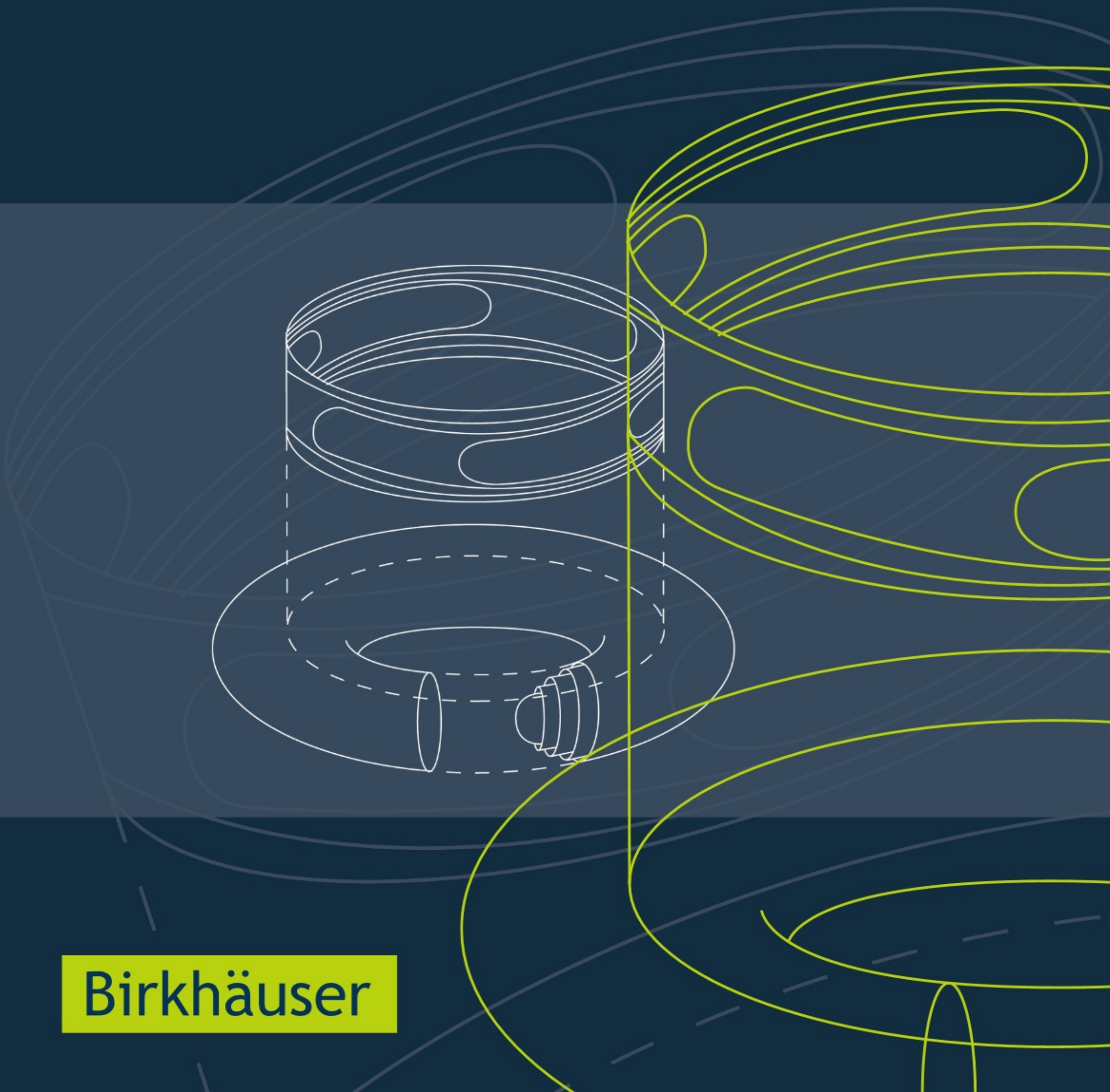
A Geometric Approach to Differential Forms

DAVID BACHMAN



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To Sebastian and Simon

Contents

Preface

The present work is not meant to contain any new material about differential forms. There are many good books out there which give complete treatments of the subject. Rather, the goal here is to make the topic of differential forms accessible to the sophomore level undergraduate, while still providing material that will be of interest to more advanced students.

There are three tracks through this text. The first is a course in *Multivariable Calculus*, suitable for the third semester in a standard calculus sequence. The second track is a sophomore level *Vector Calculus* class. The last track is for advanced undergraduates, or even beginning graduate students. At many institutions, a course in linear algebra is not a prerequisite for either multivariable calculus or vector calculus. Consequently, this book has been written so that the earlier chapters do not require many concepts from linear algebra. What little is needed is covered in the first section.

The book begins with basic concepts from multivariable calculus such as partial derivatives, gradients and multiple integrals. All of these topics are introduced in an informal, pictorial way to quickly get students to the point where they can do basic calculations and understand what they mean. The second chapter focuses on parameterizations of curves, surfaces and three-dimensional regions. We spend considerable time here developing tools which will help students find parameterizations on their own, as this is a common stumbling block.

Chapter 3 is purely motivational. It is included to help students understand why differential forms arise naturally when integrating over parameterized domains.

The heart of this text is Chapters 4 through 7. In these chapters, the entire machinery of differential forms is developed from a geometric standpoint. New ideas are always introduced with a picture. Verbal descriptions of geometric actions are set out in boxes.

Chapter 7 focuses on the development of the generalized Stokes' Theorem. This is really the centerpiece of the text. Everything that precedes it is there for the sole purpose of its development. Everything that follows is an application.

The equation is simple:

$$\int_{\partial C} \omega = \int_{C} d\omega.$$

Yet it implies, for example, all integral theorems of classical vector analysis. Its simplicity is precisely why it is easier for students to understand and remember than these classical results.

Chapter 7 concludes with a discussion on how to recover all of vector calculus from the generalized Stokes' Theorem. By the time students get through this they tend to be more proficient at vector integration than after traditional classes in vector calculus. Perhaps this will allay some of the concerns many will have in adopting this textbook for traditional classes.

Chapter 8 contains further applications of differential forms. These include Maxwell's equations and an introduction to the theory of foliations and contact structures. This material should be accessible to anyone who has worked through Chapter 7.

Chapter 9 is intended for advanced undergraduate and beginning graduate students. It focuses on generalizing the theory of differential forms to the setting of abstract manifolds. The final section contains a brief introduction to DeRham cohomology.

We now describe the three primary tracks through this text.

Track 1. Multivariable Calculus (Calculus III). For such a course, one should focus on the definitions of n-forms on $\mathbb{R}^m$, where n and m are at most 3. The following Chapters/Sections are suggested:

- Chapter 1, perhaps supplementing Section 1.5 with additional material on max/min problems,
- Chapter 2,
- Chapter 4, excluding Sections 4.4 and 4.5 due to time constraints,
- Chapters 5–7,
- Appendix A.

Track 2. Vector Calculus. In this course, one should mention that for n-forms on $\mathbb{R}^m$ the numbers n and m could be anything, although in practice it is difficult to work examples when either is bigger than 4. The following Chapters/Sections are suggested:

- Section 1.1 (unless Linear Algebra is a prerequisite),
- Chapter 2,
- Chapter 3 (one lecture),
- Chapters 4–7,
- Chapter 8, as time permits.

Track 3. Upper Division Course. Students should have had linear algebra, and perhaps even basic courses in group theory and topology.

- Chapter 3 (perhaps as a reading assignment),
- Chapters 4–7 (relatively quickly),
- Chapters 8 and 9.

The original motivation for this book came from [GP74], the text I learned differential topology from as a graduate student. In that text, differential forms are defined in a highly algebraic manner, which left me craving something more intuitive. In searching for a more geometric interpretation, I came across Chapter 7 of Arnold's text on classical mechanics [Arn97], where there is a wonderful introduction to differential forms given from a geometric viewpoint. In some sense, the present work is an expansion of the presentation given there. Hubbard and Hubbard's text [HH01] was also a helpful reference during the preparation of this manuscript.

The writing of this book began with a set of lecture notes from an introductory course on differential forms, given at Portland State University, during the summer of 2000. The notes were then revised for subsequent courses on multivariable calculus and vector calculus at California Polytechnic State University, San Luis Obispo and Pitzer College.

I thank several people. First and foremost, I am grateful to all those students who survived the earlier versions of this book. I would also like to thank several of my colleagues for giving me helpful comments. Most notably, Don Hartig, Matthew White and Jim Hoste had several comments after using earlier versions of this text for vector or multivariable calculus courses. John Etnyre and Danny Calegari gave me feedback regarding Chapter 8 and Saul Schleimer suggested Example 27. Other helpful suggestions were provided by Ryan Derby–Talbot. Alvin Bachman suggested some of the formatting of the text. Finally, the idea to write this text came from conversations with Robert Ghrist while I was a graduate student at the University of Texas at Austin.

Claremont, CA *David Bachman*
March, 2006

Guide to the Reader

It often seems like there are two types of students of mathematics: those who prefer to learn by studying equations and following derivations, and those who prefer pictures. If you are of the former type, this book is not for you. However, it is the opinion of the author that the topic of differential forms is inherently geometric, and thus should be learned in a visual way. Of course, learning mathematics in this way has serious limitations: how can one visualize a 23-dimensional manifold? We take the approach that such ideas can usually be built up by analogy to simpler cases. So the first task of the student should be to really understand the simplest case, which CAN often be visualized.

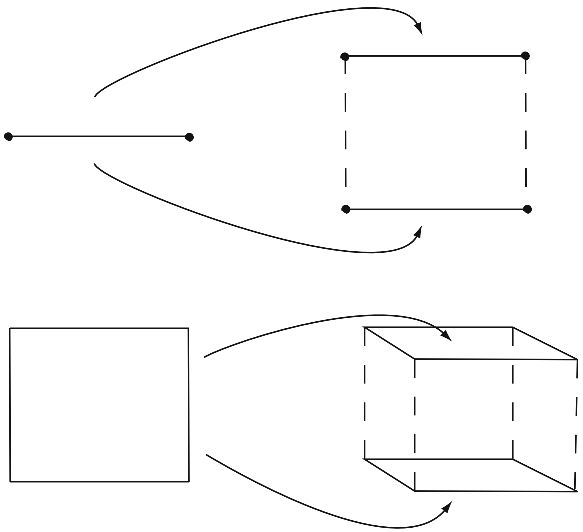

Fig. 0.1. The faces of the n-dimensional cube come from connecting the faces of two copies of an $(n - 1)$-dimensional cube.

For example, suppose one wants to understand the combinatorics of the n-dimensional cube. We can visualize a 1-D cube (i.e., an interval), and see just from our mental picture that it has two boundary points. Next, we can visualize a 2-D cube (a square), and see from our picture that this has four intervals on its boundary. Furthermore, we see that we can construct this 2-D cube by taking two parallel copies of our original 1-D cube and connecting the endpoints. Since there are two endpoints, we get two new intervals, in addition to the two we started with (see Fig. 0.1). Now, to construct a 3-D cube, we place two squares parallel to each other, and connect up their edges. Each time we connect an edge of one square to an edge of the other, we get a new square on the boundary of the 3-D cube. Hence, since there were four edges on the boundary of each square, we get four new squares, in addition to the two we started with, making six in all. Now, if the student understands this, then it should not be hard to convince him/her that every time we go up a dimension, the number of lower-dimensional cubes on the boundary is the same as in the previous dimension, plus two. Finally, from this we can conclude that there are $2n$ $(n-1)$-dimensional cubes on the boundary of the n-dimensional cube.

Note the strategy in the above example: we understand the "small" cases visually, and use them to generalize to the cases we cannot visualize. This will be our approach in studying differential forms.

Perhaps this goes against some trends in mathematics in the last several hundred years. After all, there were times when people took geometric intuition as proof, and later found that their intuition was wrong. This gave rise to the formalists, who accepted nothing as proof that was not a sequence of formally manipulated logical statements. We do not scoff at this point of view. We make no claim that the above derivation for the number of $(n-1)$-dimensional cubes on the boundary of an n-dimensional cube is actually a proof. It is only a convincing argument, that gives enough insight to actually produce a proof. Formally, a proof would still need to be given. Unfortunately, all too often the classical math book begins the subject with the proof, which hides all of the geometric intuition that the above argument leads to.

A Geometric Approach to Differential Forms

1

Multivariable Calculus

1.1 Vectors

A *vector* is a lot like a point in space. The primary difference is that we do not usually think about doing algebra with points, while algebra with vectors is common.

When one switches from talking about points like $(1, 2)$ to vectors like $\langle 1, 2\rangle$, both the language and notation change. We will be very consistent in this text about using parentheses to denote points and brackets to denote vectors. When discussing the point $(1, 2)$ we say the numbers 1 and 2 are its *coordinates*. If we are discussing the vector $\langle 1, 2\rangle$ then 1 and 2 are its *components*.

One often visualizes a vector $\langle a, b\rangle$ as an arrow from the point $(0, 0)$ to the point (a, b). This has some pleasant features. First, it immediately follows from the Pythagorean Theorem that the length of the arrow representing the vector $\langle a, b\rangle$ is

$$|\langle a, b\rangle| = \sqrt{a^2 + b^2}.$$

We add vectors just as one would hope:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle.$$

Geometrically, adding a vector V_1 to a vector V_2 is equivalent to sliding V_2 along V_1 until its “tail" is at the “tip" of V_1. The vector which represents the sum $V_1 + V_2$ is then the one which connects the tail of V_1 to the tip of V_2. See Figure 1.1.

Multiplication is a bit trickier. The most basic kind of multiplication involves a number and a vector, as follows:

$$c\langle a, b\rangle = \langle ca, cb\rangle.$$

1.1. Use similar triangles to show that $c\langle a, b\rangle$ is a vector that points in the same direction as $\langle a, b\rangle$, but has a length that is c times as large.

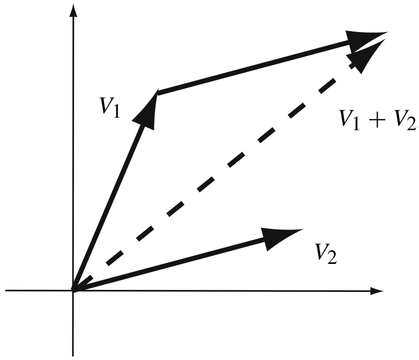

Fig. 1.1. Adding vectors.

1.2. Find a vector that points in the same direction as $\langle 3, 4\rangle$, but has length one. (Such a vector is called a *unit vector*.)

To define the product of two vectors the simplest thing to do is to define the product as follows:

$$\langle a, b\rangle\langle c, d\rangle = \langle ac, bd\rangle.$$

There is nothing wrong with this, but it does not turn out to be terribly useful. Perhaps the reason is that this definition does not lend itself to a good geometric interpretation.

A more useful way to multiply vectors is called the *dot product*. The trick with the dot product is to define the product of two vectors to be the number

$$\langle a, b\rangle \cdot \langle c, d\rangle = ac + bd.$$

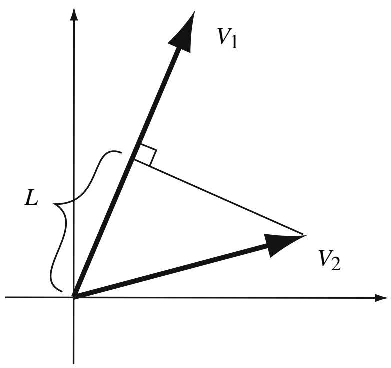

Fig. 1.2. The dot product of V_1 and V_2 is L times the length of V_1.

There are two noteworthy things that immediately follow from this definition. First, notice that if $V_1 = \langle a, b\rangle$, then $V_1 \cdot V_1 = a^2 + b^2 = |V_1|^2$. Second, notice that the slope of the line containing V_1 is $\frac{b}{a}$. If $V_2 = \langle c, d\rangle$ is perpendicular to V_1 then $\frac{d}{c} = -\frac{a}{b}$. Cross-multiplying then gives $bd = -ac$, and hence, $ac + bd = 0$. We conclude the dot product of perpendicular vectors is zero.

Both of these facts also follow from the geometric interpretation of the dot product shown in Figure 1.2. In this figure, we see that $V_1 \cdot V_2$ is the length of the projection of V_2 onto V_1, times the length of V_1. Letting θ be the angle between these two vectors leads to an alternate way to compute dot products:

$$V_1 \cdot V_2 = |V_1||V_2| \cos\theta.$$

To see this, note that the length L of the projection of V_2 onto V_1 is given by $|V_2| \cos\theta$.

1.3. Suppose $V_1 = \langle a, b\rangle$, $V_2 = \langle c, d\rangle$ and θ is the angle between them. Show that

$$ac + bd = |V_1||V_2| \cos\theta.$$

1.4. Use the dot product to compute the cosine of the angle between the vectors $\langle 1, 2\rangle$ and $\langle 4, 2\rangle$.

Another geometric quantity that we will need is the area of the parallelogram spanned by two vectors.

1.5. Suppose $V_1 = \langle a, b\rangle$ and $V_2 = \langle c, d\rangle$. Show that the area of the parallelogram spanned by these vectors is $|ad - bc|$.

One common way to denote a set of vectors is by writing a *matrix* where the vectors appear as columns (or rows). The *determinant* of such a matrix is then defined to be the (signed) area of the parallelogram spanned by its column vectors. So, from the last exercise we have:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Notice that this answer may be negative. This is because the determinant not only tells us area, but also something about the order of the vectors $\langle a, b\rangle$ and $\langle c, d\rangle$.

Everything we have discussed above generalizes to higher dimensions. For example, if $V_1 = \langle a, b, c\rangle$, then the length of V_1 is given by

$$|V_1| = \sqrt{a^2 + b^2 + c^2}.$$

If $V_2 = \langle d, e, f\rangle$, then

$$V_1 + V_2 = \langle a + d, b + e, c + f\rangle.$$

The same geometric interpretation of addition (sliding V_2 until its tail ends up at the tip of V_1) holds in higher dimensions as well. The dot product also works as expected in higher dimensions:

$$V_1 \cdot V_2 = ad + be + cf.$$

and its geometric interpretation as the projected length of V_2, times the length of V_1, holds.

It is a bit harder to show, but if $V_3 = \langle g, h, i \rangle$, then the volume of the parallelpiped spanned by V_1, V_2 and V_3 is given by the absolute value of:

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = (aei + dhc + gbf) - (ahf + dbi + gec).$$

This is the formula for the determinant of a 3×3 matrix.

1.6. Find the volume of the parallelpiped spanned by the vectors $\langle 1, 0, 1 \rangle$, $\langle 1, 2, 3 \rangle$ and $\langle 2, 5, 3 \rangle$.

1.2 Functions of multiple variables

We denote by $\mathbb{R}^n$ the set of points with n-coordinates. If n is between 1 and 3 these spaces are very familiar. For example, $\mathbb{R}^1$ is just the number line whose depiction hangs above every elementary school blackboard. The space $\mathbb{R}^2$ is just the xy-plane that we employ so often in precalculus and calculus. And $\mathbb{R}^3$ is, of course, familiar as the three-dimensional space that we feel like we experience every day (mathematicians and physicists debate whether or not this is *really* the space we live in).

The space $\mathbb{R}^4$ is probably less familiar. One can think of the extra coordinate as time, or color, or anything else that gives more information. At some point we must just give up on visualization. There is no way to picture $\mathbb{R}^{20}$. This does not mean it is useless. To model the stock market, for example, one may want to represent its state at a particular point in time as a point with as many coordinates as there are stocks.

Fortunately for us, if you really understand differential forms in dimensions up to three, then very little needs to be addressed to generalize to higher dimensions.

In this text we will often represent functions abstractly by saying how many numbers go into the function, and how many come out. So, if we write $f : \mathbb{R}^n \to \mathbb{R}^m$, we mean f is a function whose input is a point with n coordinates and whose output is a point with m coordinates.

Some cases of this are familiar. For example, if $y = f(x)$ is a typical function from Calculus I, then $f : \mathbb{R}^1 \to \mathbb{R}^1$.

In this chapter, we focus on functions of the form $f : \mathbb{R}^2 \to \mathbb{R}^1$. These functions look something like $z = f(x, y)$. To graph such a function we draw x-, y-, and z-axes in $\mathbb{R}^3$ and plot all the points where the equation $z = f(x, y)$ is true.

Computers can really help one visualize such graphs. It is worthwhile to play with any software package that will graph such functions. But it is equally worthwhile to learn a few techniques to sketch such graphs by hand.

The easiest way to begin to get a feel for a graph is by drawing its intersection with the coordinate planes. To sketch the intersection with the xz-plane, for example, set y equal to zero and graph the resulting function. Similarly, to sketch the intersection with the yz-plane, set x equal to zero.

A similar approach involves sketching *level curves*. These are just the intersections of horizontal planes of the form $z = n$ with the graph. To sketch such a curve, one simply plots the graph of $n = f(x, y)$.

Putting all of this information together on one set of axes can be a challenge (see Figure 1.3). Some artistic ability and some ability to visualize three-dimensional shapes is helpful, but nothing substitutes for lots of practice.

1.7. Sketch the graphs of

1. $z = 2x - 3y$.
2. $z = x^2 + y^2$.
3. $z = xy$ (compare with Figure 1.3).
4. $z = \sqrt{x^2 + y^2}$.
5. $z = \frac{1}{\sqrt{x^2+y^2}}$.
6. $z = \sqrt{x^2 + y^2 + 1}$.
7. $z = \sqrt{x^2 + y^2 - 1}$.
8. $z = \cos(x + y)$.
9. $z = \cos(xy)$.
10. $z = \cos(x^2 + y^2)$.
11. $z = e^{-(x^2+y^2)}$.

1.8. Find functions whose graphs are

1. A plane through the origin at 45° to both the x- and y-axes.
2. The top half of a sphere of radius two.
3. The top half of a torus centered around the z-axis (i.e., the tube of radius one, say, centered around a circle of radius two in the xy-plane).
4. The top half of the cylinder of radius one which is centered around the line where the plane $y = x$ meets the plane $z = 0$.

You may find it helpful to check your answers to the above exercises with a computer graphing program.

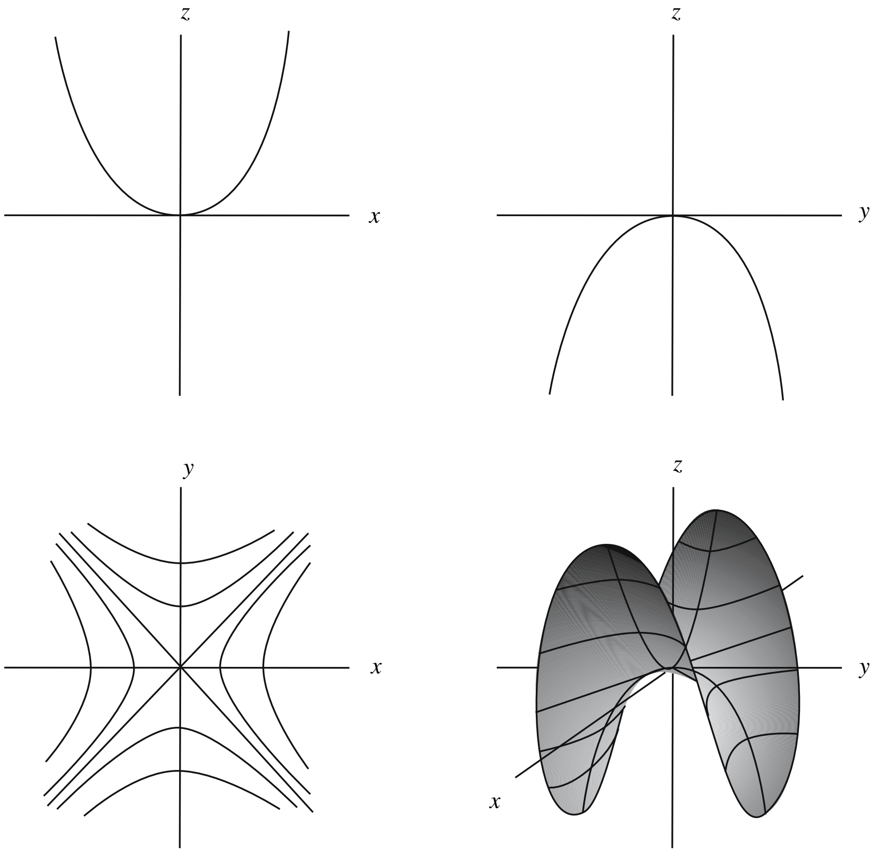

Fig. 1.3. Several views of the graph of $z = x^2 - y^2$. The top two figures are the intersections with the xz- and yz-planes. The bottom left shows several level curves.

1.3 Multiple integrals

We now address the question of how to find the volume under the graph of a function $f(x, y)$ of two variables. Recall from Calculus I that we define the integral of a function $g(x)$ of one variable on the interval $[0, a]$ by the following steps:

1. Choose a sequence of evenly spaced points $\{x_i\}_{i=0}^{n}$ in $[0, a]$ such that $x_0 = 0$ and $x_n = a$.
2. Let $\Delta x = x_{i+1} - x_i$.
3. For each i compute $g(x_i)\Delta x$.
4. Sum over all i.
5. Take the limit as n goes to ∞.

The intuition is that each term in Step 3 above gives the area of a rectangle. Piecing all of the rectangles together gives an approximation for the function $g(x)$, so the result of Step 4 is an approximation for the desired area. As n goes to ∞ in Step 5, this approximation gets better and better.

Similar steps define the volume under $f(x, y)$. Let R be the rectangle in the xy-plane with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b). We now perform the following steps:

1. Choose sequences of evenly spaced points $\{x_i\}_{i=0}^n$ and $\{y_j\}_{j=0}^m$ such that $x_0 = y_0 = 0$, $x_n = a$ and $y_m = b$. This gives a *lattice* of points of the form (x_i, y_j) in R.
2. Let $\Delta x = x_{i+1} - x_i$ and $\Delta y = y_{j+1} - y_j$.
3. For each i and j compute $f(x_i, y_j)\Delta x \Delta y$.
4. Sum over all i and j.
5. Take the limit as n and m go to ∞.

These steps define $\int\limits_R f(x, y)dx\ dy$. The intuition as to why this represents the desired volume is similar to that in the one variable case. In Step 3 we are computing the volume of a box whose base is a Δx by Δy rectangle, and whose height is $f(x_i, y_j)$ (see Figure 1.4). Putting these boxes together approximates the function $f(x, y)$, and this approximation gets better and better when n and m go to ∞.

It is important to understand the above definition from a theoretical point of view. Later in this text we will come back to it many times. Unfortunately, it is almost impossible to use this definition to compute any integrals. For this, we need an alternate point of view.

Instead of approximating $f(x, y)$ with boxes as above, we will now approximate it by “slabs" whose profiles look like slices by planes parallel to one of the coordinate planes (see Figure 1.6). To do this we carry out the following steps:

1. Choose a sequence of evenly spaced points $\{x_i\}_{i=0}^n$ such that $x_0 = 0$ and $x_n = a$.
2. Let $\Delta x = x_{i+1} - x_i$.
3. For each i compute $\left[\int\limits_0^b f(x_i, y)dy\right]\Delta x$.
4. Sum over all i.
5. Take the limit as n goes to ∞.

Note that in Step 3 the quantity $\int\limits_0^b f(x_i, y)dy$ is exactly the area under the curve that you get when you slice the graph of $f(x, y)$ by the plane parallel to the yz-plane at $x = x_i$ (see Figure 1.5). Multiplying by Δx then gives the volume of a slab of thickness Δx, with the same profile as this slice. Putting these

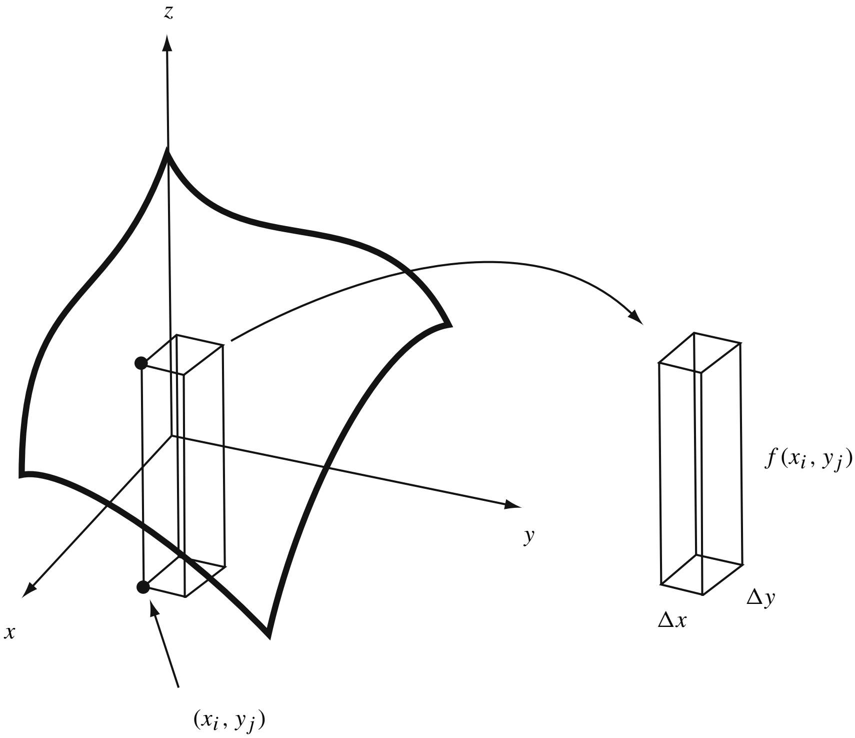

Fig. 1.4. Using boxes to approximate a function.

slabs together still approximates the function $f(x, y)$, and this approximation gets better and better as n goes to ∞ (see Figure 1.6). The result is the following:

$$\int_R f(x, y)dx\, dy = \int_0^a \left[\int_0^b f(x, y)dy \right] dx.$$

Of course, we could have added up volumes of the slabs that were parallel to the xz-plane instead. This process would have produced the following equality:

$$\int_R f(x, y)dx\, dy = \int_0^b \left[\int_0^a f(x, y)dx \right] dy.$$

Hence we see that *Fubini's theorem* must be true:

$$\int_0^a \int_0^b f(x, y)dy\, dx = \int_0^b \int_0^a f(x, y)dx\, dy.$$

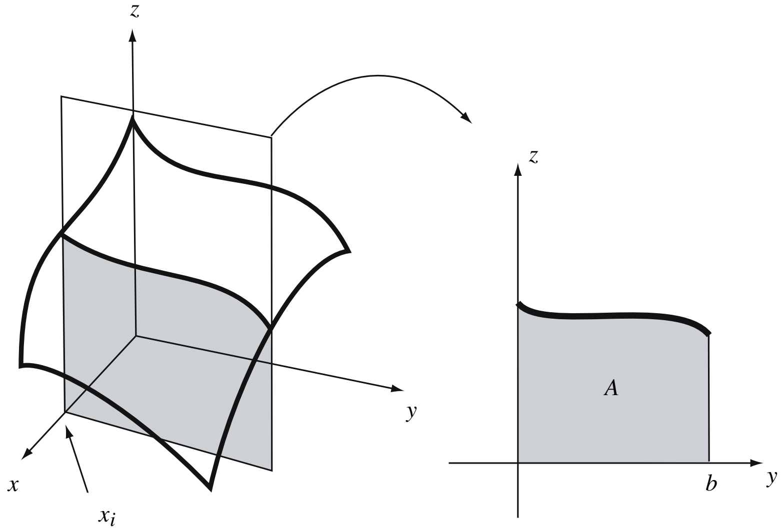

Fig. 1.5. The area A of the slice through $x = x_i$ is given by $\int_0^b f(x_i, y)dy$.

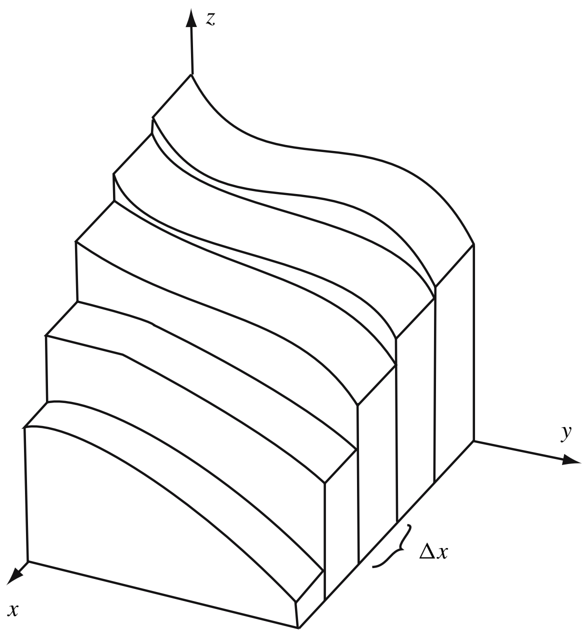

Fig. 1.6. Putting slabs together approximates the function $f(x, y)$.

Note 1. Be aware that we have avoided very technical issues here such as continuity and convergence. For a rigorous treatment, see any standard text in multivariable calculus.

Example 1. To find the volume under the graph of $f(x, y) = xy^2$ and above the rectangle R with vertices at $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(2, 3)$ we compute:

$$\begin{aligned}\int_R xy^2\, dx\, dy &= \int_0^3 \int_0^2 xy^2\, dx\, dy \\ &= \int_0^3 \left[\frac{1}{2}x^2y^2 \Big|_{x=0}^{2} \right] dy \\ &= \int_0^3 2y^2\, dy \\ &= 18.\end{aligned}$$

1.9. Let R be the rectangle in the xy-plane with vertices at $(1, 0)$, $(2, 0)$, $(1, 3)$ and $(2, 3)$. Integrate the following functions over R.

1. x^2y^2.
2. 1.
3. $x^2 + y^2$.
4. $\sqrt{x + \frac{2}{3}y}$.

1.4 Partial derivatives

In this section, we begin to discuss tangent lines to the graph of a function of the form $f : \mathbb{R}^2 \to \mathbb{R}^1$. If we slice the graph of such a function with the plane parallel to the yz-plane, through the point (x_0, y_0), then we get a curve which represents some function of y. We can then ask, "What is the slope of the tangent line to this curve when $y = y_0$?" The answer to this question is precisely the definition of $\frac{\partial f}{\partial y}(x_0, y_0)$ (see Figure 1.7).

Example 2. Suppose $f(x, y) = xy^2$. We wish to compute $\frac{\partial f}{\partial y}(2, 3)$. The slice of the graph of $f(x, y)$, parallel to the yz-plane, through the point $(2, 3)$, is given by substituting 2 for x. This gives us the function $2y^2$. Differentiating with respect to y then gives $4y$. Plugging in 3 for y yields 12.

If we instead wish to compute $\frac{\partial f}{\partial y}(4, 3)$, we could go through the same steps. The slice through the point is the graph of $4y^2$. Differentiating with respect to y gives $8y$. Evaulating at $y = 3$ yields 24.

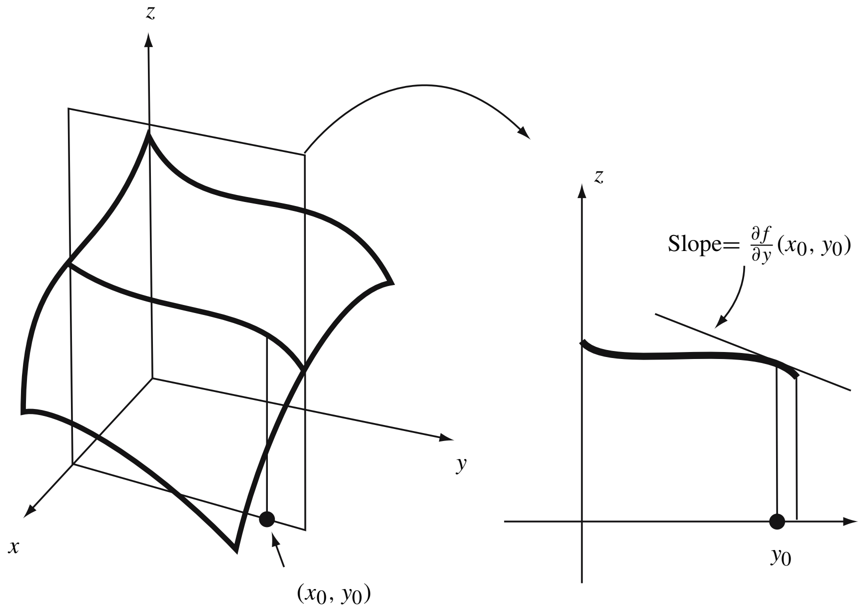

Fig. 1.7. The partial derivative with respect to y.

If we wish to repeat this many more times, it will be easier to leave the variable x in, but think of it as a constant. Hence, differentiating xy^2 with respect to y gives $2xy$, and we can now plug in whatever numbers we want for x and y to obtain a final answer immediately.

Partial derivatives with respect to x are just as easy to compute. Geometrically, we think of this as giving the slope of a line tangent to the graph which is the slice parallel to the xz-plane. Algebraically, we think of y as a constant and take the derivative with respect to x.

