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Messaoud Bounkhel

Regularity Concepts in Nonsmooth Analysis

Theory and Applications

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Regularity Concepts in Nonsmooth Analysis

Theory and Applications

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To the soul of my dear father, Ahmed, who died in 1994 and who always believed in me and encouraged me to study mathematics.

To my dear mother Fatma, who has always stood behind all my successes.

To my wife Leila, my children Saged, Kawtar, and Yakine, and to my brothers and sisters and all the members of my family.

Preface

The term *nonsmooth analysis theory* had been used in the 1970s by F. Clarke when he studied and applied the differential properties of functions and sets that are not differentiable in the usual sense. Since Clarke's work, the field of nonsmooth analysis theory has known a considerable expansion, namely with the appearence of an important concept which is the concept of "*regularity* " (regularity of functions and regularity of sets). The primary motivation for introducing regularity notions is to obtain equalities in calculus rules involving various constructs in nonsmooth analysis. The first notion of regularity appeared in Clarke's work (in the 1970s) to ensure equality form in the calculus rules of the Clarke subdifferential for Lipschitz continuous functions.

Many investigators (Rockafellar, Mordukhovich, Thibault, Poliquin et al.) have since then introduced and used many other notions of regularity in the development of nonsmooth analysis theory.

In the last decades, regularity concepts played an increasing role in the applications of nonsmooth analysis such as differential inclusions, optimization, variational inequalities, as well as in nonsmooth analysis itself. Consequently, it is becoming more and more desirable to introduce regularity, at an early stage of study, to graduate students and young researchers in order to familiarize them with the basic concepts and their applications. This book is devoted to the study of various regularity notions in nonsmooth analysis and their applications. To the best of my knowledge, the present work is the first thorough study of the regularity of functions, sets, and multifunctions as well as their important applications to differential inclusions and variational inequalities.

This book is divided into three parts. In the first part, we present an accessible and thorough introduction to nonsmooth analysis theory. Main concepts and some useful results are stated and illustrated through examples and exercises.

In Part II, the most important and recent results of various regularity concepts of sets, functions, and set-valued mappings, in nonsmooth analysis theory are presented. These results include some that have been demonstrated in different works that were published either singly (see [39, 44, 45, 48]), or in collaboration with Thibault (see [58–63]).

Part III contains six chapters, each of which addresses a different application of nonsmooth analysis theory. These applications are the fruit of research that I conducted either singly (see [42, 43]) or in collaboration with various researchers in the field (see [53–55, 58, 64]).

Batna, Algeria Messaoud Bounkhel

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After my family members and King Saud University, my gratitude goes to Boris Mordukhovich who has encouraged me at all stages of this project and who has advised me to write a book on regularity concepts in nonsmooth analysis theory and has lent me his full support to achieve such a project. His suggestions on the first versions of the book, over the years of preparation, have always been most pertinent and valuable. Similarly, I am very grateful to Alex Kruger for his accurate remarks and suggestions concerning the preliminary version of the book. Special thanks are addressed to Lionel Thibault, my doctoral thesis adviser, from whom I learned nonsmooth analysis theory and much more.

I finally would like to thank my dear friend Mustapha Bouchareb, a teacher and writer by trade, for taking the time to thoroughly read and correct the English of the preliminary version of the book and for patiently answering a tremendous amount of questions on the English language.

Contents

List of Figures

Part I
Nonsmooth Analysis Theory

Chapter 1
Nonsmooth Concepts

1.1 Introduction

This book assumes a basic knowledge of topological vector space and functional analysis. Moreover, we recall in this section several concepts and fundamental preliminaries which will be used in what follows. The following notation is used throughout this book.

X is a real topological vector space or a real normed vector space or a Banach space with norm $\|\cdot\|$ and $\mathbf{H}$ is a real Hilbert space. The inner product between elements of $\mathbf{H}$ is denoted by $\langle\cdot,\cdot\rangle$, the same notation is also employed for the pairing between X and its topological dual space X^* (the space of continuous linear functionals defined on X). The closed unit ball in X or $\mathbf{H}$ centered at some point $\bar{x}$ and with radius $r>0$ is denoted by $B(\bar{x},r)$. For $\bar{x}=0$ and $r=1$ we will use the standard notation $\mathbf{B}$ instead of $B(0,1)$. The notation $\mathbf{B}_*$ is used for the closed unit ball in X^* centered at the origin and with radius 1. Whenever needed, we use the notation $\mathbf{B}_Z$ for the closed unit ball centered at the origin of a given normed vector space Z. We will denote by $\mathscr{N}(\bar{x})$ the set of all neighborhoods of $\bar{x}$. For a given set S, the following expressions: $\operatorname{int} S, \operatorname{cl} S, \operatorname{bd} S$, signify the interior, closure, and boundary of S, respectively.

Definition 1.1. Let X be a real vector space. A set S is said to be *convex* provided that for every pair of element (x,y) of S the segment $[x,y]=\{\alpha y+(1-\alpha)x : \alpha \in [0,1]\}$ is contained in S. The *convex hull* of a nonconvex set S is defined as the intersection of all the sets containing S. It is denoted by $co\,S$ and has the following characterization:

$$co\,S=\left\{\sum_{i=1}^{n}\alpha_i x_i : \quad n\in\mathbf{N},\ \sum_{i=1}^{n}\alpha_i=1,\ \alpha_i\geq 0,\ x_i\in S\right\}.$$

The closure of $co\,S$ is called the *closed convex hull* and denoted by $\overline{co}\,S$.

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Definition 1.2. Let f be an extended real valued function, i.e., $f : X \to \mathbf{R} \cup \{+\infty\}$. We call the sets

$$\operatorname{dom} f = \{x \in X : \ f(x) < +\infty\} \text{ and } \operatorname{epi} f = \{(x,r) \in X \times \mathbf{R} : \ f(x) \leq r\},$$

the *effective domain* of f and the *epigraph* of f, respectively.

1. f is said to be a *convex function* on an open convex set $\Omega \subset X$ provided that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \text{ for all } x, y \in \Omega, \text{ and all } \alpha \in [0,1].$$

   When Ω is the whole space X we will say that f is convex.
2. f is said to be *lower semicontinuous* (in short l.s.c.) at some point $\bar{x}$ in $\operatorname{dom} f$ provided that

$$f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x).$$

   We will say that f is l.s.c. on X if it is l.s.c. at any point of X.

Exercise 1.1.

1. Prove that f is l.s.c. on X if and only if its epigraph $\operatorname{epi} f$ is closed in $X \times \mathbf{R}$.
2. Prove that f is l.s.c. on X if and only if the r-level set $\{x \in X : \ f(x) \leq r\}$ is closed for any $r \in \mathbf{R}$.
3. Prove that f is convex if and only if its epigraph $\operatorname{epi} f$ is convex. As a consequence the effective domain of convex functions is always convex.
4. Prove that the convexity of f implies the convexity of all the r-level sets. Prove by giving a counter example that the converse in the last question is not true in general.

1.2 From Derivatives to Subdifferentials

In this section, we begin with some classical concepts of differentiability (directional, Gâteaux, and Fréchet) and we will try via optimization problems to explain the evolution of the concept of differentiability from the Fréchet derivative to the generalized gradient concept (also called Clarke subdifferential).

Let X be a real topological vector space, $f : X \to \mathbf{R} \cup \{+\infty\}$ be an extended real valued function and $\bar{x} \in \operatorname{dom} f$.

1. The *directional derivative* of f at $\bar{x}$ in the direction $v \in X$ is given by

$$f'(\bar{x}; v) = \lim_{\delta \downarrow 0} \delta^{-1} \left[f(\bar{x} + \delta v) - f(\bar{x}) \right], \tag{1.1}$$

   when the limit exists.

2. We say that f is *Gâteaux differentiable* at $\bar{x}$ provided $f'(\bar{x};v)$ exists for all $v \in X$ and $f'(\bar{x};\cdot)$ is linear continuous, that is, there exists an element (necessarily unique) $f'_{\mathrm{G}}(\bar{x}) \in X^*$ (called the *Gâteaux derivative*) satisfying

$$\left\langle f'_{\mathrm{G}}(\bar{x}), v \right\rangle = f'(\bar{x};v), \text{ for all } v \in X. \tag{1.2}$$

3. If the convergence in (1.1) is uniform with respect to v in bounded subsets of X, we say that f is *Fréchet differentiable* at $\bar{x}$, and we write $f'(\bar{x})$ instead of $f'_{\mathrm{G}}(\bar{x})$.

Remark 1.1.

1. A function may admit a directional derivative $f'(\bar{x};v)$ at $\bar{x}$ in every direction $v \in X$, but fails to admit a Gâteaux derivative $f'_{\mathrm{G}}(\bar{x})$ at $\bar{x}$. For example, let X be a Banach space, $f(x) = \|x\|$, and $\bar{x} = 0$. This function has a directional derivative $f'(\bar{x};v)$ for every direction $v \in X$ and $f'(\bar{x};v) = \|v\|$, while the Gâteaux derivative of this function at $\bar{x}$ does not exist because the function $v \mapsto f'(\bar{x};v) = \|v\|$ is not linear.
2. The Fréchet and Gâteaux differentiability concepts are not equivalent in general even in finite dimensional cases. It is not hard to check that Fréchet differentiability at a point implies its continuity at that point, which is not the case for Gâteaux differentiability. For example, a l.s.c. function f (which is not necessarily continuous) may have a Gâteaux derivative f'_{G} at a point of discontinuity.
3. If X is a normed vector space and f is a locally Lipschitz, that is, for any point $\bar{x} \in X$ there exists some neighborhood V of $\bar{x}$ and some constant $L > 0$ such that

$$|f(x) - f(y)| \leq L\|y - x\|, \text{ for all } x, y \in V,$$

then the two above concepts are equivalent.

1.2.1 Unconstrained Minimization Problems

In most situations in optimization, we begin by considering the following abstract minimization problem: minimize $f(x)$ subject to $x \in S$ where $f : S \to \mathbf{R}$ is defined on S which is a subset of a real vector space X. If we redefine the function f so that $f(x) = +\infty$ for $x \notin S$, then minimizing f over S is equivalent to minimizing the new f over all of X. So, no generality is lost in this paragraph if we restrict our attention to the case where $S = X$. Let $f : X \to \mathbf{R}$ be a function and $\bar{x}$ be a point in X. Thus, let us consider the following unconstrained minimization problem:

$$(\text{UP}) \qquad \begin{cases} \text{Minimize} \quad f(x) \\ \text{subject to } x \in X. \end{cases}$$

Definition 1.3. We will say that

1. f has a local minimum at $\bar{x}$ if and only if there exists a neighborhood V of $\bar{x}$ such that $f(\bar{x}) \leq f(x)$, for all $x \in V$.
2. f has a global minimum at $\bar{x}$ over X if and only if $f(\bar{x}) \leq f(x)$, for all $x \in X$.

Assume that f is Gâteaux differentiable at $\bar{x} \in X$.

Fact 1. If f has a local minimum at $\bar{x}$, then there exists some $\varepsilon > 0$ such that

$$\left\langle f'_{\mathrm{G}}(\bar{x}), x - \bar{x} \right\rangle \geq 0, \text{ for all } x \in \bar{x} + \varepsilon \mathbf{B}. \tag{1.3}$$

Proof. Assume that f has a local minimum at $\bar{x}$, then there exists some $\alpha > 0$ such that

$$f(\bar{x}) \leq f(x), \quad \text{for all } x \in \bar{x} + \alpha \mathbf{B}. \tag{1.4}$$

Fix $\varepsilon \in (0, \alpha)$ and $\delta \in (0, \frac{\alpha}{\varepsilon})$, and fix any $x \in \bar{x} + \varepsilon \mathbf{B}$. Hence,

$$\bar{x} + \delta(x - \bar{x}) \in \bar{x} + \delta \varepsilon \mathbf{B} \subset \bar{x} + \alpha \mathbf{B}$$

and so we get by (1.4)

$$f(\bar{x} + \delta(x - \bar{x})) - f(\bar{x}) \geq 0,$$

for all $\delta \in (0, \frac{\alpha}{\varepsilon})$ and for all $x \in \bar{x} + \varepsilon \mathbf{B}$. Therefore, as f is Gâteaux differentiable at $\bar{x}$, the limit

$$\lim_{\delta \downarrow 0} \delta^{-1} \left[f(\bar{x} + \delta(x - \bar{x})) - f(\bar{x}) \right]$$

exists and so $\left\langle f'_{\mathrm{G}}(\bar{x}), x - \bar{x} \right\rangle \geq 0$ for all $x \in \bar{x} + \varepsilon \mathbf{B}$. □

Exercise 1.2.

1. Prove that the converse in Fact 1 is not true in general. This ensures that (1.3) is only a necessary optimality condition for (UP).
2. Prove that (1.3) is equivalent to

$$f'_{\mathrm{G}}(\bar{x}) = 0. \tag{1.5}$$

Assume now that the function f is not Gâteaux differentiable and f is convex. Take for instance $f(x) = \|x\|$. For this function, $f'_{\mathrm{G}}(0)$ does not exist and it is clear that f has a global minimum over X at $\bar{x} = 0$. But we cannot make use of Fact 1 to derive necessary optimality conditions like relations (1.3) or (1.5) for problem (UP), because f is not Gâteaux differentiable at $\bar{x}$. So it is a natural question to ask what could replace f'_{G} in those relations? One could think of making use of the directional derivative instead of the Gâteaux derivative as follows:

$$f'(\bar{x}; v) = 0, \text{ for all } v \in X. \tag{1.6}$$

However, the relation (1.6) does not hold for the above function, although $\bar{x}$ is a global minimum. Indeed, we can check that $f'(\bar{x}; v) = \|v\|$, for all $v \in X$ and so

$f'(\bar{x};v) = 0$ only for $v = 0$ and $f'(\bar{x};v) \neq 0$ for every $v \neq 0$. Therefore, we have to propose, something else to replace f'_G which is the subdifferential of f that we define below.

Definition 1.4. Let f be a convex continuous function on X and let $\bar{x} \in X$. We define the subdifferential of f at $\bar{x}$ as follows

$$\partial^{\text{conv}} f(\bar{x}) = \{\zeta \in X^* : \langle \zeta, v \rangle \leq f'(\bar{x};v), \text{ for all } v \in X\}. \tag{1.7}$$

Exercise 1.3. For every convex continuous function f, every $x \in X$, and every direction $v \in X$ one has:

1. The function $\delta \mapsto \delta^{-1}[f(x+\delta v) - f(x)]$ is nondecreasing for δ small enough.
2. The directional derivative $f'(\bar{x};v)$ exists and is positively homogeneous and subadditive on X with respect to v.
3. $$\partial^{\text{conv}} f(\bar{x}) = \{\zeta \in X^* : \langle \zeta, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \text{ for all } x \in X\}. \tag{1.8}$$
4. Calculus rules:
$$\partial^{\text{conv}}(f+g)(\bar{x}) = \partial^{\text{conv}} f(\bar{x}) + \partial^{\text{conv}} g(\bar{x}) \text{ and } \partial^{\text{conv}}(\alpha f)(\bar{x}) = \alpha \partial^{\text{conv}} f(\bar{x}), \tag{1.9}$$
whenever $\alpha \in \mathbf{R}$ and g is a convex continuous function on X.

Using the subdifferential concept we can derive an analogue to Fact 1 for convex continuous functions, i.e., necessary optimality conditions.

