of the creators of the geocentric theory.

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Aristotle was the unquestioned authority in the domain of philosophy and mechanics; nonetheless, he made a fundamental error that adversely affected the development of mechanics. First of all, he assumed that the laws governing the motion of bodies are different for the Earth than for other celestial bodies. It was only Galileo Galilei[7] who, over twenty centuries later, pointed out the incorrectness of the Aristotle's viewpoint.

The heliocentric model, in which the Sun is the center of the world, was introduced by Nicolaus Copernicus[8] in his fundamental work *De revolutionibus orbium coelestium* (*On the Revolutions of Heavenly Spheres*). This view was subsequently modified by Giordano Bruno,[9] who maintained that the solar system was but one of an infinite number of such systems in the Universe.

Problems connected with the motion of bodies were raised for the first time by Galileo Galilei, a dedicated proponent of Copernicus's theory. To Galileo is also attributed the discovery of the law of the pendulum (1583) and the law of freely falling objects (1602).

A great contribution to the development of mechanics was made by Johannes Kepler,[10] who formulated the following three laws of planetary motion on the basis of empirical observations previously made by Tycho Brahe.[11]

1. All the planets move in elliptical orbits with the Sun at one focus.
2. The position vector of any planet attached at this focus of an orbit where the Sun is located sweeps equal areas in equal times.
3. The squares of the orbital periods of the planets are proportional to the cubes of the semimajor axes of their orbits.

Kepler's three laws served as the foundation of the mechanics of Isaac Newton,[12] who assumed that space was homogeneous and isotropic and that phenomena are uniform with respect to the choice of the time instant. The equations derived by Newton are invariant with respect to Galilean transformation. Classical mechanics is also called Newtonian mechanics.

From Newton's point of view (Newtonian mechanics) *time*, *space*, and *mass* are absolute attributes that are independent of each other. These concepts cannot be a priori defined and are rather motivated by our intention and experience. The concept of mass allows us to compare the behavior of bodies. For instance, we say that two bodies have the same mass if they are attracted by the Earth in the same manner and they exhibit the same resistance to changes in translational motion.

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[7]Galileo Galilei (1564–1642), Italian philosopher, astronomer, astrologer, and physicist who acknowledged the supremacy of the heliocentric theory of Copernicus.

[8]Nicolaus Copernicus (1473–1543), Polish astronomer and mathematician, creator of the heliocentric theory.

[9]Giordano Bruno (1548–1600), Italian Catholic cleric, philosopher.

[10]Johannes Kepler (1571–1630), German mathematician, astronomer, and physicist.

[11]Tycho Brahe (1546–1601), Danish astronomer.

[12]Isaac Newton (1642–1727) English physicist, mathematician, philosopher, and astronomer.

A point mass (particle) position and a body position require an introduction of the concept of space. It is necessary first to define an event. Newton also introduced the concept of force. It may depend on the mass of the body on which it acts and on changes in the velocity of the body over time. Therefore, force cannot be treated as an absolute, independent attribute of mechanics.

Mechanics can also be defined as the *science of the motion of bodies*. Instead of using real objects, mechanics makes use of their *models*. In general, the model of a given object (body) is an image reflecting only those attributes of the object that are essential to investigate the phenomena of interest for a particular branch of science. To the basic models applied in mechanics belong the following ones:

A particle (material point): A body possessing mass but having such small dimensions that it can be treated as a point in a geometric sense. However, in practice, bodies whose angular velocities are zero by assumption or whose rotational motion can be neglected are treated as particles regardless of their dimensions;

A system of particles: A collection of particles;

A rigid body: The distances between elements of such a body remain constant for arbitrarily large magnitudes of forces acting on the body.

In reality, structures, machines, and mechanisms are deformable bodies. However, usually their deformations are small, and hence in many cases their effect on the statics/dynamics of the studied bodies can be neglected.

A system of rigid bodies: A collection of rigid bodies.

The laws of mechanics introduced by Newton serve to illuminate the motions of material systems. They enable us to create a *mathematical model*, that is, to formulate *equations of motion* of particles and bodies.

The main goal of mechanics is to formulate the laws of motion suitable for the investigation of a variety of real bodies. It turns out that any real body, solid, liquid, or gaseous, can be modeled as a collection of particles. The following branches of mechanics deal with problems in the previously mentioned fields:

1. *Mechanics of rigid bodies* (statics and dynamics).
2. *Mechanics of deformable bodies* (strength of materials, elasticity theory, plastic theory, or rheology).
3. *Mechanics of fluids*: Incompressible (mechanics of liquids) and compressible (mechanics of gases, aeromechanics); the mechanics of incompressible fluids such as water is known as *hydraulics*.

In technical mechanics, during the modeling process we deal with the geometry (decomposition) of mass and the description of materials from which bodies are formed. In rigid-body mechanics, we assume that the distance between any two points of a body does not change. We can talk about a completely different problem when there is a possibility of changing the distance between the points of a body. The load of bodies in this last case leads to the change in the distance between body atoms, and interatomic forces (internal) will balance the external load. Bodies and material systems made of metal as encountered in technology have regular structures of arranged atomic networks on the order of 10^{30}. With regard to the large amount of atoms, analysis is performed on the micro scale, which leads to the averaging of anisotropy of microcrystal systems. Generally, most technical materials, after having a cubicoid cut out of them with sides of around 10^{-3} m, have the same properties irrespective of the orientation of the "cutting out," and such materials are called isotropic (of the same direction). There are also anisotropic materials (of different directions) in technology whose enduring properties depend on the orientation in which the cube of material is cut out (e.g., rolled plates, timber, fabrics, and paper).

The laws originally formulated by Newton generated a set of other fundamental laws of mechanics such as the conservation of linear momentum, the conservation of angular momentum, and the conservation of kinetic energy.

Below are the laws formulated by Newton, which are valid for particles.

First law. A body at rest not acted upon by an external force (the resultant force acting on a particle is zero) will remain at rest, and a body in motion moving at a constant speed along a straight line will remain in motion unless acted upon by an external force.

Second law. The acceleration of a particle is proportional to the net force acting on the particle; the direction and the sense of acceleration are identical to those of the force.

Third law. The mutual forces of action and reaction between two bodies are equal, opposite, and collinear.

The first two laws are true in an inertial system, whereas the third law is binding in any system. It can be shown that Newton's first law is a particular case of his second law.

It should be noted that Newton's laws are based on a concept of *force* as a vector quantity. Force appears here as a primitive notion and requires the introduction of at least two bodies. Correlation of reactions between bodies results from Newton's third law, where the action (forces) causes immediate reaction, which is graphically characterized by the description presented by Newton: "If I put pressure with a finger upon a stone with a certain force, then the stone also puts pressure upon my finger with the same force." The interaction of bodies can be implemented by the direct pressure of one body on another or by indirect reaction at a distance.

The latter case is connected with Newton's law of gravitation since, if we consider two particles of masses m_1 and m_2, the gravity force $\mathbf{F}_{12}$ with which the particle of the mass m_2 attracts the particle of the mass m_1 is given by

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{r^3}\mathbf{r}_{12}, \tag{1.1}$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{Nm^2\,kg^{-2}}$, and $\mathbf{r}_{12}$ is a vector joining these two points and directed from point 1 toward point 2.

The gravitational constant G is used for describing the gravitational field and was determined for the first time by Henry Cavendish.[13] It should be noted, however, that there exists a certain arbitrariness in the definition of force. Nobel laureate Richard Feynman[14] draws attention to the fact that the definition of force in a strict sense is difficult. This is due to the approximate character of Newton's second law and generally due to the approximate character of the laws of physics.

A concept of mass can also be introduced based on Newton's second law. Let us consider an arbitrary particle and apply to it, in turn, forces of various magnitudes $\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3, \ldots,\ \mathbf{F}_N$. Each of the forces produces motion of the particle with accelerations $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3, \ldots,\ \mathbf{a}_N$, respectively. These accelerations, according to Newton's second law, are proportional to the magnitudes of the forces, i.e.,

$$\frac{\mathbf{F}_1}{\mathbf{a}_1} = \frac{\mathbf{F}_2}{\mathbf{a}_2} = \frac{\mathbf{F}_3}{\mathbf{a}_3} = \cdots = \frac{\mathbf{F}_N}{\mathbf{a}_N}. \tag{1.2}$$

The foregoing ratios describe the *inertia* of a body (particle) and define the *mass* of the body. Recall that the *weight* of a body is a product of body mass m and acceleration of gravity $\mathbf{g}$. The mass defined in that way is called a *gravitational mass*. Empirical research conducted by Hungarian physicist Roland Eötvös[15] proved that the *inertial* mass (defining the inertia of a particle) and the *gravitational* mass (being a measure of the gravitation) are identical. In other words, if we take a particle located on the Earth's surface, then we may use (1.1) to define the weight $\mathbf{G}$ of a particle of mass m. That is, introducing $r = R$ (R is now the Earth's radius) and introducing $g = \frac{Gm_1}{R^2}$, $m_2 = m$, the weight of a particle of mass m is $\mathbf{G} = m\mathbf{g}$. Observe that R depends on the particle elevation and on its latitude (the Earth is not perfectly spherical), and hence the value of g varies with the particle position.

Newton's second law can be formulated in the following form:

$$m\mathbf{a} = \mathbf{F}. \tag{1.3}$$

Newton's third law is also known as the *law of action and reaction*. It is valid both for bodies in contact and for bodies interacting at a distance ($\mathbf{F}_{12} = -\mathbf{F}_{21}$).

Finally, it should be noted that Newton's three laws were presented in a modified form. Newton's original text from his 1687 work *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*) is slightly different. For instance, Newton does not use the notion of a particle but that of a body. The concept of force was defined by him through a series of axioms and not in vector notation.

It is worth emphasizing that historically the concept of force was a very subjective notion as it was connected with the individual sensation of the exertion of muscles. Thanks to the efforts of Newton and other scholars, the concept of force

[13] Henry Cavendish (1731–1810), British physician and chemist.

[14] Richard Feynman (1918–1988), American physicist and creator of quantum electrodynamics.

[15] Roland Eötvös (1848–1919), Hungarian mathematician and physicist.

obtained its objective character. Nowadays, one can even observe certain feedback, i.e., through an objective understanding of force, scientists seek to deepen fully the notion of the so-called biological force connected with the ability of the muscular nervous system to, e.g., lift (lowering) material objects [1, 2]. In this case the force depends on the properties of fast twitch and slow twitch of muscle fibers as well as age, sex, etc.

Forces can be divided into several classes:

1. Mass (gravitational and inertial).
2. Surface and volumetric (pressure and hydrostatic pressure).
3. Electromagnetic and electrostatic.
4. Muscular (of humans or animals).
5. Contact: Compressive, acting on a surface or along a line.
6. Tensile: Such as the forces in threads, cables, strings.
7. Passive (reactive), i.e., counteracting the active forces.
8. External and internal.
9. Interaction of bodies.

Apart from the described laws, it is possible to introduce several *principles* of mechanics. While the laws describe relationships between mechanical quantities often leading to solutions (e.g., through the first integrals of *momentum, angular momentum*, or *energy*), the principles only support the formulation of equations of motion. The principles possess the value of universality since they can be applied, for example, in the theory of relativity, quantum mechanics, and some branches of physics. One can divide them into *differential principles* and *integral principles*. The principles used in classical mechanics are a part of so-called *analytical mechanics*.

The principle of *independent force of action* is a generalization of Newton's second law. If several forces act upon a particle, the acceleration of this particle is a result of a geometric sum of the accelerations produced by each of the forces acting separately (superposition principle).

Let us recall, finally, that the description of the behavior of electromagnetic fields introduced by Maxwell's[16] equations was in disagreement with Newton's idea of particle motion. It turned out that electromagnetic waves could propagate in a vacuum. This contradicts a purely mechanical approach whereby waves can propagate only in a material medium filling up space. Moreover, Maxwell's equations were invariant with respect to the Lorentz[17] transformation, whereas Newton's equations are invariant with respect to the Galilean transformation.

Albert Einstein[18] succeeded in resolving that problem thanks to the introduction of the so-called special theory of relativity in 1905. He introduced space-time as an *invariant quantity*, creating the foundations of so-called *relativistic*

[16] James Clerk Maxwell (1831–1879), Scottish mathematician and physicist.

[17] Hendrik Antoon Lorentz (1853–1928), Dutch physicist and Nobel laureate.

[18] Albert Einstein (1879–1955), distinguished German physicist, creator of the special and general theories of relativity.

mechanics. In this way, two deductive systems became unified, i.e., mechanics and electrodynamics (relativistic mechanics, like electrodynamics, is invariant with respect to the Lorentz transformation). In relativistic mechanics, space, time, and mass depend on each other and cannot be treated as *absolute independent attributes*.

Fortunately, the differences between relativistic mechanics and Newton's mechanics appear at particles speeds close to the speed of light or in the analysis of large distances. Neither of these cases will be considered in this book.

The four fundamental concepts of classical mechanics discussed so far, i.e., *space*, *time*, *mass*, and *force*, allow us to introduce the so-called *kinetic units*. However, in order to satisfy Newton's second law they cannot be taken arbitrarily, and they will be further referred to as *base units*. The remaining fourth unit will be referred to as a *derived unit*. Then the kinetic units will create the so-called *consistent system of units*. In what follows we further address only the universal system of units (SI units). In this system the base units are the units of length (meter, m), mass (kilogram, kg), and time (second, s). A meter is here defined as 1650763.73 wavelengths of orange-red light corresponding to a certain transition in an atom of krypton-86 (originally defined as one ten-millionth of the distance from the equator to either pole). A kilogram is equal to a mass of $10^{-3}\,\mathrm{m}^3$ of water, and the mass of a platinum–iridium standard kilogram is kept at the International Bureau of Weights and Measures in Sèvres in France. A second is defined as the radiation corresponding to the transition between two levels of the fundamental state of the cesium-133 atom (originally defined to represent $\frac{1}{86{,}400}$ of the mean solar day). Equation (1.3) yields the derived unit of force $1\mathrm{N} = 1\ \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$. The *weight* of a body or the *force of gravity* is $\mathbf{G} = m\mathbf{g}$, and for a body with a mass of 1 kg, its weight is 9.81 N. There are numerous multiples and submultiples of the fundamental SI units as follows: 10^{12} (tera-T), 10^9 (giga-G), 10^6 (mega-M), 10^3 (kilo-k), 10^2 (hecto-h), 10 (deka-da), 10^{-2} (deci-d), 10^{-2} (centi-c), 10^{-3} (milli-m), 10^{-6} (micro-μ), 10^{-9} (nano-n), 10^{-12} (pico-p), 10^{-15} (femto-f), 10^{-18} (atto-a). For instance $1\,\mathrm{km} = 1{,}000\,\mathrm{m}$, $1\,\mu\mathrm{m} = 10^{-6}\,\mathrm{m}$, $1\,\mathrm{Mg} = 1{,}000\,\mathrm{kg}$, $1\,\mathrm{g} = 10^{-6}\,\mathrm{kg}$, $1\,\mathrm{MN} = 10^6\,\mathrm{N}$, etc.

In the case of time units, we have the minute (min) and the hour (h), and $1\,\mathrm{min} = 60\,\mathrm{s}$, whereas $1\,\mathrm{h} = 60\,\mathrm{min}$.

We may introduce also units of area and volume. The *square meter* (m^2) is the unit of area representing the area of a square of side 1 m. The *cubic meter* is the unit of volume equal to the volume of a cube of side 1 m.

In general, the following principal SI units are applied in mechanics:

1. Acceleration ($\frac{\mathrm{m}}{\mathrm{s}^2}$).
2. Angle [radian (rad)].
3. Angular acceleration ($\frac{\mathrm{rad}}{\mathrm{s}^2}$).
4. Angular velocity ($\frac{\mathrm{rad}}{\mathrm{s}}$).
5. Area (m^2).
6. Density ($\frac{\mathrm{kg}}{\mathrm{m}^3}$).
7. Energy and work [Joule (J)].
8. Force [Newton (N)].

9. Frequency [Hertz (Hz)].
10. Impulse [Newton · second ($\mathrm{N \cdot s = kg \cdot \frac{m}{s}}$)].
11. Length [meter (m)].
12. Mass [kilogram (kg)].
13. Moment of a force [Newton · meter (N·m)].
14. Power [Watt ($\mathrm{W = \frac{J}{s}}$)].
15. Pressure and stress [Pascal ($\mathrm{Pa = \frac{N}{m^2}}$)].
16. Time [second (s)].
17. Velocity [meter per second ($\mathrm{\frac{m}{s}}$)].
18. Solid volume [cubic meter ($\mathrm{m^3}$)].
19. Liquid volume [liter ($\mathrm{10^{-3}\, m^3}$)].

In general, in classical mechanics one may adhere to the following fundamental steps yielding the solution to a stated (given) problem. First, one needs to define the statement of a problem clearly and precisely. Diagrams indicating the force acting on each body considered known as *free-body diagrams* should be constructed. Then the fundamental principles and laws of mechanics should be used to derive the governing equations holding the condition of statics (rest) or dynamics (motion) of the bodies studied.

The short historical outline of the development of mechanics presented above reveals its deep roots in ancient times, and the reader will not make the mistake of thinking that only today in the field of general mechanics do many coursebooks and monographs exist. It is almost impossible to present a complete bibliography in the field of classical mechanics. Therefore, a few sources in English are given to make the book more readable, especially for students. Therefore, no attempt was made to provide an exhaustive list of references; only those works are included that were either used by the author [3–12] or are important competitors to this book [13–25].

1.2 D'Alembert's Principle

Let us consider a constrained material system (subjected to constraints) consisting of particles, described by the following equations of motion based on Newton's second law:

$$m_n \mathbf{a}_n = \mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R, \quad n = 1, \ldots, N, \tag{1.4}$$

where $\mathbf{F}_n^e$, $\mathbf{F}_n^i$, and $\mathbf{F}_n^R$ denote, respectively, external forces, internal forces, and reactions, which follow directly from Newton's second law. Every particle, numbered n, can be subjected to the action of forces $\mathbf{F}_n^i$ coming from other (even all) particles of the considered system of particles. The external forces $\mathbf{F}_n^e$, in turn, represent the action of the environment on our material system isolated from that environment or from other isolated system parts.

If $\mathbf{F}_n^e = \mathbf{0}$ (absence of external influence), then such a system in mechanics is known as *autonomous (isolated)*. Moreover, in a general case, a system of particles (SoP) can be *free* or *constrained*. The reaction forces $\mathbf{F}_n^R$ are reactions of the *constraints*, that is, of the restrictions imposed on the particles, i.e., on their displacements and velocities. By the free system we will understand either the SoP on which the *constraints* are not imposed or one for which the reaction of the constraints can be determined *explicitly* in the form of reaction forces, i.e., they will not require solving additional so-called *equations of constraints*, and then the forces $\mathbf{F}_n^R$ can be treated as $\mathbf{F}_n^e$. Otherwise, SoP will be called *constrained*. The forces that occur on the right-hand side of (1.4) and concerning material point n in a general case may depend on the position and velocity of other particles of the SoP as well as explicitly on time, i.e., $\mathbf{F}_n^e = \mathbf{F}_n^e(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$, $\mathbf{F}_n^i = \mathbf{F}_n^i(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$, $\mathbf{F}_n^R = \mathbf{F}_n^R(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$.

Let every particle undergo a *virtual displacement* $\delta\mathbf{r}_n$, where $\mathbf{r}_n$ is a *radius vector* of the particle n. Multiplying (scalar product) (1.4) by $\delta\mathbf{r}_n$ and adding by sides, we obtain

$$\sum_{n=1}^{N}(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R - m_n\mathbf{a}_n) \circ \delta\mathbf{r}_n = 0. \tag{1.5}$$

Assuming that only ideal constraints are considered, which by definition satisfy the relation

$$\sum_{n=1}^{N}\mathbf{F}_n^R \circ \delta\mathbf{r}_n = 0, \tag{1.6}$$

(1.5) will take the form

$$\sum_{n=1}^{N}(\mathbf{F}_n^e + \mathbf{F}_n^i - m_n\mathbf{a}_n) \circ \delta\mathbf{r}_n = 0. \tag{1.7}$$

The equation just obtained enables us to formulate *d'Alembert's principle*, which reads:

The sum of scalar products of virtual displacements and external forces, internal forces and vectors $(-m_n\mathbf{a}_n)$ *of particles of a material system equals zero.*

One may conclude from (1.7) that d'Alembert's principle transforms a problem of dynamic equilibrium to that of a static equilibrium by adding the inertia force terms $(-m_n\mathbf{a}_n)$ and extends the *principle of virtual work* to dynamics.

Performing a projection of the vectors appearing in (1.7) on the axes of the adopted Cartesian coordinate system $(OX_1X_2X_3)$, we obtain

$$\sum_{n=1}^{N}\Big[(F_{nx_1}^e + F_{nx_1}^i - m_n\ddot{x}_{1n})\delta x_{1n} + (F_{nx_2}^e + F_{nx_2}^i - m_n\ddot{x}_{2n})\delta x_{2n} + (F_{nx_3}^e + F_{nx_3}^i - m_n\ddot{x}_{3n})\delta x_{3n}\Big] = 0, \tag{1.8}$$

where $\mathbf{a}_n = \ddot{x}_{1n}\mathbf{E}_1 + \ddot{x}_{2n}\mathbf{E}_2 + \ddot{x}_{3n}\mathbf{E}_3$, and $\mathbf{E}_i$, $i = 1, 2, 3$, are *unit vectors* of the coordinate system $OX_1X_2X_3$.

The equation just obtained is often called a *general equation of mechanics*. D'Alembert's principle and the general equation of mechanics are sometimes difficult in applications because they refer to coordinates of the particles. In Hamilton's and Lagrange's mechanics the introduced scalar energy functions allow one to omit the foregoing problem. Because we are considering a free system, all virtual displacements are independent. This means that the general equation is satisfied only if the expressions in brackets equal zero.

In this way we obtain three second-order differential equations of the following form:

$$\sum_{n=1}^{N}(F^e_{nx_1} + F^i_{nx_1} - m_n\ddot{x}_{1n}) = 0,$$

$$\sum_{n=1}^{N}(F^e_{nx_2} + F^i_{nx_2} - m_n\ddot{x}_{2n}) = 0,$$

$$\sum_{n=1}^{N}(F^e_{nx_3} + F^i_{nx_3} - m_n\ddot{x}_{3n}) = 0. \tag{1.9}$$

The preceding equations are simplified even more when the sum of internal forces equals zero taking the form of

$$\sum_{n=1}^{N}(F^e_{nx_1} - m_n\ddot{x}_{1n}) = 0,$$

$$\sum_{n=1}^{N}(F^e_{nx_2} - m_n\ddot{x}_{2n}) = 0,$$

$$\sum_{n=1}^{N}(F^e_{nx_3} - m_n\ddot{x}_{3n}) = 0. \tag{1.10}$$

The three equations above can be rewritten in the vector form

$$\sum_{n=1}^{N}(\mathbf{F}_{Bn} + \mathbf{F}^e_n) = \mathbf{0}. \tag{1.11}$$

It was assumed above that the force $\mathbf{F}_{Bn} = -m_n\mathbf{a}_n$. That force is also known as inertia force or d'Alembert's force acting on a particle n. Its sense is opposite to the active force $\mathbf{F}^e_n$.

Let us recall that position vectors $\mathbf{r}_n$ determine the position of material point n measured from the origin of the coordinate system. After vector premultiplication (cross product) of (1.11) by $\mathbf{r}_n$ we obtain

$$\sum_{n=1}^{N}(\mathbf{r}_n \times \mathbf{F}_{Bn} + \mathbf{r}_n \times \mathbf{F}^e_n) = \mathbf{0}. \tag{1.12}$$

Let us note that the sums of vector products occurring above represent the main moment of force vectors of the system of external forces $\mathbf{F}_n^e$ and of the system of inertia forces $\mathbf{F}_{Bn}$, that is,

$$\mathbf{M_O} = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^e), \tag{1.13}$$

$$\mathbf{M_{BO}} = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_{Bn}). \tag{1.14}$$

Let us introduce the notions of main force vector of external forces, inertia forces and reactions, and the main moment of a force vector of reaction in the following form:

$$\mathbf{F}^e = \sum_{n=1}^{N} \mathbf{F}_n^e, \quad \mathbf{F_B} = \sum_{n=1}^{N} \mathbf{F}_{Bn},$$
$$\mathbf{F}^R = \sum_{n=1}^{N} \mathbf{F}_n^R, \quad \mathbf{M_{RO}} = \sum_{n=1}^{N} (\mathbf{R}_n \times \mathbf{F}_n^r). \tag{1.15}$$

They were introduced to the system after being released from constraints. In this way the material system remains in equilibrium under the action of inertia forces, active forces and reactions, and the torques (moments of forces) due to the aforementioned forces only if

$$\mathbf{F}^e + \mathbf{F_B} + \mathbf{F}^R = \mathbf{0}, \tag{1.16}$$

$$\mathbf{M_O} + \mathbf{M_{BO}} + \mathbf{M_{RO}} = \mathbf{0}. \tag{1.17}$$

The obtained result [(1.16) and (1.17)] is summarized in the following principle:

A system of vectors consisting of inertia forces, external forces, reactions constraining the movement of this system, and their torques is equivalent to zero.

In the case of free systems (no constraints and therefore no reactions) (1.16) and (1.17) are reduced to

$$\mathbf{F}^e + \mathbf{F_B} = \mathbf{0}, \tag{1.18}$$

$$\mathbf{M_O} + \mathbf{M_{BO}} = \mathbf{0}. \tag{1.19}$$

Thus, we obtain the following principle for free material systems: *A system of external forces and torques produced by the forces acting on particles of a free material system is in every time instant balanced by a system of inertia forces and torques produced by these inertia forces.*

1.3 Principle of Virtual Work

Let us consider a material system composed of N particles at rest. Since the system is at rest, the accelerations of all its particles equal zero. From (1.5) we obtain

$$\sum_{n=1}^{N} \left(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R\right) \circ \delta \mathbf{r}_n = 0. \tag{1.20}$$

Because the scalar product of force and virtual displacement represents a virtual work of the force, (1.20) can be interpreted in the following way:

In an equilibrium position of a material system, the sum of virtual works of all external forces, internal forces, and reactions equals zero.

The foregoing principle was formulated on the basis of equilibrium equations and is a necessary condition of equilibrium. Now, let us assume that the forces acting on the system would do the work that would result in a change of kinetic energy ΔE_{kn} of every particle of the system. The kinetic energy would be created, however, from the work of the mentioned forces, and in view of that we have

$$\sum_{n=1}^{N} \left(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R\right) \circ \delta \mathbf{r}_n = \sum_{n=1}^{N} \delta E_{kn} = 0. \tag{1.21}$$

This means that the increment of the kinetic energy of the system of particles is zero, and therefore the system is not moving. That is a sufficient condition of equilibrium. Thus, the principle of virtual work shows the necessary and sufficient condition of system equilibrium. In the case of ideal constraints (the sum of works produced by reaction forces equals zero) and rigid systems (the sum of works produced by internal forces equals zero), the stated principle is simplified and takes the form of

$$\sum_{n=1}^{N} \mathbf{F}_n^e \circ \delta \mathbf{r}_n = \sum_{n=1}^{N} \left(F_{x_{1n}}^e \delta x_{1n} + F_{x_{2n}}^e \delta x_{2n} + F_{x_{3n}}^e \delta x_{3n}\right) = 0.$$

The principle of virtual work in this case reads:

In an equilibrium position of a material system, the sum of virtual works of all external forces through the virtual displacements allowed by kinematics (compatible with the constraints) of the system equals zero.

In applications, the foregoing principle has some advantageous consequences, which are listed below:

1. Reaction forces (for smooth surfaces without friction) and internal forces can be removed from consideration (this will be shown in Example 1.1).
2. The problem of statics, after application of that principle, can be solved as a problem of kinematics.

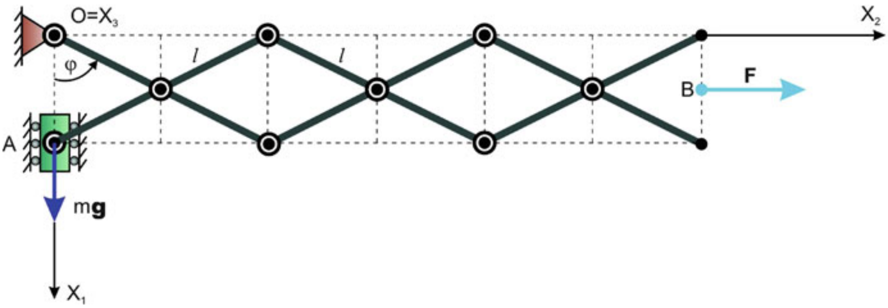


Fig. 1.1 Mechanism remaining in static equilibrium

3. The problem can be directly formulated in so-called *generalized coordinates* q_n of the form

$$\sum_{n=1}^{N} Q_n \delta q_n = 0.$$

It should be noted that in the case where the virtual work principle is related to d'Alembert's principle of the form (1.4), then item (3) is no longer valid since, in general, the latter cannot be directly formulated in terms of generalized coordinates, which makes its application much more difficult.

Example 1.1. Determine the magnitude of force $\mathbf{F}$ so that the flat mechanism depicted in Fig. 1.1 remains in static equilibrium, where the weight of slide block is denoted by $\mathbf{G} = m\mathbf{g}$.

After introducing the Cartesian coordinate system $OX_1X_2X_3$, the kinematics of points A and B is defined by the following equations:

$$x_{1A} = 2l \cos\varphi,$$

$$x_{2B} = 6l \sin\varphi.$$

According to the principle of virtual work and making use of the preceding geometric relations, we obtain

$$F\delta x_{2B} + mg\delta x_{1A} = 0.$$

Since

$$\delta x_{1A} = -2l \sin\varphi\delta\varphi,$$

$$\delta x_{2B} = 6l \cos\varphi\delta\varphi,$$

we have

$$(3F \cos\varphi - mg \sin\varphi)\delta\varphi = 0,$$

which holds true for an arbitrary $\delta\varphi$.

At the change of $\delta\varphi$ (clockwise or counterclockwise), the mechanism remains in static equilibrium when

$$F = \frac{1}{3} mg \tan\varphi. \qquad \square$$

1.4 Increment of a Function and Variation of a Function

In traditional mechanics textbooks, the presentation usually starts with a so-called *geometric* approach, based on the application of vector calculus and Newton's laws of momentum and angular momentum. Sometimes, however, it is virtually inconceivable how one should bring about the release from constraints and consider all internal and reaction forces for each single particle of a system composed of a large number of particles. Therefore, a natural question arises as to whether there exists a possibility of simplifying the problem provided that the considered system is in static or dynamic equilibrium and that internal forces in the considered system cancel each other (actions and reactions). It turns out that such a possibility exists based on the concepts of virtual work and virtual displacement, which were the subject of consideration in the previous section. In the present section, some basic information will be presented regarding a function variation in connection with the concept of virtual work, which is widely used in mechanics.

Let us first introduce the notion of virtual displacement (Fig. 1.2) after adopting the Cartesian coordinate system $OX_1X_2X_3$.

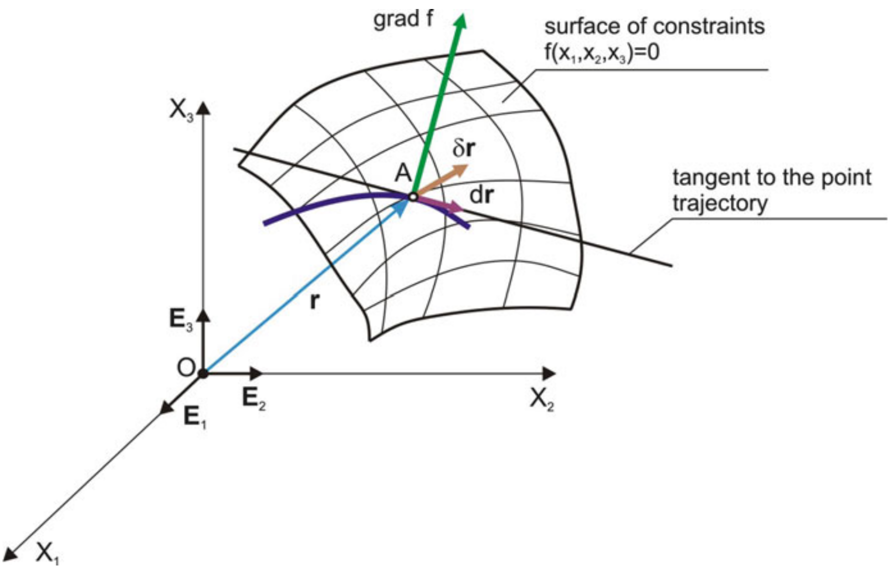


Fig. 1.2 Real ($d\mathbf{r}$) and virtual ($\delta\mathbf{r}$) displacement of a material point A

Let point A be moving in an arbitrary fashion on the surface of constraints around its position described by the radius vector $\mathbf{r}$. The arbitrariness regards both the displacement direction of point A and the length of a vector connecting the current position of point A with its position after a small displacement on the surface of constraints $f(x_1, x_2, x_3) = 0$ (a time-dependent surface can be considered as well). We shall note that the arbitrariness of the direction is associated with removal of the point motion dependency on acting forces, whereas the arbitrariness of the length $\delta\mathbf{r}$ means removing the dependency on time.

In reality, the elementary displacement of the particle takes place in the direction of the vector $d\mathbf{r}$ along the curve indicated in Fig. 1.2.

However, there exists a certain law defining the virtual displacement of point A. That point moves on the surface of constraints $f(x_1, x_2, x_3) = 0$, which means that

$$\operatorname{grad} f \circ \delta\mathbf{r} = 0, \tag{1.22}$$

where the vector

$$\operatorname{grad} f = \frac{\partial f}{\partial \mathbf{r}} = \sum_{i=1}^{3} \mathbf{E}_i \frac{\partial f}{\partial x_i} \tag{1.23}$$

is a gradient vector (normal to the surface of constraints) at the current position of the particle (for a "frozen" moment in time).

Observe that point A subjected to a virtual displacement, i.e., when its position is defined by a radius vector $\mathbf{r} + \delta\mathbf{r}$, also satisfies the equation of constraints.

Since we have

$$f(\mathbf{r} + \delta\mathbf{r}) = f(\mathbf{r}) + \frac{\partial f}{\partial \mathbf{r}} \circ \delta\mathbf{r} + O(\delta\mathbf{r})^2, \tag{1.24}$$

and because after displacement $\delta\mathbf{r}$ the point still lies on the surface of constraints, we have

$$f(\mathbf{r} + \delta\mathbf{r}) = 0. \tag{1.25}$$

Taylor's[19] expansion about point A, and on the assumption of a small variation $\delta\mathbf{r}$, showed that a point in a new position also satisfies equation of constraints (1.25). Moreover, the mentioned operation shows that $\frac{\partial f}{\partial \mathbf{r}} \equiv \operatorname{grad} f$ must be a vector since the result of the product $\frac{\partial f}{\partial \mathbf{r}} \circ \delta\mathbf{r}$ must be a scalar.

Let us note that a number of geometric constraints determines a number of additional conditions of the type (1.22) imposed on the considered material system.

According to the assumption that $\mathbf{E}_n$ are unit vectors of axes of the introduced coordinate system, we obtain

$$\delta\mathbf{r} = \sum_{n=1}^{3} \mathbf{E}_n \delta x_n. \tag{1.26}$$

[19] Brook Taylor (1685–1731), English mathematician.

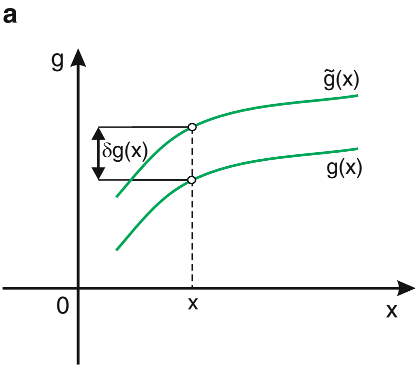


Fig. 1.3 Geometric interpretation of isochronous variation of a function (**a**) and derivative of a function (**b**) regarding point x

Let us note that $\mathbf{r} = \mathbf{r}(t)$ and the notion of virtual displacement was introduced for the "frozen" moment in time, that is, $\delta \mathbf{r}$ does not depend on time, as remarked earlier.

In a general case the notion of virtual displacement is connected with a mathematically motivated concept of function variation, which will be briefly recalled now based on the monograph [3]. At first, we will consider a so-called *isochronous* variation of a function

$$g = g(x). \tag{1.27}$$

The following function is called a variation of function $g(x)$:

$$\delta g(x) = \tilde{g}(x) - g(x) \tag{1.28}$$

where $\delta g(x) \ll 1$. It is shown that a function variation (in contrast to a function derivative) is calculated for a fixed x, whereas a function derivative about x makes use of an increment $x + \Delta x$ (Fig. 1.3).

A derivative of the variation of a function is as follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[\delta g(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[\tilde{g}(x) - g(x)] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\tilde{g}(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x) = \tilde{g}'(x) - g'(x), \end{aligned} \tag{1.29}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x}$.

Let us now introduce the notion of variation of derivative of $\tilde{g}'(x)$ and $g'(x)$. From the definition of isochronous variation we have

$$\delta g'(x) = \tilde{g}'(x) - g'(x) = \delta\left[\frac{\mathrm{d}\tilde{g}(x)}{\mathrm{d}x} - \frac{\mathrm{d}g(x)}{\mathrm{d}x}\right]. \tag{1.30}$$

By comparing (1.29) with (1.30) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[\delta g(x)] = \delta\left[\frac{\mathrm{d}\tilde{g}(x)}{\mathrm{d}x} - \frac{\mathrm{d}g(x)}{\mathrm{d}x}\right]. \tag{1.31}$$

This means that the derivative of an isochronous variation of a function is equal to the isochronous variation of a derivative of a function. Let us consider now a composite function of the form

$$f \equiv f(g, g', x). \tag{1.32}$$

For a fixed x we perform variations of the functions g and g' with values δg and $\delta g'$, respectively, which means that the function f will undergo the variation δf. Owing to (1.32) we have

$$\begin{aligned} f + \delta f &= f(g + \delta g, g' + \delta g', x) \\ &= f + \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial g}\delta g' + O\big((\delta g)^2 + (\delta g')^2\big). \end{aligned} \tag{1.33}$$

From (1.33) we obtain

$$\delta f = \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial g'}\delta g'. \tag{1.34}$$

Let us go back now to our function g and assume that this time we have

$$g = g(x, t). \tag{1.35}$$

Let us introduce now the following time variation:

$$\delta t = \tilde{t} - t. \tag{1.36}$$

A total variation $\delta^* g(x, t)$ of the function (1.35) can be determined from the equation

$$g + \delta^* g = g + \delta g + \frac{\mathrm{d}g}{\mathrm{d}t}\delta t, \tag{1.37}$$

which means that

$$\delta^* g = \delta g + \frac{\mathrm{d}g}{\mathrm{d}t}\delta t = \delta g + \dot{g}\delta t. \tag{1.38}$$

It can be shown easily that also in the case of an independent variable such as time, we have

$$\mathrm{d}(\delta t) = \delta(\mathrm{d}t). \tag{1.39}$$

Let us calculate now the total variation of a derivative of the function g with respect to time [see (1.38), where instead of g we take $\dot{g}$]:

$$\delta^* \dot{g} = \delta \dot{g} + \frac{\mathrm{d}\dot{g}}{\mathrm{d}t}\delta t = \delta \dot{g} + \ddot{g}\delta t. \tag{1.40}$$

After differentiation of (1.38) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta^* g\right) = \delta \dot{g} + \ddot{g}\delta t + \dot{g}\frac{\mathrm{d}(\delta t)}{\mathrm{d}t}. \tag{1.41}$$

Equations (1.40) and (1.41) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta^* g\right) = \delta^* \dot{g} + \dot{g}\frac{\mathrm{d}(\delta t)}{\mathrm{d}t}, \tag{1.42}$$

which means that *total* variation is not commutative with differentiation. Although the calculations were conducted regarding a scalar function, they are also valid for a vector-valued function.

Example 1.2. Three rigid bodies of masses m_i $(i = 1, 2, 3)$ are attached to a massless inextensible cable wrapped around three pulleys of negligible masses (Fig. 1.4). The bodies are in translatory motion. Determine the accelerations of the bodies.

Let us associate the virtual displacements δx_1, δx_2, and δx_3 with the respective coordinates. According to (1.7) we have

$$\sum_{i=n}^{3} (\mathbf{F}_n^e - m_n \mathbf{a}_n) \circ \delta \mathbf{r}_n = 0.$$

In the present case the weights of the bodies play the role of external forces

$$\mathbf{F}_1 = m_1 \mathbf{g}, \quad \mathbf{F}_2 = m_2 \mathbf{g}, \quad \mathbf{F}_3 = m_3 \mathbf{g},$$

and, moreover, $\delta \mathbf{r}_n = \delta x_n \mathbf{E}_n$, $n = 1, 2, 3$.

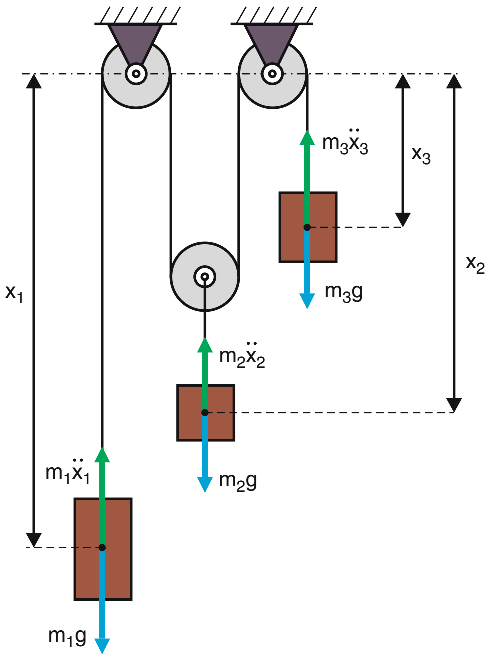
We obtain the following *equilibrium state equation*:

$$\left(m_1 g - m_1 \ddot{x}_1\right)\delta x_1 + \left(m_2 g - m_2 \ddot{x}_2\right)\delta x_2 + +\left(m_3 g - m_3 \ddot{x}_3\right)\delta x_3 = 0.$$

Because the cable is inextensible, we have

$$x_1 + 2x_2 + x_3 = C \equiv \text{const.}$$

Fig. 1.4 Three bodies of masses m_i, $i = 1, 2, 3$ in translatory motion



Let us assume that the coordinates x_1 and x_3 are independent. From the last equation we obtain

$$x_2 = -\frac{1}{2}(x_1 + x_3 - C),$$

that is,

$$\ddot{x}_2 = -\frac{1}{2}(\ddot{x}_1 + \ddot{x}_3),$$
$$\delta x_2 = -\frac{1}{2}\left(\delta x_1 + \delta x_3\right).$$

Substituting the preceding equation into the equilibrium state equation we have

$$m_1\left(g - \ddot{x}_1\right)\delta x_1 - \frac{1}{2}m_2\left[g + \frac{1}{2}(\ddot{x}_1 + \ddot{x}_3)\right](\delta x_1 + \delta x_3) + m_3\left(g - \ddot{x}_3\right)\delta x_3 = 0,$$

or equivalently

$$\delta x_1\left[g\left(m_1 - \frac{1}{2}m_2\right) - \ddot{x}_1\left(m_1 + \frac{1}{4}m_2\right) - \ddot{x}_3\frac{1}{4}m_2\right]$$
$$+ \delta x_3\left[g\left(m_3 - \frac{1}{2}m_2\right) - \ddot{x}_3\left(m_3 + \frac{1}{4}m_2\right) - \ddot{x}_1\frac{1}{4}m_2\right] = 0.$$

Since the virtual displacements δx_1 and δx_3 are independent, we obtain

$$\left(m_1 + \frac{m_2}{4}\right)\ddot{x}_1 + \frac{m_2}{4}\ddot{x}_3 = g\left(-\frac{m_2}{2} + m_1\right),$$
$$\left(m_3 + \frac{m_2}{4}\right)\ddot{x}_3 + \frac{m_2}{4}\ddot{x}_1 = g\left(-\frac{m_2}{2} + m_3\right).$$

The determinant of the foregoing system of equations is equal to

$$W = \begin{vmatrix} m_1 + \frac{m_2}{4} & \frac{m_2}{4} \\ \frac{m_2}{4} & m_3 + \frac{m_2}{4} \end{vmatrix} = m_1 m_3 + \frac{m_2}{4}(m_1 + m_3),$$

and the remaining determinants have the form

$$W_{\ddot{x}_1} = \begin{vmatrix} g\left(-\frac{m_2}{2} + m_1\right) & \frac{m_2}{4} \\ g\left(-\frac{m_2}{2} + m_3\right) & m_3 + \frac{m_2}{4} \end{vmatrix}$$
$$= g\left[-\frac{3}{4}m_2 m_3 + m_1\left(m_3 + \frac{1}{4}m_2\right)\right],$$

$$W_{\ddot{x}_3} = \begin{vmatrix} m_1 + \frac{m_2}{4} & g\left(-\frac{m_2}{2} + m_1\right) \\ \frac{m_2}{4} & g\left(-\frac{m_2}{2} + m_3\right) \end{vmatrix}$$
$$= g\left[-\frac{3}{4}m_1 m_2 + m_3\left(m_1 + \frac{m_2}{4}\right)\right].$$

Eventually, the desired accelerations are as follows:

$$\ddot{x}_1 = \frac{W_{\ddot{x}_1}}{W} = \frac{g\left[-\frac{3}{4}m_2 m_3 + m_1\left(m_3 + \frac{1}{4}m_2\right)\right]}{m_1 m_3 + \frac{m_2(m_1+m_3)}{4}},$$
$$\ddot{x}_3 = \frac{W_{\ddot{x}_3}}{W} = \frac{g\left[-\frac{3}{4}m_1 m_2 + m_3\left(m_1 + \frac{1}{4}m_2\right)\right]}{m_1 m_3 + \frac{m_2(m_1+m_3)}{4}}.$$ □

Example 1.3. A rod of weight **G** and length l is hinged at point O and a pin connected to a rod has length $2l$ and weight 2**G** (Fig. 1.5). Determine the configuration of rods as a result of the action of the weight forces (neglect the friction).

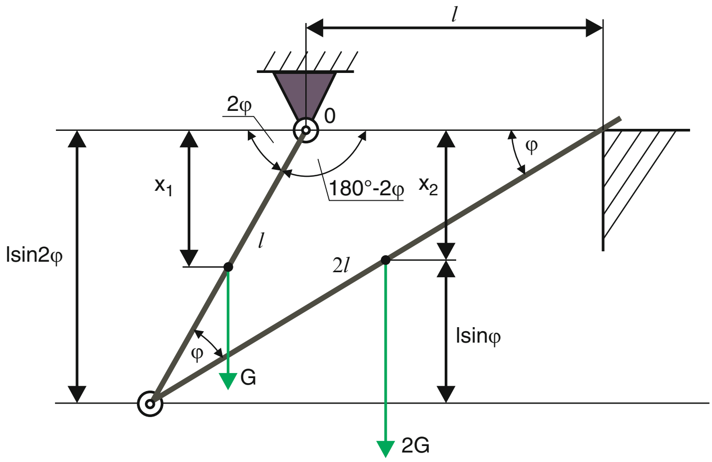


Fig. 1.5 Configuration of homogeneous rods of lengths l and $2l$ loaded with their weights

As a generalized coordinate we will take an angle φ because the analyzed system has one degree of freedom. In other words, a possible system movement can be described by only one coordinate φ. From Fig. 1.5 it follows that

$$x_1 = \frac{l}{2}\sin 2\varphi,$$

$$x_2 = l\sin 2\varphi - l\sin\varphi.$$

Because the constraints of the system are ideal, only the weight forces perform the work through the virtual displacements. The work done by the forces **G** and 2**G** through the displacements δx_n $(n = 1, 2)$ is equal to

$$G\delta x_1 + 2G\delta x_2 = 0.$$

A slight shake of the system, which remains in static equilibrium, will produce the displacements δx_1, δx_2 and change in the coordinate φ by $\delta\varphi$. We will determine the relations between δx_n and $\delta\varphi$ by applying the first two equations:

$$\delta x_1 = l\cos 2\varphi\delta\varphi, \quad \delta x_2 = (2\cos 2\varphi - \cos\varphi)l\delta\varphi,$$

and the obtained variations are substituted into the second equation, yielding

$$Gl\cos 2\varphi + 2Gl(2\cos 2\varphi - \cos\varphi) = 0.$$

Because

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1,$$

our problem is reduced to the following second-order algebraic equation:

$$\cos^2\varphi - \frac{1}{5}\cos\varphi - \frac{1}{2} = 0.$$

Solving the preceding quadratic equation we obtain

$$\cos\varphi_1 = 0.814, \quad \cos\varphi_2 = -0.614.$$

We select the physically feasible solution for which $\varphi = \varphi_1 \approx 35°30'$. □

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Chapter 2
Statics

2.1 A Concept of Equilibrium

If the velocity and acceleration of every particle in a material system are equal to zero, such a system is at rest. On the other hand, as we recall, from Newton's first law it follows that a particle having mass to which no force is applied or the applied forces are balanced is either at rest or in uniform motion along a straight line.

If a material system acted on by a force system does not change its position during an arbitrarily (infinitely) long time, we say that it is in equilibrium under the action of that force system.

Statics is the branch of mechanics that focuses on the equilibrium of material bodies (particles) under the action of forces (moments of force). It deals with the analysis of forces acting on material systems at rest or moving in uniform motion along a straight line and, as will become clear later, may be treated as a special case of *dynamics*. This chapter, devoted to statics, might be extended by material presented in books devoted to classical mechanics like, for instance, [1–25].

It turns out that for a material system to remain in equilibrium under the action of a certain force system, that force system must satisfy the so-called *equilibrium conditions*.

In order to emphasize the lack of time influence on equilibrium conditions, in statics the term *state of static equilibrium* is often used. In general, statics can be divided into *elementary statics* and *analytical statics*. In the case of elementary statics, during analysis of static equilibrium states vector algebra and graphical methods are applied. On the other hand, in the case of *analytical statics* the concepts of *virtual displacement* and *principle of virtual work* are used, which in part was discussed in Chap. 1.

Let us assume that the nth particle is subjected to K forces $\mathbf{F}_k$, L reactions $\mathbf{F}_l^R$, and M internal forces $\mathbf{F}_m^i$ (Fig. 2.1). The equilibrium condition has the following form:

$$\mathbf{F}_n + \mathbf{F}_n^R + \mathbf{F}_n^i = \mathbf{0}, \tag{2.1}$$

J. Awrejcewicz, *Classical Mechanics: Kinematics and Statics*, Advances in Mechanics and Mathematics 28, DOI 10.1007/978-1-4614-3791-8_2,

where

$$\mathbf{F}_n = \sum_{k=1}^{K} \mathbf{F}_k, \quad \mathbf{F}_n^R = \sum_{l=1}^{L} \mathbf{F}_l^R, \quad \mathbf{F}_n^i = \sum_{m=1}^{M} \mathbf{F}_m^i. \tag{2.2}$$

After multiplying both sides of (2.1) by the unit vectors $\mathbf{E}_j$ of the coordinate system $OX_1X_2X_3$ (scalar product) and taking into account (2.2), we obtain the so-called *analytical conditions of equilibrium* of the form

$$\sum_{k=1}^{K} F_{kx_j} + \sum_{l=1}^{L} F_{lx_j}^R + \sum_{m=1}^{M} F_{mx_j}^i = 0, \quad j = 1, 2, 3. \tag{2.3}$$

Because in general the forces may be projected onto the axes of any curvilinear coordinate system, the following theorem is valid.

Theorem 2.1. *A particle is in equilibrium if the sum of projections of external, internal, and reaction forces (acting on this particle) onto axes of the adopted coordinate system is equal to zero.*

The three conditions (2.3) are necessary, but not sufficient, for the particle to remain at rest, as follows from Newton's first law.

In order to formulate the equilibrium conditions for a whole material system or a body (infinite number of particles), one should formulate such equations for every particle $n \in [1, N]$ ($N = \infty$ in the case of the body) and add them together, obtaining

$$\mathbf{S} = \sum_{n=1}^{N} (\mathbf{F}_n + \mathbf{F}_n^R + \mathbf{F}_n^i) = \mathbf{0}, \tag{2.4}$$

where N is the number of particles of the material system.

A system of particles will be in equilibrium if the sum of projections of external, internal, and reaction forces, acting on every particle of the system, onto three axes of the adopted coordinate system is equal to zero. After projection we obtain $3N$ analytical equilibrium conditions for the system of N particles.

According to Newton's third law, the internal forces are the effect of action and reaction and they are *pairs of opposite forces*, which means that they cancel out one another, that is, $\sum_{n=1}^{N} \mathbf{F}_n^i = \mathbf{0}$.

Observe that the result of action of a given force on a rigid body remains unchanged when that force is applied at any point of its line of action (the so-called *principle of transmissibility*), and hence the forces acting on a rigid body can be represented by *sliding vectors*. In other words, if instead of a force $\mathbf{F}$ at a given point of the rigid body we apply a force $\mathbf{F}'$ of the same magnitude and direction at a point A' ($A \neq A'$), then the equilibrium state, or motion of a rigid body, is not affected provided that the two forces have the same line of action. The principle of transmissibility is based on experimental evidence, and the mentioned forces $\mathbf{F}$ and $\mathbf{F}'$ are called *equivalent*.

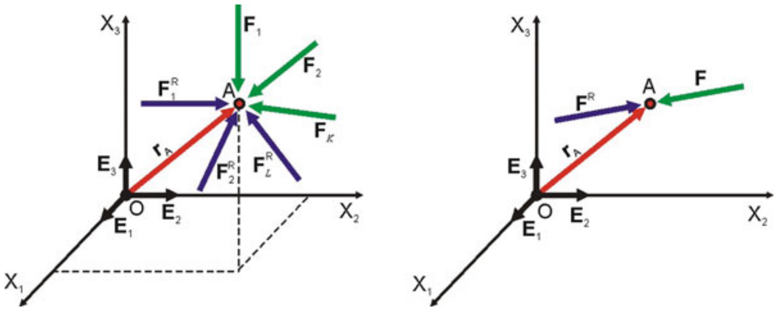


Fig. 2.1 A particle n (point A) under the action of forces $\mathbf{F}_k$ and reactions $\mathbf{F}_l^R$, and the same particle under the action of the resultant forces $\mathbf{F}$ and reactions $\mathbf{F}^R$ (internal forces not shown)

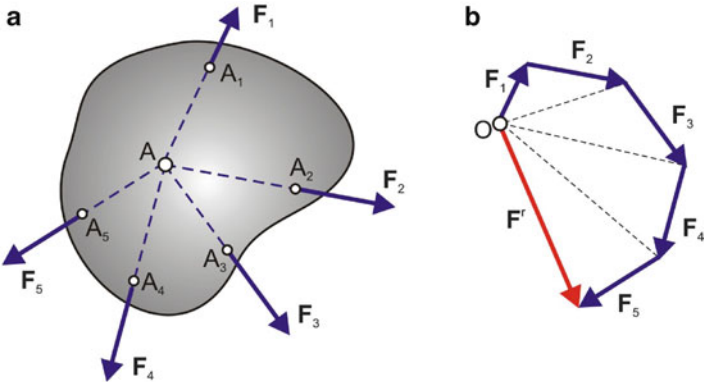


Fig. 2.2 Concurrent forces acting on a rigid body (**a**) and a force polygon (**b**)

In Fig. 2.1 it was shown that all forces are applied at a single point A. Now we will consider a more general case of a rigid body loaded with a discrete system of forces applied at points A_n of magnitudes F_n, $n = 1, \ldots, N$ and of lines of action passing through a certain point A (Fig. 2.2).

The force system that we call *concurrent* and those forces need not lie in a common plane. Because, by assumption, the lines of action of the forces intersect at point A, the system in Fig. 2.2a is equivalent to the force system depicted in Fig. 2.2b. Adding successively the force vectors and using the "triangle rule," that is, replacing every two forces by their resultant force (marked by a dotted line) we obtain a so-called *force polygon*. The action of the resultant vector $\mathbf{F}^r$ is equivalent to the simultaneous action of all forces, that is,

$$\mathbf{F}^r = \sum_{i=n}^{N} \mathbf{F}_n. \tag{2.5}$$

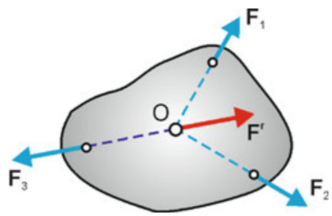


Fig. 2.3 Geometrical interpretation of the three-forces theorem

The method of construction of a force polygon indicates that in order to obtain vector $\mathbf{F}^r$ we can add the vectors $\mathbf{F}_n$ together directly (i.e., the vectors denoted by dotted lines can be omitted during addition).

The vectors $\mathbf{F}_n$ are called *sides* of a force polygon and the vector $\mathbf{F}^r$ is called a *closing vector* of a force polygon. Moreover, the sign of the sum (sigma) denotes addition of vectors from left to right, that is, from $\mathbf{F}_1, \mathbf{F}_2, \ldots$ to $\mathbf{F}_N$.

The presented construction is easy in the case of a planar force system. In space, the force polygon is a broken line whose sides are the force vectors and the resultant $\mathbf{F}^r$ connects the tail of the first force vector to the tip of the last force vector of the given force system. In this section we will take up the analysis of the planar force polygon.

From the foregoing considerations it follows directly that because the action of many forces can be replaced by the action of one resultant force, a rigid body remains in equilibrium under the action of a concurrent force system if $\mathbf{F}^r = \mathbf{0}$, that is, according to (2.5) if

$$\sum_{n=1}^{N} \mathbf{F}_n = \mathbf{0}. \tag{2.6}$$

If the above equation is satisfied, the polygon of forces $\mathbf{F}_n$ is closed, that is, the tail of the first force vector coincides with the tip of the last force vector.

Theorem 2.2. *(On three forces) If a body remains in equilibrium under the action of only three non-parallel coplanar forces, then their lines of action must intersect at a single point, that is, the system of forces must be concurrent.*

To prove the above theorem it is enough to observe that if we have three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, then the action of any two of them, e.g., $\mathbf{F}_1$ and $\mathbf{F}_2$, can be replaced with their resultant $\mathbf{F}^r$. According to Newton's first law, the body is in equilibrium if $\mathbf{F}^r = -\mathbf{F}_3$ and these vectors are collinear. Therefore, the lines of action of the three forces must intersect at a single point (Fig. 2.3).

Let us note that if we are dealing with a system of three concurrent forces and a body under the action of these forces is in equilibrium, all of these forces must be coplanar (i.e., lie in one plane).

Later we will show (by making use of Theorem 2.2) how to reduce an arbitrary three-dimensional system of non-concurrent forces to a so-called *equivalent system of two skew forces*.

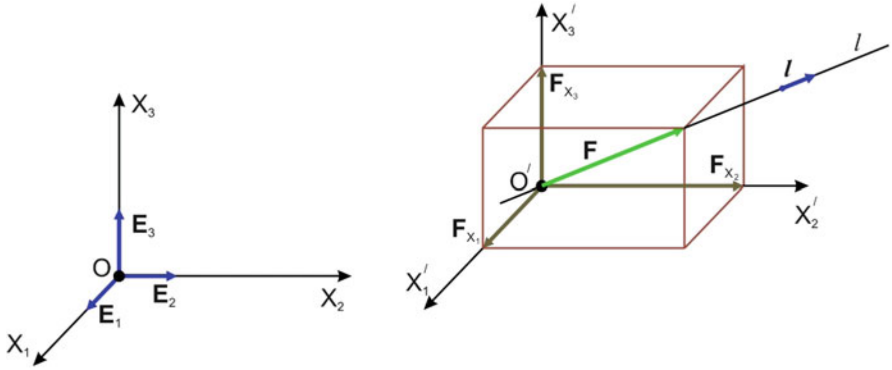


Fig. 2.4 Coordinate systems $OX_1X_2X_3$ and $O'X'_1X'_2X'_3$ and the force vector **F**

In practice many problems of statics boil down to the construction of the force polygon. The approach is like we call a *geometrical* approach and is complemented by the use of trigonometric functions to find unknown quantities. In order to use this method the magnitudes, the lines of action, and the senses of forces must be known.

Apart from the geometrical, the *analytical* approach is commonly applied. Let us introduce the fixed Cartesian coordinate system $OX_1X_2X_3$. In order to analytically describe a force **F** one should know its point of application $O'(x_1, x_2, x_3)$ and magnitudes of its projections onto axes OX_1, OX_2, and OX_3 of the form F_{x_1}, F_{x_2}, and F_{x_3}.

In Fig. 2.4 the location of point O' in the coordinate system $O'X'_1X'_2X'_3$ and the vectors **F**, $\mathbf{F}_{X_i}$ $(i = 1, 2, 3)$, are shown.

Let the vector of force **F** lie on the l axis of unit vector $\boldsymbol{l}$, and let $\mathbf{E}_i$ and $\mathbf{E}'_i$ be unit vectors respectively of the axes OX_i and $O'X'_i$, and because the axes of the systems $OX_1X_2X_3$ and $O'X'_1X'_2X'_3$ are parallel, we have $\mathbf{E}_i \parallel \mathbf{E}'_i$. According to the introduced notation we have

$$\mathbf{F} \equiv \boldsymbol{l}F = F_{x_1}\mathbf{E}_1 + F_{x_2}\mathbf{E}_2 + F_{x_3}\mathbf{E}_3 = \mathbf{F}_{x_1} + \mathbf{F}_{x_2} + \mathbf{F}_{x_3}. \tag{2.7}$$

In other words, a force **F** is said to have been resolved into three rectangular components $\mathbf{F}_{x_i}$, $i = 1, 2, 3$ if they are perpendicular to each other and directed along the coordinate axes.

After multiplication (scalar product) of the above equation in turn by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ we obtain

$$F_{x_1} = F\boldsymbol{l} \circ \mathbf{E}_1 = F\cos(\boldsymbol{l}, \mathbf{E}_1),$$
$$F_{x_2} = F\boldsymbol{l} \circ \mathbf{E}_2 = F\cos(\boldsymbol{l}, \mathbf{E}_2),$$
$$F_{x_3} = F\boldsymbol{l} \circ \mathbf{E}_3 = F\cos(\boldsymbol{l}, \mathbf{E}_3), \tag{2.8}$$

because

$$|\boldsymbol{l}| = |\mathbf{E}_1| = |\mathbf{E}_2| = |\mathbf{E}_3| = 1.$$

If we know the vector **F**, that is, its magnitude F and direction defined by the unit vector $\boldsymbol{l}$, the coordinates of the components of the vector are described by (2.8).

After squaring (2.7) by sides we obtain

$$F = \sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}, \tag{2.9}$$

and after taking into account (2.9) in (2.8) we calculate the cosines of the angles formed by the force vector with the axes of the coordinate system (called direction cosines)

$$\cos(\boldsymbol{l}, \mathbf{E}_1) = \frac{F_{x_1}}{F} = \frac{F_{x_1}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}},$$

$$\cos(\boldsymbol{l}, \mathbf{E}_2) = \frac{F_{x_2}}{F} = \frac{F_{x_2}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}},$$

$$\cos(\boldsymbol{l}, \mathbf{E}_3) = \frac{F_{x_3}}{F} = \frac{F_{x_3}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}}. \tag{2.10}$$

If F_{x_1}, F_{x_2} and F_{x_3} are known, then on the basis of (2.9) and (2.10) we can determine vector **F**, that is, its magnitude $|\mathbf{F}| = F$ and its direction defined by the direction cosines. It follows directly from (2.10) that $\cos^2(\boldsymbol{l}, \mathbf{E}_1) + \cos^2(\boldsymbol{l}, \mathbf{E}_2) + \cos^2(\boldsymbol{l}, \mathbf{E}_3) = 1$, and hence angles describing a position of the force **F** in relation to the Cartesian axes depend on each other.

Let us now return to Fig. 2.2, where the system of concurrent forces acts solely on the rigid body. If such a body is in equilibrium, according to (2.5), we have $\mathbf{F}^r = \mathbf{0}$, and after multiplying (2.5) by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ we obtain

$$F_{1x_1} + F_{2x_1} + \cdots + F_{Nx_1} = 0,$$

$$F_{1x_2} + F_{2x_2} + \cdots + F_{Nx_2} = 0,$$

$$F_{1x_3} + F_{2x_3} + \cdots + F_{Nx_3} = 0. \tag{2.11}$$

The equilibrium condition of the rigid body subjected to the action of a three-dimensional system of concurrent forces is described by three algebraic (2.11), in view of the fact that the number of unknowns should not exceed three (or, in the case of the planar system of forces, two) for the considered force system to be *statically determinate*. In Fig. 2.5 a symmetrical system of concurrent forces formed after suspending the body of weight **G** from three ropes is shown.

After introducing the Cartesian coordinate system and including the system geometry we can (through some trigonometric relations) determine three unknowns

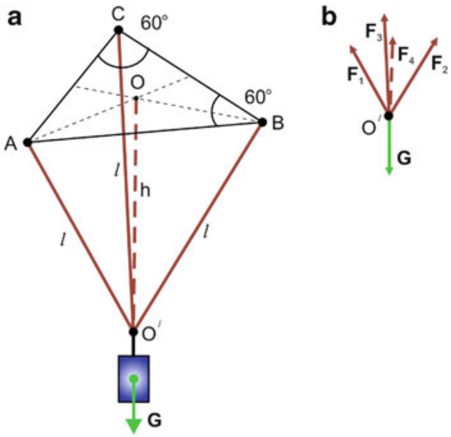


Fig. 2.5 A weight **G** suspended from three (four) ropes in space (**a**) and a system of concurrent forces (**b**)

from three equations and, consequently, describe the forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. Let us assume now that at the center O of the triangle ABC an extra fourth rope was attached. If the length of that rope differs even slightly from the distance h, either that rope exclusively carries the total weight **G** (if slightly shorter) or it carries no load at all (if slightly longer).

If we assume that the weight **G** is carried by four ropes, it is impossible to determine the four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ in the ropes; such a problem is *statically indeterminate*. This appears also in the planar system of concurrent forces if point C becomes coincident with point B and then $AO = OB$. The problem is *statically determinate* if the weight **G** is suspended from two ropes AO' and BO' and statically indeterminate if, additionally, we introduce the third rope OO' and all three ropes are loaded.

One deals with a statically indeterminate problem when the number of unknown values of forces (torques) denoted by N is larger than the number of equilibrium equations N_r. The difference $N - N_r$ is called a *degree of static indeterminacy*. As will be shown later, additional equilibrium equations are obtained after taking into account deformations of the investigated mechanical system. In general, while solving a statically indeterminate problem, the *method of forces* and the *method of displacements* are often applied.

The first approach consists of three stages [26]:

1. Determination of degree of static indeterminacy.
2. Transformation of a statically indeterminate system to a statically determinate one with unknown values of loads but known character and point of application.
3. Determination of a set of desired force values from condition of displacement continuity at the force application points.

The second method, i.e., the *displacements method*, is to use the relationship between external forces, displacement nodes of the construction, displacements of the ends of particular ropes (rods) and their geometric and material properties.

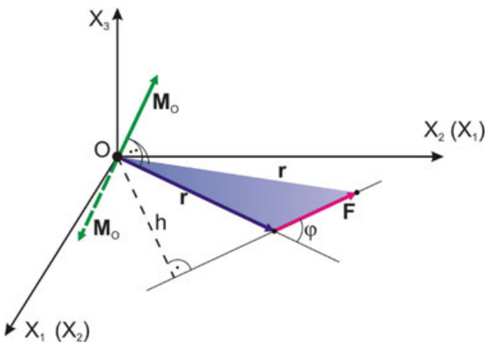


Fig. 2.6 Graphical representation of a moment of force **F** about a point O

It takes advantage of the emergence of movements of rod ends in a strict dependence resulting from the continuity of the structure.

The displacement method will be described in Example 2.5.

The static indeterminacy of a mechanical system is often due to the introduction of the so-called technologically justified assembling stress (e.g., stresses resulting from the initial stretch of the ropes supporting the structure). In addition, the state of stress may appear as a result of non-uniform heating of the system. The loading state of a mechanical system following from the introduction of assembly and thermal stresses is an independent and additional loading of the considered mechanical system.

In a general case a non-concurrent force system may act on a rigid body. In order to describe the equilibrium conditions in this case it is necessary to introduce the notions of moment of force about a point and moment of force about an axis (which was discussed in Sect. 1.2).

The notion of moment of force about a point was first used by Archimedes, but it concerned planar systems.

The moment of force **F** about point O is defined by a vector product of the form

$$\mathbf{M}_O \equiv \mathbf{M}_O(\mathbf{F}) = \mathbf{r} \times \mathbf{F}, \tag{2.12}$$

which after introduction of the Cartesian coordinate system $OX_1X_2X_3$ is illustrated in Fig. 2.6.

According to the definition of vector product [see (1.13), (1.14)] and Fig. 2.6, we have

$$\mid \mathbf{M}_O \mid \equiv M_O = rF \sin\varphi = Fh, \tag{2.13}$$

that is, the magnitude of the moment of force is equal to the doubled area of a triangle (in blue) of sides r and F. Its direction is given by the right-hand rule, that is, the arrow of the vector $\mathbf{M}_O$ points toward an eye of a person looking if the vector **r** is rotated toward the vector **F** counterclockwise (positive sense) provided that the coordinate system is right-handed. However, if the system of coordinates is left-handed (its axes in parentheses in Fig. 2.6), the sense of the vector $\mathbf{M}_O$ changes (vector marked by a dashed line in Fig. 2.6).

Let us note that vectors describing a real physical quantity (e.g., force, velocity, acceleration) do not change when the adopted coordinate system is changed. In the considered case the vector of the moment of force $\mathbf{M}_O$ changes when the coordinate system is changed from right-handed to left-handed. Vectors having such a property are called *pseudovectors*.

Theorem 2.3. *(Varignon[1]) The moment of a resultant force of a system of concurrent forces about an arbitrary point O (a pole) is equal to the sum of individual moments of each force of the system about that pole.*

Proof of the above theorem is obvious if one uses the property of distributivity of a vector product with respect to addition.

Let us note that Varignon's theorem includes also the special case of concurrent forces when the point of intersection of their lines of action is situated in infinity (in that case we are dealing with a system of parallel forces). Moreover, let us note that the introduction of any force $\mathbf{F}'$ of the direction along that of $\mathbf{r}$ does not change the moment since we have

$$\mathbf{M}_O = \mathbf{r} \times (\mathbf{F} + \mathbf{F}') = \mathbf{r} \times \mathbf{F} + \mathbf{r} \times \mathbf{F}' = \mathbf{r} \times \mathbf{F}, \tag{2.14}$$

because $\mathbf{r} \times \mathbf{F}' = \mathbf{0}$.

This trivial observation will be exploited later during the reduction of an arbitrary three-dimensional force system to two skew forces in space (forces that do not lie in one plane).

If by a resultant force we understand the force replacing the action of the system of concurrent forces, such a notion loses its meaning in the case of an arbitrary force system in space. Then the closing vector of a three-dimensional polygon of forces is called a *main force vector*.

The components of the moment of force vector about a pole O is obtained directly from the definition using the determinant, that is,

$$\begin{aligned}\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = &\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ r_{x_1} & r_{x_2} & r_{x_3} \\ F_{x_1} & F_{x_2} & F_{x_3} \end{vmatrix} \\ &= \mathbf{E}_1(r_{x_2}F_{x_3} - r_{x_3}F_{x_2}) + \mathbf{E}_2(-r_{x_1}F_{x_3} + r_{x_3}F_{x_1}) + \mathbf{E}_3(r_{x_1}F_{x_2} - r_{x_2}F_{x_1}) \\ &\equiv M_{Ox_1}\mathbf{E}_1 + M_{Ox_2}\mathbf{E}_2 + M_{Ox_3}\mathbf{E}_3. \end{aligned} \tag{2.15}$$

Let us introduce now the concept of the moment of force $\mathbf{F}$ with respect to an axis l of unit vector $\boldsymbol{l}$ (Fig. 2.7).

[1]Pierre Varignon (1654–1722), French mathematician (friend of Newton, Leibniz, and the Bernoulli family).

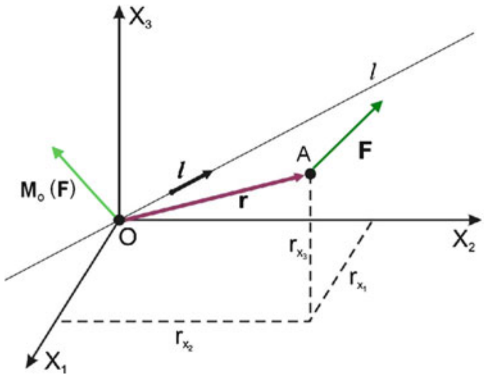


Fig. 2.7 Moment of force **F** about an axis l

Definition 2.1. *The magnitude of the moment of force about an axis l is equal to the scalar product of the moment of force about an arbitrary point O on that axis and a unit vector $\boldsymbol{l}$ of the axis (thus it is a scalar that respects the sign determined by the sense in agreement or opposite to the unit vector $\boldsymbol{l}$).*

According to the above definition, and after taking into account (2.15), we have

$$\begin{aligned} M_l(\mathbf{F}) &= \boldsymbol{l} \circ \mathbf{M}_0(\mathbf{F}) \\ &= M_{Ox_1}\cos(\boldsymbol{l},\mathbf{E}_1) + M_{Ox_2}\cos(\boldsymbol{l},\mathbf{E}_2) + M_{Ox_3}\cos(\boldsymbol{l},\mathbf{E}_3) \\ &= (r_{x_2}F_{x_3} - r_{x_3}F_{x_2})\cos(\boldsymbol{l},\mathbf{E}_1) + (-r_{x_1}F_{x_3} + r_{x_3}F_{x_1})\cos(\boldsymbol{l},\mathbf{E}_2) \\ &\quad + (r_{x_1}F_{x_2} - r_{x_2}F_{x_1})\cos(\boldsymbol{l},\mathbf{E}_3) \\ &= \begin{vmatrix} \cos(\boldsymbol{l},\mathbf{E}_1) & \cos(\boldsymbol{l},\mathbf{E}_2) & \cos(\boldsymbol{l},\mathbf{E}_3) \\ r_{x_1} & r_{x_2} & r_{x_3} \\ F_{x_1} & F_{x_2} & F_{x_3} \end{vmatrix}. \end{aligned} \tag{2.16}$$

Observe that $M_l(\mathbf{F})$ is the magnitude of the moment of force about axis l, that is, it depends on the versor $\boldsymbol{l}$ sense.

Let us resolve $\mathbf{F}(\mathbf{r})$ into two rectangular components $_{\parallel}\mathbf{F}(_{\parallel}\mathbf{r})$ and $_{\perp}\mathbf{F}(_{\perp}\mathbf{r})$ (lying in a plane perpendicular to $\boldsymbol{l}$). Then from (2.16) we obtain $M_l(\mathbf{F}) = \boldsymbol{l} \circ (_{\parallel}\mathbf{r} + {}_{\perp}\mathbf{r}) \times (_{\parallel}\mathbf{F} + {}_{\perp}\mathbf{F}) = \boldsymbol{l} \circ (_{\perp}\mathbf{r} \times {}_{\perp}\mathbf{F})$. In other words the moment $M_l(\mathbf{F})$ describes the tendency of the force **M** to give to the rigid body a rotation about the fixed axis l.

At the end of this section we will present definitions of a couple and a moment of a couple and introduce the basic properties of a couple.

Definition 2.2. *A system of two parallel forces with opposite senses and equal values we call a **couple of forces**. A plane in which these forces lie is called a **plane of a couple of forces**.*

In Fig. 2.8 a couple of forces and a moment of a couple of forces are presented graphically.

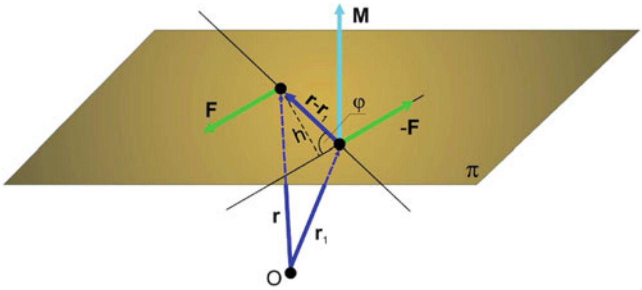


Fig. 2.8 A couple of forces and its moment about a point O

The moment of a couple of forces about a point O is the geometric sum of moments of each of the forces about point O, that is,

$$\mathbf{M} = \mathbf{M}_O(\mathbf{F}) + \mathbf{M}_O(-\mathbf{F}) = \mathbf{r} \times \mathbf{F} + \mathbf{r}_1 \times (-\mathbf{F})$$
$$= \mathbf{r} \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F} = (\mathbf{r} - \mathbf{r}_1) \times \mathbf{F}. \tag{2.17}$$

From (2.17) it follows that the moment $\mathbf{M}$ of a couple of forces about point O does not depend on the location of this point, but only on the vector $\mathbf{r}-\mathbf{r}_1$ describing the relative position of points of application of forces forming the couple and lying in a plane of the couple of forces π.

The magnitude of moment of a couple of forces is equal to

$$M = F|\mathbf{r} - \mathbf{r}_1| \sin\varphi = Fh, \tag{2.18}$$

where h is the distance between lines of action of the couple and is called the *arm of a couple of forces*. Thus, the magnitude of a moment of a couple of forces corresponds to the area of a rectangle with sides F and h. It is clear that a couple applied to a body tends to rotate it. Since $\mathbf{r} - \mathbf{r}_1$ is independent of the choice of the origin O, the moment $\mathbf{M}$ of a couple is a *free vector*.

Let us note that $\mathbf{M}$ is perpendicular to the plane π and its sense is determined by senses of vectors of the couple of forces. The static action of a couple is equivalent to the moment of a couple. What follows are some theorems regarding a couple of forces (proofs are left to the reader).

Theorem 2.4. *The action of an arbitrary couple of forces on a rigid body is invariant with respect to the rotation of the plane of the couple through an arbitrary angle.*

Theorem 2.5. *The action of an arbitrary couple of forces on a rigid body is invariant with respect to the choice of an arbitrary plane parallel to the plane of the couple.*

Theorem 2.6. *The action of an arbitrary couple of forces on a rigid body is invariant if the product Fh [see (2.18)] remains unchanged, that is, it is possible to vary the magnitude of force and its arm as long as Fh remains constant.*

Theorem 2.7. *An arbitrary system of couples of forces in R^3 space is statically equivalent to a single couple of forces whose moment is the geometrical (vector) sum of moments coming from each couple of forces in the system.*

The reader is encouraged to prove that two couples possessing the same moment $\mathbf{M}$ are equivalent. Since the couples can be presented by vectors, they can be summed up in a geometrical manner. In addition, any given force $\mathbf{F}$ acting on a rigid body can be resolved into a force at an arbitrary given pole O and a couple, that is, at point O we have an equivalent force–couple $(\mathbf{F} - \mathbf{M}_O)$ system provided that the couple's moment is equal to the moment of $\mathbf{F}$ about O. In what follows we apply the statements and comments introduced thus far.

Let us emphasize that the net result of a couple relies on the production of a moment $\mathbf{M}$ (couple vector). Since $\mathbf{M}$ is independent of the point about which it takes place (free vector), then in practice it should be computed about a most convenient point for analysis. One may add two or more couples in a geometric way. One may also replace a force with (a) an equivalent force couple at a specified point; (b) a single equivalent force provided that $\mathbf{F} \perp \mathbf{M}$, which is satisfied in all two-dimensional problems.

In Fig. 2.1 the position of a particle is described by the vector $\mathbf{r}_n$ in the adopted Cartesian coordinate system. In the case of a system of particles, each particle n of the system is described by a radius vector $\mathbf{r}_n$, $n = 1, \ldots, N$. After multiplying equilibrium condition (2.1) by $\mathbf{r}_n$ (cross product) and adding together the obtained equations (after assuming $\mathbf{F}_n^i = \mathbf{0}$) we obtain

$$\mathbf{M}_O = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n) + \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^R) = \mathbf{0}. \tag{2.19}$$

Let us note that (2.19) is also valid for $\mathbf{F}_n^i \neq \mathbf{0}$ because $\sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^i) = \mathbf{0}$ (internal forces of the system exist in pairs that are equal, act along the same line of action, but have opposite senses; therefore, taken together they all yield zero moment about any point).

According to (2.4) and (2.19) a material system is in equilibrium if the system of forces and reactions, and the main moment produced by these forces and reactions, is equal to zero, that is, we have

$$\mathbf{S} = \mathbf{0}, \quad \mathbf{M}_O = \mathbf{0}. \tag{2.20}$$

Let us note that the above condition is a necessary condition, but in general it is not a sufficient one. According to Newton's first law, particles can move along

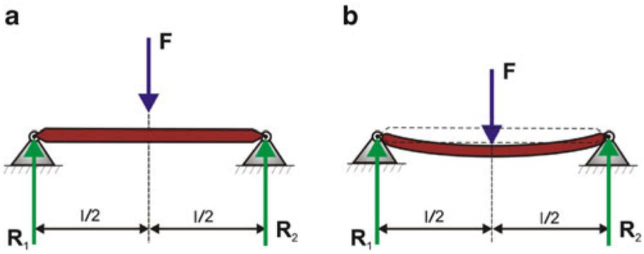


Fig. 2.9 A rigid (**a**) and flexible (**b**) beam with a concentrated force applied in the middle of their lengths

straight lines in uniform motion, that is, they can change their relative position in spite of being subjected to the action of balanced forces. If we are dealing with a rigid material system or a rigid body, then the relative motion of the particles of the body is impossible and conditions (2.20) are both necessary and sufficient conditions.

If, however, we consider a continuous material system (CMS), such as deformable solid bodies, gases, or liquids, then conditions (2.20) are necessary conditions, but not sufficient ones.

In a general case for an arbitrary material system the necessary and sufficient equilibrium conditions are described by the following equations for every particle n:

$$\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R = \mathbf{0}, \quad n = 1, 2 \ldots, N. \tag{2.21}$$

Although (2.21) seems to be simpler, in fact it is more complicated, since it involves determination of internal forces $\mathbf{F}_n^i$. In order to do so one should release from constraints the internal points of the material system and consider the equilibrium of external, internal, and reaction forces.

As an example we will consider a beam of length l loaded at the midpoint with the force $\mathbf{F}$ on the assumption that it is a rigid beam (Fig. 2.9a) and a flexible beam (Fig. 2.9b) [9].

The absence of deflection in the first case (Fig. 2.9a) follows from the fact that the beam is a rigid body. The presence of deflection in the second case (Fig. 2.9b) is caused by the beam's flexibility. The conditions of equilibrium are the same in both cases, although the equilibrium positions are different. In both cases internal forces are developed, but only the second case is accompanied by the beam's deformation. Observe that in the case of a rigid body (beam), each of the forces may be shifted in an arbitrary way along its line of action, but this approach is prohibited for a flexible body (beam), that is, the transmissibility principle is violated.

The above example is associated with the so-called *freezing principle*. Equilibrium of forces acting on a flexible body is not violated when the flexible beam becomes "frozen."

2.2 Geometrical Equilibrium Conditions of a Planar Force System

As has already been mentioned, in a general case formulation of geometrical equilibrium conditions is not an easy problem, and usually this approach is applied to systems of forces lying in a plane. As was shown earlier, according to the first equation of (2.20) a polygon constructed from force vectors should be closed.

In the case where a material system is acted upon by a system of three non-parallel forces equivalent to zero, these forces are coplanar and their lines of action intersect at one point, forming a triangle (Theorem 2.2).

In a general case for an arbitrary plane force system geometrical equilibrium conditions boil down to the *force polygon* becoming closed and to satisfying the condition of the so-called *funicular polygon*. As an introduction let us first consider three arbitrary forces $\mathbf{F}_n$, $n = 1, 2, 3$ lying in one plane (Fig. 2.10).

From the construction depicted in Fig. 2.10 it follows that, at first, any two of the three forces (in this case $\mathbf{F}_1$ and $\mathbf{F}_2$) are moved to the point of intersection of their lines of action O' and then added geometrically, yielding the force $\mathbf{F}_1 + \mathbf{F}_2$. One proceeds similarly with the vector $\mathbf{F}_1 + \mathbf{F}_2$ and the vector $\mathbf{F}_3$. The lines of action of these vectors, again, determine the point O to which both vectors are moved. Next, using the parallelogram law we obtain a resultant vector $\mathbf{F}^r$ replacing the action of those forces. We also obtain the line of action of $\mathbf{F}^r$ ($\mathbf{F}^r$ is a *sliding vector*).

A similar procedure may be applied for any number of force vectors acting in a plane (coplanar and non-parallel vectors). However, in practice sometimes it turns out that the task is not easy. One reason for this is that the forces may intersect at the points outside the drawing space. Moreover, their small angles of intersection may lead to large inaccuracies.

Below a different graphical method called *funicular polygon method* will be presented.

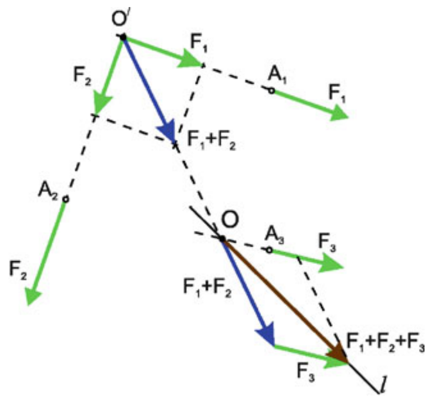


Fig. 2.10 Determination of magnitude, sense, and direction of a resultant $\mathbf{F}^r = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ for the case of a planar system of three non-parallel forces

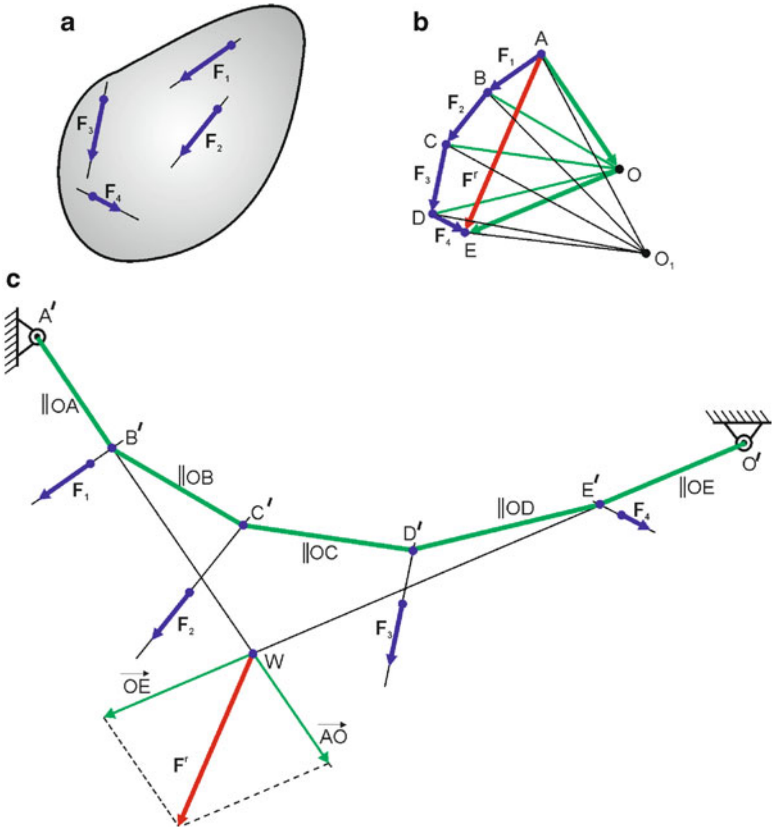


Fig. 2.11 An arbitrary system of four forces (**a**), a force polygon (**b**), and a funicular polygon (**c**)

Let us consider now an arbitrary system of forces acting in a plane on a certain rigid body (Fig. 2.11). In order to preserve clarity of the drawing we limit ourselves to consideration of four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$.

The procedure of construction of a force polygon is as follows. In a plane we take an arbitrary point A, being the tail of force vector $\mathbf{F}_1 = \overrightarrow{AB}$. Next, at the tip of that vector we attach the vector $\mathbf{F}_2 = \overrightarrow{BC}$, and so on. Next, we take an arbitrary point O called a *pole* and connect it with points $A,\ B,\ C,\ D$, and E. Let us note that

$$
\begin{aligned}
\mathbf{F}^r &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \\
&= \left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \left(\overrightarrow{OC} - \overrightarrow{OB}\right) + \left(\overrightarrow{OD} - \overrightarrow{OC}\right) + \left(\overrightarrow{OE} - \overrightarrow{OD}\right) \\
&= \overrightarrow{OE} - \overrightarrow{OA} = \overrightarrow{AO} + \overrightarrow{OE}. \qquad (2.22)
\end{aligned}
$$

The segments OA, OB, OC, OD, and OE are called *rays*, but in (2.10) the vectors of their lengths appeared as forces, which means $|\overrightarrow{OA}| = OA$, and so on.

The force polygon allows for determination of the resultant $\mathbf{F}^r$, that is, the force replacing the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$. However, additionally we have to determine the line of action of the resultant force (it must be parallel to the direction of the force $\mathbf{F}^r$ obtained using the force polygon method, see Fig. 2.11b). In practice this means that one should determine a point W through which the force $\mathbf{F}^r$ would pass.

To this end we make use of the so-called *funicular polygon*. Let us take an arbitrary point A' lying on the line of action of the force $\mathbf{F}_1$ (construction is temporarily conducted in a separate drawing). Next, we draw a line passing through that point and parallel to OA (cf. the force polygon). Then, we step off on that line a segment $A'B'$ of length $|\overrightarrow{AO}|$ (one may limit oneself to the determination of the direction of that force). Next, through point B' we draw a line parallel to OB and step off the segment $B'C' = |\overrightarrow{OB}|$. Through the obtained point C' we draw a line parallel to OC and step off the segment $C'D' = |\overrightarrow{OC}|$, obtaining in this way point D'. Through that point we draw a line parallel to OD and step off the segment $D'E' = |\overrightarrow{OD}|$. Through the obtained point E' we draw a line parallel to OE and step off the segment $E'O' = |\overrightarrow{EO}|$. According to (2.22) the resultant can be determined using the parallelogram law, since the force vectors $\overrightarrow{AO}$ and $\overrightarrow{OE}$ are known [the forces represented by the other vectors (rays) cancel each other]. One may also move the vectors $\overrightarrow{AO}$ and $\overrightarrow{OB}$, $\overrightarrow{BO}$ and $\overrightarrow{OC}$, $\overrightarrow{CO}$ and $\overrightarrow{OD}$ respectively to points B', C', D', and E'. It is easy to notice that after their geometrical addition only the vectors $\overrightarrow{AO}$ and $\overrightarrow{OE}$ remain. It follows that extending the lines passing through points A' and B', and through O' and E', leads to determination of point W, through which passes the line of action of the resultant $\mathbf{F}^r$. It follows also that it is possible to determine the magnitude of that force (after geometrical addition of $|\overrightarrow{A'B'}|$ and $|\overrightarrow{O'E'}|$).

If we took a perfectly flexible cable of negligible weight and length $AO + OB + OC + OD + OE$ and fixed it at points A' and O', then after application of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ at points B', C', D', and E' (Fig. 2.11c), the cable would remain in equilibrium. That is where the name of funicular polygon comes from. At each of these points act three forces that are in equilibrium. It is easy to check that the choice of some other pole (point O_1 in Fig. 2.11b) leads to the determination of a different point of intersection, but it must lie on the line of action of the force $\mathbf{F}^r$.

Let us now consider the particular case of the force polygon depicted in Fig. 2.11b, namely, when $\mathbf{F}^r \to \mathbf{0}$, that is, when $E \to A$, which means that $|\overrightarrow{EO}| = |\overrightarrow{OA}|$ and the vectors $\overrightarrow{OE}$ and $\overrightarrow{AO}$ become collinear at the limit while retaining the opposite senses. Since the polygon $ABCDA$ is closed, points A and E are coincident, that is, $A \equiv E$.

Such a situation is depicted in Fig. 2.12a, and the corresponding force polygon is shown in Fig. 2.12b.

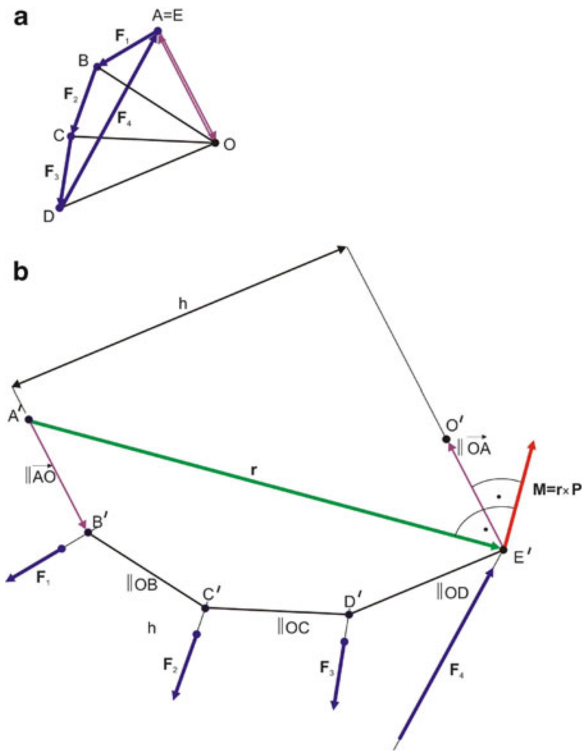


Fig. 2.12 A closed polygon of four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ (**a**) and the corresponding funicular polygon (**b**) (vector **r** is not a part of the funicular polygon but was drawn only for interpretation of a moment of force vector $\mathbf{M} = \mathbf{r} \times \mathbf{P}$ perpendicular to the drawing's plane)

From the figure it follows that the vectors of forces $\overrightarrow{A'B'} = \overrightarrow{AO}$ and $\overrightarrow{E'O'} = \overrightarrow{OE}$ possess the same magnitudes and opposite senses, so they produce the moment of a force of magnitude $|\mathbf{M}| = |\mathbf{r} \times \mathbf{P}| = Pr \sin\varphi = Ph$, where $\mathbf{P} = \overrightarrow{A'B'}$ and φ denotes the angle formed between vectors **r** and **P**. It is impossible to determine the point of application of the force $\mathbf{F}^r$ because it is a singular case where $\mathbf{F}^r = \mathbf{0}$ and $A'B' \parallel O'E'$. In this case the force system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is equivalent to the couple of forces.

Let us now consider another special case of the force polygon and funicular polygon relying on the considerations regarding Fig. 2.12. Let point $E' \to A'$, which means that $\mathbf{r} \to \mathbf{0}$, that is, $\mathbf{r} \times \mathbf{P} = \mathbf{M} \to \mathbf{0}$. When the endpoints of the funicular polygon are coincident, that is, $E' \equiv A'$, the moment $\mathbf{M} = \mathbf{0}$ (this case is presented in Fig. 2.13). Because the force polygon remains unchanged (Fig. 2.12a), in Fig. 2.13 only the funicular polygon is shown. In this case the force system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ remains in equilibrium.

From the considerations and drawings in Figs. 2.11–2.13 the following conclusions result:

1. If neither the force polygon nor the funicular polygon is closed, then the investigated planar force system is equivalent to a resultant force.

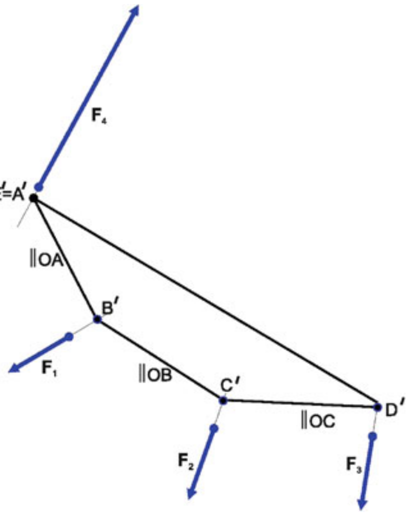


Fig. 2.13 Closed funicular polygon ($\mathbf{M} = \mathbf{0}$)

2. If the funicular polygon is not closed but the force polygon is closed, then planar force system is equivalent to a moment of force (couple).
3. If both force and funicular polygons are closed, then the force system is equivalent to zero.

The final geometrical equilibrium condition for a two-dimensional force system reads:

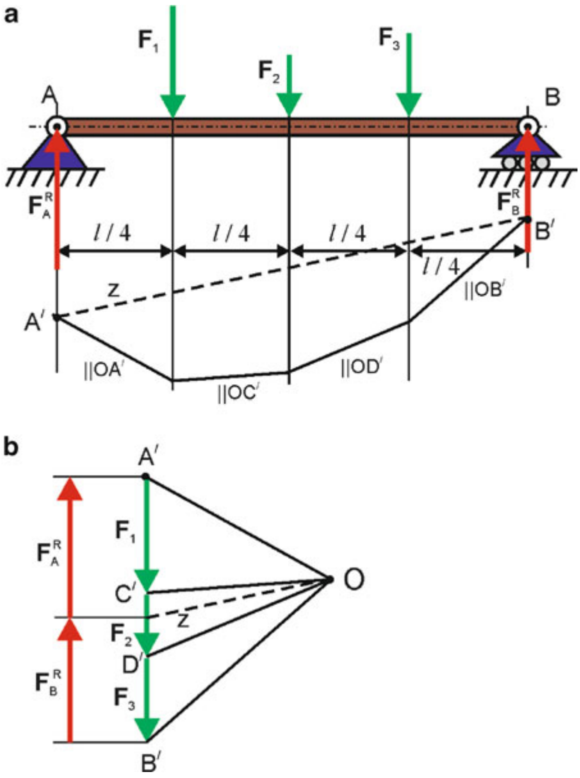
A planar force system is equivalent to zero if both the force polygon and funicular polygon are closed.

Example 2.1. Three vertical forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act on a horizontal beam of length l depicted in Fig. 2.14. The beam is pin supported at point A and has a roller support at point B. Determine the reactions in beam supports.

In this case the forces $\mathbf{F}_n$ are parallel, so the unknown reactions $\mathbf{F}_A^R$ and $\mathbf{F}_B^R$ are parallel as well and lie on the vertical lines through points A and B. According to the previous considerations let us first construct the force polygon in a certain assumed scale of the drawing. Next, after taking pole O we draw the rays from the pole to the tail and the tip of every force vector. We take an arbitrary point A' on the vertical line passing through A. After connecting points A' and B' we obtain the closing line of the force polygon (dashed line). Next, we translate it in parallel, so that it passes through the pole. Then its point of intersection with segment $A'B'$ in the force polygon defines the unknown reactions.

Since the tip of reaction $\mathbf{F}_A^R$ must coincide with the tail of the force $\mathbf{F}_1$, we complement the force polygon with reactions $\mathbf{F}_A^R$ and $\mathbf{F}_B^R$ and, as can be seen, $\mathbf{F}_A^R + \mathbf{F}_B^R + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$. □

Fig. 2.14 A horizontal beam loaded with forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ (**a**) and a force polygon (**b**)



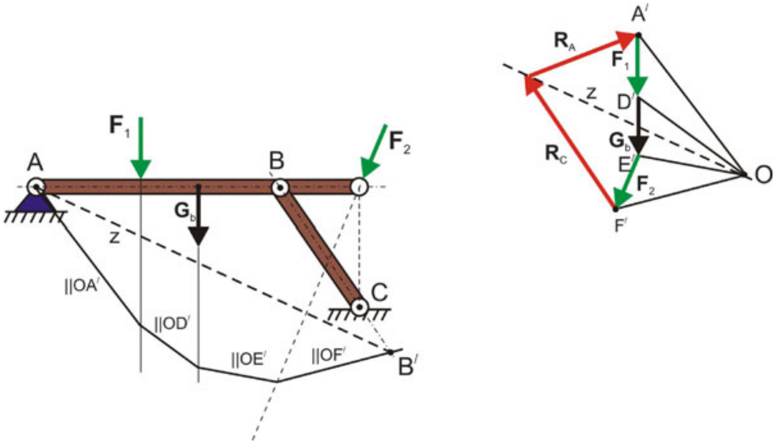


Fig. 2.15 Schematic for calculations of a supported beam, a force polygon, and a funicular polygon

Example 2.2. A horizontal beam is loaded with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (Fig. 2.15). The beam is pin supported at point A and at point B supported by the rod with a pin joint. Determine the reactions in pin joints A and C assuming that the beam has a homogeneous mass distribution and weight $\mathbf{G}_b$.

Because in this case only the vectors $\mathbf{F}_1$ and $\mathbf{G}_b$ are parallel to each other but are not parallel to $\mathbf{F}_2$, the reactions $\mathbf{F}_A^R$ and $\mathbf{F}_C^R$ are not parallel (the direction of reaction $\mathbf{F}_C^R$ is defined by the axis of rod BC). We construct the force polygon for $\mathbf{F}_1, \mathbf{G}_b, \mathbf{F}_2$.

The construction of the funicular polygon we start from point A (reaction $\mathbf{F}_A^R$ must pass through point A, but its direction is unknown). Doing the construction in the way described earlier, the line parallel to OF' intersects segment BC at point B' and segment AB' is the closing line of the funicular polygon. After drawing a line parallel to AB' that passes through pole O (line z) and drawing from the tip of the vector $\mathbf{F}_2$ the line parallel to BC, the point of their intersection determines the unknown reactions $\mathbf{F}_A^R$ and $\mathbf{F}_C^R$. □

2.3 Geometrical Equilibrium Conditions of a Space Force System

As distinct from the previously analyzed case of the concurrent force system shown in Fig. 2.2, we will consider now a non-concurrent force system.

Our aim is to reduce the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ to an arbitrary chosen point O of the body (or rigidly connected to the body) called a *pole*.

The method given below was presented already by Poinsot[2] and henceforth is called *Poinsot's method*. We will show that, according to Poinsot's method, the action of the force $\mathbf{F}_1$ on a rigid body with respect to pole O is equivalent to the action of the force $\mathbf{F}_1$ applied at point O and a couple $\mathbf{F}_1$ and $\mathbf{F}_1^{(2)}$ applied at points A_1 and O, respectively, and $\mathbf{F}_1^{(2)} = -\mathbf{F}_1$.

In other words at point O we apply the forces $\mathbf{F}_1^{(1)}$ and $\mathbf{F}_1^{(2)}$, where $\mathbf{F}_1^{(1)}$ denotes the vector $\mathbf{F}_1$ moved in a parallel translation to point O, and $\mathbf{F}_1^{(2)} = -\mathbf{F}_1^{(1)}$. Point O is in equilibrium under the action of two vectors of the directions along the direction of $\mathbf{F}_1$ and having the same magnitudes but opposite senses.

The action of force at point A_1 manifests at point O as the action of the force $\mathbf{F}_1$ at that point and the couple $\mathbf{F}_1$ (applied at point A_1) and $-\mathbf{F}_1$ (applied at point O). In turn, the action of the couple $\mathbf{F}_1$ and $-\mathbf{F}_1$ is equivalent to the action of a moment of the couple, which according to (2.17) is equal to $\mathbf{M}_1 = \overrightarrow{OA_1} \times \mathbf{F}_1$.

In Fig. 2.16a at point O only the two forces $\mathbf{F}_1^{(1)}$ and $\mathbf{F}_1^{(2)}$ canceling one another are marked, and in Fig. 2.16b reduction of the whole force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ is

[2] Louis Poinsot (1777–1859), French mathematician and physicist, precursor of geometrical mechanics.

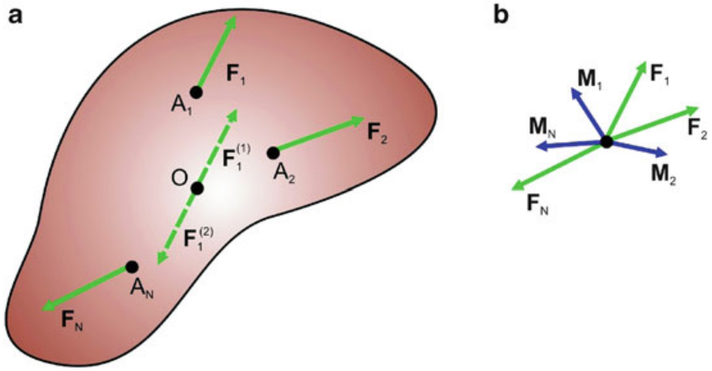


Fig. 2.16 Space force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ acting on a rigid body (**a**) and its reduction to a pole O (**b**)

shown. The latter boils down to applying at point O all the vectors of forces $\mathbf{F}_1, \ldots, \mathbf{F}_N$ and moments $\mathbf{M}_1, \ldots, \mathbf{M}_N$ determined in a way that is analogous to that shown for the case of force $\mathbf{F}_1$.

The application of Poinsot's method led to obtaining the concurrent force and moment system, which is schematically depicted in Fig. 2.16b. With the aid of the conducted construction we can develop a force polygon and a moment of the force polygon, which leads to the determination of the *main force vector* and *main moment of the force vector* of the forms

$$\mathbf{S} = \sum_{i=n}^{N} \mathbf{F}_n, \tag{2.23}$$

$$\mathbf{M}_O = \sum_{n=1}^{N} \mathbf{M}_n. \tag{2.24}$$

The technique described above is equivalent to the *theorem on parallel translation of force*.

Let us note that if the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ was concurrent and acted, e.g., at point A_1, the vector $\mathbf{S}$ according to (2.23) would be the resultant of forces applied at point A_1. The process of reduction regards pole O, and the notion of the resultant of forces at that point concerns the sum of forces $\mathbf{F}_1^{(1)}, \mathbf{F}_2^{(1)}, \ldots, \mathbf{F}_N^{(1)}$, and therefore the notion of the main moment of force vector was introduced instead of using the notion of resultant of forces.

Let us now proceed to the analysis of (2.24). Let us consider the moment of the couple $\mathbf{F}_1$ and $\mathbf{F}_1^{(2)}$, which is equal to

$$\mathbf{M}_1 = \mathbf{M}_O(\mathbf{F}_1) + \mathbf{M}_O(\mathbf{F}_1^{(2)}) = \mathbf{M}_O(\mathbf{F}_1), \tag{2.25}$$

because the force $\mathbf{F}_1^{(2)}$ passes through pole O. Similar considerations concern the remaining forces, and finally (2.24) takes the following form:

$$\mathbf{M}_O = \sum_{n=1}^{N} \mathbf{M}_O(\mathbf{F}_n). \tag{2.26}$$

The result of the considerations carried out above leads to the formulation of the general theorem of statics of a rigid body.

Theorem 2.8. *An arbitrary system of non-concurrent forces in space acting on a rigid body is statically equivalent to the action of the main force vector (2.23) applied at an arbitrary point (pole) and the main moment of force vector (2.24).*

2.4 Analytical Equilibrium Conditions

An analytical form of equilibrium conditions relies on the vectorial (2.20). Because in the Euclidean space each of the vectors possesses three projections, the equilibrium conditions boil down to the following three conditions concerning the forces:

$$S_{x_j} = \sum_{n=1}^{N} F_{x_j n} = 0, \quad j = 1, 2, 3, \tag{2.27}$$

and because

$$\begin{aligned}
\mathbf{M}_O &= \sum_{n=1}^{N} (r_n \times \mathbf{F}_n) = \sum_{n=1}^{N} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ F_{x_{1n}} & F_{x_{2n}} & F_{x_{3n}} \end{vmatrix} \\
&= \sum_{n=1}^{N} \Big[\mathbf{E}_1 \big(x_{2n} F_{x_3 n} - x_{3n} F_{x_2 n} \big) + \mathbf{E}_2 \big(x_{3n} F_{x_1 n} - x_{1n} F_{x_3 n} \big) \\
&\quad + \mathbf{E}_3 \big(x_{1n} F_{x_2 n} - x_{2n} F_{x_1 n} \big) \Big],
\end{aligned} \tag{2.28}$$

we have an additional three equilibrium equations of the form

$$\begin{aligned}
M_{O x_1} &\equiv \sum_{n=1}^{N} \big(x_{2n} F_{x_3 n} - x_{3n} F_{x_2 n} \big) = 0, \\
M_{O x_2} &\equiv \sum_{n=1}^{N} \big(x_{3n} F_{x_1 n} - x_{1n} F_{x_3 n} \big) = 0, \\
M_{O x_3} &\equiv \sum_{n=1}^{N} \big(x_{1n} F_{x_2 n} - x_{2n} F_{x_1 n} \big) = 0,
\end{aligned} \tag{2.29}$$

where $\mathbf{M}_O = \sum_{j=1}^{3} \mathbf{E}_j M_{Ox_j}$. In general, the main moment changes with a change of the reduction point (the pole).

From the equations above it follows that the force system of the lines of action laid out arbitrarily in space is in equilibrium if the algebraic equations (2.27) and (2.29) are satisfied. From the mentioned equations it is easy to obtain the relations regarding the forces lying in the selected planes $(O - X_1 - X_2)$ or $(O - X_1 - X_3)$. In the equations above the forces $\mathbf{F}_n$ denote external forces and reactions.

External forces can be divided into *concentrated forces* (applied to points of a body), *surface forces* (applied over certain areas), and *volume forces* (applied to all particles of a body). An example of volume forces are the forces caused by the body weight, and of surface forces—the forces generated by the surface of contact between the bodies being loaded. A set of external forces consists of known forces (*active forces*) and the forces that are subject to determination (*passive forces*).

The set of external (active and passive) forces is called the *loading of a mechanical system*.

Let us consider a wheeled vehicle (a car with four wheels) with an extra load placed on its roof. Here the active forces are the car weight $\mathbf{G}_c$ and load weight $\mathbf{G}_l$. At the points of contact between each wheel and road surface appear four reactions $\mathbf{F}_i^R$. Here the external forces are the forces $\mathbf{G}_c$ and $\mathbf{G}_l$ (active forces) and reactions $\mathbf{F}_i^R$ (passive forces). The remaining forces acting within the system isolated from its environment (i.e., the car and load) are called *internal forces*. According to Newton's third law these forces mutually cancel each other, and for their determination it is necessary to employ the so-called *imaginary cut technique*. For instance, in order to examine the action of the load on the car one should "cut" the system at the points of contact of the load with the car and replace their interaction with reactions, which are the internal forces. Thus we carry out the imaginary division of the analyzed mechanical system into two subsystems in static equilibrium. Then, from the equilibrium equations of either subsystem we determine the desired internal forces.

In order to determine the analytical equilibrium conditions one may also make use of the so-called *three-moments theorem*.

Theorem 2.9. *If we choose three different non-collinear points A_j, then the equilibrium conditions of a material system are*

$$\mathbf{M}_{A_j} = \mathbf{M}_{A_j}\left(\sum_{n=1}^{N} \mathbf{F}_n\right) = \mathbf{0}, \qquad j = 1, 2, 3. \tag{2.30}$$

This means that the main moment of force vectors of the force system about three arbitrary but non-collinear points are equal to zero. The proof follows. Let the equation be satisfied for A_1 (the moment about that point equals zero) and equations for two other points not be satisfied. This means that the system is not equivalent to the couple but to the resultant force that must pass through point A_1.

Now let two equations of moments about points A_1 and A_2 be satisfied. Because it is possible to draw a line through points A_1 and A_2, a non-zero resultant force must lie on that line. However, if additionally the third equation related to point A_3 is satisfied and points A_1, A_2, and A_3 are not collinear (by assumption), then the resultant force must be equal to zero. That completes the proof.

Projecting the moments of forces (2.30) on the three axes of Cartesian coordinate system we obtain nine algebraic equations of the form

$$M_{x_n A_j} = 0, \quad n, j = 1, 2, 3. \tag{2.31}$$

Here, we are dealing with the apparent contradiction because there are nine (2.31) and six (2.27) and (2.29). However, for (2.31), for each force nine coordinates defining its distance from the chosen points A_1, A_2, and A_3 are needed. In the rigid body the distances A_1A_2, A_2A_3, and A_1A_3 are constant, which reduces the number of independent coordinates to six. So, one may proceed in a different way. The six mutually independent axes should be taken in such a way as to formulate six independent equations only.

The equilibrium conditions (2.27) and (2.29) consisting of six equations resulted from projecting the main force vector $\mathbf{S}$, and the main moment of force vector $\mathbf{M}_O$ reduced to an arbitrary point O (Sect. 2.3), where the point of reduction O is the origin of the Cartesian coordinate system.

The state of equilibrium of the considered rigid body means that it does not make any *displacement*, that is, neither translation nor rotation about point O and consequently with respect to the adopted coordinate system $OX_1X_2X_3$. We assume that in the absence of forces $\mathbf{F}_1, \ldots, \mathbf{F}_N$ the rigid body does not move with respect to the adopted coordinate system $OX_1X_2X_3$, and also remains unmoved under the action of the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ if (2.27) and (2.29) are satisfied. If under the action of an arbitrary force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ the rigid body remains in equilibrium with respect to $OX_1X_2X_3$, these forces must satisfy (2.27) and (2.29). Since, if $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$, point O would be subjected to an action of the force $\mathbf{S}$ leading to the loss of static equilibrium state.

If we had $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$, then the main moment $\mathbf{M}_O$ would cause the rotation of the rigid body, leading to the loss of its static equilibrium. Equilibrium equations of the form

$$\mathbf{S} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_N = \mathbf{0},$$
$$\mathbf{M}_O = \mathbf{M}_O(\mathbf{F}_1) + \mathbf{M}_O(\mathbf{F}_2) + \ldots + \mathbf{M}_O(\mathbf{F}_N) = \mathbf{0} \tag{2.32}$$

represent the necessary and sufficient equilibrium conditions for a free (unconstrained) rigid body subjected to the action of an arbitrary three-dimensional force system. The above conditions transform into conditions (2.27) and (2.29) after the introduction of the Cartesian coordinate system of origin at O.

In other words, if we reduce an arbitrary three-dimensional system of forces $\mathbf{F}_n$ applied at the points $A_n(x_{1n}, x_{2n}, x_{3n})$, $n = 1, \ldots, N$ to an arbitrary point (the reduction pole) O, we obtain the main force vector $\mathbf{S}$ and main moment of force vector $\mathbf{M}_O$, and after adopting the Cartesian coordinate system at point O, we have

$$\begin{aligned}\mathbf{S} &= \mathbf{E}_1 \sum_{n=1}^{N} F_{1n} + \mathbf{E}_2 \sum_{n=1}^{N} F_{2n} + \mathbf{E}_3 \sum_{n=1}^{N} F_{3n} \\ &= \mathbf{E}_1 S_1 + \mathbf{E}_2 S_2 + \mathbf{E}_3 S_3,\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\mathbf{M}_O &= \mathbf{E}_1 \sum_{n=1}^{N} [F_{3n}(x_{2n} - x_{20}) - F_{2n}(x_{3n} - x_{30})] \\ &\quad + \mathbf{E}_2 \sum_{n=1}^{N} [F_{1n}(x_{3n} - x_{30}) - F_{3n}(x_{1n} - x_{10})] \\ &\quad + \mathbf{E}_3 \sum_{n=1}^{N} [F_{2n}(x_{1n} - x_{10}) - F_{1n}(x_{2n} - x_{20})] \\ &\equiv \mathbf{E}_1 M_{01} + \mathbf{E}_2 M_{02} + \mathbf{E}_3 M_{03}.\end{aligned} \tag{2.34}$$

In the case of reduction of a three-dimensional force system there exist two *reduction invariants*. The first is the main vector $\mathbf{S}$ and the second the projection of vector $\mathbf{M}_O$ onto the direction of the vector $\mathbf{S}$. The first invariant means that the reduction of spatial forces being in fact a geometrical addition of vectors gives the same result for an arbitrarily chosen point of reduction. The second invariant means that for the arbitrarily chosen point of reduction the projection of the main moment vector onto the direction of the main force vector is constant.

In the latter case it is possible to find such a direction (a straight line) where if the points of reduction lie on that line, the magnitude of $\mathbf{M}_O$ is minimum. On that line we can place the vector $\mathbf{S}$, which as the invariant may be freely moved in space. Such a line, after assigning to it the sense defined by the sense of $\mathbf{S}$, we call the central axis (axis of a wrench). Every set of force vectors and moment of force vectors can have only one central axis. The system of two vectors $\mathbf{S}$ and $\mathbf{M}_O$ lying on the central axis we call a *wrench*.

The general equilibrium conditions (2.27) and (2.29) may be simplified and then boil down to the special cases of the field of forces considered earlier, which we will briefly describe below.

A concurrent force system in space. Taking pole O at the point of intersection of the lines of action of these forces, (2.29) are identically equal to zero, and the equilibrium conditions are described only by three equations (2.27).

An arbitrary force system in a plane. An arbitrary planar force system may be reduced to the main force vector $\mathbf{S}$ and main moment of force vector $\mathbf{M}_O$. After choosing the axis OX_3 perpendicular to the plane of action of the forces the problem of determination of static equilibrium reduces to the analysis of algebraic equations of the form

$$S_1 = \sum_{n=1}^{N} F_{1n} = 0, \quad S_2 = \sum_{n=1}^{N} F_{2n} = 0,$$

$$M_{O_3} = \sum_{n=1}^{N} M_{O_{3n}}(\mathbf{F}_n) = 0. \tag{2.35}$$

A system of parallel forces in space. Let the axis OX_3 be perpendicular to the vector field of parallel forces. From (2.27) and (2.29) for the considered case we obtain

$$\sum_{n=1}^{N} F_{3n} = 0, \quad \sum_{n=1}^{N} M_{O_{1n}}(\mathbf{F}_n) = 0, \quad \sum_{n=1}^{N} M_{O_{2n}}(\mathbf{F}_n) = 0. \tag{2.36}$$

The last case will be analyzed in more detail because we are going to refer to it by the introduction of the notion of a mass center of a system of particles and a mass center of a rigid body.

In the general case an arbitrary force system acting on a rigid body is equivalent to either an action of main force vector or main moment of force vector.

Up to this point the equivalence of two force systems (sets of forces) was formulated in a descriptive way. The criterion of equivalence can, however, be stated as a theorem.

Theorem 2.10. *The necessary and sufficient condition of equivalence of two force systems, acting on a rigid body, with respect to a certain point (pole) is that these systems have identical main force vectors and main moment of force vectors with respect to that point.*

The above theorem can be proved based on the principle of virtual work [27].

The system of parallel forces in three dimensions includes four special cases:

1. The forces may lie on parallel lines and possess opposite senses and different magnitudes.
2. The forces may lie on parallel lines and possess opposite senses but the same magnitudes.
3. The forces may lie on parallel lines and possess the same senses but different magnitudes.
4. The forces may lie on parallel lines and possess the same senses and magnitudes.

Let us limit our considerations to two forces representing the cases listed above. These forces always lie in one plane as they are parallel.

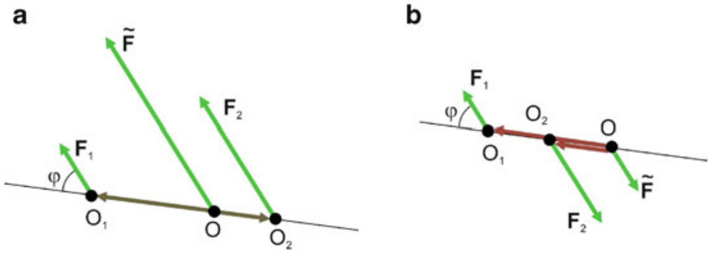


Fig. 2.17 Sketches illustrating Theorem 2.12 (**a**) and Theorem 2.13 (**b**)

We have proved already that if we have two forces of the same magnitudes and opposite senses, we are dealing with a couple. The action of a couple is equivalent to the action of a *moment of a couple*. If these forces were collinear, they would have no effect on the body because they would cancel one another (case 2). If the forces are parallel, had opposite senses and different or identical magnitudes, and arenot collinear, these cases will be considered below. For that purpose we first introduce the notion of an *equivalent equilibrant force*.

If the force system $(\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N)$ applied to a rigid body is equivalent to only one force $\mathbf{F}$, we will call such a force an *equivalent equilibrant force*.

Theorem 2.11. *If a system of forces possesses an equivalent equilibrant force, then the vector of this force* $\tilde{\mathbf{F}}$ *is equal to the main force vector* $\mathbf{F}$ *of the force system, and its moment about an arbitrary pole is equal to the main moment of force of the force system about that pole.*

We will define now an equivalent equilibrant force for the case of two forces of different magnitudes and arbitrary senses [27].

Theorem 2.12. *A system of two parallel forces* $(\mathbf{F}_1, \mathbf{F}_2)$ *having the same senses applied to a rigid body possesses an equivalent equilibrant force* $\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2$. *The force* $\tilde{\mathbf{F}}$ *lies in a plane defined by the forces* $(\mathbf{F}_1, \mathbf{F}_2)$, *is parallel to them, and its line of action divides in the inside a segment* O_1O_2, *which connects the tails of vectors of those forces, into two parts inversely proportional to their magnitudes* F_1 *and* F_2 *(Fig. 2.17a).*

Theorem 2.13. *A system of two parallel forces* $(\mathbf{F}_1, \mathbf{F}_2)$ *having opposite senses and different magnitudes applied to a rigid body possesses an equivalent equilibrant force* $\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2$. *The force* $\tilde{\mathbf{F}}$ *lies in a plane defined by the forces* $(\mathbf{F}_1, \mathbf{F}_2)$, *is parallel to them, and its line of action divides on the outside a segment* O_1O_2, *which connects the tails of vectors of those forces, into two parts inversely proportional to their magnitudes* F_1 *and* F_2 *(Fig. 2.17b).*

We will prove Theorem 2.13. According to Varignon's theorem for point O we have

$$\overrightarrow{OO_1} \times \mathbf{F}_1 - \overrightarrow{OO_2} \times \mathbf{F}_2 = \mathbf{0} \times \tilde{\mathbf{F}} \equiv \mathbf{0}$$

and

$$\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2.$$

From the first equation it follows for both cases that after passing to scalars we have

$$|\overrightarrow{OO_1}| \cdot |\mathbf{F}_1| \sin\varphi - |\overrightarrow{OO_2}| \cdot |\mathbf{F}_2| \sin(180^\circ - \varphi) = 0,$$

that is,

$$|OO_1| \cdot |\mathbf{F}_1| = |OO_2| \cdot |\mathbf{F}_2|.$$

Let us consider now the special case that follows from Fig. 2.17b, that is, when $\mathbf{F}_1$ is applied at point O_1 and at point O_2 the force $\mathbf{F}_2 = -\mathbf{F}_1$, that is, we are dealing with a couple of forces. It is easy to observe that a couple does not possess an equivalent equilibrant force because $\tilde{\mathbf{F}} = \mathbf{F}_1 - \mathbf{F}_1 = \mathbf{0}$. The couple $(\mathbf{F}_1, \mathbf{F}_2)$ does have, however, a different interesting property, which we have already mentioned. The moment produced by the couple depends not on the choice of the pole in space, but only on the distance of the points of application of the forces $\mathbf{F}_1$ and $\mathbf{F}_2 = -\mathbf{F}_1$. It is possible to prove that couples having the same moment of force are *equivalent couples*.

Let us now consider the mixed (hybrid) case of a system of parallel forces in space acting on a rigid body, where some forces have opposite senses and some the same senses.

As was considered earlier on examples of two parallel forces having either the same or opposite senses, we can reduce the whole system of parallel forces to two parallel forces $\tilde{\mathbf{F}}_1$ and $\tilde{\mathbf{F}}_2$, called an *equivalent equilibrant force*.

The force vectors $\mathbf{F}_n$, $n = 1, \ldots, N$ we will treat as bound vectors (as distinct from the free vectors used so far), and the force $\tilde{\mathbf{F}}_1$ represents an equivalent equilibrant force for the forces having senses opposite to the force $\tilde{\mathbf{F}}_2$, replacing the action of the second group of forces. Finally, the problem of reduction in this case is reduced to the problem presented in Fig. 2.17b, which means that we are able to determine the location of the point $O = C$, where an equivalent equilibrant force $(\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2)$ is applied.

The case depicted in Fig. 2.17b enables us to draw certain further conclusions.

Let us observe that according to the proof of Theorem 2.13 at the point $O = C$, the vector of force $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_1 + \tilde{\mathbf{F}}_2$ (Fig. 2.18) is applied and the location of that point depends exclusively on the magnitudes of force vectors, that is, from F_1/F_2. It follows that the locations of points C, O_1, and O_2 do not change at the rotation of force vectors through the same angle α. Let, after the rotation through the same angle α, the lines of action of the mentioned forces be parallel to the axis OX_2 of the adopted Cartesian coordinate system, that is, all the previously mentioned forces after reduction lie in a plane parallel to the plane OX_1X_2.

Let us write an equation of moments about an axis OX_3 (perpendicular to the plane of the drawing in which the forces lie).

If the number of parallel forces is N, then we will denote them as $\mathbf{F}_1, \mathbf{F}_2, \ldots,$ $\mathbf{F}_N$, where those having senses opposite to the positive direction of the axis OX_2

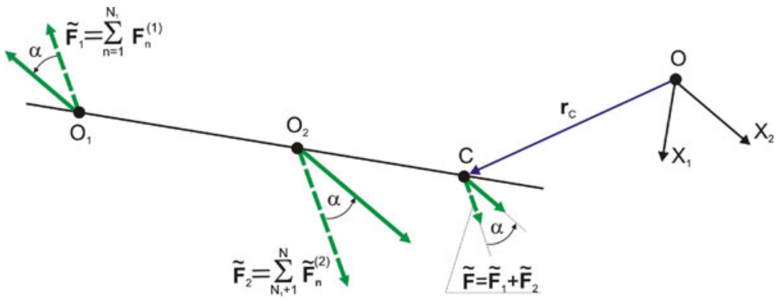


Fig. 2.18 Rotation of forces $\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2$, and $\tilde{\mathbf{F}}$ through an angle α

are defined as $\mathbf{F}_n \circ \mathbf{E}_2 = -F_n$. The sum of moments about a point O is equal to

$$\mathbf{M}_0 = \mathbf{r}_C \times \tilde{\mathbf{F}} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1C} & x_{2C} & x_{3C} \\ 0 & \tilde{F} & 0 \end{vmatrix} = -\mathbf{E}_1 \tilde{F} x_{3C} + \mathbf{E}_3 \tilde{F} x_{1C},$$

$$\mathbf{M}_1 = \sum_{n=1}^{N_1} \left(\mathbf{r}_n \times \tilde{\mathbf{F}}_n^{(1)}\right) = \sum_{n=1}^{N_1} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ 0 & -F_n & 0 \end{vmatrix}$$

$$= \mathbf{E}_1 \sum_{n=1}^{N_1} F_n x_{3n} - \mathbf{E}_3 \sum_{n=1}^{N_1} F_n x_{1n},$$

$$\mathbf{M}_2 = \sum_{n=N_1+1}^{N} \left(\mathbf{r}_n \times \tilde{\mathbf{F}}_n^{(2)}\right) = \sum_{n=N_1+1}^{N} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ 0 & F_n & 0 \end{vmatrix}$$

$$= -\mathbf{E}_1 \sum_{n=N_1+1}^{N} F_n x_{3n} + \mathbf{E}_3 \sum_{n=N_1+1}^{N} F_n x_{1n}, \tag{2.37}$$

and hence we obtain

$$\tilde{F} x_{1C} = \sum_{n=N_1+1}^{N} F_n x_{1n} - \sum_{n=1}^{N_1} F_n x_{1n},$$

$$\tilde{F} x_{3C} = \sum_{n=N_1+1}^{N} F_n x_{3n} - \sum_{n=1}^{N_1} F_n x_{3n}, \tag{2.38}$$

where

$$\tilde{F} = \sum_{n=N_1+1}^{N} F_n - \sum_{n=1}^{N_1} F_n. \tag{2.39}$$

If we deal only with N parallel forces (we assume $N_1 = 0$) of the same senses, from (2.38) we obtain

$$\left(\sum_{n=1}^{N} F_n\right) x_{1C} = \sum_{n=1}^{N} F_n x_{1n},$$
$$\left(\sum_{n=1}^{N} F_n\right) x_{3C} = \sum_{n=1}^{N} F_n x_{3n}, \tag{2.40}$$

which defines the position of the center C of the parallel forces having the same senses consistent with the sense of $\mathbf{E}_2$ in the adopted coordinate system.

The third missing equation is

$$\left(\sum_{n=1}^{N} F_n\right) x_{2C} = \sum_{n=1}^{N} F_n x_{2n},$$

which allows us, through the suitable choice of two out of the three presented equations, to determine the position of the center of parallel forces in each of the planes OX_1X_2, OX_2X_3, and OX_1X_3.

If now in the selected points $n = 1, \ldots, N$ we apply the vectors of parallel forces $m_n\mathbf{g}$ (weights), where $\mathbf{g}$ is the acceleration of gravity, we can determine the center of gravity of those forces after introducing the Cartesian coordinate system such that $\mathbf{F}_n = m_n g\mathbf{E}_3$.

The gravity center is coincident with the mass center of the given discrete mechanical system in the gravitational field.

Equation (2.40) listed above can be obtained from the following equation:

$$\mathbf{r}_C \times \sum_{n=1}^{N} \mathbf{F}_n = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{F}_n. \tag{2.41}$$

2.5 Mechanical Interactions, Constraints, and Supports

In Chap. 1 the concept of material system as a collection of particles was introduced. In many cases such simplification is not sufficient. Particles are a special case of rigid bodies whose geometrical dimensions were reduced to zero and only their masses were left. A natural consequence of expansion of the concept of system of particles is a *system of rigid bodies*. Such bodies can act on each other depending on how they are connected through *forces* and *moments of forces*. As was discussed earlier, forces in the problems of statics are treated as *sliding vectors* and moments of forces as *free vectors*.

The introduced system of rigid bodies is a system isolated from its surroundings in the so-called *modeling process*. In view of that, the surroundings act mechanically on the isolated system of rigid bodies. Such modeling leads naturally to the introduction of the notions of *active* and *passive* mechanical interactions.

Active mechanical interactions are forces and moments coming from the surroundings and acting on the considered system of bodies (they include the interactions produced by gravitational fields, by various pneumatic or hydraulic actuators, by engines, etc.).

Active mechanical interactions produce, according to Newton's third law, passive mechanical interactions, that is, interactions between the rigid bodies in the considered system. The passive interactions (forces and moments of reactions) we determine by performing a mental release from constraints of the bodies of the system interacting mechanically.

According to the other classification criterion of mechanical interactions we divide them into *external* (the counterpart of active) and *internal* (the counterpart of passive). The classification of the system of bodies depending on their number in the system is also introduced. If we are dealing with a single body (many bodies), the system is called *simple* (*complex*, *multibody*).

The simple system is one rigid body that can be in static equilibrium under the action of either external interactions exclusively or *hybrid* interactions, i.e., *external* and *internal*. Let us consider two bodies and assume that one of them was fixed. Next, the internal interaction of these bodies is replaced by the reaction forces and reaction moments of forces coming from the interactions (supports) of the fixed body (now called the base) on the free body, and now the mentioned supports are treated as external.

In the case of simple problems (one rigid body), the problem of statics boils down to releasing the body from supports, and the introduction of the mentioned reaction forces and reaction moments of forces, and then to application of (2.27) and (2.29) and their solution. During the solution of statics equations the following three cases may occur:

1. The number of equations is equal to the number of unknowns (then the problem is statically determinate and the solution of linear algebraic equations for the forces and moments of forces can be done easily).
2. The number of equations is smaller than the number of unknowns (then the problem is statically indeterminate; in order to solve the problem it is necessary to know the relation between deformation and force (stress) fields).
3. The number of equations is greater than the number of unknowns (then the body becomes a *mechanism* and the excessive forces and moments of forces can be treated as driving ones).

If we are dealing with complex system of bodies and we have a base isolated in this system, we first detach the system from the base and after that we proceed similarly as in the case of the simple system. Equilibrium conditions of such an inextricable force system are *necessary equilibrium conditions*. In the next step we mentally divide the system into subsystems detaching one by one the bodies interacting mechanically until we isolate the body and have the forces and moments of forces coming from the interactions with other bodies clearly defined. The equilibrium conditions of each of the isolated bodies now constitute the necessary and sufficient conditions for the equilibrium of the isolated body.

Let us note that the solution of the statically determinate problem is reduced to the determination of both the equilibrium position (equilibrium configuration) of the system and the unknown forces and moments of forces keeping the system in an equilibrium position.

The topic of constraints and degrees of freedom of a system will be covered in detail later; however, here we will introduce some basic notions essential to solving problems of statics.

A particle in space has three possibilities of motion (three degrees of freedom), whereas a rigid body in space has six possibilities of motion (six degrees of freedom). Since, if we introduce the geometry of the "point" (the dimension), the body gains the possibility of three independent rotations. Such a body (a particle) we call *free*. Its contact with another body occurs by means of constraints that in the analyzed problems of *statics* are called *supports*.

Now, if a rigid body (particle) during its motion is in contact with a plane at all times, the rigid body has three degrees of freedom (two translations and one rotation), whereas the particle has two degrees of freedom (two translations in the adopted coordinate system).

The mentioned plane plays the role of support, which in the case of a rigid body eliminates three degrees of freedom, and in the case of a particle, one degree of freedom. Because the particle and the rigid body by definition possess mass and mass moments of inertia, an arbitrary support (in our case the plane) produces a mechanical (supporting) interaction, that is, support forces and moments of support forces.

Depending on the geometry (shape) of the supports, they can eliminate different types of body (bodies) motion and thus create different reactions and support moments.

Below we will briefly characterize some of the supports often used in mechanical systems. Because we have already distinguished plane and space force systems, we will also classify the supports in accordance with the generation of planar and spatial force systems by them.

In Fig. 2.19 a few examples of supports with a two-dimensional *force* system are presented.

In Fig. 2.19a a sliding support is shown. The unknowns are the magnitude of reaction $\mathbf{F}^R$, because its direction is perpendicular to the radius of curvature of a base, and the reaction torque $\mathbf{M}_R$ (rotation is not possible). A similar role is played by a sleeve and telescope. In the case of Fig. 2.19b, the pin support is also present (see also Fig. 2.19c), where we have two unknowns (magnitude and direction of reaction $\mathbf{F}^R$). In order to diminish the resistance to motion (friction) between the systems in contact often rollers treated as massless elements and moving with no

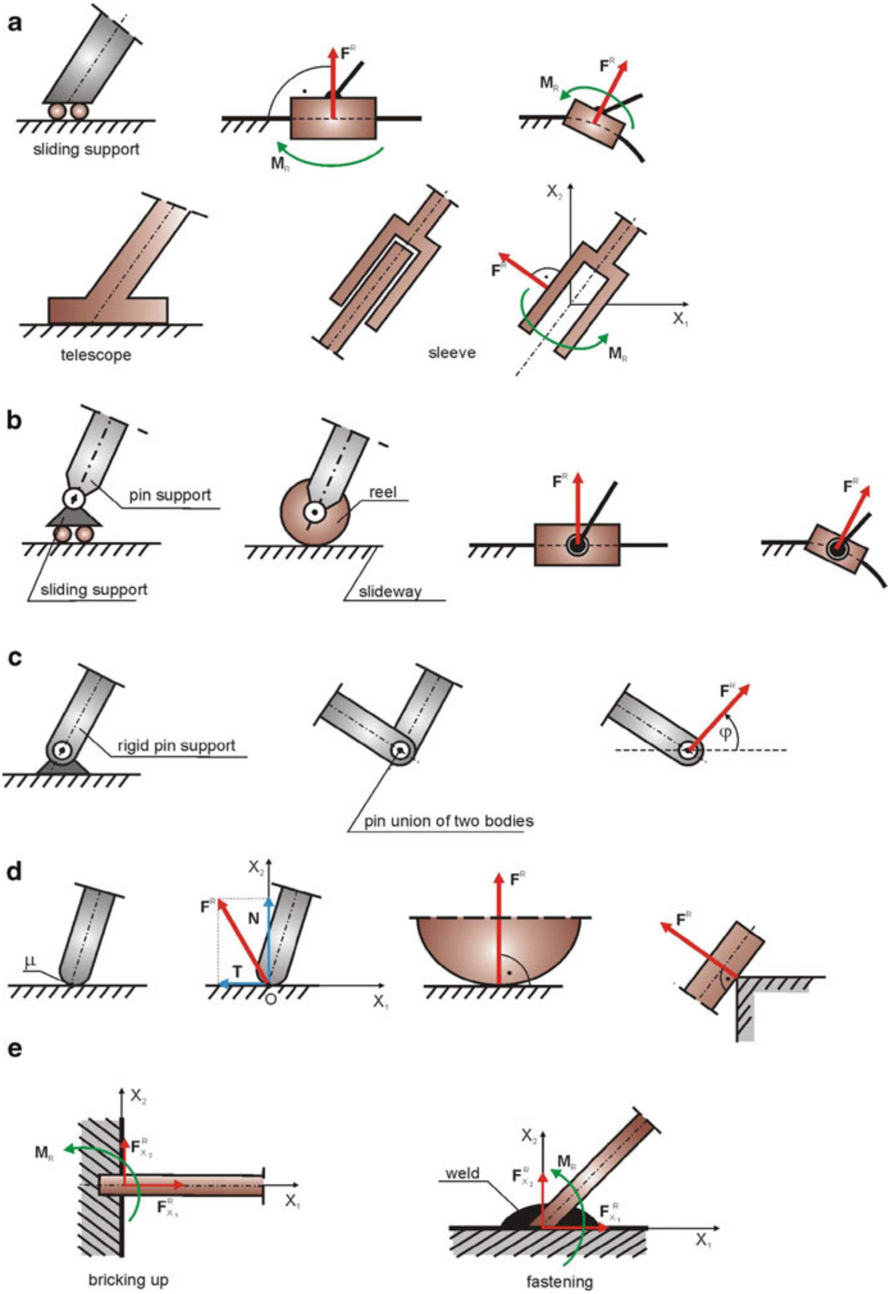


Fig. 2.19 Examples of supports with two-dimensional systems of forces, their schematic designations, and unknown reactions (moments)

resistance to motion are used (Fig. 2.19c). Rollers are often introduced in order to decrease the motion resistance (friction) between contacting bodies, and they are treated as massless bodies that are movable without motion resistance. In Fig. 2.19d the contact supports with and without friction are shown. In the latter case we are dealing with one unknown, that is, the magnitude of reaction $\mathbf{F}^R$, since its direction is defined.

In the case of a rigidly clamped edge and restraint (Fig. 2.19e), we have three unknowns, that is, two components of reaction and the reaction torque.

The supports discussed above that enable the point (the element) of a rigid body in contact with a base to move in the specific (assumed in advance) direction (technical and technological manufacture of the elements of bodies in contact) we call *directional support*. In this case the motion of the body can take place along a straight line or a curve.

Until now, our considerations have not taken into account the phenomenon of friction between the elements of a body in contact. The phenomenon of friction is the subject of the next section; here it is only necessary to emphasize that in the case of so-called fully developed friction ($T = \mu N$) we are dealing with one unknown, and in case where $T < \mu N$ (not developed friction), two unrelated forces T and N have to be determined.

In Fig. 2.20a, a ball-and-socket joint and a point contact are shown, and if a rigid body makes an arbitrary motion while in contact with a plane, we are to determine three unknown reactions $F^R_{x_1}$, $F^R_{x_2}$, and $F^R_{x_3}$. If the connection between the bodies is realized by ball-and-socket support, it prevents the translation of a body in every direction but allows for rotation about any axis passing through the center of the support (ball), and after introduction of the Cartesian coordinate system, three magnitudes of support reaction undergo determination. If the body moves on a rough surface with undeveloped friction, where $\sqrt{(T_1^2 + T_2^2)} < \mu N$, then three unknowns, T_1, T_2, and N, must also be determined. Finally, let us note that if the contact surface between the contacting parts of the bodies is from one side a sphere and from the other a cylindrical groove (the guide), the support is a ball-cylinder joint (two unknown reactions). If, instead, the contact of bodies takes place over a cylindrical surface, such a support we call a cylindrical joint (two unknown reaction forces and two reaction moments).

If a rigid body is fixed in three dimensions (Fig. 2.20b), such a support prevents any translation and any rotation of the body, and the determination of six unknown quantities, that is, three support reactions and three reaction moments, is necessary.

If the contact of a body with a plane takes place by means of a roller moving on a rough surface with undeveloped friction, two forces T and N are subject to determination. If the roller moves along the guide, this kind of support generates two unknown reactions as well (Fig. 2.20c).

In Fig. 2.20d–g, short and long radial and angular bearings are depicted along with the corresponding reaction forces and reaction moments; their diagrams are also shown.

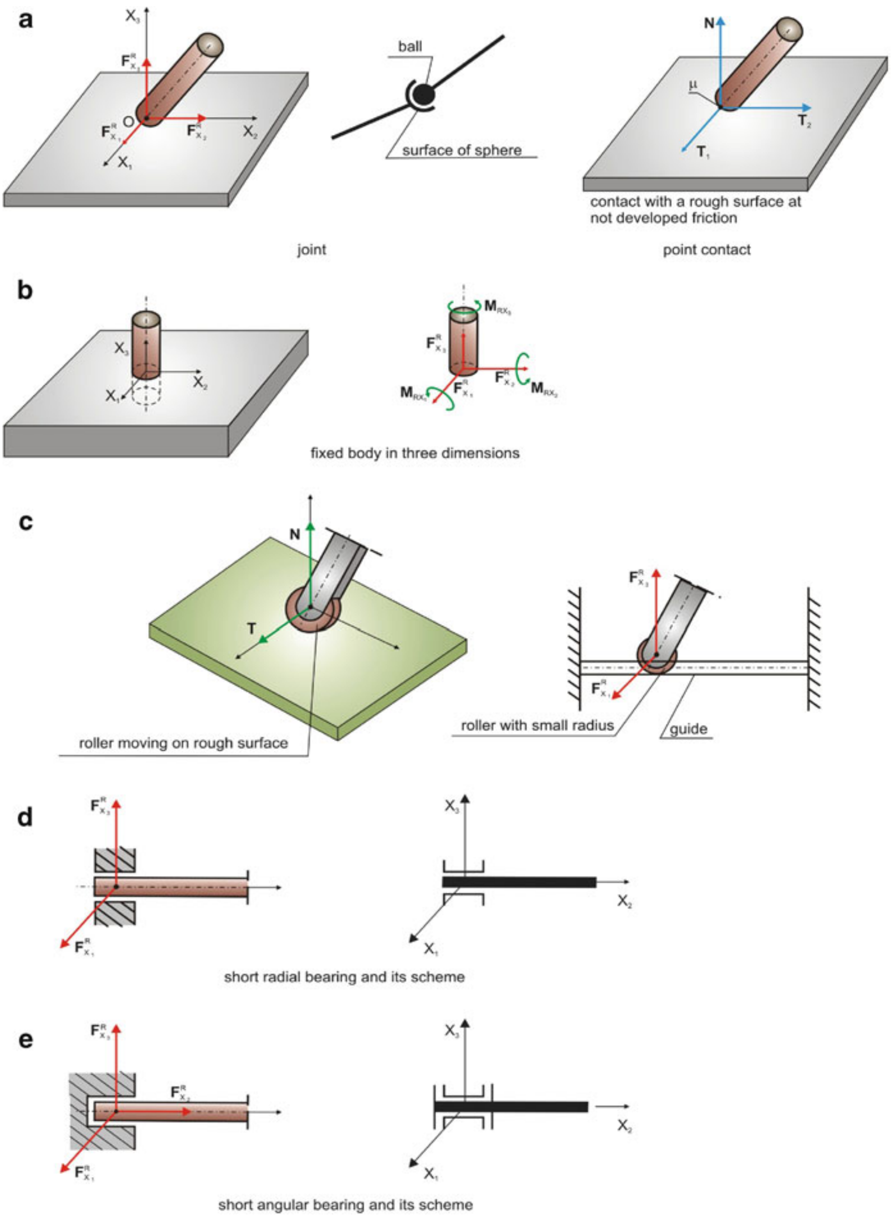
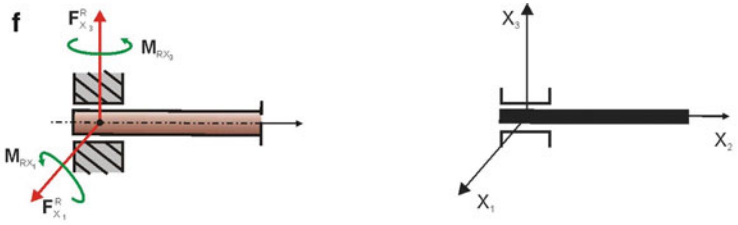
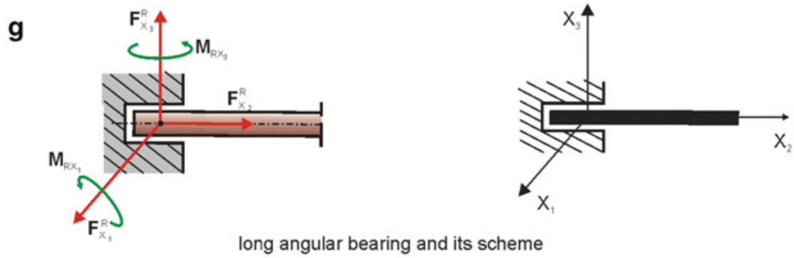


Fig. 2.20 Examples of supports carrying three-dimensional systems of forces

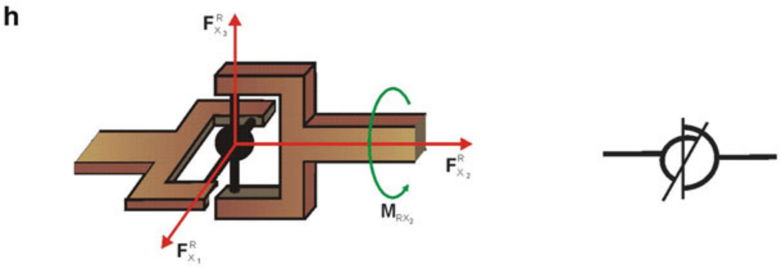
In Fig. 2.20h, the Cardan universal joint is shown. The quantities $F_{x_1}^R$, $F_{x_3}^R$, and M_{Rx_2}, which act on the right-hand side of the joint and undergo determination, are also marked.