1.10. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

1. x^2y^3.
2. $\sin(x^2y^3)$.
3. $x \sin(xy)$.

Notice that when you take a partial derivative you get another function of x and y. You can then do it again to find the *second partials*. These are denoted by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

1.11. Find all second partials for each of the functions in the previous exercise.

Note that amazingly, the “mixed" partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are always equal. This is not a coincidence! Somehow the mixed partials measure the “twisting" of the graph, and this is the same from every direction.

1.5 Gradients

Let’s look back to Figure 1.7. What if we sliced the graph of $f(x, y)$ with some vertical plane through the point (x_0, y_0) that was *not* parallel to the xz- or yz-planes, as in Figure 1.8? How could we compute the slope then?

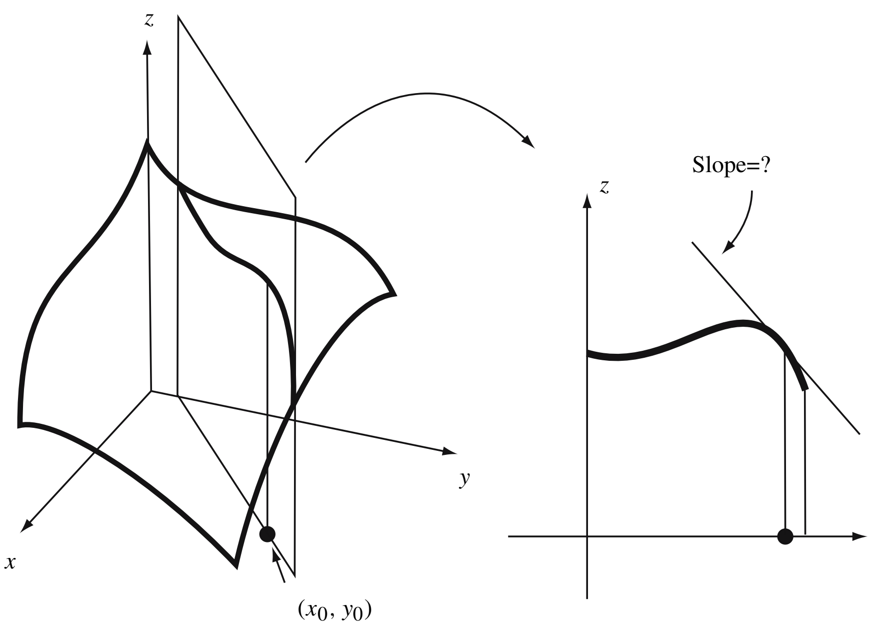

Fig. 1.8. A directional derivative.

To answer this, visualize the set of *all* lines tangent to the graph of $f(x, y)$ at the point (x_0, y_0). The is a tangent *plane*.

The equation for a plane through the origin in $\mathbb{R}^3$ is of the form $z = m_x x + m_y y$. Notice that the intersection of such a plane with the xz-plane is the graph of $z = m_x x$. Hence, m_x is the slope of this line of intersection. Similarly, the quantity m_y is the slope of the line which is the intersection with the yz-plane.

To get a plane through the point $(x_0, y_0, f(x_0, y_0))$, we can translate the origin to this point by replacing x with $x - x_0$, y with $y - y_0$ and z with $z - f(x_0, y_0)$:

$$z - f(x_0, y_0) = m_x(x - x_0) + m_y(y - y_0).$$

Since we want this to actually be a tangent plane, it follows that m_x must be equal to $\frac{\partial f}{\partial x}$ and m_y must be $\frac{\partial f}{\partial y}$. Hence, the equation of the tangent plane T is given by

$$T(x, y) = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + f$$

where $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and f are all evaluated at the point (x_0, y_0).

Now, suppose P is the vertical plane through the point (x_0, y_0) depicted in Figure 1.9. Let l denote the line where P intersects the xy-plane. The tangent line L to the graph of f, which lies above l, is also the line contained in T, which lies above l. To figure out the slope of L we will simply compute "rise over run."

Suppose l contains the vector $V = \langle a, b\rangle$, where $|V| = 1$. Then two points on l, a distance of 1 apart, are (x_0, y_0) and $(x_0 + a, y_0 + b)$. Thus the "run" will be equal to 1. The "rise" is the difference between $T(x_0, y_0)$ and $T(x_0 + a, y_0 + b)$, which we compute as follows:

$$\begin{aligned}
&T(x_0 + a, y_0 + b) - T(x_0, y_0)\\
&\qquad = \left[\frac{\partial f}{\partial x}(x_0 + a - x_0) + \frac{\partial f}{\partial y}(y_0 + b - y_0) + f\right]\\
&\qquad\quad - \left[\frac{\partial f}{\partial x}(x_0 - x_0) + \frac{\partial f}{\partial y}(y_0 - y_0) + f\right]\\
&\qquad = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}.
\end{aligned}$$

Since the slope of L is "rise" over "run," and the "run" equals 1, we conclude the slope of L is $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are evaluated at the point (x_0, y_0).

1.12. Suppose $f(x, y) = x^2y^3$. Compute the slope of the line tangent to $f(x, y)$, at the point $(2, 1)$, in the direction $\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\rangle$.

1.13. Let $f(x, y) = xy + x - 2y + 4$. Find the slope of the tangent line to the graph of $f(x, y)$, in the direction of $\langle 1, 2\rangle$, at the point $(0, 1)$.

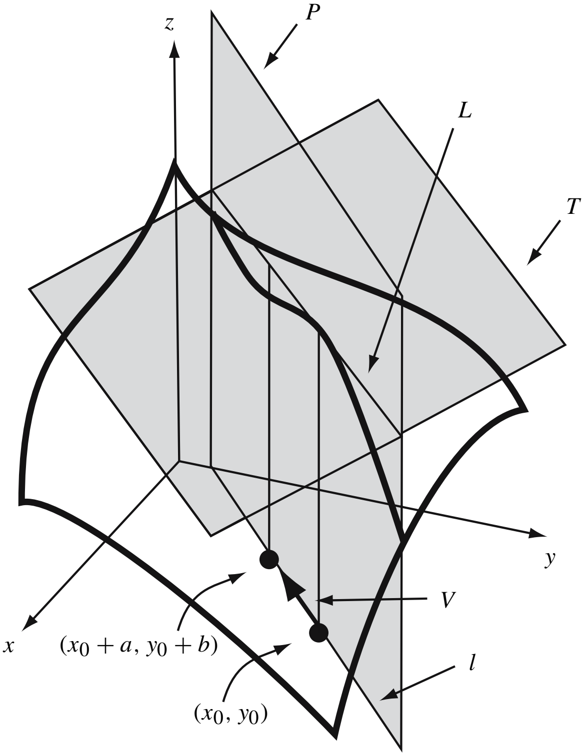

Fig. 1.9. Computing the slope of the tangent line L.

The quantity $a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$ is defined to be the *directional derivative* of f, at the point (x_0, y_0), in the direction V. We will adopt the notation $\nabla_V f(x_0, y_0)$ for this quantity.

Let $f(x, y) = xy^2$. Let's compute the directional derivative of f, at the point $(2, 3)$, in the direction $V = \langle 1, 5\rangle$. We compute:

$$\begin{aligned}\nabla_V f(2, 3) &= 1\frac{\partial f}{\partial x}(2, 3) + 5\frac{\partial f}{\partial y}(2, 3)\\ &= 1 \cdot 3^2 + 5 \cdot 2 \cdot 2 \cdot 3\\ &= 69.\end{aligned}$$

Is 69 the slope of the tangent line to some curve that we get when we intersect the graph of xy^2 with some plane? What this number represents is the rate of change of f, as we walk along the line l of Figure 1.9, *with speed* $|V|$. To find the desired slope we would have to walk with speed one. Hence, the directional derivative only represents a slope when $|V| = 1$. Let's at least see if this agrees with what we previously found.

If we stand at the point (x_0, y_0), walk in the direction $\langle 1, 0\rangle$ and ask what the rate of change of f is, we obtain the following answer:

$$\nabla_{\langle 1,0\rangle} f(x_0, y_0) = 1\frac{\partial f}{\partial x}(x_0, y_0) + 0\frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0).$$

This certainly agrees with our interpretation of $\frac{\partial f}{\partial x}$ as a slope. If we repeat this with the vector $\langle 2, 0\rangle$, then we find out how fast f changes when we walk twice as fast in the same direction:

$$\nabla_{\langle 2,0\rangle} f(x_0, y_0) = 2\frac{\partial f}{\partial x}(x_0, y_0) + 0\frac{\partial f}{\partial y}(x_0, y_0) = 2\frac{\partial f}{\partial x}(x_0, y_0).$$

As expected, f now changes twice as fast.

To proceed further, we write the definition of $\nabla_V f$ as a dot product:

$$\nabla_{\langle a,b\rangle} f(x_0, y_0) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \cdot \langle a, b\rangle.$$

The vector $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ is called the *gradient* of f, and is denoted ∇f. Using this notation we obtain the following formula:

$$\nabla_V f(x_0, y_0) = \nabla f(x_0, y_0) \cdot V.$$

Note that this dot product is greatest when V points in the same direction as ∇f. This fact leads us to the geometric significance of the gradient vector. Think of $f(x, y)$ as a function which represents the altitude in some mountain range, given a location in longitude x and latitude y. Now, if all you know is f and your location x and y, and you want to figure out which way "uphill" is, all you have to do is point yourself in the direction of ∇f.

What if you wanted to know what the slope was in the direction of steepest ascent? You would have to compute the directional derivative, using a vector of length one which points in the same direction as ∇f. Such a vector is easy to find: $U = \frac{\nabla f}{|\nabla f|}$. Now we compute this slope:

$$\begin{aligned}
\nabla_U f &= \nabla f \cdot U \\
&= \nabla f \cdot \frac{\nabla f}{|\nabla f|} \\
&= \frac{1}{|\nabla f|}(\nabla f \cdot \nabla f) \\
&= \frac{1}{|\nabla f|}|\nabla f|^2 \\
&= |\nabla f|.
\end{aligned}$$

Hence, the magnitude of the gradient vector represents the largest slope of a tangent line through a particular point.

1.14. Let $f(x, y) = xy^2$.

1. Compute ∇f.
2. Use your answer to the previous question to compute $\nabla_{\langle 1,5\rangle} f(2, 3)$.
3. Find a vector of length one that points in the direction of steepest ascent, at the point $(2, 3)$.
4. What is the largest slope of a tangent line to the graph of f when $(x, y) = (2, 3)$?

1.15. Suppose (x_0, y_0) is a point where ∇f is non-zero and let $n = f(x_0, y_0)$. Show that the vector $\nabla f(x_0, y_0)$ is perpendicular to the set of points (x, y) such that $f(x, y) = n$ (i.e., a level curve).

1.16. For each of the following functions $f(x, y)$:

- Compute $\nabla f(0, 0)$.
- What does this answer tell you about the slope of the lines tangent to the graph of f at $(0, 0)$?
- Compute all second partials at $(0, 0)$.
- At the point $(0, 0)$ compute

$$D(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 f}{\partial y \partial x}.$$

- Describe the shape of the graph of $f(x, y)$ near $(0, 0)$.

1. $x^2 + y^2$.
2. $-x^2 - y^2$.
3. $x^2 - y^2$.
4. xy.

1.17. A function $f(x, y)$ is said to have a *critical point* at (x_0, y_0) if $\nabla f(x_0, y_0) = \langle 0, 0\rangle$. Based on the previous problem, hypothesize about whether the graph of $z = f(x, y)$ has a maximum, minimum, or saddle at (x_0, y_0) if $f(x, y)$ has a critical point at (x_o, y_0),

1. $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$.
2. $D(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$.
3. $D(x_0, y_0) < 0$.

1.18. Find functions $f(x, y)$ such that $D(0, 0) = 0$ and at $(0, 0)$ the graph of $z = f(x, y)$ has a

1. Minimum.
2. Maximum.
3. Saddle.

2

Parameterizations

2.1 Parameterized curves in $\mathbb{R}^2$

Given a curve C in $\mathbb{R}^2$, a *parameterization* for C is a (one-to-one, onto, differentiable) function of the form $\phi : \mathbb{R}^1 \to C$.

Example 3. The function $\phi(t) = (\cos t, \sin t)$, where $0 \le t < 2\pi$, is a parameterization for the circle of radius 1. Another parameterization for the same circle is $\psi(t) = (\cos 2t, \sin 2t)$, where $0 \le t < \pi$. The difference between these two parameterizations is that as t increases, the image of $\psi(t)$ moves twice as fast around the circle as the image of $\phi(t)$.

2.1. A function of the form $\phi(t) = (at + c, bt + d)$ is a parameterization of a line.

1. What is the slope of the line parameterized by ϕ?
2. How does this line compare to the one parameterized by $\psi(t) = (at, bt)$?

2.2. Draw the curves given by the following parameterizations:

1. (t, t^2), where $0 \le t \le 1$.
2. (t^2, t^3), where $0 \le t \le 1$.
3. $(2\cos t, 3\sin t)$, where $0 \le t \le 2\pi$.
4. $(\cos 2t, \sin 3t)$, where $0 \le t \le 2\pi$.
5. $(t\cos t, t\sin t)$, where $0 \le t \le 2\pi$.

Given a curve, it can be very difficult to find a parameterization. There are many ways of approaching the problem, but nothing which always works. Here are a few hints:

1. If C is the graph of a function $y = f(x)$, then $\phi(t) = (t, f(t))$ is a parameterization of C. Notice that the y-coordinate of every point in the image of this parameterization is obtained from the x-coordinate by applying the function f.

2. If one has a polar equation for a curve like $r = f(\theta)$, then, since $x = r\cos\theta$ and $y = r\sin\theta$, we get a parameterization of the form $\phi(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$.

Example 4. The top half of a circle of radius one is the graph of $y = \sqrt{1-x^2}$. Hence a parameterization for this is $(t, \sqrt{1-t^2})$, where $-1 \le t \le 1$. This figure is also the graph of the polar equation $r = 1, 0 \le \theta \le \pi$, hence the parameterization $(\cos t, \sin t)$, where $0 \le t \le \pi$.

2.3. Sketch and find parameterizations for the curves described by:

1. The graph of the polar equation $r = \cos\theta$.
2. The graph of $y = \sin x$.

2.4. Find a parameterization for the line segment which connects the point $(1, 1)$ to the point $(2, 5)$.

Parameterized curves may be familiar from a second semester calculus class. Often in these classes one learns how to calculate the slope of a tangent line to the curve. But usually one does not discuss the derivative of the parameterization itself. One reason is that the derivative is actually a vector. If $\phi(t) = (f(t), g(t))$, then

$$\phi'(t) = \frac{d\phi}{dt} = \frac{d}{dt}(f(t), g(t)) = \langle f'(t), g'(t)\rangle.$$

This vector has important geometric significance. The slope of a line containing this vector when $t = t_0$ is the same as the slope of the line tangent to the curve at the point $\phi(t_0)$. The magnitude (length) of this vector gives one a concept of the *speed* of the point $\phi(t)$ as t is increases through t_0. For convenience, one often draws the vector $\phi'(t_0)$ based at the point $\phi(t_0)$ (see Figure 2.1).

2.5. Let $\phi(t) = (\cos t, \sin t)$ (where $0 \le t \le \pi$) and $\psi(t) = (t, \sqrt{1-t^2})$ (where $-1 \le t \le 1$) be parameterizations of the top half of the unit circle. Sketch the vectors $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$ at the points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(0, 1)$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

2.6. Let C be the set of points in $\mathbb{R}^2$ that satisfies the equation $x = y^2$.

1. Find a parameterization for C.
2. Find a tangent vector to C at the point $(4, 2)$.

2.2 Cylindrical and spherical coordinates

There are several ways to specify the location of a point in $\mathbb{R}^3$. The most common is to give the lengths of the projections onto the x-, y- and z-axes. These are, of course, the x-, y- and z-coordinates. We often call the (x, y, z) coordinate system

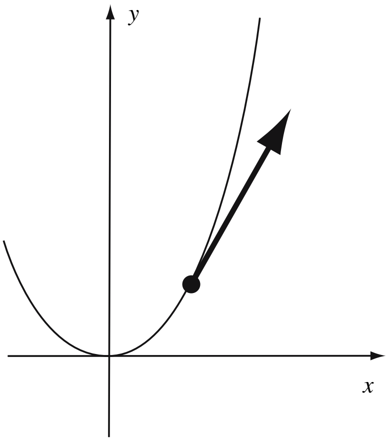

Fig. 2.1. The derivative of the parameterization $\phi(t) = (t, t^2)$ is the vector $\langle 1, 2t \rangle$. When $t = 1$ this is the vector $\langle 1, 2 \rangle$, which we picture based at the point $\phi(1) = (1, 1)$.

Cartesian coordinates (after the mathematician René Descartes), or *rectangular coordinates*.

A second method of describing the location of a point is to use polar coordinates (r, θ) to describe the projection onto the xy-plane, and the quantity z to describe the height off of the xy-plane (see Figure 2.2). It follows that the relationships between r, θ, x and y are the same as for polar coordinates:

$$\begin{array}{r|l} x = r\cos\theta & r = \sqrt{x^2 + y^2} \\ y = r\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right). \end{array}$$

The (r, θ, z) coordinates are called *cylindrical coordinates*.

The third most common coordinate system is called *spherical coordinates*. In this system, one specifies the distance ρ from the origin, the same angle θ from cylindrical coordinates and the angle ϕ that a ray to the origin makes with the z-axis (see Figure 2.3). A little basic trigonometry yields the relationships:

$$\begin{array}{l|l} x = \rho\sin\phi\cos\theta & \rho = \sqrt{x^2 + y^2 + z^2} \\ y = \rho\sin\phi\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = \rho\cos\phi & \phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right). \end{array}$$

2.7. Find all of the relationships between the quantities r, θ and z from cylindrical coordinates and the quantities ρ, θ and ϕ from spherical coordinates.

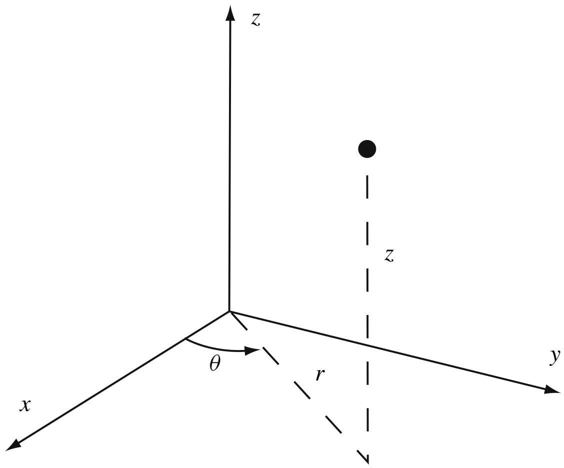

Fig. 2.2. Cylindrical coordinates.

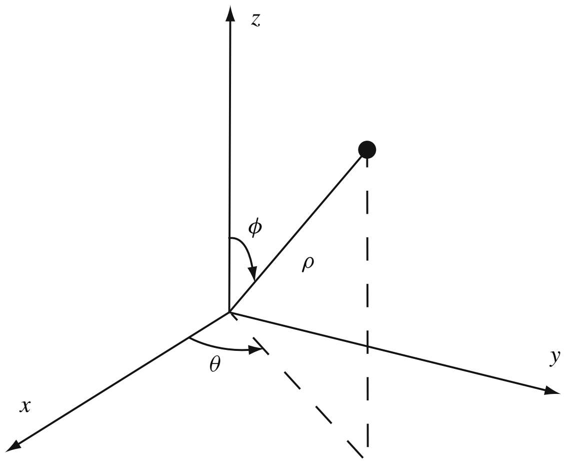

Fig. 2.3. Spherical coordinates.

Each coordinate system is useful for describing different graphs, as can be seen in the following examples.

Example 5. A cylinder of radius one, centered on the z-axis, can be described by equations in each coordinate system as follows:

- Rectangular: $x^2 + y^2 = 1$
- Cylindrical: $r = 1$
- Spherical: $\rho \sin \phi = 1$.

Example 6. A sphere of radius one is described by the equations:

- Rectangular: $x^2 + y^2 + z^2 = 1$
- Cylindrical: $r^2 + z^2 = 1$
- Spherical: $\rho = 1$.

2.8. Sketch the shape described by the following equations:

1. $\theta = \frac{\pi}{4}$.
2. $z = r^2$.
3. $\rho = \phi$.
4. $\rho = \cos\phi$.
5. $r = \cos\theta$.
6. $z = \sqrt{r^2 - 1}$.
7. $z = \sqrt{r^2 + 1}$.
8. $r = \theta$.

2.9. Find rectangular, cylindrical and spherical equations that describe the following shapes:

1. A right, circular cone centered on the z-axis, with vertex at the origin.
2. The xz-plane.
3. The xy-plane.
4. A plane that is at an angle of $\frac{\pi}{4}$ with both the x- and y-axes.
5. The surface found by revolving the graph of $z = x^3$ (where $x \geq 0$) around the z-axis.

2.3 Parameterized surfaces in $\mathbb{R}^3$

A parameterization for a surface S in $\mathbb{R}^3$ is a (one-to-one, onto, differentiable) function from some subset of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is S.

Example 7. The function $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, where (u, v) lies inside a disk of radius one, is a parameterization for the top half of the unit sphere.

One of the best ways to parameterize a surface is to find an equation *in some coordinate system* which can be used to eliminate one unknown coordinate. Then translate back to rectangular coordinates.

Example 8. An equation for the top half of the sphere in cylindrical coordinates is $r^2 + z^2 = 1$. Solving for z then gives us $z = \sqrt{1 - r^2}$. Translating to rectangular coordinates we have:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = \sqrt{1 - r^2}.$$

Hence, a parameterization is given by the function

$$\phi(r, \theta) = (r \cos\theta, r \sin\theta, \sqrt{1 - r^2})$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

Example 9. The equation $\rho = \phi$ describes some surface in spherical coordinates. Translating to rectangular coordinates then gives us:

$$x = \rho \sin\rho \cos\theta$$
$$y = \rho \sin\rho \sin\theta$$
$$z = \rho \cos\rho.$$

Hence, a parameterization for this surface is given by

$$\phi(\rho, \theta) = (\rho \sin\rho \cos\theta, \rho \sin\rho \sin\theta, \rho \cos\rho).$$

2.10. Find parameterizations of the surfaces described by the equations in Problem 2.8.

2.11. Find a parameterization for the graph of an equation of the form $z = f(x, y)$.

2.12. Use the rectangular, cylindrical and spherical equations found in Problem 2.9 to parameterize the surfaces described there.

2.13. Use spherical coordinates to find a parameterization for the portion of the sphere of radius two, centered at the origin, which lies below the graph of $z = r$ and above the xy-plane.

2.14. Sketch the surfaces given by the following parameterizations:

1. $\psi(\theta, \phi) = (\phi \sin\phi \cos\theta, \phi \sin\phi \sin\theta, \phi \cos\phi)$, $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$.
2. $\phi(r, \theta) = (r \cos\theta, r \sin\theta, \cos r)$, $0 \leq r \leq 2\pi$, $0 \leq \theta \leq 2\pi$.

Just as we could differentiate parameterizations of curves in $\mathbb{R}^2$, we can also differentiate parameterizations of surfaces in $\mathbb{R}^3$. In general, such a parameterization for a surface S can be written as

$$\phi(u, v) = (f(u, v), g(u, v), h(u, v)).$$

Thus there are two variables we can differentiate with respect to: u and v. Each of these gives a vector which is tangent to the parameterized surface:

$$\frac{\partial \phi}{\partial u} = \left\langle \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right\rangle$$
$$\frac{\partial \phi}{\partial v} = \left\langle \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right\rangle.$$

The vectors $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ determine a plane which is tangent to the surface S at the point $\phi(u, v)$.

2.15. Suppose some surface is described by the parameterization

$$\phi(u, v) = (2u, 3v, u^2 + v^2).$$

Find two (non-parallel) vectors which are tangent to this surface at the point $(4, 3, 5)$.

2.4 Parameterized curves in $\mathbb{R}^3$

We begin with an example which demonstrates a parameterization of a curve in $\mathbb{R}^3$.

Example 10. The function $\phi(t) = (\cos t, \sin t, t)$ parameterizes a curve that spirals upward around a cylinder of radius one.

2.16. Describe the difference between the curves with the following parameterizations:

1. $(\cos t^2, \sin t^2, t^2)$.
2. $(\cos t, \sin t, t^2)$.
3. $(t \cos t, t \sin t, t)$.
4. $(\cos \frac{1}{t}, \sin \frac{1}{t}, t)$.

2.17. Describe the lines given by the following parameterizations:

1. $(t, 0, 0)$.
2. $(0, 0, t)$.
3. $(0, t, t)$.
4. (t, t, t).

In the previous section, we found parameterizations of surfaces by finding an equation for the surface (in some coordinate system), solving for a variable and then translating to rectangular coordinates. To find a parameterization of a curve in $\mathbb{R}^3$, an effective strategy is to find some way to "eliminate" *two* coordinates (in some system), and then translate into rectangular coordinates. By "eliminating" a coordinate we mean either expressing it as some constant, or expressing it as a function of the third, unknown coordinate.

Example 11. We demonstrate two ways to parameterize one of the lines that is at the intersection of the cone $z^2 = x^2 + y^2$ and the plane $y = 2x$. The coordinate y is already expressed as a function of x. To express z as a function of x, we substitute $2x$ for y in the first equation. This gives us $z^2 = x^2 + (2x)^2 = 5x^2$, or $z = \sqrt{5}x$ (the negative root would give us the other intersection line). Hence, we get the paramaterization

$$\phi(x) = (x, 2x, \sqrt{5}x).$$

Another way to describe this line is with spherical coordinates. Note that for every point on the line $\phi = \frac{\pi}{4}$ (from the first equation) and $\theta = \tan^{-1} 2$ (because $\tan\theta = \frac{y}{x} = 2$, from the second equation). Converting to rectangular coordinates then gives us

$$\phi(\rho) = \left(\rho \sin\frac{\pi}{4}\cos(\tan^{-1} 2), \rho \sin\frac{\pi}{4}\sin(\tan^{-1} 2), \rho\cos\frac{\pi}{4}\right)$$

which simplifies to

$$\psi(\rho) = \left(\frac{\sqrt{10}\rho}{10}, \frac{\sqrt{10}\rho}{5}, \frac{\sqrt{2}\rho}{2}\right).$$

Note that dividing the first parameterization by $\sqrt{10}$ and simplifying yields the second parameterization.

2.18. Find a parameterization for the curve that is at the intersection of the plane $x + y = 1$ and the cone $z^2 = x^2 + y^2$.

2.19. Find two parameterizations for the circle that is at the intersection of the cylinder $x^2 + y^2 = 4$ and the paraboloid $z = x^2 + y^2$.

2.5 Parameterized regions in $\mathbb{R}^2$ and $\mathbb{R}^3$

In Section 1.3, we learned how to integrate functions of multiple variables over rectangular regions. Eventually we will learn how to integrate such functions over regions of any shape. The trick will be to parameterize such regions by functions whose domain is a rectangle. Some cases of this are already familiar.

Example 12. A parameterization for the disk of radius one (that is, the set of points in $\mathbb{R}^2$ which are at a distance of at most one from the origin) is given using polar coordinates:

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

2.20. Let B be the ball of radius one in $\mathbb{R}^3$ (i.e., the set of points satisfying $x^2 + y^2 + z^2 \le 1$).

1. Use spherical coordinates to find a parameterization for B.
2. Find a parameterization for the intersection of B with the first octant.

2.21. The "solid cylinder" of height one and radius r in $\mathbb{R}^3$ is the set of points inside the cylinder $x^2 + y^2 = r^2$, and between the planes $z = 0$ and $z = 1$.

1. Use cylindrical coordinates to find a parameterization for the solid cylinder of height one and radius one.

2. Find a parameterization for the region that is inside the solid cylinder of height one and radius two and outside the cylinder of radius one.

Example 13. A common type of region to integrate over is one that is bounded by the graphs of two functions. Suppose R is the region in $\mathbb{R}^2$ above the graph of $y = g_1(x)$, below the graph of $y = g_2(x)$ and between the lines $x = a$ and $x = b$. A parameterization for R (check this!) is given by

$$\phi(x,t) = (x, tg_2(x) + (1-t)g_1(x)), \ a \le x \le b, \ 0 \le t \le 1.$$

2.22. Let R be the region between the (polar) graphs of $r = f_1(\theta)$ and $r = f_2(\theta)$, where $a \le \theta \le b$. Find a parameterization for R.

2.23. Find a parameterization for the region in $\mathbb{R}^2$ bounded by the ellipse whose x-intercepts are 3 and -3 and y-intercepts are 2 and -2. (*Hint*: Start with the parameterization given in Example 12.)

2.24. Sketch the region in $\mathbb{R}^2$ parameterized by the following:

$$\phi(r,\theta) = (2r\cos\theta, r\sin\theta)$$

where $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$.

3

Introduction to Forms

3.1 So what *is* a differential form?

A differential form is simply this: an integrand. In other words, it is a thing which can be integrated over some (often complicated) domain. For example, consider the following integral: $\int_0^1 x^2 dx$. This notation indicates that we are integrating x^2 over the interval [0, 1]. In this case, $x^2 dx$ is a differential form. If you have had no exposure to this subject this may make you a little uncomfortable. After all, in calculus we are taught that x^2 is the integrand. The symbol "dx" is only there to delineate when the integrand has ended and what variable we are integrating with respect to. However, as an object in itself, we are not taught any meaning for "dx." Is it a function? Is it an operator on functions? Some professors call it an "infinitesimal" quantity. This is very tempting. After all, $\int_0^1 x^2 dx$ is defined to be the limit, as $n \to \infty$, of $\sum_{i=1}^{n} x_i^2 \Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval [0, 1], and $\Delta x = 1/n$. When we take the limit, the symbol "$\sum$" becomes "$\int$," and the symbol "Δx" becomes "dx." This implies that $dx = \lim_{\Delta x \to 0} \Delta x$, which is absurd. $\lim_{\Delta x \to 0} \Delta x = 0$!! We are not trying to make the argument that the symbol "dx" should be eliminated. It does have meaning. This is one of the many mysteries that this book will reveal.

One word of caution here: not all integrands are differential forms. In fact, in the appendix we will see how to calculate arc length and surface area. These calculations involve integrands which are not differential forms. Differential forms are simply natural objects to integrate, and also the first that one should study. As we shall see, this is much like beginning the study of all functions by understanding linear functions. The naive student may at first object to this, since linear functions are a very restrictive class. On the other hand, eventually we learn that any differentiable function (a much more general class) can be locally approximated by a linear function. Hence, in some sense, the linear functions are the most important ones. In the same way, one can make the argument that differential forms are the most important integrands.

3.2 Generalizing the integral

Let's begin by studying a simple example, and trying to figure out how and what to integrate. The function $f(x, y) = y^2$ maps $\mathbb{R}^2$ to $\mathbb{R}$. Let M denote the top half of the circle of radius one, centered at the origin. Let's restrict the function f to the domain, M, and try to integrate it. Here we encounter our first problem: The given description of M is not particularly useful. If M were something more complicated, it would have been much harder to describe it in words as we have just done. A parameterization is far easier to communicate, and far easier to use to determine which points of $\mathbb{R}^2$ are elements of M, and which are not. But there are lots of parameterizations of M. Here are two which we shall use:

$\phi_1(a) = (a, \sqrt{1-a^2})$, where $-1 \leq a \leq 1$, and

$\phi_2(t) = (\cos(t), \sin(t))$, where $0 \leq t \leq \pi$.

Here is the trick: integrating f over M is difficult. It may not even be clear as to what this means. But perhaps we can use ϕ_1 to translate this problem into an integral over the interval $[-1, 1]$. After all, an integral is a big sum. If we add up all the numbers $f(x, y)$ for all the points, (x, y), of M, shouldn't we get the same thing as if we added up all the numbers $f(\phi_1(a))$, for all the points, a, of $[-1, 1]$ (see Fig. 3.1)?

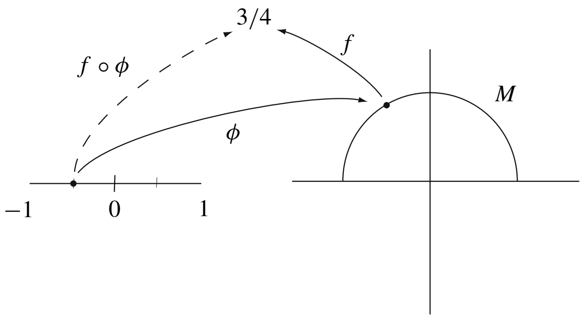

Fig. 3.1. Shouldn't the integral of f over M be the same as the integral of $f \circ \phi$ over $[-1, 1]$?

Let's try it. $\phi_1(a) = (a, \sqrt{1-a^2})$, so $f(\phi_1(a)) = 1 - a^2$. Hence, we are saying that the integral of f over M should be the same as $\int_{-1}^{1} (1-a^2)da$. Using a little calculus, we can determine that this evaluates to $4/3$.

Let's try this again, this time using ϕ_2. Using the same argument, the integral of f over M should be the same as $\int_0^{\pi} f(\phi_2(t))dt = \int_0^{\pi} \sin^2(t)dt = \pi/2$.

But hold on! The problem was stated *before* any parameterizations were chosen. Shouldn't the answer be independent of which one was picked? It would not be a very meaningful problem if two people could get different correct answers, depending on how they went about solving it. Something strange is going on!

3.3 Interlude: a review of single variable integration

In order to understand what happened, we must first review the definition of the Riemann integral. In the usual definition of the integral the first step is to divide the interval up into n evenly spaced subintervals. Thus, $\int_a^b f(x)dx$ is defined to be the limit, as $n \to \infty$, of $\sum_{i=1}^{n} f(x_i)\Delta x$, where $\{x_i\}$ are n evenly spaced points in the interval $[a, b]$, and $\Delta x = (b - a)/n$. But what if the points $\{x_i\}$ are not evenly spaced? We can still write down a reasonable sum: $\sum_{i=1}^{n} f(x_i)\Delta x_i$, where now $\Delta x_i = x_{i+1} - x_i$. In order to make the integral well-defined, we can no longer take the limit as $n \to \infty$. Instead, we must let $\max\{\Delta x_i\} \to 0$. It is a basic result of analysis that if this limit converges, then it does not matter how we picked the points $\{x_i\}$; the limit will converge to the same number. It is this number that we define to be the value of $\int_a^b f(x)dx$.

3.4 What went wrong?

We are now ready to figure out what happened in Section 3.2. Obviously, $\int_{-1}^{1} f(\phi_1(a))da$ was not what we wanted. But let's not give up on our general approach just yet; it would still be great if we could use ϕ_1 to find *some* function that we can integrate on $[-1, 1]$ that will give us the same answer as the integral of f over M. For now, let's call this mystery function "$F(a)$."

Let's look at the Riemann sum that we get for $\int_{-1}^{1} F(a)da$, when we divide the interval up into n pieces, each of width Δa: $\sum_{i=1}^{n} F(a_i)\Delta a$. Examine Figure 3.2 to see what happens to the points, a_i, under the function, ϕ_1. Notice that the points $\{\phi_1(a_i)\}$ are not evenly spaced along M. To use these points to estimate

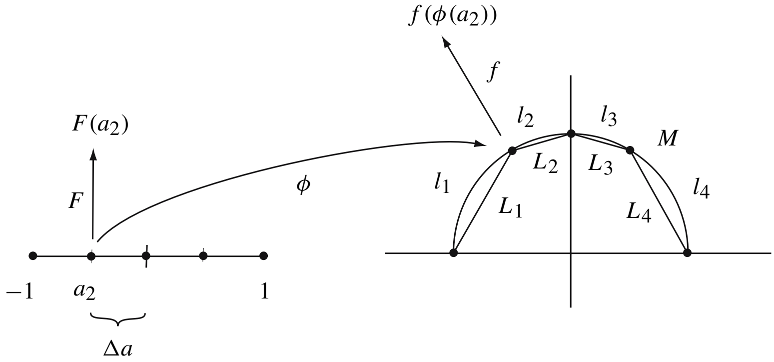

Fig. 3.2. We want $\sum_{i=1}^{n} F(a_i)\Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i$.

the integral of f over M, we would have to use the approach from the previous section. A Riemann sum for f over M would be $\sum_{i=1}^{n} f(\phi_1(a_i))l_i$, where the l_i represent the arc length, along M, between $\phi_1(a_i)$ and $\phi_1(a_{i+1})$.

This is a bit problematic, however, since arc-length is generally hard to calculate. Instead, we can approximate l_i by substituting in the length of the line segment which connects $\phi_1(a_i)$ to $\phi_1(a_{i+1})$, which we shall denote as L_i. Note that this approximation gets better and better as we let $n \to \infty$. Hence, when we take the limit, it does not matter if we use l_i or L_i.