Proposition 1.1. *Let f be a convex continuous function on X and let $\bar{x} \in X$. If f has a local minimum over X at $\bar{x}$, then*

$$0 \in \partial^{\text{conv}} f(\bar{x}). \tag{1.10}$$

Proof. It follows the same lines as in the proof of Fact 1, by using the definition of the subdifferential in (1.7) or it follows directly from (1.8). □

In fact, for convex continuous functions we have a stronger version of Fact 1. Indeed, we can prove that (1.10) is a necessary and sufficient optimality condition for (UP). Further, any local minimum is a global minimum.

Proposition 1.2. *Let f be a convex continuous function on X and let $\bar{x} \in X$. The relation (1.10) is equivalent to each one of the following assertions:*

1. *f has a local minimum over X at $\bar{x}$;*
2. *f has a global minimum over X at $\bar{x}$.*

Proof. It follows from the relation (1.8). □

Proposition 1.3. *If f is a convex continuous and Gâteaux differentiable function at $\bar{x}$, then $\partial^{\text{conv}} f(\bar{x}) = \{f'_G(\bar{x})\}$ and so the relation (1.5) becomes a necessary and sufficient optimality condition for (UP).*

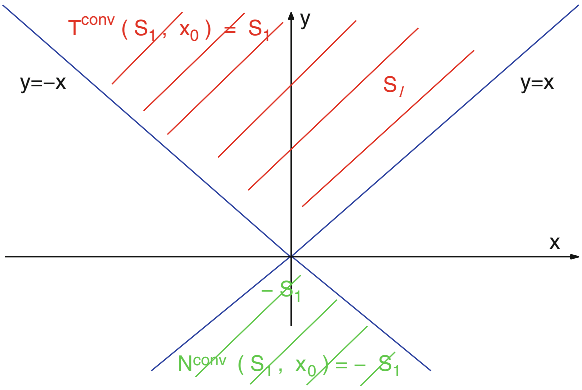


Fig. 1.1 Tangent and normal cones to convex sets

Proof. Let ζ be any element of $\partial^{\text{conv}} f(\bar{x})$. Then, $\langle \zeta, v \rangle \leq f'(\bar{x}; v)$ for all $v \in X$. On the other hand, by the Gâteaux differentiability of f at $\bar{x}$ one has $f'(\bar{x}; v) = \langle f'_{\text{G}}(\bar{x}), v \rangle$ for all $v \in X$. Consequently, we get $\langle \zeta, v \rangle \leq \langle f'_{\text{G}}(\bar{x}), v \rangle$, for all $v \in X$, which ensures that $\zeta = f'_{\text{G}}(\bar{x})$ and so $\partial^{\text{conv}} f(\bar{x}) = \{ f'_{\text{G}}(\bar{x}) \}$. The second part of the proposition follows from Proposition 1.2 and the first part of this proposition. □

1.2.2 Constrained Minimization Problems

Consider now the following constrained minimization problem:

$$\text{(CP)} \qquad \begin{cases} \text{Minimize} \quad f(x) \\ \text{subject to } x \in S, \end{cases}$$

where f is a convex continuous function and S is a closed convex set in X. First, we define the tangent cone and the normal cone for closed convex sets by

$$T^{\text{conv}}(S; \bar{x}) = \text{cl}\,[\mathbf{R}_+(S - \bar{x})] = \text{cl}\,\{\lambda(s - \bar{x}) : \lambda \geq 0, s \in S\}$$

and $N^{\text{conv}}(S; \bar{x})$ is the negative polar cone[1] of $T^{\text{conv}}(S; \bar{x})$, i.e.,

$$N^{\text{conv}}(S; \bar{x}) = \{\zeta \in X^* : \langle \zeta, v \rangle, \text{ for all } v \in T^{\text{conv}}(S; \bar{x})\}.$$

[1]For a closed nonempty set $L \subset X$, the negative polar of L is denoted by L^0 and defined as

$$L^0 = \{\zeta \in X^* : \langle \zeta, v \rangle, \text{ for all } v \in L\}.$$

Example 1.1. Let $S_1 = \{(x,y) \in \mathbf{R}^2 : y \geq |x|\}$ and $\bar{x} = (0,0)$ (see Fig. 1.1). This set is a closed convex cone and

$$T^{\mathrm{conv}}(S_1;\bar{x}) = \mathrm{cl}\,[\mathbf{R}_+(S_1 - \bar{x})] = \mathrm{cl}\,[\mathbf{R}_+(S_1)] = \mathrm{cl}\,[S_1] = S_1$$

and

$$N^{\mathrm{conv}}(S_1;\bar{x}) = -S_1 = \{(x,y) \in \mathbf{R}^2 : y \leq -|x|\}.$$

Exercise 1.4. Prove the following assertions for closed convex sets S and $\bar{x} \in S$:

1. $N^{\mathrm{conv}}(S;\bar{x}) = \{\zeta \in X : \langle \zeta, x - \bar{x} \rangle \leq 0, \text{ for all } x \in S\}$.
2. The distance function d_S is convex if and only if S is convex.
3. $\partial^{\mathrm{conv}} d_S(\bar{x}) = N^{\mathrm{conv}}(S;\bar{x}) \cap \mathbf{B}_*$.
4. $T^{\mathrm{conv}}(S;\bar{x})$ is a closed convex cone containing the vector zero.

Exercise 1.5. Prove the following:

1. Every l.s.c. convex function is continuous over $int(\mathrm{dom} f)$ the interior of the effective domain of f.
2. Assume that X is a real normed vector space. Every convex function which is finite on an open convex set Ω and bounded around some point $\bar{x} \in \Omega$, is locally Lipschitz on Ω.
3. For any closed subset S of X, and f is Lipschitz with ratio $k > 0$ on an open convex set Ω containing S, then any global minimum $\bar{x}$ of f over S is a global minimum of the function $f + kd_S$ over the whole space X.

We derive in the following proposition a necessary and sufficient optimality condition for (CP).

Proposition 1.4. *Let f be a convex continuous function on a closed convex set S and let $\bar{x} \in int(S)$. Then the following assertions are equivalent:*

1. *f has a local minimum over S at $\bar{x}$, i.e., there exists a neighborhood V of $\bar{x}$ such that $f(\bar{x}) \leq f(x)$, for all $x \in S \cap V$;*
2. *f has a global minimum over S at $\bar{x}$, i.e., $f(\bar{x}) \leq f(x)$, for all $x \in S$;*
3. $$0 \in \partial^{\mathrm{conv}} f(\bar{x}) + N^{\mathrm{conv}}(S;\bar{x}).$$

Proof. The implication $(1) \Rightarrow (2)$ is left to the reader as an exercise. We prove the implication $(2) \Rightarrow (3)$. Assume that f has a global minimum over S at $\bar{x}$. First, by the second part of Exercise 1.5, f is locally Lipschitz at $\bar{x}$ with some constant $k > 0$. Then by the third part of Exercise 1.5 the function $f + kd_S$ has a global minimum over X at $\bar{x}$, that is, $(f + kd_S)(\bar{x}) \leq (f + kd_S)(x)$ for all $x \in X$. This ensures by (1.9), (1.10) and the third part of Exercise 1.4, that $0 \in \partial^{\mathrm{conv}}(f + kd_S)(x) = \partial^{\mathrm{conv}} f(\bar{x}) + k\partial^{\mathrm{conv}} d_S(\bar{x}) \subset \partial^{\mathrm{conv}} f(\bar{x}) + N^{\mathrm{conv}}(S;\bar{x})$. The converse $(3) \Rightarrow (2)$ follows directly from the characterization of the normal cone in the first part of Exercise 1.4. $\square$

Assume now that the function f is neither convex nor Gâteaux differentiable. In this case, the directional derivative $f'(\bar{x};v)$ does not exist necessarily. Take for instance $f(x) = -\|x\|$ or $f(x) = x^2 \sin(1/x)$, for $x \neq 0$ and $f(0) = 0$, and take $\bar{x} = 0$. Even if $f'(\bar{x};v)$ exists, it may not preserve its important properties cited in Exercise 1.3. Consequently, the subdifferential $\partial^{\mathrm{conv}} f$ loses almost all of its properties, and in particular relation (1.8) as well as the characterization of the global minimum given in Propositions 1.1 and 1.4. Thus, it would be interesting to ask *what could possibly replace both the Gâteaux derivative (for Gâteaux differentiable functions) in Fact 1 and the subdifferential (for convex continuous functions) in Propositions 1.1 and 1.4*. The answer to this question was given by Clarke in [86] when he introduced a generalized gradient (also known as the Clarke subdifferential) for nondifferentiable nonconvex functions and developed a new theory that he called *Nonsmooth Analysis Theory*. Our primary goal in this book is to focus upon this theory and its applications.

1.3 Subdifferentials

In this section, we will assume that X is a normed vector space and $f : X \to \mathbf{R}$ is a locally Lipschitz function at $\bar{x} \in X$ with ratio $k > 0$.

1.3.1 The Generalized Gradient (Clarke Subdifferential)

We have seen that for convex continuous functions the subdifferential was defined in terms of the directional derivative $f'(\bar{x}, \cdot)$ (see Definition 1.4). Following the same idea, we define the the generalized gradient by using a new concept of directional differentiability because, as we have mentioned in the end of the previous section, the directional derivative $f'(\bar{x}, \cdot)$ loses almost all of its properties and it is not the appropriate directional derivative that can be used to define the generalized gradient (Clarke subdifferential). The new concept of directional derivative is called *the generalized directional derivative* (also known as *Clarke directional derivative*) and is defined by

$$f^0(\bar{x};v) = \limsup_{\substack{x \to \bar{x} \\ t \downarrow 0}} t^{-1}[f(x+tv) - f(x)]. \tag{1.11}$$

The generalized gradient (Clarke subdifferential) of f at $\bar{x}$ is defined then as

$$\partial^{\mathrm{C}} f(\bar{x}) = \{\zeta \in X^* : \langle \zeta, v \rangle \leq f^0(\bar{x};v), \text{ for all } v \in X\}. \tag{1.12}$$

The following proposition summarizes the most important properties of the generalized directional derivative and the generalized gradient for locally Lipschitz functions.

Proposition 1.5.

1. *The function* $v \mapsto f^0(\bar{x};v)$ *is finite, positively homogeneous, subadditive, and satisfies*
$$|f^0(\bar{x};v)| \le k\|v\|, \text{ for all } v \in X. \tag{1.13}$$
2. $(f+g)^0(\bar{x};v) \le f^0(\bar{x};v) + g^0(\bar{x};v)$, *where* g *is a locally Lipschitz function at* $\bar{x}$.
3. *Sum rules:*
$$\partial^C(f+g)(\bar{x}) \subset \partial^C f(\bar{x}) + \partial^C g(\bar{x}),$$
where g *is a locally Lipschitz function at* $\bar{x}$.
4. *For every* $\alpha \in \mathbf{R}$ *one has* $(\alpha f)^0(\bar{x};v) = \alpha f^0(\bar{x};v)$ *and hence* $\partial^{\mathrm{C}}(\alpha f)(\bar{x}) = \alpha \partial^{\mathrm{C}} f(\bar{x})$.
5. *If* f *has a local minimum or maximum at* $\bar{x}$, *then* $0 \in \partial^{\mathrm{C}} f(\bar{x})$.
6. *The generalized gradient* $\partial^{\mathrm{C}} f(\bar{x})$ *is a nonempty, convex,* w^**-compact subset in* X^* *and satisfies* $\partial^{\mathrm{C}} f(\bar{x}) \subset k\mathbf{B}_*$.
7. *If* x_n *and* ζ_n *are two sequences in* X *and* X^* *respectively such that* $\zeta_n \in \partial^{\mathrm{C}} f(x_n)$ *and* x_n *strongly converges to* x *and* ζ_n w^**-converges to* ζ, *then we have* $\zeta \in \partial^{\mathrm{C}} f(x)$.
8. *Mean Value Theorem: If* f *is locally Lipschitz on an open neighborhood containing the segment* $[x,y]$, *then there exists* $z \in [x,y]$ *and* $\xi \in \partial^{\mathrm{C}} f(z)$ *satisfying*
$$f(y) - f(x) = \langle \xi, y - x \rangle.$$
9. *Chain rule: Let* $F : \mathbf{H} \to \mathbf{R}^n$ *be locally Lipschitz*[2] *at* $\bar{x}$ *and let* $g : \mathbf{R}^n \to \mathbf{R}$ *be locally Lipschitz at* $F(\bar{x})$. *Then the function* $g \circ F$ *is locally Lipschitz at* $\bar{x}$ *and*
$$\partial^{\mathrm{C}}(g \circ F)(\bar{x}) \subset \overline{co}\{\partial^{\mathrm{C}}(\langle \xi, F(\cdot) \rangle)(\bar{x}) : \ \xi \in \partial^{\mathrm{C}} g(F(\bar{x}))\}.$$
10. *Pointwise maximum rule: Let* f *be a pointwise maximum of a finite number of locally Lipschitz functions at* $\bar{x}$, *that is,* $f(x) = \max_{1 \le n \le N} f_n(x)$ *with each* f_n *locally Lipschitz at* $\bar{x}$. *Then* f *is locally Lipschitz at* $\bar{x}$ *and satisfies*
$$\partial^{\mathrm{C}} f(\bar{x}) \subset co\{\partial^{\mathrm{C}} f_n(\bar{x}) : n \in I(\bar{x})\},$$
where $I(\bar{x})$ *denotes the set of indices* n *for which* $f(\bar{x}) = f_n(\bar{x})$.

Proof.

1. By the local Lipschitz property of f at $\bar{x}$, we get for $t > 0$ small enough and for x sufficiently close to $\bar{x}$
$$|t^{-1}[f(x+tv) - f(x)]| \le k\|v\|, \text{ for all } v \in X.$$

[2] $F : \mathbf{H} \to \mathbf{R}^n$ is locally Lipschitz at $\bar{x}$ means that $F = (f_1, f_2, \ldots, f_n)$ and each $f_i : \mathbf{H} \to \mathbf{R}$ $(i = 1, 2, \ldots, n)$ is locally Lipschitz at $\bar{x}$.

Thus, the inequality in Part (1) is proved and so $f^0(\bar{x};v)$ is finite. The fact that $f^0(\bar{x};v)$ is positively homogeneous and subadditive follows from the definition and the fact that the upper limit of a sum is bounded above by the sum of the upper limits.

2. It follows also from the fact that the upper limit of a sum is bounded above by the sum of the upper limits.
3. It is a consequence of Part (2) and the definition of the generalized gradient.
4. For $\alpha \geq 0$, it is easy to check that $(\alpha f)^0(\bar{x};v) = \alpha f^0(\bar{x};v)$ and hence $\partial^{\mathrm{C}}(\alpha f)(\bar{x}) = \alpha \partial^{\mathrm{C}} f(\bar{x})$, for every $\alpha \geq 0$. So it suffices to prove this equality for $\alpha = -1$, that is, $\partial^{\mathrm{C}}(-f)(\bar{x}) = -\partial^{\mathrm{C}} f(\bar{x})$. Observe first that

$$(-f)^0(\bar{x};v) = f^0(\bar{x};-v).$$

Indeed, by the definition of the generalized directional derivative one has

$$\begin{aligned}(-f)^0(\bar{x};v) &= \limsup_{\substack{x\to\bar{x}\\ t\downarrow 0}} t^{-1}[-f(x+tv)-(-f)(x)]\\ &= \limsup_{\substack{x'\to\bar{x}\\ t\downarrow 0}} t^{-1}[f(x'-tv)-f(x')]\\ &= f^0(\bar{x};-v).\end{aligned}$$

Thus, fix any element ζ in $\partial^{\mathrm{C}}(-f)(\bar{x})$. Then for all $v \in X$ one has

$$\langle \zeta, v\rangle \leq (-f)^0(\bar{x};v) = f^0(\bar{x};-v),$$

which is equivalent to $\langle -\zeta, w\rangle \leq f^0(\bar{x};w)$, for all $w \in X$. This along with the definition of the generalized gradient, ensures that $\zeta \in -\partial^{\mathrm{C}} f(\bar{x})$, thus completing the proof of this part.