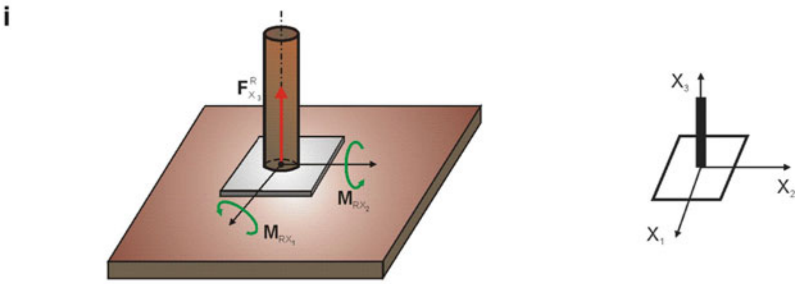
long radial bearing and its scheme



long angular bearing and its scheme



Cardan universal joint



three dimensional telescope and its scheme

Fig. 2.20 (continued) Examples of supports carrying three dimensional systems of forces

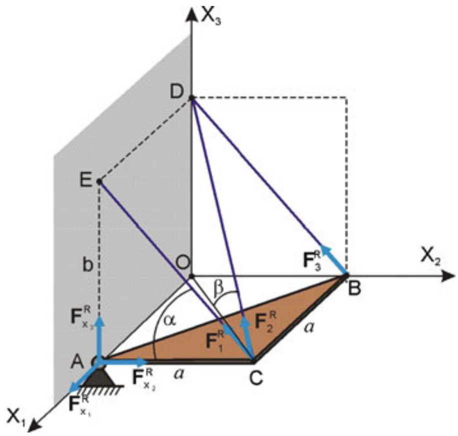


Fig. 2.21 Triangular plate suspended by three massless rods CE, CD, and BD

In Fig. 2.20i, a three-dimensional telescope is depicted along with three unknown quantities that are subject to determination. Figures 2.19 and 2.20 were prepared based on the textbooks [28, 29].

In practice, in order to determine the reactions of the base on a rigid body we choose systems of axes suitable for the particular problem and perform the projections of reaction forces and reaction moments on these axes. Let us note that the axes do not have to be mutually perpendicular, but not all of them can lie in one plane or be parallel.

In this way we solve a problem of statics in several stages.

Example 2.3. A homogeneous steel plate of the shape of a right, isosceles triangle and of the weight **G** is fixed by means of three steel weightless cables and a ball-and-socket joint at a point A (Fig. 2.21), and $AO = OB = BC = CA = a$, $AE = OD = b$. Determine reactions at point A and forces in the cables.

From equations of equilibrium with respect to the axes of the coordinate system we have

$$\begin{aligned} OX_1: &\quad F^R_{x_1} - F^R_{2x_1} = 0,\\ OX_2: &\quad F^R_{x_2} - F^R_{1x_2} - F^R_{2x_1} - F^R_{3x_2} = 0,\\ OX_3: &\quad F^R_{x_3} + F^R_{1x_3} + F^R_{2x_3} + F^R_{3x_3} - G = 0, \end{aligned}$$

and from equilibrium conditions of the moments of forces we obtain

$$\begin{aligned} M_{x_1}: &\quad a\left(F^R_{1x_3} + F^R_{2x_3} + F^R_{3x_3}\right) - \frac{2}{3}aG = 0,\\ M_{x_2}: &\quad -a\left(F^R_{1x_3} + F^R_{2x_3} - F^R_{x_3}\right) + \frac{2}{3}aG = 0,\\ M_{x_3}: &\quad a\left(-F^R_{1x_2} + F^R_{x_2}\right) = 0, \end{aligned}$$

because the gravity center of the plate is situated at the distance marked in Fig. 2.22a.

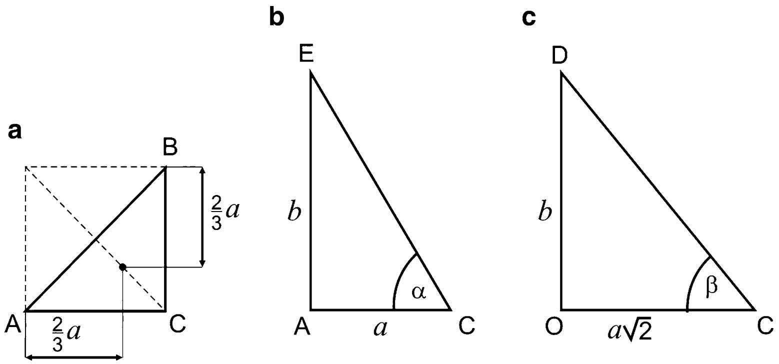


Fig. 2.22 Auxiliary sketches: (**a**) location of gravity center of plate; geometry of triangles related to rod CE (**b**) and CD (**c**)

The angles α and β are determined from the equations $\tan\beta = b/(a\sqrt{(2)})$ and $\tan\alpha = b/a$ and the equilibrium equations assume the following form:

$$\begin{cases} F_{x_1}^R - \frac{\sqrt{2}}{2} F_2^R \cos\beta = 0, \\ F_{x_2}^R - F_1^R \cos\alpha - \frac{\sqrt{2}}{2} F_2^R \cos\beta - F_3^R \cos\alpha = 0, \\ F_{x_3}^R + F_1^R \sin\alpha + F_2^R \sin\beta + F_3^R \sin\alpha - G = 0, \\ a(F_1^R \sin\alpha + F_2^R \sin\beta + F_3^R \sin\alpha) - \frac{2}{3} aG = 0, \\ -a(F_1^R \sin\alpha + F_2^R \sin\beta) - aF_{x_3}^R + \frac{2}{3} aG = 0, \\ a(-F_1^R \cos\alpha + F_{x_2}^R) = 0. \end{cases} \qquad (*)$$

From the third equation of the system $(*)$ we obtain

$$-F_{x_3}^R + G = F_1^R \sin\alpha + F_2^R \sin\beta + F_3^R \sin\alpha,$$

and substituting this into the fourth equation $(*)$ we obtain

$$-F_{x_3}^R + G - \frac{2}{3} G = 0,$$

that is,

$$F_{x_3}^R = \frac{1}{3} G.$$

Substituting $F_{x_3}^R$ into the fifth equation of $(*)$ we obtain

$$F_1^R \sin\alpha + F_2^R \sin\beta = \frac{1}{3}G, \qquad (**)$$

and substituting into the third equation of $(*)$ we have

$$F_3^R = \frac{G}{3\sin\alpha}.$$

According to the sixth equation of the system $(*)$, the second equation of $(*)$ takes the form

$$\frac{\sqrt{2}}{2}F_2^R\cos\beta + F_3^R\cos\alpha = 0,$$

and hence after substituting the already known F_3^R we obtain

$$F_2^R = -\frac{2G\cos\alpha}{3\sqrt{2}\sin\alpha\cos\beta} = -\frac{2}{3\sqrt{2}}G\frac{\cot\alpha}{\cos\beta},$$

and substituting F_2^R into $(**)$ we obtain

$$F_1^R = \frac{G}{3\sin\alpha}\left(1 + \sqrt{2}\cot\alpha\tan\beta\right).$$

Using the sixth equation of the system $(*)$ we have

$$F_{x_2}^R = \frac{G\cot\alpha}{3}\left(1 + \sqrt{2}\cot\alpha\tan\beta\right),$$

and on the basis of the first equation of $(*)$ we finally arrive at

$$F_{x_1}^R = -\frac{G}{3}\cot\alpha. \qquad \square$$

Example 2.4. Determine the reactions in support bearings of a rigid body supported along the vertical axis at two points A and B by means of ball bearings (Fig. 2.23) and loaded with an arbitrary force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$.

As a result of the projection of forces on the axes of the adopted coordinate system we obtain

$$F_{Ax_1}^R + F_{Bx_1}^R + \sum_{n=1}^{N} F_{nx_1} = 0,$$

$$F_{Ax_2}^R + F_{Bx_2}^R + \sum_{n=1}^{N} F_{nx_2} = 0,$$

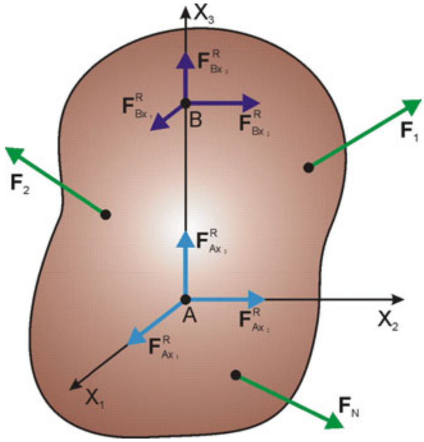


Fig. 2.23 Rigid body supported by ball bearings

$$F^R_{Ax_3} + F^R_{Bx_3} + \sum_{n=1}^{N} F_{nx_3} = 0.$$

The equations of moments about axes AX_1 and AX_2 have the following form:

$$-F^R_{Bx_2} AB + \sum_{n=1}^{N} M_{x_1}(\mathbf{F}_n) = 0,$$

$$F^R_{Bx_1} AB + \sum_{n=1}^{N} M_{x_2}(\mathbf{F}_n) = 0.$$

From the two equations above we determine $F^R_{Bx_1}$ and $F^R_{Bx_2}$, and from the first and second equations of the previous system of equations we determine $F^R_{Ax_1}$ and $F^R_{Ax_2}$.

The equation of moments about the axis AX_3 reads

$$\sum_{n=1}^{N} M_{x_3}(\mathbf{F}_n) = 0,$$

and it defines the relations between the force vectors so that the body remains in static equilibrium.

From the considerations above it follows that the third equation of the first system of equations allows for the determination of the sum of reactions $F^R_{Ax_3}$ and $F^R_{Bx_3}$, but it does not enable us to determine the individual magnitudes of $F^R_{Ax_3}$ and $F^R_{Bx_3}$. The problem becomes statically determinate if at point B the short radial bearing is applied (Fig. 2.20d). Then $F^R_{Bx_3} = 0$, and it is possible to determine the reaction $F^R_{Ax_3}$.

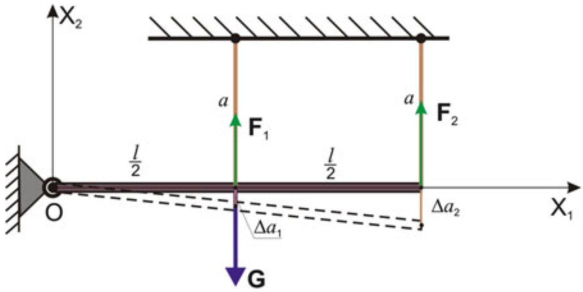


Fig. 2.24 A beam of weight **G** suspended by two steel ropes

We will now show using an example in what way statically indeterminate problems require a knowledge of kinematics. Without knowing the basic relationships of kinematics it is impossible to solve this category of problems. In Chap. 6 we generalize our observations, i.e., we solve the problem of kinematics of an elastic body (a body deformable within the limits of the linear theory). □

Example 2.5. A homogeneous beam of weight **G** and length l was suspended from two steel ropes of lengths a and cross sections S, shown in Fig. 2.24.

The sum of moments about point O leads to the following equation:

$$F_1\frac{l}{2} + F_2 l = G\frac{l}{2},$$

that is,

$$F_1 + 2F_2 = G. \qquad (*)$$

From the equation of moments relative to the center of beam, the vertical reaction at point O is equal to F_2. Let the rigid beam be in equilibrium after deformation of both ropes. According to Hooke's law and displacement compatibility (the same angle of rotation of the beam $\Delta\varphi$) we have

$$\sigma_1 = E\varepsilon_1, \qquad \sigma_2 = E\varepsilon_2,$$

where E is Young's modulus. In turn,

$$\frac{\sigma_1}{E} \equiv \varepsilon_1 = \frac{\Delta a_1}{a} = \frac{l\Delta\varphi}{2a}, \qquad \frac{\sigma_2}{E} \equiv \varepsilon_2 = \frac{\Delta a_2}{a} = \frac{l\Delta\varphi}{a}.$$

Thus

$$\frac{F_1}{E} = \frac{l\Delta\varphi S}{2a}, \qquad \frac{F_2}{E} = \frac{l\Delta\varphi S}{a},$$

and hence

$$\frac{2aF_1}{ElS} = \frac{aF_2}{ElS},$$

that is, $2F_1 = F_2$. After substituting this relation into (∗) we obtain successively $F_1 = \frac{1}{5}G$, $F_2 = \frac{2}{5}G$. □

2.6 Reduction of a Space Force System to a System of Two Skew Forces

We have already mentioned the possibility of reduction of a space force system into a so-called *equivalent system of three forces*, and then to a system of two skew forces, i.e., the forces of non-intersecting lines of action.

Let a rigid body be loaded at points B_n with forces $\mathbf{F}_n$, $n = 1, \ldots, N$. Let us take one of the points on the line of action of the force $\mathbf{F}_n$ and denote it O_n. Let us take three arbitrary, non-collinear points B_1, B_2, and B_3 and link them with point O_n (Fig. 2.25).

We attach the free vector $\mathbf{F}_n$ at point O_n and resolve it into components along the axes $\overrightarrow{O_n B_1}$, $\overrightarrow{O_n B_2}$, and $\overrightarrow{O_n B_3}$, that is,

$$\mathbf{F}_n = \mathbf{F}_{nB_1} + \mathbf{F}_{nB_2} + \mathbf{F}_{nB_3}. \tag{2.42}$$

We will proceed similarly regarding other forces, i.e., on each of their lines of action we will take a point, where we will attach the mentioned force vector, and again we will resolve it into components that link the selected point with the fixed points B_1, B_2, and B_3. In that way we will reduce all forces to three forces attached at the three chosen points B_1, B_2, and B_3, i.e., we have

$$\mathbf{F}_{B_1} = \sum_{n=1}^{N} \mathbf{F}_{nB_1}, \quad \mathbf{F}_{B_2} = \sum_{n=1}^{N} \mathbf{F}_{nB_2}, \quad \mathbf{F}_{B_3} = \sum_{n=1}^{N} \mathbf{F}_{nB_3}. \tag{2.43}$$

In a general case these three forces are situated in space and do not intersect each other. Now we will show how to reduce those forces to two skew forces (Fig. 2.26).

At the mentioned points B_1, B_2, and B_3 in Fig. 2.26 we attach the force vectors $\mathbf{F}_{B_1}$, $\mathbf{F}_{B_2}$, and $\mathbf{F}_{B_3}$ determined earlier.

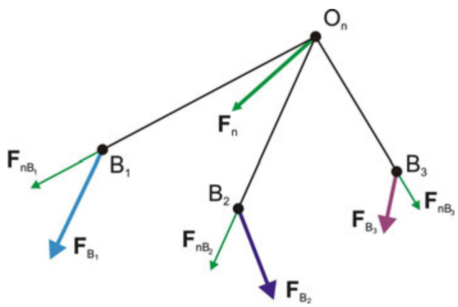


Fig. 2.25 Reduction of a space force system to forces $\mathbf{F}_{B_1}$, $\mathbf{F}_{B_2}$, and $\mathbf{F}_{B_3}$

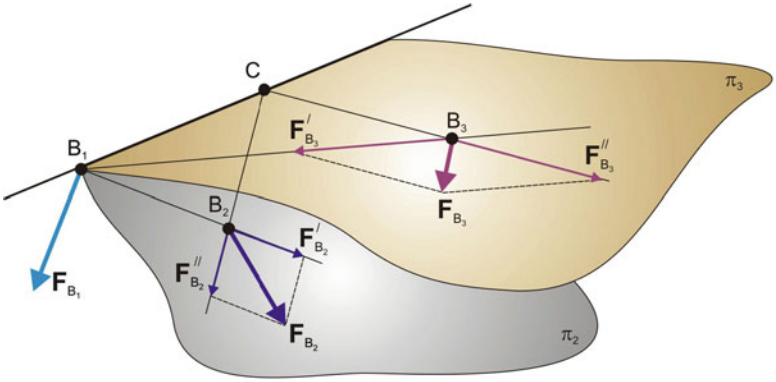


Fig. 2.26 Reduction of a force system to two skew forces

Point B_1 and vector $\mathbf{F}_{B_2}$ determine the plane π_2, and point B_1 and vector $\mathbf{F}_{B_3}$ determine the plane π_3. These planes intersect along an edge on which lies point B_1. Let us take now point C on that edge and connect both points B_1 and C lying on the edge with points B_2 and B_3. We will resolve the vector of force $\mathbf{F}_{B_2}$ in the plane π_2 into components along the rays CB_2 and B_1B_2, and the vector $\mathbf{F}_{B_3}$ in the plane π_3 into components along the rays CB_3 and B_1B_3 using the parallelogram law.

Because we are dealing with a rigid body, we will attach vectors $\mathbf{F}'_{B_2}$ and $\mathbf{F}'_{B_3}$ at point B_1 and vectors $\mathbf{F}''_{B_2}$ and $\mathbf{F}''_{B_3}$ at point C. Let us note that at points B_1 and C we obtain three-dimensional force systems that are concurrent. According to the considerations made earlier, the action of concurrent forces at those points can be replaced with resultant forces

$$\begin{aligned} \mathbf{F}^r_{B_1} &= \mathbf{F}_{B_1} + \mathbf{F}'_{B_2} + \mathbf{F}'_{B_3}, \\ \mathbf{F}^r_C &= \mathbf{F}''_{B_2} + \mathbf{F}''_{B_3}. \end{aligned} \tag{2.44}$$

These forces do not intersect and their lines of action are not parallel. The system of parallel forces $\mathbf{F}_n$, $n = 1, \ldots, N$ acting on a rigid body will be in equilibrium if these forces are collinear and $\mathbf{F}^r_{B_1} + \mathbf{F}^r_C = \mathbf{0}$. It is difficult to satisfy such a condition proceeding with a geometrical construction.

On the other hand, according to the previous construction, taking an arbitrary pole O we can reduce these forces to the main force vector $\mathbf{F} = \mathbf{F}^r_{B_1} + \mathbf{F}^r_C$ and main moment of force vector $\mathbf{M}_O = \mathbf{M}_O(\mathbf{F}^r_{B_1}) + \mathbf{M}_O(\mathbf{F}^r_C)$.

2.7 Reduction of a Space Force System to a Wrench

Expanding the earlier considerations concerning a plane force system, one can perform the reduction of an arbitrary space force system $\mathbf{F}_n$ attached at the points $A_n(x_{1n}, x_{2n}, x_{3n})$, $n = 1, \ldots, N$ to a certain arbitrarily chosen reduction pole $O(x_{10}, x_{20}, x_{30})$.

Let us consider the rigid body depicted in Fig. 2.11a; now the force system $\mathbf{F}_n$ is three-dimensional. In that figure we did not introduce the moments of force as an external load (they can be generated by the force vectors of the same lengths and opposite senses acting in parallel directions). In this case (2.27) will remain unchanged (now $\mathbf{S} \neq \mathbf{0}$), but (2.29) will assume the form

$$M_{x_1} = \sum_{n=1}^{N} \left[(x_{2n} - x_{20}) F_{x_3 n} - (x_{3n} - x_{30}) F_{x_2 n} \right],$$

$$M_{x_2} = \sum_{n=1}^{N} \left[(x_{3n} - x_{30}) F_{x_1 n} - (x_{1n} - x_{10}) F_{x_3 n} \right],$$

$$M_{x_3} = \sum_{n=1}^{N} \left[(x_{1n} - x_{10}) F_{x_2 n} - (x_{2n} - x_{20}) F_{x_1 n} \right]. \tag{2.45}$$

In general, the magnitudes of the components of the main moment of force vector depend on the choice of pole O, but the main force vector $\mathbf{S}$ does not change with the position of pole O (it is the *first invariant* of reduction in statics). Often the notion of that invariant would be introduced in the form $I_1 = S_{x_1}^2 + S_{x_2}^2 + S_{x_3}^2$. It turns out that the *second invariant* of reduction is the projection of the main moment of force vector $\mathbf{M}$ on the direction of the main force vector $\mathbf{S}$, which then has the form $I_2 = \mathbf{M} \circ \mathbf{S}$.

Let us consider now the projection of vector $\mathbf{M}$ on the direction of the vector $\mathbf{S}$ of the following form:

$$\mathbf{M} \circ \mathbf{S} = |\mathbf{M}||\mathbf{S}| \cos\varphi = M_S |\mathbf{S}|. \tag{2.46}$$

Bearing in mind that $\dfrac{\mathbf{S}}{|\mathbf{S}|}$ is the unit vector, we have

$$\mathbf{M}_S = M_S \frac{\mathbf{S}}{|\mathbf{S}|} = \frac{\mathbf{M} \circ \mathbf{S}}{|\mathbf{S}|^2} \mathbf{S}, \tag{2.47}$$

where $\frac{\mathbf{M} \circ \mathbf{S}}{|\mathbf{S}|^2} = p$ is called the pitch of the wrench. Now, let us change the pole from point O to some other point O' (Fig. 2.27).

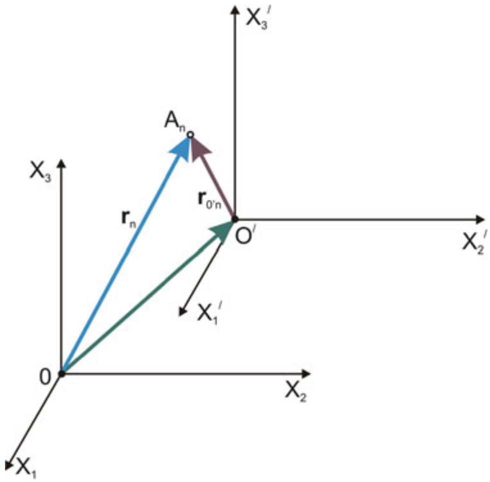


Fig. 2.27 Two poles O and O' with corresponding Cartesian coordinate systems

According to Fig. 2.27, the main moment of force vector $\mathbf{M}_O$ about pole O' is equal to

$$\begin{aligned}\mathbf{M}_{O'} &= \sum_{n=1}^{N} (\mathbf{r}_{O'n} \times \mathbf{F}_n) = \sum_{n=1}^{N} \left(\mathbf{r}_n + \overrightarrow{O'O}\right) \times \mathbf{F}_n \\ &= \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n) + \sum_{n=1}^{N} \left(\overrightarrow{O'O} \times \mathbf{F}_n\right) \\ &= \mathbf{M} + \overrightarrow{O'O} \times \sum_{n=1}^{N} \mathbf{F}_n = \mathbf{M} + \overrightarrow{O'O} \times \mathbf{S},\end{aligned} \tag{2.48}$$

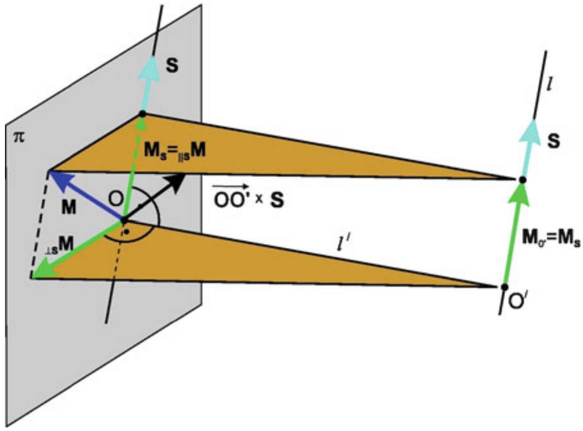
where $\mathbf{M}$ is the main moment about point O.

After projection of the vector $\mathbf{M}_{O'}$ on the direction of $\mathbf{S}$ we obtain

$$\begin{aligned}\mathbf{M}_{O'} \circ \mathbf{S} &= \left[\mathbf{M} + \left(\overrightarrow{O'O} \times \mathbf{S}\right)\right] \circ \mathbf{S} \\ &= \mathbf{M} \circ \mathbf{S} + \left(\overrightarrow{O'O} \times \mathbf{S}\right) \circ \mathbf{S} = \mathbf{M} \circ \mathbf{S}\end{aligned} \tag{2.49}$$

because the vector $\overrightarrow{O'O} \times \mathbf{S}$ is perpendicular to vector $\mathbf{S}$ (the scalar product $(\overrightarrow{O'O} \times \mathbf{S}) \circ \mathbf{S} = 0$). Thus, we showed that the change of the pole does not change the projection of a main moment of force vector on the direction of a main force vector. Let us take once more some other arbitrary pole. Let this be point O to which we reduced, in the already described fashion, the force system. Through point O we will draw a line l parallel to vector $\mathbf{S}$ and attach at that point vector $\mathbf{M}$. Vectors $\mathbf{M}$ and $\mathbf{S}$ will determine a certain plane π (Fig. 2.28).

Fig. 2.28 A construction leading to the determination of an axis of a wrench l



In that plane we will resolve vector $\mathbf{M}$ into two components: ${}_{\|\mathbf{S}}\mathbf{M}$ and ${}_{\perp\mathbf{S}}\mathbf{M}$. Now, through point O we will draw a line l' perpendicular to plane π and we choose as the new pole O' (belonging to both l and l') with respect to which we will perform the reduction of $\mathbf{M}$ and $\mathbf{S}$. Let us note that

$$\mathbf{M}_{O'} = \mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}, \tag{2.50}$$

and since

$$\mathbf{M} = {}_{\|\mathbf{S}}\mathbf{M} + {}_{\perp\mathbf{S}}\mathbf{M}, \tag{2.51}$$

we have

$$\begin{aligned} \mathbf{M}_{O'} &= {}_{\|\mathbf{S}}\mathbf{M} + {}_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S} \\ &= \mathbf{M}_{\mathbf{S}} + {}_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}. \end{aligned} \tag{2.52}$$

The component ${}_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}$ undergoes a change with the change in position of point O'. Let us take point O' so that $\mathbf{M}'_O = \mathbf{M}_S$. From (2.52) it follows that

$$\begin{aligned} \mathbf{M}_{\mathbf{S}} &= {}_{\perp\mathbf{S}}\mathbf{M} + \mathbf{M}_{\mathbf{S}} + \overrightarrow{O'O} \times \mathbf{S}, \\ {}_{\perp\mathbf{S}}\mathbf{M} &= \overrightarrow{OO'} \times \mathbf{S}. \end{aligned} \tag{2.53}$$

From (2.53) it follows that

$$|\overrightarrow{OO'}| = \frac{|{}_{\perp\mathbf{S}}\mathbf{M}|}{|\mathbf{S}|}, \tag{2.54}$$

which defines the distance of point O from point O'.

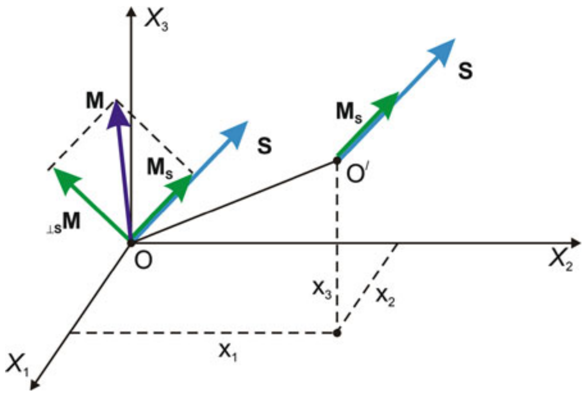


Fig. 2.29 Force S, main moment of force, and location of central axis

The axis passing through point O' and parallel to vector **S** we call a *central axis*. By the choice of an arbitrary pole O it was shown how to determine the location of the wrench axis by means of construction through the proper choice of point O'.

From the foregoing considerations we can draw an important conclusion.

The action of an arbitrary space force (and moments) system can be replaced by the action of a single force (being the geometric sum of all forces) and one main moment of force, and both vectors lie on a common line called the *central axis*.

Equations (2.53) allow for the determination of the central axis in space. We obtain from them the following equation:

$$\mathbf{M}-\mathbf{M_S}=\sum_{i=1}^{3}\left(M_{x_i}-M_{S_{x_i}}\right)\mathbf{E}_i=\begin{vmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\ x_1 & x_2 & x_3\\ S_{x_1} & S_{x_2} & S_{x_3}\end{vmatrix}$$

$$=\mathbf{E}_1\left(x_2S_{x_3}-x_3S_{x_2}\right)+\mathbf{E}_2\left(x_3S_{x_1}-x_1S_{x_3}\right)+\mathbf{E}_3\left(x_1S_{x_2}-x_2S_{x_1}\right), \quad (2.55)$$

and hence

$$\begin{aligned}M_{x_1}-M_{S_{x_1}}&=x_2S_{x_3}-x_3S_{x_2},\\ M_{x_2}-M_{S_{x_2}}&=x_3S_{x_1}-x_1S_{x_3},\\ M_{x_3}-M_{S_{x_3}}&=x_1S_{x_2}-x_2S_{x_1}.\end{aligned} \quad (2.56)$$

The method of determination of the central axis is illustrated in Fig. 2.29.

If we treated the coordinates of the point $O'(x_1, x_2, x_3)$ in the coordinate system $OX_1X_2X_3$ as that determined from (2.56), then knowing the coordinates of vector **S**, we may find the equation of the central axis.

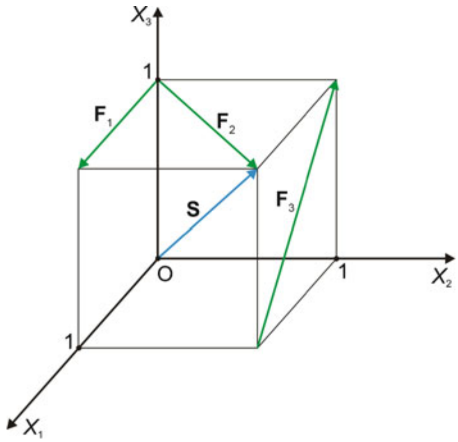


Fig. 2.30 Three forces and location of a pole O

If the coordinate system is introduced at some other point of the body and the coordinates of the pole in that system are denoted as (x_{10}, x_{20}, x_{30}), then the point belonging to the central axis can be found from the following equations:

$$\begin{aligned} M_{x_1} - M_{S_{x_1}} &= (x_2 - x_{20})S_{x_3} - (x_3 - x_{30})S_{x_2}, \\ M_{x_2} - M_{S_{x_2}} &= (x_3 - x_{30})S_{x_1} - (x_1 - x_{10})S_{x_3}, \\ M_{x_3} - M_{S_{x_3}} &= (x_1 - x_{10})S_{x_2} - (x_2 - x_{20})S_{x_1}. \end{aligned} \tag{2.57}$$

Let us assume that we reduce the space force system about pole O. We are dealing with the following special cases:

1. $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$ and $\mathbf{M}_O \perp \mathbf{S}$.
   Then $\mathbf{M}_S = \mathbf{M}_O \circ \mathbf{S} = \mathbf{0}$. The resultant force $\mathbf{F}^r$ is parallel to $\mathbf{S}$ but cannot pass through point O. It is located on the central axis.
2. $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$.
   The resultant $\mathbf{F}^r = \mathbf{S}$ and passes through pole O (because it does not produce a moment).
3. $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$.
   The moment of force vector is a free vector, so it can be attached at an arbitrary point in space.
4. $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$.
   The analyzed rigid body is in equilibrium.

Example 2.6. Reduce the three forces of magnitudes $F_1 = 1$, $F_2 = \sqrt{2}$, and $F_3 = \sqrt{2}$ that act as shown in Fig. 2.30.

The axes of the coordinate system are taken along the edges of the cube. We aim to determine the components of vector $\mathbf{S}$ and moment $\mathbf{M}$, which are equal to:

$$
\begin{aligned}
S_{x_1} &\equiv F_1 + \frac{\sqrt{2}}{2} F_2 - F_3 \frac{\sqrt{2}}{2} = 1,\\
S_{x_2} &= F_2 \frac{\sqrt{2}}{2} = 1,\\
S_{x_3} &= F_3 \frac{\sqrt{2}}{2} = 1,\\
M_{x_1} &= -F_2 \frac{\sqrt{2}}{2} \cdot 1 + F_3 \frac{\sqrt{2}}{2} \cdot 1 = 0,\\
M_{x_2} &= F_1 \cdot 1 - F_3 \frac{\sqrt{2}}{2} \cdot 1 + F_2 \frac{\sqrt{2}}{2} = 1,\\
M_{x_3} &= F_3 \frac{\sqrt{2}}{2} \cdot 1 = 1.
\end{aligned}
$$

In order to make use of formula (2.56) we need to know additionally the values of M_{Sx_1}, M_{Sx_2}, and M_{Sx_3}. For the determination of the coordinates of vector $\mathbf{M}_S$ we will multiply (2.47) in turn by $\mathbf{E}_i$, $i = 1, 2, 3$.

Since

$$
\begin{aligned}
\mathbf{M} \circ \mathbf{S} &= M_{x_1} S_{x_1} + M_{x_2} S_{x_2} + M_{x_3} S_{x_3} = 2,\\
|\mathbf{S}|^2 &= S_{x_1}^2 + S_{x_2}^2 + S_{x_3}^2 = 3,
\end{aligned}
$$

after using (2.47) we obtain

$$
M_{S_{x_1}} = \frac{2}{3}, \quad M_{S_{x_2}} = \frac{2}{3}, \quad M_{S_{x_3}} = \frac{2}{3}.
$$

According to (2.56) we have

$$
x_2 - x_3 = -\frac{2}{3}, \quad -x_1 + x_3 = \frac{1}{3}, \quad x_1 - x_2 = \frac{1}{3}.
$$

In order to determine the point of intersection of three planes we will calculate the determinant of the system

$$
\Delta = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0 + 1 \cdot (-1)^3(-1) - 1 \cdot (-1)^4(1) = 1 - 1 = 0,
$$

and the minors

$$\Delta_{x_1} = \begin{vmatrix} -\frac{2}{3} & 1 & -1 \\ \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & -1 & 0 \end{vmatrix} = \frac{1}{3}\cdot(-1)^3(-1) + 1\cdot(-1)^5\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3} - \frac{1}{3} = 0,$$

$$\Delta_{x_2} = \begin{vmatrix} 0 & -\frac{2}{3} & -1 \\ -1 & \frac{1}{3} & 1 \\ 1 & \frac{1}{3} & 0 \end{vmatrix} = -\frac{2}{3}(-1)^3(-1) - 1\cdot(-1)^4\left(-\frac{2}{3}\right) = -\frac{2}{3} + \frac{2}{3} = 0,$$

$$\Delta_{x_3} = \begin{vmatrix} 0 & 1 & -\frac{2}{3} \\ -1 & 0 & \frac{1}{3} \\ 1 & -1 & \frac{1}{3} \end{vmatrix} = 1\cdot(-1)^3\left(-\frac{2}{3}\right) - \frac{2}{3}\cdot(-1)^4(1) = \frac{2}{3} - \frac{2}{3} = 0.$$

Recall that if $\Delta \neq 0$, then the three planes intersect at one point. If $\Delta = \Delta x_1 = \Delta x_2 = \Delta x_3 = 0$, then the three planes have a common line.

Let us take $x_2 = 0$. We calculate $x_1 = \frac{1}{3}$ and $x_3 = \frac{2}{3}$. Point $A(\frac{1}{3}, 0, \frac{2}{3})$ belongs to the line constituting the central axis. We determine the central axis by drawing a line parallel to $\mathbf{S}$ through point A.

If the planes intersect at one point, we determine that point solving the system of three algebraic non-homogeneous equations. Let the point of intersection have the coordinates x_{1A}, x_{2A}, x_{3A}. The central axis passes through that point and is parallel to the vector $\mathbf{S} = \mathbf{E}_1 S_{x1} + \mathbf{E}_2 S_{x2} + \mathbf{E}_3 S_{x3}$. In view of that, the equation of central axis has the following form:

$$\frac{x_1 - x_{1A}}{S_{x_1}} = \frac{x_2 - x_{2A}}{S_{x_2}} = \frac{x_3 - x_{3A}}{S_{x_3}}. \qquad \square$$

The steps illustrated in Sects. 2.1–2.7 that aimed to reduce and simplify force systems (providing a lack of friction) can be briefly summarized in the following manner:

(a) Reducing a force system to a resultant force $\mathbf{F} = \sum_{n=1}^{N} \mathbf{F}_n$ and a couple at given point O (resultant moment):

$$\mathbf{M}_O = \sum_{n=1}^{N} (\mathbf{r} \times \mathbf{F}_n).$$

(b) Moving a force–couple system from point O_1 to point O_2, expressed by the formula

$$\mathbf{M}_{O_2} = \mathbf{M}_{O_1} + \overrightarrow{O_2O_1} \times \mathbf{F}.$$

(c) By equivalency checking, reducing to the one point O two systems of forces, which are equivalent if the two force–couple systems are identical, i.e.,

$$\sum_{n=1}^{N} \mathbf{F}_n = \sum_{k=1}^{K} \mathbf{F}_k,$$

$$\sum_{n=1}^{N} (\mathbf{r_n} \times \mathbf{F_n}) = \sum_{k=1}^{K} (\tilde{\mathbf{r}}_k \times \tilde{\mathbf{F}}_k).$$

(d) Reducing a given force system to a single force; this takes place only if $\mathbf{S} \perp \mathbf{M}_O$, i.e., when force $\mathbf{S}$ and couple vector $\mathbf{M}_O$ are mutually perpendicular (which happens for concurrent coplanar and parallel forces). If position vector $\mathbf{r}$ depicted from point O to any point on the line of action of the single force $\mathbf{F}$ satisfies the equation $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.
(e) Applying force and funicular polygons (Sect. 2.2).
(f) Applying the three-moments theorem (Sect. 2.4).
(g) Reducing a system of parallel forces (Sect. 2.4).
(h) Reducing a three-dimensional force system to two skew forces (Sect. 2.6).
(i) Reducing a given force system to a wrench.

In general for a three-dimensional force system (not concurrent, coplanar, or parallel) one may introduce the reduction on a basis of items (f), (h), and (i). Observe that the equivalent force system consisting of the resultant force $\mathbf{S}$ and the couple vector $\mathbf{M}_O$ are not mutually perpendicular and hence, although they cannot be reduced to a single force, they can be reduced to a wrench since they constitute the combination of the force $\mathbf{S}$ and a couple vector $\mathbf{M}_S$ $(\mathbf{S} \parallel \mathbf{M}_S)$ that lie on the wrench (case i). In other words, after reduction of a given force–couple system $(\mathbf{S}, \mathbf{M}_O)$ and after determination of the pitch $p = \frac{\mathbf{S}_O \circ \mathbf{M}_O}{S^2}$, $\mathbf{M}_s$ satisfies the equation $\mathbf{M}_S + \mathbf{r} \times \mathbf{S} = \mathbf{M}_O$. It allows one to find that point where the line of action of the wrench intersects a special plane ($\mathbf{r}$ is directed from O to that point).

2.8 Friction

The phenomenon of friction is common both in mechanics and in everyday life. Many scientific works, including monographs, cover this topic (citation is omitted here). Also, the present author and his coworkers have published monographs on the problems of classical friction [30–35].

In general, friction is a force that opposes a body's motion and possesses a sense opposite to that of the velocity of the body's relative motion. In order to ensure sliding of one surface with respect to another if both remain continuously in contact, it is necessary to act all the time on the moving body with a certain force. This requirement is connected with the resistance to motion called *sliding friction*. This is because it turns out that even apparently very smooth surfaces possess irregularities that cause resistance to motion. Friction is divided into *static* and *dynamic* friction. We deal with the former when we want to move one body with respect to another, e.g., remaining at rest. In the majority of cases, static friction is greater than dynamic friction. Sliding friction depends on the condition of the contacting surfaces (i.e., whether they are dry or lubricated), their smoothness and wear resistance abilities, and humidity and temperature.

In cases where two bodies roll with respect to one another, the friction is inversely proportional to the radius of the object that rolls. Such friction is called *rolling friction*.

Moreover, we distinguish *aero* and *hydrodynamic* friction. The former concerns the motion of objects in gases, while the latter concerns that in liquids. For small speeds v of objects motion in a fluid (liquid or gas), the force of resistance is proportional to the speed, whereas for medium (high) speeds of motion, the force is proportional to v^2 (v^3). It should be emphasized that the character of the mentioned dependencies presents the problem rather qualitatively since the described phenomenon of friction depends on the temperature, viscosity, object shape, and fluid density. In general the friction phenomenon is not yet fully understood because it is a very complex process connected with motion resistance and accompanied by the heat generation and wear of contacting bodies. Friction can be treated, then, also as a certain energetic process.

In most cases the kinetic energy of objects subjected to the action of friction forces is turned into heat energy. For instance, it is widely known that objects moving in air such as airplanes, rockets, or spaceships are very hot, which demands application of new materials of high quality and high heat resistance. Also, the rims of wheeled vehicles heat and the striking of two firestones or rubbing of two pieces of wood leads to igniting a fire. In this case we are dealing with *energy dissipation*, which means that the mechanical energy (kinetic and potential) of the object is not conserved since part of it is converted into *internal energy* (in this case into heat).

The friction between two solid bodies, as can be seen from the preceding description, is a complicated physical phenomenon. Apart from the mentioned heating of bodies, often we are dealing with their electrification, changes in the surface of the rubbing bodies, or diffusion phenomena, i.e., migration of the molecules of one body into another body.

Generally speaking, it is possible to distinguish three basic types of frictional interactions during the contact of two bodies: (a) sliding friction connected with the translational motion (no rotation) of two bodies; (b) sliding friction connected with rotational motion; (c) rolling resistance, e.g., a train wheel on a rail; and (d) sliding torsion friction.

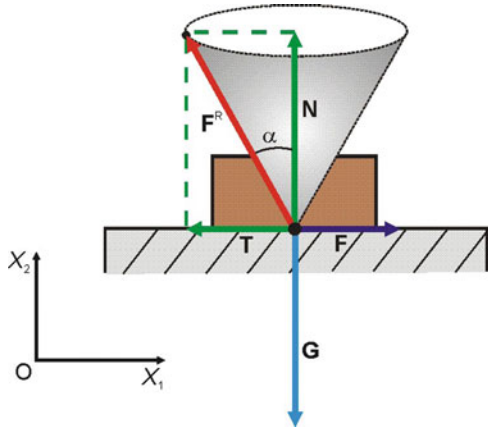


Fig. 2.31 A sliding friction and a cone of friction

Of the many researchers dealing with the friction phenomenon, the French physicist Amontons[3] should be counted as a pioneer in the field. He proved in the course of empirical investigations a lack of dependency of the friction force on the area of contact of rubbing bodies. The fundamental laws of friction were formulated by Coulomb.[4]

From the considerations above it follows that investigations aimed at finding the connection of these phenomena with the atomic scale are of great importance for a deeper understanding of the phenomenon. In this direction are moving such sciences as tribology, physics, and nanomechanics. Those investigations are important in the age of common miniaturization, nanotechnology, and mechatronics.

Friction depends on the pressure of one surface rubbing against another, but also on the relative speed of the bodies in contact. In this textbook we will consider only the classical laws of friction, but the reader particularly interested in this topic can refer to the works cited above.

Let us consider a classical example treated in many textbooks (Fig. 2.31).

A rigid body of rectangular cross section (its height is neglected) and weight $\mathbf{G}$ lies on a horizontal surface and is subjected to an action of a certain force $\mathbf{F}$. If the body remains at rest, after projection of the existing forces onto the axes OX_1 and OX_2 we obtain that $\mathbf{T} = -\mathbf{F}$ and $\mathbf{G} = -\mathbf{N}$, where $\mathbf{N}$ is the normal force and $\mathbf{T}$ the friction force (static reaction) that results from the contact between the bodies.

It is easy to imagine, and it follows also from everyday experiences, that the increase of the magnitude of force $\mathbf{F}$ will cause the increase of magnitude of tangential reaction (for $\mathbf{F} = \mathbf{0}$ there is no tangential reaction). Such a process will continue up to a certain threshold magnitude of force $\mathbf{F}_{th}$. After crossing this magnitude of force the body will start to slide over the other fixed body (the base),

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[3] Guillaume Amontons (1663–1705), self-taught French scholar dealing with, among other things, thermodynamics and friction.

[4] Charles Augustin de Coulomb (1736–1806), French physicist.

and we will be dealing with kinetic friction. The friction corresponding to $\mathbf{F}_{\text{th}}$ will be called the *limiting friction force* or a *fully developed friction force*.

The reaction $\mathbf{F}^R$ caused by the forces $\mathbf{F}$ and $\mathbf{G}$ can be determined based on the normal force $\mathbf{N}$ and the friction force $\mathbf{T}$ after adding both vectors, i.e.,

$$\mathbf{F}^R = \mathbf{T} + \mathbf{N}. \tag{2.58}$$

It is easy to observe that if $\mathbf{T} \to \mathbf{T}_{\text{th}}$, the magnitude of the resultant force vector increases. The angle of inclination α of the reaction to the vertical line increases as well, and for $\mathbf{T} = \mathbf{T}_{\text{th}}$ it reaches a value called the *angle of repose*.

Applying the classical Coulomb and Morin[5] laws of friction one should remember that they are only a certain approximation of the complex phenomena characterizing the friction process. According to Fig. 2.31 we have $T_{\text{th}} = N \tan\alpha$ and after introducing the coefficient of sliding friction we finally obtain

$$T_{\text{th}} = \mu N. \tag{2.59}$$

This moment is well worth emphasizing because for the first time mechanics has to refer to the experiment in this case (it will happen a second time during the analysis of impact phenomena and will be connected with the determination of the so-called *coefficient of restitution*). It is not possible to determine the friction force without knowing the coefficient μ, which is obtained as a result of experimental research. For rough surfaces the coefficient is big and diminishes for smooth and lubricated surfaces. The coefficient of friction may depend on the manufacturing processes connected with a material's production, e.g., the value of coefficient of friction in the case of rolled steel depends on the direction of rolling. For certain materials (e.g., wood) friction has an anisotropic character, i.e., the coefficient μ depends on the direction of motion.

The Coulomb–Morin model distinguishes between *static* and *kinetic* (*sliding*) *friction*—the forces lying in a tangent plane at the point of contact of the bodies when the (tangential) relative velocity of the contacting bodies at that point (slip velocity) is respectively equal to zero (no slip) or greater than zero (slip). The magnitude of the static friction force can change from zero up to a certain maximum value proportional to the normal force, which is usually written as $T = \mu_{\text{s}} N$ (μ_{s} is the *coefficient of static friction*). This definition includes both the introduced "tangent reaction" and the "threshold friction force" (the tangent reaction is the friction force as well). The magnitude of the static friction force is usually not known a priori (the reaction of the friction constraints) and its sense is determined as being opposite to the tendency of sliding—sense of slip at the contact point for perfectly smooth surfaces. The force of kinetic friction is defined as $T = \mu_{\text{k}} N$ (μ_{k} is the *coefficient of kinetic friction*) and its sense is opposite to that of slip velocity, and $\mu_{\text{k}} < \mu_{\text{s}}$. A separate and little researched (not adequately mathematically described) problem is the transient phenomena for small slip velocities.

[5] Arthur Jules Morin (1795–1880), French general and professor of mechanics.

Although in Fig. 2.31 we have considered so far the plane system, we may consider friction in space as well. Then force $\mathbf{F}^R$ will change depending on the change in force $\mathbf{F}$. The vector $\mathbf{F}^R$ at the change of $\mathbf{F} \in [\mathbf{0}, \mathbf{F}_{\text{th}}]$ will be located inside the cone whose generatrix and normal force form an angle of friction α. In the case of fully developed friction, the opening angle of the cone $2\alpha_{\text{th}}$ is the greatest. If the reaction $\mathbf{F}^R$ is situated outside the cone, the state of rest of the investigated body will be interrupted, and the body will start sliding over the surface of the base body. A shift from a state of rest to motion, i.e., from statics to dynamics, will occur. The body will remain at relative rest (because in a general case both bodies can move with respect to the adopted fixed coordinate system) if the following inequality is satisfied:

$$T \le \mu_{\text{s}} N. \tag{2.60}$$

The cross section of a cone with a plane perpendicular to vector $\mathbf{N}$, depicted in Fig. 2.31, is circular. In the case of anisotropic friction, the cross section is not circular, but it might be, for example, an ellipse.

In further considerations we will make use of the three laws connected with friction, which had already been formulated by Coulomb.

1. The threshold friction force is proportional to the normal force, i.e., $T_{\text{th}} = \mu_{\text{s}} N$, where μ_{s} is the *coefficient of threshold static friction (fully developed)*.
2. The friction force does not depend on the size of area of contact between the bodies.
3. The coefficient of friction depends on the kind of material and the condition of surfaces of the bodies in contact.

If the body is in motion, usually the coefficient of friction μ_{k} (so-called kinetic friction) is smaller than the coefficient of developed friction μ_{s}, that is, $\mu_{\text{k}} \le \mu_{\text{s}}$. This difference leads to certain theoretical difficulties connected with the transition from rest to motion (and vice versa) and with the mathematical modeling of the coefficient of friction because of the need to distinguish one state of the investigated system from another.

In Table 2.1 the average values of the coefficient of sliding friction are given for various materials of bodies being at rest or in motion for different surface conditions (oiled and dry). In textbooks on mechanics the so-called *rolling resistance* is considered as a separate problem. This topic will be described in accordance with the textbook [9].

We will consider two cases of a heavy cylinder lying on rigid (Fig. 2.32a) and flexible (Fig. 2.32b) planes.

In the first case (a), the cylinder is in contact with the plane along a line (the generatrix of the cylinder). Along this generatrix of the cylinder the normal and tangential forces will appear (in the cross section perpendicular to the cylinder's axis we will obtain vectors $\mathbf{N}$ and $\mathbf{T}$). The horizontal force $\mathbf{F}$ acts on the cylinder at distance h from the plane on which the cylinder is rolling. Writing the equation of moments about point A we have

$$M_A = Fh. \tag{2.61}$$

Table 2.1 Coefficients of sliding friction [9]

	At rest (μ_s)		At motion (μ_k)	
Material of contacting bodies	Dry	Oiled	Dry	Oiled
Steel vs. steel	0.15	0.1	0.1	0.01
Steel vs. cast iron or bronze	0.18	0.1	0.16	0.01
Cast iron vs. cast iron	0.45	0.25	0.2	0.05
Bronze vs. cast iron or bronze	0.21	–	0.18	–
Metal vs. wood	0.5–0.6	0.1	0.2–0.5	0.02–0.08
Wood vs. wood	0.65	0.2	0.2–0.4	0.04–0.16
Leather vs. metal	0.6	0.25	0.25	0.12
Steel vs. ice	0.03	–	0.02	–

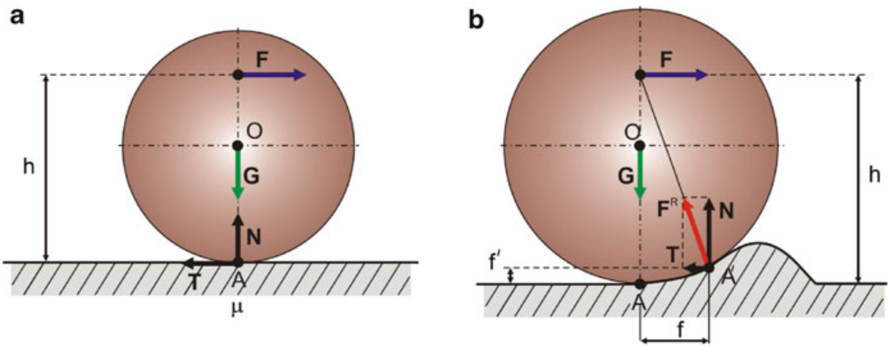

Fig. 2.32 Rolling of a heavy cylinder on a rigid (**a**) and a flexible (**b**) plane

The moment $\mathbf{M}_A = \mathbf{0}$ if $\mathbf{F} = \mathbf{0}$ or $h = 0$. Otherwise, an arbitrarily small, but non-zero, force $\mathbf{F}$ for $h > 0$ will initiate the cylinder motion. If the force $F > \mu N$, then the cylinder will slide on the plane. If $F < \mu N$, then the cylinder will roll on the plane.

The considered case is idealized since from the experiment it follows that a heavy cylinder causes deformation of the horizontal surface (Fig. 2.32b) and the action of force $\mathbf{F}$ creates a reaction connected with the force of rolling resistance $\mathbf{T}_{\text{res}}$ and the moment of force of rolling resistance $\mathbf{M}_{\text{res}}$. Force $\mathbf{F}$, which forces the cylinder to move, causes displacement of the point of application of the reaction of plane from point A to point A'. Writing the equation of moments about a point A we obtain now

$$M_A = Tf' + Nf - Fh. \tag{2.62}$$

Because $f' \ll f$ and $T < N$, the moment Tf' can be neglected, and from (2.62) we obtain

$$M_A = Nf - Fh. \tag{2.63}$$

Table 2.2 Rolling resistance coefficients [9]

Cylinder	Basis	$f(10^{-2}\text{ m})$
Cast iron	Cast iron	0.005
Steel	Steel rail	0.005
Wooden	Wood	0.8
Roller bearings and balls (hardened and ground)	Hardened and ground steel	0.0005–0.001

The motion will start when the moment Fh overcomes the moment $Nf = M_{\text{res}}$, called a *moment of rolling resistance*. The quantity f we call a *coefficient of rolling resistance* or an *arm of rolling resistance*. Let us note that, as distinct from the dimensionless coefficient μ, the quantity f possesses a dimension of length and characterizes the maximum distance at which the point of application of reaction can move preserving the state of rest of the body, i.e., when $Fh = Nf$. Several average values of the coefficient of rolling resistance f for various materials of the cylinder and ground are given in Table 2.2.

The third case of the classical division of friction, i.e., rotational friction, remains to be described. In the case of sliding friction the problem can be reduced to the point of contact. If, however, to one (or both) of the contacting bodies we apply a moment of force, the assumption of their contact only at the point is not valid. The moment of force is carried by the moment of friction forces that must occur over a certain (although very small) area. Thus, the moment of force is balanced not by a single force but by a couple of forces. After overcoming the moment of rotational friction, the moment of force will cause rotation of the body on which it acts (on the assumption that the other body remains fixed) about an axis perpendicular to the surface of contact of the bodies. The threshold magnitude of the moment of rotational friction is usually assumed to be proportional to the force **N** (which is the force compressing the bodies, normal to the surface of contact), and that threshold magnitude equals $M_{\text{F}} = \mu' N$, where now μ' denotes a *rotational friction coefficient*, and this time the coefficient μ' possesses a dimension of length. The problem is relatively complex since the moment of friction force depends on the normal stress in the area of contact (area size greater than zero) distribution, which must be uniform over the mentioned area. Additionally, the rotational friction coefficient depends on the coefficient of forward sliding friction because it is caused by the normal force **N**. For example, from the theoretical considerations concerning the contact problem of the cylinder of radius r making contact with the fixed surface over the area πr^2, the coefficient of rotational friction $\mu' = 0.25\pi r\mu$, where μ denotes the sliding friction.

Let us consider a classical case of a rigid body having the shape of a rectangular prism (Fig. 2.33), but in the present case we assume that the dimensions of its cross section with a vertical plane, i.e., width $2b$ and height $2h$ of rectangle, are known. We will also assume that its geometrical center coincides with the center of mass (point C) and that at point C the gravity force **G** and horizontal force **F** are applied (Fig. 2.33).

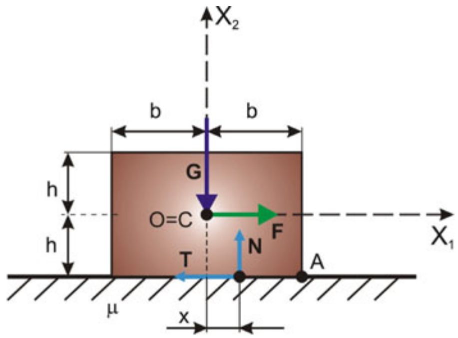

Fig. 2.33 Schematic of forces for static equilibrium of a block

The action of forces **G** and **F** will cause the reaction of the block in the form of the friction force **T** and normal force **N** as in Fig. 2.33. For the force system to be in equilibrium, on the assumption of finite area of contact, the point of application of force **N** must lie at a certain distance x from vector **G**.

From the projection of the forces onto the axes of the adopted coordinate system we obtain

$$G = N, \qquad T = F, \tag{2.64}$$

and writing the equation of moments about point C we have

$$Th = Nx, \tag{2.65}$$

which allows us to determine the unknown quantity

$$x = \frac{Th}{N} = \frac{Fh}{G}. \tag{2.66}$$

According to the law of friction $T = \mu N$, we obtain that $F < \mu G$, and next we have another inequality, that is, $x = \frac{Fh}{G} \leq b$, i.e., $F \leq \frac{b}{h}G$. Let us trace now the phenomena that accompany the process of increasing the magnitude of force **F**. If the coefficient of friction $\mu < b/h$, then the block will remain in equilibrium until the friction force reaches its threshold magnitude $T = \mu N = \frac{b}{h}N = \frac{b}{h}G$. After exceeding this value, the block will start sliding on the plane. If the coefficient of friction $\mu > b/h$, if only the magnitude of horizontal force $F > \frac{b}{h}G$, then the block will rotate about point (edge) A.

In four subsequent examples we will consider the case of a block lying on an inclined plane, this time neglecting its geometrical dimensions (Example 2.7), a homogeneous cylinder rolling on an inclined plane (Example 2.8), the more complex case of the block (wardrobe) taking into account its geometry and assuming the presence of friction along two edges (points) of contact (Example 2.9), and, finally, a shaft-bearing frictional problem (Example 2.10).

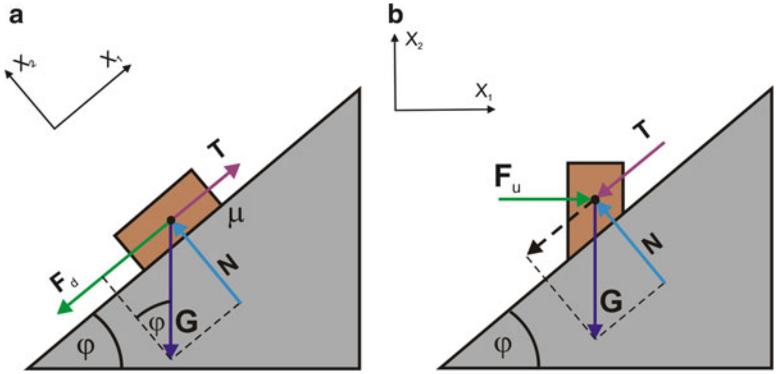

Fig. 2.34 Boundary conditions of equilibrium of bodies of masses m under the action of forces $\mathbf{F}_d$ (**a**) and $\mathbf{F}_u$ (**b**)

Example 2.7. Determine the minimum magnitude of the force $\mathbf{F}_d$ ($\mathbf{F}_u$) at the moment of block motion down (up) the inclined plane (Fig. 2.34).

In Fig. 2.34a, the equations of equilibrium read

$$-F_d + T - G\sin\varphi = 0,$$
$$N - G\cos\varphi = 0.$$

On the verge of equilibrium loss the friction is fully developed, so $T = \mu N$, and from the first equation we obtain

$$F_d = G(\mu\cos\varphi - \sin\varphi).$$

In this case, the condition $F_d = 0$ is possible, which is equivalent to $\tan\varphi = \mu$. For $\tan\varphi \leq \mu$, the block will remain motionless on the inclined plane, and upon increasing the angle of inclination, i.e., for $\tan\varphi > \mu$, the block will slide down.

In the second case (Fig. 2.34b), the equations of motion are as follows:

$$F_u - T\cos\varphi - N\sin\varphi = 0,$$
$$-G + N\cos\varphi - T\sin\varphi = 0.$$

Because $T = \mu N$, the equations take the form

$$F_u = N(\mu\cos\varphi + \sin\varphi),$$
$$G = N(\cos\varphi - \mu\sin\varphi),$$

so

$$F_u = G\frac{\mu\cos\varphi + \sin\varphi}{\cos\varphi - \mu\sin\varphi} = G\frac{\mu + \tan\varphi}{1 - \mu\tan\varphi}.$$

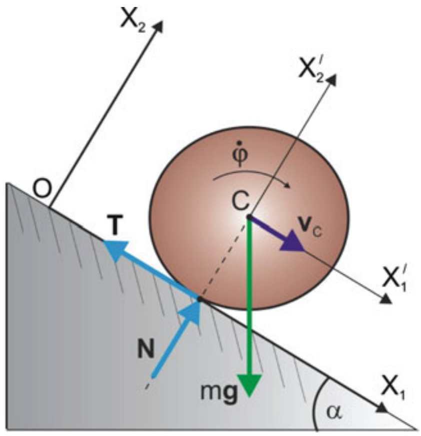

Fig. 2.35 Rolling of cylinder on inclined plane

If $\tan\varphi \to 1/\mu$, then $F_u \to \infty$. In such a case, no force is able to move the block up. We may say that a binding of the body occurs. □

Example 2.8. A homogeneous cylinder with mass m and radius r is rolling on an inclined plane without slipping. The plane forms an angle α with a horizontal line (Fig. 2.35). Determine the cylinder center acceleration and the angle value α, where a slip begins.

The equations of motion have the following form:

$$\begin{aligned} m\ddot{x}_{1C} &= mg\sin\alpha - T, \\ m\ddot{x}_{2C} &= -mg\cos\alpha + N, \\ I_C\ddot{\varphi} &= -Tr. \end{aligned} \qquad (*)$$

Because $\ddot{x}_{2C} = 0$, or $x_{2C} = r$, $N = mg\cos\alpha$.

In turn, $\dot{x}_{1C} = -r\dot{\varphi}$ because the $\dot{\varphi}$ value is negative, and hence, after substituting into the third equation (*), we have

$$\frac{mr^2}{2}\left(-\frac{\ddot{x}_{1C}}{r}\right) = -Tr,$$

or $m\ddot{x}_{1C} = 2T$.

From the first equations $(*)$ we calculate

$$\ddot{x}_{1C} = \frac{2}{3}g\sin\alpha,$$

and then $T = \frac{1}{3}mg\sin\alpha$. For $\alpha = \alpha_{th}$, we obtain $\tan\alpha = 3\mu$. □

Example 2.9. Determine the range of equilibrium of a rigid body (a wardrobe) of weight **G** situated on an inclined plane (Fig. 2.36) with (without) friction present at points A and B.

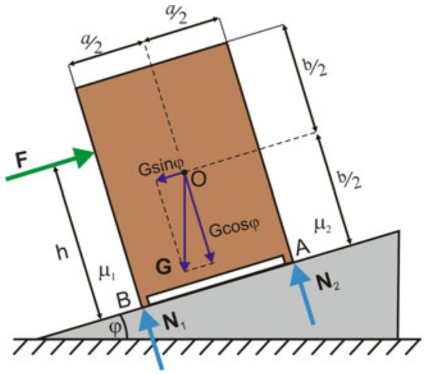

Fig. 2.36 A wardrobe standing on an inclined plane

Let us consider three cases.

(i) The case of no friction ($\mu_1 = \mu_2 = 0$):

$$F = G\sin\varphi,$$
$$N_1 + N_2 = G\cos\varphi,$$
$$M_A \equiv -Fh - N_1 a + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi = 0.$$

In addition, the equation of moments about point A was written above. From the equations we determine

$$N_1 = \frac{1}{a}\left[-Fh + \frac{1}{2}G(b\sin\varphi + a\cos\varphi)\right],$$
$$N_2 = \frac{1}{a}\left[Fh + \frac{1}{2}G(a\cos\varphi - b\sin\varphi)\right].$$

If we want to move the wardrobe standing on a horizontal floor ($\varphi = 0$), from the preceding equations we obtain

$$F = 0, \quad N_1 = N_2 = \frac{G}{2},$$

that is, the wardrobe will move under the action of an arbitrarily small force $F \neq 0$.

(ii) Let us introduce now the friction forces $\mathbf{T}_1$ and $\mathbf{T}_2$ and let us find the force $\mathbf{F}_{\min}$ such that the wardrobe starts to move up the inclined plane.

The projections of force vectors on the OX_1 and OX_2 axes give

$$F = G\sin\varphi + T_1 + T_2,$$
$$N_1 + N_2 = G\cos\varphi.$$

The equation of moments about the center of mass of the wardrobe reads

$$M_O \equiv -F\left(h-\frac{b}{2}\right)-(T_1+T_2)\frac{b}{2}-N_1\frac{a}{2}+N_2\frac{a}{2}=0.$$

From the preceding equation, after assuming $F = T_1 = T_2$ and $\varphi = 0$, we obtain $N_1 = N_2 = \frac{G}{2}$, which agrees with the result obtained earlier (case i).

If we assume that $T_1 \in [0, \mu_1 N_1]$ and $T_2 \in [0, \mu_2 N_2]$, then the problem cannot be solved because we have four unknowns N_1, N_2, T_1, and T_2 but only three equations.

In this case, we have to assume in advance the possibility of motion of the wardrobe after equilibrium loss. While increasing the force **F** for certain parameters of the system the wardrobe can first either slide or rotate about point A. The problem, so far statically indeterminate, can be solved after assuming one of two possible options. Let us assume that there occurs a rotation about point A under the action of force **F**. This means that $\mathbf{N}_1 = \mathbf{T}_1 = \mathbf{0}$. The equations of equilibrium of forces and moments for this case are as follows:

$$\begin{aligned}
F &= G\sin\varphi + T_2,\\
N_2 &= G\cos\varphi,\\
M_A &\equiv -Fh + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi = 0.
\end{aligned}$$

Taking the angle $\varphi = 0$, we obtain

$$F = T_2, \quad N_2 = G, \quad Fh = G\frac{a}{2}.$$

If we assume now that the friction force $\mathbf{T}_2$ is not developed and is bounded to the range $0 < T_2 < \mu G$, we obtain the condition for rotation on horizontal ground, which can be realized for an arbitrary $\mathbf{T}_2$ from the aforementioned range if

$$T_2 h = G\frac{a}{2},$$

that is,

$$\mu h = \frac{a}{2}.$$

After transformation for $\varphi \neq 0$ we have

$$\begin{aligned}
F &= G(\sin\varphi + \mu_2\cos\varphi),\\
Fh &= \frac{G}{2}(b\sin\varphi + a\cos\varphi),
\end{aligned}$$

and, in turn,

$$(\sin\varphi + \mu_2 \cos\varphi) = \frac{1}{2h}(b \sin\varphi + a \cos\varphi),$$

that is,

$$\sin\varphi\left(1 - \frac{b}{2h}\right) + \cos\varphi\left(\mu_2 - \frac{a}{2h}\right) = 0.$$

Note that

$$\begin{aligned} a\cos(\varphi - \psi) &= a\cos\psi\cos\varphi + a\sin\psi\sin\varphi \\ &= A\cos\varphi + B\sin\varphi. \end{aligned}$$

For our case

$$A = a\cos\psi = \mu_2 - \frac{2}{2h}, \quad B = a\sin\psi = 1 - \frac{b}{2h}.$$

The equation that we are going to solve has the form

$$\cos(\varphi - \psi) = 0,$$

that is,

$$\varphi - \psi = (2k+1)\frac{\pi}{2}, \qquad k = 0, 1, 2.$$

For $k = 0$ we have $\varphi_0 = \psi + \frac{\pi}{2}$, which means that

$$\varphi_0 = \arctan\left(\frac{2h - b}{2h\mu_2 - a}\right) + \frac{\pi}{2}.$$

For $k = 1$, such a problem cannot be physically realized in the considered case.

Let us assume now that there will be a loss of equilibrium caused by moving the wardrobe up the inclined plane. In this case the equations will take the following form ($N_1 \neq 0$):

$$\begin{aligned} &F = G\sin\varphi + T_1 + T_2, \\ &N_1 + N_2 = G\cos\varphi, \\ &M_A \equiv -Fh + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi - N_1 a = 0. \end{aligned}$$

For the angle $\varphi = 0$ we obtain

$$F = T_1 + T_2,$$
$$N_1 + N_2 = G,$$
$$-Fh + G\frac{a}{2} - N_1 a = 0.$$

From the last two equations we determine (it can be seen that the sum of friction forces is canceled out by force **F**):

$$N_1 = \frac{G}{2} - F\frac{h}{a},$$

$$N_2 = \frac{G}{2} + F\frac{h}{a}.$$

The displacement will occur when $N_1 > 0$, that is, when

$$\frac{G}{2} > F\frac{h}{a}.$$

For the case $\mu_1 = \mu_2 = \mu$ we have

$$\frac{G}{2} > \mu G\frac{h}{a}, \quad \text{or} \quad \frac{a}{2\mu} > h.$$

Now, let us assume that $\varphi \neq 0$, and to simplify the analysis, let us take $\mu_1 = \mu_2 = \mu$. From the equations in question we obtain

$$F = G(\sin\varphi + \mu\cos\varphi),$$
$$N_1 = \frac{1}{2}G\left(\frac{b}{a}\sin\varphi + \cos\varphi\right) - F\frac{h}{a}.$$

In this case, force **F** cancels the action of friction forces and the component of weight vector parallel to the surface of the inclined plane.

The condition for displacement reads

$$\frac{1}{2}\left(\frac{b}{a}\sin\varphi + \cos\varphi\right) > (\sin\varphi + \mu\cos\varphi)\frac{h}{a}.$$

(iii) Let us introduce now the friction forces $\mathbf{T}_1$ and $\mathbf{T}_2$ and consider the case where we should determine the force $\mathbf{F}_{\min}$ such that the wardrobe does not move down the inclined plane.

In the present case, the senses of friction forces will change and the equations will take the following form:

$$F = G\sin\varphi - T_1 - T_2,$$
$$N_1 + N_2 = G\cos\varphi.$$

Let us assume that as a result of action of force **F** the rotation of the wardrobe about point B will occur. The equation of moment is

$$M_B \equiv -Fh - \frac{a}{2}G\cos\varphi + \frac{b}{2}G\sin\varphi + N_2 a = 0.$$

On the assumption that $\mu_1 = \mu_2 = \mu$, from the equations above we determine

$$F = G(\sin\varphi - \mu\cos\varphi),$$
$$N_2 = F\frac{h}{a} + \frac{1}{2}G\left(\cos\varphi - \frac{b}{a}\sin\varphi\right).$$

Let us check if the rotation about point B is possible for $\varphi = 0$.

From the preceding equations ($N_2 = 0$) we obtain

$$F = -T_1,$$
$$N_1 = G,$$
$$Fh + G\frac{a}{2} = 0,$$

that is,

$$h = \frac{a}{2\mu}.$$

From the foregoing considerations it follows that there exists a possibility of equilibrium loss by the rotation of the wardrobe, and in the present case the boundary value of this condition is the same as the corresponding value for translation.

Let us assume now that $\varphi \neq 0$ and check if there possibly exists an angle for which the loss of equilibrium by rotation about point B takes place. In this case we have ($N_2 = 0$)

$$F = G\sin\varphi - T_1,$$
$$N_1 = G\cos\varphi,$$

$$-Fh - \frac{a}{2}G\cos\varphi + \frac{b}{2}G\sin\varphi = 0.$$

Then we have

$$h(\sin\varphi - \mu\cos\varphi) = \frac{1}{2}(-a\cos\varphi + b\sin\varphi),$$

and after transformations we obtain

$$\left(-h + \frac{b}{2}\right)\sin\varphi + \left(h\mu - \frac{a}{2}\right)\cos\varphi = 0.$$

Let us assume now that the equilibrium loss will occur by the motion of the wardrobe down the inclined plane, so the loss of equilibrium will result in sliding. The condition of the motion is $N_2 \geq 0$ and in the boundary case $N_2 = 0$. This condition is identical with the one for the rotation about point B.

Let us note that the current example allows for the analysis of two special cases. For $\varphi = 0$ we are dealing with the problem of equilibrium loss of the wardrobe situated on a horizontal surface (already presented). Let us assume now that $a = b = h = 0$, that is, we will be considering a particle of mass m situated on the inclined plane ($\mathbf{N}_1 + \mathbf{N}_2 = \mathbf{N}$).

From the considerations conducted for case (i) we obtain

$$F = mg\sin\varphi,$$
$$N = mg\cos\varphi.$$

This means that for the particle to remain at rest on the inclined plane one should apply to it force $\mathbf{F}$ of magnitude $mg\sin\varphi$.

In case (ii) we have

$$F^*_{\min} = G(\sin\varphi + \mu\cos\varphi),$$

whereas in case (iii) we have

$$F_{\min} = G(\sin\varphi - \mu\cos\varphi).$$

Let us note that for the forces within the range $F \in [F_{\min}, F^*_{\min}]$ the particle remains in equilibrium. Range $[F_{\min}, F^*_{\min}]$ we will call the *range of equilibrium loading*. □

Example 2.10. Determine the angle of friction φ and so-called *circle of friction* of radius r during contact of the shaft journal of radius R and the bearing shell on the assumption that the radius of the bearing shell is slightly larger than R and was omitted in Fig. 2.37.

Let us assume that the shaft journal is loaded with a couple of moment $\mathbf{M}$ and a force $\mathbf{F}$. The action of the moment causes the displacement of the point of contact A between the journal and the bearing shell upward, i.e., we are dealing with the phenomenon of shaft climbing.

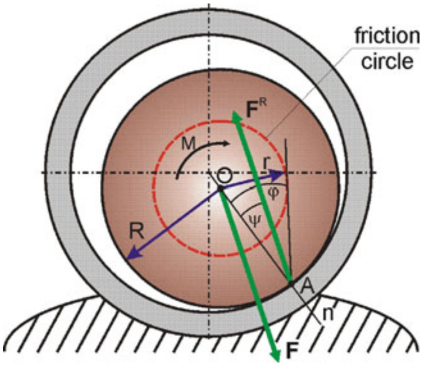

Fig. 2.37 Contact of a journal (rotor) with a bearing shell (plain bearing) with radii and forces marked

Let us consider the idealized case where there is no dependency of the friction force on the speed and no bearing lubrication; the rolling resistance can be neglected.

Because the shaft journal is in equilibrium under the action of the active force **F** and the couple moment **M** and we are dealing with a plane force system, it can be balanced with only one force (reaction) $\mathbf{F}^R$, which is equal to $-\mathbf{F}$ and is attached at point A. Then the value of the main vector of force is equal to $F^R - F = 0$. Moreover, the moment coming from the couple $(\mathbf{F}, \mathbf{F}^R)$ should be balanced with the moment coming from the pair of active forces **M**, that is

$$F^R(R\sin\psi) = M,$$

which leads to the relation

$$M = FR\sin\psi.$$

This means that the magnitudes of M and F are related by the formula above through the angle ψ, i.e., the angle between the normal to the contact surface of the journal and the bearing shell and the reaction $\mathbf{F}_R$. If the magnitude of moment **M** is increased, for the boundary case defined by the angle of friction φ the equilibrium will be lost and the journal will start to rotate, sliding on the surface of the bearing shell.

In other words, the static equilibrium condition of the shaft journal is described by the following inequality:

$$M \leq FR\sin\varphi,$$

where $\psi \leq \varphi$.

Denoting $R\sin\varphi = r$ we will introduce the notion of a friction circle of radius r and the center at point O (center of shaft journal). From Fig. 2.37 it follows that static equilibrium takes place when the reaction $\mathbf{F}^R$ crosses the friction circle and in the boundary case is tangent to it. □

2.9 Friction and Relative Motion

As was mentioned earlier in Sect. 2.8 (see description of cone of friction), in fact, the friction belongs to phenomena that should be modeled in three-dimensional space (3D). Moreover, in reality we are dealing with friction during the impact of two bodies treated as solids, and these two phenomena of mechanics are closely related to one another. The description of the rolling of two bodies is more connected with two-dimensional modeling. In the present section we will limit ourselves to the modeling of friction as a one-dimensional phenomenon, indicating certain mathematical difficulties already at this stage of elementary modeling. However, it should be emphasized that from a formal point of view, reduction of the friction phenomenon to a one-dimensional problem is justified only for "small" rotational motions of the body and in cases where it might be possible to separate the translational motion of the contacting bodies from their rotational motion.

The models of kinetic friction can be represented by the dependencies of the friction force on relative velocity (*static version*) or described by the first-order differential (*dynamic version*).

The dynamic version of friction modeling attempts to incorporate the results obtained during tribological research and allows for, e.g., an explanation of the so-called small displacements just before onset of sliding, which are observable for small speeds of motion.

We will now present some examples to show how the relative motion of bodies in contact with one another with friction may even lead to a paradoxical behavior.

1. *Supply of energy by means of friction.*

Let us consider the case of a rigid body lying on a stationary base, as was already discussed in Sect. 2.8, and subjected to the action of force **F** (Fig. 2.38).

Let the transfer of a force interaction to a block of weight m take place by means of a massless spring of stiffness k one of whose ends (point A) moves with speed v_A. The motion of point A causes an increase in force **F** acting on the block. After attaining the value $kx'_S = F_S$, the block will start to slide with respect to the stationary base in the fixed (environment) coordinate system OX. It turns out that

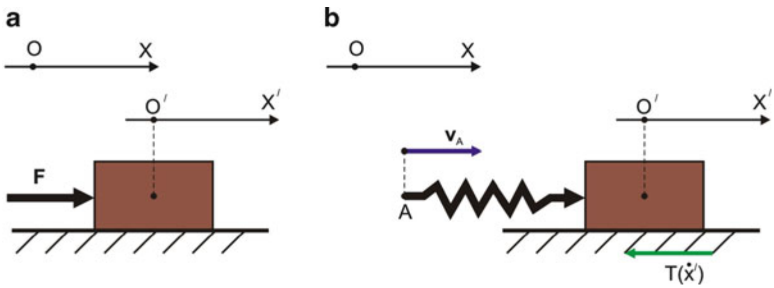

Fig. 2.38 Action of a force **F** on a block: directly (**a**) and through a spring (**b**)

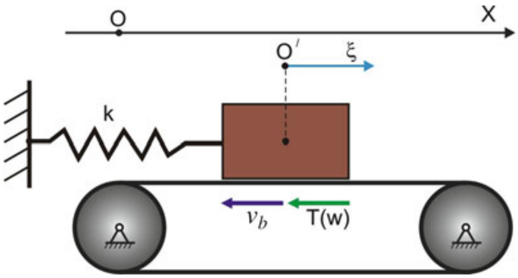

Fig. 2.39 A body of mass m lying on a conveyor moving at constant speed v_b

in many cases upon onset of motion after "breaking away" the static friction the sudden "forward jump" of the block follows. This phenomenon can be explained through the so-called decreasing part of the Coulomb friction versus the relative velocity of the moving bodies.

The analyzed system from Fig. 2.38b is analogous to the system depicted in Fig. 2.39, where the block lies on a belt moving at speed v_b and the spring in the static equilibrium position is compressed.

The equation of motion of a block (Fig. 2.39) in a fixed coordinate system reads

$$m\ddot{x} = -kx + T(w), \tag{2.67}$$

where

$$w = v_b - \dot{x} \tag{2.68}$$

denotes the relative velocity. In the case of static equilibrium $\dot{x} = \ddot{x} = 0$ and from (2.67) we obtain

$$kx_{\text{st}} = T(v_b). \tag{2.69}$$

In the coordinate system connected to the point of the static equilibrium position of the block $\dot{x} = \ddot{x} = 0$ we have

$$m\ddot{\xi} = -k(x_{\text{st}} + \xi) + T(v_b - \dot{\xi}), \tag{2.70}$$

where

$$\begin{aligned} x &= x_{\text{st}} + \xi, \\ \dot{x} &= \dot{\xi}, \\ w &= v_b - \dot{x} = v_b - \dot{\xi}. \end{aligned} \tag{2.71}$$

Taking into account (2.69), (2.71) yields

$$m\ddot{\xi} + k\xi = T(v_b - \dot{\xi}) - T(v_b) \equiv Q(\dot{\xi}), \tag{2.72}$$

and it can be seen that the friction force $T(v_b)$ carries the static deflection of the spring.

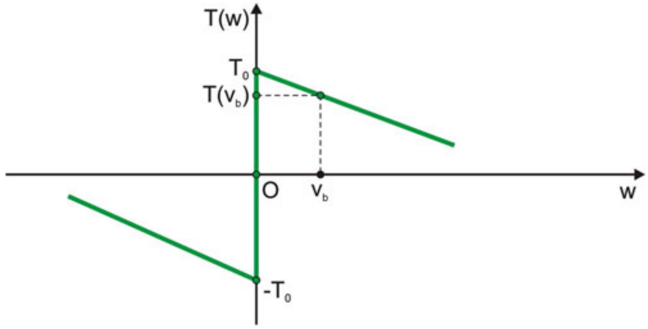

Fig. 2.40 Characteristic of friction $T(w)$

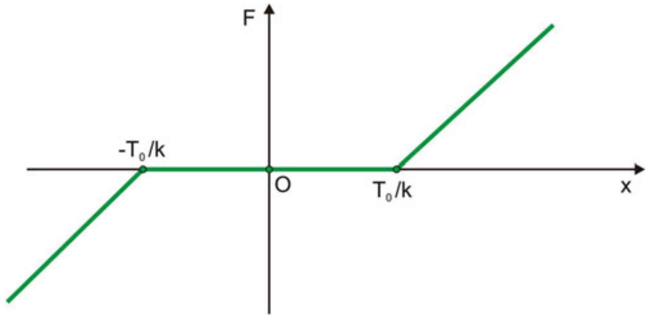

Fig. 2.41 A force in a spring $F(x)$ with so-called clearance interval marked: $\langle -T_0/k, T_0/k \rangle$ for $w = 0$

In both considered cases the vibrations of the block are observed and the only known way to explain this phenomenon thus far is by adoption of the so-called *decreasing characteristic* $T(w)$, assumed here to be a linear one (Fig. 2.40).

From the plot it can be seen that for $w = 0$, i.e., $\dot{x} = v_b$, there is no functional dependency since in the interval $\langle -T_0, T_0 \rangle$ we have infinitely many points, i.e., there appears an indeterminability often understood as *functional inclusion*.

It turns out, however, that the magnitude of friction force $T(0) \in \langle -T_0, T_0 \rangle$ is determined uniquely by the deflection of the spring on the assumption that $\ddot{\xi} = 0$, which is schematically shown in Fig. 2.41.

That characteristic does not exactly represent the real block behavior since when the spring tension reaches $kx = T_0$, we suddenly have $w = v_b$, that is, at that point (moment) the friction force is equal to $T(v_b)$. As can be seen in Fig. 2.40, the difference $T_0 - T(v_b)$, where $T_0 = T(0)$, is responsible for the appearance of the experimentally observed jump (acceleration) of the block.

We will now show that the characteristic of friction shown in Fig. 2.40 allows for the supply of energy to the system from its "reservoir" represented by the constant velocity v_b.

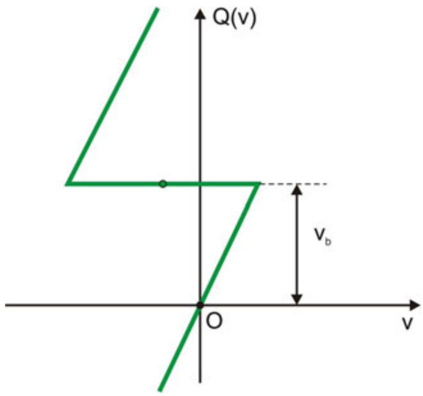

Fig. 2.42 Plot of a function $Q(v)$

From (2.72), after multiplying through by $\dot{\xi} = v$, we obtain

$$m\dot{v}v + k\xi v = Q(v)v. \tag{2.73}$$

The left-hand side of (2.73) can be cast in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_k + V) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv^2}{2} + \frac{k\xi^2}{2}\right) = mv\dot{v} + k\xi v, \tag{2.74}$$

where E_k, V denotes kinetic and potential energy, respectively. After taking into account (2.74), (2.73) will assume the following form:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = Q(v)v, \tag{2.75}$$

where E is the total energy of the considered one-degree-of-freedom system.

The function $Q(v)$ is presented in Fig. 2.42.

From (2.75) and Fig. 2.42 it follows that in certain ranges we have $Q(v)v > 0$, and in the other $Q(v)v < 0$, that is, the energy is supplied/taken away to/from the system, which was the subject of a detailed analysis of the works [31, 36].

2. *Coulomb friction as a force exciting the motion of a rigid body.*

Let us consider a one-dimensional problem, where a flat steel slat lies on two cylinders rotating in opposite directions (Fig. 2.43a) and the dependency of coefficient of friction on speed $\mu(\dot{x})$ has the form shown in Fig. 2.43b.

The equation of motion of the slat reads

$$m\ddot{x} = T_A - T_B, \tag{2.76}$$

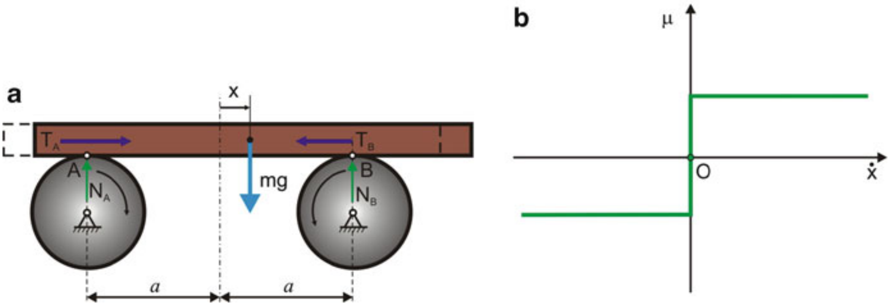

Fig. 2.43 A slat situated on two rotating cylinders (**a**) and a dependency $\mu(\dot{x})$ (**b**)

where

$$T_A = \mu N_A, \quad T_B = \mu N_B.$$

The equations of moments of forces about points A and B allow for the determination of the unknown normal reactions

$$N_A = \frac{(a-x)mg}{2a}, \quad N_B = \frac{(a+x)mg}{2a}.$$

From the equation of motion we obtain

$$m\ddot{x} = \frac{\mu mg}{2a}[a - x - (a+x)],$$

or

$$\ddot{x} + \frac{\mu g}{a}x = 0.$$

This means that the system vibrates harmonically with frequency $\alpha = \sqrt{\mu g a^{-1}}$. The equation obtained is often used to determine the coefficient of friction experimentally.

The careful reader may be somewhat confused by having the Coulomb friction presented in two different forms as in Figs. 2.40 and 2.43. In the first case it is easy to observe the self-excited vibration caused by friction, e.g., if one spreads a rosin film on the rubbing surfaces or plays on a string instrument. In the second case we are dealing with typical dry friction.

3. *Coulomb friction playing the role of viscous damping.*

Let a particle of mass m move in a plane at speed v (Fig. 2.44).

The velocity of particle v was resolved into two components:

$$\mathbf{v} = v_{x_1}\mathbf{E}_1 + v_{x_2}\mathbf{E}_2;$$

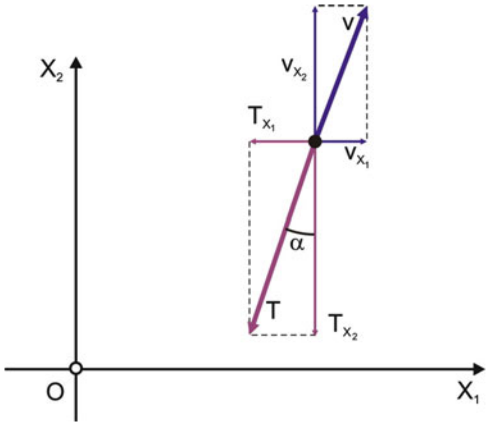

Fig. 2.44 Motion of a particle in plane OX_1X_2

the motion of the particle has two degrees of freedom and is described by two equations:

$$m\ddot{x}_1 = -T_{x1},$$
$$m\ddot{x}_2 = -T_{x2},$$

where

$$\frac{v_{x1}}{v} = \frac{T_{x1}}{T} = \sin\alpha, \quad \frac{v_{x2}}{v} = \frac{T_{x2}}{T} = \cos\alpha.$$

From the relationships above it follows that

$$T_{x_1} = T\alpha, \quad T_{x_2} = T,$$

on the assumption that α is a small angle; this means $\sin\alpha \approx \tan\alpha \approx \alpha$, $\cos\alpha \approx 1$.

This means that the particle moves at a high speed in the direction of the vertical axis and a low speed in the direction of the horizontal axis. In the considered case, the equations of motion assume the following form:

$$m\ddot{x}_1 = -T\alpha = -T\frac{v_{x1}}{v} = -T\frac{v_{x1}}{v_{x2}} = -cv_{x_1} = -c\dot{x}_1,$$
$$m\ddot{x}_2 = -T.$$

Assuming that only the component $\mathbf{v}_{x_1}$ of vector $\mathbf{v}$ is changed (keeping $\mathbf{v}_{x2}$ constant), one may introduce a resistant coefficient $c = T/v_{x_2}$. The resistant force in the direction OX_1 depends on the velocity v_{x_1}, and coefficient c is called a *viscous damping coefficient*.

2.10 Friction of Strings Wrapped Around a Cylinder

Let us consider a string (rope) wrapped around a cylinder of circular cross section (Fig. 2.45).

Let the cylinder and string have a rough surface. The weight of the string is negligible.

It follows from Fig. 2.45a that string ends are loaded by the forces $\mathbf{F}_a$ (active force) and $\mathbf{F}_p$ (passive force). Assume that force $\mathbf{F}_a$ achieved its critical value, i.e., infinitely small increase yields the string displacement coinciding with the $\mathbf{F}_a$ sense. Beginning from that time instant we are dealing with the so-called developed friction between the string and the cylinder.

Let us consider an element of the string defined by the angle $\mathrm{d}\varphi$ (Fig. 2.45a). After performing the projection of forces onto the axes of the adopted coordinate system (Fig. 2.45b), we obtain

$$\begin{aligned}(F+\mathrm{d}F)\cos\frac{\mathrm{d}\varphi}{2}-F\cos\frac{\mathrm{d}\varphi}{2}-\mathrm{d}T&=0,\\ \mathrm{d}N-(F+\mathrm{d}F)\sin\frac{\mathrm{d}\varphi}{2}-F\sin\frac{\mathrm{d}\varphi}{2}&=0,\end{aligned}\tag{2.77}$$

where T denotes the friction between belt and cylinder. After assuming $\sin\frac{\mathrm{d}\varphi}{2}\cong\frac{\mathrm{d}\varphi}{2}$, $\cos\frac{\mathrm{d}\varphi}{2}\cong 1$ we have

$$\begin{aligned}\mathrm{d}F-\mathrm{d}T&=0,\\ \mathrm{d}N-F\mathrm{d}\varphi&=0.\end{aligned}\tag{2.78}$$

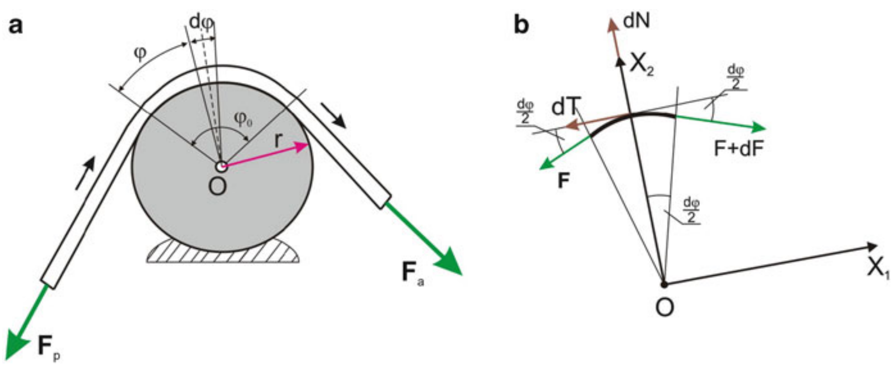

Fig. 2.45 Active ($\mathbf{F}_a$) and passive ($\mathbf{F}_p$) force acting on a string (**a**) and the computational scheme (**b**)

In what follows we take for $F = F_{\mathrm{p}}$ ($\varphi = 0$) and $F = F_{\mathrm{a}}$ ($\varphi = \varphi_0$) Because we are dealing with the fully developed friction $T = \mu N$, from the considerations above we will obtain

$$\int_{F_{\mathrm{p}}}^{F_{\mathrm{a}}} \frac{\mathrm{d}F}{F} = \mu \int_{0}^{\varphi_0} \mathrm{d}\varphi, \tag{2.79}$$

that is,

$$\ln F_{\mathrm{a}} - \ln F_{\mathrm{p}} = \mu\varphi_0, \tag{2.80}$$

where $F_{\mathrm{a}} > F_{\mathrm{p}}$ (meaning F_{a} overcomes friction resistance).

From the preceding equation we obtain the so-called Euler's belt formula of the form

$$F_{\mathrm{a}} = F_{\mathrm{p}} e^{\mu\varphi_0}. \tag{2.81}$$

From (2.81) it follows that force $\mathbf{F}_{\mathrm{a}}$ (active) balancing the action of force $\mathbf{F}_{\mathrm{p}}$ (passive), e.g., the weight hung at the end of the rope, does not depend on the radius of the cylinder on which it is pulled. For instance, assuming $\mu = 0.1$ (0.2) and the angle $\varphi_0 = \frac{\pi}{2}$ we obtain from the formula (2.81) the magnitude of active force $\mathbf{F}_{\mathrm{a}}$ equal to $2.19\,\mathbf{F}_{\mathrm{p}}$ ($4.76\,\mathbf{F}_{\mathrm{p}}$).

Observe that (2.81) can be interpreted in the following way. If the active force $\mathbf{F}_{\mathrm{a}}$ is going to keep only the weight $\mathbf{G}$ in equilibrium (i.e., its infinitely small decrease pushes the string displacement in the direction coinciding with the sense of vector $\mathbf{G}$), then the friction resistances help this force, and in this case this force value is found from the equation $F_{\mathrm{a}} = F_{\mathrm{p}} e^{-\mu\varphi_0}$.

We are dealing with a slightly different problem in the case of application of Euler's formula to conveyor systems with belts wrapped around the rotating drums, where transported material such as coal can be placed on the belt. It is desirable, then, to determine the maximum moment carried by the drum without slipping of the belt with respect to the drum. The friction of the strings is applied in belt transmissions because at the slight preliminary tension of the belt ($\mathbf{F}_{\mathrm{p}}$) the transmission even for a small angle of contact can carry significant torques.

We will show now a method to derive Euler's belt formula, which is different from that presented so far [37]. The considered contact of the drum with the rope can be approximated by means of the broken line shown in Fig. 2.46.

Let us cut out an element $OA_{m-1}A_mA_{m+1}$ from the drum and consider the equilibrium of the rope lying on $\Delta OA_{m-1}A_mA_{m+1}$, shown in Fig. 2.47. In the case depicted in Fig. 2.47a the problem cannot be solved. After neglecting the forces of friction between the rope and the drum in sections $A_{m-1}A_m$ and A_mA_{m+1}, the rope reaction will reduce to its reaction $\mathbf{F}^R$ caused by friction at the point A_m. However, at that point we do not know the direction of reaction because we do not know the direction of the normal to the surface at the contact point A_m (singular point) and $\mathbf{F}^R = \mathbf{N} + \mathbf{T}$. Therefore, we cannot apply the three-forces theorem to the forces $\mathbf{F}_{\mathrm{p}}, \mathbf{F}_{\mathrm{a}}$, and $\mathbf{F}^R$ in order to determine $\mathbf{F}_{\mathrm{a}}$. In this case we will treat the force $\mathbf{F}_{\mathrm{a}}$ as a force needed to keep the passive force $\mathbf{F}_{\mathrm{p}}$, that is, $|\mathbf{F}_{\mathrm{a}}| < |\mathbf{F}_{\mathrm{p}}|$. In order to avoid the

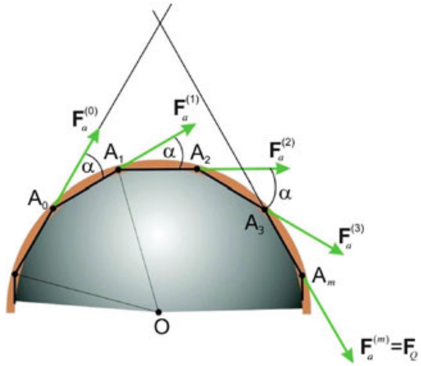

Fig. 2.46 Approximation of a rope by a broken line

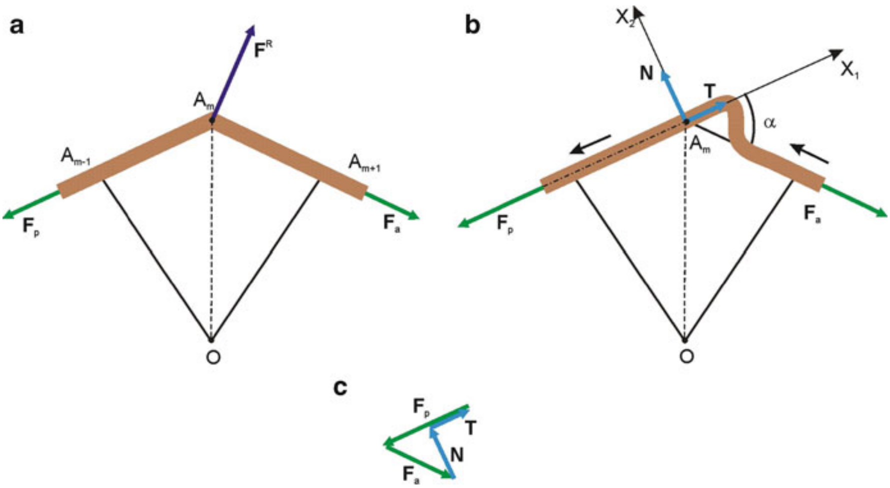

Fig. 2.47 A rope and reactions in a statically indeterminate case (**a**), a statically determinate case (**b**), and a force polygon (**c**)

singularity at the point A_m we can introduce the assumption of a high resistance to the bending of the rope and consequently assume that the rope will take the shape depicted in Fig. 2.47b around point A_m. Now we know the direction of the normal **N**, being perpendicular to $A_{m-1}A_m$, which enables us to construct the force polygon as shown in Fig. 2.47c.

Introducing the Cartesian coordinate system $A_m X_1 X_2$ (Fig. 2.47b), after projection of the force vectors we obtain

$$F_{\mathrm{p}} - T = F_{\mathrm{a}} \cos\alpha, \qquad N = F_{\mathrm{a}} \sin\alpha. \tag{2.82}$$

Considering the boundary case of fully developed friction we have $T = \mu N$. For a large number of sections M the angle α is small, and therefore we can assume that

$$\sin\alpha \cong \alpha, \qquad \cos\alpha \cong 1. \tag{2.83}$$

Consequently, from (2.82) we obtain

$$F_p = F_a(\cos\alpha + \mu \sin\alpha) = F_a(1 + \mu\alpha). \tag{2.84}$$

We will apply the equation just obtained several times to the polygon constructed from M sections.

From Fig. 2.47 it follows that

$$\begin{aligned}
F_a^{(m-1)} &= F_a^{(m)}(1 + \mu\alpha) = F_a(1 + \mu\alpha),\\
F_a^{(m-2)} &= F_a^{(m-1)}(1 + \mu\alpha) = F_a(1 + \mu\alpha)^2,\\
F_a^{(m-k)} &= F_a^{(m-k-1)}(1 + \mu\alpha) = F_a(1 + \mu\alpha)^k,\\
&\dots\\
F_a^{(1)} &= F_a(1 + \mu\alpha) = F_a(1 + \mu\alpha)^{M-1},\\
F_a^{(0)} &= F_p = F_a^{(1)}(1 + \mu\alpha) = F_a(1 + \mu\alpha)^M.
\end{aligned} \tag{2.85}$$

Because $\varphi = M\alpha$, from the last equation of (2.85) we obtain

$$F_p = F_a(1 + \mu\alpha)^M = F_a\left(1 + \mu\frac{\varphi}{M}\right)^M. \tag{2.86}$$

Moving to the exact approximation of a circle with linear sections, i.e., for $M \to \infty$, from (2.86) we obtain Euler's formula:

$$F_p = F_a \lim_{M\to\infty}\left(1 + \mu\frac{\varphi}{M}\right)^M = F_a e^{\mu\varphi}, \tag{2.87}$$

where e is the base of a natural logarithm.

From (2.87) we obtain $F_a = F_p e^{-\mu\varphi}$, which means that to support, for example, the weight $G = F_p$ hung at one end of the rope, one must use the force of $F_p e^{-\mu\varphi}$, which is smaller than the magnitude of the weight G. The situation will change if we want to pull the weight $G = F_p$ over the drum with friction. Then one should use the force of magnitude $Ge^{\mu\varphi}$. □

Example 2.11. Find the relationship between forces G and F in an equilibrium position for the braking system depicted in Fig. 2.48.

The massless rope is wrapped around a cylinder of radius r and pulled through massless pulleys 1 and 2 of negligible radii. The cylinder is connected to the braking drum of radius R, which is wrapped around by the braking belt attached to lever AB. The coefficient of friction on the braking drum is equal to μ.

After releasing from constraints we obtain two subsystems shown in Fig. 2.49.

Let us formulate the equilibrium condition for the lever writing the equation of moments about point O_3. Pulley 2 will be in equilibrium if $\mathbf{Q}_1 + \mathbf{Q}_2 = \mathbf{F}^i$ and for the equation of moments about O_3 the vertical component of $\mathbf{F}^i$, equal to $Q_1 + Q_2\cos\varphi$, is needed. The aforementioned equation of moments reads

$$Fl + F_a a_1 = (Q_1 + Q_2\cos\varphi)l_1 + F_p a_2.$$

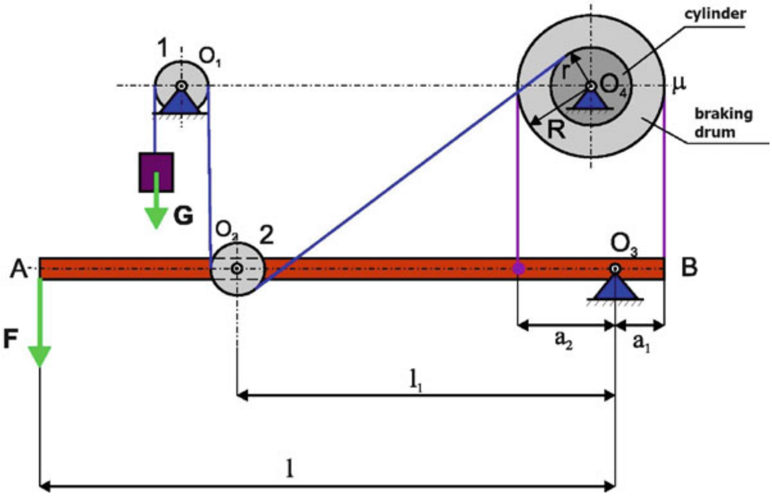

Fig. 2.48 A braking mechanism

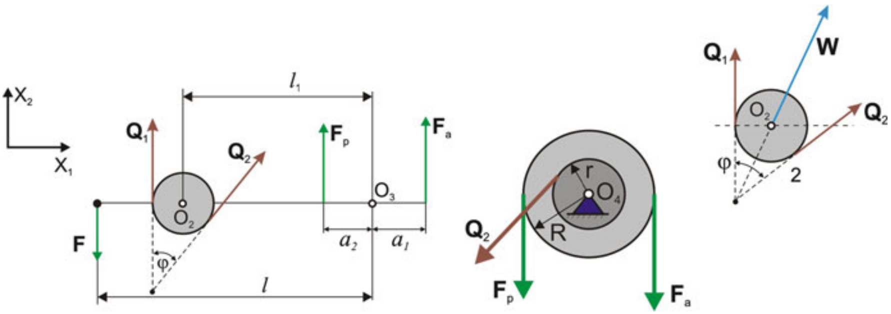

Fig. 2.49 Computational schemes

Because $Q_1 = Q_2 = G$, the equation of moments about point O_3 gives

$$Gr + F_p R = F_a R.$$

The relationship between forces F_a and F_p is defined by Euler's formula:

$$F_a = F_p e^{\mu\pi}.$$

From the preceding equation we have

$$Gr = F_p R(e^{\mu\pi} - 1),$$

that is,

$$F_{\mathrm{p}} = \frac{Gr}{(e^{\mu\pi} - 1)R}.$$

Coming back to the equation with force F we obtain

$$F = \frac{G}{l}\left[\frac{r(a_2 - a_1 e^{\mu\pi})}{R(e^{\mu\pi} - 1)} + (1 + \cos\varphi)l_1\right]. \qquad \square$$

2.11 Friction Models

2.11.1 Introduction

The aim of the present section is not an in-depth description of a friction phenomenon but only an emphasis on its complexity and a presentation of its engineering models. As has already been mentioned, the model of sliding friction $T = \mu N$ allows us to determine this force during macroscopic sliding of one of the analyzed surfaces on the other. In the general case the coefficient of friction depends on the contact pressure p (normal force N), the relative speed of sliding v_r, and temperature of the contact area θ, i.e., it is a function of the form

$$\mu = \mu(p, v_r, \theta). \tag{2.88}$$

In [38] many investigations of the dry friction phenomenon were conducted and computational methods serving to estimate this friction were proposed. In the cited works there are references to results of experimental investigations on the dependency of the friction coefficient on the speed of slippage. Various characteristics of that dependency were found, including the minimal and maximal values and with a monotonal decrease in value, as well as characteristics displaying a constant coefficient of friction for varying slip speed (Fig. 2.50).

Most frequently during the analyses the linear models of friction are used, which corresponds to a straight line 1 in Fig. 2.50.

The behavior of friction in the case of lubricated surfaces as a function of slip speed and in the case of stationary conditions is best described by friction characteristics called the Stribeck[6] curve. For low slip speeds, the friction force depends mainly on mechanical, structural, and physiochemical material properties of the rubbing surfaces (dry friction). For moderate slip speeds oil wedges form, and the resistance of a hybrid (fluid and dry) friction decreases. With further increases in slip speed complete separation of the rubbing surfaces takes place. Then only the fluid friction exists whose magnitude increases with increasing speed.

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[6] Richard Stribeck (1861–1950), German engineer working mainly in Dresden, Köln, and Stuttgart.

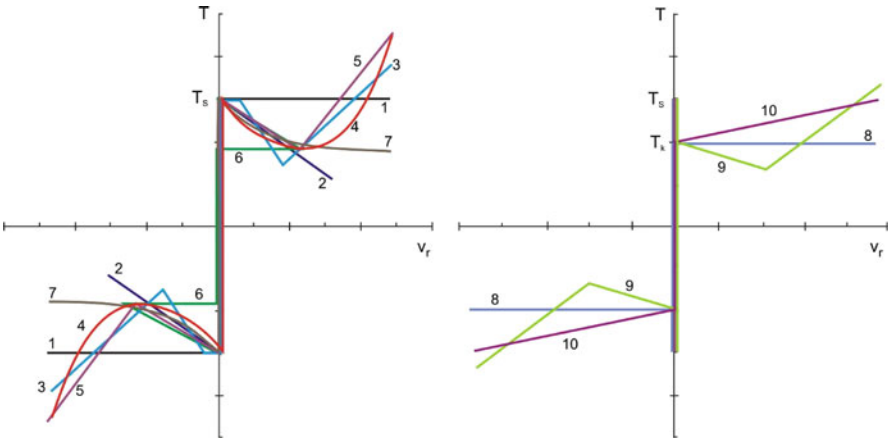

Fig. 2.50 Certain models of friction forces encountered in the literature

The Stribeck curve (curve 4 in Fig. 2.50) was used, for instance, in [32]. Simplified Stribeck curves (3 and 9 in Fig. 2.50) were the objects of analysis in [39]. The models of friction force represented by curves 2, 5, 6, 7, 8, and 10 were described among others in [36, 40]. In [33], in the numerical investigations of the dynamics of wheeled vehicles, the model of the *Magnum* type brake was used. This model relies on friction between road surface and tire. It pictures the mutual relations between the forces of adhesive friction (forces of adhesion), Coulomb friction, and viscous friction and the slip speed of the wheel, which are graphically represented by the Stribeck curve. For some road surfaces the modeling of friction with curve 7 (Fig. 2.50) is recommended.

The dependency describing curve 4 according to [41] possesses the following form:

$$\mu = \operatorname{sgn}(v_r)\Big(\big(a + b|v_r|\big)\exp\left(-c|v_r|\right) + d\Big), \tag{2.89}$$

and the dependency approximated by curve 7 [42] is expressed by

$$\mu = \operatorname{sgn}(v_r)\left(\frac{f_0 - f_{\min}}{1 + c|v_r|} + f_{\min}\right), \tag{2.90}$$

where $a, b, c, d, f_0, f_{\min}$ are constant parameters.

The Stribeck curve [43] (Fig. 2.51) attains its minimum at $v_r = v_{\min}$, and for $v_r < v_{\min}$ a characteristic drop in value of the coefficient of friction can be observed.

Starting from the classical works of Coulomb there has been much attention devoted to the problem of friction, which was treated as a classical process within the framework of Coulomb hypotheses and as a complex process involving wear and heat exchange between rubbing bodies [35]. Problems connected to one-dimensional Coulomb friction and the chaotic dynamics of simple mechanical

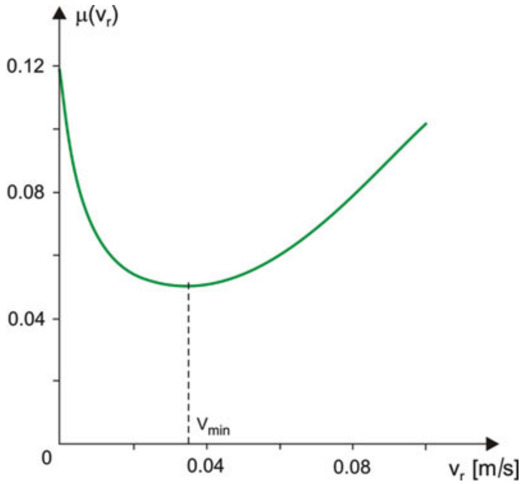

Fig. 2.51 A plot of the coefficient of kinetic friction against the relative speed of rubbing bodies

systems are also described in monographs [32] and [34], where an extensive literature on the topic was presented. However, the latest works of Contensou [44] and Zhuravlev [45, 46] point to the fact that the classical Coulomb model seen through the motion of contacting bodies in space can explain, to a large extent, many phenomena encountered in everyday life and in engineering associated with friction.

Let an arbitrary rigid body be in contact with another fixed body (a base), and the contact between the bodies takes place at a point or over a circular patch of a very small radius. The rigid body moving on the base has five degrees of freedom, and its dynamics is described by the translational motion of the mass center (vector $\mathbf{v}$) and the rotational motion about the mass center (vector $\boldsymbol{\omega}$). As we will show below, the friction model from the perspective of simultaneous translational and rotational motion of a rigid body allows for the explanation of many phenomena, and especially allows for the elimination of the classical, often incorrect, results associated with the interpretation of motion of a rigid body as a combination of translational and rotational motion on the assumption that translational friction $\mathbf{T}$ (treated as a force of resistance to translational motion) and torsional friction $\mathbf{M}_T$ (treated as a moment of resistance to rotational motion) are seen as independent of each other. In the presented model further called CCZ (Coulomb–Contensou–Zhuravlev) $T = T(\mathbf{v}, \boldsymbol{\omega})$, $M_T = M_T(\mathbf{v}, \boldsymbol{\omega})$, and at the point $\mathbf{v} = \boldsymbol{\omega} = \mathbf{0}$ functions T and M_T do not possess a limit. The introduction of even a very small quantity $\boldsymbol{\omega}$ in the case $T(\mathbf{v}, \boldsymbol{\omega})$ and a very small quantity $\mathbf{v}$ in the case $M_T(\mathbf{v}, \boldsymbol{\omega})$ (which is close to real phenomena) leads, in many cases, to the elimination of non-holonomic constraints, which complicate many problems of classical mechanics.

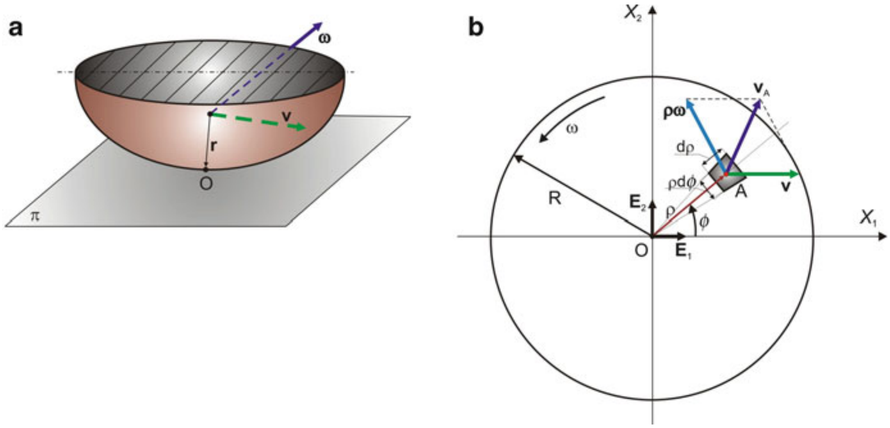

Fig. 2.52 Circular contact of rigid body and fixed plane (**a**) and velocity of an arbitrary point A of body contact patch (**b**)

2.11.2 A Modified Model of Coulomb Friction (CCZ Model)

In Fig. 2.52a the contact of a body with a fixed surface is shown, whereas Fig. 2.52b shows a body contact patch of circular shape in the neighborhood of point O.

The starting point for the construction of the CCZ model is the application of Coulomb's law. It is assumed that the differential of an elementary friction force d$\mathbf{T}$ and elementary friction moment d$\mathbf{M}_T$ is directed against the relative motion at point A according to the equations

$$\mathrm{d}\mathbf{M}_T = -\mu\sigma(\boldsymbol{\rho})\frac{\boldsymbol{\rho}\times\mathbf{v}_A}{|\mathbf{v}_A|}\mathrm{d}S, \qquad \mathrm{d}\mathbf{T} = -\mu\sigma(\boldsymbol{\rho})\frac{\mathbf{v}_A}{|\mathbf{v}_A|}\mathrm{d}S, \tag{2.91}$$

where $\mathbf{v}_A$ is the velocity of sliding at point A, that is, $\mathbf{v}_A = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}$ (Fig. 2.52a), (ρ, ϕ) are polar coordinates of point A (Fig. 2.52b), $\boldsymbol{\sigma}(\rho)\mathrm{d}S$ is a normal force dependent on the radius ρ describing the distance of point A from point O, and the elementary surface is equal to $\mathrm{d}S = \rho\mathrm{d}\rho\mathrm{d}\phi$.

In the adopted Cartesian coordinate system OX_1X_2 the velocity of point A is equal to

$$\mathbf{v}_A = (v - \omega\rho\sin\phi)\mathbf{E}_1 + \omega\rho\cos\phi\mathbf{E}_2. \tag{2.92}$$

In the general case a moment of force is calculated from the formula

$$\mathbf{M} = \boldsymbol{\rho}\times\mathbf{F} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \rho_1 & \rho_2 & \rho_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \mathbf{E}_1(\rho_2F_3 - \rho_3F_2) - \mathbf{E}_2(\rho_1F_3 - \rho_3F_1) + \mathbf{E}_3(\rho_1F_2 - \rho_2F_1), \tag{2.93}$$

and hence, in the considered case, the differential of the friction moment is equal to

$$\mathrm{d}\mathbf{M}_T = \mathbf{E}_3(\rho_2 \mathrm{d}T_1 - \rho_1 \mathrm{d}T_2). \tag{2.94}$$

According to (2.91) and (2.92) we have

$$\begin{aligned}
\mathrm{d}T_1 &= -\frac{\mu\sigma(\rho)}{v_A}(v - \omega\rho\sin\phi)\rho\mathrm{d}\rho\mathrm{d}\phi,\\
\mathrm{d}T_2 &= -\frac{\mu\sigma(\rho)}{v_A}\omega\rho\cos\phi\rho\mathrm{d}\rho\mathrm{d}\phi,\\
v_A &= |\mathbf{v}_A| = \sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2},
\end{aligned} \tag{2.95}$$

and in view of that

$$\mathrm{d}M_T = \frac{\mu\sigma(\rho)}{v_A}(\omega\rho^2 - v\rho\sin\phi)\rho\mathrm{d}\rho\mathrm{d}\phi. \tag{2.96}$$

We obtain the desired friction force $\mathbf{T}$ and friction moment $\mathbf{M}_T$ by means of integration of (2.91) and (2.96), and they are equal to

$$\begin{aligned}
\mathbf{T} &= -\mu\mathbf{E}_1 \int_0^R \int_0^{2\pi} \frac{(v - \omega\rho\sin\phi)\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}}\\
&\quad -\mu\mathbf{E}_2 \int_0^R \int_0^{2\pi} \frac{\omega\rho\cos\phi\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}},\\
\mathbf{M}_T &= -\mu\mathbf{E}_3 \int_0^R \int_0^{2\pi} \frac{(\omega\rho^2 - v\rho\sin\phi)\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}},
\end{aligned} \tag{2.97}$$

where in the considered case the second term of friction force in the first equation is equal to zero.

After the introduction of new variables $\rho^* = \frac{\rho}{R}$ and $u = \omega R$, where R is the radius of the circular contact patch between the bodies, (2.97) assume the following scalar form:

$$\begin{aligned}
T &= \mu R^2 \int_0^1 \rho^*\sigma(\rho^*) \int_0^{2\pi} \frac{(v - u\rho^*\sin\phi)\mathrm{d}\rho^*\mathrm{d}\phi}{\sqrt{u^2\rho^{*2} - 2uv\rho^*\sin\phi + v^2}},\\
M &= \mu R^3 \int_0^1 \rho^*\sigma(\rho^*) \int_0^{2\pi} \frac{(u\rho^{*2} - v\rho^*\sin\phi)\mathrm{d}\rho^*\mathrm{d}\phi}{\sqrt{u^2\rho^{*2} - 2uv\rho^*\sin\phi + v^2}}.
\end{aligned} \tag{2.98}$$

It is well worth it to return to Fig. 2.52a and discuss the choice of the pole (point O) about which the reduction of the friction force and friction moment were carried out. In the general case, for the aforementioned choice of the reduction pole, integrals (2.98) cannot be expressed in terms of elementary functions but only in terms of elliptic integrals, which is inconvenient during the calculations. However, it turns out that physical observations associated with the kinematics and statics of the problem allow for (by the proper choice of the pole) the reduction of the problem to integrals expressed in terms of elementary functions. As will be presented, these integrals in the considered cases are expressed by simple analytical functions. An arbitrary choice of the reduction pole leads to the necessity of taking the force and moment of friction into account. However, it is possible to choose the pole at the point that is an *instantaneous center of rotation* (this case will be given a detailed treatment subsequently) and then, even though we are also dealing with the force and moment of friction, integrals (2.98) assume the form of elementary integrals. One can also choose the point of reduction such that the moment arm of friction force is equal to zero. Because in the general case $M = M(v, \omega)$, let us choose the reduction point such that $M(v, 0) = 0$. The physical interpretation of the pole like that coincides with the notion of mass center of the contact patch, where the role of mass is played by the normal stress $\sigma = \sigma(x_1, x_2)$ in the coordinate system rigidly connected to the contact surface. This point is defined by two algebraic equations:

$$\iint_D x_i \sigma(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = 0, \qquad i = 1, 2, \tag{2.99}$$

where D is the area of contact of the bodies.

In the general case, (2.98) in the Cartesian coordinate system assume the form

$$\begin{aligned} T_1 &= -\mu R^2 \iint_D \frac{\sigma(x_1, x_2)(v - u x_2)\mathrm{d}x_1 \mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \\ T_2 &= -\mu R^2 \iint_D \frac{\sigma(x_1, x_2) u x_1 \mathrm{d}x_1 \mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \\ M &= -\mu R^3 \iint_D \frac{\sigma(x_1, x_2)[(x_1^2 + x_2^2)u - v x_2]\mathrm{d}x_1 \mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \end{aligned} \tag{2.100}$$

where the designation $u = \omega R$ was preserved.

As was emphasized earlier, in the case where region D is symmetrical with respect to the pole (e.g., a disc), the problem is two-dimensional since $T = T(u, v)$ and $M = M(u, v)$, and $T_2 = 0$. However, in the general case (for non-symmetrical regions), we have at our disposal three (2.100) and three kinematic quantities u, v_1, and v_2, that is, $T = T(u, v_1, v_2)$ and $M = M(u, v_1, v_2)$. Let us recall that earlier for the case of the circular contact patch we chose the axis of Cartesian coordinates as parallel to the velocity $\mathbf{v}$.

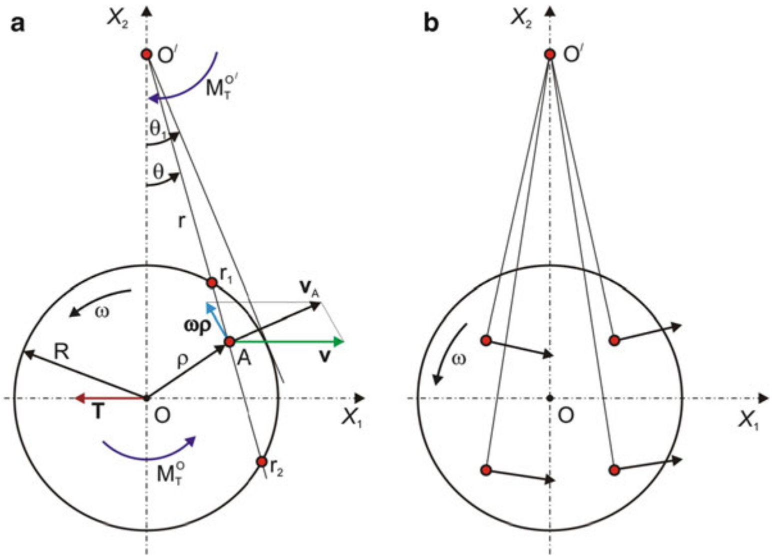

Fig. 2.53 A circular contact in the case of coordinates (r, θ) (**a**) and distributions of the velocity field of selected points of the contact patch (**b**)

Our next aim will be to determine the force and moment of friction in simple cases of symmetrical patches of contact between bodies. To this end we explore the following two cases, known from the literature, of contact stresses [47]:

(i) Circular contact patch of a disc with a plane, where the dependency of stress on the radius ρ is governed by the equation

$$\sigma(\rho) = \frac{3N}{2\pi R^2 \sqrt{1 - \left(\frac{\rho}{R}\right)^2}}; \tag{2.101}$$

(ii) Hertzian point contact, where

$$\sigma(\rho) = \frac{3N\sqrt{1 - \left(\frac{\rho}{R}\right)^2}}{2\pi R^2}. \tag{2.102}$$

In the preceding discussion, N is the normal force pressing the bodies against each other, R is the radius of the contact patch circle, and $\rho = OA$ (Figs. 2.52 and 2.53).

Case (i)

In order to avoid elliptic integrals we introduce the polar coordinate system (r, θ) shown in Fig. 2.53a.

Point O' is the instantaneous center of rotation, and $O'O = H$. In turn, from the velocity distribution in Fig. 2.53b it follows that in this case the friction component $T_2 = 0$.

Thus, by definition we have

$$\mathrm{d}T = -\mu\sigma(\rho)\frac{v_A \cos\theta}{v_A}\mathrm{d}S, \tag{2.103}$$

that is,

$$T = -\mu \iint_D \sigma(\rho)\cos\theta\, r\,\mathrm{d}\theta\mathrm{d}r. \tag{2.104}$$

In turn, on the basis of Fig. 2.53a one can formulate the following relationship:

$$\rho^2 = r^2\sin^2\theta + (H - r\cos\theta)^2. \tag{2.105}$$

From (2.104) and (2.101) we obtain

$$\begin{aligned} T &= -\frac{3N\mu}{2\pi}\iint_D \frac{\cos\theta\, r\,\mathrm{d}\theta\mathrm{d}r}{R^2\sqrt{1-\left(\frac{r^2}{R^2}\sin^2\theta + \frac{H^2}{R^2} - 2\frac{H}{R}\frac{r}{R}\cos\theta + \frac{r^2}{R^2}\cos^2\theta\right)}} \\ &= -\frac{3N\mu}{2\pi}\iint_D \frac{\left(\frac{r}{R}\right)\cos\theta\mathrm{d}\theta\left(\frac{\mathrm{d}r}{R}\right)}{\sqrt{1-q^2\sin^2\theta - k^2 + 2qk\cos\theta - q^2\cos^2\theta}} \\ &= -\frac{3N\mu}{2\pi}\iint_D \frac{q\cos\theta\mathrm{d}\theta\mathrm{d}q}{\sqrt{1-q^2+2qk\cos\theta - k^2\sin^2\theta - k^2\cos^2\theta}} \\ &= -\frac{3N\mu}{2\pi}\iint_D \frac{q\cos\theta\mathrm{d}\theta\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}}, \end{aligned} \tag{2.106}$$

where we set $q = \frac{r}{R}$ and $k = \frac{H}{R}$.

The friction moment with respect to point O' is equal to

$$M_T^{O'} = HT + M_T^O. \tag{2.107}$$

From (2.107) it follows that by calculating $M_T^{O'}$ and with H known it is possible to determine M_T^O.

During the calculation of integral (2.106) one should consider two cases, that is, one where point O' lies outside the circle of radius R (then $k > 1$), and another where it lies inside or at the boundary of that circle (then $k \le 1$). Thus the problem boils down to the calculation of the following two integrals:

$$T = -\frac{3N\mu}{4\pi}\int_0^{\pi}\cos\theta\mathrm{d}\theta\int_{q_1}^{q_2}\frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}}, \quad (k \le 1),$$

$$T = -\frac{3N\mu}{2\pi}\int_0^{\theta_1}\cos\theta\mathrm{d}\theta\int_{q_1}^{q_2}\frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}}, \quad (k > 1), \tag{2.108}$$

where

$$q_{1,2} = k\cos\theta \mp \sqrt{1-k^2\sin^2\theta}, \qquad \theta_1 = \arcsin\left(\frac{1}{k}\right).$$

Now, we will show how to calculate integrals (2.108). At first, let us calculate the indefinite integral

$$\begin{aligned} J^* &= \int \frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}} \\ &= \int \frac{q\mathrm{d}q}{\sqrt{-q^2+2qk\cos\theta+1-k^2}} = \int \frac{q\mathrm{d}q}{\sqrt{-q^2+2qa+b}}, \end{aligned} \tag{2.109}$$

where $a = k\cos\theta$, $b = 1-k^2$. Because the parameters a and b do not depend on q, in the course of the calculations they can be treated as constants. We successively have

$$\begin{aligned} J^* &= -\frac{1}{2}\int \frac{-2q\mathrm{d}q}{\sqrt{-q^2+2qa+b}} = -\frac{1}{2}\int \frac{(-2q+2a-2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}} \\ &= -\frac{1}{2}\int \frac{(-2q+2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}} + a\int \frac{\mathrm{d}q}{\sqrt{-q^2+2qa+b}}. \end{aligned} \tag{2.110}$$

After setting $J^* = J_1 + J_2$ we have

$$J_1 = -\frac{1}{2}\int \frac{(-2q+2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}} = -\frac{1}{2}\int \frac{\mathrm{d}(-q^2+2qa+b)}{\sqrt{-q^2+2qa+b}}, \tag{2.111}$$

and the above integral is calculated with the aid of the substitution $t = -q^2 + 2qa + b$. Thus we have

$$J_1 = -\frac{1}{2}\int t^{-\frac{1}{2}}\mathrm{d}t = -\frac{1}{2}\frac{t^{-\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{t} = -\sqrt{-q^2+2qa+b}. \tag{2.112}$$

Now as we proceed to calculate the integral

$$J_2 = a\int \frac{\mathrm{d}q}{\sqrt{-q^2+2qa+b}}. \tag{2.113}$$

We transform the denominator of the preceding expression into the form

$$\sqrt{-q^2+2qa+b} = \sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta} \\ = \sqrt{-(q-a)^2+c}, \tag{2.114}$$

where $c = 1 - k^2\sin^2\theta$. Thus we obtain

$$J_2 = a\int \frac{\mathrm{d}q}{\sqrt{c-(q-a)^2}} = a\int \frac{\mathrm{d}(q-a)}{\sqrt{\left(\sqrt{c}\right)^2-(q-a)^2}} \\ = a\int \frac{\mathrm{d}z}{\sqrt{\left(\sqrt{c}\right)^2-z^2}} = a\arcsin\frac{z}{\sqrt{c}} = a\arcsin\frac{q-a}{\sqrt{c}}. \tag{2.115}$$

According to (2.110) we obtain

$$J^* = J_1 + J_2 = -\sqrt{-q^2+2qa+b} + a\arcsin\frac{q-a}{\sqrt{c}}. \tag{2.116}$$

Eventually, we obtain the desired definite integral

$$J_0^* = \int_{q_1}^{q_2} \frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}} = J^*\Big|_{q_1}^{q_2},$$

where $q_{1,2} = a \mp \sqrt{c}$. We have then

$$J_0^* = -\sqrt{-q_2^2+2q_2a+b} + \arcsin\frac{q_2-a}{\sqrt{c}} + \sqrt{-q_1^2+2q_1a+b} \\ - \arcsin\frac{q_1-a}{\sqrt{c}} = -\sqrt{-a^2-2a\sqrt{c}-c+2a^2+2a\sqrt{c}+b} \\ + a\arcsin\frac{a+\sqrt{c}-a}{\sqrt{c}} + \sqrt{-a^2+2a\sqrt{c}-c+2a^2-2a\sqrt{c}+b} \\ - a\arcsin\frac{a-\sqrt{c}-a}{\sqrt{c}} = -\sqrt{a^2+b-c} + a\arcsin(1) \\ + \sqrt{a^2+b-c} - a\arcsin(-1) = a\frac{\pi}{2} - a\left(-\frac{\pi}{2}\right) = \pi a. \tag{2.117}$$

The desired definite integral is equal to

$$\int_{q_1}^{q_2} \frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}} = \pi k\cos\theta. \tag{2.118}$$

Eventually, the unknown indefinite integral reads

$$J = \int \pi k \cos\theta \cos\theta \mathrm{d}\theta = \pi k \int \cos^2\theta \mathrm{d}\theta$$
$$= \frac{\pi k}{2} \int (1 + \cos 2\theta)\mathrm{d}\theta = \frac{\pi k}{2}\theta + \frac{\pi k}{4}\sin 2\theta, \tag{2.119}$$

and its corresponding definite form is

$$J_0^{(1)} = \int_0^{\pi} \cos\theta \mathrm{d}\theta \int_{q_1}^{q_2} \frac{q\mathrm{d}q}{\sqrt{-(q - k\cos\theta)^2 + 1 - k^2\sin^2\theta}}$$
$$= \left[\frac{\pi k}{2}\theta + \frac{\pi k}{4}\sin 2\theta\right]_0^{\pi} = \frac{\pi^2 k}{2}. \tag{2.120}$$

The following integral remains to be calculated:

$$J_0^{(2)} = \int_0^{\theta_1} \cos\theta \mathrm{d}\theta \int_{q_1}^{q_2} \frac{q\mathrm{d}q}{\sqrt{-(q - k\cos\theta)^2 + 1 - k^2\sin^2\theta}}$$
$$= \left[\frac{\pi k}{2}\theta + \frac{\pi k}{4}\sin 2\theta\right]_0^{\theta_1} = \frac{\pi k}{2}\theta_1 + \frac{\pi k}{4}\sin 2\theta_1. \tag{2.121}$$

Since

$$\theta_1 = \arcsin\left(\frac{1}{k}\right), \tag{2.122}$$

we have

$$\sin 2\theta_1 = 2\sin\theta_1\cos\theta_1 = 2\sin\theta_1\sqrt{1 - \sin^2\theta_1}$$
$$= 2\sin\left(\arcsin\left(\frac{1}{k}\right)\right)\sqrt{1 - \sin^2\left(\arcsin\left(\frac{1}{k}\right)\right)}$$
$$= \frac{2}{k}\sqrt{1 - \frac{1}{k^2}} = \frac{2\sqrt{k^2 - 1}}{k^2}, \tag{2.123}$$

and finally

$$J_0^{(2)} = \frac{\pi k}{2}\arcsin\left(\frac{1}{k}\right) + \frac{\pi\sqrt{k^2 - 1}}{2k}. \tag{2.124}$$

After taking into account the results of the calculations above in (2.108) we obtain

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \frac{\pi^2 k}{4}, & k \leq 1, \\ \frac{\pi k}{2}\arcsin\left(\frac{1}{k}\right) + \frac{\pi}{2k}\sqrt{k^2-1}, & k > 1. \end{cases} \tag{2.125}$$

Case (ii)

In this case the problem boils down to the calculation of the friction force by means of the following definite integrals:

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \frac{1}{2}\int\limits_0^\pi \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}\, q\mathrm{d}q, \\ \int\limits_0^{\theta_1} \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}\, q\mathrm{d}q, \end{cases} \tag{2.126}$$

where further we adopt designations analogous to case (i), that is, $q_{1,2} = k\cos\theta \mp \sqrt{1-k^2\sin^2\theta}$, $\theta_1 = \arcsin\left(\frac{1}{k}\right)$.

At first, we calculate the indefinite integral

$$J = \int \sqrt{-q^2 + 2qa + b}\, q\mathrm{d}q = -\frac{1}{2}\int \sqrt{-q^2 + 2qa + b}\,(-2q)\mathrm{d}q$$
$$= -\frac{1}{2}\int \sqrt{-q^2 + 2qa + b}\,(-2q + 2a - 2a)\mathrm{d}q. \tag{2.127}$$

Next, we have

$$J = J_1 + J_2, \tag{2.128}$$

where

$$J_1 = -\frac{1}{2}\int \sqrt{-q^2 + 2qa + b}\,(-2q + 2a)\mathrm{d}q,$$
$$J_2 = a\int \sqrt{-q^2 + 2qa + b}\,\mathrm{d}q. \tag{2.129}$$

Because

$$(-2q + 2a)\mathrm{d}q = \mathrm{d}\left(-q^2 + 2aq\right) = \mathrm{d}\left(-q^2 + 2aq + b\right),$$

we have

$$J_1 = -\frac{1}{2}\int \left(-q^2 + 2aq + b\right)^{\frac{1}{2}} \mathrm{d}\left(-q^2 + 2aq + b\right). \tag{2.130}$$

In turn, after introducing the substitution

$$t = -q^2 + 2aq + b, \tag{2.131}$$

we obtain

$$J_1 = -\frac{1}{2}\int t^{\frac{1}{2}}\mathrm{d}t = -\frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} = -3t^{\frac{3}{2}}, \tag{2.132}$$

that is, eventually we have

$$J_1 = -3\left(\sqrt{-q^2 + 2aq + b}\right)^3. \tag{2.133}$$

We calculate the integral J_2 by parts:

$$\begin{aligned}
J_2 &= a\left[q\sqrt{-q^2+2aq+b} - \int q\mathrm{d}\left(\sqrt{-q^2+2aq+b}\right)\right] \\
&= a\left(q\sqrt{-q^2+2aq+b} - \int \frac{q(-2q+2a)\mathrm{d}q}{2\sqrt{-q^2+2aq+b}}\right) \\
&= a\left(q\sqrt{-q^2+2aq+b} + \int \frac{q^2\mathrm{d}q}{\sqrt{-q^2+2aq+b}} - a\int \frac{q\mathrm{d}q}{\sqrt{-q^2+2aq+b}}\right) \\
&= a\left(q\sqrt{-q^2+2aq+b} + J_3 - aJ_4\right),
\end{aligned} \tag{2.134}$$

where

$$\begin{aligned}
J_3 &= \int \frac{q^2\mathrm{d}q}{\sqrt{-q^2+2aq+b}}, \\
J_4 &= \frac{q\mathrm{d}q}{\sqrt{-q^2+2aq+b}}.
\end{aligned} \tag{2.135}$$

According to (2.109) $J_4 = J^*$, that is,

$$J_4 = -\sqrt{-q^2+2aq+b} + \arcsin\frac{q-a}{\sqrt{c}}, \tag{2.136}$$

where $c = 1 - k^2\sin^2\theta$.

In the case of the integral J_3 we have

$$J_3 = -\int \frac{-q^2\mathrm{d}q}{\sqrt{-q^2+2aq+b}} = -\int \frac{(-q^2+2qa+b-2qa-b)\mathrm{d}q}{\sqrt{-q^2+2aq+b}}$$

$$= -\int \frac{\left(\sqrt{-q^2+2aq+b}\right)^2 \mathrm{d}q}{\sqrt{-q^2+2aq+b}} + 2a\int \frac{q\mathrm{d}q}{\sqrt{-q^2+2aq+b}}$$
$$+b\int \frac{\mathrm{d}q}{\sqrt{-q^2+2aq+b}} = -\int \sqrt{-q^2+2aq+b}\,\mathrm{d}q + 2aJ_4 + bJ_5. \tag{2.137}$$

From (2.115) it follows that the integral J_5 is equal to

$$J_5 = \arcsin\frac{q-a}{\sqrt{c}}. \tag{2.138}$$

Let us note that the first integral in (2.137) is equal to $-\frac{1}{a}J_2$ [see (2.129)], that is, (2.137) assumes the form

$$J_3 = -\frac{1}{a}J_2 + 2aJ_4 + bJ_5, \tag{2.139}$$

where J_2 is given by (2.134).

From (2.134) and after taking into account (2.139) we obtain

$$J_2 = a\left(q\sqrt{-q^2+2aq+b} - \frac{1}{a}J_2 + 2aJ_4 + bJ_5 - aJ_4\right), \tag{2.140}$$

hence we obtain

$$J_2 = \frac{aq}{2}\sqrt{-q^2+2aq+b} + \frac{a^2}{2}J_4 + \frac{ab}{2}J_5. \tag{2.141}$$

Let us now proceed to the calculation of the definite integral $\tilde{J}$ corresponding to the indefinite integral (2.127). According to the Newton–Leibniz formula we have

$$\tilde{J} = \int_{q_1}^{q_2} \sqrt{-q^2+2aq+b}\,q\mathrm{d}q = (J_1+J_2)\Big|_{q_1}^{q_2} = J_1\Big|_{q_1}^{q_2} + J_2\Big|_{q_1}^{q_2}. \tag{2.142}$$

We successively calculate

$$J_1\Big|_{q_1}^{q_2} = -3\left(\sqrt{-q^2+2aq+b}\right)^3\Big|_{q_1}^{q_2} = -3\left(\sqrt{-q^2+2aq+b}\right)^3\Big|_{a-\sqrt{c}}^{a+\sqrt{c}}$$
$$= -3\left[\left(\sqrt{-(a+\sqrt{c})^2+2a(a+\sqrt{c})+b}\right)^3\right.$$
$$\left. - \left(\sqrt{-(a-\sqrt{c})^2+2a(a-\sqrt{c})+b}\right)^3\right]$$
$$= -3\left[\left(\sqrt{a^2+b-c}\right)^3 - \left(\sqrt{a^2+b-c}\right)^3\right] \equiv 0, \tag{2.143}$$

$$J_2\Big|_{q_1}^{q_2} = \frac{a}{2}\left(q\sqrt{-q^2+2aq+b}\right)\Big|_{q_1}^{q_2} + \frac{a^2}{2}J_4\Big|_{q_1}^{q_2} + \frac{ab}{2}J_5\Big|_{q_1}^{q_2}$$

$$= \frac{a}{2}\left[(a+\sqrt{c})\sqrt{a^2+b-c} - (a-\sqrt{c})\sqrt{a^2+b-c}\right]$$

$$+ \frac{a^2}{2}J_4\Big|_{q_1}^{q_2} + \frac{ab}{2}J_5\Big|_{q_1}^{q_2}. \tag{2.144}$$

The definite integral $J_4\Big|_{q_1}^{q_2}$ [see (2.136)] has already been obtained [see (2.117)] and is equal to

$$J_4\Big|_{q_1}^{q_2} = \pi a, \tag{2.145}$$

and the integral $J_5\Big|_{q_1}^{q_2}$ [see (2.138)] has been also calculated earlier (see (2.117)) and reads

$$J_5\Big|_{q_1}^{q_2} = \pi. \tag{2.146}$$

From (2.144) and after taking into account (2.145) and (2.146) we obtain

$$J_2\Big|_{q_1}^{q_2} = a\sqrt{c}\sqrt{a^2+b-c} + \frac{\pi}{2}a^3 + \frac{ab\pi}{2}. \tag{2.147}$$

Returning to the original notation we have

$$\begin{aligned}
\sqrt{a^2+b-c} &= \sqrt{k^2\cos^2\theta + 1 - k^2 - \left(1-k^2\sin^2\theta\right)},\\
\frac{\pi}{2}a^3 &= \frac{\pi k^3}{2}\cos^3\theta,\\
\frac{ab\pi}{2} &= \frac{\pi}{2}\left(1-k^2\right)k\cos\theta,
\end{aligned} \tag{2.148}$$

and in view of that

$$J_2\Big|_{q_1}^{q_2} = \frac{\pi k^3}{2}\cos^3\theta + \frac{\pi}{2}(1-k^2)k\cos\theta.$$

The problem then boils down to the calculation of the integral

$$\int \tilde{J}\cos\theta \mathrm{d}\theta = \frac{\pi k^3}{2}\int \cos^4\theta \mathrm{d}\theta + \frac{\pi}{2}(1-k^2)k\int \cos^2\theta \mathrm{d}\theta. \tag{2.149}$$

The elementary integrals are calculated in the following way:

$$\int \cos^2\theta \mathrm{d}\theta = \frac{1}{2}\int (1+\cos 2\theta)\mathrm{d}\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta,$$

$$\begin{aligned}\int \cos^4\theta \mathrm{d}\theta &= \frac{1}{4}\int (1+\cos 2\theta)^2\mathrm{d}\theta \\ &= \frac{1}{4}\int (1+2\cos 2\theta+\cos^2 2\theta)\mathrm{d}\theta \\ &= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\int (1+\cos 4\theta)\mathrm{d}\theta \\ &= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{\theta}{8} + \frac{1}{32}\sin 4\theta. \end{aligned} \tag{2.150}$$

In order to shorten the notation we calculate the following definite integrals:

$$\int_0^{\pi} \cos^2\theta \mathrm{d}\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi} = \frac{\pi}{2},$$

$$\begin{aligned}\int_0^{\theta_1} \cos^2\theta \mathrm{d}\theta &= \left(\frac{\theta}{2} + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}\right)\Bigg|_0^{\arcsin\frac{1}{k}} \\ &= \frac{1}{2}\arcsin\frac{1}{k} + \frac{\sqrt{k^2-1}}{2k^2},\end{aligned}$$

$$\int_0^{\pi} \cos^4\theta \mathrm{d}\theta = \left(\frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Bigg|_0^{\pi} = \frac{3\pi}{8},$$

$$\begin{aligned}\int_0^{\theta_1} \cos^4\theta \mathrm{d}\theta &= \Bigg(\frac{3\theta}{8} + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} \\ &\quad + \frac{1}{8}\sin\theta\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)\Bigg)\Bigg|_0^{\arcsin\frac{1}{k}} \\ &= \frac{3}{8}\arcsin\frac{1}{k} + \frac{\sqrt{k^2-1}}{2k^2} + \frac{(k^2-1)\sqrt{k^2-1}}{8k^4} \\ &= \frac{3}{8}\arcsin\frac{1}{k} + \frac{(5k^2-2)\sqrt{k^2-1}}{8k^4}.\end{aligned} \tag{2.151}$$

From (2.126) we eventually obtain

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \int\limits_0^{\pi} \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}\, q\mathrm{d}q \\ = \frac{\pi^2 k}{16}\left(4-k^2\right); \qquad (k \le 1) \\ \int\limits_0^{\theta_1} \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}\, q\mathrm{d}q \\ = \frac{\pi k}{16}\left(4-k^2\right)\arcsin\frac{1}{k} + \frac{\pi\left(2+k^2\right)\sqrt{k^2-1}}{16k}; \qquad (k > 1). \end{cases} \tag{2.152}$$

Let us now return to the friction model presented in Figs. 2.52 and 2.53. The friction moment about point O' (the instantaneous center of rotation) we obtain by exploiting (2.106), where additionally the numerator of the integrand is multiplied by $\frac{Rr}{R} = Rq$ to obtain

$$M_T^{O'} = \mu N R \begin{cases} \frac{1}{2}\int\limits_0^{\pi} \mathrm{d}\theta \int\limits_{q_1}^{q_2} \frac{q^2\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}}, \qquad (k \le 1), \\ \int\limits_0^{\theta_1} \mathrm{d}\theta \int\limits_{q_1}^{q_2} \frac{q^2\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}}, \qquad (k > 1). \end{cases} \tag{2.153}$$

After using the previous calculations [see (2.135)] we obtain

$$M_T^{O'} = \mu N R \begin{cases} \frac{\pi^2(2+k^2)}{8}, \qquad k \le 1, \\ \frac{\pi^2(2+k^2)}{4}\arcsin\frac{1}{k} + \frac{3\pi}{4}\sqrt{k^2-1}, \qquad k > 1. \end{cases} \tag{2.154}$$

Our aim is to determine the moment of friction forces with respect to point O, which, according to the designations presented in Fig. 2.53a, is equal to

$$M_T^{O'} - M_T^{O} - HT = 0, \tag{2.155}$$

that is,

$$M_T^{O} = M_T^{O'} - HT, \tag{2.156}$$

where the friction force T is given by (2.106).