So our goal is to find a function, $F(a)$, on the interval $[-1, 1]$, so that

$$\sum_{i=1}^{n} F(a_i)\Delta a = \sum_{i=1}^{n} f(\phi_1(a_i))L_i.$$

Of course this equality will hold if $F(a_i)\Delta a = f(\phi_1(a_i))L_i$. Solving, we get $F(a_i) = \frac{f(\phi_1(a_i))L_i}{\Delta a}$.

What happens to this function as $\Delta a \to 0$? First, note that $L_i = |\phi_1(a_{i+1}) - \phi_1(a_i)|$. Hence,

$$\begin{aligned}
\lim_{\Delta a \to 0} F(a_i) &= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))L_i}{\Delta a} \\
&= \lim_{\Delta a \to 0} \frac{f(\phi_1(a_i))|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \lim_{\Delta a \to 0} \frac{|\phi_1(a_{i+1}) - \phi_1(a_i)|}{\Delta a} \\
&= f(\phi_1(a_i)) \left| \lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1}) - \phi_1(a_i)}{\Delta a} \right|.
\end{aligned}$$

But $\lim_{\Delta a \to 0} \frac{\phi_1(a_{i+1})-\phi_1(a_i)}{\Delta a}$ is precisely the definition of the derivative of ϕ_1 at a_i, $\frac{d\phi_1}{da}(a_i)$. Hence, we have $\lim_{\Delta a \to 0} F(a_i) = f(\phi_1(a_i))|\frac{d\phi_1}{da}(a_i)|$. Finally, this means that the integral we want to compute is $\int\limits_{-1}^{1} f(\phi_1(a))|\frac{d\phi_1}{da}|da$.

3.1. Check that $\int\limits_{-1}^{1} f(\phi_1(a))|\frac{d\phi_1}{da}|da = \int\limits_{0}^{\pi} f(\phi_2(t))|\frac{d\phi_2}{dt}|dt$, using the function, f, defined in Section 3.2.

Recall that $\frac{d\phi_1}{da}$ is a vector, based at the point $\phi(a)$, tangent to M. If we think of a as a time parameter, then the length of $\frac{d\phi_1}{da}$ tells us how fast $\phi_1(a)$ is moving along M. How can we generalize the integral, $\int\limits_{-1}^{1} f(\phi_1(a))|\frac{d\phi_1}{da}|da$? Note that the bars $|\cdot|$ denote a function that "eats" vectors, and "spits out" real numbers. So we can generalize the integral by looking at other such functions. In other words, a more general integral would be $\int\limits_{-1}^{1} f(\phi_1(a))\omega(\frac{d\phi_1}{da})da$, where f is a function of points and ω is a function of vectors.

It is not the purpose of the present work to undertake a study of integrating with respect to all possible functions, ω. However, as with the study of functions of real variables, a natural place to start is with *linear* functions. This is the study of differential forms. A differential form is precisely a linear function which eats vectors, spits out numbers and is used in integration. The strength of differential forms lies in the fact that their integrals do not depend on a choice of parameterization.

3.5 What about surfaces?

Let's repeat the previous discussion (faster this time), bumping everything up a dimension. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = z^2$. Let M be the top half of the sphere of radius one, centered at the origin. We can parameterize M by the function, ϕ, where $\phi(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1-r^2})$, $0 \le r \le 1$, and $0 \le \theta \le 2\pi$. Again, our goal is not to figure out how to actually integrate f over M, but to use ϕ to set up an equivalent integral over the rectangle, $R = [0, 1] \times [0, 2\pi]$.

Let $\{x_{i,j}\}$ be a lattice of evenly spaced points in R. Let $\Delta r = x_{i+1,j} - x_{i,j}$, and $\Delta\theta = x_{i,j+1} - x_{i,j}$. By definition, the integral over R of a function, $F(x)$, is equal to $\lim_{\Delta r, \Delta\theta \to 0} \sum F(x_{i,j})\Delta r \Delta\theta$.

To use the mesh of points, $\phi(x_{i,j})$, in M to set up a Riemann sum, we write down the following sum: $\sum f(\phi(x_{i,j}))\text{Area}(L_{i,j})$, where $L_{i,j}$ is the rectangle spanned by the vectors $\phi(x_{i+1,j}) - \phi(x_{i,j})$ and $\phi(x_{i,j+1}) - \phi(x_{i,j})$. If we want

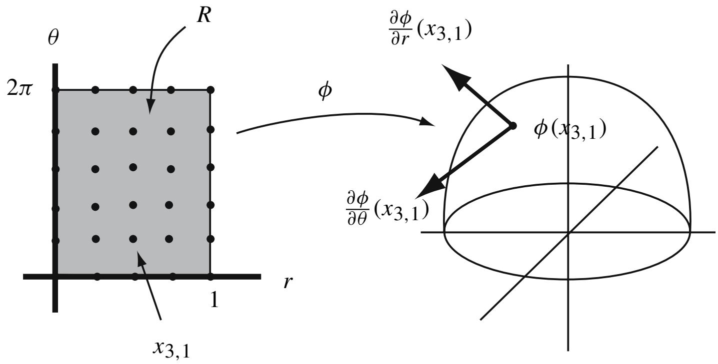

Fig. 3.3. Setting up the Riemann sum for the integral of z^2 over the top half of the sphere of radius one.

our Riemann sum over R to equal this sum, then we end up with $F(x_{i,j}) = \frac{f(\phi(x_{i,j}))\text{Area}(L_{i,j})}{\Delta r \Delta \theta}$.

We now leave it as an exercise to show that as Δr and $\Delta \theta$ get small, $\frac{\text{Area}(L_{i,j})}{\Delta r \Delta \theta}$ converges to the area of the parallelogram spanned by the vectors $\frac{\partial \phi}{\partial r}(x_{i,j})$ and $\frac{\partial \phi}{\partial \theta}(x_{i,j})$. The upshot of all this is that the integral we want to evaluate is the following:

$$\int_R f(\phi(r,\theta))\text{Area}\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}\right) dr d\theta.$$

3.2. Compute the value of this integral for the function $f(x, y, z) = z^2$.

The point of all this is not the specific integral that we have arrived at, but the *form* of the integral. We integrate $f \circ \phi$ (as in the previous section), times a function which takes *two* vectors and returns a real number. Once again, we can generalize this by using other such functions:

$$\int_R f(\phi(r,\theta))\omega\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}\right) dr d\theta.$$

In particular, if we examine linear functions for ω, we arrive at a differential form. The moral is that if we want to perform an integral over a region parameterized by $\mathbb{R}$, as in the previous section, then we need to multiply by a function which takes a vector and returns a number. If we want to integrate over something parameterized by $\mathbb{R}^2$, then we need to multiply by a function which takes *two* vectors and returns a number. In general, an n-form is a linear function which takes n vectors and returns a real number. One integrates n-forms over regions that can be parameterized by $\mathbb{R}^n$. Their strength is that the value of such an integral does not depend on the choice of parameterization.

4

Forms

4.1 Coordinates for vectors

Before we begin to discuss functions of vectors, we first need to learn how to specify a vector. And before we can answer that, we must first learn where vectors live. In Figure 4.1 we see a curve, C, and a tangent line to that curve. The line can be thought of as the set of all tangent vectors at the point, p. We denote that line as T_pC, the *tangent space* to C at the point p.

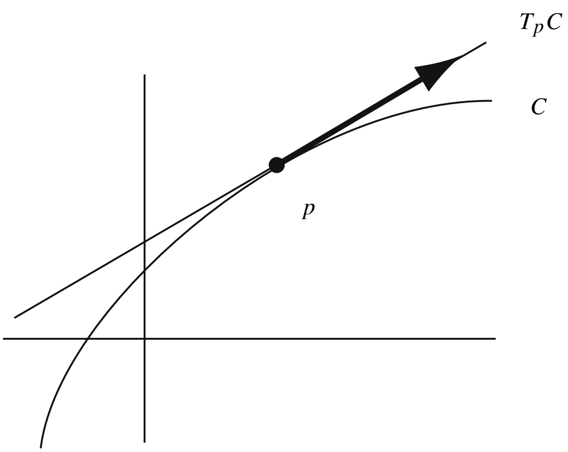

Fig. 4.1. T_pC is the set of all vectors tangents to C at p.

What if C is actually a straight line? Will T_pC be the same line? To answer this, let's instead think about the real number line, $L = \mathbb{R}^1$. Suppose p is the point corresponding to the number 2 on L. We would like to understand T_pL, the set of all vectors tangent to L at the point p. For example, where would you draw a vector of length three? Would you put its base at the origin on L? Of course not. You would put its base at the point p. This is really because the origin for T_pL is different than the origin for L. We are thus thinking about L and T_pL as two different lines, placed right on top of each other.

The key to understanding the difference between L and T_pL is their *coordinate systems*. Let's pause here for a moment to look a little more closely. What are "coordinates" anyway? They are a way of assigning a number (or, more generally, a set of numbers) to a point in space. In other words, coordinates are functions which take points of a space and return (sets of) numbers. When we say that the x-coordinate of p in $\mathbb{R}^2$ is 5, we really mean that we have a function, $x : \mathbb{R}^2 \to \mathbb{R}$, such that $x(p) = 5$.

Of course we need two numbers to specify a point in a plane, which means that we have two coordinate functions. Suppose we denote the plane by P and $x : P \to \mathbb{R}$ and $y : P \to \mathbb{R}$ are our coordinate functions. Then, saying that the coordinates of a point, p, are $(2, 3)$ is the same thing as saying that $x(p) = 2$ and $y(p) = 3$. In other words, the coordinates of p are $(x(p), y(p))$.

So what do we use for coordinates in the tangent space? Well, first we need a *basis* for the tangent space of P at p. In other words, we need to pick two vectors which we can use to give the relative positions of all other points. Note that if the coordinates of p are (x, y) then $\frac{d(x+t,y)}{dt} = \langle 1, 0\rangle$ and $\frac{d(x,y+t)}{dt} = \langle 0, 1\rangle$. We have switched to the notation "$\langle \cdot, \cdot \rangle$" to indicate that we are not talking about points of P anymore, but rather vectors in T_pP. We take these two vectors to be a basis for T_pP. In other words, any point of T_pP can be written as $dx\langle 0, 1\rangle + dy\langle 1, 0\rangle$, where $dx, dy \in \mathbb{R}$. Hence, "dx" and "dy" are coordinate functions for T_pP. Saying that the coordinates of a vector V in T_pP are $\langle 2, 3\rangle$, for example, is the same thing as saying that $dx(V) = 2$ and $dy(V) = 3$. In general, we may refer to the coordinates of an arbitrary vector in T_pP as $\langle dx, dy\rangle$, just as we may refer to the coordinates of an arbitrary point in P as (x, y).

It will be helpful in the future to be able to distinguish between the vector $\langle 2, 3\rangle$ in T_pP and the vector $\langle 2, 3\rangle$ in T_qP, where $p \neq q$. We will do this by writing $\langle 2, 3\rangle_p$ for the former and $\langle 2, 3\rangle_q$ for the latter.

Let's pause for a moment to address something that may have been bothering you since your first term of calculus. Let's look at the tangent line to the graph of $y = x^2$ at the point $(1, 1)$. We are no longer thinking of this tangent line as lying in the same plane that the graph does. Rather, it lies in $T_{(1,1)}\mathbb{R}^2$. The horizontal axis for $T_{(1,1)}\mathbb{R}^2$ is the "dx" axis and the vertical axis is the "dy" axis (see Fig. 4.2). Hence, we can write the equation of the tangent line as $dy = 2dx$. We can rewrite this as $\frac{dy}{dx} = 2$. Look familiar? This is one explanation of why we use the notation $\frac{dy}{dx}$ in calculus to denote the derivative.

4.1.

1. Draw a vector with $dx = 1$, $dy = 2$ in the tangent space $T_{(1,-1)}\mathbb{R}^2$.
2. Draw $\langle -3, 1\rangle_{(0,1)}$.

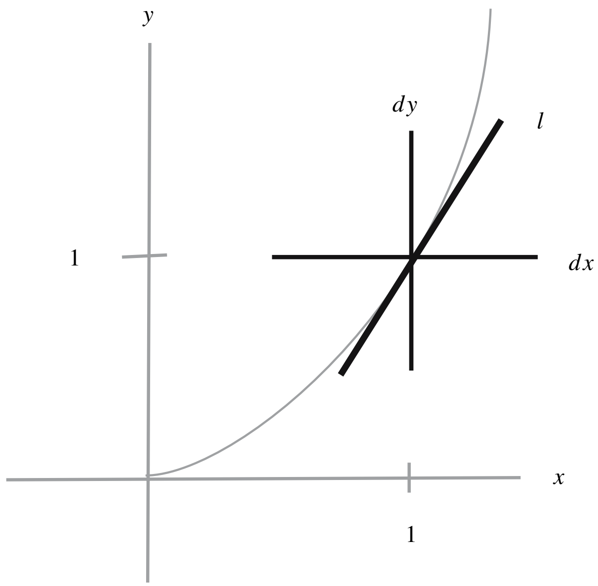

Fig. 4.2. The line, l, lies in $T_{(1,1)}\mathbb{R}^2$. Its equation is $dy = 2dx$.

4.2 1-forms

Recall from the previous chapter, that a 1-form is a linear function which acts on vectors and returns numbers. For the moment let's just look at 1-forms on $T_p\mathbb{R}^2$ for some fixed point, p. Recall that a linear function, ω, is just one whose graph is a plane through the origin. Hence, we want to write down an equation of a plane though the origin in $T_p\mathbb{R}^2 \times \mathbb{R}$, where one axis is labelled dx, another dy and the third, ω (see Fig. 4.3). This is easy: $\omega = a\,dx + b\,dy$. Hence, to specify a 1-form on $T_p\mathbb{R}^2$ we only need to know two numbers: a and b.

Here is a quick example. Suppose $\omega(\langle dx, dy\rangle) = 2dx + 3dy$, then

$$\omega(\langle -1, 2\rangle) = 2 \cdot -1 + 3 \cdot 2 = 4.$$

The alert reader may see something familiar here: the dot product. That is, $\omega(\langle -1, 2\rangle) = \langle 2, 3\rangle \cdot \langle -1, 2\rangle$. Recall the geometric interpretation of the dot product; you project $\langle -1, 2\rangle$ onto $\langle 2, 3\rangle$ and then multiply by $|\langle 2, 3\rangle| = \sqrt{13}$. In other words:

Evaluating a 1-form on a vector is the same as projecting onto some line and then multiplying by some constant.

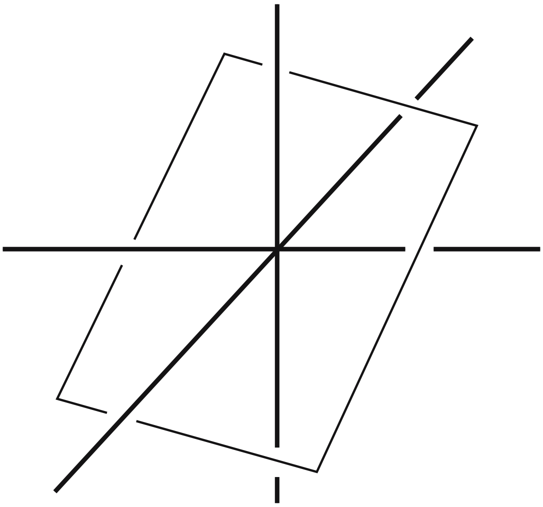

Fig. 4.3. The graph of ω is a plane though the origin.

In fact, we can even interpret the act of multiplying by a constant geometrically. Suppose ω is given by $a\ dx + b\ dy$. Then the value of $\omega(V_1)$ is the length of the projection of V_1 onto the line, l, *where* $\frac{\langle a,b\rangle}{|\langle a,b\rangle|^2}$ *is a basis vector for* l.

This interpretation has a huge advantage... it is coordinate free. Recall from the previous section that we can think of the plane, P, as existing independent of our choice of coordinates. We only pick coordinates so that we can communicate to someone else the location of a point. Forms are similar. They are objects that exist independently of our choice of coordinates. This is one key as to why they are so useful outside of mathematics.

There is still another geometric interpretation of 1-forms. Let's first look at the simple example $\omega(\langle dx, dy\rangle) = dx$. This 1-form simply returns the first coordinate of whatever vector you feed into it. This is also a projection; it's the projection of the input vector onto the dx-axis. This immediately gives us a new interpretation of the action of a general 1-form, $\omega = a\ dx + b\ dy$.

Evaluating a 1-form on a vector is the same as projecting onto each coordinate axis, scaling each by some constant and adding the results.

Although this interpretation is more cumbersome, it is the one that will generalize better when we get to n-forms.

Let's move on now to 1-forms in n dimensions. If $p \in \mathbb{R}^n$, then we can write p in coordinates as $(x_1, x_2, ..., x_n)$. The coordinates for a vector in $T_p\mathbb{R}^n$ are

$\langle dx_1, dx_2, ..., dx_n\rangle$. A 1-form is a linear function, ω, whose graph (in $T_p\mathbb{R}^n \times \mathbb{R}$) is a plane through the origin. Hence, we can write it as $\omega = a_1dx_1 + a_2dx_2 + ... + a_ndx_n$. Again, this can be thought of as either projecting onto the vector $\langle a_1, a_2, ..., a_n\rangle$ and then multiplying by $|\langle a_1, a_2, ..., a_n\rangle|$ or as projecting onto each coordinate axis, multiplying by a_i, and then adding.

4.2. Let $\omega(\langle dx, dy\rangle) = -dx + 4dy$.

1. Compute $\omega(\langle 1, 0\rangle)$, $\omega(\langle 0, 1\rangle)$ and $\omega(\langle 2, 3\rangle)$.
2. What line does ω project vectors onto?

4.3. Find a 1-form which computes the length of the projection of a vector onto the indicated line, multiplied by the indicated constant c.

1. The dx-axis, $c = 3$.
2. The dy-axis, $c = \frac{1}{2}$.
3. Find a 1-form that does both of the two preceding operations and adds the result.
4. The line $dy = \frac{3}{4}dx$, $c = 10$.

4.4. If ω is a 1-form show

1. $\omega(V_1 + V_2) = \omega(V_1) + \omega(V_2)$, for any vectors V_1 and V_2.
2. $\omega(cV) = c\omega(V)$, for any vector V and constant c.

4.3 Multiplying 1-forms

In this section we would like to explore a method of multiplying 1-forms. You may think, "What is the big deal? If ω and ν are 1-forms can't we just define $\omega \cdot \nu(V) = \omega(V) \cdot \nu(V)$?" Well, of course we *can*, but then $\omega \cdot \nu$ is not a linear function, so we have left the world of forms.

The trick is to define the product of ω and ν to be a 2-form. So as not to confuse this with the product just mentioned, we will use the symbol "$\wedge$" (pronounced "wedge") to denote multiplication. So how can we possibly define $\omega \wedge \nu$ to be a 2-form? We must define how it acts on a pair of vectors, (V_1, V_2).

Note first that there are four ways to combine all the ingredients:

$$\omega(V_1), \quad \nu(V_1), \quad \omega(V_2), \quad \nu(V_2).$$

The first two of these are associated with V_1 and the second two with V_2. In other words, ω and ν together give a way of taking each vector and returning a *pair* of numbers. And how do we visualize pairs of numbers? In the plane, of course! Let's define a new plane with one axis as the ω-axis and the other as the ν-axis. So, the coordinates of V_1 in this plane are $[\omega(V_1), \nu(V_1)]$ and the coordinates of V_2 are $[\omega(V_2), \nu(V_2)]$. Note that we have switched to the notation "$[\cdot, \cdot]$" to indicate that we are describing points in a new plane. This may seem a little confusing at first. Just keep in mind that when we write something like (1, 2) we are describing the location of a point in the xy-plane, whereas $\langle 1, 2\rangle$ describes a vector in the $dxdy$-plane and [1, 2] is a vector in the $\omega\nu$-plane.

Let's not forget our goal now. We wanted to use ω and ν to take the pair of vectors, (V_1, V_2), and return a number. So far all we have done is to take this pair of vectors and return another pair of vectors. But do we know of a way to take these vectors and get a number? Actually, we know several, but the most useful one turns out to be the area of the parallelogram that the vectors span. This is precisely what we define to be the value of $\omega \wedge \nu(V_1, V_2)$ (see Fig. 4.4).

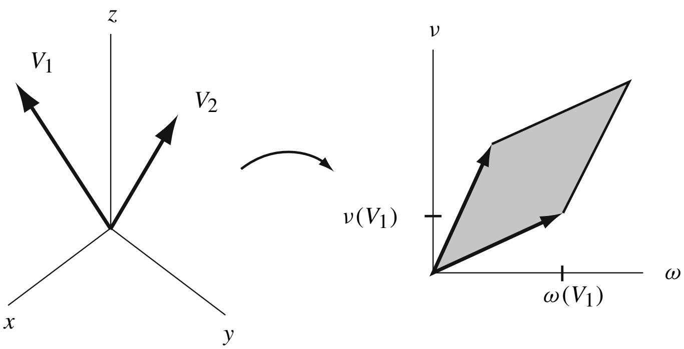

Fig. 4.4. The product of ω and ν.

Example 14. Let $\omega = 2dx - 3dy + dz$ and $\nu = dx + 2dy - dz$ be two 1-forms on $T_p\mathbb{R}^3$ for some fixed $p \in \mathbb{R}^3$. Let's evaluate $\omega \wedge \nu$ on the pair of vectors, $(\langle 1, 3, 1\rangle, \langle 2, -1, 3\rangle)$. First we compute the $[\omega, \nu]$ coordinates of the vector $\langle 1, 3, 1\rangle$.

$$\begin{aligned}[\omega(\langle 1,3,1\rangle), \nu(\langle 1,3,1\rangle)] &= [2\cdot 1 - 3\cdot 3 + 1\cdot 1, 1\cdot 1 + 2\cdot 3 - 1\cdot 1] \\ &= [-6, 6].\end{aligned}$$

Similarly, we compute $[\omega(\langle 2, -1, 3\rangle), \nu(\langle 2, -1, 3\rangle)] = [10, -3]$. Finally, the area of the parallelogram spanned by $[-6, 6]$ and $[10, -3]$ is

$$\begin{vmatrix} -6 & 10 \\ 6 & -3 \end{vmatrix} = 18 - 60 = -42.$$

Should we have taken the absolute value? Not if we want to define a *linear* operator. The result of $\omega \wedge \nu$ is not just an area, it is a *signed* area; it can either be positive or negative. We will see a geometric interpretation of this soon. For now, we define:

$$\omega \wedge \nu(V_1, V_2) = \begin{vmatrix} \omega(V_1) & \omega(V_2) \\ \nu(V_1) & \nu(V_2) \end{vmatrix}.$$

4.5. Let ω and ν be the following 1-forms:

$$\omega(\langle dx, dy \rangle) = 2dx - 3dy$$

$$\nu(\langle dx, dy \rangle) = dx + dy.$$

1. Let $V_1 = \langle -1, 2 \rangle$ and $V_2 = \langle 1, 1 \rangle$. Compute $\omega(V_1)$, $\nu(V_1)$, $\omega(V_2)$ and $\nu(V_2)$.
2. Use your answers to the previous question to compute $\omega \wedge \nu(V_1, V_2)$.
3. Find a constant c such that $\omega \wedge \nu = c\, dx \wedge dy$.

4.6. $\omega \wedge \nu(V_1, V_2) = -\omega \wedge \nu(V_2, V_1)$ ($\omega \wedge \nu$ is *skew-symmetric*).

4.7. $\omega \wedge \nu(V, V) = 0$. (This follows immediately from the previous exercise. It should also be clear from the geometric interpretation.)

4.8. $\omega \wedge \nu(V_1 + V_2, V_3) = \omega \wedge \nu(V_1, V_3) + \omega \wedge \nu(V_2, V_3)$ and $\omega \wedge \nu(cV_1, V_2) = \omega \wedge \nu(V_1, cV_2) = c\, \omega \wedge \nu(V_1, V_2)$, where c is any real number ($\omega \wedge \nu$ is *bilinear*).

4.9. $\omega \wedge \nu(V_1, V_2) = -\nu \wedge \omega(V_1, V_2)$.

It is interesting to compare Problems 4.6 and 4.9. Problem 4.6 says that the 2-form, $\omega \wedge \nu$, is a skew-symmetric operator on pairs of vectors. Problem 4.9 says that $\wedge$ can be thought of as a skew-symmetric operator on 1-forms.

4.10. $\omega \wedge \omega(V_1, V_2) = 0$.

4.11. $(\omega + \nu) \wedge \psi = \omega \wedge \psi + \nu \wedge \psi$ ($\wedge$ is distributive).

There is another way to interpret the action of $\omega \wedge \nu$ which is much more geometric. First let $\omega = a\, dx + b\, dy$ be a 1-form on $T_p\mathbb{R}^2$. Then we let $\langle \omega \rangle$ be the vector $\langle a, b \rangle$.

4.12. Let ω and ν be 1-forms on $T_p\mathbb{R}^2$. Show that $\omega \wedge \nu(V_1, V_2)$ is the area of the parallelogram spanned by V_1 and V_2, times the area of the parallelogram spanned by $\langle \omega \rangle$ and $\langle \nu \rangle$.

4.13. Use the previous problem to show that if ω and ν are 1-forms on $\mathbb{R}^2$ such that $\omega \wedge \nu = 0$ then there is a constant c such that $\omega = c\nu$.

There is also a more geometric way to think about $\omega \wedge \nu$ if ω and ν are 1-forms on $T_p\mathbb{R}^3$, although it will take us some time to develop the idea. Suppose $\omega = a\, dx + b\, dy + c\, dz$. Then we will denote the vector $\langle a, b, c \rangle$ as $\langle \omega \rangle$. From the previous section, we know that if V is any vector, then $\omega(V) = \langle\omega\rangle \cdot V$, and that this is just the projection of V onto the line containing $\langle\omega\rangle$, times $|\langle\omega\rangle|$.

Now suppose ν is some other 1-form. Choose a scalar x so that $\langle\nu - x\omega\rangle$ is perpendicular to $\langle\omega\rangle$. Let $\nu_\omega = \nu - x\omega$. Note that $\omega \wedge \nu_\omega = \omega \wedge (\nu - x\omega) = \omega \wedge \nu - x\omega \wedge \omega = \omega \wedge \nu$. Hence, any geometric interpretation we find for the action of $\omega \wedge \nu_\omega$ is also a geometric interpretation of the action of $\omega \wedge \nu$.

Finally, we let $\overline{\omega} = \frac{\omega}{|\langle\omega\rangle|}$ and $\overline{\nu_\omega} = \frac{\nu_\omega}{|\langle\nu_\omega\rangle|}$. Note that these are 1-forms such that $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$ are perpendicular unit vectors. We will now present a geometric interpretation of the action of $\overline{\omega} \wedge \overline{\nu_\omega}$ on a pair of vectors, (V_1, V_2).

First, note that since $\langle\overline{\omega}\rangle$ is a unit vector then $\overline{\omega}(V_1)$ is just the projection of V_1 onto the line containing $\langle\overline{\omega}\rangle$. Similarly, $\overline{\nu_\omega}(V_1)$ is given by projecting V_1 onto the line containing $\langle\overline{\nu_\omega}\rangle$. As $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$ are perpendicular, we can think of the quantity

$$\overline{\omega} \wedge \overline{\nu_\omega}(V_1, V_2) = \begin{vmatrix} \overline{\omega}(V_1) & \overline{\omega}(V_2) \\ \overline{\nu_\omega}(V_1) & \overline{\nu_\omega}(V_2) \end{vmatrix}$$

as the area of parallelogram spanned by V_1 and V_2, projected onto the plane containing the vectors $\langle\overline{\omega}\rangle$ and $\langle\overline{\nu_\omega}\rangle$. This is the same plane as the one which contains the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$.

Now observe the following:

$$\overline{\omega} \wedge \overline{\nu_\omega} = \frac{\omega}{|\langle\omega\rangle|} \wedge \frac{\nu_\omega}{|\langle\nu_\omega\rangle|} = \frac{1}{|\langle\omega\rangle||\langle\nu_\omega\rangle|}\omega \wedge \nu_\omega.$$

Hence,

$$\omega \wedge \nu = \omega \wedge \nu_\omega = |\langle\omega\rangle||\langle\nu_\omega\rangle|\overline{\omega} \wedge \overline{\nu_\omega}.$$

Finally, note that since $\langle\omega\rangle$ and $\langle\nu_\omega\rangle$ are perpendicular, the quantity $|\langle\omega\rangle||\langle\nu_\omega\rangle|$ is just the area of the rectangle spanned by these two vectors. Furthermore, the parallelogram spanned by the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$ is obtained from this rectangle by skewing. Hence, they have the same area. We conclude

Evaluating $\omega \wedge \nu$ on the pair of vectors (V_1, V_2) gives the area of parallelogram spanned by V_1 and V_2 projected onto the plane containing the vectors $\langle\omega\rangle$ and $\langle\nu\rangle$, and multiplied by the area of the parallelogram spanned by $\langle\omega\rangle$ and $\langle\nu\rangle$.

CAUTION: While every 1-form can be thought of as projected length *not* every 2-form can be thought of as projected area. The only 2-forms for which this interpretation is valid are those that are the product of 1-forms. See Problem 4.18.

Let's pause for a moment to look at a particularly simple 2-form on $T_p\mathbb{R}^3$, $dx \wedge dy$. Suppose $V_1 = \langle a_1, a_2, a_3\rangle$ and $V_2 = \langle b_1, b_2, b_3\rangle$. Then

$$dx \wedge dy(V_1, V_2) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This is precisely the (signed) area of the parallelogram spanned by V_1 and V_2 projected onto the $dxdy$-plane.

4.14. $\omega \wedge \nu(\langle a_1, a_2, a_3\rangle, \langle b_1, b_2, b_3\rangle) = c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dy \wedge dz$, for some real numbers c_1, c_2 and c_3.

The preceding comments, and this last exercise, give the following geometric interpretation of the action of a 2-form on the pair of vectors, (V_1, V_2):

Every 2-form projects the parallelogram spanned by V_1 and V_2 onto each of the (2-dimensional) coordinate planes, computes the resulting (signed) areas, multiplies each by some constant, and adds the results.

This interpretation holds in all dimensions. Hence, to specify a 2-form we need to know as many constants as there are 2-dimensional coordinate planes. For example, to give a 2-form in 4-dimensional Euclidean space we need to specify six numbers:

$$c_1 dx \wedge dy + c_2 dx \wedge dz + c_3 dx \wedge dw + c_4 dy \wedge dz + c_5 dy \wedge dw + c_6 dz \wedge dw.$$

The skeptic may argue here. Problem 4.14 only shows that a 2-form which is a product of 1-forms can be thought of as a sum of projected, scaled areas. What about an arbitrary 2-form? Well, to address this, we need to know what an arbitrary 2-form is! Up until now we have not given a complete definition. Henceforth, we shall define a 2-form to be a bilinear, skew-symmetric, real-valued function on $T_p\mathbb{R}^n \times T_p\mathbb{R}^n$. That is a mouthful. This just means that it is an operator which eats pairs of vectors, spits out real numbers, and satisfies the conclusions of Problems 4.6 and 4.8. Since these are the only ingredients necessary to do Problem 4.14, our geometric interpretation is valid for all 2-forms.

4.15. If $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$, and $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$, compute

$$\omega \wedge \nu(\langle 1, 2, 3\rangle, \langle -1, 4, -2\rangle).$$

4.16. Let $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$ and $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$. Find constants c_1, c_2 and c_3, such that

$$\omega \wedge \nu = c_1 dx \wedge dy + c_2 dy \wedge dz + c_3 dx \wedge dz.$$

4.17. Express each of the following as the product of two 1-forms:

1. $3dx \wedge dy + dy \wedge dx$.
2. $dx \wedge dy + dx \wedge dz$.
3. $3dx \wedge dy + dy \wedge dx + dx \wedge dz$.
4. $dx \wedge dy + 3dz \wedge dy + 4dx \wedge dz$.

4.4 2-forms on $T_p\mathbb{R}^3$ (optional)

4.18. Find a 2-form which is *not* the product of 1-forms.

In doing this exercise, you may guess that, in fact, all 2-forms on $T_p\mathbb{R}^3$ *can* be written as a product of 1-forms. Let's see a proof of this fact that relies heavily on the geometric interpretations we have developed.

Recall the correspondence introduced above between vectors and 1-forms. If $\alpha = a_1dx + a_2dy + a_3dz$ then we let $\langle\alpha\rangle = \langle a_1, a_2, a_3\rangle$. If V is a vector, then we let $\langle V\rangle^{-1}$ be the corresponding 1-form.

We now prove two lemmas:

Lemma 1. *If α and β are 1-forms on $T_p\mathbb{R}^3$ and V is a vector in the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$, then there is a vector, W, in this plane such that $\alpha \wedge \beta = \langle V\rangle^{-1} \wedge \langle W\rangle^{-1}$.*

Proof. The proof of the above lemma relies heavily on the fact that 2-forms which are the product of 1-forms are very flexible. The 2-form $\alpha\wedge\beta$ takes pairs of vectors, projects them onto the plane spanned by the vectors $\langle\alpha\rangle$ and $\langle\beta\rangle$, and computes the area of the resulting parallelogram times the area of the parallelogram spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$. Note, that for every non-zero scalar c, the area of the parallelogram spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$ is the same as the area of the parallelogram spanned by $c\langle\alpha\rangle$ and $1/c\langle\beta\rangle$. (This is the same thing as saying that $\alpha \wedge \beta = c\alpha \wedge \frac{1}{c}\beta$.) The important point here is that we can scale one of the 1-forms as much as we want at the expense of the other and get the same 2-form as a product.

Another thing we can do is apply a rotation to the pair of vectors $\langle\alpha\rangle$ and $\langle\beta\rangle$ in the plane which they determine. As the area of the parallelogram spanned by these two vectors is unchanged by rotation, their product still determines the same 2-form. In particular, suppose V is any vector in the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$. Then we can rotate $\langle\alpha\rangle$ and $\langle\beta\rangle$ to $\langle\alpha'\rangle$ and $\langle\beta'\rangle$ so that $c\langle\alpha'\rangle = V$, for some scalar c. We can then replace the pair $(\langle\alpha\rangle, \langle\beta\rangle)$ with the pair $(c\langle\alpha'\rangle, 1/c\langle\beta'\rangle) = (V, 1/c\langle\beta'\rangle)$. To complete the proof, let $W = 1/c\langle\beta'\rangle$.

Lemma 2. *If $\omega_1 = \alpha_1 \wedge \beta_1$ and $\omega_2 = \alpha_2 \wedge \beta_2$ are 2-forms on $T_p\mathbb{R}^3$, then there exist 1-forms, α_3 and β_3, such that $\omega_1 + \omega_2 = \alpha_3 \wedge \beta_3$.*

Proof. Let's examine the sum, $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$. Our first case is that the plane spanned by the pair $(\langle\alpha_1\rangle, \langle\beta_1\rangle)$ is the same as the plane spanned by the pair, $(\langle\alpha_2\rangle, \langle\beta_2\rangle)$. In this case it must be that $\alpha_1 \wedge \beta_1 = C\alpha_2 \wedge \beta_2$, and hence, $\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = (1 + C)\alpha_1 \wedge \beta_1$.

If these two planes are not the same, then they intersect in a line. Let V be a vector contained in this line. Then by the preceding lemma there are 1-forms γ and γ' such that $\alpha_1 \wedge \beta_1 = \langle V\rangle^{-1} \wedge \gamma$ and $\alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma'$. Hence,

$$\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = \langle V\rangle^{-1} \wedge \gamma + \langle V\rangle^{-1} \wedge \gamma' = \langle V\rangle^{-1} \wedge (\gamma + \gamma').$$

Now note that *any* 2-form is the sum of products of 1-forms. Hence, this last lemma implies that any 2-form on $T_p\mathbb{R}^3$ is a product of 1-forms. In other words:

> *Every 2-form on $T_p\mathbb{R}^3$ projects pairs of vectors onto some plane and returns the area of the resulting parallelogram, scaled by some constant.*

This fact is precisely why all of classical vector calculus works. We explore this in the next few exercises, and further in Section 7.3.

4.19. Use the above geometric interpretation of the action of a 2-form on $T_p\mathbb{R}^3$ to justify the following statement: For every 2-form ω on $T_p\mathbb{R}^3$ there are non-zero vectors V_1 and V_2 such that V_1 is not a multiple of V_2, but $\omega(V_1, V_2) = 0$.

4.20. Does Problem 4.19 generalize to higher dimensions?

4.21. Show that if ω is a 2-form on $T_p\mathbb{R}^3$, then there is a line l in $T_p\mathbb{R}^3$ such that if the plane spanned by V_1 and V_2 contains l, then $\omega(V_1, V_2) = 0$.

Note that the conditions of Problem 4.21 are satisfied when the vectors that are perpendicular to both V_1 and V_2 are also perpendicular to l.

4.22. Show that if all you know about V_1 and V_2 is that they are vectors in $T_p\mathbb{R}^3$ that span a parallelogram of area A, then the value of $\omega(V_1, V_2)$ is maximized when V_1 and V_2 are perpendicular to the line l of Problem 4.21.

Note that the conditions of this exercise are satisfied when the vectors perpendicular to V_1 and V_2 are parallel to l.