5. First, we observe that the generalized directional derivative satisfies

$$\liminf_{t\downarrow 0} t^{-1}[f(\bar{x}+tv)-f(\bar{x})] \leq f^0(\bar{x};v) \text{ for all } v \in X.$$

This liminf is called *the lower Dini directional derivative* and is denoted by $f^-(\bar{x};v)$, i.e.,

$$f^-(\bar{x};v) := \liminf_{t\downarrow 0} t^{-1}[f(\bar{x}+tv)-f(\bar{x})]. \tag{1.14}$$

Assume now that f has a local minimum (the case of local maximum may be done similarly) at $\bar{x}$, that is, there exists $\varepsilon > 0$ such that

$$f(x) \geq f(\bar{x}) \text{ for all } x \in \bar{x}+\varepsilon\mathbf{B}.$$

Let v be any direction in X and fix $\delta > 0$ such that $\delta\|v\| \leq \varepsilon$. Then, by the above inequality we get for any $t \in (0,\delta)$

$$t^{-1}\left[f(\bar{x}+tv)-f(\bar{x})\right]\geq 0.$$

Therefore, we get $f^0(\bar{x};v)\geq f^-(\bar{x};v)\geq 0$, which ensures, by the definition of the generalized gradient, that $0\in\partial^{\mathrm{C}}f(\bar{x})$.

6. It is a consequence of Part (1).
7. Let x_n be a sequence in X and let ζ_n be a sequence in X^* such that $\zeta_n\in\partial^{\mathrm{C}}f(x_n)$ and x_n strongly converges to x and ζ_n w^*-converges to ζ. Then,

$$\left\langle\zeta_n,v\right\rangle\leq f^0(x_n;v)\text{ for all }v\in X.$$

Clearly, the upper semicontinuity of the function $x\mapsto f^0(x;v)$ completes the proof of this part. So we have to show

$$\limsup_n f^0(x_n;v)\leq f^0(x;v).$$

By the definition of the upper limit, there exists $y_n\in X$ and $t_n>0$ such that

$$\|y_n-x_n\|+t_n<\frac{1}{n}\text{ and }f^0(x_n;v)\leq t_n^{-1}\left[f(y_n+t_nv)-f(y_n)\right]+\frac{1}{n}.$$

Upon letting $n\to+\infty$ we get the desired inequality of the u.s.c.

8. Let $g:[0,1]\to\mathbf{R}$ be a function defined by

$$g(t)=f(x+t(y-x))+t(f(x)-f(y)),\text{ for all }t\in[0,1].$$

Since f is locally Lipschitz on an open neighborhood containing the segment $[x,y]$, it is easy to check that g is locally Lipschitz on $[0,1]$ and satisfies $g(0)=g(1)$. Then, by the classical intermediate value theorem, there exists at least one point $t\in(0,1)$, where g attains its local maximum or minimum. Therefore, by the Part (5) of this theorem we have $0\in\partial^{\mathrm{C}}g(t)$ and so by the sum rule in the part (4) we get

$$0\in\partial^{\mathrm{C}}h(t)+f(x)-f(y),\text{ i.e., }f(y)-f(x)\in\partial^{\mathrm{C}}h(t),$$

where $h(t)=f(x+t(y-x))$. On the other hand, we can verify that any element ζ of $\partial^{\mathrm{C}}h(t)$ can be written in the form $\zeta=\left\langle\xi,y-x\right\rangle$, with $\xi\in\partial^{\mathrm{C}}f(x+t(y-x))$. Indeed, let $\zeta\in\partial^{\mathrm{C}}h(t)$. Then,

$$\zeta v\leq h^0(t;v).$$

Let $t_n\to t$ and $\lambda_n\downarrow 0$ be sequences realizing the limsup in the definition of $h^0(t;v)$, i.e.,

$$h^0(t;v)=\lim_{n\to\infty}\lambda_n^{-1}\left[h(t_n+\lambda_nv)-h(t_n)\right].$$

Then for $z_n = x + t_n(y-x) \to x + t(y-x)$ we have

$$\begin{aligned} h^0(t;v) &= \lim_{n\to\infty} \lambda_n^{-1} \left[f(z_n + \lambda_n v(y-x)) - f(z_n) \right] \\ &\le \limsup_{\substack{\lambda \downarrow 0 \\ z \to x}} \lambda^{-1} \left[f(z + \lambda v(y-x)) - f(z) \right] \\ &= f^0(x + t(y-x); v(y-x)). \end{aligned}$$

Now we note that

$$f^0(x+t(y-x);y-x) = \max_{\xi \in \partial^C f(x+t(y-x))} \left\{ \langle \xi, y-x \rangle \right\}$$

and

$$-f^0(x+t(y-x);-(y-x)) = \min_{\xi \in \partial^C f(x+t(y-x))} \left\{ \langle \xi, y-x \rangle \right\},$$

and we recall that the generalized directional derivative is positively homogenous with respect to the direction. Consequently, simple calculations yield

$$\zeta \le f^0(x+t(y-x);y-x) = \max_{\xi \in \partial^C f(x+t(y-x))} \left\{ \langle \xi, y-x \rangle \right\}$$

and

$$\zeta \le -f^0(x+t(y-x);-(y-x)) = \min_{\xi \in \partial^C f(x+t(y-x))} \left\{ \langle \xi, y-x \rangle \right\}.$$

These two inequalities and the convexity of $\langle \partial^C f(x+t(y-x)), y-x \rangle$ ensure that $\zeta \in \langle \partial^C f(x+t(y-x)), y-x \rangle$. Finally, we get an element $z = x + t(y-x) \in [x,y]$ and an element $\xi \in \partial^C f(z)$ such that

$$f(y) - f(x) = \langle \xi, y-x \rangle,$$

which is the desired relation.

9. First we calculate the generalized directional derivative of f. Let $x_n \to \bar{x}$ and $t_n \downarrow 0$ be sequences realizing the limsup in the definition of $f^0(t;v)$, i.e.,

$$f^0(\bar{x};v) = \lim_{n\to\infty} t_n^{-1} \left[f(x_n + t_n v) - f(x_n) \right].$$

By the Mean Value Theorem proved in Part (8), there exists a sequence z_n in the open segment $(F(x_n), F(x_n + t_n v))$ which converges to $F(x)$ and $\xi_n \in \partial^C g(z_n)$ such that

$$g(F(x_n + t_n v)) - g(F(x_n)) = \langle \xi_n, F(x_n + t_n v) - F(x_n) \rangle.$$

Observe first that the sequence ξ_n is bounded in $\mathbf{R}^n$ and so we may extract a subsequence which converges to some limit ξ. By Part (7) of this theorem, this limit must lie in $\partial^{\mathrm{C}} g(F(\bar{x}))$. We apply now the Mean Value Theorem with the function $\langle \xi, F(\cdot)\rangle$ on the segment $[x_n, x_n + t_n v]$ and we get a sequence y_n in the open segment $(x_n, x_n + t_n v)$ which converges to $\bar{x}$ and a sequence $\zeta_n \in \partial^{\mathrm{C}}\left[\langle \xi, F(\cdot)\rangle\right](y_n)$ such that

$$\langle \xi, F(x_n + t_n v)\rangle - \langle \xi, F(x_n)\rangle = \langle \zeta_n, t_n v\rangle.$$

Observe that the sequence ζ_n is bounded in X^* and so we may extract a subsequence which w^*-converges to some limit ζ. Part (7) ensures once again that $\zeta \in \partial^{\mathrm{C}}\left[\langle \xi, F(\cdot)\rangle\right](\bar{x})$. Thus, we have

$$\begin{aligned} t_n^{-1}\left[f(x_n + t_n v) - f(x_n)\right] &= t_n^{-1}\left[g(F(x_n + t_n v)) - g(F(x_n))\right] \\ &= t_n^{-1}\left[\langle \xi_n, F(x_n + t_n v) - F(x_n)\rangle\right] \\ &= \langle \zeta_n, v\rangle + t_n^{-1}\left[\langle \xi - \xi_n, F(x_n + t_n v) - F(x_n)\rangle\right]. \end{aligned}$$

Finally, as $t_n^{-1}\left[F(x_n + t_n v) - F(x_n)\right]$ is bounded because F is Lipschitz and as $\xi_n \to \xi$ we have $t_n^{-1}\left[\langle \xi - \xi_n, F(x_n + t_n v) - F(x_n)\rangle\right] \to 0$ as $n \to \infty$ and so letting $n \to \infty$ in the last system of equalities we obtain

$$f^0(\bar{x}; v) = \langle \zeta, v\rangle.$$

Take now any element $w \in \partial^{\mathrm{C}} f(\bar{x})$. Then, by what precedes we have $\langle w, v\rangle \le f^0(\bar{x}; v) = \langle \zeta, v\rangle$, for all $v \in X$ and so $w = \zeta$, where ζ is the w^*-limit of a sequence $\zeta_n \in \partial^{\mathrm{C}}\left[\langle \xi, F(\cdot)\rangle\right](y_n)$ with $y_n \to \bar{x}$ and $\xi \in \partial^{\mathrm{C}} g(F(\bar{x}))$. This completes the proof of the chain rule.

10. To prove this part, we use the chain rule just proved in Part (9) and so it is left to the reader as an exercise. □

Exercise 1.6. Let f be locally Lipschitz at $\bar{x}$. Prove the following:

1. If f is Gâteaux differentiable at $\bar{x}$, then $f'_{\mathrm{G}}(\bar{x}) \in \partial^{\mathrm{C}} f(\bar{x})$;
2. If f is continuously differentiable at $\bar{x}$, then $\partial^{\mathrm{C}} f(\bar{x}) = \{f'_{\mathrm{G}}(\bar{x})\}$;
3. Compute $f^0(0;\cdot)$ and $\partial^{\mathrm{C}} f(0)$ for $f : \mathbf{R}^N \to \mathbf{R}$ given by $f(x) = \|x\|$. Same questions for $f(x) = \max\{0, x\}$.

1.3.2 Other Concepts of Subdifferentials

Many other concepts of subdifferentiability for nonconvex functions have been introduced since the generalized gradient. We state in this subsection some of them.

We start with the *Dini subdifferential* which is defined in terms of the lower Dini directional derivative $f^-(\bar{x};\cdot)$ (see (1.14) for the definition) in the same way the generalized gradient is expressed. It is defined as

$$\partial^- f(\bar{x}) = \{\zeta \in X^* : \langle \zeta, v \rangle \le f^-(\bar{x}; v), \text{ for all } v \in X\}.$$

The other concepts of subdifferential that we state here are defined otherwise.

- The *Fréchet subdifferential* of f at $\bar{x}$ is denoted by $\widehat{\partial} f(\bar{x})$ (also denoted by $\partial^F f(\bar{x})$) and defined as the set of all $\zeta \in X^*$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \le f(x) - f(\bar{x}) + \varepsilon \|x - \bar{x}\|, \text{ for all } x \in \bar{x} + \delta \mathbf{B}.$$

- The *basic subdifferential* (also called *limiting subdifferential or Mordukhovich subdifferential*) of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) = \{\zeta \in X^* : \exists x_n \to \bar{x}, \exists \zeta_n \to^{w^*} \zeta \text{with} \zeta_n \in \widehat{\partial} f(x_n)\}.$$

  The set in the right side of this equality is denoted by $\limsup_{x \to \bar{x}} \widehat{\partial} f(x)$.

- The *proximal subdifferential* of f at $\bar{x}$ is denoted by $\partial^P f(\bar{x})$ and defined as the set of all $\zeta \in X^*$ for which there exist two real numbers $\sigma > 0$ and $\delta > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \le f(x) - f(\bar{x}) + \sigma \|x - \bar{x}\|^2, \text{ for all } x \in \bar{x} + \delta \mathbf{B}.$$

Exercise 1.7. Show that one always has the following inclusions:

$$\partial^P f(\bar{x}) \subset \widehat{\partial} f(\bar{x}) \subset \partial f(\bar{x}) \subset \partial^C f(\bar{x}) \quad \text{and} \quad \partial^P f(\bar{x}) \subset \widehat{\partial} f(\bar{x}) \subset \partial^- f(\bar{x}) \subset \partial^C f(\bar{x}).$$

Note that in general there is no relation between the Dini subdifferential $\partial^- f(\bar{x})$ and the basic subdifferential $\partial f(\bar{x})$. The inclusions in the previous exercise may be strict in general as Parts (2) and (3) in the next exercise prove it.

It is very important to point out that the Dini subdifferential $\partial^- f$ and the Fréchet subdifferential $\widehat{\partial} f$ coincide whenever X is assumed to be a finite dimensional space. In the infinite dimensional space we can find functions f for which the inclusion $\widehat{\partial} f \subset \partial^- f$ is strict (see for example [61]).

Exercise 1.8.

1. For any convex continuous function f one has

$$\partial^P f(\bar{x}) = \widehat{\partial} f(\bar{x}) = \partial f(\bar{x}) = \partial^C f(\bar{x}) = \partial^{\text{conv}} f(\bar{x}) \neq \emptyset.$$

2. For $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = -|x|$ with $\bar{x} = 0$, one has

$$\partial^P f(\bar{x}) = \widehat{\partial} f(\bar{x}) = \emptyset, \partial f(\bar{x}) = \{-1, +1\}, \text{and} \partial^C f(\bar{x}) = [-1, +1].$$

3. For $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = -(|x|)^{3/2}$ one has $\widehat{\partial} f(0) = \partial^C f(0) = \{0\}$ while $\partial^P f(0) = \emptyset$.

1.4 Tangent Cones

In this section, X will be a *normed vector* space.

There are several ways to define general concepts of tangent cones of a subset S at some point $\bar{x} \in S$ in Nonsmooth Analysis Theory. We state here the definition of Clarke tangent cone $T^{\mathrm{C}}(S;\bar{x})$ (also called regular tangent cone in Mordukhovich [192]) and the contingent cone $K(S;\bar{x})$ (also called Bouligand tangent cone) which play a crucial role in Nonsmooth Analysis Theory and will be needed for our study in the present book.

Definition 1.5. Let S be a nonempty closed subset of X and $\bar{x} \in S$.

- The contingent cone $K(S;\bar{x})$ to S at $\bar{x}$ is the set of all $v \in X$ for which one has $d_S^-(\bar{x}; v) = 0$.
- The Clarke tangent cone $T^{\mathrm{C}}(S;\bar{x})$ to S at $\bar{x}$ is the set of all $v \in X$ for which one has $d_S^0(\bar{x}; v) = 0$.

We begin by stating sequential characterizations of $T^{\mathrm{C}}(S;\bar{x})$ and $K(S;\bar{x})$.