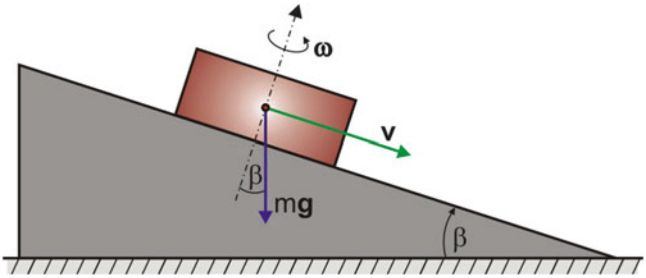

Fig. 2.54 Motion of a rigid body on an inclined plane with velocity of rotation (ω) and velocity of slip (v) included simultaneously

As an example of the application of the introduced theoretical considerations we will examine the motion of a disc on an inclined and a horizontal plane.

Let us consider the classical case of motion of a rigid body of mass m and mass moment of inertia I about the principal centroidal axis of inertia perpendicular to the surface of contact being inclined at an angle β to the horizontal surface (Fig. 2.54).

The equations of motion of the body have the form

$$m\dot{v} = -T + mg\sin\beta,$$
$$I\dot{\omega} = -M_T. \tag{2.157}$$

The calculations will be conducted for the previously considered cases (i) and (ii) using the Padé approximation.

Let us now return to (2.98). As has already been mentioned, integrals (2.98) can be expressed in terms of elliptic integrals that can be checked easily, for instance, by employing the *Mathematica* software. However, it turns out that they can be approximated using the Padé approximation after introducing $\sigma(r)$ described by (2.101) and (2.102), which was demonstrated by Zhuravlev [46]. The calculations below were conducted using a *Mathematica* application for symbolic calculations. Let us apply the simplest form of the Padé approximation

$$M_T[u,v] = M_0\frac{u}{u+av},$$
$$T[u,v] = T_0\frac{v}{v+bu}, \tag{2.158}$$

where T_0, M_0, a, and b require determination.

The quantities T_0 and M_0 are determined in the following way:

$$T_0 = T[0,v], \qquad M_0 = M[u,0], \tag{2.159}$$

and from (2.98) we find

$$T_0 = \mu R^2 \int_0^1 \rho^* \sigma(\rho^*) \int_0^{2\pi} \mathrm{d}\rho^* \mathrm{d}\phi = 2\pi \mu R^2 \int_0^1 \rho^* \sigma(\rho^*) \mathrm{d}\rho^*,$$

$$M_0 = \mu R^3 \int_0^1 \rho^* \sigma(\rho^*) \int_0^{2\pi} \rho^* \mathrm{d}\rho^* \mathrm{d}\phi = 2\pi \mu R^3 \int_0^1 \rho^{*2} \sigma(\rho^*) \mathrm{d}\rho^*. \tag{2.160}$$

The quantities a and b are obtained after the differentiation of (2.158) and (2.98) with respect to variables u and v. After adequate differentiation of the equations we have

$$\frac{\partial T}{\partial v}[u, 0] = \frac{T_0}{bu}, \qquad \frac{\partial M}{\partial u}[0, v] = \frac{M_0}{av}, \tag{2.161}$$

and after differentiation of (2.98) we obtain

$$\frac{\partial T}{\partial v}[u, 0] = \frac{R\pi}{u} \int_0^1 \sigma(\rho^*) \mathrm{d}\rho^*,$$

$$\frac{\partial M}{\partial u}[0, v] = \frac{\pi}{Rv} \int_0^1 \rho^{*3} \sigma(\rho^*) \mathrm{d}\rho^*. \tag{2.162}$$

After equating (2.161) and (2.162) by sides we obtain the unknown coefficients

$$a = \frac{2 \int_0^1 \rho^{*2} \sigma(\rho^*) \mathrm{d}\rho^*}{\int_0^1 \rho^{*3} \sigma(\rho^*) \mathrm{d}\rho^*}, \qquad b = \frac{2 \int_0^1 \rho^* \sigma(\rho^*) \mathrm{d}\rho^*}{\int_0^1 \sigma(\rho^*) \mathrm{d}\rho^*}. \tag{2.163}$$

The desired coefficients were found using *Mathematica*.

Equations (2.157), after taking into account the Padé approximation, assume the form

$$m \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{T_0 v}{|v| + bu} + mg \sin\beta,$$

$$I \frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{R M_0 u}{|u| + av}, \tag{2.164}$$

where $u = R\omega$.

In the case of a contact over the circular patch (case i) R denotes the radius, and in the case of the point contact $R \to 0$.

Following the transformation of (2.164) we obtain

$$\frac{m}{T_0}\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v}{|v| + bu} + \frac{mg\sin\beta}{T_0},$$
$$\frac{I}{RM_0}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{u}{|u| + av}. \tag{2.165}$$

The preceding equations assume the following form:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{v}{|v| + bu} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\frac{eu}{|u| + av}, \tag{2.166}$$

where the following substitution for time and two dimensionless parameters were introduced:

$$\tau = \frac{T_0}{m}t, \qquad p = \frac{mg\sin\beta}{T_0}, \qquad e = \frac{RM_0m}{T_0I}. \tag{2.167}$$

For case (i) we assume $m = \rho_0\pi R^2$ (ρ_0 is the surface density) and $I = \frac{1}{2}mR^2$, and from (2.160) and (2.163) we obtain

$$M_0 = \frac{1}{4}N\pi R\mu = \frac{1}{4}\pi RT_0, \qquad a = \frac{3\pi}{4},\ b = \frac{4}{\pi},$$

and for case (ii) we have

$$M_0 = \frac{3}{16}N\pi R\mu = \frac{3}{16}\pi RT_0, \qquad a = \frac{15\pi}{16},\ b = \frac{8}{3\pi}.$$

For case (i) $e = \frac{2R\pi RT_0m}{4T_0mR^2} = \frac{\pi}{2}$ [for case (ii) we have $M_0 = 0$ because $R = 0$].

Equations (2.166) for case (i) take the form

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{v\pi}{|v|\pi + 4u} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\frac{2\pi u}{4|u| + 3\pi v}, \tag{2.168}$$

and for case (ii) (2.166) are as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{3v\pi}{3|v|\pi + 8u} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 0. \tag{2.169}$$

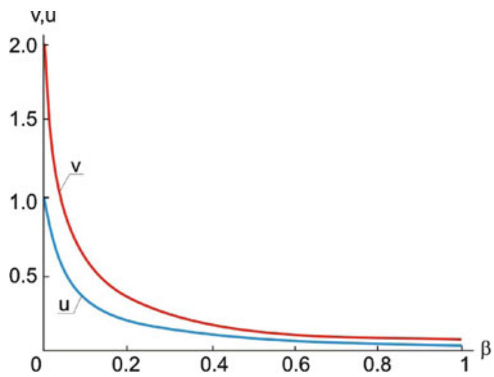

Fig. 2.55 Graphs of $v(\beta)$ and $u(\beta)$ dependencies

Let us consider (2.168) in more detail. After dividing these equations by sides we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{[(p-1)v + pbu](u+av)}{eu(v+bu)}, \tag{2.170}$$

where $a = \frac{3\pi}{4}$, $b = \frac{4}{\pi}$.

Mathematica cannot integrate (2.170) directly, although it is possible after the introduction of adequate substitution for the coordinates. For the case $p = 0$ (motion of a cylindrical disc on a horizontal surface), (2.170) is integrable by quadratures.

In the general case the numerical integration of (2.168) is made more difficult because of singularities of these equations (the denominators may tend to zero or to infinity). On the example of the case $p = 0$ we will show how to avoid the singularities of those equations by applying the substitution of the independent variable (time) according to the equation

$$\tau = \int_0^\beta (3\pi v + 4u)(\pi v + 4u)\mathrm{d}\beta. \tag{2.171}$$

The obtained equations

$$\frac{\mathrm{d}v}{\mathrm{d}\beta} = -v(3\pi v + 4u),$$
$$\frac{\mathrm{d}u}{\mathrm{d}\beta} = -2u(\pi v + 4u) \tag{2.172}$$

do not have singularities at the point $u = v = 0$ and can be easily integrated.

Below some illustrative results are shown.

From Fig. 2.55 it can be seen that $u(\beta) \to 0$, $v(\beta) \to 0$ for $\beta \to \infty$. In view of that, according to (2.171) τ attains a finite value. For certain finite

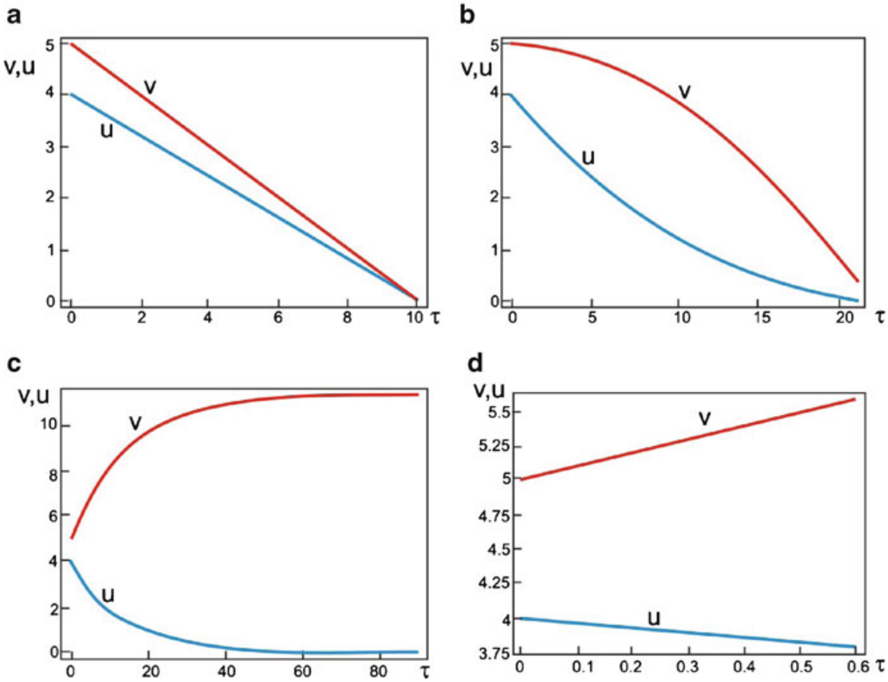

Fig. 2.56 Dependencies $u(\tau)$ and $v(\tau)$ for the following cases: (**a**) $p = 0$; (**b**) $p = 0.5$; (**c**) $p = 1$; (**d**) $p = 1.5$ for the circular contact

values of τ_* (which will be shown later with the aid of numerical calculations) $u(\tau_*) = v(\tau_*) = 0$. *This means that in the case of the applied Padé approximations both the sliding velocity v and the rotation velocity ω reach zero values at the same time instant.*

In Fig. 2.56 some illustrative graphs of $v(\tau)$ and $u(\tau)$ for $p = 0$, $p = 0.5$, $p = 1$, and $p = 1.5$ are presented. From the qualitative analysis conducted in [46] it follows that the behavior of the disc differs in a qualitative sense for $p < 1$ and for $p > 1$ (the critical case of $p = 1$).

In all the presented cases identical initial conditions are assumed: $v_0 = 5$, $u_0 = 4$. The time plots for $p = 0$ and $p = 1.5$ do not deviate much from straight lines, whereas for $p = 1.5$ we have $\omega(\tau) \to 0$ and $v(\tau) \to \infty$. In the case $p = 0.5$ it can be seen that $v(\tau)$ and $\omega(\tau)$ attain zero and their graphs are curves. In the critical case of $\tau \to \infty$ we have $\omega(\tau) \to 0$, whereas $v(\tau)$ reaches a steady value v_*, which can be easily estimated in an analytical way.

A natural and interesting question arises as to whether the introduction of the initial condition only for one variable causes the failure of the other variable to appear. This case is considered in a manner analogous to that of the previously analyzed example, and we assume respectively $v_0 = 0$, $u_0 = 0$, $v = 5$.

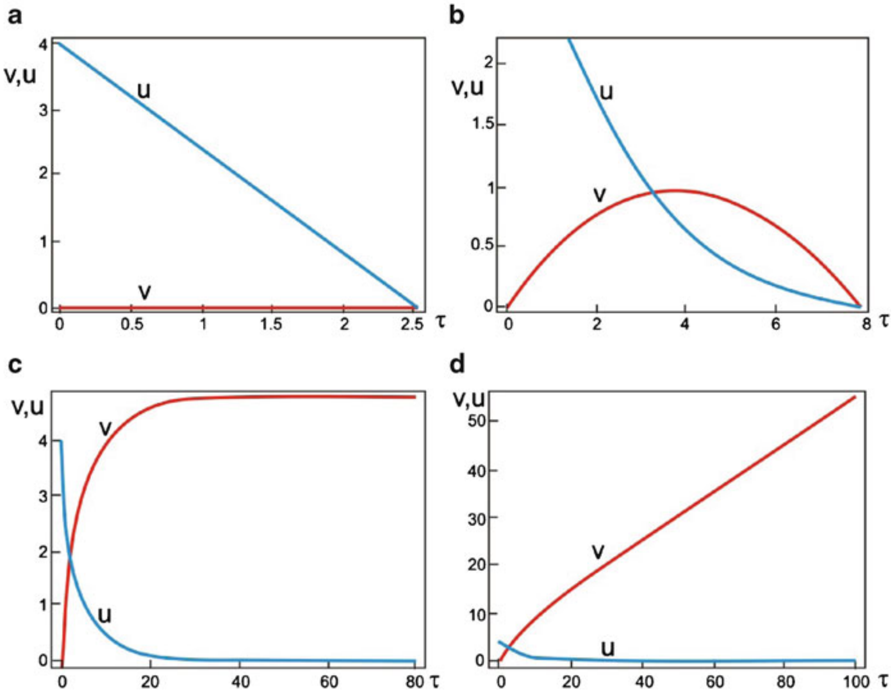

Fig. 2.57 Graphs of time characteristics $u(\tau)$ and $v(\tau)$ for the initial conditions $v_0 = 0$ and $u_0 = 4$ and cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

From Fig. 2.57 it follows that during motion on the horizontal plane the disc only rotates with the rotation becoming gradually slower until it reaches a state of rest with no contribution from the slip velocity. In the $p = 0.5$ case the angular velocity attains zero along the curve, while the sliding velocity initially increases, reaches the maximum, and decreases to zero. In the critical case $p = 1$ the velocity $\omega(\tau)$ drops quickly to zero, and the sliding velocity reaches a steady value.

In the case $p = 0$ ($p = 0.5$) the velocity $\omega(\tau) \to 0$ for $\tau_* = 5$ ($\tau_* = 10$) at $v(\tau) = 0$. In the critical case $p = 1$ we have $v(\tau) = v_0$, $\omega(\tau) = \omega_0$, and in the case $p = 1.5$ we have $v(\tau) = 0$, while $v(\tau) \to \infty$ (Fig. 2.58).

Let us now return to the exact equations of disc motion for the case $p = 0$, which have the following form:

(i) The instantaneous center of rotation lies within the contact patch of the disc and inclined plane ($\omega R \geq v$):

$$\dot{v} = -\frac{\pi \mu p_0}{4\rho_0 R^3}\frac{v}{\omega},$$

$$\dot{\omega} = -\frac{\pi \mu p_0}{4\rho_0 R^3}\frac{\omega^2 R^2 - v^2}{\omega^2 R^2}; \tag{2.173}$$

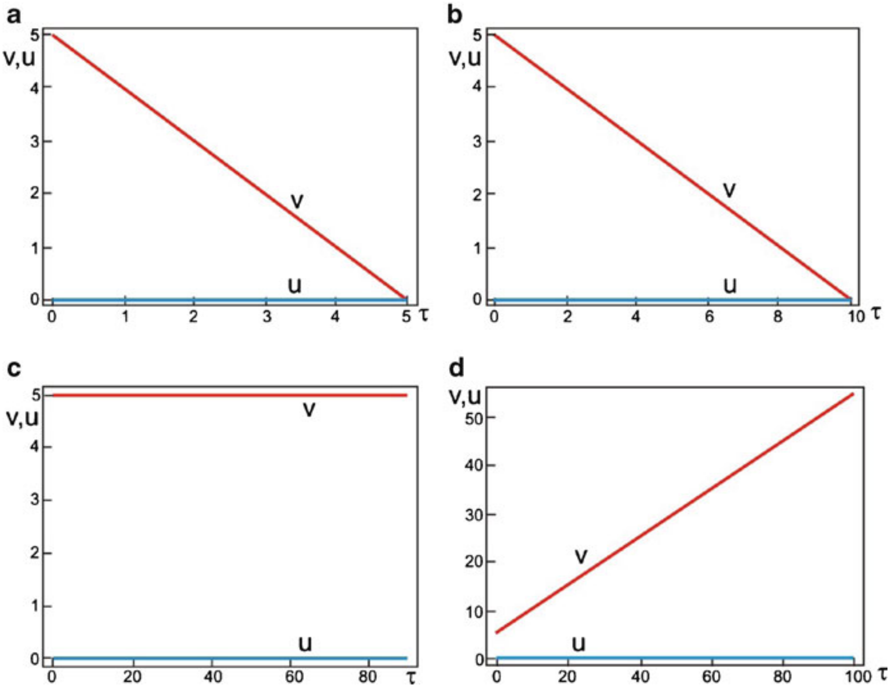

Fig. 2.58 Graphs of time characteristics $u(\tau)$ and $v(\tau)$ for initial conditions $v_0 = 5$ and $u_0 = 0$ and cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

(ii) The instantaneous center of rotation lies outside the contact patch of the disc and inclined plane ($\omega R < v$):

$$\dot{v} = -\frac{\mu p_0}{2\rho_0 R^2}\left(\frac{v}{\omega R}\arcsin\frac{\omega R}{v} + \frac{\sqrt{v^2 - \omega^2 R^2}}{v}\right),$$

$$\dot{\omega} = -\frac{\mu p_0}{2\rho_0 R^2}\left(\frac{\omega^2 R^2 - v^2}{\omega^2 R^2}\arcsin\frac{\omega R}{v} + \frac{\sqrt{v^2 - \omega^2 R^2}}{\omega R}\right). \tag{2.174}$$

Above we have set $m = \rho_0 \pi R^2$, $I = \frac{1}{2}\rho_0 \pi R^4$.

In Fig. 2.59 three illustrative results of calculations are shown, where besides $v(t)$ and $\omega(t)$ the graph of $\omega(t)/v(t)$ is presented and where (we recall) the position of the instantaneous center of rotation is described by the equation $k = \frac{v(t)}{\omega(t)R} = \frac{r(t)}{R}$.

The position of the instantaneous center of rotation for $v(t) \to 0$ and $\omega(t) \to 0$ was analyzed in [48].

After dividing (2.173) and (2.174) respectively by sides we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}\omega} = R\frac{k}{2-k^2}, \qquad (k \le 1), \tag{2.175}$$

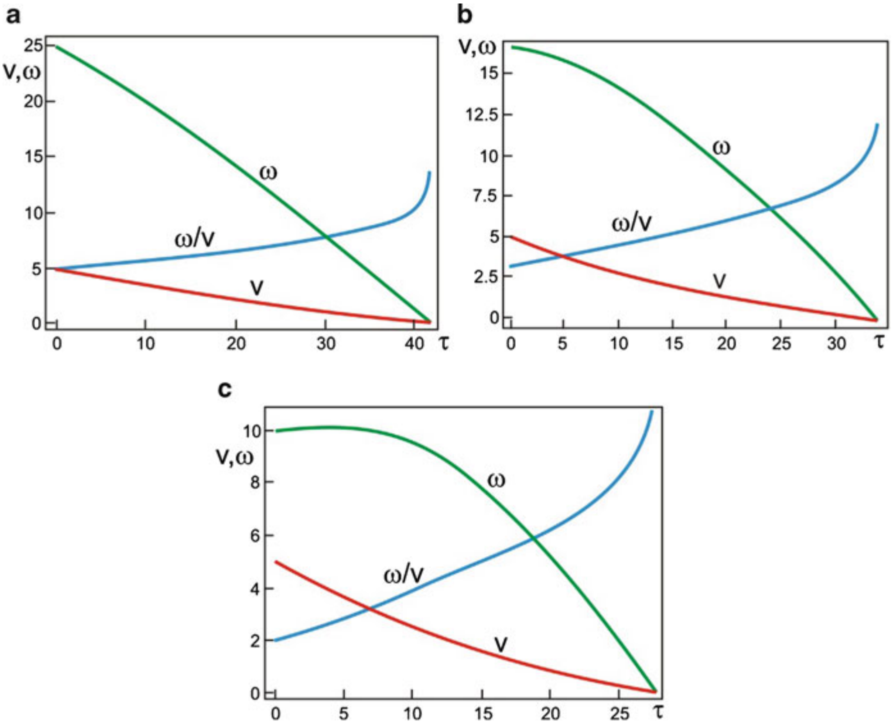

Fig. 2.59 Time histories of $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for (**a**) $\omega_0 = 25$, (**b**) $\omega_0 = 16.666$, (**c**) $\omega_0 = 10$

$$\frac{dv}{d\omega} = R\frac{k\arcsin\left(\frac{1}{k}\right) + \frac{\sqrt{k^2-1}}{k}}{(2-k^2)\arcsin\left(\frac{1}{k}\right) + \sqrt{k^2-1}}, \qquad (k > 1). \tag{2.176}$$

The equation of a tangent to the trajectory of the system has the form $v/\omega = Rk$, and from (2.175) and (2.176) we obtain

$$k = \frac{k}{2-k^2}, \qquad (k \leq 1),$$

$$k = \frac{k\arcsin\left(\frac{1}{k}\right) + \frac{\sqrt{k^2-1}}{k}}{(2-k^2)\arcsin\left(\frac{1}{k}\right) + \sqrt{k^2-1}}, \qquad (k > 1). \tag{2.177}$$

The first equation possesses the solution $k = 1$, while the second one does not have any solutions.

In the end, let us return to case (ii) and (2.169). Some illustrative calculations for the point contact are given in Fig. 2.60 for different values of the parameter p. It can be seen that for $p \geq 1$ the motion changes in a qualitative sense.

As was mentioned, (2.158) at the points $u = 0$ and $v = 0$ are singular and the limits $\lim_{u,v\to 0} M_T$ and $\lim_{u,v\to 0} T$ do not exist. After the introduction of the designations $m = |M_T|/M_0$, $n = |T|/T_0$ (2.158) take the form

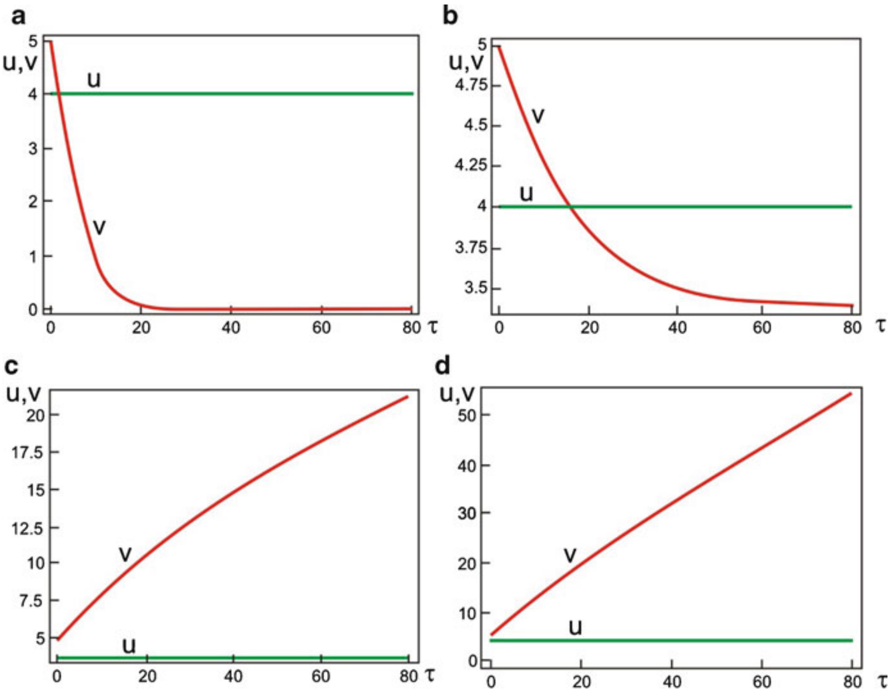

Fig. 2.60 Dependencies $u(\tau)$ and $v(\tau)$ for the point contact for the cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

$$m = \frac{u}{u + av}, \qquad n = \frac{v}{v + bu}. \tag{2.178}$$

Because, as the numerical calculations showed, u and v tend to zero attaining this singular point simultaneously, let us introduce the relationship $v = \alpha u$. After taking into account this relation in (2.178) we obtain

$$m = \frac{u}{u + a\alpha v}, \qquad n = \frac{\alpha u}{\alpha u + bu}. \tag{2.179}$$

From (2.179) we obtain

$$m(1 + \alpha a) = 1, \qquad n\left(1 + \frac{b}{\alpha}\right) = 1. \tag{2.180}$$

From the first of (2.180) we calculate

$$\alpha = \frac{1 - m}{ma}, \tag{2.181}$$

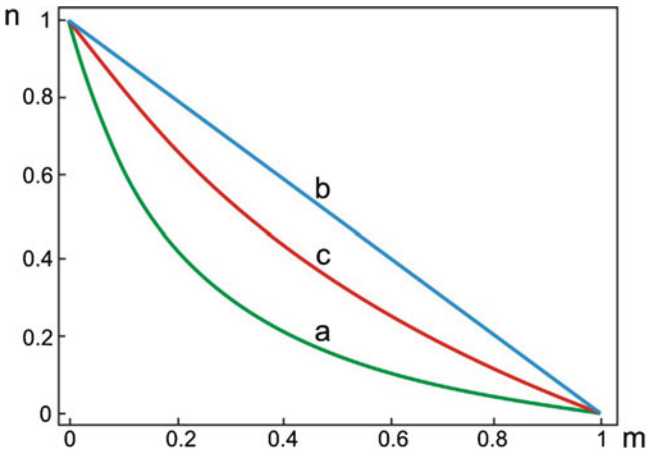

Fig. 2.61 Illustrative graphs of (2.182) for (**a**) $ab = 2$, (**b**) $ab = 17/3$, (**c**) $ab = 1$

and then, from the second one, after taking into account (2.181) we have

$$m + n[1 + (ab - 1)m] = 1, \tag{2.182}$$

and after transformations we obtain

$$n = \frac{1 - m}{(ab - 1)m + 1}.$$

In Fig. 2.61 the graphs of (2.182) are shown for the cases (a) for $ab = 2$, (b) for $ab = 17/3$, and for $ab = 1$ as a critical case (c).

From the graphs in Fig. 2.61 follows a very interesting conclusion, which is often observed in everyday life. In the case where a rigid body is at rest, the action of even a very small moment M_T leads to a rapid drop in magnitude of friction force T, which greatly facilitates the start of movement of a body. It is also associated with the frequently observed dangerous skid during cornering of a wheeled vehicle.

The CCZ model allows one also to pinpoint many characteristics of bodies moving with frictional contact. We will show that the friction does depend largely on the area represented by the radius R. To this end let us return to (2.173), which are solved for two values of R (Fig. 2.62).

When comparing the results presented in Figs. 2.59a and 2.62 it is clear that with increasing radius R the time needed for the disc to stop increases in a non-linear fashion. A similar situation can be observed also in the case of $\omega_0 = 10$ and two different values of R on the basis of (2.174) (Fig. 2.63).

Let us now analyze the effect of a two-dimensional friction on relative motion and self-excited vibrations.

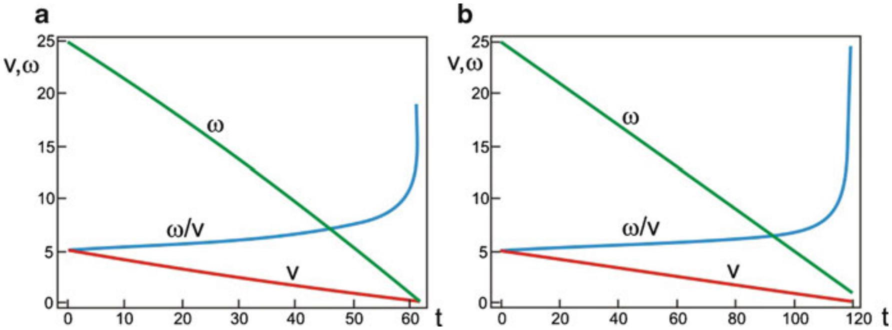

Fig. 2.62 Time histories $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for $\omega_0 = 25$ and $R = 0.35$ (**a**) and $R = 0.45$ (**b**) (the remaining parameters as in Fig. 2.58)

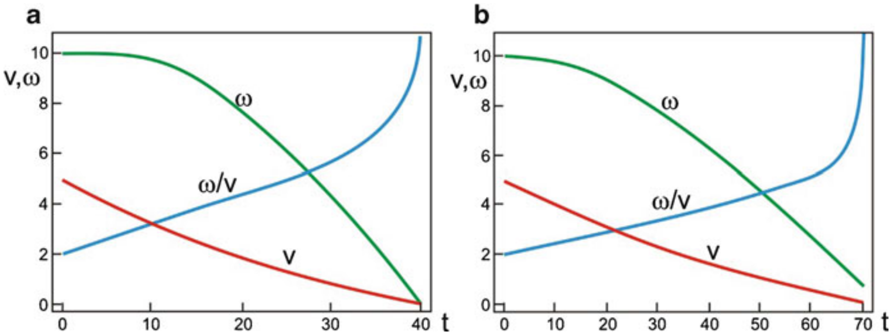

Fig. 2.63 Time histories $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for $\omega_0 = 10$ and $R = 0.35$ (**a**) and $R = 0.45$ (**b**) (the remaining parameters as in Fig. 2.58)

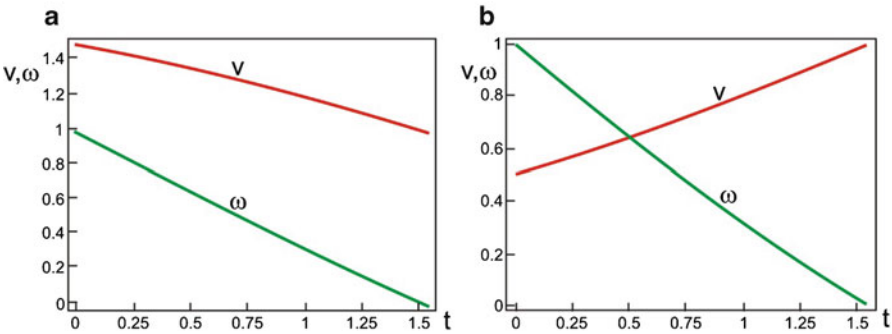

Fig. 2.64 Motion of disc determined by $\omega(t)$ and $v(t)$ for the cases $u_0 = 1$ and $v_0 = 1.5$ (**a**) and $v_0 = 0.5$ (**b**)

Let us consider the motion of a disc on an inclined plane, where now the motion takes place on an inclined plane remaining in contact with the moving disc and running with a constant velocity v_0. Equations of motion (2.168) assume the form

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}\tau} &= -\frac{\pi(v-v_0)}{\pi|v-v_0|+4|u|}+p,\\ \frac{\mathrm{d}u}{\mathrm{d}\tau} &= -\frac{2\pi u}{3\pi|v-v_0|+4|u|}.\end{aligned} \tag{2.183}$$

In Fig. 2.64 an example of calculations for the case $u_0 = 1$ and for two different initial velocities: (a) $v_0 = 1.5$ and (b) $v_0 = 0.5$.

From Fig. 2.64 it is seen that in both cases the velocities $\omega(t) \to 0$ and $v(t) \to 1$ attaining these values simultaneously in finite time.

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Chapter 3
Geometry of Masses

3.1 Basic Concepts

Let us consider a system of particles A_n $(n = 1, \ldots, N)$ of masses m_n and radius vectors $\mathbf{r}_n$ with respect to a certain adopted coordinate system $OX_1X_2X_3$ (see also [1–9]).

We will call the point C whose position is determined by the following radius vector a *mass center* C of a system of particles:

$$\mathbf{r}_c = \frac{\sum_{n=1}^{N} m_n \mathbf{r}_n}{M}, \tag{3.1}$$

where M denotes the *mass of the system* given by the formula

$$M = \sum_{n=1}^{N} m_n. \tag{3.2}$$

Multiplying the numerator and denominator of (3.1) by the acceleration of gravity $\mathbf{g}$ the problem boils down to the determination of a center of gravity in a uniform gravitational field.

Note that the mass center is sometimes called the *inertia center*.

The previously introduced notion of body mass is a measure of inertia of a body in translational motion. In rotation of the body about an axis a measure of inertia of the body is its moment of inertia with respect to that axis.

Above we defined the position of the mass center of a discrete (lumped) system of particles with respect to the origin of the Cartesian coordinate system $OX_1X_2X_3$.

A moment of inertia of a system of particles with respect to a plane Γ is defined by the following formula:

$$I_\Gamma = \sum_{n=1}^{N} m_n s_n^2, \tag{3.3}$$

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where s_n denotes the distance of point A_n from the plane Γ. It is a scalar quantity equal to the sum of products of particle mass and the square of its distance to the plane for each particle.

If as planes we choose those of the Cartesian coordinate system $OX_1X_2X_3$, the moment of inertia with respect to those three planes is given by the equations

$$I_{OX_2X_3} = \sum_{n=1}^{N} m_n x_{1n}^2, \quad I_{OX_1X_3} = \sum_{n=1}^{N} m_n x_{2n}^2,$$
$$I_{OX_1X_2} = \sum_{n=1}^{N} m_n x_{3n}^2. \tag{3.4}$$

It is also possible to define the notion of the *moment of inertia of a system of particles with respect to an axis* l, namely:

$$I_l = \sum_{n=1}^{N} m_n d_n^2, \tag{3.5}$$

where d_n denotes the distance of point A_n from axis l.

The moment of inertia about axis l we can also represent in the form

$$I_l = Md^2, \tag{3.6}$$

where M (mass of the system) is determined by (3.2) and we will call quantity d a *radius of gyration* of a system of particles about an axis l.

The radius of gyration d describes the distance from axis l of a certain point at which the whole mass M of a system of particles should be concentrated so as to obtain equality of moments of inertia with respect to axis l of the system and of the mass M at that point.

We can define the moment of inertia of a system of particles with respect to an axis as a scalar quantity equal to the sum of products of the mass and square of the distance from the axis for each particle.

If as axes we choose those of the Cartesian coordinate system, moments of inertia with respect to those three axes are described by the equations

$$I_{X_1} = \sum_{n=1}^{N} m_n (x_{2n}^2 + x_{3n}^2), \quad I_{X_2} = \sum_{n=1}^{N} m_n (x_{1n}^2 + x_{3n}^2),$$
$$I_{X_3} = \sum_{n=1}^{N} m_n (x_{1n}^2 + x_{2n}^2). \tag{3.7}$$

The *moment of inertia* of a system of particles with respect to an arbitrary pole O, the so-called *polar moment of inertia*, is given by the formula

$$I_O = \sum_{n=1}^{N} m_n \mathbf{r}_n^2 = \sum_{n=1}^{N} m_n r_n^2. \tag{3.8}$$

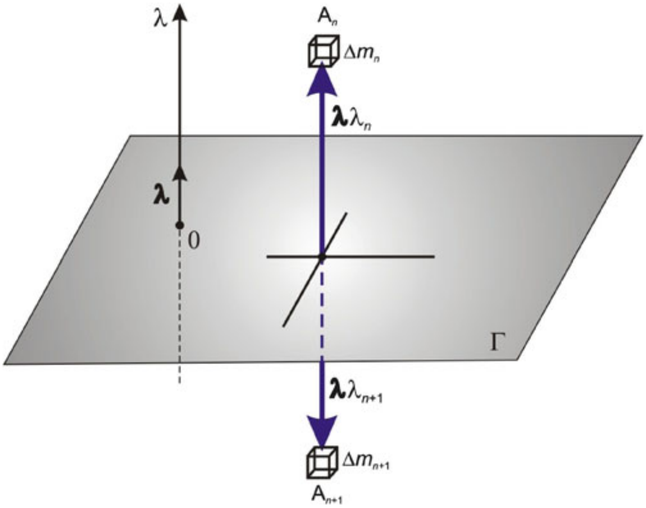


Fig. 3.1 Static moment with respect to the plane Γ ($\lambda_n > 0$ and $\lambda_{n+1} < 0$)

This is a scalar quantity equal to the sum of products of the mass and square of the distance from the chosen pole for each point of the system of particles.

Observe that

$$I_O = \sum_{n=1}^{N} m_n \left(x_{1n}^2 + x_{2n}^2 + x_{3n}^2\right) = I_{OX_2X_3} + I_{OX_1X_3} + I_{OX_1X_2}, \tag{3.9}$$

where (3.8) and (3.7) were used.

In order to investigate the distribution of masses of a discrete system of particles we can make use of the plane Γ.

The quantity defined by the following equation we call a *static moment of a system of particles about a plane* Γ (Fig. 3.1):

$$I_\Gamma^S = \sum_{n=1}^{N} m_n \lambda_n, \tag{3.10}$$

where λ_n are coordinates of points A_n measured along the chosen axis perpendicular to the plane (therefore they are positive or negative).

If we are dealing with a continuous system of particles, the symbols of sum used earlier are replaced by integrals.

If, as in the last case, we consider a continuous body, the transition from the discrete system of particles to the continuous system is connected with an increase in the number of particles $N \to \infty$ with the masses of those particles simultaneously tending to zero, that is,

$$I_\Gamma^S = \lim_{\substack{N \to \infty \\ \Delta m_n \to 0}} \sum_{n=1}^{N} \Delta m_n \lambda_n = \int_M \lambda(x_{1n}, x_{2n}, x_{3n}) \mathrm{d}m. \tag{3.11}$$

The integral above corresponds to an infinite sum of the products of elementary points $A_n(x_{1n}, x_{2n}, x_{3n})$ and their coordinates with respect to the mentioned plane.

The distance λ (with plus or minus sign) is a function of the position of point A, which was emphasized in (3.11).

The following integral we call the *geometric static moment of the volume with respect to a plane* Γ of a continuous system (a solid):

$$I_\Gamma^{SG} = \int_V \lambda(A)\mathrm{d}V, \tag{3.12}$$

where V denotes the volume of a solid. It represents a distribution of the volume of the solid with respect to the plane Γ.

Let us note that $\mathrm{d}m = \rho \mathrm{d}V$ and from (3.11) we obtain

$$I_\Gamma^{S} = \int_V \rho(A)\lambda(A)\mathrm{d}V. \tag{3.13}$$

The quantities ρ and λ depend on the position of point A, that is, on the position of a particle of elementary mass $\mathrm{d}m$.

If the solid body under consideration is homogeneous, that is, $\rho(A) \equiv \rho = \mathrm{const}$, then from (3.12) and (3.13) we obtain the relationship between the first moment of mass and the first moment of the volume (geometric first moment) of the form

$$I_\Gamma^{S} = \rho I_\Gamma^{G}. \tag{3.14}$$

During our discussions regarding the kinematics of a rigid body we have often used the Cartesian coordinate system $OX_1X_2X_3$. Coordinates of this system are determined by the intersection of three planes, OX_1X_2, OX_2X_3, and OX_1X_3. The first moments of a discrete system of particles (DMS) and a continuous one (CMS) with respect to the aforementioned planes are equal to

$$\begin{aligned} I_{OX_1X_2}^{S} &= \textstyle\sum_{n=1}^{N} m_n x_{3n}, \\ I_{OX_1X_3}^{S} &= \textstyle\sum_{n=1}^{N} m_n x_{2n}, \\ I_{OX_2X_3}^{S} &= \textstyle\sum_{n=1}^{N} m_n x_{1n}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} I_{OX_1X_2}^{S} &= \int_M x_3 \mathrm{d}m = \int_V x_3 \rho(x_1, x_2, x_3)\mathrm{d}V, \\ I_{OX_1X_3}^{S} &= \int_M x_2 \mathrm{d}m = \int_V x_2 \rho(x_1, x_2, x_3)\mathrm{d}V, \\ I_{OX_2X_3}^{S} &= \int_M x_1 \mathrm{d}m = \int_V x_1 \rho(x_1, x_2, x_3)\mathrm{d}V. \end{aligned} \tag{3.16}$$

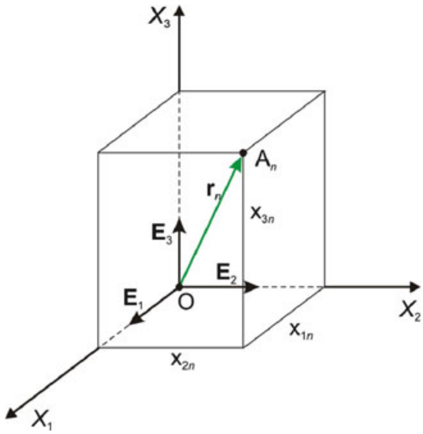


Fig. 3.2 Representation of a radius vector of point A_n

Let us now return to (3.1) determining the position of mass center of DMS. Let us note that any point A_n described by the radius vector $\mathbf{r}_n$ can also be represented by the components of this vector (Fig. 3.2).

According to (3.1) we have

$$x_{1C}\mathbf{E}_1 + x_{2C}\mathbf{E}_2 + x_{3C}\mathbf{E}_3 = \frac{\sum_{n=1}^{N} m_n(x_{1n}\mathbf{E}_1 + x_{2n}\mathbf{E}_2 + x_{3n}\mathbf{E}_3)}{M}. \tag{3.17}$$

Multiplying (3.17) in turn by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ and taking into account (3.15) we obtain

$$x_{1C} = \frac{I^S_{OX_2X_3}}{M}, \quad x_{2C} = \frac{I^S_{OX_1X_3}}{M}, \quad x_{3C} = \frac{I^S_{OX_1X_2}}{M}, \tag{3.18}$$

and taking into account (3.16) we obtain

$$\begin{aligned} x_{1C} &= \frac{\int_V x_1\rho(x_1, x_2, x_3)\mathrm{d}V}{M}, \\ x_{2C} &= \frac{\int_V x_2\rho(x_1, x_2, x_3)\mathrm{d}V}{M}, \\ x_{3C} &= \frac{\int_V x_3\rho(x_1, x_2, x_3)\mathrm{d}V}{M}, \end{aligned} \tag{3.19}$$

that is, we determined the position of mass center of DMS (3.18) and mass center of CMS (3.19) using first moments with respect to planes.

It is easy to notice that a mass center C of a solid and a centroid of that solid C^G do not have to be coincident. They coincide only in the case of a homogeneous body since we have

$$x_{jC} = \frac{\int\limits_V \rho(x_1, x_2, x_3) x_j \mathrm{d}V}{\int\limits_V \rho(x_1, x_2, x_3) \mathrm{d}V} = \frac{\rho \int\limits_V x_j \mathrm{d}V}{\rho \int\limits_V \mathrm{d}V} = x_{jC}^G, \quad j = 1, 2, 3, \tag{3.20}$$

which can be treated as a sufficient condition for the coincidence.

The CMS can be three-, two-, or one-dimensional, homogeneous or nonhomogeneous. The density of such bodies is related to the volume (kg/m^3), the surface (kg/m^2), and the curve (kg/m), respectively. In the case of homogeneous bodies, if they possess a plane or an axis of symmetry, their mass center lies on that plane or axis. If the homogeneous body possesses a center of symmetry, it coincides with the mass center.

Because the dimensions of bodies with respect to Earth's radius are small, the forces of attraction to Earth's center, that is, the central forces, are approximated with the field of parallel gravitational forces. The main force vector is equal to the sum of forces acting on each particle of a body, and the center of those parallel forces of identical senses is called the *center of gravity* of a body.

Let us consider a homogeneous volume V, which we will divide into certain volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_N$, and assume that the positions of gravity centers of each of the elementary volumes ΔV_N are known. If one denotes the specific weight of the body by $\boldsymbol{\gamma}$, its density by ρ ($\boldsymbol{\gamma} = \rho \mathbf{g}$), and the elementary mass corresponding to an elementary volume by Δm_C, then the resultant force in such a case is equal to

$$\mathbf{G} = \sum_{n=1}^{N} \Delta \mathbf{G}_n = \mathbf{g} \sum_{n=1}^{N} \rho \Delta V_n = \mathbf{g} \sum_{n=1}^{N} \Delta m_n, \tag{3.21}$$

or

$$\mathbf{G} = \boldsymbol{\gamma} \sum_{n=1}^{N} \Delta V_n = V \boldsymbol{\gamma}. \tag{3.22}$$

We have then determined the resultant force (main force vector), and determining, using the described method, the position of the center of parallel forces C, from (3.1) we obtain its position:

$$\begin{aligned} x_{1C} &= \frac{\sum_{n=1}^{N} \Delta G_n x_{1n}}{G} = \frac{\sum_{n=1}^{N} \Delta m_n x_{1n}}{M}, \\ x_{2C} &= \frac{\sum_{n=1}^{N} \Delta G_n x_{2n}}{G} = \frac{\sum_{n=1}^{N} \Delta m_n x_{2n}}{M}, \\ x_{3C} &= \frac{\sum_{n=1}^{N} \Delta G_n x_{3n}}{G} = \frac{\sum_{n=1}^{N} \Delta m_n x_{3n}}{M}. \end{aligned} \tag{3.23}$$

In a similar way we can determine the position of mass centers of certain surfaces or one-dimensional mechanical systems (e.g., rope, wire).

Let us consider, e.g., a homogeneous shell of a constant thickness h and a specific weight of material $\boldsymbol{\gamma}$ (Fig. 3.3).

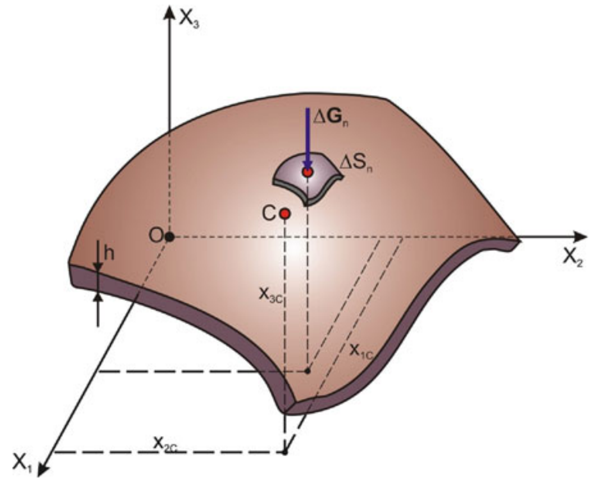


Fig. 3.3 Determination of the mass center of a shell having a thickness *h*

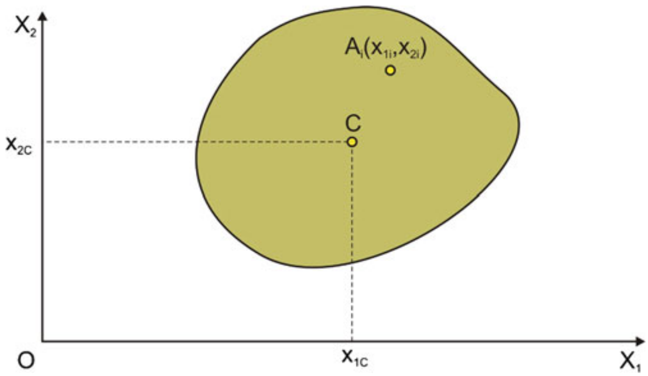


Fig. 3.4 Determination of the center of gravity of a plane figure

We proceed in a way similar to that described earlier. Because we are dealing with a continuous surface and the number of elementary cut-outs of the surface ΔS_n tends to infinity and their area tends to zero, we replace the signs of sum with integrals.

We obtain the following values of the coordinates of point C (the mass center of the shell) in the adopted Cartesian coordinate system $OX_1X_2X_3$:

$$x_{iC} = \frac{\int\limits_G \mathrm{d}G x_i}{\int\limits_G \mathrm{d}G} = \frac{\gamma h \int\limits_S x_i \mathrm{d}S}{\gamma h \int\limits_S \mathrm{d}S} = \frac{\int\limits_S x_i \mathrm{d}S}{\int\limits_S \mathrm{d}S}, \qquad i = 1, 2, 3. \tag{3.24}$$

Let us introduce the notion of the center of a plane figure (Fig. 3.4). Any plane body of small thickness can be treated as a plane material figure. If we assume a

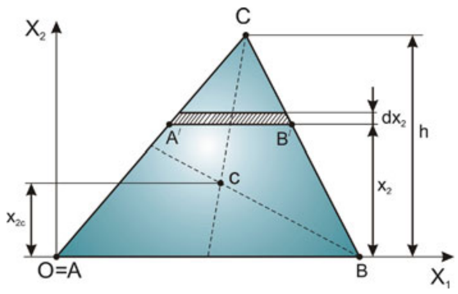


Fig. 3.5 Determination of the mass center of a triangle

unit weight $\gamma\left[\frac{N}{m^2}\right]$ and an area of the figure S, then its weight $G = S\gamma$. Dividing the area into elementary ΔS_n, the weight of the elementary area may be written as $\Delta G_n = \gamma\Delta S_n$. According to the earlier derived (3.24), the coordinates of the center of gravity for this figure are described by the equations

$$x_{iC} = \frac{\sum_{n=1}^{N} x_{1n}\Delta G_n}{G} = \frac{\sum_{n=1}^{N} x_{1n}\gamma\Delta S_n}{\gamma S} = \frac{\sum_{n=1}^{N} x_{in}\Delta S_n}{S}, \qquad i = 1, 2. \tag{3.25}$$

From the obtained equation it follows that the location of the center of gravity of the considered (any) plane figure does not depend on the γ constant. That is why the center of gravity of a homogeneous plate is called the *center of gravity of a surface of the plate*.

Definition 3.1. *The sum of products of elementary areas into which an area of a plane figure was divided and their distances from a certain axis is called the* ***static moment of the plane figure with respect to the axis****.*

Knowing the static moments of the area of a plane figure, it is possible to determine the coordinates of the center of gravity of this figure from (3.25).

The reader can solve an example regarding the determination of mass center of, e.g., a rope, that is, a one-dimensional system in R^3 space.

In practice we calculate the mass centers of solids by introducing the division of the given solid into a certain number of component solids in order to make the calculations more convenient.

Let us consider some examples.

Example 3.1. Determine the position of a centroid of a triangle ΔABC with respect to the base AB (Fig. 3.5).

We will determine the position of a centroid of a triangle as the ratio of the first moment about the axis OX_1 to the area of the triangle, that is,

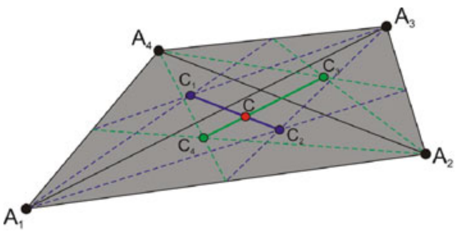


Fig. 3.6 Determination of the mass center of a quadrilateral

$$x_{2C} = \frac{I_{OX_1}}{S_\Delta} = \frac{\int\limits_{F_\Delta} x_2 \mathrm{d}S_\Delta}{S_\Delta} = \frac{\int\limits_0^h x_2 a' \mathrm{d}x_2}{\frac{1}{2}ha} = \frac{2\int\limits_0^h x_2(h-x_2)\mathrm{d}x_2}{h^2}$$

$$= \frac{2\left[\frac{1}{2}h^3 - \frac{1}{3}h^3\right]}{h^2} = \frac{2(3-2)}{6}h = \frac{1}{3}h\,,$$

because

$$\frac{a'}{a} = \frac{h - x_2}{h},$$

where $a = AB$, $a' = A'B'$. □

The preceding example was aimed at the presentation of the method of integration since it is well known that the centroid of a homogeneous triangle lies at the intersection of any two of its medians.

Example 3.2. Determine the position of a mass center of the quadrilateral (homogeneous plate) $A_1A_2A_3A_4$ shown in Fig. 3.6.

Let us connect points A_1 and A_3 and then determine the centers of triangles $\Delta A_1A_2A_3$ and $\Delta A_1A_3A_4$, obtaining the points C_1 and C_2, which we connect to one another. Similarly, we proceed in the case of triangles $\Delta A_1A_2A_4$ and $\Delta A_2A_3A_4$, and as a result we obtain the segment C_3C_4. The point of intersection of the segments C_1C_2 and C_3C_4 determines point C, which was to be found. □

Example 3.3. Calculate the position of the mass center of a sector of a disc of radius r and of the angle-at-the-center 2α shown in Fig. 3.7.

The area of the circular sector can be treated as the area composed of elementary triangles of altitudes r (one of them marked with a hatching line). According to Example 3.1 the centroids of these triangles will lie at a distance of $\frac{1}{3}$ from the base, that is, on a circle of radius $\frac{2}{3}r$. The problem, then, boils down to the determination of the centroid of a one-dimensional line, that is, the arc of a circle of radius $\frac{2}{3}r$. On account of the symmetry of the problem, the centroid of that arc lies on the axis OX_2.

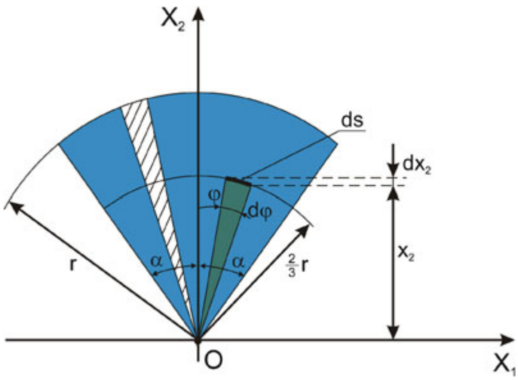


Fig. 3.7 Determination of the centroid of a circular sector

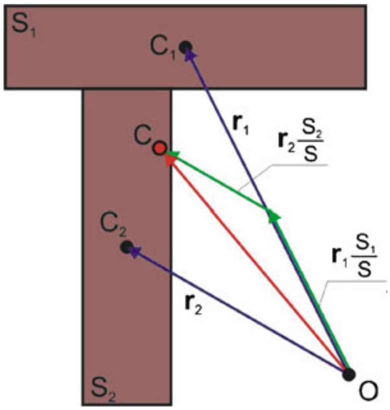


Fig. 3.8 Graphical method of determining the centroid of a figure having the shape of an asymmetric T

The centroid of the arc of a circle of length L is given by the formula

$$\begin{aligned} x_{2C} &= \frac{\int\limits_L x_2 \mathrm{d}s}{L} = \frac{\frac{2}{3}\int\limits_L x_2 r \mathrm{d}\varphi}{2\alpha r} = \frac{\frac{2}{3}r^2 \int\limits_{-\alpha}^{\alpha} \cos\varphi \mathrm{d}\varphi}{2\alpha r} \\ &= \frac{2}{3}\frac{r}{\alpha}\int\limits_0^{\alpha} \cos\varphi \mathrm{d}\varphi = \frac{2}{3}\frac{r}{\alpha}\Big[\sin\varphi\Big]_0^{\alpha} = \frac{2r\sin\alpha}{3\alpha}. \end{aligned}$$ □

Example 3.4. Determine the radius vector of the centroid of a figure composed of two parts having areas S_1 and S_2, as shown in Fig. 3.8.

From the definition we have

$$\mathbf{r}_C = \frac{\sum\limits_{n=1}^{2} S_n \mathbf{r}_n}{S_1 + S_2} = \frac{S_1\mathbf{r}_1 + S_2\mathbf{r}_2}{S} = \frac{S_1}{S}\mathbf{r}_1 + \frac{S_1}{S}\mathbf{r}_2,$$

and the vector sum resulting from the equation above is presented in Fig. 3.8. □

It has been shown that the coordinates of the centroid of an area are found by dividing the first moments of that area by the area itself. Observe that if the centroid of an area lies on a coordinate axis, then the first moment of the area around that axis is zero, and vice versa.

If an area or a line has an axis of symmetry, then its first moment with respect to the axis of symmetry is zero, and its centroid is located on that axis. Furthermore, if an area or line has two axes of symmetry, then its centroid is located at the intersection of the two axes.

It can be easily shown that the centroid of the area coincides with its center of symmetry. Recall that an area is *symmetric with respect to a center* if for every element of area $\mathrm{dS}(x_1, x_2)$ there is an element $\mathrm{dS}'(-x_1, -x_2)$ such that $\mathrm{dS} = \mathrm{dS}'$.

We have considered so far the general rules for locating the centers of gravity of two-dimensional bodies (wires) and the centroids of plane areas and lines. We have shown that if the considered bodies are homogeneous, their centers of gravity and centroids coincide. If one is dealing with composite plates and wires, then such objects should be divided into known parts (rectangles, triangles, arcs, etc.), and one may then proceed in a standard way (see also Example 3.7).

Table 3.1 gives some examples of centroids location for common shapes of selected plane figures and curves [4].

In the case of the solids of revolution, that is, the solids generated as a result of the revolution of a certain curve about an axis lying in the plane of that curve but not intersecting it (Fig. 3.9), there exists a relation between the lateral surface and the volume of the solid that is expressed by the following two Pappus–Guldinus rules (theorems).

Theorem 3.1 (Pappus–Guldinus[1] Rule I). *The area of a surface created through a full revolution of a plane curve about an axis OX_3 not intersecting the curve and lying in its plane is equal to the product of the length of that curve and the circumference of a circle described by the centroid of the curve while the surface is being generated.*

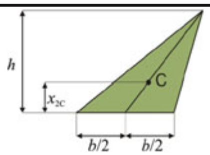
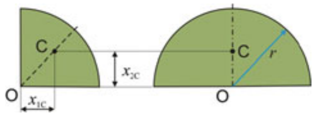
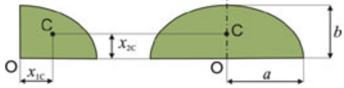
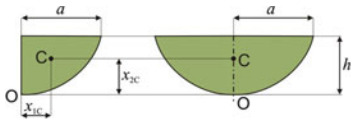
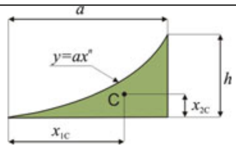
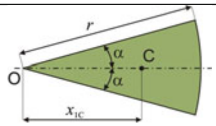
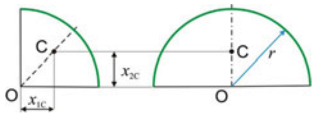
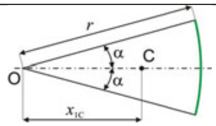
Theorem 3.2 (Pappus–Guldinus Rule II). *The volume of a body created through a full revolution of a plane figure about an axis not intersecting the figure and lying in its plane is equal to the product of the area of the figure and the circumference of a circle described by the centroid of the figure while the volume is being generated.*

According to Fig. 3.9a the area of the surface of revolution is equal to

$$S = \int_l 2\pi r(s)\mathrm{d}s = 2\pi \int_l r(s)\mathrm{d}s = 2\pi I^S_{OX_3}, \tag{3.26}$$

[1]The theorems are named after the mathematicians Pappus, who worked in Alexandria and lived in the third or fourth century AD, and Guldin (Guldinus), who in 1635 formulated the results of his work on the center of gravity.

Table 3.1 Centroids of common shapes of areas and lines (see also [5])

Shape		x_{1C}	x_{2C}	Area/curve length
Triangle			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular and semicircular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic and parabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Area under curve ax^n		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc and semicircular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
		0	$\frac{2r}{\pi}$	πr
Arc of a circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

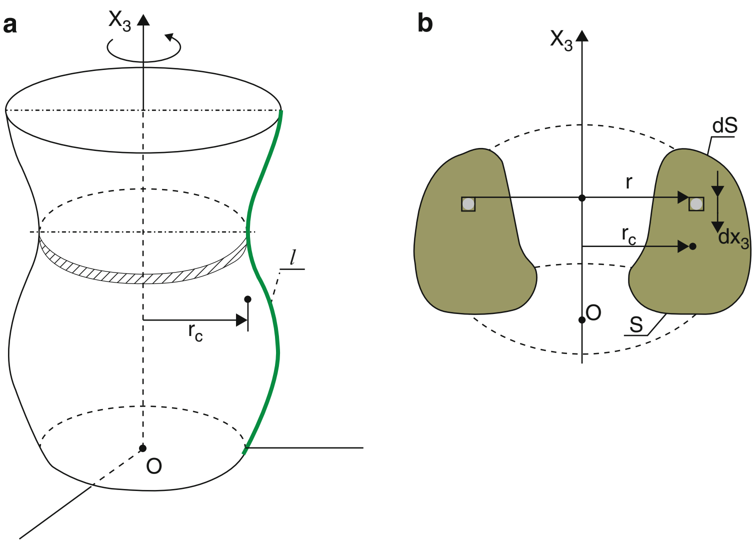


Fig. 3.9 The lateral surface and volume of a body of revolution obtained as a result of the revolution of a curve l (**a**) and a figure S (**b**) about an axis OX_3 (see Pappus–Guldinus rules I and II)

where $I^S_{OX_3}$ is the first moment with respect to the axis OX_3. The first moment is equal to

$$I^S_{OX_3} = l r_C, \tag{3.27}$$

where r_C defines the position of the mass center of a curve of length S.

Substituting (3.27) into (3.26) we obtain

$$S = 2\pi l r_C. \tag{3.28}$$

According to Fig. 3.9b the volume of a revolution is equal to

$$V = 2\pi \int_S r\mathrm{d}S = 2\pi I^S_{OX_3}, \tag{3.29}$$

where the first moment $I^S_{OX_3}$ of a plane figure can be determined from the equation

$$I^S_{OX_3} = S r_C, \tag{3.30}$$

that is,

$$V = 2\pi S r_C. \tag{3.31}$$

In what follows we show, using examples, how the Pappus–Guldinus rules can be used to (1) compute the area of the surface of revolution and volumes of bodies of revolution, (2) determine the centroid of a plane curve when the area of the surface generated by the curve is known, or (3) determine the centroid of a plane area when the volume of the body generated by this area is known.

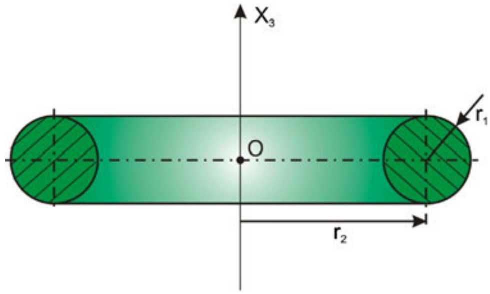


Fig. 3.10 Determination of the volume of a torus

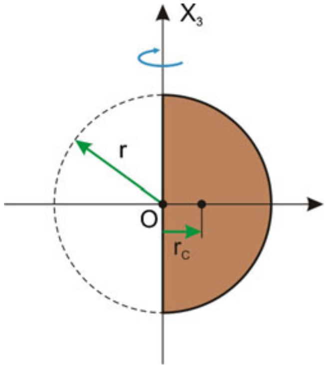


Fig. 3.11 Schematic leading to the determination of a mass center of a half-disc

Example 3.5. Determine the volume of a solid (torus) created as a result of the revolution of a circle about an axis OX_3 (Fig. 3.10).

Directly from the second Pappus-Guldinus rule we obtain

$$V = \pi r_1^2 \cdot 2\pi r_2 = 2\pi^2 r_1^2 r_2.$$ □

Example 3.6. Calculate the position of a mass center of a half-disc of radius r.

Let us note that for the determination of the distance of the mass center of a half-disc from its diameter lying on the axis OX_3 (Fig. 3.11) one may use the second Pappus–Guldinus rule if one knows the volume of a ball given by $V_b = \frac{4}{3}\pi r^3$. Since we have

$$V_b = \frac{1}{2}\pi r^2 \cdot 2\pi r_C,$$

and from that

$$r_C = \frac{V_b}{\pi^2 r^2} = \frac{4}{3}\frac{\pi r^3}{\pi^2 r^2} = \frac{4}{3}\frac{r}{\pi}.$$

The result just obtained is identical to the one from Example 3.3 for $\alpha = \pi/2$. □

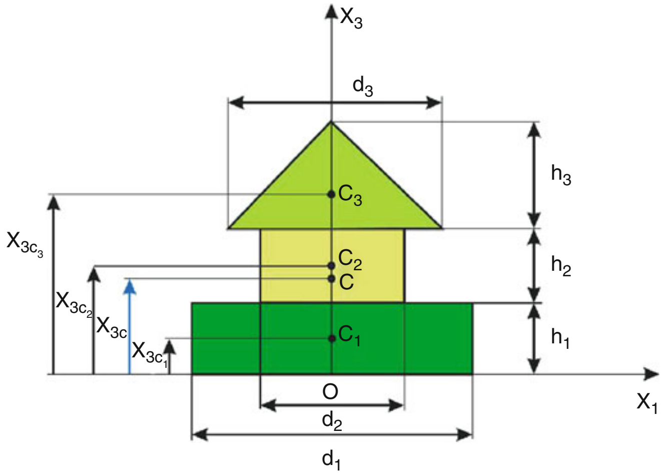


Fig. 3.12 Determination of the position of a mass center of a figure consisting of two rectangles and a triangle

Example 3.7. Determine the position of a mass center of the plane figure depicted in Fig. 3.12.

The position of mass center of the figure is given by

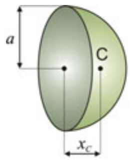
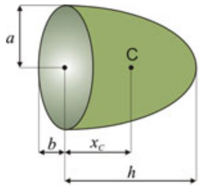
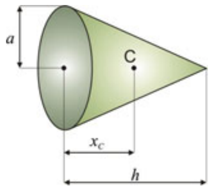
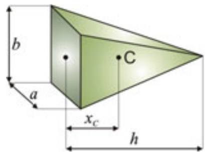
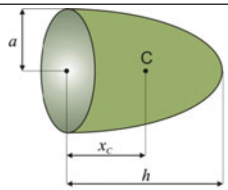
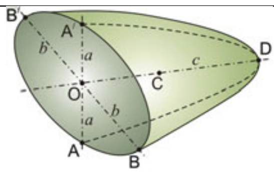
$$
\begin{aligned}
x_{3C} &= \frac{S_1 x_{3C_1} + S_2 x_{3C_2} + S_3 x_{3C_3}}{S_1 + S_2 + S_3} \\
&= \frac{d_1 h_1 \frac{h_1}{2} + d_2 h_2 \left(h_1 + \frac{h_2}{2} \right) + \frac{1}{2} d_3 h_3 \left(h_1 + h_2 + \frac{h_3}{3} \right)}{d_1 h_1 + d_2 h_2 + \frac{1}{2} d_3 h_3}. \qquad \square
\end{aligned}
$$

In general, a procedure to determine the location of the centers of gravity of three-dimensional bodies or the centroids of their volumes is similar to that of the thus far studied two-dimensional shapes and flat lines and hence will not be further addressed. However, centroids of common shapes and volumes are reported in Table 3.2.

3.2 Second Moments

Static moments (mass or area/volume) are called *first moments*. The moments of inertia (with respect to a plane, axis, and pole) and products of inertia are called *second moments*.

Table 3.2 Centroid location of common shapes and volumes (see also [5,7])

Shape	Scheme	x_c	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Elliptic paraboloid		$\frac{h}{3}$	$\frac{1}{2}\pi abh$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}\pi abh$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Three-axis nonsymmetric ellipsoid		$\frac{3c}{8}$	$\frac{2}{3}\pi abc$

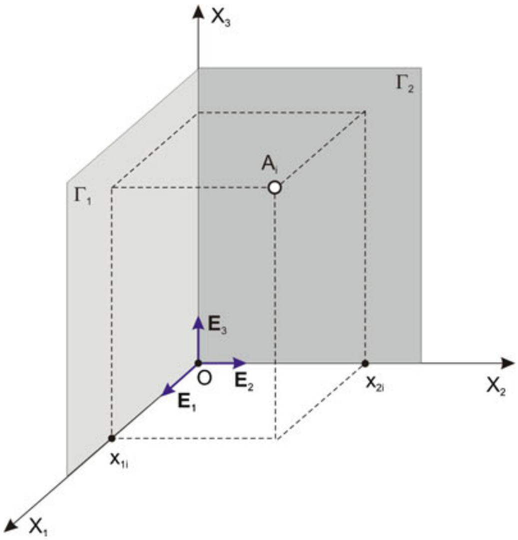


Fig. 3.13 Product of inertia of point A_i with respect to two perpendicular planes Γ_1 and Γ_2

We have already introduced the notion of moment of inertia of a particle about a plane, axis, or pole.

The position of a particle A_i can be defined also with respect to two arbitrary perpendicular planes (Fig. 3.13).

We call the following product $I_{\Gamma_1\Gamma_2}$ a product of inertia of a point A_i with respect to planes $\Gamma_1(OX_1X_3)$ and $\Gamma_2(OX_2X_3)$:

$$I_{\Gamma_1\Gamma_2} = m_i x_{1i} x_{2i}, \tag{3.32}$$

where m_i denotes the mass of the point and x_{1i} and x_{2i} the distances to the planes Γ_2 and Γ_1, respectively.

In the case of the CMS the moments of inertia and the products of inertia are defined as follows:

$$I_\Gamma = \int_M s^2 \mathrm{d}m, \tag{3.33}$$

$$I_l = \int_M d^2 \mathrm{d}m, \tag{3.34}$$

$$I_O = \int_M r^2 \mathrm{d}m, \tag{3.35}$$

$$I_{\Gamma_1\Gamma_2} = \int_M x_1 x_2 \mathrm{d}m, \tag{3.36}$$

where s, d, and r are defined in Sect. 3.1.

Let us introduce now the stationary right-handed Cartesian coordinate system $OX_1X_2X_3$ and calculate the second moments defined using (3.33)–(3.36).

Moments of inertia with respect to planes OX_1X_2, OX_1X_3, and OX_2X_3 are equal to

$$I_{OX_1X_2} = \int_M x_3^2 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_3^2 \mathrm{d}V,$$

$$I_{OX_1X_3} = \int_M x_2^2 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_2^2 \mathrm{d}V,$$

$$I_{OX_2X_3} = \int_M x_1^2 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_1^2 \mathrm{d}V. \tag{3.37}$$

Moments of inertia with respect to axes OX_1, OX_2, and OX_3 are equal to

$$I_{OX_1} \equiv I_{X_1} = \int_M \left(x_2^2 + x_3^2\right) \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) \left(x_2^2 + x_3^2\right) \mathrm{d}V,$$

$$I_{OX_2} \equiv I_{X_2} = \int_M \left(x_1^2 + x_3^2\right) \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) \left(x_1^2 + x_3^2\right) \mathrm{d}V,$$

$$I_{OX_3} \equiv I_{X_3} = \int_M \left(x_1^2 + x_2^2\right) \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) \left(x_1^2 + x_2^2\right) \mathrm{d}V. \tag{3.38}$$

The moment of inertia of a rigid body with respect to the origin O of the coordinate system is equal to

$$\begin{aligned} I_O &= \int_M \left(x_1^2 + x_2^2 + x_3^2\right) \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) \left(x_1^2 + x_2^2 + x_3^2\right) \mathrm{d}V \\ &= \frac{1}{2}\left(I_{X_1} + I_{X_2} + I_{X_3}\right). \end{aligned} \tag{3.39}$$

The products of inertia of a rigid body with respect to the planes of the coordinate system are equal to

$$I_{\Gamma_1\Gamma_2} \equiv I_{X_1X_2} = \int_M x_1 x_2 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_1 x_2 \mathrm{d}V,$$

$$I_{\Gamma_1\Gamma_3} \equiv I_{X_1X_3} = \int_M x_1 x_3 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_1 x_3 \mathrm{d}V,$$

$$I_{\Gamma_2\Gamma_3} \equiv I_{X_2X_3} = \int_M x_2 x_3 \mathrm{d}m = \int_V \rho(x_1, x_2, x_3) x_2 x_3 \mathrm{d}V. \tag{3.40}$$

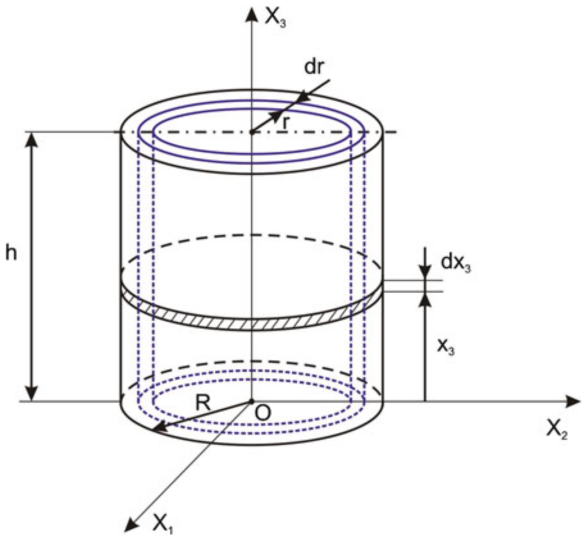


Fig. 3.14 Computational scheme for calculation of the moments of inertia of a homogeneous cylinder

The second moments of the volume of a rigid body in the coordinate system $OX_1X_2X_3$ are obtained from (3.37)–(3.40) after omitting the quantity $\rho(x_1, x_2, x_3)$.

In the case where $\rho(x_1, x_2, x_3) = \text{const}$, we obtain the relation between the second moments of mass and volume in the form

$$I = I^G \rho, \tag{3.41}$$

where I denotes any of the moments described earlier, I^G denotes the moment of the volume corresponding to I, and ρ is the density of any particle of the continuous system.

Example 3.8. Determine the moment of inertia of a homogeneous cylinder of mass M, radius R, and height h with respect to a base plane, an axis of the cylinder, and a line passing through a diameter of the cylinder base (Fig. 3.14).

The mass moment of inertia of a cylinder with respect to the plane OX_1X_2 is equal to

$$I_{OX_1X_2} = \int\limits_V \rho x_3^2 \mathrm{d}V = \pi\rho R^2 \int\limits_0^h x_3^2 \mathrm{d}x_3 = \frac{1}{3}\rho\pi R^2 h^3 = \frac{1}{3}Mh^2 \left[\mathrm{kg}\cdot\mathrm{m}^2\right],$$

and the corresponding moment of the volume reads

$$I^G_{OX_1X_2} = \frac{1}{3}\pi R^2 h^3 \left[\mathrm{m}^5\right].$$

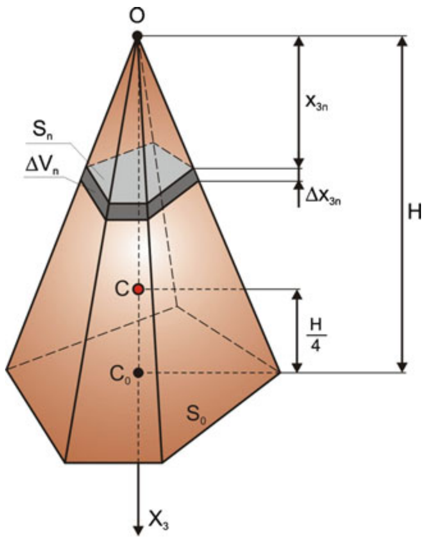


Fig. 3.15 Determination of mass center C of a pentagonal pyramid

Let us now calculate the mass moment of inertia of the cylinder about the axis OX_3. In this case the elementary volume $\mathrm{d}V$ is determined by the volume of the cylindrical shell of radius r, thickness $\mathrm{d}r$, and height h, that is,

$$I_{OX_3} = \int\limits_V (x_1^2 + x_2^2)\rho(x_1, x_2, x_3)\mathrm{d}V = \rho \int\limits_0^R r^2 2\pi r h \mathrm{d}r$$

$$= 2\pi\rho h \frac{R^4}{4} = \frac{1}{2}\pi R^2 h\rho R^2 = \frac{1}{2}MR^2 .$$

The corresponding moment of the volume reads

$$I^V_{OX_3} = \frac{1}{2}\pi R^4 h.$$

For the determination of the mass moment of inertia of the cylinder about the axis OX_1 let us note that

$$I_{OX_1} = I_{OX_1X_3} + I_{OX_1X_2}, \quad I_{OX_3} = I_{OX_1X_3} + I_{OX_2X_3}.$$

From the symmetry of the cylinder we have $I_{OX_1X_3} = I_{OX_2X_3}$, that is, $I_{OX_3} = 2I_{OX_1X_3}$, hence

$$I_{OX_1} = I_{OX_1X_2} + \frac{1}{2}I_{OX_3} = \frac{1}{3}Mh^2 + \frac{1}{4}MR^2 = M\left(\frac{h^2}{3} + \frac{R^2}{4}\right). \qquad \square$$

Example 3.9. Determine the position of the mass center of the pentagonal pyramid depicted in Fig. 3.15.

First, we determine the center of the pyramid base C_0 and prove that point C will lie on a line OC_0 at a distance $\frac{1}{4}H$ from the base, where H is the height of the pyramid.

We are dealing with a solid whose elementary volumes ΔV_n, lying at a distance x_{3n} from the origin of the axis OX_3, are equal to $\Delta V_n = S_n \Delta x_{3n} = S_n \frac{H}{N}$, where N denotes the number of elementary volumes.

From proportion it follows that $\frac{S_n}{S_0} = \frac{x_{3n}^2}{H^2}$, where S_0 is the area of the pentagon of the pyramid base.

The position of the mass center of the pyramid is

$$\begin{aligned} x_{3C} &= \frac{1}{V}\sum_{n=1}^{N} x_{3n}\Delta V_n = \frac{S_0}{VH^2}\sum_{n=1}^{N} x_{3n}^3 \Delta x_{3n} \\ &\cong \frac{S_0}{VH^2}\int_0^H x_{3n}^3 \mathrm{d}x_{3n} = \frac{S_0 H^2}{4V}. \end{aligned}$$

Because $V = \frac{1}{3}S_0 H$, eventually $x_{3C} = \frac{3}{4}H$. □

The (geometric) moments of inertia for selected plane figures are reported in Table 3.3. The moments of inertia of selected three-dimensional bodies are shown in Table 3.4.

3.3 The Inertia Matrix and Its Transformations

We will call an array of numbers of the following form marked with superscript V the *geometric matrix of inertia* (mass is not taken into account):

$$\mathbf{I}^V = \begin{bmatrix} I_{X_1}^V & -I_{X_1X_2}^V & -I_{X_1X_3}^V \\ -I_{X_1X_2}^V & I_{X_2}^V & -I_{X_2X_3}^V \\ -I_{X_1X_3}^V & -I_{X_2X_3}^V & I_{X_3}^V \end{bmatrix}, \tag{3.42}$$

and the elements (numbers) of this matrix are moments of inertia of a set of points with respect to the axes of the coordinate system $OX_1X_2X_3$ and products of inertia with respect to three planes of that coordinate system. The matrix is a representation of the symmetric tensor of the second order for point O (the origin of the coordinate system $OX_1X_2X_3$). Dropping the superscript V we obtain the inertia matrix of a rigid body.

Table 3.3 Moments of inertia of common geometric figures

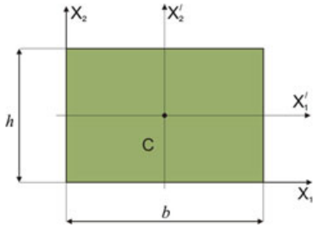
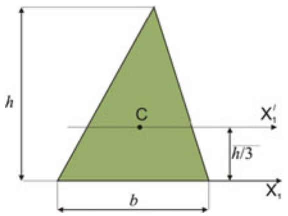
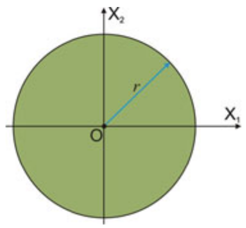
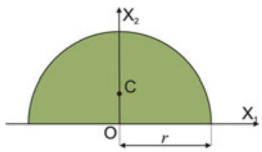
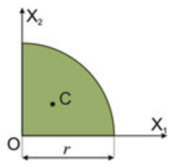
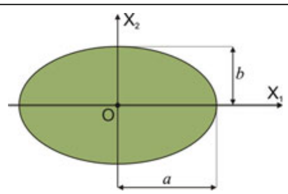
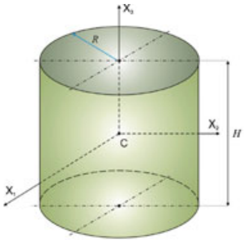
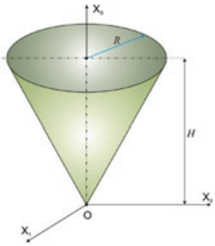
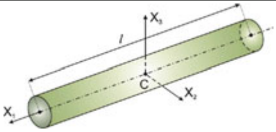
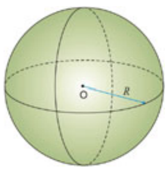
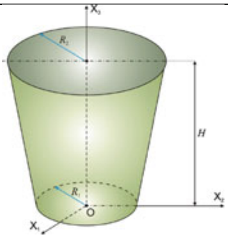
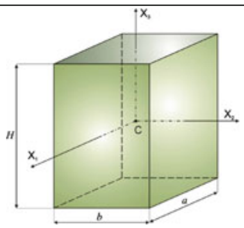
Shape	Schematic diagram	Moment of inertia
Rectangle		$I_{X_1'} = \frac{1}{12}bh^3$ $I_{X_2'} = \frac{1}{12}b^3h$ $I_{X_1} = \frac{1}{3}bh^3$ $I_{X_2} = \frac{1}{3}b^3h$ $I_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$I_{X_1'} = \frac{1}{36}bh^3$ $I_{X_1} = \frac{1}{12}bh^3$
Circle		$I_{X_1} = I_{X_2} = \frac{1}{4}\pi r^4$ $I_O = \frac{1}{2}\pi r^4$
Semicircle		$I_{X_1} = I_{X_2} = \frac{1}{8}\pi r^4$ $I_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_{X_1} = I_{X_2} = \frac{1}{16}\pi r^4$ $I_O = \frac{1}{8}\pi r^4$
Ellipse		$I_{X_1} = \frac{1}{4}\pi ab^3$ $I_{X_2} = \frac{1}{4}\pi a^3b$ $I_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Table 3.4 Moments of inertia of three-dimensional homogeneous bodies with mass m

Body	Schematic diagram	Moment of inertia
Cylinder (vertical)		$I_{X_3} = \frac{1}{2} m R^2$ $I_{X_1} = I_{X_2} = \frac{1}{4} m R^2 + \frac{1}{12} m H^2$ $I_C = \frac{1}{2} m R^2 + \frac{1}{12} m H^2$
Cone		$I_{X_3} = \frac{3}{10} m R^2$ $I_{X_1} = I_{X_2} = \frac{3}{20} m R^2 + \frac{3}{5} m H^2$ $I_O = \frac{3}{10} m R^2 + \frac{3}{5} m H^2$
Thin cylidrical rod		$I_{X_2} = I_{X_3} = \frac{ml^2}{12}$ $I_{X_1} = 0$
Ball		$I_{X_1} = I_{X_2} = I_{X_3} = \frac{2}{5} m R^2$ $2 I_O = I_{X_1} + I_{X_2} + I_{X_3}$
Frustum of cone		$I_{X_3} = \frac{3m(R_2^5 - R_1^5)}{10(R_2 - R_1)(R_1^2 + R_2^2 + R_1 R_2)}$ $I_{X_1} = I_{X_2} = \frac{3m(R_2^5 - R_1^5)}{20(R_2 - R_1)(R_1^2 + R_2^2 + R_1 R_2)}$ $+ \frac{3mH^2}{(R_1^2 + R_2^2 + R_1 R_2)} [\frac{1}{2} R_1 (R_2 - R_1) H^3$ $+ \frac{1}{3} R_1^2 + \frac{1}{5} (R_2 - R_1)^2 H^3]$ $I_O = I_{X_1} + \frac{1}{2} I_{X_3}$
Rectangular cuboid		$I_{X_1} = \frac{m}{12}(b^2 + H^2)$ $I_{X_2} = \frac{m}{12}(a^2 + H^2)$ $I_{X_3} = \frac{m}{12}(a^2 + b^2)$

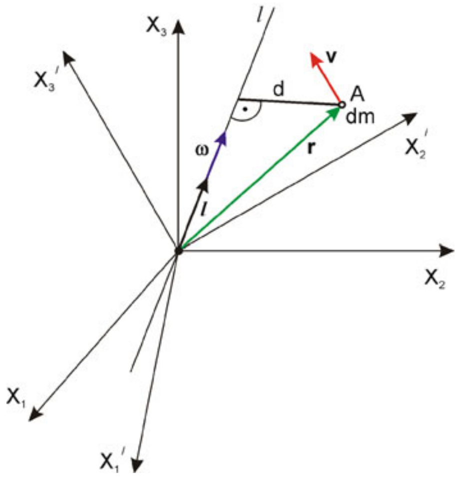


Fig. 3.16 Determination of kinetic energy of a body rotating about a certain stationary axis l

An arbitrary position of a rigid body in space can be obtained from its initial position in the adopted absolute coordinate system through translation to point O', and the subsequent rotation of the coordinate system $OX_1X_2X_3$ can be made to coincide with the system $OX_1'X_2'X_3'$.

At this point the question arises as to how the inertia matrix will change as a result of translation and rotation of the coordinate system.

In order to derive formally the inertia matrix (3.42) we will consider the geometry of mass of a rigid body and the notion, introduced earlier, of moment of inertia with respect to a certain stationary axis l (Fig. 3.16).

The kinetic energy of a rigid body equals

$$T = \frac{1}{2}\int_M \mathbf{v} \circ \mathbf{v} \mathrm{d}m. \tag{3.43}$$

Because the elementary particle of a mass dm is situated at distance d from the axis of revolution, from (3.43) we obtain

$$T = \frac{\omega^2}{2}\int_M d^2 \mathrm{d}m = \frac{\omega^2}{2} I_l, \tag{3.44}$$

where I_l is the moment of inertia with respect to axis l.

Because the position of a point A of mass dm is determined by a radius vector $\mathbf{r}$, the velocity of point A equals $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ and from formula (3.43) we obtain

$$T = \frac{1}{2}\int_M (\boldsymbol{\omega} \times \mathbf{r}) \circ (\boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}m. \tag{3.45}$$

Recall that both vectors $\boldsymbol{\omega}$ and $\mathbf{r}$ can be expressed in the coordinates related to a non-stationary $(OX_1'X_2'X_3')$ or stationary $(OX_1X_2X_3)$ coordinate system. In turn, as will be shown later in Chap. 5,

$$\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\Omega}\boldsymbol{\omega}, \tag{3.46}$$

where $\boldsymbol{\Omega}$ is the tensor of angular velocity, which is a skew-symmetric tensor, that is, $\Omega_{ij} = -\Omega_{ji}$ for $i \neq j$.

From (3.46) we obtain

$$\begin{vmatrix} \mathbf{E}_1' & \mathbf{E}_2' & \mathbf{E}_3' \\ \omega_1' & \omega_2' & \omega_3' \\ r_1' & r_2' & r_3' \end{vmatrix} = \left[\mathbf{E}_1' \; \mathbf{E}_2' \; \mathbf{E}_3' \right] \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}. \tag{3.47}$$

Next we calculate

$$\begin{aligned} &\mathbf{E}_1'\left(\omega_2' r_3' - r_2'\omega_3'\right) + \mathbf{E}_2'\left(\omega_3' r_1' - \omega_1' r_3'\right) + \mathbf{E}_3'\left(\omega_1' r_2' - \omega_2' r_1'\right) \\ &\quad = \mathbf{E}_1'\left(a_{12}\omega_2' + a_{13}\omega_3'\right) + \mathbf{E}_2'\left(-a_{12}\omega_1' + a_{23}\omega_3'\right) + \mathbf{E}_3'\left(-a_{13}\omega_1' - a_{23}\omega_2'\right), \end{aligned}$$

and finally we obtain

$$a_{12} = r_3', \quad a_{13} = -r_2', \quad a_{23} = r_1'.$$

The scalar product of two vectors

$$(\boldsymbol{\omega} \times \mathbf{r}) \circ (\boldsymbol{\omega} \times \mathbf{r}) = (\boldsymbol{\Omega}\boldsymbol{\omega})^T \circ (\boldsymbol{\Omega}\boldsymbol{\omega}) = \boldsymbol{\omega}^T \boldsymbol{\Omega}^T \boldsymbol{\Omega}\boldsymbol{\omega}, \tag{3.48}$$

because $\boldsymbol{\Omega}\boldsymbol{\omega}$ is a vector.

Substituting (3.48) into (3.45) we obtain

$$T = \frac{1}{2}\boldsymbol{\omega} \circ \mathbf{I}\boldsymbol{\omega}, \tag{3.49}$$

where tensor I associated with mass equals

$$\mathbf{I} = \int_M \boldsymbol{\Omega}^T \boldsymbol{\Omega}\, \mathrm{d}m. \tag{3.50}$$

We will show that the obtained tensor $\boldsymbol{\Omega}^T\boldsymbol{\Omega}$ is a geometric matrix of inertia (3.42). Since we have

$$\boldsymbol{\Omega}^T\boldsymbol{\Omega} = \begin{bmatrix} 0 & -r'_3 & r'_2 \\ r'_3 & 0 & -r'_2 \\ -r'_2 & r'_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & r'_3 & -r'_2 \\ -r'_3 & 0 & r'_1 \\ r'_2 & -r'_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} r'^2_3 + r'^2_2 & -r'_1 r'_2 & -r'_1 r'_3 \\ -r'_1 r'_2 & r'^2_1 + r'^2_3 & -r'_2 r'_3 \\ -r'_1 r'_3 & -r'_2 r'_3 & r'^2_1 + r'^2_2 \end{bmatrix}$$

$$= \begin{bmatrix} I^V_{X'_1} & -I^V_{X'_1 X'_2} & -I^V_{X'_1 X'_3} \\ -I^V_{X'_1 X'_2} & I^V_{X'_2} & -I^V_{X'_2 X'_3} \\ -I^V_{X'_1 X'_3} & -I^V_{X'_2 X'_3} & I^V_{X'_3} \end{bmatrix} = \mathbf{I}^V, \tag{3.51}$$

this explains the introduction of minus signs in definition (3.42).

The ($'$) symbol denotes the coordinates of the non-stationary system. Dropping the primes we obtain the inertia matrix in the stationary system.

From (3.51) it follows that the inertia matrix of a rigid body is equal to

$$\mathbf{I} = \int_M \mathbf{I}^V \mathrm{d}m = \rho\big(x'_1, x'_2, x'_3\big) \int_V \mathbf{I}^V \mathrm{d}V, \tag{3.52}$$

on the assumption that $\rho = \text{const}$ at each point of the body.

In the general case by application of tensor notation to a perfectly rigid body the coefficients of inertia matrix can be determined from the following equation:

$$I_{ij} = \int_V \rho\big(\delta_{mn} x_m x_n \delta_{ij} - x_{ij}\big)\big(x_1, x_2, x_3\big)\mathrm{d}V, \tag{3.53}$$

where δ_{mn} and δ_{ij} are the Kronecker symbols and ρ is the density of the body.

Figure 3.17 contains one rigid body (in the general case there may be many bodies; then we introduce a numbering scheme for the bodies, e.g., if it was a body of the number i, then its mass center would be denoted C_i) and three coordinate systems. The first one, $OX_1X_2X_3$, is the absolute one, whereas the second and third ones are the body's coordinate systems rigidly connected with the body at the point $O' = O''$ for the description of the position of an arbitrary point A of the body. Note that the axes of the coordinate system $OX'_1X'_2X'_3$ are parallel to the axes of the absolute system $OX_1X_2X_3$, the system $OX''_1X''_2X''_3$ is the body system, and all systems are right-handed Cartesian systems.

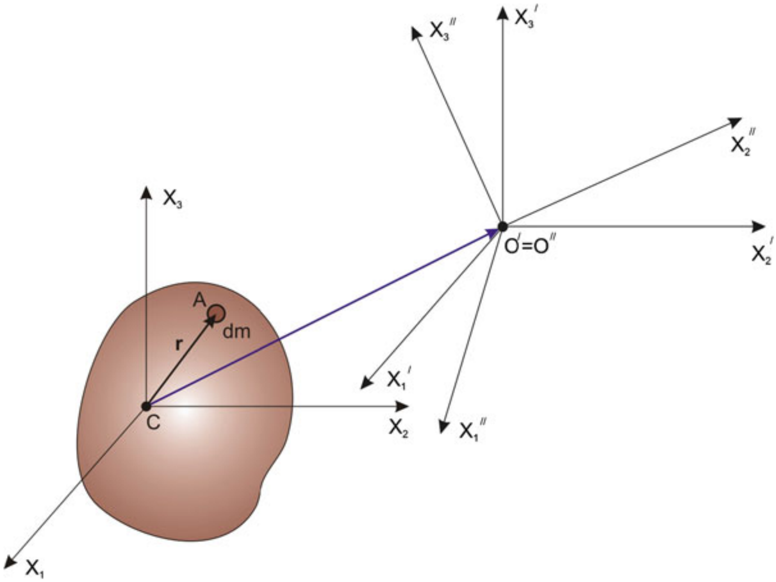


Fig. 3.17 Two coordinate systems of parallel axes and a system with rotated axes $OX_1''X_2''X_3''$

The position of point A defined by the vector $\mathbf{r}$ can be described as follows in the coordinate system $OX_1'X_2'X_3'$ and $OX_1''X_2''X_3''$, where the latter is rotated with respect to $OX_1'X_2'X_3'$:

$$\mathbf{r} = \mathbf{E}_1'x_1' + \mathbf{E}_2'x_2' + \mathbf{E}_3'x_3' = \mathbf{E}_1''x_1'' + \mathbf{E}_2''x_2'' + \mathbf{E}_3''x_3''. \tag{3.54}$$

In order to determine the coordinates of the vector $\mathbf{r} = \mathbf{r}[x_1'', x_2'', x_3'']$ in the system $OX_1'X_2'X_3'$ one should multiply (3.54) successively by the unit vectors $\mathbf{E}_i'$ (scalar product), which leads to the following result:

$$\begin{aligned}
x_1' &= \mathbf{r} \circ \mathbf{E}_1' = x_1''\mathbf{E}_1'' \circ \mathbf{E}_1' + x_2''\mathbf{E}_2'' \circ \mathbf{E}_1' + x_3''\mathbf{E}_3'' \circ \mathbf{E}_1' \\
&= x_1'' \cos(\mathbf{E}_1'', \mathbf{E}_1') + x_2'' \cos(\mathbf{E}_2'', \mathbf{E}_1') + x_3'' \cos(\mathbf{E}_3'', \mathbf{E}_1'), \\
x_2' &= \mathbf{r} \circ \mathbf{E}_2' = x_1''\mathbf{E}_1'' \circ \mathbf{E}_2' + x_2''\mathbf{E}_2'' \circ \mathbf{E}_2' + x_3''\mathbf{E}_3'' \circ \mathbf{E}_2' \\
&= x_1'' \cos(\mathbf{E}_1'', \mathbf{E}_2') + x_2'' \cos(\mathbf{E}_2'', \mathbf{E}_2') + x_3'' \cos(\mathbf{E}_3'', \mathbf{E}_2'), \\
x_3' &= \mathbf{r} \circ \mathbf{E}_3' = x_1''\mathbf{E}_1'' \circ \mathbf{E}_3' + x_2''\mathbf{E}_2'' \circ \mathbf{E}_3' + x_3''\mathbf{E}_3'' \circ \mathbf{E}_3' \\
&= x_1'' \cos(\mathbf{E}_1'', \mathbf{E}_3') + x_2'' \cos(\mathbf{E}_2'', \mathbf{E}_3') + x_3'' \cos(\mathbf{E}_3'', \mathbf{E}_3'),
\end{aligned} \tag{3.55}$$

and can be written in matrix form:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cos(\mathbf{E}_1'', \mathbf{E}_1') & \cos(\mathbf{E}_2'', \mathbf{E}_1') & \cos(\mathbf{E}_3'', \mathbf{E}_1') \\ \cos(\mathbf{E}_1'', \mathbf{E}_2') & \cos(\mathbf{E}_2'', \mathbf{E}_2') & \cos(\mathbf{E}_3'', \mathbf{E}_2') \\ \cos(\mathbf{E}_1'', \mathbf{E}_3') & \cos(\mathbf{E}_2'', \mathbf{E}_3') & \cos(\mathbf{E}_3'', \mathbf{E}_3') \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}, \tag{3.56}$$

or $x_i' = a_{ij} x_j''$, retaining the Einstein summation convention where $a_{ij} = \cos(\mathbf{E}_j'', \mathbf{E}_i')$.

The expression above means that, knowing the coordinates of the point $A(x_1'', x_2'', x_3'')$, one can, from the relationship above, determine the coordinates of that point in the coordinate system $OX_1'X_2'X_3'$, provided that the direction cosines are known.

Similarly, we can determine the coordinates of the vector $\mathbf{r}$ in the coordinate system $OX_1''X_2''X_3''$. After multiplication of (3.53) in turn by $\mathbf{E}_i'$ (scalar product) we obtain

$$\begin{aligned} x_1'' &= x_1' \cos(\mathbf{E}_1', \mathbf{E}_1'') + x_2' \cos(\mathbf{E}_2', \mathbf{E}_1'') + x_3' \cos(\mathbf{E}_3', \mathbf{E}_1''), \\ x_2'' &= x_1' \cos(\mathbf{E}_1', \mathbf{E}_2'') + x_2' \cos(\mathbf{E}_2', \mathbf{E}_2'') + x_3' \cos(\mathbf{E}_3', \mathbf{E}_2''), \\ x_3'' &= x_1' \cos(\mathbf{E}_1', \mathbf{E}_3'') + x_2' \cos(\mathbf{E}_2', \mathbf{E}_3'') + x_3' \cos(\mathbf{E}_3', \mathbf{E}_3''), \end{aligned} \tag{3.57}$$

which in matrix notation will take the following form:

$$\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} \cos(\mathbf{E}_1', \mathbf{E}_1'') & \cos(\mathbf{E}_2', \mathbf{E}_1'') & \cos(\mathbf{E}_3', \mathbf{E}_1'') \\ \cos(\mathbf{E}_1', \mathbf{E}_2'') & \cos(\mathbf{E}_2', \mathbf{E}_2'') & \cos(\mathbf{E}_3', \mathbf{E}_2'') \\ \cos(\mathbf{E}_1', \mathbf{E}_3'') & \cos(\mathbf{E}_2', \mathbf{E}_3'') & \cos(\mathbf{E}_3', \mathbf{E}_3'') \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}, \tag{3.58}$$

or we have that $x_i'' = a_{ji} x_j'$.

Because

$$\cos(\mathbf{E}_i', \mathbf{E}_j'') = \cos(\mathbf{E}_j'', \mathbf{E}_i'), \; i, j = 1, 2, 3,$$

so

$$[a_{ij}]^{-1} = [a_{ij}]^T = [a_{ij}].$$

We will prove now that the matrix of direction cosines a_{ij} is an orthogonal matrix.

According to the definition of the orthogonal matrix we have

$$[a_{ij}][a_{ij}]^T = [a_{ij}]^T[a_{ij}] = \mathbf{E}, \tag{3.59}$$

where $\mathbf{E}$ is an identity matrix.

In our case we calculate

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{11}a_{31}+a_{12}a_{32}+a_{13}a_{33} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2 & a_{21}a_{31}+a_{22}a_{32}+a_{23}a_{33} \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} & a_{31}a_{21}+a_{32}a_{22}+a_{33}a_{23} & a_{31}^2+a_{32}^2+a_{33}^2 \end{bmatrix}. \tag{3.60}$$

Let us note that each of the unit vectors of one coordinate system can be expressed by the unit vectors of another coordinate system. For example, we obtain representation of the vectors $\mathbf{E}_i''$ in the system $OX_1'X_2'X'_3$ from (3.58) after substituting the column of unit vectors $[\mathbf{E}_1'\mathbf{E}_2'\mathbf{E}_3']^T$:

$$\mathbf{E}_1'' = [a_{11}\ a_{21}\ a_{31}] \begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix},$$
$$\mathbf{E}_2'' = [a_{12}\ a_{22}\ a_{32}] \begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix},$$
$$\mathbf{E}_3'' = [a_{13}\ a_{23}\ a_{33}] \begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix}. \tag{3.61}$$

After scalar multiplication of the equations above by each other we obtain

$$\begin{aligned}
\mathbf{E}_1'' \circ \mathbf{E}_2'' &= (a_{11}\mathbf{E}_1' + a_{21}\mathbf{E}_2' + a_{31}\mathbf{E}_3') \circ (a_{12}\mathbf{E}_1' + a_{22}\mathbf{E}_2' + a_{32}\mathbf{E}_3') \\
&= a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \\
\mathbf{E}_1'' \circ \mathbf{E}_3'' &= (a_{11}\mathbf{E}_1' + a_{21}\mathbf{E}_2' + a_{31}\mathbf{E}_3') \circ (a_{13}\mathbf{E}_1' + a_{23}\mathbf{E}_2' + a_{33}\mathbf{E}_3') \\
&= a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0, \\
\mathbf{E}_2'' \circ \mathbf{E}_3'' &= (a_{12}\mathbf{E}_1' + a_{22}\mathbf{E}_2' + a_{32}\mathbf{E}_3') \circ (a_{13}\mathbf{E}_1' + a_{23}\mathbf{E}_2' + a_{33}\mathbf{E}_3') \\
&= a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, \\
\mathbf{E}_1'' \circ \mathbf{E}_1'' &= (a_{11}\mathbf{E}_1' + a_{21}\mathbf{E}_2' + a_{31}\mathbf{E}_3') \circ (a_{11}\mathbf{E}_1' + a_{21}\mathbf{E}_2' + a_{31}\mathbf{E}_3') \\
&= a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, \\
\mathbf{E}_2'' \circ \mathbf{E}_2'' &= (a_{12}\mathbf{E}_1' + a_{22}\mathbf{E}_2' + a_{32}\mathbf{E}_3') \circ (a_{12}\mathbf{E}_1' + a_{22}\mathbf{E}_2' + a_{32}\mathbf{E}_3') \\
&= a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, \\
\mathbf{E}_3'' \circ \mathbf{E}_3'' &= (a_{13}\mathbf{E}_1' + a_{23}\mathbf{E}_2' + a_{33}\mathbf{E}_3') \circ (a_{13}\mathbf{E}_1' + a_{23}\mathbf{E}_2' + a_{33}\mathbf{E}_3') \\
&= a_{13}^2 + a_{23}^2 + a_{33}^2 = 1,
\end{aligned} \tag{3.62}$$

which proves (3.59).

Below we will present a formula allowing for easy determination of the inertia matrix in stationary and non-stationary coordinate systems [9]. Recall that the unit vectors $\mathbf{E}_i$ ($\mathbf{E}_i'$) ($i = 1, 2, 3$) are associated with the non-stationary (stationary) coordinate system, and in view of that we obtain [see (3.61)]

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix} = \begin{bmatrix} a_1^{1'} & a_2^{1'} & a_3^{1'} \\ a_1^{2'} & a_2^{2'} & a_3^{2'} \\ a_1^{3'} & a_2^{3'} & a_3^{3'} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{3.63}$$

and the coefficient $a_j^{i'}$ denotes the cosine of an angle formed by the X_i' axis with the X_j axis. For example, $a_3^{2'} = (\mathbf{E}_2' \circ \mathbf{E}_3) = \cos(\mathbf{E}_2', \mathbf{E}_3)$.

Although we have nine direction cosines altogether, only three are independent. Because we can select the latter arbitrarily, we will keep all nine coefficients.

Let us introduce new notation for the elements of the inertia matrix:

$$\begin{aligned}
&I_{X_1} = I_{11},\ I_{X_2} = I_{22},\ I_{X_3} = I_{33}, \\
&I_{X_1X_2} = -I_{12},\ I_{X_2X_3} = -I_{23},\ I_{X_1X_3} = -I_{13}.
\end{aligned}$$

Now the elements of the inertia matrix are equal to

$$I_{ij} = \int\limits_M \left[\left(\sum_{k=1}^{3} r_k^2 \right) \delta_{ij} - r_i r_j \right] \mathrm{d}m, \tag{3.64}$$

where δ_{ij} is the Kronecker delta. For example,

$$\begin{aligned} I_{23} &= \int\limits_M \left[\left(r_1^2 + r_2^2 + r_3^2 \right) \cdot 0 - r_2 r_3 \right] \mathrm{d}m = \int\limits_M I_{23}^V \mathrm{d}m, \\ I_{22} &= \int\limits_M \left[\left(r_1^2 + r_2^2 + r_3^2 \right) \cdot 1 - r_2^2 \right] \mathrm{d}m = \int\limits_M I_{22}^V \mathrm{d}m. \end{aligned}$$

In the rotated system the geometrical inertia matrix has the form

$$I' = \begin{bmatrix} I_{1'1'} & I_{1'2'} & I_{1'3'} \\ I_{2'1'} & I_{2'2'} & I_{2'3'} \\ I_{3'1'} & I_{3'2'} & I_{3'3'} \end{bmatrix}, \tag{3.65}$$

and dropping the $(')$ symbols we obtain matrix I in the stationary coordinate system.

The relation transforming the coefficients of the inertia matrix in the stationary and non-stationary systems has the form

$$I_{k'l'} = \sum_{i,j=1}^{3} I_{ij} a_i^{k'} a_j^{l'}, \tag{3.66}$$

and the application of this formula will be illustrated in Example 3.10.

We now consider the translational displacement. Let us introduce the absolute coordinate system $CX_1X_2X_3$ at the mass center C of the body and perform the parallel translation of this system by a vector $\overrightarrow{CO'}$ (Fig. 3.17).

In the absolute system $CX_1X_2X_3$, let the inertia matrix be known, that is, the moments of inertia with respect to the axes $I_{X_1}, I_{X_2}, I_{X_3}$ and products of inertia $I_{X_1X_2}$, $I_{X_1X_3}$, and $I_{X_2X_3}$.

As an example we will calculate the moment with respect to the X_2' axis, that is,

$$I_{X_2'} = \int\limits_M \left(x_1'^2 + x_3'^2 \right) \mathrm{d}m. \tag{3.67}$$

According to Fig. 3.17 we have

$$\mathbf{r} = \overrightarrow{CO'} + \overrightarrow{O'A}, \tag{3.68}$$

that is,

$$\begin{aligned}
&\mathbf{E}_1 x_{1A} + \mathbf{E}_2 x_{2A} + \mathbf{E}_3 x_{3A} \\
&\quad = \mathbf{E}_1 x_{1O'} + \mathbf{E}_2 x_{2O'} + \mathbf{E}_3 x_{3O'} + \mathbf{E}'_1 x'_{1A} + \mathbf{E}'_2 x'_{2A} + \mathbf{E}'_3 x'_{3A}.
\end{aligned} \tag{3.69}$$

Because for (3.67) we need the quantities x'_1 and x'_3, after dropping A in the subscripts of (3.69) and multiplying it successively by $\mathbf{E}'_1$ and $\mathbf{E}'_2$, we obtain

$$\begin{aligned}
x'_1 &= x_1 - x_{1O'}, \\
x'_3 &= x_3 - x_{3O'}.
\end{aligned} \tag{3.70}$$

Substituting (3.70) into (3.67) we have

$$\begin{aligned}
I_{X'_2} &= \int_M [(x_1 - x_{1O'})^2 + (x_3 - x_{3O'})^2]\mathrm{d}m \\
&= \int_M \left(x_1^2 + x_3^2\right)\mathrm{d}m - 2\int_M x_1 x_{1O'}\mathrm{d}m - 2\int_M x_3 x_{3O'}\mathrm{d}m \\
&\quad + \int_M \left(x_{1O'}^2 + x_{3O'}\right)^2 \mathrm{d}m = I_{X_2} + \left(x_{1O'}^2 + x_{3O'}^2\right) M,
\end{aligned} \tag{3.71}$$

because the first moments with respect to the mass center are equal to zero. The quantity $x_{1O'}^2 + x_{3O'}^2$ is a square of the projection of the vector $\overrightarrow{CO'}$ onto the plane CX_1X_3, that is, it denotes the square of the distance of the axis $O'X'_2$ from the axis CX_2.

Theorem 3.3 (Steiner's First Theorem[2]). *A moment of inertia with respect to an axis passing through the mass center is the smallest compared to the moment of inertia with respect to any other parallel axis, and the difference of these moments of inertia is equal to the product of the square of the distance between the axes and the body mass.*

As an example we will calculate the product of inertia:

$$\begin{aligned}
I_{X'_1X'_3} &= \int_M x'_1 x'_3 \mathrm{d}m = \int_M (x_1 - x_{1O'})(x_3 - x_{3O'})\mathrm{d}m \\
&= \int_M x_1 x_3 \mathrm{d}m - x_{1O'}\int_M x_3 \mathrm{d}m - x_{3O'}\int_M x_1 \mathrm{d}m \\
&\quad + M x_{1O'} x_{3O'} = I_{X_1X_3} + x_{1O'} x_{3O'} M,
\end{aligned} \tag{3.72}$$

[2] Jakob Steiner (1796–1863), Swiss mathematician and outstanding geometer, working mainly at Humboldt University in Berlin.

because of the fact that the second and third integrals are equal to zero (first moments with respect to the mass center).

Theorem 3.4 (Steiner's Second Theorem: Parallel-Axis Theorem for Products of Inertia). *We obtain the product of inertia of a body with respect to planes $O'X_1'X_2'$, $O'X_1'X_3'$, and $O'X_2'X_3'$, after adding to the products of inertia of the body with respect to planes OX_1X_2, OX_1X_3, and OX_2X_3, respectively, parallel to the primed planes and passing through the mass center of the body, a product of the body mass and the distances between pairs of the parallel planes (the product can be positive or negative).*

Both parallel-axis theorems of Steiner follow from a general theorem about the moments of inertia in Cartesian coordinate systems of parallel axes.

Theorem 3.5 (Moments of Inertia in Parallel Cartesian Coordinate Systems). *The mass moments of inertia matrices in Cartesian coordinate systems $OX_1X_2X_3$ and $O'X_1'X_2'X_3'$ of parallel axes (Fig. 3.17) are related to one another by the following relationship:*

$$\begin{bmatrix} I_{X_1'} & -I_{X_1'X_2'} & -I_{X_1'X_3'} \\ -I_{X_1'X_2'} & I_{X_2'} & -I_{X_2'X_3'} \\ -I_{X_1'X_3'} & -I_{X_2'X_3'} & I_{X_3'} \end{bmatrix} = \begin{bmatrix} I_{X_1} & -I_{X_1X_2} & -I_{X_1X_3} \\ -I_{X_1X_2} & I_{X_2} & -I_{X_2X_3} \\ -I_{X_1X_3} & -I_{X_2X_3} & I_{X_3} \end{bmatrix}$$
$$+ M \begin{bmatrix} x_{2O'}^2 + x_{3O'}^2 & -x_{1O'}x_{2O'} & -x_{1O'}x_{3O'} \\ -x_{1O'}x_{2O'} & x_{1O'}^2 + x_{3O'}^2 & -x_{2O'}x_{3O'} \\ -x_{1O'}x_{3O'} & -x_{2O'}x_{3O'} & x_{1O'}^2 + x_{2O'}^2 \end{bmatrix}, \tag{3.73}$$

where $(x_{1O'}, x_{2O'}, x_{3O'})$ are the coordinates of the origin of the coordinate system $O'X_1'X_2'X_3'$ of the axes parallel to the axes of the system $OX_1X_2X_3$, whose origin coincides with the mass center of the body, i.e., $O = C$.

Finally, let us consider the case where the axes of two right-handed coordinate systems possess a common origin and are rotated with respect to one another (Fig. 3.16). Our aim is the determination of relationships between the inertia matrices expressed in the coordinate systems $OX_1X_2X_3$ and $OX_1'X_2'X_3'$. We will assume that the rigid body with which these two coordinate systems are associated rotates with the angular velocity $\boldsymbol{\omega}$ about a certain stationary axis l.

The momentum of a rigid body in the system $OX_1X_2X_3$ is equal to

$$\{I\omega\} = [I]\{\omega\}, \tag{3.74}$$

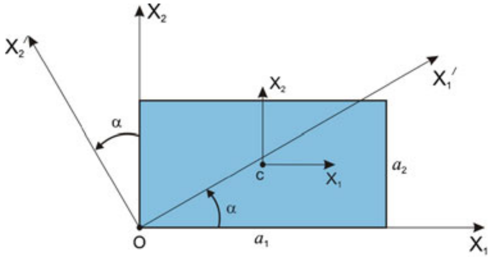


Fig. 3.18 A rectangle and its coordinate axes

and in the system $OX_1'X_2'X_3'$

$$\{I'\omega'\} = [I']\{\omega'\}, \tag{3.75}$$

where vectors are denoted by curly brackets.

According to (3.58) we have

$$\{I'\omega'\} = [a_{ij}]\{I\omega\}, \tag{3.76}$$

$$\{\omega'\} = [a_{ij}]\{\omega\}. \tag{3.77}$$

Substituting relationships (3.76) and (3.77) into (3.75) we obtain

$$[a_{ij}]\{I\omega\} = [I'][a_{ij}]\{\omega\}. \tag{3.78}$$

Then, substituting (3.74) into (3.78) we obtain

$$[a_{ij}][I]\{\omega\} = [I'][a_{ij}]\{\omega\}, \tag{3.79}$$

which eventually enables us to determine the desired relationship of the form

$$[I'] = [a_{ij}][I][a_{ij}]^T. \tag{3.80}$$

Example 3.10. Determine the geometrical inertia matrix of the rectangle of dimensions a_1 and a_2 depicted in Fig. 3.18 in the coordinate system $OX_1'X_2'$.

In the considered case the geometrical inertia matrices in the systems OX_1X_2 and $OX_1'X_2'$ assume the forms (dropping superscripts V)

$$\mathbf{I} = \begin{bmatrix} I_{X_1} & -I_{X_1X_2} \\ -I_{X_1X_2} & I_{X_2} \end{bmatrix},$$

$$\mathbf{I}' = \begin{bmatrix} I_{X_1'} & -I_{X_1'X_2'} \\ -I_{X_1'X_2'} & I_{X_2'} \end{bmatrix}.$$

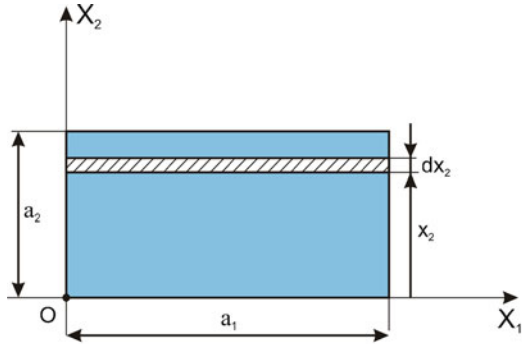


Fig. 3.19 Determination of a (geometric) moment of inertia of a rectangle with respect to the axis OX_1

Let us calculate the moment of inertia with respect to the axis OX_1:

$$I_{X_1} = \int_S x_2^2 \mathrm{d}F = \int_S x_2^2 a_1 \mathrm{d}x_2 = a_1 \int_0^{a_2} x_2^2 \mathrm{d}x_2 = \frac{1}{3} a_1 a_2^3,$$

and the method of integration is illustrated in Fig. 3.19.

Similarly we obtain

$$I_{X_2} = \frac{1}{3} a_2 a_1^3.$$

Let us note that the elements of the inertia matrix of the rectangle in the system CX_1X_2 will not be needed for the problem solution. The coordinate system CX_1X_2 will be used only to determine the product of inertia $I_{X_1X_2}$ from the parallel-axis theorem, that is,

$$I_{X_1X_2} = I^C_{X_1X_2} + a_1 a_2 \left(\frac{a_1}{2}\right)\left(\frac{a_2}{2}\right) = \frac{1}{4} a_1^2 a_2^2,$$

because $I^C_{X_1X_2} = 0$.

In order to determine the elements of the matrix $\mathbf{I}'$ we will use (3.66), and now all the indices assume a value of either 1 or 2, because we are dealing with the two-dimensional system

$$I_{1'1'} = I_{11} a_1^{1'} a_1^{1'} + I_{12} a_1^{1'} a_2^{1'} + I_{21} a_2^{1'} a_1^{1'} + I_{22} a_2^{1'} a_2^{1'}.$$

Let us note that

$$a_1^{1'} = \cos(\mathbf{E}'_1, \mathbf{E}_1) = \cos\alpha,$$

$$a_2^{1'} = \cos(\mathbf{E}'_1, \mathbf{E}_2) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha.$$

Therefore,

$$\begin{aligned} I_{1'1'} = I_{X_1'} &= I_{X_1}\cos^2\alpha - 2I_{X_1X_2}\sin\alpha\cos\alpha + I_{X_2}\sin^2\alpha \\ &= I_{X_1}\cos^2\alpha + I_{X_2}\sin^2\alpha - I_{X_1X_2}\sin 2\alpha. \end{aligned}$$

Next, according to (3.66) we have

$$I_{1'2'} = \sum_{i,j=1}^{2} I_{ij}a_i^{1'}a_j^{2'} = I_{11}a_1^{1'}a_1^{2'} + I_{12}a_1^{1'}a_2^{2'} + I_{22}a_2^{1'}a_2^{2'} + I_{21}a_2^{1'}a_1^{2'}.$$

Because

$$\begin{aligned} a_1^{2'} &= \cos(\mathbf{E}_2', \mathbf{E}_1) = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha, \\ a_2^{2'} &= \cos(\mathbf{E}_2', \mathbf{E}_2) = \cos\alpha, \end{aligned}$$

we arrive at

$$\begin{aligned} I_{1'2'} &\equiv -I_{X_1'X_2'} = -I_{X_1}\cos\alpha\sin\alpha \\ &\qquad -I_{X_1X_2}\cos^2\alpha + I_{X_2}\sin\alpha\cos\alpha + I_{X_2X1}\sin^2\alpha, \end{aligned}$$

that is,

$$I_{X_1'X_2'} = \frac{I_{X_1} - I_{X_2}}{2}\sin 2\alpha + I_{X_1X_2}\cos 2\alpha.$$

Using the quantities I_{X_1}, I_{X_2} and $I_{X_1X_2} = I_{X_2X_1}$ obtained earlier we calculate the quantities $I_{X_1'}$, $I_{X_2'}$, $I_{X_1'X_2'}$, that is, we determine elements of the matrix $\mathbf{I}'$. □

3.4 Principal Axes and Principal Moments on a Plane

In Fig. 3.20 we show the area S and two rectangular coordinates OX_1X_2 and $OX_1'X_2'$, where Θ is the rotation angle between axes OX_1 and OX_1'.

Assuming a knowledge of the moments and product of inertia regarding coordinates OX_1X_2 of the form

$$I_{X_1} = \int x_2^2 \mathrm{d}S, \qquad I_{X_2} = \int x_1^2 \mathrm{d}S, \qquad I_{X_1X_2} = \int x_1x_2 \mathrm{d}S, \tag{3.81}$$

we will determine $I_{X_1'}$, $I_{X_2'}$, and $I_{X_1'X_2'}$. Observe that between the coordinates x_1', x_2', and x_1, x_2 of an element of area $\mathrm{d}S$ the following relations hold:

$$\begin{aligned} x_1' &= x_1\cos\Theta + x_2\sin\Theta, \\ x_2' &= x_2\cos\Theta - x_1\sin\Theta. \end{aligned} \tag{3.82}$$

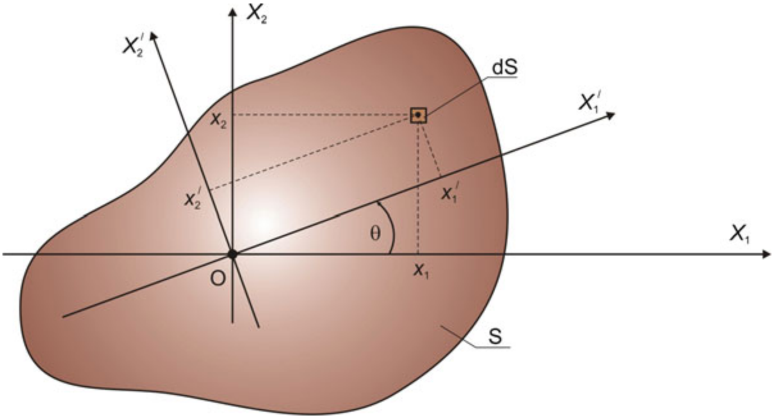


Fig. 3.20 Area S and two systems of rectangular coordinates OX_1X_2 and $OX_1'X_2'$

Therefore, we have

$$\begin{aligned} I_{X_1'} &= \int (x_2')^2 \mathrm{d}S = \int (x_2 \cos\Theta - x_1 \sin\Theta)^2 \mathrm{d}S \\ &= \cos^2\Theta \int x_2^2 \mathrm{d}S - \sin 2\Theta \int x_1 x_2 \mathrm{d}S + \sin^2\Theta \int x_1^2 \mathrm{d}S \\ &= I_{X_1}\cos^2\Theta - I_{X_1X_2}\sin 2\Theta + I_{X_2}\sin^2\Theta \\ &= \frac{I_{X_1}+I_{X_2}}{2} + \frac{I_{X_1}-I_{X_2}}{2}\cos 2\Theta - I_{X_1X_2}\sin 2\Theta, \end{aligned} \tag{3.83}$$

and proceeding in a similar way one obtains

$$I_{X_2'} = \frac{I_{X_1}+I_{X_2}}{2} - \frac{I_{X_1}-I_{X_2}}{2}\cos 2\Theta + I_{X_1X_2}\sin 2\Theta, \tag{3.84}$$

$$I_{X_1'X_2'} = \frac{I_{X_1}-I_{X_2}}{2}\sin 2\Theta + I_{X_1X_2}\cos 2\Theta. \tag{3.85}$$

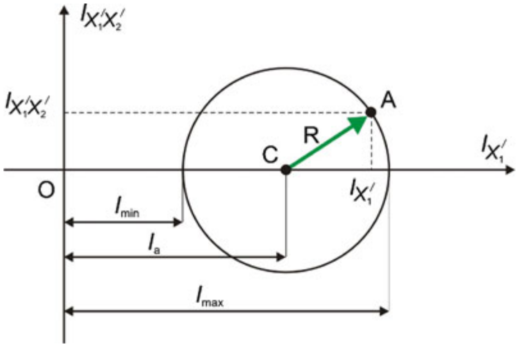
Observe that $I_O = I_{X_1} + I_{X_2} = I_{X_1'} + I_{X_2'}$. We show that if we take a point $A(I_{X_1'}, I_{X_1'X_2'})$ for any given value of the parameter Θ, then all of the points will lie on a circle (Fig. 3.21). That is, eliminating Θ from (3.83)–(3.85) yields

$$(I_{X_1'} - I_a)^2 + I_{X_1'X_2}^2 = R^2, \tag{3.86}$$

where

$$I_a = \frac{I_{X_1}+I_{X_2}}{2}, \qquad R^2 = \left(\frac{I_{X_1}-I_{X_2}}{2}\right)^2 + I_{X_1X_2}^2. \tag{3.87}$$

Fig. 3.21 Plane inertia circle (Mohr's circle)



The circle governed by (3.86) is shown in Fig. 3.21 and is called *Mohr's circle*.[3] The points corresponding to $I_{\min}$ and $I_{\max}$ can be determined from (3.83) assuming $I_{X_1'X_2'} = 0$, and hence

$$\tan 2\Theta = -\frac{2I_{X_1X_2}}{I_{X_1} - I_{X_2}}. \tag{3.88}$$

Equation (3.88) defines two values Θ that are 90° apart, that is, to those of $I_{\min}$ and $I_{\max}$. The two axes defined by the two values of Θ determined so far are perpendicular to each other and are called the principal axes of area S about point O, whereas the corresponding values $I_{\min}$ and $I_{\max}$ are called the *principal moments of inertia of area S about point O*. Observe that the product of inertia corresponding to the principal axes $I_{X_1'X_2'} = 0$. Taking into account Fig. 3.21 and (3.87) one obtains

$$I_{\max,\,\min} = I_a \pm R = \frac{I_{X_1} + I_{X_2}}{2} \pm \sqrt{\left(\frac{I_{X_1} - I_{X_2}}{2}\right)^2 + I_{X_1X_2}^2}. \tag{3.89}$$

Note that the obtained properties are valid for any point O, that is, inside or outside of area S. However, if point O coincides with the centroid of S, then any axis through O is a centroidal axis. Furthermore, if two principal axes of area S have their origin in S centroid, then they are referred *to as the principal centroidal axes of area S*. In what follows we show how Mohr's circle can be used to determine principal axes and principal moments of inertia about a point O assuming a knowledge of the moments and product of inertia with respect to the axes OX_1 and OX_2. In Fig. 3.22 we present a given area S with a given origin and three systems of rectangular coordinates OX_1X_2, $OX_1'X_2'$, and $OX_1^pX_2^p$ (*a*) and the corresponding points lying on Mohr's circle (b).

[3]Otto Mohr (1835–1918), German engineer.

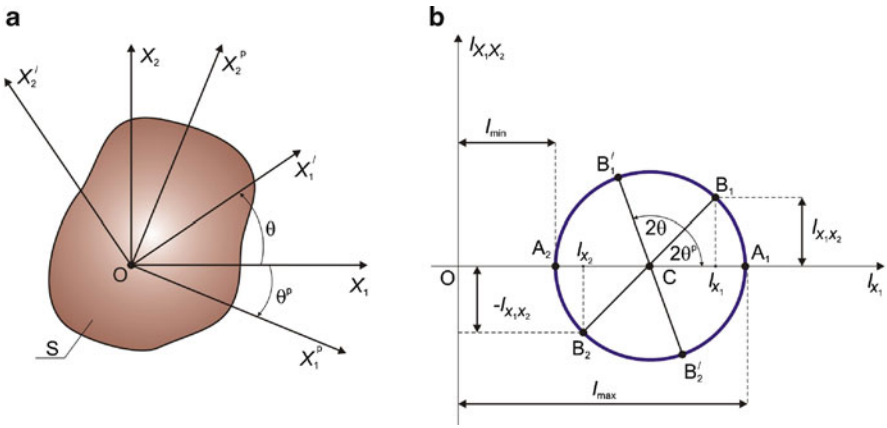


Fig. 3.22 An area S with three systems of rectangular coordinates (**a**) and the corresponding Mohr's circle (**b**)

Let $B_1 = B_1(I_{X_1}, I_{X_1X_2})$, and $B_2 = B_2(I_{X_2}, -I_{X_1X_2})$. An intersection of B_1B_2 with OI_{X_1} yields point C and then we draw the circle of diameter B_1B_2 having center C. Comparing our construction in Fig. 3.22b and (3.87) we conclude that the obtained circle is Mohr's circle for the given area S (Fig. 3.22a) about point O, and hence the principal moments of inertia $I_{\min}$ and $I_{\max}$ are determined. Observe that $\tan(B_1CA_1) = \frac{2I_{X_1X_2}}{I_{X_1}-I_{X_2}} = \tan(2\Theta^p)$, and hence owing to (3.88) the angle Θ^p defines the principal axis corresponding to point A_1. Since $I_{X_1} > I_{X_2}$ and $I_{X_1X_2} > 0$, the rotation of CB_1 into CA_1 is clockwise, and also a clockwise rotation through Θ^p is required to bring OX_1 into the corresponding principal axis OX_1^p.

Furthermore, assume that we need to determine the moment and product of inertia $I_{X_1'}$, $I_{X_2'}$, and $I_{X_1'X_2'}$ regarding the axis OX_1' and OX_2' rotated counterclockwise in comparison to OX_1, as is shown in Fig. 3.22b. We rotate B_1B_2 counterclockwise through an angle 2Θ, and the coordinates of points B_1' and B_2' define $I_{X_1'}$, $I_{X_2'}$, and $I_{X_1'X_2'}$.

3.5 Inertia Tensor, Principal Axes of Inertia, and an Ellipsoid of Inertia

Let us return to Fig. 3.16 and consider the moment of inertia of a discrete system of particles with respect to an axis l.

Observe that a projection of vector $\mathbf{r}(x_{1n}, x_{2n}, x_{3n})$ onto axis l of unit vector $\boldsymbol{l}$ can be determined by applying the scalar product

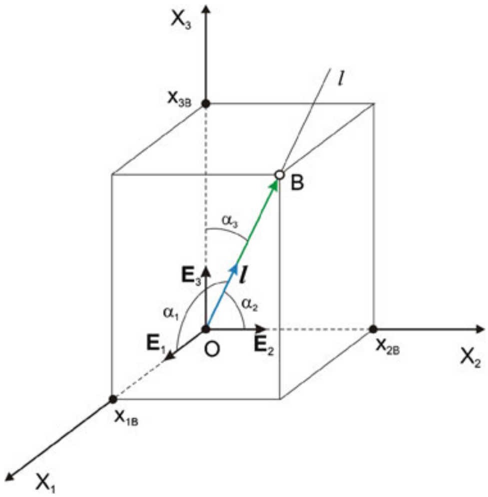


Fig. 3.23 Projection of a vector $\overrightarrow{OB}$ of a coordinate system $OX_1X_2X_3$

$$\begin{aligned}\mathbf{r} \circ \boldsymbol{l} &= (\mathbf{E}_1 x_{1n} + \mathbf{E}_2 x_{2n} + \mathbf{E}_3 x_{3n}) \circ (\mathbf{E}_1 \cos\alpha_1 + \mathbf{E}_2 \cos\alpha_2 + \mathbf{E}_3 \cos\alpha_3) \\ &= \cos\alpha_1 x_{1n} + \cos\alpha_2 x_{2n} + \cos\alpha_3 x_{3n} \\ &= a_1^{l'} x_{1n} + a_2^{l'} x_{2n} + a_3^{l'} x_{3n} \equiv a_{1l} x_{1n} + a_{2l} x_{2n} + a_{3l} x_{3n},\end{aligned} \tag{3.90}$$

which is confirmed by the auxiliary drawing in Fig. 3.23 (during the calculations $l = 1$).

According to (3.66) we obtain (assuming $I_l = I_{X'_l} = I_{l'l'}$)

$$\begin{aligned}I_l \equiv I_{1'1'} = \sum_{i,j=1}^{3} I_{ij} a_i^{1'} a_j^{1'} &= I_{11} a_1^{1'} a_1^{1'} + I_{12} a_1^{1'} a_2^{1'} + I_{13} a_1^{1'} a_3^{1'} \\ &+ I_{21} a_2^{1'} a_1^{1'} + I_{22} a_2^{1'} a_2^{1'} + I_{23} a_2^{1'} a_3^{1'} + I_{31} a_3^{1'} a_1^{1'} + I_{32} a_3^{1'} a_2^{1'} \\ &+ I_{33} a_3^{1'} a_3^{1'} = I_{11}(a_{11})^2 + I_{22}(a_{21})^2 + I_{33}(a_{31})^2 + 2I_{12} a_{11} a_{21} \\ &+ 2I_{13} a_{11} a_{31} + 2I_{23} a_{21} a_{31} = I_{X_1} a_{11}^2 + I_{X_2} a_{21}^2 + I_{X_3} a_{31}^2 \\ &- 2I_{X_1X_2} a_{11} a_{21} - 2I_{X_1X_3} a_{11} a_{31} - 2I_{X_2X_3} a_{21} a_{31}.\end{aligned} \tag{3.91}$$

According to Fig. 3.23, based on the coordinates of an arbitrarily chosen point $B(x_{1B}, x_{2B}, x_{3B})$ on axis l, we have

$$a_{11} \equiv \cos\alpha_1 = \frac{x_{1B}}{\sqrt{x_{1B}^2 + x_{2B}^2 + x_{3B}^2}},$$

$$
\begin{aligned}
a_{21} &\equiv \cos\alpha_2 = \frac{x_{2B}}{\sqrt{x_{1B}^2 + x_{2B}^2 + x_{3B}^2}},\\
a_{31} &\equiv \cos\alpha_3 = \frac{x_{3B}}{\sqrt{x_{1B}^2 + x_{2B}^2 + x_{3B}^2}},
\end{aligned}
\tag{3.92}
$$

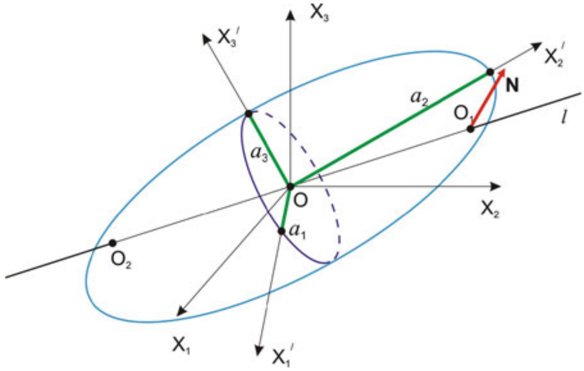
that is,

$$
a_{11}^2 + a_{21}^2 + a_{31}^2 = 1. \tag{3.93}
$$

Let us note that relationships (3.91) can be also directly obtained from the interpretation of the moment of inertia with respect to axis l (Fig. 3.16) and (3.90). We have

$$
\begin{aligned}
I_l &= \int_M \rho d^2 \mathrm{d}m = \int_M \rho\{\mathbf{r}^2 - [(\mathbf{r}\circ \boldsymbol{l})\boldsymbol{l}]^2\}\mathrm{d}m\\
&= \int_M \rho\{\mathbf{r}^2 - (\mathbf{r}\circ \boldsymbol{l})^2\}\mathrm{d}m\\
&= \int_M \rho\left\{\left(x_{1n}^2 + x_{2n}^2 + x_{3n}^2\right) - (a_{11}x_{1n} + a_{21}x_{2n} + a_{31}x_{3n})^2\right\}\mathrm{d}m\\
&= \int_M \rho\Big[(1-a_{11}^2)x_{1n}^2 + (1-a_{21}^2)x_{2n}^2 + (1-a_{31}^2)x_{3n}^2\\
&\qquad -2a_{11}a_{21}x_{1n}x_{2n} - 2a_{11}a_{31}x_{1n}x_{3n} - 2a_{21}a_{31}x_{2n}x_{3n}\Big]\mathrm{d}m\\
&= \int_M \rho(x_1,x_2,x_3)\Big[(a_{21}^2 + a_{31}^2)x_{1n}^2 + (a_{11}^2 + a_{31}^2)x_{2n}^2 + (a_{11}^2 + a_{21}^2)x_{3n}^2\\
&\qquad -2a_{11}a_{21}x_{1n}x_{2n} - 2a_{11}a_{31}x_{1n}x_{3n} - 2a_{21}a_{31}x_{2n}x_{3n}\Big]\mathrm{d}m\\
&= \int_M \rho(x_1,x_2,x_3)\Big[a_{11}^2(x_{2n}^2 + x_{3n}^2) + a_{21}^2(x_{1n}^2 + x_{3n}^2) + a_{31}^2(x_{1n}^2 + x_{2n}^2)\\
&\qquad -2a_{11}a_{21}x_{1n}x_{2n} - 2a_{11}a_{31}x_{1n}x_{3n} - 2a_{21}a_{31}x_{2n}x_{3n}\Big]\mathrm{d}m\\
&= a_{11}^2\int_M \rho(x_{2n}^2 + x_{3n}^2)\mathrm{d}m + a_{21}^2\int_M \rho(x_{1n}^2 + x_{3n}^2)\mathrm{d}m
\end{aligned}
$$

Fig. 3.24 Ellipsoid of inertia



$$+a_{31}^2\int\limits_M \rho(x_{1n}^2+x_{2n}^2)\mathrm{d}m-2a_{11}a_{21}\int\limits_M \rho x_{1n}x_{2n}\mathrm{d}m$$

$$-2a_{11}a_{31}\int\limits_M \rho x_{1n}x_{3n}\mathrm{d}m-2a_{21}a_{31}\int\limits_M \rho x_{2n}x_{3n}\mathrm{d}m$$

$$=a_{11}^2I_{X_1}+a_{21}^2I_{X_2}+a_{31}^2I_{X_3}-2a_{11}a_{21}I_{X_1X_2}-2a_{11}a_{31}I_{X_1X_3}$$
$$-2a_{21}a_{31}I_{X_2X_3}=a_{11}^2I_{11}+a_{21}^2I_{22}+a_{31}^2I_{33}+2a_{11}a_{21}I_{12}$$
$$+2a_{11}a_{31}I_{13}+2a_{21}a_{31}I_{23}, \tag{3.94}$$

where (3.93) was used during transformations.

According to (3.90) the change in position of axis l is accompanied by a change in $\cos\alpha_1, \cos\alpha_2$, and $\cos\alpha_3$, that is, the direction cosines of that axis in the adopted stationary coordinate system. On each line l from a pencil of straight lines at point O we will step off the segments OO_1 and OO_2, which are equal to

$$OO_1=OO_2=\frac{\mu}{d}=\mu\sqrt{\frac{M}{I_l}}, \tag{3.95}$$

where d is the notion of radius of gyration of the body introduced by (3.6) and μ is a certain coefficient.

Points described by (3.95) will lie on the ellipsoid presented in Fig. 3.24. Let us assume, as earlier, that line l forms with the axes of the stationary coordinate system the angles α_1, α_2, and α_3. Therefore the coordinates of the points lying on the surface of the ellipsoid are equal to

$$x_i=\pm\mu\sqrt{\frac{M}{I_l}}\cos\alpha_i, \quad i=1,2,3. \tag{3.96}$$

According to (3.91) we have

$$\begin{aligned} I_l &= I_{X_1}\cos^2\alpha_1 + I_{X_2}\cos^2\alpha_2 \\ &\quad + I_{X_3}\cos^2\alpha_3 - 2I_{X_1X_2}\cos\alpha_1\cos\alpha_2 \\ &\quad - 2I_{X_1}X_3\cos\alpha_1\cos\alpha_3 - 2I_{X_2X_3}\cos\alpha_2\cos\alpha_3, \end{aligned} \tag{3.97}$$

and after substitution of (3.96) we obtain

$$\begin{aligned} I_l &= \Big(I_{X_1}x_1^2 + I_{X_2}x_2^2 + I_{X_3}x_3^2 - 2I_{X_1X_2}x_1x_2 \\ &\quad - 2I_{X_1X_3}x_1x_3 - 2I_{X_2X_3}x_2x_3 \Big) \frac{I_l}{M\mu^2}, \end{aligned}$$

that is,

$$\begin{aligned} & I_{X_1}x_1^2 + I_{X_2}x_2^2 + I_{X_3}x_3^2 - 2I_{X_1X_2}x_1x_2 \\ & \quad - 2I_{X_1X_3}x_1x_3 - 2I_{X_2X_3}x_2x_3 = M\mu^2. \end{aligned} \tag{3.98}$$

Rescaling the coordinates $x_i \mapsto \mu\sqrt{M}x_i$ we obtain

$$\begin{aligned} & I_{X_1}x_1^2 + I_{X_2}x_2^2 + I_{X_3}x_3^2 - 2I_{X_1X_2}x_1x_2 \\ & \quad - 2I_{X_1X_3}x_1x_3 - 2I_{X_2X_3}x_2x_3 = 1. \end{aligned} \tag{3.99}$$

The obtained quadric surface is a locus of the endpoints of all segments of length inversely proportional to the radius of gyration of the body with respect to the axes passing through the chosen point of the rigid body O; it will be called an *ellipsoid of inertia of the body* at point O.

Let us note that for every point of the body we have a different ellipsoid of inertia. Moreover, for a given point of the body the number of ellipsoids depends on the choice of the parameter μ (there can be infinitely many of them).

If the chosen point is the mass center of a rigid body, such an ellipsoid will be called the *centroidal ellipsoid of inertia* (or the *ellipsoid of inertia*) of the body.

The obtained ellipsoid possesses the semiaxes a_1, a_2 and a_3 and can be represented in the coordinates corresponding to its axes in the following form:

$$A_1 x_1'^2 + A_2 x_2'^2 + A_3 x_3'^2 = k^2, \tag{3.100}$$

where $A_j = I_{X_j'}$, $j = 1, 2, 3$, $k^2 = M\mu^2$, and the products of inertia in this coordinate system are equal to zero. We will call the axes OX_j', $j = 1, 2, 3$ the *principal axes of inertia*, and if $O = C$, where C is the mass center, then we call these axes *principal centroidal axes of inertia*.

On the other hand, the principal axes of ellipsoid (3.100) coincide with the axes of the ellipsoid of inertia. Because according to (3.95) the radius of the ellipsoid

OO_1 is inversely proportional to the square root of I_l ($OO_1 \sim \frac{1}{\sqrt{I_l}}$), the shortest (longest) axis of the ellipsoid is the principal axis about which the moment of inertia reaches its maximum (minimum).

In Fig. 3.24 the point lying on the surface of the ellipsoid and the normal vector **N**, that is, the vector perpendicular to the surface at the point $(x_{1O_1}, x_{2O_1}, x_{3O_1})$, are marked. Recall that this vector **l** can be described by the following equation:

$$\mathbf{N} = \nabla f(x_1, x_2, x_3) = \frac{\partial f}{\partial x_1}\mathbf{E}_1 + \frac{\partial f}{\partial x_2}\mathbf{E}_2 + \frac{\partial f}{\partial x_3}\mathbf{E}_3, \tag{3.101}$$

and $f(x_1, x_2, x_3) = M\mu^2 \equiv$ const describes the left-hand side of (3.98).

Next we calculate

$$\begin{aligned} N_{x_1} &\equiv \frac{\partial f}{\partial x_1} = 2I_{X_1}x_1 - 2I_{X_1X_2}x_2 - 2I_{X_1X_3}x_3, \\ N_{x_2} &\equiv \frac{\partial f}{\partial x_2} = -2I_{X_1X_2}x_1 + 2I_{X_2}x_2 - 2I_{X_2X_3}x_3, \\ N_{x_3} &\equiv \frac{\partial f}{\partial x_3} = -2I_{X_1X_3}x_1 - 2I_{X_2X_3}x_2 + 2I_{X_3}x_3. \end{aligned} \tag{3.102}$$

If point O_1 lies at the end of each of three ellipsoid axes (they are perpendicular to each other), then $\mathbf{N} \parallel \overrightarrow{OO_1}$, and the components of both vectors must be proportional to one another, that is,

$$N_{x_j} = 2\sigma x_j, \quad j = 1, 2, 3, \tag{3.103}$$

where 2σ denotes the proportionality factor.

Substituting (3.103) into (3.102) we obtain

$$\begin{aligned} (I_{X_1} - \sigma)x_1 - I_{X_1X_2} - I_{X_1X_3}x_3 &= 0, \\ -I_{X_1X_2}x_1 + (I_{X_2} - \sigma)x_2 - I_{X_2X_3}x_3 &= 0, \\ -I_{X_1X_3}x_1 - I_{X_2X_3}x_2 + (I_{X_3} - \sigma)x_3 &= 0. \end{aligned} \tag{3.104}$$

The unknown point (x_1, x_2, x_3) does not coincide with the origin of the coordinate system, that is, it is not the point $x_1 = x_2 = x_3 = 0$, if

$$\begin{vmatrix} \sigma - I_{X_1} & I_{X_1X_2} & I_{X_1X_3} \\ I_{X_1X_2} & \sigma - I_{X_2} & I_{X_2X_3} \\ I_{X_1X_3} & I_{X_2X_3} & \sigma - I_{X_3} \end{vmatrix} = 0. \tag{3.105}$$

Expanding the determinant above we have

$$\begin{aligned}
&(\sigma - I_{X_1})[(\sigma - I_{X_2})(\sigma - I_{X_3}) - I^2_{X_2X_3}] - I_{X_1X_2}[I_{X_1X_2}(\sigma - I_{X_3}) \\
&\quad - I_{X_2X_3}I_{X_1X_3}] + I_{X_1X_3}[I_{X_1X_2}I_{X_2X_3} - I_{X_1X_3}(\sigma - I_{X_2})] \\
&= (\sigma - I_{X_1}[\sigma^2 - \sigma(I_{X_2} + I_{X_3}) + I_{X_2X_3} - I^2_{X_2X_3}] \\
&\quad - I^2_{X_1X_2}(\sigma - I_{X_3}) + I_{X_1X_2}I_{X_2X_3}I_{X_1X_3} + I_{X_1X_3}I_{X_1X_2}I_{X_2X_3} \\
&\quad - I^2_{X_1X_3}(\sigma - I_{X_2}) = \sigma^3 - \sigma^2(I_{X_2} + I_{X_3}) - I_{X_1}\sigma^2 + I_{X_1}\sigma(I_{X_2} + I_{X_3}) \\
&\quad +(\sigma - I_{X_1})(I_{X_2}I_{X_3} - I^2_{X_2X_3}) - I^2_{X_1X_2}\sigma + I_{X_3}I^2_{X_1X_2} \\
&\quad +2I_{X_1X_2}I_{X_2X_3}I_{X_1X_3} - I^2_{X_1X_3}\sigma + I_{X_2}I^2_{X_1X_3} = \sigma^3 \\
&\quad -(I_{X_1} + I_{X_2} + I_{X_3})\sigma^2 + \sigma(I_{X_1}I_{X_2} + I_{X_1}I_{X_3} + I_{X_2}I_{X_3} \\
&\quad -I^2_{X_1X_2} - I^2_{X_1X_3} - I^2_{X_2X_3}) - I_{X_1}I_{X_2}I_{X_3} \\
&\quad +2I_{X_1X_2}I_{X_2X_3}I_{X_1X_3} + I_{X_3}I^2_{X_1X_2} + I_{X_2}I^2_{X_1X_3} + I_{X_1}I^2_{X_2X_3} \\
&= \sigma^3 - s_I\sigma^2 + s_{II}\sigma - s_{III} = 0,
\end{aligned} \tag{3.106}$$

where we call

$$\begin{aligned}
s_I &= I_{X_1} + I_{X_2} + I_{X_3}, \\
s_{II} &= I_{X_1}I_{X_2} + I_{X_1}I_{X_3} + I_{X_2}I_{X_3} - I^2_{X_1X_2} - I^2_{X_1X_3} - I^2_{X_2X_3}, \\
s_{III} &= I_{X_1}I_{X_2}I_{X_3} - 2I_{X_1X_2}I_{X_2X_3}I_{X_1X_3} \\
&\quad - I_{X_3}I^2_{X_1X_2} - I_{X_2}I^2_{X_1X_3} - I_{X_1}I^2_{X_2X_3}
\end{aligned} \tag{3.107}$$

the *invariants of the inertia tensor* at point O.