4.23. Let N be a vector perpendicular to V_1 and V_2 in $T_p\mathbb{R}^3$ whose length is precisely the area of the parallelogram spanned by these two vectors. Show that there is a vector V_ω in the line l of Problem 4.21 such that the value of $\omega(V_1, V_2)$ is precisely $V_\omega \cdot N$.

Remark. You may have learned that the vector N of the previous exercise is precisely the cross product of V_1 and V_2. Hence, the previous problem implies that if ω is a 2-form on $T_p\mathbb{R}^3$ then there is a vector V_ω such that $\omega(V_1, V_2) = V_\omega \cdot (V_1 \times V_2)$.

4.24. Show that if $\omega = F_x\, dy \wedge dz - F_y\, dx \wedge dz + F_z\, dx \wedge dy$, then $V_\omega = \langle F_x, F_y, F_z \rangle$.

4.5 2-forms and 3-forms on $T_p\mathbb{R}^4$ (optional)

Many of the techniques of the previous section can be used to prove results about 2- and 3-forms on $T_p\mathbb{R}^4$.

4.25. Show that any 3-form on $T_p\mathbb{R}^4$ can be written as the product of three 1-forms. (*Hint*: Two three-dimensional subspaces of $T_p\mathbb{R}^4$ must meet in at least a line.)

We now give away an answer to Problem 4.18. Let $\omega = dx \wedge dy + dz \wedge dw$. Then an easy computation shows that $\omega \wedge \omega = 2dx \wedge dy \wedge dz \wedge dw$. But if ω were equal to $\alpha \wedge \beta$, for some 1-forms α and β, then $\omega \wedge \omega$ would be zero (why?). This argument shows that in general, if ω is any 2-form such that $\omega \wedge \omega \neq 0$, then ω cannot be written as the product of 1-forms.

4.26. Let ω be a 2-form on $T_p\mathbb{R}^4$. Show that ω can be written as the sum of exactly two products. That is, $\omega = \alpha \wedge \beta + \delta \wedge \gamma$. (*Hint*: Given three planes in $T_p\mathbb{R}^4$ there are at least two of them that intersect in more than a point.)

Above, we saw that if ω is a 2-form such that $\omega \wedge \omega \neq 0$, then ω is not the product of 1-forms. We now use the previous exercise to show the converse:

Theorem 1. *If ω is a 2-form on $T_p\mathbb{R}^4$ such that $\omega \wedge \omega = 0$, then ω can be written as the product of two 1-forms.*

Our proof of this again relies heavily on the geometry of the situation. By the previous exercise, $\omega = \alpha \wedge \beta + \delta \wedge \gamma$. A short computation then shows

$$\omega \wedge \omega = 2\alpha \wedge \beta \wedge \delta \wedge \gamma.$$

If this 4-form is the zero 4-form, then it must be the case that the (4-dimensional) volume of the parallelepiped spanned by $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\delta\rangle$ and $\langle\gamma\rangle$ is zero. This, in turn, implies that the plane spanned by $\langle\alpha\rangle$ and $\langle\beta\rangle$ meets the plane spanned by $\langle\delta\rangle$ and $\langle\gamma\rangle$ in at least a line (show this!). Call such an intersection line $\mathcal{L}$.

As in the previous section, we can now rotate $\langle\alpha\rangle$ and $\langle\beta\rangle$, in the plane they span, to vectors $\langle\alpha'\rangle$ and $\langle\beta'\rangle$ such that $\langle\alpha'\rangle$ lies in the line $\mathcal{L}$. The 2-form $\alpha' \wedge \beta'$ must equal $\alpha \wedge \beta$ since they determine the same plane, and span a parallelogram of the same area. Similarly, we rotate $\langle\delta\rangle$ and $\langle\gamma\rangle$ to vectors $\langle\delta'\rangle$ and $\langle\gamma'\rangle$ such that $\langle\delta'\rangle \subset \mathcal{L}$. It follows that $\delta \wedge \gamma = \delta' \wedge \gamma'$.

Since $\langle\alpha'\rangle$ and $\langle\delta'\rangle$ lie on the same line, there is a constant c such that $c\alpha' = \delta'$. We now put all of this information together:

$$\begin{aligned}\omega &= \alpha \wedge \beta + \delta \wedge \gamma \\ &= \alpha' \wedge \beta' + \delta' \wedge \gamma' \\ &= (c\alpha') \wedge (\frac{1}{c}\beta') + \delta' \wedge \gamma' \\ &= \delta' \wedge (\frac{1}{c}\beta') + \delta' \wedge \gamma' \\ &= \delta' \wedge (\frac{1}{c}\beta' + \gamma').\end{aligned}$$

4.6 n-forms

Let's think a little more about our multiplication operator, $\wedge$. If it is really going to be anything like multiplication, we should be able to take three 1-forms, ω, ν and ψ, and form the product $\omega \wedge \nu \wedge \psi$. How can we define this? A first guess might be to say that $\omega \wedge \nu \wedge \psi = \omega \wedge (\nu \wedge \psi)$, but $\nu \wedge \psi$ is a 2-form and we have not defined the product of a 2-form and a 1-form. We take a different approach and define $\omega \wedge \nu \wedge \psi$ directly.

This is completely analogous to the previous section. ω, ν and ψ each act on a vector, V, to give three numbers. In other words, they can be thought of as coordinate functions. We say the coordinates of V are $[\omega(V), \nu(V), \psi(V)]$. Hence, if we have three vectors, V_1, V_2 and V_3, we can compute the $[\omega, \nu, \psi]$ coordinates of each. This gives us three new vectors. The signed volume of the parallelepiped which they span is what we define to be the value of $\omega \wedge \nu \wedge \psi(V_1, V_2, V_3)$.

There is no reason to stop at three dimensions. Suppose $\omega_1, \omega_2, ..., \omega_n$ are 1-forms and $V_1, V_2, ..., V_n$ are vectors. Then we define the value of

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, V_2, ..., V_n)$$

to be the signed (n-dimensional) volume of the parallelepiped spanned by the vectors $[\omega_1(V_i), \omega_2(V_i), ..., \omega_n(V_i)]$. Algebraically,

$$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, V_2, ..., V_n) = \begin{vmatrix} \omega_1(V_1) & \omega_1(V_2) & ... & \omega_1(V_n) \\ \omega_2(V_1) & \omega_2(V_2) & ... & \omega_2(V_n) \\ \vdots & \vdots & & \vdots \\ \omega_n(V_1) & \omega_n(V_2) & ... & \omega_n(V_n) \end{vmatrix}.$$

Note that, just as in Problem 4.12, if α, β and γ are 1-forms on $T_p\mathbb{R}^3$, then $\alpha \wedge \beta \wedge \gamma(V_1, V_2, V_3)$ is the (signed) volume of the parallelepiped spanned by V_1, V_2 and V_3 times the volume of the parallelepiped spanned by $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\gamma\rangle$.

4.27. Suppose ω is a 2-form on $T_p\mathbb{R}^3$ and ν is a 1-form on $T_p\mathbb{R}^3$. Show that if $\omega \wedge \nu = 0$, then there is a 1-form γ such that $\omega = \nu \wedge \gamma$. (*Hint*: Combine the above geometric interpretation of a 3-form, which is the product of 1-forms on T_pR^3, with the results of Section 4.4.)

It follows from linear algebra that if we swap any two rows or columns of this matrix, the sign of the result flips. Hence, if the n-tuple, $\mathbf{V}' = (V_{i_1}, V_{i_2}, ..., V_{i_n})$ is obtained from $\mathbf{V} = (V_1, V_2, ..., V_n)$ by an even number of exchanges, then the sign of $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(\mathbf{V}')$ will be the same as the sign of $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(\mathbf{V})$. If the number of exchanges is odd, then the sign is opposite. We sum this up by saying that the n-form, $\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n$ is *alternating*.

The wedge product of 1-forms is also *multilinear*, in the following sense:

$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i + V_i', ..., V_n) = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i, ..., V_n) + \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i', ..., V_n)$,
and
$\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., cV_i, ..., V_n) = c\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n(V_1, ..., V_i, ..., V_n)$,
for all i and any real number, c.

In general, we define an n-form to be any alternating, multilinear real-valued function which acts on n-tuples of vectors.

4.28. Prove the following geometric interpretation: (*Hint*: All of the steps are completely analogous to those in the last section.)

> *An m-form on $T_p\mathbb{R}^n$ can be thought of as a function which takes the parallelepiped spanned by m vectors, projects it onto each of the m-dimensional coordinate planes, computes the resulting areas, multiplies each by some constant, and adds the results.*

4.29. How many numbers do you need to give to specify a 5-form on $T_p\mathbb{R}^{10}$?

We turn now to the simple case of an n-form on $T_p\mathbb{R}^n$. Notice that there is only one n-dimensional coordinate plane in this space, namely, the space itself. Such a form, evaluated on an n-tuple of vectors, must therefore give the n-dimensional volume of the parallelepiped which it spans, multiplied by some constant. For this reason such a form is called a *volume form* (in 2-dimensions, an *area form*).

Example 15. Consider the forms, $\omega = dx + 2dy - dz$, $\nu = 3dx - dy + dz$ and $\psi = -dx - 3dy + dz$, on $T_p\mathbb{R}^3$. By the above argument $\omega \wedge \nu \wedge \psi$ must be a volume form. But which volume form is it? One way to tell is to compute its value on a set of vectors which we *know* span a parallelepiped of volume one, namely $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$ and $\langle 0, 0, 1\rangle$. This will tell us how much the form scales volume.

$$\omega \wedge \nu \wedge \psi(\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle) = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 1 \\ -1 & -3 & 1 \end{vmatrix} = 4.$$

So, $\omega \wedge \nu \wedge \psi$ must be the same as the form $4dx \wedge dy \wedge dz$.

4.30. Let $\omega(\langle dx, dy, dz\rangle) = dx + 5dy - dz$, $\nu(\langle dx, dy, dz\rangle) = 2dx - dy + dz$ and $\gamma(\langle dx, dy, dz) = -dx + dy + 2dz$.

1. If $V_1 = \langle 1,0,2\rangle$, $V_2 = \langle 1,1,2\rangle$ and $V_3 = \langle 0,2,3\rangle$, compute $\omega \wedge \nu \wedge \gamma(V_1, V_2, V_3)$.
2. Find a constant, c, such that $\omega \wedge \nu \wedge \gamma = c\, dx \wedge dy \wedge dz$.
3. Let $\alpha = 3dx \wedge dy + 2dy \wedge dz - dx \wedge dz$. Find a constant, c, such that $\alpha \wedge \gamma = c\, dx \wedge dy \wedge dz$.

4.31. Simplify:

$$dx \wedge dy \wedge dz + dx \wedge dz \wedge dy + dy \wedge dz \wedge dx + dy \wedge dx \wedge dy.$$

4.32. Let ω be an n-form and ν an m-form.

1. Show that
$$\omega \wedge \nu = (-1)^{nm} \nu \wedge \omega.$$
2. Use this to show that if n is odd then $\omega \wedge \omega = 0$.

4.7 Algebraic computation of products

In this section, we break with the spirit of the text briefly. At this point, we have amassed enough algebraic identities that multiplying forms becomes similar to multiplying polynomials. We quickly summarize these identities and work a few examples.

Let ω be an n-form and ν be an m-form. Then we have the following identities

$$\begin{aligned} \omega \wedge \nu &= (-1)^{nm} \nu \wedge \omega \\ \omega \wedge \omega &= 0 \text{ if } n \text{ is odd} \\ \omega \wedge (\nu + \psi) &= \omega \wedge \nu + \omega \wedge \psi \\ (\nu + \psi) \wedge \omega &= \nu \wedge \omega + \psi \wedge \omega. \end{aligned}$$

Example 16.

$$\begin{aligned} (x\, dx + y\, dy) \wedge (y\, dx + x\, dy) &= \cancel{xy\, dx \wedge dx} + x^2\, dx \wedge dy + y^2\, dy \wedge dx \\ &\quad + \cancel{yx\, dy \wedge dy} \\ &= x^2\, dx \wedge dy + y^2\, dy \wedge dx \\ &= x^2\, dx \wedge dy - y^2\, dx \wedge dy \\ &= (x^2 - y^2)\, dx \wedge dy. \end{aligned}$$

Example 17.

$$
\begin{aligned}
(x\,dx + y\,dy) \wedge (xz\,dx \wedge dz + yz\,dy \wedge dz)
&= \cancel{x^2 z\,dx \wedge dx \wedge dz} + xyz\,dx \wedge dy \wedge dz \\
&\quad + yxz\,dy \wedge dx \wedge dz + \cancel{y^2 z\,dy \wedge dy \wedge dz} \\
&= xyz\,dx \wedge dy \wedge dz + yxz\,dy \wedge dx \wedge dz \\
&= xyz\,dx \wedge dy \wedge dz - xyz\,dx \wedge dy \wedge dz \\
&= 0.
\end{aligned}
$$

4.33. Expand and simplify:

1. $[(x-y)\,dx + (x+y)\,dy + z\,dz] \wedge [(x-y)\,dx + (x+y)\,dy]$.
2. $(2dx + 3dy) \wedge (dx - dz) \wedge (dx + dy + dz)$.

5

Differential Forms

5.1 Families of forms

Let us now go back to the example in Chapter 3. In the last section of that chapter, we showed that the integral of a function, $f : \mathbb{R}^3 \to \mathbb{R}$, over a surface parameterized by $\phi : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ is

$$\int_R f(\phi(r,\theta))\text{Area}\left[\frac{\partial\phi}{\partial r}(r,\theta), \frac{\partial\phi}{\partial\theta}(r,\theta)\right] dr d\theta.$$

This gave one motivation for studying differential forms. We wanted to generalize this integral by considering functions other than "Area$(\cdot,\cdot)$" that eat pairs of vectors and return numbers. But in this integral the point at which such a pair of vectors is based, changes. In other words, Area$(\cdot,\cdot)$ does *not* act on $T_p\mathbb{R}^3 \times T_p\mathbb{R}^3$ for a *fixed* p. We can make this point a little clearer by re-examining the above integrand. Note that it is of the form $f(\star)\text{Area}(\cdot,\cdot)$. For a fixed point, $\star$, of $\mathbb{R}^3$, this is an operator on $T_\star\mathbb{R}^3 \times T_\star\mathbb{R}^3$, much like a 2-form is.

But so far all we have done is to define 2-forms at fixed points of $\mathbb{R}^3$. To really generalize the above integral we must start to consider entire families of 2-forms, $\omega_p : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$, where p ranges over all of $\mathbb{R}^3$. Of course, for this to be useful such a family must have some "niceness" properties. For one thing, it should be *continuous*. That is, if p and q are close, then ω_p and ω_q should be similar.

An even stronger property is that the family, ω_p, is *differentiable*. To see what this means recall that for a fixed p, a 2-form ω_p can always be written as $a_p dx \wedge dy + b_p dy \wedge dz + c_p dx \wedge dz$, where a_p, b_p, and c_p are constants. But if we let our choice of p vary over all of $\mathbb{R}^3$, then so will these constants. In other words, a_p, b_p and c_p are all functions from $\mathbb{R}^3$ to $\mathbb{R}$. To say that the family, ω_p, is differentiable we mean that each of these functions is differentiable. If ω_p is differentiable, then we will refer to it as a *differential* form. When there can be no confusion we will suppress the subscript, p.

Example 18. $\omega = x^2y\, dx \wedge dy - xz\, dy \wedge dz$ is a differential 2-form on $\mathbb{R}^3$. On the space $T_{(1,2,3)}\mathbb{R}^3$ it is just the 2-form $2dx \wedge dy - 3dy \wedge dz$. We will denote vectors in $T_{(1,2,3)}\mathbb{R}^3$ as $\langle dx, dy, dz\rangle_{(1,2,3)}$. Hence, the value of $\omega(\langle 4, 0, -1\rangle_{(1,2,3)}, \langle 3, 1, 2\rangle_{(1,2,3)})$ is the same as the 2-form, $2dx \wedge dy + dy \wedge dz$, evaluated on the vectors $\langle 4, 0, -1\rangle$ and $\langle 3, 1, 2\rangle$, which we compute:

$$\begin{aligned}\omega(\langle 4, 0, -1\rangle_{(1,2,3)} &, \langle 3, 1, 2\rangle_{(1,2,3)}) \\ &= 2dx \wedge dy - 3dy \wedge dz\,(\langle 4, 0, -1\rangle, \langle 3, 1, 2\rangle) \\ &= 2\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 5.\end{aligned}$$

Suppose ω is a differential 2-form on $\mathbb{R}^3$. What does ω act on? It takes a pair of vectors at each point of $\mathbb{R}^3$ and returns a number. In other words, it takes two *vector fields* and returns a function from $\mathbb{R}^3$ to $\mathbb{R}$. A vector field is simply a choice of vector in $T_p\mathbb{R}^3$, for each $p \in \mathbb{R}^3$. In general, a differential n-form on $\mathbb{R}^m$ acts on n vector fields to produce a function from $\mathbb{R}^m$ to $\mathbb{R}$ (see Fig. 5.1).

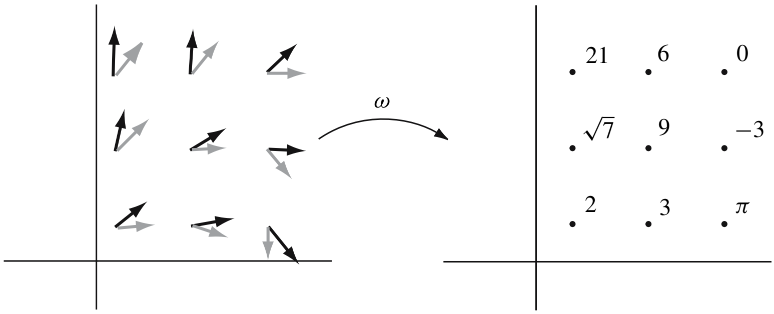

Fig. 5.1. A differential 2-form, ω, acts on a pair of vector fields, and returns a function from $\mathbb{R}^n$ to $\mathbb{R}$.

Example 19. $V_1 = \langle 2y, 0, -x\rangle_{(x,y,z)}$ is a vector field on $\mathbb{R}^3$. For example, it contains the vector $\langle 4, 0, -1\rangle \in T_{(1,2,3)}\mathbb{R}^3$. If $V_2 = \langle z, 1, xy\rangle_{(x,y,z)}$ and ω is the differential 2-form, $x^2y\, dx \wedge dy - xz\, dy \wedge dz$, then

$$\begin{aligned}\omega(V_1, V_2) &= x^2y\, dx \wedge dy - xz\, dy \wedge dz(\langle 2y, 0, x\rangle_{(x,y,z)}, \langle z, 1, xy\rangle_{(x,y,z)}) \\ &= x^2y\begin{vmatrix} 2y & z \\ 0 & 1 \end{vmatrix} - xz\begin{vmatrix} 0 & 1 \\ -x & xy \end{vmatrix} = 2x^2y^2 - x^2z,\end{aligned}$$

which is a function from $\mathbb{R}^3$ to $\mathbb{R}$.

Notice that V_2 contains the vector $\langle 3, 1, 2\rangle_{(1,2,3)}$. So, from the previous example we would expect that $2x^2y^2 - x^2z$ equals 5 at the point (1, 2, 3), which is indeed the case.

5.1. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\, dx \wedge dy + x^2 z\, dy \wedge dz - y\, dx \wedge dz.$$

Let V_1 and V_2 be the following vector fields:

$$V_1 = \langle y, z, x^2 \rangle_{(x,y,z)},\ V_2 = \langle xy, xz, y \rangle_{(x,y,z)}.$$

1. What vectors do V_1 and V_2 contain at the point $(1, 2, 3)$?
2. Which 2-form is ω on $T_{(1,2,3)}\mathbb{R}^3$?
3. Use your answers to the previous two questions to compute $\omega(V_1, V_2)$ at the point $(1, 2, 3)$.
4. Compute $\omega(V_1, V_2)$ at the point (x, y, z). Then plug in $x = 1$, $y = 2$ and $z = 3$ to check your answer against the previous question.

5.2 Integrating differential 2-forms

Let's now recall the steps involved with integration of functions on subsets of $\mathbb{R}^2$, which we learned in Section 1.3. Suppose $R \subset \mathbb{R}^2$ and $f : R \to \mathbb{R}$. The following steps define the integral of f over R:

1. Choose a lattice of points in R, $\{(x_i, y_j)\}$.
2. For each i, j define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$ and $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$ (See Fig. 5.2). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are both vectors in $T_{(x_i,y_j)}\mathbb{R}^2$.
3. For each i, j compute $f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j})$, where $\text{Area}(V, W)$ is the function which returns the area of the parallelogram spanned by the vectors V and W.
4. Sum over all i and j.
5. Take the limit as the maximal distance between adjacent lattice points goes to zero. This is the number that we define to be the value of $\int\limits_R f\, dx\, dy$.

Let's focus on Step 3. Here we compute $f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j})$. Notice that this is exactly the value of the differential 2-form $\omega = f(x, y)dx \wedge dy$, evaluated on the vectors $V^1_{i,j}$ and $V^2_{i,j}$ at the point (x_i, y_j). Hence, in Step 4 we can write this sum as $\sum\limits_i \sum\limits_j f(x_i, y_j)\text{Area}(V^1_{i,j}, V^2_{i,j}) = \sum\limits_i \sum\limits_j \omega_{(x_i,y_j)}(V^1_{i,j}, V^2_{i,j})$. It is reasonable, then, to adopt the shorthand "$\int\limits_R \omega$" to denote the limit in Step 5. The upshot of all this is the following:

> *If* $\omega = f(x, y)dx \wedge dy$, *then* $\int\limits_R \omega = \int\limits_R f\, dx\, dy$.

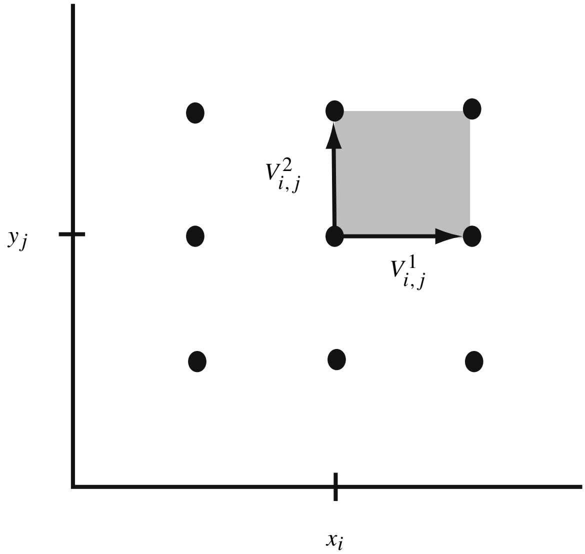

Fig. 5.2. The steps toward integration.

Since all differential 2-forms on $\mathbb{R}^2$ are of the form $f(x, y)dx \wedge dy$ we now know how to integrate them.

CAUTION! When integrating 2-forms on $\mathbb{R}^2$ it is tempting to always drop the "$\wedge$" and forget you have a differential form. This is only valid with $dx \wedge dy$. It is NOT valid with $dy \wedge dx$. This may seem a bit curious since Fubini's theorem gives us

$$\int f\ dx \wedge dy = \int f\ dx\ dy = \int f\ dy\ dx.$$

All of these are equal to $-\int f\ dy \wedge dx$. We will revisit this issue in Example 27.

5.2. Let $\omega = xy^2\ dx \wedge dy$ be a differential 2-form on $\mathbb{R}^2$. Let D be the region of $\mathbb{R}^2$ where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Calculate $\int\limits_D \omega$.

What about integration of differential 2-forms on $\mathbb{R}^3$? As remarked at the end of Section 3.5 we do this only over those subsets of $\mathbb{R}^3$ which can be parameterized by subsets of $\mathbb{R}^2$. Suppose M is such a subset, like the top half of the unit sphere. To define what we mean by $\int\limits_M \omega$ we just follow the steps above:

1. Choose a lattice of points in M, $\{p_{i,j}\}$.
2. For each i, j define $V_{i,j}^1 = p_{i+1,j} - p_{i,j}$ and $V_{i,j}^2 = p_{i,j+1} - p_{i,j}$. Notice that $V_{i,j}^1$ and $V_{i,j}^2$ are both vectors in $T_{p_{i,j}}\mathbb{R}^3$ (see Fig. 5.3).
3. For each i, j compute $\omega_{p_{i,j}}(V_{i,j}^1, V_{i,j}^2)$.

4. Sum over all i and j.
5. Take the limit as the maximal distance between adjacent lattice points goes to 0. This is the number that we define to be the value of $\int_M \omega$.

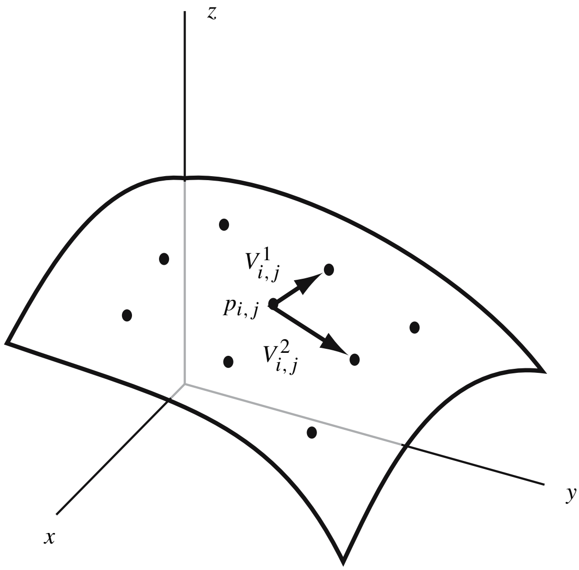

Fig. 5.3. The steps toward integrating a 2-form.

Unfortunately these steps are not so easy to follow. For one thing, it is not always clear how to pick the lattice in Step 1. In fact, there is an even worse problem. In Step 3, why did we compute $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$? After all, $V^1_{i,j}$ and $V^2_{i,j}$ are two randomly oriented vectors in $T\mathbb{R}^3_{p_{i,j}}$. There is no reasonable way to decide which should be first and which second. There is nothing to be done about this. At some point we just have to make a choice and make it clear which choice we have made. Such a decision is called an *orientation*. We will have much more to say about this later. For now, we simply note that a different choice will only change our answer by changing its sign.

While we are on this topic, we also note that we would end up with the same number in Step 5 if we had calculated $\omega_{p_{i,j}}(-V^1_{i,j}, -V^2_{i,j})$ in Step 4, instead. Similarly, if it turns out later that we should have calculated $\omega_{p_{i,j}}(V^2_{i,j}, V^1_{i,j})$, then we could have also arrived at the right answer by computing $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. In other words, there are really only two possibilities: either $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ gives the correct answer or $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$ does. Which one will depend on our choice of orientation.

Despite all the difficulties with using the above definition of $\int_M \omega$, all hope is not lost. Remember that we are only integrating over regions which can be parameterized by subsets of $\mathbb{R}^2$. The trick is to use such a parameterization to translate the problem into an integral of a 2-form over a region in $\mathbb{R}^2$. The steps are analogous to those in Section 3.5.

Suppose $\phi : R \subset \mathbb{R}^2 \to M$ is a parameterization. We want to find a 2-form, $f(x, y)\, dx \wedge dy$, such that a Riemann sum for this 2-form over R gives the same result as a Riemann sum for ω over M. Let's begin:

1. Choose a rectangular lattice of points in R, $\{(x_i, y_j)\}$. This also gives a lattice, $\{\phi(x_i, y_j)\}$, in M.
2. For each i, j, define $V^1_{i,j} = (x_{i+1}, y_j) - (x_i, y_j)$, $V^2_{i,j} = (x_i, y_{j+1}) - (x_i, y_j)$, $\mathcal{V}^1_{i,j} = \phi(x_{i+1}, y_j) - \phi(x_i, y_j)$, and $\mathcal{V}^2_{i,j} = \phi(x_i, y_{j+1}) - \phi(x_i, y_j)$ (see Fig. 5.4). Notice that $V^1_{i,j}$ and $V^2_{i,j}$ are vectors in $T_{(x_i,y_j)}\mathbb{R}^2$ and $\mathcal{V}^1_{i,j}$ and $\mathcal{V}^2_{i,j}$ are vectors in $T_{\phi(x_i,y_j)}\mathbb{R}^3$.
3. For each i, j compute $f(x_i, y_j)\, dx \wedge dy(V^1_{i,j}, V^2_{i,j})$ and $\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$.
4. Sum over all i and j.

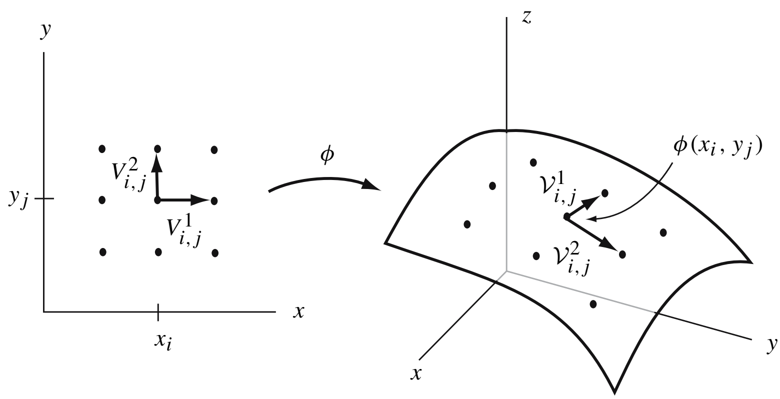

Fig. 5.4. Using ϕ to integrate a 2-form.

At the conclusion of Step 4 we have two sums, $\sum_i \sum_j f(x_i, y_j)\, dx \wedge dy(V^1_{i,j}, V^2_{i,j})$ and $\sum_i \sum_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})$. In order for these to be equal, we must have:

$$f(x_i, y_j)\, dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j}).$$

And so,

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{dx \wedge dy(V^1_{i,j}, V^2_{i,j})}.$$

But, since we are using a rectangular lattice in R we know $dx \wedge dy(V^1_{i,j}, V^2_{i,j}) = \text{Area}(V^1_{i,j}, V^2_{i,j}) = |V^1_{i,j}| \cdot |V^2_{i,j}|$. We now have

$$f(x_i, y_j) = \frac{\omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j})}{|V^1_{i,j}| \cdot |V^2_{i,j}|}.$$

Using the bilinearity of ω this reduces to

$$f(x_i, y_j) = \omega_{\phi(x_i,y_j)}\left(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\right).$$

But, as the distance between adjacent points of our partition tends toward 0,

$$\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|(x_{i+1}, y_j) - (x_i, y_j)|} = \frac{\phi(x_{i+1}, y_j) - \phi(x_i, y_j)}{|x_{i+1} - x_i|} \to \frac{\partial \phi}{\partial x}(x_i, y_j).$$

Similarly, $\frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}$ converges to $\frac{\partial \phi}{\partial y}(x_i, y_j)$.

Let's summarize what we have so far. We defined $f(x, y)$ so that

$$\sum_i \sum_j \omega_{\phi(x_i,y_j)}(\mathcal{V}^1_{i,j}, \mathcal{V}^2_{i,j}) = \sum_i \sum_j f(x_i, y_j)\, dx \wedge dy(V^1_{i,j}, V^2_{i,j})$$

$$= \sum_i \sum_j \omega_{\phi(x_i,y_j)}\left(\frac{\mathcal{V}^1_{i,j}}{|V^1_{i,j}|}, \frac{\mathcal{V}^2_{i,j}}{|V^2_{i,j}|}\right) dx \wedge dy(V^1_{i,j}, V^2_{i,j}).$$

We have also shown that when we take the limit as the distance between adjacent partition points tends toward zero this sum converges to the sum

$$\sum_i \sum_j \omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right) dx \wedge dy(V^1_{i,j}, V^2_{i,j}).$$

Hence, it must be that

$$\int_M \omega = \int_R \omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right) dx \wedge dy. \tag{5.1}$$

At first glance, this seems like a very complicated formula. Let's break it down by examining the integrand on the right. The most important thing to notice is that this is just a differential 2-form on R, even though ω is a 2-form on $\mathbb{R}^3$. For each pair of numbers, (x, y), the function $\omega_{\phi(x,y)}\left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y)\right)$ just returns some real number. Hence, the entire integrand is of the form $g\ dx \wedge dy$, where $g : R \to \mathbb{R}$.

The only way to *really* convince oneself of the usefulness of this formula is to actually use it.

Example 20. Let M denote the top half of the unit sphere in $\mathbb{R}^3$. Let $\omega = z^2 dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Calculating $\int\limits_M \omega$ directly by setting up a Riemann sum would be next to impossible. So we employ the parameterization $\phi(r, t) = (r \cos t, r \sin t, \sqrt{1 - r^2})$, where $0 \le t \le 2\pi$ and $0 \le r \le 1$.

$$\begin{aligned}
\int\limits_M \omega &= \int\limits_R \omega_{\phi(r,t)}\left(\frac{\partial \phi}{\partial r}(r, t), \frac{\partial \phi}{\partial t}(r, t)\right) dr \wedge dt \\
&= \int\limits_R \omega_{\phi(r,t)}\left(\langle \cos t, \sin t, \frac{-r}{\sqrt{1 - r^2}}\rangle, \langle -r \sin t, r \cos t, 0\rangle\right) dr \wedge dt \\
&= \int\limits_R (1 - r^2) \begin{vmatrix} \cos t & -r \sin t \\ \sin t & r \cos t \end{vmatrix} dr \wedge dt \\
&= \int\limits_R (1 - r^2)(r) dr \wedge dt \\
&= \int\limits_0^{2\pi} \int\limits_0^1 r - r^3 dr\ dt = \frac{\pi}{2}.
\end{aligned}$$

Notice that as promised, the term $\omega_{\phi(r,t)}\left(\frac{\partial \phi}{\partial r}(r, t), \frac{\partial \phi}{\partial t}(r, t)\right)$ in the second integral above simplified to a function from R to $\mathbb{R}$, $r - r^3$.

5.3. Integrate the 2-form

$$\omega = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$$

over the top half of the unit sphere using the following parameterizations from cylindrical and spherical coordinates:

1. $(r, \theta) \to (r \cos \theta, r \sin \theta, \sqrt{1 - r^2})$, where $0 \le \theta \le 2\pi$ and $0 \le r \le 1$.
2. $(\theta, \phi) \to (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$, where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$.

5.4. Let ω be the 2-form from the previous problem. Integrate ω over the surface parameterized by the following:

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, \cos r), \ 0 \le r \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi.$$

5.5. Let S be the surface in $\mathbb{R}^3$ parameterized by

$$\Psi(\theta, z) = (\cos\theta, \sin\theta, z)$$

where $0 \le \theta \le \pi$ and $0 \le z \le 1$. Let $\omega = xyz\, dy \wedge dz$. Calculate $\int\limits_S \omega$.

5.6. Let ω be the differential 2-form on $\mathbb{R}^3$ given by

$$\omega = xyz\, dx \wedge dy + x^2 z\, dy \wedge dz - y\, dx \wedge dz.$$

1. Let P be the portion of the plane $3 = 2x + 3y - z$ in $\mathbb{R}^3$ that lies above the square $\{(x, y) | 0 \le x \le 1, 0 \le y \le 1\}$. Calculate $\int\limits_P \omega$.
2. Let M be the portion of the graph of $z = x^2 + y^2$ in $\mathbb{R}^3$ that lies above the rectangle $\{(x, y) | 0 \le x \le 1, 0 \le y \le 2\}$. Calculate $\int\limits_M \omega$.

5.7. Let D be some region in the xy-plane. Let M denote the portion of the graph of $z = g(x, y)$ that lies above D.

1. Let $\omega = f(x, y)\, dx \wedge dy$ be a differential 2-form on $\mathbb{R}^3$. Show that

$$\int\limits_M \omega = \int\limits_D f(x, y)\, dx\, dy.$$

   Notice the answer does not depend on the function $g(x, y)$.
2. Now suppose $\omega = f(x, y, z)\, dx \wedge dy$. Show that

$$\int\limits_M \omega = \int\limits_D f(x, y, g(x, y))\, dx\, dy.$$

5.8. Let S be the surface obtained from the graph of $z = f(x) = x^3$, where $0 \le x \le 1$, by rotating around the z-axis. Integrate the 2-form $\omega = y\, dx \wedge dz$ over S. (*Hint*: use cylindrical coordinates to parameterize S.)

5.3 Orientations

What would have happened in Example 20 if we had used the parameterization $\phi'(r, t) = (-r \cos t, r \sin t, \sqrt{1 - r^2})$ instead? We leave it to the reader to check that we end up with the answer $-\pi/2$ rather than $\pi/2$. This is a problem. We defined $\int\limits_M \omega$ before we started talking about parameterizations. Hence, the value which we calculate for this integral should not depend on our choice of parameterization. So what happened?