Proposition 1.6.

1. $T^{\mathrm{C}}(S;\bar{x}) = \{v \in X : \forall t_n \downarrow 0, \forall x_n \to^S \bar{x}, \exists v_n \to v \text{ s.t. } x_n + t_n v_n \in S\ \forall n\}$;
2. $K(S;\bar{x}) = \{v \in X : \exists t_n \downarrow 0 \exists v_n \to v \text{ s.t. } \bar{x} + t_n v_n \in S\ \forall n\}$.
Here, $x_n \to^S \bar{x}$ *means that* $x_n \to \bar{x}$ *and* $x_n \in S$ *for all* n.

Proof. We prove only the sequential characterizations of $T^{\mathrm{C}}(S;\bar{x})$ and the other one may be conducted likewise. We start with the inclusion

$$T^{\mathrm{C}}(S;\bar{x}) \subset \{v \in X : \forall t_n \downarrow 0, \forall x_n \to^S \bar{x}, \exists v_n \to v \text{ s.t. } x_n + t_n v_n \in S\ \forall n\}.$$

Let $v \in T^{\mathrm{C}}(S;\bar{x})$. Then $d_S^0(\bar{x}; v) = 0$, i.e.,

$$\limsup_{\substack{x \to \bar{x} \\ t \downarrow 0}} t^{-1} [d_S(x + tv) - d_S(x)] = 0.$$

By the definition of the limsup we get

$$\inf_{\substack{V \in \mathcal{N}(\bar{x}) \\ \delta > 0}} \sup_{\substack{x \in V \\ 0 < t < \delta}} t^{-1} [d_S(x + tv) - d_S(x)] = 0.$$

Then, for any $\varepsilon > 0$ there exist $V \in \mathcal{N}(\bar{x})$ and $\delta > 0$ such that

$$|d_S(x+tv) - d_S(x)| < \varepsilon, \text{ for all } x \in V, \text{ and all } t \in (0,\delta).$$

So,

$$d_S(x+tv) < \varepsilon, \text{ for all } x \in V \cap S, \text{ and all } t \in (0,\delta).$$

Then, for any sequence $t_n \downarrow 0$ and any sequence $x_n \rightarrow^S \bar{x}$ we have

$$d_S(x_n + t_n v) < t_n^2,$$

which ensures the existence of some $v_n \rightarrow v$ such that $x_n + t_n v_n \in S$, for all n. Indeed, by the definition of the inf and the last equality, there exists $y_n \in S$ such that

$$\|x_n + t_n v - y_n\| < d_S(x_n + t_n v) + t_n^2 = 2t_n^2.$$

Put $v_n = t_n^{-1}(y_n - x_n)$. Hence,

$$\|v - v_n\| = t_n^{-1}\|x_n + t_n v - y_n\| < \frac{t_n}{2} \rightarrow 0.$$

Thus, completing the proof of the desired inclusion. Now we prove the converse. Let v be the limit of a sequence v_n which satisfies $x_n + t_n v_n \in S$, for all n and for any sequence $t_n \downarrow 0$ and any sequence $x_n \rightarrow^S \bar{x}$. We have to show that this point v belongs to $T^C(S;\bar{x})$. Let $t_n \downarrow 0$ and $x_n \rightarrow \bar{x}$ (not necessarily in S) realizing the limsup in the definition of $d_S^0(\bar{x};v)$, i.e.,

$$d_S^0(\bar{x};v) = \lim_n t_n^{-1}\left[d_S(x_n + t_n v) - d_S(x_n)\right].$$

By the definition of the inf, there exists for every n an element $y_n \in S$ satisfying

$$\|y_n - x_n\| < d_S(x_n) + t_n^2.$$

It follows that $y_n \rightarrow^S \bar{x}$ and so we have $y_n + t_n v_n \in S$, for all n. Therefore,

$$\begin{aligned} d_S(x_n + t_n v) - d_S(x_n) &= d_S(y_n + t_n v) + \|y_n - x_n\| - d_S(x_n) \\ &= \|y_n - x_n\| - d_S(x_n) \\ &< t_n^2, \end{aligned}$$

and so

$$t_n^{-1}\left[d_S(x_n + t_n v) - d_S(x_n)\right] < t_n.$$

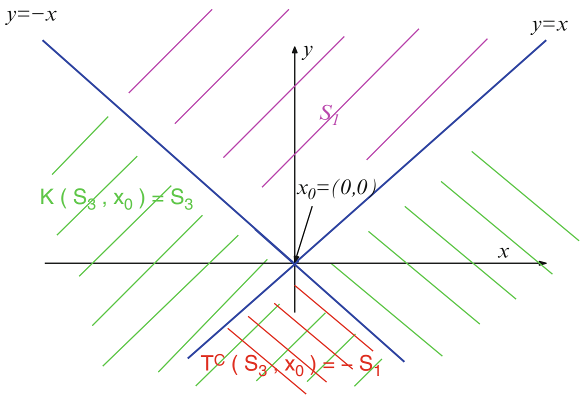


Fig. 1.2 Clarke and Bouligand tangent cones to S_3

Passing to the limit when $n \to \infty$ we get

$$d_S^0(\bar{x}; v) \le 0,$$

and as the reverse inequality is always true we conclude that $d_S^0(\bar{x}; v) = 0$, that is, $v \in T^C(S; \bar{x})$ and so the proof is complete. □

Exercise 1.9.

1. Show that one always has $0 \in T^C(S; \bar{x}) \subset K(S; \bar{x})$ and $T^C(S; \bar{x})$ is a closed convex cone and that $K(S; \bar{x})$ is a closed cone. Give an example showing that $K(S; \bar{x})$ may be nonconvex.
2. Prove that for closed convex sets S and any $\bar{x} \in S$ one has $T^C(S; \bar{x}) = K(S; \bar{x}) = T^{\text{conv}}(S; \bar{x})$.

Exercise 1.10.

1. Let S_3 be the closure of the complement of the set S_1 given in Example 1.1 and $x_0 = (0,0)$. Show that $T^C(S; x_0) = -S_1$ and $K(S; x_0) = S_3$ (see Fig. 1.2).
2. Let $S_4 = \{(x,y) \in \mathbf{R}^2 : \|(x-1,y)\| \le 1\} \cup \{(x,y) \in \mathbf{R}^2 : \|(x+1,y)\| \le 1\}$ and $\bar{x} = (0,0)$(see Fig. 1.3). Show that $T^C(S_4; \bar{x}) = \{(0,0)\}$ and $K(S_4; \bar{x}) = \mathbf{R}^2$.

1.5 Normal Cones

As for tangent cones we may consider several concepts of normal cones to nonconvex sets. We present here some with different examples to illustrate the notions.

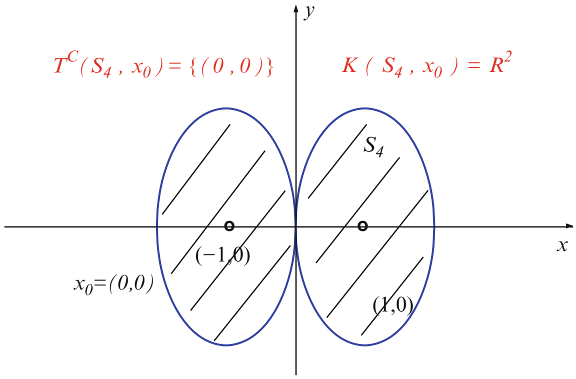


Fig. 1.3 Clarke and Bouligand tangent cones to S_4

1.5.1 The Convexified (Clarke) Normal Cone

Let X be a normed vector space and S be a nonempty closed subset in X, and $\bar{x} \in S$. There are many ways to define the *Convexified Normal Cone* (also called *Clarke normal cone*) to S at $\bar{x}$. As we have defined in the previous section the Clarke tangent cone $T^C(S;\bar{x})$, the likely candidate for the convexified (Clarke) normal cone is the one obtained from $T^C(S;\bar{x})$ by polarity. So, we define $N^C(S;\bar{x})$, the convexified (Clarke) normal cone to S at $\bar{x}$, as follows:

$$N^C(S;\bar{x}) = \{\zeta \in X^* : \langle \zeta, v \rangle \leq 0, \forall v \in T^C(S;\bar{x})\}.$$

Example 1.2.

1. For any closed convex set S and $\bar{x} \in S$ one has $N^C(S;\bar{x}) = N^{\text{conv}}(S;\bar{x})$.
2. Let $S = \{(x,y) \in \mathbf{R}^2 : y \leq |x|\}$ and $\bar{x} = (0,0)$. In the previous section, (see Exercise 1.10 Part (2)), we have seen that $T^C(S;\bar{x}) = \{(v_1,v_2) \in \mathbf{R}^2 : v_2 \leq -|v_1|\}$ (see Fig. 1.4). Using the definition of the convexified (Clarke) normal cone we can prove
$$N^C(S;\bar{x}) = \{(\xi_1,\xi_2) \in \mathbf{R}^2 : \xi_2 \geq |\xi_1|\}.$$

1.5.2 The Proximal Normal Cone

First, we give the definition of the proximal normal cone in Hilbert spaces. Let S be a nonempty closed set of $\mathbf{H}$.

Definition 1.6. Let $\bar{u} \notin S$. We define $\text{Proj}_S(\bar{u})$ the projection of $\bar{u}$ on S (may be empty) as the set of all $\bar{x} \in S$ whose distance to $\bar{u}$ is minimal, that is, $\|\bar{u} - \bar{x}\| = d_S(\bar{u})$. So
$$\text{Proj}_S(\bar{u}) = \{\bar{x} \in S : d_S(\bar{u}) = \|\bar{u} - \bar{x}\|\}.$$

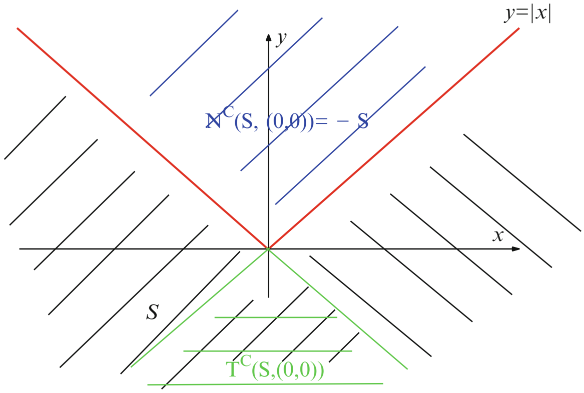


Fig. 1.4 Clarke tangent cone and convexified (Clarke) normal cone to nonconvex sets

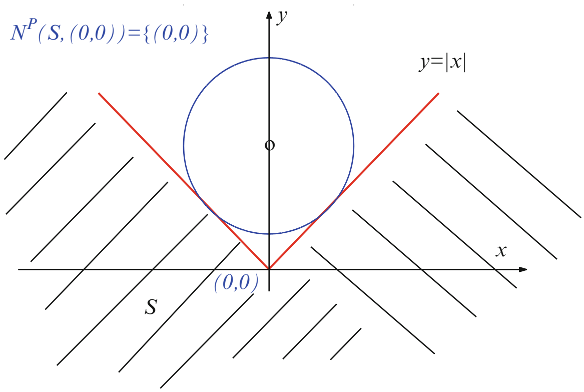


Fig. 1.5 $N^P(S;\bar{x}) = \{(0,0)\}$ with $\bar{x} \in S$

Let $\bar{x} \in S$. We define the proximal normal cone to S at $\bar{x}$ as the set of all elements $\xi \in \mathbf{H}$ for which there exists a positive number $r > 0$ such that

$$d_S(\bar{x} + r\xi) = r\|\xi\|, \text{ i.e. } \bar{x} \text{ is the projection of } \bar{x} + r\xi \text{ on } S.$$

So

$$N^P(S;\bar{x}) = \{\xi \in \mathbf{H} : \exists r > 0 : d_S(\bar{x} + r\xi) = r\|\xi\|\}. \tag{1.15}$$

Remark 1.2.

1- When $\bar{x} \notin S$, the proximal normal cone $N^P(S;\bar{x})$ is undefined.
2- When $\bar{x}$ belongs to S and is such that $\bar{x} \notin \mathrm{Proj}_S(u)$, for all $u \notin S$ (i.e., there is no point u outside of S such that $\bar{x} \in \mathrm{Proj}_S(u)$ (which is the case when $\bar{x} \in \mathrm{int}\, S$) we set $N^P(S;\bar{x}) = \{0\}$. The following example illustrates this fact.

Example 1.3. Let $S = \{(x,y) \in \mathbf{R}^2 : y \leq |x|\}$ and $\bar{x} = (0,0)$ (see Fig. 1.5). There are no points outside S whose projection on S is $\bar{x}$. So, $N^P(S;\bar{x}) = \{(0,0)\}$, while $\bar{x} \in bd\, S$ (not in $\mathrm{int}\, S$).

Proposition 1.7. *The following analytic characterizations of the proximal normal cone hold:*

$$\xi \in N^P(S;\bar{x}) \;\Leftrightarrow\; \exists\ \sigma,\delta > 0 \ \text{ s. t.} \begin{cases} \langle \xi, x-\bar{x}\rangle \le \sigma\|x-\bar{x}\|^2, \\ \text{for all } \ x \in (\bar{x}+\delta\mathbf{B}) \cap S; \end{cases}$$

$$\Leftrightarrow\; \exists\ \sigma = \sigma(\xi,\bar{x}) > 0 \ \text{ s. t.} \begin{cases} \langle \xi, x-\bar{x}\rangle \le \sigma\|x-\bar{x}\|^2, \\ \text{for all } \ x \in S. \end{cases} \tag{1.16}$$

Proof. We start by proving the first characterization. Suppose $\varphi \in N^P(S;\bar{x})$. Then,

$$\begin{aligned}
\varphi \in N^P(S;\bar{x}) &\Leftrightarrow d_S(\bar{x}+\alpha\varphi) = \alpha\|\varphi\|, \text{ for some } \alpha > 0 \\
&\Leftrightarrow \|(\bar{x}+\alpha\varphi)-\bar{x}\|^2 \le \|(\bar{x}+\alpha\varphi)-x\|^2, \text{ for all } x \in S \\
&\Leftrightarrow \|\bar{x}+\alpha\varphi\|^2 - 2\langle \bar{x}+\alpha\varphi, \bar{x}\rangle + \|\bar{x}\|^2 \\
&\le \|\bar{x}+\alpha\varphi\|^2 - 2\langle \bar{x}+\alpha\varphi, x\rangle + \|x\|^2, \text{ for all } x \in S \\
&\Leftrightarrow 2\langle \bar{x}+\alpha\varphi, x-\bar{x}\rangle \le \|x\|^2 - \|\bar{x}\|^2, \text{ for all } x \in S \\
&\Leftrightarrow 2\alpha\langle \varphi, x-\bar{x}\rangle \le \|x\|^2 - \|\bar{x}\|^2 - 2\langle \bar{x}, x\rangle + 2\|\bar{x}\|^2, \text{ for all } x \in S \\
&\Leftrightarrow 2\alpha\langle \varphi, x-\bar{x}\rangle \le \|x\|^2 - 2\langle \bar{x}, x\rangle + \|\bar{x}\|^2, \text{ for all } x \in S \\
&\Leftrightarrow \langle \varphi, x-\bar{x}\rangle \le \frac{1}{2\alpha}\|x-\bar{x}\|^2, \text{ for all } x \in S.
\end{aligned}$$

The proof of the first part is complete by putting $\sigma = \frac{1}{2\alpha}$. Now we prove the equivalence between the two characterizations of the proximal normal cone.