It is possible to demonstrate that the obtained characteristic equation of third order always possesses three real roots, although they are not always distinct.

Each of the roots $\sigma_n,\ n = 1, 2, 3$ has a corresponding vector of eigenvalues obtained from the equations in the matrix notation [see (3.104)]

$$\begin{bmatrix} I_{X_1} - \sigma_n & -I_{X_1X_2} & -I_{X_1X_3} \\ -I_{X_1X_2} & I_{X_2} - \sigma_n & -I_{X_2X_3} \\ -I_{X_1X_3} & -I_{X_2X_3} & I_{X_3} - \sigma_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \tag{3.108}$$

and the eigenvalues of a real symmetric matrix are real.

In the expanded form according to (3.104) and taking into account (3.96) and (3.92) we obtain

$$\begin{aligned}(I_{X_1}-\sigma_n)a_{11}-I_{X_1X_2}a_{21}-I_{X_1X_3}a_{31} &= 0,\\ -I_{X_1X_2}a_{11}+(I_{X_2}-\sigma_n)a_{21}-I_{X_2X_3}a_{31} &= 0,\\ -I_{X_1X_3}a_{11}-I_{X_2X_3}a_{21}+(I_{X_3}-\sigma_n)a_{31} &= 0.\end{aligned} \tag{3.109}$$

Let us multiply these equations in turn by a_{11}, a_{21}, a_{31}, and adding them we obtain

$$\begin{aligned}&(I_{X_1}-\sigma_n)a_{11}^2+(I_{X_2}-\sigma_n)a_{21}^2+(I_{X_3}-\sigma_n)a_{31}^2\\ &\quad -2a_{11}a_{21}I_{X_1X_2}-2a_{11}a_{31}I_{X_1X_3}-2a_{21}a_{31}I_{X_2X_3}=0,\end{aligned} \tag{3.110}$$

and using (3.93) we have

$$\begin{aligned}\sigma_n &= I_{X_1}a_{11}^2+I_{X_2}a_{21}^2+I_{X_3}a_{31}^2\\ &\quad -2a_{11}a_{21}I_{X_1X_2}-2a_{11}a_{31}I_{X_1X_3}-2a_{21}a_{31}I_{X_2X_3}.\end{aligned} \tag{3.111}$$

From a comparison of (3.91) and (3.111) it can be seen that $\sigma_n = I_l$. Thus, we have demonstrated that the roots of the characteristic equation are equal to the moments of inertia of the principal axes of inertia passing through the point $O_1(x_1, x_2, x_3)$, that is, of direction cosines a_{11}, a_{21}, and a_{31} (we have three such axes).

This line of reasoning justifies the validity of (3.66) already applied earlier but presented without proof.

Let us assume that axis l chosen in such a way will be coincident after rotation with one of the axes a_i of the ellipsoid (Fig. 3.24). If that is the case, the products of inertia with respect to that axis will be equal to zero. Let us take the coordinate system $OX_1'X_2'X_3'$ such that the axes of the ellipsoid a_1, a_2, and a_3 coincide with the axes of that system. The equation of ellipsoid in such a coordinate system [see (3.100)] takes the form

$$I_{X_1'}{x_1'}^2+I_{X_2'}{x_2'}^2+I_{X_3'}{x_3'}^2=k^2. \tag{3.112}$$

Three such axes are called the *principal axes of inertia of a body* at point O (at the mass center of the body).

If the principal axes of inertia and the corresponding moments of inertia A_i are known, it is very easy to determine the moment of inertia with respect to axis l, which is equal to [see (3.97)]

$$I_l = A_1\cos^2\alpha_1 + A_2\cos^2\alpha_2 + A_3\cos^2\alpha_3, \tag{3.113}$$

where the angles $\alpha_1, \alpha_2, \alpha_3$ define the position of axis l with respect to the principal centroidal axes of inertia of the body.

If the principal moments of inertia are equal (triple root of the characteristic equation), then

$$I_l = A(\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha^3) = A, \tag{3.114}$$

because $A_1 = A_2 = A_3 = A$.

This means that all lines passing through the mass center are principal axes and the ellipsoid becomes a ball. The body whose ellipsoid of inertia is a ball we call a *ball-type body* (the shape of such a body can be a ball or cube).

In cases where two roots of the characteristic equation are equal (double root), from (3.113) we obtain ($A_1 = A_2 = A \neq A_3$)

$$I_l = A(1 - \cos^2\alpha_3) + A_3\cos^2\alpha_3. \tag{3.115}$$

Having defined the position of the axis by the angle α_3 and the values of moments of inertia A_3 and A, we determine the moment of inertia I_l. An ellipsoid defined in that way we call an *ellipsoid of revolution* because an arbitrary axis perpendicular to the one defined by the angle α_3 is the principal axis.

If we take the equation of ellipsoid of the form (3.100) of arbitrary coefficients $I_{X_1'}$, $I_{X_2'}$, and $I_{X_3'}$, it does not have to represent the ellipsoid of inertia. We are dealing with an ellipsoid of inertia if the mentioned coefficients satisfy the triangle inequality of the form

$$I_{X_1'} + I_{X_2'} \geq I_{X_3'}, \quad I_{X_1'} + I_{X_3'} \geq I_{X_2'}, \quad I_{X_2'} + I_{X_3'} \geq I_{X_1'}. \tag{3.116}$$

Let us note that

$$\begin{aligned} I_{X_1'} + I_{X_2'} &= \int_M \left(X_{2'}^2 + x_{3'}^2 + x_{1'}^2 + x_{3'}^2\right) \mathrm{d}m \\ &= 2\int_M x_{3'}^2 \mathrm{d}m + I_{X_3}' \geq I_{X_3}'. \end{aligned} \tag{3.117}$$

In a similar way we can prove the other inequalities (3.116).

3.6 Properties of Principal and Principal Centroidal Axes of Inertia

Let us assume that axis X_3 is a principal centroidal axis of inertia at point O, and the remaining axes of the Cartesian coordinate system $OX_1X_2X_3$ are arbitrary. Each point $O_1(x_1, x_2, x_3)$ has a counterpart $O_2(-x_1, -x_2, x_3)$ symmetrical with respect to the axis X_3. Inserting the coordinates of points O_1 and O_2 into (3.99) we obtain

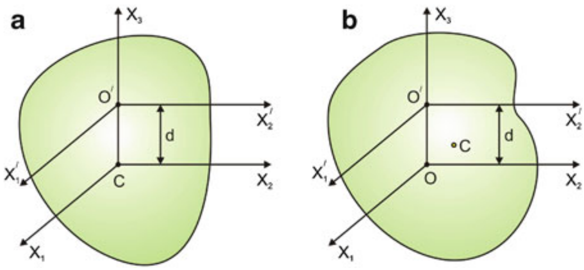


Fig. 3.25 Principal centroidal axis of inertia CX_3 (**a**) and principal axis of inertia OX_3 (**b**)

$$
\begin{aligned}
&I_{X_1}x_1^2 + I_{X_2}x_2^2 + I_{X_3}x_3^2 - 2I_{X_1X_2}x_1x_2 \\
&\qquad - 2I_{X_1X_3}x_1x_3 - 2I_{X_2X_3}x_2x_3 = 1, \\
&I_{X_1}x_1^2 + I_{X_2}x_2^2 + I_{X_3}x_3^2 - 2I_{X_1X_2}x_1x_2 \\
&\qquad + 2I_{X_1X_3}x_1x_3 + 2I_{X_2X_3}x_2x_3 = 1.
\end{aligned}
\tag{3.118}
$$

Combining those equations we obtain

$$
x_3(I_{X_2X_3}x_2 - I_{X_1X_3}x_1) = 0, \tag{3.119}
$$

and because $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$, from (3.119) it follows that

$$
I_{X_2X_3} = 0, \qquad I_{X_1X_3} = 0. \tag{3.120}
$$

If any of the axes of the Cartesian coordinate system of origin at a given point is the principal axis of inertia at that point, then the products of inertia in which there appear coordinates of this axis are equal to zero.

In Fig. 3.25a an axis CX_3 is shown; it is the principal centroidal axis of inertia of a body. In Fig. 3.25b an axis OX_3 is presented, and this axis is the principal axis of inertia of the body at point O.

Both axes are principal axes of inertia, that is,

$$
I_{X_2X_3} = \sum_{n=1}^{N} m_n x_{2n} x_{3n} = 0, \qquad I_{X_1X_3} = \sum_{n=1}^{N} m_n x_{1n} x_{3n} = 0. \tag{3.121}
$$

On both axes let us take points $O'(CO' = d = OO')$ and draw through them the axes $O'X_1' \parallel OX_1$, $O'X_2' \parallel OX_2$, $O'X_1' \parallel CX_1$, and $O'X_2' \parallel CX_2$. Let us calculate the following products of inertia:

$$
I_{X_2'X_3} = \sum_{n=1}^{N} m_n x_{2n}(x_{3n} - d)
$$

$$= \sum_{n=1}^{N} m_n x_{2n} x_{3n} - \mathrm{d} \sum_{n=1}^{N} m_n x_{2n} = -\mathrm{d} m x_{2C},$$

$$I_{X_1' X_3} = -\mathrm{d} m x_{1C}. \tag{3.122}$$

In the case of the principal centroidal axis of inertia $x_{1C} = x_{2C} = 0$ and in view of that, $I_{X_2' X_3} = I_{X_1' X_3} = 0$. The conclusion follows that the axis CX_3 is the principal axis of inertia not only at point C but also at point O'. Because O' has been arbitrarily chosen, the principal centroidal axis of inertia is the principal axis of inertia for all its points.

If an axis is the principal axis and it does not pass through the mass center of the body (Fig. 3.25b), then $x_{1C} \neq 0$ and $x_{2C} \neq 0$, that is, $I_{X_2' X_3} \neq 0$ and $I_{X_1' X_3} \neq 0$, which means that the axis OX_3 is not the principal axis of inertia at the point O'.

From that conclusion it can be deduced that if the principal axis of inertia does not pass through the mass center of a rigid body, then it is the principal axis at only one of its points.

The reader is advised to prove the following two observations:

1. If a homogeneous rigid body has an axis of symmetry, then that axis is its principal centroidal axis of inertia.
2. If a homogeneous rigid body has a plane of symmetry, then at all points of this plane one of principal axes of inertia is perpendicular to that plane.

3.7 Determination of Moments of Inertia of a Rigid Body

3.7.1 Determination of Moments of Inertia of a Body with Respect to an Arbitrary Axis

A moment like that can be determined easily if the directions of the principal centroidal axes of inertia and the moments of inertia of a body about those axes are known.

Let an axis l pass through the mass center of the body (Fig. 3.26).

In this case we have

$$I_l = \sum_{i=1}^{3} I_i \cos^2 \alpha_i. \tag{3.123}$$

Let axis l not pass through the mass center of a body. In this case, first the moment of inertia with respect to an axis $l' \parallel l$ passing through point C (the mass center) is determined, and then, by Steiner's theorem, I_l is obtained:

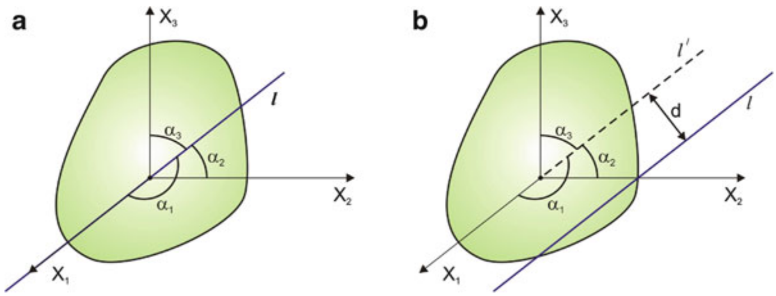


Fig. 3.26 Axis l passes (**a**) and does not pass (**b**) through the mass center of a rigid body

$$I_{l'} = \sum_{i=1}^{3} I_i \cos^2 \alpha_i \,,$$

$$I_l = I_{l'} + md^2, \tag{3.124}$$

since the distance between l and l' is denoted d.

3.7.2 Determination of Mass Moments of Inertia of a Rigid Body

The mass moments of inertia of a rigid body can be determined if the directions of the principal centroidal axes of inertia and the moments of inertia about those axes are known.

Case 1. The axes of coordinate systems $CX_1X_2X_3$ and $O'X_1'X_2'X_3'$ are mutually parallel (Fig. 3.27).

From Fig. 3.27 it follows that

$$\overrightarrow{O'A_n} = \overrightarrow{O'C} + \overrightarrow{CA_n}, \tag{3.125}$$

which leads to the determination of relations between the coordinates of point A in both coordinate systems:

$$x_{1n}' = x_{1C}' + x_{1n},$$
$$x_{2n}' = x_{2C}' + x_{2n},$$
$$x_{3n}' = x_{3C}' + x_{3n}. \tag{3.126}$$

As an example, we determine the product of inertia

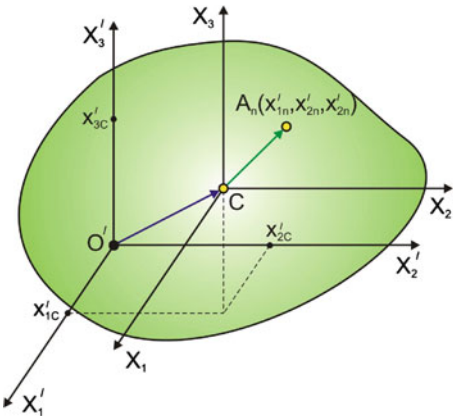


Fig. 3.27 Determination of mass moments of inertia in the system $O'X_1'X_2'X_3'$, where $O'X_i' \parallel CX_i$, $i = 1, 2, 3$

$$
\begin{aligned}
I_{X_1'X_2'} &= \sum_{n=1}^{\infty} m_n x_{1n}' x_{2n}' = \sum_{n=1}^{\infty} m_n (x_{1C}' + x_{1n})(x_{2C}' + x_{2n}) \\
&= \sum_{n=1}^{\infty} m_n x_{1C}' x_{2C}' + \sum_{n=1}^{\infty} m_n x_{1C}' x_{2n} + \sum_{n=1}^{\infty} m_n x_{1n} x_{2C}' \\
&\quad + \sum_{n=1}^{\infty} m_n x_{1n} x_{2n} = x_{1C}' x_{2C}' \sum_{n=1}^{\infty} m_n + x_{1C}' \sum_{n=1}^{\infty} m_n x_{2n} \\
&\quad + x_{2C}' \sum_{n=1}^{\infty} m_n x_{1n} + \sum_{n=1}^{\infty} m_n x_{1n} x_{2n}.
\end{aligned} \tag{3.127}
$$

We have

$$
\sum_{n=1}^{\infty} m_n = M, \qquad \sum_{n=1}^{\infty} m_n x_{1n} x_{2n} = I_{X_1X_2},
$$

$$
\sum_{n=1}^{\infty} m_n x_{1n} = 0, \qquad \sum_{n=1}^{\infty} m_n x_{2n} = 0, \tag{3.128}
$$

and the last two equations follow from the observation that axes CX_1 and CX_2 are the principal centroidal axes. Proceeding in an analogous way with the remaining axes we eventually obtain

$$
\begin{aligned}
I_{X_1'X_2'} &= I_{X_1X_2} + M x_{1C}' x_{2C}', \\
I_{X_2'X_3'} &= I_{X_2X_3} + M x_{2C}' x_{3C}', \\
I_{X_3'X_1'} &= I_{X_3X_1} + M x_{3C}' x_{1C}'.
\end{aligned} \tag{3.129}
$$

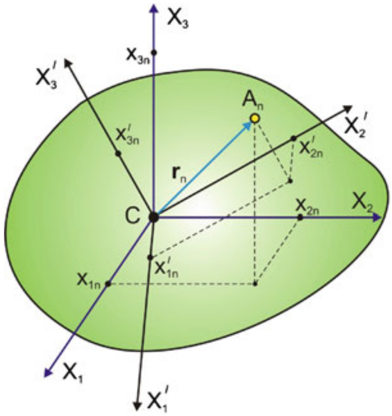


Fig. 3.28 Determination of products of inertia in case where axes of systems $CX_1'X_2'X_3'$ and $CX_1X_2X_3$ are rotated with respect to each other

Case 2. The coordinate axes of the system $CX_1'X_2'X_3'$ pass through the mass center of a rigid body and form known angles α_i, $i = 1, 2, 3$, with the system $CX_1X_2X_3$ (Fig. 3.28).

From Fig. 3.28 it follows that

$$\begin{aligned} a_{11} &= \cos(\mathbf{E}_1', \mathbf{E}_1) = \mathbf{E}_1' \circ \mathbf{E}_1, \quad a_{21} = \cos(\mathbf{E}_2', \mathbf{E}_1) = \mathbf{E}_2' \circ \mathbf{E}_1, \\ a_{31} &= \cos(\mathbf{E}_3', \mathbf{E}_1) = \mathbf{E}_3' \circ \mathbf{E}_1, \\ a_{12} &= \cos(\mathbf{E}_1', \mathbf{E}_2) = \mathbf{E}_1' \circ \mathbf{E}_2, \quad a_{22} = \cos(\mathbf{E}_2', \mathbf{E}_2) = \mathbf{E}_2' \circ \mathbf{E}_2, \\ a_{32} &= \cos(\mathbf{E}_3', \mathbf{E}_2) = \mathbf{E}_3' \circ \mathbf{E}_2, \\ a_{13} &= \cos(\mathbf{E}_1', \mathbf{E}_3) = \mathbf{E}_1' \circ \mathbf{E}_3, \quad a_{23} = \cos(\mathbf{E}_2', \mathbf{E}_3) = \mathbf{E}_2' \circ \mathbf{E}_3, \\ a_{33} &= \cos(\mathbf{E}_3', \mathbf{E}_3) = \mathbf{E}_3' \circ \mathbf{E}_3. \end{aligned} \tag{3.130}$$

Because the vector

$$\mathbf{r}_n = \mathbf{E}_1' x_{1n}' + \mathbf{E}_2' x_{2n}' + \mathbf{E}_3' x_{3n}' = \mathbf{E}_1 x_{1n} + \mathbf{E}_2 x_{2n} + \mathbf{E}_3 x_{3n}, \tag{3.131}$$

the above equations are successively multiplied by $\mathbf{E}_1'$, $\mathbf{E}_2'$, and $\mathbf{E}_3'$, and taking into account (3.130) we obtain

$$\mathbf{r}_n' = \mathbf{A}\mathbf{r}_n, \tag{3.132}$$

or, in expanded matrix form,

$$\begin{bmatrix} x_{1n}' \\ x_{2n}' \\ x_{3n}' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \end{bmatrix}, \tag{3.133}$$

which defines the relation between coordinates of a point in coordinate systems $CX_1X_2X_3$ and $CX_1'X_2'X_3'$. As an example, we determine the product of inertia

$$\begin{aligned}
I_{X_2'X_3'} &= \sum_{n=1}^{\infty} m_n x_{2n}' x_{3n}' = \sum_{n=1}^{\infty} m_n (a_{21}x_{1n} + a_{22}x_{2n} \\
&\quad + a_{23}x_{3n})(a_{31}x_{1n} + a_{32}x_{2n} + a_{33}x_{3n}) \\
&= \sum_{n=1}^{\infty} m_n (a_{21}a_{31}x_{1n}^2 + a_{21}a_{32}x_{1n}x_{2n} + a_{21}a_{33}x_{3n}x_{1n} \\
&\quad + a_{22}a_{31}x_{1n}x_{2n} + a_{22}a_{32}x_{2n}^2 + a_{22}a_{33}x_{2n}x_{3n} \\
&\quad + a_{23}a_{31}x_{3n}x_{1n} + a_{23}a_{32}x_{2n}x_{3n} + a_{23}a_{33}x_{3n}^2) \\
&= a_{21}a_{31}\sum_{n=1}^{\infty} m_n x_{1n}^2 + a_{22}a_{32}\sum_{n=1}^{\infty} m_n x_{2n}^2 \\
&\quad + a_{23}a_{33}\sum_{n=1}^{\infty} m_n x_{3n}^2 + (a_{21}a_{32} + a_{22}a_{31})\sum_{n=1}^{\infty} m_n x_{1n}x_{2n} \\
&\quad + (a_{21}a_{33} + a_{23}a_{31})\sum_{n=1}^{\infty} m_n x_{3n}x_{1n} \\
&\quad + (a_{22}a_{33} + a_{23}a_{32})\sum_{n=1}^{\infty} m_n x_{2n}x_{3n}.
\end{aligned} \tag{3.134}$$

In the above equation

$$\sum_{n=1}^{\infty} m_n x_{1n}x_{2n} = 0, \quad \sum_{n=1}^{\infty} m_n x_{3n}x_{1n} = 0, \quad \sum_{n=1}^{\infty} m_n x_{2n}x_{3n} = 0, \tag{3.135}$$

because the axes CX_1, CX_2, and CX_3 are the principal centroidal axes of inertia of a body.

From (3.133) it follows that

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}, \tag{3.136}$$

from which we determine

$$\mathbf{E}_2' = a_{21}\mathbf{E}_1 + a_{22}\mathbf{E}_2 + a_{23}\mathbf{E}_3,$$
$$\mathbf{E}_3' = a_{31}\mathbf{E}_1 + a_{32}\mathbf{E}_2 + a_{33}\mathbf{E}_3, \quad (3.137)$$

and calculate

$$\begin{aligned}\mathbf{E}_2' \circ \mathbf{E}_3' &= (a_{21}\mathbf{E}_1 + a_{22}\mathbf{E}_2 + a_{23}\mathbf{E}_3) \circ (a_{31}\mathbf{E}_1 + a_{32}\mathbf{E}_2 + a_{33}\mathbf{E}_3)\\ &= a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0, \end{aligned} \quad (3.138)$$

and hence obtain

$$a_{23}a_{33} = -(a_{21}a_{31} + a_{22}a_{32}). \quad (3.139)$$

Substituting (3.139) into (3.134) we obtain

$$\begin{aligned} I_{X_2'X_3'} &= a_{21}a_{31}\sum_{n=1}^{\infty} m_n(x_{1n}^2 - x_{3n}^2) + a_{22}a_{32}\sum_{n=1}^{\infty} m_n(x_{2n}^2 - x_{3n}^2)\\ &= a_{21}a_{31}\Big[\sum_{n=1}^{\infty} m_n(x_{1n}^2 + x_{2n}^2) - \sum_{n=1}^{\infty} m_n(x_{2n}^2 + x_{3n}^2)\Big]\\ &\quad + a_{22}a_{32}\Big[\sum_{n=1}^{\infty} m_n(x_{1n}^2 + x_{2n}^2) - \sum_{n=1}^{\infty} m_n(x_{3n}^2 + x_{1n}^2)\Big]. \end{aligned} \quad (3.140)$$

Proceeding in a similar way with the remaining products of inertia we eventually obtain

$$\begin{aligned} I_{X_2'X_3'} &= a_{21}a_{31}(I_{X_3} - I_{X_1}) + a_{22}a_{32}(I_{X_3} - I_{X_2}),\\ I_{X_3'X_1'} &= a_{11}a_{31}(I_{X_3} - I_{X_1}) + a_{12}a_{32}(I_{X_3} - I_{X_2}),\\ I_{X_1'X_2'} &= a_{11}a_{21}(I_{X_3} - I_{X_1}) + a_{12}a_{22}(I_{X_3} - I_{X_2}). \end{aligned} \quad (3.141)$$

Case 3. The axes of the coordinate system $OX_1''X_2''X_3''$ pass through an arbitrary point O of a rigid body (Fig. 3.29).

In order to determine the products of inertia $I_{X_2''X_3''}$, $I_{X_3''X_1''}$, and $I_{X_1''X_2''}$ we introduce at point C the coordinate system $OX_1'X_2'X_3'$ of axes mutually parallel to the axes of the system $OX_1''X_2''X_3''$. Knowing the direction cosines of angles between the axes of systems $CX_1'X_2'X_3'$ and $CX_1X_2X_3$, we can make use of (3.141) to determine $I_{X_2'X_3'}$, $I_{X_3'X_1'}$, and $I_{X_1'X_2'}$. Knowing the position of the mass center C in the coordinate system $CX_1''X_2''X_3''$, according to second Steiner's theorem, we obtain

$$I_{X_2''X_3''} = I_{X_2'X_3'} + Mx_{2C}''x_{3C}''$$

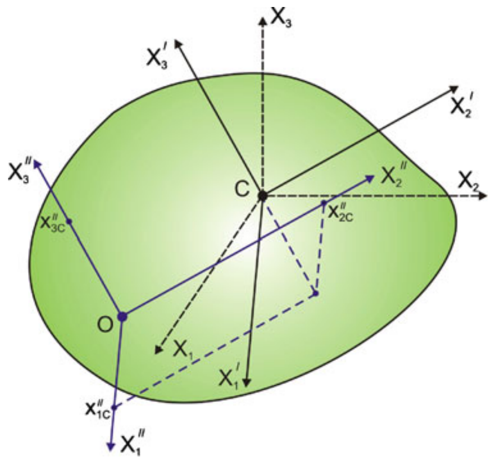


Fig. 3.29 An arbitrary position of the coordinate system $OX_1''X_2''X_3''$

$$
\begin{aligned}
&= Mx_{2C}''x_{3C}'' + a_{21}a_{31}(I_{X_3} - I_{X_1}) + a_{22}a_{32}(I_{X_3} - I_{X_2}),\\
I_{X_3''X_1''} &= I_{X_3'X_1'} + Mx_{3C}''x_{1C}''\\
&= Mx_{3C}''x_{1C}'' + a_{11}a_{31}(I_{X_3} - I_{X_1}) + a_{12}a_{32}(I_{X_3} - I_{X_2}),\\
I_{X_1''X_2''} &= I_{X_1'X_2'} + Mx_{1C}''x_{2C}''\\
&= Mx_{1C}''x_{2C}'' + a_{11}a_{21}(I_{X_3} - I_{X_1}) + a_{12}a_{22}(I_{X_3} - I_{X_2}).
\end{aligned}
\tag{3.142}
$$

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Chapter 4
Particle Kinematics and an Introduction to the Kinematics of Rigid Bodies

4.1 Particle Motion on a Plane

First of all, the word *kinematics* comes from the Greek word for *motion*. As was already mentioned, kinematics is the branch of mechanics that deals with the analysis of motion of particles and rigid bodies in space, but from a geometric point of view, that is, neglecting the forces and torques that produce the motion.

Because the motion of a particle or a mechanical system takes place in time and space, in classical mechanics (as distinct from relativity theory), the notions of absolute time and absolute space are introduced.

In mechanics the notion of *absolute time* indicates a constantly changing quantity whose value increases from the past to the future. It is assumed that this quantity is identical at all points of matter. It is homogeneous and does not depend on the motion of matter.

In turn, *absolute space* denotes a three-dimensional, homogeneous, and isotropic Euclidean space. Results obtained in relativistic physics show that the assumption of such a model of space is justified for relatively small regions of physical space (of the Universe).

The concept of motion has a relative character. We say that one rigid body moves with respect to another if distances between certain points of the bodies change.

In order to investigate the geometrical properties of motion, in kinematics one introduces a certain rigid body that is fixed. The motion of other bodies with respect to that body is called *absolute motion*. One introduces the *system of absolute coordinates* rigidly connected to the fixed body, and the motion of some other body is described with respect to those coordinates.

During analysis or synthesis of motion on Earth or in its vicinity, as the system of *absolute coordinates* one takes the system rigidly connected to Earth. The body is in a state of relative rest if it does not move with respect to the chosen coordinate system.

Although in calculations in Euclidean three-dimensional space time is taken into account and has an approximate character, the calculations are sufficiently close to

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the real kinematic states of bodies and particles, provided that the considered speeds are significantly smaller than the speed of light.

Among the pioneers of kinematics one should count Euler. Galileo was the first to introduce the notion of acceleration, which was then extended to notions of tangential and normal accelerations by Huygens.[1]

A breakthrough development of kinematics took place in the nineteenth century due to the development of machines and mechanisms, which led to the emergence of a branch of science called the *kinematics of machines and mechanisms*.

In particle kinematics only three units are used: 1 m, 1 s, and 1 rad, and all the characteristics of motion, i.e., displacement, speed, and acceleration, are described with the aid of these three units only.

In kinematics, the time t can be considered in the interval $-\infty < t < \infty$, since while "being" in a particular moment of time it is possible to go back to the past by means of a "return through time."

In kinematics it is assumed that the notion of a particle is the same as the notion of a geometric point.

A set of successive positions of a particle in Euclidean space is called the *trajectory of motion* of that point. When the trajectory of motion of a particle in the time interval $t_k < t < t_{k+1}$ is a straight line, then the motion of this particle is called *rectilinear motion*. Otherwise it is called *curvilinear motion*. If the trajectory of motion is closed, i.e., the motion is repetitive, then the motion can proceed, e.g., along a rectangle, an ellipse, or a circle.

Kinematics deals with two basic problems:

1. Prescription of motion with properties defined in advance.
2. Determination of displacements, velocities, and accelerations of the particles of mechanical systems.

The current chapter was elaborated on the basis of the classical works [1–4] and Polish/Russian monographs [5–11].

4.1.1 Motion of a Particle and Trajectory (Path) of Motion

As was already mentioned, the trajectory of motion of a particle can be a straight or curved line. Let us assume that we have some trajectory given in advance along which moves a particle A (Fig. 4.1).

---

[1]Christian Huygens (1629–1695), Dutch mathematician, physicist and astronomer; he worked on the development of differential–integral calculus, formulated a theory of dice games, and was the first to calculate the speed of light.

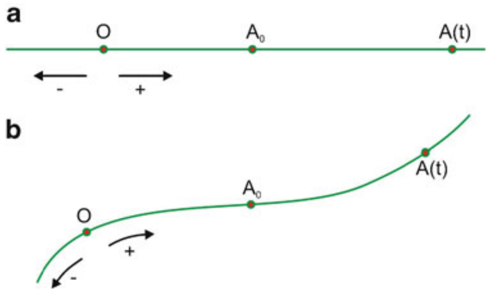


Fig. 4.1 Motion of a particle A on a *straight line* (**a**) and on a *curve* (**b**)

Thus, in the general case, the motion of a particle on a trajectory takes place along the curve $A_0A = s$, where $A_0 = A(0)$, $A = A(t)$, that is

$$s = f(t), \tag{4.1}$$

and the foregoing equation shows the change in point position along the path.

The equation of motion for a particle is defined if its trajectory of motion, origin O of motion, and direction and the function $s = s(t)$ are known.

The adopted arc coordinate s should not be confused with the distance traveled by the particle because we deal with the latter only when the motion of particle A begins at point O and proceeds to the right, that is, in the positive direction.

The distance traveled by a particle in the time interval $[t_0, t]$ is equal to

$$|A_0A| = |OA - OA_0| = |s - s_0|. \tag{4.2}$$

The total differential of an arc coordinate corresponding to time interval dt reads

$$\mathrm{d}s = f'(t)\mathrm{d}t, \tag{4.3}$$

where d$s > 0$ (d$s < 0$) if the motion takes place to the right (to the left) with respect to point O.

An elementary increment of the distance is equal to

$$|\mathrm{d}s| = |f'(t)|\mathrm{d}t, \tag{4.4}$$

and the distance traveled in the time interval $[0, t]$ reads

$$|s| = \int_0^t |f'(t)|\mathrm{d}t, \tag{4.5}$$

and quantities s and $|s|$ are expressed in meters, assuming that $s(t)$ is a monotonic function.

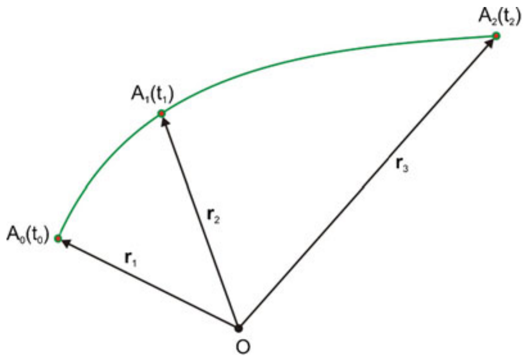


Fig. 4.2 Tracking of particle A's motion with the aid of a position vector $\mathbf{r} = \mathbf{r}(t)$

The presented method of describing particle motion is called a *natural way*.

Let us consider another way to describe the motion of a particle A in Euclidean space (Fig. 4.2).

Let us take one fixed point O in this space, called the center. The equation of motion of particle A is given by the vector function

$$\mathbf{r} = \mathbf{r}(t). \tag{4.6}$$

The trajectory of motion of particle A is a set of tips of the radius vector $\mathbf{r}(t)$ in his consecutive positions.

In general, a curve formed by such a set of tips of a vector attached at a fixed point is called a *hodograph of a vector*.

The particle trajectory shown in Fig. 4.2 is the hodograph of the radius vector of that particle. The above method of describing a particle's motion is called a *vectorial way*.

The position of particle A in time can also be observed after some system of rectilinear or curvilinear coordinates is chosen. In Fig. 4.3 the motion of a particle A is shown in the Cartesian coordinate system.

The motion of this particle is described by three equations:

$$x_i(t) = f_i(t), \qquad i = 1, 2, 3. \tag{4.7}$$

Equation (4.7) can be treated as parametric equations of the trajectory of a particle's motion. From the first equation of (4.7) we can determine the time:

$$t = t(x_1), \tag{4.8}$$

and inserting (4.8) into the two remaining (4.7) we obtain

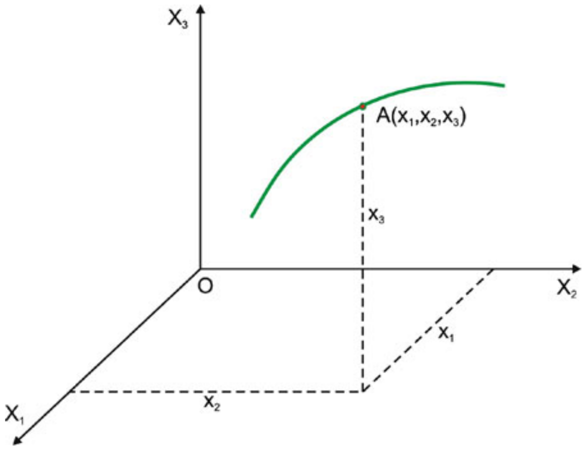


Fig. 4.3 Motion of a particle A in Cartesian coordinate system

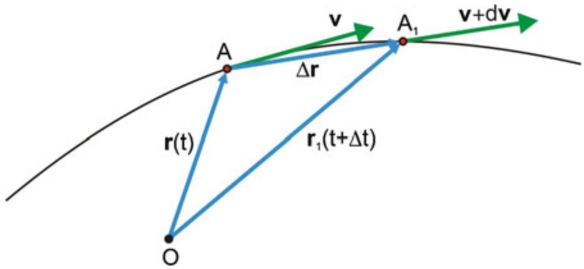


Fig. 4.4 Average and instantaneous velocity of a particle A

$$x_2 = f_2(x_1),$$
$$x_3 = f_3(x_1), \tag{4.9}$$

which corresponds to the elimination of time, and the two equations of (4.9) describe a curve in three-dimensional space also called the *path of a particle*.

4.1.2 Velocity of a Particle

The vector quantity characterizing the direction of particle motion and rate of change in its position in the adopted way of measurement and observation is called the *velocity of a particle*.

In we wish to track the motion of particle A using the radius vector $\mathbf{r} = \mathbf{r}(t)$, we make use of the vector equation obtained on the basis of Fig. 4.4.

On a trajectory let us choose two positions of particle A described by the radii vectors $\mathbf{r}(t)$ and $\mathbf{r}_1(t + \Delta t)$. From ΔOAA_1 it follows that

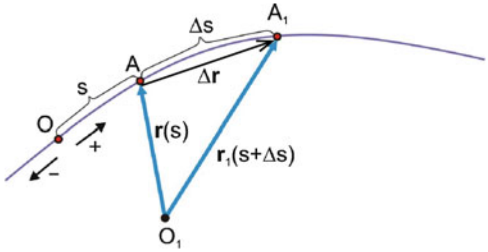


Fig. 4.5 Velocity of a particle in natural coordinates

$$\mathbf{r}_1(t + \Delta t) = \mathbf{r}(t) + \Delta\mathbf{r}, \tag{4.10}$$

and dividing by Δt we obtain

$$\mathbf{v}_{av} = \frac{\Delta\mathbf{r}}{\Delta t}, \tag{4.11}$$

and the direction and sense of average velocity $\mathbf{v}_{av}$ of particle A coincide with the vector $\Delta\mathbf{r}$.

If we take $\Delta t \to 0$ in (4.11), we obtain

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \equiv \mathbf{v}. \tag{4.12}$$

The velocity vector of particle A is tangent to its trajectory of motion, and its sense is determined by the particle motion, that is, by positions in successive time instants.

In everyday speech, instead of the term velocity vector (acceleration vector), the notion of velocity (acceleration) is used.

We will now describe the velocity vector of particle A using natural coordinates (Fig. 4.5). The trajectory of particle motion, origin O, and equation of motion for the particle $s = f(t)$ are known. The positions of points $A^{(s)}$ and $A_1^{(s+\Delta s)}$ correspond to time instants t and $t + \Delta t$, and we have $\Delta s = \widehat{AA_1}$. We choose the arbitrary center O_1 and introduce two radius vectors describing the positions of points A and A_1. We successively have

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \boldsymbol{\tau}\frac{\mathrm{d}s}{\mathrm{d}t}, \tag{4.13}$$

where $\boldsymbol{\tau}$ is a unit vector of velocity, that is, it is tangent to the trajectory at point A and its sense is determined by increasing the arc coordinate.

On the other hand,

$$\tau = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \lim_{\Delta s \to 0} \frac{\Delta\mathbf{r}}{\mathrm{d}s}, \tag{4.14}$$

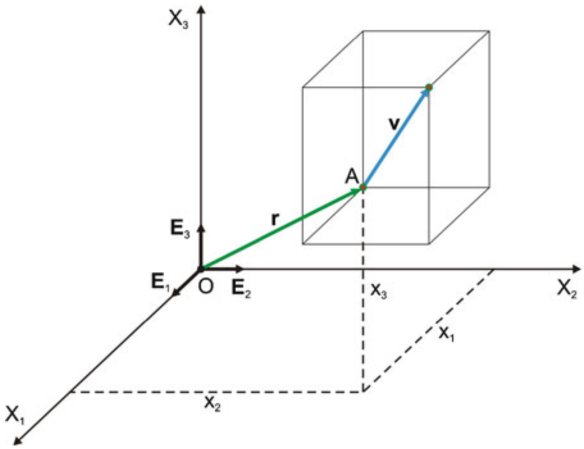


Fig. 4.6 Velocity of particle *A* in Cartesian coordinates

and its magnitude is equal to

$$\lim_{\Delta s \to 0} \left| \frac{\Delta \mathbf{r}}{\Delta s} \right| = \lim_{A_1 \to A} \left(\frac{AA_1}{\widehat{AA_1}} \right) = 1. \tag{4.15}$$

In turn, the second factor in (4.13) $\frac{\mathrm{d}s}{\mathrm{d}t}$ is an algebraic quantity of velocity, that is, it describes the projection of the velocity vector $\mathbf{v}$ onto the tangent line, that is,

$$v = \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right|, \tag{4.16}$$

which means that the magnitude of the velocity is equal to the absolute value of the derivative of the particle's arc coordinate with respect to time.

If $\frac{\mathrm{d}s}{\mathrm{d}t} > 0$ ($\frac{\mathrm{d}s}{\mathrm{d}t} < 0$), then the function $f(s)$ increases (decreases) and the sense of velocity $\mathbf{v}$ is in agreement with (opposite to) the sense of the unit vector $\boldsymbol{\tau}$. If the direction of motion changes, then it means that $\frac{\mathrm{d}s}{\mathrm{d}t}(t_0) = 0$ at a certain time instant.

Eventually, we will describe the velocity vector of particle A in the system of Cartesian coordinates (Fig. 4.6).

According to Fig. 4.6 we have

$$\mathbf{r} = \mathbf{E}_1 x_1 + \mathbf{E}_2 x_2 + \mathbf{E}_3 x_3, \tag{4.17}$$

and differentiation with respect to time

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{E}_1 \dot{x}_1 + \mathbf{E}_2 \dot{x}_2 + \mathbf{E}_3 \dot{x}_3 = \mathbf{v}_{x_1} + \mathbf{v}_{x_2} + \mathbf{v}_{x_3}. \tag{4.18}$$

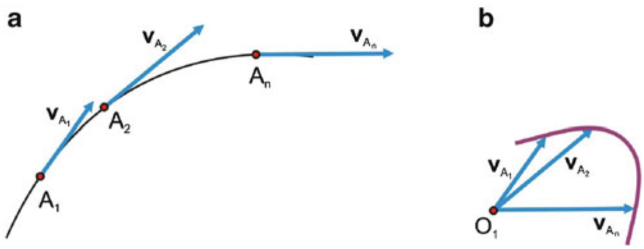


Fig. 4.7 Velocity vectors at points $A_1, A_2, \ldots, A_n$ (**a**) and velocity hodograph (**b**)

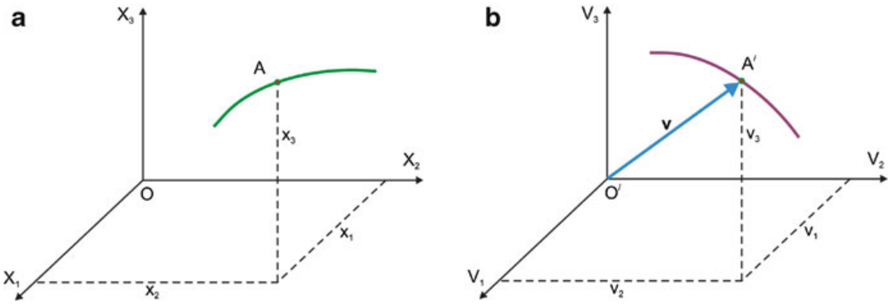


Fig. 4.8 Trajectory of motion (**a**) and velocity hodograph (**b**) of a particle A in space

The magnitude and direction of velocity of the particle is described by the equations

$$v = \sqrt{v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2},$$

$$\cos(\mathbf{v}, \mathbf{E}_i) = \frac{v_{x_i}}{v}, \qquad i = 1, 2, 3. \tag{4.19}$$

Let the motion of the point be a plane, i.e., let it be on the X_1OX_2 flat surface and let $x_i = f_i(t)$. Thus we have

$$v = \sqrt{v_{x_1}^2 + v_{x_2}^2},$$

$$\cos(\mathbf{v}, \mathbf{E}_i) = \frac{v_{x_i}}{v}, \qquad i = 1, 2. \tag{4.20}$$

In the case of rectilinear motion $x_1 = f(t)$, $v = |\frac{\mathrm{d}x}{\mathrm{d}t}|$ and $v_x > 0$ ($v_x < 0$) if particle A moves in agreement with (opposite to) the sense of the axis OX_1.

If the particle moves on a curve in a non-uniform way, as was shown in Fig. 4.7a, then at the points $A_1, A_2, \ldots, A_n$ we have different velocity vectors $\mathbf{v}_{A_1}, \mathbf{v}_{A_2}, \ldots, \mathbf{v}_{A_n}$; at this point its tips will form a curve called a velocity hodograph (Fig. 4.7b).

Figure 4.8 presents the trajectory and a velocity hodograph of particle A in space.

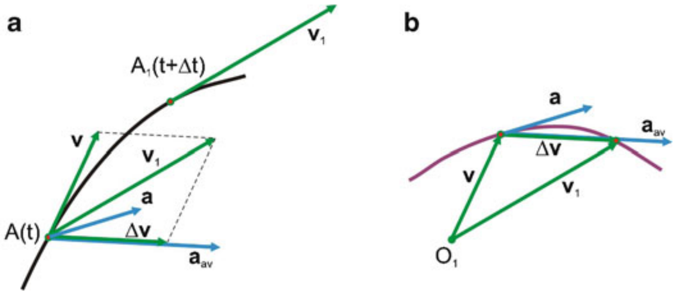


Fig. 4.9 Velocities of particles A and A_1 (**a**) and velocity hodograph (**b**)

From Fig. 4.8b it follows that

$$\mathbf{v} = v_1\mathbf{E}_1 + v_2\mathbf{E}_2 + v_3\mathbf{E}_3 = \dot{x}_1\mathbf{E}_1 + \dot{x}_2\mathbf{E}_2 + \dot{x}_3\mathbf{E}_3, \tag{4.21}$$

and hence we obtain three scalar equations,

$$\dot{x}_i = v_i, \qquad i = 1, 2, 3, \tag{4.22}$$

called *the parametric equations* of a velocity hodograph.

4.1.3 Acceleration of a Particle

Figure 4.9 presents the motion of a particle A along a curve and the velocity hodograph of this particle.

From Fig. 4.9a it follows that the increment of velocity $\Delta\mathbf{v}$ during time Δt is equal to

$$\Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}, \tag{4.23}$$

and dividing both sides of the above equation by Δt we have

$$\mathbf{a}_{av} = \frac{\Delta\mathbf{v}}{\Delta t}, \tag{4.24}$$

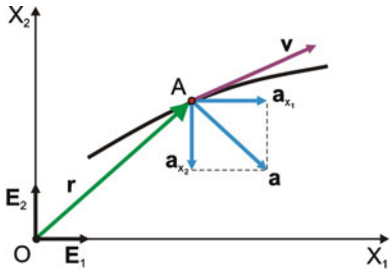
and successively

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}. \tag{4.25}$$

The acceleration vector is equal to the first derivative of the velocity vector with respect to time or to the second derivative of the radius vector with respect to time.

The acceleration of the particle characterizes the rate of change in magnitude and direction of the velocity vector.

Fig. 4.10 Motion of a particle *A* on a planar curve



As can be seen from the construction presented in Fig. 4.9, the velocity vector is tangent to the velocity hodograph.

In the general case, the trajectory of motion of particle A is not a planar curve. The vector $\mathbf{a}_{\mathrm{av}}$ lies in the plane determined by the tangent to the trajectory at point A and the line passing through A and parallel to $\mathbf{v}_1$.

If the curve is planar, then the plane in which it lies is the osculating plane. The velocity vector lies in the osculating plane and points to the inside of the bend of the curve. We will describe the acceleration vector of the particle in the Cartesian coordinate system by giving its magnitude and direction. We have

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{E}_1\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + \mathbf{E}_2\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + \mathbf{E}_3\frac{\mathrm{d}^2x_3}{\mathrm{d}t^2}$$
$$= \mathbf{E}_1\frac{\mathrm{d}v_1}{\mathrm{d}t} + \mathbf{E}_2\frac{\mathrm{d}v_2}{\mathrm{d}t} + \mathbf{E}_3\frac{\mathrm{d}v_3}{\mathrm{d}t}. \tag{4.26}$$

From (4.26) it follows that projections of the acceleration vector on the axes of the Cartesian coordinate system are equal to the second derivatives of the respective coordinates with regard to time or to the first derivatives of projections of the velocity vector on the respective axes with regard to time (Fig. 4.10).

The magnitude and orientation of the acceleration vector are described by the equations

$$a = \sqrt{a_{x_1}^2 + a_{x_2}^2 + a_{x_3}^2}, \quad \cos(\mathbf{a}, \mathbf{E}_i) = \frac{a_{x_i}}{a}, \quad i = 1, 2, 3. \tag{4.27}$$

If we are dealing with planar motion, then

$$a = \sqrt{a_{x_1}^2 + a_{x_2}^2}, \quad \cos(\mathbf{a}, \mathbf{E}_i) = \frac{a_{x_i}}{a}, \quad i = 1, 2. \tag{4.28}$$

Finally, in the case of rectilinear motion we have $x = f(t)$ and the acceleration $\mathbf{a}$ is in agreement with (opposite to) the sense of the axis if $a_x > 0$ ($a_x < 0$).

Let us now consider how to determine the acceleration vector in natural coordinates (Fig. 4.11).

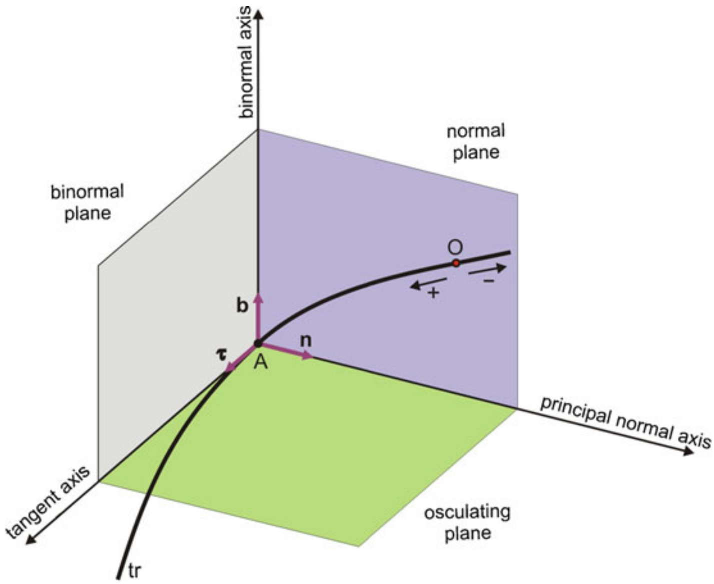


Fig. 4.11 Natural coordinates

At point A of the curve we draw the osculating plane, the normal plane perpendicular to it, and the third plane perpendicular to both of those planes; this is called the binormal plane. The following three mutually perpendicular axes are called natural axes:

1. *Tangent axis* directed in agreement with the increment of the arc coordinate.
2. *Principal normal axis* directed to the inside of the bend of the curvature.
3. *Binormal axis* chosen in such a way that the system $(\boldsymbol{\tau}, \mathbf{n}, \mathbf{b})$ is right-handed.

The system of natural coordinates moves on the curve simultaneously changing orientation in space.

In Fig. 4.12 are shown two positions of particles $A = s$ and $A_1 = s + \Delta s$. Although the magnitude of the unit vector $|\boldsymbol{\tau}| = 1$, this vector is not constant because it changes direction. From Fig. 4.12 it follows that

$$\Delta\boldsymbol{\tau} = \boldsymbol{\tau}_1 - \boldsymbol{\tau}, \tag{4.29}$$

and after division by Δs we obtain the vector of average curvature of the trajectory χ_{av}, which characterizes the change in position of $\boldsymbol{\tau}$ on the arc AA_1.

If $\Delta s \to 0$, then we obtain the definition of the vector of curvature of a curve (trajectory) at point A of the form

$$\boldsymbol{\chi} = \lim_{\Delta s \to 0} \frac{\Delta\boldsymbol{\tau}}{\Delta s} = \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s}. \tag{4.30}$$

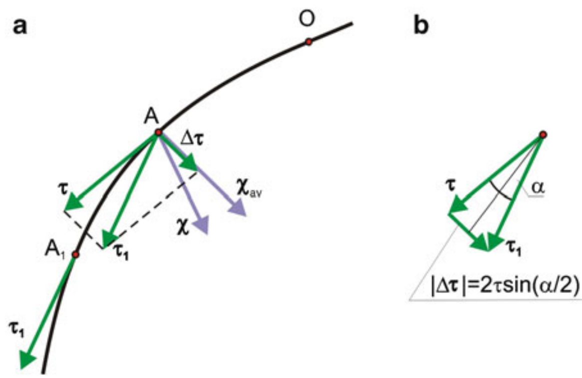


Fig. 4.12 Vector of curvature $\boldsymbol{\chi}$; vector of average curvature $\boldsymbol{\chi}_{\mathrm{av}}$ (**a**) and the magnitude of the vector $\varDelta\boldsymbol{\tau}$ (**b**)

The vector of curvature of the curve at a given point is equal to the derivative of a unit vector of the axis tangent to the curve with respect to the arc coordinate. The sense of the vector $\boldsymbol{\chi}$ is in agreement with the sense of the vector $\varDelta\boldsymbol{\tau}$. The magnitude of the vector $|\varDelta\boldsymbol{\tau}|$ is equal to (Fig. 4.12b)

$$|\varDelta\boldsymbol{\tau}| = 2\tau \sin(\alpha/2) \cong \alpha, \tag{4.31}$$

and in view of that,

$$\chi = \lim_{\varDelta s\to 0}\left|\frac{\varDelta\boldsymbol{\tau}}{\varDelta s}\right| = \lim_{\varDelta s\to 0}\frac{\alpha}{\varDelta s} = \frac{1}{\rho}. \tag{4.32}$$

From differential geometry it is known that the ratio of angle α to the increment of arc coordinate $\varDelta s$, provided that $\varDelta s \to 0$, is equal to the curvature of a curve ρ^{-1}, where ρ denotes the radius of curvature of a curve at point A.

The vector $\boldsymbol{\chi}$ lies in the osculating plane and is directed along the principal normal toward the center of curvature and can be represented in the form of the vector equation

$$\boldsymbol{\chi} = \mathbf{n}\frac{1}{\rho}, \tag{4.33}$$

where ρ is the radius of curvature at point A.

The acceleration **a** of particle A is equal to

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} + \boldsymbol{\tau}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}. \tag{4.34}$$

Because $\mathbf{v} = \boldsymbol{\tau}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}$, (see (4.13)) from (4.33) and (4.34) we obtain

$$\mathbf{a} = \mathbf{n}\frac{v^2}{\rho} + \boldsymbol{\tau}\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} = \mathbf{n}a_n + \boldsymbol{\tau}a_\tau, \tag{4.35}$$

where $(\frac{\mathrm{d}s}{\mathrm{d}t})^2 = v^2$.

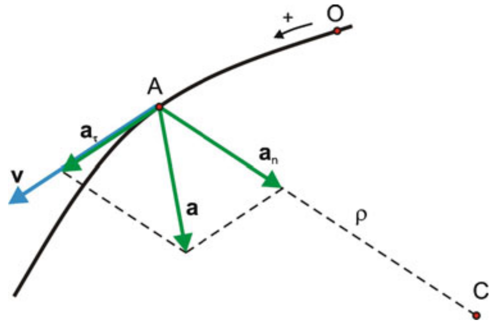


Fig. 4.13 Vector of acceleration **a** in natural coordinates

The projection of acceleration **a** of point A on the principal normal is equal to the square of magnitude of velocity of point A divided by the radius of curvature at this point. The projection of acceleration **a** of point A on the tangent axis is equal to the second derivative with respect to the time of the arc coordinate of this point or to the first derivative with respect to the time of velocity $\frac{\mathrm{d}v}{\mathrm{d}t}$ of the point.

If $\rho = \infty$, which takes place in the case of motion of the point along a straight line, we have $a_n = v^2/\rho = 0$.

Normal acceleration exists only during the motion of a particle along a curve and characterizes the change in the direction of velocity.

During uniform motion, $v = const$, and then $a_\tau = \frac{\mathrm{d}v}{\mathrm{d}t} = 0$.

Tangential acceleration exists only during the non-uniform motion of a particle and characterizes the change in the magnitude of velocity.

If we know the magnitudes of velocity and acceleration of the particle in rectilinear coordinates

$$v = \sqrt{v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2}, \qquad a = \sqrt{a_{x_1}^2 + a_{x_2}^2 + a_{x_3}^2}, \tag{4.36}$$

then we successively calculate

$$\begin{aligned} a_\tau &= \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2} = \frac{v_{x_1}\frac{\mathrm{d}v_{x_1}}{\mathrm{d}t} + v_{x_2}\frac{\mathrm{d}v_{x_2}}{\mathrm{d}t} + v_{x_3}\frac{\mathrm{d}v_{x_3}}{\mathrm{d}t}}{\sqrt{v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2}} \\ &= \frac{v_{x_1}a_{x_1} + v_{x_2}a_{x_2} + v_{x_3}a_{x_3}}{v} = \frac{\mathbf{v} \circ \mathbf{a}}{v}, \end{aligned} \tag{4.37}$$

$$a_n = \sqrt{a^2 - a_\tau^2}, \qquad \rho = \frac{v^2}{a_n}, \tag{4.38}$$

that is, we determine accelerations a_τ and a_n and the radius of curvature ρ in natural coordinates (Fig. 4.13).

4.2 Selected Problems of Planar Motion of a Particle

4.2.1 Rectilinear Motion

If the particle selected for consideration is in rectilinear motion, then after the introduction of axis OX, the position of particle A at time instant t is determined by one coordinate $x(t)$. The time occurs here as a parameter (Fig. 4.14).

The speed of that particle directed along the OX axis is equal to

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}(t). \tag{4.39}$$

If in equal time intervals Δt the particle travels equal distances Δx, then we call such motion a *uniform rectilinear motion*.

The *average velocity* in uniform rectilinear motion is equal to $v_{\mathrm{av}} = \Delta x/\Delta t$, whereas the *instantaneous velocity* of the motion is defined by (4.39). If the particle moves in agreement with (opposite to) the positive direction of the OX axis, the speed of that particle $v_x > 0$ ($v_x < 0$).

The rate of change in speed of the particle is characterized by an *acceleration*. When the particle travels along a straight line, its acceleration is given by

$$\mathrm{a}_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \ddot{x}(t). \tag{4.40}$$

If the considered particle moves with the acceleration constant as to direction and magnitude, then taking the direction of the OX axis in accordance with the direction of the acceleration, we have

$$|\mathbf{a}| = \mathrm{a}_x = \ddot{x}(t) = \mathrm{const.} \tag{4.41}$$

Integrating (4.41) we obtain

$$v_x = \dot{x}(t) = \mathrm{a}_x t + C_1, \tag{4.42}$$

where C_1 denotes the constant of integration. Integrating (4.42) we obtain

$$x = \frac{\mathrm{a}_x t^2}{2} + C_1 t + C_2. \tag{4.43}$$

In order to define the motion it is necessary to know its initial conditions, which we will assume to be

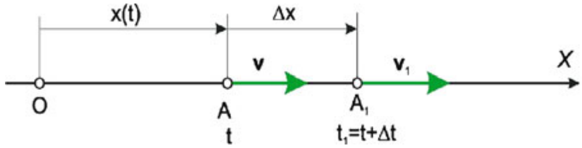


Fig. 4.14 Rectilinear motion of a particle

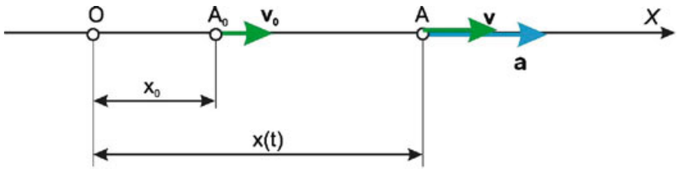


Fig. 4.15 Velocity and acceleration of a particle A at the time instant $t_0 = 0$ and t

$$x(t = 0) = x_0,$$
$$v_x(t = 0) = v_{x0}. \tag{4.44}$$

Substituting (4.44) into (4.42) and (4.43) we obtain

$$C_1 = v_{x0}, \quad C_2 = x_0. \tag{4.45}$$

Substituting (4.45) into (4.42) and (4.43) we have

$$x = \frac{\mathrm{a}_x t^2}{2} + v_{x0} t + x_0, \quad v_x = \mathrm{a}_x t + v_{x0}. \tag{4.46}$$

If the sense of initial velocity $\mathbf{v}_x$ is in agreement with the positive direction of the OX axis, then for the total duration of motion the values of x and v_x increase (Fig. 4.15). Motion like that is called *uniformly accelerated motion*.

In the case where the initial velocity $\mathbf{v}_x$ possesses a sense opposite to the assumed sense of the acceleration $\mathbf{a}_x$, (4.46) will take the form

$$x = \frac{\mathrm{a}_x t^2}{2} - v_{x0} t + x_0, \qquad v_x = \mathrm{a}_x t - v_{x0}. \tag{4.47}$$

In the time interval $0 \leq t < \frac{v_{x0}}{\mathrm{a}_x}$, the speed $v_x < 0$. In this time interval the motion proceeds in agreement with the sense of initial velocity and opposite to the sense of acceleration. Such motion is called *uniformly decelerated motion*. In that interval the speed decreases and attains a value equal to zero for the time instant $t_0 = \frac{v_{x0}}{\mathrm{a}_x}$. For the time $t > t_0$ the sense of velocity is in agreement with the sense of acceleration, that is, the change in the sense of particle motion occurs at the time instant $t = t_0$ (Fig. 4.16).

4.2.2 Rectilinear Harmonic Motion and Special Cases of Plane Curvilinear Motion

Apart from the described uniformly accelerated and decelerated motions, the motion of a particle along a straight line can be *harmonic motion*. As we will see later, such motion can also be described by the same equation if the particle trajectory is a circle.

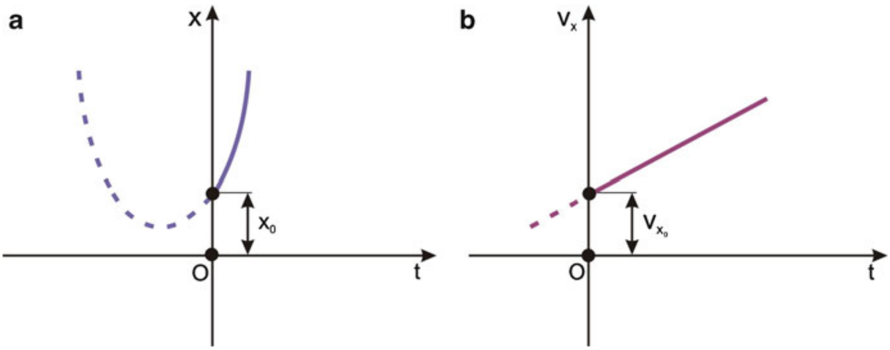


Fig. 4.16 Graphical representation of displacement (**a**) and velocity (**b**) governed by (4.46)

The harmonic motion is described by the following second-order differential equation:

$$\ddot{x} + \alpha^2 x = 0, \tag{4.48}$$

whose solution is

$$x = a \sin(\alpha t + \psi_0). \tag{4.49}$$

In the equation above a denotes an *amplitude of motion*, α the *frequency of motion* of the point, and the angle ψ_0 is called the *initial phase* since the argument $\alpha t + \theta_0$ is called a *phase of harmonic motion*. The amplitude and initial phase of motion are defined by imposing the initial conditions of the motion. We will conduct an analysis of the motion using (4.49). It is easy to notice that the motion takes place around point O, which is the origin of the adopted coordinate axis. The most remote position of a particle from the origin O is equal to a, so it is equal to the amplitude of motion. Because the motion is harmonic, let us try to determine the shortest time T after which the motion will start to repeat itself. The condition of repetition will take the form

$$\alpha(t + T) + \psi_0 = \alpha t + \psi_0 + 2\pi, \tag{4.50}$$

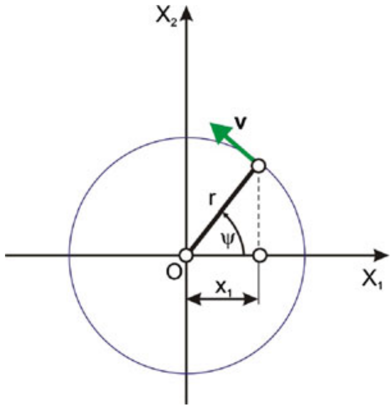
from which we find $T = 2\pi/\alpha$. From the last equation we will determine the *frequency of motion* as

$$f = T^{-1} = \frac{\alpha}{2\pi}, \tag{4.51}$$

which is understood as the number of periods per unit of time. Differentiating (4.49) we will easily determine the speed and acceleration of the point motion

$$\begin{aligned} v_x &= \dot{x} = a\alpha \cos(\alpha t + \psi_0), \\ \mathrm{a}_x &= \ddot{x} = -a\alpha^2 \sin(\alpha t + \psi_0). \end{aligned} \tag{4.52}$$

Fig. 4.17 Circular motion of a particle with constant speed $\mathbf{v}$ ($\psi = \alpha t,\ r = a$)



At the time instants in which the speed is equal to zero, the acceleration reaches its maximum value. The simple interpretation of a harmonic motion can be obtained by an analysis of the circular motion of a particle.

From Fig. 4.17 it follows that after projecting the point moving uniformly along the circle onto the OX_1 axis we obtain

$$x_1 = a\cos\alpha t = a\sin\left(\alpha t + \frac{\pi}{2}\right), \tag{4.53}$$

which means that we are dealing with the harmonic motion ($\alpha = \text{const}$).

The curvilinear motion of point along a circle is a special case of motion along an ellipse.

Let us consider the motion of a point described by the following equations:

$$x_1 = a\cos\alpha t, \qquad x_2 = b\sin\alpha t. \tag{4.54}$$

Eliminating parameter t from (4.54) we obtain the equation of an ellipse:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \tag{4.55}$$

which is depicted in Fig. 4.18.

The speed of motion of point A of coordinates x_1 and x_2 is equal to

$$\begin{aligned} v &= \sqrt{v_{x_1}^2 + v_{x_2}^2} = \sqrt{(-a\alpha\sin\alpha t)^2 + (b\alpha\cos\alpha t)^2} \\ &= \alpha\sqrt{b^2\cos^2\alpha t + a^2\sin^2\alpha t} = \alpha\sqrt{\frac{b^2}{a^2}x_1^2 + \frac{a^2}{b^2}x_2^2}. \end{aligned} \tag{4.56}$$

If $v_{x_1} = -ab^{-1}\alpha x_2$, $v_{x_2} = ba^{-1}\alpha x_1$, then the vector $\mathbf{v} = v_{x_1}\mathbf{E}_1 + v_{x_2}\mathbf{E}_2$ lies on a tangent to the ellipse at point A and possesses the sense indicated in the figure.

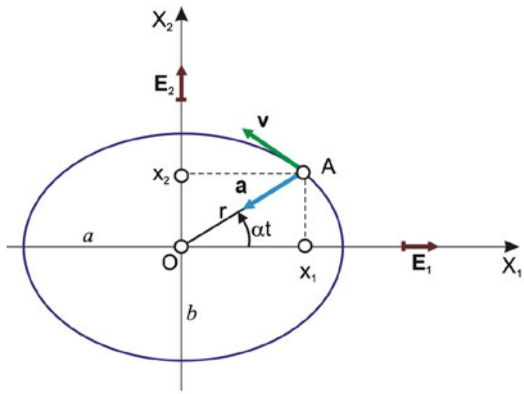


Fig. 4.18 Motion of a point along an ellipse of axes $2a$ and $2b$

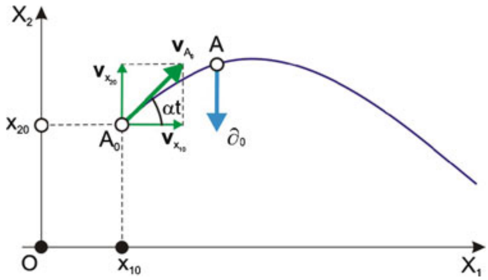


Fig. 4.19 Curvilinear planar motion of a point determined by initial conditions (4.58) and (4.59)

Let us note that both values of the components of acceleration at the position of point A (Fig. 4.18) have signs opposite to unit vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ because

$$\mathrm{a}_{x_1} = -a\alpha^2 \cos \alpha t, \quad \mathrm{a}_{x_2} = -b\alpha^2 \sin \alpha t. \tag{4.57}$$

The vector of acceleration $\mathbf{a}$ has a magnitude

$$\mathrm{a} = \sqrt{\mathrm{a}_{x_1}^2 + \mathrm{a}_{x_2}^2} = \alpha^2 \sqrt{a^2 \cos^2 \alpha t + b^2 \sin^2 \alpha t} = \alpha^2 r,$$

where $r = \sqrt{x_1^2 + x_2^2}$ (Fig. 4.18).

From the foregoing discussion one can draw the conclusion that the acceleration of a point traveling along an ellipse depends on the position of the point and possesses the direction of the radius vector and a sense toward the center of the ellipse.

Finally, let us consider the case where a particle moves in a plane with a curvilinear motion with the acceleration $|\mathbf{a}_0|$ constant as to direction and magnitude. We take the Cartesian coordinate system OX_1X_2 in such way that the axis $OX_2 \parallel \mathbf{a}_0$ (Fig. 4.19).

Let us take the following initial conditions:

$$x_1(t=0) = x_{10}, \quad x_2(t=0) = x_{20}, \tag{4.58}$$

$$v_{x_1}(t=0) = v_0 \cos\alpha, \quad v_{x_2}(t=0) = v_0 \sin\alpha. \tag{4.59}$$

According to the adopted coordinate system we have

$$\ddot{x}_1 = 0, \quad \ddot{x}_2 = -\mathrm{a}_0. \tag{4.60}$$

Integrating the equations above we obtain

$$\dot{x}_1 = C_{1x_1}, \quad \dot{x}_2 = -\mathrm{a}_0 t + C_{1x_2}, \tag{4.61}$$

$$x_1 = C_{1x_1} t + C_{2x_1}, \quad x_2 = -\frac{\mathrm{a}_0 t^2}{2} + C_{1x_2} t + C_{2x_2}. \tag{4.62}$$

Taking into account the initial conditions in (4.61) and (4.62) we obtain the following values of the introduced constants of integration

$$C_{2x_1} = x_{10}, \quad C_{2x_2} = x_{20}, \tag{4.63}$$

$$C_{1x_1} = v_0 \cos\alpha, \quad C_{1x_2} = v_0 \sin\alpha, \tag{4.64}$$

and in view of that the projections of velocity and displacement of point A are equal to

$$v_{x_1} = v_0 \cos\alpha, \quad v_{x_2} = -\mathrm{a}_0 t + v_0 \sin\alpha, \tag{4.65}$$

$$x_1 = v_0 t \cos\alpha + x_{10}, \quad x_2 = -\frac{\mathrm{a}_0 t^2}{2} + v_0 t \sin\alpha + x_{20}. \tag{4.66}$$

We determine the path of point A from (4.66) after eliminating parameter t, thereby obtaining

$$x_2 = -\frac{\mathrm{a}_0 (x_1 - x_{10})^2}{2 v_0^2 \cos^2\alpha} + (x_1 - x_{10}) \tan\alpha + x_{20}. \tag{4.67}$$

Let us consider the special case where the vector $\mathbf{a}_0 = \mathbf{g}$, where $\mathbf{g}$ is the acceleration of gravity on Earth, and where $\alpha = \frac{\pi}{2}$.

In this case, from (4.65) and (4.66) we obtain

$$v_{x_1} = 0, \quad v_{x_2} = -gt + v_0, \tag{4.68}$$

$$x_1 = x_{10}, \quad x_2 = x_{20} + v_0 t - \frac{gt^2}{2}. \tag{4.69}$$

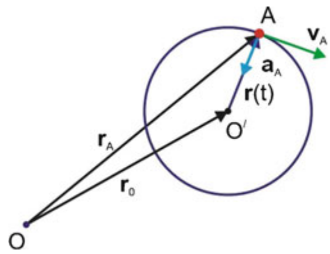


Fig. 4.20 Uniform motion of a point A along circle of radius r and center O'

If in this case the sense of velocity $\mathbf{v}_0$ was in agreement with the positive direction of the OX_2 axis, then the path of the point is a straight line and the point moves at first in uniformly decelerated and then in uniformly accelerated motion. Such motion is called a *vertical motion*, whereas the motion of the point considered earlier is called an *oblique motion*.

4.2.3 Circular, Rectilinear, and Curvilinear Motion in Vector Approach

The uniform motion of a point A along a circle of radius r and center O' is presented in Fig. 4.20.

First let us show that in such motion, $\mathbf{v}_A \perp \mathbf{r}$, where $\dot{\mathbf{r}}_A \equiv \mathbf{v}_A$. According to the conditions of motion and Fig. 4.20 we have

$$(\mathbf{r}_A - \mathbf{r}_0)^2 = r^2 \equiv \text{const.} \tag{4.70}$$

Differentiating (4.70) with respect to time we obtain

$$2(\mathbf{r}_A - \mathbf{r}_0) \circ \dot{\mathbf{r}}_A = 0, \tag{4.71}$$

which means that $\mathbf{v}_A \perp \mathbf{r}$.

Differentiating (4.71) with respect to time and taking into account that we are dealing with uniform motion we obtain

$$\dot{\mathbf{r}}_A^2 + (\mathbf{r}_A - \mathbf{r}_0) \circ \ddot{\mathbf{r}}_A = 0, \tag{4.72}$$

from which it follows that

$$(\mathbf{r}_A - \mathbf{r}_0) \circ \ddot{\mathbf{r}}_A = \text{const.} \tag{4.73}$$

Differentiating (4.73) with respect to time we obtain

$$\dot{\mathbf{r}}_A \circ \mathbf{a}_A \equiv \mathbf{v}_A \circ \mathbf{a}_A = 0, \tag{4.74}$$

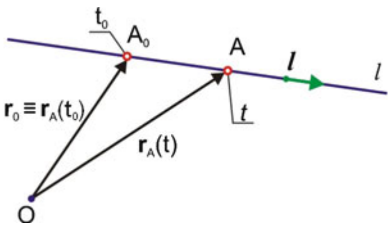


Fig. 4.21 Motion of point A along a straight line

which shows that the vectors of velocity and acceleration of point A are perpendicular to one another. In the equations above during transformations we used the relations $\dot{\mathbf{r}}_A^2 = \text{const}$ and $\ddot{\mathbf{r}}_A^2 = \text{const}$ (however, note that $\dot{\mathbf{r}}_A \neq \text{const}$ and $\ddot{\mathbf{r}}_A \neq \text{const}$).

Because we are dealing with planar motion, all vectors describing the motion of point A, i.e., $\mathbf{r}$ (position), $\mathbf{v}_A$ (velocity), and $\mathbf{a}_A$ (acceleration) lie in one plane. And now, if $\mathbf{v}_A \perp \mathbf{r}$ and $\mathbf{a}_A \perp \mathbf{v}_A$, we have $\mathbf{r} \parallel \mathbf{a}_A$. The senses of vectors $\mathbf{r}$ and $\mathbf{a}_A$ are opposite, because from (4.72) it follows that (Fig. 4.20)

$$\mathbf{r} \circ \mathbf{a}_A = -\mathbf{v}_A^2 \equiv \text{const}, \tag{4.75}$$

and the magnitude of acceleration equals

$$\mathrm{a}_A = \frac{v_A^2}{r}, \tag{4.76}$$

where r is the radius of a circle along which point A moves uniformly.

Let us now consider the uniform motion of point A along a straight line, depicted in Fig. 4.21.

The trajectory of motion of point A is a straight line, and because we will analyze uniform motion, in any time instant t we have $\dot{\mathbf{r}}_A = \mathbf{v}_0 = \text{const}$. The initial position of point A is defined by the vector $\mathbf{r}_0$, and in view of that from Fig. 4.21 we have

$$\mathbf{r}_A(t) = \mathbf{v}_0 t + \mathbf{r}_0. \tag{4.77}$$

Now, let the velocity of point A along the straight line be defined by the relationship

$$\mathbf{v}_A(t) = \mathbf{a}_0 t + \mathbf{v}_0, \tag{4.78}$$

where $\mathbf{a}_0 = \text{const}$ is the acceleration (the vectors of acceleration and velocity are tangent to the path). The motion described by (4.78) is not a uniform rectilinear motion. Integrating (4.78) we obtain

$$\mathbf{r}_A(t) = \frac{1}{2}\mathbf{a}_0 t^2 + \mathbf{v}_0 t + \mathbf{r}_0, \tag{4.79}$$

which describes the change in radius vector of point A.

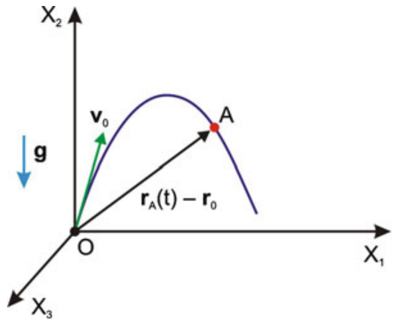


Fig. 4.22 Motion of point *A* during a projection at an angle onto the horizontal in the gravitational field

We have already assumed that the senses of vectors $\mathbf{r}_A - \mathbf{r}_0$ and $\mathbf{v}_0$ are the same. Let us assume now that the senses of vectors $\mathbf{a}_0$ and $\mathbf{v}_0$ (initial conditions of motion) are also in agreement.

The vector $\frac{1}{2}\mathbf{a}_0 t^2 + \mathbf{v}_0 t$ describes the motion of point A along a straight line in time starting from the initial instant t_0. Taking the unit vector $\boldsymbol{l}$ of axis l, (4.79) in this case will assume the form

$$\mathbf{r}_A(t) = \left(\frac{1}{2}\mathrm{a}_0 t^2 + v_0 t\right)\boldsymbol{l} + \mathbf{r}_0. \tag{4.80}$$

In kinematics we do not refer to forces, but only to observations of motion that rely on a geometrical approach. In the case just considered, the initial vectors of motion $\mathbf{a}_0$ and $\mathbf{v}_0$ were collinear, and we considered the case of uniformly accelerated motion (we are dealing with uniformly decelerated motion if $-\mathbf{a}_0$ is substituted into (4.79)).

Let us consider now the case where vectors $\mathbf{a}_0$ and $\mathbf{v}_0$ are coplanar (i.e., they lie in one plane) and are not parallel to one another.

We must deal with such a case during a projection at an angle onto a horizontal in the gravitational field, where $\mathbf{a}_A = \mathbf{g} \equiv \text{const}$ and $\mathbf{g}$ is the acceleration of gravity on Earth.

Also in this case, the motion is governed by the vectorial equation (4.79), but now the path of motion will not be a straight line. Therefore, for the purpose of analysis of this motion, we introduce the Cartesian coordinate system chosen in a special way. According to (4.79) we have

$$\mathbf{r}_A(t) - \mathbf{r}_0 = \frac{1}{2}\mathbf{a}_0 t^2 + \mathbf{v}_0 t, \tag{4.81}$$

and the introduced Cartesian coordinate system is depicted in Fig. 4.22, where now all three vectors $\mathbf{a}_0 = \mathbf{g}$, $\mathbf{r}_A - \mathbf{r}_0$, and $\mathbf{v}_0$ lie in the OX_1X_2 plane.

In order to obtain the equations of motion in this system in scalar form we will multiply (4.81) by $\mathbf{E}_i$, bearing in mind that

$$\mathbf{r}_A(t) - \mathbf{r}_0 = x_1(t)\mathbf{E}_1 + x_2(t)\mathbf{E}_2,$$
$$\mathbf{v}_0 = v_{01}\mathbf{E}_1 + v_{02}\mathbf{E}_2,$$
$$\mathbf{g} = -g\mathbf{E}_2. \tag{4.82}$$

As a result of that operation we obtain

$$x_1(t) = v_{01}t,$$
$$x_2(t) = v_{02}t - \frac{1}{2}gt^2,$$
$$x_3(t) = 0, \tag{4.83}$$

which was obtained earlier using the standard approach [see (4.68) and (4.69)].

4.3 Radius Vector and Rectangular and Curvilinear Coordinates in Space

4.3.1 Introduction

We will consider the motion of a point in Euclidean space (three-dimensional); strictly speaking we will deal with its kinematics. Because motion is defined as a change in the position of a point (body) with respect to the adopted (in this case fixed) coordinate system or the point of reference, the easiest way to determine the position of the point is to make use of the so-called *radius vector (position vector)*.

In Fig. 4.23 point A moving along the curve in space is presented. The position of that point may be described at every time instant by the vector-valued function $\mathbf{r} = \mathbf{r}(t)$. The velocity of point A is defined by the relationship $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$, and the acceleration of the point is equal to $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t = \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$.

If we are going to observe motion from point O, then we can locate there the most popular Cartesian coordinate system. The point in the successive time instants t_i occupies the positions $A_i, i = 1, 2, \ldots$. These positions are also described by the tip of the radius vector $\mathbf{r}(t_i)$.

The vector-valued function $\mathbf{r}(t)$ defines the *position of a point*, whereas the tip of vector $\mathbf{r}(t)$ draws a hodograph, which is a *path of point motion*. In turn, taking under consideration the adopted rectangular coordinate system $OX_1X_2X_3$, we define the position of point A by three scalar functions, $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$, and the position vector can be expressed as

$$\mathbf{r} = x_1(t)\mathbf{E}_1 + x_2(t)\mathbf{E}_2 + x_3(t)\mathbf{E}_3, \tag{4.84}$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are the unit vectors respectively of the axes OX_1, OX_2, and OX_3.

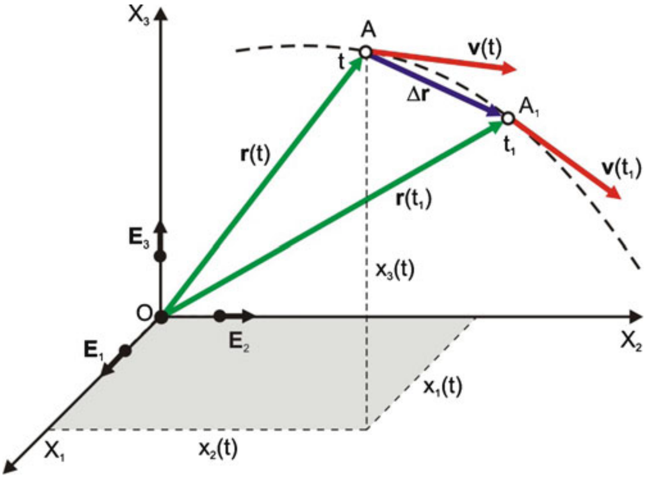


Fig. 4.23 Motion of a point in a three-dimensional space

A derivative of the vector-valued function (position vector) with respect to time $\dot{\mathbf{r}}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is called the *velocity*:

$$\mathbf{v} = v_1\mathbf{E}_1 + v_2\mathbf{E}_2 + v_3\mathbf{E}_3, \tag{4.85}$$

where $v_1 = \dot{x}_1(t)$, $v_2 = \dot{x}_2(t)$, $v_3 = \dot{x}_3(t)$.

The velocity of a point is a vector tangent to the hodograph of the position vector $\mathbf{r}(t)$ (Fig. 4.23).

On the other hand, the derivative of the velocity vector of the point is called the *acceleration*:

$$\mathbf{a}(t) = \dot{\mathbf{v}} = \mathrm{a}_1\mathbf{E}_1 + \mathrm{a}_2\mathbf{E}_2 + \mathrm{a}_3\mathbf{E}_3, \tag{4.86}$$

where $\mathrm{a}_1 = \dot{v}_1 = \frac{\mathrm{d}^2x_1}{\mathrm{d}t^2}$, $\mathrm{a}_2 = \dot{v}_2 = \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2}$, $\mathrm{a}_3 = \dot{v}_3 = \frac{\mathrm{d}^2x_3}{\mathrm{d}t^2}$.

The magnitudes of the velocity and the acceleration are equal to

$$v = |\mathbf{v}(t)| = \sqrt{v_1^2 + v_2^2 + v_3^2}, \tag{4.87}$$

$$\mathrm{a} = |\mathbf{a}(t)| = \sqrt{\mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2}, \tag{4.88}$$

having the units m·s^{-1} and m·s^{-2}, respectively. Let us note that the direction cosines of the acceleration vector are equal to

$$\cos(\mathbf{a}, \mathbf{E}_1) = \frac{\ddot{x}_1}{\mathrm{a}}, \quad \cos(\mathbf{a}, \mathbf{E}_2) = \frac{\ddot{x}_2}{\mathrm{a}}, \quad \cos(\mathbf{a}, \mathbf{E}_3) = \frac{\ddot{x}_3}{\mathrm{a}}.$$

The vectors of the velocity of the point at the instants t and t_1 are drawn in Fig. 4.23. The average velocity in the time interval $\Delta t = t_1 - t$ is defined as

$$\mathbf{v}_{av} = \frac{\mathbf{r}(t_1) - \mathbf{r}(t)}{t_1 - t} = \frac{\Delta\mathbf{r}(\Delta t)}{\Delta t}. \tag{4.89}$$

If $\Delta t \to 0$, then the direction of the vector $\mathbf{v}_{av}$ will tend toward the tangent to the motion trajectory at point A. The vectors of acceleration were not drawn in Fig. 4.23 because their determination requires slightly deeper computations. The average and instantaneous accelerations are defined in a way that is analogous to the definitions of average and instantaneous velocity.

Often, because of the convenience of calculations, interpretations, or simplifications regarding the point motion, one introduces its analytical description with the aid of *curvilinear coordinates*.

4.3.2 Classification of Particle Motion with Regard to Accelerations of Motion

Because $\mathbf{a} = \mathbf{a}_n + \mathbf{a}_\tau$, we introduce the classification of motion with respect to $\mathbf{a}_n$ and $\mathbf{a}_\tau$.

1. Let the normal acceleration $\mathbf{a}_n = \mathbf{0}$ and tangential acceleration $\mathbf{a}_\tau = \mathbf{0}$. If in a certain time interval the normal and tangential accelerations of a particle are equal to zero, then in this time interval neither the direction nor the magnitude of velocity changes and the particle is in uniform rectilinear motion and its acceleration $\mathbf{a} = \mathbf{0}$.
2. Let $\mathbf{a}_n \neq \mathbf{0}$, $\mathbf{a}_\tau = \mathbf{0}$. If in a certain time interval the normal acceleration of a particle is nonzero, but it does not have tangential acceleration, we are dealing with a change in direction of velocity without a change in its magnitude, and the particle is in uniform curvilinear motion, where $a = \frac{v^2}{\rho}$.
3. Let $\mathbf{a}_n = \mathbf{0}$ and $\mathbf{a}_\tau \neq \mathbf{0}$. If in a certain time interval the normal acceleration is equal to zero, but the tangential acceleration is nonzero, then the direction of the velocity vector does not change, but its magnitude changes, that is, the particle moves nonuniformly along a straight line. If the senses of $\mathbf{v}$ and $\mathbf{a}_\tau$ are in agreement (opposite), then the particle is in accelerated (decelerated) motion. If only in a certain time instant $a_n = \frac{v^2}{\rho} = 0$, then either the particle is not in rectilinear motion ($\rho = \infty$) or the magnitude of velocity $v = 0$, which occurs, for instance, when the sense of the particle's motion changes.
4. Let $\mathbf{a}_n \neq \mathbf{0}$ and $\mathbf{a}_\tau \neq \mathbf{0}$. If in a certain time interval during the motion of a particle $\mathbf{a}_n \neq \mathbf{0}$ and $\mathbf{a}_\tau \neq \mathbf{0}$, then both the direction and the magnitude of the velocity vector change. When the senses of vectors $\mathbf{v}$ and $\mathbf{a}_\tau$ are in agreement (opposite), then the particle is in curvilinear accelerated (decelerated) motion. If during motion $\mathbf{a}_\tau = \text{const}$, then the particle is in uniformly variable motion.

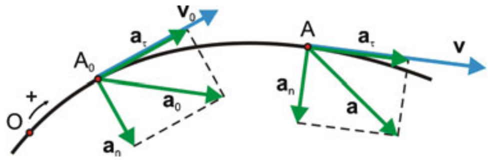


Fig. 4.24 Motion of a particle on a curve with constant tangential acceleration

In this case (Fig. 4.24) we have

$$dv = a_\tau dt. \tag{4.90}$$

Integrating (4.90) by sides we obtain

$$\frac{ds}{dt} \equiv v = v_0 + a_\tau dt, \tag{4.91}$$

and the above equation describes the velocity of the particle during its uniformly variable motion. Now, integrating (4.91) by sides we obtain

$$s(t) = s_0 + v_0 t + a_\tau \frac{t^2}{2}, \tag{4.92}$$

and (4.92) is called an equation describing the uniformly variable motion of a particle. If $v_0 > 0$ and $a_\tau > 0$, then the motion is uniformly accelerated, and if $a_\tau < 0$, then the particle's motion is uniformly decelerated.

4.3.3 Curvilinear Coordinates

In this section we will introduce superscripts on vectors and, generally, on tensors (more information about tensor calculus is presented in Chap. 6). Let us start by motivating the introduction of the superscripts (indices).

Because of the need for a concise representation of formulas and equations and simplicity of their transformations, we can represent an arbitrary vector **a** in the following form using the basis $\mathbf{E}_n$:

$$\mathbf{a} = \mathrm{a}^n \mathbf{E}_n, \tag{4.93}$$

where the twice occurring indices at the upper and lower levels denote the summation (here $n = 1, 2, 3$). However, often apart from the Euclidean space (which is planar, though, e.g., N-dimensional) we apply different spaces, e.g., Riemannian[2] space, and the index n, which can be either a superscript or a subscript, assumes the dimension of the considered space N, so it can change from 1 to N. In the equation, we denoted the vector in the Euclidean space R^3 and thus the

[2]Georg Riemann (1826–1866), German mathematician working in the field of analysis and differential geometry.

summation of the index n proceeded from 1 to 3, but if the vector were considered in the space of dimension N, then the summation would proceed from 1 up to N.

The dot product of two vectors, e.g., in Euclidean space is equal to

$$\mathbf{a} \circ \mathbf{b} = \sum_{m=1}^{3} \sum_{n=1}^{3} \delta_{mn} a^m b^n = \sum_{n=1}^{3} a_n b^n, \tag{4.94}$$

where

$$a_n = \sum_{m=1}^{3} \delta_{mn} a^m. \tag{4.95}$$

The summation in the formula above due to the application of the *Kronecker*[2] *delta* δ_{mn} led to a decrementation of the index. However, earlier, in using a dot product, we did not use superscripts, and (4.94) and (4.95) could be effectively written using the subscripts only. It follows then that $a^m = a_m$, $b^n = b_n$, and $\delta_n^m = \delta_{mn}$. However, it turns out that we are allowed to proceed like that only when using a Cartesian coordinate system. In curvilinear systems, there exists a difference between the quantities a^m and a_m.

It should be noted then that in the Cartesian coordinate system one will not make a mistake if one uses indices at only one or both levels.

Let us additionally consider the vector product of two vectors also in Euclidean space. Denoting

$$\mathbf{a} = a^i \mathbf{E}_i, \qquad \mathbf{b} = a^j \mathbf{E}_j, \tag{4.96}$$

we have

$$\mathbf{c} \equiv \mathbf{a} \times \mathbf{b} = a^i b^j (\mathbf{E}_i \times \mathbf{E}_j). \tag{4.97}$$

An arbitrary kth component of the vector $\mathbf{c}$ can be directly obtained from the formula

$$c_k = (\mathbf{a} \times \mathbf{b}) \circ \mathbf{E}_k = a^i b^j (\mathbf{E}_i \times \mathbf{E}_j) \circ \mathbf{E}_k = a^i b^j \varepsilon_{kij}, \tag{4.98}$$

which defines the tensor of rank three (alternating tensor) of the form

$$\varepsilon_{kij} = (\mathbf{E}_i \times \mathbf{E}_j) \circ \mathbf{E}_k. \tag{4.99}$$

The alternating tensor possesses the following permutation properties:

1. $\varepsilon_{kij} = 0$ if two out of three indices are identical.
2. $\varepsilon_{kij} = 1(-1)$ if the sequence of indices k, i, j is (is not) the sequence $1, 2, 3$ or its even (odd) permutation.

[2] Leopold Kronecker (1823–1891), German mathematician and logician; born into a Jewish family (among other places, he worked in Wroclaw and Legnica).

The unknown vector **c** can be represented in the following form making use of the alternating tensor:

$$\mathbf{c} = \delta^{kl} c_k \mathbf{E}_l = \delta^{kl} \varepsilon_{kij} a^i b^j \mathbf{E}_l. \tag{4.100}$$

In the case of the Cartesian system we may use the subscripts only, and then we obtain

$$c_k = \varepsilon_{kij} a_i b_j, \qquad \mathbf{c} = \mathbf{a} \times \mathbf{b} = \varepsilon_{kij} a_i b_j \mathbf{E}_k. \tag{4.101}$$

Between the Kronecker delta δ and the alternating tensor ε the following relationship exists:

$$\varepsilon_{kij} \varepsilon_{kmn} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}. \tag{4.102}$$

The validity of the preceding relationship can be proved. If $m = i$ and $j = n$, then we have $\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = 1$, and if $j = m$ and $i = n$, then we have $\delta_{in}\delta_{jn} - \delta_{in}\delta_{jm} = -1$. This is identical to the left-hand side of (4.102) after using the properties of the alternating tensor mentioned earlier.

We will show as an example how to apply the tensor notation and use the transformations making use of the introduced concepts of Kronecker delta and alternating tensor while proving the relationship known from the calculus of vectors

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{b} \circ \mathbf{c})\mathbf{a}. \tag{4.103}$$

Successively we have

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \varepsilon_{kij} a_i b_j c_m \varepsilon_{kmn} \mathbf{E}_n = \varepsilon_{kij} \varepsilon_{kmn} a_i b_j c_m \mathbf{E}_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_i b_j c_m \mathbf{E}_n = a_m c_m \mathbf{b} - b_m c_m \mathbf{a} \\
&= (\mathbf{a} \circ \mathbf{c})\mathbf{b} - (\mathbf{b} \circ \mathbf{c})\mathbf{a}.
\end{aligned} \tag{4.104}$$

Let us consider the Euclidean space of dimension N and take an arbitrary point. Moreover, let us introduce two local systems of arbitrary coordinates, not necessarily Cartesian. Let the chosen point have the coordinates x^i in the first coordinate system and x'^j in the second one. The transition from the coordinates x^i to x'^j we define as one-to-one and continuous mapping (*homeomorphism*) and write it formally as

$$x'^j = f'^j(x^1, \ldots, x^N). \tag{4.105}$$

According to the assumption made, the *Jacobian of transformation* (4.105) has the form

$$J = \left| \frac{\partial f'^j}{\partial x^k} \right| = \left| \frac{\partial x'^j}{\partial x^k} \right| = \begin{vmatrix} \dfrac{\partial x'^1}{\partial x^1} & \cdots & \dfrac{\partial x'^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x'^N}{\partial x^1} & \cdots & \dfrac{\partial x'^N}{\partial x^N} \end{vmatrix}. \tag{4.106}$$

As a result of differentiation of (4.105) with respect to x^i we obtain

$$dx'^j = \frac{\partial x'^j}{\partial x^i} dx^i. \tag{4.107}$$

The equation just obtained can be applied also in the calculus of vectors if we assume $d\mathbf{r} = d\mathbf{r}[dx^1, \ldots, dx^N]$.

In general, the set of quantities T^i defined at the given point of space is the set of components of *the contravariant tensor of the first rank* if during the change in the coordinate system its components undergo transformations similarly to differentials, i.e., when according to (4.107) the following relationship holds:

$$T'^j = \frac{\partial x'^j}{\partial x^i} T^i, \tag{4.108}$$

and the derivatives $\frac{\partial x'^j}{\partial x^i}$ are calculated at the mentioned selected point. In this book we denote the vectors by small bold letters, e.g., $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$, whereas we will denote in majority of the cases the contravariant tensors by uppercase letters with superscripts, e.g., $A^i, B^j, C^k, \ldots$.

We then define the *contravariant tensor of the second rank* in a similar way, but it possesses the following property:

$$T'^{mn} = \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^s} T^{is}. \tag{4.109}$$

Let

$$T'^i = \frac{\partial x'^i}{\partial x^m} T^m, \qquad D'^l = \frac{\partial x'^l}{\partial x^n} D^n. \tag{4.110}$$

We will demonstrate that as a result of multiplication of the foregoing contravariant tensors of the first rank we obtain as well the contravariant tensor of the second rank of the form

$$T'^i D'^l = B'^{il}. \tag{4.111}$$

From (4.111) taking into account (4.110) we obtain

$$B'^{il} = \frac{\partial x'^i}{\partial x^m} T^m \frac{\partial x'^l}{\partial x^n} D^n = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^l}{\partial x^n} T^m D^n = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^l}{\partial x^n} B^{il}. \tag{4.112}$$

From (4.111) and (4.112) it follows that the multiplication of tensors leads to an increment of tensor order, and this is the so-called *outer multiplication of tensors* and is a generalization of the cross product of vectors. In turn, the so-called *inner multiplication of tensors* is the extension of the operation of the dot product of vectors and is connected with tensor contraction.

As a result of this operation we obtain a decrementation of the rank of the tensor of the second rank by two, that is we obtain a scalar. In the case of a scalar we have the same value regardless of the choice of coordinates, i.e., the scalar is an invariant.

The notion of gradient may serve as an example of a covariant.

Let the scalar function ϕ be given. The vector grad ϕ has the following form in the two chosen coordinate systems:

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x^i}\mathbf{E}^i, \qquad \operatorname{grad}\phi = \frac{\partial\phi}{\partial x'^k}\mathbf{E}'^k. \tag{4.113}$$

The coordinates of the vector grad ϕ in the new and old coordinate systems are interrelated through the equation

$$\phi(x^1,\ldots,x^N) = \phi[x^1(x'^1,\ldots,x'^N),\ldots,x^N(x'^1,\ldots,x'^N)]. \tag{4.114}$$

After differentiation of the foregoing composite function with respect to $\frac{\partial}{\partial x'^i}$ we obtain

$$\frac{\partial\phi}{\partial x'^i} = \frac{\partial\phi}{\partial x^l}\frac{\partial x^l}{\partial x'^i}, \tag{4.115}$$

which defines the law of transformation of the components of the gradient $\frac{\partial\phi}{\partial x^l}$ in the old coordinate system to the components of the gradient $\frac{\partial\phi}{\partial x'^i}$ in the new coordinate system.

The preceding observation is expressed as

$$T_i' = \frac{\partial x^l}{\partial x'^i}T_l, \tag{4.116}$$

and T_i', T_l are the components of the covariant tensor of the first rank.

The quantities T_{ik} are the set of components of the covariant tensor of the second rank if they undergo transformation to the coordinate system ($'$) according to the following equation:

$$T'_{mn} = \frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^n}T_{ik}. \tag{4.117}$$

One may conclude that the covariant tensors are represented by subscripts.

As we already mentioned, an arbitrary vector of the components a^i in the Cartesian coordinate system is simultaneously a covariant tensor and a contravariant tensor of the first rank.

A very important notion, used especially in Chap. 6, is the notion of a *metric tensor*.

Let d**r** denote the difference between two points in the R^3 space in the rectangular coordinate system. The distance between those points will be determined from the equation in the standard notation and the tensor notation

$$(\mathrm{d}r)^2 = (\mathrm{d}s)^2 = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2,$$

$$\mathrm{d}s^2 \equiv (\mathrm{d}s)^2 = \delta_{ik}\mathrm{d}x_i\mathrm{d}x_k. \tag{4.118}$$

The square of the distance $(\mathrm{d}s)^2$ will not change its value if the coordinate system changes because it is a scalar, that is, a tensor of a zeroth rank. Let us introduce, in addition to the Cartesian system, an arbitrary system of curvilinear coordinates. Then we have

$$\mathbf{r} = \mathbf{r}(q^1, q^2, q^3) \tag{4.119}$$

or, in equivalent notation,

$$x_i = x_i(q^1, q^2, q^3), \tag{4.120}$$

from which we obtain

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial q^s}\mathrm{d}q^s. \tag{4.121}$$

Substituting (4.121) into the second equation of (4.118) we obtain

$$\mathrm{d}s^2 = \delta_{ik}\frac{\partial x_i}{\partial q^s}\frac{\partial x_k}{\partial q^r}\mathrm{d}q^s\mathrm{d}q^r. \tag{4.122}$$

On the other hand, we have

$$\mathrm{d}s^2 = dr^2 = \frac{\partial r}{\partial q^s}\frac{\partial r}{\partial q^r}\mathrm{d}q^s\mathrm{d}q^r = g_{sr}\mathrm{d}q^s\mathrm{d}q^r, \tag{4.123}$$

and hence

$$g_{sr} = \frac{\partial r}{\partial q^s}\frac{\partial r}{\partial q^r} = \delta_{ik}\frac{\partial x_i}{\partial q^s}\frac{\partial x_k}{\partial q^r}, \tag{4.124}$$

where g_{sr} are the functions of the curvilinear coordinates q^1, q^2, q^3. It is possible to demonstrate that g_{sr} is a tensor of the second rank, and in the Cartesian coordinates g_{sr} transforms into δ_{sr}.

Because $\mathrm{d}s^2$ is the tensor of zeroth rank (the scalar), for the scalar the following relationship holds:

$$g_{sr}\mathrm{d}q^s\mathrm{d}q^r = g'_{mn}\mathrm{d}q'^m\mathrm{d}q'^n. \tag{4.125}$$

In turn, because $\mathrm{d}q^i$ are the components of contravariant tensor, taking into account

$$\mathrm{d}q'^m = \frac{\mathrm{d}q'^m}{\partial q^s}\mathrm{d}q^s, \tag{4.126}$$

in (4.125) we obtain

$$\left(g_{sr} - g'_{mn}\frac{\mathrm{d}q'^m}{\partial q^s}\frac{\mathrm{d}q'^n}{\partial q^r}\right)\mathrm{d}q^s\mathrm{d}q^r = 0. \tag{4.127}$$

Changing in the preceding formula the sequence of indices s and r one obtains

$$\left(g_{rs} - g'_{mn} \frac{\mathrm{d}q'^m}{\partial q^r} \frac{\mathrm{d}q'^n}{\partial q^s} \right) \mathrm{d}q^r \mathrm{d}q^s = 0. \tag{4.128}$$

Adding by sides (4.127) and (4.128) we obtain

$$g_{sr} + g_{rs} = \left(g'_{mn} + g'_{nm} \right) \frac{\mathrm{d}q'^m}{\partial q^s} \frac{\mathrm{d}q'^n}{\partial q^r}. \tag{4.129}$$

By definition, the tensor $g_{sr} = g_{rs}$ is symmetrical; hence from (4.129) we obtain

$$g_{sr} = g'_{mn} \frac{\mathrm{d}q'^m}{\partial q^s} \frac{\mathrm{d}q'^n}{\partial q^r}. \tag{4.130}$$

We call the tensor above, which is a covariant tensor of the second rank, a *metric tensor*. The notion of metric tensor was introduced in the discussion of the transition from the rectangular coordinate system to the curvilinear coordinate system. It turns out that rectangular coordinate systems can be introduced only in flat Euclidean spaces.

The already mentioned Riemannian space is a curved space. We will present some of the properties of metric tensor g_{il} on an example of two-dimensional Riemannian space, where the position vector $\mathbf{r}$ in the rectangular coordinate system can be expressed by the coordinates of the position point of coordinates (q^1, q^2).

A two-dimensional Riemannian space is a surface, and assuming

$$\mathbf{r} = \mathbf{r}(q^1, q^2), \tag{4.131}$$

we can determine the so-called *infinitesimal displacement* of the form

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^1} \mathrm{d}q^1 + \frac{\partial \mathbf{r}}{\partial q^2} \mathrm{d}q^2. \tag{4.132}$$

From the preceding equation we obtain

$$\mathrm{d}s^2 = \frac{\partial \mathbf{r}}{\partial q^1} \frac{\partial \mathbf{r}}{\partial q^1} \mathrm{d}q^1 \mathrm{d}q^1 + \frac{\partial \mathbf{r}}{\partial q^1} \frac{\partial \mathbf{r}}{\partial q^2} \mathrm{d}q^1 \mathrm{d}q^2 + \frac{\partial \mathbf{r}}{\partial q^2} \frac{\partial \mathbf{r}}{\partial q^1} \mathrm{d}q^2 \mathrm{d}q^1 + \frac{\partial \mathbf{r}}{\partial q^2} \frac{\partial \mathbf{r}}{\partial q^2} \mathrm{d}q^2 \mathrm{d}q^2. \tag{4.133}$$

By relying on the summation convention we can write

$$\mathrm{d}s^2 = \frac{\partial \mathbf{r}}{\partial q^j} \frac{\partial \mathbf{r}}{\partial q^k} \mathrm{d}q^j \mathrm{d}q^k = q_{jk} \mathrm{d}q^j \mathrm{d}q^k, \tag{4.134}$$

where during summation the indices change from 1 to 2, since we are dealing with a two-dimensional Riemannian space.

The introduced metric tensor g_{jk} allows for the determination of the metric (length) ds. Moreover, with its aid it is possible to define the arbitrary dot product of two vectors in space of dimension N having the form

$$\mathbf{a} \circ \mathbf{b} = g_{jk} a^j b^k, \tag{4.135}$$

and additionally we have

$$g_{jk} g^{km} = \delta_j^m. \tag{4.136}$$

The basic notions of the tensor calculus introduced so far are based on the classical works [12, 13], where the reader can find plenty of additional information related to the properties of tensors and their applications.

The transformation between the coordinates of a point in Cartesian coordinate system (x^1, x^2, x^3) and curvilinear coordinate system (q^1, q^2, q^3) has the following form:

$$\mathbf{r} = \mathbf{r}\left[q^1, q^2, q^3\right] = x^1\mathbf{E}_1 + x^2\mathbf{E}_2 + x^3\mathbf{E}_3 \tag{4.137}$$

or, in equivalent notation,

$$\begin{aligned} x^1 &= x^1\left(q^1, q^2, q^3\right), \\ x^2 &= x^2\left(q^1, q^2, q^3\right), \\ x^3 &= x^3\left(q^1, q^2, q^3\right). \end{aligned} \tag{4.138}$$

It can be assumed that three numbers, q^1, q^2, and q^3, determine uniquely the position of a point in Euclidean space and can be treated as curvilinear coordinates of that point.

If the determinant of the Jacobian[3] matrix (*the Jacobian*)

$$\det(\mathbf{J}) \neq 0, \tag{4.139}$$

where $J^i_j = \partial x^i / \partial q^j$, then (4.138) enables us to express the curvilinear coordinates of the point through its Cartesian coordinates:

$$\begin{aligned} q^1 &= q^1\left(x^1, x^2, x^3\right), \\ q^2 &= q^2\left(x^1, x^2, x^3\right), \\ q^3 &= q^3\left(x^1, x^2, x^3\right). \end{aligned} \tag{4.140}$$

[3]Carl G. Jacobi (1804–1851), great Prussian mathematician, born in a Jewish family.

It should be emphasized that despite (4.140) being theoretically motivated, in practice it is very difficult to obtain their analytical form. The assumption det($\mathbf{J} \neq 0$) guarantees that the arbitrary coordinates (x^1, x^2, x^3) possess the uniquely corresponding (q^1, q^2, q^3) coordinates and vice versa.

We say that the motion of a particle will be prescribed if the coordinates $q^i = q^i(t),\ i = 1, 2, 3$, are prescribed.

If one of the coordinates is assumed to be constant, that is, for example,

$$q_0^k = q^k\left(x^1, x^2, x^3\right), \tag{4.141}$$

then that equation describes the surface q^k corresponding to the coordinate q^k. It is defined by the set of points in the Cartesian coordinates described by (4.141).

Let us take, e.g., $k = 1$. While moving over this surface the coordinates (x^1, x^2, x^3) will be changing, which implies that, according to the two remaining (4.140), q^2 and q^3 will also undergo change.

We will use the term *first coordinate curve* to refer to the curve passing through the point $A_0(q^1, q^{20}, q^{30})$ and obtained for fixed q^2 and q^3 coordinates and for varying q^1, that is, $\mathbf{r} = \mathbf{r}(q^1, q^{20}, q^{30})$. In a similar way, we describe the *second* and *third coordinate curves*.

The radius vector $\mathbf{r}$ of an arbitrary point can be expressed as a vector-valued function of the curvilinear coordinates in the following way:

$$\mathbf{r} = x^i\left(q^1, q^2, q^3\right)\mathbf{E}_i, \quad i = 1, 2, 3, \tag{4.142}$$

where $\mathbf{E}_i$ are unit vectors of the Cartesian coordinate system.

Let us introduce a *covariant* basis of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ defined as follows:

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\mathbf{E}_j. \tag{4.143}$$

Let us note that if we calculate the derivative, e.g., for $i = 2$, then we will treat q^1 and q^3 as constants. Then, vector $\mathbf{a}_2$ will be tangent to the curve q^2 with the sense corresponding to the direction of increasing values of q^2. The same is valid for the coordinates q^1 and q^3.

Knowing the functions x^1, x^2, and x^3 [see (4.138)], the relationship between a basis of covariant vectors $\mathbf{a}_i$ and a basis of Cartesian vectors $\mathbf{E}_i$ can be written in the following matrix form [see (4.143)]:

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}. \tag{4.144}$$

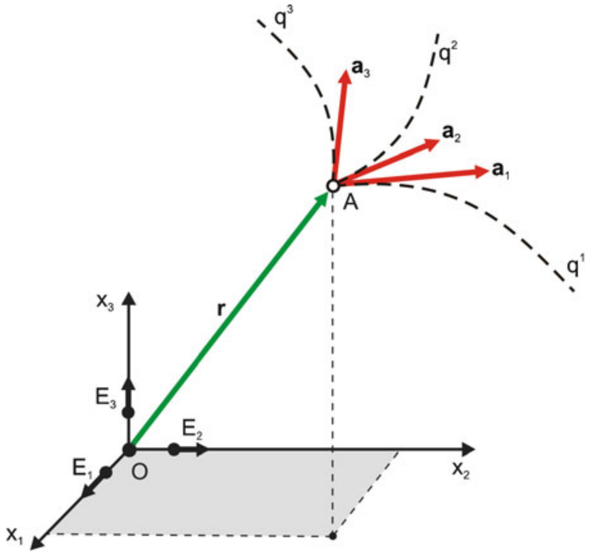


Fig. 4.25 Point A lying at an intersection of curves q^i $(i = 1, 2, 3)$ and covariant vectors $\mathbf{a}_i$

The physical meaning of (4.143) and (4.144) can be illustrated with aid of the velocity of a point described by $\mathbf{r} = x^i \mathbf{E}_i$ and $x^i = x^i(q^1, q^2, q^3)$. That is, because $\dot{x}^i = J^i_j \dot{q}^j$, we have $\mathbf{v} = \dot{x}^i \mathbf{E}_i = J^i_j \dot{q}^j \mathbf{E}_i = \dot{q}^j J^i_j \mathbf{E}_i = \dot{q}^j \mathbf{a}_j$, where $\mathbf{a}_j = J^i_j \mathbf{E}_i$, which is equivalent to its notation in matrix form (4.144). It is important that vectors $\mathbf{a}_j$ refer to the speed $\dot{q}^j$ (similar to $\mathbf{E}_i$ to $\dot{x}^i$), constitute the covariant basis for contravariant components $\dot{q}^j$, and are tangent to coordinates q^j at the actual position of the point, as illustrated in Fig. 4.25.

Let us note that just as $J^i_j = \partial x^i / \partial q^j$ is the Jacobian matrix of the expression $x^i = x^i(q^1, q^2, q^3)$, the Jacobian matrix for the relation $q^i = q^i(x^1, x^2, x^3)$ is $J'^i_j = \partial q^i / \partial x^j$. Observe that because $\dot{x}^i = J^i_j \dot{q}^j = J^i_j J'^j_k \dot{x}^k$, we have $J^i_j J'^j_k = \delta^i_k$.

With curvilinear coordinates one may also associate the triple of *basis vectors* $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ called *contravariant vectors*. Vector $\mathbf{a}^i$ is perpendicular to the surface q^i. The vectors can be described in the following way:

$$\mathbf{a}^i = J'^i_j \mathbf{E}^j, \tag{4.145}$$

which is equivalent to the matrix notation

$$\begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \end{bmatrix} = \begin{bmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \frac{\partial q^1}{\partial x^3} \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^2}{\partial x^3} \\ \frac{\partial q^3}{\partial x^1} & \frac{\partial q^3}{\partial x^2} & \frac{\partial q^3}{\partial x^3} \end{bmatrix} \begin{bmatrix} \mathbf{E}^1 \\ \mathbf{E}^2 \\ \mathbf{E}^3 \end{bmatrix}. \tag{4.146}$$

In general, neither covariant nor contravariant vectors are unit vectors or constitute an orthonormal basis, that is, they can constitute an arbitrary vector basis. However, it is easy to notice that they satisfy the following relation:

$$\begin{aligned}\mathbf{a}^j \circ \mathbf{a}_i &= J'^j_k \mathbf{E}^k J^m_i \mathbf{E}_m = J'^j_k \mathbf{E}^k \circ \mathbf{E}_m J^m_i \\ &= J'^j_k \delta^k_m J^m_i = J'^j_k J^k_i = \delta^j_i,\end{aligned} \tag{4.147}$$

because

$$\mathbf{E}^k \circ \mathbf{E}_m = \delta^k_m, \tag{4.148}$$

and δ^j_i is the Kronecker delta, i.e., $\delta^j_i = 1$, for $j = i$, $\delta^j_i = 0$ for $j \neq i$. Covariance/contravariance of the bases and coordinates is also associated with the notion of the *metric of a vector space (metric tensors)*. Obviously the metric of Cartesian space satisfies the following conditions: $\mathbf{E}^i \circ \mathbf{E}^j = \delta^{ij}$, $\mathbf{E}^i \circ \mathbf{E}_j = \delta^i_j$, and $\mathbf{E}_i \circ \mathbf{E}_j = \delta_{ij}$, which can be exploited in several branches of mechanics (but not in dynamics, where the metric additionally takes into account the inertia). Therefore, covariant and contravariant bases/coordinates of Cartesian space are often considered identical with each other.

Let us introduce *covariant unit vectors* $\mathbf{a}^0_1, \mathbf{a}^0_2, \mathbf{a}^0_3$, that is, unit vectors that are respectively tangent to $q^1(t), q^2(t), q^3(t)$:

$$\mathbf{a}^0_i = \frac{\mathbf{a}_i}{|\mathbf{a}_i|} = \frac{1}{H_i}\frac{\partial \mathbf{r}}{\partial q^i}, \tag{4.149}$$

where

$$H_i = |\mathbf{a}_i| = \left|\frac{\partial \mathbf{r}}{\partial q^i}\right| = \sqrt{\left(\frac{\partial x^1}{\partial q^i}\right)^2 + \left(\frac{\partial x^2}{\partial q^i}\right)^2 + \left(\frac{\partial x^3}{\partial q^i}\right)^2}, \tag{4.150}$$

and the quantities H_i are called *Lamé coefficients*,[4] and the assumption of $|\mathbf{a}_i| = H_i$ indicates the introduction of the unit metric.

The direction cosines of angles formed by the axes of a curvilinear coordinate system with the axes of a Cartesian coordinate system may be determined, for instance, in the following way:

$$\begin{aligned}\cos\left(\mathbf{a}^0_i, \mathbf{E}_1\right) &= \mathbf{a}^0_i \circ \mathbf{E}_1 \\ &= \frac{1}{H_i}\left(\frac{\partial x^1}{\partial q^i}\mathbf{E}_1 + \frac{\partial x^2}{\partial q^i}\mathbf{E}_2 + \frac{\partial x^3}{\partial q^i}\mathbf{E}_3\right) \circ \mathbf{E}_1 \\ &= \frac{1}{H_i}\frac{\partial x^1}{\partial q^i}.\end{aligned} \tag{4.151}$$

[4]Gabriel Lamé (1795–1870), French mathematician.

In the general case we then have

$$\cos\left(\mathbf{a}_i^0, \mathbf{E}_j\right) = \frac{1}{H_i}\frac{\partial x^j}{\partial q^i}, \quad i, j = 1, 2, 3. \tag{4.152}$$

Figure 4.25 shows a covariant basis associated with the curves q^i.

Recall that the differential operator

$$\nabla = \frac{\partial}{\partial x^1}\mathbf{E}_1 + \frac{\partial}{\partial x^2}\mathbf{E}_2 + \frac{\partial}{\partial x^3}\mathbf{E}_3 \equiv \frac{\partial}{\partial x^i}\mathbf{E}_i, \tag{4.153}$$

although not a vector, can still be treated as a so-called *symbolic (conventional) vector*. As a result of the action of ∇ on scalar function f, we obtain a vector function called a *gradient*, i.e., $\nabla f = \frac{\partial f}{\partial x^i}\mathbf{E}_i$. For example, $\mathbf{f} \circ \nabla$ and $\mathbf{f} \times \nabla$ are operators, whereas the operations $\nabla \circ \mathbf{f}$ and $\nabla \times \mathbf{f}$ lead to a scalar function and a vector function. The obtained scalar function is called a *divergence of vector function* $\operatorname{div}\mathbf{f} = \nabla \circ \mathbf{f}$.

Let $\mathbf{f} = [f_1, f_2, f_3]$. Then

$$\nabla \times f = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ f_1 & f_2 & f_3 \end{vmatrix}. \tag{4.154}$$

We call the obtained vector function a *curl* and denote it by the symbol $\operatorname{curl}\mathbf{f} = \nabla \times \mathbf{f}$.

Using the introduced notion of ∇, e.g., (4.145) may be written in the form

$$\mathbf{a}^k = \nabla q^k = \left|\nabla q^k\right| \mathbf{a}_0^k, \tag{4.155}$$

where $\mathbf{a}_0^k$ are unit vectors and

$$\left|\nabla q^k\right| = \sqrt{\left(\frac{\partial q^k}{\partial x^1}\right)^2 + \left(\frac{\partial q^k}{\partial x^2}\right)^2 + \left(\frac{\partial q^k}{\partial x^3}\right)^2} \tag{4.156}$$

denote the so-called scaling factors.

The vector ∇q^1 is perpendicular to the surface $\nabla q^1(x_1, x_2, x_3) = q_0^1$ at the chosen point.

Let us also recall that three non-coplanar vectors $\mathbf{b}_i (i = 1, 2, 3)$ form a basis if any vector can be expressed as their linear combination. The basis will be called *right-handed* (*left-handed*) if the scalar triple product of the three vectors $(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$ is positive (negative). Let us note that the scalar triple product will change sign when the right-handed coordinate system is replaced by the left-handed system or upon mirror reflection of the coordinate system (in that case, the left-handed system transforms into the right-handed one). For this reason the scalar triple product is not a real scalar but *pseudoscalar*. The scalar triple product is equal to zero (provided that none of the vectors is a zero vector) if and only if the vectors are parallel to the same plane, that is, they are coplanar.

The scalar triple product is defined in the following way:

$$(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = \mathbf{b}_1 \circ \mathbf{b}_2 \times \mathbf{b}_3 = \begin{vmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \\ b_3^1 & b_3^2 & b_3^3 \end{vmatrix}, \tag{4.157}$$

and superscripts in the determinants denote vector coordinates.

We obtain another basis $\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3$, called a *dual basis* to the chosen vectors $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$, when the following condition is satisfied:

$$\mathbf{b}_m \circ \mathbf{b}^n = \delta_m^n = \begin{cases} 1 & \text{when} \quad m = n \\ 0 & \text{when} \quad m \neq n. \end{cases} \tag{4.158}$$

The obtained property is valid for covariant and contravariant bases.

A set of three non-zero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ forms an *orthogonal basis* if they are mutually orthogonal, that is,

$$\mathbf{v}_m \circ \mathbf{v}_n = 0, \tag{4.159}$$

for all $m \neq n$.

The set of three non-zero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ forms an *orthonormal basis* if and only if it is orthogonal, i.e., condition (4.159) is satisfied, and additionally the vectors of the basis are unit vectors. The unit vectors $\mathbf{E}_1 = [1, 0, 0]^T$, $\mathbf{E}_2 = [0, 1, 0]^T$, $\mathbf{E}_3 = [0, 0, 1]^T$ used so far form the orthonormal basis.

According to the considerations above for the orthogonal system, and especially for the orthonormal system, we have

$$\mathbf{a}_i \circ \mathbf{a}_j = 0 \quad (i \neq j), \tag{4.160}$$

which after taking into account (4.149) leads to the relationship

$$\frac{\partial \mathbf{r}}{\partial q^i} \circ \frac{\partial \mathbf{r}}{\partial q^j} = 0 \quad (i \neq j). \tag{4.161}$$

Condition (4.160) narrows our considerations to orthogonal bases of the curvilinear coordinates.

Taking into account (4.143) in (4.161) we obtain

$$\frac{\partial x_1}{\partial q^i}\frac{\partial x_1}{\partial q^j} + \frac{\partial x_2}{\partial q^i}\frac{\partial x_2}{\partial q^j} + \frac{\partial x_3}{\partial q^i}\frac{\partial x_3}{\partial q^j} = 0 \quad (i \neq j), \tag{4.162}$$

and for $i = j$ we have

$$\left(\frac{\partial x_1}{\partial q^i}\right)^2 + \left(\frac{\partial x_2}{\partial q^i}\right)^2 + \left(\frac{\partial x_3}{\partial q^i}\right)^2 = H_i^2. \tag{4.163}$$

The differential of arc of a curve in a curvilinear coordinate system has the following form:

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i}\mathrm{d}q^i, \tag{4.164}$$

and multiplying the arc differentials by itself we obtain

$$(\mathrm{d}s)^2 = \mathrm{d}\mathbf{r} \circ \mathrm{d}\mathbf{r} = \sum_{i=1}^{3} H_i^2 \left(\mathrm{d}q^i\right)^2, \tag{4.165}$$

where (4.149) was used.

We determine the velocity in the curvilinear system (q^1, q^2, q^3) from the relationship

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}\left(q^1, q^2, q^3\right)}{\mathrm{d}t} = \frac{\partial \mathbf{r}}{\partial q^i}\dot{q}^i, \tag{4.166}$$

and taking into account (4.143) we have

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}q^i}\dot{q}^i = \frac{\partial x_j}{\partial q^i}\mathbf{E}_j\dot{q}^i = v_j\mathbf{E}_j. \tag{4.167}$$

Below we will show that the relationship just presented is true:

$$\begin{aligned}
\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}q^1}\dot{q}^1 + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}q^2}\dot{q}^2 + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}q^3}\dot{q}^3 \\
&= \left(\frac{\partial x_1}{\partial q^1}\mathbf{E}_1 + \frac{\partial x_2}{\partial q^1}\mathbf{E}_2 + \frac{\partial x_3}{\partial q^1}\mathbf{E}_3\right)\dot{q}^1 + \left(\frac{\partial x_1}{\partial q^2}\mathbf{E}_1 + \frac{\partial x_2}{\partial q^2}\mathbf{E}_2 + \frac{\partial x_3}{\partial q^2}\mathbf{E}_3\right)\dot{q}^2 \\
&\quad + \left(\frac{\partial x_1}{\partial q^3}\mathbf{E}_1 + \frac{\partial x_2}{\partial q^3}\mathbf{E}_2 + \frac{\partial x_3}{\partial q^3}\mathbf{E}_3\right)\dot{q}^3
\end{aligned}$$

$$= \left(\frac{\partial x_1}{\partial q^1}\dot{q}^1 + \frac{\partial x_1}{\partial q^2}\dot{q}^2 + \frac{\partial x_1}{\partial q^3}\dot{q}^3\right)\mathbf{E}_1 + \left(\frac{\partial x_2}{\partial q^1}\dot{q}^1 + \frac{\partial x_2}{\partial q^2}\dot{q}^2 + \frac{\partial x_2}{\partial q^3}\dot{q}^3\right)\mathbf{E}_2$$
$$+ \left(\frac{\partial x_3}{\partial q^1}\dot{q}^1 + \frac{\partial x_3}{\partial q^2}\dot{q}^2 + \frac{\partial x_3}{\partial q^3}\dot{q}^3\right)\mathbf{E}_3 = v_j \mathbf{E}_j, \tag{4.168}$$

where

$$v_j = \frac{\partial x_j}{\partial q^i}\dot{q}^i. \tag{4.169}$$

From (4.169) and taking into account (4.162) and (4.163) we obtain

$$v^2 = \sum_{j=1}^{3} v_j^2 = \left(\frac{\partial x_1}{\partial q^1}\right)^2 \left(\dot{q}^1\right)^2 + \left(\frac{\partial x_1}{\partial q^2}\right)^2 \left(\dot{q}^2\right)^2 + \left(\frac{\partial x_1}{\partial q^3}\right)^2 \left(\dot{q}^3\right)^2$$
$$+ \left(\frac{\partial x_2}{\partial q^1}\right)^2 \left(\dot{q}^1\right)^2 + \left(\frac{\partial x_2}{\partial q^2}\right)^2 \left(\dot{q}^2\right)^2 + \left(\frac{\partial x_2}{\partial q^3}\right)^2 \left(\dot{q}^3\right)^2$$
$$+ \left(\frac{\partial x_3}{\partial q^1}\right)^2 \left(\dot{q}^1\right)^2 + \left(\frac{\partial x_3}{\partial q^2}\right)^2 \left(\dot{q}^2\right)^2 + \left(\frac{\partial x_3}{\partial q^3}\right)^2 \left(\dot{q}^3\right)^2$$
$$= \sum_{i=1}^{3} H_i^2 \left(\dot{q}^i\right)^2. \tag{4.170}$$

In view of that, if the coordinates q^1, q^2, and q^3 are orthogonal, then we have

$$v^2 = \sum_{i=1}^{3} H_i^2 \left(\dot{q}^i\right)^2. \tag{4.171}$$

In turn, from (4.165) and dividing both sides by $(\mathrm{d}t^2)$ we obtain

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \sum_{i=1}^{3} H_i^2 \left(\dot{q}^i\right)^2. \tag{4.172}$$

Eventually, from (4.171) and (4.172) we obtain

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{4.173}$$

We obtain the coordinates of velocity of a point in curvilinear coordinates through projection of the velocity vector onto the directions of the unit vectors

$$v_i = \mathbf{v} \circ \mathbf{a}_i^0 = \frac{1}{H_i}\mathbf{v} \circ \frac{\partial \mathbf{r}}{\partial q^i} = \frac{1}{H_i}\mathbf{v} \circ \frac{\partial \mathbf{v}}{\partial \dot{q}^i} = \frac{1}{H_i}\frac{1}{2}\frac{\partial \mathbf{v}^2}{\partial \dot{q}^i} = \frac{1}{H_i}\frac{\partial T}{\partial \dot{q}^i}, \tag{4.174}$$

where $T = \frac{v^2}{2}$. The obtained quantity T can be interpreted as the kinetic energy of a particle of unit mass. It is the consequence of geometrically unitary Cartesian space, introduced earlier.

Similarly, we determine the components of the acceleration of a point in curvilinear coordinates by projecting the acceleration vector onto the directions of the unit vectors to obtain

$$\mathrm{a}_i = \mathbf{a} \circ \mathbf{a}_i^0 = \dot{\mathbf{v}} \circ \frac{1}{H_i} \frac{\partial \mathbf{r}}{\partial q^i}. \tag{4.175}$$

Let us note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v} \circ \frac{\partial \mathbf{r}}{\partial q^i}\right) = \dot{\mathbf{v}} \circ \frac{\partial \mathbf{r}}{\partial q^i} + \mathbf{v} \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}}{\partial q^i}\right), \tag{4.176}$$

and according to (4.175) we obtain

$$\mathrm{a}_i = \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v} \circ \frac{\partial \mathbf{r}}{\partial q^i}\right) - \mathbf{v} \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}}{\partial q^i}\right)\right], \tag{4.177}$$

and additionally, according to (4.174), we have

$$\mathbf{v} \circ \frac{\partial \mathbf{r}}{\partial q^i} = \frac{\partial T}{\partial \dot{q}^i}, \tag{4.178}$$

and the second term on the right-hand side of (4.177) is equal to

$$\mathbf{v} \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}}{\partial q^i}\right) = \mathbf{v} \circ \left(\frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^j}\dot{q}^j\right). \tag{4.179}$$

In turn, differentiating expression (4.166) with respect to coordinates q^i, multiplying through by $\mathbf{v}$, and changing the index on the right-hand side of the equation from i to j we have

$$\mathbf{v} \circ \frac{\partial \mathbf{v}}{\partial q^i} = \mathbf{v} \circ \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^j}\dot{q}^j. \tag{4.180}$$

Comparing (4.179) with (4.180) we obtain

$$\mathbf{v} \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}}{\partial q^i}\right) = \mathbf{v} \circ \frac{\partial \mathbf{v}}{\partial q^i} = \frac{1}{2}\frac{\partial \mathbf{v}^2}{\partial q^i} = \frac{\partial T}{\partial q^i}. \tag{4.181}$$

Finally, substituting (4.178) and (4.181) into (4.177) we obtain

$$\mathrm{a}_i = \frac{1}{H_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i}\right], \quad i = 1, 2, 3. \tag{4.182}$$

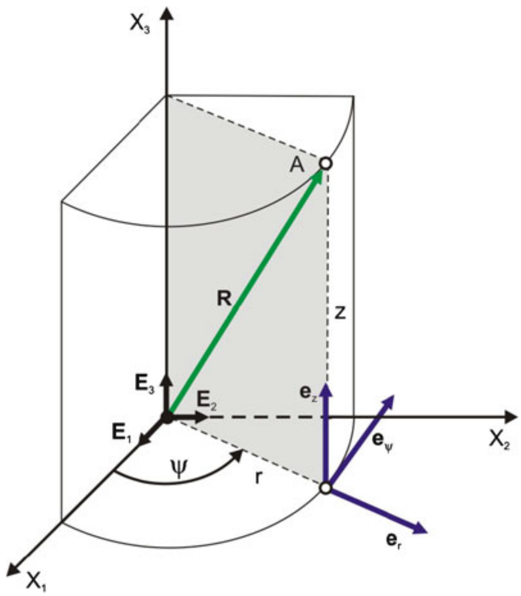


Fig. 4.26 Position of a point in Cartesian coordinates (represented by right-handed and orthonormal basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$) and in cylindrical coordinates (r, ψ, z)

Let us now consider in more detail the two most frequently used curvilinear coordinate systems, i.e., the cylindrical and the spherical coordinate systems. Figure 4.26 depicts the position of a point in cylindrical coordinates.

Knowing the Cartesian coordinates of point A of the form (x_1, x_2, x_3) it is possible to find, making use of Fig. 4.26, the position of that point in cylindrical coordinates. Namely,

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \psi = \arctan\left(\frac{x_2}{x_1}\right), \qquad z = x_3, \tag{4.183}$$

where $\psi = \left[0, 2\pi\right)$.

On the other hand, if we know the position of a point in cylindrical coordinates, the corresponding Cartesian coordinates can be determined from the following relationships:

$$x_1 = r\cos\psi, \qquad x_2 = r\sin\psi, \qquad x_3 = z. \tag{4.184}$$

From relationships (4.183) and (4.184) it follows that given the position of the point (r, ψ, z) it is possible to uniquely determine the corresponding position (x_1, x_2, x_3) and vice versa, on the condition that $x_1 \neq 0$.

If the position of the point is defined by the position vector $\mathbf{R}$, then we have

$$\mathbf{R} = \sum_{i=1}^{3} x_i \mathbf{E}_i = \mathbf{R}_{\mathbf{e_R}} = r(\cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2) + z\mathbf{E}_3 = r\mathbf{e}_r + z\mathbf{E}_3. \tag{4.185}$$

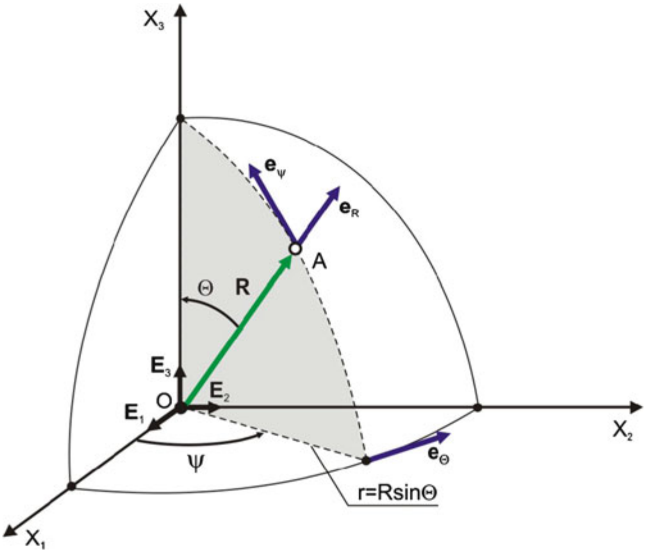


Fig. 4.27 Position of a point in Cartesian coordinates (represented by right-handed and orthonormal basis $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$) and in spherical coordinates (R, Θ, ψ)

It turns out that it is convenient to introduce the following right-handed orthonormal basis $\{\mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_z\}$ in the following way:

$$\mathbf{e}_r = \cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2, \quad \mathbf{e}_\psi = \cos\psi\mathbf{E}_2 - \sin\psi\mathbf{E}_1, \quad \mathbf{e}_z = \mathbf{E}_3. \tag{4.186}$$

Moreover, the following relationships hold true (the reader is encouraged to calculate their derivation):

$$\dot{\mathbf{e}}_r = \dot{\psi}\mathbf{e}_\psi, \qquad \dot{\mathbf{e}}_\psi = -\dot{\psi}\mathbf{e}_r. \tag{4.187}$$

Figure 4.27 shows the position of point A in spherical coordinates.

Let us note that in the case where $x_3 =$ const for the cylindrical coordinate system and $x_3 = 0$ for the spherical coordinate system (Fig. 4.27) point A can move in a plane and both systems reduce to the polar coordinate system depicted in Fig. 4.28.

In this case we have [see (4.185)]

$$\mathbf{r} = \sum_{i=1}^{2} x_i\mathbf{E}_i = r(\cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2) = r\mathbf{e}_r. \tag{4.188}$$

Knowing the Cartesian coordinates of point A in the form (x_1, x_2, x_3) we can find the position of that point in the spherical coordinates of the form

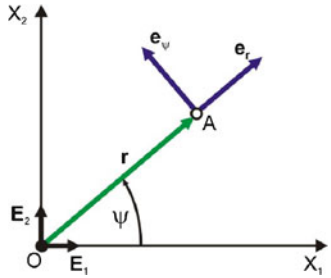


Fig. 4.28 Polar coordinate system

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \tan\psi = \frac{x_2}{x_1}, \qquad \tan\Theta = \frac{\sqrt{x_1^2 + x_2^2}}{x_3}, \tag{4.189}$$

where $\psi \in \left[0, 2\pi\right]$, $\Theta \in (0, \pi)$. Knowing the position of the point defined by the spherical coordinates (R, Θ, ψ) we have

$$\begin{aligned} x_1 &= R\cos\psi\sin\Theta, \\ x_2 &= R\sin\psi\sin\Theta, \\ x_3 &= R\cos\Theta. \end{aligned} \tag{4.190}$$

If the position of the point is described by a radius vector $\mathbf{R}$, then we have

$$\mathbf{R} = \sum_{i=1}^{3} x_i \mathbf{E}_i = R\sin\Theta[\cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2] + R\cos\Theta\mathbf{E}_3 = R\mathbf{e}_R. \tag{4.191}$$

Let us introduce the following right-handed orthonormal basis:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_\psi \end{bmatrix} = \begin{bmatrix} \cos\psi\sin\Theta & \sin\psi\sin\Theta & \cos\Theta \\ \cos\psi\cos\Theta & \sin\psi\cos\Theta & -\sin\Theta \\ -\sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}. \tag{4.192}$$

From (4.192) it follows that $\mathbf{e}_R = \sin\Theta\cos\psi\mathbf{E}_1 + \sin\Theta\sin\psi\mathbf{E}_2 + \cos\Theta\mathbf{E}_3$, etc.

It is easy to prove (the reader is encouraged to carry out the relevant calculations) that

$$\begin{bmatrix} \dot{\mathbf{e}}_R \\ \dot{\mathbf{e}}_\Theta \\ \dot{\mathbf{e}}_\psi \end{bmatrix} = \begin{bmatrix} 0 & \dot{\Theta} & \dot{\psi}\sin\Theta \\ -\dot{\Theta} & 0 & \dot{\psi}\cos\Theta \\ -\dot{\psi}\sin\Theta & -\dot{\psi}\cos\Theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\Theta \\ \mathbf{e}_\psi \end{bmatrix}. \tag{4.193}$$

Now we will determine the position, velocity, and acceleration of a particle successively in Cartesian, cylindrical, spherical, and arbitrary curvilinear coordinates q^i.

The position of a point is described by the equation

$$\sum_{i=1}^{3} x_i \mathbf{E}_i = r\mathbf{e}_r + z\mathbf{E}_3 = R\mathbf{e}_R$$
$$= \sum_{i=1}^{3} x_i \left(q^1, q^2, q^3\right) \mathbf{E}_i \, . \tag{4.194}$$

Differentiating the expression above with respect to time we obtain

$$\mathbf{v} = \sum_{i=1}^{3} \dot{x}_i \mathbf{E}_i = \dot{r}\mathbf{e}_r + r\dot{\psi}\mathbf{e}_\psi + \dot{z}\mathbf{E}_3$$
$$= \dot{R}\mathbf{e}_R + R\dot{\Theta}\mathbf{e}_\Theta + R\dot{\psi}\sin\Theta\mathbf{e}_\psi = \sum_{i=1}^{3} \dot{q}^i \frac{\partial \mathbf{r}}{\partial q^i} = \sum_{i=1}^{3} \dot{q}^i \mathbf{a}_i \, , \tag{4.195}$$

and after yet another differentiation we obtain

$$\mathbf{a} = \sum_{i=1}^{3} \ddot{x}_i \mathbf{E}_i = \left(\ddot{r} - r\dot{\psi}^2\right)\mathbf{e}_r + \left(r\ddot{\psi} + 2\dot{r}\dot{\psi}\right)\mathbf{e}_\psi + \ddot{z}\mathbf{E}_3$$
$$= \left(\ddot{R} - R\dot{\Theta}^2 - R\dot{\psi}^2\sin^2\Theta\right)\mathbf{e}_R + \left(R\ddot{\Theta} + 2\dot{R}\dot{\Theta} - R\dot{\psi}^2\sin\Theta\cos\Theta\right)\mathbf{e}_\Theta$$
$$+ \left(R\ddot{\psi}\sin\Theta + 2\dot{R}\dot{\psi}\sin\Theta + 2R\dot{\psi}\dot{\Theta}\cos\Theta\right)\mathbf{e}_\psi$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^i\mathbf{a}_i\right) = \sum_{i=1}^{3}\left(\ddot{q}^i\mathbf{a}_i + \dot{q}^i\frac{\mathrm{d}\mathbf{a}_i}{\mathrm{d}t}\right)$$
$$= \sum_{i=1}^{3} \ddot{q}^i\mathbf{a}_i + \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\dot{q}^i\dot{q}^j\frac{\partial\mathbf{a}_i}{\partial q^j}\right). \tag{4.196}$$

The kinetic energy of a particle in an arbitrary curvilinear coordinate system is given by

$$T = \frac{m\mathbf{v}^2}{2} = \frac{m}{2}\left(\sum_{i=1}^{3}\dot{q}^i\mathbf{a}_i\right)\left(\sum_{l=1}^{3}\dot{q}^l\mathbf{a}_l\right) = \frac{m}{2}\sum_{i=1}^{3}\sum_{l=1}^{3}a_{il}\dot{q}^i\dot{q}^l \, , \tag{4.197}$$

where $a_{il} = a_{li} = \mathbf{a}_l \circ \mathbf{a}_i$. The reader is advised to derive explicit expressions for the kinetic energy of a particle in Cartesian, cylindrical, and spherical coordinates.

Example 4.1. Determine the scaling factors and orthonormal covariant and contravariant bases of vectors $\mathbf{a}_i^0$ and $\mathbf{a}_0^i$ for the case of the cylindrical polar coordinate system described by the equations (Fig. 4.26)

$$x = r\cos\psi, \quad y = r\sin\psi, \quad z = z, \tag{$*$}$$

where $r > 0$, $0 \le \psi \le 2\pi$, $-\infty \le z \le \infty$.

The Jacobian of the transformation above is equal to

$$J = \frac{\partial(x, y, z)}{\partial(r, \psi, z)} = \begin{vmatrix} \cos\psi & -r\sin\psi & 0 \\ \sin\psi & r\cos\psi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2\psi + \sin^2\psi\right) = r \ne 0. \tag{$**$}$$

From ($*$) we obtain

$$q^1 = r = \sqrt{x^2 + y^2},$$
$$q^2 = \psi = \tan^{-1}\frac{y}{x},$$
$$q^3 = z.$$

The radius vector is described by the equation

$$\mathbf{r} = x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3 = r\cos\psi\mathbf{E}_1 + r\sin\psi\mathbf{E}_2 + z\mathbf{E}_3.$$

According to ($**$) we have (see also 4.186)

$$\mathbf{a}_1 = \frac{\partial\mathbf{r}}{\partial r} = \cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2 = \mathbf{e}_r,$$
$$\mathbf{a}_2 = \frac{\partial\mathbf{r}}{\partial\psi} = -r\sin\psi\mathbf{E}_1 + r\cos\psi\mathbf{E}_2 = r\mathbf{e}_\psi,$$
$$\mathbf{a}_3 = \frac{\partial\mathbf{r}}{\partial z} = \mathbf{E}_3 = \mathbf{e}_z.$$

The scaling factors are equal to

$$|\mathbf{a}_1| = \sqrt{\cos^2\psi + \sin^2\psi} = 1, \quad |\mathbf{a}_2| = r, \quad |\mathbf{a}_3| = 1.$$

The covariant basis consists of the following three vectors:

$$\mathbf{a}_1^0 = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2,$$
$$\mathbf{a}_2^0 = \frac{\mathbf{a}_2}{|\mathbf{a}_2|} = -\sin\psi\mathbf{E}_1 + \cos\psi\mathbf{E}_2,$$
$$\mathbf{a}_3^0 = \frac{\mathbf{a}_3}{|\mathbf{a}_3|} = \mathbf{E}_3.$$

To determine a contravariant basis, we calculate

$$\mathbf{a}^1 = \nabla r = \frac{\partial r}{\partial x}\mathbf{E}_1 + \frac{\partial r}{\partial y}\mathbf{E}_2 + \frac{\partial r}{\partial z}\mathbf{E}_3 = \frac{x\mathbf{E}_1 + y\mathbf{E}_2}{\sqrt{x^2 + y^2}} = \mathbf{e}_r,$$

$$\mathbf{a}^2 = \nabla \psi = \frac{\partial \psi}{\partial x}\mathbf{E}_1 + \frac{\partial \psi}{\partial y}\mathbf{E}_2 + \frac{\partial \psi}{\partial z}\mathbf{E}_3 = \frac{-y\mathbf{E}_1 + x\mathbf{E}_2}{x^2 + y^2} = \frac{1}{r}\mathbf{e}_\psi,$$

$$\mathbf{a}^3 = \nabla z = \mathbf{E}_3 = \mathbf{e}_z.$$

Scaling factors are equal to

$$|\mathbf{a}^1| = 1, \qquad |\mathbf{a}^2| = \frac{1}{r}, \qquad |\mathbf{a}^3| = 1.$$

In light of that, the contravariant vectors have the form

$$\mathbf{a}_0^1 = \frac{\nabla r}{|\nabla r|} = \mathbf{a}_1^0,$$

$$\mathbf{a}_0^2 = \frac{\nabla \psi}{|\nabla \psi|} = \mathbf{a}_2^0,$$

$$\mathbf{a}_0^3 = \frac{\nabla z}{|\nabla z|} = \mathbf{E}_3 = \mathbf{a}_3^0.$$

□

Example 4.2. The coefficients of the paraboloidal coordinate system $\{u, v, \psi\}$ are determined by the Cartesian coordinates $\{x_1, x_2, x_3\}$ in the following way:

$$q^1 = u = \pm\sqrt{x_3 + \sqrt{x_3^2 + (x_1^2 + x_2^2)}},$$

$$q^2 = v = \pm\sqrt{-x_3 + \sqrt{x_3^2 + (x_1^2 + x_2^2)}},$$

$$q^3 = \psi = \tan^{-1}\left(\frac{x_2}{x_1}\right). \qquad (*)$$

The inverse relationships to those given above have the form

$$x_1 = uv\cos\psi, \qquad x_2 = uv\sin\psi, \qquad x_3 = \frac{1}{2}\left(u^2 - v^2\right).$$

Perform the following operations:

1. Draw in plane $(\mathbf{e}_r, \mathbf{E}_3)$ some examples of surfaces u and v.
2. Draw in space $(X_1 - X_2 - X_3)$ a surface u (mark curves u and v on that surface).
3. Determine the bases of covariant and contravariant vectors.
4. Determine the singularities in the paraboloidal coordinate system and define the velocity and energy of a particle.

Recall that in the cylindrical coordinate system we have

$$\mathbf{e}_r = \cos\psi \mathbf{E}_1 + \sin\psi \mathbf{E}_2, \qquad \mathbf{e}_\psi = -\sin\psi \mathbf{E}_1 + \cos\psi \mathbf{E}_2, \qquad \mathbf{e}_z = \mathbf{E}_3,$$

and $r = \sqrt{x_1^2 + x_2^2}$.