To analyze this completely, we need to go back to the definition of $\int\limits_M \omega$ from the previous section. We noted at the time that a choice was made to calculate $\omega_{p_{i,j}}(V^1_{i,j}, V^2_{i,j})$ instead of $\omega_{p_{i,j}}(-V^1_{i,j}, V^2_{i,j})$. But was this choice correct? The answer is a resounding *maybe!* We are actually missing enough information to tell. An *orientation* is precisely some piece of information about M which we can use to make the right choice. This way we can tell a friend what M is, what ω is, and what the orientation on M is, and they are sure to get the same answer. Recall Equation 5.1:

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)} \left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y) \right) dx \wedge dy.$$

Depending on the specified orientation of M, it may be incorrect to use Equation 5.1. Sometimes we may want to use:

$$\int\limits_M \omega = \int\limits_R \omega_{\phi(x,y)} \left(-\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y) \right) dx \wedge dy.$$

Both ω and $\int$ are linear. This just means the negative sign in the integrand on the right can go all the way outside. Hence, we can write both this equation and Equation 5.1 as

$$\int\limits_M \omega = \pm \int\limits_R \omega_{\phi(x,y)} \left(\frac{\partial \phi}{\partial x}(x, y), \frac{\partial \phi}{\partial y}(x, y) \right) dx \wedge dy. \tag{5.2}$$

We define an *orientation on M* to be any piece of information that can be used to decide, for each choice of parameterization ϕ, whether to use the "+" or "−" sign in Equation 5.2, so that the integral will always yield the same answer.

We will see several ways to specify an orientation on M. The first will be geometric. It has the advantage that it can be easily visualized, but the disadvantage that it is actually much harder to use in calculations. All we do is draw a small circle on M with an arrowhead on it. To use this "oriented circle" to tell if we need the "+" or "−" sign in Equation 5.2, we draw the vectors $\frac{\partial \phi}{\partial x}(x, y)$ and $\frac{\partial \phi}{\partial y}(x, y)$ and an arc with an arrow from the first to the second. If the direction of this arrow agrees with the oriented circle, then we use the "+" sign. If they disagree, then we use the "−" sign. See Figure 5.5.

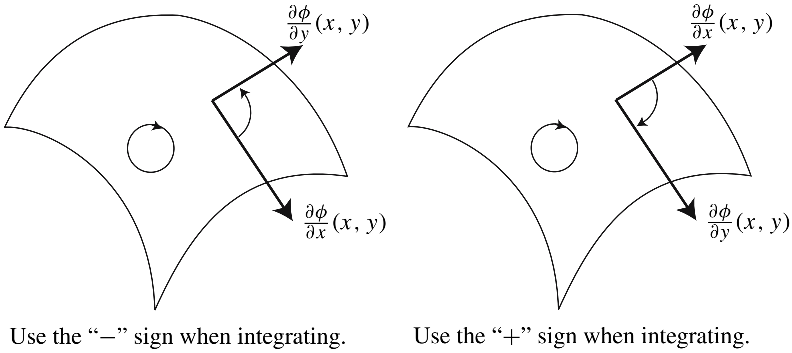

Fig. 5.5. An orientation on M is given by an oriented circle.

A more algebraic way to specify an orientation is to simply pick a point p of M and choose any 2-form ν on $T_p\mathbb{R}^3$ such that $\nu(V_p^1, V_p^2) \neq 0$ whenever V_p^1 and V_p^2 are vectors tangent to M, and V_1 is not a multiple of V_2. Do not confuse this 2-form with the *differential* 2-form, ω, of Equation 5.2. The 2-form ν is only defined at the single tangent space $T_p\mathbb{R}^3$, whereas ω is defined everywhere.

Let us now see how we can use ν to decide whether to use the "+" or "−" sign in Equation 5.2. All we must do is calculate $\nu\left(\frac{\partial\phi}{\partial x}(x_p, y_p), \frac{\partial\phi}{\partial y}(x_p, y_p)\right)$, where $\phi(x_p, y_p) = p$. If the result is positive, then we will use the "+" sign to calculate the integral in Equation 5.2. If it is negative then we use the "−" sign. Let's see how this works with an example.

Example 21. Let's revisit Example 20. The problem was to integrate the form $z^2 dx \wedge dy$ over M, the top half of the unit sphere. But no orientation was ever given for M, so the problem was not very well stated. Let's pick an easy point, p, on M: $(0, \sqrt{2}/2, \sqrt{2}/2)$. The vectors $\langle 1, 0, 0\rangle_p$ and $\langle 0, 1, -1\rangle_p$ in $T_p\mathbb{R}^3$ are both tangent to M. To give an orientation on M, all we do is specify a 2-form ν on $T_p\mathbb{R}^3$ such that $\nu(\langle 1, 0, 0\rangle, \langle 0, 1, -1\rangle) \neq 0$. Let's pick an easy one: $\nu = dx \wedge dy$.

Now, let's see what happens when we try to evaluate the integral by using the parameterization $\phi'(r, t) = (-r\cos t, r\sin t, \sqrt{1 - r^2})$. First, note that $\phi'(\sqrt{2}/2, \pi/2) = (0, \sqrt{2}/2, \sqrt{2}/2)$ and

$$\left(\frac{\partial\phi'}{\partial r}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right), \frac{\partial\phi'}{\partial t}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{2}\right)\right) = \left(\langle 0, 1, -1\rangle, \langle\frac{\sqrt{2}}{2}, 0, 0\rangle\right).$$

Now we check the value of ν when this pair is plugged in:

$$dx \wedge dy\left(\langle 0, 1, -1\rangle, \langle\frac{\sqrt{2}}{2}, 0, 0\rangle\right) = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 1 & 0 \end{vmatrix} = -\frac{\sqrt{2}}{2}.$$

The sign of this result is "−," so we need to use the negative sign in Equation 5.2 in order to use ϕ' to evaluate the integral of ω over M.

$$\int\limits_M \omega = -\int\limits_R \omega_{\phi'(r,t)}\left(\frac{\partial \phi'}{\partial r}(r,t), \frac{\partial \phi'}{\partial t}(r,t)\right) dr \wedge dt$$
$$= -\int\limits_R (1-r^2)\begin{vmatrix} -\cos t & r\sin t \\ \sin t & r\cos t \end{vmatrix} dr\ dt = \frac{\pi}{2}.$$

Very often, the surface that we are going to integrate over is given to us by a parameterization. In this case, there is a very natural choice of orientation. Just use the "+" sign in Equation 5.2! We will call this the orientation of M *induced* by the parameterization. In other words, if you see a problem phrased like this, "*Calculate the integral of the form* ω *over the surface M given by parameterization* ϕ *with the induced orientation,*" then you should just go back to using Equation 5.1 and do not worry about anything else. In fact, this situation is so common that when you are asked to integrate some form over a surface which is given by a parameterization, but no orientation is specified, then you should assume the induced orientation is the desired one.

5.9. Let M be the image of the parameterization, $\phi(a,b) = (a, a+b, ab)$, where $0 \le a \le 1$ and $0 \le b \le 1$. Integrate the form $\omega = 2z\ dx \wedge dy + y\ dy \wedge dz - x\ dx \wedge dz$ over M using the orientation induced by ϕ.

There is one subtle technical point here that should be addressed. The novice reader may want to skip this for now. Suppose someone gives you a surface defined by a parameterization and tells you to integrate some 2-form over it, using the induced orientation. But you are clever, and you realize that if you change parameterizations you can make the integral easier. Which orientation do you use? The problem is that the orientation induced by your new parameterization may not be the same as the one induced by the original parameterization.

To fix this we need to see how we can define a 2-form on some tangent space $T_p\mathbb{R}^3$, where p is a point of M, that yields an orientation of M consistent with the one induced by a parameterization ϕ. This is not so hard. If $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right)$ is positive, then we simply let $\nu = dx \wedge dy$. If it is negative, then we let $\nu = -dx \wedge dy$. In the unlikely event that $dx \wedge dy\left(\frac{\partial \phi}{\partial x}(x_p, y_p), \frac{\partial \phi}{\partial y}(x_p, y_p)\right) = 0$ we can remedy things by either changing the point p or by using $dx \wedge dz$ instead of $dx \wedge dy$. Once we have defined ν, we know how to integrate M using any other parameterization.

5.10. Let ψ be the following parameterization of the sphere of radius one:

$$\psi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Which of the following 2-forms on $T_{(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})}\mathbb{R}^3$ determine the same orientation on the sphere as that induced by ψ?

1. $\alpha = dx \wedge dy + 2dy \wedge dz$.
2. $\beta = dx \wedge dy - 2dy \wedge dz$.
3. $\gamma = dx \wedge dz$.

5.4 Integrating 1-forms on $\mathbb{R}^m$

In the previous sections we saw how to integrate a 2-form over a region in $\mathbb{R}^2$, or over a subset of $\mathbb{R}^3$ parameterized by a region in $\mathbb{R}^2$. The reason that these dimensions were chosen was because there is nothing new that needs to be introduced to move to the general case. In fact, if the reader were to go back and look at what we did, he/she would find that almost nothing would change if we had been talking about n-forms instead.

Before we jump to the general case, we will work one example showing how to integrate a 1-form over a parameterized curve.

Example 22. Let C be the curve in $\mathbb{R}^2$ parameterized by

$$\phi(t) = (t^2, t^3)$$

where $0 \le t \le 2$. Let ν be the 1-form $y\ dx + x\ dy$. We calculate $\int\limits_C \nu$.

The first step is to calculate

$$\frac{d\phi}{dt} = \langle 2t, 3t^2 \rangle.$$

So, $dx = 2t$ and $dy = 3t^2$. From the parameterization we also know $x = t^2$ and $y = t^3$. Hence, since $\nu = y\ dx + x\ dy$, we have

$$\nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) = (t^3)(2t) + (t^2)(3t^2) = 5t^4.$$

Finally, we integrate:

$$\begin{aligned}\int\limits_C \nu &= \int\limits_0^2 \nu_{\phi(t)}\left(\frac{d\phi}{dt}\right) dt \\ &= \int\limits_0^2 5t^4\ dt \\ &= t^5\Big|_0^2 \\ &= 32.\end{aligned}$$

5.11. Let C be the curve in $\mathbb{R}^3$ parameterized by $\phi(t) = (t, t^2, 1+t)$, where $0 \le t \le 2$. Integrate the 1-form $\omega = y\,dx + z\,dy + xy\,dz$ over C using the induced orientation.

5.12. Let C be the curve parameterized by the following:

$$\phi(t) = (2\cos t, 2\sin t, t^2), \quad 0 \le t \le 2.$$

Integrate the 1-form $(x^2 + y^2)\,dz$ over C.

5.13. Let C be the subset of the graph of $y = x^2$ where $0 \le x \le 1$. An orientation on C is given by the 1-form dx on $T_{(0,0)}\mathbb{R}^2$. Let ω be the 1-form $-x^4\,dx + xy\,dy$. Integrate ω over C.

5.14. Let M be the line segment in $\mathbb{R}^2$ which connects $(0,0)$ to $(4,6)$. An orientation on M is specified by the 1-form $-dx$ on $T_{(2,3)}\mathbb{R}^2$. Integrate the form $\omega = \sin y\,dx + \cos x\,dy$ over M.

Just as there was for surfaces, for parameterized curves there is also a pictorial way to specify an orientation. All we have to do is place an arrowhead somewhere along the curve, and ask whether or not the derivative of the parameterization gives a tangent vector that points in the same direction. We illustrate this in the next example.

Example 23. Let C be the portion of the graph of $x = y^2$ where $0 \le x \le 1$, as pictured in Figure 5.6. Notice the arrowhead on C. We integrate the 1-form $\omega = dx + dy$ over C with the indicated orientation.

First, parameterize C as $\phi(t) = (t^2, t)$, where $0 \le t \le 1$. Now notice that the derivative of ϕ is

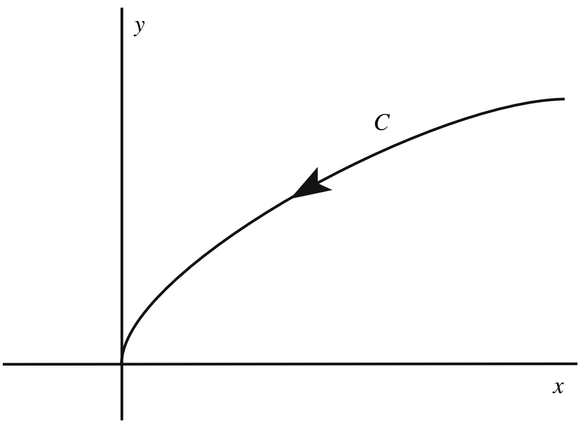

Fig. 5.6. An orientation on C is given by an arrowhead.

$$\frac{d\phi}{dt} = \langle 2t, 1\rangle.$$

At the point (0, 0) this is the vector $\langle 0, 1\rangle$, which points in a direction opposite to that of the arrowhead. This tells us to use a negative sign when we integrate, as follows:

$$\begin{aligned}\int_C \omega &= -\int_0^1 \omega_{(t^2,t)}(\langle 2t, 1\rangle) \\ &= -(2t+1)|_0^1 \\ &= -2.\end{aligned}$$

5.5 Integrating *n*-forms on $\mathbb{R}^m$

To proceed to the general case, we need to know what the integral of an n-form over a region of $\mathbb{R}^n$ is. The steps to define such an object are precisely the same as before, and the results are similar. If our coordinates in $\mathbb{R}^n$ are $(x_1, x_2, ..., x_n)$, then an n-form is always given by

$$f(x_1, ..., x_n)dx_1 \wedge dx_2 \wedge ... \wedge dx_n.$$

Going through the steps, we find that the definition of $\int\limits_{\mathbb{R}^n} \omega$ is exactly the same as the definition we learned in Chapter 1 for $\int\limits_{\mathbb{R}^n} f\, dx_1 dx_2 ... dx_n$.

5.15. Let Ω be the cube in $\mathbb{R}^3$

$$\{(x, y, z) |\ 0 \le x, y, z \le 1\}.$$

Let γ be the 3-form $x^2 z\, dx \wedge dy \wedge dz$. Calculate $\int\limits_{\Omega} \gamma$.

Moving on to integrals of n-forms over subsets of $\mathbb{R}^m$ parameterized by a region in $\mathbb{R}^n$, we again find nothing surprising. Suppose we denote such a subset by M. Let $\phi : R \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ be a parameterization. Then we find that the following generalization of Equation 5.2 must hold:

$$\int_M \omega = \pm \int_R \omega_{\phi(x_1,...,x_n)} \left(\frac{\partial \phi}{\partial x_1}(x_1, ...x_n), ..., \frac{\partial \phi}{\partial x_n}(x_1, ...x_n) \right) dx_1 \wedge ... \wedge dx_n. \tag{5.3}$$

To decide whether or not to use the negative sign in this equation we must specify an orientation. Again, one way to do this is to give an n-form ν on $T_p\mathbb{R}^m$, where p is some point of M. We use the negative sign when the value of

$$\nu\Big(\frac{\partial \phi}{\partial x_1}(x_1, ...x_n), ..., \frac{\partial \phi}{\partial x_n}(x_1, ...x_n)\Big)$$

is negative, where $\phi(x_1, ...x_n) = p$. If M was originally given by a parameterization and we are instructed to use the induced orientation, then we can ignore the negative sign.

Example 24. Suppose $\phi(a, b, c) = (a+b, a+c, bc, a^2)$, where $0 \le a, b, c \le 1$. Let M be the image of ϕ with the induced orientation. Suppose $\omega = dy \wedge dz \wedge dw - dx \wedge dz \wedge dw - 2y\, dx \wedge dy \wedge dz$. Then,

$$\begin{aligned}
\int\limits_M \omega &= \int\limits_R \omega_{\phi(a,b,c)} \left(\frac{\partial \phi}{\partial a}(a, b, c), \frac{\partial \phi}{\partial b}(a, b, c), \frac{\partial \phi}{\partial c}(a, b, c)\right) da \wedge db \wedge dc \\
&= \int\limits_R \omega_{\phi(a,b,c)} \left(\langle 1, 1, 0, 2a\rangle, \langle 1, 0, c, 0\rangle, \langle 0, 1, b, 0\rangle\right) da \wedge db \wedge dc \\
&= \int\limits_R \left|\begin{matrix} 1 & 0 & 1 \\ 0 & c & b \\ 2a & 0 & 0 \end{matrix}\right| - \left|\begin{matrix} 1 & 1 & 0 \\ 0 & c & b \\ 2a & 0 & 0 \end{matrix}\right| - 2(a+c) \left|\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & c & b \end{matrix}\right| da \wedge db \wedge dc \\
&= \int_0^1 \int_0^1 \int_0^1 2bc + 2c^2\, da\, db\, dc = \frac{7}{6}.
\end{aligned}$$

5.6 The change of variables formula

There is a special case of Equation 5.3 which is worth noting. Suppose ϕ is a parameterization that takes some subregion, R, of $\mathbb{R}^n$ into some other subregion, M, of $\mathbb{R}^n$ and ω is an n-form on $\mathbb{R}^n$. At each point, ω is just a volume form, so it can be written as $f(x_1, ..., x_n)\, dx_1 \wedge ... \wedge dx_n$. If we let $\bar{x} = (x_1, ...x_n)$ then Equation 5.3 can be written as:

$$\int\limits_M f(\bar{x}) dx_1 ... dx_n = \pm \int\limits_R f(\phi(\bar{x})) \left|\frac{\partial \phi}{\partial x_1}(\bar{x}) ... \frac{\partial \phi}{\partial x_n}(\bar{x})\right| dx_1 ... dx_n. \tag{5.4}$$

The bars $|\cdot|$ indicate that we take the determinant of the matrix whose column vectors are $\frac{\partial \phi}{\partial x_i}(\bar{x})$.

5.6.1 1-forms on $\mathbb{R}^1$

When $n = 1$ this is just the substitution rule for integration from calculus. We demonstrate this in the following example.

Example 25. Let's integrate the 1-form $\omega = \sqrt{u}\, du$ over the interval $[1, 5]$. This would be easy enough to do directly, but using a parameterization of this interval will be instructive. Let $\phi : [0, 2] \to [1, 5]$ be the parameterization given by $\phi(x) = x^2 + 1$. Then $\frac{d\phi}{dx} = 2x$. Now we compute:

$$\begin{aligned}\int_1^5 \sqrt{u}\, du = \int_{[1,5]} \omega &= \int_{[0,2]} \omega_{\phi(x)} \left(\frac{d\phi}{dx}\right) dx \\ &= \int_{[0,2]} \omega_{x^2+1} (\langle 2x \rangle)\, dx \\ &= \int_{[0,2]} 2x\sqrt{x^2+1}\, dx \\ &= \int_0^2 2x\sqrt{x^2+1}\, dx.\end{aligned}$$

Reading this backwards is doing the integral $\int_0^2 2x\sqrt{x^2+1}\, dx$ by "u-substitution."

5.6.2 2-forms on $\mathbb{R}^2$

For other n, Equation 5.4 is the general change of variables formula.

Example 26. We will use the parameterization $\Psi(u, v) = (u, u^2 + v^2)$ to evaluate

$$\iint_R (x^2 + y)\, dA$$

where R is the region of the xy-plane bounded by the parabolas $y = x^2$ and $y = x^2 + 4$, and the lines $x = 0$ and $x = 1$.

The first step is to find out what the limits of integration will be when we change coordinates.

$$y = x^2 \Rightarrow u^2 + v^2 = u^2 \Rightarrow v = 0$$

$$y = x^2 + 4 \Rightarrow u^2 + v^2 = u^2 + 4 \Rightarrow v = 2$$

$$x = 0 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = 1.$$

Next, we will need the partial derivatives.

$$\frac{\partial \Psi}{\partial u} = \langle 1, 2u \rangle$$

$$\frac{\partial \Psi}{\partial v} = \langle 0, 2v \rangle.$$

Finally, we can integrate.

$$\begin{aligned}
\iint_R (x^2 + y)\, dA &= \int_R (x^2 + y)\, dx \wedge dy \\
&= \int_0^2 \int_0^1 u^2 + (u^2 + v^2) \begin{vmatrix} 1 & 0 \\ 2u & 2v \end{vmatrix} du\, dv \\
&= \int_0^2 \int_0^1 4vu^2 + 2v^3\, du\, dv \\
&= \int_0^2 \frac{4}{3}v + 2v^3\, dv \\
&= \frac{8}{3} + 8 = \frac{32}{3}.
\end{aligned}$$

Example 27. In our second example, we revisit Fubini's theorem, which says that the order of integration does not matter in a multiple integral. Recall from Section 5.2 the curious fact that $\int f\, dx\, dy = \int f\, dx \wedge dy$, but $\int f\, dy\, dx \neq \int f\, dy \wedge dx$. We are now prepared to see why this is.

Let's suppose we want to integrate the function $f(x, y)$ over the rectangle R in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b). We know the answer is just $\int_0^b \int_0^a f(x, y)\, dx\, dy$. We also know this integral is equal to $\int_R f\, dx \wedge dy$, where R is given the "standard" orientation (e.g., the one specified by a counter-clockwise oriented circle).

Let's see what happens if we try to compute the integral using the following parameterization:

$$\phi(y, x) = (x, y), 0 \leq y \leq b, 0 \leq x \leq a.$$

First, we need the partials of ϕ:

$$\frac{\partial \phi}{\partial y} = \langle 0, 1 \rangle$$

$$\frac{\partial \phi}{\partial x} = \langle 1, 0 \rangle.$$

Next we have to deal with the issue of orientation. The pair of vectors we just found, $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ are in an order which does not agree with the orientation of R. So we have to use the negative sign when employing Equation 5.4:

$$\begin{aligned}\int_R f(x, y)\, dx\, dy &= -\int_R f(\phi(y, x)) \left|\frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial x}\right| dy\, dx \\ &= -\int_R f(x, y) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} dy \wedge dx \\ &= -\int_R f(x, y)(-1) dy \wedge dx \\ &= \int_R f(x, y) dy\, dx.\end{aligned}$$

From the above, we see one of the reasons why Fubini's theorem is true is because when the order of integration is switched there are *two* negative signs. So, $\int_R f\ dy\ dx$ actually *does* equal $\int_R f\ dy \wedge dx$, *but only if you remember to switch the orientation of R!*

5.16. Let E be the region in $\mathbb{R}^2$ parameterized by $\Psi(u, v) = (u^2 + v^2, 2uv)$, where $0 \le u \le 1$ and $0 \le v \le 1$. Evaluate

$$\int_E \frac{1}{\sqrt{x - y}}\, dA.$$

Up until this point, we have only seen how to integrate functions $f(x, y)$ over regions in the plane which are rectangles. Let's now see how we can use parameterizations to integrate over more general regions. Suppose first, that R is the region of the plane below the graph of $y = g(x)$, above the x-axis, and between the lines $x = a$ and $x = b$.

The region R can be parameterized by

$$\Psi(u, v) = (u, vg(u))$$

where $a \le u \le b$ and $0 \le v \le 1$. The partials of this parameterization are

$$\begin{aligned}\frac{\partial \Psi}{\partial u} &= \left\langle 1, v\frac{dg(u)}{du}\right\rangle \\ \frac{\partial \Psi}{\partial v} &= \langle 0, g(u)\rangle .\end{aligned}$$

Hence,

$$dx \wedge dy = \begin{vmatrix} 1 & 0 \\ v\frac{dg(u)}{du} & g(u) \end{vmatrix} = g(u).$$

We conclude with the identity

$$\int_R f(x, y)\, dy\, dx = \int_a^b \int_0^1 f(u, vg(u))g(u)\, dv\, du.$$

5.17. Let R be the region below the graph of $y = x^2$, and between the lines $x = 0$ and $x = 2$. Calculate

$$\int_R xy^2\, dx\, dy.$$

A slight variant is to integrate over a region bounded by the graphs of equations $y = g_1(x)$ and $y = g_2(x)$, and by the lines $x = a$ and $x = b$, where $g_1(x) < g_2(x)$ for all $x \in [a, b]$. To compute such an integral we may simply integrate over the region below $g_2(x)$, then integrate over the region below $g_1(x)$, and subtract.

5.18. Let R be the region to the right of the y-axis, to the left of the graph of $x = g(y)$, above the line $y = a$ and below the line $y = b$. Find a formula for $\int_R f(x, y)\, dx\, dy$.

5.19. Let R be the region in the first quadrant of $\mathbb{R}^2$, below the line $y = x$, and bounded by $x^2 + y^2 = 4$. Integrate the 2-form

$$\omega = \left(1 + \frac{y^2}{x^2}\right) dx \wedge dy$$

over R.

5.20. Let R be the region of the xy-plane bounded by the ellipse

$$9x^2 + 4y^2 = 36.$$

Integrate the 2-form $\omega = x^2 dx \wedge dy$ over R (*Hint*: see Problem 2.23 of Chapter 1.)

5.21. Integrate the 2-form

$$\omega = \frac{1}{x} dy \wedge dz - \frac{1}{y} dx \wedge dz$$

over the top half of the unit sphere using the following parameterization from rectangular coordinates:

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2})$$

where $\sqrt{x^2 + y^2} \leq 1$. Compare your answer to Problem 5.3.

5.6.3 3-forms on $\mathbb{R}^3$

Example 28. Let $V = \{(r, \theta, z) | 1 \le r \le 2,\ 0 \le z \le 1\}$. ($V$ is the region between the cylinders of radii one and two and between the planes $z = 0$ and $z = 1$.) Let's calculate

$$\int_V z(x^2 + y^2)\, dx \wedge dy \wedge dz.$$

The region V is best parameterized using cylindrical coordinates:

$$\Psi(r, \theta, z) = (r\cos\theta, r\sin\theta, z),$$

where $1 \le r \le 2$, $1 \le \theta \le 2\pi$, and $0 \le z \le 1$.

We compute the partials:

$$\frac{\partial \Psi}{\partial r} = \langle \cos\theta, \sin\theta, 0\rangle$$
$$\frac{\partial \Psi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$$
$$\frac{\partial \Psi}{\partial z} = \langle 0, 0, 1\rangle.$$

Hence,

$$dx \wedge dy \wedge dz = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Also,

$$z(x^2 + y^2) = z(r^2\cos^2\theta + r^2\sin^2\theta) = zr^2.$$

So we have

$$\begin{aligned}
\int_V z(x^2 + y^2)\, dx \wedge dy \wedge dz &= \int_0^1 \int_0^{2\pi} \int_1^2 (zr^2)(r)\, dr\, d\theta\, dz \\
&= \int_0^1 \int_0^{2\pi} \int_1^2 zr^3\, dr\, d\theta\, dz \\
&= \frac{15}{4} \int_0^1 \int_0^{2\pi} z\, d\theta\, dz \\
&= \frac{15\pi}{2} \int_0^1 z\, dz \\
&= \frac{15\pi}{4}.
\end{aligned}$$

5.22. Integrate the 3-form $\omega = x\, dx \wedge dy \wedge dz$ over the region of $\mathbb{R}^3$ in the first octant bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and the plane $z = 2$.

5.23. Let R be the region in the first octant of $\mathbb{R}^3$ bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Integrate the 3-form $\omega = dx \wedge dy \wedge dz$ over R.

5.24. Let V be the volume *in the first octant*, inside the cylinder of radius one, and below the plane $z = 1$. Integrate the 3-form

$$2\sqrt{1 + x^2 + y^2}\, dx \wedge dy \wedge dz$$

over V.

5.25. Let V be the region inside the cylinder of radius one, centered around the z-axis, and between the planes $z = 0$ and $z = 2$. Integrate the function $f(x, y, z) = z$ over V.

5.7 Summary: How to integrate a differential form

5.7.1 The steps

To compute the integral of a differential n-form, ω, over a region, S, the steps are as follows:

1. Choose a parameterization, $\Psi : R \to S$, where R is a subset of $\mathbb{R}^n$ (see Figure 5.7).

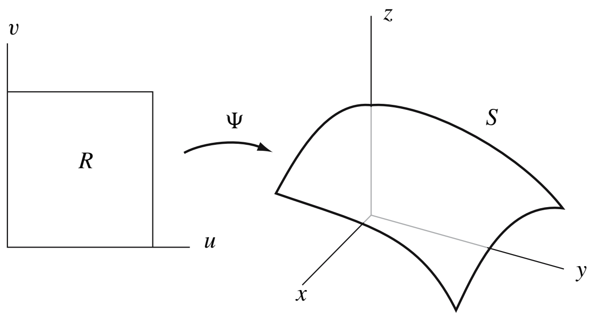

Fig. 5.7.

2. Find all n vectors given by the partial derivatives of Ψ. Geometrically, these are tangent vectors to S which span its tangent space (see Figure 5.8).
3. Plug the tangent vectors into ω at the point $\Psi(u_1, u_2, ..., u_n)$.
4. Integrate the resulting function over R.

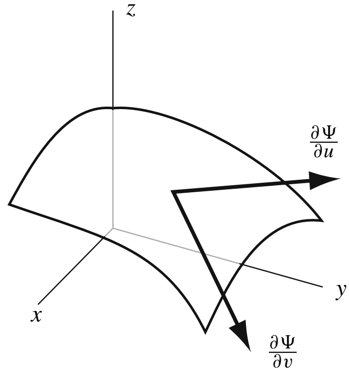

Fig. 5.8.

5.7.2 Integrating 2-forms

The best way to see the above steps in action is to look at the integral of a 2-form over a surface in $\mathbb{R}^3$. In general, such a 2-form is given by

$$\omega = f_1(x, y, z)\, dx \wedge dy + f_2(x, y, z)\, dy \wedge dz + f_3(x, y, z)\, dx \wedge dz.$$

To integrate ω over S we now follow the steps:

1. Choose a parameterization, $\Psi : R \to S$, where R is a subset of $\mathbb{R}^2$.

$$\Psi(u, v) = (g_1(u, v), g_2(u, v), g_3(u, v)).$$

2. Find both vectors given by the partial derivatives of Ψ.

$$\frac{\partial \Psi}{\partial u} = \left\langle \frac{\partial g_1}{\partial u}, \frac{\partial g_2}{\partial u}, \frac{\partial g_3}{\partial u} \right\rangle$$
$$\frac{\partial \Psi}{\partial v} = \left\langle \frac{\partial g_1}{\partial v}, \frac{\partial g_2}{\partial v}, \frac{\partial g_3}{\partial v} \right\rangle.$$

3. Plug the tangent vectors into ω at the point $\Psi(u, v)$.
   To do this, x, y and z will come from the coordinates of Ψ. That is, $x = g_1(u, v)$, $y = g_2(u, v)$ and $z = g_3(u, v)$. Terms like $dx \wedge dy$ are determinants of 2×2 matrices, whose entries come from the vectors computed in the previous step. Geometrically, the value of $dx \wedge dy$ is the area of the parallelogram spanned by the vectors $\frac{\partial \Psi}{\partial u}$ and $\frac{\partial \Psi}{\partial v}$ (tangent vectors to S), projected onto the $dxdy$-plane (see Figure 5.9).
   The result of all this is:

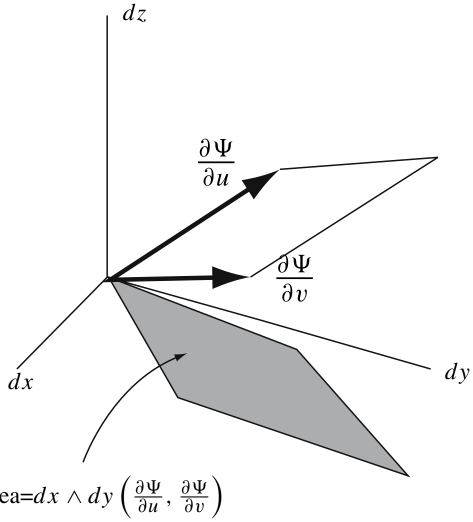

Fig. 5.9. Evaluating $dx \wedge dy$ geometrically.

$$f_1(g_1, g_2, g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} + f_2(g_1, g_2, g_3)\begin{vmatrix} \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}$$
$$+f_3(g_1, g_2, g_3)\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{vmatrix}.$$

Note that simplifying this gives a function of u and v.

4. Integrate the resulting function over R. In other words, if $h(u, v)$ is the function you ended up with in the previous step, then compute

$$\int_R h(u, v)\, du\, dv.$$

If R is not a rectangle you may have to find a parameterization of R whose domain is a rectangle and repeat the above steps to compute this integral.

5.7.3 A sample 2-form

Let $\omega = (x^2 + y^2)\, dx \wedge dy + z\, dy \wedge dz$. Let S denote the subset of the cylinder $x^2 + y^2 = 1$ that lies between the planes $z = 0$ and $z = 1$.

1. Choose a parameterization, $\Psi : R \to S$.

$$\Psi(\theta, z) = (\cos\theta, \sin\theta, z).$$

Where $R = \{(\theta, z) | 0 \le \theta \le 2\pi, 0 \le z \le 1\}$.

2. Find both vectors given by the partial derivatives of Ψ.

$$\frac{\partial \Psi}{\partial \theta} = \langle -\sin\theta, \cos\theta, 0 \rangle$$
$$\frac{\partial \Psi}{\partial z} = \langle 0, 0, 1 \rangle.$$

3. Plug the tangent vectors into ω at the point $\Psi(\theta, z)$. We get

$$(\cos^2\theta + \sin^2\theta) \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} + z \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix}.$$

This simplifies to the function $z\cos\theta$.

4. Integrate the resulting function over R.

$$\int_0^1 \int_0^{2\pi} z\cos\theta \; d\theta \; dz.$$

Note that the integrand comes from Step 3 and the limits of integration come from Step 1.

6

Differentiation of Forms

6.1 The derivative of a differential 1-form

The goal of this section is to figure out what we mean by the derivative of a differential form. One way to think about a derivative is as a function which measures the variation of some other function. Suppose ω is a 1-form on $\mathbb{R}^2$. What do we mean by the "variation" of ω? One thing we can try is to plug in a vector field V. The result is a function from $\mathbb{R}^2$ to $\mathbb{R}$. We can then think about how this function varies near a point p of $\mathbb{R}^2$. But p can vary in a lot of ways, so we need to pick one. In Section 1.5, we learned how to take another vector, W, and use it to vary p. Hence, the derivative of ω, which we shall denote "$d\omega$," is a function that acts on both V and W. In other words, it must be a 2-form!

Let's recall how to vary a function $f(x, y)$ in the direction of a vector W at a point p. This was precisely the definition of the *directional derivative*:

$$\nabla_W f(p) = \nabla f(p) \cdot W,$$

where $\nabla f(p)$ is the *gradient* of f at p:

$$\nabla f(p) = \left\langle \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right\rangle.$$

Going back to the 1-form ω and the vector field V, we take the directional derivative of the function $\omega(V)$. Let's do this now for a specific example. Suppose $\omega = y\, dx - x^2 dy$, $V = \langle 1, 2\rangle$, $W = \langle 2, 3\rangle$, and $p = (1, 1)$. Then $\omega(V)$ is the function $y - 2x^2$. Now we compute:

$$\nabla_W \omega(V) = \nabla\omega(V) \cdot W = \langle -4x, 1\rangle \cdot \langle 2, 3\rangle = -8x + 3.$$

At the point $p = (1, 1)$ this is the number -5.

What about the variation of $\omega(W)$, in the direction of V, at the point p? The function $\omega(W)$ is $2y - 3x^2$. We now compute:

$$\nabla_V \omega(W) = \nabla\omega(W) \cdot V = \langle -6x, 2\rangle \cdot \langle 1, 2\rangle = -6x + 4.$$

At the point $p = (1, 1)$ this is the number -2.

This is a small problem. We want $d\omega$, the derivative of ω, to be a 2-form. Hence, $d\omega(V, W)$ should equal $-d\omega(W, V)$. How can we use the variations above to define $d\omega$ so this is true? Simple. We just define it to be the difference in these variations:

$$d\omega(V, W) = \nabla_V \omega(W) - \nabla_W \omega(V). \tag{6.1}$$

Hence, in the above example, $d\omega(\langle 1, 2\rangle, \langle 2, 3\rangle)$, at the point $p = (1, 1)$, is the number $-2 - (-5) = 3$.

6.1. Suppose $\omega = xy^2\, dx + x^3 z\, dy - (y + z^9)\, dz$, $V = \langle 1, 2, 3\rangle$, and $W = \langle -1, 0, 1\rangle$.

1. Compute $\nabla_V \omega(W)$ and $\nabla_W \omega(V)$, at the point $(2, 3, -1)$.
2. Use your answer to the previous question to compute $d\omega(V, W)$ at the point $(2, 3, -1)$.

There are other ways to determine what $d\omega$ is rather than using Equation 6.1. Recall that a 2-form acts on a pair of vectors by projecting them onto each coordinate plane, calculating the area they span, multiplying by some constant, and adding. So the 2-form is completely determined by the constants that you multiply by after projecting. In order to figure out what these constants are, we are free to examine the action of the 2-form on any pair of vectors. For example, suppose we have two vectors that lie in the xy-plane and span a parallelogram with area one. If we run these through some 2-form and end up with the number five, then we know that the multiplicative constant for that 2-form, associated with the xy-plane is 5. This, in turn, tells us that the 2-form equals $5\, dx \wedge dy + \nu$. To figure out what ν is, we can examine the action of the 2-form on other pairs of vectors.