Assume that there exist $\sigma > 0$ and $\delta > 0$ such that

$$\langle \varphi, x-\bar{x}\rangle \le \sigma\|x-\bar{x}\|^2,$$

holds for all $x \in S \cap (\bar{x}+\delta\mathbf{B})$. Let now $x \in S$ so that $\|x-\bar{x}\| > \delta$. Then we have the following two cases.

(I.) If $\|x-\bar{x}\| \ge 1$, then

$$|\|x\| - \|\bar{x}\|| \le \|x-\bar{x}\| \le \|x-\bar{x}\|^2.$$

So,

$$\begin{aligned}
\langle \varphi, x-\bar{x}\rangle &\le \|\varphi\|\|x-\bar{x}\| \\
&\le \|\varphi\|(\|x\| + \|\bar{x}\|)
\end{aligned}$$

$$
\begin{aligned}
&= \|\varphi\|(\|x\| - \|\bar{x}\| + 2\|\bar{x}\|) \\
&\leq \|\varphi\|(|\|x\| - \|\bar{x}\|| + 2\|\bar{x}\|) \\
&\leq \|\varphi\|(\|x-\bar{x}\|^2 + 2\|\bar{x}\|) \\
&\leq \|\varphi\|(\|x-\bar{x}\|^2 + 2\|\bar{x}\|\|x-\bar{x}\|^2) \\
&\leq \|\varphi\|(1+2\|\bar{x}\|)\|x-\bar{x}\|^2.
\end{aligned}
$$

(II.) If $\|x-\bar{x}\| < 1$, then

$$\delta \leq \|x-\bar{x}\| < 1, \text{ for some } \delta > 0.$$

This ensures that $\frac{1}{\delta}\|x-\bar{x}\| \geq 1$ and $\frac{1}{\delta} > 1$. So, we have

$$
\begin{aligned}
\langle \varphi, x-\bar{x} \rangle &\leq \|\varphi\|\|x-\bar{x}\| \\
&\leq \|\varphi\|(\|x\| + \|\bar{x}\|) \\
&= \|\varphi\|(\|x\| - \|\bar{x}\| + 2\|\bar{x}\|) \\
&\leq \|\varphi\|(|\|x\| - \|\bar{x}\|| + 2\|\bar{x}\|) \\
&\leq \|\varphi\|(\|x-\bar{x}\| + 2\|\bar{x}\|) \\
&\leq \|\varphi\|\left(\left(\frac{1}{\delta}\right)\|x-\bar{x}\| + 2\|\bar{x}\|\right) \\
&\leq \|\varphi\|\left(\frac{\|x-\bar{x}\|}{\delta}\frac{\|x-\bar{x}\|}{\delta} + 2\|\bar{x}\|\right) \\
&\leq \|\varphi\|\left(\frac{\|x-\bar{x}\|^2}{\delta^2} + 2\|\bar{x}\|\right) \\
&\leq \|\varphi\|\left(\frac{\|x-\bar{x}\|^2}{\delta^2} + \frac{\|x-\bar{x}\|^2}{\delta^2}2\|\bar{x}\|\right) \\
&\leq \left(\frac{\|\varphi\|}{\delta^2}\right)(1+2\|\bar{x}\|)\|x-\bar{x}\|^2.
\end{aligned}
$$

Taking $\bar{\sigma} = \max\{\sigma, \|\varphi\|(1+2\|\bar{x}\|), \left(\frac{\|\varphi\|}{\delta^2}\right)(1+2\|\bar{x}\|)\}$ gives

$$\langle \varphi, x-\bar{x} \rangle \leq \bar{\sigma}\|\bar{x}-x\|^2, \text{ for all } x \in S.$$

Thus, completing the proof of the direct implication. The reverse implication is obvious. □

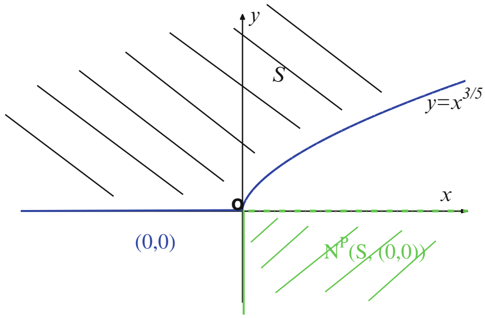


Fig. 1.6 Proximal normal cone to S in Example 1.4 Part (2)

Example 1.4.

1. For any closed convex subset S and any point $\bar{x} \in S$ one has

$$N^P(S;\bar{x}) = N^{\text{conv}}(S;\bar{x}).$$

Proof. As the inclusion $N^{\text{conv}}(S;\bar{x}) \subset N^P(S;\bar{x})$ is obvious we will prove the reverse one, i.e., $N^P(S;\bar{x}) \subset N^{\text{conv}}(S;\bar{x})$. Fix $\xi \in N^P(S;\bar{x})$ and $\sigma > 0$ as in (1.16). Let x' be any point in S. Since S is convex, the point $x = \bar{x} + t(x' - x)$ also belongs to S for each $t \in (0,1)$. Applying (1.16) with this point x yields

$$t\langle \xi, x' - x \rangle \leq \sigma t^2 \|x' - x\|^2.$$

Dividing across by t and letting $t \downarrow 0$ one obtains

$$\langle \xi, x' - x \rangle \leq 0, \text{forall} x' \in S,$$

which ensures that $\xi \in N^{\text{conv}}(S;\bar{x})$. □

2. Let $S = \{(x,y) \in \mathbf{R}^2 : y \geq 0, y \geq x^{\frac{3}{5}}\}$ and $\bar{x} = (0,0)$. For this subset one has $N^P(S;\bar{x}) = \{(0,0)\} \cup \{(\xi_1, \xi_2) \in \mathbf{R}^2 : \xi_1 \geq 0, \xi_2 < 0\}$. (see Fig. 1.6).
3. Let $S = \{(x,y) \in \mathbf{R}^2 : y \geq x \geq 0\} \cup \{(x,y) \in \mathbf{R}^2 : y \geq 0, y \geq -(-x)^{\frac{3}{2}}, x < 0\}$ and $\bar{x} = (0,0)$. One has $N^P(S;\bar{x}) = \{(0,0)\} \cup \{(\xi_1, \xi_2) \in \mathbf{R}^2 : \xi_1 > 0, \xi_2 \leq -\xi_1\}$. (See Fig. 1.7).

Observe that the definition of the proximal normal cone given by (1.15), in the beginning of this subsection, is strongly based on the Hilbert structure of the space $\mathbf{H}$. We use the characterizations given by (1.16), in the previous exercise, to define the proximal normal cone in any Banach space. We have the following definition:

Definition 1.7. Let X be a Banach space, S be a closed nonempty subset in X, and $\bar{x} \in S$. An element $\xi \in X^*$ is a proximal normal to S at $\bar{x}$ (i.e., $\xi \in N^P(S;\bar{x})$) if and only if there exist $\sigma, \delta > 0$

$$\langle \xi, x - \bar{x} \rangle \leq \sigma \|x - \bar{x}\|^2, \text{forall} x \in (\bar{x} + \delta \mathbf{B}) \cap S.$$

Fig. 1.7 Proximal normal cone to S in Example 1.4 Part (3)

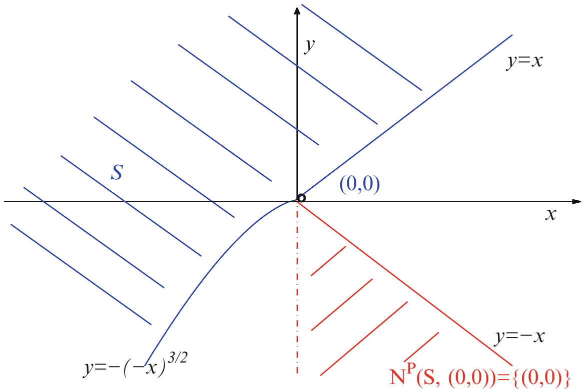
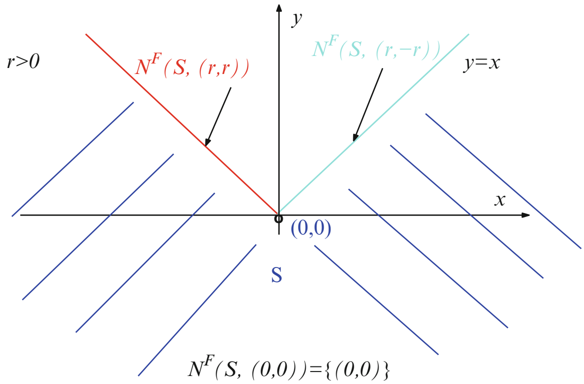


Fig. 1.8 Fréchet normal cone to S in Example 1.5 Part (2a)



1.5.3 The Fréchet Normal Cone (Prenormal cone)

Let X be a Banach space, S be a closed nonempty subset in X, and $\bar{x} \in S$. A vector $\zeta \in X^*$ is *Fréchet normal* (also called *prenormal* in Mordukhovich [192]) to S at $\bar{x}$ provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|, \text{forall} x \in (\bar{x} + \delta \mathbf{B}) \cap S.$$

We will denote the set of all these vectors by $\widehat{N}(S;\bar{x})$ (also denoted by $N^F(S;\bar{x})$ and this notation will be used in Figs. 1.8–1.11).

Example 1.5.

1. For any closed convex set S and any $\bar{x} \in S$ one has

$$\widehat{N}(S;\bar{x}) = N^{\text{conv}}(S;\bar{x}).$$

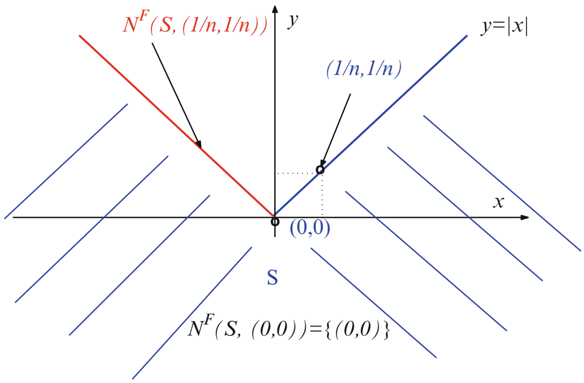


Fig. 1.9 Instability of Fréchet normal cone for S in Example 1.5 Part (2a)

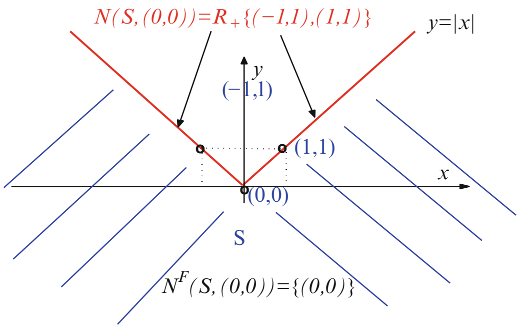


Fig. 1.10 Case of nonconvex basic normal cone

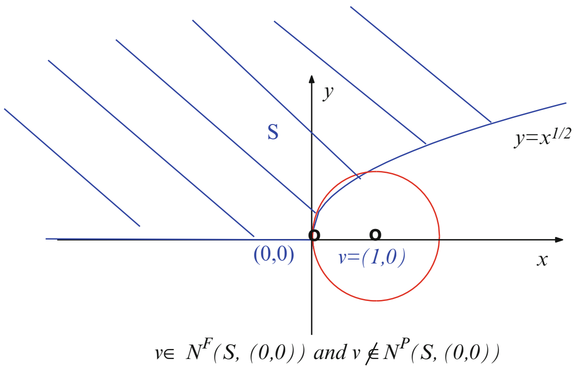


Fig. 1.11 A set with $N^P(S;\bar{x}) \not\subset \widehat{N}(S;\bar{x})$

2. Let $S = \{(x,y) \in \mathbf{R}^2 : y \leq |x|\}$ and $\bar{x} = (0,0)$.
   (a) Prove that $\widehat{N}(S;\bar{x}) = \{(0,0)\}$. (See Fig. 1.8).
   (b) For the same subset prove the following:

$$\widehat{N}(S;(x,|x|)) = \begin{cases} \mathbf{R}_+\{(-1,1)\} & \text{if} \quad x > 0; \\ \{(0,0)\} & \text{if} \quad x = 0; \\ \mathbf{R}_+\{(1,1)\} & \text{if} \quad x < 0. \end{cases}$$

3. What is the Fréchet normal cone to the following subset

$$S_m = \{(x,y) \in \mathbf{R}^2 : y \leq m|x|\}(m > 0)$$

at each one of its points?

1.5.4 The Basic Normal Cone (Limiting Normal Cone or Mordukhovich Normal Cone)

One of the serious problems of the Fréchet normal (prenormal) cone (the same problem exists with the proximal normal cone) is its instability, i.e., the Fréchet normal cone may vary widely as its point base varies (see example below). Many applications of Nonsmooth Analysis require that such instability be excluded. To illustrate this, consider the subset S as in Example 1.5 Part (2a). For this subset one has $\widehat{N}(S;(0,0)) = \{(0,0)\}$ while $\widehat{N}(S;(\frac{1}{n},\frac{1}{n})) = \mathbf{R}_+\{(-1,1)\}$, for all $n \geq 1$. (See Fig. 1.8).

In order to obtain a stable normal cone, in the above sense, we define (see, e.g., [180, 192, 194]) the basic normal cone (limiting normal cone or Mordukhovich normal cone) as follows

$$\begin{aligned} N(S;\bar{x}) &= \limsup_{x \to^S \bar{x}} \widehat{N}(S;x) \\ &= \{\zeta \in \mathbf{H} : \exists x_n \to^S \bar{x}, \exists \zeta_n \to^w \zeta \,\text{with}\, \zeta_n \in \widehat{N}(S;x_n)\}. \end{aligned}$$

One can define the limiting proximal normal cone in the same way, i.e.,

$$N^{PL}(S;\bar{x}) = \limsup_{x \to^S \bar{x}} N^P(S;x).$$

It has been proved (see, for instance, [140]) that in any Hilbert space these two limiting normal cones (the basic normal cone and the limiting proximal cone) coincide. So, we will only work with the basic normal cone.

Example 1.6.

1. For any closed convex subset S and any point $\bar{x} \in S$ one has $N(S;\bar{x}) = N^{\text{conv}}(S;\bar{x})$.
2. Let $S = \{(x,y) \in \mathbf{R}^2 : y \leq |x|\}$ and $\bar{x} = (0,0)$. We wish to prove that $N(S;\bar{x}) = \{(r,|r|) : r \in \mathbf{R}\}$. (See Fig. 1.9).

Let $x_n = (p_n, q_n) \to^S \bar{x} = (0,0)$. If $x_n \in \operatorname{int} S$, that is $q_n < |p_n|$ one obviously has $N(S;x_n) = \{(0,0)\}$. Now, if $x_n \in bd\, S$ that is, $q_n = |p_n|$, one gets (by Example 1.5 Part (2b)

$$N(S;x_n) = \begin{cases} \{(-r,r) : r \geq 0\} \text{if} p_n > 0; \\ \{(r,r) : r \geq 0\} \text{if} p_n < 0. \end{cases}$$

Thus,

$$N(S;\bar{x}) = \{(0,0)\} \cup \{(-r,r) : r \geq 0\} \cup \{(r,r) : r \geq 0\} = \{(r,|r|) : r \in \mathbf{R}\}.$$

Now, we summarize some properties of these normal cones in the following proposition.