1. In order to obtain the surface u we will take $u = \bar{u} = \text{const}$. From the first equation of (*) we obtain an equation with respect to x_3, which is the quadratic equation with respect to r, of the form

$$\bar{u}^2 = x_3 + \sqrt{x_3^2 + r^2}, \qquad \left(\bar{u}^2 - x_3\right)^2 = x_3^2 + r^2,$$

$$\bar{u}^4 - 2x_3\bar{u}^2 + x_3^2 = x_3^2 + r^2, \qquad x_3 = -\frac{1}{2\bar{u}^2}\left(r^2 - \bar{u}^4\right). \qquad (**)$$

For $r = 0$, $x_3 = \frac{1}{2}\bar{u}^2$, and for $x_3 = 0$ we have $r = \bar{u}^2$. In the plane $(\mathbf{e}_r, \mathbf{E}_3)$ surface u is given by (**) representing a curve.

That curve is presented in Fig. 4.29 for the values $u = -1,\ 1,\ 1.5,\ 2$. Rotating the presented curves through the angle 2π about the X_3 axis we obtain the surface u in the space $(X_1 - X_2 - X_3)$.

In order to obtain the surface v one should assume $v = \bar{v} = \text{const}$. From (*) we obtain

$$\bar{v}^2 = -x_3 + \sqrt{x_3^2 + r^2}, \qquad \left(\bar{v}^2 + x_3\right)^2 = x_3^2 + r^2,$$

$$\bar{v}^4 + 2x_3\bar{v}^2 + x_3^2 = x_3^2 + r^2, \qquad x_3 = \frac{1}{2\bar{v}^2}\left(r^2 - \bar{v}^4\right).$$

For $r = 0$, $\bar{v}^2 = 0$, and for $x_3 = 0$, $r = \bar{v}^2$. In the space $(\mathbf{e}_r, \mathbf{E}_3)$, the surface v is described by (4.2). In Fig. 4.30 the curves of this surface are plotted for $\bar{v} = -1,\ 1.5,\ 2$. In the space $(X_1 - X_2 - X_3)$, rotating the curve through the angle 2π about the X_3 axis we obtain the surfaces v.

From Figs. 4.29 and 4.30 it follows that by varying u and v the whole plane $(\mathbf{e}_r, \mathbf{E}_3)$ is covered by the surfaces u and v. Moreover, after rotation through angle ψ about the X_3 axis the entire three-dimensional space will be covered by the surfaces u and v. An arbitrary point of that space will be represented by three numbers (u, v, ψ).

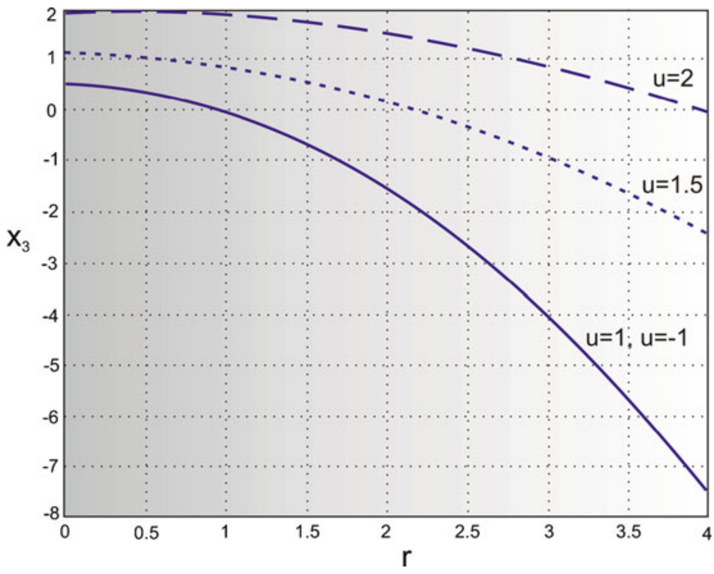


Fig. 4.29 Surfaces *u*

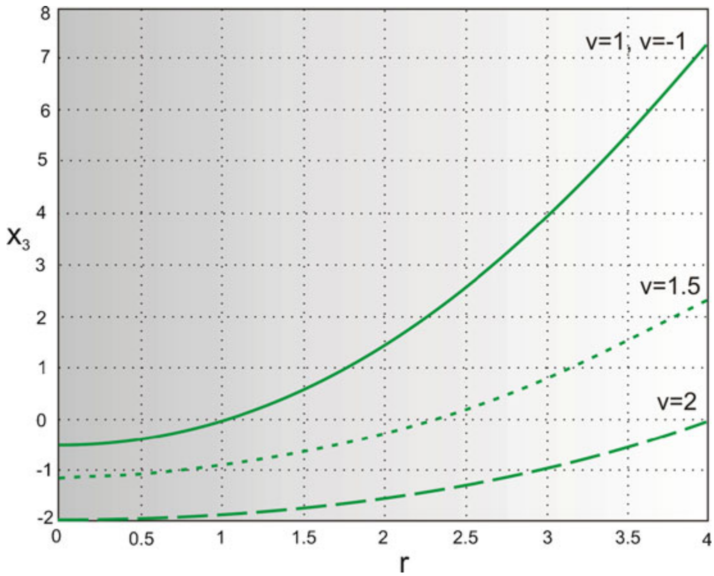


Fig. 4.30 Surfaces *v*

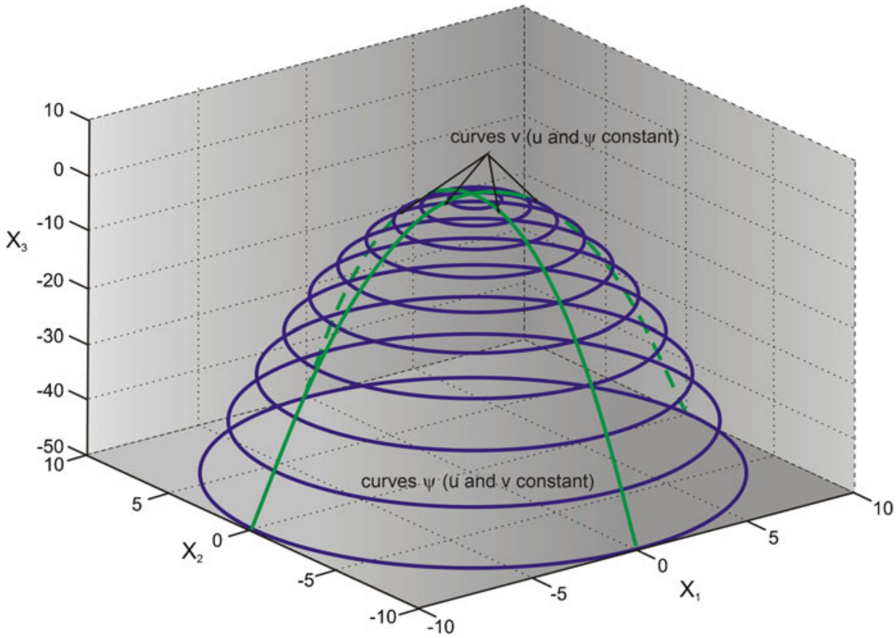


Fig. 4.31 Example of the surface $\overline{u} = 1$

2. At first, the relationship between r, u, v should be determined in the form

$$r^2 = x_1^2 + x_2^2 = (uv\cos\psi)^2 + (uv\sin\psi)^2 = u^2v^2, \qquad r = |u||v|.$$

Let us take the surface $\bar{u} = 1$ in Fig. 4.31. This coordinate is presented in the space $X_1 - X_2 - X_3$. Each circle represents a value of v; therefore we call it a ψ curve. Each arc-shaped curve represents a certain value of ψ; therefore it is called a v curve. The curves v and ψ together constitute a skeleton of the surface u.

3. The position of a point determined by the radius vector in paraboloidal coordinates is described by the formula

$$\mathbf{r} = uv\cos\psi\mathbf{E}_1 + uv\sin\psi\mathbf{E}_2 + \frac{1}{2}\left(u^2 - v^2\right)\mathbf{E}_3.$$

By definition, a covariant basis is formed by the vectors $\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial q^i}$. An example of a set of a covariant vector basis for $u = 2$, $v = 1$, and $\psi = 60°$ is shown in Fig. 4.32. We calculate successively

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}}{\partial q^1} = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial}{\partial u}\left(uv\cos\psi\mathbf{E}_1 + uv\sin\psi\mathbf{E}_2 + \frac{1}{2}(u^2 - v^2)\mathbf{E}_3\right)$$
$$= v(\cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2) + u\mathbf{E}_3 = v\mathbf{e}_r + u\mathbf{E}_3,$$

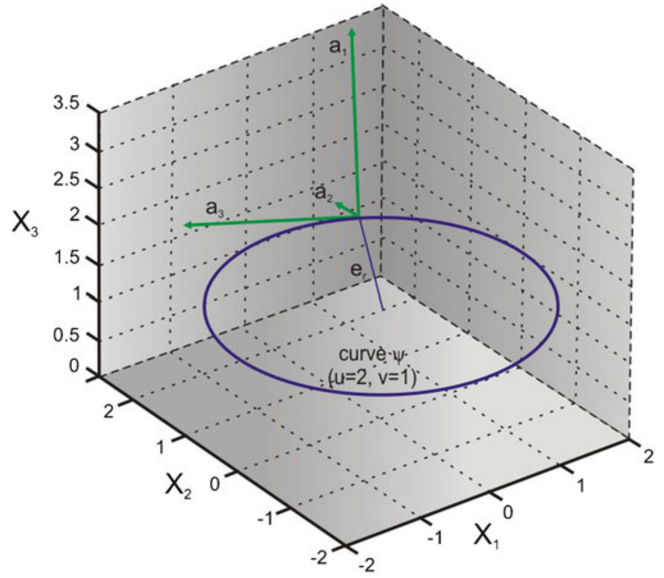


Fig. 4.32 Example of a covariant basis

$$\mathbf{a}_2 = \frac{\partial \mathbf{r}}{\partial q^2} = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial}{\partial v}\left(uv\cos\psi\mathbf{E}_1 + uv\sin\psi\mathbf{E}_2 + \frac{1}{2}(u^2 - v^2)\mathbf{E}_3\right)$$
$$= u(\cos\psi\mathbf{E}_1 + \sin\psi\mathbf{E}_2) - v\mathbf{E}_3 = u\mathbf{e}_r - v\mathbf{E}_3,$$
$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial q^3} = \frac{\partial \mathbf{r}}{\partial \psi} = \frac{\partial}{\partial \psi}\left(uv\cos\psi\mathbf{E}_1 + uv\sin\psi\mathbf{E}_2 + \frac{1}{2}(u^2 - v^2)\mathbf{E}_3\right)$$
$$= uv(-\sin\psi\mathbf{E}_1 + \cos\psi\mathbf{E}_2) = uv\mathbf{e}_\psi.$$

The contravariant basis is defined as $\mathbf{a}^i = \nabla q^i$. An example of a contravariant basis for $u = 2$, $v = 1$, and $\psi = 60°$ is shown in Fig. 4.33 ($\mathbf{a}^2$ is very small and is indicated by an arrow).

For the purpose of obtaining $\mathbf{a}^i$ we will make use of the relationship

$$u^2 + v^2 = 2\sqrt{x_3^2 + x_1^2 + x_2^2},$$

$$\mathbf{a}^1 = \nabla q^1 = \frac{\partial q^1}{\partial x_1}\mathbf{E}_1 + \frac{\partial q^1}{\partial x_2}\mathbf{E}_2 + \frac{\partial q^1}{\partial x_3}\mathbf{E}_3$$
$$= \frac{1}{2}\left(x_3 + \sqrt{x_3^2 + x_1^2 + x_2^2}\right)^{-\frac{1}{2}}\left(x_3^2 + x_1^2 + x_2^2\right)^{-\frac{1}{2}} x_1\mathbf{E}_1$$

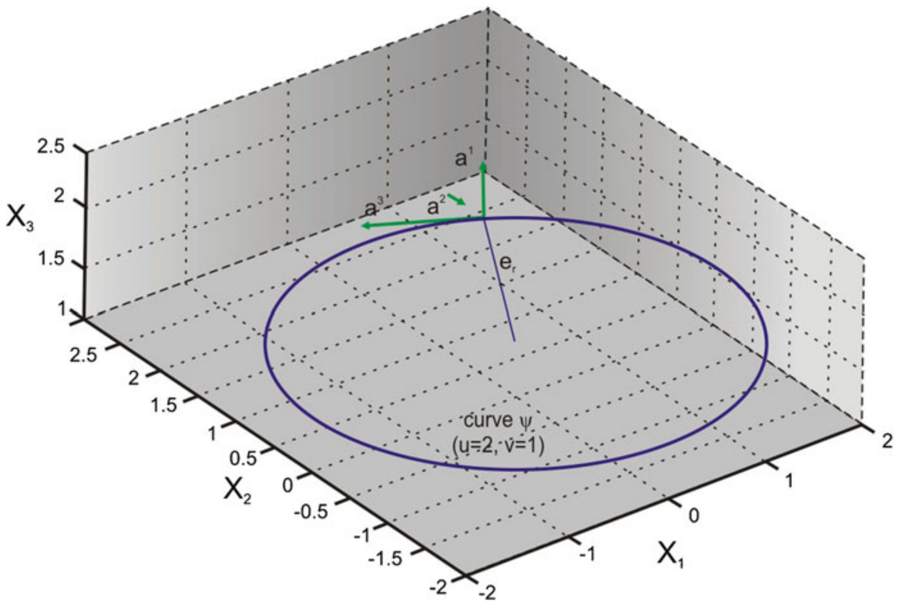


Fig. 4.33 Example of a contravariant basis

$$+\frac{1}{2}\left(x_3+\sqrt{x_3^2+x_1^2+x_2^2}\right)^{-\frac{1}{2}}\left(x_3^2+x_1^2+x_2^2\right)^{-\frac{1}{2}}x_2\mathbf{E}_2$$

$$+\frac{1}{2}\left(x_3+\sqrt{x_3^2+x_1^2+x_2^2}\right)^{-\frac{1}{2}}\left[1+\left(x_3^2+x_1^2+x_2^2\right)^{-\frac{1}{2}}x_3\right]\mathbf{E}_3.$$

Substituting the relationships derived earlier into the equation above we obtain

$$\begin{aligned}\mathbf{a}^1 &= \frac{uv\cos\psi}{u(u^2+v^2)}\mathbf{E}_1+\frac{uv\sin\psi}{u(u^2+v^2)}\mathbf{E}_2+\frac{1}{2u}\left[1+\frac{u^2-v^2}{(u^2+v^2)}\right]\mathbf{E}_3\\ &= \frac{1}{u^2+v^2}\left[v\left(\cos\psi\mathbf{E}_1+\sin\psi\mathbf{E}_2\right)+u\mathbf{E}_3\right]\\ &= \frac{1}{u^2+v^2}\left(v\mathbf{e}_r+u\mathbf{E}_3\right)=\frac{1}{u^2+v^2}\mathbf{a}_1.\end{aligned}$$

Proceeding similarly we obtain

$$\begin{aligned}\mathbf{a}^2 = \nabla q^2 &= \frac{\partial q^2}{\partial x_1}\mathbf{E}_1+\frac{\partial q^2}{\partial x_2}\mathbf{E}_2+\frac{\partial q^2}{\partial x_3}\mathbf{E}_3\\ &= \frac{1}{2}\left(-x_3+\sqrt{x_3^2+x_1^2+x_2^2}\right)^{-\frac{1}{2}}\left(x_3^2+x_1^2+x_2^2\right)^{-\frac{1}{2}}x_1\mathbf{E}_1\end{aligned}$$

$$+\frac{1}{2}\left(-x_3+\sqrt{x_3^2+x_1^2+x_2^2}\right)^{-\frac{1}{2}}\left(x_3^2+x_1^2+x_2^2\right)^{-\frac{1}{2}}x_2\mathbf{E}_2$$

$$+\frac{1}{2}\left(-x_3+\sqrt{x_3^2+x_1^2+x_2^2}\right)^{-\frac{1}{2}}\left[-1+\left(x_3^2+x_1^2+x_2^2\right)^{-\frac{1}{2}}x_3\right]\mathbf{E}_3$$

$$=\frac{uv\cos\psi}{v(u^2+v^2)}\mathbf{E}_1+\frac{uv\sin\psi}{v(u^2+v^2)}\mathbf{E}_2+\frac{1}{2u}\left[-1+\frac{u^2-v^2}{(u^2+v^2)}\right]\mathbf{E}_3$$

$$=\frac{1}{u^2+v^2}\left[u(\cos\psi\mathbf{E}_1+\sin\psi\mathbf{E}_2)-v\mathbf{E}_3\right]$$

$$=\frac{1}{u^2+v^2}\left[u\mathbf{e}_r-v\mathbf{E}_3\right]=\frac{1}{u^2+v^2}\mathbf{a}_2,$$

$$\mathbf{a}^3=\nabla q^3=\frac{\partial q^3}{\partial x_1}\mathbf{E}_1+\frac{\partial q^3}{\partial x_2}\mathbf{E}_2+\frac{\partial q^3}{\partial x_3}\mathbf{E}_3$$

$$=\frac{1}{\left(\frac{x_2}{x_1}\right)^2+1}\frac{-x_2}{x_1^2}\mathbf{E}_1+\frac{1}{\left(\frac{x_2}{x_1}\right)^2+1}\frac{1}{x_1}\mathbf{E}_2+0\mathbf{E}_3$$

$$=\frac{1}{x_1^2+x_2^2}\left[-x_2\mathbf{E}_1+x_1\mathbf{E}_2\right]$$

$$=\frac{1}{u^2v^2}\left[-uv\sin\psi\mathbf{E}_1+uv\cos\psi\mathbf{E}_2\right]=\frac{1}{uv}\left[-\sin\psi\mathbf{E}_1+\cos\psi\mathbf{E}_2\right]$$

$$=\frac{1}{uv}\mathbf{e}_\psi=\frac{1}{u^2v^2}\mathbf{a}_3.$$

4. In order to determine singularities one should examine the Jacobian of the system

$$J=\frac{\partial(x_1,x_2,x_3)}{\partial(q^1,q^2,q^3)}=\begin{vmatrix}\frac{\partial x_1}{\partial q^1}&\frac{\partial x_2}{\partial q^1}&\frac{\partial x_3}{\partial q^1}\\\frac{\partial x_1}{\partial q^2}&\frac{\partial x_2}{\partial q^2}&\frac{\partial x_3}{\partial q^2}\\\frac{\partial x_1}{\partial q^3}&\frac{\partial x_2}{\partial q^3}&\frac{\partial x_3}{\partial q^3}\end{vmatrix}=\begin{vmatrix}v\cos\psi&v\sin\psi&u\\u\cos\psi&u\sin\psi&-v\\-uv\sin\psi&uv\cos\psi&0\end{vmatrix}$$

$$=-uv\sin\psi(-v^2\sin\psi-u^2\sin\psi)-uv\cos\psi(-v^2\cos\psi-u^2\cos\psi)$$

$$=uv(v^2\sin^2\psi+u^2\sin^2\psi+v^2\cos^2\psi+u^2\cos^2\psi)=uv(v^2+u^2).$$

If $u=0$ or $v=0$, then $J=0$. This means that either $u=0$ or $v=0$ is the singularity of the paraboloidal system of coordinates. If $u=0$ or $v=0$, then $\mathbf{a}^i$ are undefined because the expressions $1/(v^2+u^2)$ and $1/(vu)$ are not defined. The velocity of the particle is equal to

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{u}\mathbf{a}_1 + \dot{v}\mathbf{a}_2 + \dot{\psi}\mathbf{a}_3$$
$$= \dot{u}\left(u^2 + v^2\right)\mathbf{a}^1 + \dot{v}\left(u^2 + v^2\right)\mathbf{a}^2 + \dot{\psi}\left(u^2v^2\right)\mathbf{a}^3.$$

The kinetic energy T reads

$$T = \frac{1}{2}m\mathbf{v} \circ \mathbf{v} = \frac{1}{2}m\left(\dot{u}\mathbf{a}_1 + \dot{v}\mathbf{a}_2 + \dot{\psi}\mathbf{a}_3\right)$$
$$\circ \left[\dot{u}\left(u^2 + v^2\right)\mathbf{a}^1 + \dot{v}\left(u^2 + v^2\right)\mathbf{a}^2 + \dot{\psi}\left(u^2v^2\right)\mathbf{a}^3\right]$$
$$= \frac{1}{2}m\left[\left(\dot{u}^2 + \dot{v}^2\right)\left(u^2 + v^2\right) + \dot{\psi}^2\left(u^2v^2\right)\right]. \qquad \square$$

4.4 Natural Coordinates

4.4.1 Introduction

The introduced notion of a *vector* will be used for the determination of the position of a particle in space. Let us recall that motion is defined as the change in position of a particle (the body) with respect to the considered coordinate system. In turn, by the change in body position we will mean the changes of its configuration in time and in a subspace of Euclidean space. Moreover, these changes will be treated as a mapping called *homeomorphism*, i.e., transforming successive points in a one-to-one naturally unique way.

4.4.2 Basic Notions

Let us use the previously considered Euclidean space, where we will introduce the Cartesian coordinate system $OX_1X_2X_3$ of basis $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$. The position of an arbitrary particle is known if the position vector $\mathbf{r}$ is known in the mentioned coordinate system. Formulas (4.84)–(4.86) describe the changes respectively of a particle's position vector, a velocity vector, and acceleration vector. According to (4.85) we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = v_1(t), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = v_2(t), \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = v_3(t). \tag{4.198}$$

We call the preceding equations *kinematic equations of motion*. Recall that *dynamic equations of motion* result from Newton's laws. The notion of *kinetic equations* is also used. The differential equations of motion describe bodies possessing mass (mass moments of inertia) and subjected to the action of forces (moments of forces). Observe that whereas dynamics deals with the science of forces and moments of forces and its special case is statics, kinetics involves only the dynamics of bodies in motion (not at rest).

Solving (4.198) we obtain

$$\begin{aligned} x_1(t) &= x_{10} + \int_0^t v_1(t)\mathrm{d}t, \\ x_2(t) &= x_{20} + \int_0^t v_2(t)\mathrm{d}t, \\ x_3(t) &= x_{30} + \int_0^t v_3(t)\mathrm{d}t. \end{aligned} \tag{4.199}$$

To determine the equation of the path of a particle one should eliminate time from (4.199). Equation (4.199) describes, then, the trajectory of motion in parametric form. It is worth emphasizing that rarely is it possible to successfully carry out the integration, that is, to determine the equation of a particle's path in analytical form. In most cases the integration is carried out using computational methods.

4.4.3 Velocities and Accelerations in Natural Coordinates

So far we have considered the motion of a particle using the vector-valued function $\mathbf{r} = \mathbf{r}(t)$. We have a prescribed curve (path) along which the particle moves. The motion along the path is described by $s = s(t)$.

Note the difference between the notions of path and trajectory. The *trajectory* of a particle conveys more information about its motion because every position is described additionally by the corresponding time instant. The *path* is rather a purely geometrical notion. Complete paths or their parts may consist of, e.g., line segments, arcs, circles, ellipses, hyperbolas, helical curves.

According to Fig. 4.34, let

$$\mathbf{r} = \mathbf{r}(s) = x_1(s)\mathbf{E}_1 + x_2(s)\mathbf{E}_2 + x_3(s)\mathbf{E}_3, \tag{4.200}$$

where s is a certain parameter (in this case the length of the arc) (Fig. 4.34).

If we take an arbitrary point O (in this case the origin of the coordinate system) and we connect this point to a point lying on the path and describing the motion of the particle, then the created vector will be a *position vector*. The limit of the

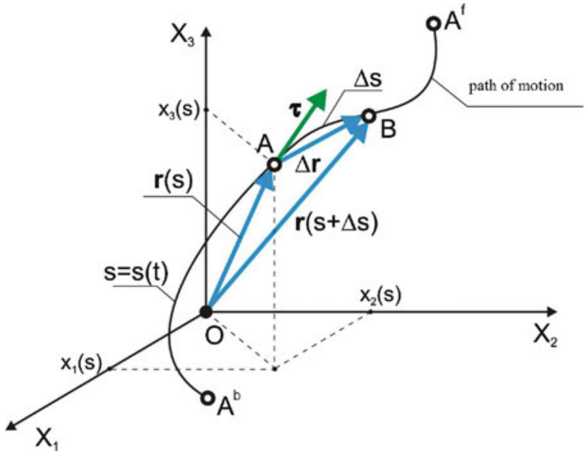


Fig. 4.34 Illustration of a vector's derivative, where $A^b(A^f)$ is the starting (final) point of the motion of point A

ratio of increment of the vector to the increment Δs for $\Delta s \to 0$ will be called the *derivative of the vector with respect to parameter* s (in this case an arc), that is,

$$\mathbf{r}'(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta s} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(s + \Delta s) - \mathbf{r}(s)}{\Delta s}. \tag{4.201}$$

The chord AB becomes the tangent when $\Delta s \to 0$.

We then come to the conclusion that the derivative of a vector is the vector tangent to the curve described by the radius vector $\mathbf{r}(s)$. The sense of $\mathbf{r}'(s)$ is determined by the sense of the curve arc s.

If in Euclidean space as a basis we choose three vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then

$$\mathbf{r} = x_1(t)\mathbf{a}_1 + x_2(t)\mathbf{a}_2 + x_3(t)\mathbf{a}_3 \tag{4.202}$$

and its derivative

$$\mathbf{r}' = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}x_1}{\mathrm{d}s}\mathbf{a}_1 + \frac{\mathrm{d}x_2}{\mathrm{d}s}\mathbf{a}_2 + \frac{\mathrm{d}x_3}{\mathrm{d}s}\mathbf{a}_3. \tag{4.203}$$

The method of differentiation presented above is true only for a stationary basis of vectors $\mathbf{a}_i$, $i = 1, 2, 3$. In the general case, rule (4.203) should be complemented by the derivatives of the basis vectors. In a special case, instead of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, we can in (4.202) and (4.203), use the basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$.

According to (4.203), we define the differential of a vector as

$$\mathrm{d}\mathbf{r} = \mathbf{r}'(s)\mathrm{d}s. \tag{4.204}$$

The differential is a vector having the same direction as the vector of derivative, and $|\mathrm{d}\mathbf{r}|$ is called the *linear arc element*.

Let us now consider the following case.

Let the vector $\mathbf{r}$ move in three-dimensional space and satisfy the condition

$$\mathbf{r} \circ \mathbf{r} = \rho^2, \tag{4.205}$$

where $\rho^2 = \text{const}$. Equation (4.205) describes the motion of the *point on a sphere*. We will trace the motion of vector $\mathbf{r}$ in a Cartesian coordinate system of unit vectors $\mathbf{E}_i$, $i = 1, 2, 3$.

Those vectors satisfy the relationships

$$\mathbf{E}_i^2 = 1, \quad \mathbf{E}_i \circ \mathbf{E}_k = 0. \tag{4.206}$$

According to (4.205) we have

$$(x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3)^2 = \rho^2, \tag{4.207}$$

hence we obtain the equation of a sphere

$$x_1^2 + x_2^2 + x_3^2 = \rho^2. \tag{4.208}$$

Let $\mathbf{r} = \mathbf{r}(s)$. Differentiating (4.205) we obtain

$$\mathbf{r} \circ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = 0. \tag{4.209}$$

This means that these two vectors are perpendicular to one another, i.e., the tangent to the sphere is perpendicular to the radius at that point.

From the preceding analysis we can draw two conclusions. First, if the vector $|\mathbf{r}| = \text{const}$, then $\mathrm{d}\mathbf{r}/\mathrm{d}s \perp \mathbf{r}$. Second, if $\mathbf{r}$ has a fixed direction, then the derivative $\mathrm{d}\mathbf{r}/\mathrm{d}s$ has the same direction as $\mathbf{r}$ (the reader is advised to prove this conclusion).

If we introduce now a unit vector $\boldsymbol{\tau}$ tangent to the curve (Fig. 4.34), then

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \boldsymbol{\tau}, \tag{4.210}$$

and in view of that,

$$\boldsymbol{\tau}^2 = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right)^2 = 1. \tag{4.211}$$

From the last equation we have

$$\mathrm{d}\mathbf{r} = \boldsymbol{\tau}\,\mathrm{d}s, \tag{4.212}$$

which implies that the differential

$$\mathrm{d}\mathbf{r}^2 = \mathrm{d}\mathbf{s}^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2. \tag{4.213}$$

If a vector is a function of more than one parameter, its differentiation can be expanded to a partial differentiation. If, e.g., $\mathbf{a} = \mathbf{a}(x_1, x_2, x_3)$, then the Cartesian coordinates can be treated as parameters. If we treat x_2 and x_3 as constants and x_1 as a variable, then the operator $\frac{\partial \mathbf{a}}{\partial x_1}$ is the limit of the ratio of vector increment to x_1 increment when the latter tends to zero.

If x_1, x_2, x_3 change simultaneously, then the total differential

$$\mathrm{d}\mathbf{a} = \frac{\partial \mathbf{a}}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \mathbf{a}}{\partial x_2}\mathrm{d}x_2 + \frac{\partial \mathbf{a}}{\partial x_3}\mathrm{d}x_3 \tag{4.214}$$

describes the total increment of the vector.

Example 4.3. (See [5]) Determine the type of curve described by a radius vector $\mathbf{r} = \alpha s \cos s\mathbf{E}_1 + \alpha s \sin s\mathbf{E}_2 + \beta s\mathbf{E}_3$. Find equations of planes normal and tangent to the curve at the point $s = 0$.

In a Cartesian coordinate system we have

$$x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3 = \alpha s \cos s\mathbf{E}_1 + \alpha s \sin s\mathbf{E}_2 + \beta s\mathbf{E}_3,$$

and in view of that,

$$x_1 = \alpha s \cos s, \qquad x_2 = \alpha s \sin s, \qquad x_3 = \beta s.$$

Eliminating parameter s we obtain the equation of the helix surface

$$x_2 x_1^{-1} = \tan\left(x_3\beta^{-1}\right)$$

and the cone equation

$$x_1^2 + x_2^2 - \alpha^2\beta^{-2}x_3^2 = 0.$$

The intersection of the found helix surface and the conical surface determines the helix wrapped around the cone (called *canonical helix curve*).
Recall that the equation of a tangent to a curve at the point $\mathbf{r}(s_0)$ is given by

$$\mathbf{r} = \mathbf{r}(s_0) + \lambda\mathbf{r}'(s_0).$$

Since the plane normal to the curve is perpendicular to the tangent, its equation is given by the following dot product:

$$[\mathbf{r} - \mathbf{r}(s_0)] \circ \mathbf{r}'(s_0) = 0.$$

Let us note that

$$\mathbf{r}'(s_0) = (\alpha \cos s - \alpha s \sin s)\mathbf{E}_1 + (\alpha \sin s + \alpha s \cos s)\mathbf{E}_2 + \beta\mathbf{E}_3.$$

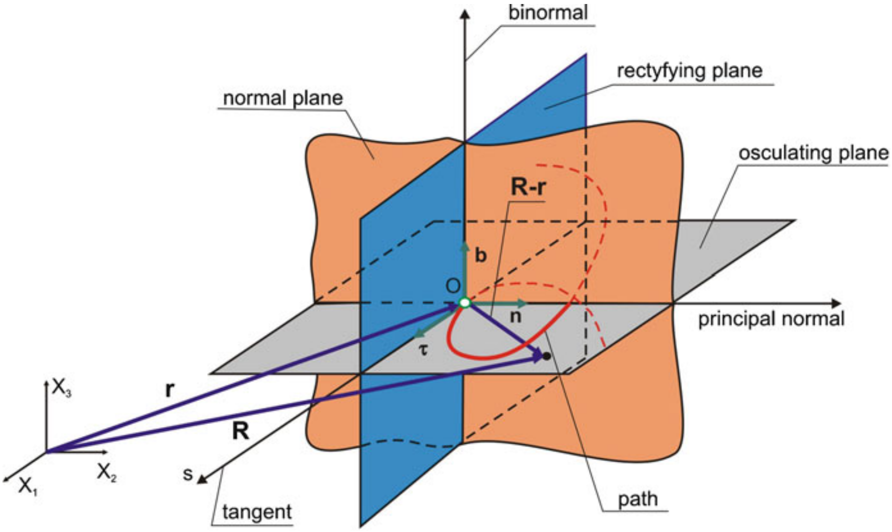


Fig. 4.35 Normal coordinate system

For the parameter $s = 0$ we have $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}'(0) = \alpha\mathbf{E}_1 + \beta\mathbf{E}_3$, and in view of that,

$$\mathbf{r} = \lambda(\alpha\mathbf{E}_1 + \beta\mathbf{E}_3),$$

defines the equation of the tangent to a curve for $s = 0$.

The equation of a plane normal to a curve at $s = 0$ has the form

$$(\alpha\mathbf{E}_1 + \beta\mathbf{E}_3) \circ (x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3) = 0,$$

hence

$$\alpha x_1 + \beta x_2 = 0. \qquad \square$$

For the purpose of further analysis of motion, velocity, and acceleration in Euclidean space, we will recall some basic notions regarding the geometry of curves [5].

With every point of a three-dimensional curve (here point O is chosen) it is possible to associate a certain Cartesian coordinate system determined by the so-called *accompanying basis (accompanying tripod)* consisting of three unit vectors $\boldsymbol{\tau}$, $\mathbf{n}$, $\mathbf{b}$ (Fig. 4.35).

We construct the aforementioned unit vectors in the following way. The vector $\boldsymbol{\tau} = \mathrm{d}\mathbf{r}/\mathrm{d}s$ lies on a tangent to the curve. The normal vector $\mathbf{n} \perp \boldsymbol{\tau}$ is defined by

$$\mathbf{n} = \frac{\mathrm{d}\boldsymbol{\tau}/\mathrm{d}s}{|\mathrm{d}\boldsymbol{\tau}/\mathrm{d}s|}, \qquad (4.215)$$

and we call it a *principal normal vector*.

The third vector is perpendicular to the two previous vectors, i.e.,

$$\mathbf{b} = \boldsymbol{\tau} \times \mathbf{n}, \tag{4.216}$$

and it is called a *binormal vector*. The vectors $\mathbf{b}, \boldsymbol{\tau}, \mathbf{n}$ form a right-handed basis, where the vector $\boldsymbol{\tau}$ is called a *unit tangent vector*.

The osculating plane determined by the vectors $\boldsymbol{\tau}$ and $\mathbf{n}$ is tangent to the path at point O. If vector $\mathbf{r}$ describes the position of point O and $\mathbf{R}$ determines an arbitrary point in the osculating plane, then the three vectors $\boldsymbol{\tau}, \mathbf{n}$, and $\mathbf{R} - \mathbf{r}$ lie in the osculating plane of the equation

$$(\mathbf{R} - \mathbf{r}, \boldsymbol{\tau}, \mathbf{n}) = 0, \tag{4.217}$$

where the left-hand side of the preceding expression denotes the scalar triple product of the three vectors [5].

Since $\mathbf{r}' = \mathrm{d}\mathbf{r}/\mathrm{d}s = \boldsymbol{\tau}$ and $\mathbf{r}'' = \rho\,\mathrm{d}^2\mathbf{r}/\mathrm{d}s^2$ (which will be demonstrated later), (4.217) can be written in the form

$$(\mathbf{R} - \mathbf{r}, \mathbf{r}', \mathbf{r}'') = 0. \tag{4.218}$$

If point R lies on the principal normal, then

$$\mathbf{R} \equiv \mathbf{r} + \lambda\mathbf{n} = \mathbf{r} + \lambda\mathbf{r}'', \tag{4.219}$$

and hence

$$\frac{X_1 - x_1}{x_1''} = \frac{X_2 - x_2}{x_2''} = \frac{X_3 - x_3}{x_3''}, \tag{4.220}$$

where $R = R(X_1, X_2, X_3)$, $r = r(x_1, x_2, x_3)$.

Let the particle O be moving along a path with velocity

$$\mathbf{v} = \boldsymbol{\tau} v. \tag{4.221}$$

The vectors of the basis move together with the particle, and the basis performs the translational and rotational motion while traveling along the path.

Our aim is to determine the vector of rotation called a *Darboux vector*[5]:

$$\boldsymbol{\omega}_{\boldsymbol{D}} = \alpha_1\boldsymbol{\tau} + \alpha_2\mathbf{n} + \alpha_3\mathbf{b}. \tag{4.222}$$

Recall that if a body rotates about a certain axis with angular velocity $\boldsymbol{\omega}$, then the linear velocity of an arbitrary point of that body is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}, \tag{4.223}$$

where $\mathbf{r}$ is a vector describing an arbitrary point of interest of the body.

[5]Jean Darboux (1842–1917), French mathematician who worked in the field of differential geometry of curves and surfaces.

The linear velocities of the tips of the basis vectors in our right-handed system are equal to

$$\frac{d\boldsymbol{\tau}}{dt} = \dot{\boldsymbol{\tau}} = \boldsymbol{\omega_D} \times \boldsymbol{\tau} = \alpha_3 \mathbf{n} - \alpha_2 \mathbf{b},$$
$$\frac{d\mathbf{n}}{dt} = \dot{\mathbf{n}} = \boldsymbol{\omega_D} \times \mathbf{n} = -\alpha_3 \boldsymbol{\tau} + \alpha_1 \mathbf{b},$$
$$\frac{d\mathbf{b}}{dt} = \dot{\mathbf{b}} = \boldsymbol{\omega_D} \times \mathbf{b} = \alpha_2 \boldsymbol{\tau} - \alpha_1 \mathbf{n}. \tag{4.224}$$

Multiplying the first of (4.224) by $\mathbf{b}$ we obtain $\alpha_2 = 0$, and the Darboux vector lies in the plane determined by vectors $\boldsymbol{\tau}$ and $\mathbf{b}$. From the first and third equations of (4.224) we determine

$$\alpha_1 = -\mathbf{n} \circ \frac{d\mathbf{b}}{dt}, \quad \alpha_3 = \mathbf{n} \circ \frac{d\boldsymbol{\tau}}{dt}, \tag{4.225}$$

from which it follows that α_1 is positive when $\mathbf{n}$ and $d\mathbf{b}$ have opposite senses.

Since, as was mentioned, the Darboux vector lies in the plane $\boldsymbol{\tau}$, $\mathbf{b}$, the coefficients α_1 and α_3 can be related to the arc measure.

From the definition of the arc measure of the angle formed by two neighboring tangents it follows that it is equal to $|d\boldsymbol{\tau}|$. In turn, the angle between two neighbouring binormals is equal to $\pm|d\mathbf{b}|$.

We call the ratio

$$\frac{|d\boldsymbol{\tau}|}{ds} = \frac{1}{\rho}, \tag{4.226}$$

where ds is an element of the arc between the neighboring tangents, the *first curvature of a curve* (always positive). We call the ratio

$$\pm\left|\frac{d\mathbf{b}}{ds}\right| = \frac{1}{k} \tag{4.227}$$

a *torsion* or a *second curvature of a curve* (positive or negative).

Let us assume that the origin of the basis $v = ds/dt = 1$, that is, $ds = dt$. Then $d\boldsymbol{\tau}/ds = d\boldsymbol{\tau}/dt$ and $\pm|d\mathbf{b}|/ds = \pm|d\mathbf{b}|/dt$, and from (4.225) we have $\alpha_1 = k^{-1}$ and $\alpha_3 = \rho^{-1}$, and the Darboux vector is equal to

$$\boldsymbol{\omega_D} = \frac{\boldsymbol{\tau}}{k} + \frac{\mathbf{b}}{\rho}. \tag{4.228}$$

In turn, from (4.224) we obtain so-called *Frenet–Serret*[6] *formulas* of the form

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{\mathbf{n}}{\rho}, \qquad \frac{d\mathbf{n}}{ds} = -\frac{\boldsymbol{\tau}}{\rho} + \frac{\mathbf{b}}{k}, \qquad \frac{d\mathbf{b}}{ds} = -\frac{\mathbf{n}}{k}. \tag{4.229}$$

[6]Both scientists came up with preceding formulas independently, J. Frenet in 1847 and J. Serret in 1851.

Let us note that, defining the so-called *third curvature of the curve* as

$$\frac{1}{\chi} = \frac{|\mathrm{d}\mathbf{n}|}{\mathrm{d}s}, \tag{4.230}$$

from the second equation of (4.229), we obtain the so-called *Lancret formula* [14] of the form

$$\chi^{-2} = \rho^{-2} + k^{-2}. \tag{4.231}$$

From the preceding equation and (4.228) it follows that

$$\frac{1}{\chi} = \sqrt{\frac{1}{\rho^2} + \frac{1}{k^2}} = |\boldsymbol{\omega}_{\boldsymbol{D}}|, \tag{4.232}$$

i.e., that the third or total curvature of the curve is equal to the magnitude of the Darboux vector.

According to the definition of the tangent vector $\boldsymbol{\tau} = \mathrm{d}\mathbf{r}/\mathrm{d}s$ and taking into account (4.229) we have

$$\mathbf{n} = \rho\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} = \rho\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{\rho}{(\mathbf{r}' \circ \mathbf{r}')^2}\left[(\mathbf{r}'\circ\mathbf{r}')\mathbf{r}'' - (\mathbf{r}'\circ\mathbf{r}'')\mathbf{r}'\right]. \tag{4.233}$$

In turn, we easily calculate that

$$\mathbf{b} = \boldsymbol{\tau} \times \mathbf{n} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \times \rho\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \rho\frac{\mathbf{r}' \times \mathbf{r}''}{(\sqrt{\mathbf{r}'\circ\mathbf{r}'})^3}. \tag{4.234}$$

It is also possible to derive the formulas for the first and second curvatures as expressed in terms of a radius vector $\mathbf{r} = \mathbf{r}(s)$, where s is the arc length.

From (4.234) we calculate the first curvature

$$\frac{1}{\rho^2} = \frac{(\mathbf{r}' \times \mathbf{r}'')^2}{(\mathbf{r}' \circ \mathbf{r}')^3}. \tag{4.235}$$

Multiplying the third equation of (4.229) by $\mathbf{n}$ we obtain

$$\frac{1}{k} = -\mathbf{n} \circ \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{(\mathbf{r}' \times \mathbf{r}'')^2} \tag{4.236}$$

The preceding formulas can be expressed by the coordinates of a vector

$$\mathbf{r}(s) = x_1(s)\mathbf{E}_1 + x_2(s)\mathbf{E}_2 + x_3(s)\mathbf{E}_3. \tag{4.237}$$

By definition of $\boldsymbol{\tau}$ and from (4.233) and (4.234) we have

$$\boldsymbol{\tau} = \frac{\mathrm{d}x_1}{\mathrm{d}s}\mathbf{E}_1 + \frac{\mathrm{d}x_2}{\mathrm{d}s}\mathbf{E}_2 + \frac{\mathrm{d}x_3}{\mathrm{d}s}\mathbf{E}_3,$$

$$\mathbf{n} = \rho\frac{\mathrm{d}^2x_1}{\mathrm{d}s^2}\mathbf{E}_1 + \rho\frac{\mathrm{d}^2x_2}{\mathrm{d}s^2}\mathbf{E}_2 + \rho\frac{\mathrm{d}^2x_3}{\mathrm{d}s^2}\mathbf{E}_3,$$

$$\mathbf{b} = \rho\begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \frac{\mathrm{d}x_1}{\mathrm{d}s} & \frac{\mathrm{d}x_2}{\mathrm{d}s} & \frac{\mathrm{d}x_3}{\mathrm{d}s} \\ \frac{\mathrm{d}^2x_1}{\mathrm{d}s^2} & \frac{\mathrm{d}^2x_2}{\mathrm{d}s^2} & \frac{\mathrm{d}^2x_3}{\mathrm{d}s^2} \end{vmatrix}. \tag{4.238}$$

In turn, the values of the radius of curvature $\frac{1}{\rho}$ and curvature k of the path at the given point are equal to

$$\frac{1}{\rho} = \left[\left(\frac{\mathrm{d}^2x_1}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2x_2}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2x_3}{\mathrm{d}s^2}\right)^2\right]^{\frac{1}{2}},$$

$$\frac{1}{k} = \left[\left(\frac{\mathrm{d}^2x_1}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2x_2}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2x_3}{\mathrm{d}s^2}\right)^2\right]^{-1}\left(\begin{vmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}s} & \frac{\mathrm{d}x_2}{\mathrm{d}s} & \frac{\mathrm{d}x_3}{\mathrm{d}s} \\ \frac{\mathrm{d}^2x_1}{\mathrm{d}s^2} & \frac{\mathrm{d}^2x_2}{\mathrm{d}s^2} & \frac{\mathrm{d}^2x_3}{\mathrm{d}s^2} \\ \frac{\mathrm{d}^3x_1}{\mathrm{d}s^3} & \frac{\mathrm{d}^3x_2}{\mathrm{d}s^3} & \frac{\mathrm{d}^3x_3}{\mathrm{d}s^3} \end{vmatrix}\right). \tag{4.239}$$

According to the basic notions introduced in Sect. 4.4.2 we have

$$\dot{\mathbf{r}} = \mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\boldsymbol{\tau},$$

$$\ddot{\mathbf{r}} = \mathbf{a} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} + v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t}. \tag{4.240}$$

The acceleration $\ddot{\mathbf{r}} = \mathbf{a}$ defined by the second of (4.240) is the sum of the tangential and normal accelerations, i.e.,

$$\mathbf{a} = \mathbf{a}_\tau + \mathbf{a}_n, \tag{4.241}$$

where

$$\mathbf{a}_\tau = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau},$$

$$\mathbf{a}_n = v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \equiv \frac{v^2}{\rho}\mathbf{n}, \tag{4.242}$$

and in the calculations of $\mathbf{a}_n$ (4.233) was taken into account.

The acceleration vector **a** lies in the osculating plane and its magnitude equals

$$\mathrm{a} = \sqrt{\dot{v}^2 + v^4\rho^{-2}}. \tag{4.243}$$

Finally, we will consider two examples excerpted from [5].

Example 4.4. Let the helix be described by the equation $\mathbf{r} = \alpha\cos u\mathbf{E}_1 + \alpha\sin u\mathbf{E}_2 + \beta u\mathbf{E}_3$, where u is a parameter to be expressed by the arc length. Determine the first and second curvatures of this curve.

The arc length equals

$$s = \int_0^u \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| \mathrm{d}u = \int_0^u \sqrt{\alpha^2 + \beta^2}\mathrm{d}u = \sqrt{\alpha^2 + \beta^2}u.$$

We substitute $u = s/\sqrt{\alpha^2 + \beta^2}$ into the equation describing the helix obtaining

$$\mathbf{r}(s) = \alpha\cos\left(\frac{s}{\sqrt{\alpha^2 + \beta^2}}\right)\mathbf{E}_1 + \alpha\sin\left(\frac{s}{\sqrt{\alpha^2 + \beta^2}}\right)\mathbf{E}_2 + \frac{\beta s}{\sqrt{\alpha^2 + \beta^2}}\mathbf{E}_3.$$

According to (4.235) we have

$$\frac{1}{\rho^2} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} \circ \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}.$$

Since

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = -\frac{\alpha}{\alpha^2 + \beta^2}\cos\left(\frac{s}{\sqrt{\alpha^2 + \beta^2}})\right)\mathbf{E}_1 - \frac{\alpha}{\alpha^2 + \beta^2}\sin\left(\frac{s}{\sqrt{\alpha^2 + \beta^2}})\right)\mathbf{E}_2,$$

we have

$$\frac{1}{\rho} = \frac{\alpha}{\alpha^2 + \beta^2}.$$

In turn, according to (4.239) we have

$$\frac{1}{k} = \frac{(\alpha^2 + \beta^2)^2}{\alpha^2}\begin{vmatrix} -\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \\ -\frac{\alpha}{\alpha^2+\beta^2}\cos\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & -\frac{\alpha}{\alpha^2+\beta^2}\sin\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & 0 \\ \frac{\alpha}{(\alpha^2+\beta^2)^{3/2}}\sin\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & -\frac{\alpha}{(\alpha^2+\beta^2)^{3/2}}\cos\left(\frac{s}{\sqrt{\alpha^2+\beta^2}}\right) & 0 \end{vmatrix}$$

$$= \frac{\beta}{\alpha^2 + \beta^2}.$$

□

Example 4.5. Demonstrate that if the position vector of a particle has the form

$$\mathbf{r} = \mathbf{r}(s) = \alpha s\mathbf{e} + \mathbf{e} \times \mathbf{a}(s),$$

where s is a length of the curve arc, α is a scalar, and $\mathbf{e}$ is a fixed unit vector, then:

(i) Tangents to the curve $\mathbf{r}(s)$ form a constant angle with $\mathbf{e}$.
(ii) The principal normal and the vector $\mathbf{e}$ are perpendicular.
(iii) The ratio ρ/k is the same for every point of the curve.

According to the definition of a vector tangent to a curve we have

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \boldsymbol{\tau} = \alpha\mathbf{e} + \mathbf{e} \times \frac{\mathrm{d}\mathbf{a}(s)}{\mathrm{d}s}.$$

In order to determine the angle we calculate

$$\boldsymbol{\tau} \circ \mathbf{e} = \left(\alpha\mathbf{e} + \mathbf{e} \times \frac{\mathrm{d}\mathbf{a}(s)}{\mathrm{d}s}\right) \circ \mathbf{e} = \alpha + (\mathbf{e} \times \mathbf{e}) \circ \frac{\mathrm{d}\mathbf{a}(s)}{\mathrm{d}s} = \alpha.$$

Note that $\mathbf{n} \parallel d\boldsymbol{\tau}/\mathrm{d}s$, and hence

$$\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} = \mathbf{e} \times \frac{\mathrm{d}^2\mathbf{a}(s)}{\mathrm{d}s^2}.$$

In turn,

$$\mathbf{e} \circ \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} = \mathbf{e} \circ \left(\mathbf{e} \times \frac{\mathrm{d}^2\mathbf{a}(s)}{\mathrm{d}s^2}\right) = (\mathbf{e} \times \mathbf{e}) \circ \frac{\mathrm{d}^2\mathbf{a}(s)}{\mathrm{d}s^2} = 0,$$

which proves that $\mathbf{n} \perp \mathbf{e}$.

From the foregoing considerations it follows that

$$\mathbf{e} \circ \mathbf{n} = 0,$$

and after differentiation with respect to s we have

$$\mathbf{e} \circ \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \mathbf{e} \circ \left(-\frac{\boldsymbol{\tau}}{\rho} + \frac{\mathbf{b}}{k}\right) = 0, \qquad (*)$$

where the second equation of (4.229) was taken into account.

If $\mathbf{n} \perp \mathbf{e}$, vectors $\boldsymbol{\tau}$, $\mathbf{b}$, and $\mathbf{e}$ lie in one plane, i.e., they are coplanar.

Because earlier it was shown that $\mathbf{e} \circ \boldsymbol{\tau} = \alpha$ and from the complementarity property it follows that $\mathbf{e} \circ \boldsymbol{\tau} = \cos\varphi$, we therefore have $\alpha = \cos\varphi$ (Fig. 4.36).

From Fig. 4.36 it follows that

$$(\mathbf{e} \circ \mathbf{b})^2 = \left[\cos\left(\frac{\pi}{2} - \varphi\right)\right]^2 = \sin^2\varphi = 1 - \cos^2\varphi = 1 - \alpha^2,$$

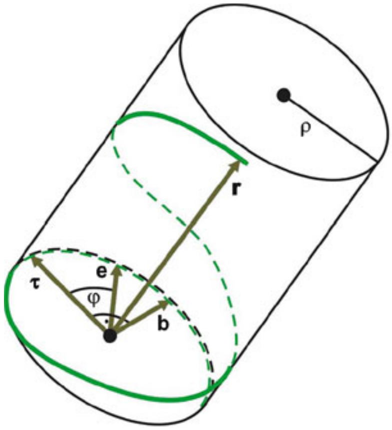


Fig. 4.36 Helix wrapped around a cylinder (vectors $\boldsymbol{\tau}$, $\mathbf{b}$, and $\mathbf{e}$ lie in one plane)

hence

$$\mathbf{e} \circ \mathbf{b} = \pm\sqrt{1-\alpha^2}.$$

From the equation (∗) we have

$$\mathbf{e} \circ \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\frac{\mathbf{e} \circ \boldsymbol{\tau}}{\rho} + \frac{\mathbf{e} \circ \mathbf{b}}{k} = -\frac{\alpha}{\rho} \pm \frac{\sqrt{1-\alpha^2}}{k} = 0,$$

that is

$$\frac{\rho}{k} = \pm\frac{\alpha}{\sqrt{1-\alpha^2}} = \text{const.}$$

The obtained properties and the property $\mathbf{e} \circ \boldsymbol{\tau} = \alpha = \text{const}$ are characteristic of the helix. □

Eventually, the results of the conducted calculations were reduced to the following (see also Fig. 4.37).

The three unit vectors $\boldsymbol{\tau}, \mathbf{n}, \mathbf{b}$ form the Frenet trihedron. Vectors $\boldsymbol{\tau}$ and $\mathbf{n}$ determine the plane tangent to the path at point O. Vector $\boldsymbol{\tau}$ possesses a sense in agreement with the positive increments of the arc length. The sense of vector $\mathbf{n}$ is directed toward the center of curvature of the path. Vector $\mathbf{b}$ is perpendicular to the plane determined by vectors $\boldsymbol{\tau}$ and $\mathbf{n}$.

We showed earlier that

$$\boldsymbol{\tau}(s) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}, \qquad \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s} = \frac{1}{\rho}\mathbf{n}(s),$$

where ρ denotes the radius of curvature of the trajectory at point O.

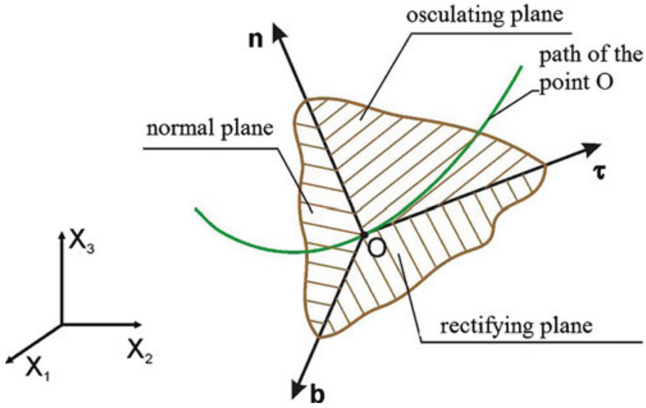


Fig. 4.37 Normal, tangent, and rectifying plane

According to the previous calculations we also have

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{s}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = v_{\tau}\boldsymbol{\tau},$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}v_{\tau}}{\mathrm{d}t}\boldsymbol{\tau} + v_{\tau}\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \ddot{s}\boldsymbol{\tau} + \frac{\dot{s}}{\rho}\mathbf{n}.$$

Since the acceleration of a particle lies always in the osculating plane, i.e., the plane determined by vectors $\boldsymbol{\tau}$ and $\mathbf{n}$, it is possible to resolve it into two components of the form

$$\mathbf{a} = \mathbf{a}_{\tau} + \mathbf{a}_{n},$$

where $\mathbf{a}_{\tau} = \ddot{s}\boldsymbol{\tau}$, $\mathbf{a}_{n} = \frac{\dot{s}^2}{\rho}\mathbf{n}$, which is in agreement with the so-called *Huygens theorem*.

The tangent acceleration is responsible for the changes in the acceleration magnitude, whereas the normal acceleration determines the direction of the particle acceleration.

We will call the motion of a particle in space *uniform* when its velocity $(\dot{s}) =$ const. Note that $\mathrm{d}(\dot{s}^2)/\mathrm{d}t = 2\dot{s}\ddot{s}$. If $\dot{s}\ddot{s} > 0$, then we will call the motion of a particle *accelerated*, otherwise *decelerated*. Moreover, if in a certain time interval $\ddot{s} = 0$, then in this time interval particle motion is uniform. Further, if in a certain time interval $\dot{s} \neq 0$ and $\mathrm{a}_n = 0$, then we are dealing with rectilinear motion because $\rho \to \infty$.

4.5 Kinematic Pairs and Chains, Joint Variables, and the Denavit–Hartenberg Convention

4.5.1 Kinematic Pairs and Chains

In the theory of machines and mechanisms, the theory of manipulators and robotics, the kinematics of the mutually connected rigid bodies plays the major role because it allows for the design of various machines, mechanisms, robots or manipulators. Most often the moving isolated bodies in one such mechanical system can be treated as rigid and pairwise connected to each other, and such bodies are called *links* (we call the fixed body a *base* (*frame*)). The connection of two links allowing for their relative motion is called a *kinematic pair*. In turn, we call the group of links connected by means of kinematic pairs a *kinematic chain*.

The lower-order kinematic pairs are presented in Table 4.1.

From an analysis of Table 4.1 it follows that a *cylindrical pair* can be replaced with a *revolute pair* and a *prismatic pair* of the same axis of action. In turn, a *universal pair* and a *spherical pair* are respectively equivalent to two and three revolute pairs of the intersecting axes. Such equivalence is exploited during introduction of the so-called *Denavit–Hartenberg*[7] (D–H) *parameters* serving the purpose of transforming coordinates from the pair number i to $i+1$ or $i-1$, $i=1,\ldots,N$, where $i=0$ corresponds to the base.

We call all kinematic pairs summarized in Table 4.1 the *lower-order kinematic pairs* because the contact of the connected links takes place over a certain surface. In the case of *line contact* or *point contact* between the links we introduce the notion of *higher-order kinematic pairs*. The *unary kinematic pairs* and *binary kinematic pairs* correspond to unilateral and bilateral constraints, discussed earlier.

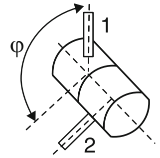
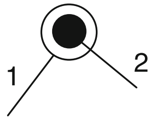
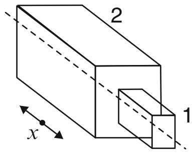
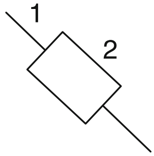
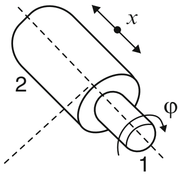
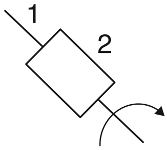
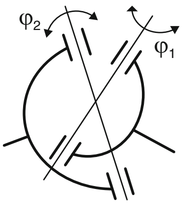
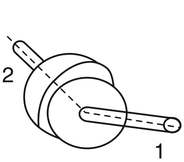
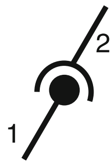
If we treat two rigid bodies as a discrete material system (DMS), the number of degrees of freedom of such a system is equal to $2\times 6=12$. Now we will join these two bodies with a revolute pair. As a single rigid body (composed of two bodies temporarily connected to one another), such a system has 6 degrees of freedom, but the additional possibility of motion admits the revolute pair, that is, the number of degrees of freedom of the two-body system connected by the revolute pair is equal to seven. The mentioned pair was taken from the equivalent system of $12-7=5$ degrees of freedom.

Definition 4.1 (Class of a kinematic pair). *The number of degrees of freedom lost by a system of two unconstrained rigid bodies after their connection by means of a given kinematic pair is called the class number of this kinematic pair.*

Definition 4.2 (A mechanism). *A kinematic chain whose number of degrees of freedom is equal to the number of its driven links is called a mechanism.*

[7] The Denavit–Hartenberg convention, parameters, and matrices were introduced by Jaques Denavit and Richard S. Hartenberg.

Table 4.1 Lower-order kinematic pairs

Pair	Example of realization	Symbol	Relative coordinates
Rotational			φ
Prismatic			x
Cylindrical			x, φ
Universal			φ_1, φ_2
Spherical			$\varphi_1, \varphi_2, \varphi_3$ e.g. Euler's angels

The number of degrees of freedom of a mechanism (the movability) is determined by the structural formulas

$$w = 5n - 4p_1 - 3p_2 - 2p_3 - p_4,$$

$$w = 4n - 3p_1 - 2p_2 - p_3,$$

$$w = 3n - 2p_1 - p_2,$$
$$w = 2n - p_1$$
$$w = n, \tag{4.244}$$

where n is the number of moveable links and p_i denotes the number of kinematic pairs of the class "i" in the analyzed mechanism.

The formulas are valid for the fifth, fourth, third, second, and first mechanism families, respectively. The third (second) family concerns plane (wedge) mechanisms, whereas the first family ($w = n$) contains rotor mechanisms.

Definition 4.3 (A group). *A kinematic chain with free kinematic pairs whose mobility is equal to zero after its connection to a base is called a structural group.*

We will classify kinematic chains in accordance with their topology. One can distinguish, for instance, *simple chains*, *tree-structured chains*, and *chains with a kinematic loop*. Chains with kinematic loops will be called *closed kinematic chains*, whereas simple and tree-structured chains will be called *open kinematic chains*. An example of the open kinematic chain is a manipulator.

4.5.2 Joint Variables and the Denavit–Hartenberg Convention

Let us consider an open kinematic chain consisting of N links, with its ith link situated between joints of the numbers $i - 1$ and i. According to the previous considerations we will assume that the joint transmits either rotational motion or translational motion. It follows that the number of joints determines the number of links. Link number 1 is connected to the base by means of one of its nodes, and the other node is used to connect to link number 2 (see also [15, 16]).

In order to explain the *D–H convention* we will consider two successive links of the numbers $i - 1$ and i depicted in Fig. 4.38.

From that figure it can be seen that the following procedure (algorithm) was adopted:

1. The sense of the $X_3^{(i)}$ axis can be arbitrarily taken.
2. The $X_1^{(i)}$ axis is perpendicular to the axes of joints i and $i + 1$ and is common for both joints. Its sense is directed toward the joint having larger number (the $X_2^{(i)}$ axis, not drawn in the figure, can be easily determined using the right-hand rule).

The advantage of the D–H notation [17] consists in its using only four control parameters (joint variables) to uniquely determine the position of the system $O_i X_1^i X_2^i X_3^i$ with respect to $O_{i-1} X_1^{i-1} X_2^{i-1} X_3^{i-1}$. They are as follows:

– The angle of rotation of the link α_i describing the rotation about the $X_1^{(i)}$ axis and formed by the $X_3^{(i-1)}$ and $X_3^{(i)}$ axes.

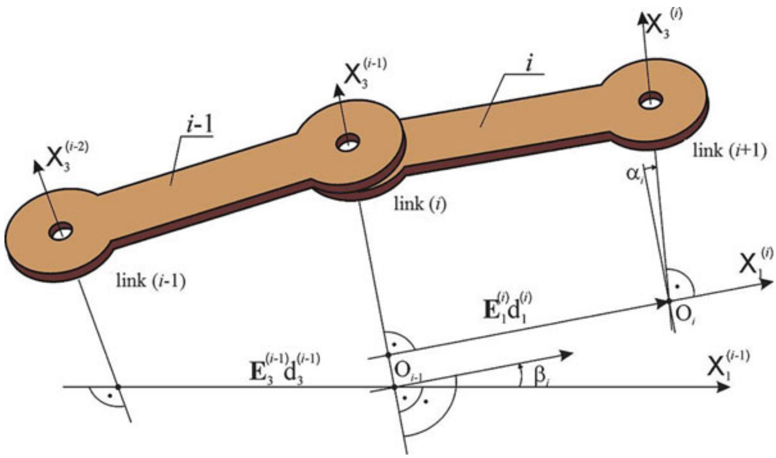


Fig. 4.38 Links $i-1$ and i and their associated local coordinate systems

– The length of link l_i measured along the $X_1^{(i)}$ axis between the $X_3^{(i-1)}$ and $X_3^{(i)}$ axes.
– The linear translation in the ith joint d_i measured along the $X_3^{(i-1)}$ axis between $X_1^{(i-1)}$ and $X_1^{(i)}$.
– The angle of rotation of the ith joint β_i about the $X_3^{(i-1)}$ axis formed by the $X_1^{(i-1)}$ and $X_1^{(i)}$ axes.

In the case of a revolute joint, the joint variable is β_i and $d_i = \text{const}$. In contrast, in the case of a prismatic joint, the joint variable is the translation d_i and $\beta_i = \text{const}$.

The transition from the system $O_{i-1}X_1^{i-1}X_2^{i-1}X_3^{i-1}$ to the system $O_i X_1^i X_2^i X_3^i$ is realized successively by means of the following four transformations:

1. The rotation about the $X_3^{(i-1)}$ axis through the angle β_i.
2. The translation along the $X_3^{(i-1)}$ axis by d_i.
3. The translation along $X_1^{(i)}$ by l_i.
4. The rotation about the $X_1^{(i)}$ axis through the angle α_i.

The successive application of the four preceding transformations is equivalent to 4×4 actions, which as a result give matrix $\mathbf{A}^{(i)}$, which is the transformation matrix between the coordinate systems of the origins at points O_{i-1} and O_i. An arbitrary point A has coordinates determined by the relationship

$$\mathbf{r}^{(i-1)} = \mathbf{A}^{(i)}\mathbf{r}^{(i)}, \tag{4.245}$$

where the homogeneous transformation matrix $\mathbf{A}^{(i)}$ includes the four transformations mentioned earlier and has the form

$$
\begin{aligned}
\mathbf{A}^{(i)} &= \boldsymbol{\beta}\left(\mathbf{E}_3^{(i-1)}\right)\mathbf{D}\left(\mathbf{E}_3^{(i-1)}\right)\mathbf{D}\left(\mathbf{E}_3^{(i)}\right)\boldsymbol{\alpha}(\mathbf{E}_3^{(i)}) \\
&= \begin{bmatrix} \cos\beta_i & -\sin\beta_i & 0 & 0 \\ \sin\beta_i & \cos\beta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot
\begin{bmatrix} 1 & 0 & 0 & d_1^{(i)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta_i & -\sin\beta_i\cos\alpha_i & \sin\beta_i\sin\alpha_i & d_1^{(i)}\cos\beta_i \\ \sin\beta_i & \cos\beta_i\cos\alpha_i & -\cos\beta_i\sin\alpha_i & d_1^{(i)}\sin\beta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_3^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{4.246}
$$

4.6 Classification of Kinematic Problems

In the case of the connections of rigid bodies by means of *constraints* and after introduction of the coordinate system of the environment (e.g., Cartesian system), with each of the successive bodies we will also associate the Cartesian coordinate system of the body. The transmission of motion from the first body to the second, then from the second to the third, etc., will proceed through the constraints understood as the connection between the bodies. We will consider this problem on an example of constraints (connections) between the bodies realized, e.g., by some human engineering activity. In the theory of machines and mechanisms, such constraints will be called *nodes* and in the theory of manipulators and in robotics, *joints*.

Selecting an arbitrary rigid body of number i we associate with it the local coordinate system $O_i X_1^i X_2^i X_3^i$. The forward problem of kinematics consists in the introduction of the successive transformations between the systems of bodies. These transformations allow for the determination of the *homogeneous matrix of transformation* for the whole system of bodies with respect to the coordinate system of the environment $OX_1X_2X_3$, on the assumption that they form the so-called *kinematic chain* (Sect. 4.5). The homogeneous matrix is formed by the resultant vector of the displacement $\mathbf{r}$ and the matrix of composite rotation $\mathbf{A}$. According to the adopted model we can also introduce *constraint variables* (node or joint variables). Recall that a node eliminates a certain number of movement possibilities (degrees of freedom) of the bodies, which it joins (from that is also derived the term *joint*). The joint variables are also subjected to the transformation mentioned above. As a result the *vector of joint variables* $\mathbf{q}$ expressed in the coordinate system $OX_1X_2X_3$ can be obtained. Prescribing various values of the joint variables one may easily determine vector $\mathbf{r}$ and matrix $\mathbf{A}$ as vector functions of $\mathbf{q}$, that is, $\mathbf{r} = \mathbf{r}(\mathbf{q})$ and $\mathbf{A} = \mathbf{A}(\mathbf{q})$. In general, vector $\mathbf{q}$ belongs to the space $\mathbf{R}^N$, where N is the number of rigid bodies connected together, i.e., $\mathbf{q} \in \mathbf{R}^N$. Then, the position ($\mathbf{r}^i$) and the orientation ($\mathbf{A}^i$) of any body i is a subset of $\mathbf{R}^6$ space, also called a *configuration space* of body i.

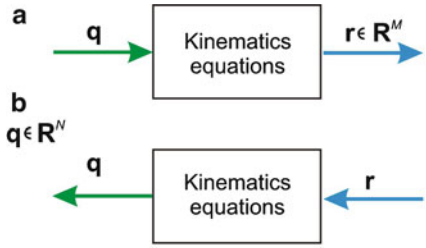


Fig. 4.39 Forward (**a**) and inverse (**b**) kinematics problem ($M \leq N$)

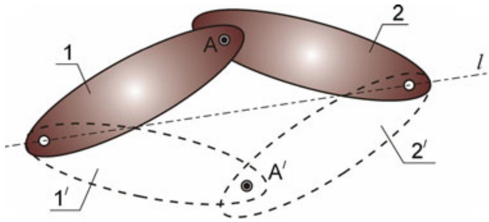


Fig. 4.40 Two different possible positions of rigid bodies connected by a revolute node

Definition 4.4. *Determination of the transformation of the joint space of connected rigid bodies into a configuration space of number N* ($\mathbf{q} \rightarrow \mathbf{r}$) *is called a forward problem of kinematics.*

Definition 4.5. *Determination of the transformation of the configuration space of the body of number N into a joint space, that is,* $\mathbf{r} \rightarrow \mathbf{q}$*, is called an inverse problem of kinematics.*

Both definitions are illustrated in Fig. 4.39.

In general, the forward kinematics problem does not lead to serious difficulties while being solved, whereas the inverse kinematic problem is generally a difficult problem.

Let us assume that we demand from a body of number N to move in a plane, i.e., to have three degrees of freedom. The joint space of the system of bodies should possess at least a dimension of three. The preselection of the space of possible movements of body N is a complex problem because it depends on the possibility and the range of movements allowed by the successive nodes (joints) of the system of bodies.

Only in special cases is the solution of the inverse kinematic problem unique. It is clearly visible on an example of revolute nodes. Already the position of the kinematic pair of the first class (two rigid bodies connected by a revolute node) allows for two possibilities for the location of such a pair that are symmetrical with respect to the axis passing through the remaining two joints of these two bodies (Fig. 4.40).

The kinematic relationships between those bodies are determined by trigonometric functions, which exhibit singularities of various types (their values can be equal to zero). Moreover, these relationships are *non-linear* and, therefore, usually the considerations regarding the motion of the connected rigid bodies cannot be used for analysis of the motion of other (even of a similar structure) systems of connections of such bodies are as well.

In the case of the analysis of an open kinematic chain of rigid bodies (links), a *forward kinematics problem* reduces to the determination of the position of the origin O^N of the coordinate system associated with body N and the determination of the orientation of that system with respect to the coordinate system of the environment $OX_1X_2X_3$.

In the case of the considered chain consisting of N bodies we have

$$\mathbf{r} = \mathbf{A}'(q_1)\mathbf{r}', \quad \mathbf{r}' = \mathbf{A}''(q_2)\mathbf{r}'', \ldots, \mathbf{r}^{N-1} = \mathbf{A}^N(q_N)\mathbf{r}^N. \tag{4.247}$$

The unknown relationship has the form

$$\mathbf{r} = \mathbf{A}'(q_1)\mathbf{A}''(q_2)\ldots\mathbf{A}^N(q_N)\mathbf{r}^N = \mathbf{A}(\mathbf{q})\mathbf{r}^N, \tag{4.248}$$

where $\mathbf{q} = [q_1, q_2, \ldots, q_N]^{\mathrm{T}}$.

The position and orientation of body N with respect to the coordinate system of the environment is determined by relationship (4.248), where $\mathbf{r}^N = \mathbf{r}^N(\mathbf{q}) \in \mathbf{R}^6$. The first three components of this vector determine the position of point O^N, whereas the three remaining ones describe its orientation with respect to $OX_1X_2X_3$.

The kinematics of connections of N rigid bodies into an open kinematic chain has very wide application in the theory of manipulators and in robotics. The changes in the vector of joint variables $\mathbf{q}^{(n)}$, $n = 1, \ldots, N$ can be realized by the mechanical and electrical interactions on nodes (joints). Not without reason did we describe in Sect. 4.4 the possibility of transforming a point's coordinates from a Cartesian system into a curvilinear system (and vice versa) using the Jacobian of the transformation. The singularities of the Jacobian of the transformation prevented the uniqueness of that transformation.

In a general case, the space in which body N moves is described by a vector $\mathbf{r}_A \in \mathbf{R}^M$, $M \le 6$, where A denotes a point of body N. In manipulator theory and in robotics, body N is a tool and the space $\mathbf{R}^M$ is called a *task space*. If the tool is axisymmetrical, the dimension of the task space is equal to five.

As a result of solution to the forward kinematics problem we obtain the relationship

$$\mathbf{r}_A = \mathbf{f}(\mathbf{q}), \tag{4.249}$$

where $\mathbf{q} = \mathbf{q}(t)$ and $\mathbf{r}_A = \mathbf{r}_A(t)$.

Coming back to our interpretation of body N as a tool, it seems obvious that the motion of that body (the tool) is imposed in advance by the requirements of, e.g., machining, the manufacturing process. In other words, having the form of $\mathbf{r}_A(t)$ imposed in advance we would like to determine the vector of joint variables $\mathbf{q} = \left[q^1, \ldots, q^N\right]^{\mathrm{T}}$ to guarantee realization of the necessary form $\mathbf{r}_A(t)$. This means that, using (4.249), we determine the unknown vector

$$\mathbf{q} = \mathbf{f}^{-1}(\mathbf{r}_A), \quad \mathbf{q} \in \mathbf{R}^N, \quad \mathbf{r}_A \in \mathbf{R}^M, \quad M \le N, \tag{4.250}$$

which indicates the realization of the inverse kinematics problem. The condition $M < N$ indicates the *redundant problem*, that is, a system of connected bodies possesses more degrees of freedom than needed to complete the task. In such a case, the inverse kinematics problem (4.250) leads to non-uniqueness of the solutions. Every point of the path can be obtained by (infinitely) many configurations of the multi-body system. In order to choose the "proper" configuration, one should exploit one of the possible optimization criteria.

Equation (4.249) is equivalent to the following matrix form:

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{bmatrix} = \begin{bmatrix} f_1(q_1, \ldots, q_N) \\ f_2(q_1, \ldots, q_N) \\ \vdots \\ f_M(q_1, \ldots, q_N) \end{bmatrix}, \tag{4.251}$$

and hence we obtain the total differential of the preceding expression (index A is omitted):

$$\begin{aligned} \mathrm{d}\mathbf{r} \equiv \begin{bmatrix} \mathrm{d}r_1 \\ \mathrm{d}r_2 \\ \vdots \\ \mathrm{d}r_M \end{bmatrix} &= \begin{bmatrix} \frac{\partial f_1}{\partial q_1}\mathrm{d}q_1 & \frac{\partial f_1}{\partial q_2}\mathrm{d}q_2 & \cdots & \frac{\partial f_1}{\partial q_N}\mathrm{d}q_N \\ \frac{\partial f_2}{\partial q_1}\mathrm{d}q_1 & \frac{\partial f_2}{\partial q_2}\mathrm{d}q_2 & \cdots & \frac{\partial f_2}{\partial q_N}\mathrm{d}q_N \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial q_1}\mathrm{d}q_1 & \frac{\partial f_M}{\partial q_2}\mathrm{d}q_2 & \cdots & \frac{\partial f_M}{\partial q_N}\mathrm{d}q_N \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_N} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_N} \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial q_1} & \frac{\partial f_M}{\partial q_2} & \cdots & \frac{\partial f_M}{\partial q_N} \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \vdots \\ \mathrm{d}q_N \end{bmatrix} \equiv \mathbf{J}(\mathbf{q})\mathrm{d}\mathbf{q}. \end{aligned} \tag{4.252}$$

We obtained the Jacobian matrix (in the theory of manipulators and in robotics the matrix $\mathbf{J}(\mathbf{q})$ is referred to as an *analytic Jacobian of a manipulator*).

Dividing both sides of (4.252) by $\mathrm{d}t$ one obtains

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \equiv \dot{\mathbf{r}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}, \tag{4.253}$$

and the preceding equation represents the *forward kinematics problem for velocity*, and $\dot{\mathbf{q}} \in \mathbf{R}^N$, $\dot{\mathbf{r}} \in \mathbf{R}^M$.

Employing the foregoing reasoning, in the case of applications, we often face the requirement of solving the *inverse kinematics problem for velocity* of the form

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\dot{\mathbf{r}}. \tag{4.254}$$

Unfortunately, this is the application side, where problems connected with the determination of the inverse Jacobian matrix $\mathbf{J}^{1}(\mathbf{q})$ emerge. Transformation (4.254) can be realized only when $\det(\mathbf{J}^{-1}(\mathbf{q})) \neq 0$. The square Jacobian matrix is singular when at least one of its rows can be expressed as a linear combination of the remaining rows (then the rank of the matrix is decremented). The singularities appear when, e.g., the axes of at least two revolute joints coincide. In the case of a manipulator, it then loses its smooth ability to move and, despite the movements of joints, the position and the orientation of the tool remain unchanged.

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Chapter 5
Kinematics of a Rigid Body and Composite Motion of a Point

5.1 Translational and Rotational Motion

5.1.1 Rigid Body in a Three-Dimensional Space and Degrees of Freedom

We will show that to determine the position of a rigid body in three-dimensional space, it is sufficient to know the position of any three of its points, but the points must not be collinear.

Figure 5.1 shows the stationary Cartesian coordinate system $OX_1X_2X_3$ and positions of a rigid body at two different time instants of motion. We assume that the positions of the points A, B, and C are known, and we will show that if so, then it is possible to determine the position of any other point of the rigid body (in the present case point D). To this end one should construct the tetrahedron $ABCD$, and then, knowing the edges CD, AD, and BD, construct the spheres of radii equal to the lengths of those edges. An intersection of those three spherical surfaces will be two points located symmetrically with respect to the plane of triangle ABC. By continuous evolution in time from the original positions t_0 one is able to make the proper choice of point D. Knowledge of the positions of points A, B, and C implies knowledge of their coordinates (x_{1A}, x_{2A}, x_{3A}), (x_{1B}, x_{2B}, x_{3B}), and (x_{1C}, x_{2C}, x_{3C}). The relative distances of points A, B, and C do not change in time, thus

$$\begin{aligned}
(x_{1A} - x_{1B})^2 + (x_{2A} - x_{2B})^2 + (x_{3A} - x_{3B})^2 &= r_{AB}^2, \\
(x_{1A} - x_{1C})^2 + (x_{2A} - x_{2C})^2 + (x_{3A} - x_{3C})^2 &= r_{AC}^2, \\
(x_{1B} - x_{1C})^2 + (x_{2B} - x_{2C})^2 + (x_{3B} - x_{3C})^2 &= r_{BC}^2,
\end{aligned} \tag{5.1}$$

where $r_{AB} = AB, r_{AC} = AC, r_{BC} = BC$.

J. Awrejcewicz, *Classical Mechanics: Kinematics and Statics*, Advances in Mechanics and Mathematics 28, DOI 10.1007/978-1-4614-3791-8_5,

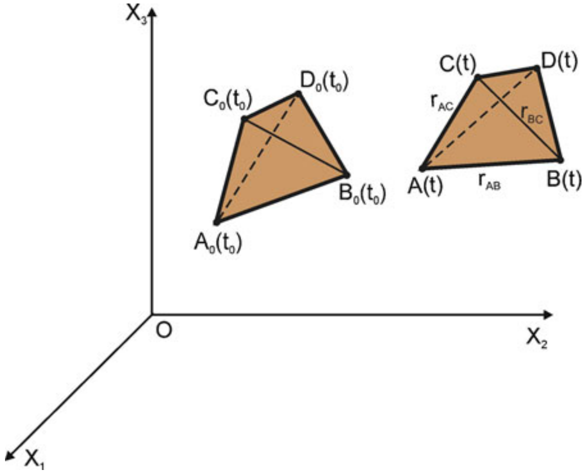


Fig. 5.1 Positions of points of a rigid body at the time instant t_0 and t

Let us note that three out of nine coordinates defining the position of points A, B, and C can be determined from (5.1), thus only six can be chosen arbitrarily.

It follows that in order to uniquely define the position of a rigid body moving freely (without constraints) in space, one should know six independent parameters. They uniquely describe the instantaneous position of the rigid body and are called its *degrees of freedom*.

As we will see later, the choice of parameters that determine the number of degrees of freedom need not be associated with the selection of three non-collinear points of the rigid body.

For instance, let us choose three coordinates of point A_1. The position of the second point A_2 can be determined with the aid of only two coordinates with respect to point A_1 because point A_2 can move on the surface of the sphere of the center A_1 (the motion of the point on the surface is determined by two coordinates). Points A_1 and A_2 determine the line A_1A_2. The position of point A_3 is determined by one coordinate with respect to that line by means of drawing the plane passing through point A_3 and perpendicular to that line. The path of that point will be a circle lying in the mentioned plane.

The motion of point A_3 is uniquely described by the radius of that circle.

Let us assume now that one of the points of the considered rigid body (e.g., point A) becomes fixed. All the remaining points of that body move along the paths lying on the concentric spheres of centers at point A (Fig. 5.2).

We will call the motion of a rigid body in space with one of the points fixed (this point may not belong to the body) the *motion about a point*. Now the position of the rigid body is determined by the position of points B and C, that is, by six parameters (coordinates). However, three equations (5.1) must be satisfied, and in light of that, the motion about a point of a rigid body with one point fixed has three degrees of freedom.

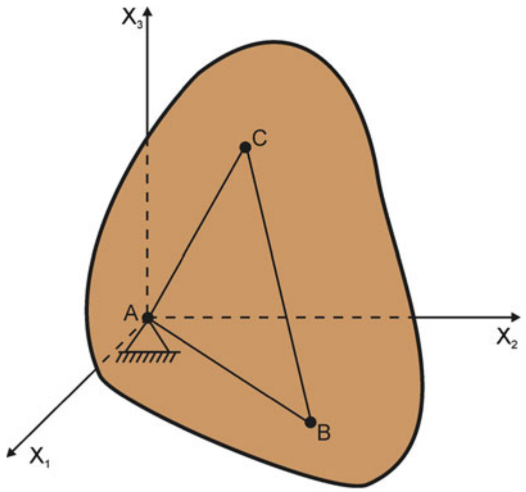


Fig. 5.2 A rigid body in motion about a point

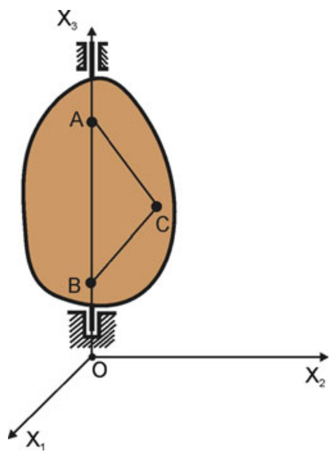


Fig. 5.3 Rotational motion of a rigid body

Let us further constrain the motion of the considered body, i.e., we now fix its two points A and B. The line determined by these two points is called an *axis of rotation* of a rigid body, and the motion is called the *rotational motion* (Fig. 5.3).

In this case, in order to determine the instantaneous position of the considered rigid body, it suffices to know the position of any point of the body (e.g., point C) not lying on the axis of rotation. The position of points A and B does not play any role here (they lie on the axis of rotation), and in view of that the distance of point C from points A and B is determined by the last two equations of the system (5.1). Because point C has three coordinates (x_{1C}, x_{2C}, x_{3C}), and we have two equations, only one of those coordinates can be chosen arbitrarily. This means that a rigid body with two points fixed has one degree of freedom.

The number of degrees of freedom determines the number of equations necessary for a complete description of rigid-body motion.

If our rigid body is, for example, a rigid rod, it will have five degrees of freedom (one degree of freedom associated with the rotation about the rod axis is neglected because $d \ll l$, where d is the rod cross-sectional dimension and l its length).

The investigation of the kinematics of a rigid body reduces to the knowledge connected with the requirement of realization of the necessary rigid-body motion and to the determination of all the kinematic characteristics of arbitrary points of a rigid body based on the knowledge of certain (a few) kinematic characteristics of the body.

Let us consider two states of a rigid body, i.e., initial and final state. We call the transition from the initial to the final state the *displacement of a rigid body*. One of the tasks of kinematics is the transformation of the position of a body from the initial to the final state, but neglecting the real process of body motion and its duration.

We distinguish the following three types of displacement of a rigid body.

Translation of a rigid body is a kind of displacement where all the points of the body perform a geometrically identical motion (rectilinear or curvilinear).

Rotation of a rigid body is a kind of displacement where the final state of the body is obtained from its initial state as a result of rotation about a certain line called the axis of rotation.

Screw displacement of a rigid body is the combination of translation and rotation such that the direction of rotation vector is coincident with the direction of translation.

5.1.2 Velocity of Points of a Rigid Body

Let us consider the relationships between velocities of two arbitrary points of a rigid body (Fig. 5.4).

The positions of the points are described by radius vectors $\mathbf{r}_{A_1}$ and $\mathbf{r}_{A_2}$ attached at the origin of the Cartesian coordinate system $OX_1X_2X_3$. The velocities of those points are equal to

$$\mathbf{v}_{A_1} = \frac{\mathrm{d}\mathbf{r}_{A_1}}{\mathrm{d}t}, \quad \mathbf{v}_{A_2} = \frac{\mathrm{d}\mathbf{r}_{A_2}}{\mathrm{d}t}, \tag{5.2}$$

and according to Fig. 5.4 we have

$$\mathbf{r}_{A_1A_2} = \mathbf{r}_{A_2} - \mathbf{r}_{A_1}. \tag{5.3}$$

Let us note that although the vector $\mathbf{r}_{A_1A_2}$ (of the tail at A_1 and tip at A_2) changes direction and position, its magnitude is always the same, i.e.,

$$\mathbf{r}_{A_1A_2} \circ \mathbf{r}_{A_1A_2} = r^2_{A_1A_2}. \tag{5.4}$$

Differentiating (5.4) with respect to time we obtain

$$\mathbf{r}_{A_1A_2} \circ \frac{\mathrm{d}\mathbf{r}_{A_1A_2}}{\mathrm{d}t} = 0. \tag{5.5}$$

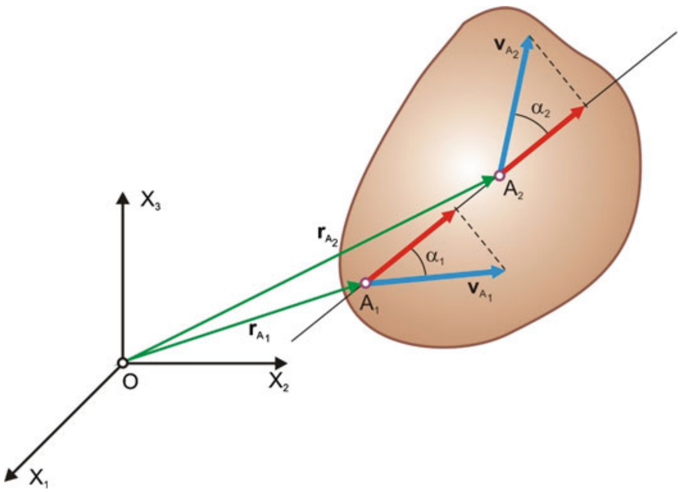


Fig. 5.4 Projections of velocities of points A_1 and A_2 onto the line passing through these points

Differentiating (5.3) with respect to time and taking into account (5.2) we obtain

$$\frac{\mathrm{d}\mathbf{r}_{A_1A_2}}{\mathrm{d}t} = \mathbf{v}_{A_2} - \mathbf{v}_{A_1}. \tag{5.6}$$

Substituting (5.6) into (5.5) we have

$$\mathbf{r}_{A_1A_2} \circ \mathbf{v}_{A_2} = \mathbf{r}_{A_1A_2} \circ \mathbf{v}_{A_1}. \tag{5.7}$$

Taking into account the definition of scalar product of vectors and Fig. 5.4 we obtain

$$v_{A_2} \cos \alpha_2 = v_{A_1} \cos \alpha_1. \tag{5.8}$$

The obtained relationship enables us to formulate the following theorem.

Theorem 5.1. *The projections of velocities of two arbitrary points of a rigid body onto the line passing through these points are equal.*

5.1.3 Translational Motion

If during the motion of a rigid body an arbitrary line that belongs to that body remains continuously parallel to its previous positions, we call such motion the *translational motion*.

As can be seen in Fig. 5.5, line AB (corresponding to time instant t) is parallel to line A_1B_1 ($t + \Delta t$), and segments AB and A_1B_1 are equal. Segment AB underwent *translation* (parallel movement) and assumed the position A_1B_1. In the figure, paths

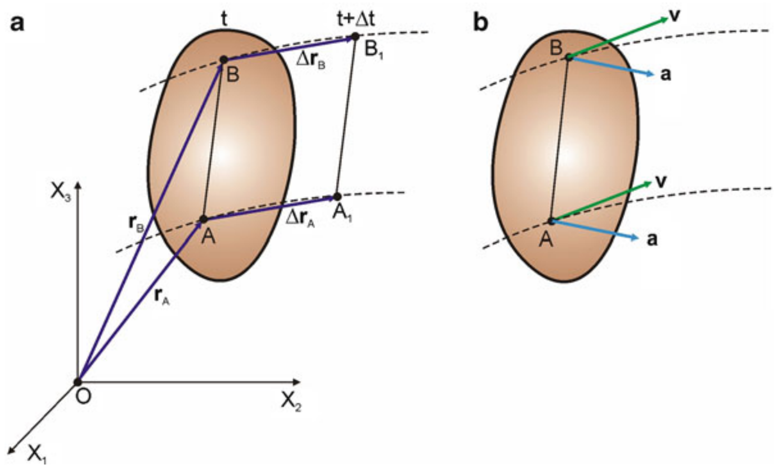


Fig. 5.5 Translational motion of a rigid body ($AB \parallel A_1B_1$ and $AB = A_1B_1$) (**a**) and velocities and accelerations of points A and B (**b**)

of two arbitrary points are indicated by dotted lines (the paths can be curvilinear). We can determine velocities of points A and B from the following formulas:

$$\mathbf{v}_A = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}_A}{\Delta t}, \quad \mathbf{v}_B = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}_B}{\Delta t}. \tag{5.9}$$

However, as we have already mentioned, $\Delta \mathbf{r}_A = \Delta \mathbf{r}_B$, and from (5.9) we obtain

$$\mathbf{v}_A = \mathbf{v}_B = \mathbf{v}. \tag{5.10}$$

Recall that points A and B are arbitrary; therefore, the following theorem is valid.

Theorem 5.2. *During translational motion of a rigid body the velocities of all its points are equal.*

We will denote such a velocity by the symbol $\mathbf{v}$ and call it the *velocity of translational motion*.

The accelerations of points A and B are equal to

$$\mathbf{a}_A = \frac{\mathrm{d}\mathbf{v}_A}{\mathrm{d}t}, \quad \mathbf{a}_B = \frac{\mathrm{d}\mathbf{v}_B}{\mathrm{d}t}. \tag{5.11}$$

Taking into account (5.10) we obtain

$$\mathbf{a}_A = \mathbf{a}_B = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{a}. \tag{5.12}$$

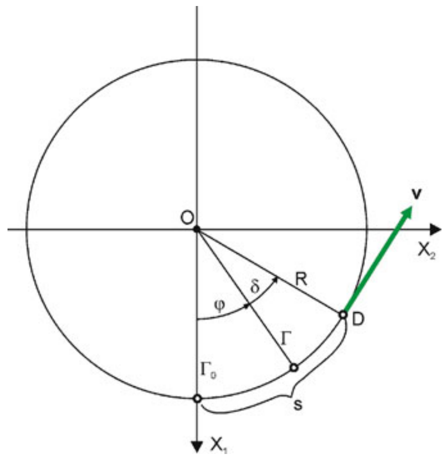


Fig. 5.6 Circular motion of point D

During translational motion of a rigid body accelerations of all its points are equal.

We will call the introduced acceleration **a** the *acceleration of translational motion*.

In translational motion, the paths can be parallel lines, either straight or curved. In the textbook [1], examples of such motion encountered in technology are presented. For instance, the motion of a piston of an internal combustion engine takes place along a straight line, whereas the paths of points associated with the motion of a rigid horizontal beam suspended by means of two cranks (at its ends) are curvilinear.

5.1.4 Rotational Motion

In this section we will consider the case presented in Fig. 5.3 in more detail. Let the cylindrical rigid body be supported in the thrust slide bearing and radial slide bearing through two coaxial cylindrical journals. The axis OX_3 of the adopted coordinate system coincides with the axis of rotation of the body. The points belonging to that body describe concentric circles lying in the planes perpendicular to the axis of rotation, and the radii of these circles are determined by distances of points from the axis of rotation. The body has one degree of freedom, and we will describe its motion with the aid of only one parameter. In order to describe it we will choose the stationary plane Γ_O coincident with the plane OX_1X_3. Next, we will take an arbitrary point lying on the plane Γ and determined by the angle φ (Fig. 5.6), henceforth called the *angle of body rotation*.

Looking at the arrow of the OX_3 axis (from positive to negative values), the directed angle φ will be positive if the motion of the point proceeds in a counterclockwise direction. In the general case, the angle changes in time $\varphi = \varphi(t)$

and is expressed in the arc measure, that is, radians. Let us take two time instants t and $t + \Delta t$, and let the corresponding angle values be $\varphi(t)$ and $\varphi_1(t + \Delta t)$, respectively. The following ratio of rotation angle increment to time increment is called the *average angular velocity of a body*:

$$\omega_{\mathrm{av}} = \frac{\varphi_1(t + \Delta t) - \varphi(t)}{\Delta t} = \frac{\Delta\varphi}{\Delta t}. \tag{5.13}$$

If $\Delta t \to 0$, then $\omega_{\mathrm{av}} \to \omega$, and from (5.13) we obtain

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}. \tag{5.14}$$

The *instantaneous angular velocity* of a body rotating about a fixed axis is defined by the time derivative of an angle of rotation of that body.

A unit of angular velocity is 1 rad/s. In the general case, the angular velocity depends on time, i.e., $\omega = \omega(t)$. If $\omega > 0$ ($\omega < 0$), then the rotation angle φ increases (decreases) in time.

In an analogous way we introduce the notion of average angular acceleration. Let us take two time instants t and $t + \Delta t$, and let the corresponding values of angular velocity be $\omega(t)$ and $\omega_1(t + \Delta t)$, respectively.

The ratio of angular velocity increment to time increment (time in which the angular velocity changes) is called the *average angular acceleration* of a body':

$$\varepsilon_{\mathrm{av}} = \frac{\Delta\left(\omega_1 - \omega\right)}{\Delta t} = \frac{\Delta\omega}{\Delta t}. \tag{5.15}$$

If $\Delta t \to 0$, then $\varepsilon_{\mathrm{av}} \to \varepsilon$, and from (5.15) we obtain

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}. \tag{5.16}$$

The *instantaneous angular acceleration* of a body rotating about a fixed axis is defined by the time derivative of its angular velocity, or by the second derivative with respect to time of its angle of rotation, and the unit of angular acceleration is 1 rad/s^2.

The special cases of the rotational motion of a rigid body include uniform rotational motion and uniformly variable (accelerated, decelerated) rotational motion. In the case of uniform rotational motion, from (5.14) we obtain

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathrm{const}, \tag{5.17}$$

that is,

$$\varphi = \omega t + \varphi_0, \tag{5.18}$$

where $\varphi_0 = \varphi(t_0)$.

In the case of this motion, often instead of the velocity expressed in radians per second, the number of revolutions n made by the body in 1 min is given. According to (5.13) and because

$$\frac{\Delta\varphi}{2\pi} = n, \quad t = 60\,\mathrm{s}, \tag{5.19}$$

we have

$$\omega = \frac{\pi n}{30} \frac{\mathrm{rad}}{\mathrm{s}}. \tag{5.20}$$

In the case of uniform motion $\omega = \mathrm{const}$, and so we have $\varepsilon = 0$ according to (5.16).

The *uniformly accelerated (decelerated) motion* is characterized by a constant magnitude of angular acceleration. According to (5.16) we have

$$\varepsilon = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \mathrm{const}. \tag{5.21}$$

Integrating the preceding equation with respect to time we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega = \varepsilon t + \omega_0, \tag{5.22}$$

where $\omega_0 = \omega(t_0)$.

In turn, integrating (5.22) with respect to time we have

$$\varphi = \varepsilon \frac{t^2}{2} + \omega_0 t + \varphi_0, \tag{5.23}$$

where $\varphi_0 = \varphi(t_0)$.

If the considered body has one degree of freedom, then knowing the velocity and acceleration of rotational motion allows for the determination of the velocity and acceleration of any point of the body. Choosing an arbitrary point D of the body through that point, we will draw a plane perpendicular to the axis of rotation. Connecting the point of intersection of the plane and the rotation axis with point D we obtain the radius R of the circle on which point D moves. The arc measure s of the position of point D is equal to

$$s = R(\varphi + \delta), \tag{5.24}$$

where φ denotes the directed angle between the planes Γ_O and Γ, which varies, whereas the angle δ is constant and determines the position of point D with respect to the plane Γ (Fig. 5.6). Differentiating (5.24) we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = R\frac{\mathrm{d}\varphi}{\mathrm{d}t} = R\omega, \tag{5.25}$$

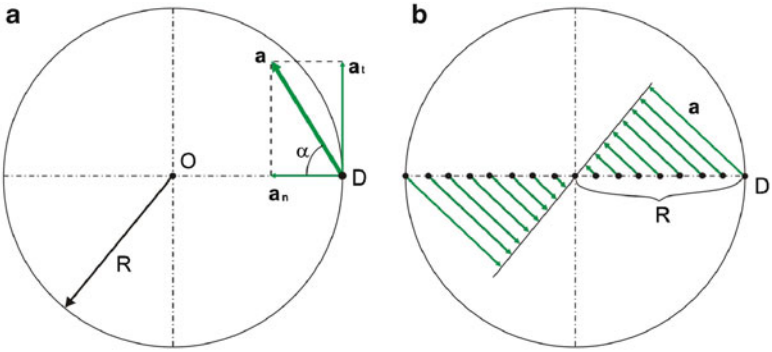


Fig. 5.7 Resolution of acceleration **a** into normal and tangential components (**a**) and distribution of accelerations along the radius (**b**)

which describes the magnitude of velocity v of point D. In order to determine the vector of acceleration of point D we will resolve it into tangential $\mathbf{a}_t$ and normal $\mathbf{a}_n$ components, that is,

$$\mathbf{a}_t = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \quad |\mathbf{a}_n| = \frac{\mathbf{v}^2}{R}. \tag{5.26}$$

The magnitudes of accelerations a_t and a_n are equal to (Fig. 5.7a)

$$\mathrm{a}_t = \varepsilon R, \quad \mathrm{a}_n = \omega^2 R. \tag{5.27}$$

According to Fig. 5.7a, using the introduced angle α one may determine the *total acceleration* of point D, which reads

$$\mathrm{a} = \sqrt{\mathrm{a}_t^2 + \mathrm{a}_n^2} = R\sqrt{\varepsilon^2 + \omega^4}, \tag{5.28}$$

$$\tan\alpha = \frac{\mathrm{a}_t}{\mathrm{a}_n} = \frac{\varepsilon}{\omega^2}. \tag{5.29}$$

The angle α defining the direction of the total acceleration vector is independent of the position of point D, but the magnitude of acceleration depends on the position described by the radius R. The dependency of the acceleration magnitude on the distance of point D from the rotation axis is shown in Fig. 5.7b. The graph of the velocity distribution along the distance is analogous, but the velocity vectors are perpendicular to the radius R.

Finally, we will determine the components of the velocity and the acceleration of an arbitrary point D based on the diagrams depicted in Fig. 5.8.

According to Fig. 5.8a we obtain

$$v_{x_1} = -v\sin(\varphi + \delta) = -\omega R\sin(\varphi + \delta), \tag{5.30}$$

$$v_{x_2} = v\cos(\varphi + \delta) = \omega R\cos(\varphi + \delta). \tag{5.31}$$

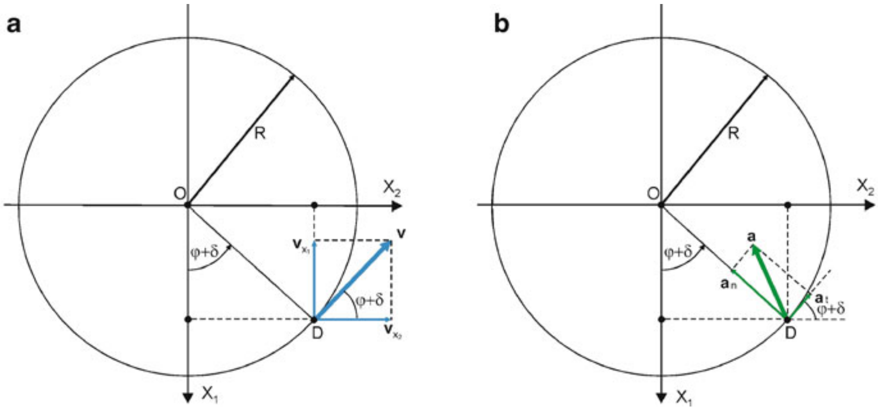


Fig. 5.8 Rectangular components of the velocity vector (**a**) and of the acceleration vector (**b**)

It can be easily noticed that because

$$x_2 = R\sin(\varphi + \delta), \quad x_1 = R\cos(\varphi + \delta), \tag{5.32}$$

(5.30) and (5.31) will take the form

$$v_{x_1} = -\omega x_2, \quad v_{x_2} = \omega x_1. \tag{5.33}$$

Analogous considerations lead to the determination of the rectangular components of acceleration. According to Fig. 5.8b we have

$$\begin{aligned} \mathrm{a}_{x_1} &= -\mathrm{a}_t \sin(\varphi + \delta) - \mathrm{a}_n \cos(\varphi + \delta), \\ \mathrm{a}_{x_2} &= \mathrm{a}_t \cos(\varphi + \delta) - \mathrm{a}_n \sin(\varphi + \delta). \end{aligned} \tag{5.34}$$

Taking into account (5.32) and (5.27) we obtain

$$\mathrm{a}_{x_1} = -\varepsilon x_2 - \omega^2 x_1, \quad \mathrm{a}_{x_2} = \varepsilon x_1 - \omega^2 x_2. \tag{5.35}$$

5.1.5 Angular Velocities and Angular Accelerations as Vectors and the Vector of Small Rotation

Let us consider a rigid body rotating about a fixed vertical axis OX_3 (Fig. 5.9).

We will attach the vector $\boldsymbol{\omega}$ of magnitude $|\omega| = \dot{\varphi}$ at an arbitrary point of the OX_3 axis. This vector is a sliding vector because it can be freely moved along the OX_3 axis. If $\dot{\varphi} > 0$, then $\boldsymbol{\omega}$ has the same direction and sense as does the OX_3 axis. According to the previous considerations

$$v = \omega R = \omega r \sin\alpha. \tag{5.36}$$

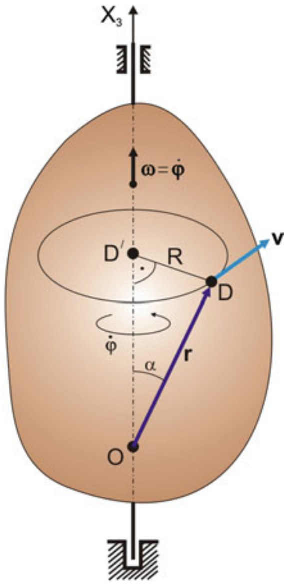


Fig. 5.9 Rigid body rotating with angular velocity $\boldsymbol{\omega}$

From Fig. 5.9 one may notice that vector $\mathbf{v}$ is perpendicular to the plane determined by vectors $\mathbf{r}$ and $\boldsymbol{\omega}$, which express vector $\mathbf{v}$ in the following way:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{5.37}$$

Because the acceleration $\mathbf{a}$ of point D is a geometric derivative of the velocity $\mathbf{v}$ with respect to time, we have

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \mathbf{r}) = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \tag{5.38}$$

where $\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boldsymbol{\varepsilon}$ is the angular acceleration.

The vector of angular velocity $\boldsymbol{\omega}$ may change the magnitude, but not the direction. It follows that the vector of angular acceleration also lies on the axis of rotation OX_3. If $\boldsymbol{\omega}$ increases, then $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ have the same senses. If $\boldsymbol{\omega}$ decreases, then $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ have opposite senses. We can write (5.38) in the form

$$\mathbf{a} = \boldsymbol{\varepsilon} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}. \tag{5.39}$$

From Fig. 5.10 it follows that

$$\mathbf{a}_t = \boldsymbol{\varepsilon} \times \mathbf{r}. \tag{5.40}$$

The vector $\boldsymbol{\varepsilon} \times \mathbf{r}$ is simultaneously perpendicular to $\boldsymbol{\varepsilon}$ and $\mathbf{r}$, and its magnitude $|\boldsymbol{\varepsilon} \times \mathbf{r}| = \varepsilon r \sin\alpha = \varepsilon R$. The second vector is simultaneously perpendicular to $\boldsymbol{\omega}$ and $\mathbf{v}$, that is, directed along the radius R. Moreover, since $\boldsymbol{\omega} \perp \mathbf{v}$, we have $|\boldsymbol{\omega} \times \mathbf{v}| = \omega v = \omega^2 R$. The latter vector is the normal component of the acceleration

$$\mathbf{a}_n = \boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{5.41}$$

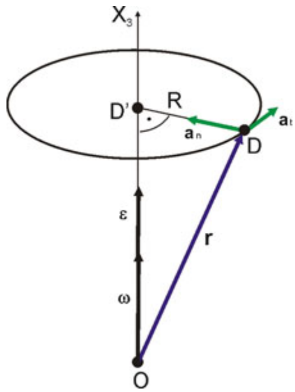


Fig. 5.10 Normal and tangential accelerations of point D

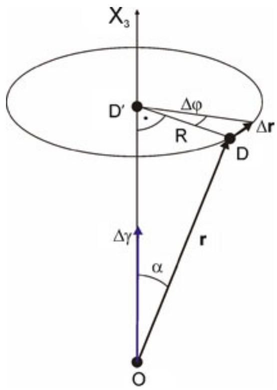


Fig. 5.11 Vector of small rotation angle $\Delta\boldsymbol{\gamma}$

At the end of this section, we will introduce the notion of the *small rotation vector*. Its geometrical interpretation is depicted in Fig. 5.11.

A small displacement Δr caused by a rotation $\Delta\boldsymbol{\gamma}$ for small $\Delta\varphi$ is equal to

$$\Delta r = R\Delta\boldsymbol{\varphi} = r\Delta\varphi \sin\alpha. \tag{5.42}$$

For small $\Delta\boldsymbol{\varphi}$ one can assume that vector $\Delta\mathbf{r}$ is tangent to a circle of radius R, and is equal to

$$\Delta\mathbf{r} = \Delta\boldsymbol{\gamma} \times \mathbf{r}. \tag{5.43}$$

We will demonstrate next that small rotation vectors can be added geometrically. Let the position of point D be described by the radius vector $\mathbf{r}$, and let us impose on the body two small rotations $\Delta\boldsymbol{\gamma}_1$ and $\Delta\boldsymbol{\gamma}_2$ along certain intersecting axes. After the first rotation $\Delta\boldsymbol{\gamma}_1$ point D will undergo displacement by

$$\Delta\mathbf{r}_1 = \Delta\boldsymbol{\gamma}_1 \times \mathbf{r}, \tag{5.44}$$

and its new position will be determined by the radius vector

$$\mathbf{r}_1 = \mathbf{r} + \Delta\mathbf{r}_1 = \mathbf{r} + \Delta\boldsymbol{\gamma}_1 \times \mathbf{r}. \tag{5.45}$$

In turn, the point thus obtained will, after the first rotation, take the position described by the radius vector $\mathbf{r}_1$ and then be subjected to the rotation $\Delta\boldsymbol{\gamma}_2$. It will undergo the following displacement:

$$\Delta\mathbf{r}_2 = \Delta\boldsymbol{\gamma}_2 \times \mathbf{r}_1 = \Delta\boldsymbol{\gamma}_2 \times (\mathbf{r} + \Delta\boldsymbol{\gamma}_1 \times \mathbf{r}) = \Delta\boldsymbol{\gamma}_2 \times \mathbf{r} \tag{5.46}$$

because we assumed that the vector product $\Delta\boldsymbol{\gamma}_2 \times (\Delta\boldsymbol{\gamma}_1 \times \mathbf{r})$ was a second-order differential. The total displacement of point D is equal to

$$\Delta\mathbf{r}_t = \Delta\mathbf{r}_1 + \Delta\mathbf{r}_2 = \Delta\boldsymbol{\gamma}_1 \times \mathbf{r} + \Delta\boldsymbol{\gamma}_2 \times \mathbf{r} = (\Delta\boldsymbol{\gamma}_1 + \Delta\boldsymbol{\gamma}_2) \times \mathbf{r}. \tag{5.47}$$

We will obtain a similar equation if we first apply the small rotation angle $\Delta\boldsymbol{\gamma}_2$ and then subject the obtained point to displacement associated with rotation $\Delta\boldsymbol{\gamma}_1$. Formula (5.47) has the following physical interpretation.

Two small rotations of a rigid body about two intersecting straight lines can be replaced with one vector whose resulting rotation is the geometrical sum of the small rotation vectors.

The introduced notion of small rotation vector will now serve to define the angular velocity.

Recall that the velocity of an arbitrary point of a body is given by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\gamma} \times \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\gamma}}{\Delta t} \times \mathbf{r}. \tag{5.48}$$

In the preceding formula, (5.43) was used. Taking into account (5.37) we obtain

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\gamma}}{\Delta t}. \tag{5.49}$$

The angular velocity $\boldsymbol{\omega}$ of a rigid body is equal to the limit to which the ratio of a small rotation vector and the time increment tend, given a time increment that is tending to zero.

5.2 Planar Motion

5.2.1 Introduction

Figure 5.12 shows a rigid body moving in planar motion. There are two planes Γ and Γ_O in the figure. The plane Γ_O was taken arbitrarily and is fixed since motion is measured with respect to that plane. We call such a plane a *reference plane* [2]. We dealt with a similar situation in the case of particle motion where a certain point O was taken as fixed and the distance of the particle from that point was measured by the radius vector.

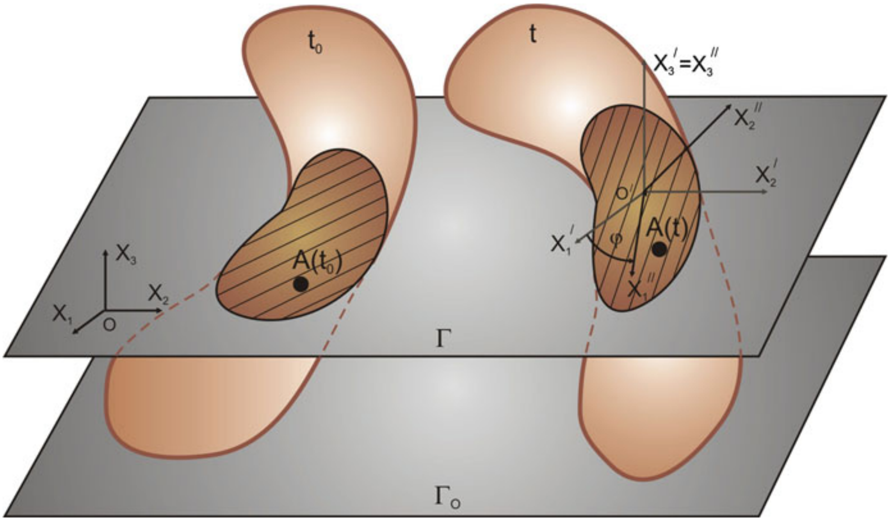


Fig. 5.12 Position of a rigid body with respect to reference plane Γ_O at time instant t_0 and t (notice translation and rotation of the figure marked by hatching line)

Let us take an arbitrary point of a rigid body and investigate its motion. To this end we introduce two Cartesian coordinate systems of parallel axes: $OX_1X_2X_3$ (absolute system) and $O'X_1'X_2'X_3'$. The first system is stationary and arbitrarily taken. The second system moves together with the investigated rigid body and is called a non-stationary system. Its origin (point O') lies on the plane figure obtained from the intersection of the body with the plane Γ. The planes $X_1 - X_2$ and $X_1' - X_2'$ lie in the plane Γ. It follows that the axes OX_3 and $O'X_3'$ are perpendicular to the chosen planes.

The motion of a rigid body during which all the points of the body move in planes parallel to the reference plane are called the *planar motion*.

An arbitrary point of a rigid body moves only in a plane passing through that point and parallel to the reference plane.

If all the points lying on an arbitrary line perpendicular to the reference plane move along identical paths and their velocities and accelerations are also identical, then the description of planar motion of the rigid body boils down to the description of the planar motion of the figure obtained as a result of the intersection of that body by a plane parallel to the reference plane.

The letter subscripts of the coordinates concern the point, e.g., $x_{1O'}, x_{2O'}$ are the coordinates of point O' in the system OX_1X_2, whereas x_{1A}', x_{2A}' are the coordinates of point A in the system $O'X_1'X_2'$. Moreover, the points denoted by the letter O are the origins of Cartesian coordinate systems, e.g., O', O'', O'''.

Knowing the motion of such a plane figure in its plane is sufficient to describe the planar motion of a rigid body.

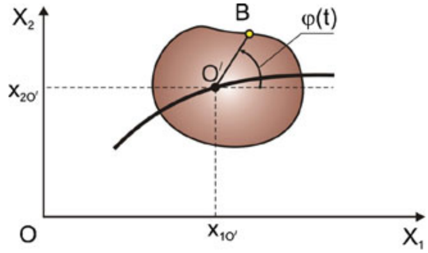


Fig. 5.13 Motion of a planar figure in the plane OX_1X_2 with the chosen pole O'

The region where the plane Γ intersects the analyzed rigid body is marked by a hatching line. All the points belonging to the X_3' axis move in translational motion so they possess the same velocities and accelerations. It follows that the velocity of any points of that body (e.g., point O') and its velocity of rotation completely describe the velocity field of the body. In turn, knowing the acceleration of an arbitrary point of the body and the angular velocity and angular acceleration of the body enables us to determine the acceleration field of the planar figure (e.g., of point A). Next, by drawing lines perpendicular to the plane of the figure passing through arbitrary points of that region we can determine the velocities and accelerations of other points of the rigid body lying on these lines.

As was mentioned earlier, the problem of planar motion boils down to an analysis of the motion of a planar figure in some stationary plane parallel to the reference plane. The position of the planar figure in the plane is described by the position of its two points, which we may connect to obtain a line segment. Thus an analysis of the kinematics of a planar figure can be reduced to an analysis of the kinematics of a segment in a plane.

We will now show how to determine the position, velocity, and acceleration of an arbitrary point from the region marked by a hatching line with respect to the stationary coordinate system $OX_1X_2X_3$ introduced earlier. Let us note that the position of the marked figure is uniquely determined with respect to the adopted coordinate system by two coordinates of an arbitrary point $x_1(t)$, $x_2(t)$ and an angle $\varphi(t)$. The angle φ denotes the rotation of the coordinate system $O'X_1'X_2'X_3'$ to the position $O''X_1''X_2''X_3''$ defined exactly by the angle φ (Fig. 5.12). The point $O' = O''$ is an arbitrary pole of the planar figure, and the system $O''X_1''X_2''X_3''$ is rigidly connected to that planar figure. We can always obtain the motion of an arbitrary point A of the marked figure by the parallel translation of the coordinate system $OX_1X_2X_3$ to the position $O'X_1'X_2'X_3'$ and subsequent rotation of that system through the angle φ. The angle φ is commonly called the *angle of rotation of a planar figure*.

Figure 5.13 shows an example of the motion of a planar figure.

The motion of a pole and, consequently, the translational motion of a planar figure are described by the equations $x_{1O'} = x_{1O'}(t)$ and $x_{2O'} = x_{2O'}(t)$. To describe the rotational motion of a planar figure, let us take a ray $O'A'$ that does not belong to this planar figure but moves in translational motion with the pole O'. Let us take point B of the figure, and let a ray $O'B$ belong to the planar figure

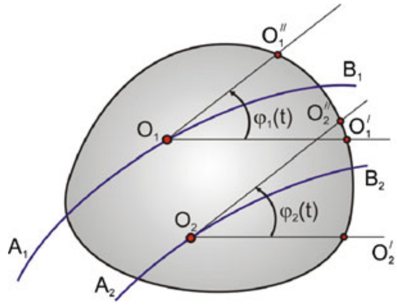


Fig. 5.14 Translational motion of poles O_1 and O_2 and rotational motion with respect to those poles

during the whole time of motion. It follows that the angle $A'O'B = \varphi$ describes the rotational motion of the planar figure $\varphi = \varphi(t)$. Eventually, the motion of the planar figure is uniquely described by the equations

$$x_{1O'} = x_{1O'}(t), \qquad x_{2O'} = x_{2O'}(t), \qquad \varphi = \varphi(t), \tag{5.50}$$

which are called *equations of planar motion of a rigid body*.

We will prove that the form of the equation $\varphi = \varphi(t)$ does not depend on the choice of the pole. Let us consider the motion of the planar figure depicted in Fig. 5.14.

As the two poles we choose two points O_1 and O_2 of the planar figure that during its motion move along trajectories A_1B_1 and A_2B_2. Through the poles we draw two parallel rays O_1O_1' and O_2O_2', which move in translational motion with the poles O_1 and O_2 and during motion are always parallel to one another. We draw in a plane of the figure two parallel rays, O_1O_1'' and O_2O_2'', that belong to the figure and are parallel to one another in an arbitrary figure position, that is,

$$\varphi_1(t) = \varphi_2(t) = \varphi(t). \tag{5.51}$$

This means that the angle $\varphi(t)$ does not depend on the choice of the pole (i.e., the angle is the same although $O_1 \neq O_2$).

The angular velocities $\omega_i(t)$ and angular accelerations $\varepsilon_i(t)$ about poles O_i are equal to

$$\omega_1 = \frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = \frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \omega_2, \qquad \varepsilon_1 = \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{\mathrm{d}\omega_2}{\mathrm{d}t} = \varepsilon_2. \tag{5.52}$$

It follows that the angular velocities and accelerations also do not depend on the choice of poles. They are identical for all points of a plane figure, and therefore vectors

$$\boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}, \qquad \boldsymbol{\varepsilon} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{\varphi}}{\mathrm{d}t}, \tag{5.53}$$

are respectively called the *angular velocity* $\boldsymbol{\omega}$ and the *angular acceleration* $\boldsymbol{\varepsilon}$ *of a planar figure*.

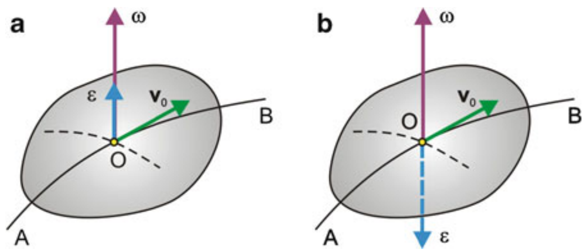


Fig. 5.15 Angular velocities $\boldsymbol{\omega}$ and angular accelerations $\boldsymbol{\varepsilon}$ during motion of a planar figure having identical (**a**) and opposite senses (**b**)

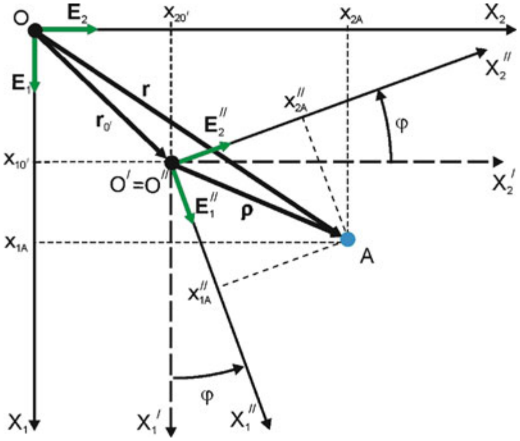


Fig. 5.16 Position of point A of a rigid body in plane Γ

The preceding vectors are parallel to one another and perpendicular to the plane of the figure and can have the same or opposite senses (Fig. 5.15).

Recall that throughout this book, uniform terminology and notation has been used. In the case of the Cartesian coordinate system (used most frequently), we denote the coordinates of, e.g., point A using lowercase letters x_{1A}, x_{2A}, x_{3A}. In turn, we will denote the axes of a stationary Cartesian coordinate system using capital letters $OX_1X_2X_3$, and we will denote Cartesian coordinate systems after successive transformations by, e.g., $O'X_1'X_2'X_3'$, $O''X_1''X_2''X_3''$.

In order to determine the position of point A in the coordinate system $OX_1X_2X_3$ we will make use of Fig. 5.16.

From this figure it follows that

$$\mathbf{r} = \mathbf{r}_{O'} + \boldsymbol{\rho}, \tag{5.54}$$

where $\boldsymbol{\rho}$ is described in the coordinate system $O''X_1''X_2''$.

The angle between the unit vectors $\mathbf{E}_1'$, $\mathbf{E}_1''$ and $\mathbf{E}_2'$, $\mathbf{E}_2''$ is equal to φ. In view of that we have

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}_1''\cos\varphi - \mathbf{E}_2''\sin\varphi,\\ \mathbf{E}_2 &= \mathbf{E}_1''\sin\varphi + \mathbf{E}_2''\cos\varphi.\end{aligned} \tag{5.55}$$

Multiplying the preceding equations respectively by $\cos\varphi$ and $\sin\varphi$, and then by $-\sin\varphi$ and $\cos\varphi$, and adding them by sides we obtain

$$\begin{aligned}\mathbf{E}_1'' &= \mathbf{E}_1\cos\varphi + \mathbf{E}_2\sin\varphi,\\ \mathbf{E}_2'' &= -\mathbf{E}_1\sin\varphi + \mathbf{E}_2\cos\varphi.\end{aligned} \tag{5.56}$$

According to (5.54) and taking into account (5.55) we have

$$\begin{aligned}x_{1A}\mathbf{E}_1 + x_{2A}\mathbf{E}_2 = x_{1O'}\mathbf{E}_1 + x_{2O'}\mathbf{E}_2 &+ x_{1A}''(\mathbf{E}_1\cos\varphi + \mathbf{E}_2\sin\varphi)\\ &+ x_{2A}''(-\mathbf{E}_1\sin\varphi + \mathbf{E}_2\cos\varphi).\end{aligned} \tag{5.57}$$

Multiplying this equation successively by $\mathbf{E}_1$ and $\mathbf{E}_2$ we obtain

$$\begin{aligned}x_{1A}(t) &= x_{1O'}(t) + x_{1A}''\cos\varphi(t) - x_{2A}''\sin\varphi(t),\\ x_{2A}(t) &= x_{2O'}(t) + x_{1A}''\sin\varphi(t) + x_{2A}''\cos\varphi(t).\end{aligned} \tag{5.58}$$

Returning to the introduced Cartesian coordinate systems of three axes, (5.58) for the case of axes' parallel translation ($\varphi = 0$) can be written in the following form:

$$\begin{bmatrix} x_{1A}\\ x_{2A}\\ x_{3A}\end{bmatrix} = \begin{bmatrix} x_{1O'}\\ x_{2O'}\\ x_{3O'}\end{bmatrix} + \begin{bmatrix} x_{1A}'\\ x_{2A}'\\ x_{3A}'\end{bmatrix}, \tag{5.59}$$

where in our plane case we have $x_{3A} = x_{3O'} = x'_{3A} = 0$ and additionally in the absence of rotation $x_{iA}'' = x'_{iA}$, $i = 1, 2, 3$.

Vector (5.54) describes in this case the position of point A in two coordinate systems of parallel axes OX_1X_2 and $OX_1'X_2'$ (in the general case $OX_1X_2X_3$ and $O'X_1'X_2'X_3'$).

Now let us assume that we are dealing only with the rotation of the coordinate system, say, about point O'.

Assuming that the rotation takes place through the angle φ in the plane parallel to $X_1 - X_2$ and using (5.58) we have

$$\mathbf{r} = \boldsymbol{\varphi}(\mathbf{E}_3)\boldsymbol{\rho}, \tag{5.60}$$

where

$$\mathbf{r} = \begin{bmatrix} x_{1A}'\\ x_{2A}'\\ x_{3A}'\end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} x_{1A}''\\ x_{2A}''\\ x_{3A}''\end{bmatrix}, \quad \boldsymbol{\varphi}(\mathbf{E}_3) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{5.61}$$

The vector $\boldsymbol{\rho} = \overrightarrow{O''A}$ is expressed in the moving coordinates $O''X_1''X_2''X_3''$, and the vector $\mathbf{r} = \overrightarrow{O'A}$ in the coordinates $O'X_1'X_2'X_3'$.

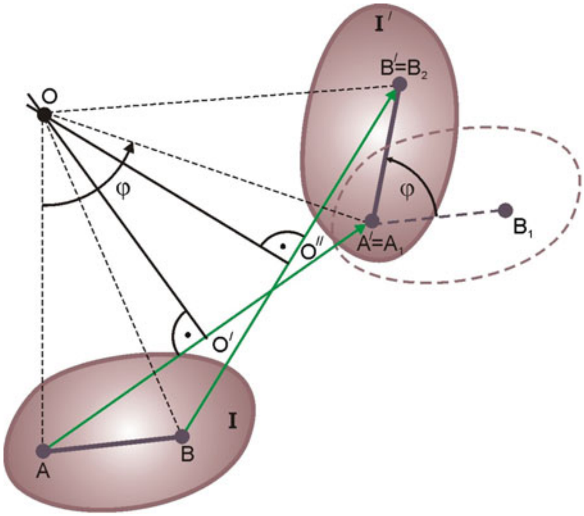


Fig. 5.17 Plane displacement of segment AB into position $A'B'$ and center of rotation O

What is more, the matrix $\boldsymbol{\varphi}(\mathbf{E}_3)$ is the matrix of elementary rotation about the X_3 axis through angle φ. One may check that the matrices of rotation about the X_1 and X_2 axes through angle φ have respectively the forms

$$\boldsymbol{\varphi}(\mathbf{E}_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, \quad \boldsymbol{\varphi}(\mathbf{E}_2) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}. \tag{5.62}$$

It is easy to demonstrate that $\boldsymbol{\varphi}^{-1} = \boldsymbol{\varphi}^{\mathrm{T}}$ and $\det(\boldsymbol{\varphi}) = 1$ or, equivalently, that $-\boldsymbol{\varphi}(\mathbf{E}_i) = \boldsymbol{\varphi}^{\mathrm{T}}(\mathbf{E}_i)$.

The coordinates x''_{1A} and x''_{2A} are time independent. Vector $\mathbf{r}_{O'}$ denotes the displacement, whereas the time change of vector $\boldsymbol{\rho}$ is connected with the rotation through angle $\varphi(t)$. Vector $\boldsymbol{\rho}$ rotates with the angular speed $\dot{\varphi}$ (it changes the direction but not the magnitude). Thus the conclusion can be drawn that *planar motion can be treated as a composition (geometric sum) of two motions, i.e., the translational motion of an arbitrary point (in our case point O') and the rotational motion about the axis perpendicular to the plane Γ and passing through point O'.*

In order to attribute the physical interpretation to this observation let us consider two arbitrary positions I and I' of a plane figure (Fig. 5.17).

At first we perform a translation of the figure with segment AB to point A_1. Next, we rotate the obtained segment A_1B_1 through angle φ obtaining segment $A'B'$. It is also possible to move segment AB to point B' and then rotate this segment also through angle φ, but with the opposite sense. From the foregoing considerations it follows that any non-translational displacement of a planar figure in its plane can be considered a composition of two displacements and a translational displacement including the point chosen as a pole, and then rotation about that pole. Moreover, the translational displacement of a planar figure depends on the choice of the pole, whereas the magnitude of angle of rotation and its sense are independent of the choice of pole.

We will show now that segment AB can be transformed into position $A'B'$ solely by rotation about an axis passing through point O and perpendicular to the drawing's plane. Such a point is called a *center of rotation*.

The perpendicular bisectors OO' and OO'' of, respectively, segments AA' and BB' divide the segments in halves, i.e., $AO' = O'A'$, $BO'' = O''B'$. Note that $OA = OA'$ and $OB = OB'$ and $AB = A'B'$, that is, triangles OAB and $OA'B'$ are congruent, and the former can be rotated about point O through angle φ in order to obtain the latter.

If positions I and I' are close to one another, then φ is the small rotation angle, $B' \to B$ and $A' \to A$. The directions of perpendicular bisectors OO' and OO'' for $\Delta t \to 0$ tend to the instantaneous velocities of points A and B, and point O tends to the instantaneous center of rotation, which coincides with the instantaneous center of velocity. The latter is created as a result of the intersection of the lines perpendicular to the vectors of instantaneous velocity $\mathbf{v}_A$ and $\mathbf{v}_B$ passing through the tails of these vectors.

In the first special case, figures I and I' can be situated with respect to one another in such way that they possess a symmetry axis. Then lines AB and A_1B_1 and the bisector intersect at one point called the *instantaneous center of rotation* C, and figures I and I' coincide only after rotation through a certain angle.

In the second special case, the figures become coincident only after displacement by a certain distance, because $I \parallel I'$, and then point C lies in infinity.

In the book [2] there is also a discussion of the case where the position of segments is such that $AA' \parallel BB'$. The preceding observations lead to the following theorem.

Theorem 5.3 (Euler's first theorem). *An arbitrary displacement of a planar figure can be realized through rotation about a certain fixed point, called the center of rotation, lying in the plane of the figure.*

To sum up, the planar motion of a rigid body can be replaced with the motion of a certain plane figure.

5.2.2 Instantaneous Center of Velocities

Taking an arbitrary plane that is parallel to the reference plane, and intersecting the analyzed rigid body, we obtain as a result a planar figure. Knowing the velocity of an arbitrary point of this figure, i.e., the linear velocity and angular velocity $\boldsymbol{\omega}$, enables us to determine the velocity of any point of the considered rigid body.

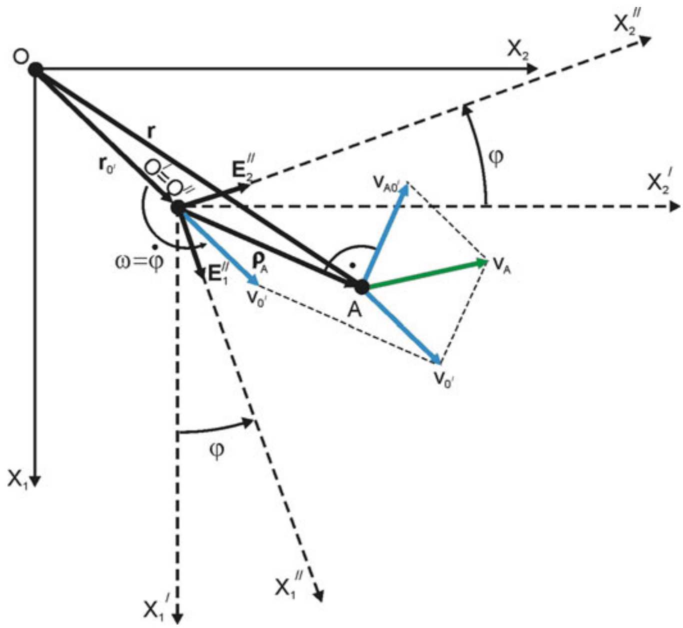


Fig. 5.18 Velocity $\mathbf{v}_A$ of point A

The velocity of an arbitrary point A of a rigid body is equal to (Fig. 5.18)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_A) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_{O'} + \boldsymbol{\rho}_A) \tag{5.63}$$

or

$$\mathbf{v}_A = \mathbf{v}_{O'} + \mathbf{v}_{AO'}, \tag{5.64}$$

where $\mathbf{v}_A$ is the velocity vector of point A, $\mathbf{v}_{O'}$ is the velocity vector of point O', and $\mathbf{v}_{AO'}$ is the velocity vector of point A with respect to point O', where

$$\dot{\boldsymbol{\rho}}_A \equiv \mathbf{v}_{AO'} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A \equiv \boldsymbol{\omega} \times (\mathbf{r}_A - \mathbf{r}_{O'}) . \tag{5.65}$$

The vector $\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}}$ is perpendicular to the plane of the figure.

Theorem 5.4. *If the motion of a planar figure that takes place in its plane at the given time instant is not the instantaneous translational motion, then there exists one point C belonging to the plane of the figure such that its velocity is equal to zero. The velocities of other points of the plane figure result from its instantaneous rotation about point C.*

Proof. The velocity of the point $\mathbf{v}_C = \mathbf{0}$ and, by assumption, $\boldsymbol{\omega} \neq \mathbf{0}$. According to (5.64) in the system $O'X_1'X_2'X_3'$ we have

$$\mathbf{v}_C = \mathbf{0} = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \overrightarrow{O'C}, \tag{$*$}$$

where

$$\mathbf{v}_{O'} = \begin{bmatrix} \dot{x}_{1O'} \\ \dot{x}_{2O'} \\ 0 \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix}, \quad \overrightarrow{O'C} = \begin{bmatrix} x'_{1C} \\ x'_{2C} \\ 0 \end{bmatrix}.$$

Because

$$\boldsymbol{\omega} \times \overrightarrow{O'C} = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ 0 & 0 & \dot{\varphi} \\ x'_{1C} & x'_{2C} & 0 \end{vmatrix} = -\mathbf{E}'_1\dot{\varphi}x'_{2C} + \mathbf{E}'_2\dot{\varphi}x'_{1C},$$

we have

$$\dot{x}_{1O'} - x'_{2C}\dot{\varphi} = 0,$$
$$\dot{x}_{2O'} + x'_{1C}\dot{\varphi} = 0.$$

The preceding equations allow for the determination of the position of point C given knowledge of $\dot{\varphi} = \omega$.

Premultiplying both sides of $(*)$ by $\boldsymbol{\omega}$ we obtain

$$\boldsymbol{\omega} \times \mathbf{v}_{O'} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \overrightarrow{O'C}\right) = \mathbf{0},$$

and hence

$$\boldsymbol{\omega} \times \mathbf{v}_{O'} = \boldsymbol{\omega} \times \left(\overrightarrow{O'C} \times \boldsymbol{\omega}\right) = \omega^2\overrightarrow{O'C} - \left(\overrightarrow{O'C} \circ \boldsymbol{\omega}\right)\boldsymbol{\omega}.$$

Finally, the unknown vector is equal to

$$\overrightarrow{O'C} = \frac{\boldsymbol{\omega} \times \mathbf{v}_{O'}}{\omega^2}.$$

The preceding formula enables us to construct the vector $\overrightarrow{O'C}$ (and consequently to determine point C). Looking in the direction of vector $\boldsymbol{\omega}$, we rotate the vector $\mathbf{v}_{O'}$ in the positive direction through the angle $\pi/2$, and on the obtained ray we lay off a segment of length $v_{O'}/\omega$, i.e., we mark the location of point C. □

5.2.2.1 Instantaneous Center of Velocities

We will prove that at every time instant during the motion of a planar figure there exists its point C such that its velocity $\mathbf{v}_C = \mathbf{0}$, and we will call it the *instantaneous center of velocities*.

At any time instant let the velocity of a certain point O of the plane figure (Fig. 5.19) be known, with the considered time instant having angular velocity $\boldsymbol{\omega}$.

Let point O be the pole. We will determine the velocity of an arbitrary other point after adding vector $\mathbf{v}_O$ and the velocity vector of this point with respect to pole O.

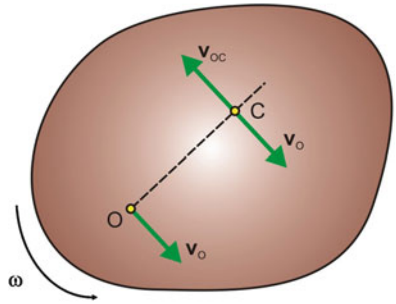


Fig. 5.19 Determination of instantaneous center of velocities

Let $OC \perp \mathbf{v}_O$. We will search for the position of point C such that $\mathbf{v}_{OC} = -\mathbf{v}_O$. The velocity of point C is equal to

$$\mathbf{v}_C = \mathbf{v}_O + \mathbf{v}_{OC} = \mathbf{v}_O - \mathbf{v}_O = \mathbf{0}, \tag{5.66}$$

and in view of that point C is the instantaneous center of velocities. Because

$$v_{OC} = OC \cdot \omega = v_O, \tag{5.67}$$

we have

$$OC = \frac{v_O}{\omega}. \tag{5.68}$$

To sum up, the instantaneous center of velocities lies on a ray $OC \perp \mathbf{v}_O$ at a distance from pole O equal to v_O/ω.

We will now determine the velocities of arbitrary points of a planar figure using the introduced notion of instantaneous center of velocities. Let us take three arbitrary points A, A_1, and A_2 (Fig. 5.20).

We successively have

$$\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{CA}, \quad \mathbf{v}_{A_1} = \mathbf{v}_C + \mathbf{v}_{CA_1}, \quad \mathbf{v}_{A_2} = \mathbf{v}_C + \mathbf{v}_{CA_2}, \tag{5.69}$$

and because $\mathbf{v}_C = \mathbf{0}$, we also have

$$\begin{aligned} |\mathbf{v}_A| &= |\mathbf{v}_{CA}| = CA \cdot \omega, && \mathbf{v}_A \perp CA, \\ |\mathbf{v}_{A_1}| &= |\mathbf{v}_{CA_1}| = CA_1 \cdot \omega, && \mathbf{v}_{A_1} \perp CA_1, \\ |\mathbf{v}_{A_2}| &= |\mathbf{v}_{CA_2}| = CA_2 \cdot \omega, && \mathbf{v}_{A_2} \perp CA_2. \end{aligned} \tag{5.70}$$

From (5.70) the following conclusion can be drawn.

The velocity vector of an arbitrary point of a planar figure at every time instant possesses a magnitude equal to the product of the angular velocity of this figure ω and the length of the segment connecting this point with the instantaneous center of velocities, and its sense is in agreement with the sense of ω.

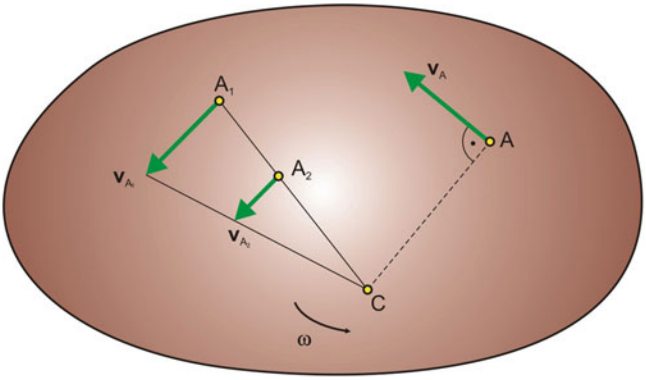
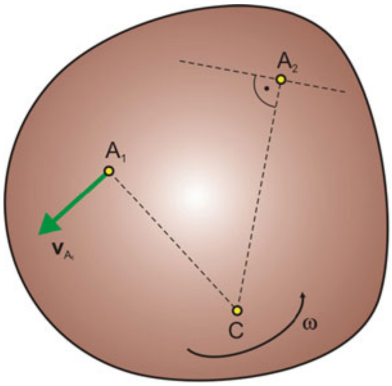


Fig. 5.20 Determination of velocities $\mathbf{v}_A, \mathbf{v}_{A_1}$, and $\mathbf{v}_{A_2}$ using the instantaneous center of velocities C

Fig. 5.21 Determination of position of instantaneous center of velocities if $\mathbf{v}_{A_1}$ and direction of $\mathbf{v}_{A_2}$ are known



From Fig. 5.20 also follow the relationships

$$\frac{v_A}{v_{A_1}} = \frac{CA}{CA_1}, \qquad \frac{v_{A_1}}{v_{A_2}} = \frac{CA_1}{CA_2}. \tag{5.71}$$

Let us now consider certain, qualitatively distinct, cases for determining the position of the instantaneous center of velocities.

I. Let the velocity $\mathbf{v}_{A_1}$ and the direction of velocity of point A_2 be given (Fig. 5.21).

Two rays A_1C and A_2C intersect at point C, that is, at the instantaneous center of velocities, and $\omega = v_{A_1}/CA_1$. Then we determine the magnitude of velocity v_{A_2} from the equation $v_{A_2}/v_{A_1} = CA_2/CA_1$ or the equation $v_{A_2} = CA_2 \cdot \omega$.

II. Let the velocities $\mathbf{v}_{A_1} \parallel \mathbf{v}_{A_2}$ and perpendicular to A_1A_2. To determine point C, the magnitudes of velocities v_{A_1} and v_{A_2} must be known (Fig. 5.22).

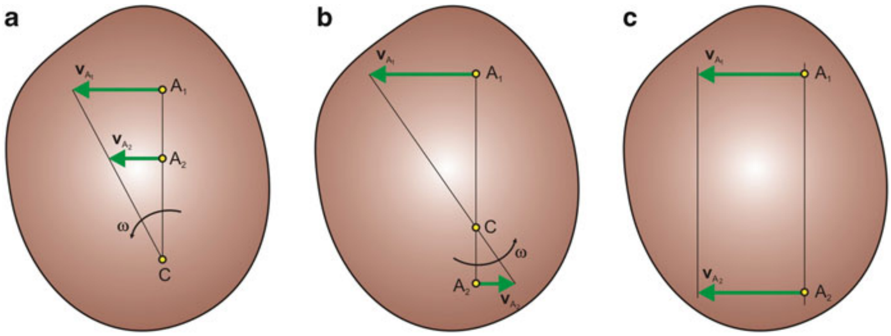


Fig. 5.22 Determination of instantaneous center of velocities C in the case where $\mathbf{v}_{A_1} \parallel \mathbf{v}_{A_2}$: $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ have the same senses (**a**), opposite senses (**b**), and the same senses and equal magnitudes (**c**)

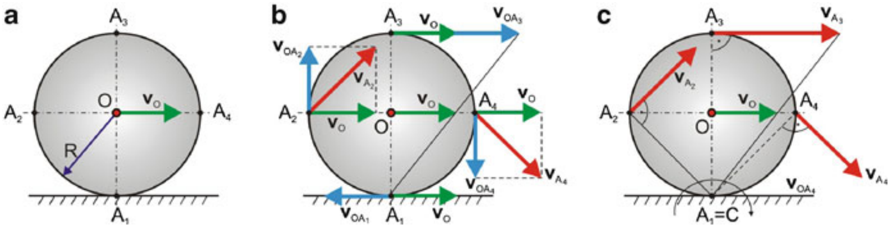


Fig. 5.23 (**a**) Disk rolling without sliding on a straight line. (**b**) Velocities of points A_1, A_2, A_3, and A_4 determined from the equation with respect to pole O and (**c**) with respect to the instantaneous center of velocities

From Fig. 5.22a, b follows the proportion

$$\frac{v_{A_1}}{v_{A_2}} = \frac{CA_1}{CA_2}, \tag{5.72}$$

and in the case shown in Fig. 5.22c we have $A_1C = \infty$, that is, $\frac{v_{A_1}}{A_1C} = \frac{v_{A_1}}{\infty} = 0$.

III. Let $\mathbf{v}_{A_1} \parallel \mathbf{v}_{A_2}$, but the vectors are not perpendicular to A_1A_2. The instantaneous center of velocities lies in infinity, that is, $A_1C = \infty$ and $\omega = \frac{v_{A_1}}{\infty} = 0$. Moreover, we have $A_1C = A_2C = \infty$, and therefore $v_{A_1} = v_{A_2}$. We deal with such a case in the translational motion of a plane figure because then the velocities of all its points are equal in the geometrical sense and the instantaneous center of velocities lies in infinity. On the other hand, if $v_{A_1} = v_{A_2}$ in a certain time interval, and not at a certain time instant only, then the plane figure moves in translational motion.

Example 5.1. Circular disk of radius R rolls without sliding on a straight line (Fig. 5.23). The velocity of the center of the disk $\mathbf{v}_O = \text{const}$. Determine the velocities of points A_1, A_2, A_3, and A_4 lying on a circle of radius R.

The disk moves without sliding on a certain stationary straight line. In this case the instantaneous center of velocities C lies at the point of contact of the disk with the line. The velocity of point C of the disk is equal to zero because it is in contact with the point of a fixed line and there is no sliding.

The velocity of point C can be determined with respect to pole O. Then we have

$$\mathbf{v}_C = \mathbf{0} = \mathbf{v}_O + \mathbf{v}_{OC},$$

that is, at point C we have $\mathbf{v}_O = -\mathbf{v}_{OC}$, which is illustrated in Fig. 5.23b.

The velocities of the remaining points A_2, A_3, and A_4 with respect to pole O are equal to

$$v_{OA_1} = v_{OA_2} = v_{OA_3} = v_{OA_4} = v_O.$$

Geometric adding vectors $\mathbf{v}_{OA_i}$, $i = 1, 2, 3, 4$, and $\mathbf{v}_O$ we obtain the desired vectors of velocities of points A_i, and their magnitudes read

$$v_{A_2} = \sqrt{v_O^2 + v_{OA_2}^2} = \sqrt{v_O^2 + v_O^2} = \sqrt{2}v_O,$$
$$v_{A_3} = v_O + v_{OA_4} = v_O + v_O = 2v_O,$$
$$v_{A_4} = \sqrt{v_O^2 + v_{OA_4}^2} = \sqrt{v_O^2 + v_O^2} = \sqrt{2}v_O.$$

The desired velocities of points A_i can be determined if as a pole we take the instantaneous center of velocities C (Fig. 5.23c). We successively have

$$v_{A_2} = v_O \frac{SA_2}{SO} = v_O \frac{R\sqrt{2}}{R} = \sqrt{2}v_O,$$
$$v_{A_3} = v_O \frac{SA_3}{SO} = 2v_O,$$
$$v_{A_4} = v_O \frac{SA_4}{SO} = v_O \frac{R\sqrt{2}}{R} = \sqrt{2}v_O. \qquad \square$$

5.2.3 Moving and Fixed Centrodes

As was mentioned earlier, the analysis of motion of a plane figure in its plane can be reduced to the analysis of an arbitrarily chosen segment AB of this figure at time instants $t, t + \Delta t, t + 2\Delta t, \ldots$. This segment will assume the following positions: $A_1B_1, A_2B_2, A_3B_3, \ldots$. To each of those positions correspond instantaneous centers of rotation $O_1,\ O_2,\ O_3, \ldots$, such that rotation through angle $\Delta\varphi_1$ about point O_1 moves segment A_1B_1 to A_2B_2, rotation through $\Delta\varphi_2$ about O_2 moves segment A_2B_2 to A_3B_3, etc. At the given time instants the positions of $O_1, O_2, O_3, \ldots$ are fixed and can be joined to form a broken line $O_1O_2O_3\ldots$, which belongs to the fixed plane.