Let's try this with a general differential 2-form on $\mathbb{R}^3$. Such a form always looks like $d\omega = a(x, y, z)dx \wedge dy + b(x, y, z)dy \wedge dz + c(x, y, z)dx \wedge dz$. To figure out what $a(x, y, z)$ is, for example, all we need to do is determine what $d\omega$ does to the vectors $\langle 1, 0, 0\rangle_{(x,y,z)}$ and $\langle 0, 1, 0\rangle_{(x,y,z)}$. Let's compute this using Equation 6.1, assuming $\omega = f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$.

$$\begin{aligned} d\omega(\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle) &= \nabla_{\langle 1,0,0\rangle}\omega(\langle 0, 1, 0\rangle) - \nabla_{\langle 0,1,0\rangle}\omega(\langle 1, 0, 0\rangle) \\ &= \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \cdot \langle 1, 0, 0\rangle - \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle 0, 1, 0\rangle \\ &= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}. \end{aligned}$$

Similarly, direct computation shows:

$$d\omega(\langle 0,1,0\rangle, \langle 0,0,1\rangle) = \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}$$

and,

$$d\omega(\langle 1,0,0\rangle, \langle 0,0,1\rangle) = \frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}.$$

Hence, we conclude that

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dx \wedge dz.$$

6.2. Suppose $\omega = f(x,y)dx + g(x,y)dy$ is a 1-form on $\mathbb{R}^2$. Show that $d\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dx \wedge dy$.

6.3. If $\omega = y\,dx - x^2\,dy$, find $d\omega$. Verify that $d\omega(\langle 1,2\rangle, \langle 2,3\rangle) = 3$ at the point $(1,1)$.

Technical Note: Equation 6.1 defines the value of $d\omega$ as long as the vector fields V and W are constant. If non-constant vector fields are used, then the answer provided by Equation 6.1 will involve partial derivatives of the components of V and W, and hence will not be a differential form. Despite this Equation 6.1 does lead to the correct formulas for $d\omega$, as in Exercise 6.2 above. Once such formulas are obtained then any vector fields can be plugged in.

6.2 Derivatives of *n*-forms

Before jumping to the general case, let's look at the derivative of a 2-form. A 2-form, ω, acts on a pair of vector fields, V and W, to return a function. To find a variation of ω we can examine how this function varies in the direction of a third vector, U, at some point p. Hence, whatever $d\omega$ turns out to be, it will be a function of the vectors U, V, and W at each point p. So, we would like to define it to be a 3-form.

Let's start by looking at the variation of $\omega(V,W)$ in the direction of U. We write this as $\nabla_U \omega(V,W)$. If we were to define this as the value of $d\omega(U,V,W)$, we would find that, in general, it would not be alternating. That is, usually $\nabla_U \omega(V,W) \neq -\nabla_V \omega(U,W)$. To remedy this, we simply define $d\omega$ to be the alternating sum of *all* the variations:

$$d\omega(U,V,W) = \nabla_U \omega(V,W) - \nabla_V \omega(U,W) + \nabla_W \omega(U,V).$$

We leave it to the reader to check that $d\omega$ is alternating and multilinear (assuming U, V, and W are constant vector fields).

It should not be hard for the reader to now jump to the general case. Suppose ω is an n-form and $V^1, ..., V^{n+1}$ are $n+1$ vector fields. Then we define

$$d\omega(V^1, ..., V^{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}\nabla_{V^i}\ \omega(V^1, ..., V^{i-1}, V^{i+1}, ..., V^{n+1}).$$

In other words, $d\omega$, applied to $n+1$ vectors, is the alternating sum of the variations of ω applied to n of those vectors in the direction of the remaining one. Note that we can think of "d" as an operator which takes n-forms to $(n+1)$-forms.

6.4. Show that $d\omega$ is alternating.

6.5. Show that $d(\omega + \nu) = d\omega + d\nu$ and $d(c\omega) = cd\omega$, for any constant c.

6.6. Suppose $\omega = f(x, y, z)\ dx \wedge dy + g(x, y, z)\ dy \wedge dz + h(x, y, z)\ dx \wedge dz$. Find $d\omega$ (*Hint*: Compute $d\omega(\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle)$). Compute $d(x^2y\ dx \wedge dy + y^2z\ dy \wedge dz)$.

6.3 Interlude: 0-forms

Let's go back to Section 4.1, when we introduced coordinates for vectors. At that time, we noted that if C was the graph of the function $y = f(x)$ and p was a point of C, then the tangent line to C at p lies in $T_p\mathbb{R}^2$ and has equation $dy = m\ dx$, for some constant, m. Of course, if $p = (x_0, y_0)$, then m is just the derivative of f evaluated at x_0.

Now, suppose we had looked at the graph of a function of 2-variables, $z = f(x, y)$, instead. At some point, $p = (x_0, y_0, z_0)$, on the graph we could look at the tangent plane, which lies in $T_p\mathbb{R}^3$. Its equation is $dz = m_1dx + m_2dy$. Since $z = f(x, y)$, $m_1 = \frac{\partial f}{\partial x}(x_0, y_0)$ and $m_2 = \frac{\partial f}{\partial y}(x_0, y_0)$, we can rewrite this as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Notice that the right-hand side of this equation is a differential 1-form. This is a bit strange; we applied the "d" operator to something and the result was a 1-form. However, we know that when we apply the "d" operator to a differential n-form we get a differential $(n+1)$-form. So, it must be that $f(x, y)$ is a *differential 0-form* on $\mathbb{R}^2$!

In retrospect, this should not be so surprising. After all, the input to a differential n-form on $\mathbb{R}^m$ is a point, and n vectors based at that point. So, the input to a differential 0-form should be a point of $\mathbb{R}^m$, and *no* vectors. In other words, a 0-form on $\mathbb{R}^m$ is just another word for a real-valued function on $\mathbb{R}^m$.

Let's extend some of the things we can do with forms to 0-forms. Suppose f is a 0-form, and ω is an n-form (where n may also be 0). What do we mean by $f \wedge \omega$? Since the wedge product of an n-form and an m-form is an $(n+m)$-form, it must be that $f \wedge \omega$ is an n form. It is hard to think of any other way to define this as just the product, $f\omega$.

What about integration? Remember that we integrate n-forms over subsets of $\mathbb{R}^m$ that can be parameterized by a subset of $\mathbb{R}^n$. So 0-forms get integrated over things parameterized by $\mathbb{R}^0$. In other words, we integrate a 0-form over a *point*. How do we do this? We do the simplest possible thing; define the value of a 0-form, f, integrated over the point, p, to be $\pm f(p)$. To specify an orientation we just need to say whether or not to use the $-$ sign. We do this just by writing "$-p$" instead of "p" when we want the integral of f over p to be $-f(p)$.

One word of caution here...beware of orientations! If $p \in \mathbb{R}^n$, then we use the notation "$-p$" to denote p with the negative orientation. So if $p = -3 \in \mathbb{R}^1$, then $-p$ is not the same as the point, 3. $-p$ is just the point, -3, with a negative orientation. So, if $f(x) = x^2$, then $\int_{-p} f = -f(p) = -9$.

6.7. If f is the 0-form x^2y^3, p is the point $(-1, 1)$, q is the point $(1, -1)$, and r is the point $(-1, -1)$, then compute the integral of f over the points $-p$, $-q$, and $-r$, with the indicated orientations.

Let's go back to our exploration of derivatives of n-forms. Suppose $f(x, y)\, dx$ is a 1-form on $\mathbb{R}^2$. Then we have already shown that $d(f\, dx) = \frac{\partial f}{\partial y}\, dy \wedge dx$. We now compute:

$$\begin{aligned} df \wedge dx &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) \wedge dx \\ &= \frac{\partial f}{\partial x}dx \wedge dx + \frac{\partial f}{\partial y}dy \wedge dx \\ &= \frac{\partial f}{\partial y}dy \wedge dx \\ &= d(f\, dx). \end{aligned}$$

6.8. If f is a 0-form, show that $d(f\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n) = df \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$.

6.9. Prove: $d(d\omega) = 0$.

6.10. If ω is an n-form, and μ is an m-form, then show that $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^n \omega \wedge d\mu$.

6.4 Algebraic computation of derivatives

As in Section 4.7 we break with the spirit of the text to list the identities we have acquired, and work a few examples.

Let ω be an n-form, μ an m-form, and f a 0-form. Then we have the following identities:

$$\begin{aligned}
d(d\omega) &= 0\\
d(\omega+\mu) &= d\omega + d\mu\\
d(\omega\wedge\mu) &= d\omega\wedge\mu + (-1)^n\omega\wedge d\mu\\
d(f\,dx_1\wedge dx_2\wedge\ldots\wedge dx_n) &= df\wedge dx_1\wedge dx_2\wedge\ldots\wedge dx_n\\
df &= \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n.
\end{aligned}$$

Example 29.

$$\begin{aligned}
&d\left(xy\,dx - xy\,dy + xy^2z^3\,dz\right)\\
&\quad= d(xy)\wedge dx - d(xy)\wedge dy + d(xy^2z^3)\wedge dz\\
&\quad= (y\,dx + x\,dy)\wedge dx - (y\,dx + x\,dy)\wedge dy\\
&\qquad+(y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz)\wedge dz\\
&\quad= \cancel{y\,dx\wedge dx} + x\,dy\wedge dx - y\,dx\wedge dy - \cancel{x\,dy\wedge dy}\\
&\qquad+y^2z^3\,dx\wedge dz + 2xyz^3\,dy\wedge dz + \cancel{3xy^2z^2\,dz\wedge dz}\\
&\quad= x\,dy\wedge dx - y\,dx\wedge dy + y^2z^3\,dx\wedge dz + 2xyz^3\,dy\wedge dz\\
&\quad= -x\,dx\wedge dy - y\,dx\wedge dy + y^2z^3\,dx\wedge dz + 2xyz^3\,dy\wedge dz\\
&\quad= (-x-y)\,dx\wedge dy + y^2z^3\,dx\wedge dz + 2xyz^3\,dy\wedge dz.
\end{aligned}$$

Example 30.

$$\begin{aligned}
&d\left(x^2(y+z^2)\,dx\wedge dy + z(x^3+y)\,dy\wedge dz\right)\\
&\quad= d(x^2(y+z^2))\wedge dx\wedge dy + d(z(x^3+y))\wedge dy\wedge dz\\
&\quad= 2x^2z\,dz\wedge dx\wedge dy + 3x^2z\,dx\wedge dy\wedge dz\\
&\quad= 5x^2z\,dx\wedge dy\wedge dz.
\end{aligned}$$

6.11. For each differential n-form, ω, find $d\omega$.

1. $\sin y\,dx + \cos x\,dy$.
2. $xy^2\,dx + x^3z\,dy - (y+z^9)\,dz$.
3. $xy^2\,dy\wedge dz + x^3z\,dx\wedge dz - (y+z^9)\,dx\wedge dy$.
4. $x^2y^3z^4\,dx\wedge dy\wedge dz$.

6.12. If f is the 0-form x^2y^3 and ω is the 1-form $x^2z\,dx + y^3z^2\,dy$ (on $\mathbb{R}^3$), then use the identity $d(f\,d\omega) = df\wedge d\omega$ to compute $d(f\,d\omega)$.

6.13. Let f, g and h be functions from $\mathbb{R}^3$ to $\mathbb{R}$. If

$$\omega = f\, dy \wedge dz - g\, dx \wedge dz + h\, dx \wedge dy,$$

then compute $d\omega$.

6.5 Antiderivatives

Just as in single-variable calculus it will be helpful to have some proficiency in recognizing antiderivatives. Nothing substitutes for practice...

6.14. Find forms whose derivatives are

1. $dx \wedge dy$.
2. $dx \wedge dy \wedge dz$.
3. $yz\, dx + xz\, dy + xy\, dz$.
4. $y^2z^2\, dx + 2xyz^2\, dy + 2xy^2z\, dz$.
5. $(y^2 - 2xy)\cos(xy^2)\, dx \wedge dy$.

6.15. Show that $\omega = xy^2\, dx$ is not the derivative of any 0-form. (*Hint*: consider $d\omega$.)

7

Stokes' Theorem

7.1 Cells and chains

Up until now, we have not been very specific as to the types of subsets of $\mathbb{R}^m$ on which one integrates a differential n-form. All we have needed is a subset that can be parameterized by a region in $\mathbb{R}^n$. To go further we need to specify the types of regions.

Definition 1. *Let $I = [0, 1]$. An n-cell, σ, is the image of a differentiable map, $\phi : I^n \to \mathbb{R}^m$, with a specified orientation. We denote the same cell with opposite orientation as $-\sigma$. We define a 0-cell to be an oriented point of $\mathbb{R}^m$.*

Example 31. Suppose $g_1(x)$ and $g_2(x)$ are functions such that $g_1(x) < g_2(x)$ for all $x \in [a, b]$. Let R denote the subset of $\mathbb{R}^2$ bounded by the graphs of the equations $y = g_1(x)$ and $y = g_2(x)$, and by the lines $x = a$ and $x = b$. In Example 13, we showed that R is a 2-cell (assuming the induced orientation).

We would like to treat cells as algebraic objects which can be added and subtracted. But if σ is a cell, it may not at all be clear what "2σ" represents. One way to think about it is as two copies of σ, placed right on top of each other.

Definition 2. *An n-chain is a formal linear combination of n-cells.*

As one would expect, we assume the following relations hold:

$$\sigma - \sigma = \emptyset$$
$$n\sigma + m\sigma = (n + m)\sigma$$
$$\sigma + \tau = \tau + \sigma$$

You may be able to guess what the integral of an n-form, ω, over an n-chain is. Suppose $C = \sum n_i \sigma_i$. Then we define

$$\int_C \omega = \sum_i n_i \int_{\sigma_i} \omega.$$

7.1. If f is the 0-form x^2y^3, p is the point $(-1, 1)$, q is the point $(1, -1)$, and r is the point $(-1, -1)$, then compute the integral of f over the following 0-chains:

1. $p - q$; $r - p$.
2. $p + q - r$.

Another concept that will be useful for us is the *boundary* of an n-chain. As a warm-up, we define the boundary of a 1-cell. Suppose σ is the 1-cell which is the image of $\phi : [0, 1] \to \mathbb{R}^m$ with the induced orientation. Then we define the boundary of σ (which we shall denote "$\partial\sigma$") as the 0-chain, $\phi(1) - \phi(0)$. We can represent this pictorially as in Figure 7.1.

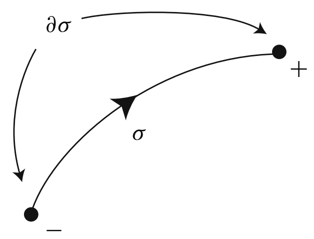

Fig. 7.1. Orienting the boundary of a 1-cell.

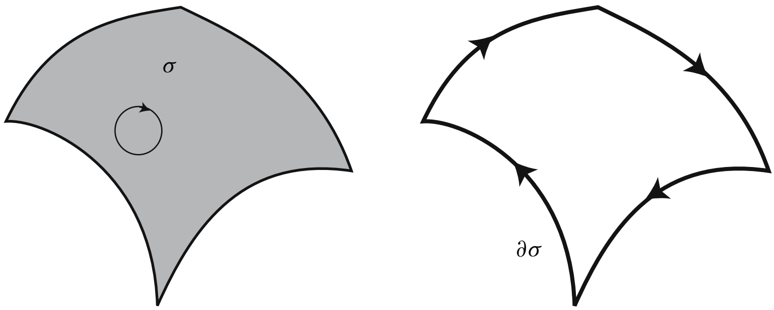

Fig. 7.2. The boundary of a 2-cell.

Figure 7.2 depicts a 2-cell and its boundary. Notice that the boundary consists of four individually oriented 1-cells. This hints at the general formula. In general, if the n-cell σ is the image of the parameterization $\phi : I^n \to \mathbb{R}^m$ with the induced orientation then

$$\partial \sigma = \sum_{i=1}^{n} (-1)^{i+1} \left(\phi|_{(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)} - \phi|_{(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)} \right).$$

So, if σ is a 2-cell, then

$$\begin{aligned} \partial \sigma &= (\phi(1, x_2) - \phi(0, x_2)) - (\phi(x_1, 1) - \phi(x_1, 0)) \\ &= \phi(1, x_2) - \phi(0, x_2) - \phi(x_1, 1) + \phi(x_1, 0). \end{aligned}$$

The four terms on the right side of this equality are the four 1-cells depicted in Figure 7.2. The signs in front of these terms guarantee that the orientations are as pictured.

If σ is a 3-cell, then

$$\begin{aligned} \partial \sigma &= (\phi(1, x_2, x_3) - \phi(0, x_2, x_3)) - (\phi(x_1, 1, x_3) - \phi(x_1, 0, x_3)) \\ &\quad + (\phi(x_1, x_2, 1) - \phi(x_1, x_2, 0)) \\ &= \phi(1, x_2, x_3) - \phi(0, x_2, x_3) - \phi(x_1, 1, x_3) + \phi(x_1, 0, x_3) \\ &\quad + \phi(x_1, x_2, 1) - \phi(x_1, x_2, 0). \end{aligned}$$

An example will hopefully clear up the confusion this all was sure to generate:

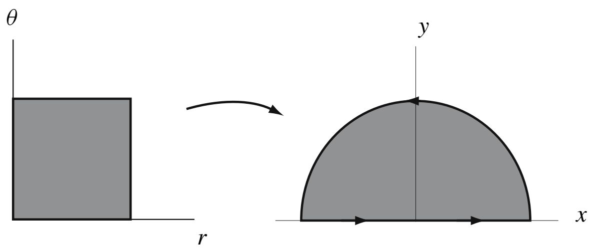

Fig. 7.3. Orienting the boundary of a 2-cell.

Example 32. Suppose $\phi(r, \theta) = (r \cos \pi\theta, r \sin \pi\theta)$. The image of ϕ is the 2-cell, σ, depicted in Figure 7.3. By the above definition,

$$\begin{aligned} \partial \sigma &= (\phi(1, \theta) - \phi(0, \theta)) - (\phi(r, 1) - \phi(r, 0)) \\ &= (\cos \pi\theta, \sin \pi\theta) - (0, 0) + (r, 0) - (-r, 0). \end{aligned}$$

This is the 1-chain depicted in Figure 7.3.

Finally, we are ready to define what we mean by the boundary of an n-chain. If $C = \sum n_i \sigma_i$, then we define $\partial C = \sum n_i \partial \sigma_i$.

Example 33. Suppose

$$\phi_1(r, \theta) = (r \cos 2\pi\theta, r \sin 2\pi\theta, \sqrt{1 - r^2}),$$

$$\phi_2(r, \theta) = (-r \cos 2\pi\theta, r \sin 2\pi\theta, -\sqrt{1 - r^2}),$$

$\sigma_1 = Im(\phi_1)$ and $\sigma_2 = Im(\phi_2)$. Then $\sigma_1 + \sigma_2$ is a sphere in $\mathbb{R}^3$. One can check that $\partial(\sigma_1 + \sigma_2) = \emptyset$.

7.2. If σ is an n-cell, show that $\partial\partial\sigma = \emptyset$. (At least show this if σ is a 2-cell and a 3-cell. The 2-cell case can be deduced pictorially from Figures 7.1 and 7.2.)

7.3. If σ is given by the parameterization

$$\phi(r, \theta) = (r \cos\theta, r \sin\theta)$$

for $0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{4}$, then what is $\partial\sigma$?

7.4. If σ is given by the parameterization

$$\phi(r, \theta) = (r \cos\theta, r \sin\theta, r)$$

for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, then what is $\partial\sigma$?

7.2 The generalized Stokes' Theorem

In calculus, we learn that when you take a function, differentiate it, and then integrate the result, something special happens. In this section, we explore what happens when we take a form, differentiate it, and then integrate the resulting form over some chain. The general argument is quite complicated, so we start by looking at forms of a particular type integrated over very special regions.

Suppose $\omega = a\, dx_2 \wedge dx_3$ is a 2-form on $\mathbb{R}^3$, where $a : \mathbb{R}^3 \to \mathbb{R}$. Let R be the unit cube, $I^3 \subset \mathbb{R}^3$. We would like to explore what happens when we integrate $d\omega$ over R. Note first that Problem 6.8 implies that $d\omega = \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$.

Recall the steps used to define $\int\limits_R d\omega$:

1. Choose a lattice of points in R, $\{p_{i,j,k}\}$. Since R is a cube, we can choose this lattice to be rectangular.
2. Define $V^1_{i,j,k} = p_{i+1,j,k} - p_{i,j,k}$. Similarly, define $V^2_{i,j,k}$ and $V^3_{i,j,k}$.
3. Compute $d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k})$.
4. Sum over all i, j and k.
5. Take the limit as the maximal distance between adjacent lattice points goes to zero.

Let's focus on Step 3 for a moment. Let t be the distance between $p_{i+1,j,k}$ and $p_{i,j,k}$, and assume t is small. Then $\frac{\partial a}{\partial x_1}(p_{i,j,k})$ is approximately equal to $\frac{a(p_{i+1,j,k})-a(p_{i,j,k})}{t}$. This approximation gets better and better when we let $t \to 0$, in Step 5.

The vectors, $V^1_{i,j,k}$ through $V^3_{i,j,k}$, form a little cube. If we say the vector $V^1_{i,j,k}$ is "vertical," and the other two are horizontal, then the "height" of this cube is t, and the area of its base is $dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$, which makes its volume $t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k})$. Putting all this together, we find that

$$\begin{aligned} d\omega_{p_{i,j,k}}(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) &= \frac{\partial a}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3(V^1_{i,j,k}, V^2_{i,j,k}, V^2_{i,j,k}) \\ &\approx \frac{a(p_{i+1,j,k}) - a(p_{i,j,k})}{t} t\ dx_2 \wedge dx_3(V^2_{i,j,k}, V^3_{i,j,k}) \\ &= \omega(V^2_{i+1,j,k}, V^3_{i+1,j,k}) - \omega(V^2_{i,j,k}, V^3_{i,j,k}). \end{aligned}$$

Let's move on to Step 4. Here we sum over all i, j and k. Suppose for the moment that i ranges between 1 and N. First, we fix j and k, and sum over all i. The result is $\omega(V^2_{N,j,k}, V^3_{N,j,k}) - \omega(V^2_{1,j,k}, V^3_{1,j,k})$. Now notice that $\sum\limits_{j,k} \omega(V^2_{N,j,k}, V^3_{N,j,k})$ is a Riemann sum for the integral of ω over the "top" of R, and $\sum\limits_{j,k} \omega(V^2_{1,j,k}, V^3_{1,j,k})$ is a Riemann sum for ω over the "bottom" of R. Lastly, note that ω, evaluated on any pair of vectors which lie in the sides of the cube, gives zero. Hence, the integral of ω over a side of R is zero. Putting all this together, we conclude:

$$\int\limits_R d\omega = \int\limits_{\partial R} \omega. \tag{7.1}$$

7.5. Prove that Equation 7.1 holds if $\omega = b\ dx_1 \wedge dx_3$, or if $\omega = c\ dx_1 \wedge dx_2$. **Caution!** Beware of signs and orientations.

7.6. Use the previous problem to conclude that if $\omega = a\ dx_2 \wedge dx_3 + b\ dx_1 \wedge dx_3 + c\ dx_1 \wedge dx_2$ is an arbitrary 2-form on $\mathbb{R}^3$, then Equation 7.1 holds.

7.7. If ω is an arbitrary $(n-1)$-form on $\mathbb{R}^n$ and R is the unit cube in $\mathbb{R}^n$, then show that Equation 7.1 still holds.

In general, if $C = \sum n_i \sigma_i$ is an n-chain, then

$$\boxed{\int\limits_{\partial C} \omega = \int\limits_C d\omega.}$$

This equation is called the *generalized Stokes' Theorem*. It is unquestionably the most crucial result of this text. In fact, everything we have done up to this point has been geared toward developing this equation and everything that follows will be applications of this equation. Technically, we have only established this theorem when integrating over cubes and their boundaries. We postpone the general proof to Section 9.1.

Example 34. Let $\omega = x\ dy$ be a 1-form on $\mathbb{R}^2$. Let σ be the 2-cell which is the image of the parameterization $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$, where $0 \le r \le R$ and $0 \le \theta \le 2\pi$. By the generalized Stokes' Theorem,

$$\int\limits_{\partial\sigma} \omega = \int\limits_{\sigma} d\omega = \int\limits_{\sigma} dx \wedge dy = \int\limits_{\sigma} dx\ dy = Area(\sigma) = \pi R^2.$$

7.8. Verify directly that $\int\limits_{\partial\sigma} \omega = \pi R^2$.

Example 35. Let $\omega = x\ dy + y\ dx$ be a 1-form on $\mathbb{R}^2$, and let σ be any 2-cell. Then $\int\limits_{\partial\sigma} \omega = \int\limits_{\sigma} d\omega = 0$.

7.9. Find a 1-chain in $\mathbb{R}^2$, which bounds a 2-cell, and integrate the form $x\ dy + y\ dx$ over this curve.

7.10. Let ω be a differential $(n-1)$-form and σ a $(n+1)$-cell. Use the generalized Stokes' Theorem in **two** different ways to show $\int\limits_{\partial\sigma} d\omega = 0$.

Example 36. Let C be the curve in $\mathbb{R}^2$ parameterized by $\phi(t) = (t^2, t^3)$, where $-1 \le t \le 1$. Let f be the 0-form x^2y. We use the generalized Stokes' Theorem to calculate $\int\limits_{C} df$.

The curve C goes from the point (1,-1) , when $t = -1$, to the point (1,1), when $t = 1$. Hence, ∂C is the 0-chain $(1,1) - (1,-1)$. Now we use Stokes:

$$\int\limits_{C} df = \int\limits_{\partial C} f = \int\limits_{(1,1)-(1,-1)} x^2y = 1 - (-1) = 2.$$

7.11. Calculate $\int\limits_{C} df$ directly.

7.12. Let C be any curve in $\mathbb{R}^3$ from $(0,0,0)$ to $(1,1,1)$. Let $\omega = y^2z^2\ dx + 2xyz^2\ dy + 2xy^2z\ dz$. Calculate $\int\limits_{C} \omega$.

Example 37. Let $\omega = (x^2 + y)dx + (x - y^2)dy$ be a 1-form on $\mathbb{R}^2$. We wish to integrate ω over σ, the top half of the unit circle, oriented clockwise. First, note that $d\omega = 0$, so that if we integrate ω over the boundary of any 2-cell, we would get zero. Let τ denote the line segment connecting (-1,0) to (1,0). Then the 1-chain $\sigma - \tau$ bounds a 2-cell. So $\int\limits_{\sigma-\tau} \omega = 0$, which implies that $\int\limits_{\sigma} \omega = \int\limits_{\tau} \omega$. This latter integral is a bit easier to compute. Let $\phi(t) = (t, 0)$ be a parameterization of τ, where $-1 \le t \le 1$. Then

$$\int\limits_{\sigma} \omega = \int\limits_{\tau} \omega = \int\limits_{[-1,1]} \omega_{(t,0)}(\langle 1, 0\rangle)\, dt = \int\limits_{-1}^{1} t^2\, dt = \frac{2}{3}.$$

7.13. Let $\omega = -y^2\, dx + x^2\, dy$. Let σ be the 2-cell in $\mathbb{R}^2$ parameterized by the following:

$$\phi(u, v) = (2u - v, u + v),\ 1 \le u \le 2,\ 0 \le v \le 1.$$

Calculate $\int\limits_{\partial\sigma} \omega$.

7.14. Let $\omega = dx - \ln x\, dy$. Let σ be the 2-cell parameterized by the following:

$$\phi(u, v) = (uv^2, u^3 v),\ 1 \le u \le 2,\ 1 \le v \le 2.$$

Calculate: $\int\limits_{\partial\sigma} \omega$.

7.15. Let σ be the 2-cell given by the following parameterization:

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta),\ 0 \le r \le 1,\ 0 \le \theta \le \pi.$$

Suppose $\omega = x^2\, dx + e^y\, dy$.

1. Calculate $\int\limits_{\sigma} d\omega$ directly.
2. Let C_1 be the horizontal segment connecting $(-1, 0)$ to $(0, 0)$, and C_2 be the horizontal segment connecting $(0, 0)$ to $(1, 0)$. Calculate $\int\limits_{C_1} \omega$ and $\int\limits_{C_1} \omega$ directly.
3. Use your previous answers to determine the integral of ω over the top half of the unit circle (oriented counter-clockwise).

7.16. Let $\omega = (x + y^3)\, dx + 3xy^2\, dy$ be a differential 1-form on $\mathbb{R}^2$. Let Q be the rectangle $\{(x, y) | 0 \le x \le 3, 0 \le y \le 2\}$.

1. Compute $d\omega$.
2. Use the generalized Stokes' Theorem to compute $\int\limits_{\partial Q} \omega$.

3. Compute $\int_{\partial Q} \omega$ directly, by integrating ω over each each edge of the boundary of the rectangle, and then adding in the appropriate manner.
4. How does $\int_{R-T-L} \omega$ compare to $\int_B \omega$?
5. Let S be any curve in the upper half plane (i.e., the set $\{(x, y)|y \geq 0\}$) that goes from the point $(3, 0)$ to the point $(0, 0)$. What is $\int_S \omega$? Why?
6. Let S be *any* curve that goes from the point $(3, 0)$ to the point $(0, 0)$. What is $\int_S \omega$? WHY?

7.17. Let ω be the following 2-form on $\mathbb{R}^3$:

$$\omega = (x^2 + y^2)dy \wedge dz + (x^2 - y^2)dx \wedge dz.$$

Let V be the region of $\mathbb{R}^3$ bounded by the graph of $y = \sqrt{1 - x^2}$, the planes $z = 0$ and $z = 2$, and the xz-plane (see Figure 7.4).

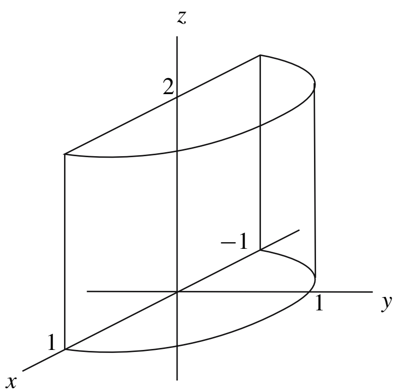

Fig. 7.4. The region V of Problem 7.17.

1. Parameterize V using cylindrical coordinates.
2. Determine $d\omega$.
3. Calculate $\int_V d\omega$.
4. The sides of V are parameterized as follows:
   a) Bottom: $\phi_B(r, \theta) = (r \cos\theta, r \sin\theta, 0)$, where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$.
   b) Top: $\phi_T(r, \theta) = (r \cos\theta, r \sin\theta, 2)$, where $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$.
   c) Flat side: $\phi_F(x, z) = (x, 0, z)$, where $-1 \leq x \leq 1$ and $0 \leq z \leq 2$.
   d) Curved side: $\phi_C(\theta, z) = (\cos\theta, \sin\theta, z)$, where $0 \leq \theta \leq \pi$ and $0 \leq z \leq 2$.

Calculate the integral of ω over the top, bottom and flat side. (Do not calculate this integral over the curved side.)

5. If C is the curved side of ∂V, use your answers to the previous questions to determine $\int_C \omega$.

7.18. Calculate the volume of a ball of radius one, $\{(\rho, \theta, \phi) | \rho \leq 1\}$, by integrating some 2-form over the sphere of radius one, $\{(\rho, \theta, \phi) | \rho = 1\}$.

7.19. Calculate

$$\int_C x^3 \, dx + \left(\frac{1}{3}x^3 + xy^2\right) dy$$

where C is the circle of radius two, centered about the origin.

7.20. Suppose $\omega = x\,dx + x\,dy$ is a 1-form on $\mathbb{R}^2$. Let C be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Determine the value of $\int_C \omega$ by integrating some 2-form over the region bounded by the ellipse.

7.21. Let $\omega = -y^2\,dx + x^2\,dy$. Let σ be the 2-cell in $\mathbb{R}^2$ parameterized by the following:

$$\phi(r, \theta) = (r \cosh\theta, r \sinh\theta)$$

where $0 \leq r \leq 1$ and $-1 \leq \theta \leq 1$. Calculate $\int_{\partial\sigma} \omega$.

$$\left(\text{Recall: } \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.\right)$$

7.22. Suppose ω is a 1-form on $\mathbb{R}^2$ such that $d\omega = 0$. Let C_1 and C_2 be the 1-cells given by the following parameterizations:

$$C_1 : \phi(t) = (t, 0), \ 2\pi \leq t \leq 6\pi$$
$$C_2 : \psi(t) = (t\cos t, t\sin t), \ 2\pi \leq t \leq 6\pi.$$

Show that $\int_{C_1} \omega = \int_{C_2} \omega$.

(Caution: Beware of orientations!)

7.3 Vector calculus and the many faces of the generalized Stokes' Theorem

Although the language and notation may be new, you have already seen the generalized Stokes' Theorem in many guises. For example, let $f(x)$ be a 0-form on

$\mathbb{R}$. Then $df = f'(x)dx$. Let $[a,b]$ be a 1-cell in $\mathbb{R}$. Then the generalized Stokes' Theorem tells us

$$\int_a^b f'(x)dx = \int_{[a,b]} f'(x)dx = \int_{\partial[a,b]} f(x) = \int_{b-a} f(x) = f(b) - f(a),$$

which is, of course, the "Fundamental Theorem of Calculus." If we let R be some 2-chain in $\mathbb{R}^2$ then the generalized Stokes' Theorem implies

$$\int_{\partial R} P\,dx + Q\,dy = \int_R d(P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

This is what we call "Green's theorem" in calculus. To proceed further, we restrict ourselves to $\mathbb{R}^3$. In this dimension, there is a nice correspondence between vector fields and both 1- and 2-forms.

$$\begin{aligned}\mathbf{F} = \langle F_x, F_y, F_z\rangle &\leftrightarrow \omega^1_{\mathbf{F}} = F_x dx + F_y dy + F_z dz\\ &\leftrightarrow \omega^2_{\mathbf{F}} = F_x dy \wedge dz - F_y dx \wedge dz + F_z dx \wedge dy.\end{aligned}$$

On $\mathbb{R}^3$ there is also a useful correspondence between 0-forms (functions) and 3-forms.

$$f(x,y,z) \leftrightarrow \omega^3_f = f\,dx \wedge dy \wedge dz.$$

We can use these correspondences to define various operations involving functions and vector fields. For example, suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a 0-form. Then df is the 1-form, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. The vector field associated to this 1-form is then $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. In calculus we call this vector field **grad** f, or ∇f. In other words, ∇f is the vector field associated with the 1-form, df. This can be summarized by the equation

$$df = \omega^1_{\nabla f}.$$

It will be useful to think of this as a diagram as well.

$$\begin{array}{ccc} f & \xrightarrow{\textbf{grad}} & \nabla f \\ \| & & \uparrow \\ f & \xrightarrow[d]{} & df \end{array}$$

Example 38. Suppose $f = x^2y^3z$. Then $df = 2xy^3z\,dx + 3x^2y^2z\,dy + x^3y^3\,dz$. The associated vector field, **grad** f, is then $\nabla f = \langle 2xy^3z, 3x^2y^2z, x^3y^3\rangle$.

Similarly, if we start with a vector field, $\mathbf{F}$, form the associated 1-form, $\omega^1_{\mathbf{F}}$, differentiate it, and look at the corresponding vector field, then the result is called **curl F**, or $\nabla\times\mathbf{F}$. So, $\nabla\times\mathbf{F}$ is the vector field associated with the 2-form, $d\omega^1_{\mathbf{F}}$. This can be summarized by the equation

$$d\omega^1_{\mathbf{F}} = \omega^2_{\nabla\times\mathbf{F}}.$$

This can also be illustrated by the following diagram.

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\textbf{curl}} & \nabla\times\mathbf{F} \\ \downarrow & & \uparrow \\ \omega^1_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^1_{\mathbf{F}} \end{array}$$

Example 39. Let $\mathbf{F} = \langle xy, yz, x^2\rangle$. The associated 1-form is then

$$\omega^1_{\mathbf{F}} = xy\,dx + yz\,dy + x^2\,dz.$$

The derivative of this 1-form is the 2-form

$$d\omega^1_{\mathbf{F}} = -y\,dy\wedge dz + 2x\,dx\wedge dz - x\,dx\wedge dy.$$

The vector field associated to this 2-form is **curl F**, which is

$$\nabla\times\mathbf{F} = \langle -y, -2x, -x\rangle.$$

Lastly, we can start with a vector field, $\mathbf{F} = \langle F_x, F_y, F_z\rangle$, and then look at the 3-form, $d\omega^2_{\mathbf{F}} = (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z})dx\wedge dy\wedge dz$ (see Problem 6.13). The function, $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ is called **div F**, or $\nabla\cdot\mathbf{F}$. This is summarized in the following equation and diagram.

$$d\omega^2_{\mathbf{F}} = \omega^3_{\nabla\cdot\mathbf{F}}$$

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\textbf{div}} & \nabla\cdot\mathbf{F} \\ \downarrow & & \uparrow \\ \omega^2_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^2_{\mathbf{F}} \end{array}$$

Example 40. Let $\mathbf{F} = \langle xy, yz, x^2\rangle$. The associated 2-form is then

$$\omega^2_{\mathbf{F}} = xy\,dy\wedge dz - yz\,dx\wedge dz + x^2\,dx\wedge dy.$$

The derivative is the 3-form

$$d\omega^2_{\mathbf{F}} = (y+z)\,dx\wedge dy\wedge dz.$$

So **div F** is the function $\nabla\cdot\mathbf{F} = y + z$.