Proposition 1.8.

1. *One always has the following inclusions:*

$$N^P(S;\bar{x}) \subset \widehat{N}(S;\bar{x}) \subset N(S;\bar{x}) \subset N^{\mathrm{C}}(S;\bar{x}), \forall \bar{x} \in S.$$

By convention, we set $N^P(S;\bar{x}) = \cdots = N^{\mathrm{C}}(S;\bar{x}) = \emptyset$, *if* $\bar{x} \notin S$ *and* $\{0\}$ *if* $\bar{x} \in int\, S$.

2. *For any nonempty closed convex subset S and any point* $\bar{x} \in S$ *all these normal cones coincide with* $N^{\mathrm{conv}}(S;\bar{x})$.
3. $\widehat{N}(S;\bar{x})$ *and* $N^{\mathrm{C}}(S;\bar{x})$ *are strongly closed convex cones in* X^*.
4. $N^P(S;\bar{x})$ *is a convex cone in* $\mathbf{H}$ *(it needs be neither open nor closed, see Example 1.4 parts (2) and (3)).*
5. $N(S;\bar{x})$ *is a strongly closed cone in* X^* *(that may be nonconvex, see Example 1.6 Part (2)).*
6. *The basic normal cone* $N(S;\bar{x})$ *has the following important property:*

$$\left.\begin{array}{l} x_n \to_S \bar{x}, \\ \zeta_n \to^{w^*} \zeta, \\ \zeta_n \in N(S;x_n) \end{array}\right\} \Longrightarrow \zeta \in N(S;\bar{x}).$$

Note that in general (even in finite dimensional spaces), this property is not true for the convexified (Clarke) normal cone (see Exercise 6.1.1 page 169 in [91]).

7. *The inclusions in the first assertion of this proposition may be strict as the subset S in Example 1.6 Part (2) proves it. For this subset one has*

$$\begin{aligned} N^P(S;\bar{x}) = \widehat{N}(S;\bar{x}) &= \{(0,0)\} \\ &\not\subset N(S;\bar{x}) = \{(r,|r|) : r \in \mathbf{R}\} \\ &\not\subset N^{\mathrm{C}}(S;\bar{x}) = \{(r,s) : s \geq |r|\}. \end{aligned}$$

Proof. The Parts (1),(2),(5)–(7) are obvious. We have to prove the properties (3) and (4).

(3)- Since $N^{\mathrm{C}}(S;\bar{x})$ is the negative polar of $T^{\mathrm{C}}(S;\bar{x})$, then it is obviously a strong closed convex cone. So we have to show the strong closedness of $\widehat{N}(S;\bar{x})$. Let

(ξ_k) be a sequence in $\widehat{N}(S;\bar{x})$ converging to ξ with respect to the norm of X^*. Let $\varepsilon > 0$. Fix an integer p such that $\|\xi_p - \xi\| \le \frac{\varepsilon}{2}$ and fix $\delta > 0$ such that

$$\langle \xi_p, x - \bar{x} \rangle \le \frac{\varepsilon}{2}\|x - \bar{x}\| \quad \text{forall} \quad x \in [\bar{x} + \delta \mathbf{B}] \cap S.$$

Then, for all $x \in [\bar{x} + \delta \mathbf{B}] \cap S$ we have

$$\begin{aligned} \langle \xi, x - \bar{x} \rangle &= \langle \xi - \xi_p, x - \bar{x} \rangle + \langle \xi_p, x - \bar{x} \rangle \\ &\le \varepsilon \|x - \bar{x}\| \end{aligned}$$

and hence $\xi \in \widehat{N}(S;\bar{x})$.

(4)- Let ξ_1, $\xi_2 \in N^P(S;\bar{x})$ and let $\alpha \in [0,1]$. Then by (1.16) there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ such that for all $x \in S$

$$\langle \xi_1, x - \bar{x} \rangle \le \sigma_1 \|x - \bar{x}\|^2$$

and

$$\langle \xi_2, x - \bar{x} \rangle \le \sigma_2 \|x - \bar{x}\|^2.$$

Therefore, we get for $\sigma = \max\{\sigma_1, \sigma_2\}$

$$\langle \alpha\xi_1 + (1-\alpha)\xi_2, x - \bar{x} \rangle \le [\alpha\sigma_1 + (1-\alpha)\sigma_2]\,\|x - \bar{x}\|^2 \le \sigma\|x - \bar{x}\|^2,$$

for all $x \in S$. This ensures that $N^P(S;\bar{x})$ is convex. □

The following example (See Fig. 1.11) shows that the inclusion $N^P(S;\bar{x}) \subset \widehat{N}(S;\bar{x})$ may be strict in general.

Example 1.7.

$$S = \{(x,y) \in \mathbf{R} : y \ge 0, y \ge x^{\frac{1}{2}}\} \text{ and } \bar{x} = (0,0).$$

We can check that the vector $v = (1,0) \notin N^P(S;\bar{x})$, because there are no points of $\{\bar{x} + \lambda v : \lambda > 0\}$ that project onto $\bar{x}$, see Fig. 1.11. On the other hand, we can easily check that this vector $v = (1,0)$ lies in $\widehat{N}(S;\bar{x})$ (it is sufficient to take for every $\varepsilon > 0$, the number $\delta = \varepsilon^2$ and check that the inequality in the definition of the Fréchet normal (prenormal) cone holds with this $\delta > 0$). Therefore, $N^P(S;\bar{x}) \not\subset \widehat{N}(S;\bar{x})$.

1.6 Commentary to Chap. 1

In this book, we restrict our attention on the nonsmooth concepts presented in this chapter that will be used, in the next chapters, to define and study various regularity concepts in Nonsmooth Analysis Theory. Many other nonsmooth concepts are introduced and studied by different authors. We refer the reader to the first two monographs on this subject [88, 91]. The reader is also refered for more details, more results, and more examples to the two-volume monograph [192, 193] of Mordukhovich, and in particular to the first volume. A very detailed and very well elaborated section is found in Volume 1 where the author presents the motivations and the early developments in Nonsmooth Analysis Theory. We especially refer the reader to that part of the book to learn more on the evolution of Nonsmooth Analysis Theory. For the case of finite dimensional spaces, we refer the reader to the excellent comprehensive books [194] of Mordukhovich and [241] of Rockafellar and Wets. Two other books have recently been published in the domain of nonsmooth and variational analysis by Borwein and Zhu [38] and Schirotzek [244]. They can profitably be consulted for more studies and examples.

The results in Propositions 1.5 and 1.6 were taken from Clarke [88, 91]. The analytic characterization of the proximal normal cone in Proposition 1.7 is stated in [91] without proof. Our proof here is taken from the recent paper [56] in which the authors proved the result in reflexive Banach spaces while the existing result is proved initially in Hilbert spaces.

The basic objects (subdifferential and normal cone) are also called (limiting or Mordukhovich) subdifferential and (limiting or Mordukhovich) normal cone, are thoroughly studied with more details in [192].

We refer the interested readers, for more nonsmooth concepts, to the excellent bibliography in [192] and for completeness here is a list of references: [22–25, 28–41, 44, 45, 48, 55–57, 60–63, 71, 72, 86, 88–91, 93, 94, 97, 98, 102, 104–106, 120, 121, 126, 133, 136, 137, 139–142, 144–146, 148–151, 159, 160, 167, 170–172, 174, 179–181, 183, 184, 190, 192–200, 203–206, 215, 224–230, 233–242, 244, 247, 250–256, 258, 261, 262].

Part II
Regularity Concepts in Nonsmooth Analysis Theory

Chapter 2
Regularity of Sets

2.1 Motivations

We present here two examples of motivations of the study of regularity concepts for sets and their importance in applications.

2.1.1 Calculus Rules

Calculus rules of the Clarke tangent cones and the convexified (Clarke) normal cones of sets is one of the mathematical domains where the regularity of sets in some sense plays a crucial role to obtain exact formulas. For instance, let X be a Banach space, S_1, S_2 be two subsets in X with $\bar{x} \in S_1 \cap S_2$ and S_2 be epi-Lipschitz[1] around $\bar{x}$. Then

$$T^{\mathrm{C}}(S_1;\bar{x}) \cap T^{\mathrm{C}}(S_2;\bar{x}) \subset T^{\mathrm{C}}(S_1 \cap S_2;\bar{x}), \qquad (*)$$

whenever the qualification condition (QC)

$$T^{\mathrm{C}}(S_1;\bar{x}) \cap \operatorname{int} T^{\mathrm{C}}(S_2;\bar{x}) \neq \emptyset,$$

is satisfied. The corresponding normal formula that holds under (QC) and some other hypothesis is

$$N^{\mathrm{C}}(S_1 \cap S_2;\bar{x}) \subset N^{\mathrm{C}}(S_1;\bar{x}) + N^{\mathrm{C}}(S_2;\bar{x}). \qquad (**)$$

[1] A closed nonempty subset S of X is said to be *epi-Lipschitz* around $\bar{x} \in S$ if it can be represented near $\bar{x}$ as the epigraph of a Lipschitz function.

M. Bounkhel, *Regularity Concepts in Nonsmooth Analysis: Theory and Applications*, Springer Optimization and Its Applications 59, DOI 10.1007/978-1-4614-1019-5_2,

In [88], Clarke showed that formulas $(*)$ and $(**)$ become equalities if both sets S_1 and S_2 are *tangentially regular* at $\bar{x}$, that is, the Clarke tangent cone $T^C(S;\bar{x})$ coincides with the Bouligand tangent cone $K(S;\bar{x})$. So, we will obtain under (QC) and the tangential regularity of both sets at $\bar{x}$

$$T^C(S_1 \cap S_2;\bar{x}) = T^C(S_1;\bar{x}) \cap T^C(S_2;\bar{x})$$

and

$$N^C(S_1 \cap S_2;\bar{x}) = N^C(S_1;\bar{x}) + N^C(S_2;\bar{x}).$$

There are many other calculus rules for tangent and normal cones that need the hypothesis of tangential regularity to become equalities, for examples: sets with constraint structure $S = A \cap F^{-1}(D)$, product subsets $S = \Pi_{i=1}^m S_i, \ldots$, etc.

2.1.2 *Differential Inclusions*

Another type of problem which is an important area of applications of our main results in these chapter is the differential inclusion problems (see for details Chaps. 5 and 6), more precisely, the first and second order sweeping process problems. Recall that the first order sweeping process problem was introduced in the 1970s by Moreau [207–210] and extensively studied by himself and other authors (Castaing [73, 75, 78, 81, 82], Valadier [255–257], and their students). Let **H** be a Hilbert space, $T > 0$ be some real positive number, and $C : [0,T] \rightrightarrows \mathbf{H}$ be a set-valued mapping taking nonempty closed values in **H**. The first order sweeping process consists in solving the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N^C(C(t);x(t)), & \text{a.e. } t \in [0,T], \\ x(t) \in C(t), & \forall t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases} \tag{SP}$$

Under the convexity assumption on $C(t)$ and other natural hypothesis, Moreau proved existence and uniqueness of a solution to (SP). A natural question, which many authors attacked, is whether similar results can be obtained if $C(t)$ in (SP) is not assumed to be convex. The most important regularity hypothesis assumed on $C(t)$ in the setting of Hilbert space, in order to obtain existence results of (SP), are: uniform prox-regularity in the sense of [229] (or equivalently proximal smoothness in the sense of [89])(see [42, 58, 248]), epi-Lipschitz property (see [92,255–257]), and Φ-convexity (see [92]). We will see with more details in Chap. 5 the important role of the uniform prox-regularity in such problems. The second order sweeping process was firstly studied by Castaing [80] with convex-valued set-valued mappings and later by the author and other investigators in some regular cases (see for instance [43, 54, 64] and the references therein). It consists in solving the following abstract second order differential inclusion:

$$\begin{cases} \ddot{x}(t) \in -N^{\mathrm{C}}(C(x(t));\dot{x}(t)), & \text{a.e. } t \in [0,T], \\ \dot{x}(t) \in C(x(t)), & \forall t \in [0,T], \\ x(0) = x_0 \in \mathbf{H} \text{ and } \dot{x}(0) \in C(x(0)), \end{cases} \qquad (SOSP)$$

where C is a set-valued mapping defined from $\mathbf{H}$ to itself. This problem (SOSP) and many of its variants will be the main purpose of Chap. 6 when the convexity of $C(t)$ is excluded and the uniform prox-regularity is employed.

2.2 Tangential Regularity of Sets

There are several ways to define general concepts of regularity of a subset S at some point $\bar{x} \in S$ in Nonsmooth Analysis Theory. One of them is the tangential regularity introduced by Clarke in [88], that is, the Clarke tangent cone $T^{\mathrm{C}}(S;\bar{x})$ coincides with the Bouligand tangent cone $K(S;\bar{x})$ (Note that one always has $T^{\mathrm{C}}(S;\bar{x}) \subset K(S;\bar{x})$). Let us recall the definition of these two classical tangent cones when the space X is assumed to be topological vector space not necessarily normed. In all this section X will be a topological vector space.

Definition 2.1. Let S be a nonempty closed subset of X and $\bar{x} \in S$.

(i) The Bouligand tangent cone $K(S;\bar{x})$ to S at $\bar{x}$ is the set of all $h \in X$ such that for every neighborhood H of h in X and for every $\varepsilon > 0$, there exists $t \in (0,\varepsilon)$ such that
$$(\bar{x} + tH) \cap S \neq \emptyset.$$

(ii) The Clarke tangent cone $T^{\mathrm{C}}(S;\bar{x})$ to S at $\bar{x}$ is the set of all $h \in X$ such that for every neighborhood H of h in X there exist a neighborhood U of $\bar{x}$ in X and a real number $\varepsilon > 0$ such that
$$(x + tH) \cap S \neq \emptyset \quad \text{for all } x \in U \cap S \text{ and } t \in (0,\varepsilon).$$

Exercise 2.1.

1. Assume that X is a normed vector space. Prove that the definitions of the Clarke tangent cone and the Bouligand tangent cone given above coincide with the ones given in Chap. 1 in Definition 1.5 in terms of the directional derivatives of the distance function d_S.
2. Show that the Parts (1) and (2) in Exercise 1.9 are still true even when X is a topological vector space.

There is a large class of subsets for which the inclusion $T^{\mathrm{C}}(S;\bar{x}) \subset K(S;\bar{x})$ has the equality form. Following Clarke [88], we get with the following definition.

Definition 2.2. We will say that S is *tangentially regular* at $\bar{x}$ provided that $T^{\mathrm{C}}(S;\bar{x}) = K(S;\bar{x})$.

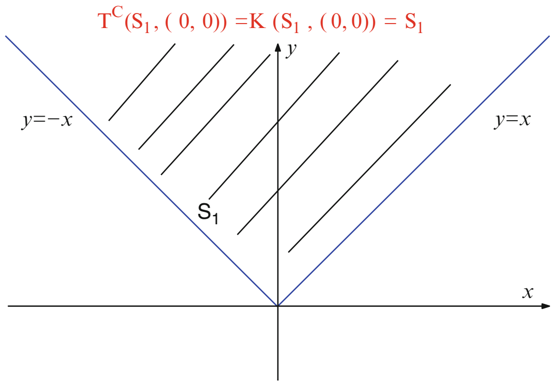


Fig. 2.1 Tangential regularity of convex sets

We give in the following example some regular and irregular subsets.