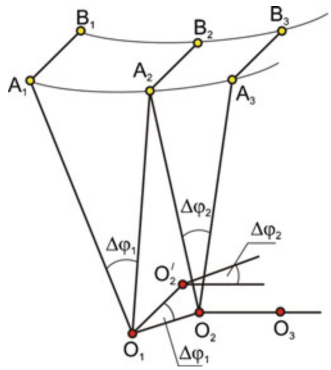


Fig. 5.24 Determination of points $O_1,\ O_2,\ O_3, \ldots$ and $O_1',\ O_2',\ O_3', \ldots$

We will now describe the broken line $O_1' O_2' O_3' \ldots$ corresponding to the broken line $O_1 O_2 O_3 \ldots$, where now the points $O_1', O_2', O_3', \ldots$ belong to the moving plane figure. We will show how to determine point O_2', which after rotation through the angle $\Delta\varphi_1$, becomes coincident with the fixed point O_2 (Fig. 5.24).

We lay off the segment $O_1 O_2' = O_1 O_2$ at the angle $\Delta\varphi_1$ opposite to the sense of rotation of a plane figure with respect to O_1 of segment $O_1 O_2$. We proceed similarly in the next step, but now we lay off $O_2' O_3'$ at the angle $\Delta\varphi_2$ with respect to a line parallel to $O_2 O_3$ and passing through point O_2. The obtained broken line $O_1 O_2' O_3' \ldots$ belongs to the moving plane figure. If $\Delta t \to 0$ and $\Delta\varphi_i \to 0$, then the broken lines will turn into smooth curves and the motion of the planar figure can be represented as a rolling of the curve $O_1 O_2' O_3' \ldots$ on the fixed curve $O_1 O_2 O_3, \ldots$ without sliding because $O_1 O_2' = O_1 O_2$, etc. The curve $O_1 O_2 O_3, \ldots$ is called a *fixed centrode*, and the curve $O_1 O_2' O_3', \ldots$ a *moving centrode* (point O_3' is omitted in Fig. 5.24). During such rolling the point of contact of a moving centrode with a fixed centrode is the instantaneous center of velocities C corresponding to the given time instant.

During planar motion the instantaneous center of velocities C moves in a plane associated with the stationary system on a curve called the *fixed centrode*. The trajectory of point C in a system rigidly connected to a moving body is called the *moving centrode*. *During motion the moving centrode rolls on the fixed centrode without sliding*. This observation is also known as the *Poinsot theorem*.

The approach based on the concept of the rolling of the moving centrode on a fixed centrode often allows for a simpler equivalent interpretation of motion.

Let us consider the rolling without sliding of a disk placed between slats moving in opposite directions with velocities v_1 and v_2 (Fig. 5.25).

We determine the position of the instantaneous center of velocities from the equation $OC = (v_1 - v_2)/((v_1 + v_2)R)$, and the problem is equivalent to a rolling of the moving centrode, i.e., circle of radius OC, on the fixed centrode, that is, on a fixed horizontal straight line.

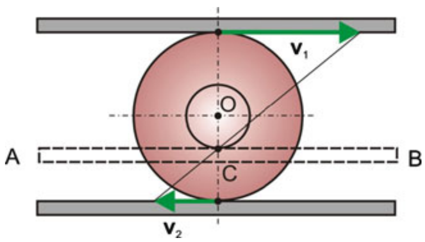


Fig. 5.25 Rolling without sliding of a disk of radius R between two slats moving with velocities v_1 and v_2 in opposite directions

In order to obtain the velocity in the coordinates associated with the unit vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, one should differentiate (5.57), which leads to the result

$$\dot{x}_{1A}\mathbf{E}_1 + \dot{x}_{2A}\mathbf{E}_2 = \dot{x}_{1O'}\mathbf{E}_1 + \dot{x}_{2O'}\mathbf{E}_2 + x''_{1A}\left(-\mathbf{E}'_1\omega\sin\varphi + \mathbf{E}'_2\omega\cos\varphi\right) + x''_{2A}\left(-\mathbf{E}'_1\omega\cos\varphi - \mathbf{E}'_2\omega\sin\varphi\right). \quad (5.73)$$

Multiplying (5.73) by sides one time by $\mathbf{E}_1$ and the other time by $\mathbf{E}_2$ we have

$$\dot{x}_{1A}(t) = \dot{x}_{1O'} - \omega x''_{1A}\sin\varphi - \omega x''_{2A}\cos\varphi,$$
$$\dot{x}_{2A}(t) = \dot{x}_{2O'} + \omega x''_{1A}\cos\varphi - \omega x''_{2A}\sin\varphi. \quad (5.74)$$

Let us return to vector $\overrightarrow{O'A}$ from Fig. 5.16. It can be expressed by the unit vectors of both coordinate systems in the following way:

$$\overrightarrow{O'A} = x'_{1A}\mathbf{E}'_1 + x'_{2A}\mathbf{E}'_2 = x''_{1A}\mathbf{E}''_1 + x''_{2A}\mathbf{E}''_2. \quad (5.75)$$

In order to obtain the coordinates of this vector in the system $OX'_1X'_2$ ($OX''_1X''_2$) one should multiply (scalar product) the preceding equation in turn by $\mathbf{E}'_1$ and $\mathbf{E}'_2$ ($\mathbf{E}''_1$ and $\mathbf{E}''_2$). After that operation we obtain

$$x'_{1A} = x''_{1A}\mathbf{E}''_1\circ\mathbf{E}'_1 + x''_{2A}\mathbf{E}''_2\circ\mathbf{E}'_1 = x''_{1A}\cos\varphi + x''_{2A}\cos\left(\frac{\pi}{2}+\varphi\right),$$
$$x'_{2A} = x''_{1A}\mathbf{E}''_1\circ\mathbf{E}'_2 + x''_{2A}\mathbf{E}''_2\circ\mathbf{E}'_2 = x''_{1A}\cos\left(\frac{\pi}{2}-\varphi\right) + x''_{2A}\cos\varphi,$$
$$x''_{1A} = x'_{1A}\mathbf{E}'_1\circ\mathbf{E}''_1 + x'_{2A}\mathbf{E}'_2\circ\mathbf{E}''_1 = x'_{1A}\cos\varphi + x'_{2A}\cos\left(\frac{\pi}{2}-\varphi\right),$$
$$x''_{2A} = x'_{1A}\mathbf{E}'_1\circ\mathbf{E}''_2 + x'_{2A}\mathbf{E}'_2\circ\mathbf{E}''_2 = x'_{1A}\cos\left(\frac{\pi}{2}+\varphi\right) + x'_{2A}\cos\varphi, \quad (5.76)$$

or, in matrix notation,

$$\begin{bmatrix} x'_{1A} \\ x'_{2A} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x''_{1A} \\ x''_{2A} \end{bmatrix}, \tag{5.77}$$

$$\begin{bmatrix} x''_{1A} \\ x''_{2A} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x'_{1A} \\ x'_{2A} \end{bmatrix}. \tag{5.78}$$

In both of the preceding cases the transformation is connected with the rotation through angle φ. Moreover, the sum of squares of the elements of any row or column equals one, and the sum of products of elements of two rows or columns equals zero. We call such a linear transformation an *orthogonal transformation* [3]. The determinant of both matrices is equal to one (if the determinant yielded -1, it would indicate rotation accompanied by the mirror reflection [3]).

The transformations described by (5.55) and (5.56) are also orthogonal and indicate rotation through angle φ.

The interpretation of vector (5.64) is as follows. The velocity of an arbitrary point A is composed of the velocity of translational motion of the system $O'X'_1X'_2$ and the rotational motion of that system about the axis perpendicular to the drawing's plane and passing through point O'. On that axis lies the vector $\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}}$. The velocity vector $\mathbf{v}_{AO'}$ coming from the motion of point A with respect to point O' is perpendicular to line segment $O'A$.

Using (5.77), (5.74) can be represented as

$$\begin{aligned} \dot{x}_{1A}(t) &= \dot{x}_{1O'} - \omega(x''_{1A}\sin\varphi + x''_{2A}\cos\varphi) \\ &= \dot{x}_{1O'} - \omega x'_{2A} = \dot{x}_{1O'} - \omega(x_{2A} - x_{2O'}), \\ \dot{x}_{2A}(t) &= \dot{x}_{2O'} + \omega(x''_{1A}\cos\varphi - x''_{2A}\sin\varphi) \\ &= \dot{x}_{2O'} + \omega x'_{1A} = \dot{x}_{2O'} + \omega(x_{1A} - x_{1O'}), \end{aligned} \tag{5.79}$$

and these are the components of the velocity of point A in the absolute system $OX_1X_2X_3$, i.e.,

$$\mathbf{v}_A = \dot{x}_{1A}\mathbf{E}_1 + \dot{x}_{2A}\mathbf{E}_2. \tag{5.80}$$

On the other hand, in one case multiplying the first equation of (5.74) by $\cos\varphi$ and the second by $\sin\varphi$ and adding by sides, and in the other case the first by $\sin\varphi$ and the second by $-\cos\varphi$ and adding by sides, we respectively obtain

$$\begin{aligned} \dot{x}_{1A}\cos\varphi + \dot{x}_{2A}\sin\varphi &= \dot{x}_{1O'}\cos\varphi + \dot{x}_{2O'}\sin\varphi - \omega x''_{2A}, \\ \dot{x}_{1A}\sin\varphi - \dot{x}_{2A}\cos\varphi &= \dot{x}_{1O'}\sin\varphi - \dot{x}_{2O'}\cos\varphi - \omega x''_{1A}. \end{aligned} \tag{5.81}$$

Multiplying the second equation through by (-1) and using (5.78) we obtain the components of velocity of point A in the non-stationary coordinate system $O''X''_1X''_2X''_3$:

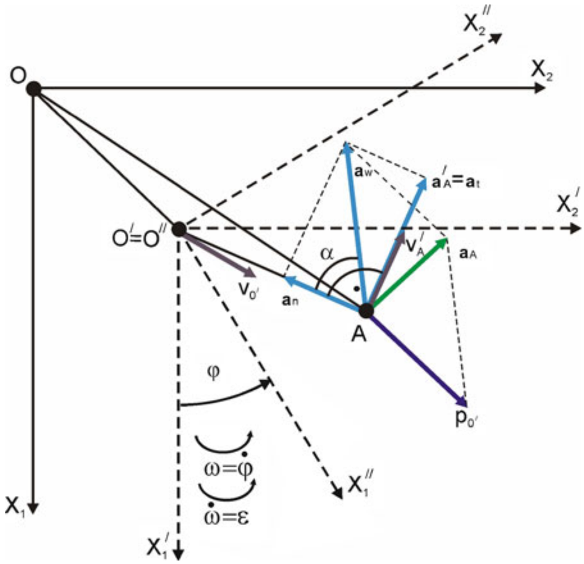


Fig. 5.26 Determination of acceleration of point A

$$\dot{x}''_{1A} = \dot{x}_{1O'} \cos\varphi + \dot{x}_{2O'} \sin\varphi - \omega x''_{2A},$$
$$\dot{x}''_{2A} = \dot{x}_{2O'} \cos\varphi - \dot{x}_{1O'} \sin\varphi + \omega x''_{1A}, \tag{5.82}$$

that is,

$$\mathbf{v}_A = \dot{x}''_{1A}\mathbf{E}''_1 + \dot{x}''_{2A}\mathbf{E}''_2. \tag{5.83}$$

Let us introduce one of the physical interpretations of an absolute system and a system rigidly connected to the moving body. The system $O''X''_1X''_2X''_3$ moving with the body does not move with respect to that body. If we associate such a system with a spacecraft, then with respect to an astronaut the points inside the spacecraft do not move because vector $\boldsymbol{\rho}$ does not change its position. The velocity vector of point A in the system $O''X''_1X''_2X''_3$ is described by (5.83), and the acceleration vector will be determined in the following section.

5.2.4 Accelerations and Center of Acceleration

In a similar way we can determine the acceleration of point A. In this case we will relate our considerations to the hodograph of velocity (Fig. 5.26).

Differentiating (5.64) with respect to time, we obtain

$$\mathbf{a}_A \equiv \dot{\mathbf{v}}_A = \dot{\mathbf{v}}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_A + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_A. \tag{5.84}$$

In the general case, $\mathbf{a}_{O'} = \dot{\mathbf{v}}_{O'}$ can have an arbitrary direction with respect to $\mathbf{v}_{O'}$ because the motion of point O' is, generally, not a rectilinear motion.

If we are dealing with rectilinear motion, vectors $\mathbf{a}_{O'}$ and $\mathbf{v}_{O'}$ are parallel. The vector $\mathbf{a}_t = \boldsymbol{\varepsilon} \times \boldsymbol{\rho}_A$, where $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ is the tangential acceleration vector resulting from the rotation of point A about point O', lies in the plane of the figure, and it is perpendicular to $O'A$ (since vector $\dot{\boldsymbol{\omega}}$ has the same direction as vector $\boldsymbol{\omega}$). According to (5.64) and (5.65) we have

$$\dot{\boldsymbol{\rho}}_A = \boldsymbol{\omega} \times \boldsymbol{\rho}_A, \tag{5.85}$$

and in view of that the last vector on the right-hand side of (5.84) can be expressed as

$$\mathbf{a}_n \equiv \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_A = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) . \tag{5.86}$$

Next we will use the law of *vector triple product expansion*, i.e., for three non-coplanar vectors we have $\boldsymbol{a} \times (\mathbf{b} \times \mathbf{c}) = (\boldsymbol{a} \circ \mathbf{c})\mathbf{b} - (\boldsymbol{a} \circ \mathbf{b})\mathbf{c}$.

Equation (5.86) assumes the form

$$\mathbf{a}_n = (\boldsymbol{\omega} \circ \boldsymbol{\rho}_A)\,\boldsymbol{\omega} - (\boldsymbol{\omega} \circ \boldsymbol{\omega})\,\boldsymbol{\rho}_A = -\omega^2 \boldsymbol{\rho}_A \tag{5.87}$$

because $\boldsymbol{\omega} \circ \boldsymbol{\rho}_A = 0$ (it follows from vectors' being perpendicular to each other).

From the preceding equation it follows that this component of the acceleration of point A is directed along the radius $\boldsymbol{\rho}_A$ toward point O' and is called *normal acceleration (centripetal acceleration)*.

From the preceding considerations it follows that all three distinct vectors of acceleration allow for the determination of the unknown acceleration of point A through their geometrical addition according to (5.84), i.e.,

$$\mathbf{a}_A = \mathbf{a}_{O'} + \mathbf{a}_t + \mathbf{a}_n, \tag{5.88}$$

where the tangential $\mathbf{a}_t$ and normal $\mathbf{a}_n$ accelerations result from the rotation of point A about point O'.

The magnitudes of the vectors $|\mathbf{a}_t| = |\overrightarrow{O'A}|\varepsilon$ and $|\mathbf{a}_n| = |\overrightarrow{O'A}|\omega^2$, so introducing angle α as shown in Fig. 5.26 we have

$$\mathbf{a} = \mathbf{a}_{O'} + \mathbf{a}_w, \tag{5.89}$$

where

$$\mathrm{a}_w = \left|\overrightarrow{O'A}\right| \sqrt{\varepsilon^2 + \omega^4}, \qquad \frac{\mathrm{a}_t}{\mathrm{a}_n} = \tan\alpha = \frac{\varepsilon}{\omega^2}. \tag{5.90}$$

5.2.4.1 Instantaneous Center of Accelerations

Let us consider a plane figure (Fig. 5.27) and assume that we know the acceleration $\mathbf{a}_O$ of a certain point O of this figure, and additionally the figure has the velocity $\boldsymbol{\omega} > 0$ and acceleration $\boldsymbol{\varepsilon} > 0$. Let us introduce angle α such that

$$\alpha = \arctan\frac{\varepsilon}{\omega^2} > 0, \tag{5.91}$$

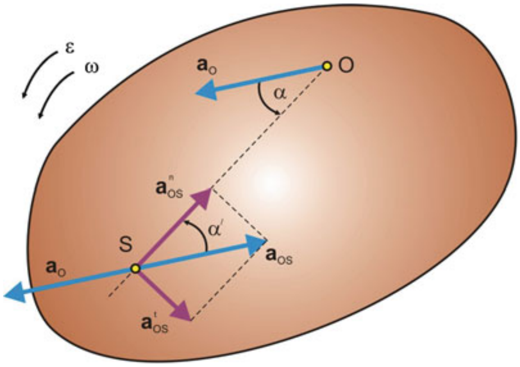


Fig. 5.27 Determination of instantaneous center of accelerations

which means that $\alpha \in [0, \pi/2]$. We lay off angle α in accordance with the sense of $\boldsymbol{\varepsilon}$.

On a ray going away from point O let us lay off the segment

$$OS = \frac{a_O}{\sqrt{\varepsilon^2 + \omega^4}}. \tag{5.92}$$

As a pole let us take point O, and because the acceleration of an arbitrary point of a plane figure is a geometric sum of acceleration of a pole and the acceleration of this point caused by its motion with respect to the pole, we have

$$\mathbf{a}_S = \mathbf{a}_O + \mathbf{a}_{OS}, \tag{5.93}$$

where $\mathbf{a}_{OS}$ is in accordance with the sense of $\boldsymbol{\varepsilon}$. Our task is finding point S such that its acceleration at the given time instant is equal to zero.

The acceleration following from the motion about pole O can be resolved into two components:

$$\mathbf{a}_{OS} = \mathbf{a}^n_{OS} + \mathbf{a}^t_{OS}, \tag{5.94}$$

and hence

$$a_{OS} = \sqrt{(a^n_{OS})^2 + (a^t_{OS})^2} = OS\sqrt{\varepsilon^2 + \omega^4} = a_O,$$

where (5.92) was taken into account. Additionally,

$$\tan\alpha' = \frac{a^t_{OS}}{a^n_{OS}} = \frac{OS\varepsilon}{OS\omega^2} = \frac{\varepsilon}{\omega^2} = \tan\alpha, \tag{5.95}$$

where (5.91) was taken into account. We have demonstrated that $\alpha = \alpha'$. From the preceding considerations it follows that $\mathbf{a}_{OS} = -\mathbf{a}_O$, and in view of that, from (5.93) we have $\mathbf{a}_S = \mathbf{0}$.

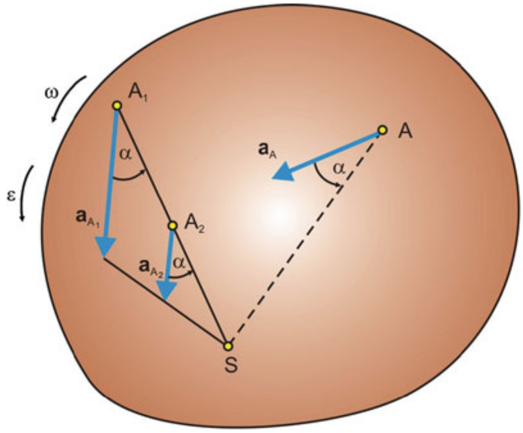


Fig. 5.28 Accelerations of points A_1, A_2, and A with respect to S

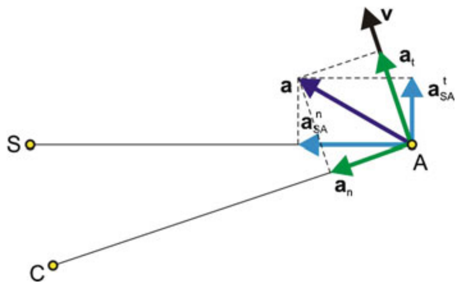


Fig. 5.29 Instantaneous center of accelerations S and instantaneous center of velocities C

If we choose as a pole the instantaneous center of accelerations found in the described way, then we easily determine the acceleration of an arbitrary point with respect to pole S, which is shown in Fig. 5.28.

From Fig. 5.28 we have

$$a_{A_1} = a_{SA_1} = SA_1\sqrt{\varepsilon^2 + \omega^4},$$
$$a_{A_2} = a_{SA_2} = SA_2\sqrt{\varepsilon^2 + \omega^4},$$
$$a_A = a_{SA} = SA\sqrt{\varepsilon^2 + \omega^4}. \tag{5.96}$$

From (5.96) follow the relationships

$$\frac{a_{A_2}}{a_{A_1}} = \frac{SA_2}{SA_1}, \qquad \frac{a_A}{a_{A_1}} = \frac{SA}{SA_1}, \tag{5.97}$$

where α and ε have the same senses.

We will show that the instantaneous center of velocities C and instantaneous center of accelerations S are different points of a planar figure (Fig. 5.29).

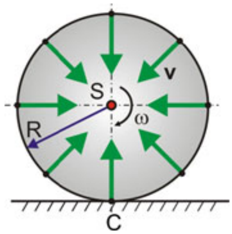


Fig. 5.30 Instantaneous center of velocities C and center of accelerations S of a disk

Let us choose an arbitrary point A of a plane figure and connect it by segments CA and SA respectively with the center of velocities C and the center of accelerations S. Point A at the given time instant has the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. The acceleration

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n = \mathbf{a}^t_{SA} + \mathbf{a}^n_{SA}, \tag{5.98}$$

and the components of the vector of acceleration are associated with the existence of distinct points S and C.

5.2.4.2 Determination of Instantaneous Centers of Acceleration

There are three basic methods to determine the position of the instantaneous center of accelerations:

I. *The point of a planar figure whose accelaration* $\boldsymbol{\varepsilon}$ *is at a certain time instant equal to zero is known on the basis of the problem conditions.*

   A disk of radius R rolls without sliding along a straight line on a surface [e.g., rolling of railway wheel on a rail with a constant velocity of the disk's center (Fig. 5.30)].

   We have $v = CS\omega = R\omega$, and hence $\omega = \frac{v}{R} =$ const. If the center of the disk moves uniformly with velocity $\mathbf{v} =$ const, then $\mathbf{a}_S = 0$, that is, the geometric center of the disk is the instantaneous center of accelerations. The acceleration of every point of the disk's circumference is directed toward point S and is equal to v^2/R. The instantaneous center of velocities C is situated at the point of contact of the disk with the ground. Although its instantaneous velocity is equal to zero, it has an acceleration of $\mathbf{a}_C$. In turn, the instantaneous center of accelerations has an instantaneous acceleration equal to zero but possesses velocity $\mathbf{v}$.

II. *The acceleration* $\mathbf{a}$ *of a point of a planar figure, as well as* $\boldsymbol{\omega}$ *and* $\boldsymbol{\varepsilon}$*, is known on the basis of the problem conditions.*

   Let us consider four cases connected with quantities ω and ε.

   (i) $\omega \neq 0$, $\varepsilon \neq 0$. In this case, the instantaneous center of accelerations is defined the angle $\alpha = \arctan(\varepsilon/\omega^2)$, where the directed angle α is laid off

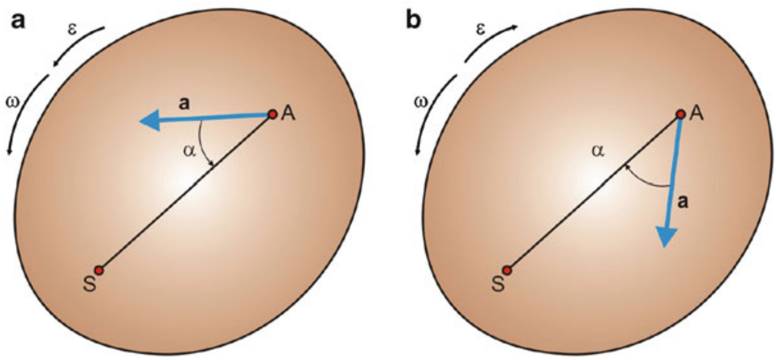


Fig. 5.31 Position of instantaneous center of accelerations S in case of the same (**a**) and opposite (**b**) senses of ω and ε

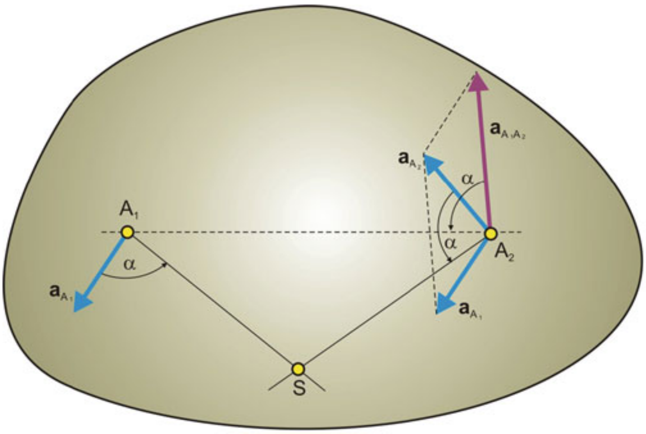


Fig. 5.32 Determination of center of accelerations S knowing $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_2}$

from the point's acceleration in accordance with the sense of ε and the distance $AS = \frac{a}{\sqrt{\varepsilon^2+\omega^4}}$ (Fig. 5.31).

(ii) $\omega \neq 0$, $\varepsilon = 0$. In this case, $\alpha = \arctan(\varepsilon/\omega^2) = 0$ and the accelerations of all points of a planar figure are directed toward the instantaneous center of accelerations, which lies at the distance $AS = \frac{a}{\omega^2}$.

(iii) $\omega = 0$, $\varepsilon \neq 0$. Because $\tan(\varepsilon/\omega^2) = \infty$, we have $\alpha = \pi/2$. The accelerations of all points are perpendicular to AS.

(iv) $\omega = 0$, $\varepsilon = 0$. The accelerations of all points of a figure are equal.

III. *The magnitudes and directions of accelerations of two points A_1 and A_2 of a planar figure (Fig. 5.32) are known.*

We have

$$\mathbf{a}_{A_2} = \mathbf{a}_{A_1} + \mathbf{a}_{A_1A_2},$$

and at point A_2 we construct a parallelogram of sides $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_1A_2}$. We determine the angle α, and then in accordance with the sense of ε we lay off the directed angle α from vectors $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_2}$. Rays forming those angles intersect at the desired point S.

For accelerations of two points $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_2}$ parallel to one another, some sketches leading to the determination of the instantaneous center of accelerations are shown in Fig. 5.33. In the case where $\mathbf{a}_{A_1} \parallel \mathbf{a}_{A_2}$, $\omega \neq 0$, $\varepsilon \neq 0$ the senses of acceleration are the same (a) and opposite (b); $\tan\alpha = (\frac{\varepsilon}{\omega^2}) = \infty$, $\omega = 0$, $\varepsilon \neq 0$, senses of accelerations the same (c) and opposite (d); $\alpha = 0$, $\tan\alpha = (\frac{\varepsilon}{\omega^2}) = 0$, $\omega \neq 0$, $\varepsilon = 0$, the senses of acceleration are the same and vectors lie on one line (e) and the senses of acceleration are opposite and vectors lie on one line (f); $\mathbf{a}_{A_1} = \mathbf{a}_{A_2}$, $A_1S = \infty$ (g).

Theorem 5.5. *If during the planar motion of a planar figure at least one of the quantities* $\boldsymbol{\omega} = \dot{\boldsymbol{\varphi}}$, $\boldsymbol{\varepsilon} = \ddot{\boldsymbol{\varphi}}$ *at a given time instant is different than zero, then there exists at this time instant point* S *belonging to the plane of motion such that its acceleration* $\mathbf{a}_S = \mathbf{0}$.

Proof. Our task is finding such a point S that $\mathbf{a}_S = \mathbf{0}$. The position of this point can be described by the radius vector $\overrightarrow{O'S}$. From (5.84) we have

$$\mathbf{a} = \mathbf{0} = \mathbf{a}_{O'} + \varepsilon \times \overrightarrow{O'S} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \overrightarrow{O'S}),$$

that is,

$$\mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \overrightarrow{O'S} - \boldsymbol{\omega}^2\overrightarrow{O'S} = \mathbf{0}. \tag{*}$$

Because

$$\boldsymbol{\varepsilon} \times \overrightarrow{O'S} = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ 0 & 0 & \ddot{\phi} \\ x'_{1S} & x'_{2S} & x'_{3S} \end{vmatrix} = -\mathbf{E}'_1\ddot{\phi}x'_{2S} + \mathbf{E}'_2\ddot{\phi}x'_{1S}\,,$$

the equation (*) takes the form

$$\mathbf{0} = \ddot{x}_{1O'}\mathbf{E}_1 + \ddot{x}_{2O'}\mathbf{E}_2 - \ddot{\varphi}x'_{2S}\mathbf{E}'_1 + \ddot{\varphi}x'_{1S}\mathbf{E}'_2 - \omega^2 x'_{1S}\mathbf{E}'_1 - \omega^2 x'_{2S}\mathbf{E}'_2.$$

Multiplying the preceding equations successively through $\mathbf{E}'_1$ and $\mathbf{E}'_2$ we obtain the following algebraic equations:

$$\ddot{x}_{1O'} = \ddot{\varphi}x'_{2S} + \dot{\varphi}^2 x'_{1S},$$

$$\ddot{x}_{2O'} = -\ddot{\varphi}x'_{1S} + \dot{\varphi}^2 x'_{2S},$$

which allow for the determination of the position of point S, which proves its existence.

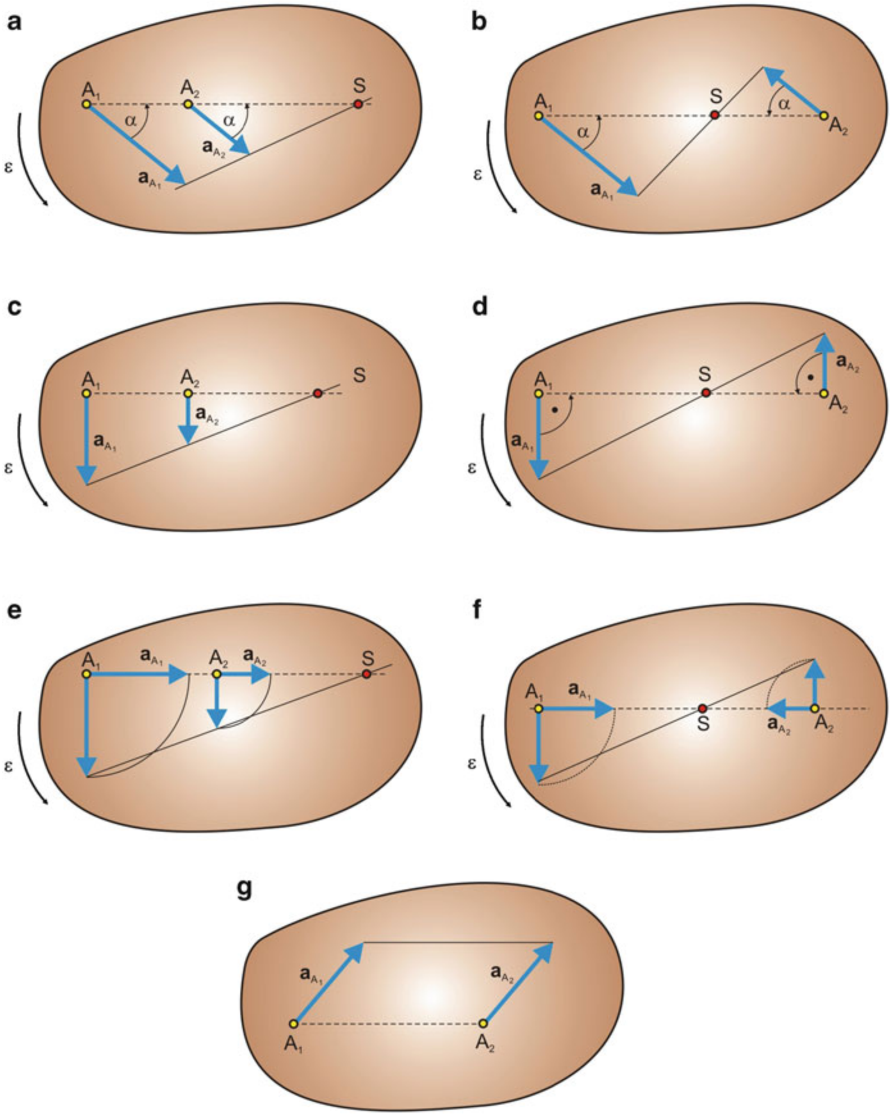


Fig. 5.33 Determination of instantaneous center of accelerations

Solving the aforementioned system of equations we obtain

$$x'_{1S} = \frac{1}{\varepsilon^2 + \omega^4}(\omega^2 \ddot{x}_{1O'} - \varepsilon \ddot{x}_{2O'}),$$

$$x'_{2S} = \frac{1}{\varepsilon^2 + \omega^4}(\varepsilon \ddot{x}_{1O'} + \omega^2 \ddot{x}_{2O'}), \tag{5.99}$$

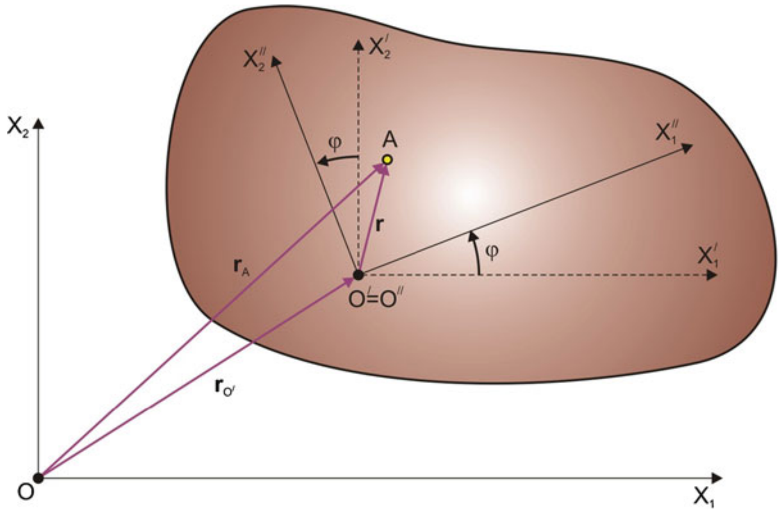


Fig. 5.34 Motion of a planar figure and non-stationary $O'X_1'X_2'$ and stationary OX_1X_2 coordinate systems

which allows for introduction of the following vector describing the position of the *center of accelerations* S:

$$\overrightarrow{O'S} = \frac{1}{\varepsilon^2 + \omega^4}(\omega^2 \mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \mathbf{a}_{O'}). \tag{5.100}$$

□

5.2.5 *Equations of Moving and Fixed Centrodes*

We will show how to determine analytically equations of moving and fixed centrodes. For the purpose of observing the motion of a plane figure in its plane we introduce the stationary OX_1X_2 and non-stationary $O'X_1'X_2'$ Cartesian coordinate system, where point O' is an arbitrary point (the pole) of a planar figure (Fig. 5.34).

The velocity of an arbitrary point A of a planar figure reads

$$\mathbf{v}_A = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}, \tag{5.101}$$

where $\mathbf{v}_{O'}$ is the velocity of the pole $O'(x_{1O'}, x_{2O'})$, $\boldsymbol{\omega}$ is the angular velocity of a planar figure, and $\mathbf{r} = \overrightarrow{O'A}$.

We will express the velocity vector $\boldsymbol{\omega} \times \mathbf{r}$ in a stationary system, that is,

$$\boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ 0 & 0 & \omega \\ x_1 - x_{1O'} & x_2 - x_{2O'} & 0 \end{vmatrix}$$
$$= -\mathbf{E}_1\omega(x_2 - x_{2O'}) + \mathbf{E}_2\omega(x_1 - x_{1O'}), \tag{5.102}$$

and from (5.101) we obtain

$$\begin{aligned} v_{Ax_1} &= v_{O'x_1} - \omega(x_2 - x_{2O'}), \\ v_{Ax_2} &= v_{O'x_2} - \omega(x_1 - x_{1O'}). \end{aligned} \tag{5.103}$$

The velocities of the desired instantaneous center of velocities C are equal to $v_{Cx_1} = v_{Cx_2} = 0$, and

$$\begin{aligned} v_{O'x_1} - \omega(x_{2C} - x_{2O'}) &= \dot{x}_{1O'} - \omega(x_{2C} - x_{2O'}) = 0, \\ v_{O'x_2} + \omega(x_{1C} - x_{1O'}) &= \dot{x}_{2O'} + \omega(x_{1C} - x_{1O'}) = 0. \end{aligned} \tag{5.104}$$

From the preceding equations we determine the desired coordinates of the instantaneous center of velocities

$$\begin{aligned} x_{1C} &= x_{1O'} - \frac{1}{\omega}\dot{x}_{2O'}, \\ x_{2C} &= x_{2O'} + \frac{1}{\omega}\dot{x}_{1O'}. \end{aligned} \tag{5.105}$$

Equations (5.105) are called *equations of a fixed centrode in parametric form in a stationary system*.

The velocity of point A is equal to

$$\mathbf{E}_A = v_{O'x_1}\mathbf{E}_1 + v_{O'x_2}\mathbf{E}_2 + \begin{vmatrix} \mathbf{E}_1'' & \mathbf{E}_2'' & \mathbf{E}_3'' \\ 0 & 0 & \omega \\ x_1'' & x_2'' & 0 \end{vmatrix}$$
$$= v_{O'x_1}\mathbf{E}_1 + v_{O'x_2}\mathbf{E}_2 - \mathbf{E}_1''\omega x_2'' + \mathbf{E}_2''\omega x_1''. \tag{5.106}$$

We will multiply (scalar product) (5.106) by $\mathbf{E}_1''$ and $\mathbf{E}_2''$, obtaining

$$\begin{aligned} v_{Ax_1''} &= v_{O'x_1}\cos(\mathbf{E}_1, \mathbf{E}_1'') + v_{O'x_2}\cos(\mathbf{E}_2, \mathbf{E}_1'') - \omega x_2'', \\ v_{Ax_2''} &= v_{O'x_1}\cos(\mathbf{E}_1, \mathbf{E}_2'') + v_{O'x_2}\cos(\mathbf{E}_2, \mathbf{E}_2'') + \omega x_1'', \end{aligned} \tag{5.107}$$

or

$$\begin{aligned} v_{Ax_1''} &= v_{O'x_1}\cos\varphi + v_{O'x_2}\sin\varphi - \omega x_2'', \\ v_{Ax_2''} &= -v_{O'x_1}\sin\varphi + v_{O'x_2}\cos\varphi + \omega x_1''. \end{aligned} \tag{5.108}$$

The desired coordinates of the instantaneous center of velocities in a non-stationary system of coordinates are described by the equations

$$\begin{aligned} \dot{x}_{1O'} \cos\varphi + \dot{x}_{2O'} \sin\varphi - \omega x''_{2C} &= 0, \\ -\dot{x}_{1O'} \sin\varphi + \dot{x}_{2O'} \cos\varphi + \omega x''_{1C} &= 0, \end{aligned} \tag{5.109}$$

from which we determine

$$\begin{aligned} x''_{1C} &= \frac{1}{\omega}(\dot{x}_{1O'} \sin\varphi - \dot{x}_{2O'} \cos\varphi), \\ x''_{2C} &= \frac{1}{\omega}(\dot{x}_{1O'} \cos\varphi + \dot{x}_{2O'} \sin\varphi). \end{aligned} \tag{5.110}$$

The preceding equations are called *equations of a moving centrode in parametric form in a non-stationary coordinate system*.

5.2.6 Vector Methods in the Kinematics of Planar Motion

5.2.6.1 Velocities

The computations conducted earlier regarding the planar motion of a rigid body lead to the following three principal conclusions:

1. The projections of velocities of any two points of a rigid body onto the line joining these points are equal.
2. At an arbitrary time instant the planar motion of a rigid body can be treated as the instantaneous rotation of that body about a point called an instantaneous center of rotation.
3. At an arbitrary time instant the planar motion of a rigid body can be treated as motion composed of the translational motion of an arbitrary point of the body (the pole) and the body's rotational motion about that point.

As we showed in Sect. 5.2.1 the figure marked by a hatching line in Fig. 5.12 moves in translational motion in the plane Γ. In Fig. 5.17 it is shown how to determine the position of the instantaneous center of rotation at the given time instant t. Let us note that the instantaneous centers of rotation in the stationary $OX_1X_2X_3$ and non-stationary $O''X''_1X''_2X''_3$ coordinate systems are coincident; however, the set of instantaneous positions of centers of rotation in the plane OX_1X_2 (stationary plane) forms a continuous curve called a *space centrode*, whereas the set of instantaneous positions of centers of rotation in the plane $OX''_1X''_2$ (moving plane) forms a curve called a *body centrode*.

It follows that the planar motion of a rigid body can be represented as rolling without sliding of the *body centrode* on the *space centrode*. If the plane Γ is moving, the aforementioned centrodes refer to the relative motion measured with respect to the stationary plane Γ_O.

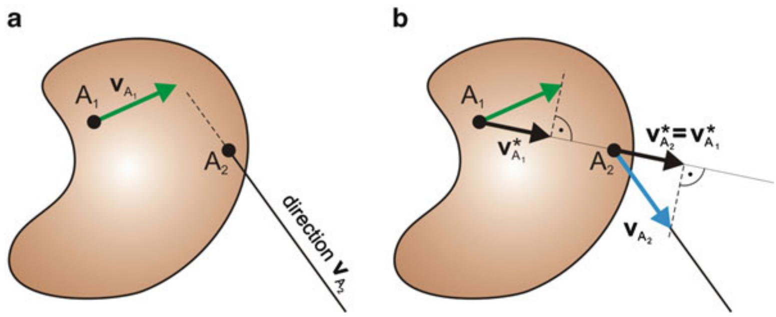


Fig. 5.35 Determination of velocity $\mathbf{v}_{A_2}$ knowing $\mathbf{v}_{A_1}$ and direction of $\mathbf{v}_{A_2}$

It is worth emphasizing that because the instantaneous motion of the planar figure (planar motion of a rigid body) is described by the motion of a pole (two coordinates) and the rotation about the pole, the number of parameters (scalar quantities) needed for the complete description of the planar motion of a rigid body is equal to three (i.e., it is equal to the number of degrees of freedom of the rigid body).

We will show using two examples, which follow from previous discussions in this chapter, how to determine the velocities of arbitrary points of a planar figure, that is, the points of a rigid body in planar motion.

(i) *Projection of velocities of two points of a planar figure onto the line joining these points*

Let the velocity $\mathbf{v}_{A_1}$ and the direction of the velocity of point A_2 be given (Fig. 5.35a). By Theorem 5.1 we find the projection of vector $\mathbf{v}_{A_1}$ onto line A_1A_2 and attach the obtained vector $\mathbf{v}^*_{A_1}$ at point A_2 (Fig. 5.35b). The line perpendicular to line A_1A_2 and passing through the tip of vector $\mathbf{v}^*_{A_1}$ intersects the line, which determines the direction of the unknown vector $\mathbf{v}_{A_2}$ and passes through point A_2 at the point that is its tip.

On the other hand (see proof of Theorem 5.4), we have

$$\mathbf{v}_{A_1} = \mathbf{v}_{A_2} + \boldsymbol{\omega} \times \overrightarrow{A_2A_1}, \tag{5.111}$$

that is,

$$\mathbf{v}_{A_2} = \mathbf{v}_{A_1} + \mathbf{v}_{A_1A_2}, \tag{5.112}$$

where

$$\mathbf{v}_{A_2A_1} = \boldsymbol{\omega} \times \overrightarrow{A_2A_1} \tag{5.113}$$

and $\mathbf{v}_{A_2A_1}$ denotes the relative velocity of point A_1 with respect to A_2. According to Theorem 5.1 we have $\mathbf{v}^*_{A_2} = \mathbf{v}^*_{A_1}$. Therefore,

$$\mathbf{v}^*_{A_2A_1} \equiv \mathbf{v}^*_{A_2} - \mathbf{v}^*_{A_1} = \mathbf{0}. \tag{5.114}$$

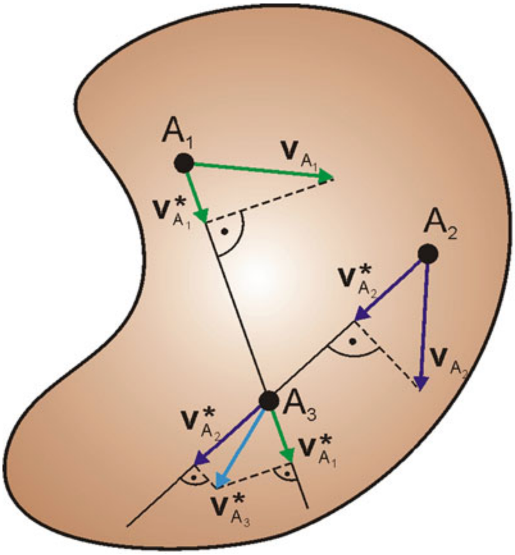


Fig. 5.36 Determination of velocity of an arbitrary point A_3 if $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ are known (case i)

If $\mathbf{v}^*_{A_2A_1}$ is the projection of $\mathbf{v}_{A_2A_1}$ onto line A_2A_1, we have $\mathbf{v}_{A_2A_1} \perp A_2A_1$, which also follows from the interpretation of the vector product given by (5.113).

We will now show how to determine the velocity of an arbitrary point A_3 if the velocities of two points $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ are known (the latter was determined in Fig. 5.35). We draw the lines through points A_1 and A_3 and through A_2 and A_3 (Fig. 5.36). Next, we project the velocity $\mathbf{v}_{A_1}$ ($\mathbf{v}_{A_2}$) onto direction A_1A_3 (A_2A_3). We move the obtained projections $\mathbf{v}^*_{A_1}$ and $\mathbf{v}^*_{A_2}$ along the mentioned directions to point A_3. Then we draw perpendicular lines from the tips of both vectors $\mathbf{v}^*_{A_1}$ and $\mathbf{v}^*_{A_2}$. The point of intersection of the lines determines the tip of the unknown vector $\mathbf{v}_{A_3}$.

(ii) *Geometric sum of motion of a pole and the rotation about that pole*

Let point A_1 of velocity $\mathbf{v}_{A_1}$ be a pole, and let the direction of the velocity of point A_2 be given (Fig. 5.37).

The velocity of point A_2 is determined by (5.112), where $\mathbf{v}_{A_1}$ denotes the velocity of pole A_1 and $\mathbf{v}_{A_2A_1}$ denotes the velocity resulting from the rotation of point A_2 with respect to A_1. Equation (5.112) can be written in the form

$$\underline{\mathbf{v}_{A_2}} = \underline{\underline{\mathbf{v}_{A_1}}} + \underline{\mathbf{v}_{A_2A_1}}, \tag{5.115}$$

where the double underline denotes that knowledge about the vector is complete, while the single underline indicates that only the vector's direction is known. Vector (5.115) allows us to determine the magnitude of vector $\mathbf{v}_{A_2}$. As can be seen in Fig. 5.37a, vector $\mathbf{v}_{A_1}$ is moved to point A_2. Then through its tip we draw a line perpendicular to A_1A_2, which intersects the known direction of vector $\mathbf{v}_{A_2}$ at the

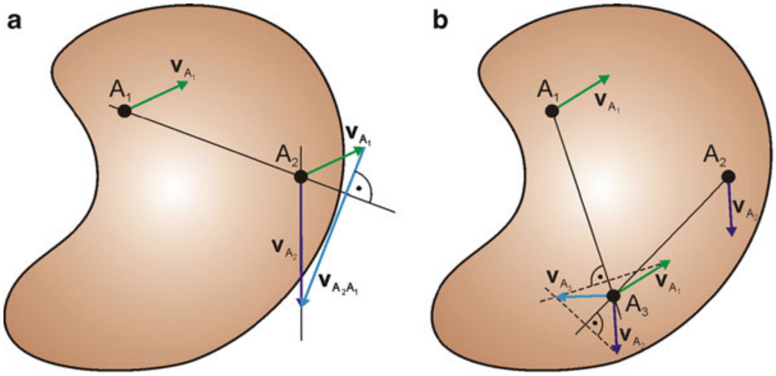


Fig. 5.37 Successive determination of velocities of points A_2 and A_3 (case ii)

point that constitutes the tip of the unknown vector $\mathbf{v}_{A_2}$. It can be seen that the presented vectors satisfy vector (5.115). Because we now know the velocities of two points of the plane figure $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$, we are able to determine the velocity of an arbitrary point A_3 of that figure not lying on line A_1A_2 from the following vector equations

$$\mathbf{v}_{A_3} = \underline{\underline{\mathbf{v}_{A_1}}} + \underline{\mathbf{v}_{A_3A_1}},$$
$$\mathbf{v}_{A_3} = \underline{\underline{\mathbf{v}_{A_2}}} + \underline{\mathbf{v}_{A_3A_2}}, \tag{5.116}$$

which leads to a single vector equation

$$\underline{\underline{\mathbf{v}_{A_1}}} + \underline{\mathbf{v}_{A_3A_1}} = \underline{\underline{\mathbf{v}_{A_2}}} + \underline{\mathbf{v}_{A_3A_2}}. \tag{5.117}$$

This equation enables us to find the solution because we have two unknowns. It becomes clear after introduction of the plane Cartesian coordinate system and projection of (5.112) onto its axes. As a result we obtain two algebraic equations with two unknowns.

Figure 5.37b shows a geometrical construction leading to the solution of (5.117). We attach vectors $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ at point A_3, and through their tips we draw lines perpendicular to A_1A_3 and A_2A_3, respectively. Their intersection point determines the magnitude and sense of vector $\mathbf{v}_{A_3}$.

The method to determine the velocity $\mathbf{v}_{A_3}$ if point A_3 lies on line A_1A_2 is presented in Fig. 5.38.

While the velocity $\mathbf{v}_{A_2}$ was being determined, vector $\mathbf{v}_{A_2A_1}$ was also determined, which allows also for the determination of $\mathbf{v}_{A_3A_1}$. Now, moving vector $\mathbf{v}_{A_1}$ to point A_3 we determine the unknown velocity $\mathbf{v}_{A_3}$ from the parallelogram law.

(iii) *The center of velocity*

Making an assumption similar to that in previous cases – that we know the velocity of point A_1 and the direction of the velocity at point A_2. From this we

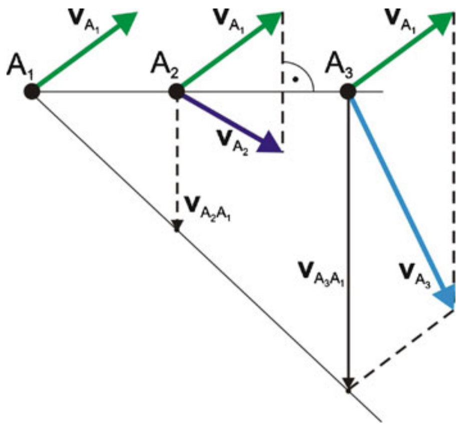


Fig. 5.38 Determination of velocity of point A_3 lying on line A_1A_2 (case ii)

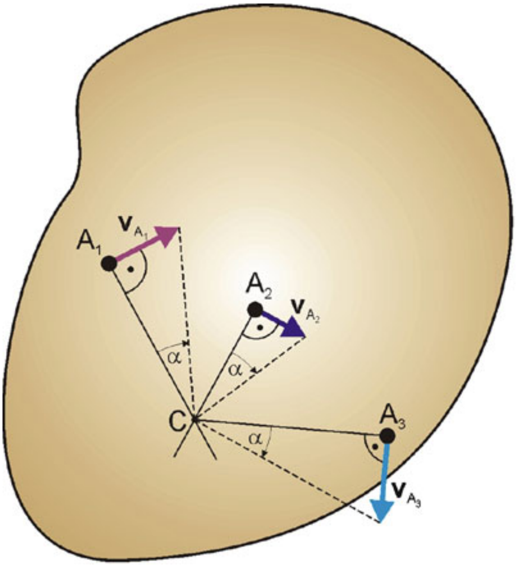


Fig. 5.39 Determination of an instantaneous center of velocity (case iii) and the velocity of an arbitrary point A_3

will determine velocity vector $\mathbf{v}_{A_2}$. The perpendicular lines drawn through the tails of velocity vectors $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ determine at their point of intersection the instantaneous center of velocity C (Fig. 5.39).

Connecting the tips of vectors $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ with point C we determine the angle $\alpha = \omega t$, where $\dot{\alpha} = \omega =$ const. Because

$$\tan\alpha_1 = \frac{v_{A_1}}{CA_1} = \omega,$$

$$\tan\alpha_2 = \frac{v_{A_2}}{CA_2} = \omega, \tag{5.118}$$

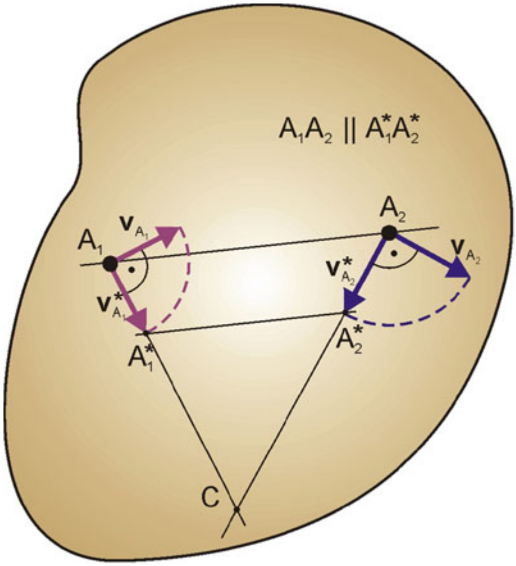


Fig. 5.40 Determination of velocity $\mathbf{v}_{A_2}$ with the aid of the method of rotated velocities (case iii)

we have $\alpha_1 = \alpha_2 = \alpha$. If we want to determine the velocity of an arbitrary point A_3, we lay off a directed angle α from segment CA_3, and the point of intersection of the line perpendicular to CA_3 (at point A_3) with the ray formed after rotation of segment CA_3 through angle α determines the tip of vector $\mathbf{v}_{A_3}$.

It is convenient to use a certain version of that method called the *method of rotated velocities*, presented in Fig. 5.40.

We join point A_1 with A_2. We rotate the known velocity $\mathbf{v}_{A_1}$ through the angle $\pi/2$ in the direction of the instantaneous rotation obtaining the velocity $\mathbf{v}^*_{A_1}$. Line $A_1^* A_2^*$ is the set of tips of the rotated velocity vectors whose tails lie on lines A_1A_2 and $A_1A_2 \parallel A_1^*A_2^*$. In order to determine $\mathbf{v}_{A_2}$ we can proceed in the following way. We rotate vector $\mathbf{v}_{A_1}$ through the angle $\pi/2$ and draw a line parallel to A_1A_2 passing through A_1^*. Its intersection with the line perpendicular to the direction of the velocity $\mathbf{v}_{A_2}$ allows for the determination of A_2^*, that is, the determination of $\mathbf{v}^*_{A_2}$. The rotation of that vector through the angle $\pi/2$ opposite to the direction of the instantaneous rotation leads to the determination of vector $\mathbf{v}_{A_2}$ (this construction does not require determination of point C).

Finally, we show how to determine the velocity of an arbitrary point A_3 using the method of rotated velocities (Fig. 5.41). To this end we rotate in the previously described way velocities $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ obtaining $\mathbf{v}^*_{A_1}$ and $\mathbf{v}^*_{A_2}$.

We draw lines A_1A_3 and A_2A_3. Through the tip of $\mathbf{v}^*_{A_1}$ we draw a line parallel to A_1A_3 and through the tip of $\mathbf{v}^*_{A_2}$ a line parallel to A_2A_3. Their point of intersection determines the tip of the vector $\mathbf{v}^*_{A_3}$, which after rotation becomes the desired vector $\mathbf{v}_{A_3}$.

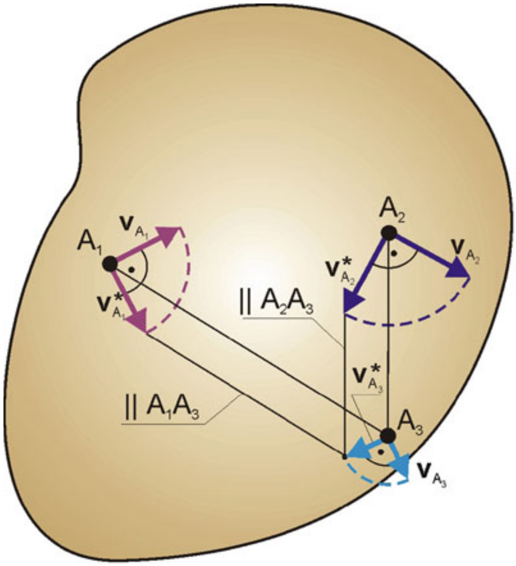


Fig. 5.41 Determination of velocity of point A_3 with the aid of the method of rotated velocities (case iii)

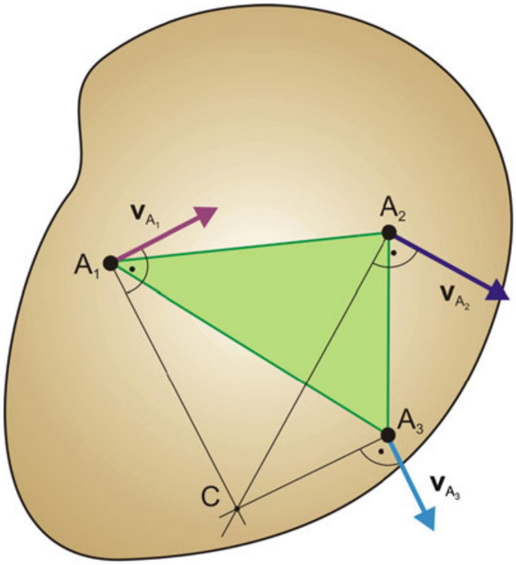


Fig. 5.42 Velocities of three points A_i, $i = 1, 2, 3$, instantaneous center of velocity C, and $\Delta A_1 A_2 A_3$

(iv) *Burmester theorem (velocity diagram)*

In Fig. 5.42, three arbitrary points of a planar figure have been chosen, and their velocity vectors are marked (from the previous computations it follows that they cannot be drawn arbitrarily).

In order to construct a velocity diagram we take an arbitrary pole C^* to which we move the (sliding) vectors $\mathbf{v}_{A_1}$, $\mathbf{v}_{A_2}$, and $\mathbf{v}_{A_3}$, as shown in Fig. 5.43.

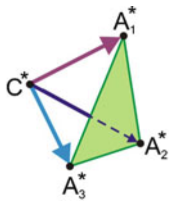


Fig. 5.43 Velocity diagram (case iv)

From the construction it follows that $\overrightarrow{C^*A_1^*} = \mathbf{v}_{A_1}$, $\overrightarrow{C^*A_2^*} = \mathbf{v}_{A_2}$, and $\overrightarrow{C^*A_3^*} = \mathbf{v}_{A_3}$. This means that vectors $\overrightarrow{C^*A_1^*}$, $\overrightarrow{C^*A_2^*}$, and $\overrightarrow{C^*A_3^*}$ shown in the velocity diagram are perpendicular respectively to lines CA_1, CA_2, and CA_3 in Fig. 5.42. The marked figure (in the present case a triangle $A_1^*A_2^*A_3^*$) in Fig. 5.43 is similar to the figure (triangle) $A_1A_2A_3$ (Fig. 5.42) and rotated with respect to it through the angle $\pi/2$ in the direction of the instantaneous rotation.

From the velocity diagram it follows that

$$\overrightarrow{C^*A_1^*} + \overrightarrow{A_1^*A_2^*} = \overrightarrow{C^*A_2^*},$$
$$\overrightarrow{C^*A_2^*} + \overrightarrow{A_2^*A_3^*} = \overrightarrow{C^*A_3^*}, \tag{5.119}$$

which after returning to the velocity interpretation means

$$\mathbf{v}_{A_2} - \mathbf{v}_{A_1} = \overrightarrow{A_1^*A_2^*} \equiv \mathbf{v}_{A_2A_1},$$
$$\mathbf{v}_{A_3} - \mathbf{v}_{A_2} = \overrightarrow{A_1^*A_3^*} \equiv \mathbf{v}_{A_3A_2}, \tag{5.120}$$

where $\mathbf{v}_{A_2A_1}$ ($\mathbf{v}_{A_3A_2}$) denotes the relative velocity of point A_2 (A_3) with respect to A_1 (A_2).

These observations can be formally stated as the following theorem.

Burmester Theorem
A figure obtained as a result of connecting the tips of vectors of a velocity diagram of points of a rigid body moving in planar motion is similar to a figure determined by those points of the body and rotated with respect to the second figure through an angle $\pi/2$ in the direction of instantaneous rotation of the body.

5.2.6.2 Accelerations

The calculations conducted in the previous section can be extended to an analysis of accelerations of a rigid body in planar motion. We will take up cases involving the determination of accelerations of the points of a rigid body relying on three methods: the method of geometric composition of accelerations of a pole's translational motion and the rotational motion about that pole (Sect. 5.2.1), the method of instantaneous center of acceleration, and the method of similarity (acceleration diagram).

(i) *Geometrical composition of translational and rotational motion*

Let the acceleration $\mathbf{a}_{A_1}$ of point A_1 (the pole; previously point O' played this role) be given. Our aim will be to determine the acceleration of point A_2 in the considered time instant. According to (5.115) we have

$$\mathbf{a}_{A_2} = \mathbf{a}_{A_1} + \mathbf{a}_{A_2A_1}, \tag{5.121}$$

where $\mathbf{a}_{A_2A_1} \equiv \mathbf{a}^W$ denotes the acceleration resulting from the rotation of point A_2 with respect to point A_1. According to (5.88) this acceleration has two components, i.e., it is a sum of two vectors

$$\mathbf{a}_{A_2A_1} = \underline{\underline{\mathbf{a}^n_{A_2A_1}}} + \underline{\mathbf{a}^t_{A_2A_1}}. \tag{5.122}$$

Because the distance $|\overrightarrow{A_2A_1}|$ is known, the acceleration $\mathbf{a}^n_{A_2A_1}$ is completely known (double underline). Its sense is determined by vector $\overrightarrow{A_2A_1}$, and its magnitude is equal to

$$\mathrm{a}^n_{A_2A_1} = \frac{v_{A_2A_1}}{\left|\overrightarrow{A_2A_1}\right|}. \tag{5.123}$$

The acceleration $\mathbf{a}^t_{A_2A_1}$ is known only as to the direction because this vector is attached at point A_2 and perpendicular to $\overrightarrow{A_2A_1}$. The total acceleration of point A_2 is equal to

$$\mathbf{a}_{A_2} = \underline{\underline{\mathbf{a}^n_{A_2}}} + \underline{\mathbf{a}^t_{A_2}}. \tag{5.124}$$

If, according to the assumption, the radius of curvature of the path of point A_2 is known, let it be equal to $|\overrightarrow{A_2O}|$, the magnitude of the normal acceleration

$$\mathrm{a}^n_{A_2} = \frac{v^2_{A_2}}{\left|\overrightarrow{A_2O}\right|}, \tag{5.125}$$

and its sense and direction be in agreement with those of the vector $\overrightarrow{A_2O}$ (it passes through point A_2). Taking into account the preceding considerations, (5.121) will assume the form

$$\underline{\underline{\mathbf{a}^n_{A_2}}} + \underline{\mathbf{a}^t_{A_2}} = \underline{\underline{\mathbf{a}_{A_1}}} + \underline{\underline{\mathbf{a}^n_{A_2A_1}}} + \underline{\mathbf{a}^t_{A_2A_1}}. \tag{5.126}$$

This equation contains two unknowns and leads to their determination.

Figure 5.44 the construction leading to the determination of the acceleration of point A_2 on the assumptions introduced earlier is presented.

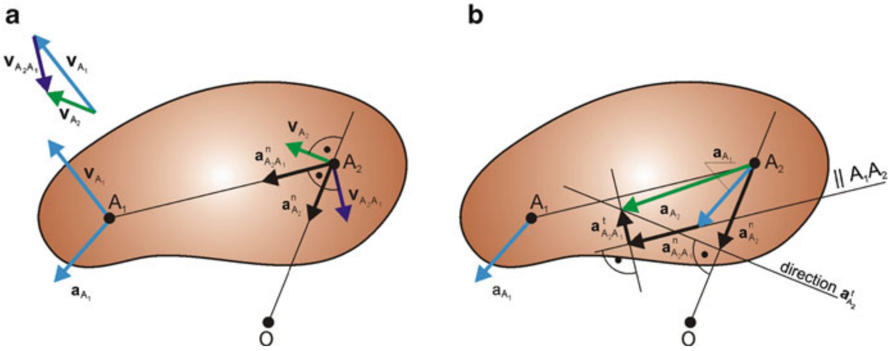


Fig. 5.44 Determination of acceleration of point A_2 (method i)

First we determine the accelerations $\mathbf{a}^n_{A_2A_1}$ and $\mathbf{a}^n_{A_2}$ (Fig. 5.44a). Next we attach vector $\mathbf{a}_{A_1}$ at point A_2 (Fig. 5.44b) and add to it vector $\mathbf{a}^n_{A_2A_1}$. Then from its tip we draw a perpendicular line on which lies vector $\mathbf{a}^t_{A_2A_1}$. On the other hand, at point A_2 we attach vector $\mathbf{a}^n_{A_2}$, and from its tip we draw a perpendicular line that intersects the direction of $\mathbf{a}^t_{A_2A_1}$ at the point representing the tip of vector $\mathbf{a}_{A_2}$.

(ii) *Method of instantaneous center of accelerations*

Let us assume (cf. the final considerations of Sect. 5.2.1) that the acceleration of an arbitrary point A belonging to a figure in planar motion is equal to

$$\mathbf{a}_A = \dot{\mathbf{v}}_A, \tag{5.127}$$

where $\mathbf{v}_A$ is the velocity of point A. If point A moves along a curvilinear path, then

$$\mathbf{a}_A = \mathbf{a}^n_A + \mathbf{a}^t_A, \tag{5.128}$$

where

$$\mathrm{a}^n_A = \frac{v_A^2}{\left|\overrightarrow{AO}\right|}, \qquad \mathrm{a}^t_A = \frac{\mathrm{d}v_A}{\mathrm{d}t} = \varepsilon\left|\overrightarrow{AO}\right|. \tag{5.129}$$

The normal acceleration $\mathbf{a}^n_A$ has a direction and sense in agreement with those of vector $\overrightarrow{AO}$. The tangential acceleration $\mathbf{a}^t_A$ has a direction tangent to the path.

As was shown earlier, the acceleration $\mathbf{a}_A$ forms the angle α with the radius of curvature, and the value of the angle is given by (5.90), i.e., $\tan\alpha = \frac{\mathrm{a}_t}{\mathrm{a}_n} = \frac{\varepsilon}{\omega^2}$, which is illustrated in Fig. 5.45.

In planar motion angular velocity $\boldsymbol{\omega}$, angular acceleration $\boldsymbol{\varepsilon}$, and angle $\boldsymbol{\alpha}$, shown in Fig. 5.45, do not depend on the choice of pole. In particular cases, as poles we can choose the instantaneous centers of velocity and acceleration. The origin of the radius of curvature (point O) determines the *instantaneous center of acceleration*, a notion that is analogous to the *instantaneous center of velocity* introduced earlier.

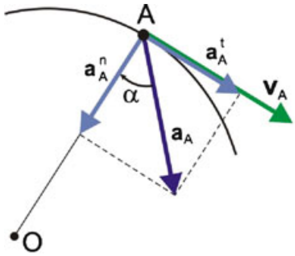


Fig. 5.45 Path of point A, radius of curvature, and acceleration $\mathbf{a}_A$ with its two components

Recall that the center of velocity is a point where the velocity is equal to zero, and the velocity of an arbitrary point A belonging to the plane Γ follows from the purely rotational motion and its magnitude is given by $v_A = \omega OA$. At a given time instant the path of point A is approximated by the arc of a circle of radius OA, where $\mathbf{v}_A \perp \overrightarrow{OA}$ at point A, and its sense is defined by the sense of $\boldsymbol{\omega}$.

Let us also assume that in order to determine the instantaneous center of velocity, it suffices to know the directions of velocities $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ at arbitrary points A_1 and A_2 (see case iii in Sect. 5.2.6.1). The point of intersection of the lines perpendicular to the directions of $\mathbf{v}_{A_1}$ and $\mathbf{v}_{A_2}$ at points A_1 and A_2 determines the desired instantaneous center of velocity C.

Although the velocity of a point is always tangent to its path (there is no notion of normal velocity), the acceleration (except for translational motion) possesses a normal component, which is why $\mathbf{a}_A$ is not tangent to the path, and its position with respect to the radius of curvature is defined by angle α.

From the foregoing discussion it follows that the instantaneous center of rotation and the instantaneous center of acceleration can coincide only in special cases. At a given time instant the planar motion of a body can be treated as an instantaneous rotational motion, as described in Sect. 5.1.4. At another time instant the position of the aforementioned center will change.

We will now show how to determine the instantaneous center of acceleration based on our knowledge of the accelerations of two arbitrary points $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_2}$. The problem boils down to the determination of angle α formed by these accelerations with the direction of the path's radius of curvature (Fig. 5.45).

Figure 5.46 shows how to graphically determine angle α (if $\boldsymbol{\varepsilon} \circ \boldsymbol{\omega} < 0$, then we lay off angle α in the opposite direction).

Let us take point A_1 as a pole. The acceleration of point A_2 with respect to A_1 can be expressed by the vector equation

$$\mathbf{a}_{A_2} = \mathbf{a}_{A_1} + \mathbf{a}_{A_2A_1}, \tag{5.130}$$

which allows for the determination of $\mathbf{a}_{A_2A_1}$ (Fig. 5.46a). An arbitrary point of a planar figure can be taken as a pole because in planar motion the angular velocity $\boldsymbol{\omega}$ and the angular acceleration $\boldsymbol{\varepsilon}$ do not depend on the chosen pole.

Figure 5.45 provides the following information. Pole O at the given instant remains at rest in the position indicated in the figure. The acceleration of point A

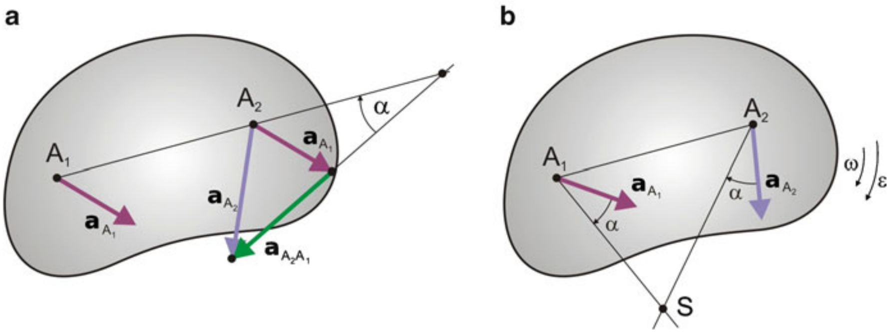


Fig. 5.46 Determination of a center of acceleration (case ii)

with respect to the stationary point O forms angle α with radius OA. The acceleration $\mathbf{a}_{A_2A_1}$ denotes the acceleration of point A_2 with respect to A_1, which by analogy leads to the determination of angle α formed between line A_1A_2 and the direction of vector $\mathbf{a}_{A_2A_1}$ (Fig. 5.46a). Next (Fig. 5.46b) we draw the lines passing through points A_1 and A_2 and forming angle α with, respectively, $\mathbf{a}_{A_1}$ and $\mathbf{a}_{A_2}$, and their point of intersection is the desired center of acceleration.

(iii) *Method of similarity*

By analogy to the velocity diagram (Figs. 5.42 and 5.43) it is possible to build an acceleration diagram whose construction is based on the similarity principle. At the arbitrary point P^* (the pole) we attach the acceleration vectors $\mathbf{a}_{A_i}$ (we will skip here a drawing of an acceleration diagram), and joining their tips we obtain the triangle $\Delta A_1^* A_2^* A_3^*$. At points A_i, $i = 1, 2, 3$, one should draw vectors $\mathbf{a}_{A_i}$, which form angles α with segments PA_i, and point P denotes the instantaneous center of acceleration. Attaching those vectors at an arbitrary pole P^* their tips will form the triangle $\Delta A_1^* A_2^* A_3^*$, which is similar to $\Delta A_1 A_2 A_3$ and rotated with respect to it through the angle $\pi - \alpha$ (and with respect to the velocity diagram through angle $\frac{\pi}{2} - \alpha$) in agreement with the sense of $\boldsymbol{\omega}$.

The proof of all the discussed principles of similarity is left to the reader.

Example 5.2. The beam AD (Fig. 5.47) moves in planar motion and is in continuous contact with points A and B, and its end A has the horizontal velocity $\mathbf{v}_A = \text{const}$. Determine the positions of instantaneous centers of velocity and acceleration during the motion of the beam. Determine also the values of angular velocity ω and angular acceleration ε of the beam.

Point B does not undergo rotation with respect to point A since it belongs to the rigid body (the beam). In view of that the velocity $\mathbf{v}_B$ is directed along the beam. Drawing lines perpendicular to the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ passing through points A and B, we will obtain point C as it is the instantaneous center of velocity. Because by assumption $\mathbf{v}_A = \text{const}$, we have $\mathbf{a}_A = \mathbf{0}$, and point A is the instantaneous center of acceleration.

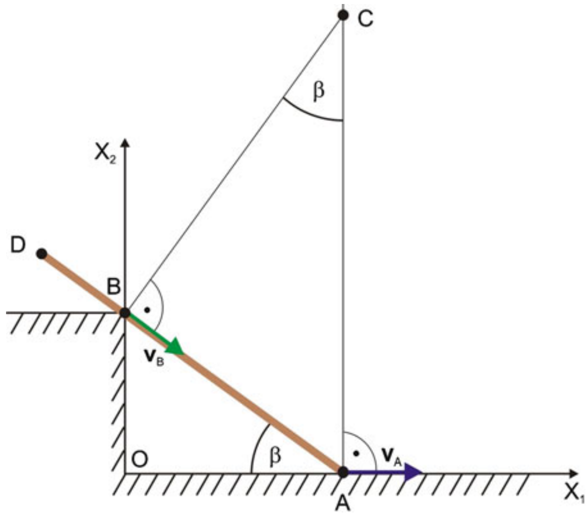


Fig. 5.47 Planar motion of a beam

Let $OA \equiv x(t) = v_A t$ (for $t = 0$ we have $OA = 0$), and let $OB = h$. The angles denoted in the figure are equal, and we calculate successively

$$AB = \sqrt{h^2 + x^2},$$
$$\tan\beta = \frac{h}{x} = \frac{AB}{BC},$$
$$AC = \sqrt{(AB)^2 + (BC)^2},$$

hence

$$BC = \frac{x}{h}\sqrt{h^2 + x^2},$$
$$AC = \sqrt{\frac{x^2(h^2 + x^2)}{h^2} + \frac{h^2(h^2 + x^2)}{h^2}} = \frac{h^2 + x^2}{h}.$$

The beam is in the instantaneous rotational motion about point C with the angular velocity

$$\omega = \frac{v_A}{AC} = \frac{hv_A}{h^2 + x^2}.$$

The angular acceleration of the beam is equal to

$$\varepsilon(t) = \dot{\omega} = hv_A \frac{\mathrm{d}}{\mathrm{d}t}(h^2 + x^2)^{-1} = -hv_A \frac{2x\dot{x}}{(h^2 + x^2)^2} = -\frac{2hv_A^3 t}{(h^2 + v_A^2 t^2)^2},$$

and because $t > 0$, the sense of $\boldsymbol{\varepsilon}$ is opposite to the sense of $\boldsymbol{\omega}$.

The velocity and acceleration of point B of the beam are equal to $v_B = \omega CB$ and $a_B = \varepsilon AB$. □

5.3 Composite Motion of a Point in a Three-Dimensional Space

We will call such a motion where the point (body) is simultaneously involved in several motions the *composite motion of a material point or a rigid body*. Apart from its own motion, a ship that crosses a river takes part in the motion of the river. A man who moves on the ship, apart from his own motion, is involved in the motion of the ship and the motion of the river. A similar situation occurs, for instance, during a person's motion in a moving elevator or airplane.

In many applications, especially in the case of the motion of spacecraft, there exists a need to conduct a kinematic analysis of a particle with respect to two coordinate systems (see, e.g., [4–11]). In the course of our calculations we have also introduced such coordinate systems: $OX_1X_2X_3$ (absolute) and $O''X_1''X_2''X_3''$ (rigidly connected to a moving body). In the latter case the position of point A was determined by the radius vector $\boldsymbol{\rho}$, where $|\boldsymbol{\rho}| = \text{const}$, which means that point A did not move with respect to $O''X_1''X_2''X_3''$.

We will now analyze the case where point A moves with respect to $O''X_1''X_2''X_3''$, and in turn $O''X_1''X_2''X_3''$ moves with respect to $OX_1X_2X_3$, but in a certain prescribed way (e.g., the motion of a spacecraft is prescribed), i.e., the motion of point O'' and the position of the axes $O''X_1''X_2''X_3''$ with respect to the axes of the absolute coordinate system $OX_1X_2X_3$ are completely known (Fig. 5.48). We will denote vector $\overrightarrow{O'A}$ describing the position of point A in the coordinate system $O''X_1''X_2''X_3''$ by $\boldsymbol{\rho}$ and in the coordinate system $OX_1X_2X_3$ by vector $\mathbf{r} = \mathbf{r}_A - \mathbf{r}_{O'}$.

The motion of point A with respect to the system $OX_1X_2X_3$ is called the *composite motion (absolute motion)*; the motion of point A with respect to the system $O''X_1''X_2''X_3''$ is called the *relative motion*, whereas the motion of the $O''X_1''X_2''X_3''$ with respect to the system $OX_1X_2X_3$ is called the *motion of transportation*.

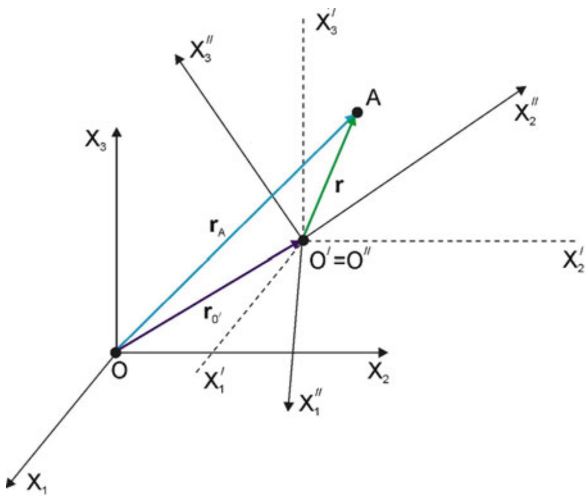


Fig. 5.48 Composite motion of point A in a three-dimensional space

Similar terminology is associated with velocity (and acceleration): absolute, relative, and transportation.

Our aim is to determine the relationships between the displacements, velocities, and accelerations of point A in the systems $OX_1X_2X_3$ and $O''X_1''X_2''X_3''$. The velocity (acceleration) of transportation of point A is the velocity (acceleration) of such a point A'' not moving with respect to the system $O''X_1''X_2''X_3''$ (i.e., the point and the system are rigidly connected), which at a given time instant is coincident with point A.

To the definitions introduced above we will give a physical meaning. Let a man be moving on a certain revolving platform. The coordinate system rigidly connected to the platform is called a non-stationary system, and the system associated with Earth is called a stationary system. The motion of the platform is the motion of transportation, the motion of the man with respect to the platform is called the relative motion, and the motion of the man with respect to Earth is called the absolute motion. The velocity (acceleration) of point C, which is associated with the platform at which the mass center of the man is situated at a given time instant, is called the velocity of transportation (acceleration of transportation) of the mass center of the man.

The analysis of composite motion relies on the determination of relations between velocities and accelerations of relative, transportation, and absolute motion.

A change in the radius vector $\mathbf{r}_A$ (Fig. 5.48) or coordinates x_{1A}, x_{2A}, x_{3A} describes the composite motion of a point. A change in the radius vector $\boldsymbol{\rho}$ or coordinates x''_{1A}, x''_{2A}, x''_{3A} describes the relative motion of point A, and a change in the radius vector $\mathbf{r}_{O'}$, that is, of coordinates of the pole $x_{1O'}$, $x_{2O'}$, $x_{3O'}$, describes the absolute motion of pole O'.

From Fig. 5.48 it follows that

$$\mathbf{r}_A = \mathbf{r}_{O'} + \mathbf{r}. \tag{5.131}$$

In turn, the vector $\overrightarrow{O'A}$ is expressed in the system $OX_1X_2X_3$ as

$$\mathbf{r} = \mathbf{A}(t)\boldsymbol{\rho}, \tag{5.132}$$

where matrix $\mathbf{A}$ is the matrix of transformation from the system $O''X_1''X_2''X_3''$ to $OX_1X_2X_3$. As will be shown further, it can be expressed by three independent angles called the *Euler angles*, which in our case would be prescribed functions of time.

An absolute velocity of point A in the system $OX_1X_2X_3$ is obtained through differentiation of (5.131) with respect to time:

$$\mathbf{v}_A \equiv \dot{\mathbf{r}}_A = \dot{\mathbf{r}}_{O'} + \dot{\mathbf{r}}. \tag{5.133}$$

Differentiating (5.132) we have

$$\dot{\mathbf{r}} = \dot{\mathbf{A}}\boldsymbol{\rho} + \mathbf{A}\dot{\boldsymbol{\rho}} = \dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{r} + \mathbf{A}\dot{\boldsymbol{\rho}}. \tag{5.134}$$

In turn, we have (to be demonstrated later)

$$\dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}, \tag{5.135}$$

and finally

$$\mathbf{v}_A = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{A}\dot{\boldsymbol{\rho}}, \tag{5.136}$$

where $\mathbf{v}_{O'} = \dot{\mathbf{r}}_{O'}$.

Eventually, the absolute velocity of point A (in the system $OX_1X_2X_3$) has the form

$$\begin{aligned} \mathbf{v}_A &= \mathbf{v}_A^t + \mathbf{v}_A^r, \\ v_A &= \sqrt{(v_A^t)^2 + (v_A^r)^2 + 2v_A^t v_A^r \cos(\mathbf{v}_A^t, \mathbf{v}_A^r)}. \end{aligned} \tag{5.137}$$

Velocity $\mathbf{v}_A^r = \mathbf{A}\dot{\boldsymbol{\rho}}$ is called *the relative velocity* of point A and is expressed in the system $O''X_1''X_2''X_3''$. In turn, vector $\boldsymbol{v}_A^t = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}$ defines the transportation velocity of point A.

In order to obtain the absolute acceleration of point A one should differentiate (5.136) with respect to time, and hence

$$\begin{aligned} \mathbf{a}_A \equiv \dot{\mathbf{v}}_A &= \dot{\mathbf{v}}_{O'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\mathbf{A}}\dot{\boldsymbol{\rho}} + \mathbf{A}\ddot{\boldsymbol{\rho}} \\ &= \mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} + \mathbf{A}\dot{\boldsymbol{\rho}}) + \dot{\mathbf{A}}\dot{\boldsymbol{\rho}} + \mathbf{A}\ddot{\boldsymbol{\rho}}, \end{aligned} \tag{5.138}$$

where $\mathbf{a}_{O'} = \mathbf{v}_{O'} = \ddot{\mathbf{r}}_{O'}$, $\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}$ and during transformations, (5.134) and (5.135) were used. Then the term

$$\dot{\mathbf{A}}\dot{\boldsymbol{\rho}} = \dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{A}\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \mathbf{A}\dot{\boldsymbol{\rho}}, \tag{5.139}$$

where during transformations we used relation (5.135).

Taking into account (5.139), (5.138) takes the form

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_A^r + \mathbf{A}\ddot{\boldsymbol{\rho}} \\ &= \mathbf{a}_A^t + \mathbf{a}_A^r + \mathbf{a}_A^C, \end{aligned} \tag{5.140}$$

where $\mathbf{a}_A^t = \mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, $\mathbf{a}_A^C = 2\boldsymbol{\omega} \times \mathbf{v}_A^r$, $\mathbf{a}_A^r = \mathbf{A}\ddot{\boldsymbol{\rho}}$.

The acceleration of transportation of point A is the acceleration of a point A'' rigidly connected to the system $O''X_1''X_2''X_3''$ with respect to the system $OX_1X_2X_3$, which at the given instant coincides with point A moving with respect to the system $\mathrm{O}''X_1''X_2''X_3''$.

The relative acceleration of the analyzed point $\mathbf{a}_A^r$ is the acceleration of point A measured with respect to the non-stationary system $O''X_1''X_2''X_3''$.

The acceleration $\mathbf{a}_A^C$ is called the *Coriolis acceleration.*[1] The emergence of this acceleration results from two aspects, which can be traced during the derivation of (5.140).

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[1] Gaspard–Gustave de Coriolis (1792–1843), French physicist and mathematician working at École Polytechnique in Paris.

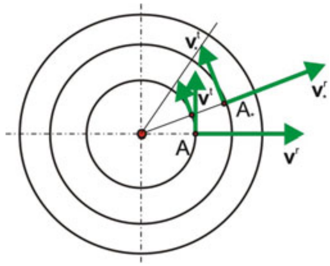


Fig. 5.49 Illustration of causes of emergence of Coriolis acceleration

The vector $\mathbf{v}_A^r$ rotates with respect to the system $OX_1X_2X_3$ as a result of the rotation of the system $O''X_1''X_2''X_3''$ with velocity $\boldsymbol{\omega}$ and yields the acceleration $\frac{1}{2}\mathbf{a}_A^C$. Moreover, the position of point A in the system $O''X_1''X_2''X_3''$ undergoes change, which leads to a change in the transportation velocity. This leads to the emergence of the acceleration $\frac{1}{2}\mathbf{a}_A^C$.

In the case of translational motion of transportation we have $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\varepsilon} = \mathbf{0}$, and then the Coriolis acceleration

$$\mathbf{a}_A^C = 2(\boldsymbol{\omega} \times \mathbf{v}_A^r) = \mathbf{0}. \tag{5.141}$$

The component of the vector of absolute acceleration of a point in composite motion, which is equal to a doubled vector product of angular velocity of transportation $\boldsymbol{\omega}$ and the relative velocity of the point $\mathbf{v}_A^r$, is called the Coriolis acceleration (return acceleration).

We will present certain characteristics of the Coriolis acceleration using the example of a circular platform (disk) on which a person moves along a radius (Fig. 5.49).

If the person (point A) moves uniformly along the radius of a uniformly rotating circular platform, then her relative velocity is the velocity v^r, and her velocity of transportation v^t is the velocity of the point of the platform where the person is at the moment. At time instant t let the person be at point A, and at time instant $t + \Delta t$ let her be at point A_*. Because the relative motion is uniform and rectilinear, we have $\mathbf{a}^r = 0$. However, in the time interval Δt, the velocity $\mathbf{v}^r$ changes direction to $\mathbf{v}_*^r$ as a result of platform rotation (motion of transportation).

In this time interval Δt a change in the magnitude of velocity of transportation also occurs from a value of $v^t = \omega \cdot OA$ to $v_*^t = \omega \cdot OA_*$ because during time Δt the person moved from point A to A_*. Those two changes in velocities $\mathbf{v}^r$ and $\mathbf{v}^t$ during time Δt caused the emergence of the Coriolis acceleration of magnitude

$$a^C = 2\omega v^r \sin(\boldsymbol{\omega}, \mathbf{v}^r). \tag{5.142}$$

From the given example it follows that the Coriolis acceleration is associated with the following changes:

1. The magnitude and direction of the velocity of transportation of the point as a result of its relative motion.
2. The direction of the relative velocity of the point as a result of the rotational motion of transportation.

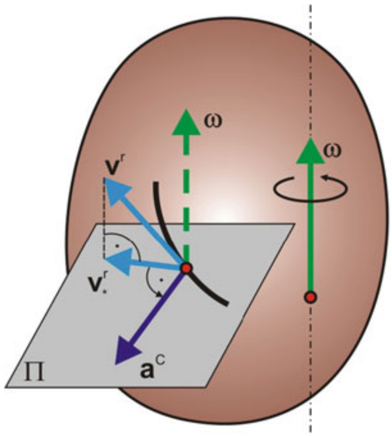


Fig. 5.50 Determination of direction and sense of $\mathbf{a}^C$

According to (5.142) the Coriolis acceleration is equal to zero in three cases:

1. $\boldsymbol{\omega} = \mathbf{0}$, which takes place in the case of translational motion of transportation or at time instants where $\omega = 0$ during non-translational motion.
2. $\mathbf{v}^r = \mathbf{0}$, which occurs in the case of a relative state of rest of the point or at time instants where $v^r = 0$ during the relative motion of the point.
3. $\sin(\boldsymbol{\omega}, \mathbf{v}^r) = 0$, which takes place when the angle between vectors $\boldsymbol{\omega}$ and $\mathbf{v}^r$ is equal to 0 or π.

We will show how to determine in practice the direction and sense of the Coriolis acceleration (Fig. 5.50).

In order to determine the direction and sense of the Coriolis acceleration one should project the relative velocity of point $\mathbf{v}^r$ onto the plane $\Pi \perp \boldsymbol{\omega}$ and then rotate vector $\mathbf{v}^r_*$ through 90° according to the sense of $\boldsymbol{\omega}$. If $\mathbf{v}^r \perp \boldsymbol{\omega}$, then $p^C = 2\omega v^r$.

Example 5.3. A rectangular plate rotates with a constant velocity $\omega =$ const about axis OX_3 (Fig. 5.51). Point A moves on the plate along a circle (Fig. 5.51a) and ellipse (Fig. 5.51b).

Determine the absolute acceleration of point A when:

(a) The axis of plate rotation OX_3 lies in its plane, the center of the circle $O' = O'(x_{2O'}, x_{3O'})$, and the point motion along the arc of the circle is governed by the equation $s = A_O A = \frac{1}{6}\pi\rho_A t^3$, where $A_O = A(t_O)$ (Fig. 5.51a);
(b) The axis of plate rotation OX_3 is perpendicular to its plane, and the motion of point A on the plate is described by the equations $x_1'' = a_1 \cos\alpha t$; $x_2'' = a_2 \sin\alpha t$ (Fig. 5.51b).

In the considered case (Fig. 5.51a) the relative velocity $v^r_A = \dot{\rho}^r_A = \frac{1}{2}\pi\rho_A t^2$. The relative tangent acceleration of point A is $\mathrm{a}^r_{A\tau} = \pi\rho_A t$, whereas the relative normal acceleration has a value of $\mathrm{a}^r_{An} = \frac{(v^r_A)^2}{\rho_A} = \frac{1}{4}\pi^2\rho_A t^4$. The transportation

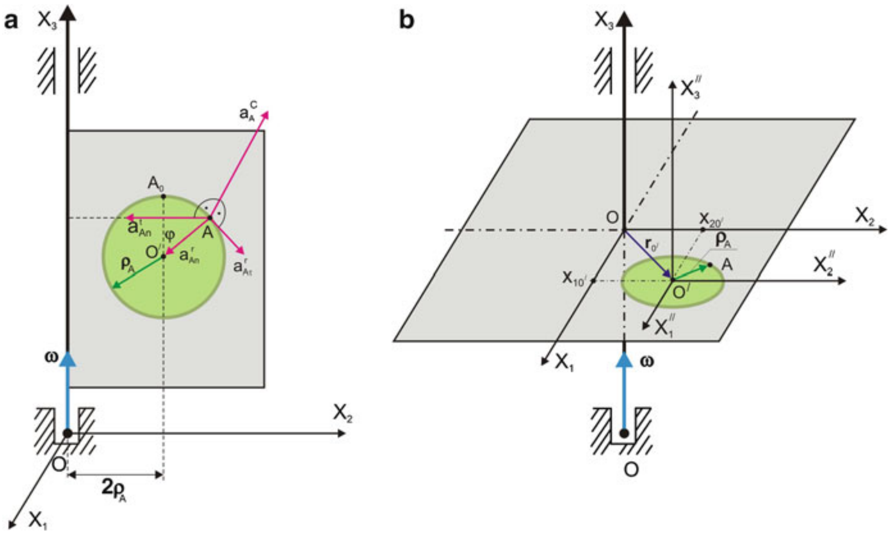


Fig. 5.51 Composite motion of a point A in three-dimensional space

acceleration of point A is $\mathbf{a}^t_A = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)$, i.e., $a^t_A = a^t_{An} = \omega^2 \rho_A (2 + \sin\varphi)$, where $\varphi = \frac{s}{\rho_A} = \frac{1}{6}\pi t^3, \varphi = \sphericalangle (A_O O' A)$. The Coriolis acceleration of point A is $\mathbf{a}^c_A = 2\boldsymbol{\omega} \times \mathbf{v}^r_A$, and hence

$$a^C_A = 2\omega v^r_A \sin\left(\frac{\pi}{2} + \varphi\right)$$
$$= 2\omega v^r_A \cos\varphi = 2\omega v^r_A \cos\left(\frac{\pi}{6} t^3\right).$$

The projections of acceleration vector $\mathbf{a}_A$ onto the axes of the coordinate system are equal to

$$a_{Ax_1} = -a^C_A = -\pi\omega\rho_A t^2 \cos\left(\frac{\pi}{6} t^3\right),$$
$$a_{Ax_2} = -a^t_A - a^r_{An} \sin\varphi + a^r_{A\tau} \cos\varphi = -\omega^2 \rho_A \left[2 + \sin\left(\frac{\pi}{6} t^3\right)\right]$$
$$- \frac{1}{4}\pi^2 \rho_A t^4 \sin\left(\frac{\pi}{6} t^3\right) + \pi\rho_A t \cos\left(\frac{\pi}{6} t^3\right),$$
$$a_{Ax_3} = -a^r_{A\tau} \sin\varphi - a^r_{An} \cos\varphi = -\pi\rho_A t \sin\left(\frac{\pi}{6} t^3\right) - \frac{1}{4}\pi^2 \rho_A t^4 \cos\left(\frac{\pi}{6} t^3\right),$$

and acceleration vector $\mathbf{a}_A$ is described by the equation

$$\mathbf{a}_A = a_{Ax_1}\mathbf{E}_1 + a_{Ax_2}\mathbf{E}_2 + a_{Ax_3}\mathbf{E}_3.$$

In the case of Fig. 5.51b, we have

$$\mathbf{r}_{O'} = \mathbf{E}_1 x_{1O'} + \mathbf{E}_2 x_{2O'},$$

where $x_{1O'} = \text{const}$ and $x_{2O'} = \text{const}$. The position of point A is governed by the following equation:

$$\mathbf{r}_A - \mathbf{r}_{O'} = \mathbf{E}_1 a_1 \cos\alpha t + \mathbf{E}_2 a_2 \sin\alpha t,$$

and because $\boldsymbol{\omega} = \text{const}$, we have $\boldsymbol{\varepsilon} = \mathbf{0}$.

The successive terms of (5.140) have the following form:

$$\mathbf{a}_{O'} = \mathbf{a}^t_{O'} = -\omega^2(x_{1O'}\mathbf{E}_1 + x_{2O'}\mathbf{E}_2), \quad \boldsymbol{\varepsilon} \times \boldsymbol{\rho}_A = \mathbf{0}, \quad \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) = -\omega^2 \boldsymbol{\rho}_A,$$

that is,

$$\begin{aligned}\mathbf{a}^t_A &= -\omega^2(x_{1O'}\mathbf{E}_1 + x_{2O'}\mathbf{E}_2) - \omega^2(\mathbf{E}_1 a_1 \cos\alpha t + \mathbf{E}_2 a_2 \sin\alpha t)\\ &= -\omega^2(a_1 \cos\alpha t + x_{1O'})\mathbf{E}_1 - \omega^2(a_2 \sin\alpha t + x_{2O'})\mathbf{E}_2.\end{aligned}$$

The relative acceleration of point A is equal to

$$\mathbf{a}^r_A = -a_1\alpha^2\mathbf{E}_1 \cos\alpha t - a_2\alpha^2\mathbf{E}_2 \sin\alpha t,$$

and its Coriolis acceleration reads

$$\begin{aligned}\mathbf{a}^C_A &= 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}^r_A = 2\begin{vmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\ 0 & 0 & \omega\\ \dot{x}''_1 & \dot{x}''_2 & 0\end{vmatrix} = -2\mathbf{E}_1\omega\dot{x}''_2 + 2\mathbf{E}_2\omega\dot{x}''_1\\ &= -2a_2\omega\alpha\mathbf{E}_1 \cos\alpha t - 2a_1\omega\alpha\mathbf{E}_2 \sin\alpha t.\end{aligned}$$

Finally, the absolute acceleration of point A is given by

$$\begin{aligned}\mathbf{a}_A = {}&[-(\omega^2 + \alpha^2)a_1 \cos\alpha t + 2a_1\omega\alpha \sin\alpha t]\mathbf{E}_1\\ &+ [-(\alpha^2 + \omega^2)a_2 \sin\alpha t - x_{20'}\omega^2 + 2a_2\omega\alpha \cos\alpha t]\mathbf{E}_2.\end{aligned}$$

□

5.4 Composite Planar Motion of a Point

This problem is a special case of the composite motion in three-dimensional space, but since it is very important in a discussion of the kinematics of plane mechanisms, we will devote some attention to it. Moreover, we apply here a method of analysis different than that described in Sect. 5.3.

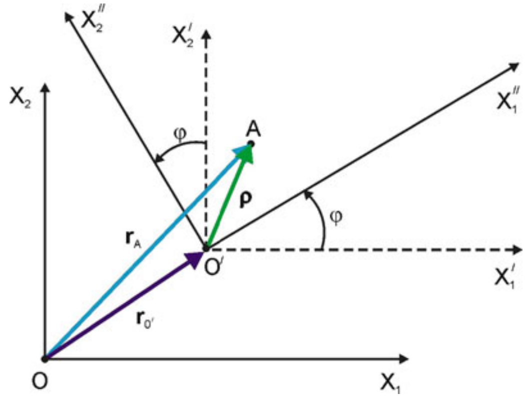


Fig. 5.52 Composite planar motion of point A

The composite planar motion of point A is depicted in Fig. 5.52, where now, as distinct from the analysis of the case presented in Figs. 5.18 (velocity) and 5.26 (acceleration), point A moves with respect to the non-stationary coordinate system $O''X_1''X_2''$.

In this case, the plane OX_1X_2 is stationary (unmovable). The motion of point A with respect to the plane OX_1X_2 is called the *composite planar motion*, and the motion of the point with respect to the plane $O''X_1''X_2''$ is called the *relative plane motion of the point*, whereas the motion of the plane $O''X_1''X_2''$ with respect to the plane OX_1X_2 is called the transportation planar motion.

From Fig. 5.52 it follows that

$$\mathbf{r}_A = \mathbf{r}_{O'} + \boldsymbol{\rho}, \tag{5.143}$$

and therefore

$$\mathbf{v}_A \equiv \dot{\mathbf{r}}_A = \mathbf{v}_{O'} + \dot{\boldsymbol{\rho}}, \tag{5.144}$$

where $\mathbf{v}_{O'} = \dot{\mathbf{r}}_{O'}$, and hence

$$\dot{\boldsymbol{\rho}} = \mathbf{E}_1''\dot{x}_{1A}'' + \mathbf{E}_2''\dot{x}_{2A}'' + \dot{\mathbf{E}}_1''x_{1A}'' + \dot{\mathbf{E}}_2''x_{2A}''. \tag{5.145}$$

Taking into account (5.145), (5.144) becomes

$$\mathbf{v}_A = \mathbf{v}_A^t + \mathbf{v}_A^r, \tag{5.146}$$

where $\mathbf{v}_A^t = \mathbf{v}_{O'} + \dot{\boldsymbol{\varphi}} \times \boldsymbol{\rho} = \mathbf{v}_{O'} + \dot{\boldsymbol{\varphi}} \times \left(x_{1A}''\mathbf{E}_1'' + x_{2A}''\mathbf{E}_2''\right)$, $\mathbf{v}_A^r = \dot{x}_{1A}''\mathbf{E}_1'' + \dot{x}_{2A}''\mathbf{E}_2''$, and in the preceding discussion we used the relation $\dot{\boldsymbol{E}}_i'' = \dot{\boldsymbol{\varphi}} \times \boldsymbol{E}_i''$.

Equation (5.146) in comparison to (5.64) contains an additional term $\mathbf{v}_A^r$, which is the relative velocity of point A in the non-stationary coordinate system $O''X_1''X_2''$.

Differentiating (5.146) with respect to time we have

$$\mathbf{a}_A \equiv \dot{\mathbf{v}}_A = \dot{\mathbf{v}}_A^t + \dot{\mathbf{v}}_A^r, \tag{5.147}$$

where

$$\begin{aligned}\mathbf{a}_A &= \mathbf{a}_{O'} + \mathbf{E}_1''\ddot{x}_{1A}'' + \mathbf{E}_2''\ddot{x}_{2A}'' + \dot{\mathbf{E}}_1''\dot{x}_{1A}'' + \dot{\mathbf{E}}_2''\dot{x}_{2A}''\\ &\quad + \ddot{\boldsymbol{\varphi}} \times (x_{1A}''\mathbf{E}_1'' + x_{2A}''\mathbf{E}_2'') + \dot{\boldsymbol{\varphi}} \times \left[(\dot{x}_{1A}''\mathbf{E}_1'' + \dot{x}_{2A}''\mathbf{E}_2'') + (x_{1A}''\dot{\mathbf{E}}_1'' + x_{2A}''\dot{\mathbf{E}}_2'')\right]\\ &= \mathbf{a}_{O'} + \mathbf{E}_1''\ddot{x}_{1A}'' + \mathbf{E}_2''\ddot{x}_{2A}'' + 2\dot{\boldsymbol{\varphi}} \times (\dot{x}_{1A}''\mathbf{E}_1'' + \dot{x}_{2A}''\mathbf{E}_2'')\\ &\quad + \boldsymbol{\varepsilon} \times (x_{1A}''\mathbf{E}_1'' + x_{2A}''\mathbf{E}_2'') + \dot{\boldsymbol{\varphi}} \times \left[\dot{\boldsymbol{\varphi}} \times (x_{1A}''\mathbf{E}_1'' + x_{2A}''\mathbf{E}_2'')\right]\\ &= \mathbf{a}_A^t + \mathbf{a}_A^r + \mathbf{a}_A^C, \end{aligned} \tag{5.148}$$

where $\mathbf{a}_A^t = \mathbf{a}_{O'} + \boldsymbol{\varepsilon} \times \boldsymbol{\rho} + \dot{\boldsymbol{\varphi}} \times \left(\dot{\boldsymbol{\varphi}} \times \boldsymbol{\rho}\right)$, $\mathbf{a}_A^r = \mathbf{E}_1''\ddot{x}_{1A}'' + \mathbf{E}_2''\ddot{x}_{2A}''$, $\mathbf{a}_A^C = 2\dot{\boldsymbol{\varphi}} \times \mathbf{v}_A^r$.

Equation (5.148) indicates that the acceleration in the system OX_1X_2 (i.e., the absolute acceleration) of point A is the geometric sum of the acceleration of point O' (the pole) in that system and the acceleration of that point in the system $O''X_1''X_2''$.

The particular terms of (5.148) are equal to

$$\begin{aligned}\mathbf{a}_{O'} &= \ddot{x}_{1O'}\mathbf{E}_1 + \ddot{x}_{20'}\mathbf{E}_2,\\ \boldsymbol{\varepsilon} \times \boldsymbol{\rho}_A &= \begin{vmatrix}\mathbf{E}_1'' & \mathbf{E}_2'' & \mathbf{E}_3''\\ 0 & 0 & \varepsilon\\ x_{1A}'' & x_{2A}'' & 0\end{vmatrix} = -\mathbf{E}_1''\varepsilon x_{2A}'' + \mathbf{E}_2''\varepsilon x_{1A}'',\\ \dot{\boldsymbol{\varphi}} \times (\dot{\boldsymbol{\varphi}} \times \boldsymbol{\rho}_A) &= (\dot{\boldsymbol{\varphi}} \circ \boldsymbol{\rho})\dot{\boldsymbol{\varphi}} - \dot{\varphi}^2\boldsymbol{\rho} = -\dot{\varphi}^2\boldsymbol{\rho}\\ &= -\dot{\varphi}^2\left(\mathbf{E}_1''x_{1A}'' + \mathbf{E}_2''x_{2A}''\right),\\ \dot{\boldsymbol{\varphi}} \times \mathbf{v}_A^r &= \begin{vmatrix}\mathbf{E}_1'' & \mathbf{E}_2'' & \mathbf{E}_3''\\ 0 & 0 & \dot{\varphi}\\ \dot{x}_{1A}'' & \dot{x}_{2A}'' & 0\end{vmatrix} = -\mathbf{E}_1''\dot{\varphi}\dot{x}_{2A}'' + \mathbf{E}_2''\dot{\varphi}\dot{x}_{1A}'', \end{aligned} \tag{5.149}$$

and in view of that the acceleration of point A is determined by the following two scalar equations:

$$\begin{aligned}\ddot{x}_{iA} &= \ddot{x}_{iO'} - \varepsilon x_{2A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right) + \varepsilon x_{1A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right) - \dot{\varphi}^2 x_{1A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right)\\ &\quad - \dot{\varphi}^2 x_{2A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right) + \ddot{x}_{1A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right) + \ddot{x}_{2A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right)\\ &\quad - 2\dot{\varphi}\dot{x}_{2A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right) + 2\dot{\varphi}\dot{x}_{2A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right),\ i = 1, 2, \end{aligned} \tag{5.150}$$

where

$$\begin{aligned}\mathbf{E}_1'' \circ \mathbf{E}_1 &= \cos\varphi,\\ \mathbf{E}_2'' \circ \mathbf{E}_1 &= \cos\left(\frac{\pi}{2} + \varphi\right) = -\sin\varphi,\\ \mathbf{E}_2'' \circ \mathbf{E}_2 &= \cos\varphi,\\ \mathbf{E}_1'' \circ \mathbf{E}_2 &= \cos\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi. \end{aligned} \tag{5.151}$$

The absolute velocity of point A is described by (5.146), whose terms are equal to

$$
\begin{aligned}
\mathbf{v}_{O'} &= \dot{x}_{1O'}\mathbf{E}_1 + \dot{x}_{2O'}\mathbf{E}_2,\\
\dot{\boldsymbol{\varphi}} \times \boldsymbol{\rho} &= \begin{bmatrix} \mathbf{E}_1'' & \mathbf{E}_2'' & \mathbf{E}_3'' \\ 0 & 0 & \dot{\varphi} \\ x_{1A}'' & x_{2A}'' & 0 \end{bmatrix} = -\mathbf{E}_1''\dot{\varphi}x_{2A}'' + \mathbf{E}_2''\dot{\varphi}x_{1A}'',\\
\mathbf{v}_A^r &= \dot{x}_{1A}''\mathbf{E}_1'' + \dot{x}_{2A}''\mathbf{E}_2'',
\end{aligned}
\tag{5.152}
$$

and in view of that the components of the absolute velocity of point A are described by the following two equations:

$$
\begin{aligned}
\dot{x}_{iA} &= \dot{x}_{iO'} - \dot{\varphi}x_{2A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right) + \dot{\varphi}x_{1A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right)\\
&\quad + \dot{x}_{1A}''\left(\mathbf{E}_1'' \circ \mathbf{E}_i\right) + \dot{x}_{2A}''\left(\mathbf{E}_2'' \circ \mathbf{E}_i\right), \quad i = 1, 2.
\end{aligned}
\tag{5.153}
$$

5.5 Motion in a Three-Dimensional Space

5.5.1 Introduction

In this section we will take up the kinematics of a rigid body. According to the definition introduced earlier, a body in mechanics is a collection of particles, which is why we take up the kinematics of a point (points) first.

Figure 5.53 presents the position of body C at the time instants t_0 $\left(C_{t_0}\right)$ and t (C_t) and are marked as three points of the body A, A_1, A_2. The minimum number of points required to determine the position of a rigid body is four (they must not lie in one plane). If the positions of the three points A, A_1, A_2 of the body and the additional pole O are known, then the position of any other point can be easily determined. The configuration of the considered body at time instant t_0 can be described by the function $\mathbf{r}_0 = \mathbf{f}_{t_0}(A)$, whereas at time instant t that function has the form $\mathbf{r} = \mathbf{f}_t(A)$ (because we take point A arbitrarily, we will additionally drop the subscript A, i.e., $\mathbf{r}_A = \mathbf{r}$). The position of arbitrarily chosen point A can undergo a change. The functions describe the mappings of particles A of body C_t at time instant t in three-dimensional Euclidean space.

Using the stationary Cartesian coordinate system introduced in Fig. 5.53 (position vectors $\mathbf{r}_0$) we can trace the position of the body at arbitrary time instants t (position vectors $\mathbf{r}$). In other words, we will describe the motion of the body by a certain vector function

$$
\mathbf{r}(t) = \mathbf{F}(\mathbf{r}_0, t). \tag{5.154}
$$

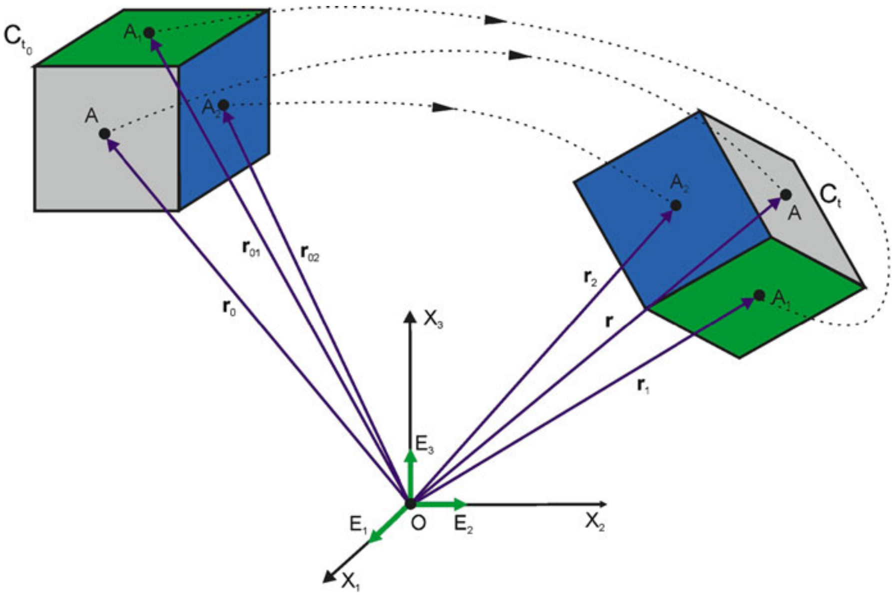


Fig. 5.53 Position of a body at time instants t_0 and t

The preceding notation means that the motion of an arbitrary point A of body C depends on its position and on time. In the case of a rigid body the distances between its points do not change, i.e., if we take any two points of numbers 1 and 2, we have

$$\|\mathbf{r}_1 - \mathbf{r}_2\| = \|\mathbf{r}_{01} - \mathbf{r}_{02}\|. \tag{5.155}$$

Moreover, the motion of an arbitrary point of a rigid body is described by the equation (to be demonstrated later)

$$\mathbf{r}(t) = \mathbf{A}(t)\mathbf{r}_0 + \mathbf{a}(t), \tag{5.156}$$

where $\mathbf{A}(t)$ is the rotation tensor (matrix) and $\mathbf{a}(t)$ is a vector.

Taking into account two particles A_1 and A_2 of a rigid body, from (5.156) we obtain

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{A}(\mathbf{r}_{01} - \mathbf{r}_{02}). \tag{5.157}$$

From the preceding equation it follows that the vector connecting two arbitrary points of a rigid body can either rotate ($\mathbf{A} \neq \mathbf{E}$) or not rotate ($\mathbf{A} = \mathbf{E}$, where $\mathbf{E}$ is the identity matrix). Thus, in order to check if the rigid body rotates during motion, one should investigate the behavior of the vector $\mathbf{r}_1 - \mathbf{r}_2$ in time. Let us first consider the simplest case, $\mathbf{A} = \mathbf{E}$. From (5.157) it follows that

$$\mathbf{r}_1 - \mathbf{r}_2 \equiv \mathbf{r}_{01} - \mathbf{r}_{02} \equiv \text{const}. \tag{5.158}$$

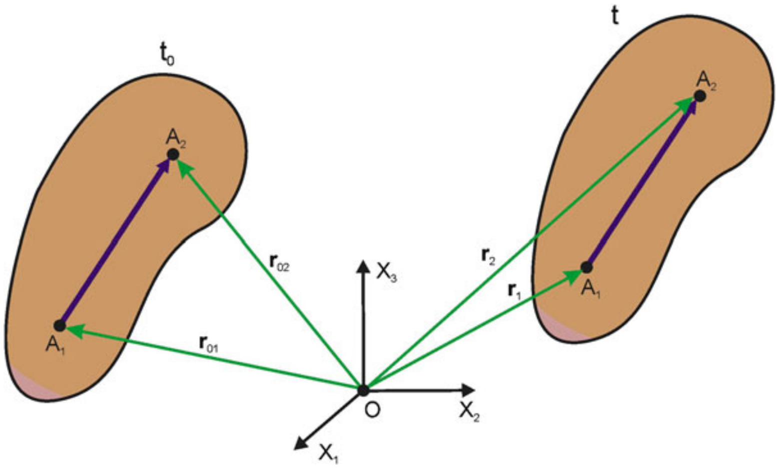


Fig. 5.54 Translational motion of a rigid body

This means that the vector $\mathbf{r}_1 - \mathbf{r}_2$ connecting points A_1 and A_2 of the rigid body is a constant vector, i.e., this vector moves in three-dimensional space in time but remains parallel to its initial position at time instant t_0 (Fig. 5.54). We will call such motion the *translational motion of a rigid body*.

The investigation of the motion of a rigid body in such a case boils down to the investigation of motion of its arbitrary point. If we take point A, then we have

$$\mathbf{v} \equiv \mathbf{v}_A = \frac{\mathrm{d}\mathbf{r}_A}{\mathrm{d}t}, \tag{5.159}$$

$$\mathbf{a} \equiv \mathbf{a}_A = \frac{\mathrm{d}^2\mathbf{r}_A}{\mathrm{d}t^2}. \tag{5.160}$$

Translational motion can be either rectilinear or curvilinear motion. From (5.159) and (5.160) it follows that in translational motion the velocities and accelerations of every point of a rigid body are the same. Recall that because the motion of a particle can be divided into uniform (constant velocity), uniformly variable (constant tangential acceleration), and non-uniformly variable (changing tangential acceleration), one can similarly divide the translational motion of a rigid body.

Let us consider now a second case associated with (5.157), i.e., $\mathbf{A} \neq \mathbf{E}$.

Recall that the *rotation tensor* $\mathbf{A}$ is an orthogonal second-order tensor possessing the following properties:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{E}, \tag{5.161}$$

$$\det(\mathbf{A}) = 1. \tag{5.162}$$

From (5.161) it follows that there are six constraints imposed on nine elements of tensor $\mathbf{A}$. Therefore, only three parameters are independent and fully describe rotation tensor **A**. Although in what follows we will mainly use matrix and vector calculus, here some basic information connected with tensor calculus will be introduced, which will enable readers to use it for their own purposes if necessary.

The property of the second-order rotation tensor ideally serves the purpose of interpreting the motion of a rigid body with one point fixed. That motion can be described by the choice of three parameters, most often the so-called *Euler angles*.

5.5.2 Angular Velocity and Angular Acceleration of a Rigid Body

If rotation tensor **A** is known, it is possible to define the *tensor of angular velocity* $\boldsymbol{\Omega}$ and the *vector of angular velocity* $\boldsymbol{\omega}$ of a rigid body in the following way [5,6,8,10]:

$$\boldsymbol{\Omega} = \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}}, \tag{5.163}$$

$$\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\varepsilon}[\boldsymbol{\Omega}], \tag{5.164}$$

where $\boldsymbol{\varepsilon}$ is the tensor of the third order called *the alternating tensor* [8].

During application of (5.164) some properties of the alternating tensor were used. Recall that an arbitrary tensor of the third order transforms the vector into second-order tensors and can transform the tensors of the second order into the vectors

$$(\boldsymbol{a} \otimes \mathbf{b} \otimes \mathbf{c})[\mathbf{d} \otimes \mathbf{e}] = \boldsymbol{a}(\mathbf{b} \circ \mathbf{d})(\mathbf{c} \circ \mathbf{e}), \tag{5.165}$$

$$(\boldsymbol{a} \otimes \mathbf{b} \otimes \mathbf{c})\mathbf{d} = \boldsymbol{a} \otimes \mathbf{b}(\mathbf{c} \circ \mathbf{d}). \tag{5.166}$$

If we take the basis $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, then the tensor of the third order has the form

$$\mathbf{Q} = Q_{ijk}\mathbf{t}_i \otimes \mathbf{t}_j \otimes \mathbf{t}_k. \tag{5.167}$$

The alternating tensor has the form

$$\begin{aligned} \boldsymbol{\varepsilon} &= \varepsilon_{ijk}\mathbf{t}_i \otimes \mathbf{t}_j \otimes \mathbf{t}_k = \mathbf{t}_1 \otimes \mathbf{t}_2 \otimes \mathbf{t}_3 + \mathbf{t}_3 \otimes \mathbf{t}_1 \otimes \mathbf{t}_2 \\ &\quad + \mathbf{t}_2 \otimes \mathbf{t}_3 \otimes \mathbf{t}_1 - \mathbf{t}_2 \otimes \mathbf{t}_1 \otimes \mathbf{t}_3 - \mathbf{t}_1 \otimes \mathbf{t}_3 \otimes \mathbf{t}_2 - \mathbf{t}_3 \otimes \mathbf{t}_2 \otimes \mathbf{t}_1 \end{aligned} \tag{5.168}$$

because $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$ and $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$.

If **B** is a symmetric tensor, then $\boldsymbol{\varepsilon}\left[\mathbf{B}\right] = 0$.

If **b** is a vector, then

$$\begin{aligned} \boldsymbol{\varepsilon}\mathbf{b} &= \varepsilon_{ijk}\mathbf{t}_i \otimes \mathbf{t}_j \mathbf{b}_k = \mathbf{b}_3(\mathbf{t}_1 \otimes \mathbf{t}_2 - \mathbf{t}_2 \otimes \mathbf{t}_1) \\ &\quad + \mathbf{b}_2(\mathbf{t}_3 \otimes \mathbf{t}_1 - \mathbf{t}_1 \otimes \mathbf{t}_3) + \mathbf{b}_1(\mathbf{t}_2 \otimes \mathbf{t}_3 - \mathbf{t}_3 \otimes \mathbf{t}_2) = -\mathbf{B}\,. \end{aligned} \tag{5.169}$$

As a result of the preceding operation we obtain the skew-symmetric tensor $\mathbf{B}$. Property (5.169) was used in (5.164). What is more, for an arbitrary vector $\boldsymbol{a}$ we have

$$\mathbf{B}\boldsymbol{a} = (-\boldsymbol{\varepsilon}\mathbf{b})\boldsymbol{a} = \mathbf{b} \times \boldsymbol{a}, \tag{5.170}$$

that is,

$$\boldsymbol{\Omega}\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{a}. \tag{5.171}$$

The angular acceleration $\dot{\boldsymbol{\omega}}$ of the rigid body is obtained from (5.163) and (5.164) as

$$\dot{\boldsymbol{\omega}} = -\frac{1}{2}\boldsymbol{\varepsilon}\left[\ddot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} + \dot{\mathbf{A}}\dot{\mathbf{A}}^{\mathrm{T}}\right] = \frac{1}{2}\boldsymbol{\varepsilon}\left[\dot{\boldsymbol{\Omega}}\right]. \tag{5.172}$$

The relative velocities and relative accelerations of two arbitrary points of a rigid body described by position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are obtained by differentiation of (5.157) with respect to time.

The first differentiation leads to the result

$$\begin{aligned}\mathbf{v}_1 - \mathbf{v}_2 &= \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = \dot{\mathbf{A}}(\mathbf{r}_{01} - \mathbf{r}_{02}) = \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{r}_{01} - \mathbf{r}_{02}) \\ &= \boldsymbol{\Omega}\mathbf{A}(\mathbf{r}_{01} - \mathbf{r}_{02}) = \boldsymbol{\Omega}(\mathbf{r}_1 - \mathbf{r}_2) = \boldsymbol{\omega} \times (\mathbf{r}_1 - \mathbf{r}_2),\end{aligned} \tag{5.173}$$

where relationships (5.157), (5.161), and (5.171) were used in the transformations. The next differentiation leads to the determination of relative accelerations of the form

$$\begin{aligned}\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2 &= \dot{\boldsymbol{\omega}} \times (\mathbf{r}_1 - \mathbf{r}_2) + \boldsymbol{\omega} \times (\mathbf{v}_1 - \mathbf{v}_2) \\ &= \dot{\boldsymbol{\omega}} \times (\mathbf{r}_1 - \mathbf{r}_2) + \boldsymbol{\omega} \times \boldsymbol{\omega} \times (\mathbf{r}_1 - \mathbf{r}_2).\end{aligned} \tag{5.174}$$

Equations (5.173) and (5.174) describe the dependency of relative velocities and relative accelerations on t, $\mathbf{r}_1$ and $\mathbf{r}_2$.

5.5.3 *Euler's Proposal*

We aim at the parameterization of the angular velocity tensor $\boldsymbol{\Omega}$ and the angular velocity vector $\boldsymbol{\omega}$ so as to obtain their possibly simplest representation. Euler demonstrated that an arbitrary rotation tensor $\mathbf{A}$ could be represented in the following way:

$$\mathbf{A} = \cos\varphi(\mathbf{E} - \mathbf{r} \otimes \mathbf{r}) - \sin\varphi(\boldsymbol{\varepsilon}\mathbf{r}) + \mathbf{r} \otimes \mathbf{r}. \tag{5.175}$$

The preceding vector $\mathbf{r}$ is a unit vector (lying on the axis of rotation) composed of two independent elements, and φ is the directed positive angle of rotation.

Recall that a second order tensor is called a rotation tensor if it satisfies conditions (5.161) and (5.162). It belongs to the class of proper orthogonal tensors, which form a subclass of the orthogonal tensors. Moreover, the determinant of an arbitrary second-order tensor $\mathbf{B}$ has the following form:

$$\det(\mathbf{B}) = [\mathbf{Bt}_1, \mathbf{Bt}_2, \mathbf{Bt}_3] = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} B_{i1}B_{j2}B_{k3}[\mathbf{t}_i, \mathbf{t}_j, \mathbf{t}_k]$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} B_{i1}B_{j2}B_{k3}, \tag{5.176}$$

where $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ is the orthonormal right-handed basis.

Taking into account the obtained value of determinant $\det(\mathbf{B})$ one may notice that we obtain the same result calculating the determinant of the following matrix:

$$\det(\mathbf{B}) = \det \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}. \tag{5.177}$$

Our task is to show that the tensor (5.175) is the proper orthogonal tensor.

To this end we will make use of the orthonormal right-handed basis $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ (introduced earlier), where $\mathbf{t}_3 = \mathbf{r}$.

Using the notions of the tensor calculus introduced earlier, the terms of (5.175) have the form

$$\mathbf{E} = \mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2 + \mathbf{t}_3 \otimes \mathbf{t}_3,$$
$$\mathbf{E} - \mathbf{r} \otimes \mathbf{r} = \mathbf{t}_1 \otimes \mathbf{t}_2 + \mathbf{t}_2 \otimes \mathbf{t}_2,$$
$$-\boldsymbol{\varepsilon}\mathbf{r} = \mathbf{t}_2 \otimes \mathbf{t}_1 - \mathbf{t}_1 \otimes \mathbf{t}_2. \tag{5.178}$$

Taking into account (5.178) in (5.175) we obtain

$$\mathbf{A} = \cos\varphi(\mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2)$$
$$+ \sin\varphi(\mathbf{t}_2 \otimes \mathbf{t}_1 - \mathbf{t}_1 \otimes \mathbf{t}_2) + \mathbf{t}_3 \otimes \mathbf{t}_3. \tag{5.179}$$

Then

$$\mathbf{A}^{\mathrm{T}} = \cos\varphi(\mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2) + \sin\varphi(\mathbf{t}_1 \otimes \mathbf{t}_2 - \mathbf{t}_2 \otimes \mathbf{t}_1) \tag{5.180}$$

because $\left(\mathbf{a} \otimes \mathbf{b}\right)^{\mathrm{T}} = \left(\mathbf{b} \otimes \mathbf{a}\right) = -\left(\mathbf{a} \otimes \mathbf{b}\right)$. It is easy to check now that $\mathbf{AA}^{\mathrm{T}} = \mathbf{E}$.

Let us note that

$$\mathbf{A} = \sum_{i=1}^{3}\sum_{k=1}^{3} A_{ik}\mathbf{t}_i \otimes \mathbf{t}_k, \tag{5.181}$$

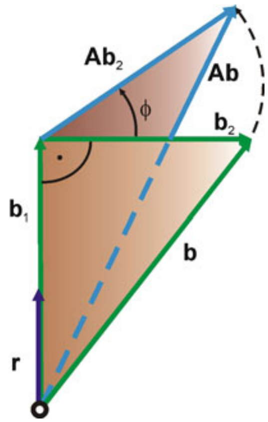


Fig. 5.55 Action of Euler's rotation tensor **A** on vector **b** ($\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b}; \mathbf{b}_1 \parallel \mathbf{r}; \mathbf{b}_2 \perp \mathbf{r}$)

where

$$A_{ik} = (\mathbf{A}\mathbf{t}_k) \circ \mathbf{t}_i. \tag{5.182}$$

The determinant of the tensor **A** is equal to

$$\det(\mathbf{A}) = \det \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1, \tag{5.183}$$

and consequently it was demonstrated that **A** is a rotation tensor.

Let us now consider the action of Euler's rotation tensor (5.175) on an arbitrary vector **b** (Fig. 5.55).

As follows from Fig. 5.55, the component $\mathbf{b}_2$ of vector **b** ($\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$) undergoes a change, whereas $\mathbf{b}_1 \parallel \mathbf{r}$ remains unchanged.

Let us note that

$$\begin{aligned} \mathbf{b}_1 &= (\mathbf{b} \circ \mathbf{r})\mathbf{r} = (\mathbf{r} \otimes \mathbf{r})\mathbf{b}, \\ \mathbf{b}_2 &= \mathbf{b} - \mathbf{b}_1 = \mathbf{b} - (\mathbf{b} \circ \mathbf{r})\mathbf{r} = (\mathbf{E} - \mathbf{r} \otimes \mathbf{r})\mathbf{b}, \end{aligned} \tag{5.184}$$

and in view of that we have [see (5.175)]

$$\begin{aligned} \mathbf{A}(\phi, \mathbf{r}) &= [\cos\phi(\mathbf{E} - \mathbf{r} \otimes \mathbf{r}) - \sin\phi(\boldsymbol{\varepsilon}\mathbf{r}) + \mathbf{r} \otimes \mathbf{r}]\,\mathbf{b} \\ &= \cos\phi(\mathbf{E} - \mathbf{r} \otimes \mathbf{r})\mathbf{b} - \sin\phi(\boldsymbol{\varepsilon}\mathbf{r})\mathbf{b} + (\mathbf{r} \otimes \mathbf{r})\mathbf{b} \\ &= \cos\phi(\mathbf{E} - \mathbf{r} \otimes \mathbf{r})\mathbf{b} + \sin\phi\mathbf{r} \times \mathbf{b} + (\mathbf{r} \circ \mathbf{b})\mathbf{r} \\ &= \cos\phi\mathbf{b}_2 + \sin\phi\mathbf{r} \times \mathbf{b}_2 + \mathbf{b}_1 = \mathbf{A}\mathbf{b}_2 + \mathbf{b}_1, \end{aligned} \tag{5.185}$$

where (5.171) was used during the transformations.

From the preceding calculations it follows that we obtain vector $\mathbf{Ab}$ by adding vector $\mathbf{Ab}_2$ to vector $\mathbf{b}_1$ (parallel to $\mathbf{r}$). The latter can be obtained easily by means of the rotation of vector $\mathbf{b}_2$ through angle ϕ in a counterclockwise direction, where $|\mathbf{Ab}_2| = |\mathbf{b}_2|$. We call (5.185) the *Euler formula*.

Euler's rotation tensor (5.175) exhibits some interesting properties:

(i) $\mathbf{A}(\phi, \mathbf{r}) = \mathbf{A}(-\phi, \mathbf{r})$;
(ii) $\mathbf{A}(\phi = 0, \mathbf{r}) = \mathbf{E}$.

The obtained results will be used to represent the tensor of angular velocity of a rigid body $\boldsymbol{\Omega}$ [see (5.163)] and the vector of angular velocity of a rigid body $\boldsymbol{\omega}$ [see (5.164)] on the assumption that $\phi = \phi(t)$ and $\mathbf{r} = \mathbf{r}(t)$.

Recall that $\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $\dot{\mathbf{E}} = \mathbf{0}$, and $\mathbf{r} \circ \dot{\mathbf{r}} = 0$ because $\mathbf{r} \perp \dot{\mathbf{r}}$.

Differentiating (5.175) with respect to time we obtain

$$\dot{\mathbf{A}}(\phi, \mathbf{r}) = -\dot{\phi}\sin\phi(\mathbf{E} - \mathbf{r}\otimes\mathbf{r}) - \dot{\phi}\cos\phi(\boldsymbol{\varepsilon}\mathbf{r}) + (1-\cos\phi)(\dot{\mathbf{r}}\otimes\mathbf{r} + \mathbf{r}\otimes\dot{\mathbf{r}}) - \sin\phi(\boldsymbol{\varepsilon}\dot{\mathbf{r}}). \tag{5.186}$$

We will further use relationships (5.178). Additionally,

$$\dot{\mathbf{r}} = a\mathbf{t}_1 + b\mathbf{t}_2, \tag{5.187}$$

where a and b are certain scalars. The foregoing result follows from the equation $\mathbf{r} \circ \dot{\mathbf{r}} = 0$, which indicates that $\dot{\mathbf{r}} \perp \mathbf{t}_3 \equiv \mathbf{r}$. Therefore, vector $\dot{\mathbf{r}}$ is expressed by $\{\mathbf{t}_1, \mathbf{t}_2\}$.

Let us calculate the cross product

$$\mathbf{r}\times\dot{\mathbf{r}} = \begin{vmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \\ 0 & 0 & 1 \\ a & b & 0 \end{vmatrix} = -b\mathbf{t}_1 + a\mathbf{t}_2, \tag{5.188}$$

and note that [relation (5.169)]

$$\boldsymbol{\varepsilon}\dot{\mathbf{r}} \equiv a(\mathbf{t}_2\otimes\mathbf{t}_3 - \mathbf{t}_3\otimes\mathbf{t}_2) + b(\mathbf{t}_3\otimes\mathbf{t}_1 - \mathbf{t}_1\otimes\mathbf{t}_3). \tag{5.189}$$

According to (5.186) we have

$$\begin{aligned}\dot{\mathbf{A}} &= -\dot{\phi}\sin\phi(\mathbf{E} - \mathbf{t}_3\otimes\mathbf{t}_3) - \dot{\phi}\cos\phi(\mathbf{t}_1\otimes\mathbf{t}_2 - \mathbf{t}_2\otimes\mathbf{t}_1) \\ &\quad + (1-\cos\phi)\left[(a\mathbf{t}_1 + b\mathbf{t}_2)\otimes\mathbf{t}_3 + \mathbf{t}_3\otimes(a\mathbf{t}_1 + b\mathbf{t}_2)\right] \\ &\quad - \sin\phi\left[a(\mathbf{t}_2\otimes\mathbf{t}_3 - \mathbf{t}_3\otimes\mathbf{t}_2) + b(\mathbf{t}_3\otimes\mathbf{t}_1 - \mathbf{t}_1\otimes\mathbf{t}_3)\right],\end{aligned} \tag{5.190}$$

$$\begin{aligned}\mathbf{A}^{\mathrm{T}} &= \cos\phi(\mathbf{t}_1\otimes\mathbf{t}_1 + \mathbf{t}_2\otimes\mathbf{t}_2) \\ &\quad + \sin\phi(\mathbf{t}_1\otimes\mathbf{t}_2 - \mathbf{t}_2\otimes\mathbf{t}_1) + \mathbf{t}_3\otimes\mathbf{t}_3,\end{aligned} \tag{5.191}$$

$$\dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} = -\dot{\phi}\boldsymbol{\varepsilon}\mathbf{t}_3 - [a(1-\cos\phi) + b\sin\phi]\boldsymbol{\varepsilon}\mathbf{t}_2 - [-b(1-\cos\phi) + a\sin\phi]\boldsymbol{\varepsilon}\mathbf{t}_1. \tag{5.192}$$

Finally, we obtain

$$\boldsymbol{\Omega} = \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} = -\dot{\phi}\boldsymbol{\varepsilon}\mathbf{r} - (1-\cos\phi)\boldsymbol{\varepsilon}(\mathbf{r}\times\dot{\mathbf{r}}) - \sin\phi\boldsymbol{\varepsilon}\dot{\mathbf{r}}, \tag{5.193}$$

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{r} - \sin\phi\dot{\mathbf{r}} + (1-\cos\phi)\mathbf{r}\times\dot{\mathbf{r}}. \tag{5.194}$$

The preceding expressions undergo significant simplification if $\mathbf{r} = \text{const}$, that is, from (5.193) and (5.194) it follows that

$$\mathbf{A} = -\dot{\phi}\boldsymbol{\varepsilon}\mathbf{r}, \tag{5.195}$$

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{r}. \tag{5.196}$$

At the end of the calculations in this section we should emphasize that there exist several different possibilities for representing the rotation tensor [8].

In the literature the following three possibilities are used most often: Euler's angles, Euler's parameters, and Rodrigues[2] parameters. All three of the aforementioned representations are described in the book [8], where for the analysis and illustration of the conducted calculation, the tensor calculus was used.

In the present book during subsequent calculations we will limit ourselves to descriptions of the most commonly used representation, i.e., Euler's angles [4–11].

5.5.4 Eulerian Angles

Recall (Sect. 5.5.2) that the arbitrary motion of a rigid body can be represented as a composition of the motion of its arbitrary point O (pole) and rotation of the body about that pole. Let us assume that point O is stationary, and let us consider the kinematics of a rigid body with one point fixed. Any four non-coplanar points of a rigid body O and A_i, $i = 1, 2, 3$, form a pyramid. With this pyramid, whose position varies in time, we will associate the stationary coordinate system $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and non-stationary coordinate system $\{\mathbf{E}'_1, \mathbf{E}'_2, \mathbf{E}'_3\}$. We assume that the apex of the pyramid lies at point O.

Euler noticed that an arbitrary position of a tetrahedron in three-dimensional space can be represented through three angles. This means that the rotations of the tetrahedron with respect to certain axes and with the aid of only three angles of rotation in certain planes will suffice to make the stationary pyramid coincident with the moving one. The Eulerian angles are presented in Fig. 5.56 (see, e.g., [10]).

Let us introduce the Cartesian coordinate system and choose the tetrahedron in such way that its one vertex lies at the origin of the coordinate system O and the remaining vertices A_1, A_2, and A_3 lie on successive axes of this coordinate system.

[2] Olinde Rodrigues (1795–1851), French banker and mathematician.

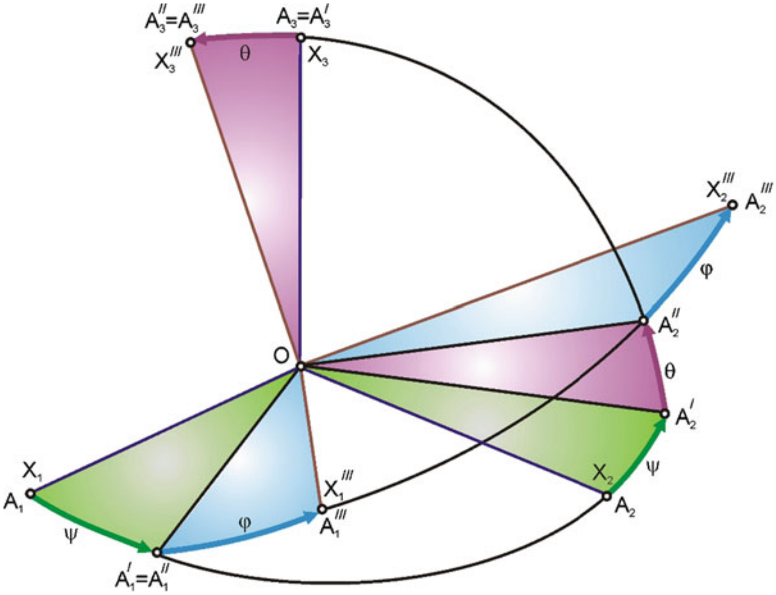


Fig. 5.56 Eulerian angles

Following the composition of three plane rotations successively through angles ψ, θ, and φ, the tetrahedron $OA_1A_2A_3$ ends up in the position $OA_1'''A_2'''A_3'''$. The first rotation through angle ψ (angle of precession) takes place in the plane OA_1A_2 about the $OA_3 \equiv X_3$ axis. The second rotation through angle θ (angle of nutation) takes place in the plane $OA_2'A_3'$ about the $OA_1' \equiv X_1'$ axis. Finally, we perform rotation through angle φ (angle of eigenrotation or spin) in the plane $OA_1''A_2''$ about the OA_3'' axis. The order of rotation of the tetrahedron can be schematically presented in the following way:

$$OA_1A_2A_3 \xrightarrow{\Psi} OA_1'A_2'A_3' \xrightarrow{\Theta} OA_1''A_2''A_3'' \xrightarrow{\varphi} OA_1'''A_2'''A_3'''. \tag{5.197}$$

The aforementioned successive transformations are schematically presented in Fig. 5.57.

Let us assume that the rigid body moves in motion about a point in the absolute coordinate system $OX_1X_2X_3$ and that the body system is denoted by $OX_1'''X_2'''X_3'''$. If we know the motion of the body system with respect to the absolute system, we also know the motion of the rigid body itself with respect to this system. Eulerian angles allow for the description of the system $OX_1'''X_2'''X_3'''$ with respect to the system $OX_1X_2X_3$. The edge of the intersection of the plane OX_1X_2 of the absolute system with the plane $OX_1'''X_2'''$ of the non-stationary system shown in Fig. 5.56 as OA_1'' is called the *line of nodes*.

Angles ψ, θ, and φ are independent and can be chosen arbitrarily. Three numbers corresponding to the values of these angles determine uniquely the position of the

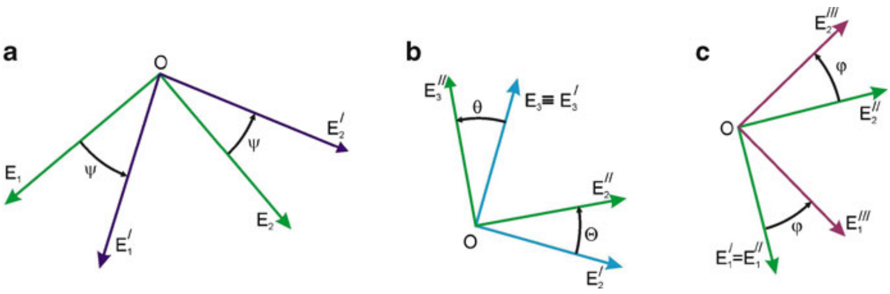


Fig. 5.57 Successive rotations through angles ψ (**a**), θ (**b**), and φ (**c**)

body in three-dimensional space. It is a common practice to assume that $0 \leq \psi < 2\pi$, $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$. Let us note that for $\theta = 0$ or $\theta = \pi$ the line of nodes OA_1' and angles ψ and φ are not uniquely defined. Therefore, often a different set of angles is introduced in order to avoid this singularity of the Eulerian angles.

An arbitrary vector $\boldsymbol{a}$ can be represented as

$$\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 = \boldsymbol{a}_1' + \boldsymbol{a}_2' + \boldsymbol{a}_3' = \boldsymbol{a}_1'' + \boldsymbol{a}_2'' + \boldsymbol{a}_3'' = \boldsymbol{a}_1''' + \boldsymbol{a}_2''' + \boldsymbol{a}_3''',$$

where $\boldsymbol{a}_i^{(*)}$, $i = 1, 2, 3$, $\quad * = ', '', '''$ denote the components of the vector relative to the appropriate axes. The following relationships are valid:

$$\begin{aligned}
a_1 &= a_1' \cos\psi - a_2' \sin\psi, \\
a_2 &= a_1' \sin\psi + a_2' \cos\psi, \\
a_3 &= a_3',
\end{aligned} \tag{5.198}$$

$$\begin{aligned}
a_1' &= a_1'', \\
a_2' &= a_2'' \cos\theta - a_3'' \sin\theta, \\
a_3' &= a_2'' \sin\theta + a_3'' \cos\theta,
\end{aligned} \tag{5.199}$$

$$\begin{aligned}
a_1'' &= a_1''' \cos\varphi - a_2''' \sin\varphi, \\
a'' &= a_1'' \sin\varphi + a_2'' \cos\varphi, \\
a'' &= a_3'''.
\end{aligned} \tag{5.200}$$

We will show how to derive these equations on example of (5.198). According to the previous representation of vector $\boldsymbol{a}$ we have

$$\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 = \boldsymbol{a}_1' + \boldsymbol{a}_2' + \boldsymbol{a}_3',$$

or

$$a_1\mathbf{E}_1 + a_2\mathbf{E}_2 + a_3\mathbf{E}_3 = a_1'\mathbf{E}_1' + a_2'\mathbf{E}_2' + a_3'\mathbf{E}_3'.$$

Multiplying the preceding equation by sides in turn by unit vectors $\mathbf{E}_i$, $(i = 1, 2, 3)$, we obtain

$$a_1 = a_1' \left(\mathbf{E}_1' \circ \mathbf{E}_1\right) + a_2' \left(\mathbf{E}_2' \circ \mathbf{E}_1\right),$$
$$a_2 = a_1' \left(\mathbf{E}_1' \circ \mathbf{E}_2\right) + a_2' \left(\mathbf{E}_2' \circ \mathbf{E}_2\right),$$
$$a_3 = a_3' \left(\mathbf{E}_3' \circ \mathbf{E}_3\right).$$

The obtained relationships enable us to define the following second-order rotation tensors:

$$\boldsymbol{\Psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.201}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{5.202}$$

$$\boldsymbol{\phi} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.203}$$

It can be easily checked that these tensors are orthogonal and possess the properties (5.161) and (5.162). Therefore, the inverse relationships can be easily found after constructing transpose matrices $\boldsymbol{\Psi}^{\mathrm{T}}$, $\boldsymbol{\Theta}^{\mathrm{T}}$, and $\boldsymbol{\phi}^{\mathrm{T}}$, which allows us to determine the following relationships inverse to (5.198) and (5.200) of the form

$$\begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{5.204}$$

$$\begin{bmatrix} a_1'' \\ a_2'' \\ a_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix}, \tag{5.205}$$

$$\begin{bmatrix} a_1''' \\ a_2''' \\ a_3''' \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1'' \\ a_2'' \\ a_3'' \end{bmatrix}. \tag{5.206}$$

The relationships described by (5.198)–(5.200) can be represented in the form

$$\begin{bmatrix} a_1 \equiv \mathbf{A}a_1''' \\ a_2 \equiv \mathbf{A}a_2''' \\ a_3 \equiv \mathbf{A}a_3''' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_1''' \\ a_2''' \\ a_3''' \end{bmatrix}, \tag{5.207}$$

where

$$\mathbf{A} = \boldsymbol{\Psi}\boldsymbol{\Theta}\boldsymbol{\phi} \tag{5.208}$$

is a rotation tensor describing the result of action of the three rotation tensors $\boldsymbol{\Psi}, \boldsymbol{\Theta}, \boldsymbol{\phi}$.

Multiplying according to (5.208) we obtain

$$\begin{aligned}
A_{11} &= \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi,\\
A_{12} &= -\sin\varphi\cos\psi - \cos\varphi\cos\theta\sin\psi,\\
A_{13} &= \sin\theta\sin\psi,\\
A_{21} &= \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi,\\
A_{22} &= -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi,\\
A_{23} &= -\sin\theta\cos\psi,\\
A_{31} &= \sin\varphi\sin\theta,\\
A_{32} &= \cos\varphi\sin\theta,\\
A_{33} &= \cos\theta.
\end{aligned} \tag{5.209}$$

Displacement of the tetrahedron $OA_1A_2A_3$ to the position $OA_1'''A_2'''A_3'''$ is performed through matrix $\mathbf{A}$, which is the product of three matrices that prescribe the rotations. Because a product of matrices is generally not commutative, the sequence of rotations is important.

The non-commutativity can be illustrated on an example of two rotations of a tetrahedron: one time initially through an angle $\psi = \pi/2$ and then through $\theta = \pi/2$ (Fig. 5.58), and the other time in the reverse order.