Two important vector identities follow from the fact that for a differential form, ω, calculating $d(d\omega)$ always yields zero (see Problem 6.9). For the first identity, consider the following diagram.

$$\begin{array}{ccccc} f & \xrightarrow{\textbf{grad}} & \nabla f & \xrightarrow{\textbf{curl}} & \nabla \times (\nabla f) \\ \Vert & & \uparrow & & \uparrow \\ f & \xrightarrow[d]{} & df & \xrightarrow[d]{} & ddf \end{array}$$

This shows that if f is a 0-form, then the vector field corresponding to ddf is $\nabla \times (\nabla f)$. But $ddf = 0$, so we conclude

$$\nabla \times (\nabla f) = 0.$$

For the second identity, consider the diagram

$$\begin{array}{ccccc} \mathbf{F} & \xrightarrow{\textbf{curl}} & \nabla \times \mathbf{F} & \xrightarrow{\textbf{div}} & \nabla \cdot (\nabla \times \mathbf{F}) \\ \downarrow & & \downarrow & & \uparrow \\ \omega^1_{\mathbf{F}} & \xrightarrow[d]{} & d\omega^1_{\mathbf{F}} & \xrightarrow[d]{} & dd\omega^1_{\mathbf{F}}. \end{array}$$

This shows that if $dd\omega^1_{\mathbf{F}}$ is written as $g\, dx \wedge dy \wedge dz$, then the function g is equal to $\nabla \cdot (\nabla \times \mathbf{F})$. But $dd\omega^1_{\mathbf{F}} = 0$, so we conclude

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

In vector calculus we also learn how to integrate vector fields over parameterized curves (1-chains) and surfaces (2-chains). Suppose first that σ is some parameterized curve. Then we can integrate the component of $\mathbf{F}$ which points in the direction of the tangent vectors to σ. This integral is usually denoted by $\int\limits_\sigma \mathbf{F} \cdot d\mathbf{s}$, and its definition is precisely the same as the definition we learned here for $\int\limits_\sigma \omega^1_{\mathbf{F}}$. A special case of this integral arises when $\mathbf{F} = \nabla f$, for some function, f. In this case, $\omega^1_{\mathbf{F}}$ is just df, so the definition of $\int\limits_\sigma \nabla f \cdot d\mathbf{s}$ is the same as $\int\limits_\sigma df$.

7.23. Let C be any curve in $\mathbb{R}^3$ from $(0, 0, 0)$ to $(1, 1, 1)$. Let $\mathbf{F}$ be the vector field $\langle yz, xz, xy \rangle$. Show that $\int\limits_C \mathbf{F} \cdot d\mathbf{s}$ does not depend on C.

We also learn to integrate vector fields over parameterized surfaces. In this case, the quantity we integrate is the component of the vector field which is normal to the surface. This integral is often denoted by $\int\limits_S \mathbf{F} \cdot d\mathbf{S}$. Its definition is precisely the same as that of $\int_S \omega^2_{\mathbf{F}}$ (see Problems 4.23 and 4.24). A special case of this is when $\mathbf{F} = \nabla \times \mathbf{G}$, for some vector field, $\mathbf{G}$. Then $\omega^2_{\mathbf{G}}$ is just $d\omega^1_{\mathbf{G}}$, so we see that $\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$ must be the same as $\int_S d\omega^1_{\mathbf{G}}$.

The most basic thing to integrate over a 3-dimensional region (i.e., a 3-chain), Ω, in $\mathbb{R}^3$ is a function $f(x, y, x)$. In calculus we denote this integral as $\int_\Omega f\ dV$. Note that this is precisely the same as $\int_\Omega \omega^3_f$. A special case is when $f = \nabla \cdot \mathbf{F}$, for some vector field $\mathbf{F}$. In this case $\int_\Omega f\ dV = \int_\Omega (\nabla \cdot \mathbf{F}) dV$. But we can write this integral with differential forms as $\int_\Omega d\omega^2_{\mathbf{F}}$.

We summarize the equivalence between the integrals developed in vector calculus and various integrals of differential forms in Table 7.1.

Vector Calculus	Differential Forms
$\int_\sigma \mathbf{F} \cdot d\mathbf{s}$	$\int_\sigma \omega^1_{\mathbf{F}}$
$\int_\sigma \nabla f \cdot d\mathbf{s}$	$\int_\sigma df$
$\int_S \mathbf{F} \cdot d\mathbf{S}$	$\int_S \omega^2_{\mathbf{F}}$
$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$	$\int_S d\omega^1_{\mathbf{F}}$
$\int_\Omega f\ dV$	$\int_\Omega \omega^3_f$
$\int_\Omega (\nabla \cdot \mathbf{F}) dV$	$\int_\Omega d\omega^2_{\mathbf{F}}$

Table 7.1. The equivalence between the integrals of vector calculus and differential forms.

Let us now apply the generalized Stokes' Theorem to various situations. First, we start with a parameterization, $\phi : [a, b] \to \sigma \subset \mathbb{R}^3$, of a curve in $\mathbb{R}^3$, and a function, $f : \mathbb{R}^3 \to \mathbb{R}$. Then we have

$$\int_\sigma \nabla f \cdot d\mathbf{s} \equiv \int_\sigma df = \int_{\partial\sigma} f = f(\phi(b)) - f(\phi(a)).$$

This shows the independence of path of line integrals of gradient fields. We can use this to prove that a line integral of a gradient field over any simple closed curve is 0, but for us there is an easier, direct proof, which again uses the generalized Stokes' Theorem. Suppose σ is a simple closed loop in $\mathbb{R}^3$ (i.e., $\partial\sigma = \emptyset$). Then $\sigma = \partial D$, for some 2-chain, D. We now have

$$\int_{\sigma} \nabla f \cdot d\mathbf{s} \equiv \int_{\sigma} df = \int_{D} ddf = 0.$$

Now, suppose we have a vector field, $\mathbf{F}$, and a parameterized surface, S. Yet another application of the generalized Stokes' Theorem yields

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \equiv \int_{\partial S} \omega^1_{\mathbf{F}} = \int_{S} d\omega^1_{\mathbf{F}} \equiv \int_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

In vector calculus we call this equality "Stokes' theorem." In some sense, $\nabla \times \mathbf{F}$ measures the "twisting" of $\mathbf{F}$ at points of S. So Stokes' theorem says that the net twisting of $\mathbf{F}$ over all of S is the same as the amount $\mathbf{F}$ circulates around ∂S.

Example 41. Suppose we are faced with a problem phrased as: "Use Stokes' theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + 1$, and $\mathbf{F}$ is the vector field $\langle -x^2 y, xy^2, z^3 \rangle$."

We will solve this problem by translating to the language of differential forms, and using the generalized Stokes' Theorem, instead. To begin, note that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \omega^1_{\mathbf{F}}$, and $\omega^1_{\mathbf{F}} = -x^2 y\, dx + xy^2\, dy + z^3\, dz$.

Now, to use the generalized Stokes' Theorem we will need to calculate

$$d\omega^1_{\mathbf{F}} = (x^2 + y^2)\, dx \wedge dy.$$

Let D denote the subset of the plane $z = x + 1$ bounded by C. Then $\partial D = C$. Hence, by the generalized Stokes' Theorem we have

$$\int_{C} \omega^1_{\mathbf{F}} = \int_{D} d\omega^1_{\mathbf{F}} = \int_{D} (x^2 + y^2)\, dx \wedge dy.$$

The region D is parameterized by $\Psi(r, \theta) = (r\cos\theta, r\sin\theta, r\cos\theta + 1)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Using this one can (and should!) show that $\int_D (x^2 + y^2)\, dx \wedge dy = \frac{\pi}{2}$.

7.24. Let C be the square with sides $(x, \pm 1, 1)$, where $-1 \le x \le 1$ and $(\pm 1, y, 1)$, where $-1 \le y \le 1$, with the indicated orientation (see Figure 7.5). Let $\mathbf{F}$ be the vector field $\langle xy, x^2, y^2 z \rangle$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

Suppose now that Ω is some volume in $\mathbb{R}^3$. Then we have

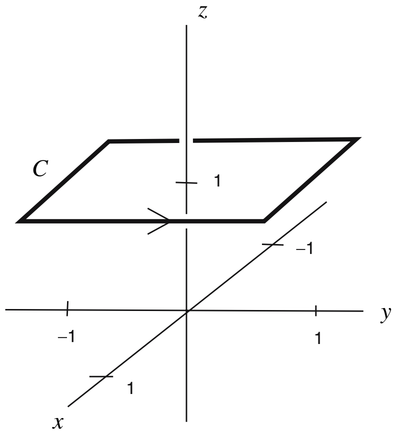

Fig. 7.5.

$$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} \equiv \int_{\partial\Omega} \omega^2_{\mathbf{F}} = \int_{\Omega} d\omega^2_{\mathbf{F}} \equiv \int_{\Omega} (\nabla \cdot \mathbf{F}) dV.$$

This last equality is called "Gauss' Divergence Theorem." $\nabla \cdot \mathbf{F}$ is a measure of how much **F** "spreads out" at a point. So Gauss' theorem says that the total spreading out of **F** inside Ω is the same as the net amount of **F** "escaping" through $\partial\Omega$.

7.25. Let Ω be the cube $\{(x, y, z)|0 \leq x, y, z \leq 1\}$. Let **F** be the vector field $\langle xy^2, y^3, x^2y^2\rangle$. Compute $\int\limits_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$.

8

Applications

8.1 Maxwell's equations

As a brief application, we show how the language of differential forms can greatly simplify the classical vector equations of Maxwell. Much of this material is taken from [MTW73], where the interested student can find many more applications of differential forms to physics.

Maxwell's equations describe the relationship between electric and magnetic fields. Classically, both electricity and magnetism are described as a 3-dimensional vector field which varies with time:

$$\mathbf{E} = \langle E_x, E_y, E_z \rangle$$

$$\mathbf{B} = \langle B_x, B_y, B_z \rangle,$$

where E_x, E_z, E_z, B_x, B_y, and B_z are all functions of x, y, z and t.

Maxwell's equations are then:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} &= -4\pi \mathbf{J}. \end{aligned}$$

The quantity ρ is called the *charge density* and the vector $\mathbf{J} = \langle J_x, J_y, J_z \rangle$ is called the *current density*.

We can make all of this look much simpler by making the following definitions. First, we define a 2-form called the *Faraday*, which simultaneously describes both the electric and magnetic fields:

$$\begin{aligned} \mathbf{F} = {} & E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt \\ & + B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy. \end{aligned}$$

Next we define the "dual" 2-form, called the *Maxwell*:

$$\begin{aligned} {}^*\mathbf{F} = E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy \\ + B_x\, dt \wedge dx + B_y\, dt \wedge dy + B_z\, dt \wedge dz. \end{aligned}$$

We also define the *4-current*, **J**, and its "dual," ${}^*\mathbf{J}$:

$$\begin{aligned} \mathbf{J} &= \langle \rho, J_x, J_y, J_z \rangle \\ {}^*\mathbf{J} &= \rho\, dx \wedge dy \wedge dz \\ &\quad - J_x\, dt \wedge dy \wedge dz \\ &\quad - J_y\, dt \wedge dz \wedge dx \\ &\quad - J_z\, dt \wedge dx \wedge dy. \end{aligned}$$

Maxwell's four vector equations now reduce to:

$$\begin{aligned} d\mathbf{F} &= 0 \\ d^*\mathbf{F} &= 4\pi^*\mathbf{J}. \end{aligned}$$

8.1. Show that the equation $d\mathbf{F} = 0$ implies the first two of Maxwell's equations.

8.2. Show that the equation $d^*\mathbf{F} = 4\pi^*\mathbf{J}$ implies the second two of Maxwell's equations.

The differential form version of Maxwell's equation has a huge advantage over the vector formulation: *it is coordinate free!* A 2-form such as **F** is an operator that "eats" pairs of vectors and "spits out" numbers. The way it acts is completely geometric ... that is, it can be defined without any reference to the coordinate system (t, x, y, z). This is especially poignant when one realizes that Maxwell's equations are laws of nature that should not depend on a man-made construction such as coordinates.

8.2 Foliations and contact structures

Everyone has seen tree rings and layers in sedimentary rock. These are examples of *foliations*. Intuitively, a foliation is when some region of space has been "filled up" with lower-dimensional surfaces. A full treatment of foliations is a topic for a much larger textbook than this one. Here we will only be discussing foliations of $\mathbb{R}^3$.

Let U be an open subset of $\mathbb{R}^3$. We say U has been *foliated* if there is a family $\phi^t : R_t \to U$ of parameterizations (where for each t the domain $R_t \subset \mathbb{R}^2$) such that every point of U is in the image of exactly one such parameterization. In other words, the images of the parameterizations ϕ^t are surfaces that fill up U, and no two overlap.

Suppose p is a point of U and U has been foliated as above. Then there is a unique value of t such that p is a point in $\phi^t(R_t)$. The partial derivatives, $\frac{\partial \phi^t}{\partial x}(p)$ and $\frac{\partial \phi^t}{\partial y}(p)$ are then two vectors that span a plane in $T_p\mathbb{R}^3$. Let's call this plane Π_p. In other words, if U is foliated, then at every point p of U we get a plane Π_p in $T_p\mathbb{R}^3$.

The family $\{\Pi_p\}$ is an example of a *plane field*. In general, a plane field is just a choice of a plane in each tangent space which varies smoothly from point to point in $\mathbb{R}^3$. We say a plane field is *integrable* if it consists of the tangent planes to a foliation.

This should remind you a little of first-term calculus. If $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a differentiable function, then at every point p on its graph we get a line in $T_p\mathbb{R}^2$ (see Figure 4.2). If we only know the lines and want the original function, then we integrate.

There is a theorem that says that every *line field* on $\mathbb{R}^2$ is integrable. The question we would like to answer in this section is whether or not this is true of plane fields on $\mathbb{R}^3$. The first step is to figure out how to specify a plane field in some reasonably nice way. This is where differential forms come in. Suppose $\{\Pi_p\}$ is a plane field. At each point p, we can define a line in $T_p\mathbb{R}^3$ (i.e., a *line field*) by looking at the set of all vectors that are perpendicular to Π_p. We can then define a 1-form ω by projecting vectors onto these lines. So, if V_p is a vector in Π_p then $\omega(V_p) = 0$. Another way to say this is that the plane Π_p is the set of all vectors which yield zero when plugged into ω. In shorthand, we write this set as Ker ω ("Ker" comes from the word "Kernel," a term from linear algebra). So all we are saying is that ω is a 1-form such that $\Pi_p = \text{Ker}\,\omega$. This is very convenient. To specify a plane field, all we have to do now is write down a 1-form!

Example 42. Suppose $\omega = dx$. Then, at each point p of $\mathbb{R}^3$, the vectors of $T_p\mathbb{R}^3$ that yield zero when plugged into ω are all those in the $dydz$-plane. Hence, Ker ω is the plane field consisting of all of the $dydz$-planes (one for every point of $\mathbb{R}^3$). It is obvious that this plane field is integrable; at each point p we just have the tangent plane to the plane parallel to the yz-plane through p.

In the above example, note that any 1-form that looks like $f(x, y, z)dx$ defines the same plane field, as long as f is non-zero everywhere. So, knowing something about a plane field (like the assumption that it is integrable) seems like it might not say much about the 1-form ω, since so many different 1-forms give the same plane field. Let's investigate this further.

First, let's see if there is anything special about the derivative of a 1-form that looks like $\omega = f(x, y, z)dx$. This is easy: $d\omega = \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial f}{\partial z}dz \wedge dx$. This is nothing special so far. What about combining this with ω? Let's compute:

$$\omega \wedge d\omega = f(x, y, z)dx \wedge \left(\frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial f}{\partial z}dz \wedge dx\right) = 0.$$

Now that is special! In fact, recall our earlier emphasis on the fact that forms are *coordinate free*. In other words, any computation one can perform with forms will give the same answer regardless of what coordinates are chosen. The wonderful thing about foliations is that near every point you can always choose coordinates so that your foliation looks like planes parallel to the yz-plane. In other words, the above computation is not as special as you might think:

Theorem 2. *If Ker ω is an integrable plane field, then $\omega \wedge d\omega = 0$ at every point of $\mathbb{R}^3$.*

It should be noted that we have only chosen to work in $\mathbb{R}^3$ for ease of visualization. There are higher-dimensional definitions of foliations and plane fields. In general, if the kernel of a 1-form ω defines an integrable plane field then $\omega \wedge d\omega^n = 0$.

Our search for a plane field that is *not* integrable (i.e., not the tangent planes to a foliation) has now been reduced to the search for a 1-form ω for which $\omega \wedge d\omega \neq 0$ *somewhere*. There are many such forms. An easy one is $x\,dy + dz$. We compute:

$$(x\,dy + dz) \wedge d(x\,dy + dz) = (x\,dy + dz) \wedge (dx \wedge dy) = dz \wedge dx \wedge dy.$$

Our answer is quite special. All we needed was a 1-form such that

$$\omega \wedge d\omega \neq 0$$

somewhere. What we found was a 1-form for which $\omega \wedge d\omega \neq 0$ *everywhere*. This means that there is not a single point of $\mathbb{R}^3$ which has a neighborhood in which the planes given by Ker $x\,dy + dz$ are tangent to a foliation. Such a plane field is called a *contact structure*.

At this point you are probably wondering, "What could Ker $x\,dy + dz$ possibly look like?!" It is not so easy to visualize this, but we have tried to give you some indication in Figure 8.1.[1] A good exercise is to stare at this picture long enough to convince yourself that the planes pictured cannot be the tangent planes to a foliation.

We have just seen how we can use differential forms to tell if a plane field is integrable. But one may still wonder if there is more we can say about a 1-form, assuming its kernel is integrable. Let's go back to the expression $\omega \wedge d\omega$. Recall that ω is a 1-form, which makes $d\omega$ a 2-form, and hence $\omega \wedge d\omega$ a 3-form.

A 3-form on $T_p\mathbb{R}^3$ measures the volume of the parallelepiped spanned by three vectors, multiplied by a constant. For example, if $\psi = \alpha \wedge \beta \wedge \gamma$ is a 3-form, then the constant it scales volume by is given by the volume of the parallelepiped

[1] Figure drawn by Stephan Schoenenberger. Taken from *Introductory Lectures on Contact Geometry* by John B. Etnyre.

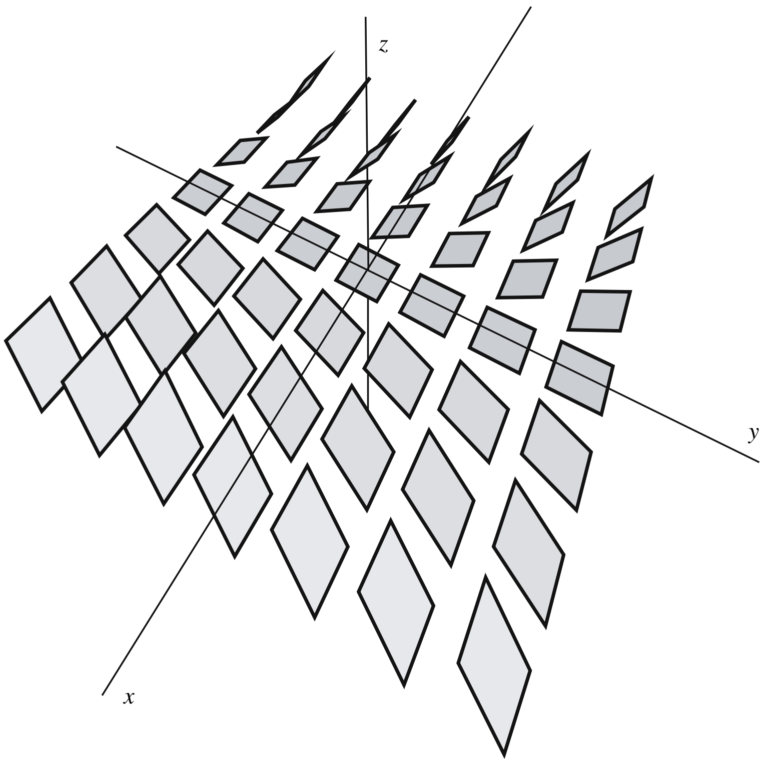

Fig. 8.1. The plane field Ker $x\ dy + dz$.

spanned by the vectors $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\gamma\rangle$ (where "$\langle\alpha\rangle$" refers to the vector dual to the 1-form α introduced in Section 4.3). If it turns out that ψ is the zero 3-form, then the vector $\langle\alpha\rangle$ must be in the plane spanned by the vectors $\langle\beta\rangle$ and $\langle\gamma\rangle$.

On $\mathbb{R}^3$ the results of Section 4.3 tell us that a 2-form such as $d\omega$ can always be written as $\alpha \wedge \beta$, for some 1-forms α and β. If ω is a 1-form with integrable kernel, then we have already seen that $\omega \wedge d\omega = \omega \wedge \alpha \wedge \beta = 0$. But this tells us that $\langle\omega\rangle$ must be in the plane spanned by the vectors $\langle\alpha\rangle$ and $\langle\beta\rangle$. Now we can invoke Lemma 1 of Chapter 4, which says that we can rewrite $d\omega$ as $\omega \wedge \nu$, for some 1-form ν. (See also Problem 4.27.)

If we start with a foliation and choose a 1-form ω whose kernel consists of planes tangent to the foliation, then the 1-form ν that we have just found is in no way canonical. We made a lot of choices to get to ν, and different choices will end up with different 1-forms. But here is the amazing fact: the integral of the 3-form $\nu \wedge d\nu$ does *not* depend on any of our choices! It is completely determined by the original foliation. Whenever a mathematician runs into a situation like this they usually throw up their hands and say, "Eureka! I've discovered an *invariant*." The quantity $\int \nu \wedge d\nu$ is referred to as the *Gobillion–Vey invariant* of the foliation. It is a topic of current research to identify exactly what information this number tells us about the foliation.

Two special cases are worth noting. First, it may turn out that $\nu \wedge d\nu = 0$ everywhere. This tells us that the plane field given by Ker ν is integrable, so we get another foliation. The other interesting case is when $\nu \wedge d\nu$ is nowhere zero. Then we get a contact structure.

8.3 How *not* to visualize a differential 1-form

There are several contemporary physics texts that attempt to give a visual interpretation of differential forms that seems quite different from the one presented here. As this alternate interpretation is much simpler than anything described in these notes, one may wonder why we have not taken this approach.

Let's look again at the 1-form dx on $\mathbb{R}^3$. Given a vector V_p at a point p, the value of $dx(V_p)$ is just the projection of V_p onto the dx axis in $T_p\mathbb{R}^3$. Now, let C be some parameterized curve in $\mathbb{R}^3$ for which the x-coordinate is always increasing. Then $\int\limits_C dx$ is just the length of the projection of C onto the x-axis. To the nearest integer, this is just the number of planes that C punctures of the form $x = n$, where n is an integer. So one way to visualize the form dx is to picture these planes.

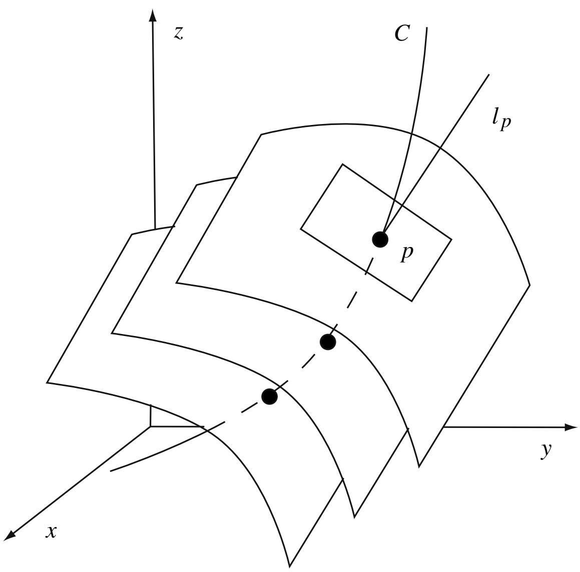

Fig. 8.2. "Surfaces" of ω?

This view is very appealing. After all, every 1-form ω, at every point p, projects vectors onto some line l_p. So can we integrate ω along a curve C (at least to the nearest integer) by counting the number of surfaces punctured by C whose tangent planes are perpendicular to the lines l_p (see Figure 8.2)? If you have read the previous section, you might guess that the answer is a categorical NO!

Recall that the planes perpendicular to the lines l_p are precisely Ker ω. To say that there are surfaces whose tangent planes are perpendicular to the lines l_p is the same thing as saying that Ker ω is an integrable plane field. But we have seen in the previous section that there are 1-forms as simple as $x\,dy + dz$ whose kernels are *nowhere* integrable.

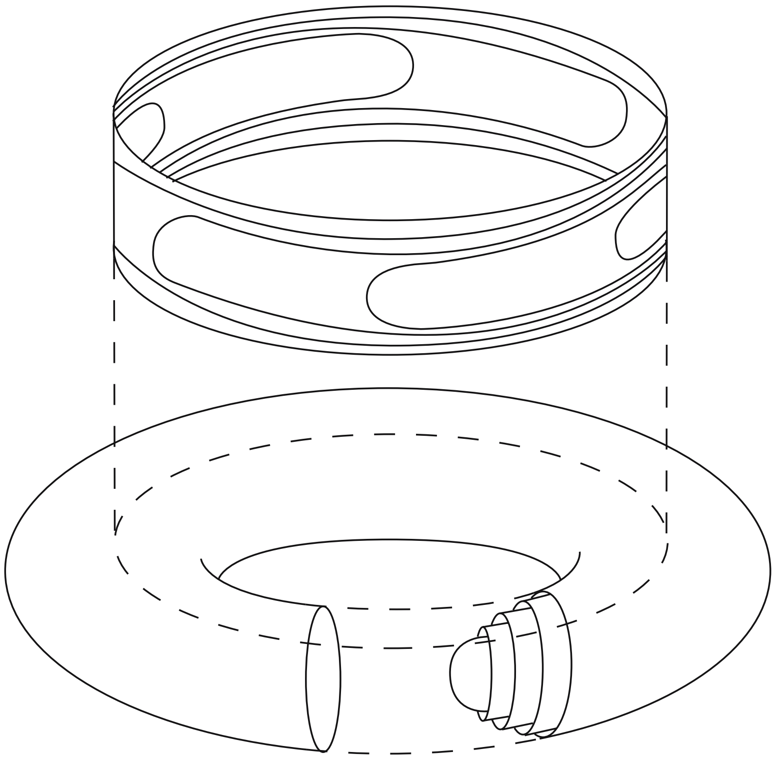

Fig. 8.3. The Reeb foliation of the solid torus.

Can we at least use this interpretation for a 1-form whose kernel *is* integrable? Unfortunately, the answer is still no. Let ω be the 1-form on the solid torus whose kernel consists of the planes tangent to the foliation pictured in Figure 8.3. (This is called the *Reeb foliation* of the solid torus.) The surfaces of this foliation spiral continually outward. So if we try to pick some number of "sample" surfaces, then they will “bunch up” near the boundary torus. This seems to indicate that if we want to integrate ω over any path that cuts through the solid torus, then we should get an infinite answer, since such a path would intersect our “sample” surfaces an infinite number of times. However, we can certainly find a 1-form ω for which this is not the case.

We do not want to end this section on such a down note. Although it is not valid in general to visualize a 1-form as a sample collection of surfaces from a foliation, we *can* visualize it as a plane field. For example, Figure 8.1 is a pretty good depiction of the 1-form $x\ dy + dz$. In this picture there are a few evenly spaced elements of its kernel, but this is enough. To get a rough idea of the value of $\int\limits_C x\ dy + dz$ we can just count the number of (transverse) intersections of the planes pictured with C. So, for example, if C is a curve whose tangents are always contained in one of these planes (a so-called *Legendrian* curve), then $\int\limits_C x\ dy + dz$ will be zero. Inspection of the picture reveals that examples of such curves are the lines parallel to the x-axis.

8.3. Show that if C is a line parallel to the x-axis, then $\int\limits_C x\ dy + dz = 0$.

9

Manifolds

9.1 Pull-backs

Before moving on to defining forms in more general contexts, we need to introduce one more concept. Let's re-examine Equation 5.3:

$$\int\limits_M \omega = \pm \int\limits_R \omega_{\phi(x_1,...,x_n)}\left(\frac{\partial \phi}{\partial x_1}(x_1, ...x_n), ..., \frac{\partial \phi}{\partial x_n}(x_1, ...x_n)\right) dx_1 \wedge ... \wedge dx_n.$$

The form in the integrand on the right was defined so as to integrate to give the same answer as the form on the left. This is what we would like to generalize. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a parameterization, and ω is a k-form on $\mathbb{R}^m$. We define the *pull-back of ω under ϕ* to be the form on $\mathbb{R}^n$ which gives the same integral over any k-cell, σ, as ω does when integrated over $\phi(\sigma)$. Following convention, we denote the pull-back of ω under ϕ as "$\phi^*\omega$."

So how do we decide how $\phi^*\omega$ acts on a k-tuple of vectors in $T_p\mathbb{R}^n$? The trick is to use ϕ to translate the vectors to a k-tuple in $T_{\phi(p)}\mathbb{R}^m$, and then plug them into ω. The matrix $D\phi$, whose columns are the partial derivatives of ϕ, is an $n \times m$ matrix. This matrix acts on vectors in $T_p\mathbb{R}^n$, and returns vectors in $T_{\phi(p)}\mathbb{R}^m$. So, we define (see Figure 9.1):

$$\phi^*\omega(V_p^1, ..., V_p^k) = \omega(D\phi(V_p^1), ..., D\phi(V_p^k)).$$

Example 43. Suppose $\omega = y\,dx + z\,dy + x\,dz$ is a 1-form on $\mathbb{R}^3$, and $\phi(a, b) = (a + b, a - b, ab)$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Then $\phi^*\omega$ will be a 1-form on $\mathbb{R}^2$. To determine which one, we can examine how it acts on the vectors $\langle 1, 0\rangle_{(a,b)}$ and $\langle 0, 1\rangle_{(a,b)}$.

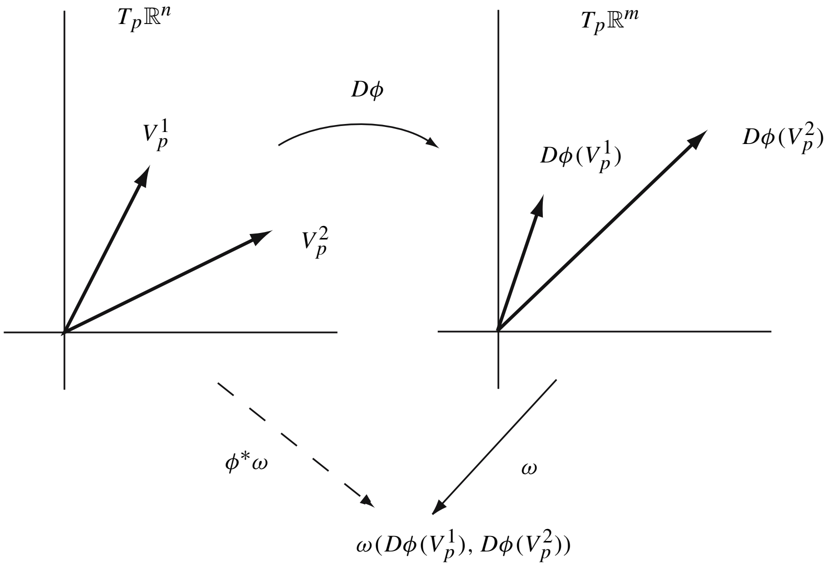

Fig. 9.1. Defining $\phi^*\omega$.

$$\begin{aligned}
\phi^*\omega(\langle 1, 0\rangle_{(a,b)}) &= \omega(D\phi(\langle 1, 0\rangle_{(a,b)})) \\
&= \omega\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ b & a \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(a,b)}\right) \\
&= \omega(\langle 1, 1, b\rangle_{(a+b,a-b,ab)}) \\
&= (a-b) + ab + (a+b)b \\
&= a - b + 2ab + b^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\phi^*\omega(\langle 0, 1\rangle_{(a,b)}) &= \omega(\langle 1, -1, a\rangle_{(a+b,a-b,ab)}) \\
&= (a-b) - ab + (a+b)a \\
&= a - b + a^2.
\end{aligned}$$

Hence,

$$\phi^*\omega = (a - b + 2ab + b^2)\, da + (a - b + a^2)\, db.$$

9.1. If $\omega = x^2 dy \wedge dz + y^2 dz \wedge dw$ is a 2-form on $\mathbb{R}^4$, and $\phi(a, b, c) = (a, b, c, abc)$, then what is $\phi^*\omega$?

9.2. If ω is an n-form on $\mathbb{R}^m$ and $\phi : \mathbb{R}^n \to \mathbb{R}^m$, then

$$\phi^*\omega = \omega_{\phi(x_1,...,x_n)}\left(\frac{\partial\phi}{\partial x_1}(x_1, ...x_n), ..., \frac{\partial\phi}{\partial x_n}(x_1, ...x_n)\right) dx_1 \wedge ... \wedge dx_n.$$

In light of the preceding exercise, Equation 5.3 can be re-written as

$$\int\limits_M \omega = \int\limits_R \phi^*\omega.$$

9.3. If σ is a k-cell in $\mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{R}^m$, and ω is a k-form on $\mathbb{R}^m$ then

$$\int\limits_\sigma \phi^*\omega = \int\limits_{\phi(\sigma)} \omega.$$

9.4. If $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and ω is a k-form on $\mathbb{R}^m$, then $d(\phi^*\omega) = \phi^*(d\omega)$.

These exercises prepare us for the proof of the generalized Stokes' Theorem (recall that in Chapter 7 we only proved this theorem when integrating over cubes and their boundaries). Suppose σ is an n-cell in $\mathbb{R}^m$, $\phi : I^n \subset \mathbb{R}^n \to \mathbb{R}^m$ is a parameterization of σ and ω is an $(n-1)$-form on $\mathbb{R}^m$. Then we can combine Problems 9.3, 9.4, and 7.7 to give us

$$\int\limits_{\partial\sigma} \omega = \int\limits_{\phi(\partial I^n)} \omega = \int\limits_{\partial I^n} \phi^*\omega = \int\limits_{I^n} d(\phi^*\omega) = \int\limits_{I^n} \phi^*(d\omega) = \int\limits_{\phi(I^n)} d\omega = \int\limits_\sigma d\omega.$$

9.2 Forms on subsets of $\mathbb{R}^n$

The goal of this chapter is to slowly work up to defining forms in a more general setting than just on $\mathbb{R}^n$. One reason for this is because the generalized Stokes' Theorem actually tells us that forms on $\mathbb{R}^n$ are not very interesting. For example, let's examine how a 1-form, ω, on $\mathbb{R}^2$, for which $d\omega = 0$ (i.e., ω is *closed*), integrates over any 1-chain, C, such that $\partial C = \emptyset$ (i.e., C is *closed*). It is a basic result of topology that any such 1-chain bounds a 2-chain, D. Hence, $\int\limits_C \omega = \int\limits_D d\omega = 0$!

Fortunately, there is no reason to restrict ourselves to differential forms which are defined on all of $\mathbb{R}^n$. Instead, we can simply consider forms which are defined on subsets, U, of $\mathbb{R}^n$. For technical reasons, we will always assume such subsets are *open*. This is a technical condition which means that for each $p \in U$, there is an ϵ such that

$$\{q \in \mathbb{R}^n | d(p, q) < \epsilon\} \subset U.$$

In this case, $TU_p = T\mathbb{R}^n_p$. Since a differential n-form is nothing more than a choice of an n-form on $T\mathbb{R}^n_p$, for each p (with some condition about differentiability), it makes sense to talk about a differential form on U.

Example 44.

$$\omega_0 = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is a differential 1-form on $\mathbb{R}^2 - (0,0)$.

9.5. Show that $d\omega_0 = 0$.

9.6. Let C be the unit circle, oriented counter-clockwise. Show that $\int\limits_C \omega_0 = 2\pi$.

Hint: Let $\omega' = -y\,dx + x\,dy$. Note that on C, $\omega_0 = \omega'$.

If C is any closed 1-chain in $\mathbb{R}^2 - (0,0)$, then the quantity $\frac{1}{2\pi}\int\limits_C \omega_0$ is called the *winding number* of C, since it computes the number of times C winds around the origin.