Example 2.1.

1. Any closed convex subset is tangentially regular at each of its points. This follows directly from the Part (2) in the previous exercise. For example, let $S_1 = \{(x,y) \in \mathbf{R}^2 : \quad y \geq |x|\}$ and $\bar{x} = (0,0)$ (see Fig. 2.1). This subset is convex and hence it is tangentially regular at $\bar{x}$ with
$$T^{\mathrm{C}}(S_1;\bar{x}) = K(S_1;\bar{x}) = T^{\mathrm{conv}}(S_1;\bar{x}) = \mathrm{cl}\,[\mathbf{R}_+(S_1 - \bar{x})] = \mathrm{cl}\,[\mathbf{R}_+(S_1)] = S_1.$$
The last equality follows from the fact that S_1 is a closed cone.
2. Let $S_2 = \{(x,y) \in \mathbf{R}^n : \quad f(x) = 0\}$ with $f \in C^1$. This subset is tangentially regular at each point $\bar{x} \in S$ satisfying $\nabla f(\bar{x}) \neq 0$ and one has
$$T^{\mathrm{C}}(S_2;\bar{x}) = K(S_2;\bar{x}) = \{v \in \mathbf{R}^n : \quad \nabla f(\bar{x})v = 0\}.$$
More generally, the constraint set $S = F^{-1}(D) = \{x \in X : \quad F(x) \in D\}$ is tangentially regular at each point $\bar{x} \in S$, whenever F is a C^1 mapping, D is tangentially regular at $F(\bar{x})$, and under some natural conditions (such as Robinson qualification condition see for instance [35, 233]) and in this case one has
$$R(S;\bar{x}) = \nabla F(\bar{x})^{-1} R(D;F(\bar{x}))$$
where $R(S;u) = T^{\mathrm{C}}(S;u) = K(S;u)$. Note that in the general case when D is not tangentially regular one only has the following inclusions
$$K(S;\bar{x}) \subset \nabla F(\bar{x})^{-1} K(D;F(\bar{x})) \ \text{ and } \ \nabla F(\bar{x})^{-1} T^{\mathrm{C}}(D;F(\bar{x})) \subset T^{\mathrm{C}}(S;\bar{x}).$$
3. Let S_3 be the closure of the complement of the set S_1 (see Fig. 1.2) and $\bar{x} = (0,0)$. Show that $T^{\mathrm{C}}(S;\bar{x}) = -S_1$ and $K(S;\bar{x}) = S_3$? and hence S_3 is not tangentially regular at $\bar{x}$.

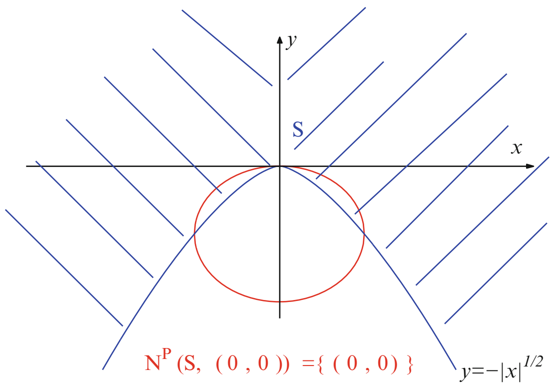


Fig. 2.2 A set which is (FNR) but not (PNR)

4. Let $S_4 = \{(x,y) \in \mathbf{R}^2 : \ \|(x-1,y)\| \leq 1\} \cup \{(x,y) \in \mathbf{R}^2 : \ \|(x+1,y)\| \leq 1\}$ and $\bar{x} = (0,0)$ (see Fig. 1.3). Show that $T^C(S_4;\bar{x}) = \{(0,0)\}$ and $K(S_4;\bar{x}) = \mathbf{R}^2$? and hence one gets that S_4 is not tangentially regular at $\bar{x}$.

2.3 Fréchet and Proximal Normal Regularity of Sets

Another natural concept of regularity of a subset S at some point $\bar{x} \in S$, that needs to be considered is the normal regularity. This means that the convexified (Clarke) normal cone $N^C(S;\bar{x})$ of S at $\bar{x}$ coincides with a prescribed normal cone $N^\#(S;\bar{x})$ of S at $\bar{x}$. We state here the case of the Fréchet or the proximal normal cones.

Definition 2.3. Let S be a nonempty closed subset of X and let $\bar{x} \in S$. We will say that S is *Fréchet normally* (resp. *proximal normally) regular* at $\bar{x}$ if one has $\widehat{N}(S;\bar{x}) = N^C(S;\bar{x})$ (resp. $N^P(S;\bar{x}) = N^C(S;\bar{x})$).

Remark 2.1. As one always has $N^P(S;\bar{x}) \subset \widehat{N}(S;\bar{x})$, one sees that the proximal normal regularity (PNR) always implies the Fréchet normal regularity (FNR). The converse is not true. Indeed, we take $S = \{(x,y) \in \mathbf{R}^2 : \ y \geq -|x|^\alpha\}$, with $1 < \alpha < 2$ (for instance $\alpha = \frac{3}{2}$) and $\bar{x} = (0,0)$. One has $N^P(S;\bar{x}) = \{(0,0)\}$, because no ball whose interior fails intersect S can have $\bar{x}$ on its boundary. While all other normal cones coincide and are equal to $\{(0,-r) : r \in \mathbf{R}_+\}$ (Fig. 2.2).

2.4 Scalar Regularity of Sets

One more but not less natural notion of regularity for a subset S at $\bar{x} \in S$ is the *scalar regularity*, which means the regularity of an associated scalar function with the subset S. The scalar function which will be used here is the associated distance function d_S. Note that d_S is always globally 1-Lipschitz function on X. So, we will need some preliminaries and some regularity concepts for locally Lipschitz functions.

Remark 2.2. Note that a general study about various notions of regularity of lower semicontinuous (non necessarily locally Lipschitz) functions will be given in Chap. 3. Because of the use of the distance function here, we restrict our attention on the following preliminaries in the case of locally Lipschitz functions.

Let $f : X \to \mathbf{R}$ be a locally Lipschitz function and $\bar{x} \in X$. Like in the case of subsets, there are many ways to define general concepts of regularity for functions in Nonsmooth Analysis Theory. The well known is the directional regularity defined as follows.

Definition 2.4. Following Clarke [88], we will say that f is *directionally regular* at $\bar{x}$ if one has

$$f^0(\bar{x};\cdot) = f^-(\bar{x};\cdot)$$

where $f^0(\bar{x};\cdot)$ and $f^-(\bar{x};\cdot)$ are, respectively, the generalized directional derivative and the lower Dini directional derivative of f at $\bar{x}$.

Example 2.2.

1. Any convex continuous function f is directionally regular and
$$f^0(\bar{x};v) = f^-(\bar{x};v) = f'(\bar{x};v).$$
2. Any continuously differentiable function f (i.e. $f \in C^1$) is directionally regular and one has
$$f^0(\bar{x};v) = f^-(\bar{x};v) = \nabla f(\bar{x})(v).$$
3. Let $f : X \to \mathbf{R}$ be defined by $f(x) = -\|x\|$ and $\bar{x} = 0$. Then for all $v \in X$ one has $f^0(\bar{x};v) = \|v\|$ and $f^-(\bar{x};v) = -\|v\|$. Hence, f is not directionally regular at $\bar{x}$.
4. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 \sin(\frac{1}{x})$ and $\bar{x} = 0$. This function is differentiable at $\bar{x}$ and its derivative $\nabla f(\bar{x}) = 0$, but it is not directionally regular at $\bar{x}$. Indeed, for all $v \in X$ one has $f^0(\bar{x};v) = \|v\|$ and $f^-(\bar{x};v) = 0$.

Remark 2.3. The Part (4) in the last example shows that the differentiability of f at $\bar{x}$ is not sufficient for f to be directionally regular at $\bar{x}$.

Another natural concept of regularity for functions is *the subdifferential regularity*. This means that the generalized gradient (Clarke subdifferential) $\partial^C f(\bar{x})$ of f at $\bar{x}$ coincides with a prescribed subdifferential $\partial^\# f(\bar{x})$ (for example: $\partial^\# = \widehat{\partial}$ (for Fréchet), $\partial^\# = \partial^P$ (for proximal), $\partial^\# = \partial$ (for basic),...) of f at $\bar{x}$. We restrict our attention here only on the Fréchet and the proximal cases. The following proposition states a very important relationship between the subdifferential concept and the normal cone concept which will be used in our study. It cannot be proved for a general subdifferential and a general normal cone. Each case has been proved separately and we refer the reader to different works, in each one we can find the proof of a particular case (see [88, 91, 140, 180, 196, 198]).

Proposition 2.1. *Let* $(\partial^{\#};N^{\#}) \in \{(\partial^P;N^P),(\widehat{\partial};\widehat{N}),(\partial^{\mathrm{C}};N^{\mathrm{C}}),(\partial;N)\}$. *Then one has*

$$\partial^{\#} f(\bar{x}) = \{x^* \in X^* : \quad (x^*,-1) \in N^{\#}(\operatorname{epi} f;(\bar{x},f(\bar{x})))\}. \tag{2.1}$$

Now we are in position to introduce the second type of scalar regularity.

Definition 2.5. We will say that f is *Fréchet* (resp. *proximal*) *subdifferentially regular* at $\bar{x}$ provided that $\widehat{\partial} f(\bar{x})$ (resp. $\partial^P f(\bar{x})$) coincides with $\partial^C f(\bar{x})$.

Remark 2.4. Obviously, one has proximal subdifferential regularity implies Fréchet subdifferential regularity. The converse does not hold. Indeed, take $f(x) = -(|x|)^{\frac{3}{2}}$, $f \in C^1$. One has $\widehat{\partial} f(0) = \partial^C f(0) = \{0\}$, while $\partial^P f(0) = \emptyset$ (because $N^P(\operatorname{epi} f;(0,0)) = \{(0,0)\}$).

Conclusion. We have considered three types of regularity for locally Lipschitz functions. Consequently, we will obtain three additional types of regularity for subsets by taking $f = d_S$ and then we will say that

- S is (DR) at $\bar{x} \in S$ if and only if d_S is directionally regular at $\bar{x}$.
- S is (FSR) at $\bar{x} \in S$ if and only if d_S is Fréchet subdifferentially regular at $\bar{x}$;
- S is (PSR) at $\bar{x} \in S$ if and only if d_S is proximally subdifferentially regular at $\bar{x}$.

Now, we summarize all what we have considered as a concept of regularity for a subset S at a point $\bar{x} \in S$:

(i) Tangential regularity (TR), i.e., $T^{\mathrm{C}}(S,\bar{x}) = K(S,\bar{x})$;
(ii) Proximal Normal Regularity (PNR), i.e., $N^P(S,\bar{x}) = N^{\mathrm{C}}(S,\bar{x})$;
(iii) Fréchet Normal Regularity (FNR), i.e., $\widehat{N}(S,\bar{x}) = N^{\mathrm{C}}(S,\bar{x})$;
(iv) Directional Regularity (DR), i.e., $d_S^0(\bar{x};\cdot) = d_S^-(\bar{x};\cdot)$;
(v) Proximal Subdifferential Regularity (PSR), i.e., $\partial^P d_S(\bar{x}) = \partial^{\mathrm{C}} d_S(\bar{x})$;
(vi) Fréchet Subdifferential Regularity (FSR), i.e., $\widehat{\partial} d_S(\bar{x}) = \partial^{\mathrm{C}} d_S(\bar{x})$.

Our main goal in the sequel of this Chapter is to study the relationships between all of these notions of regularity. We will proceed as follows:

1. (TR)$\Longleftrightarrow$(DR)? (can be seen as a scalarization of (TR));
2. (FNR) $\Longleftrightarrow$(FSR)? (can be seen as a scalarization of (FNR));
3. (PNR) $\Longleftrightarrow$(PSR)? (can be seen as a scalarization of (PNR));
4. (TR) $\Longleftrightarrow$(FNR)? (We can see this equivalence as a bridge between the primal notion of regularity (TR) and the dual notion of regularity (FNR)).

2.5 Scalarization of Tangential Regularity: [(TR)⇔(DR)?]

In this section, we prove the following main theorem due to Burke et al. [71].

Theorem 2.1. *Let S be a closed nonempty subset of a normed vector space X and let $\bar{x} \in S$.*

1. *If d_S is directionally regular at $\bar{x} \in S$, then S is tangentially regular (TR) at $\bar{x} \in S$;*
2. *If, in addition, $\dim X < +\infty$, then the converse holds, i.e., (TR) $\Longleftrightarrow$(DR).*

Proof.

1. By Definition 1.5 in Chap. 1 and the directional regularity of d_S at $\bar{x}$ we have
$$K(S;\bar{x}) = \{v \in X : d_S^-(\bar{x},v) = 0\} = \{v \in X : d_S^0(\bar{x},v) = 0\} = T^{\mathrm{C}}(S;\bar{x})$$
and hence S is tangentially regular at $\bar{x}$.
2. Assume that $\dim X < +\infty$, and S is tangentially regular at $\bar{x}$. In order to make clear the idea of the proof of this Part (2), we omit the following facts:

Fact 1.
$$d_S^0(\bar{x},v) \le d_{T^{\mathrm{C}}(S;\bar{x})}(v), \qquad \text{for all } v \in X.$$

Fact 2.
$$d_S^-(\bar{x},v) = d_{K(S;\bar{x})}(v), \qquad \text{for all } v \in X.$$

We will give the proof of these two facts after completing the proof of the Part (2).

Then one has for all $v \in X$
$$\begin{aligned} d_S^0(\bar{x},v) &\le d_{T^{\mathrm{C}}(S;\bar{x})}(v) && \text{(by Fact 1)} \\ &= d_{K(S;\bar{x})}(v) && \text{(by (TR))} \\ &= d_S^-(\bar{x},v). && \text{(by Fact 2)} \end{aligned}$$

Hence, $d_S^0(\bar{x},v) \le d_S^-(\bar{x},v)$ and as the reverse inequality is always true, one gets the equality, i.e., the directional regularity of d_S at $\bar{x}$. This completes the proof of the Part (2). □

Now let us prove Fact 1 and Fact 2.