Let us demonstrate that matrix $\mathbf{A}$, being the product of three matrices of elementary orthonormal rotations, is also an orthonormal matrix:

$$\begin{aligned}
\mathbf{A}\mathbf{A}^{\mathrm{T}} &= (\boldsymbol{\Psi}\boldsymbol{\Theta}\boldsymbol{\phi})(\boldsymbol{\Psi}\boldsymbol{\Theta}\boldsymbol{\phi})^{\mathrm{T}}\\
&= \boldsymbol{\Psi}\boldsymbol{\Theta}\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\Psi}\boldsymbol{\Theta})^{\mathrm{T}} = \boldsymbol{\Psi}\boldsymbol{\Theta}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\Psi}^{\mathrm{T}} = \mathbf{E}_{3\times 3}.
\end{aligned} \tag{5.210}$$

It can also be shown easily that the composition of any number of elementary rotation matrices is an orthonormal matrix.

From the preceding calculations it follows that after taking two arbitrary positions of two Cartesian coordinate systems in three-dimensional Euclidean space with a common origin, in general, it is possible to achieve their coincidence by applying the *Euler rotations* (in the present case, the three angles $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}$, and $\boldsymbol{\phi}$).

The first rotation can occur about any one of the three axes of the system $OX_1X_2X_3$ (in our case, X_3). Next, in the system $OX_1'X_2'X_3'$ we have two axes available about which to perform the rotation (in our case, X_1'). Subsequently, in the system $OX_1''X_2''X_3''$ obtained after the last rotation, we perform the rotation

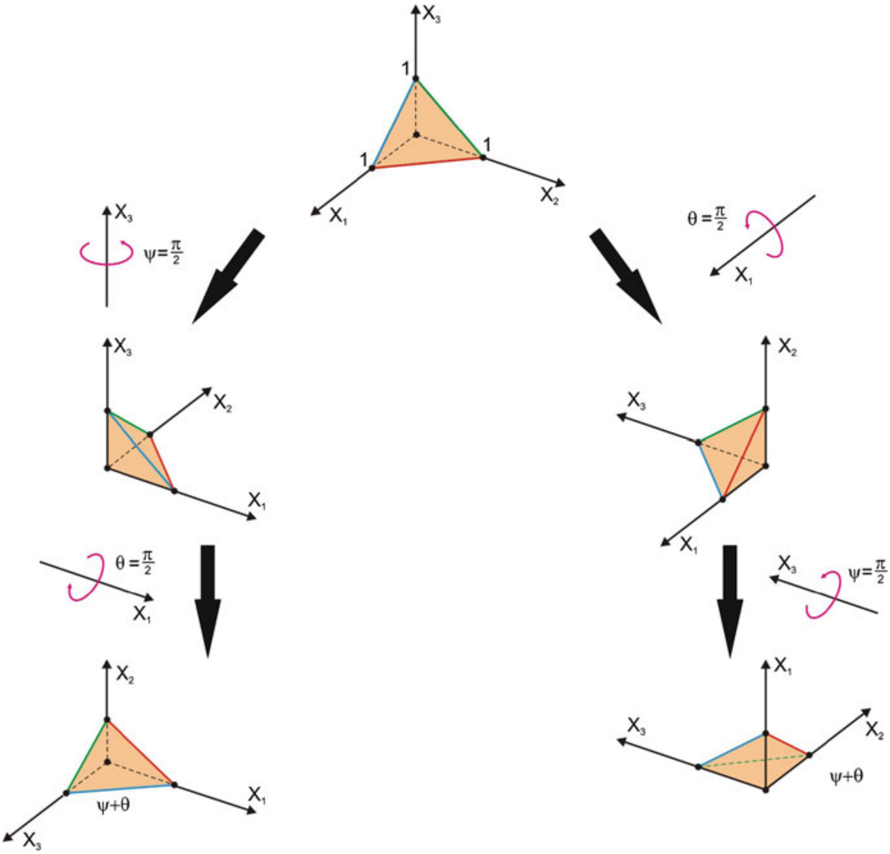


Fig. 5.58 Non-commutativity of two rotations of tetrahedron

about one of the remaining axes, i.e., axes lying in the plane perpendicular to the previous axis of rotation (in our case, X_3'''). Applying the notation already used in Chap. 3, (5.208) can be represented in the form

$$\mathbf{A}_{\Psi\Theta\phi}(X_3, X_1', X_3'') = \boldsymbol{\Psi}(\mathbf{E}_3)\boldsymbol{\Theta}(\mathbf{E}_1')\boldsymbol{\phi}(\mathbf{E}_3''), \tag{5.211}$$

and the preceding notation contains the information concerning the axis about which and the angle through which the transformation of rotation was performed.

According to those considerations, a number of possible Euler rotations is equal to $3 \cdot 2 \cdot 2 = 12$, that is, there exist 12 possibilities for choosing three subsequent axes and angles through which we perform the rotations.

The matrices following from the composition of Eulerian rotations are distinct since, as was mentioned previously, the product of matrices is not commutative.

Let us make use of the calculations conducted earlier where we defined matrices of rotation about each of the three axes of the Cartesian coordinate system.

Let us perform the rotations in turn about the X_1, X_2', and X_3''' axes respectively through angles ψ_1, ψ_2, and ψ_3. The resulting matrix has the form

$$\begin{aligned}
\mathbf{A}_{\psi_1\psi_2\psi_3}(\mathbf{E}_1, \mathbf{E}_2', \mathbf{E}_3'') &= \boldsymbol{\Psi}_1(\mathbf{E}_1)\boldsymbol{\Psi}_2(\mathbf{E}_2')\boldsymbol{\Psi}_3(\mathbf{E}_3'') \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & \sin\psi_1 & \cos\psi_1 \end{bmatrix} \begin{bmatrix} \cos\psi_2 & 0 & \sin\psi_2 \\ 0 & 1 & 0 \\ -\sin\psi_2 & 0 & \cos\psi_2 \end{bmatrix} \begin{bmatrix} \cos\psi_3 & -\sin\psi_3 & 0 \\ \sin\psi_3 & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \cos\psi_2\cos\psi_3 & -\cos\psi_2\sin\psi_3 & \sin\psi_2 \\ \sin\psi_1\sin\psi_2\cos\psi_3 + \cos\psi_1\sin\psi_3 & -\sin\psi_1\sin\psi_2\sin\psi_3 + \cos\psi_1\cos\psi_2 & -\sin\psi_1\cos\psi_2 \\ -\cos\psi_1\sin\psi_2\cos\psi_3 + \sin\psi_1\sin\psi_3 & \cos\psi_1\sin\psi_2\sin\psi_3 + \sin\psi_1\cos\psi_3 & \cos\psi_1\cos\psi_2 \end{bmatrix}. \tag{5.212}$$

Now, performing the rotations in the reverse order we obtain

$$\begin{aligned}
\mathbf{A}_{\psi_3\psi_2\psi_1}(\mathbf{E}_3'', \mathbf{E}_2', \mathbf{E}_1) &= \boldsymbol{\Psi}_3(\mathbf{E}_3'')\boldsymbol{\Psi}_2(\mathbf{E}_2')\boldsymbol{\Psi}_1(\mathbf{E}_1) \\
&= \begin{bmatrix} \cos\psi_3 & -\sin\psi_3 & 0 \\ \sin\psi_3 & \cos\psi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi_2 & 0 & \sin\psi_2 \\ 0 & 1 & 0 \\ -\sin\psi_2 & 0 & \cos\psi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_1 & -\sin\psi_1 \\ 0 & \sin\psi_1 & \cos\psi_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\psi_2\cos\psi_3 & \sin\psi_1\sin\psi_2\cos\psi_3 - \cos\psi_1\sin\psi_3 & \cos\psi_1\sin\psi_2\cos\psi_3 + \sin\psi_1\sin\psi_3 \\ \cos\psi_2\sin\psi_3 & \sin\psi_1\sin\psi_2\sin\psi_3 + \cos\psi_1\cos\psi_3 & \cos\psi_1\sin\psi_2\sin\psi_3 - \sin\psi_1\cos\psi_3 \\ -\sin\psi_2 & \sin\psi_1\cos\psi_2 & \cos\psi_1\cos\psi_2 \end{bmatrix}.
\end{aligned} \tag{5.213}$$

The obtained matrices (5.212) and (5.213) differ significantly from each other.

In Sect. 5.1.5 we introduced the notion of small rotation. If the angles of Euler rotations are small, then $\cos(\mathrm{d}\psi_i) = 1$, $\sin(\mathrm{d}\psi_i) = \mathrm{d}\psi_i$. Thus from (5.212) and (5.213) we obtain the same matrix

$$\mathbf{A}_{\mathrm{d}\Psi_1\mathrm{d}\Psi_2\mathrm{d}\Psi_3} = \mathbf{A}_{\mathrm{d}\Psi_3\mathrm{d}\Psi_2\mathrm{d}\Psi_1} = \begin{bmatrix} 1 & -\mathrm{d}\psi_3 & \mathrm{d}\psi_2 \\ \mathrm{d}\psi_3 & 1 & -\mathrm{d}\psi_1 \\ -\mathrm{d}\psi_2 & \mathrm{d}\psi_1 & 1 \end{bmatrix}, \tag{5.214}$$

where small terms of second order were neglected. Observe that now the sequence of rotations through small angles is not important, which leads to the conclusion that small rotations are vector quantities.

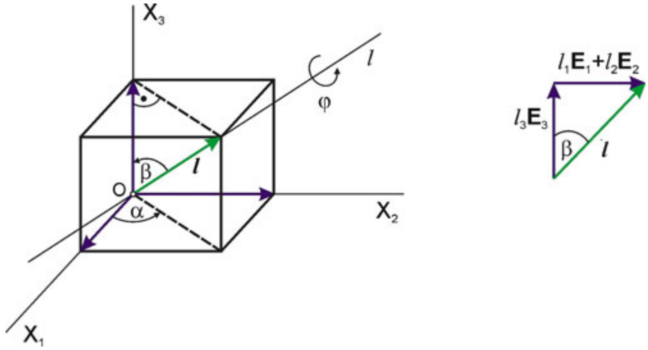


Fig. 5.59 Rotation of a rigid body about an arbitrary axis l through angle φ

Knowledge of the kinematics of a rigid body finds wide application in robotics and in the theory and practice of the design and production of manipulators. For instance, in the case of rigid manipulators, where the description of motion of particular links of a kinematic chain is made on the assumption that they are rigid bodies, in order to describe the position of any of the links it is sufficient to introduce a single stationary absolute Cartesian coordinate system and then associate a local Cartesian coordinate system with every link (e.g., all systems are right-handed). Apart from the aforementioned Cartesian coordinate systems, the previously discussed spherical and cylindrical coordinate systems are also used (Chap. 4).

One of the fundamental problems in robotics and in the theory of manipulators is the formulation of the condition of rotation of a rigid body through a prescribed angle φ about a prescribed axis l by means of rotations about mutually perpendicular axes (Fig. 5.59).

According to Fig. 5.59 we have

$$\boldsymbol{l} = l_1\mathbf{E}_1 + l_2\mathbf{E}_2 + l_3\mathbf{E}_3, \tag{5.215}$$

where $|\boldsymbol{l}| = 1$.

Note that

$$\begin{aligned}
\cos\beta &= l_3\mathbf{E}_3 \circ \boldsymbol{l} = l_3,\\
\sin\beta &= \sqrt{\frac{(l_1\mathbf{E}_1 + l_2\mathbf{E}_2)^2}{\boldsymbol{l}^2}} = \sqrt{l_1^2 + l_2^2},\\
\cos\alpha &= \sqrt{\frac{(l_1\mathbf{E}_1)^2}{(l_1\mathbf{E}_1 + l_2\mathbf{E}_2)^2}} = \frac{l_1}{\sqrt{l_1^2 + l_2^2}},\\
\sin\alpha &= \sqrt{\frac{(l_2\mathbf{E}_2)^2}{(l_1\mathbf{E}_1 + l_2\mathbf{E}_2)^2}} = \frac{l_2}{\sqrt{l_1^2 + l_2^2}}.
\end{aligned} \tag{5.216}$$

During rotations we will use only three angles, α, β, and φ. However, the rotations through angles α and β will be performed about the axes obtained as a result of the preceding rotation.

A composite rotation matrix has the following form:

$$\boldsymbol{\varphi}(\boldsymbol{l}) = \boldsymbol{\alpha}(\mathbf{E}_3)\boldsymbol{\beta}(\mathbf{E}_2')\boldsymbol{\varphi}(\mathbf{E}_3'')(-\boldsymbol{\beta}(\mathbf{E}_2'''))(-\boldsymbol{\alpha}(\mathbf{E}_3'''')). \tag{5.217}$$

As can be seen from the preceding relationship, the rotation about the l axis is obtained after realization of five consecutive rotations. At first we rotate the system $OX_1X_2X_3$ about the X_3 axis through angle α, obtaining the system $OX_1'X_2'X_3'$. Next we rotate $OX_1'X_2'X_3'$ about the X_2' axis through angle β. Following the composition of these two rotations the X_3' axis will coincide with the l axis. Then we rotate the obtained system $OX_1''X_2''X_3''$ about the X_3'' axis through angle φ. Then we rotate the obtained system $OX_1'''X_2'''X_3'''$ through angle $(-\beta)$ about the X_3''' axis, obtaining the system $OX_1''''X_2''''X_3''''$. We rotate this last system (for the fifth time) about the X_3'''' axis through angle $(-\alpha)$. Because we applied Euler rotations, the composite rotation matrix $\boldsymbol{\varphi}(\boldsymbol{l})$ will be orthonormal, i.e., $\boldsymbol{\varphi}^{-1}(\boldsymbol{l}) = \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{l})$.

Multiplying the matrices according to (5.217) and taking into account (5.216) and the relevant transformations we obtain

$$\boldsymbol{\varphi}(\boldsymbol{l}) = \boldsymbol{l}\cdot\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi) + \begin{bmatrix} \cos\varphi & -l_3\sin\varphi & l_2\sin\varphi \\ l_3\sin\varphi & \cos\varphi & -l_1\sin\varphi \\ -l_2\sin\varphi & l_1\sin\varphi & \cos\varphi \end{bmatrix}, \tag{5.218}$$

where

$$\boldsymbol{l}\cdot\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi) = \begin{bmatrix} l_1^2 & l_1l_2 & l_1l_3 \\ l_1l_2 & l_2^2 & l_2l_3 \\ l_1l_3 & l_2l_3 & l_3^2 \end{bmatrix}(1-\cos\varphi).$$

It is easy to check that matrix $\boldsymbol{\varphi}(\boldsymbol{l})$ has the following properties:

$$-\boldsymbol{\varphi}(\boldsymbol{l}) = \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{l}), \quad -\boldsymbol{\varphi}(-\boldsymbol{l}) = \boldsymbol{\varphi}(\boldsymbol{l}).$$

Equation (5.217) can be written in the following form:

$$\boldsymbol{\varphi}(\boldsymbol{l}) = \boldsymbol{\alpha}(\mathbf{E}_3)\boldsymbol{\beta}(\mathbf{E}_2')\boldsymbol{\varphi}(\mathbf{E}_3'')\boldsymbol{\beta}^{\mathrm{T}}(\mathbf{E}_2''')\boldsymbol{\alpha}^{\mathrm{T}}(\mathbf{E}_3''''). \tag{5.219}$$

Let us perform the rotation about the $\mathbf{E}_3$ axis through angle φ.

According to (5.219) we have

$$\boldsymbol{\varphi}(\mathbf{E}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}(1-\cos\varphi) + \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\cos\varphi \end{bmatrix} + \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \cos\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.220}$$

which is in agreement with the elementary rotation about the $\mathbf{E}_3$ axis [see (5.61)].

One can expect, as we showed earlier (Fig. 5.56), that an arbitrary matrix

$$\boldsymbol{\varphi}_{3\times 3} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix} \tag{5.221}$$

will have representation by a certain axis $\boldsymbol{l}$ and an angle ' as long as it has the property $\boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}} = \mathbf{E}_{3\times 3}$ and $\det\boldsymbol{\varphi} = 1$, i.e., it is an orthonormal matrix. A matrix trace of (5.218) reads

$$\begin{aligned} \mathrm{Tr}(\boldsymbol{\varphi}) &= \left[l_1^2(1-\cos\varphi)+\cos\varphi\right] + \left[l_2^2(1-\cos\varphi)+\cos\varphi\right] \\ &+ \left[l_3^2(1-\cos\varphi)+\cos\varphi\right] = l_1^2+l_2^2+l_3^2+3\cos\varphi \\ &- (l_1^2+l_2^2+l_3^2)\cos\varphi = 1+2\cos\varphi \end{aligned} \tag{5.222}$$

because $\boldsymbol{l}^2 = l_1^2+l_2^2+l_3^2 = 1$.

From (5.222) we can determine the angle of rotation φ representing matrix (5.221), which is equal to

$$\varphi = \arccos\frac{\mathrm{Tr}(\boldsymbol{\varphi})-1}{2} = \arccos\frac{\varphi_{11}+\varphi_{22}+\varphi_{33}-1}{2}. \tag{5.223}$$

In the next step we will determine the axis about which the rotation will be performed. Equating the terms in matrices (5.218) and (5.221) we obtain

$$\begin{aligned} l_1l_2(1-\cos\varphi) - l_3\sin\varphi &= \varphi_{12}, \\ l_1l_3(1-\cos\varphi) + l_2\sin\varphi &= \varphi_{13}, \\ l_2l_3(1-\cos\varphi) - l_1\sin\varphi &= \varphi_{23}, \\ l_1l_2(1-\cos\varphi) + l_3\sin\varphi &= \varphi_{21}, \\ l_1l_3(1-\cos\varphi) - l_2\sin\varphi &= \varphi_{31}, \\ l_2l_3(1-\cos\varphi) + l_1\sin\varphi &= \varphi_{32}, \end{aligned} \tag{5.224}$$

and after their appropriate subtraction we have

$$\begin{aligned}\varphi_{13}-\varphi_{31}&=2l_2\sin\varphi,\\ \varphi_{21}-\varphi_{12}&=2l_3\sin\varphi,\\ \varphi_{32}-\varphi_{23}&=2l_1\sin\varphi.\end{aligned} \tag{5.225}$$

Multiplying (5.225) by sides by the appropriate unit vectors $\mathbf{E}_i$ and adding the obtained equations to each other we obtain

$$\begin{aligned}&\frac{1}{2\sin\varphi}\left[\mathbf{E}_1(\varphi_{32}-\varphi_{23})+\mathbf{E}_2(\varphi_{13}-\varphi_{31})+\mathbf{E}_3(\varphi_{21}-\varphi_{12})\right]\\ &\quad=\mathbf{E}_1l_1+\mathbf{E}_2l_2+\mathbf{E}_3l_3\equiv\boldsymbol{l},\end{aligned} \tag{5.226}$$

which defines for us the unknown unit vector of the l axis.

Let us note that we are dealing with the singularities for $\varphi=0$ and $\varphi=\pi$.

Matrix $\boldsymbol{\varphi}(\boldsymbol{l})$ can be written in the equivalent form

$$\boldsymbol{\varphi}(\boldsymbol{l})=\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)+\mathbf{E}\cos\varphi+\mathbf{S}(\boldsymbol{l})\sin\varphi, \tag{5.227}$$

where

$$\mathbf{S}(\boldsymbol{l})=\begin{bmatrix}0&-l_3&l_2\\ l_3&0&-l_1\\ -l_2&l_1&0\end{bmatrix}, \tag{5.228}$$

and all matrices have the dimensions 3×3.

Matrix $\mathbf{S}$ is the skew-symmetric matrix ($\mathbf{S}+\mathbf{S}^{\mathrm{T}}=\mathbf{0}$).

Theorem 5.6. *The rotation matrix $\boldsymbol{\varphi}(\boldsymbol{l})$ described by (5.227) satisfies the following relationship:*

$$\frac{\mathrm{d}\boldsymbol{\varphi}(\boldsymbol{l})}{\mathrm{d}\varphi}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{l})=\mathbf{S}(\boldsymbol{l}). \tag{5.229}$$

Proof. Differentiating (5.227) with respect to φ we obtain

$$\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}\varphi}=\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\sin\varphi-\mathbf{E}\sin\varphi+\mathbf{S}(\boldsymbol{l})\cos\varphi.$$

The left-hand side of (5.229) takes the form

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\varphi}(\boldsymbol{l})}{\mathrm{d}\varphi}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{l})&=(\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\sin\varphi-\mathbf{E}\sin\varphi+\mathbf{S}\cos\varphi)(\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)+\mathbf{E}\cos\varphi+\mathbf{S}\sin\varphi)^{\mathrm{T}}\\ &=(\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\sin\varphi-\mathbf{E}\sin\varphi+\mathbf{S}\cos\varphi)(\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)+\mathbf{E}\cos\varphi-\mathbf{S}\sin\varphi)\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{ll}^{\mathrm{T}}\boldsymbol{ll}^{\mathrm{T}}(1-\cos\varphi)\sin\varphi + \boldsymbol{ll}^{\mathrm{T}}\mathbf{E}\sin\varphi\cos\varphi \\
&\quad - \boldsymbol{ll}^{\mathrm{T}}\mathbf{S}\sin^2\varphi + \mathbf{E}\boldsymbol{ll}^{\mathrm{T}}(1-\cos\varphi)\sin\varphi - \mathbf{E}^2\sin\varphi\cos\varphi \\
&\quad + \mathbf{ES}\sin^2\varphi + \mathbf{S}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)\cos\varphi + \mathbf{SE}\cos^2\varphi - \mathbf{S}^2\sin\varphi\cos\varphi,
\end{aligned}
$$

where the relationship $\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$ and the symmetry of matrix $\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}$ were used. □

Because

$$
\mathbf{S}(\boldsymbol{l})\boldsymbol{l} = \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 - l_2 l_3 + l_2 l_3 \\ l_1 l_3 + 0 - l_1 l_3 \\ -l_1 l_2 + l_1 l_2 + 0 \end{bmatrix} = \mathbf{0} = \boldsymbol{l}^{\mathrm{T}}\mathbf{S}(\boldsymbol{l})
$$

and

$$
\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} = \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_2 l_1 & l_2^2 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 \end{bmatrix},
$$

$$
\boldsymbol{l}^{\mathrm{T}}\boldsymbol{l} = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = l_1^2 + l_2^2 + l_3^2 = \boldsymbol{l}^2 = 1,
$$

and therefore the transformed left-hand side of (5.229) takes the form

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\varphi}(\boldsymbol{l})}{\mathrm{d}\varphi}\boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{l}) &= \boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)\sin\varphi - \boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}(1-\cos\varphi)\sin\varphi \\
&\quad + \boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\sin\varphi\cos\varphi - \mathbf{0} - \mathbf{E}\sin\varphi\cos\varphi + \mathbf{S}\sin^2\varphi + \mathbf{0} + \mathbf{S}\cos^2\varphi \\
&\quad - \mathbf{S}^2\sin\varphi\cos\varphi = (\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} - \mathbf{S}^2 - \mathbf{E})\sin\varphi\cos\varphi + \mathbf{S}.
\end{aligned}
$$

Because

$$
\begin{aligned}
\mathbf{S}^2 = \mathbf{SS} &= \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -l_3^2 - l_2^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & -l_3^2 - l_1^2 & l_2 l_3 \\ l_1 l_3 & l_2 l_3 & -l_2^2 - l_1^2 \end{bmatrix},
\end{aligned}
$$

we have

$$\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} - \mathbf{S}^2 = \begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_2 l_1 & l_2^2 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 \end{bmatrix} + \begin{bmatrix} l_3^2 + l_2^2 & -l_1 l_2 & -l_1 l_3 \\ -l_1 l_2 & l_1^2 + l_3^2 & -l_2 l_3 \\ -l_1 l_3 & -l_2 l_3 & l_1^2 + l_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E},$$

and in view of that the matrix

$$(\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}} - \mathbf{S}^2 - \mathbf{E}) = (\mathbf{E} - \mathbf{E}) = \mathbf{0}.$$

Eventually we obtained (5.229), which was what we had wanted to demonstrate.

5.5.5 Kinematic Eulerian Equations

Let the vector $\boldsymbol{\omega} = \dot{\varphi}\boldsymbol{l}$ act along the $\boldsymbol{l}$ axis. Calculating the time derivative of the rotation matrix and taking into account (5.229) we obtain

$$\frac{\mathrm{d}\boldsymbol{\varphi}(\boldsymbol{l})}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\varphi}(\boldsymbol{l})}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \mathbf{S}(\boldsymbol{l})\boldsymbol{\varphi}(\boldsymbol{l})\dot{\varphi} = \mathbf{S}(\dot{\varphi}\boldsymbol{l})\boldsymbol{\varphi}(\boldsymbol{l}), \tag{5.230}$$

where according to (5.228) the matrix

$$\boldsymbol{\Omega} \equiv \mathbf{S}(\dot{\varphi}\boldsymbol{l}) = \begin{bmatrix} 0 & -l_3\dot{\varphi} & l_2\dot{\varphi} \\ l_3\dot{\varphi} & 0 & -l_1\dot{\varphi} \\ -l_2\dot{\varphi} & l_1\dot{\varphi} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{5.231}$$

where the skew-symmetric matrix $\boldsymbol{\Omega} = \mathbf{S}(\boldsymbol{\omega})$ uniquely corresponds to the vector $\boldsymbol{\omega} = \omega_1\mathbf{E}_1 + \omega_2\mathbf{E}_2 + \omega_3\mathbf{E}_3$ and can be interpreted as a skew-symmetric matrix representing the angular velocity (angular velocity tensor) of a rotating rigid body with respect to the absolute system $OX_1X_2X_3$.

The matrix (tensor) of the angular velocity according to (5.230) can be expressed as

$$\boldsymbol{\Omega} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{\varphi}(\boldsymbol{l})\right]\boldsymbol{\varphi}^{-1}(\boldsymbol{l}). \tag{5.232}$$

Thus we proved relationship (5.163). To illustrate how to determine, from the time-dependent matrix of rotation $\boldsymbol{\varphi}(\boldsymbol{l},\, t)$ the angular velocity matrix $\boldsymbol{\Omega}$, the angular velocity vector of a rigid body $\boldsymbol{\omega}$, and the vector $\boldsymbol{l}$ on which the $\boldsymbol{\omega}$ lies from the time-dependent rotation matrix $\boldsymbol{\varphi}(\boldsymbol{l}, t)$, we will consider the following matrix:

$$\boldsymbol{\varphi}(t) = \begin{bmatrix} -\sin 2t & -\frac{1}{2}\sin 3t + \frac{1}{2}\sin t & \frac{1}{2}\cos 3t + \frac{1}{2}\cos t \\ 0 & \cos t & \sin t \\ \cos 2t & \frac{1}{2}\cos 3t - \frac{1}{2}\cos t & \frac{1}{2}\sin 3t + \frac{1}{2}\sin t \end{bmatrix}. \tag{5.233}$$

The preceding matrix is an orthonormal matrix because $\boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}} = \mathbf{E}$. Differentiation of rotation matrix (5.233) with respect to time reduces to a differentiation of their elements. As a result of this procedure we obtain

$$\frac{\mathrm{d}\boldsymbol{\varphi}(t)}{\mathrm{d}t} = \begin{bmatrix} -2\cos 2t & -\frac{3}{2}\cos 3t + \frac{1}{2}\cos t & -\frac{3}{2}\sin 3t - \frac{1}{2}\sin t \\ 0 & -\sin t & \cos t \\ -2\sin 2t & -\frac{3}{2}\sin 3t + \frac{1}{2}\sin t & \frac{3}{2}\cos 3t + \frac{1}{2}\cos t \end{bmatrix}. \tag{5.234}$$

According to (5.233) the angular velocity matrix is equal to

$$\boldsymbol{\Omega} \equiv \mathbf{S}(\dot{\varphi}\boldsymbol{l}) = \begin{bmatrix} 0 & -\cos 2t & -2 \\ \cos 2t & 0 & \sin 2t \\ 2 & -\sin 2t & 0 \end{bmatrix}. \tag{5.235}$$

Using the matrix form (5.231) we determine the angular velocity vector

$$\begin{aligned} \boldsymbol{\omega} &= \omega\boldsymbol{l}(t) = \omega l_1(t)\mathbf{E}_1 + \omega l_2(t)\mathbf{E}_2 + \omega l_3(t)\mathbf{E}_3 \\ &= \omega_1\mathbf{E}_1 + \omega_2\mathbf{E}_2 + \omega_3\mathbf{E}_3 = -\sin 2t\mathbf{E}_1 - 2\mathbf{E}_2 + \cos 2t\mathbf{E}_3, \end{aligned} \tag{5.236}$$

and hence we calculate

$$\boldsymbol{\omega}^2 = \omega^2 = \sin^2 2t + \cos^2 2t + 4 = 5,$$

that is, $\omega = \sqrt{5}$. The preceding equation indicates that the magnitude of vector $\boldsymbol{\omega}$ remains constant in time. According to (5.236) vector $\mathbf{l}$ changes in time because its direction alters, although its magnitude is constantly equal to one. The normalization of a matrix corresponds to the normalization of a vector $\mathbf{l}$.

From (5.236) we calculate

$$l_1(t) = \frac{\omega_1(t)}{\omega} = -\frac{\sin 2t}{\sqrt{5}},$$

$$l_2(t) = \frac{\omega_2(t)}{\omega} = -\frac{2}{\sqrt{5}},$$

$$l_3(t) = \frac{\omega_3(t)}{\omega} = -\frac{\cos 2t}{\sqrt{5}},$$

and clearly $\sqrt{l_1^2 + l_2^2 + l_3^2} = 1$.

Let the constant angular velocity vectors of the body along the OX_3', OX_1'', and OX_3'' axes be prescribed (Fig. 5.60).

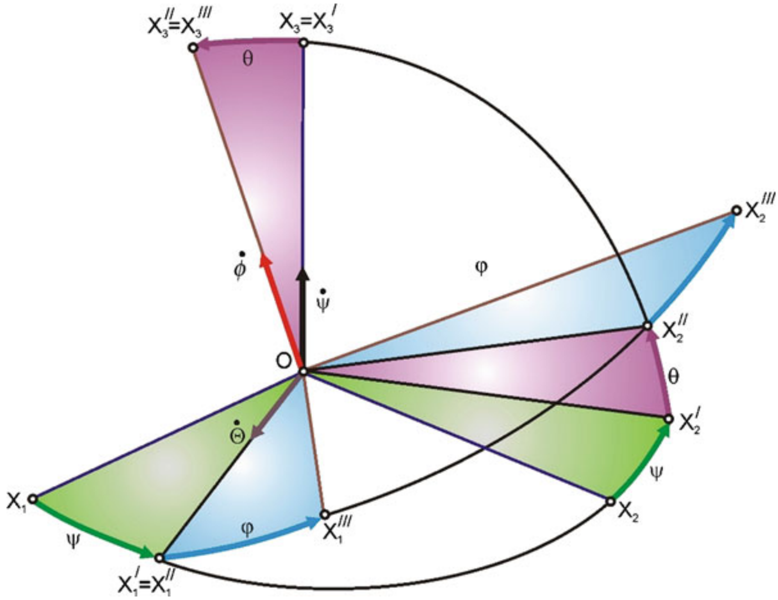


Fig. 5.60 Eulerian angles and the angular velocity vectors $\dot{\boldsymbol{\psi}}$, $\dot{\boldsymbol{\Theta}}$, $\dot{\boldsymbol{\phi}}$

From Fig. 5.60 it follows that

$$\dot{\boldsymbol{\Psi}} = \dot{\psi}\mathbf{E}_3', \tag{5.237}$$

$$\dot{\boldsymbol{\Theta}} = \dot{\theta}\mathbf{E}_1'', \tag{5.238}$$

$$\dot{\boldsymbol{\phi}} = \dot{\phi}\mathbf{E}_3'''. \tag{5.239}$$

We wish to determine the angular velocity $\boldsymbol{\omega}$ of a body in the coordinate system $OX_1'''X_2'''X_3'''$. Therefore, vectors (5.237)–(5.239) should be projected onto the axes of that system. Assuming

$$\boldsymbol{\omega} = \omega_1'''\mathbf{E}_1''' + \omega_2'''\mathbf{E}_2''' + \omega_3\mathbf{E}_3''' \tag{5.240}$$

we obtain

$$\begin{bmatrix}\omega_1'''\\ \omega_2'''\\ \omega_3'''\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot{\psi}\end{bmatrix}$$
$$+\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dot{\phi}\end{bmatrix}, \tag{5.241}$$

and in expanded form we have

$$\omega_1''' = \dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi,$$
$$\omega_2''' = \dot{\psi}\cos\phi\sin\theta - \dot{\theta}\sin\phi,$$
$$\omega_3''' = \dot{\psi}\cos\Theta + \dot{\phi}. \tag{5.242}$$

From (5.242) we obtain the magnitude of angular velocity

$$\omega = \sqrt{\left(\omega_1'''\right)^2 + \left(\omega_2'''\right)^2 + \left(\omega_3'''\right)^2}$$
$$= \sqrt{\dot{\psi}^2 + \dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\psi}\dot{\phi}\cos\theta}. \tag{5.243}$$

The magnitude of angular velocity in the absolute coordinate system can be obtained easily from (5.207). Substituting (5.239) into (5.207) we have

$$\omega_1^{(\dot{\phi})} = \dot{\phi}\sin\theta\sin\psi,$$
$$\omega_2^{(\dot{\phi})} = -\dot{\phi}\sin\theta\cos\psi,$$
$$\omega_3^{(\dot{\phi})} = \dot{\phi}\cos\theta. \tag{5.244}$$

Vector (5.238) passes to the absolute system after one rotation. Assuming the values $\theta = 0$ and $\varphi = 0$ in matrix **A** we obtain

$$\omega_1^{(\dot{\theta})} = \dot{\theta}\cos\psi,$$
$$\omega_2^{(\dot{\theta})} = \dot{\theta}\sin\psi,$$
$$\omega_3^{(\dot{\theta})} = 0. \tag{5.245}$$

Vector (5.239) is already given in the absolute system since $\mathbf{E}_3' = \mathbf{E}_3$. From (5.244), (5.245), and (5.239) we obtain

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi,$$
$$\omega_2 = -\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi,$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}. \tag{5.246}$$

5.5.6 Displacement of a Rigid Body with One Point Fixed

Recall that a rigid body with one point fixed is in motion about a point. Such a body has three degrees of freedom, and for a description of its motion three angles, $\psi = \psi(t)$, $\theta = \theta(t)$, and $\phi = \phi(t)$, can be used. With the fixed point, also called

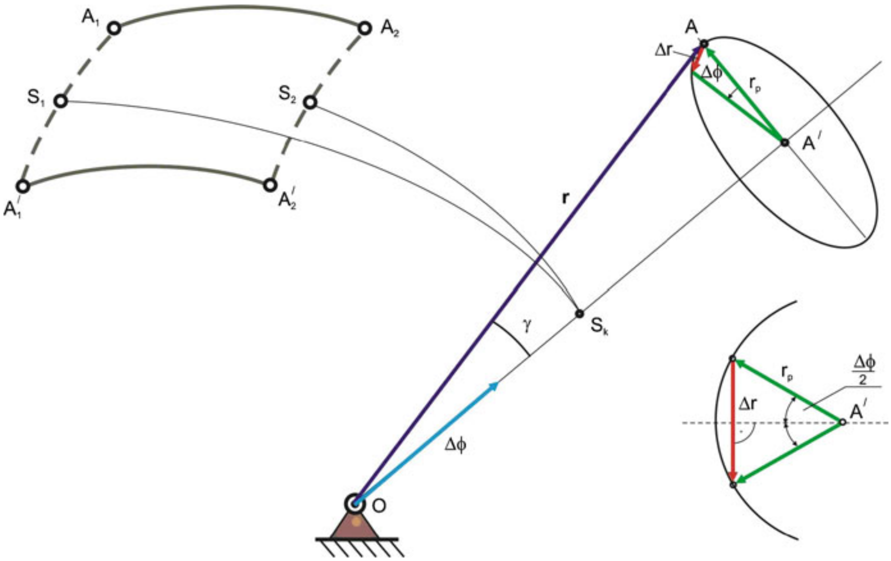


Fig. 5.61 Construction of axis of rotation of a rigid body in motion about a point ($A_1A_2 = A_1'A_2'$)

the *center of rotation of motion about point* O, we will associate the origins of the stationary and non-stationary Cartesian coordinate systems. Point A is situated at a distance $|\overrightarrow{OA}|$ from the center of rotation, which is equal to the radius of a sphere of center at O. The paths of point A are closed or open curves.

On the other hand, the position of a rigid body in space can be uniquely described after choosing its three non-collinear points. If we are dealing with the motion of a rigid body with one point fixed, then the motion of this body can be described after choosing its two points provided they do not lie on a common line with the fixed point O.

In technology as well as in everyday life we encounter many examples of motion about a point. A spinning top (until recently a popular children's toy) is in motion about a point; it rotates about its own symmetry axis and simultaneously performs rotation about other axes. A spherical pendulum, consisting of, e.g., a point mass attached to a weightless rod that is fixed by a ball-and-socket joint at the opposite end, also performs motion about a point. A situation occurs in the case of motion of a differential pinion gear rolling on a stationary differential side gear [1]. The intersection of a sphere of radius OA with the analyzed rigid body determines a certain region of the spherical surface. The motion of this region on the sphere uniquely describes the body motion about a point.

As in the case of planar motion, which we were able to reduce to simply rotational motion about the center of rotation, so in the case of motion about a point the arbitrary displacement of a rigid body can be realized through rotation about a certain axis passing through the fixed point of body O. Figure 5.61 shows how to determine the aforementioned axis.

Let segment A_1A_2, lying on a sphere of radius OA_1 at time instant t_0, reach position $A_1'A_2'$ at time instant t_1. In order to determine the second point belonging to the considered axis of rotation we determine the centers of arcs A_1A_1' and A_2A_2', and then through these centers S_1 and S_2 we draw perpendicular arcs that belong to large circles of the spheres, which will intersect at the desired point S_k. It can be demonstrated that the triangle $A_1S_kA_2$ will coincide with $A_1'S_kA_2'$ after rotation about the OS_k axis through a certain angle ϕ. The OS_k axis will be called an *instantaneous axis of rotation* of a rigid body for $\Delta t \to 0$. If we have determined the instantaneous axis of rotation, then we can easily find the displacement of an arbitrary point A. Drawing through that point a line perpendicular to the axis we will obtain point A' on the axis of rotation. Therefore, point A during the motion about a point of a rigid body will move along a circle of radius $|\mathbf{r}_p|$. A small displacement $\Delta\mathbf{r}$ of point A is equal to (Fig. 5.61)

$$\Delta r = 2r_p \sin\frac{\Delta\phi}{2} \approx r_p\Delta\phi = r\Delta\phi\sin\gamma. \tag{5.247}$$

Let us note that

$$\Delta\mathbf{r} = \mathbf{r}\times\Delta\boldsymbol{\phi}. \tag{5.248}$$

We will show that successive small rotations of a rigid body can be replaced with one resultant small rotation of this body. In our calculations we will limit ourselves to two rotations.

After the first rotation point A will reach the position described by the radius vector

$$\mathbf{r}_1 = \mathbf{r} + \Delta\mathbf{r} = \mathbf{r} + (\mathbf{r}\times\Delta\boldsymbol{\phi}_1). \tag{5.249}$$

After the second rotation point A will reach the position described by the radius vector

$$\begin{aligned}\mathbf{r}_2 &= \mathbf{r}_1 + (\mathbf{r}\times\Delta\boldsymbol{\phi}_2) = \mathbf{r} + (\mathbf{r}\times\Delta\boldsymbol{\phi}_1) + (\mathbf{r}\times\Delta\phi_2)\\ &= \mathbf{r} + \mathbf{r}\times(\Delta\boldsymbol{\phi}_1 + \Delta\boldsymbol{\phi}_2) = \mathbf{r} + (\mathbf{r}\times\Delta\boldsymbol{\phi}),\end{aligned} \tag{5.250}$$

where

$$\Delta\boldsymbol{\phi} = \Delta\boldsymbol{\phi}_1 + \Delta\boldsymbol{\phi}_2. \tag{5.251}$$

The limit to which the ratio $\Delta\phi/\Delta t$ for $\Delta t \to 0$ tends is called the angular velocity of a rigid body at time t and is equal to

$$\omega = \lim_{\Delta\to 0}\frac{\Delta\phi}{\mathrm{d}t}. \tag{5.252}$$

In turn, the position of the axis of rotation OA' for $\Delta t \to 0$ is called the instantaneous axis of rotation of a rigid body corresponding to time t. The sense of $\boldsymbol{\omega}$ is such that when we look along the instantaneous axis of rotation, the arrow points toward the eye of the viewer if the body rotation is counterclockwise (Fig. 5.62).

The instantaneous axis of rotation is the set of points that, at a given time instant t, have a velocity equal to zero.

Fig. 5.62 Instantaneous axis of rotation **l** and vector of angular velocity $\boldsymbol{\omega}$

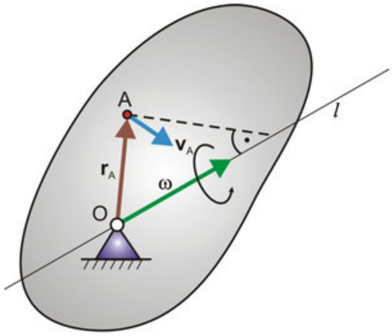
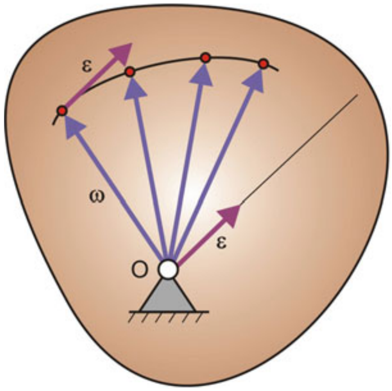


Fig. 5.63 Hodograph of angular velocity and axis of angular acceleration



During motion about a point of a rigid body the position of the instantaneous axis of rotation undergoes a change, so both the magnitude and direction of vector $\boldsymbol{\omega}$ change. If $\omega_1 = \omega_1(t + \Delta t)$ and $\omega = \omega(t)$, then it is possible to describe the average angular acceleration

$$\varepsilon_{\mathrm{av}} = \frac{\omega_1 - \omega}{\Delta t} = \frac{\Delta\omega}{\Delta t}, \tag{5.253}$$

and then the angular acceleration vector

$$\boldsymbol{\varepsilon} = \lim_{\Delta t \to 0} = \frac{\Delta\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}. \tag{5.254}$$

If at a fixed point we successively lay off vectors $\boldsymbol{\omega}$ corresponding to consecutive time instants, then the curve connecting their tips will be called a *hodograph of angular velocity vector* (Fig. 5.63).

The vector of angular acceleration is at any time instant tangent to the hodograph of the angular velocity, that is, it is equal to the translational velocity of the tip of the vector of angular velocity. We move this vector to the fixed point O, and the line passing through point O on which lies vector $\boldsymbol{\varepsilon}$ is called the *axis of angular acceleration*.

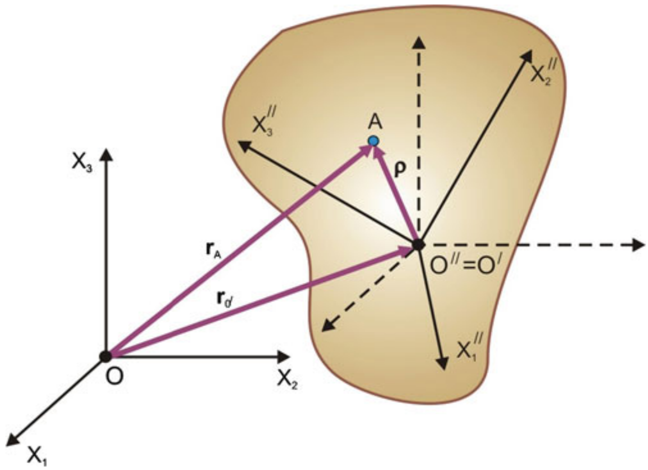


Fig. 5.64 Coordinate systems of space and body, and vectors describing position of point A of a rigid body

5.5.7 *Displacement and Rotation of a Rigid Body (Basic Theorems)*

According to the convention adopted earlier, we will introduce the principal absolute Cartesian coordinate system $OX_1X_2X_3$. We will connect the body rigidly to the coordinate system $OX_1''X_2''X_3''$, where O'' is an arbitrary point of a rigid body. The system $OX_1X_2X_3$ is sometimes called a *space system*, and the system $OX_1''X_2''X_3''$ a *body system*. The motion of the rigid body is represented by the motion of the coordinate system $OX_1''X_2''X_3''$.

The rigid body in R^3 space, generally, can be in translation and rotation. If in a sense we hold the body still at an arbitrary time instant t (we take a "photo") and we have the position of the body given at the initial time instant t_0 (at t_0 the pole $O = O'$), then the body position at time t can be obtained by translation of the coordinate system in parallel to the axes $OX_1X_2X_3$ (dashed axes in Fig. 5.64) up to point O', and then in the general case performing three Eulerian elementary rotations (or only one after determining the axis of rotation $\boldsymbol{l}$ and the angle of composite rotation $\boldsymbol{\varphi}$). In Fig. 5.64 are shown the body and space systems, and the position of one arbitrary point A of the body is characterized.

Recall that according to the adopted convention the vector $\overrightarrow{O''A} = \boldsymbol{\rho}$ has known coordinates in the system $OX_1''X_2''X_3''$. Recall that vectors are added geometrically. If the vectors possess the matrix representation, that is, scalar notation (in terms of coordinates), then the vectors from Fig. 5.64 should be expressed in the same coordinate system, i.e., in either $OX_1X_2X_3$ or $OX_1''X_2''X_3''$. Further we will describe the position of point A in absolute coordinates, and because vector $\boldsymbol{\rho}$ is expressed in the coordinates $OX_1''X_2''X_3''$, it should be represented in the coordinates $OX_1X_2X_3$

as $\mathbf{r} = \mathbf{A}\boldsymbol{\rho}$, where the matrix $\mathbf{A} = \mathbf{A}(t)$ (different at every time instant of the motion) is the transformation matrix from the system $OX_1''X_2''X_3''$ to $OX_1X_2X_3$ expressed in the coordinates of the system $OX_1X_2X_3$. The position of point A in the absolute coordinates has the form

$$\mathbf{r}_A(t) = \mathbf{r}_{O'}(t) + \mathbf{A}(t)\boldsymbol{\rho}. \tag{5.255}$$

The transformation matrix is an orthogonal matrix, i.e., $\mathbf{A}\mathbf{A}^{-1} = \mathbf{E}$. Its elements are connected with each other through six equations [the sum of squares of elements in any row (column) is equal to one, and the sum of products of the remaining elements in the columns (rows) is equal to zero]. Therefore, it has only three independent elements (Sect. 5.2.2).

In the case of motion about a point, point O' is fixed, that is, vector $\mathbf{r}_{O'} = \text{const}$ (Fig. 5.64). Let us take the system $OX_1X_2X_3$ so that it coincides with $OX_1''X_2''X_3''$ at the instant t_0, which means that $\mathbf{r} = \boldsymbol{\rho}$ ($\mathbf{A} = \mathbf{E}$ at the instant t_0 and $\mathbf{r}_{O'}(t_0) = \mathbf{0}$).

Next, in the motion about a point vector $\overrightarrow{OA}$ will rotate around the point $O = O'$. In the absolute coordinates $OX_1X_2X_3$ the position of vector $\overrightarrow{OA}$ is described by vector $\mathbf{A}(t)\boldsymbol{\rho}$. Matrix $\mathbf{A}(t)$ is at any time instant an orthogonal matrix, i.e., $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{E}$. In the case of the orthogonal matrix $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{E}$ and $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$. Moreover, $\det\mathbf{A}^{\mathrm{T}} = \det\mathbf{A}$.

Thus we have

$$\det(\mathbf{A})\det\left(\mathbf{A}^{\mathrm{T}}\right) = (\det(\mathbf{A}))^2 = \det(\mathbf{E}) = 1.$$

The motion about a point of a rigid body in the absolute system can be described by the orthogonal transformation $\mathbf{A}(t)\boldsymbol{\rho}$, where $\det(\mathbf{A}) = 1$.

Theorem 5.7 (Euler's theorem). *An arbitrary displacement of a rigid body with one point fixed can be realized through rotation about a certain axis passing through that point.*

Proof. (See Markeev [6])

As was demonstrated earlier, at any time instant the matrix $\mathbf{A} = \mathbf{A}(t)$ is an orthogonal matrix. Let the vector $\mathbf{r}_A$ lie on the axis of rotation, i.e., $\mathbf{r}_A = \mathbf{A}\mathbf{r}_A$. Vector $\mathbf{r}_A$ does not change under the action of the matrix $\mathbf{A} = \mathbf{A}(t)$. If we take an arbitrary point A not lying on the axis of rotation, the vector $\mathbf{r}_A(t) = \mathbf{E}_1x_{1A}(t) + \mathbf{E}_2x_{2A}(t) + \mathbf{E}_3x_{3A}(t)$ rotates in the system $OX_1X_2X_3$. However, if point A lies on the axis of rotation, the vector $\mathbf{r}_A$ does not rotate. Further, this means that $(\mathbf{A} - \mathbf{E})\mathbf{r}_A = \mathbf{0}$. That equation possesses the solution if $\det(\mathbf{A} - \sigma\mathbf{E}) \equiv f(\sigma) = 0$, where σ denotes the eigenvalue of the square matrix $\mathbf{A}$ and $f(\sigma)$ is a polynomial of the order $n = 3$ in σ. For $\sigma = 1$ we have $\mathbf{r}_A = \mathbf{A}\mathbf{r}_A$, and eigenvector $\mathbf{r}_A$ corresponds to the eigenvalue $\sigma = 1$ of matrix $\mathbf{A}$. The proof boils down to a demonstration that matrix $\mathbf{A}$ possesses the eigenvalue $\sigma = 1$. □

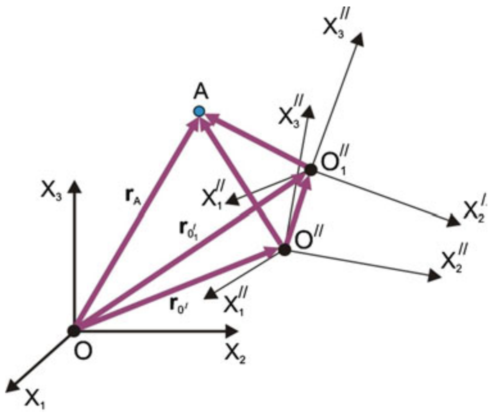


Fig. 5.65 Sketch used during proof of Theorem 5.8

We have

$$\begin{aligned}
f(1) &= \det(\mathbf{A}-\mathbf{E}) = \det(\mathbf{A}^{\mathrm{T}}-\mathbf{E}^{\mathrm{T}}) = \det(\mathbf{A}^{-1}-\mathbf{E}) \\
&= \det\mathbf{A}\det(\mathbf{A}^{-1}-\mathbf{E}) = \det(\mathbf{A}(\mathbf{A}^{-1}-\mathbf{E})) \\
&= \det(\mathbf{E}-\mathbf{A}) = \det(-(\mathbf{A}-\mathbf{E})) \\
&= \det(-\mathbf{E}(\mathbf{A}-\mathbf{E})) = \det(-\mathbf{E})\det(\mathbf{A}-\mathbf{E}) \\
&= (-1)^3\det(\mathbf{A}-\mathbf{E}) = -f(1),
\end{aligned}$$

where the properties $\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$, $\det\mathbf{A} = 1$, were used successively.

Note that $f(1) = -f(1)$ only if $f(1) = 0$, which means that $\sigma = 1$ is the eigenvalue of matrix $\mathbf{A}$ because $f(\sigma) = (\sigma - 1)P_{n-1}(\sigma)$, where $P_{n-1}(\sigma)$ is the polynomial of the order $n-1$ in σ.

Theorem 5.8. *The most complex displacement of a rigid body can be decomposed into translation (an arbitrary pole is subjected to translation from its initial to final position) and rotation about a certain axis passing through the pole. The direction and length of translation change depend on the choice of the pole, but the direction of the rotation axis and the rotation angle are independent of the choice of pole.*

Proof. (See Markeev [6])

In Fig. 5.65 the absolute coordinate system $OX_1X_2X_3$, along with two other coordinate systems $O'X_1''X_2''X_3''$ and $O_1''X_1''X_2''X_3''$ with origins at two different poles O'' and O_1'', is presented.

The system $\mathrm{O}'X_1'X_2'X_3'$ is obtained after the translation of the $\mathrm{O}X_1X_2X_3$ by vector $\mathbf{r}_{O'}$, and the system $\mathrm{O}_1'X_1'X_2'X_3'$ is obtained after the translation of $\mathrm{O}X_1X_2X_3$ by vector $\mathbf{r}_{O_1'}$ [the systems denoted by (') are not shown]. Vectors $\mathbf{r}_{O'}$, $\mathbf{r}_{O_1'}$, and $\mathbf{r}_A$ (radius vector of point A) have their coordinates in the absolute system. The three remaining vectors drawn in the figure have coordinates in the coordinate systems rigidly connected to the body of origins at points O'' and O_1''. □

According to Fig. 5.65 we have

$$\mathbf{r}_A = \mathbf{r}_{O'} + \mathbf{A}\overrightarrow{O''_A A} = \mathbf{r}_{O'} + \mathbf{A}\left(\overrightarrow{O''O''_1} + \overrightarrow{O'_A A}\right)$$
$$= \mathbf{r}_{O'} + \mathbf{A}\overrightarrow{O''O''_1} + \mathbf{A}\overrightarrow{O''_1 A} = \mathbf{r}_{O'_1} + \mathbf{A}\overrightarrow{O''_1 A},$$

where the vectors $\overrightarrow{O''O''_1}$ and $\overrightarrow{O''_1 A}$ are given in the coordinates of $O''X''_1X''_2X''_3$ and $O''_1X''_1X''_2X''_3$ and $\mathbf{A}$ is (as previously) the transformation matrix from systems $OX'_1X'_2X'_3$ and $O'_1X'_1X'_2X'_3$ to systems $O''X''_1X''_2X''_3$ and $O''_1X''_1X''_2X''_3$ (the same rotation), which completes the proof.

From the preceding equality it can be seen that the introduction of different poles O' and O'_1 requires different translations $\mathbf{r}_{O'}$ and $\mathbf{r}_{O'_1} = \mathbf{r}_{O'} + \mathbf{A}\overrightarrow{O''O''_1}$. The position of the rotation axis and the value of the rotation angle are prescribed by matrix $\mathbf{A}$, which according to Euler's theorem does not depend on the choice of pole.

It is worth emphasizing that in this case the final position of the rigid body does not depend on the order in which translation and rotation are carried out.

Theorem 5.9. *The most general displacement of a rigid body is a screw displacement.*

The proof of this theorem can be found in [6] and is omitted here.

From that theorem follows the next one.

Theorem 5.10. *The most general displacement of a plane figure in its plane can be realized through translation or rotation about a certain point (the center of rotation).*

5.5.8 Geometric Interpretation of General Motion of a Rigid Body

For the purpose of observing the positions of a rigid body in space it is sufficient to choose its arbitrary three points A, B, and C and connect them to obtain a triangle. The problem of motion of a rigid body in space reduces to the analysis of motion of $\triangle ABC$ in space.

In Fig. 5.66 is shown the position of a rigid body in space at time instant t represented by $\triangle ABC$ and the position of the body at time instant t_1 represented by $\triangle A_1B_1C_1$.

At first we perform translation of $\triangle ABC$ such that $A = A_1$ and obtain $\triangle A'_1B'_1C'_1 \parallel \triangle ABC$. Next, according to Euler's (rotation) theorem we find axis l passing through A_1 and perform rotation with respect to this axis through angle φ_1, keeping point A_1 fixed. Thus, it has been demonstrated that $\triangle ABC$ became coincident with $\triangle A'_1B'_1C'_1$ at two independent displacements, that is, the translation of pole A and subsequent rotation about the pole with respect to a certain axis l_1 through angle φ_1.

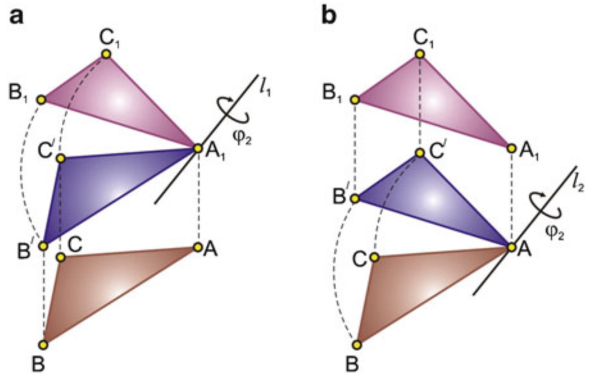


Fig. 5.66 Displacement of triangle ABC to the position $A_1B_1C_1$ through translation and rotation (**a**) and through rotation and translation (**b**)

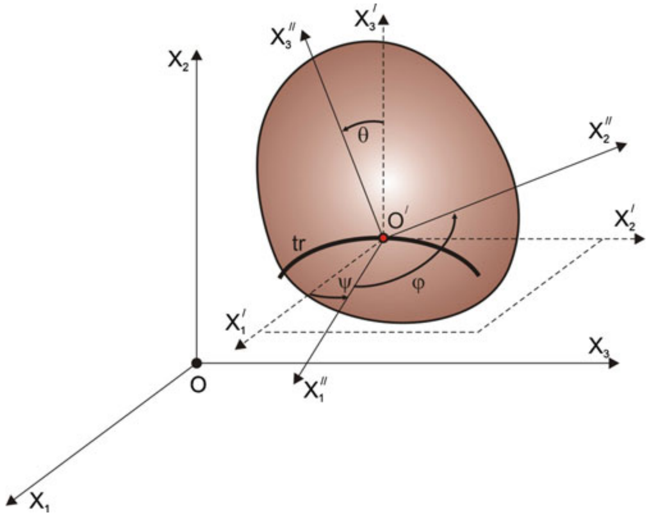


Fig. 5.67 Motion of a rigid body as composition of motion of a pole and spherical motion about that pole

We will show that the same result can be obtained by the exchange of the sequence of displacements. First we perform the translation of $\triangle A_1B_1C_1$, taking point O_1 as a pole, and then we perform rotation through angle φ_2 with respect to a certain axis l_2 ($l_1 = l_2$, $\varphi_1 = \varphi_2$).

In reality, the motion of a rigid body takes place in such a way that it is possible to treat it as a composition of two simultaneous motions: the motion of its arbitrary pole and the simultaneous spherical motion (or motion about a point) with respect to that pole, which is shown in Fig. 5.67.

The arbitrarily chosen point O' (the pole) moves on trajectory tr, and its motion in the stationary system is described by three equations:

$$x_{1O'} = x_{1O'}(t), \qquad x_{2O'} = x_{2O'}(t), \qquad x_{3O'} = x_{3O'}(t). \tag{*}$$

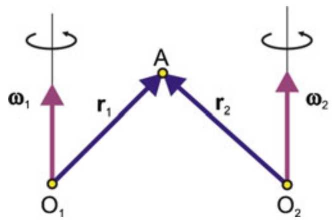


Fig. 5.68 Angular velocities with respect to two poles O_1 and O_2 and the position of point A

The coordinate system $O'X_1'X_2'X_3'$ has during motion axes that are mutually parallel to the axes of the stationary coordinate system and moves in translational motion together with the pole O'. The position of a rigid body with respect to the axes $O'X_1'X_2'X_3'$ is described by three Euler angles, $\psi(t)$, $\varphi(t)$, and $\theta(t)$. First three equations describe the motion of the pole and depend on the choice of pole; the three remaining equations describe the motion about a point of a rigid body with respect to the pole and do not depend on the choice of pole.

The description of this motion is associated with the choice of six independent coordinates, and therefore a free rigid body has six degrees of freedom.

Theorem 5.11. *The vector of angular velocity* $\boldsymbol{\omega}$ *and the vector of angular acceleration* $\boldsymbol{\varepsilon}$ *of a rigid body are independent of the choice of pole.*

Proof. Let us choose two poles O_1 and O_2 of a rigid body (Fig. 5.68).

Let us choose an arbitrary point of body A and determine its position with respect to pole O_1 (O_2) by $\mathbf{r}_1$ ($\mathbf{r}_2$). Velocity of this point is equal to

$$\mathbf{v}_A = \mathbf{v}_{O_1} + \boldsymbol{\omega}_1 \times \mathbf{r}_1,$$
$$\mathbf{v}_A = \mathbf{v}_{O_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_2. \tag{5.256}$$

□

Then,

$$\mathbf{v}_{O_2} = \mathbf{v}_{O_1} + \boldsymbol{\omega}_1 \times \overrightarrow{O_1O_2}, \tag{5.257}$$

and substituting into (5.256) we have

$$\mathbf{v}_{O_1} + \boldsymbol{\omega}_1 \times \mathbf{r}_1 = \mathbf{v}_{O_1} + \boldsymbol{\omega}_1 \times \overrightarrow{O_1O_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_2, \tag{5.258}$$

that is,

$$\boldsymbol{\omega}_1 \times (\mathbf{r}_1 - \overrightarrow{O_1O_2}) = \boldsymbol{\omega}_2 \times \mathbf{r}_2, \tag{5.259}$$

which can be rewritten as

$$(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \times \mathbf{r}_2 = \mathbf{0}, \tag{5.260}$$

which means that $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$.

Differentiating (5.260) we obtain

$$\boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2, \tag{5.261}$$

which completes the proof.

The conclusion follows that at the given time instant the instantaneous axis of rotation is described in the same way for each point of the body. Distinct poles differ only as to the magnitude of velocity of the translational motion of the pole.

5.5.9 Parallel Translation and Rotation of a Rigid Body and Homogeneous Transformations

Let us return to the motion of a rigid body whose state at any time instant can be obtained as a result of parallel translation of the absolute system $OX_1X_2X_3$ by vector $\mathbf{r}_{O'}$ and then the rotation realized by means of rotation matrix $\mathbf{A}$ to the position of the body described in the $O''X_1''X_2''X_3''$ system. The position of point A is described by (5.255).

The notation in the form of vector (5.255), the matrix (scalar) representation of the vector equation in the absolute system, is sometimes inconvenient to use, and the equivalent matrix notation of the following form is introduced:

$$\mathbf{r}_j = \mathbf{A}_j \boldsymbol{\rho}_j, \tag{5.262}$$

where $\mathbf{A}_j$ is a homogeneous matrix.

Let us note that the newly introduced homogeneous transformation matrix

$$\mathbf{A}_{j4\times 4} = \begin{bmatrix} \mathbf{A} & \mathbf{r}_{O'} \\ \mathbf{0} & 1 \end{bmatrix} \tag{5.263}$$

consists of the following block matrices $\mathbf{A}_{3\times 3}$ (where $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{E}_{3\times 3}$), $\det(\mathbf{A}) = 1, \mathbf{0} = [0, 0, 0]$. Moreover

$$\mathbf{r}_j = \begin{bmatrix} \mathbf{r}_A \\ 1 \end{bmatrix}_{4\times 1}, \quad \boldsymbol{\rho}_j = \begin{bmatrix} \boldsymbol{\rho} \\ 1 \end{bmatrix}_{4\times 1}. \tag{5.264}$$

In other words, in (5.263) matrix $\mathbf{A}$ is the matrix of transformation through rotation from (5.255).

The special cases of that matrix are the following matrices:

1. Homogeneous transformation matrix of rotation

$$\mathbf{ROT}(\boldsymbol{l}, \varphi) = \begin{bmatrix} \boldsymbol{\varphi}(\boldsymbol{l}) & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{5.265}$$

2. Homogeneous transformation matrix of translation

$$\mathbf{TRANS}(\mathbf{r}_{O'}) = \begin{bmatrix} \mathbf{E} & \mathbf{r}_{O'} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{5.266}$$

Let us calculate the inverse of matrix (5.263). According to the definition we have

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{r}_{O'} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & 1 \end{bmatrix}, \tag{5.267}$$

and after multiplication we obtain

$$\mathbf{A}_j^{-1} = \begin{bmatrix} \mathbf{A}^{\mathrm{T}} & -\mathbf{A}^{\mathrm{T}}\mathbf{r}_{O'} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{5.268}$$

5.5.10 Kinematic States of a Rigid Body

In general, during the motion of a rigid body the velocities and accelerations of its points change and are usually different for different points.

If at a certain time instant the velocities of all points of the rigid body are the same and equal to $\mathbf{v}$, we say that the rigid body is in *instantaneous translational motion* with velocity $\mathbf{v}$ (if $\mathbf{v} = 0$, then the body is instantaneously at rest). During instantaneous rest the accelerations of the points of the body do not have to be equal to zero.

If at a certain time instant there exist points lying on the line of zero velocities, we say that the rigid body is in *instantaneous rotation* about that line, and that line is called the *instantaneous axis of rotation*.

In general, the instantaneous axis of rotation at different time instants can assume various positions in the absolute coordinate system as well as in the body system.

A body at a given time instant may participate simultaneously in two instantaneous motions, i.e., in translational motion with respect to an axis and in rotational motion about that axis. In that case we say that the body performs *instantaneous screw motion*.

5.5.11 Velocity and Acceleration in Translational Motion

The object of interest of our previous discussions was the displacements of a rigid body from its initial to final position without taking into account the velocities and accelerations. In other words, the action of the researcher (subject) takes the rigid body to the final state. In reality the object under investigation (the rigid body) passes from the initial state to the final state in time, and in this case all the points of the body possess velocities and accelerations, which usually undergo change at every time instant.

We will call the *motion of a rigid body* in a given time interval (t_1, t_2) *translational motion* if for any time instants t_1' and t_2' the body, taken from this time interval, can be guided from state t_1' to t_2' by means of translational displacement.

An elevator (rectilinear trajectory) or a car in an amusement park (the trajectory of motion is a circle) connected to a uniformly rotating "big wheel" moves in translational motion.

If during the translation of a rigid body its two arbitrary points A_1 and A_2 have equal free vectors of displacement $\Delta\mathbf{r}_{A_1}$ and $\Delta\mathbf{r}_{A_2}$, then during translational motion all points of the rigid body have identical velocities and accelerations. Therefore, the use of the notions of velocity and acceleration of translational motion of a body is justified. Those notions have no meaning in the case of other motions of a rigid body because then the body's points have different velocities and accelerations.

5.5.12 Velocity and Acceleration in Motion About a Point

Let us now move to determining the velocity and acceleration in motion about a point, where we assume that one point of a rigid body is fixed. In Fig. 5.53 let this point be point A_2, i.e., $\mathbf{r}_2 = \mathbf{0}$ and $A_2 = 0$, which means that the origin of the space coordinate system is situated at point A_2 associated with the rigid body (also, the non-stationary coordinate system is rigidly connected to the body), which is shown in Fig. 5.64. Vector $\boldsymbol{\omega}$ lies on the instantaneous axis of rotation, and the position of an arbitrary point A of a rigid body in a stationary (non-stationary) system is described by the position vector $\mathbf{r} = \mathbf{r}'$.

The velocity of point A is obtained from (5.173) for $\mathbf{r}_2 = \mathbf{0}$. In the stationary and non-stationary systems it is respectively equal to

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r}, \\ \mathbf{v}' &= \boldsymbol{\omega}' \times \mathbf{r}'. \end{aligned} \tag{5.269}$$

Let us assume that the angular velocity vector possesses the following components in the stationary and non-stationary systems:

$$\boldsymbol{\omega} = \omega_1\mathbf{E}_1 + \omega_2\mathbf{E}_2 + \omega_3\mathbf{E}_3 = \omega_1'\mathbf{E}_1' + \omega_2'\mathbf{E}_2' + \omega_3'\mathbf{E}_3'. \tag{5.270}$$

Similarly, the radius vector of point A

$$\mathbf{r} = r_1\mathbf{E}_1 + r_2\mathbf{E}_2 + r_3\mathbf{E}_3 = r_1'\mathbf{E}_1' + r_2'\mathbf{E}_2' + r_3'\mathbf{E}_3', \tag{5.271}$$

where $\mathbf{E}_i' = \mathbf{E}_i'(t)$ because for every time instant during the motion of the body the position of these unit vectors changes.

The scalar quantities r_i' are constant, and the magnitude of vector $|\mathbf{r}|$ remains unchanged, but its direction changes in time, which is expressed by the relationship $\mathbf{E}_i' = \mathbf{E}_i'(t)$. Although both vectors describing the position of point A are equal $\left(\mathbf{r} = \mathbf{r}'\right)$, in general $r_1 \neq r_1'$, $r_2 \neq r_2'$, and $r_3 \neq r_3'$ because we have different bases $\mathbf{E}_i$ and $\mathbf{E}_i'$ $(i = 1, 2, 3)$, and we pass from the system $OX_1X_2X_3$ to $O'X_1'X_2'X_3'$ using the rotation matrix.

The velocity of point A in the coordinate system $\{\mathbf{E}'_1, \mathbf{E}'_2, \mathbf{E}'_3\}$ is equal to

$$\mathbf{v}' = \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t} = r'_1\dot{\mathbf{E}}_1 + r'_2\dot{\mathbf{E}}_2 + r'_3\dot{\mathbf{E}}_3. \tag{5.272}$$

Then

$$\dot{\mathbf{E}}'_i = \boldsymbol{\omega} \times \mathbf{E}'_i, \tag{5.273}$$

and in view of that we obtain

$$\begin{aligned}
\mathbf{E}'_2 \circ \dot{\mathbf{E}}'_1 &= \mathbf{E}'_2 \circ \left(\boldsymbol{\omega} \times \mathbf{E}'_1\right) = \boldsymbol{\omega} \circ \mathbf{E}'_1 \times \mathbf{E}'_2 = \boldsymbol{\omega} \cdot \mathbf{E}'_3 = \omega'_3,\\
\mathbf{E}'_3 \circ \dot{\mathbf{E}}'_2 &= \mathbf{E}'_3 \circ \left(\boldsymbol{\omega} \times \mathbf{E}'_2\right) = \boldsymbol{\omega} \circ \mathbf{E}'_2 \times \mathbf{E}'_3 = \boldsymbol{\omega} \cdot \mathbf{E}'_1 = \omega'_1,\\
\mathbf{E}'_1 \circ \dot{\mathbf{E}}'_3 &= \mathbf{E}'_1 \circ \left(\boldsymbol{\omega} \times \mathbf{E}'_3\right) = \boldsymbol{\omega} \circ \mathbf{E}'_3 \times \mathbf{E}'_1 = \boldsymbol{\omega} \cdot \mathbf{E}'_2 = \omega'_2,
\end{aligned} \tag{5.274}$$

where the so-called *cyclic permutation of factors* was applied, i.e., $\boldsymbol{a} \circ \mathbf{b} \times \mathbf{c} = \mathbf{b} \circ \mathbf{c} \times \boldsymbol{a} = \mathbf{c} \circ \boldsymbol{a} \times \mathbf{b}$.

From (5.273) we obtain

$$\dot{\mathbf{E}}'_1 = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ 1 & 0 & 0 \end{vmatrix}, \quad \dot{\mathbf{E}}'_2 = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ 0 & 1 & 0 \end{vmatrix}, \quad \dot{\mathbf{E}}'_3 = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ 0 & 0 & 1 \end{vmatrix}, \tag{5.275}$$

that is,

$$\begin{aligned}
\dot{\mathbf{E}}'_1 &= \mathbf{E}'_2\omega'_3 - \mathbf{E}'_3\omega'_2,\\
\dot{\mathbf{E}}'_2 &= -\mathbf{E}'_1\omega'_3 + \mathbf{E}'_3\omega'_1,\\
\dot{\mathbf{E}}'_3 &= \mathbf{E}'_1\omega'_2 - \mathbf{E}'_2\omega'_1.
\end{aligned} \tag{5.276}$$

Substituting (5.276) into (5.272) we obtain

$$\begin{aligned}
\mathbf{v}' = {} & r'_1\left(\mathbf{E}'_2\omega'_3 - \mathbf{E}'_3\omega'_2\right) + r'_2\left(-\mathbf{E}'_1\omega'_3 + \mathbf{E}'_3\omega'_1\right)\\
& + r'_3\left(\mathbf{E}_1\omega'_2 - \mathbf{E}'_2\omega'_1\right) = \mathbf{E}'_1\left(-r'_2\omega'_3 + r'_3\omega'_2\right)\\
& + \mathbf{E}'_2\left(r'_1\omega'_3 - r'_3\omega'_1\right) + \mathbf{E}'_3\left(-r'_1\omega'_2 + r'_2\omega_1\right).
\end{aligned} \tag{5.277}$$

Then, according to (5.269), we have

$$\mathbf{v}' = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ r'_1 & r'_2 & r'_3 \end{vmatrix} = \mathbf{E}'_1\left(\omega'_2 r'_3 - \omega'_3 r'_2\right) + \mathbf{E}'_2\left(r'_1\omega'_3 - \omega'_1 r'_3\right) + \mathbf{E}'_3\left(\omega'_1 r'_2 - r'_1\omega'_2\right). \tag{5.278}$$

It can be seen that (5.277) and (5.278) are the same, which means that vector $\boldsymbol{\omega}$ in the non-stationary system has coordinates ω'_1, ω'_2, and ω'_3.

In the case of the stationary system according to (5.271) we have

$$\dot{\mathbf{r}} \equiv \mathbf{v} = \dot{r}_1\mathbf{E}_1 + \dot{r}_2\mathbf{E}_2 + \dot{r}_3\mathbf{E}_3. \tag{5.279}$$

According to (5.269) we have

$$\mathbf{v} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \mathbf{E}_1\left(\omega_2 r_3 - \omega_3 r_2\right) + \mathbf{E}_2\left(r_1\omega_3 - \omega_1 r_3\right) + \mathbf{E}_3\left(\omega_1 r_2 - \omega_2 r_1\right). \tag{5.280}$$

Comparing (5.279) with (5.280) we obtain

$$\begin{aligned} v_1 &\equiv \dot{r}_1 = \omega_2 r_3 - \omega_3 r_2, \\ v_2 &\equiv \dot{r}_2 = \omega_3 r_1 - \omega_1 r_3, \\ v_3 &\equiv \dot{r}_3 = \omega_1 r_2 - \omega_2 r_1, \end{aligned} \tag{5.281}$$

where clearly

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{E}_i, \tag{5.282}$$

and this equation is identical with (5.279).

As was mentioned earlier, for every time instant there exist infinitely many points of a rigid body lying on the instantaneous axis of rotation whose velocities are equal to zero.

If the position of the instantaneous axis of angular velocity and the velocity of an arbitrary point of the body $\mathbf{v}_A$ not lying on this axis are known, then $\omega = v_A/d$, where d is the distance of the point from the axis.

According to (5.209), in the non-stationary system for points lying on the instantaneous rotation axis we obtain

$$\boldsymbol{\omega}' \times \mathbf{r}' = \mathbf{0}, \tag{5.283}$$

which indicates that vectors $\boldsymbol{\omega}'$ and $\mathbf{r}'$ are parallel.

In view of that they can be expressed by the relationship

$$\mathbf{r}' = \sigma'\boldsymbol{\omega}', \tag{5.284}$$

where σ' is a scalar.

From (5.284) it follows that

$$r'_i = \sigma'\omega'_i. \tag{5.285}$$

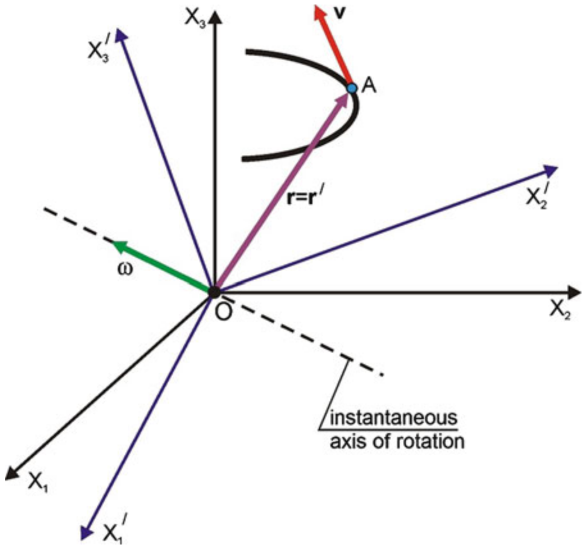


Fig. 5.69 Angular velocity $\boldsymbol{\omega}$ of a body in motion about a point, velocity **v** of an arbitrary point A, and the Cartesian coordinate systems of space $OX_1X_2X_3$ and body $O'X_1'X_2'X_3'$

In turn from (5.278) and (5.285) we obtain

$$\frac{r_1'}{\omega_1'} = \frac{r_3'}{\omega_3'}, \quad \frac{r_2'}{\omega_2'} = \frac{r_3'}{\omega_3'}, \quad \frac{r_1'}{\omega_1'} = \frac{r_2'}{\omega_2'}, \tag{5.286}$$

which implies [see (5.285)]

$$\frac{r_1'}{\omega_1'} = \frac{r_2'}{\omega_2'} = \frac{r_3'}{\omega_3'} = \sigma'. \tag{5.287}$$

Similar calculations conducted for the stationary system lead to the relationship

$$\frac{r_1}{\omega_1} = \frac{r_2}{\omega_2} = \frac{r_3}{\omega_3} = \sigma, \tag{5.288}$$

where σ is a parameter (scalar) assuming real values.

Equations (5.287) describe the instantaneous rotation axis in a non-stationary coordinate system, whereas (5.288) describe it in a stationary coordinate system.

If the vector of angular velocity of a rigid body $\boldsymbol{\omega}$ is known, then according to Fig. 5.69 the velocity of an arbitrary point of the rigid body is described by (5.269).

Let us consider the motion about a point of point A situated at a distance **r** from the origin of the coordinate system, i.e., from the center O of the motion about a point. The path of point A is located on the surface of a ball of radius **r**. The position of the instantaneous axes of rotation in the body and space coordinate systems varies in time, but all the axes must always pass through the center of motion about a point. Instantaneous axes of rotation intersect a sphere of radius **r** at certain points. Sets of these points in the body and space coordinate systems

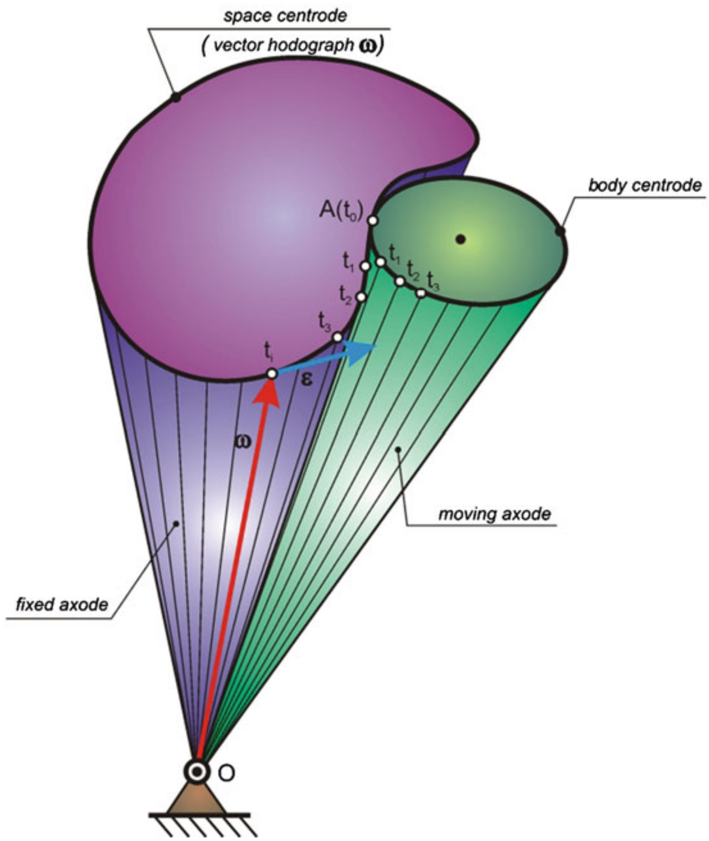


Fig. 5.70 Axodes and centrodes associated with motion of point A

constitute the trajectories of motion of point A respectively in the body and space coordinate systems. Point A belongs simultaneously to both trajectories at the given time instant. The lines passing through points A and O at the time instants $t_0, t_1, t_2, \ldots$ form the surface called the *stationary cone of instantaneous axes (fixed axode)* in the space coordinate system and the surface called the *moving cone of instantaneous axes (moving axode)* in the body coordinate system (Fig. 5.70).

The path of point A lies on a sphere and is described by the curve called the *body centrode (non-stationary)* in the body coordinate system (non-stationary). These curves are in contact at point A since it belongs simultaneously to both of them. The motion about a point can be illustratively represented as the rolling of a moving axode on a fixed axode. Both axodes have contact along the generating line, which is the instantaneous axis of rotation, and do not slide with respect to one another. The hodograph of vector $\boldsymbol{\omega}$ lies on the fixed axode. Because $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$, the angular acceleration is tangent to the space centrode, and it does not necessarily have to lie on the axis of vector $\boldsymbol{\omega}$. Moving and fixed axodes can be non-closed surfaces.

Analysis of the acceleration of the particle will be performed based on (5.174), from which we obtain

$$\mathbf{a} = \mathbf{a}_O + \mathbf{a}_c, \tag{5.289}$$

where

$$\mathbf{a}_O = \dot{\boldsymbol{\omega}} \times \mathbf{r} = \boldsymbol{\varepsilon} \times \mathbf{r}, \tag{5.290}$$

$$\mathbf{a}_c = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \tag{5.291}$$

and it was assumed that $\mathbf{r} = \mathbf{r}_1$ and $\mathbf{r}_2 = \mathbf{0}$.

In the preceding formulas, $\mathbf{a}_O$ is the component of acceleration of the body point in motion about a point called the *rotational acceleration*. Vector $\mathbf{a}_d$ represents the *centripetal acceleration*, also called the *axial acceleration*, directed toward the instantaneous axis of rotation.

Let us resolve the angular acceleration $\boldsymbol{\varepsilon}$ (Fig. 5.70) into two components, $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. Because $\boldsymbol{\omega} = \widehat{\boldsymbol{\omega}}\omega$, we have $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \dot{\omega}\widehat{\boldsymbol{\omega}} + \dot{\widehat{\boldsymbol{\omega}}}\omega = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2$. Vector $\boldsymbol{\varepsilon}_1$ characterizes the change in the magnitude of angular velocity and vector $\boldsymbol{\varepsilon}_2$ the change in the direction of angular velocity. Moreover, vector $\boldsymbol{\varepsilon}_1$ is directed along the instantaneous axis of rotation, whereas vector $\boldsymbol{\varepsilon}_2$ is perpendicular to it. If the instantaneous axis of rotation of the body in motion about a point rotates about point O with velocity $\boldsymbol{\Omega}$, then $\boldsymbol{\varepsilon}_2 = \boldsymbol{\Omega} \times \boldsymbol{\omega}$.

Recall that for three arbitrary vectors the following relation holds true: $\mathbf{a}_1 \times (\mathbf{a}_2 \times \mathbf{a}_3) = \mathbf{a}_2(\mathbf{a}_1 \circ \mathbf{a}_3) - \mathbf{a}_3(\mathbf{a}_1 \circ \mathbf{a}_2)$. Applying the preceding formula to (5.291) we obtain

$$\mathbf{a}_c = \boldsymbol{\omega}(\boldsymbol{\omega} \circ \mathbf{r}) - \omega^2\mathbf{r}. \tag{5.292}$$

Bearing in mind that we associated the instantaneous axis of rotation with unit vector $\widehat{\boldsymbol{\omega}}$, that is, $\boldsymbol{\omega} = \widehat{\boldsymbol{\omega}}\omega$, from (5.292) we obtain

$$\mathbf{a}_c = \omega^2\mathbf{d}, \tag{5.293}$$

where

$$\mathbf{d} = \widehat{\boldsymbol{\omega}}(\widehat{\boldsymbol{\omega}} \cdot \mathbf{r}) - \mathbf{r}. \tag{5.294}$$

Vector $\mathbf{d} \perp \boldsymbol{\omega}$ and is directed toward the instantaneous axis of rotation (Fig. 5.71). In turn, $\widehat{\boldsymbol{\omega}} \circ \mathbf{r}$ is the projection of vector $\mathbf{r}$ onto the instantaneous axis of rotation. From Fig. 5.71 it can be seen that the rotational acceleration $\boldsymbol{\epsilon}$ is associated with the rotation of vector $\mathbf{r}$ about the instantaneous axis of rotation and $\mathbf{a}_O \perp \mathbf{r}$. The centripetal acceleration is perpendicular to the instantaneous axis of rotation $\boldsymbol{\omega}$. It should be emphasized that the quantity $\boldsymbol{\omega}$ is not a result of differentiation of a certain angle $\alpha(t)$ because there exists no such axis of rotation of a rigid body about which the rotation through angle α would take place.

Finally, let us emphasize a certain analogy between centripetal acceleration and normal acceleration, and between rotational acceleration and tangential acceleration, of a point in curvilinear motion. Only in special cases does rotational acceleration coincide with tangential acceleration, and centripetal acceleration with acceleration normal to the path of a point.

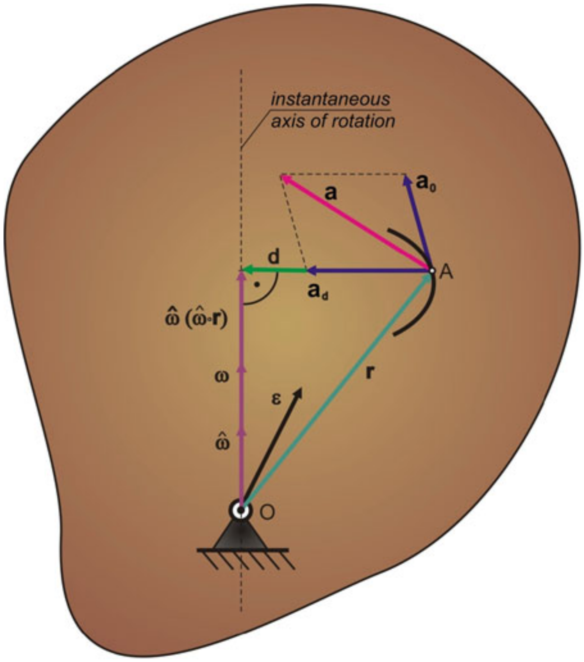


Fig. 5.71 Rotational acceleration $\mathbf{a}_O$, centripetal acceleration $\mathbf{a}_d$, and the resultant acceleration of point A in motion about a point of a rigid body

If the vector of acceleration $\mathbf{a}_A = \mathbf{a}$ is known, then we can project it directly onto the axes of the stationary and non-stationary systems:

$$\mathbf{a} = \sum_{i=1}^{3} \mathrm{a}_i \mathbf{E}_i = \sum_{i=1}^{3} \mathrm{a}'_i \mathbf{E}'_i. \tag{5.295}$$

The components of acceleration in the space (body) system will be obtained through scalar multiplication of (5.295) respectively by $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ ($\mathbf{E}'_1$, $\mathbf{E}'_2$, $\mathbf{E}'_3$). Let us write the result in matrix form:

$$\begin{bmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \\ \mathrm{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mathrm{a}'_1 \\ \mathrm{a}'_2 \\ \mathrm{a}'_3 \end{bmatrix}. \tag{5.296}$$

From (5.296) it follows that if the components of acceleration in the body coordinate system (non-stationary) a'_1, a'_2, a'_3 are known, then through coefficients a_{ij} we can calculate the acceleration components in the space coordinate system a_1, a_2, a_3. According to (5.296) we obtain the relationships between the unit vectors in the stationary and non-stationary coordinate systems of the form

$$\begin{aligned} \mathbf{E}_1 &= a_{11}\mathbf{E}'_1 + a_{12}\mathbf{E}'_2 + a_{13}\mathbf{E}'_3, \\ \mathbf{E}_2 &= a_{21}\mathbf{E}'_1 + a_{22}\mathbf{E}'_2 + a_{23}\mathbf{E}'_3, \\ \mathbf{E}_3 &= a_{31}\mathbf{E}'_1 + a_{32}\mathbf{E}'_2 + a_{33}\mathbf{E}'_3, \end{aligned} \tag{5.297}$$

where a_{ij} denotes the cosines of the angles formed by the axis X_i of the stationary system with the axis X'_j of the non-stationary system. Let us recall that the following relationships are valid for the direction cosines:

$$\begin{aligned}
&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,\\
&a_{21}^2 + a_{22}^2 + a_{23}^2 = 1,\\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1,\\
&a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,\\
&a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,\\
&a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.
\end{aligned} \tag{5.298}$$

Therefore we can choose three arbitrary independent angles between the coordinate axes. The remaining six cosines of the direction angles should be determined from (5.298).

According to (5.290) the rotational acceleration is expressed by

$$\mathbf{a}_O = \begin{vmatrix} \mathbf{E}'_1 & \mathbf{E}'_2 & \mathbf{E}'_3 \\ \dot{\omega}'_1 & \dot{\omega}'_2 & \dot{\omega}'_3 \\ r'_1 & r'_2 & r'_3 \end{vmatrix} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \dot{\omega}_1 & \dot{\omega}_2 & \dot{\omega}_3 \\ r_1 & r_2 & r_3 \end{vmatrix}, \tag{5.299}$$

from which we obtain

$$\begin{aligned}
\mathbf{a}_O &= \mathbf{E}'_1\left(\dot{\omega}'_2 r'_3 - \dot{\omega}'_3 r'_2\right) + \mathbf{E}'_2\left(\dot{\omega}'_3 r'_1 - \dot{\omega}'_1 r'_3\right) + \mathbf{E}'_3\left(\dot{\omega}'_1 r'_2 - \dot{\omega}'_2 r'_1\right)\\
&= \mathbf{E}_1\left(\dot{\omega}_2 r_3 - \dot{\omega}_3 r_2\right) + \mathbf{E}_2\left(\dot{\omega}_3 r_1 - \dot{\omega}_1 r_3\right) + \mathbf{E}_3\left(\dot{\omega}_1 r_2 - \dot{\omega}_2 r_1\right).
\end{aligned} \tag{5.300}$$

We will calculate the centripetal acceleration using (5.292). But first, let us note that

$$\boldsymbol{\omega} \circ \mathbf{r} = \omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3 = \omega'_1 r'_1 + \omega'_2 r'_2 + \omega'_3 r'_3. \tag{5.301}$$

Finally, according to (5.292) we obtain

$$\begin{aligned}
\mathbf{a}_c &= \left[\omega_1\left(\omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3\right) - \omega^2 r_1\right]\mathbf{E}_1\\
&\quad + \left[\omega_2\left(\omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3\right) - \omega^2 r_2\right]\mathbf{E}_2\\
&\quad + \left[\omega_3\left(\omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3\right) - \omega^2 r_3\right]\mathbf{E}_3\\
&= \left[\omega'_1\left(\omega'_1 r'_1 + \omega'_2 r'_2 + \omega'_3 r'_3\right) - \omega^2 r'_1\right]\mathbf{E}'_1\\
&\quad + \left[\omega'_2\left(\omega'_1 r'_1 + \omega'_2 r'_2 + \omega'_3 r'_3\right) - \omega^2 r'_2\right]\mathbf{E}'_2\\
&\quad + \left[\omega'_3\left(\omega'_1 r'_1 + \omega'_2 r'_2 + \omega'_3 r'_3\right) - \omega^2 r'_3\right]\mathbf{E}'_3.
\end{aligned} \tag{5.302}$$

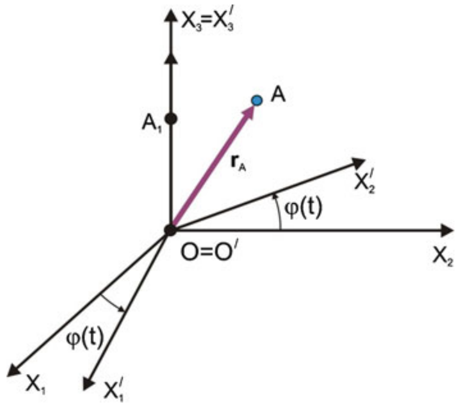


Fig. 5.72 Systems of absolute coordinates $OX_1X_2X_3$ and of coordinates rigidly connected to a body $O'X_1'X_2'X_3'$

Resolving vectors $\mathbf{a}_O$ and $\mathbf{a}_c$ into components in the non-stationary and stationary systems, we will obtain easily the projections of this acceleration in terms of ω_i, r_i and ω as well as ω_i', r_i' and ω.

5.5.13 Velocities and Accelerations in Body Motion About a Fixed Axis

Let us take two fixed points O and A_1 of a rigid body, which determine the rotation axis of this body (Fig. 5.72).

The position of the body with respect to the absolute coordinate system is traced by the angle $\varphi(t)$ between the OX_1 and $O'X_1'$ axes. Any point of the body (let us take point A) not lying on the rotation axis (this axis is collinear with the axes OX_3 and $O'X_3'$) moves along the circle in the plane passing through that point and perpendicular to the rotation axis. Let the radius vector $\boldsymbol{\rho}_A = \overrightarrow{OA}$ have the coordinates given in the body system (i.e., the system rigidly connected to the body).

According to the previous calculations vector $\mathbf{r}_A$ expressed in the coordinates $OX_1X_2X_3$ will be calculated through vector $\overrightarrow{OA}$ expressed in the coordinates $O'X_1'X_2'X_3'$ using the transformation matrix $\mathbf{A}$ according to equation

$$\mathbf{r} \equiv \mathbf{r}_A = \mathbf{A}\boldsymbol{\rho}, \quad \mathbf{A} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.303}$$

Differentiating (5.303) with respect to time we obtain

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \dot{\mathbf{A}}\boldsymbol{\rho} + \mathbf{A}\boldsymbol{\rho} = \dot{\mathbf{A}}\boldsymbol{\rho} = \dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{r} \tag{5.304}$$

because $\dot{\boldsymbol{\rho}} = \mathbf{0}$.

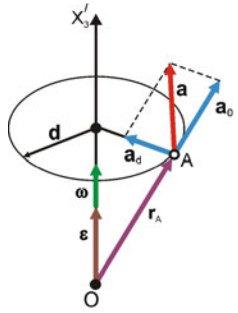


Fig. 5.73 Kinematics of a point in motion of a rigid body about a fixed axis

For the considered case of motion of a rigid body about a fixed axis we obtain

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\varphi} \end{bmatrix}, \quad \dot{\mathbf{A}}\mathbf{A}^{-1} = \begin{bmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.305}$$

The reader is advised to derive the last relationship of (5.305).

From the obtained results it is possible to conclude that vector $\boldsymbol{\omega}$ lies on the axis of rotation ($\boldsymbol{\omega} = \dot{\varphi}\mathbf{E}_3$). The angular acceleration vector also lies on the rotation axis, but its sense is in agreement with the sense of $\boldsymbol{\omega}$ if $\dot{\varphi}\ddot{\varphi} > 0$, and the rotation of a rigid body is accelerated in this case. If $\dot{\varphi}\ddot{\varphi} < 0$, then the rotation is decelerated, and the senses of vectors $\boldsymbol{\omega}$ and **ffl** are opposite.

The velocity of an arbitrary point not lying on the rotation axis

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{5.306}$$

and because generally vector **v** lies in a plane perpendicular to the rotation axis, the magnitude of velocity is $v = |\dot{\varphi}|d$, where d is the radius of the circle along which point A moves (Fig. 5.73).

Differentiating the preceding equation with respect to time we obtain the following acceleration of point A:

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} = \mathbf{a}_O + \mathbf{a}_c, \tag{5.307}$$

where $\mathbf{a}_O = \boldsymbol{\varepsilon} \times \mathbf{r}$ is the rotational acceleration and $\mathbf{a}_c = \boldsymbol{\omega} \times \mathbf{v}$ is the centripetal acceleration.

The magnitudes of these accelerations are equal to

$$|\mathbf{a}_O| = \varepsilon d, \quad |\mathbf{a}_c| = \omega^2 d, \quad |\mathbf{a}| = d\sqrt{\varepsilon^2 + \omega^4}. \tag{5.308}$$

5.5.14 *Velocities of a Point of a Rigid Body in Various Coordinate Systems*

According to the homogeneous transformation (5.262) introduced earlier, the tip of homogeneous position vector $\mathbf{r}_j(t)$ determines the trajectory of motion of point A through the homogeneous transformation matrix of displacement $\mathbf{A}_j(t)$ described by relationship (5.263).

Based on the calculations of Sect. 5.5.9, we can introduce the notion of the homogeneous transformation matrix of the velocity of point A of a rigid body in the space coordinate system of the form

$$\mathbf{v}_{O'}(t) = \frac{\mathrm{d}\mathbf{A}_j(t)}{\mathrm{d}t}\mathbf{A}_j^{-1}(t), \tag{5.309}$$

where the subscript indicates that matrix $\mathbf{v}_{O'}$ describes the velocity of the system $O'X_1'X_2'X_3'$ with respect to the system $OX_1X_2X_3$.

According to (5.263) and (5.268), from (5.309) we have

$$\begin{aligned}\mathbf{v}_{O'} &= \begin{bmatrix}\dot{\mathbf{A}} & \dot{\mathbf{r}}_{O'}\\ \mathbf{0} & 0\end{bmatrix}\begin{bmatrix}\mathbf{A}^{\mathrm{T}} & -\mathbf{A}^{\mathrm{T}}\mathbf{r}_{O'}\\ \mathbf{0} & 1\end{bmatrix}\\ &= \begin{bmatrix}\dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} & \dot{\mathbf{r}}_{O'} - \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}}\mathbf{r}_{O'}\\ \mathbf{0} & 0\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Omega}_{O'} & \mathbf{v}_{O'} - \boldsymbol{\Omega}_{O'}\mathbf{r}_{O'}\\ \mathbf{0} & 0\end{bmatrix},\end{aligned} \tag{5.310}$$

where (5.232) was used, and $\dot{\mathbf{r}}_{O'} = \mathbf{v}_{O'}$ was assumed.

Recall that angular velocity matrix $\boldsymbol{\Omega}$ described by relationship (5.232) enables us to determine the angular velocity of rotational motion of a rigid body since according to (5.171) we have

$$\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}. \tag{5.311}$$

If we use matrix (5.232) on the left-hand side of the preceding equation, it takes the form

$$\begin{bmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\end{bmatrix}\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\end{bmatrix} = \begin{vmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\ \omega_1 & \omega_2 & \omega_3\\ r_1 & r_2 & r_3\end{vmatrix}. \tag{5.312}$$

The left-hand side of (5.312) has the form

$$\begin{bmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\end{bmatrix}\begin{bmatrix}-\omega_3 r_2 + \omega_2 r_3\\ \omega_3 r_1 - \omega_1 r_3\\ -\omega_2 r_1 + \omega_1 r_2\end{bmatrix} = \begin{vmatrix}\mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3\\ \omega_1 & \omega_2 & \omega_3\\ r_1 & r_2 & r_3\end{vmatrix}, \tag{5.313}$$

which proves relationship (5.311).

The velocity matrix described by relationship (5.310) contains a non-zero block describing the angular velocity matrix $\boldsymbol{\Omega}_{O'}$ of motion of the system $O''X_1''X_2''X_3''$ with respect to $OX_1X_2X_3$, and from relationship (5.311) we can also determine the angular velocity vector of the rotational motion of the rigid body.

From Fig. 5.64 it follows that

$$\mathbf{r}_A = \mathbf{A}\boldsymbol{\rho} + \mathbf{r}_{O'}. \tag{5.314}$$

This means that knowing the coordinates of point A in the coordinate system $O''X_1''X_2''X_3''$ given by radius vector $\boldsymbol{\rho}$ we can determine the radius vector of that point in the system $OX_1X_2X_3$. Conversely, knowing the position of point A in the system $OX_1X_2X_3$ from the preceding equation we can determine the position of point A in the system $O''X_1''X_2''X_3''$. Premultiplying (5.314) by $\mathbf{A}^{\mathrm{T}}$ we obtain

$$\boldsymbol{\rho} = \mathbf{A}^{\mathrm{T}}(\mathbf{r}_A - \mathbf{r}_{O'}). \tag{5.315}$$

We determine the velocity of point A in the system $OX_1X_2X_3$ by differentiating, with respect to time, relationship (5.314), obtaining

$$\begin{aligned}
\mathbf{v}_A \equiv \dot{\mathbf{r}}_A &= \dot{\mathbf{A}}\boldsymbol{\rho} + \mathbf{A}\dot{\boldsymbol{\rho}} + \dot{\mathbf{r}}_{O'} \\
&= \mathbf{v}_{O'} + \mathbf{S}(\boldsymbol{\omega}_{O'})\boldsymbol{\rho} + \mathbf{A}\dot{\boldsymbol{\rho}} \\
&= \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{A}\boldsymbol{\rho} + \mathbf{A}\dot{\boldsymbol{\rho}} \\
&= \mathbf{v}_{O'} + \boldsymbol{\omega} \times (\mathbf{r}_A - \mathbf{r}_{O'}) + \mathbf{A}\dot{\boldsymbol{\rho}} \\
&= \mathbf{v}_{O'} + \boldsymbol{\omega} \times \overrightarrow{OA} + \mathbf{A}\dot{\boldsymbol{\rho}},
\end{aligned} \tag{5.316}$$

where relationships (5.231) and (5.311) for operator $\boldsymbol{\Omega} \equiv \mathbf{S}$ and (5.314) were used.

From relationship (5.316) we read that the velocity of point A in the system $OX_1X_2X_3$ is equal to the sum of the velocity vector of point O' with respect to point O, vector $\boldsymbol{\omega}_{O'} \times \overrightarrow{OA}$ (vector $\overrightarrow{OA}$ is expressed in the coordinates $OX_1X_2X_3$), and vector $\mathbf{A}\dot{\boldsymbol{\rho}}$.

If the vector $\dot{\boldsymbol{\rho}} = \mathrm{const}$, then it is easy to determine the velocity of point A in the system $O''X_1''X_2''X_3''$, which is equal to

$$\mathbf{v}_A'' = \mathbf{v}_{O'}'' + \boldsymbol{\omega}'' \times \boldsymbol{\rho}. \tag{5.317}$$

We will now show that the preceding equation can be obtained from (5.316) if we assume $\dot{\boldsymbol{\rho}} = \mathbf{0}$.

Let us premultiply (5.316) by $\mathbf{A}^{\mathrm{T}}$ obtaining

$$\begin{aligned}
\mathbf{A}^{\mathrm{T}}\mathbf{v}_A &= \mathbf{A}^{\mathrm{T}}\mathbf{v}_{O'} + \mathbf{A}^{\mathrm{T}}\left(\boldsymbol{\omega} \times \overrightarrow{OA}\right) \\
&= \mathbf{A}^{\mathrm{T}}\mathbf{v}_{O'} + (\mathbf{A}^{\mathrm{T}}\boldsymbol{\omega}) \times \left(\mathbf{A}^{\mathrm{T}}\overrightarrow{OA}\right),
\end{aligned} \tag{5.318}$$

where the relationship $\mathbf{Q}(\boldsymbol{a}_1 \times \boldsymbol{a}_2) = (\mathbf{Q}\boldsymbol{a}_1) \times (\mathbf{Q}\boldsymbol{a}_2)$ was used.

An arbitrary vector including a vector denoting displacement, velocity, or acceleration is transformed from one system to another through the transformation matrix, in this case the rotation matrix.

Let vector $\boldsymbol{a}$ be expressed in the system $OX_1X_2X_3$. Then vector $\boldsymbol{a} = \mathbf{A}\boldsymbol{a}''$, and $\boldsymbol{a}''$ is expressed in the system $O''X_1''X_2''X_3''$. In view of that, $\boldsymbol{a}'' = \mathbf{A}^{\mathrm{T}}\boldsymbol{a} = \mathbf{A}'\boldsymbol{a}$, which implies that (5.315) is written in the system $O''X_1''X_2''X_3''$ in the form

$$\mathbf{v}_A'' = \mathbf{v}_{O'}'' + \boldsymbol{\omega}'' \times \overrightarrow{OA'}, \tag{5.319}$$

which is in agreement with (5.317), because $\overrightarrow{OA''}$ is vector $\overrightarrow{OA}$ expressed in the system $O''X_1''X_2''X_3''$.

The same vector of angular velocity can be expressed in the space system as $\boldsymbol{\omega}$ and in the body system as $\boldsymbol{\omega}''$. Recall that according to (5.163) we have

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}}, \tag{5.320}$$

and postmultiplying the preceding equation by $\mathbf{A}'^{\mathrm{T}}$ and premultiplying by $\mathbf{A}'$ we obtain

$$\mathbf{A}'\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{A}'^{\mathrm{T}} = \mathbf{A}'(\dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}})\mathbf{A}'^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\dot{\mathbf{A}}, \tag{5.321}$$

because as it has been shown $\mathbf{A}' = \mathbf{A}^{\mathrm{T}}$.

We will show that for an arbitrary matrix function $\boldsymbol{\Omega}(\boldsymbol{\omega})$ and for any rotation matrix $\mathbf{A}'_{3\times3}$ the following relationship holds true:

$$\mathbf{A}'\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{A}'^{\mathrm{T}} = \boldsymbol{\Omega}(\mathbf{A}'\boldsymbol{\omega}). \tag{5.322}$$

Let us examine the action of the left-hand side of equality (5.322) on an arbitrary vector $\boldsymbol{a}$. We have

$$\begin{aligned}\mathbf{A}'\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{A}'^{\mathrm{T}}\boldsymbol{a} &= \mathbf{A}'\left(\boldsymbol{\omega} \times \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{a}\right)\right) = \left(\mathbf{A}'\boldsymbol{\omega}\right) \times \left(\mathbf{A}'\mathbf{A}'^{\mathrm{T}}\boldsymbol{a}\right) \\ &= \left(\mathbf{A}'\boldsymbol{\omega}\right) \times \boldsymbol{a} = \boldsymbol{\Omega}\left(\mathbf{A}'\boldsymbol{\omega}\right)\boldsymbol{a},\end{aligned} \tag{5.323}$$

where relationship (5.311) was used.

From (5.321) and (5.323) we have

$$\boldsymbol{\Omega}\left(\boldsymbol{\omega}''\right) = \mathbf{A}^{\mathrm{T}}\dot{\mathbf{A}}, \tag{5.324}$$

where $\boldsymbol{\omega}'' = \mathbf{A}'\boldsymbol{\omega}$.

The obtained value of the matrix of velocity of rotational motion of the rigid body $\boldsymbol{\Omega}(\boldsymbol{\omega}'')$ is expressed in the system $O''X_1''X_2''X_3''$, whereas $\boldsymbol{\Omega}(\boldsymbol{\omega})$, described by (5.320), is expressed in the system $OX_1X_2X_3$.

The relationship between these matrices can be easily determined based on (5.320) and (5.324). From (5.324) we have

$$\dot{\mathbf{A}} = \mathbf{A}\boldsymbol{\Omega}\left(\boldsymbol{\omega}''\right), \tag{5.325}$$

and from (5.320) we have

$$\dot{\mathbf{A}} = \boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{A}, \tag{5.326}$$

hence

$$\boldsymbol{\Omega}\left(\boldsymbol{\omega}''\right) = \mathbf{A}^{\mathrm{T}}\boldsymbol{\Omega}(\boldsymbol{\omega})\mathbf{A}. \tag{5.327}$$

5.5.15 Regular Precession of a Rigid Body

Let us return to the motion about a point of a rigid body presented in Fig. 5.60. Regular precession takes place when $\dot{\Theta} = 0$ ($\Theta = \Theta_0 = \text{const}$), and the body rotates about the spin axis X_3''' with angular speed $\dot{\phi} = \text{const}$, and this axis, in turn, rotates about the axis X_3' with angular speed $\dot{\psi} = \text{const}$. The angle between the X_3''' and X_3' axes equals Θ_0, and we call the motion of the body in this case *precession*.

According to (5.243) we have

$$\omega = \sqrt{\dot{\Psi}^2 + \dot{\phi}^2 + 2\dot{\Psi}\dot{\phi}\cos\Theta_0}, \tag{5.328}$$

and the resultant vector of the angular velocity of the rigid body reads

$$\boldsymbol{\omega} = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\phi}}. \tag{5.329}$$

We determine the projections of this resultant angular velocity onto the axes of the stationary and non-stationary coordinate systems by expressing the same vector $\boldsymbol{\omega}$ in the coordinates of those two systems, namely,

$$\boldsymbol{\omega} = \omega_1\mathbf{E}_1 + \omega_2\mathbf{E}_2 + \omega_3\mathbf{E}_3 = \dot{\Psi}\mathbf{E}_3' + \dot{\phi}\mathbf{E}_3'''. \tag{5.330}$$

Multiplying this equation successively by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ we obtain

$$\begin{aligned}
\omega_1 &= \dot{\Psi}\mathbf{E}_3' \circ \mathbf{E}_1 + \dot{\phi}\mathbf{E}_3'' \circ \mathbf{E}_1 = \dot{\Psi}\mathbf{E}_3 \circ \mathbf{E}_1 \\
&\quad + \dot{\phi}\left[-(-\sin\Psi\mathbf{E}_1 + \cos\Psi\mathbf{E}_2)\sin\Theta_0 + \mathbf{E}_3\cos\Theta_0\right] \circ \mathbf{E}_1 \\
&= \dot{\phi}\sin\Theta_0\sin\dot{\Psi}t, \\
\omega_2 &= \dot{\Psi}\mathbf{E}_3' \circ \mathbf{E}_2 + \dot{\phi}\mathbf{E}_3'' \circ \mathbf{E}_2 = \dot{\Psi}\mathbf{E}_3 \circ \mathbf{E}_2 + \dot{\phi}\left[-\cos\Psi\sin\Theta_0\mathbf{E}_2\right] \circ \mathbf{E}_2 \\
&= -\dot{\phi}\sin\Theta_0\cos\dot{\Psi}t, \\
\omega_3 &= \dot{\Psi}\mathbf{E}_3' \circ \mathbf{E}_3 + \dot{\phi}\mathbf{E}_3'' \circ \mathbf{E}_3 = \dot{\Psi} + \dot{\phi}\cos\Theta_0,
\end{aligned} \tag{5.331}$$

where in the preceding transformations the scalar multiplication of the vectors was conducted using (5.204)–(5.206).

In order to determine the components of vector $\boldsymbol{\omega}$ in the non-stationary coordinate system let us use (5.242), from which we obtain

$$\begin{aligned}\omega_1''' &= \dot{\Psi}\sin\Theta_0\sin\dot{\phi}t,\\ \omega_2''' &= \dot{\Psi}\sin\Theta_0\cos\dot{\phi}t,\\ \omega_3''' &= \dot{\Psi}\cos\Theta_0 + \dot{\phi}.\end{aligned} \tag{5.332}$$

The instantaneous axes of rotation in the non-stationary and stationary systems will be obtained from (5.287) and (5.288) as well as from (5.331) and (5.332). They are described by the equations

$$\frac{r_1'''}{\dot{\Psi}\sin\Theta_0\sin\dot{\phi}t} = \frac{r_2'''}{\dot{\Psi}\sin\Theta_0\cos\dot{\phi}t} = \frac{r_3'''}{\dot{\phi}+\dot{\Psi}\cos\Theta_0}, \tag{5.333}$$

$$\frac{r_1}{\dot{\phi}\sin\Theta_0\sin\dot{\Psi}t} = -\frac{r_2}{\dot{\phi}\sin\Theta_0\cos\dot{\Psi}t} = \frac{r_3}{\dot{\Psi}+\dot{\phi}\cos\Theta_0}, \tag{5.334}$$

where $(r_1''',\ r_2''',\ r_3''')$ and $(r_1,\ r_2,\ r_3)$ are coordinates of a point respectively in the non-stationary and stationary system.

From (5.334) we obtain

$$\frac{r_1^2}{\left(\dot{\phi}\sin\Theta_0\right)^2} = \frac{r_3^2}{\left(\dot{\Psi}+\dot{\phi}\cos\Theta_0\right)^2}\sin^2\dot{\Psi}t, \tag{5.335}$$

$$\frac{r_2^2}{\left(\dot{\phi}\sin\Theta_0\right)^2} = \frac{r_3^2}{\left(\dot{\Psi}+\dot{\phi}\cos\Theta_0\right)^2}\cos^2\dot{\Psi}t. \tag{5.336}$$

Adding by sides (5.335) and (5.336) we obtain the equation of fixed axode of the form

$$\frac{r_1^2+r_2^2}{\dot{\phi}^2\sin^2\Theta_0} = \frac{r_3^2}{\left(\dot{\Psi}+\dot{\phi}\cos\Theta_0\right)^2}. \tag{5.337}$$

Similarly, we proceed with (5.333) obtaining

$$\frac{\left(r_1'''\right)^2}{\left(\dot{\Psi}\sin\Theta_0\right)^2} = \frac{\left(r_3'''\right)^2}{\left(\dot{\phi}+\dot{\Psi}\cos\Theta_0\right)^2}\sin^2\dot{\phi}t, \tag{5.338}$$

$$\frac{\left(r_2'''\right)^2}{\left(\dot{\Psi}\sin\Theta_0\right)^2} = \frac{\left(r_3'''\right)^2}{\left(\dot{\phi}+\dot{\Psi}\cos\Theta_0\right)^2}\cos^2\dot{\phi}t. \tag{5.339}$$

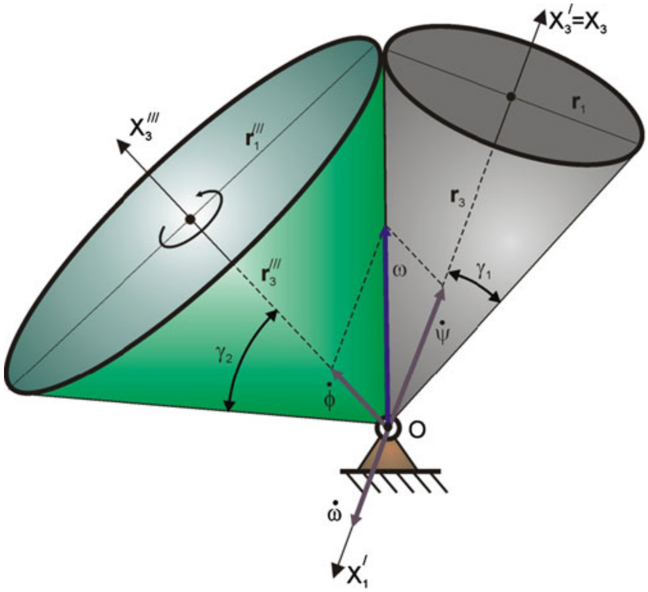


Fig. 5.74 Cone of a circular base and symmetry axis OX_3''' (moving axode) rolls without sliding on a cone of symmetry axis OX_3' (fixed axode)

Adding by sides (5.338) and (5.339) we obtain the equation of moving axode of the form

$$\frac{\left(r_1'''\right)^2 + \left(r_2'''\right)^2}{\dot{\Psi}^2 \sin^2 \Theta_0} = \frac{\left(r_3'''\right)^2}{\left(\dot{\phi} + \dot{\Psi}\cos\Theta_0\right)^2}. \tag{5.340}$$

The moving and fixed axodes are the cones of a circular base, as shown in Fig. 5.74.

Angular velocity vector $\boldsymbol{\omega}$ rotates about the OX_3' axis with angular speed $\dot{\psi}$, and its magnitude is constant [see (5.328)].

The angles γ_1 and γ_2 can be determined from relationships (5.337) and (5.340):

$$\tan\gamma_1 = \left(\frac{r_1}{r_3}\right)_{r_2=0} = \frac{\dot{\phi}\sin\Theta_0}{\dot{\Psi} + \dot{\phi}\cos\Theta_0}, \tag{5.341}$$

$$\tan\gamma_2 = \left(\frac{r_1'''}{r_3'''}\right)_{N_2'''=0} = \frac{\dot{\Psi}\sin\Theta_0}{\dot{\phi} + \dot{\Psi}\cos\Theta_0}. \tag{5.342}$$

We determine the angular acceleration from the equation

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\Psi}} \times \boldsymbol{\omega} = \dot{\boldsymbol{\Psi}} \times \left(\dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\Psi}}\right) = \dot{\boldsymbol{\Psi}} \times \dot{\boldsymbol{\phi}}, \tag{5.343}$$

and its magnitude reads

$$\dot{\omega} = \dot{\Psi}\dot{\phi} \sin \Theta_0. \tag{5.344}$$

If the angle between $\dot{\phi}$ and $\dot{\psi}$ is an obtuse angle, the motion is called a *retrograde precession*.

Example 5.4. The motion about a point of a rigid body was described by Eulerian angles of the form $\phi = \phi_0 + \dot{\phi}t$, $\psi = \psi_0 + \dot{\psi}t$, and $\theta = \theta_0 + \dot{\theta}t$. Determine the angular velocity $\boldsymbol{\omega}$ and the angular acceleration $\boldsymbol{\varepsilon}$ of a body and the acceleration of point $A(x_1, x_2, x_3)$. Additionally, determine the equations of the moving and fixed axodes. Solve the problem for the following two cases:

(i) $\phi_0 = 0$, $\psi_0 = \frac{\pi}{4}$, $\theta_0 = \frac{\pi}{4}$, $\dot{\phi} = \frac{\pi}{4}$, $\dot{\psi} = \frac{\pi}{6}$, $\dot{\theta} = 0$, $A(0, 0, x_{30})$;
(ii) $\phi_0 = \frac{\pi}{2}$, $\psi_0 = 0$, $\theta_0 = \frac{\pi}{3}$, $\dot{\phi} = -\pi$, $\dot{\psi} = \frac{\pi}{2}$, $\dot{\theta} = 0$, $A(x_{10}, 0, x_{30})$.

In the considered case we are dealing with the regular precession ($\dot{\theta} = 0$), so for the solution we will use (5.246) and (5.280).

First, let us calculate the components of the angular velocity in a stationary coordinate system. From (5.246) we obtain

$$\omega_1^{(i)} = \dot{\phi} \sin\theta_0 \sin\dot{\psi}t = \frac{\pi}{4}\sin\frac{\pi}{4}\sin\left(-\frac{\pi}{6}\right)t = -\frac{\pi}{4\sqrt{2}}\sin\frac{\pi}{6}t,$$

$$\omega_1^{(ii)} = -\pi\sin\frac{\pi}{3}\sin\frac{\pi}{2}t = -\frac{\sqrt{3}\pi}{2}\sin\frac{\pi}{2}t,$$

$$\omega_2^{(i)} = -\dot{\phi}\sin\theta_0\cos\dot{\psi}t = -\frac{\pi}{4\sqrt{2}}\cos\frac{\pi}{6}t,$$

$$\omega_2^{(ii)} = \frac{\sqrt{3}\pi}{2}\cos\frac{\pi}{2}t,$$

$$\omega_3^{(i)} = \dot{\psi} + \dot{\phi}\cos\theta_0 = -\frac{\pi}{6} + \frac{\pi}{4}\cos\frac{\pi}{4}$$

$$= -\frac{\pi}{6} + \frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}\left(\frac{\sqrt{2}}{4} - \frac{1}{3}\right) = \frac{\pi}{24}\left(3\sqrt{2} - 4\right),$$

$$\omega_3^{(ii)} = \frac{\pi}{2} - \pi\cos\frac{\pi}{3} = 0.$$

According to (5.280) the velocity of an arbitrary point A not lying on the rotation axis is equal to

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_A = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= \mathbf{E}_1\left(\omega_2 x_3 - \omega_3 x_2\right) + \mathbf{E}_2\left(\omega_3 x_1 - \omega_1 x_3\right) + \mathbf{E}_3\left(\omega_1 x_2 - \omega_2 x_1\right),$$

that is, the velocity components for point A are equal to

$$v_1 = \omega_2 x_3 - \omega_3 x_2, \quad v_2 = \omega_3 x_1 - \omega_1 x_3, \quad v_3 = \omega_1 x_2 - \omega_2 x_1. \qquad (*)$$

The components of velocity for the considered cases are equal to

$$\begin{aligned}
v_1^{(i)} &= \omega_2^{(i)} x_3^{(i)} - x_3^{(i)} \omega_2^{(i)} = -x_{30} \frac{3\sqrt{2}}{8} \cos \frac{\pi}{6} t, \\
v_1^{(ii)} &= x_{30} \frac{\pi\sqrt{3}}{2} \cos \frac{\pi}{2} t \ , \\
v_2^{(i)} &= \omega_3^{(i)} x_1^{(i)} - \omega_1^{(i)} x_3^{(i)} = x_{30} \frac{\pi\sqrt{2}}{8} \sin \frac{\pi}{6} t, \\
v_2^{(ii)} &= x_{30} \frac{\pi\sqrt{3}}{2} \sin \frac{\pi}{2} t, \\
v_3^{(i)} &= \omega_1^{(i)} x_2^{(i)} - \omega_2^{(i)} x_1^{(i)} = 0, \\
v_3^{(ii)} &= -x_{10} \frac{\pi\sqrt{3}}{2} \cos \frac{\pi}{2} t.
\end{aligned}$$

Differentiation of velocity with respect to time allows us to determine the acceleration components of point A, which are equal to

$$\begin{aligned}
&\mathrm{a}_1^{(i)} = \dot{v}_1^{(i)} = x_{30} \frac{\pi^2\sqrt{2}}{48} \sin \frac{\pi}{6} t, \quad && \mathrm{a}_1^{(ii)} = \dot{v}_1^{(ii)} = -x_{30} \frac{\pi^2\sqrt{3}}{4} \sin \frac{\pi}{2} t, \\
&\mathrm{a}_2^{(i)} = \dot{v}_2^{(i)} = x_{30} \frac{\pi^2\sqrt{2}}{48} \cos \frac{\pi}{6} t, \quad && \mathrm{a}_2^{(ii)} = \dot{v}_2^{(ii)} = x_{30} \frac{\pi^2\sqrt{3}}{4} \cos \frac{\pi}{2} t, \\
&\mathrm{a}_3^{(i)} = \dot{v}_3(i) = 0, && \mathrm{a}_3^{(ii)} = \dot{v}_3^{(ii)} = x_{10} \frac{\pi^2\sqrt{3}}{4} \sin \frac{\pi}{2} t.
\end{aligned}$$

The results obtained lead to the determination of the angular velocity of the body:

$$\begin{aligned}
\omega^{(i)} &= \sqrt{\omega_1^{(i)2} + \omega_2^{(i)2} + \omega_3^{(i)2}} \\
&= \left[\frac{\pi^2}{32} \sin \frac{\pi}{6} t^{\,2} + \frac{\pi^2}{32} \cos \frac{\pi}{6} t^{\,2} + \frac{\pi^2}{576} \left(-4 + 3\sqrt{2}\right)^2 \right]^{\frac{1}{2}} \\
&= \pi \sqrt{\frac{1}{32} + \frac{\left(-4 + 3\sqrt{3}\right)^2}{576}} = \frac{\pi}{24} \sqrt{18 + \left(-4 + 3\sqrt{2}\right)^2} \\
&= \frac{\pi}{24} \sqrt{18 + 16 + 18 - 24\sqrt{2}} = \frac{\pi}{24} \sqrt{52 - 24\sqrt{2}}
\end{aligned}$$

$$= \frac{\pi}{12}\sqrt{13 - 6\sqrt{2}},$$

$$\omega^{(ii)} = \left(\frac{3\pi^2}{4}\sin\frac{\pi}{2}t^2 + \frac{3\pi^2}{4}\cos\frac{\pi}{2}t^2\right)^{\frac{1}{2}} = \frac{\pi\sqrt{3}}{2},$$

and subsequently also its angular acceleration

$$\varepsilon = \sqrt{\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2}.$$

In the considered cases we have

$$\dot{\omega}_1^{(i)} = -\frac{\pi^2}{24\sqrt{2}}\cos\frac{\pi}{6}t, \quad \dot{\omega}_1^{(ii)} = -\frac{\pi^2\sqrt{3}}{4}\cos\frac{\pi}{2}t,$$

$$\dot{\omega}_2^{(i)} = -\frac{\pi^2}{24\sqrt{2}}\sin\frac{\pi}{6}t, \quad \dot{\omega}_2^{(ii)} = -\frac{\pi^2\sqrt{3}}{4}\sin\frac{\pi}{2}t,$$

$$\dot{\omega}_3^{(i)} = 0, \qquad \dot{\omega}_3^{(ii)} = 0,$$

that is,

$$\varepsilon^{(i)} = \frac{\pi^2}{24\sqrt{2}}, \quad \varepsilon^{(ii)} = \frac{\pi^2\sqrt{3}}{4}.$$

We calculate the acceleration of point A from the equation

$$\mathrm{a} = \sqrt{\dot{v}_1^2 + \dot{v}_2^2 + \dot{v}_3^2}.$$

We will now show how to determine the *fixed axode* taking into account point $A^0(x_1, x_2, x_3)$ on the instantaneous axis of rotation. All the points lying on the instantaneous axis of rotation have a velocity equal to zero, that is, from equation (*) we obtain

$$\omega_2 x_3 - \omega_3 x_2 = 0, \qquad \omega_3 x_1 - \omega_1 x_3 = 0, \qquad \omega_1 x_2 - \omega_2 x_1 = 0,$$

where x_1, x_2, and x_3 are the coordinates of point A^0. From those equations we obtain

$$\frac{x_2}{x_3} = \frac{\omega_2}{\omega_3}, \qquad \frac{x_1}{x_3} = \frac{\omega_1}{\omega_3}, \qquad \frac{x_1}{x_2} = \frac{\omega_1}{\omega_2},$$

from which also result (5.283), derived earlier.

In the considered case (i) we then have

$$\frac{x_2}{x_3} = \frac{3}{2\sqrt{2}-3}\cos\frac{\pi}{6}t, \qquad \frac{x_1}{x_3} = \frac{3}{2\sqrt{2}-3}\sin\frac{\pi}{6}t, \qquad \frac{x_1}{x_2} = \tan\frac{\pi}{6}t.$$

From the preceding equations we obtain

$$\left(\frac{x_1}{x_3}\right)^2 + \left(\frac{x_2}{x_3}\right)^2 = \left(\frac{3}{2\sqrt{2}-3}\right)^2,$$

and after transformation

$$x_1^2 + x_2^2 - \left(\frac{3}{2\sqrt{2}-3}\right)^2 x_3^2 = 0.$$

The preceding equation (*fixed axode*) describes a circular cone whose opening angle is equal to $2\gamma_1$, where $\tan\gamma_1 = \frac{3-2\sqrt{2}}{3}$, and its axis coincides with the OX_3 axis.

In order to determine the equation of the moving axode one should determine the components of the angular velocity in the non-stationary coordinate system. A point lying on the instantaneous axis of rotation satisfies the equations

$$\frac{x_2'''}{x_3'''} = \frac{\omega_2'''}{\omega_3'''}, \qquad \frac{x_1'''}{x_3'''} = \frac{\omega_1'''}{\omega_3'''}, \qquad \frac{x_1'''}{x_2'''} = \frac{\omega_1'''}{\omega_2'''},$$

where

$$\omega_1''' = \dot{\psi}\sin\theta_0 \sin\dot{\phi}t = -\frac{\pi}{6}\sin\frac{\pi}{4}\sin\frac{\pi}{4}t = -\frac{\pi}{6\sqrt{2}}\sin\frac{\pi}{4}t,$$

$$\omega_2''' = \dot{\psi}\sin\theta_0 \cos\dot{\phi}t = -\frac{\pi}{6\sqrt{2}}\cos\frac{\pi}{4}t,$$

$$\omega_3''' = \dot{\psi}\cos\theta_0 + \dot{\phi} = -\frac{\pi}{6}\cos\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{6\sqrt{2}}$$

$$= \frac{\pi}{2}\left[\frac{1}{2} - \frac{1}{3\sqrt{2}}\right] = \frac{\pi}{2}\frac{\left(3\sqrt{2}-2\right)}{6\sqrt{2}} = \frac{\pi}{24}\left(6-2\sqrt{2}\right) = \frac{\pi}{12}\left(3-\sqrt{2}\right).$$

From the preceding equations we obtain

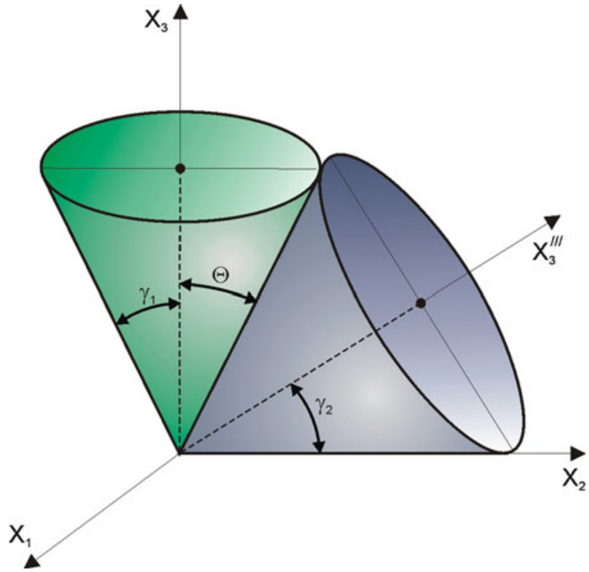
$$\frac{x_2'''}{x_3'''} = \frac{\left(3\sqrt{2}+2\right)}{21}\sin\frac{\pi}{4}t, \qquad \frac{x_1'''}{x_2'''} = -\frac{\left(3\sqrt{2}+2\right)}{21}\cos\frac{\pi}{4}t,$$

$$\frac{x_1'''}{x_3'''} = -\frac{\pi}{6\sqrt{2}}\frac{12}{\pi\left(3-\sqrt{2}\right)}\cos\frac{\pi}{4}t = -\frac{2}{3\left(3\sqrt{2}-2\right)}\cos\frac{\pi}{4}t,$$

and hence we obtain the equation of a moving axode

$$\left(x_1'''\right)^2 + \left(x_2'''\right)^2 - \left(\frac{2+3\sqrt{2}}{21}\right)^2\left(x_3'''\right)^2 = 0.$$

Fig. 5.75 Motion of moving axode on fixed axode for case (i)



The equation of a moving axode describes the cone of opening angle $2\gamma_2$, where $\tan\gamma_2 = \frac{21}{2+3\sqrt{2}}$.

The motion of a moving axode on a fixed axode for case (i) is shown in Fig. 5.75.

At the end of this example we will consider rolling without sliding of a moving axode on a fixed axode for case (ii).

In this case we have $\omega_3^{(ii)} = 0$, so the fixed axode is the plane OX_1X_2, and the instantaneous axis of rotation lies in that plane. Let us derive, then, the equation of a moving axode. We calculate successively

$$\frac{x_2'''}{x_3'''} = \frac{\omega_2'''}{\omega_3'''} = -\frac{\left(2+3\sqrt{2}\right)}{7}\cos\frac{\pi}{4}t,$$

$$\frac{x_1'''}{x_3'''} = \frac{\omega_1'''}{\omega_3'''} = -\frac{\left(2+3\sqrt{2}\right)}{7}\sin\frac{\pi}{4}t,$$

and hence we find the equation of a body cone of the form

$$\left(x_1'''\right)^2 + \left(x_2'''\right)^2 - \frac{\left(2+3\sqrt{2}\right)}{49}\left(x_3'''\right)^2 = 0.$$

The motion of a cone describing a moving axode on a fixed axode (the plane OX_1X_2) is shown in Fig. 5.76. □

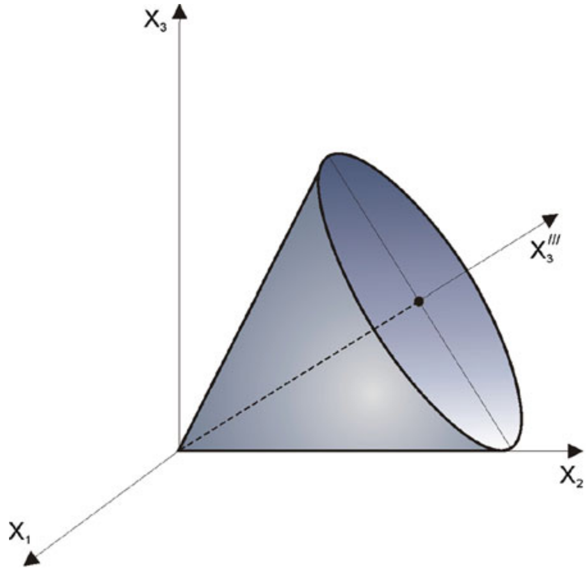


Fig. 5.76 Motion of moving axode on fixed axode for case (ii)

5.5.16 Screw Motion

The screw motion of a point was considered in Chap. 4. Now we will consider the *screw motion* of a rigid body. This motion is a composite motion because the rigid body moves in translational motion along a fixed line with velocity $\mathbf{v}_3$, with simultaneous rotation about that axis described by vector $\boldsymbol{\omega}$ (Fig. 5.77). Vectors $\boldsymbol{\omega}$ and $\mathbf{v}_3$ lie on one axis called the *wrench axis*.

Motion along the X_3 axis is described by the position of point A of coordinate $x_3 \equiv s$, and the rotation of the body is characterized by angle ψ. Let us introduce the *parameter of the screw motion* in the following way:

$$\lambda = \frac{\mathbf{v}_3}{\boldsymbol{\omega}} = \frac{v_3}{\omega} = \frac{\mathrm{d}s}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\psi} = \frac{\mathrm{d}s}{\mathrm{d}\psi}. \tag{5.345}$$

From (5.345) after integration we obtain

$$s = \int_0^{\Theta} \lambda \mathrm{d}\psi = \lambda\psi, \tag{5.346}$$

on the assumption that λ is a constant.

If ψ varies from zero to 2π, then

$$s \equiv p = 2\pi\lambda, \tag{5.347}$$

where p is called the *pitch of screw motion*.

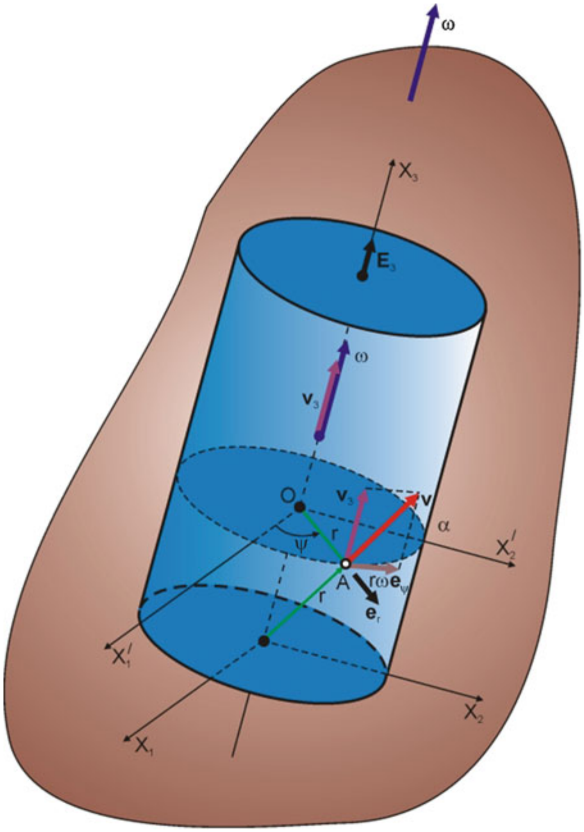


Fig. 5.77 Screw motion of a rigid body

Using the relationships introduced earlier for the velocity in a cylindrical coordinate system [see (4.195)] we obtain

$$\dot{\mathbf{r}} \equiv \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\psi}\mathbf{e}_\psi + \dot{z}\mathbf{E}_3 = r\omega\mathbf{e}_\psi + v_3\mathbf{E}_3. \tag{5.348}$$

Because $\mathbf{E}_3 \perp \mathbf{e}_\psi$, we have

$$v = \sqrt{r^2\omega^2 + v_3^2}, \tag{5.349}$$

and taking into account (5.345) we obtain

$$v = \omega\sqrt{r^2 + \lambda^2}. \tag{5.350}$$

From Fig. 5.77 it follows that

$$\tan\alpha = \frac{v_3}{r\omega} = \frac{\lambda}{r}. \tag{5.351}$$

Let us note that for all points lying on a cylindrical surface of radius r the angle $\alpha = \text{const}$, that is, the path of point A permanently forms a constant angle with the plane perpendicular to the axis of rotation. From the figure it follows that the projections of velocities of all points of a rigid body onto a wrench axis are equal.

5.5.17 Geometrical Interpretation of Velocity and Acceleration of a Point of a Rigid Body in General Motion

In previous sections we conducted analyses of special cases of the motion of a rigid body, i.e., motion about a point, planar motion, screw motion. Now we will take up a more detailed analysis of the general motion of a rigid body (with emphasis on the geometrical interpretation of such motion).

In order to determine the velocity and acceleration of an arbitrary point of a body with the aid of the vector calculus we will proceed in such a way that is analogous to that described earlier in our analysis of the planar motion of a rigid body.

According to Fig. 5.78 we have

$$\mathbf{r}_A = \mathbf{r}_{O'} + \mathbf{r}'', \tag{5.352}$$

and the relationships between the unit vectors in the non-stationary and stationary coordinate systems are given by (5.297), where a_{ij} denote the cosines of the angles respectively between the unit vectors $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ and $\mathbf{E}''_1$, $\mathbf{E}''_2$, $\mathbf{E}''_3$ (i indices are associated with the unit vectors $\mathbf{E}_i$ and j with $\mathbf{E}''_j$), i.e., $a_{ij} = \mathbf{E}_i \circ \mathbf{E}''_j$, $i, j = 1, 2, 3$. Observe that earlier we introduced vector $\boldsymbol{\rho} = \mathbf{r}''$.

From (5.352) it follows that

$$x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3 = x_{1O'}\mathbf{E}_1 + x_{2O'}\mathbf{E}_2 + x_{3O'}\mathbf{E}_3 + x''_1\mathbf{E}''_1 + x''_2\mathbf{E}''_2 + x''_3\mathbf{E}''_3, \tag{5.353}$$

where (x_1, x_2, x_3) denote the coordinates of point A in the stationary coordinate system $OX_1X_2X_3$; $(x_{1O'}, x_{2O'}, x_{3O'})$ denote the coordinates of point O' in the same system; and (x''_1, x''_2, x''_3) denote the coordinates of point A in the moving coordinate system $O''X''_1X''_2X''_3$.

We will obtain the coordinates namely, of point A in a stationary coordinate system, multiplying (5.353) successively by $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, are gets.

$$\begin{aligned}
x_1 &= x_{1O'} + x''_1\left(\mathbf{E}''_1 \circ \mathbf{E}_1\right) + x''_2\left(\mathbf{E}''_2 \circ \mathbf{E}_1\right) + x_3\left(\mathbf{E}''_3 \circ \mathbf{E}_1\right),\\
x_2 &= x_{2O'} + x''_1\left(\mathbf{E}''_1 \circ \mathbf{E}_2\right) + x''_2\left(\mathbf{E}''_2 \circ \mathbf{E}_2\right) + x_3\left(\mathbf{E}''_3 \circ \mathbf{E}_2\right),\\
x_3 &= x_{3O'} + x''_1\left(\mathbf{E}''_1 \circ \mathbf{E}_3\right) + x''_2\left(\mathbf{E}''_2 \circ \mathbf{E}_3\right) + x_3\left(\mathbf{E}''_3 \circ \mathbf{E}_3\right),
\end{aligned} \tag{5.354}$$

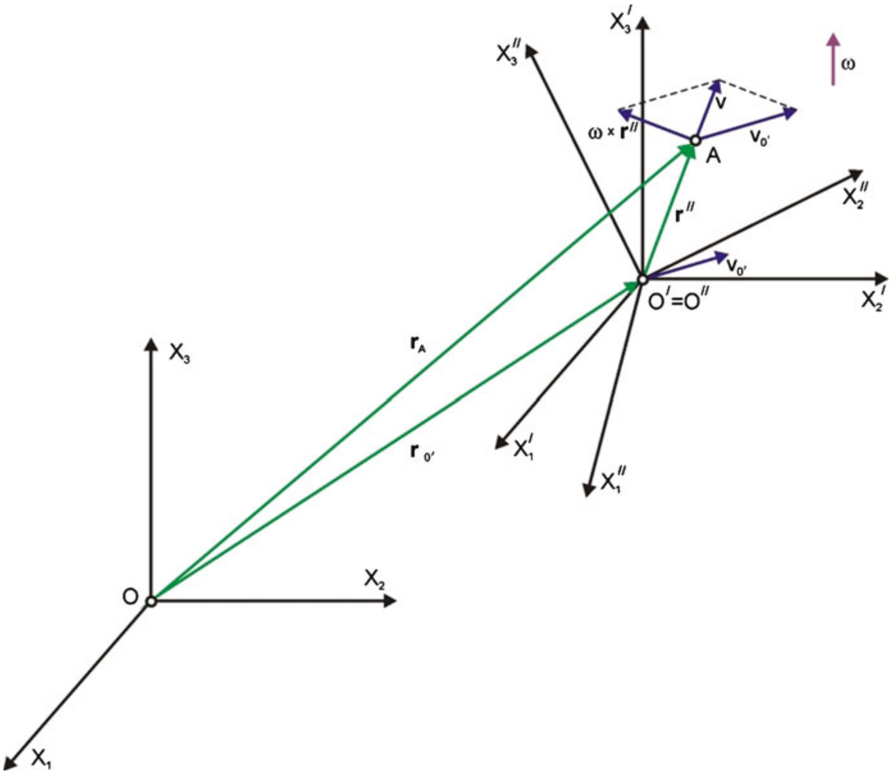


Fig. 5.78 Position of point A in non-stationary and stationary coordinate systems and its velocity $\mathbf{v}_A \equiv \mathbf{v}$

and taking into account (5.297) we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_{1O'} \\ x_{2O'} \\ x_{3O'} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}, \tag{5.355}$$

which is the expanded form of the symbolic matrix notation (5.255).

We will obtain the coordinates of point A in a non-stationary coordinate system from (5.353) after multiplying it successively by $\mathbf{E}_1''$, $\mathbf{E}_2''$, $\mathbf{E}_3''$:

$$x_1''\mathbf{E}_1''+x_2''\mathbf{E}_2''+x_3''\mathbf{E}_3'' = (x_1 - x_{1O'})\,\mathbf{E}_1+(x_2 - x_{2O'})\,\mathbf{E}_2+(x_3 - x_{3O'})\,\mathbf{E}_3, \tag{5.356}$$

that is,

$$x_1'' = (x_1 - x_{1O'})\left(\mathbf{E}_1 \circ \mathbf{E}_1''\right) + (x_2 - x_{2O'})\left(\mathbf{E}_2 \circ \mathbf{E}_1''\right) + (x_3 - x_{3O'})\left(\mathbf{E}_3 \circ \mathbf{E}_1''\right),$$

$$x_2'' = (x_1 - x_{1O'})\left(\mathbf{E}_1 \circ \mathbf{E}_2''\right) + (x_2 - x_{2O'})\left(\mathbf{E}_2 \circ \mathbf{E}_2''\right) + (x_3 - x_{3O'})\left(\mathbf{E}_3 \circ \mathbf{E}_2''\right),$$
$$x_3'' = (x_1 - x_{1O'})\left(\mathbf{E}_1 \circ \mathbf{E}_3''\right) + (x_2 - x_{2O'})\left(\mathbf{E}_2 \circ \mathbf{E}_3''\right) + (x_3 - x_{3O'})\left(\mathbf{E}_3 \circ \mathbf{E}_3''\right), \tag{5.357}$$

and taking into account (5.237) we have

$$\begin{aligned} x_1'' &= a_{11}(x_1 - x_{1O'}) + a_{21}(x_2 - x_{2O'}) + a_{31}(x_3 - x_{3O'}), \\ x_2'' &= a_{12}(x_1 - x_{1O'}) + a_{22}(x_2 - x_{2O'}) + a_{32}(x_3 - x_{3O'}), \\ x_3'' &= a_{13}(x_1 - x_{1O'}) + a_{23}(x_2 - x_{2O'}) + a_{33}(x_3 - x_{3O'}). \end{aligned} \tag{5.358}$$

Let us note that the coefficients a_{ji}, which appear in (5.358) are obtained through the transposition of matrix $[a_{ij}]$.

We call matrix $[a_{ij}]$ orthogonal (Chap. 4) because the sum of the squares of the elements in the rows is equal to one, the sum of the products of the remaining elements of the rows (three combinations) is equal to zero [see (5.298)], and the linear transformations (5.355) and (5.358) are associated with rotation.

According to the vector function (5.352) the motion of an arbitrary point of a rigid body can be treated as a composition of the translational motion of a certain point of that body (point O') and the motion about a point taking place about the moving point O'.

This observation is confirmed also by the formulas obtained earlier describing the velocity of motion in the case of the non-stationary (5.277) and stationary (5.280) coordinate systems. This means that the general motion of a rigid body can be obtained as a result of the geometrical composition of the translational motion and the motion about a point.

The velocity of point A (Fig. 5.78) is equal to

$$\mathbf{v} = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}'', \tag{5.359}$$

where the preceding equation is a vector equation.

That result can be obtained in the following formal way.

Vector $\overrightarrow{OA}$ has the coordinates in the coordinate system $OX_1X_2X_3$ and will be denoted by $\mathbf{r}$. Because the system $O''X_1''X_2''X_3''$ is rigidly connected to the body, vector $\overrightarrow{O''A}$ in the coordinates $O''X_1''X_2''X_3''$ will be denoted by $\mathbf{r}''$, and according to (5.355), vector $\mathbf{r}_A$ takes the form

$$\mathbf{r}_A = \mathbf{r}_{O'} + \mathbf{A}\mathbf{r}'', \tag{5.360}$$

where after differentiation of the preceding equation with respect to time we obtain the vector (5.359).