9.7. Let x^+ denote the positive x-axis in $\mathbb{R}^2 - (0,0)$, and let C be any closed 1-chain. Suppose V_p is a basis vector of TC_p which agrees with the orientation of C at p. A *positive* (respectively, *negative*) intersection of C with x^+ is one where V_p has a component which points "up" (respectively, "down"). Assume all intersections of C with x^+ are either positive or negative. Let P denote the number of positive ones and N the number of negative ones. Show that $\frac{1}{2\pi}\int\limits_C \omega_0 = P - N$.

Hint: Use the generalized Stokes' Theorem.

9.3 Forms on parameterized subsets

Recall that at each point, a differential from is simply an alternating, multi-linear map on a tangent plane. So all we need to define a differential form on a more general space is a well-defined tangent space. One case in which this happens is when we have a parameterized subset of $\mathbb{R}^m$. Let $\phi : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ be a (one-to-one) parameterization of M. Then recall that TM_p is defined to be the span of the partial derivatives of ϕ at $\phi^{-1}(p)$, and is a n-dimensional Euclidean space, regardless of the point, p. Hence, we say the *dimension* of M is n.

A differential k-form on M is simply an alternating, multilinear, real-valued function on TM_p, for each $p \in M$, *which varies differentiably with* p. In other words, a differential k-form on M is a whole family of k-forms, each one acting on TM_p, for different points, p. It is not so easy to say precisely what we mean when we say the form varies in a differentiable way with p. Fortunately, we have already introduced the tools necessary to do this. Let's say that ω is a family of k-forms, defined on TM_p, for each $p \in M$. Then $\phi^*\omega$ is a family of k-forms, defined on $T\mathbb{R}^n_{\phi^{-1}(p)}$, for each $p \in M$. We say that ω is a *differentiable* k-form on M, if $\phi^*\omega$ is a differentiable family on U.

This definition illustrates an important technique which is often used when dealing with differential forms on manifolds. Rather than working in M directly, we use the map ϕ^* to translate problems about forms on M into problems about forms on U. These are nice because we already know how to work with forms which are defined on open subsets of R^n. We will have much more to say about this later.

Example 45. The infinitely long cylinder, L, of radius one, centered along the z-axis, is given by the parameterization, $\phi(a, b) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, \ln\sqrt{a^2+b^2}\right)$, whose domain is $\mathbb{R}^2 - (0, 0)$. We can use ϕ^* to solve any problem about forms on L, by translating it back to a problem about forms on U.

9.8. Consider the 1-form, $\tau' = -y\,dx + x\,dy$, on $\mathbb{R}^3$. In particular, this form acts on vectors in TL_p, where L is the cylinder of the previous example, and p is any point in L. Let τ be the restriction of τ' to vectors in TL_p. So, τ is a 1-form on L. Compute $\phi^*\tau$. What does this tell you that τ measures?

If ω is a k-form on M, then what do we mean by $d\omega$? Whatever the definition, we clearly want $d\phi^*\omega = \phi^* d\omega$. So why do we not use this to define $d\omega$? After all, we know what $d\phi^*\omega$ is, since $\phi^*\omega$ is a form on $\mathbb{R}^n$. Recall that $D\phi_p$ is a map from $T\mathbb{R}^n_p$ to $T\mathbb{R}^m_p$. However, if we restrict the range to TM_p, then $D\phi_p$ is one-to-one, so it makes sense to refer to $D\phi_p^{-1}$. We now define

$$d\omega(V_p^1, ..., V_p^{k+1}) = d\phi^*\omega(D\phi_p^{-1}(V_p^1), ..., D\phi_p^{-1}(V_p^{k+1})).$$

9.9. If τ' and τ are the 1-forms on $\mathbb{R}^3$ and L, respectively, defined in the previous section, compute $d\tau'$ and $d\tau$.

9.4 Forms on quotients of $\mathbb{R}^n$ (optional)

This section requires some knowledge of topology and algebra. It is not essential for the flow of the text.

While we are on the subject of differential forms on subsets of $\mathbb{R}^n$, there is a very common construction of a topological space for which it is very easy to define what we mean by a differential form. Let's look again at the cylinder, L, of the previous section. One way to construct L is to start with the plane, $\mathbb{R}^2$, and "roll it up." More technically, we can consider the map, $\mu(\theta, z) = (\cos\theta, \sin\theta, z)$. In general, this is a many-to-one map, so it is not a parameterization, in the strict sense. To remedy this, one might try and restrict the domain of μ to $\{(\theta, z) \in \mathbb{R}^2 | 0 \le \theta < 2\pi\}$, however this set is not open.

Note that for each point, $(\theta, z) \in \mathbb{R}^2$, $D\mu$ is a one-to-one map from $T\mathbb{R}^2_{(\theta,z)}$ to $TL_{\mu(\theta,z)}$. This is all we need in order for $\mu^*\tau$ to make sense, where τ is the form on L defined in the previous section.

9.10. Show that $\mu^*\tau = d\theta$.

In this case, we say that μ is a *covering map*, $\mathbb{R}^2$ is a *cover* of L, and $d\theta$ is the *lift* of τ to $\mathbb{R}^2$.

9.11. Suppose ω_0 is the 1-form on $\mathbb{R}^2$ which we used to define the winding number. Let $\mu(r, \theta) = (r\cos\theta, r\sin\theta)$. Let $U = \{(r, \theta)|r > 0\}$. Then $\mu : U \to \{\mathbb{R}^2 - (0, 0)\}$ is a covering map. Hence, there is a one-to-one correspondence between a quotient of U and $\mathbb{R}^2 - (0, 0)$. Compute the lift of ω_0 to U.

Let's go back to the cylinder, L. Another way to look at things is to ask: How can we recover L from the θz-plane? The answer is to view L as a quotient space. Let's put an equivalence relation, R, on the points of $\mathbb{R}^2$: $(\theta_1, z_1) \sim (\theta_2, z_2)$ if and only if $z_1 = z_2$, and $\theta_1 - \theta_2 = 2n\pi$, for some $n \in \mathbb{Z}$. We will denote the quotient of $\mathbb{R}^2$ under this relation as $\mathbb{R}^2/R$. μ now induces a one-to-one map, $\bar{\mu}$, from $\mathbb{R}^2/R$ onto L. Hence, these two spaces are homeomorphic.

Let's suppose now that we have a form on U, an open subset of $\mathbb{R}^n$, and we would like to know when it *descends* to a form on a quotient of U. Clearly, if we begin with the lift of a form, then it will descend. Let's try and see why. In general, if $\mu : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ is a many-to-one map, differentiable at each point of U, then the sets, $\{\mu^{-1}(p)\}$, partition U. Hence, we can form the quotient space, U/μ^{-1}, under this partition. For each $x \in \mu^{-1}(p)$, $D\mu_x$ is a one-to-one map from TU_x to TM_p, and hence, $D\mu_x^{-1}$ is well-defined. If x and y are both in $\mu^{-1}(p)$, then $D\mu_y^{-1} \circ D\mu_x$ is a one-to-one map from TU_x to TU_y. We will denote this map as $D\mu_{xy}$. We say a k-form, ω, on $\mathbb{R}^n$ *descends* to a k-form on U/μ^{-1} if and only if $\omega(V_x^1, ..., V_x^k) = \omega(D\mu_{xy}(V_x^1), ..., D\mu_{xy}(V_x^1))$, for all $x, y \in U$ such that $\mu(x) = \mu(y)$.

9.12. If τ is a differential k-form on M, then $\mu^*\tau$ (the lift of τ) is a differential k-form on U which descends to a differential k-form on U/μ^{-1}.

Now suppose that we have a k-form, $\tilde{\omega}$, on U which descends to a k-form on U/μ^{-1}, where $\mu : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^m$ is a covering map. How can we get a k-form on M? As we have already remarked, $\bar{\mu} : U/\mu^{-1} \to M$ is a one-to-one map. Hence, we can use it to *push forward* the form, ω. In other words, we can define a k-form on M as follows: Given k vectors in TM_p, we first choose a point, $x \in \mu^{-1}(p)$. We then define

$$\mu_*\omega(V_p^1, ..., V_p^k) = \tilde{\omega}(D\mu_x^{-1}(V_p^1), ..., D\mu_x^{-1}(V_p^k)).$$

It follows from the fact that $\tilde{\omega}$ descends to a form on U/μ^{-1} that it does not matter which point, x, we choose in $\mu^{-1}(p)$. Note that although μ is not one-to-one, $D\mu_x$ is, so $D\mu_x^{-1}$ makes sense.

If we begin with a form on U, there is a slightly more general construction of a form on a quotient of U, which does not require the use of a covering map. Let Γ be a group of transformations of U. We say Γ acts *discretely* if for each $p \in U$, there exists an $\epsilon > 0$ such that $N_\epsilon(p)$ does not contain $\gamma(p)$, for any non-identity element, $\gamma \in \Gamma$. If Γ acts discretely, then we can form the quotient of U by Γ, denoted U/Γ, as follows: $p \sim q$ if there exists $\gamma \in \Gamma$ such that $\gamma(p) = q$. (The fact that Γ acts discretely is what guarantees a "nice" topology on U/Γ.)

Now, suppose $\tilde{\omega}$ is a k-form on U. We say $\tilde{\omega}$ descends to a k-form, ω, on U/Γ, if and only if $\tilde{\omega}(V_p^1, ..., V_p^k) = \tilde{\omega}(D\gamma(V_p^1), ..., D\gamma(V_p^1))$, for all $\gamma \in \Gamma$.

Now that we have decided what a form on a quotient of U is, we still have to define n-chains, and what we mean by integration of n-forms over n-chains. We say an n-chain, $\tilde{C} \subset U$, descends to an n-chain, $C \subset U/\Gamma$, if $\gamma(\tilde{C}) = \tilde{C}$, for all $\gamma \in \Gamma$. The n-chains of U/Γ are simply those which are descendants of n-chains in U.

Integration is a little more subtle. For this we need the concept of a *fundamental domain for* Γ. This is nothing more than a closed subset of U, whose interior does not contain two equivalent points. Furthermore, for each equivalence class, there is at least one representative in a fundamental domain. Here is one way to construct a fundamental domain: First, choose a point, $p \in U$. Now, let $D = \{q \in U | d(p, q) \leq d(\gamma(p), q)$, for all $\gamma \in \Gamma\}$.

Now, let $\tilde{C}$ be an n-chain on U which descends to an n-chain, C, on U/Γ, and let $\tilde{\omega}$ be an n-form that descends to an n-form, ω. Let D be a fundamental domain for Γ in U. Then we define

$$\int\limits_C \omega \equiv \int\limits_{\tilde{C}\cap D} \tilde{\omega}.$$

Technical note: In general, this definition is invariant of which point was chosen in the construction of the fundamental domain, D. However, a VERY unlucky choice will result in $\tilde{C} \cap D \subset \partial D$, which could give a different answer for the above integral. Fortunately, it can be shown that the set of such "unlucky" points has *measure zero*. That is, if we were to choose the point at random, then the odds of picking an "unlucky" point are 0%. Very unlucky indeed!

Example 46. Suppose Γ is the group of transformations of the plane generated by $(x, y) \to (x + 1, y)$, and $(x, y) \to (x, y + 1)$. The space $\mathbb{R}^2/\Gamma$ is often denoted T^2, and referred to as a *torus*. Topologists often visualize the torus as the surface of a donut. A fundamental domain for Γ is the unit square in $\mathbb{R}^2$. The 1-form, dx, on $\mathbb{R}^2$ descends to a 1-form on T^2. Integration of this form over a closed 1-chain, $C \subset T^2$, counts the number of times C wraps around the "hole" of the donut.

9.5 Defining manifolds

As we have already remarked, a differential n-form on $\mathbb{R}^m$ is just an n-form on $T_p\mathbb{R}^m$, for each point $p \in \mathbb{R}^m$, along with some condition about how the form varies in a differentiable way as p varies. All we need to define a form on a space other than $\mathbb{R}^m$ is some notion of a tangent space at every point. We call such a space a *manifold*. In addition, we insist that at each point of a manifold the tangent space has the same dimension, n, which we then say is the dimension of the manifold.

How do we guarantee that a given subset of $\mathbb{R}^m$ is a manifold? Recall that we defined the tangent space to be the span of some partial derivatives of a parameterization. However, insisting that the whole manifold is capable of being parameterized is very restrictive. Instead, we only insist that every point of a manifold lies in a subset that can be parameterized. Hence, if M is an n-manifold in $\mathbb{R}^m$ then there is a set of open subsets, $\{U_i\} \subset \mathbb{R}^n$, and a set of differentiable maps, $\{\phi_i : U_i \to M\}$, such that $\{\phi_i(U_i)\}$ is a *cover* of M. (That is, for each point, $p \in M$, there is an i, and a point, $q \in U_i$, such that $\phi_i(q) = p$.)

Example 47. S^1, the unit circle in R^2, is a 1-manifold. Let $U_i = (-1, 1)$, for $i = 1, 2, 3, 4$, $\phi_1(t) = (t, \sqrt{1-t^2})$, $\phi_2(t) = (t, -\sqrt{1-t^2})$, $\phi_3(t) = (\sqrt{1-t^2}, t)$ and $\phi_4(t) = (-\sqrt{1-t^2}, t)$. Then $\{\phi_i(U_i)\}$ is certainly a cover of S^1 with the desired properties.

9.13. Show that S^2, the unit sphere in $\mathbb{R}^3$, is a 2-manifold.

9.6 Differential forms on manifolds

Basically, the definition of a differential n-form on an m-manifold is the same as the definition of an n-form on a subset of $\mathbb{R}^m$ which was given by a single parameterization. First and foremost it is just an n-form on T_pM, for each $p \in M$.

Let's say M is an m-manifold. Then we know there is a set of open sets, $\{U_i\} \subset \mathbb{R}^m$, and a set of differentiable maps, $\{\phi_i : U_i \to M\}$, such that $\{\phi_i(U_i)\}$ covers M. Now, let's say that ω is a family of n-forms, defined on T_pM, for each $p \in M$. Then we say that the family, ω, is a *differentiable* n-form on M if $\phi_i^*\omega$ is a differentiable n-form on U_i, for each i.

Example 48. In the previous section, we saw how S^1, the unit circle in $\mathbb{R}^2$, is a 1-manifold. If (x, y) is a point of S^1, then $TS^1_{(x,y)}$ is given by the equation $dy = -\frac{x}{y}dx$, in $T\mathbb{R}^2_{(x,y)}$, as long as $y \neq 0$. If $y = 0$, then $TS^1_{(x,y)}$ is given by $dx = 0$. We define a 1-form on S^1, $\omega = -y\, dx + x\, dy$. (Actually, ω is a 1-form on all of $\mathbb{R}^2$. To get a 1-form on just S^1, we restrict the domain of ω to the tangent lines to S^1.) To check that this is really a differential form, we must compute all pull-backs:

$$\phi_1^*\omega = \frac{-1}{\sqrt{1-t^2}}\, dt\ ,\ \phi_2^*\omega = \frac{1}{\sqrt{1-t^2}}\, dt,$$
$$\phi_3^*\omega = \frac{1}{\sqrt{1-t^2}}\, dt\ ,\ \phi_4^*\omega = \frac{-1}{\sqrt{1-t^2}}\, dt.$$

Since all of these are differentiable on $U_i = (-1, 1)$, we can say that ω is a differential form on S^1.

We now move on to integration of n-chains on manifolds. The definition of an n-chain is no different than before; it is just a formal linear combination of n-cells in M. Let's suppose that C is an n-chain in M, and ω is an n-form. Then how do we define $\int_C \omega$? If C lies entirely in $\phi_i(U_i)$, for some i, then we could define the value of this integral to be the value of $\int_{\phi_i^{-1}(C)} \phi_i^*\omega$. But it may be that part of C lies in both $\phi_i(U_i)$ and $\phi_j(U_j)$. If we define $\int_C \omega$ to be the sum of the two integrals we get when we pull-back ω under ϕ_i and ϕ_j, then we end up "double counting" the integral of ω on $C \cap \phi_i(U_i) \cap \phi_j(U_j)$. Somehow, as we move from $\phi_i(U_i)$ into $\phi_j(U_j)$, we want the effect of the pull-back of ω under ϕ_i to "fade out," and the effect of the pull-back under ϕ_j to "fade in." This is accomplished by a *partition of unity*.

The technical definition of a *partition of unity subordinate to the cover,* $\{\phi_i(U_i)\}$ is a set of differentiable functions, $f_i : M \to [0, 1]$, such that $f_i(p) = 0$ if $p \notin \phi_i(U_i)$, and $\sum_i f_i(p) = 1$, for all $p \in M$. We refer the reader to any book on differential topology for a proof of the existence of partitions of unity.

We are now ready to give the full definition of the integral of an n-form on an n-chain in an m-manifold.

$$\int_C \omega \equiv \sum_i \int_{\phi_i^{-1}(C)} \phi_i^*(f_i\omega).$$

We illustrate this with a simple example.

Example 49. Let M be the manifold which is the interval $(1, 10) \subset \mathbb{R}$. Let $U_i = (i, i+2)$, for $i = 1, ..., 8$. Let $\phi_i : U_i \to M$ be the identity map. Let $\{f_i\}$ be a partition of unity, subordinate to the cover, $\{\phi_i(U_i)\}$. Let ω be a 1-form on M. Finally, let C be the 1-chain which consists of the single 1-cell, $[2, 8]$. Then we have

$$\int_C \omega \equiv \sum_{i=1}^{8} \int_{\phi_i^{-1}(C)} \phi_i^*(f_i\omega) = \sum_{i=1}^{8} \int_C f_i\omega = \int_C \sum_{i=1}^{8} (f_i\omega) = \int_C \left(\sum_{i=1}^{8} f_i\right)\omega = \int_C \omega$$

as one would expect!

Example 50. Let S^1, U_i, ϕ_i and ω be defined as in Examples 47 and 48. A partition of unity subordinate to the cover $\{\phi_i(U_i)\}$ is as follows:

$$f_1(x, y) = \begin{cases} y^2 & y \geq 0 \\ 0 & y < 0 \end{cases} \quad f_2(x, y) = \begin{cases} 0 & y > 0 \\ y^2 & y \leq 0 \end{cases}$$

$$f_3(x,y) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad f_4(x,y) = \begin{cases} 0 & x > 0 \\ x^2 & x \leq 0 \end{cases}.$$

(Check this!) Let $\mu : [0, \pi] \to S^1$ be defined by $\mu(\theta) = (\cos\theta, \sin\theta)$. Then the image of μ is a 1-cell, σ, in S^1. Let's integrate ω over σ:

$$\begin{aligned}
\int_\sigma \omega &\equiv \sum_{i=1}^{4} \int_{\phi_i^{-1}(\sigma)} \phi_i^*(f_i\omega) \\
&= \int_{-(-1,1)} -\sqrt{1-t^2}\,dt + 0 + \int_{[0,1)} \sqrt{1-t^2}\,dt + \int_{-[0,1)} -\sqrt{1-t^2}\,dt \\
&= \int_{-1}^{1} \sqrt{1-t^2}\,dt + 2\int_{0}^{1} \sqrt{1-t^2}\,dt \\
&= \pi.
\end{aligned}$$

CAUTION: Beware of orientations!

9.7 Application: DeRham cohomology

One of the predominant uses of differential forms is to give *global* information about manifolds. Consider the space $\mathbb{R}^2 - (0,0)$, as in Example 44. Near every point of this space we can find an open set which is identical to an open set around a point of $\mathbb{R}^2$. This means that all of the *local* information in $\mathbb{R}^2 - (0,0)$ is the same as the local information in $\mathbb{R}^2$. The fact that the origin is missing is a global property.

For the purposes of detecting global properties, certain forms are interesting, and certain forms are completely uninteresting. We will spend some time discussing both. The interesting forms are the ones whose derivative is zero. Such forms are said to be *closed*. An example of a closed 1-form was ω_0, from Example 44 of the previous chapter. For now, let's just focus on closed 1-forms so that you can keep this example in mind.

Let's look at what happens when we integrate a closed 1-form ω_0 over a 1-chain C such that $\partial C = 0$ (i.e., C is a *closed* 1-chain). If C bounds a disk D then Stokes' theorem says

$$\int_C \omega_0 = \int_D d\omega_0 = \int_D 0 = 0.$$

In a sufficiently small region of every manifold, every closed 1-chain bounds a disk. So integrating closed 1-forms on "small" 1-chains gives us no information. In other words, closed 1-forms give no local information.

Suppose now that we have a closed 1-form ω_0 and a closed 1-chain C such that $\int\limits_C \omega_0 \neq 0$. Then we know C does not bound a disk. The fact that there exists such a 1-chain is global information. This is why we say that the closed forms are the ones that are interesting, from the point of view of detecting only global information.

Now let's suppose that we have a 1-form ω_1, that is the derivative of a 0-form f (i.e., $\omega_1 = df$). We say such a form is *exact*. Again, let C be a closed 1-chain. Let's pick two points, p and q, on C. Then $C = C_1 + C_2$, where C_1 goes from p to q and C_2 goes from q back to p. Now let's do a quick computation:

$$\begin{aligned}\int\limits_C \omega_1 &= \int\limits_{C_1+C_2} \omega_1 \\ &= \int\limits_{C_1} \omega_1 + \int\limits_{C_2} \omega_1 \\ &= \int\limits_{C_1} df + \int\limits_{C_2} df \\ &= \int\limits_{p-q} f + \int\limits_{q-p} f \\ &= 0.\end{aligned}$$

So integrating an exact form over a closed 1-chain always gives zero. This is why we say the exact forms are completely uninteresting. Unfortunately, in Problem 6.9 we learned that every exact form is also closed. This is a problem, since this would say that all of the completely uninteresting forms are also interesting! To remedy this we define an *equivalence relation*.

We pause here for a moment to explain what this means. An equivalence relation is just a way of taking one set and creating a new set by declaring certain objects in the original set to be "the same." This is the idea behind telling time. To construct the clock numbers, start with the integers and declare two to be "the same" if they differ by a multiple of 12. So $10 + 3 = 13$, but 13 is the same as one, so if it's now 10 o'clock then in three hours it will one o'clock.

We play the same trick for differential forms. We will restrict ourselves to the closed forms, but we will consider two of them to be "the same" if their difference is an exact form. The set which we end up with is called the *cohomology* of the manifold in question. For example, if we start with the closed 1-forms, then, after our equivalence relation, we end up with the set which we will call H^1, or the *first* cohomology (see Figure 9.2).

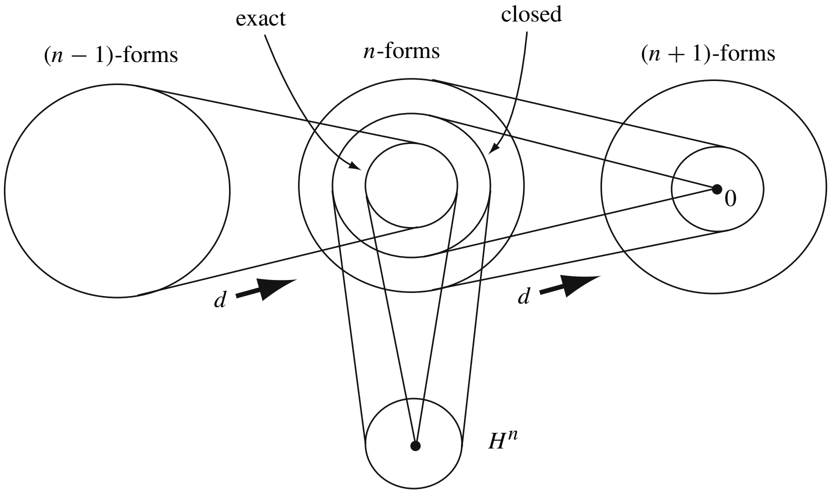

Fig. 9.2. Defining H^n.

Note that the difference between an exact form and the form which always returns the number zero is an exact form. Hence, every exact form is equivalent to 0 in H^n, as in the figure.

For each n the set H^n contains a lot of information about the manifold in question. For example, if $H^1 \cong \mathbb{R}^1$ (as it turns out is the case for $\mathbb{R}^2 - (0, 0)$), then this tells us that the manifold has one "hole" in it. Studying manifolds via cohomology is one topic of a field of mathematics called *Algebraic Topology*. For a complete treatment of this subject, see [BT95].

A

Non-linear forms

A.1 Surface area

Now that we have developed some proficiency with differential forms, let's see what *else* we can integrate. A basic assumption that we used to come up with the definition of an n-form was the fact that at every point it is a *linear* function which "eats" n vectors and returns a number. But what about the non-linear functions?

Let's go all the way back to Section 3.5. There we decided that the integral of a function f over a surface R in $\mathbb{R}^3$ should look something like:

$$\int_R f(\phi(r,\theta))\text{Area}\left[\frac{\partial\phi}{\partial r}(r,\theta), \frac{\partial\phi}{\partial\theta}(r,\theta)\right] dr d\theta. \tag{A.1}$$

At the heart of the integrand is the *Area* function, which takes two vectors and returns the area of the parallelogram that it spans. The 2-form $dx \wedge dy$ does this for two vectors in $T_p\mathbb{R}^2$. In $T_p\mathbb{R}^3$ the right function is the following:

$$\text{Area}(V_p^1, V_p^2) = \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

(The reader may recognize this as the magnitude of the cross product between V_p^1 and V_p^2.) This is clearly non-linear!

Example 51. The area of the parallelogram spanned by $\langle 1, 1, 0\rangle$ and $\langle 1, 2, 3\rangle$ can be computed as follows:

$$\begin{aligned}\text{Area}(\langle 1,1,0\rangle, \langle 1,2,3\rangle) &= \sqrt{\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}^2} \\ &= \sqrt{3^2 + 3^2 + 1^2} \\ &= \sqrt{19}.\end{aligned}$$

The thing that makes (linear) differential forms so useful is the generalized Stokes' Theorem. We do not have anything like this for non-linear forms, but that is not to say that they do not have their uses. For example, there is no differential 2-form on $\mathbb{R}^3$ that one can integrate over arbitrary surfaces to find their surface area. For that we would need to compute the following:

$$\text{Area}(R) = \int_S \sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

For relatively simple surfaces, this integrand can be evaluated by hand. Integrals such as this play a particularly important role in certain applied problems. For example, if one were to dip a loop of bent wire into a soap film, the resulting surface would be the one of minimal area. Before one can even begin to figure out what surface this is for a given piece of wire, one must be able to know how to compute the area of an arbitrary surface, as above.

Example 52. We compute the surface area of a sphere of radius r in $\mathbb{R}^3$. A parameterization is given by

$$\Phi(\theta, \phi) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

Now we compute:

$$\begin{aligned}
&\text{Area}\left(\tfrac{\partial\Phi}{\partial\theta}, \tfrac{\partial\Phi}{\partial\phi}\right) \\
&= \text{Area}\left(\langle -r \sin\phi \sin\theta, r \sin\phi \cos\theta, 0\rangle, \langle r \cos\phi \cos\theta, r \cos\phi \sin\theta, -r \sin\phi\rangle\right) \\
&= \sqrt{(-r^2 \sin^2\phi \cos\theta)^2 + (r^2 \sin^2\phi \sin\theta)^2 + (-r^2 \sin\phi \cos\phi)^2} \\
&= r\sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi} \\
&= r \sin\phi.
\end{aligned}$$

And so the desired area is given by

$$\begin{aligned}
&\int_S \text{Area}\left(\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial\phi}\right) d\theta\, d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} r \sin\phi\, d\theta\, d\phi \\
&= 4\pi r.
\end{aligned}$$

A.1. Compute the surface area of a sphere of radius r in $\mathbb{R}^3$ using the parameterizations

$$\Phi(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, \pm\sqrt{r^2 - \rho^2})$$

for the top and bottom halves, where $0 \le \rho \le r$ and $0 \le \theta \le 2\pi$.

Let's now go back to Equation A.1. Classically, this is called a *surface integral*. It might be a little clearer how to compute such an integral if we write it as follows:

$$\int_R f(x, y, z)\, dS = \int_R f(x, y, z)\sqrt{(dy \wedge dz)^2 + (dx \wedge dz)^2 + (dx \wedge dy)^2}.$$

A.2 Arc length

Lengths are very similar to areas. In calculus you learn that if you have a curve C in the plane, for example, parameterized by the function $\phi(t) = (x(t), y(t))$, where $a \le t \le b$, then its arc length is given by

$$\text{Length}(C) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We can write this without making reference to the parameterization by employing a non-linear 1-form:

$$\text{Length}(C) = \int_C \sqrt{dx^2 + dy^2}.$$

Finally, we can define what is classically called a *line integral* as follows:

$$\oint_C f(x, y)\, ds = \int_C f(x, y)\sqrt{dx^2 + dy^2}.$$

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[HH01] John Hubbard and Barbara Hubbard. *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach*. Prentice Hall, 2001.

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Index

Solutions

Chapter 1

1.2 $\langle \frac{3}{5}, \frac{4}{5} \rangle$

1.4 $\frac{8\sqrt{5}}{15}$

1.6 -8

1.8

1. $x + y$
2. $\sqrt{4 - x^2 - y^2}$
3. $\sqrt{1 - (2 - x^2 - y^2)^2}$
4. $\sqrt{1 - (y - x)^2}$

1.9

1. 21
2. 1
3. 3
4. 16
5. $\frac{2}{5}(33 - \sqrt{2}^5 - \sqrt{3}^5)$

1.10

1. $\frac{\partial f}{\partial x} = 2xy^3$, $\frac{\partial f}{\partial y} = 3x^2y^2$
2. $\frac{\partial f}{\partial x} = 2xy^3 \cos(x^2y^3)$, $\frac{\partial f}{\partial y} = 3x^2y^2 \cos(x^2y^3)$
3. $\frac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy)$, $\frac{\partial f}{\partial y} = x^2 \cos(xy)$

1.12 $-4\sqrt{2}$

1.13 $\frac{-2\sqrt{5}}{5}$

1.14

1. $\left\langle y^2, 2xy \right\rangle$
2. 69
3. $\left\langle \frac{9}{15}, \frac{12}{15} \right\rangle$
4. 15

Chapter 2

2.1

1. $\frac{b}{a}$
2. They are parallel. The one parameterized by ϕ can be obtained from the other by shifting c units to the right and d units up.

2.3

1. $(\cos^2\theta, \cos\theta\sin\theta)$
2. $(x, \sin x)$

2.4 $\phi(t) = (t, 4t - 3)$, $1 \le t \le 2$ (There are many other answers.)

2.6

1. (t^2, t)
2. $\langle 4, 1 \rangle$

2.7

$$\begin{array}{l|l} r = \rho\sin\phi & \rho = \sqrt{r^2 + z^2} \\ \theta = \theta & \theta = \theta \\ z = \rho\cos\phi & \phi = \tan^{-1}\left(\frac{r}{z}\right) \end{array}$$

2.9

1. $z = \sqrt{x^2 + y^2}$, $z = r$, $\phi = \frac{\pi}{4}$
2. $y = 0$, $\theta = 0$, $\theta = 0$
3. $z = 0$, $z = 0$, $\phi = \frac{\pi}{2}$
4. $z = x + y$, $z = r(\sin\theta + \cos\theta)$, $\cot\phi = \sin\theta + \cos\theta$
5. $z = (x^2 + y^2)^3$, $z = r^3$, $z = (\rho\sin\phi)^3$

2.10

1. $\phi(u, z) = (u, u, z)$
2. $\phi(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$
3. $\psi(\theta, \phi) = (\phi\sin\phi\cos\theta, \phi\sin\phi\sin\theta, \phi\cos\phi)$
4. $\psi(\theta, \phi) = (\cos\phi\sin\phi\cos\theta, \cos\phi\sin\phi\sin\theta, \cos\phi\cos\phi)$
5. $\phi(\theta, z) = (\cos^2\theta, \sin\theta\cos\theta, z)$
6. $\phi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{r^2 - 1})$

7. $\phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{r^2+1})$
8. $\phi(\theta,z) = (\theta\cos\theta, \theta\sin\theta, z)$

2.11 $\phi(x,y) = (x, y, f(x,y))$

2.13 $\psi(\theta,\phi) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $0 \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

2.15 $\langle 2, 0, 4\rangle$, $\langle 0, 3, 2\rangle$

2.17

1. The x-axis
2. The z-axis
3. The line $y = z$ and $x = 0$
4. The line $x = y = z$

2.18 $\phi(t) = (t, 1-t, \sqrt{1-2t+2t^2})$

2.19

1. $\phi(\theta) = (2\cos\theta, 2\sin\theta, 4)$, $0 \le \theta \le 2\pi$
2. $\psi(t) = (t, \pm\sqrt{4-t^2}, 4)$, $-2 \le t \le 2$

2.20

1. $\psi(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$
2. $\psi(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$, $0 \le \rho \le 1$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

2.21

1. $\phi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $0 \le z \le 1$
2. $\phi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, $1 \le r \le 2$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$

2.22

$$\phi(t,\theta) = ([tf_2(\theta) + (1-t)f_1(\theta)]\cos\theta, [tf_2(\theta) + (1-t)f_1(\theta)]\sin\theta)$$

$0 \le t \le 1$, $a \le \theta \le b$

2.23 $\phi(r,\theta) = (3r\cos\theta, 2r\sin\theta)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

Chapter 4

4.2

1. $-1, 4, 10$
2. $dy = -4dx$

4.3

1. $3dx$
2. $\frac{1}{2}dy$
3. $3dx + \frac{1}{2}dy$
4. $8dx + 6dy$

4.5

1. $\omega(V_1) = -8$, $\nu(V_1) = 1$, $\omega(V_2) = -1$ and $\nu(V_2) = 2$.
2. -15
3. 5

4.15 -127

4.16 $c_1 = -11$, $c_2 = 4$, and $c_3 = 3$

4.17

1. $2dx \wedge dy$
2. $dx \wedge (dy + dz)$
3. $dx \wedge (2dy + dz)$
4. $(dx + 3dz) \wedge (dy + 4dz)$

4.29 252

4.30

1. -87
2. -29
3. 5

4.31 $dx \wedge dy \wedge dz$

4.33

1. $z(x - y)dz \wedge dx + z(x + y)dz \wedge dy$
2. $-4dx \wedge dy \wedge dz$

Chapter 5

5.1

1. $\langle 2, 3, 1\rangle, \langle 2, 3, 2\rangle$
2. $6dx \wedge dy + 3dy \wedge dz - 2dx \wedge dz$
3. 5
4. $x^2yz^2 - x^5z^2 - y^3 + x^3y^2$

5.2 $\frac{1}{6}$

5.3 $\pm 4\pi$. (You don't have enough information yet to properly tell the sign.)

5.4 4π

5.5 $\frac{1}{3}$

5.6

1. $-\frac{17}{12}$
2. $-\frac{29}{6}$

5.8 $\frac{3\pi}{5}$

5.9 $\frac{1}{6}$

5.10

1. Opposite orientation
2. Same orientation
3. Does not determine an orientation

5.11 16

5.12 16

5.13 $\frac{1}{5}$

5.14 $\frac{2}{3}\cos 6 - \frac{3}{2}\sin 4 - \frac{2}{3}$

5.15 $\frac{1}{6}$

5.16 4

5.17 $\frac{32}{3}$

5.19 2

5.20 6π

5.22 $\frac{14}{3}$

5.23 $\frac{-7\pi}{6}$

5.24 $\frac{\pi}{3}\left(2^{3/2} - 1\right)$

5.25 2π

Chapter 6

6.1

1. $\nabla_V \omega(W) = -62$, $\nabla_W \omega(V) = 4$
2. -66

6.3 $d\omega = (-2x - 1)dx \wedge dy$

6.6 $d(x^2 y\, dx \wedge dy + y^2 z\, dy \wedge dz) = 0$

6.7 $-1, 1, 1$

6.11

1. $(-\sin x - \cos y)dx \wedge dy$
2. $(3x^2 z - 2xy)dx \wedge dy - (x^3 + 1)dy \wedge dz$
3. $(y^2 - 9z^8)dx \wedge dy \wedge dz$
4. 0

6.12 $(3x^4 y^2 - 4xy^6 z)dx \wedge dy \wedge dz$

6.14

1. $x\, dy$
2. $x\, dy \wedge dz$
3. xyz
4. $xy^2 z^2$
5. $\sin(xy^2)dx + \sin(xy^2)dy$

Chapter 7

7.1

1. $2, -2$
2. 1

7.12 1

7.13 27

7.14 $\frac{35}{3}$

7.15

1. 0
2. $\int\limits_{C_1} \omega = \int\limits_{C_2} \omega = \frac{1}{3}$
3. $-\frac{2}{3}$

7.16

1. 0
2. 0
3. If L, R, T, and B represent the 1-cells that are the left, right, top and bottom of Q, then

$$\int_{\partial Q} \omega = \int_{(R-L)-(T-B)} \omega = \int_R \omega - \int_L \omega - \int_T \omega + \int_B \omega = 24 - 0 - 28\frac{1}{2} + 4\frac{1}{2} = 0$$

4. Opposite
5. $4\frac{1}{2}$
6. $4\frac{1}{2}$

7.19 8π

7.20 6π

7.21 $\frac{2}{3}\left(e - \frac{1}{e}\right)$

7.24 0

7.25 $\frac{4}{3}$

Chapter 9

9.9 $d\tau' = 2\,dx \wedge dy$ and $d\tau = 0$