Theorem 5.12. *There exists one and only one vector* $\boldsymbol{\omega}$*, called the angular velocity of a rigid body, by means of which it is possible to describe the velocity of an arbitrary point A of a rigid body through the following equality:*

$$\mathbf{v} = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r},$$

where $\mathbf{v}_{O'}$ *is the velocity of a pole* O' *and* $\boldsymbol{\omega}$ *is independent of the choice of the pole.*

Proof. Let vector $\mathbf{r}''$ be constant. Differentiating (5.360) with respect to time we obtain

$$\mathbf{v} = \mathbf{v}_{O'} + \dot{\mathbf{A}}\mathbf{r}'' = \mathbf{v}_{O'} + \dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{r},$$

where $\mathbf{r} = \mathbf{A}\mathbf{r}''$.

Because $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{E}$, differentiating that equality we obtain $\dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} + \mathbf{A}\dot{\mathbf{A}}^{\mathrm{T}} = 0$, and hence we obtain $\dot{\mathbf{A}}\mathbf{A}^{-1} = -\mathbf{A}\dot{\mathbf{A}}^{\mathrm{T}}$. Transposing both sides of the resulting equation we obtain

$$\left(\dot{\mathbf{A}}\mathbf{A}^{-1}\right)^{\mathrm{T}} = -\dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} = -\dot{\mathbf{A}}\mathbf{A}^{-1}.$$

Because we demonstrated that

$$\left(\dot{\mathbf{A}}\mathbf{A}^{-1}\right)^{\mathrm{T}} = -\left(\dot{\mathbf{A}}\mathbf{A}^{-1}\right),$$

the matrix $\dot{\mathbf{A}}\mathbf{A}^{-1}$ is a skew-symmetric matrix.

Let us choose this matrix in the following way:

$$\dot{\mathbf{A}}\mathbf{A}^{-1} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} \dot{\mathbf{A}}\mathbf{A}^{-1}\mathbf{r} &= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \left(\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} -\omega_3 r_2 + \omega_2 r_3 \\ \omega_3 r_1 - \omega_1 r_3 \\ -\omega_2 r_1 + \omega_1 r_2 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \end{bmatrix} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \\ &= \boldsymbol{\omega} \times \mathbf{r}, \end{aligned}$$

which proves the form of the equation in the theorem.

Then the elements of vector $\boldsymbol{\omega}$ are determined by the elements of matrix $\mathbf{A}$, which does not depend on the choice of the pole. This completes the proof. □

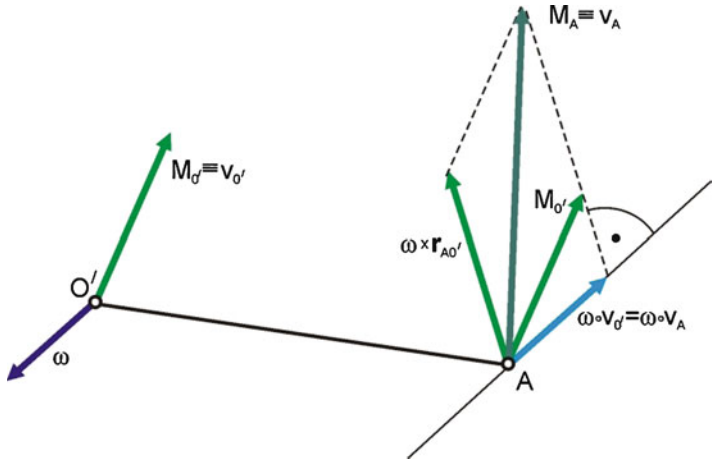


Fig. 5.79 Vector $\boldsymbol{\omega}$ and three main moments

Let us note [1] that after the change of pole (point O') vector $\boldsymbol{\omega}$ will not change (it is the so-called first invariant of system I_1). The second invariant of the system is the projection of the velocity of pole O' onto the instantaneous axis of rotation, i.e., the scalar product $I_2 = \widehat{\boldsymbol{\omega}} \circ \mathbf{v}_{O'} = \text{const}$, where $\widehat{\boldsymbol{\omega}}$ is the unit vector of the instantaneous axis of rotation. The projections of velocities of each of the points of the rigid body onto the direction of the instantaneous axis of rotation are identical.

Figure 5.79 shows points O' and A as well as vector $\boldsymbol{\omega}$ and three main moments. According to this figure we have

$$\mathbf{M}_A = \mathbf{M}_{O'} + \boldsymbol{\omega} \times \mathbf{r}_{AO'} = \mathbf{M}_{O'} - \boldsymbol{\omega} \times \mathbf{r}_{O'A}. \tag{5.361}$$

Projecting the vectors from (5.361) onto the direction of vector $\boldsymbol{\omega}$ we obtain

$$\boldsymbol{\omega} \circ \mathbf{M}_A = \boldsymbol{\omega} \circ \mathbf{M}_{O'} + \boldsymbol{\omega} \circ (\boldsymbol{\omega} \times \mathbf{r}_{AO'}) = \boldsymbol{\omega} \circ \mathbf{M}_{O'} \equiv \boldsymbol{\omega} \circ \mathbf{v}_{O'} \tag{5.362}$$

because vector $\boldsymbol{\omega} \perp \boldsymbol{\omega} \times \mathbf{r}_{AO'}$. Because points O' and A were arbitrarily chosen, one can conclude that the invariants of the system are vector $\boldsymbol{\omega}$ and the scalar product of the angular velocity vector and the velocity vector of an arbitrary point (in the present case, pole O').

By changing pole O' we change the moment $\mathbf{M}_{O'} \equiv \mathbf{v}_{O'}$. It was shown, however, that the projection of vector $\mathbf{v}_{O'}$ onto the axis $\boldsymbol{\omega}$ would not change [see (5.362)]. Because the main moment $\mathbf{M}_{O'}$ can be resolved into two components, the first one parallel to the direction of $\boldsymbol{\omega}$ in the form $_{\|}\mathbf{M}_{O'}$ and the second one perpendicular to it $_{\perp}\mathbf{M}_{O'}$, only the component $_{\perp}\mathbf{M}_{O'}$ can undergo change. One might expect that there would exist a special point C at which $_{\perp}\mathbf{M}_C = \mathbf{O}$. We will show how to find such a point C through which passes the central axis (instantaneous screw axis), which is parallel to the axis passing through pole O' (Fig. 5.80).

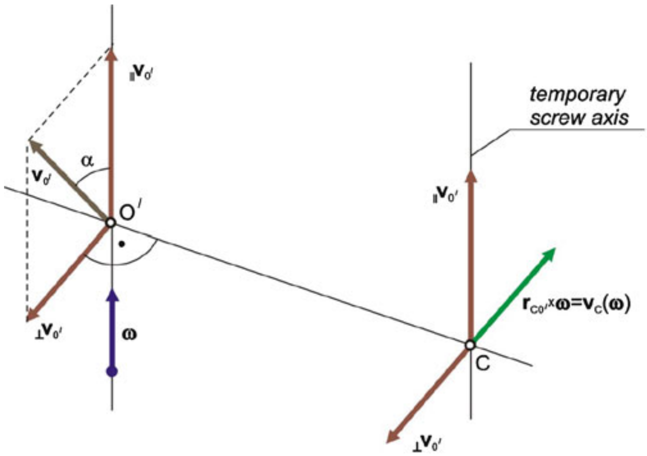


Fig. 5.80 Geometrical interpretation of the procedure leading to the determination of the position of point C (temporary screw axis)

Vectors $\boldsymbol{\omega}$ and $\mathbf{v}_{O'}$ determine a certain plane. The vector $\mathbf{v}_{O'} \equiv \mathbf{M}_{O'}$ we resolve into the components $_{\parallel}\mathbf{M}_{O'}$ and $_{\perp}\mathbf{M}_{O'}$ and draw a line perpendicular to the aforementioned plane and passing through point O'.

The desired point C will lie on that line at one of the sides of the aforementioned plane.

Let that point satisfy the following condition:

$$\mathbf{v}_C = \mathbf{v}_{O'} + \mathbf{v}_C(\boldsymbol{\omega}) = {}_{\perp}\mathbf{v}_{O'} + {}_{\parallel}\mathbf{v}_{O'} + \mathbf{v}_C(\boldsymbol{\omega}) = {}_{\parallel}\mathbf{v}_{O'}. \tag{5.363}$$

This means that

$$\mathbf{v}_C(\boldsymbol{\omega}) + {}_{\perp}\mathbf{v}_{O'} = \mathbf{r}_{CO'} \times \boldsymbol{\omega} + {}_{\perp}\mathbf{v}_{O'} = \mathbf{0},$$

that is, as can be seen from Fig. 5.79, the moments $_{\perp}\mathbf{v}_{O'} = -\mathbf{v}_C(\boldsymbol{\omega})$. This means that at point C vectors $_{\perp}\mathbf{v}_{O'}$ and $\mathbf{v}_C$ lie on one line and have opposite senses and the same magnitudes, which implies that they cancel out one another.

Finally, the vector of moment calculated about point C is parallel to vector $\boldsymbol{\omega}$ because $\boldsymbol{\omega} \parallel {}_{\parallel}\mathbf{v}_{O'}$. Then we take the magnitude of vector $\mathbf{r}_{CO'}$ according to the following condition:

$$|\mathbf{v}_C(\boldsymbol{\omega})| = |{}_{\perp}\mathbf{v}_{O'}| = r_{CO'}\omega. \tag{5.364}$$

The changes in the normal components of points O' and A while approaching point C are presented in Figs. 5.81 and 5.82.

From (5.364) it follows that

$$r_{CO'} = \frac{v \sin\alpha}{\omega}, \tag{5.365}$$

where v is the velocity of an arbitrary pole.

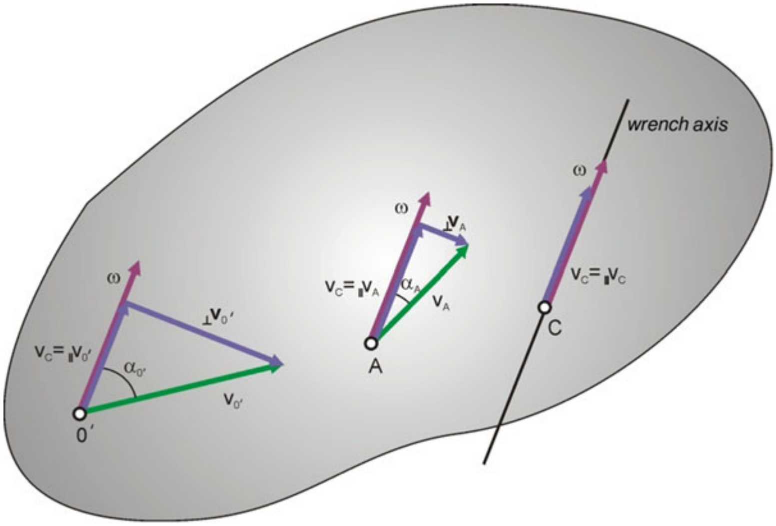


Fig. 5.81 A central axis (temporary screw axis) with point C ($_{\perp}\mathbf{v}_C = 0$)

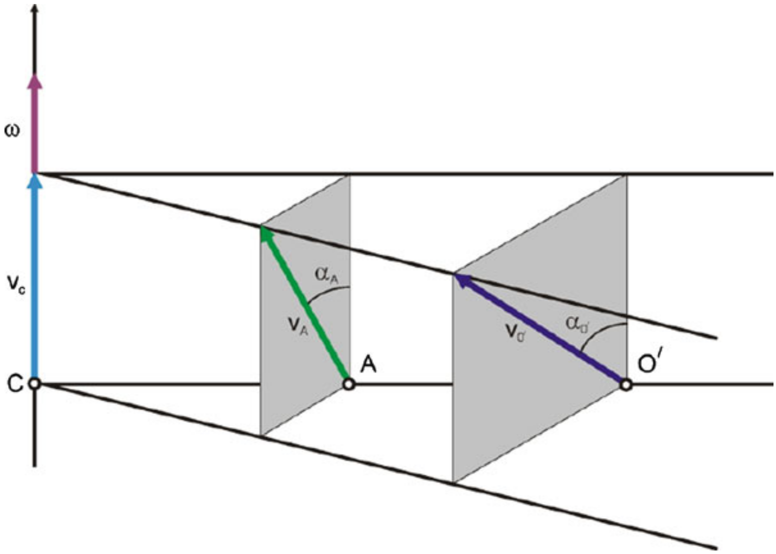


Fig. 5.82 Main moments (velocities) of points O', A, and C while approaching a wrench axis (a wrench axis has the smallest main moment, i.e., velocity vector $\mathbf{v}_C$)

The equation of a central axis in a stationary and non-stationary coordinate system in the case of motion about a point is described by (5.287) and (5.288). Figure 5.81 shows the velocities of poles O' and A of the main moments while approaching the central axis. At point C vectors $\boldsymbol{\omega}$ and $\mathbf{v}_C$ are parallel to one

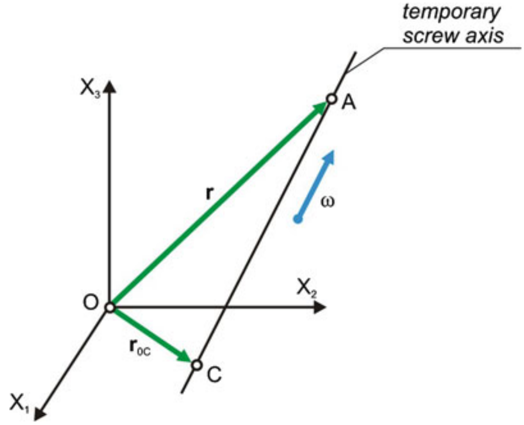


Fig. 5.83 Sketch leading to determination of a wrench axis

another, and at that position $\mathbf{v}_C$ reaches the minimum value. The axis on which lie vectors $\mathbf{v}_C$ and $\boldsymbol{\omega}$ is called the *central axis*, *wrench axis*, or *temporary screw axis*.

Having determined the position of point C the problem leading to the determination of the position of the central axis passing through that point is solved since the direction of that axis is determined by vector $\boldsymbol{\omega}$ (Fig. 5.83).

From the previous calculations we know that vector $\boldsymbol{\omega}$ is parallel to the central axis. It follows that

$$\mathbf{r} = \mathbf{r}_{OC} + \lambda\boldsymbol{\omega}. \tag{5.366}$$

The parameter λ is the proportionality coefficient between vectors $\mathbf{r}_{CA}$ and $\boldsymbol{\omega}$, i.e., $\mathbf{r}_{CA} = \lambda\boldsymbol{\omega}$. From (5.366) we obtain

$$\lambda\boldsymbol{\omega} = \mathbf{r} - \mathbf{r}_{OC}, \tag{5.367}$$

and multiplying successively by $\mathbf{E}_i''$ and $\mathbf{E}_i$ we obtain

$$\begin{aligned} \frac{x_1'' - x_{1C}''}{\omega_1''} &= \frac{x_2'' - x_{2C}''}{\omega_2''} = \frac{x_3'' - x_{3C}''}{\omega_3''} = \lambda, \\ \frac{x_1 - x_{1C}}{\omega_1} &= \frac{x_2 - x_{2C}}{\omega_2} = \frac{x_3 - x_{3C}}{\omega_3} = \lambda, \end{aligned} \tag{5.368}$$

where $\boldsymbol{\omega} \equiv \mathbf{E}_1\omega_1 + \mathbf{E}_2\omega_2 + \mathbf{E}_3\omega_3 = \mathbf{E}_1''\omega_1'' + \mathbf{E}_2''\omega_2'' + \mathbf{E}_3''\omega_3''$.

According to (5.368) the central axis will be described analytically if we determine the coordinates of point C, i.e., x_{1C}, x_{2C}, x_{3C}.

If point O' tends to point C, the angle $\alpha \to 0$, i.e., ${}_{\perp}\mathbf{v}_{O'} \to 0$ (Figs. 5.79–5.81).

In other words, the velocity $\mathbf{v}_{O'}$ will attain its minimum magnitude at the point of coordinates (x_{1C}, x_{2C}, x_{3C}) on the central axis. This condition can be used to determine the coordinates of point C:

$$
\begin{aligned}
x''_{1C} &= \frac{\omega''_2 v_{3O'} - \omega''_3 v_{2O'}}{\omega^2},\\
x''_{2C} &= \frac{\omega''_3 v_{1O'} - \omega''_1 v_{3O'}}{\omega^2},\\
x''_{3C} &= \frac{\omega''_1 v_{2O'} - \omega''_2 v_{1O'}}{\omega^2}.
\end{aligned}
\tag{5.369}
$$

The obtained equation of the central axis is an example of a line equation in slope-intercept form since knowing point C and vector $\boldsymbol{\omega}$ enables us to draw in three-dimensional space the desired line passing through that point and parallel to vector $\boldsymbol{\omega}$.

In the stationary coordinate system the velocity of point C is equal to

$$
\begin{aligned}
\mathbf{v} &= \mathbf{v}_{O'} + \mathbf{v}_{O'C} = \mathbf{v}_{O'} + \boldsymbol{\omega} \times (\mathbf{r}_C - \mathbf{r}_{OO'})\\
&= \dot{x}_1\mathbf{E}_1 + \dot{x}_2\mathbf{E}_2 + \dot{x}_3\mathbf{E}_3 = v_{1O'}\mathbf{E}_1 + v_{2O'}\mathbf{E}_2 + v_{3O'}\mathbf{E}_3\\
&\quad + \mathbf{E}_1 \left[\omega_2 (x_{3C} - x_{3O'}) - \omega_3 (x_{2C} - x_{2O'})\right]\\
&\quad + \mathbf{E}_2 \left[\omega_3 (x_{1C} - x_{1O'}) - \omega_1 (x_{3C} - x_{3O'})\right]\\
&\quad + \mathbf{E}_3 \left[\omega_1 (x_{2C} - x_{2O'}) - \omega_2 (x_{1C} - x_{1O'})\right]
\end{aligned}
\tag{5.370}
$$

because

$$
\boldsymbol{\omega} \times (\mathbf{r}_C - \mathbf{r}_{OO'}) = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ x_{1C} - x_{1O'} & x_{2C} - x_{2O'} & x_{3C} - x_{3O'} \end{vmatrix}.
\tag{5.371}
$$

The coordinates of point C in the stationary coordinate system are equal to

$$
\begin{aligned}
x_{1C} &= x_{1O'} + \frac{\omega'_2 v_{3O'} - \omega'_3 v_{2O'}}{\omega^2},\\
x_{2C} &= x_{2O'} + \frac{\omega'_3 v_{1O'} - \omega'_1 v_{3O'}}{\omega^2},\\
x_{3C} &= x_{3O'} + \frac{\omega'_1 v_{2O'} - \omega'_2 v_{1O'}}{\omega^2}.
\end{aligned}
\tag{5.372}
$$

The velocities of the points of a rigid body possess certain properties. Below we will give a few theorems whose proofs are omitted.

Theorem 5.13. *The velocities of three arbitrary non-collinear points of a rigid body completely describe the velocity of any point of that body.*

Theorem 5.14. *If the velocity vectors of three arbitrary non-collinear points of a rigid body are equal at a certain time instant, the body is in instantaneous translational motion.*

Theorem 5.15. *If at a certain time instant the velocities of two points of a rigid body are equal to zero, then this body is either at instantaneous rest or in instantaneous rotational motion about an axis passing through these points.*

Theorem 5.16. *If the velocity of a certain point of a rigid body is equal to zero at a certain time instant, then this body is at instantaneous rest or is rotating about an instantaneous axis of rotation passing through that point.*

Theorem 5.17. *The instantaneous motion of a rigid body in the general case is a composite motion composed of two motions, i.e., translational motion of an arbitrary pole and rotational motion about an axis passing through that pole.*

The carried out calculations and discussions in this section can be summed up by the following observations.

The general motion of a rigid body can be treated as its motion along the central axis with simultaneous rotation about that axis. Such motion we call the *screw motion*.

The general motion of a rigid body can be treated as consecutive sequences of screw motion in the successive time instants $t_0, t_1, t_2, \ldots$.

Also, the general motion of a rigid body is characterized by the motion of the moving axode with respect to the fixed axode, but now the rolling of the moving axode on the fixed axode takes place with sliding (along the generatrix).

So far we have used (5.359) to determine the velocity of an arbitrary point of a rigid body in general motion. Differentiating that formula with respect to time it is also possible to derive the relationships describing the acceleration of an arbitrary point of a given body. However, here we will make use of the previous considerations regarding the motion about a point and of the obtained (5.300) and (5.302), which allow for the determination of, respectively, the rotational and centripetal accelerations.

Because the general motion of a rigid body can be treated at a given instant as the composition of translational motion and motion about a point, to accelerations $\mathbf{a}_O$ and $\mathbf{a}_c$ obtained earlier one should geometrically add acceleration $\mathbf{a}_t$, associated with the translational motion of point O', which has the form

$$\mathbf{a}_t = \ddot{x}_{1O'}\mathbf{E}_1 + \ddot{x}_{2O'}\mathbf{E}_2 + \ddot{x}_{3O'}\mathbf{E}_3. \tag{5.373}$$

If we wish to express the components of $\mathbf{a}$ in the stationary coordinate system through components in the non-stationary system (or vice versa), then we should multiply those equations by $\mathbf{E}_i$ and $\mathbf{E}''_i$ (Sect. 5.5.12).

The acceleration of the considered point of a rigid body in general motion is equal to

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_O + \mathbf{a}_c, \tag{5.374}$$

so it is the geometrical sum of vectors $\mathbf{a}_t$, $\mathbf{a}_O$, and $\mathbf{a}_c$.

5.6 Composite Motion of a Rigid Body

The arbitrary motion of a rigid body encountered in a real (nature) or artificial (technology) setting is composed of a series of four basic instantaneous motions of that body, i.e., instantaneous rest, instantaneous translation, instantaneous rotation, and instantaneous screw motion. Below we will briefly characterize three of these instantaneous motions of a rigid body.

Let a rigid body be in motion with respect to the coordinate system $O'X_1'X_2'X_3'$. Then, let this system be moving with respect to the absolute system $OX_1X_2X_3$. The body will be in composite motion with respect to the coordinate system $OX_1X_2X_3$.

If a passenger on Earth, for example, is traveling on a moving bus, then the bus (a rigid body) is in motion with respect to the system $O'X_1'X_2'X_3'$ associated with Earth. The absolute system can be associated with the Sun or any other star. If an astronaut is in weightless conditions and moves with respect to a spacecraft with which we associate the system $O'X_1'X_2'X_3'$, then Earth can be the absolute system.

(i) Composition of two instantaneous translational motions

Let a body be moving in translational motion, and let it have at a certain time instant velocity $\mathbf{v}_1$ with respect to $O'X_1'X_2'X_3'$. Let the system $O'X_1'X_2'X_3'$ have at the same time instant velocity $\mathbf{v}_2$ with respect to $OX_1X_2X_3$. The absolute velocity of an arbitrary point of the body is given by

$$\mathbf{v}_a = \mathbf{v}_1 + \mathbf{v}_2; \tag{5.375}$$

$\mathbf{v}_1$ will be the relative velocity of that point and $\mathbf{v}_2$ the velocity of transportation.

(ii) Composition of two instantaneous rotational motions about parallel axes

Let a rigid body at a given time instant have the angular velocity $\boldsymbol{\omega}_1$ with respect to $O'X_1'X_2'X_3'$, and let the system $O'X_1'X_2'X_3'$ have the angular velocity $\boldsymbol{\omega}_2$ with respect to $OX_1X_2X_3$. Let $\boldsymbol{\omega}_1 \parallel \boldsymbol{\omega}_2$. In this case the velocities of the rigid body at the given instant will be the same as in the case of the planar motion of this body. The points of the body lying on an arbitrary line parallel to the angular velocity vector will have the same velocities. It suffices then to consider any plane $\pi \perp \boldsymbol{\omega}_1$ and draw a plane π' through $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. These planes will intersect along the line on which points A, B, and O will lie. At these points are respectively attached vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, and $\boldsymbol{\Omega}$ (Fig. 5.84).

If $\boldsymbol{\omega}_1 \circ \boldsymbol{\omega}_2 > 0$, then the instantaneous composite motion of the body is described by the vector $\boldsymbol{\Omega}$, where

$$\Omega = \omega_1 + \omega_2, \tag{5.376}$$

$$\omega_1 AO = \omega_2 BO. \tag{5.377}$$

We have

$$\begin{aligned} |\mathbf{v}_0| &= |\boldsymbol{\omega}_1 \times \overrightarrow{AO} + \boldsymbol{\omega}_2 \times \overrightarrow{AO}| \\ &= -\omega_1 AO + \omega_2 BO = 0. \end{aligned} \tag{5.378}$$

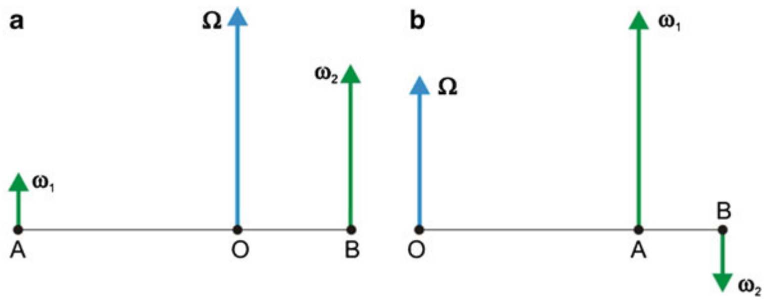


Fig. 5.84 Determination of angular velocity $\boldsymbol{\Omega}$ of composite motion of a rigid body when velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have identical (**a**) and opposite senses (**b**)

The instantaneous composite motion of a rigid body in this case can be treated as a motion with angular velocity $\boldsymbol{\Omega}$ given by (5.376) about point O described by (5.378).

The velocity of point B is equal to

$$\mathbf{v}_B = \boldsymbol{\Omega} \times \overrightarrow{OB} = \boldsymbol{\omega}_1 \times \overrightarrow{AB}, \tag{5.379}$$

which indicates that $\boldsymbol{\omega}_1 \parallel \boldsymbol{\Omega}$ and $\boldsymbol{\omega}_1 \circ \boldsymbol{\Omega} > 0$. From (5.379) we obtain

$$\Omega\,\mathrm{OB} = \omega_1 AB. \tag{5.380}$$

According to Fig. 5.84a we have

$$AB = AO + BO = \left(\frac{\omega_2}{\omega_1} + 1\right) BO = \frac{\omega_2 + \omega_1}{\omega_1} BO. \tag{5.381}$$

Substituting (5.381) into (5.380) we obtain (5.376).

In a similar way it can be shown that for the case from Fig. 5.84b we have

$$\Omega = \omega_1 - \omega_2, \tag{5.382}$$

$$\omega_1 AO = \omega_2 BO, \tag{5.383}$$

on the assumption that $\omega_1 > \omega_2$.

(iii) Couple of (instantaneous) rotations

A special case, depicted in Fig. 5.84b, where $\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2$, defines a *couple of (instantaneous) rotations*. The plane determined by vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ is called the *plane of a couple of rotations*, the distance AB is called the *arm of a couple of rotations*, and $\overrightarrow{AB} \times \boldsymbol{\omega}_2$ is called the moment of a couple of rotations. We will show that if a rigid body is subjected to the action of a couple of rotations at a given time instant, then it is in instantaneous translational motion with velocity equal to the the moment of a couple of rotations.

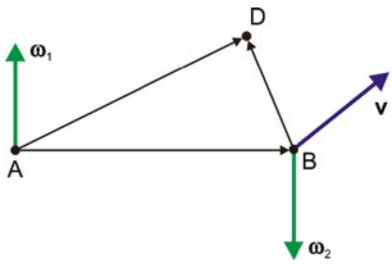


Fig. 5.85 Determination of velocity of point D of a rigid body and a couple of rotations

Let us take an arbitrary point of the rigid body D. The velocity of that point (Fig. 5.85) is equal to

$$\begin{aligned}\mathbf{v} &= \boldsymbol{\omega}_1 \times \overrightarrow{AD} + \boldsymbol{\omega}_2 \times \overrightarrow{BD} \\ &= \overrightarrow{AD} \times \boldsymbol{\omega}_2 - \overrightarrow{DB} \times \boldsymbol{\omega}_2 \\ &= \left(\overrightarrow{AD} - \overrightarrow{BD}\right) \times \boldsymbol{\omega}_2 = \overrightarrow{AB} \times \boldsymbol{\omega}_2.\end{aligned} \tag{5.384}$$

In (5.384) the subscript D in the velocity was dropped because velocity vector $\mathbf{v}$ is a free vector. All points of the body have the same instantaneous translational velocity equal to $\mathbf{v}$. Vectors $\overrightarrow{AB}$, $\boldsymbol{\omega}_2$, and $\mathbf{v}$ form a right-handed triad of vectors. If $|\boldsymbol{\omega}_1| = |\boldsymbol{\omega}_2| = \omega$, then the velocity of an arbitrary point of the body is equal to $v = \omega AB$.

If the body is in instantaneous translational motion with velocity $\mathbf{v}$, then this motion is equivalent to a body moving under the the action of a couple of rotations situated in a plane perpendicular to $\mathbf{v}$, of arm AB, and with magnitude of angular velocity ω such that the equation $v = \omega AB$ is satisfied, which can be realized in infinitely many ways.

(iv) Composition of rotational motions of a rigid body about intersecting axes

The rotational motion of a rigid body can be represented by the *instantaneous axis of rotation*, that is, the locus of points whose velocities at a given time instant are equal to zero.

The angular velocity vector is the vector sliding along this axis; if we look along its direction at its arrow, then the rotation occurs counterclockwise.

Figure 5.86 presents a rigid body (disk) rotating with angular velocity ω_2 about the CO axis (it is the relative rotation), and this motion is transported by a beam rotating with angular velocity ω_1 (rotation of transportation). Both axes of rotation intersect at one point O.

Figure 5.87 shows one method for determining the absolute velocity of body 1.

Theorem 5.18. *The angular velocity of absolute rotation* $\boldsymbol{\omega}$ *of a rigid body in composite motion associated with rotations about two intersecting axes is equal to the geometric sum of angular velocities with respect to those axes:*

$$\boldsymbol{\omega} = \boldsymbol{\omega}^t + \boldsymbol{\omega}^r. \tag{5.385}$$

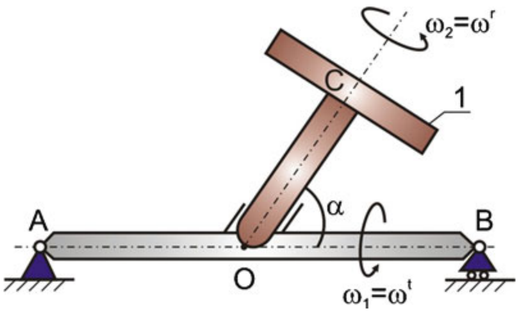


Fig. 5.86 Disk 1 mounted to OC axle rotating with angular velocity $\omega_2 = \omega^r$, which in turn is mounted to AB axle rotating with angular velocity $\omega_1 = \omega^t$

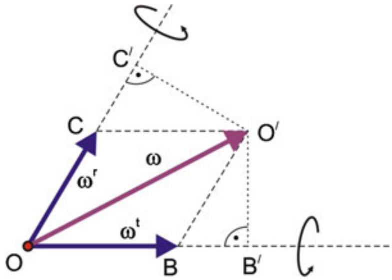


Fig. 5.87 Geometric composition of angular velocities of a rigid body

We will prove the validity of vector (5.385). Point O, being the vertex of a parallelogram (Fig. 5.87), simultaneously belongs to two instantaneous axes of rotation OC and OB, and therefore $v_O = 0$. Point O' belongs to the body and simultaneously participates in two rotations, that is, about the OC and OB axes.

The velocities

$$\begin{aligned} v^r_{O'} &= \omega^r O'C' = 2F_{\Delta OCO'}, \\ v^t_{O'} &= \omega^t O'C' = 2F_{\Delta OBO'}, \end{aligned} \tag{5.386}$$

and because the areas of the triangles on the right-hand sides of (5.386) are equal, we have $\mathbf{v}^r_{O'} = -\mathbf{v}^t_{O'}$, and, finally,

$$\mathbf{v}_{O'} = \mathbf{v}^r_{O'} + \mathbf{v}^t_{O'} = \mathbf{0}. \tag{5.387}$$

We have demonstrated that the instantaneous velocity of point O' is equal to zero and in view of that the OO' axis is the instantaneous axis of absolute rotation of a rigid body.

Let us take in Fig. 5.87 an arbitrary point A of a rigid body such that $\overrightarrow{OA} = \mathbf{r}$. The absolute velocity of this point

$$\mathbf{v}_A = \mathbf{v}^r_A + \mathbf{v}^t_A. \tag{5.388}$$

The absolute velocity of point A reads

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}, \tag{5.389}$$

where $\boldsymbol{\omega}$ is the sliding vector of the instantaneous axis of rotation OO' and in turn

$$\mathbf{v}_A^r = \boldsymbol{\omega}^r \times \mathbf{r},$$
$$\mathbf{v}_A^t = \boldsymbol{\omega}^t \times \mathbf{r}. \tag{5.390}$$

Substituting (5.389) and (5.390) into (5.388) we obtain

$$\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega}^r \times \mathbf{r} + \boldsymbol{\omega}^t \times \mathbf{r} = (\boldsymbol{\omega}^r + \boldsymbol{\omega}^t) \times \mathbf{r}, \tag{5.391}$$

which proves the validity of (5.385).

If a rigid body simultaneously rotates about N instantaneous axes of rotation which intersect, then the angular velocity $\boldsymbol{\omega}$ of the absolute rotation is equal to

$$\boldsymbol{\omega} = \sum_{n=1}^{N} \boldsymbol{\omega}_n. \tag{5.392}$$

Theorem 5.19. *The absolute angular acceleration of a rigid body rotating non-uniformly about two intersecting axes is equal to the geometric sum of the following angular accelerations: of transportation $\boldsymbol{\varepsilon}^t$, relative $\boldsymbol{\varepsilon}^r$, and rotational $\boldsymbol{\varepsilon}^c$.*

Proof. In order to prove the equation

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^r + \boldsymbol{\varepsilon}^c$$

we will represent (5.385) in the form

$$\boldsymbol{\omega} = \boldsymbol{\omega}^t + \mathbf{E}_1'\omega_1'^r + \mathbf{E}_2'\omega_2'^r + \mathbf{E}_3'\omega_3'^r,$$

where the axes of the system $O'X_1'X_2'X_3'$ are rigidly connected with a rigid body during its relative motion.

Differentiating the last equation with respect to time we obtain

$$\begin{aligned}
\boldsymbol{\varepsilon} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{\omega}^t}{\mathrm{d}t} + \dot{\mathbf{E}}_1'\omega_1'^r + \dot{\mathbf{E}}_2'\omega_2'^r + \dot{\mathbf{E}}_3'\omega_3'^r + \mathbf{E}_1'\dot{\omega}_1'^r + \mathbf{E}_2'\dot{\omega}_2'^r + \mathbf{E}_3'\dot{\omega}_3'^r \\
&= \boldsymbol{\varepsilon}^t + \boldsymbol{\omega}^t \times (\mathbf{E}_1'\omega_1'^r + \mathbf{E}_2'\omega_2'^r + \mathbf{E}_3'\omega_3'^r) + \mathbf{E}_1'\dot{\omega}_1'^r + \mathbf{E}_2'\dot{\omega}_2'^r + \mathbf{E}_3'\dot{\omega}_3'^r \\
&= \boldsymbol{\varepsilon}^t + \boldsymbol{\omega}^t \times \boldsymbol{\omega}^r + \boldsymbol{\varepsilon}^r = \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^r + \boldsymbol{\varepsilon}^c,
\end{aligned}$$

which proves the theorem.

We call the foregoing $\boldsymbol{\varepsilon}^c = \boldsymbol{\omega}^t \times \boldsymbol{\omega}^r$ the angular rotational acceleration.

Note that $\boldsymbol{\omega}^r = \boldsymbol{\omega} - \boldsymbol{\omega}^t$, and therefore

$$\begin{aligned}\boldsymbol{\varepsilon}^c &= \boldsymbol{\omega}^t \times \boldsymbol{\omega}^r = \boldsymbol{\omega}^t \times (\boldsymbol{\omega} - \boldsymbol{\omega}^t)\\ &= \boldsymbol{\omega}^t \times \boldsymbol{\omega} - \boldsymbol{\omega}^t \times \boldsymbol{\omega}^t = \boldsymbol{\omega}^t \times \boldsymbol{\omega}.\end{aligned}$$ □

The component of absolute angular acceleration equal to the vector product of the angular velocity of transportation and the relative angular velocity or to the vector product of the angular velocity of transportation and the absolute rotational velocity of a body is called the rotational angular acceleration of a rigid body.

The acceleration $\boldsymbol{\varepsilon}^c$ characterizes the change in direction of angular velocity $\boldsymbol{\omega}^r$ caused by the rotational motion of transportation of the body.

Let us consider three simple cases of addition of vectors of angular accelerations being the result of the composition of the rotational motions of a rigid body about two intersecting axes.

1. Uniform rotation of transportation:

$$\boldsymbol{\omega}^t = \text{const}; \quad \boldsymbol{\varepsilon}^t = \mathbf{0}; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^r + \boldsymbol{\varepsilon}^c.$$

2. Uniform relative rotation:

$$\boldsymbol{\omega}^r = \text{const}; \quad \boldsymbol{\varepsilon}^r = \mathbf{0}; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^c.$$

3. Uniform relative and transportation rotations:

$$\boldsymbol{\varepsilon}^t = \mathbf{0}; \quad \boldsymbol{\varepsilon}^r = \mathbf{0}; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^c.$$

(v) Composition of instantaneous translational and rotational motion

Let a rigid body rotate with respect to the coordinate system $O'X_1'X_2'X_3'$ with instantaneous angular velocity $\boldsymbol{\omega}$, and let the coordinate system $O'X_1'X_2'X_3'$ move with instantaneous translational velocity $\mathbf{v}$ with respect to the space system $OX_1X_2X_3$. Let us take an arbitrary point of body A and attach the vectors $\mathbf{v}$ and $\boldsymbol{\omega}$ at that point (Fig. 5.88).

We will resolve velocity $\mathbf{v}$ into two components, $\mathbf{v}_1$ and $\mathbf{v}_2$, and replace the component $\mathbf{v}_2$ with the couple of rotations of arm AB, and velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2 = -\boldsymbol{\omega}_1$ ($\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ lie in a plane perpendicular to $\mathbf{v}_2$). Because $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \mathbf{0}$, in Fig. 5.88a there remain only two vectors, $\mathbf{v}_1$ and $\boldsymbol{\omega}_2$. The instantaneous motion of the body in this case is equivalent to instantaneous rotation with velocity $\boldsymbol{\omega}_2$ and instantaneous translation with velocity $\mathbf{v}_1$, where $\mathbf{v}_1 \parallel \boldsymbol{\omega}_2$.

Because $v_1 = v\cos\alpha$ and vector $\mathbf{v}_1$ is the free vector, we can move it to point B (Fig. 5.88b). The composition of the instantaneous translational and rotational motions is the instantaneous screw motion. The instantaneous axis of the screw motion is parallel to the angular velocity $\boldsymbol{\omega}$ of the body, and they lie at the distance $AB = (v\sin\alpha)/\omega$. The pitch of the *kinematic screw* is equal to $\lambda = (v\cos\alpha)/\omega$.

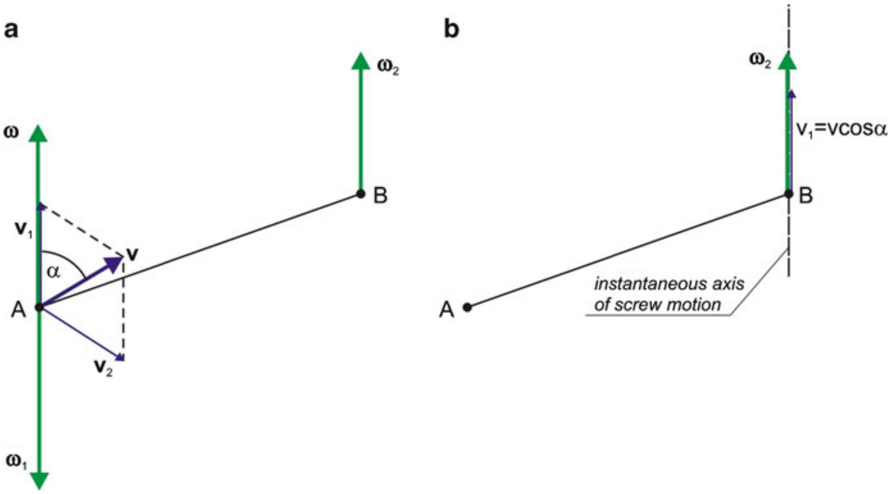


Fig. 5.88 Composition of instantaneous translational motion with velocity **v** and rotational motion with angular velocity $\boldsymbol{\omega}$ (**a**) and the schematic diagram of a kinematic screw (**b**)

If the angle between **v** and $\boldsymbol{\omega}$ is equal to $\alpha = \pi/2$, then $\mathbf{v} \perp \boldsymbol{\omega}$ and $\lambda = 0$, and the instantaneous composite motion of the body will be equivalent to the instantaneous rotation with velocity $\boldsymbol{\omega}$ passing through point B, where the distance $AB = v/\omega$.

If $\alpha = 0$, then vectors **v** and $\boldsymbol{\omega}$ immediately form the *kinematic screw*.

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Chapter 6
Kinematics of a Deformable Body

6.1 Tensors in Mechanics

The notion of a tensor appears not only in kinematics, but also in statics, and the term was intentionally used several times in previous chapters of the book (see also [1, 2]).

Chapter 2 presented an example of a flexible beam (Fig. 2.2) that, despite being deformed, is in static equilibrium. The deformation of that beam is described by a deformation tensor. The application of a tensor is often indispensable also when solving statically indeterminate problems where, as was mentioned in Chap. 2 (Example 2.5), we must know the relationships between the stresses (forces) and deformations (displacements) in deformable bodies, that is, we must know the so-called stress tensor, which is a two-dimensional tensor. The application of tensor calculus appears in a natural way in the mechanics of deformable bodies and continuum mechanics [3–8].

There are many books on classical mechanics, and all authors justify their own way of presenting the material, motivating it by the didactic requirements of the book. In the present book we try, as far as possible, to preserve a certain natural and historical view of the development of mechanics while at the same time striving to remain within the framework of Newtonian mechanics.

The fundamental attribute in mechanics for Newton was the notion of *force*. His dream was the discovery and explanation of the nature of force and an explanation of the properties of bodies' motion with the aid of the introduced by him notions of forces. So far, in the statics of particles and rigid bodies the attributes of the force are the *magnitude* (the intensity expressed by a number), *direction* of action, and *sense*, which allow for a description of force by means of the mathematical notion of a *force vector*.

However, such an approach has its drawbacks. These consist of the lack of information regarding the nature (physical aspect) of a force. We use the same vector to represent the force occurring in the point of contact of two material bodies, the gravity force (contactless), the electrostatic force, etc. Until now, in discussions on

J. Awrejcewicz, *Classical Mechanics: Kinematics and Statics*, Advances in Mechanics and Mathematics 28, DOI 10.1007/978-1-4614-3791-8_6,

the statics of a perfectly rigid body, the force vector was a sliding vector, that is, it was possible to move it along its line of action. We will show, however, in the following sections that this is not always the case. We cannot proceed in that way in cases of a deformable body or a body in a field of parallel forces. Examples of scalar and vector quantities in mechanics were presented earlier (Chap. 1).

The scalar in Euclidean space requires the definition of one quantity (a number), whereas the vector requires the knowledge of three quantities (numbers). However, both scalar and vector notions in mechanics have physical meaning and denote an *objective* physical (mechanical) quantity.

For instance, temperature (scalar) or direction distributions, points of application, and force magnitudes, after all, do not depend on the type of introduced coordinates (Cartesian or curvilinear). Therefore, the main goal of mechanics is the introduction of mathematical apparatus that would best reflect (model) the physical (mechanical) quantities.

Neither temperature nor force depends on the choice of coordinates or axes introduced to carry out the analysis of a problem since they are physically objective quantities.

In the case of an arbitrary scalar quantity in a certain subspace of Euclidean space we can build a functional dependency between a point of that subspace described by three numbers (x_1, x_2, x_3) and a function value. For example, in the case of a scalar quantity such as temperature we are dealing with the function $T = T(x_1, x_2, x_3)$, that is, with a *uniquely* established relationship between the point of the space and the scalar (temperature). If in such a subspace we assign the value of that function to every one of its points, it is said that a scalar field of the considered physical quantity, in our case *temperature field*, is defined.

For any scalar field the function defining that field is *invariant*, that is, it does not depend on the choice of coordinate system. Such property, however, does not hold for a *vector field*. A change in the type of coordinate system, and even its position in the considered bounded space, leads to a change in the projections of the given vector (e.g., the aforementioned force) onto the axes of the coordinate system. In this sense, the projection of a vector onto axes does not retain *invariance* and is a *variant* process. Nevertheless, any choice of coordinate system preserves the *length of a vector*, in this case the magnitude of the force. Thus, the length of a vector is an invariant quantity.

Not without reason did we use earlier a certain logical numbering scheme for the coordinates (x_1, x_2, x_3), the coordinate axes $OX_1X_2X_3$, and for new coordinates $OX_1'X_2'X_3'$, $OX_1''X_2''X_3''$, etc. The introduction of the preceding numbering scheme allows for an uncomplicated notation of the relationships between the coordinates of points in old $(OX_1X_2X_3)$ and new $(OX_1'X_2'X_3')$ coordinate systems through the introduction of a rotation tensor.

Let us return for a while to the analysis presented in Chap. 5 and as an example consider the relationship between "old" and "new" coordinates described, for instance, by (5.296).

The aforementioned algebraic relationships can be obtained very quickly using the following Table 6.1 [9].

Table 6.1 Direction cosines

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

where a_{ij}, $(i, j = 1, 2, 3)$ are direction cosines.

Using the *rows* of the preceding table we write

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \tag{6.1}$$

and then, using the table *columns* we obtain

$$\begin{aligned} x_1 &= a_{11}x'_1 + a_{21}x'_2 + a_{31}x'_3, \\ x_2 &= a_{12}x'_1 + a_{22}x'_2 + a_{32}x'_3, \\ x_3 &= a_{13}x'_1 + a_{23}x'_2 + a_{33}x'_3. \end{aligned} \tag{6.2}$$

The systems of (6.1) and (6.2) are respectively equivalent to the following shortened notation form:

$$x'_i = \sum_{j=1}^{3} a_{ij}x_j, \quad x_j = \sum_{i=1}^{3} a_{ij}x'_i. \tag{6.3}$$

The foregoing notation can be simplified even more by dropping the sigma sign, and now (6.3) take the following form:

$$x'_i = a_{ij}x_j \quad (i = 1, 2, 3), \qquad x_j = a_{ij}x'_i \quad (j = 1, 2, 3). \tag{6.4}$$

In this notational convention, summation is done with respect to the repeated indices (in the first case j and in the second i), which disappear in the process of summation and are called *summation indices*. The remaining indices (in the first case i, in the second j) are called *free indices*.

Let vector $\mathbf{r}$ have non-zero coordinates of the tail, that is, we have

$$\begin{aligned} \mathbf{r} &= \mathbf{E}_1(x_1 - x_{10}) + \mathbf{E}_2(x_2 - x_{20}) + \mathbf{E}_3(x_3 - x_{30}) \\ &= \mathbf{E}'_1(x'_1 - x'_{10}) + \mathbf{E}'_2(x'_2 - x'_{20}) + \mathbf{E}'_3(x'_3 - x'_{30}). \end{aligned} \tag{6.5}$$

Using relationship (6.4) we have

$$\begin{aligned} x'_i &= a_{ij}x_j, \qquad x_j = a_{ij}x'_i, \\ x'_{i0} &= a_{ij}x_{j0}, \quad x_{j0} = a_{ij}x'_{i0}, \end{aligned} \tag{6.6}$$

and subtracting by sides the second equation from the first one we obtain

$$x'_i - x'_{i0} = a_{ij}(x_j - x_{j0}), \qquad x_j - x_{j0} = a_{ij}(x'_i - x'_{i0}), \tag{6.7}$$

that is,

$$r'_i = a_{ij} r_j, \qquad r_j = a_{ij} r'_i, \tag{6.8}$$

where now $\mathbf{r}[r_1, r_2, r_3] = \mathbf{r}'[r'_1, r'_2, r'_3]$.

Relationships (6.8) concern the coordinates of a vector and have a form analogous to the relationships regarding the coordinates of a point [(6.4)].

We will show now that relationships (6.8) preserve the physical aspect of a vector quantity.

Using the summation convention introduced earlier we have

$$r'_m r'_m = a_{mk} r_k a_{ml} r_l = a_{mk} a_{ml} r_k r_l. \tag{6.9}$$

From the properties of coefficients a_{mk} describing the cosines of angles between axes m and k, coefficients a_{ml} describing the cosines of angles between axes m and l, and their summation $a_{mk} a_{ml}$ (m plays the role of summation index), and from relationships (3.53) derived earlier we obtain

$$r'_m r'_m = r_l r_l, \tag{6.10}$$

that is, in expanded form,

$$(r'_1)^2 + (r'_2)^2 + (r'_3)^2 = (r_1)^2 + (r_2)^2 + (r_3)^2, \tag{6.11}$$

which proves the *invariance* of the vector length.

The greatest role in mechanics, however, is played by a tensor of the second rank. The work of Cauchy,[1] published in 1827, regarding the formulation of the so-called stress formulas in a deformable body, points to the need to apply such a tensor. In mechanics, apart from the stress tensor there also appears, e.g., the moment of inertia tensor (see Chap. 3), the strain tensor, deformation velocity tensor.

6.2 Body Kinematics and Stresses

So far we have introduced the idealized notions of a particle and system of particles. If there are infinitely many particles and they are distributed very close to each other, then we are dealing with the dense mass distribution, and a *discrete* mechanical

[1] Augustin L. Cauchy (1789–1857), French mathematician of genius, his collected works were published in 27 volumes.

system (e.g., of lumped mass) transforms into the *continuous* system (it can be treated as a process of mass spreading). Also the mechanical properties of such a system transform from discrete to continuous.

The simplest (idealized) example of a continuous system is a perfectly rigid body, with which we have already dealt many times. It turns out that the methods and the concept of equations of statics obtained on the basis of the idealized perfectly rigid body can be applied also to flexible bodies that after deformation remain in static equilibrium. If, for instance, one initially stretches a leather belt, and afterward the system remains in equilibrium acted on by tension forces, then in that position we can increase the stiffness of the belt (body) and consider it a perfectly rigid body and, consequently, apply the *methods and laws of statics* discussed earlier. However, in order to calculate the mentioned elongation of the belt it is necessary to exploit the theory of deformable bodies. This observation will be proved later.

Until now we have used the statement that "the forces $\mathbf{F}_i$ are applied at points A_i of a rigid body." Already here the idealization appears since it is hard to imagine the action of real, natural interactions between bodies, especially at their direct contact, by means of the point force. Rather, the "transmission" of force takes place over a certain (often very small) surface. Earlier we applied, especially while solving statics examples, the method of sections (intersecting) to the massless elements of cross sections tending to a point (e.g., massless models of rods, strings, ropes), and we replaced the interaction caused by the "cut away" part with the action of external force $\mathbf{F}^e$ (e.g., see Chap. 1).

If we expand the notion of force to a surface (which is closer to reality) instead of a point, then we should introduce the notion of surface force (stress). If we have a bounded surface of area S $[m^2]$, then the average density of distribution of surface forces over the mentioned surface is described by

$$\boldsymbol{\sigma}_{\text{av}} = \frac{\mathbf{F}_S}{S} \quad \left[\frac{\text{N}}{\text{m}^2}\right], \tag{6.12}$$

and vector $\boldsymbol{\sigma}_{\text{av}}$ is expressed in pascals. This means that at every point of surface S the stress vector $\boldsymbol{\sigma}_{\text{av}}$ is applied. Now, if we consider the "contraction" of the surface to a point (i.e., when $S \to 0$), from (6.12) we obtain a stress vector at the point of the form

$$\boldsymbol{\sigma} = \lim_{S \to 0} \frac{\mathbf{F}^{(S)}}{S} = \frac{\delta \mathbf{F}_S}{\delta S}. \tag{6.13}$$

The symbol δ denotes the infinitely small quantity on a surface and differs from the notion of differential d, which signifies the increment of a certain physical quantity related to the increment of an independent variable.

If we take a certain infinitesimal surface δS, then we calculate the surface force acting on it from the formula

$$\delta \mathbf{F}^{(S)} = \boldsymbol{\sigma} \delta S, \tag{6.14}$$

and further it is called the elementary surface force. In this way we "stretched" the force concentrated at a point onto the surface.

Surface forces are a model of thickly “packed” forces laid out continuously and densely over a certain (mentally) isolated surface. If an object (a body) lies on another and contact takes place over a certain surface, we are dealing with surface gravity forces that are related to the given surface. If a plate is immersed in liquid, the forces of liquid pressure act on both its sides. These surface forces are perpendicular to the plate’s surface, and their magnitude is determined by the distance of the horizontally oriented plate from the surface of the liquid in a container.

The physical quantity that is most frequently encountered and described in textbooks characterizing continuous mechanical systems is the continuity of mass distribution in a certain volume, that is, in isolated three-dimensional space. If we consider a volume V $[\mathrm{m}^3]$, the density of mass distribution over the whole volume is given by

$$\rho_{\mathrm{av}} = \frac{M}{V} \quad \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \tag{6.15}$$

and we obtain the density of mass distribution at point A from (6.15) moving on to the limit $V \to 0$, that is,

$$\rho_A = \lim_{V \to 0} \frac{M}{V}. \tag{6.16}$$

Introducing the notion of infinitesimal mass δM and volume δV we have

$$\rho_A = \frac{\delta M}{\delta V}. \tag{6.17}$$

In a similar way we define the force distribution with respect to the volume of a continuous system. The *averaged volume force* is defined by the equation

$$\mathbf{F}_{\mathrm{av}}^{(V)} = \frac{\mathbf{F}}{M} = \frac{\mathbf{F}}{\rho_{\mathrm{av}} V} \quad \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right], \tag{6.18}$$

where $\mathbf{F}$ is a main vector (geometric sum) of forces applied at all points inside volume V of mass M. Note that $\mathbf{F}_{\mathrm{av}}^{(V)}$ has a dimension of acceleration.

Moving on with the volume $V \to 0$ we obtain a definition of volume force at a point described by the equation

$$\mathbf{F}^{(V)} = \lim_{V \to 0} \frac{\mathbf{F}}{\rho_{\mathrm{av}} V} = \frac{\delta \mathbf{F}}{\rho \delta V}. \tag{6.19}$$

Knowing the volume force at a point, we calculate the main vector of forces at that point from the equation

$$\delta \mathbf{F} = \rho \mathbf{F}^{(V)} \delta V. \tag{6.20}$$

According to (6.18) we can calculate the volume force in the field of gravity forces, which is equal to

$$\mathbf{F}_{\mathrm{av}}^{(V)} = \frac{M\mathbf{g}}{M} = \mathbf{g} \quad \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]. \tag{6.21}$$

However, it should be noted that also here we are dealing with a certain approximation since the gravity forces are actually not parallel. If we take two points on the surface of the Earth separated by a distance of 1 km and lying on the meridian, and we place there identical masses, an angle between vectors of gravity forces will be equal to $32''$.

If one of the dimensions of a surface tends to zero, then we obtain a spatial curve (one-dimensional body), for example, a rope, and the notion of force related to a unit of length can be introduced $\mathrm{N/m}\left[\mathrm{kg/s^2}\right]$.

The summation of elementary surface forces over the whole surface under consideration leads to the determination of a *main vector of surface forces*, and the summation of *elementary volume forces* leads to obtaining the *main vector of volume forces*. In the case of mechanical systems, such a summation is replaced with integration over the surface or the volume.

The forces distributed along one dimension (curve) and over the surface and volume of a solid were discussed in Chap. 3 by the introduction of the notions of mass center, centroid of a plane figure (three dimensional figure), and center of gravity.

Often in practice, because of the application of Cartesian coordinate systems, right triangles or squares are introduced as elementary surfaces, and tetrahedra or cubes as elementary volumes. Tetrahedra are such that three of their faces are right triangles whose vertices coincide and are located at the origin of coordinate system $OX_1X_2X_3$.

Let us take an arbitrary point O of a (rigid or deformable) body and then take the coordinate system $OX_1X_2X_3$. Then let us cut out from our body the aforementioned tetrahedron bounded by the faces lying in the planes $O - X_1 - X_3$, $O - X_2 - X_3$, and $O - X_1 - X_2$. Subsequently, near point O we draw an arbitrary plane such that it intersects the axes of the coordinate system at points O_1, O_2, and O_3, respectively (Fig. 6.1).

We will replace the action of the body (the part left after cutting out the pyramid) on the faces of the tetrahedron with the action of a surface force. Because we are dealing with three elementary right triangles ΔOO_1O_3, ΔOO_2O_3, and ΔOO_1O_2 constituting the rear faces of the pyramid, $\Delta O_1O_2O_3$ constitutes its front face.

Since all three triangles are very small, we are going to assume that the density of distribution of surface forces is the same for particular faces of triangles and is described by one direction and sense.

The areas of the aforementioned triangles we respectively denote by $S_{\Delta OO_2O_3} = \delta S_1$, $S_{\Delta OO_1O_3} = \delta S_2$, $S_{\Delta OO_1O_2} = \delta S_3$, and $S_{\Delta O_1O_2O_3} = \delta S$. The surface forces acting on the faces of the triangle and coming from the part of the material that remained after cutting are as follows: $-\boldsymbol{\sigma}_1\delta S_1$, $-\boldsymbol{\sigma}_2\delta S_2$, $-\boldsymbol{\sigma}_3\delta S_3$, $\boldsymbol{\sigma}^{(n)}\delta S$.

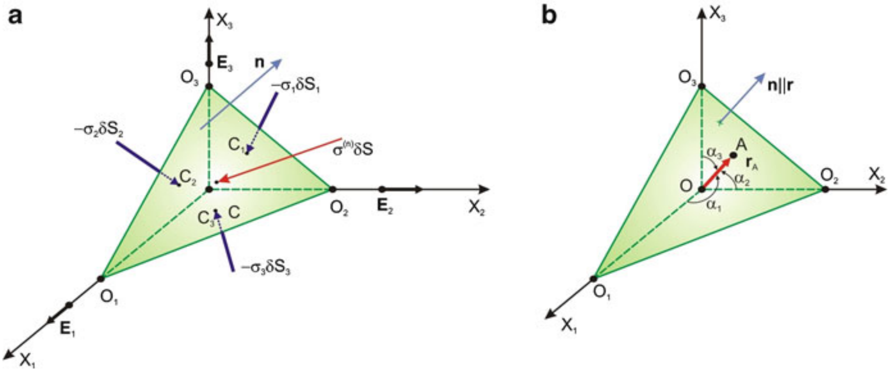

Fig. 6.1 A tetrahedron $OO_1O_2O_3$ cut out from a body; main vectors of surface forces and normal vectors to pyramid's faces

The convention of applied signs adopted for the named surface force vectors follows from the observation that if point O approached face $O_1O_2O_3$ and finally rested on it, the surface force on the reverse of $\Delta O_1O_2O_3$ would have a sense opposite to the surface force on the of $\Delta O_1O_2O_3$ (the aforementioned $\Delta O_1O_2O_3$ is in equilibrium if these forces cancel out). The sense of the surface force acting on the obverse of $\Delta O_1O_2O_3$ is assumed positive. Figure 6.1b illustrates how to easily write the equation of a plane in which triangle $\Delta O_1O_2O_3$ lies after choosing a certain point A belonging to the surface of that triangle. A vector $\overrightarrow{OA} = \mathbf{r}_A$ (which must be perpendicular to the plane $\Delta O_1O_2O_3$) is now the position vector of the plane, and the direction cosines of $\mathbf{r}(x_1, x_2, x_3)$ are known. The equation of the mentioned plane, also called a *normal equation of a plane*, has the form

$$x_1 \cos\alpha_1 + x_2 \cos\alpha_2 + x_3 \cos\alpha_3 - |\mathbf{r}| = 0. \tag{6.22}$$

Because surface forces are distributed with identical density over each of the triangles, they can be replaced with the main vectors applied at centroids C_1, C_2, C_3, C of the considered triangles. The centroids are found as a result of the intersection of medians of every triangle.

Thus, we have the main force vector attached at the geometrical center of every face of the pyramid. The direction of the main force vector usually does not coincide with the direction of the normal vector of the face. Apart from the surface forces acting on four faces of the tetrahedron, the volume force $\rho\mathbf{F}dV$ also acts, but it is not plotted in Fig. 6.1. The surface forces are "small," of the second order, because the areas of right triangles are equal to magnitudes of $1/2\ \delta x_1\delta x_2$, $1/2\ \delta x_2\delta x_3$, and $1/2\ \delta x_1\delta x_3$, and hence the volume of the tetrahedron is equal to $1/3\ S_{\Delta O_1O_2O_3}OC$, which is "small," of a third-order magnitude, and therefore it was neglected.

The static equilibrium of the tetrahedron requires, then, that the polygon of the main vectors of the four surface forces be closed, and additionally the main moment resulting from these forces and calculated with respect to point O should be equal to zero.

In the first case the condition of zeroing of the main vector of surface forces has the following form:

$$\boldsymbol{\sigma}^{(n)}\delta S = \boldsymbol{\sigma}_1\delta S_1 + \boldsymbol{\sigma}_2\delta S_2 + \boldsymbol{\sigma}_3\delta S_3. \tag{6.23}$$

The rear faces of the tetrahedron are projections of the front face onto the planes $O - X_1 - X_2$, $O - X_2 - X_3$, and $O - X_1 - X_3$, and in view of that (Fig. 6.1a)

$$\begin{aligned}
\frac{\delta S_1}{\delta S} &= \mathbf{E}_1 \circ \mathbf{n} = \cos(\mathbf{E}_1, \mathbf{n}) \equiv n_1,\\
\frac{\delta S_2}{\delta S} &= \mathbf{E}_2 \circ \mathbf{n} = \cos(\mathbf{E}_2, \mathbf{n}) \equiv n_2,\\
\frac{\delta S_3}{\delta S} &= \mathbf{E}_3 \circ \mathbf{n} = \cos(\mathbf{E}_3, \mathbf{n}) \equiv n_3.
\end{aligned} \tag{6.24}$$

Substituting (6.23) into (6.22) we obtain

$$\boldsymbol{\sigma}^{(n)} = n_1\boldsymbol{\sigma}_1 + n_2\boldsymbol{\sigma}_2 + n_3\boldsymbol{\sigma}_3. \tag{6.25}$$

Let us stop for a moment at the physical interpretation of the preceding vectorial equation. Vector $\boldsymbol{\sigma}^{(n)}$ represents the physical quantity of the stress applied to an arbitrary elementary front face. The situation was similar with other vector quantities in mechanics considered earlier, for example, with the force vector. The components of that vector did not have any physical interpretation either and were dependent on the choice of coordinate system.

Surface forces similar to stress vector $\boldsymbol{\sigma}^{(n)}$ do not possess the invariance property. The change in position of a front plane involves a change in the magnitude of $\boldsymbol{\sigma}^{(n)}$, so the stress vectors $\boldsymbol{\sigma}^{(n)}$ do not form a *vector field*. The situation is even worse with the surface forces vectors of the rear faces of a tetrahedron, that is, with vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$. Not only do they depend on the choice of coordinate system but they also possess no physical interpretation and therefore are often called *pseudo vectors*.

Let us project vectorial (6.25) onto the axes of the $OX_1X_2X_3$ coordinate system, which, in practice, boils down to the multiplication of this equation by unit vectors $\mathbf{E}_i$, $i = 1, 2, 3$.

We know that

$$\begin{aligned}
\boldsymbol{\sigma}_n^{(n)} &= \sigma_1^{(n)}\mathbf{E}_1 + \sigma_2^{(n)}\mathbf{E}_2 + \sigma_3^{(n)}\mathbf{E}_3,\\
\boldsymbol{\sigma}_1 &= \sigma_{11}\mathbf{E}_1 + \sigma_{12}\mathbf{E}_2 + \sigma_{13}\mathbf{E}_3,\\
\boldsymbol{\sigma}_2 &= \sigma_{21}\mathbf{E}_1 + \sigma_{22}\mathbf{E}_2 + \sigma_{23}\mathbf{E}_3,\\
\boldsymbol{\sigma}_3 &= \sigma_{31}\mathbf{E}_1 + \sigma_{32}\mathbf{E}_2 + \sigma_{33}\mathbf{E}_3,
\end{aligned} \tag{6.26}$$

and as a result of premultiplication we obtain the following system of three scalar equations representing a tensor:

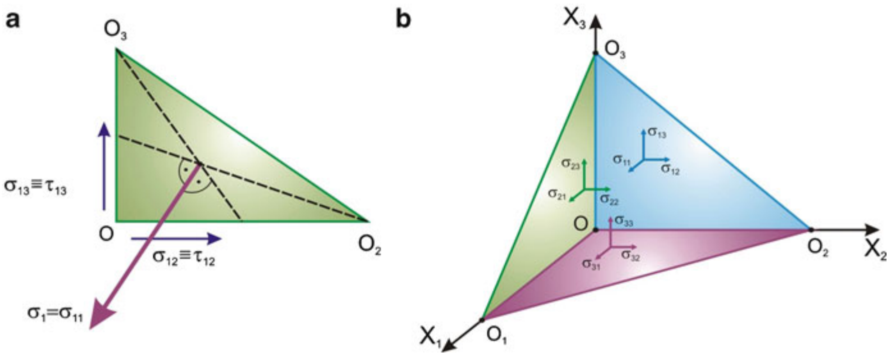

Fig. 6.2 Distributions of normal and shear stresses for the face $\Delta O O_2 O_3$ (**a**) and for three rear faces (**b**)

$$\begin{aligned}\sigma_1^{(n)} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31},\\ \sigma_2^{(n)} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32},\\ \sigma_3^{(n)} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33};\end{aligned} \tag{6.27}$$

because of the scalar multiplication the equation k resulted from the projection of the OX_k axis, which corresponds to $\sigma_k^{(n)}$.

Knowing the position of the triangle (face) with respect to the adopted coordinate system $OX_1X_2X_3$, that is, knowing the direction cosines n_1, n_2, and n_3 and nine values of σ_{kl}, $k, l = 1, 2, 3$, we are able to calculate $\sigma_1^{(n)}$, $\sigma_2^{(n)}$, and $\sigma_3^{(n)}$ and as a result determine the stress vector $\boldsymbol{\sigma}^{(n)}$ acting on the chosen face $\Delta O_1O_2O_3$. The table of these nine coordinates describes in a unique and invariant way the stress $\boldsymbol{\sigma}^{(n)}$, although its magnitude will now depend on the choice of plane $\Delta O_1O_2O_3$ defined by the normal $\mathbf{n}$. Moreover, the obtained values σ_{kl} occurring in (6.26) have a physical meaning. According to the adopted numeration convention, at σ_{kl} the index l denotes the projection onto the OX_l axis of a stress acting on a face (triangle) perpendicular to the OX_k axis. For instance, σ_{21} denotes the projection of the stress acting on a face perpendicular to the OX_2 axis (i.e., of vector $\boldsymbol{\sigma}_2$) onto the OX_1 axis.

Quantities with identical indices, that is, σ_{11}, σ_{22}, and σ_{33}, denote the projections of the stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ onto the directions of vectors normal (perpendicular) to the rear faces of the tetrahedron, and we call them *normal stresses*. We call the remaining six elements of the stress, that is, σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, and σ_{32} *shear stresses*. For example, according to the introduced notation the stresses σ_{12} and σ_{13} are tangent to the face of the triangle perpendicular to the OX_1 axis (i.e., $\Delta O O_2 O_3$) and acting along the OX_2 and OX_3 axes. The latter is illustrated in Fig. 6.2a, and all the components of the stress tensor are shown in Fig. 6.2b, where the direction of senses is changed with respect to the senses from Fig. 6.1a.

In the branch of mechanics regarding the *mechanics of materials*, normal stresses are customarily denoted by $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, and $\sigma_{33} = \sigma_3$, whereas the remaining tangential stresses are denoted by $\sigma_{12} = \tau_{12}$, $\sigma_{13} = \tau_{13}$, $\sigma_{21} = \tau_{21}$, $\sigma_{23} = \tau_{23}$, $\sigma_{31} = \tau_{31}$, and $\sigma_{32} = \tau_{32}$.

Let us now proceed with the interpretation of the obtained system of (6.27), bearing in mind that

$$\mathbf{n} = n_1\mathbf{E}_1 + n_2\mathbf{E}_2 + n_3\mathbf{E}_3,$$
$$\boldsymbol{\sigma}_n^{(n)} = \sigma_1^{(n)}\mathbf{E}_1 + \sigma_2^{(n)}\mathbf{E}_2 + \sigma_3^{(n)}\mathbf{E}_3. \tag{6.28}$$

Equation (6.27) describes the transition from the components of vector $\mathbf{n}$ to the components of vector $\boldsymbol{\sigma}^{(n)}$ through linear transformation of the form

$$\boldsymbol{\sigma}^{(n)} = \mathbf{n}\sigma \tag{6.29}$$

with the aid of the second-order tensor σ whose matrix has the following form:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}. \tag{6.30}$$

In (6.29) we exploited postmultiplication of vector $\mathbf{n}$ by tensor σ.

Because the rank of a tensor is determined by the number of indices that describe it, in our case the order of tensor equals two as we have two indices. Applying the tensor notation introduced earlier, the system of (6.27) can be written now in a very simple form:

$$\sigma_k^{(n)} = \mathbf{n}_l\sigma_{lk}, \quad k = 1,\ 2,\ 3. \tag{6.31}$$

Now the use of the tensor notation and linear transformation (transition) from one vector ($\mathbf{n}$) into the other ($\boldsymbol{\sigma}_n$) is applied in accordance with (6.29).

What is more, both vectors $\boldsymbol{\sigma}_n$ and $\mathbf{n}$ are connected with the chosen elementary front face and both have a physical meaning.

Recall that, for example, in the case of the force vector, its invariant quantity was the length, although its components were dependent on the choice of coordinate system. Also, now the values of coefficients in matrix representation of a tensor σ of the form (6.30), that is, the coefficients σ_{kl}, vary with the choice of coordinate system, but the set of all nine coefficients possesses one physical meaning representing the stress state of a deformable body at its chosen point. For the given coordinate system, that is, knowing the stresses acting on the side faces of the tetrahedron and knowing vector $\mathbf{n}$, that is, having the front face chosen, we are able to determine the stress vector acting on the front face.

The reader has surely noticed that the matrix representation (6.30) possesses a slightly different structure than the matrices encountered so far where the first index should indicate the row number to which the particular element belongs.

We call the tensor σ^* the conjugated (transposed) tensor with tensor σ, and it has the form

$$\sigma^* = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \tag{6.32}$$

where the following equality holds true:

$$\sigma^*_{kl} = \sigma_{lk}, \quad k,\ l = 1, 2, 3. \tag{6.33}$$

For an arbitrary vector $\boldsymbol{a}_i$, for an arbitrary tensor of the second order R represented by a matrix of the form $\left[r_{kl}\right]_{3\times 3}$ we have

$$R\boldsymbol{a} = \boldsymbol{a}R^T, \qquad R^T\boldsymbol{a} = \boldsymbol{a}R. \tag{6.34}$$

We call a tensor with the property

$$R^T = R, \qquad r_{kl} = r_{lk} \tag{6.35}$$

a *symmetric tensor*.

We will demonstrate subsequently that the analyzed stress tensor is a symmetrical tensor, that is, that the notation $\boldsymbol{\sigma}_n = \mathbf{n}\sigma$ used in (6.29) was also correct. The relationships (6.35) lead to the relationships $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{23} = \sigma_{32}$.

In order to demonstrate the aforementioned property of the stress tensor let us consider the second necessary condition of keeping the tetrahedron in the static equilibrium position, that is, we will write the equation of moments caused by the surface forces $\sigma_n \delta S$, $\sigma_1 \delta S_1$, $\sigma_2 \delta S_2$, and $\sigma_3 \delta S_3$.

As was mentioned previously, the vectors of these surface forces will be applied at the geometrical centers of the triangular faces of the tetrahedron which, in turn, are the points of intersection of triangles' medians. Therefore, we must define, with the aid of position vectors, the four previously mentioned points in which the vectors of external loads are applied. For this purpose we will use the diagram shown in Fig. 6.3.

Setting $\mathbf{r}_1 = \overrightarrow{OC_1}$, $\mathbf{r}_2 = \overrightarrow{OC_2}$, $\mathbf{r}_3 = \overrightarrow{OC_3}$, and $\mathbf{r} = \overrightarrow{OC}$, the equation of moments with respect to point O assumes the following form:

$$\mathbf{r} \times \boldsymbol{\sigma}^{(n)}\delta S = \mathbf{r}_1 \times \boldsymbol{\sigma}_1 \delta S_1 + \mathbf{r}_2 \times \boldsymbol{\sigma}_2 \delta S_2 + \mathbf{r}_3 \times \boldsymbol{\sigma}_3 \delta S_3, \tag{6.36}$$

and taking into account (6.23) and (6.25) in it we obtain

$$(\mathbf{r} - \mathbf{r}_1) \times \boldsymbol{\sigma}_1 n_1 + (\mathbf{r} - \mathbf{r}_2) \times \boldsymbol{\sigma}_2 n_2 + (\mathbf{r} - \mathbf{r}_3) \times \boldsymbol{\sigma}_3 n_3 = \mathbf{0}. \tag{6.37}$$

The projections of point C onto the rear planes of the tetrahedron are the points C_1, C_2, and C_3, which are the centers of the respective rear faces (triangles) of the tetrahedron. According to Fig. 6.3 we have

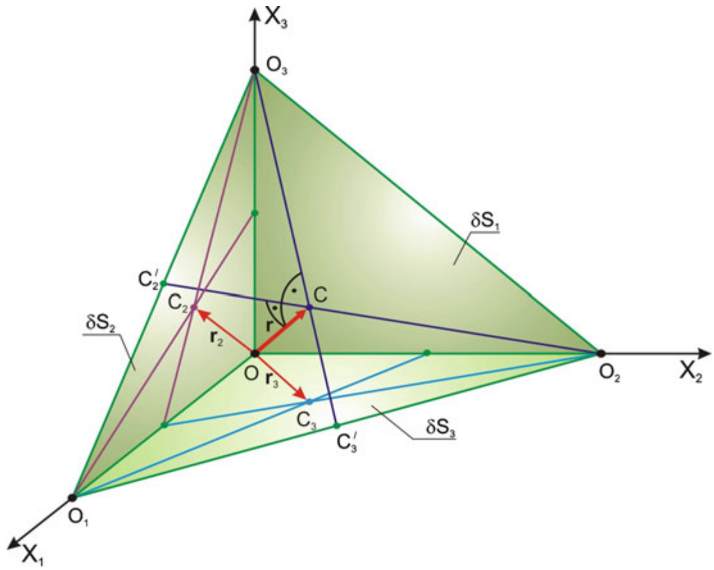

Fig. 6.3 A tetrahedron with marked face centers C, C_i $(i = 1, 2, 3)$ and their position vectors $\mathbf{r}_{C_i}$, $\mathbf{r}$

$$
\begin{aligned}
\overrightarrow{C_3C} &= \overrightarrow{OC} - \overrightarrow{OC_3} = \mathbf{r} - \mathbf{r}_3 \equiv \lambda_3\mathbf{E}_3,\\
\overrightarrow{C_2C} &= \overrightarrow{OC} - \overrightarrow{OC_2} = \mathbf{r} - \mathbf{r}_2 \equiv \lambda_2\mathbf{E}_2,\\
\overrightarrow{C_1C} &= \overrightarrow{OC} - \overrightarrow{OC_1} = \mathbf{r} - \mathbf{r}_1 \equiv \lambda_1\mathbf{E}_1.
\end{aligned} \tag{6.38}
$$

This means that vectors $\overrightarrow{C_iC}$ are parallel to unit vectors $\mathbf{E}_i$, where the roles of scaling factors are played by scalars λ_i. Substituting (6.38) into (6.37) we obtain

$$
\lambda_1 n_1(\mathbf{E}_1 \times \boldsymbol{\sigma}_1) + \lambda_2 n_2(\mathbf{E}_2 \times \boldsymbol{\sigma}_2) + \lambda_3 n_3(\mathbf{E}_3 \times \boldsymbol{\sigma}_3) = \mathbf{0}. \tag{6.39}
$$

The terms of the equation of the form $\mathbf{E}_i \times \boldsymbol{\sigma}_i$ can be determined from (6.38) after their scalar multiplication by $\mathbf{n}$, because we have

$$
\begin{aligned}
(\mathbf{r} - \mathbf{r}_1) \circ \mathbf{n} &= \mathbf{r} \circ \mathbf{n} - \mathbf{r}_1 \circ \mathbf{n},\\
(\mathbf{r} - \mathbf{r}_2) \circ \mathbf{n} &= \mathbf{r} \circ \mathbf{n} - \mathbf{r}_2 \circ \mathbf{n},\\
(\mathbf{r} - \mathbf{r}_3) \circ \mathbf{n} &= \mathbf{r} \circ \mathbf{n} - \mathbf{r}_3 \circ \mathbf{n}.
\end{aligned} \tag{6.40}
$$

Let us note now that

$$
\mathbf{r} \circ \mathbf{n} = |\mathbf{r}|, \tag{6.41}
$$

and $|\mathbf{r}| = r$ denotes the distance of point O from the plane $O_1O_2O_3$. Extending the segments OC_1, OC_2, and OC_3 until they intersect with the sides of $\Delta O_1O_2O_3$, we obtain points C_1', C_2', and C_3' (in Fig. 6.3 only the last two points are marked). The projections of vectors $\overrightarrow{OC_1'}$, $\overrightarrow{OC_2'}$, and $\overrightarrow{OC_3'}$ onto the normal vector $\mathbf{n}$, that is, onto the direction of vector $\mathbf{r}$, are identical and equal to

$$\overrightarrow{OC_1'} \circ \mathbf{n} = \overrightarrow{OC_2'} \circ \mathbf{n} = \overrightarrow{OC_3'} \circ \mathbf{n} = \frac{3}{2}\mathbf{r}_1 \circ \mathbf{n} = \frac{3}{2}\mathbf{r}_2 \circ \mathbf{n} = \frac{3}{2}\mathbf{r}_3 \circ \mathbf{n} = r, \quad (6.42)$$

because $\overrightarrow{OC_1} = 2/3\overrightarrow{OC_1'}$, $\overrightarrow{OC_2} = 2/3\overrightarrow{OC_2'}$, $\overrightarrow{OC_3} = 2/3\overrightarrow{OC_3'}$, which follows from the properties of the midperpendicular of the triangle.

Taking into account (6.41) and (6.42) in (6.40), and then taking into account (6.37) we obtain

$$\begin{aligned} \mathbf{r} \circ \mathbf{n} - \mathbf{r}_1 \circ \mathbf{n} &= \frac{1}{3}r = \lambda_1 \mathbf{E}_1 \circ \mathbf{n} = \lambda_1 n_1, \\ \mathbf{r} \circ \mathbf{n} - \mathbf{r}_2 \circ \mathbf{n} &= \frac{1}{3}r = \lambda_2 \mathbf{E}_2 \circ \mathbf{n} = \lambda_2 n_2, \\ \mathbf{r} \circ \mathbf{n} - \mathbf{r}_3 \circ \mathbf{n} &= \frac{1}{3}r = \lambda_3 \mathbf{E}_3 \circ \mathbf{n} = \lambda_3 n_3, \end{aligned} \quad (6.43)$$

which allows for the determination of the desired quantities λ_i, $i = 1, 2, 3$.

Substituting (6.43) into (6.39) we obtain

$$\mathbf{E}_1 \times \boldsymbol{\sigma}_1 + \mathbf{E}_2 \times \boldsymbol{\sigma}_1 + \mathbf{E}_3 \times \boldsymbol{\sigma}_1 = \mathbf{0}. \quad (6.44)$$

Now, let us premultiply (6.44) by $\mathbf{E}_i$ (vector product) to obtain

$$\begin{aligned} \mathbf{E}_1 \times (\mathbf{E}_1 \times \boldsymbol{\sigma}_1) + \mathbf{E}_1 \times (\mathbf{E}_2 \times \boldsymbol{\sigma}_2) + \mathbf{E}_1 \times (\mathbf{E}_3 \times \boldsymbol{\sigma}_3) &= \mathbf{0}, \\ \mathbf{E}_2 \times (\mathbf{E}_1 \times \boldsymbol{\sigma}_1) + \mathbf{E}_2 \times (\mathbf{E}_2 \times \boldsymbol{\sigma}_2) + \mathbf{E}_2 \times (\mathbf{E}_3 \times \boldsymbol{\sigma}_3) &= \mathbf{0}, \\ \mathbf{E}_3 \times (\mathbf{E}_1 \times \boldsymbol{\sigma}_1) + \mathbf{E}_3 \times (\mathbf{E}_2 \times \boldsymbol{\sigma}_2) + \mathbf{E}_3 \times (\mathbf{E}_3 \times \boldsymbol{\sigma}_3) &= \mathbf{0}, \end{aligned} \quad (6.45)$$

and then, using the following property of the vector product

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \circ \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \circ \boldsymbol{b})\boldsymbol{c}, \quad (6.46)$$

(6.45) can be written in the form

$$\begin{aligned} (\mathbf{E}_1 \circ \boldsymbol{\sigma}_1)\mathbf{E}_1 + (\mathbf{E}_1 \circ \boldsymbol{\sigma}_2)\mathbf{E}_2 + (\mathbf{E}_1 \circ \boldsymbol{\sigma}_3)\mathbf{E}_3 &= \boldsymbol{\sigma}_1, \\ (\mathbf{E}_2 \circ \boldsymbol{\sigma}_1)\mathbf{E}_1 + (\mathbf{E}_2 \circ \boldsymbol{\sigma}_2)\mathbf{E}_2 + (\mathbf{E}_2 \circ \boldsymbol{\sigma}_3)\mathbf{E}_3 &= \boldsymbol{\sigma}_2, \\ (\mathbf{E}_3 \circ \boldsymbol{\sigma}_1)\mathbf{E}_1 + (\mathbf{E}_3 \circ \boldsymbol{\sigma}_2)\mathbf{E}_2 + (\mathbf{E}_3 \circ \boldsymbol{\sigma}_3)\mathbf{E}_3 &= \boldsymbol{\sigma}_3. \end{aligned} \quad (6.47)$$

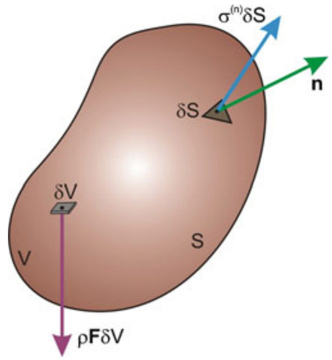

Fig. 6.4 The considered region of surface S cut out of a deformable body and volume V with elementary main vectors of surface and volume forces

Performing the scalar multiplications using the definitions of vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ introduced earlier [see (6.26)] we obtain

$$\begin{aligned} \sigma_{11}\mathbf{E}_1 + \sigma_{21}\mathbf{E}_2 + \sigma_{31}\mathbf{E}_3 &= \sigma_{11}\mathbf{E}_1 + \sigma_{12}\mathbf{E}_2 + \sigma_{13}\mathbf{E}_3, \\ \sigma_{12}\mathbf{E}_1 + \sigma_{22}\mathbf{E}_2 + \sigma_{32}\mathbf{E}_3 &= \sigma_{21}\mathbf{E}_1 + \sigma_{22}\mathbf{E}_2 + \sigma_{23}\mathbf{E}_3, \\ \sigma_{13}\mathbf{E}_1 + \sigma_{23}\mathbf{E}_2 + \sigma_{33}\mathbf{e}_3 &= \sigma_{31}\mathbf{E}_1 + \sigma_{32}\mathbf{E}_2 + \sigma_{33}\mathbf{E}_3, \end{aligned} \tag{6.48}$$

from which it follows that

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{31} = \sigma_{13}, \tag{6.49}$$

which proves the symmetry of the stress tensor. Because of the previously mentioned symmetry of the stress tensor six components of the tensor are determined.

From the conducted calculations regarding the analysis of equilibrium state of the elementary tetrahedron cut out from a deformable body it follows that the static equilibrium conditions of this tetrahedron are *necessary* but not *sufficient conditions*. At this stage of our study we are not in a position to completely solve the stated problem, that is, to determine the six components of the stress tensor.

Let us recall that the elementary tetrahedron chosen by us was small enough for the volume forces to be neglected. Now we will try to proceed in a different way, that is, in a continuous deformable body we will isolate a volume V bounded by the surface S large enough that this time the volume forces (which correspond to internal forces in the body) cannot be neglected.

Figure 6.4 shows the region of volume V "cut out" of the considered body and bounded by the surface S.[2]

[2]Subsequently we also denote by S a symmetric tensor, but these notions are made distinct in the text.

If after deformation of a body its cut-out volume is in equilibrium, then internal forces must have arisen that, in the general case, cannot be determined without taking into account the stress-strain state of the body. This was already indicated by previous cases of so-called *statically indeterminate mechanical systems*.

In a deformed static equilibrium state vectors of volume and surface forces calculated over the total surface and volume of the cut out volume must be balanced by each other, that is,

$$\int_S \boldsymbol{\sigma}^{(n)}\delta S + \int_V \rho \mathbf{F} \mathrm{d}V = \mathbf{0}. \tag{6.50}$$

In order to enable further calculations we must pass from the surface integral to the volume integral based on the so-called Gauss[3]–Ostrogradski[4] integral formula:

$$\int_V \boldsymbol{\nabla}(.)\delta V = \int_S \mathbf{n}(.)\delta S, \tag{6.51}$$

where in a Cartesian coordinate system

$$\boldsymbol{\nabla} = \nabla_i \mathbf{E}_i = \frac{\partial}{\partial x_1}\mathbf{E}_1 + \frac{\partial}{\partial x_2}\mathbf{E}_2 + \frac{\partial}{\partial x_3}\mathbf{E}_3 \tag{6.52}$$

is “nabla,” that is, a differential operator that, after acting on a scalar quantity, generates a vector.

Let the scalar quantity be, for instance, the quantity γ. Then (6.50) takes the form

$$\int_V \boldsymbol{\nabla}\gamma\delta v = \int_S \mathbf{n}\gamma\delta S, \tag{6.53}$$

and, as is easily noticed, we have

$$\boldsymbol{\nabla}\gamma = \frac{\partial \gamma}{\partial x_1}\mathbf{E}_1 + \frac{\partial \gamma}{\partial x_2}\mathbf{E}_2 + \frac{\partial \gamma}{\partial x_3}\mathbf{E}_3 = \mathrm{grad}\gamma. \tag{6.54}$$

Notation (6.53) indicates the one-to-one correspondence between the operator $\boldsymbol{\nabla}$ (“nabla” vector) acting in R^3 and vector $\mathbf{n}$ in R^2 since it is normal to surface S.

However, the Gauss–Ostrogradski integral formula (6.51) possesses a broader interpretation because one may use scalar, vector, or tensor functions as the operator (.) In (6.29) we multiplied the vector by the tensor. In the general case, such a multiplication is not commutative, and only regarding the symmetric tensor is the relationship $\mathbf{n}\sigma = \sigma\mathbf{n}$ valid [see also (6.34) and (6.35)].

---

[3] Carl F. Gauss (1777–1855), German mathematician who studied number theory, analysis, statistics, differential geometry, astronomy, and optics.

[4] Michail W. Ostrogradski (1801–1862), Ukrainian mathematician who studied algebra, number theory, analysis, and probability calculus.

We will introduce now a postmultiplication operation of a tensor R by a vector $\boldsymbol{a}$,

$$R\boldsymbol{a} = \boldsymbol{b}, \tag{6.55}$$

defined in the following way:

$$b_l = (R\boldsymbol{a})_l = r_{l1}a_1 + r_{l2}a_2 + r_{l3}a_3 \equiv r_{lk}a_k, \quad l = 1, 2, 3. \tag{6.56}$$

In a similar way we introduce a premultiplication operation of tensor R by vector $\boldsymbol{a}$:

$$c_l = (\boldsymbol{a}R)_l = r_{1l}a_1 + r_{2l}a_2 + r_{3l}a_3 \equiv a_k r_{kl}, \quad l = 1, 2, 3. \tag{6.57}$$

Depending on the mode of multiplication, this operation generates either vector $\boldsymbol{b}$ or vector $\boldsymbol{c}$.

The simplest example of the Gauss–Ostrogradski formula, that is, for one-dimensional cases related to the three axes of a Cartesian coordinate system, is

$$\begin{aligned}
\int_S \frac{\partial\varphi}{\partial x_1}\partial S &= \int_V n_1\varphi\partial V,\\
\int_S \frac{\partial\varphi}{\partial x_2}\partial S &= \int_V n_2\varphi\partial V,\\
\int_S \frac{\partial\varphi}{\partial x_3}\partial S &= \int_V n_3\varphi\partial V.
\end{aligned} \tag{6.58}$$

Multiplying the equations respectively by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ and adding them together we obtain

$$\int_S \left(\frac{\partial\varphi}{\partial x_1}\mathbf{E}_1 + \frac{\partial\varphi}{\partial x_2}\mathbf{E}_2 + \frac{\partial\varphi}{\partial x_3}\mathbf{E}_3\right)\delta S = \int_V (n_1\varphi\mathbf{E}_1 + n_2\varphi\mathbf{E}_2 + n_3\varphi\mathbf{E}_3)\,\delta V, \tag{6.59}$$

which yields (6.51), where in place of dots we put φ.

In the case of tensor ϕ of elements $\phi_{kl}(k, l = 1, 2, 3)$, we have

$$\int_S \frac{\partial\phi_{kl}}{\partial x_k}\delta S = \int_V n_k\phi_{kl}\delta V, \quad l = 1, 2, 3, \tag{6.60}$$

and indices k are summation indices. The preceding notation is equivalent to

$$\int_S \nabla_k\phi_{kl}\delta S = \int_V \mathbf{n}_k\phi_{kl}\delta V, \quad l = 1, 2, 3. \tag{6.61}$$

In turn, according to (6.58) we have

$$\int_S (\nabla\phi)_l \delta S = \int_V (\mathbf{n}\phi)_l \delta V, \quad l = 1, 2, 3. \tag{6.62}$$

Multiplying (6.62) respectively by $\mathbf{E}_l$, that is, $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, and adding them by sides we obtain

$$\int_S \nabla\phi \delta S = \int_V \mathbf{n}\phi \delta V, \tag{6.63}$$

that is, again, we obtained (6.51) where instead of dots, tensor ϕ was inserted.

According to the previous transformations we applied the following projection along the l axis:

$$(\nabla\phi)_l = \nabla_k \phi_{kl} = \nabla_k \phi_{kl} = \frac{\partial \phi_{kl}}{\partial x_k} \equiv (\mathrm{div}\phi)_l. \tag{6.64}$$

It follows that the vector $\nabla\phi = \mathrm{div}\phi$ after returning to (6.63) leads to the expression

$$\int_S \mathbf{div}\phi \delta S = \int_V \mathbf{n}\phi \delta V, \tag{6.65}$$

where projections of vector $\mathbf{div}\phi$, that is, the *divergence of tensor* ϕ, have the form

$$\begin{aligned}
(\mathrm{div}\phi)_1 &= (\nabla\phi)_1 = \frac{\partial \phi_{11}}{\partial x_1} + \frac{\partial \phi_{21}}{\partial x_2} + \frac{\partial \phi_{31}}{\partial x_3}, \\
(\mathrm{div}\phi)_2 &= (\nabla\phi)_2 = \frac{\partial \phi_{12}}{\partial x_1} + \frac{\partial \phi_{22}}{\partial x_2} + \frac{\partial \phi_{32}}{\partial x_3}, \\
(\mathrm{div}\phi)_3 &= (\nabla\phi)_3 = \frac{\partial \phi_{13}}{\partial x_1} + \frac{\partial \phi_{23}}{\partial x_2} + \frac{\partial \phi_{33}}{\partial x_3}.
\end{aligned} \tag{6.66}$$

The vector of divergence of tensor ϕ can be represented in a Cartesian coordinate system in the form

$$\begin{aligned}
\mathbf{div}\phi &= \mathbf{E}_1 (\nabla\phi)_1 + \mathbf{E}_2 (\nabla\phi)_2 + \mathbf{E}_3 (\nabla\phi)_3 \\
&= \mathbf{E}_1 \left(\frac{\partial \phi_{11}}{\partial x_1} + \frac{\partial \phi_{21}}{\partial x_2} + \frac{\partial \phi_{31}}{\partial x_3} \right) + \mathbf{E}_2 \left(\frac{\partial \phi_{12}}{\partial x_1} + \frac{\partial \phi_{22}}{\partial x_2} + \frac{\partial \phi_{32}}{\partial x_3} \right) \\
&\quad + \mathbf{E}_3 \left(\frac{\partial \phi_{13}}{\partial x_1} + \frac{\partial \phi_{23}}{\partial x_2} + \frac{\partial \phi_{33}}{\partial x_3} \right).
\end{aligned} \tag{6.67}$$

Finally, the equivalent tensor form (6.65) of the Gauss–Ostrogradski formula (6.51) was derived.

Now, let us return to the state of equilibrium of a deformable body described by (6.50). Using (6.65) the equation takes the form

$$\int_V (\mathbf{div}\sigma + \rho\mathbf{F})\,\delta V = \mathbf{0}. \tag{6.68}$$

The preceding integral is calculated over volume V of the cut out volume of the body, and also in a special case can be applied to the elementary volume δV. In this case from (6.68) we obtain

$$\int_0^{\delta V} (\mathbf{div}\sigma + \rho\mathbf{F})\,\delta V = [\mathbf{div}\sigma + \rho\mathbf{F}]_0^{\delta V} = (\mathbf{div}\sigma + \rho\mathbf{F})\,\delta V = 0. \tag{6.69}$$

Because by assumption δV is arbitrary and different than zero, from (6.69) we finally obtain

$$\mathbf{div}\sigma + \rho\mathbf{F} = \mathbf{0}. \tag{6.70}$$

Multiplying the preceding equations respectively by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, and taking into account (6.67), where instead of ϕ one should substitute σ, we obtain the following three scalar equations:

$$\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + \rho F_1 = 0,\\
&\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} + \rho F_2 = 0,\\
&\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho F_3 = 0.
\end{aligned} \tag{6.71}$$

Because we already proved earlier that the stress tensor was symmetrical, (6.71) can be represented in the form

$$\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho F_1 = 0,\\
&\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho F_2 = 0,\\
&\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho F_3 = 0.
\end{aligned} \tag{6.72}$$

At this point of our discussion let us note that we have three algebraic equations (6.72) at our disposal for the purpose of determining the six unknowns σ_{11}, σ_{12}, σ_{13}, σ_{22}, σ_{23}, and σ_{33}, and in view of that the system of equations (6.72) constitutes the necessary equilibrium conditions, but not sufficient conditions.

Let us now consider the infinitesimal displacement of an elementary volume cut out of a deformable body. Let us recall that in the course of previous considerations it was shown that an arbitrary force system acting on a rigid body could be reduced to a wrench. In this case the main moment of forces would cause a “small” rotation of the rigid body about the axis of a wrench, and the main force vector would cause a translation along the wrench axis (central axis). A rigid body, by definition, is a body whose every point is subjected to the same translation and rotation through the same angle measured in the plane perpendicular to the mentioned axis.

That observation, related to the whole arbitrary volume of a body, has a *global* character. However, in the case of a deformable body, that is, for a purely mathematical model allowing for the possibility of displacement of internal points of the body, static equilibrium has a local character. The deformation of any chosen volume of a deformable body can occur in a similar way determined by the translation and rotation of a rigid body reflected in translational deformation (connected with the diagonal part of of the stress) and the rotational deformation connected with the non-diagonal elements of the deformation tensor.

Already H. Helmholtz[5] in 1858 had shown that the deformation (displacement) of an arbitrary point of elementary volume of a deformable continuous system consisted of a so-called *quasistiff deformation*, including translational and rotational displacement, as well as of *deformational displacement* (displacement due to deformation).

Helmholtz formulated his observation in the form of a theorem today known as the Helmholtz theorem.

We will now show how the two aforementioned deformation components of a deformable medium (body) should be interpreted. However, it should be indicated that the Helmholtz theorem regards the *elementary volume of deformable medium*, and therefore, as distinct from a rigid body, it has a *local character*.

Since in a deformable medium (as distinct from a rigid one) the possibility of relative displacement of two arbitrary points of the body is allowed, let us take in an elementary volume of such a body δV two arbitrary points B and B_1 whose positions are described by position vectors $\mathbf{r}$ and $\mathbf{r}_1$ with respect to a certain fixed point O (Fig. 6.5).

Let the elementary displacements (infinitely small) of points B and B_1 be equal to $\mathbf{u} = \mathrm{d}\mathbf{r}$ and $\mathbf{u}_1 = \mathrm{d}\mathbf{r}_1$. Because we want to estimate the relative displacement of the points, we introduce a radius vector with respect to pole B denoted by $\delta\mathbf{r}$. Before deformation $B = B'$ and $B_1 = B_1'$, that is, $\mathbf{u} = 0$ and $\mathbf{u}_1 = 0$.

Let us recall the operation of differentiation of a vector along the direction of the second vector known from mathematical analysis. Let us return to symbolical vector operator $\boldsymbol{\nabla}$, which has coordinates in the adopted Cartesian coordinate system of the form

$$\nabla_k = \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3. \tag{6.73}$$

[5]Hermann Helmholtz (1821–1894), a German physicist and mathematician who studied acoustics and thermodynamics.

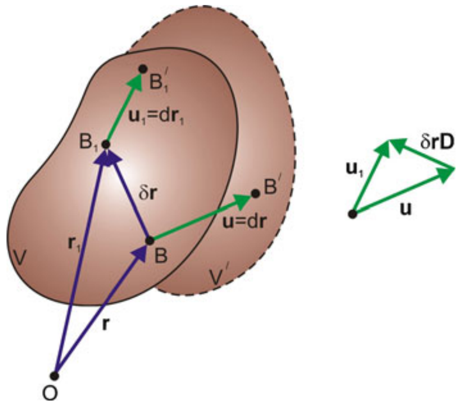

Fig. 6.5 Position vectors **r** and $\mathbf{r}_1$ and displacements **u** and $\mathbf{u}_1$ of points B and B_1 (see formula (6.110))

A gradient of scalar function φ can be represented in the form of a product of vector ∇ and the scalar φ of the form

$$\mathrm{grad}\varphi = \nabla\varphi. \tag{6.74}$$

In turn, a derivative of scalar function φ in the direction of axis $\boldsymbol{l}$ is represented in the form

$$\frac{\partial\varphi}{\partial l} = \boldsymbol{l} \circ \nabla\varphi = \boldsymbol{l} \circ \mathrm{grad}\varphi, \tag{6.75}$$

where as previously a dot denotes the scalar product of two vectors.

Let us note that

$$\boldsymbol{l} \circ \nabla = l_1 \frac{\partial}{\partial x_1} + l_2 \frac{\partial}{\partial x_2} + l_3 \frac{\partial}{\partial x_3} = l_k \frac{\partial}{\partial x_k} = \frac{\partial}{\partial l}, \tag{6.76}$$

where $l_k = \cos(\boldsymbol{l}, \mathbf{E}_k)$.

We obtain the projections of the gradient of scalar function $\varphi(x_1, x_2, x_3)$ onto the coordinate axes from (6.74) after multiplying by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ and after substituting the unit vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ into (6.75) instead of $\boldsymbol{l}$, that is,

$$\mathbf{E}_1 \circ \mathrm{grad}\varphi = (\mathrm{grad}\varphi)_1 = \mathbf{E}_1 \circ \nabla\varphi = \frac{\partial}{\partial x_1}\varphi,$$

$$\mathbf{E}_2 \circ \mathrm{grad}\varphi = (\mathrm{grad}\varphi)_2 = \mathbf{E}_2 \circ \nabla\varphi = \frac{\partial}{\partial x_2}\varphi,$$

$$\mathbf{E}_3 \circ \mathrm{grad}\varphi = (\mathrm{grad}\varphi)_3 = \mathbf{E}_3 \circ \nabla\varphi = \frac{\partial}{\partial x_3}\varphi. \tag{6.77}$$

The multiplication operation $\boldsymbol{l} \circ \nabla = \frac{\partial}{\partial l}$ can also be applied to vectorial functions (e.g., $\boldsymbol{a} = \boldsymbol{a}(x_1, x_2, x_3)$) and tensor functions T in the following way:

$$\frac{\partial \boldsymbol{a}}{\partial l} = (\boldsymbol{l} \circ \nabla)\,\boldsymbol{a}, \quad \frac{\partial T}{\partial l} = (\boldsymbol{l} \circ \nabla)\,T. \tag{6.78}$$

The first (6.78) defines a notion of projection of vector $\boldsymbol{a}$ onto the direction of vector $\boldsymbol{l}$. Since we have for two vectorial functions $\boldsymbol{l} = \boldsymbol{l}(x_1, x_2, x_3)$ and $\boldsymbol{a} = \boldsymbol{a}(x_1, x_2, x_3)$

$$(\boldsymbol{l} \circ \nabla)\,\boldsymbol{a} = l\left(\frac{\boldsymbol{l}}{l} \circ \nabla\right)\boldsymbol{a} = l\frac{\partial \boldsymbol{a}}{\partial l}, \tag{6.79}$$

because $\frac{\boldsymbol{l}}{l}$ is a unit vector of the l axis. In (6.79) the symbol $\frac{\partial \boldsymbol{a}}{\partial l}$ denotes the derivative of vector $\boldsymbol{a}$ on the direction of vector $\boldsymbol{l}$.

With the aid of the introduced symbolic operator-vector nabla ∇, apart from the aforementioned vector derivative, we can additionally define the following two operations of *spatial differentiation* related to the vectorial function $\boldsymbol{a} = \boldsymbol{a}(x_1, x_2, x_3)$.

We call the scalar product of the form

$$\operatorname{div}\boldsymbol{a} = \nabla \circ \boldsymbol{a} = \nabla_k a_k = \frac{\partial a_k}{\partial x_k} \equiv \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \tag{6.80}$$

a *divergence* of a vector field (a vector).

We call the vector product of the form

$$\operatorname{curl}\boldsymbol{a} = \nabla \times \boldsymbol{a} \tag{6.81}$$

a *curl* of a vector field (a vector) $\boldsymbol{a}$.

The vector curl $\boldsymbol{a}$ can be represented in tensor form using the notion, introduced earlier, of the third-order tensor ε_{pqr} (alternating tensor) of the form

$$\operatorname{curl}\boldsymbol{a} = \varepsilon_{pqr}\mathbf{E}_p\frac{\partial a_r}{\partial x_q}, \tag{6.82}$$

where according to (5.140) the only non-zero elements of that tensor are $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$ and $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$, and the summation in (6.82) from 1 to 3 is done with respect to the three introduced repeating indices.

From (6.81) it follows directly that

$$\begin{aligned}\operatorname{curl}\boldsymbol{a} &= \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \mathbf{E}_1\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) + \mathbf{E}_2\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) + \mathbf{E}_3\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) \\ &= \mathbf{E}_1\operatorname{curl}_1\boldsymbol{a} + \mathbf{E}_2\operatorname{curl}_2\boldsymbol{a} + \mathbf{E}_3\operatorname{curl}_3\boldsymbol{a}. \end{aligned} \tag{6.83}$$

Once the essential information on the tensor calculus is presented, we will explain the notion of *dyad* referring to two physical vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of the form

$$\boldsymbol{ab} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}, \tag{6.84}$$

where we do not place a dot between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, the dyad is, apart from the scalar product and the vector product, the third operation of multiplication of two vectors.

For an arbitrary Cartesian coordinate system described by vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ it is possible to form nine dyads of the form $\mathbf{E}_p\mathbf{E}_q$, where $p, q = 1, 2, 3$. The elements of a dyad matrix can be determined using the following equation:

$$\left(\mathbf{E}_p\mathbf{E}_q\right)_{rs} = \left(\mathbf{E}_p \circ \mathbf{E}_r\right)\left(\mathbf{E}_q \circ \mathbf{E}_s\right). \tag{6.85}$$

Let us note that once a vector in a Cartesian coordinate system is represented by unit vectors as

$$\boldsymbol{a} = a_1\mathbf{E}_1 + a_2\mathbf{E}_2 + a_3\mathbf{E}_3, \tag{6.86}$$

a two-dimensional tensor T of elements T_{qr} is represented by dyads

$$T = T_{qr}\mathbf{E}_q\mathbf{E}_r. \tag{6.87}$$

In turn, a unit tensor I_0 represented by a matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.88}$$

has the form

$$I_0 = \mathbf{E}_1\mathbf{E}_1 + \mathbf{E}_2\mathbf{E}_2 + \mathbf{E}_3\mathbf{E}_3 = \delta_{pq}\mathbf{E}_p\mathbf{E}_q. \tag{6.89}$$

If in (6.84) $\boldsymbol{a} = \nabla$, then we obtain a tensor of the form

$$D = \nabla\boldsymbol{b} = \mathbf{E}_p\mathbf{E}_q\frac{\partial b_q}{\partial x_p}, \tag{6.90}$$

and then the matrix representation of tensor D has the form

$$\begin{bmatrix} \frac{\partial b_1}{\partial x_1} & \frac{\partial b_2}{\partial x_1} & \frac{\partial b_3}{\partial x_1} \\ \frac{\partial b_1}{\partial x_2} & \frac{\partial b_2}{\partial x_2} & \frac{\partial b_3}{\partial x_2} \\ \frac{\partial b_1}{\partial x_3} & \frac{\partial b_2}{\partial x_3} & \frac{\partial b_3}{\partial x_3} \end{bmatrix}, \tag{6.91}$$

and the elements of tensor D are as follows:

$$D_{pq} = \nabla_p b_q = \frac{\partial b_q}{\partial x_p}. \tag{6.92}$$

So far the operator $\boldsymbol{\nabla}$ acting on a scalar φ defined for us

$$\boldsymbol{\nabla}\varphi = \mathrm{grad}\varphi, \tag{6.93}$$

where gradφ is a vector.

In turn, based on (6.90) we introduce the notion of a gradient of vectorial function $\boldsymbol{b} = \boldsymbol{b}(x_1, x_2, x_3)$, which will be denoted by

$$D = \boldsymbol{\nabla b} = \mathrm{Grad}\boldsymbol{b}. \tag{6.94}$$

However, if D^* is a tensor conjugated with D, then

$$\mathrm{d}\boldsymbol{b} = D^*\mathrm{d}\mathbf{r} = \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}\mathbf{r}}\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r}D = \mathrm{d}\mathbf{r}\,\mathrm{Grad}\boldsymbol{b}, \tag{6.95}$$

where

$$\mathrm{d}b_p = \frac{\partial b_{qp}}{\partial x_q}, \quad p = 1, 2, 3. \tag{6.96}$$

An arbitrary non-symmetric tensor T can be represented in the form of symmetric tensor ${}^{(S)}T \equiv S$ and antisymmetric (skew-symmetric) tensor ${}^{(A)}T \equiv A$. Since

$$T \equiv \frac{1}{2}\left(T + T^*\right) + \frac{1}{2}\left(T - T^*\right), \tag{6.97}$$

and taking into account the introduced notations

$${}^{(S)}T = \frac{1}{2}\left(T + T^*\right), \qquad {}^{(A)}T = \frac{1}{2}\left(T - T^*\right), \tag{6.98}$$

we have

$$T =^{(S)} T +^{(A)} T. \tag{6.99}$$

The notion of symmetric tensor was discussed on an example of a physical stress tensor [see (6.33)], and its basic property is

$$^{(S)}T =^{(S)} T^*, \tag{6.100}$$

where the symbol (*) denotes a transposition, that is, replacement of the rows by columns in the matrix representation of tensor T.

An arbitrary skew-symmetric tensor A has, in turn, the property

$$A^* = -A, \qquad A_{rs} = -A_{sr}, \tag{6.101}$$

which means that

$$\begin{aligned} A_{12} &= -A_{21}, \quad A_{23} = -A_{32}, \quad A_{31} = -A_{13}, \\ A_{11} &= A_{22} = A_{33} = 0, \end{aligned} \tag{6.102}$$

and finally the matrix of tensor A has the form

$$\begin{bmatrix} 0 & A_{12} & -A_{31} \\ -A_{12} & 0 & A_{23} \\ A_{31} & -A_{23} & 0 \end{bmatrix}. \tag{6.103}$$

The tensor has nine elements, only three of which are normally independent.

As can be seen, tensor A consists of only three elements: A_{12}, A_{31}, and A_{23}. The set of three elements (numbers) may represent a vector in three-dimensional space:

$$\boldsymbol{c} = c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + c_3\mathbf{E}_3, \tag{6.104}$$

where $c_1 = A_{23}$, $c_2 = A_{31}$, and $c_3 = A_{12}$.

Let us choose an arbitrary vector $\boldsymbol{a}$ and consider the case of pre- and postmultiplication of this vector by tensor A. In the former case we have successively

$$\begin{aligned} (A\boldsymbol{a})_1 &= A_{11}a_1 + A_{12}a_2 + A_{13}a_3 \\ &= A_{12}a_2 - A_{31}a_3 = c_3a_2 - c_2a_3, \\ (A\boldsymbol{a})_2 &= A_{21}a_1 + A_{22}a_2 + A_{23}a_3 \\ &= -A_{12}a_1 + A_{23}a_3 = c_1a_3 - c_3a_1, \\ (A\boldsymbol{a})_3 &= A_{31}a_1 + A_{32}a_2 + A_{33}a_3 \\ &= A_{31}a_1 - A_{23}a_2 = c_2a_1 - c_1a_2. \end{aligned} \tag{6.105}$$

In the latter case we have successively

$$\begin{aligned}(\boldsymbol{a}A)_1 &= a_1A_{11} + a_2A_{21} + a_3A_{31}\\ &= -A_{12}a_2 + A_{31}a_3 = c_2a_3 - c_3a_2,\\ (\boldsymbol{a}A)_2 &= a_1A_{12} + a_2A_{22} + a_3A_{32}\\ &= A_{12}a_1 - A_{23}a_3 = c_3a_1 - c_1a_3,\\ (\boldsymbol{a}A)_3 &= a_1A_{13} + a_2A_{23} + a_3A_{33}\\ &= A_{23}a_2 - A_{31}a_1 = c_1a_2 - c_2a_1.\end{aligned} \tag{6.106}$$

From (6.105) and (6.106) we obtain

$$A\boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{c}, \qquad \boldsymbol{a}A = \boldsymbol{c} \times \boldsymbol{a}. \tag{6.107}$$

From (6.107) it follows that

$$A\boldsymbol{a} \neq \boldsymbol{a}A. \tag{6.108}$$

If vector $\boldsymbol{c}$ is associated with tensor A, after the introduction of a vector $\boldsymbol{c}^* = -\boldsymbol{c}$ associated with tensor A^*, the following equalities are valid:

$$\begin{aligned}A\boldsymbol{a} &= \boldsymbol{a}A^* = \boldsymbol{c}^* \times \boldsymbol{a},\\ \boldsymbol{a}A &= A^*\boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{c}^*.\end{aligned} \tag{6.109}$$

Introducing spatial derivatives of a vector function and on the basis of Fig. 6.5 we can write

$$\mathbf{u}_1 = \mathbf{u} + (\delta\mathbf{r} \circ \nabla)\,\mathbf{u} = \mathbf{u} + \delta\mathbf{r}\,(\nabla\mathbf{u}) = \mathbf{u} + \delta\mathbf{r}D, \tag{6.110}$$

where according to (6.90) and (6.91) we have the following matrix representation of tensor D:

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{6.111}$$

In turn, according to the decomposition of tensor D into symmetrical ${}^{(S)}D$ and antisymmetrical tensor ${}^{(A)}D$, (6.110) takes the form

$$\mathbf{u}_1 = \mathbf{u} + \delta\mathbf{r}^{(A)}D + \delta\mathbf{r}^{(S)}D. \tag{6.112}$$

According to (6.104) and (6.107) we have

$$\delta \mathbf{r}^{(A)} D = \boldsymbol{c} \times \delta \mathbf{r}, \tag{6.113}$$

where

$$c_1 = {}^A D_{23}, \quad c_2 = {}^A D_{31}, \quad c_3 = {}^A D_{12}. \tag{6.114}$$

In order to determine the values c_1, c_2, and c_3 associated with the asymmetric tensor we use the following relationships [see (6.110) and (6.111)]:

$$D = \nabla \mathbf{u}, \quad D^* = (\nabla \mathbf{u})^*, \quad {}^{(A)}D = \frac{1}{2}\left[D - D^*\right]. \tag{6.115}$$

The desired coordinates of vector $\boldsymbol{c}$ are equal to

$$\begin{aligned} c_1 &= \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \frac{1}{2}\left(\operatorname{curl} \mathbf{u}\right)_1, \\ c_2 &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\operatorname{curl} \mathbf{u}\right)_2, \\ c_3 &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}\left(\operatorname{curl} \mathbf{u}\right)_3. \end{aligned} \tag{6.116}$$

Multiplying (6.116) respectively by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ and then adding by sides we obtain

$$\boldsymbol{c} = \frac{1}{2}\operatorname{curl} \mathbf{u}. \tag{6.117}$$

We write the original (6.110) in the equivalent form

$$\mathbf{u}_1 = \mathbf{u}_1^s + \mathbf{u}_1^e, \tag{6.118}$$

where

$$\mathbf{u}_1^s = \mathbf{u} + \boldsymbol{\theta} \times \delta \mathbf{r}, \qquad \boldsymbol{\theta} = \frac{1}{2}\operatorname{curl} \mathbf{u}, \tag{6.119}$$

and $\boldsymbol{\theta}$ is the infinitesimal small rotation introduced into the study regarding the displacement of a rigid body in Sect. 5.1.5.

Note that $\mathbf{u}_1^s$ describes a displacement of point B_1 with respect to point B just as would occur in a rigid body. It is the geometric sum of translation vector $\mathbf{u}$ and the displacement resulting from rotation about an axis $\delta \mathbf{r}$ through an angle θ, and hence this term of the equation $\mathbf{u}_1^s$ is marked with a superscript abbreviation s indicating a behavior similar to that of a rigid body.

The remaining term $\mathbf{u}_1^e$ describes the displacement of point B_1 with respect to point B as a result of translatory deformation of the chosen cutout of a deformable body.

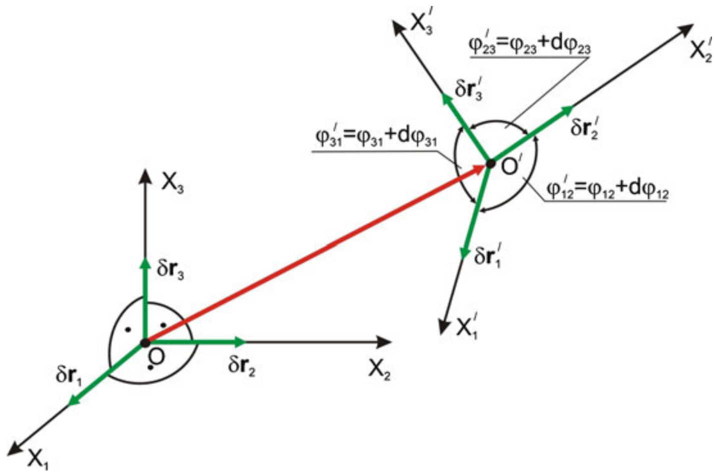

Fig. 6.6 Presentation of the action of a symmetric tensor consisting of six elements onto elastic displacements of point $O' \equiv B'$ with respect to $O \equiv B$

Let us note that, because of the symmetry of tensor ${}^{(S)}D$, we have

$$\mathbf{u}_1^e = \delta\mathbf{r}^{(s)}D = {}^{(s)}D\delta\mathbf{r}. \tag{6.120}$$

Using the second equation of (6.115) we determine the matrix of symmetric tensor ${}^{(S)}D = \varepsilon$ of the form

$$[\varepsilon] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}, \tag{6.121}$$

where

$$\varepsilon_{km} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right). \tag{6.122}$$

Now we will show that the diagonal elements of the symmetric tensor represent the relative strain of the infinitely small segments $\delta\mathbf{r}_1[\delta x_1, 0, 0]$, $\delta\mathbf{r}_2[0, \delta x_2, 0]$ and $\delta\mathbf{r}_3[0, 0, \delta x_3]$ and their relative angular displacement. Let us look at Fig. 6.6, where at the points $B = O$ and $B' = O'$ (Fig. 6.5) the origins of the coordinate systems $OX_1X_2X_3$ and $OX_1'X_2'X_3'$ were introduced; note that the second system is not rectangular.

According to the previous discussion we have

$$\mathbf{u} = \mathrm{d}\mathbf{r}, \tag{6.123}$$

where the operator d is understood as the increment of radius vector $\mathbf{r}$ in time since $(\mathrm{d}/\mathrm{d}t)\mathrm{d}t = d$. Multiplying both sides of (6.123) by the operator δ we obtain

$$\delta\mathbf{u} = \delta\left(\mathrm{d}\mathbf{r}\right) = \mathbf{u}_1 - \mathbf{u} = \mathrm{d}\mathbf{r}_1 - \mathrm{d}\mathbf{r} = \mathrm{d}\left(\mathbf{r}_1 - \mathbf{r}\right) = \mathrm{d}\left(\delta\mathbf{r}\right). \tag{6.124}$$

Hence

$$\mathbf{u}_1 - \mathbf{u} = \delta\mathbf{u} = \varepsilon\delta\mathbf{r} = \mathrm{d}\left(\delta\mathbf{r}\right), \tag{6.125}$$

where the symbol δ denotes an arbitrary infinitely small increment of the given quantity during transition from point O to O' at the given (fixed) time instant.

Multiplying (6.125) (that is, the part consisting of the two last terms) in turn by $\delta\mathbf{r}_1$, $\delta\mathbf{r}_2$, and $\delta\mathbf{r}_3$ we obtain

$$\varepsilon_{11}\delta x_1 = \mathrm{d}\delta x_1, \quad \varepsilon_{22}\delta x_2 = \mathrm{d}\delta x_2, \quad \varepsilon_{33}\delta x_3 = \mathrm{d}\delta x_3. \tag{6.126}$$

From the preceding equations it follows that

$$\varepsilon_{kk} = \frac{\mathrm{d}\delta x_k}{\delta x_k} = \varepsilon_k = \frac{\partial u_k}{\partial x_k}, \tag{6.127}$$

where the elements of ε_{11}, ε_{22}, and ε_{33} denote the relative elongations ε_1, ε_2, and ε_3 of the infinitely small lengths $\delta r_1 = |\delta\mathbf{r}_1|$, $\delta r_2 = |\delta\mathbf{r}_2|$, and $\delta r_3 = |\delta\mathbf{r}_3|$.

Let φ_{kl} denote the angles between the axes k and l and $\varphi'_{k'l'}$ denote the angles between the axes k' and l' after deformation, that is,

$$\varphi'_{k'l'} = \varphi_{kl} + \mathrm{d}\varphi_{kl}. \tag{6.128}$$

In turn, the cosines of the direction angles are equal to

$$\cos\varphi_{kl} = \left(\frac{\delta\mathbf{r}_k}{\delta x_k} \circ \frac{\delta\mathbf{r}_l}{\delta x_l}\right) = \frac{1}{\delta x_k \delta x_l}\left(\delta\mathbf{r}_k \circ \delta\mathbf{r}_l\right). \tag{6.129}$$

Using operator d on both sides of (6.129) we obtain

$$-\sin\varphi_{kl}\mathrm{d}\varphi_{kl} = \frac{1}{\delta x_k \delta x_l}\mathrm{d}\left(\delta\mathbf{r}_k \circ \delta\mathbf{r}_l\right) + \mathrm{d}\left(\frac{1}{\delta x_k \delta x_l}\right)\left(\delta\mathbf{r}_k \circ \delta\mathbf{r}_l\right). \tag{6.130}$$

Because before deformation for $k \neq l$ we have

$$\varphi_{kl} = \frac{\pi}{2}, \qquad \sin\varphi_{kl} = 1, \qquad \delta\mathbf{r}_k \circ \delta\mathbf{r}_l = 0, \tag{6.131}$$

from (6.131) we obtain

$$- \mathrm{d}\varphi_{kl} \equiv \gamma_{kl} = \frac{1}{\delta x_k \delta x_l} \mathrm{d}\left(\delta \mathbf{r}_k \circ \delta \mathbf{r}_l\right). \tag{6.132}$$

Following the transformations of (6.132) we obtain

$$\gamma_{kl} = \frac{1}{\delta x_k \delta x_l} \left(\mathrm{d}\delta \mathbf{r}_k \circ \delta \mathbf{r}_l + \delta \mathbf{r}_k \circ \mathrm{d}\delta \mathbf{r}_l\right), \tag{6.133}$$

and taking into account equalities (6.125) in it we obtain

$$\gamma_{kl} = \frac{1}{\delta x_k \delta x_l} \left[(\varepsilon \delta \mathbf{r}_k) \circ \delta \mathbf{r}_l + (\varepsilon \delta \mathbf{r}_l \circ \delta \mathbf{r}_k)\right]. \tag{6.134}$$

In order to perform scalar multiplication of vectors one should take the values $k = 1, 2, 3$ and $l = 1, 2, 3$; as a result we obtain the relationships

$$\begin{aligned}
&(\delta \mathbf{r}_1)_1 = \delta x_1, \qquad &(\delta \mathbf{r}_1)_2 = 0, \qquad &(\delta \mathbf{r}_1)_3 = 0,\\
&(\delta \mathbf{r}_2)_1 = 0, \qquad &(\delta \mathbf{r}_2)_2 = \delta x_2, \qquad &(\delta \mathbf{r}_2)_3 = 0,\\
&(\delta \mathbf{r}_3)_1 = 0, \qquad &(\delta \mathbf{r}_3)_2 = 0, \qquad &(\delta \mathbf{r}_3)_3 = \delta x_3.
\end{aligned} \tag{6.135}$$

In turn, we conduct a summation over the repeating indices k and l in (6.134) for $k \neq l$, obtaining

$$\begin{aligned}
(\varepsilon \delta \mathbf{r}_1) \circ \delta \mathbf{r}_2 + (\varepsilon \delta \mathbf{r}_2) \circ \delta \mathbf{r}_1 &= (\varepsilon \delta \mathbf{r}_1)_2 \cdot \delta x_2 + (\varepsilon \delta \mathbf{r}_2)_1 \cdot \delta x_1\\
&= (\varepsilon_{12} + \varepsilon_{21})\, \delta x_1 \delta x_2,\\
(\varepsilon \delta \mathbf{r}_2) \circ \delta \mathbf{r}_3 + (\varepsilon \delta \mathbf{r}_3) \circ \delta \mathbf{r}_2 &= (\varepsilon \delta \mathbf{r}_2)_3 \cdot \delta x_3 + (\varepsilon \delta \mathbf{r}_3)_2 \cdot \delta x_2\\
&= (\varepsilon_{23} + \varepsilon_{32})\, \delta x_2 \delta x_3,\\
(\varepsilon \delta \mathbf{r}_3) \circ \delta \mathbf{r}_1 + (\varepsilon \delta \mathbf{r}_1) \circ \delta \mathbf{r}_3 &= (\varepsilon \delta \mathbf{r}_3)_1 \cdot \delta x_1 + (\varepsilon \delta \mathbf{r}_1)_3 \cdot \delta x_3\\
&= (\varepsilon_{31} + \varepsilon_{13})\, \delta x_1 \delta x_3.
\end{aligned} \tag{6.136}$$

Substituting (6.136) into (6.134) and taking into account that $\varepsilon_{12} = \varepsilon_{21}$, $\varepsilon_{23} = \varepsilon_{32}$, and $\varepsilon_{13} = \varepsilon_{31}$ we obtain

$$\gamma_{12} = 2\varepsilon_{12}, \qquad \gamma_{23} = 2\varepsilon_{23}, \qquad \gamma_{13} = 2\varepsilon_{13}. \tag{6.137}$$

The values γ_{kl} $(k \neq l)$, taking into account (6.137) and (6.121), are equal to

$$\gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1},$$

$$\begin{aligned}\gamma_{23} &= 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2},\\ \gamma_{13} &= 2\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}.\end{aligned} \tag{6.138}$$

The obtained relationships (6.138) allow for a physical interpretation of non-diagonal elements of the symmetric tensor ε that are equal to

$$\varepsilon_{kl} = \frac{1}{2}\gamma_{kl}, \qquad k, l = 1, 2, 3, \qquad k \neq l. \tag{6.139}$$

This means that the non-diagonal elements of tensor ε represent shear strain angles between the coordinate axes in planes determined by the axes of the numbers k and l.

If an elastic body undergoes deformation, we may speak of the velocity of this deformation. Let us return then to our basic vectorial equation describing the deformation of two points in an elastic body, that is, to (6.118) and (6.119). Dividing the foregoing relationships by dt we obtain

$$\dot{\mathbf{u}}_1 = \dot{\mathbf{u}} + \dot{\boldsymbol{\theta}} \times \delta\mathbf{r} + \dot{\varepsilon}\delta\mathbf{r}, \tag{6.140}$$

or, in equivalent form,

$$\mathbf{v}_1 = \mathbf{v} + \boldsymbol{\omega} \times \delta\mathbf{r} + \varepsilon\dot{\delta}\mathbf{r}, \tag{6.141}$$

where $\dot{\varepsilon}$ denotes a tensor describing the velocity of body deformation that, according to (6.121), has the following matrix representation:

$$[\dot{\varepsilon}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) & \frac{\partial v_3}{\partial x_3} \end{bmatrix}. \tag{6.142}$$

From (6.141) we obtain

$$\delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v} = \boldsymbol{\omega} \times \delta\mathbf{r} + \dot{\varepsilon}\delta\mathbf{r}. \tag{6.143}$$

From the kinematics of a point the observation follows that if the distance between points B and B_1 remained constant (rigid body), the following condition would have to be satisfied:

$$\delta\mathbf{v} \circ \delta\mathbf{r} = 0. \tag{6.144}$$

Let us demand that the condition be satisfied for the case (6.143). Multiplying (6.143) through by $\delta\mathbf{r}$ we obtain

$$(\boldsymbol{\omega} \times \delta\mathbf{r}) \circ \delta\mathbf{r} + \dot{\varepsilon}\,(\delta r)^2 = \dot{\varepsilon}\,(\delta r)^2 = 0, \tag{6.145}$$

which indicates that matrix (6.142) $[\dot{\varepsilon}] = [0]$, that is, all elements of that matrix should be equal to zero.

We aim at drawing conclusions from the fact that the elements of tensor $\dot{\varepsilon}$ became zeros.

Let us first consider the elements lying on the diagonal of tensor (6.142). They are equal to

$$\frac{\partial v_1}{\partial x_1} = 0, \qquad \frac{\partial v_2}{\partial x_2} = 0, \qquad \frac{\partial v_3}{\partial x_3} = 0, \tag{6.146}$$

which after integration yields

$$v_1 = v_1\,(x_2, x_3, t)\,, \quad v_2 = v_2\,(x_1, x_3, t)\,, \quad v_3 = v_3\,(x_1, x_2, t)\,. \tag{6.147}$$

Equating the non-diagonal elements to zero we obtain

$$\frac{\partial v_1}{\partial x_2} = -\frac{\partial v_2}{\partial x_1}, \quad \frac{\partial v_1}{\partial x_3} = -\frac{\partial v_3}{\partial x_1}, \quad \frac{\partial v_2}{\partial x_3} = -\frac{\partial v_3}{\partial x_2}. \tag{6.148}$$

We multiply the preceding equation by sides respectively by $\partial/\partial x_3$, $\partial/\partial x_2$, and $\partial/\partial x_1$ to obtain

$$\frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_3}\right) = -\frac{\partial}{\partial x_1}\left(\frac{\partial v_2}{\partial x_3}\right), \quad \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_2}\right),$$
$$\frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_1}\right) = -\frac{\partial}{\partial x_2}\left(\frac{\partial v_3}{\partial x_1}\right). \tag{6.149}$$

For instance, let

$$\frac{\partial v_1}{\partial x_3} = \psi_{13}\,(x_2, t)\,, \qquad \frac{\partial v_1}{\partial x_2} = \psi_{12}\,(x_3, t)\,, \tag{6.150}$$

and in general,

$$\frac{\partial v_i}{\partial x_j} = \psi_{ij}\,(x_k, t)\,. \tag{6.151}$$

Differentiating the first from the preceding equations with respect to x_2 and the second with respect to x_3 we have

$$\frac{\partial^2 v_1}{\partial x_2 \partial x_3} = \frac{\partial \psi_{13}\,(x_2, t)}{\partial x_2}, \qquad \frac{\partial^2 v_1}{\partial x_2 \partial x_3} = \frac{\partial \psi_{12}\,(x_3, t)}{\partial x_3}, \tag{6.152}$$

which means that

$$\frac{\partial\psi_{13}\left(x_2,t\right)}{\partial x_2}=\frac{\partial\psi_{12}\left(x_3,t\right)}{\partial x_3}=\gamma_{132}(t)=\gamma_{123}(t). \tag{6.153}$$

After integration of the preceding equation we obtain

$$\begin{aligned}\psi_{13}\left(x_2,t\right) &= \gamma_{132}x_2+\omega_{13}(t),\\ \psi_{12}\left(x_3,t\right) &= \gamma_{123}x_3+\omega_{12}(t),\end{aligned} \tag{6.154}$$

which can be generalized as

$$\frac{\partial\psi_{ij}}{\partial x_k}=\gamma_{ijk}(t), \tag{6.155}$$

so generally we have $\gamma_{ikj}(t)=\gamma_{ijk}(t)$.

From (6.148) it follows directly that $\psi_{ij}=-\psi_{ji}$. The integration conducted earlier allows for the introduction of a general expression of the form

$$\psi_{ij}\left(x_k,t\right)=\gamma_{ijk}x_k+\omega_{ij}(t). \tag{6.156}$$

In a similar way we can write

$$\psi_{ji}\left(x_k,t\right)=\gamma_{jik}x_k+\omega_{ji}(t). \tag{6.157}$$

Adding the two preceding equations by sides we obtain

$$\left(\gamma_{ijk}+\gamma_{jik}\right)x_k+\omega_{ij}+\omega_{ji}=0. \tag{6.158}$$

Equation (6.158) is satisfied when

$$\gamma_{ijk}=-\gamma_{jik},\qquad \omega_{ij}=-\omega_{ji} \tag{6.159}$$

for $i\neq j,\ j\neq k,\ i\neq k$.

We noted earlier that $\gamma_{ikj}=\gamma_{ijk}$ and $\gamma_{ijk}=-\gamma_{jik}$; therefore, it is possible only if $\gamma_{ijk}=0$ for $i\neq j,\ j\neq k,\ i\neq k$. This means that $\psi_{ij}=\omega_{ij}$ and $\psi_{ji}=\omega_{ji}$.

From a condition of lack of deformations, that is, from the zeroing $\dot{\varepsilon}=0$, it follows that the velocity field in three-dimensional space has the property $\omega_{ij}=-\omega_{ji}$, that is, $\omega_{12}=-\omega_{21}$, $\omega_{13}=-\omega_{31}$, and $\omega_{23}=-\omega_{32}$. It can be seen that we are dealing with the skew-symmetric tensor ω. We take

$$\omega_1=-\omega_{32}=\omega_{23},\qquad \omega_2=-\omega_{13}=\omega_{31},\qquad \omega_3=-\omega_{21}=\omega_{12}, \tag{6.160}$$

which can be represented by means of an alternating tensor as

$$\omega_r=\frac{1}{2}\varepsilon_{rik}\omega_{ik}, \tag{6.161}$$

and, in turn, introducing

$$\boldsymbol{\omega}=\mathbf{E}_1\omega_1+\mathbf{E}_2\omega_2+\mathbf{E}_3\omega_3 \tag{6.162}$$

we have

$$\delta \mathbf{v} = \boldsymbol{\omega} \times \delta \mathbf{r} = \dot{\boldsymbol{\theta}} \times \delta \mathbf{r}. \tag{6.163}$$

The zeroing of the tensor $\dot{\varepsilon}$ causes the appearance of the antisymmetric tensor ω, and we have

$$\delta \mathbf{r}\omega = \boldsymbol{\omega} \times \delta \mathbf{r} = \delta \mathbf{v}, \tag{6.164}$$

where ω is now the tensor ω_{kl}.

Figure 6.5 depicts the volumes of the element cut out of a deformable body before deformation V and after deformation V'. We will now show how the elementary volume changes after deformation. To this end we calculate the so-called *relative coefficient of velocity expansion* of the form

$$\chi = \frac{\delta V' - \delta V}{\delta V}, \tag{6.165}$$

where

$$\delta V = \delta x_1 \delta x_2 \delta x_3. \tag{6.166}$$

The increment of the volume [numerator in (6.165)] on the assumption of infinitely small deformation has the form

$$\begin{aligned} \delta V' - \delta V \equiv \mathrm{d}(\delta V) &= \mathrm{d}(\delta x_1 \delta x_2 \delta x_3) \\ &= \mathrm{d}(\delta x_1)\, \delta x_2 \delta x_3 + \delta x_1 \mathrm{d}(\delta x_2)\, \delta x_3 + \delta x_1 \delta x_2 \mathrm{d}(\delta x_3) \\ &= (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\, \delta x_1 \delta x_2 \delta x_3 \\ &= (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\, \delta V, \end{aligned} \tag{6.167}$$

where (6.126) and (6.166) were used during the transformations.

From (6.165), taking into account (6.127), we obtain

$$\chi = \frac{(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\, \delta V}{\delta V} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \operatorname{div} \mathbf{u}. \tag{6.168}$$

Dividing the foregoing equation through by $\mathrm{d}t$ we obtain

$$\dot{\chi} = \operatorname{div} \mathbf{v}, \tag{6.169}$$

which determines the speed of the relative volume dilation (expansion) of the considered cutout of a deformable body of volume δV.

Earlier, in our analysis of statically indeterminate problems, we mentioned that it is not possible to solve a problem without knowing the deformations, which requires the introduction of kinematics or, in the case of the calculation conducted in this chapter, the tensor of deformations. Knowing the tensor of deformation in the case of a perfectly elastic isotropic (its mechanical properties are independent of the choice of direction of the coordinate axes) body we can express the desired stress tensor in

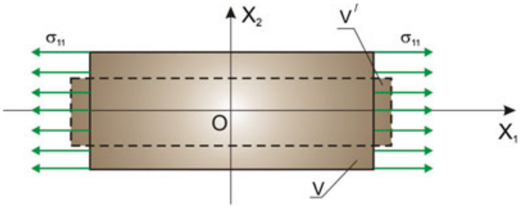

Fig. 6.7 Uniaxial stress state

the following way:

$$\sigma_{ik} = 2G\varepsilon_{ik} + \delta_{ik}\lambda\delta_{mn}\varepsilon_{mn}. \tag{6.170}$$

The preceding equation is called a state equation, where G is a shear modulus, δ_{ik} a Kronecker delta, and λ a Lamé[6] constant.

Finally, let us note that the constants G and λ can be expressed in terms of two other constants, namely, Young's[7] modulus E and Poisson's[8] ratio ν:

$$G = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{6.171}$$

Let us now consider the special cases resulting from the equation of state (6.170), which will be transformed into the form

$$\varepsilon_{ik} = \frac{1}{E}\left[(1+\nu)\sigma_{ik} - \nu\sigma_{mn}\delta_{mn}\delta_{ik}\right], \tag{6.172}$$

and then the special cases that follow from the state (6.172).

(i) Inserting $i = k = 1$ into (6.172) we obtain

$$\varepsilon_{11} = \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\sigma_{11}\right] = \frac{\sigma_{11}}{E}, \tag{6.173}$$

which is illustrated in Fig. 6.7.

Equation (6.172) describes the classical Hooke's[9] law, which we used in Example 2.5. In Fig. 6.7 the volume of a deformed homogeneous rod of a constant cross section is indicated by V'. The presented deformation in the directions of axes OX_2 and OX_3 is caused by the uniaxial stress σ_{11}. From (6.172) we calculate

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[6]Gabriel Lamé (1795–1870), French mathematician.

[7]Thomas Young (1733–1829), English mathematician and mechanician who also studied medicine and physiology.

[8]Simeon D. Poisson (1781–1840), French mathematician and physicist.

[9]Robert Hooke (1635–1702) formulated this law in 1676; on the basis of experimental research he observed that the deformation caused by a load is proportional to that load.

$$\varepsilon_{22} = \frac{1}{E}\left[(1+\nu)\,\sigma_{22} - \nu\,(\sigma_{11}+\sigma_{22})\right] = -\frac{\nu\sigma_{11}}{E}, \tag{6.174}$$

and in a similar way we calculate $\varepsilon_{33} = \varepsilon_{22}$.

(ii) Let us now consider the case of a so-called pure shear, that is, when we have shear stresses $\sigma_{12} = \sigma_{21}$ exclusively. From (6.172) it follows that the only non-zero deformations are

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}, \qquad \varepsilon_{21} = \frac{1+\nu}{E}\sigma_{21}, \tag{6.175}$$

and in this case the matrices of the stress tensor and deformation tensor are as follows:

$$[\sigma_{ik}] = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\varepsilon_{ik}] = \begin{bmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.176}$$

From (6.170) we have

$$\sigma_{12} = 2G\varepsilon_{12}, \tag{6.177}$$

and because from (6.138) it follows that $\gamma_{12} = 2\varepsilon_{12}$, taking into account this relationship in (6.177), we obtain

$$\sigma_{12} = G\gamma_{12}, \tag{6.178}$$

which indicates pure shear.

In general, during deformation the positions in space R^3 of both elementary surfaces and elementary volumes undergo change, and what follows is that also in the distribution of surface forces and volume internal (or sometimes even external) forces are subjected to change. However, by the problems discussed so far and by the derivation of stress tensors and the speeds of their changes, that problem was simplified because of the assumption that the so-called freezing hypothesis was satisfied. The mathematical model of the problem was reduced, then, to an analysis of algebraic equations and differential linear equations. Such an assumption enables the superposition principle to be applied.

Let us return to the case discussed earlier of the loading of an elastic (deformable) body obeying Hooke's law with an arbitrary system of concentrated forces and moments whose effects are linear and angular displacements. Further we will not distinguish between forces and moments, but we introduce a notion of generalized forces Q_k and corresponding generalized displacements q_k. This problem was considered in Chap. 1.

The action of generalized forces $Q_k, k = 1, \ldots, K$ on the considered mechanical system generates a reaction of the system in the form of displacements (deformations) of the points of this system according to a linear transformation of the form

$$q_l = r_{lk} Q_k, \qquad k = 1, \ldots, K, \tag{6.179}$$

where the matrix r_{lk} is called the *matrix of influence (transmittance)*. This approach can also be applied in the case of system dynamics on the assumption of linearity of the considered model.

If we take, for example, a matrix element r_{lk}, it transmits the force Q_k to point q_l, which leads to the determination of a linear displacement or a rotation angle at that point depending on whether we are dealing with a force or a moment.

We call mechanical systems in relation to which we can apply (6.179) *Clapeyron*[10] *systems*, after the scientist who introduced this line of argumentation.

An elastic system subjected to the action of generalized forces Q_k undergoes displacements q_l, that is, the following elastic potential energy accumulates in the system:

$$V = \frac{1}{2} q_l Q_l, \qquad l = 1, \ldots, K. \tag{6.180}$$

We will briefly describe a superposition principle leading to the formulation of Betti's[11] reciprocity theorem.

Let us divide the system of generalized forces into two groups.

Theorem 6.1 (Betti's theorem). *If the Clapeyron system is subjected to two sets of forces, then the work done by the first set of forces through the displacements generated by the second set of forces is equal to the work done by the second set of forces through the displacements generated by the first set of forces.*

Proof. The work of the first group $Q_l^{(1)}$ done through the displacement is equal to

$$\tilde{W}^{(1)} = \frac{1}{2} \sum_{l=1}^{L} q_l Q_l^{(1)},$$

where L denotes the number of forces of the first group.

Now, to the body already loaded with the forces $Q_l^{(1)}$, $l = 1, \ldots, L$, we apply additionally the forces $Q_m^{(2)}$, $m = 1, \ldots, M$ of the second group.

In this case the work is done by the forces of both the first and second groups, and therefore

$$\tilde{\tilde{W}}^{(1)} = \frac{1}{2} \left[\sum_{m=1}^{M} q_m^{(1)} Q_m^{(2)} + \sum_{l=1}^{L} q_l^{(2)} Q_l^{(1)} \right],$$

where $q_l^{(2)}$ denotes displacement at the points of application of the forces from the first group generated by the forces from the second group, and $q_m^{(1)}$ denotes displacement at the points of application of forces from the second group produced

[10]Benoit Clapeyron (1799–1864), French mathematician and physicist, founder of modern thermodynamics.

[11]Enrico Betti (1823–1892), Italian mathematician.

by the forces from the first group. The total work of the forces is equal to

$$W^{(1)} = \tilde{W}^{(1)} + \tilde{\tilde{W}}^{(1)}.$$

Now, assuming that at first that the body is acted upon by the forces from the second group and then forces from the first group, we obtain

$$W^{(2)} = \tilde{W}^{(2)} + \tilde{\tilde{W}}^{(2)},$$

where

$$\begin{aligned} \tilde{W}^{(2)} &= \frac{1}{2}\sum_{l=1}^{L} q_l Q_l^{(2)}, \\ \tilde{\tilde{W}}^{(2)} &= \frac{1}{2}\sum_{m=1}^{L} q_m^{(2)} Q_m^{(1)} + \sum_{l=1}^{M} q_l^{(1)} Q_l^{(2)}. \end{aligned}$$

According to the superposition principle we have

$$W^{(1)} = W^{(2)},$$

from which it follows that

$$\sum_{m=1}^{L} q_m^{(2)} Q_m^{(1)} = \sum_{l=1}^{M} q_l^{(1)} Q_l^{(2)},$$

which proves the theorem. □

It is also possible to demonstrate that the transmittance matrix is a symmetrical matrix, that is, that $r_{lk} = r_{kl}$ for every k and l. Knowing the elastic (potential) energy V of an elastic body (system) and the generalized forces Q allows us to determine the generalized displacements q caused by these forces.

Theorem 6.2 (Castigliano's[12] theorem). *Generalized displacements q_k produced by generalized forces Q_k at an arbitrary point of a Clapeyron system are derivatives of the strain energy V of the system with respect to the generalized force acting on the generalized displacement, that is, the following equation is satisfied*

$$q_i = \frac{\partial V}{\partial Q_i}. \tag{6.181}$$

Proof. Substituting (6.179) into (6.180) we obtain

$$V = \frac{1}{2} r_{lk} Q_l Q_k.$$

[12]Carlo A. Castigliano (1847–1884), Italian engineer who completed his studies in Turin; the theorem formulated in his diploma thesis was later named after him.

Differentiating V with respect to Q_i we obtain

$$\begin{aligned}\frac{\partial V}{\partial Q_i} &= \frac{1}{2} r_{lk}\left(\frac{\partial Q_l}{\partial Q_i} Q_k + Q_l \frac{\partial Q_k}{\partial Q_i}\right) = \frac{1}{2} r_{lk}\delta_l^i Q_k + \frac{1}{2} r_{lk} Q_l \delta_k^i \\ &= \frac{1}{2} r_{ik} Q_k + \frac{1}{2} r_{li} Q_l,\end{aligned}$$

since $\frac{\partial Q_l}{\partial Q_i} = \delta_l^i$ and $\frac{\partial Q_k}{\partial Q_i} = \delta_k^i$.

Finally, because of the matrix transmittance symmetry $r_{ik} = r_{ki}$ we obtain

$$\frac{\partial V}{\partial Q_i} = \frac{1}{2}\left(r_{ik} Q_k + r_{li} Q_l\right) = q_i,$$

which completes the proof. □

Finally, we return to the example of a beam of length l loaded in the middle with a force (Fig. 2.9). In this case we will determine its deflection at the point of application of the force.

The beam underwent deflection as a result of action of the force **F** and in effect the internal (potential) energy due to bending was accumulated in it. A differential of this energy is equal to

$$dV = \frac{1}{2} M \, d\varphi, \tag{6.182}$$

where M is a bending moment and the differential $d\varphi$ is described by

$$d\varphi = \frac{M \, dx}{EI}, \tag{6.183}$$

where EI denotes beam rigidity.

The bending moment, after introducing axis OX whose origin is at the left end of the beam, is equal to $M = \frac{1}{2} F x$, that is, $\frac{\partial M}{\partial F} = \frac{1}{2} x$.

Because $V = \frac{1}{2} \int\limits_0^l \frac{M^2}{EI} dx$, from (6.181) it follows that the displacement at the point of application of the force **F** is equal to

$$q = \frac{\partial V}{\partial F} = \int\limits_0^l \frac{M}{EI} \frac{\partial M}{\partial F} dx = \frac{F}{2EI} \int\limits_0^{\frac{l}{2}} x^2 dx = \frac{F l^3}{48 EI}, \tag{6.184}$$

where because of the symmetry the integration was conducted over the interval $\left[0, \frac{l}{2}\right]$.

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