Proof of Fact 1. Fix any $v \in X$ and any $\varepsilon > 0$. There exists (by the definition of the infimum) some $\bar{v} \in T^{\mathrm{C}}(S;\bar{x})$ such that
$$\|v - \bar{v}\| \le d_{T^{\mathrm{C}}(S;\bar{x})}(v) + \varepsilon. \tag{2.2}$$

Consider a sequence (t_n, x_n) in $(0,+\infty) \times S$ converging to $(0,\bar{x})$ satisfying
$$d_S^0(\bar{x};v) = \lim_n t_n^{-1} d_S(x_n + t_n v) = \lim_n d_{t_n^{-1}(S-x_n)}(v). \tag{2.3}$$

By the sequential characterization of the Clarke tangent cone $T^{C}(S;\bar{x})$, there exists a sequence $v_n \to \bar{v}$ such that $x_n + t_n v_n \in S$, i.e., $v_n \in t_n^{-1}(S - x_n)$, for all n. Thus,

$$d_{t_n^{-1}(S-x_n)}(v) \le d_{t_n^{-1}(S-x_n)}(v_n) + \|v - v_n\| \le \|v - v_n\|.$$

By letting $n \to +\infty$, one gets by (2.2) and (2.3)

$$d_S^0(\bar{x};v) \le \|v - \bar{v}\| \le d_{T^C(S;\bar{x})}(v) + \varepsilon.$$

This completes the proof of Fact 1. □

Proof of Fact 2. With the same method as in the proof of Fact 1, we can prove the inequality

$$d_S^-(\bar{x};v) \le d_{K(S;\bar{x})}(v), \quad \text{for all} \quad v \in X.$$

So, we proceed now to prove the reverse inequality, i.e.,

$$d_{K(S;\bar{x})}(v) \le d_S^-(\bar{x};v), \quad \text{for all} \quad v \in X. \tag{2.4}$$

Fix any $v \in X$. Let us consider a sequence of real positive numbers $t_n \downarrow 0$ such that

$$d_S^-(\bar{x};v) = \lim_n t_n^{-1} d_S(\bar{x} + t_n v) = \lim_n d_{t_n^{-1}(S-\bar{x})}(v). \tag{2.5}$$

For each $n \in \mathbf{N}$, we choose $v_n \in t_n^{-1}(S - \bar{x})$ with

$$\|v - v_n\| \le d_{t_n^{-1}(S-\bar{x})}(v) + t_n.$$

Then, by (2.5) one gets

$$\lim_n \|v - v_n\| \le d_S^-(\bar{x};v). \tag{2.6}$$

This ensures that the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded. Hence, as $\dim X < +\infty$, some subsequence converges to some vector $\bar{v} \in X$. Consequently, the sequential characterization of $K(S;\bar{x})$ and the choice of v_n ensures that $\bar{v}$ must lie in $K(S;\bar{x})$. It follows then, by (2.6) that

$$d_{K(S;\bar{x})} \le \|v - \bar{v}\| \le d_S^-(\bar{x};v).$$

This completes the proof of (2.4) and hence the proof of Fact 2 is finished. □

Remark 2.5. Note that Fact 1 and its corresponding inequality for $d_S^-(\bar{x};\cdot)$ and $d_{K(S;\bar{x})}(\cdot)$ are true for any closed nonempty subset S in any normed vector space X, without the hypothesis $\dim X < \infty$.

It is well known, in convex analysis theory, that the bridge formula between the normal cone $N^{\text{conv}}(S;\bar{x})$ to a nonempty closed subset S at $\bar{x} \in S$ and the subdifferential $\partial^{\text{conv}} d_S(\bar{x})$ of its distance function d_S at $\bar{x}$ is the following:

$$\partial^{\text{conv}} d_S(\bar{x}) = N^{\text{conv}}(S;\bar{x}) \cap \mathbf{B}_*. \tag{2.7}$$

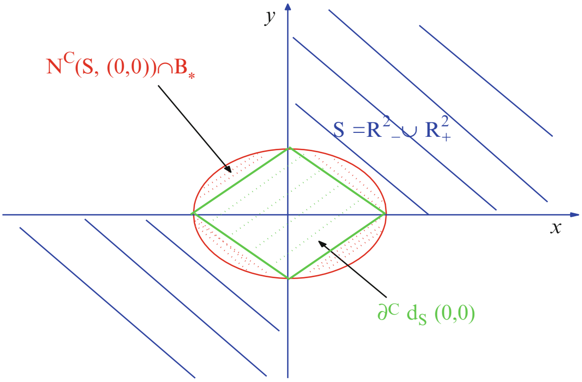


Fig. 2.3 A set with a strict inclusion in (2.8) (Example 2.3)

In the nonconvex case, this formula may fail for the convexified (Clarke) normal cone $N^{\mathrm{C}}(S;\bar{x})$ and the generalized gradient $\partial^C d_S(\bar{x})$ and one only has the direct inclusion, i.e.,

$$\partial^C d_S(\bar{x}) \subset N^{\mathrm{C}}(S;\bar{x}) \cap \mathbf{B}_*. \tag{2.8}$$

The proof of this inclusion will be given later, see the proof of Theorem 2.2.

We recall the following example, given in [71], proving this fact.

Example 2.3. Let $S = \mathbf{R}^2_- \cup \mathbf{R}^2_+$, $X = \mathbf{R}^2$ endowed with the Euclidean norm and let $\bar{x} = (0,0)$. It is not hard to prove that $N^{\mathrm{C}}(S;\bar{x}) = \mathbf{R}^2$ and hence

$$N^{\mathrm{C}}(S;\bar{x}) \cap \mathbf{B}_* = \{(\xi_1, \xi_2) : \quad \xi_1^2 + \xi_2^2 \leq 1\}$$

and one can also check that $d^0(\bar{x};(v_1,v_2)) = \max\{|v_1|,|v_2|\}$, which ensures that

$$\partial^C d_S(\bar{x}) = \{(\xi_1, \xi_2) : \quad |\xi_1| + |\xi_2| \leq 1\}.$$

This shows that $\partial^C d_S(\bar{x})$ is strictly included in $N^{\mathrm{C}}(S;\bar{x}) \cap \mathbf{B}_*$ (see Fig. 2.3).

In the following theorem, we show that the inclusion (2.8) becomes equality whenever S is tangentially regular at $\bar{x}$ and $\dim X < \infty$.

Theorem 2.2. *Let S be a closed subset of a finite dimensional vector space X and let $\bar{x} \in S$. Assume that S is tangentially regular at $\bar{x}$. Then one has*

1. $d_S^0(\bar{x}, v) = d_{T^{\mathrm{C}}(S;\bar{x})}(v)$, *for all $v \in X$ and*
2. $\partial^{\mathrm{C}} d_S(\bar{x}) = N^{\mathrm{C}}(S;\bar{x}) \cap \mathbf{B}_*$.

Proof.

1. Assume that S is tangentially regular at $\bar{x}$. Then by Theorem 2.1, the function d_S is directionally regular at $\bar{x}$ (because $\dim X < \infty$), i.e., $d_S^0(\bar{x}, v) = d_S^-(\bar{x}, v)$, for all $v \in X$. Therefore, by Theorem 2.1 once again and by the tangential regularity of S at $\bar{x}$ one gets for all $v \in X$

$$d_S^0(\bar{x}, v) = d_S^-(\bar{x}, v) = d_{K(S;\bar{x})}(v) = d_{T^{\mathrm{C}}(S;\bar{x})}(v).$$

2. First, we prove that the direct inclusion, $\partial^C d_S(\bar{x}) \subset N^C(S;\bar{x}) \cap \mathbf{B}_*$, holds in every normed vector space X (without the finite dimensional assumption of X and the tangential regularity of S at $\bar{x}$).

Fix any $x^* \in \partial^C d_S(\bar{x})$. Then, by the relations (1.12) and (1.13) and by the fact that d_S is 1-Lipschitz, one gets

$$\langle x^*, v\rangle \le d_S^0(\bar{x}, v) \le \|v\|, \quad \text{for all} \quad v \in X. \tag{2.9}$$

This ensures that $\|x^*\| \le 1$. Now, we wish to show that $x^* \in N^C(S;\bar{x})$. By Fact 1 in Theorem 2.1 and the relation (1.12) one gets

$$\langle x^*, v\rangle \le d_S^0(\bar{x}, v) \le d_{T^C(S;\bar{x})}(v), \quad \text{for all} \quad v \in X$$

and hence $\langle x^*, v\rangle \le 0$ for all $v \in T^C(S;\bar{x})$. This ensures that $x^* \in (T^C(S;\bar{x}))^0 = N^C(S;\bar{x})$.

Now, we use the hypothesis $\dim X < \infty$ and the tangential regularity of S at $\bar{x}$ to prove the reverse inclusion. Fix any $x^* \in N^C(S;\bar{x}) = (T^C(S;\bar{x}))^0$ with $\|x^*\| \le 1$. Then for any $v \in T^C(S;\bar{x})$ one has $\langle x^*, v\rangle \le 0$. Thus, the function $h : X \to \mathbf{R}$ defined by $h(v) = -\langle x^*, v\rangle$ is 1-Lipschitz (because $\|x^*\| \le 1$) and satisfies $h(v) \ge h(0) = 0$ for all $v \in T^C(S;\bar{x})$. Using the exact penalization in the Part (3) of Exercise 1.5, one obtains

$$0 \le h(v) + d_{T^C(S;\bar{x})}(v), \quad \text{for all} \quad v \in X$$

and hence by the Part (1) of the theorem one gets

$$\langle x^*, v\rangle \le d_{T^C(S;\bar{x})}(v) = d_S^0(\bar{x}; v), \quad \text{for all} \quad v \in X.$$

This ensures that $x^* \in \partial^C d_S(\bar{x})$ and hence the proof of the theorem is complete. □

Another interesting relationship between the normal cone concept and the subdifferential concept of the distance function, which will be used hereafter, is the following

$$N^{\#}(S;x) = \mathrm{cl}_{w^*}\left(\mathbf{R}_+ \partial^{\#} d_S(x)\right), \tag{2.10}$$

for $(N^{\#}, \partial^{\#}) \in \{(\widehat{N}, \widehat{\partial}), (N^C, \partial^C)\}$.

Exercise 2.2. Prove that the relation (2.10) is always true for $(N^{\#}, \partial^{\#}) = (N^C, \partial^C)$, whenever X is a real normed space.

Observe that the relation (2.10) is weaker than the equality

$$\partial^{\#} d_S(x) = N^{\#}(S;x) \cap \mathbf{B}_*, \tag{2.11}$$

i.e., (2.11) ⇒ (2.10). The reverse is not true in general as Exercise 2.2 and Example 2.3 prove it.

2.6 Scalarization of Fréchet Normal Regularity: [(FNR)⇔(FSR)]?

We begin this subsection by presenting a simple and different proof given by Bounkhel and Thibault [61] of a result due to Ioffe [140] and Kruger [166] (It can also be found in [167]). It proves that the inclusion (2.8) becomes equality when we replace the convexified (Clarke) normal cone and the generalized gradient (Clarke subdifferential) by their corresponding in the Fréchet case.

Theorem 2.3. *Let X be a normed vector space, S be a nonempty closed subset of X and x be a point in S. Then*

$$\widehat{\partial} d_S(x) = \widehat{N}(S;x) \cap \mathbf{B}_*.$$

Proof. We begin by proving the inclusion

$$\widehat{\partial} d_S(x) \subset \widehat{N}(S;x) \cap \mathbf{B}_*. \tag{2.12}$$

Fix $x^* \in \widehat{\partial} d_S(x)$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in x + \delta\mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|.$$

Hence

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|, \quad \text{for all} \quad x' \in S \cap (x + \delta\mathbf{B}),$$

which ensures that $x^* \in \widehat{N}(S;x)$. Then as one always has $\widehat{\partial} d_S(x) \subset \partial^C d_S(x) \subset \mathbf{B}_*$, the inclusion (2.12) is proved.

Now, we prove the reverse inclusion, i.e.

$$\widehat{N}(S;x) \cap \mathbf{B}_* \subset \widehat{\partial} d_S(x).$$

Let $x^* \in \widehat{N}(S;x)$ with $\|x^*\| \leq 1$ and let $\varepsilon > 0$. Fix $0 < \varepsilon' < \frac{\varepsilon}{2}$. Then, there exists $\delta > 0$ such that, for all $x' \in S \cap (x + \delta\mathbf{B})$

$$\langle x^*, x' - x \rangle \leq \varepsilon' \|x' - x\|. \tag{2.13}$$

Fix $r > 0$ such that $0 < 2r < \delta$. Since the function $x' \mapsto \langle x^*, x - x' \rangle + \varepsilon' \|x - x'\|$ is L-Lipschitz with $L = \|x^*\| + \varepsilon'$, then using (2.13) and the exact penalization in the Part (3) of Exercise 1.5, we obtain for all $x' \in X$

$$\begin{aligned}\langle x^*, x' - x \rangle &\leq \varepsilon' \|x' - x\| + (\|x^*\| + \varepsilon') d_{S \cap (x + \delta\mathbf{B})}(x') \\ &\leq \varepsilon' \|x' - x\| + d_{S \cap (x + \delta\mathbf{B})}(x') + \varepsilon' d_{S \cap (x + \delta\mathbf{B})}(x')\end{aligned}$$

$$\le \varepsilon'\|x'-x\| + d_{S\cap(x+\delta\mathbf{B})}(x') + \varepsilon'\|x'-x\|$$

$$\le 2\varepsilon'\|x'-x\| + d_{S\cap(x+\delta\mathbf{B})}(x').$$

On the other hand, we can easily check that for all $x' \in x + r\mathbf{B}$

$$d_{S\cap(x+2r\mathbf{B})}(x') = d_S(x').$$

Thus, for all $\varepsilon > 0$, there exists $r > 0$ such that for all $x' \in x + r\mathbf{B}$

$$\langle x^*, x'-x\rangle \le \varepsilon\|x'-x\| + d_S(x') - d_S(x).$$

Therefore, $x^* \in \widehat{\partial} d_S(x)$ and hence the proof is finished. □

Now, we can state the first result on the relationship between the Fréchet normal regularity (FNR) of a set and the Fréchet subdifferential regularity (FSR) of its distance function.

Theorem 2.4. *Let X be any Banach space and let S be a nonempty closed subset of X with $x \in S$. Suppose that S is Fréchet normally regular at x. Then, d_S is Fréchet subdifferentially regular at x. If, in addition, X is reflexive, then one has the equivalence.*

Proof. Assume that S is Fréchet normally regular at x, i.e., $N^C(S;x) = \widehat{N}(S;x)$. Then, by the relations (2.12) and (2.8) one has

$$\partial^C d_S(x) \subset N^C(S;x)\cap\mathbf{B}_* = \widehat{N}(S;x)\cap\mathbf{B}_* = \widehat{\partial} d_S(x),$$

which ensures the Fréchet subdifferential regularity of d_S at x.

Let us prove the reverse implication under the additional hypotheses of the theorem. Assume that X is reflexive and d_S is Fréchet subdifferentially regular at x. The definition of Fréchet subdifferential regularity ensures that $\widehat{\partial} d_S(x) = \partial^C d_S(x)$ and hence by Exercise 2.2 and Theorem 2.3 one gets

$$N^C(S;x) = \mathrm{cl}_{w^*}\left(\mathbf{R}_+\widehat{\partial} d_S(x)\right) = \mathrm{cl}_{w^*}\left(\widehat{N}(S;x)\right).$$

By Part (3) in Proposition 1.8, the set $\widehat{N}(S;x)$ is strongly closed convex in X^* and hence it is weak star closed in X^* since X is reflexive. Thus, $N^C(S;x) = \mathrm{cl}_{w^*}\left(\widehat{N}(S;x)\right) = \widehat{N}(S;x)$, which ensures the Fréchet normal regularity of S at x. The proof of the theorem is then complete. □

Remark 2.6. The equivalence in the previous theorem is still true for Asplund–Banach spaces which are more general than reflexive spaces, if we assume that the subset S is compactly epi-Lipschitz at x. For more details on this result and the definition of compactly epi-Lipschitz property we refer the reader to [61] and the references therein.

Corollary 2.1. *Let S be a nonempty closed subset of* $\mathbf{R}^N$ *and let* $x \in S$. *Then the following assertions are equivalent:*

(i) *S is Fréchet normally regular at x;*
(ii) d_S *is Fréchet subdifferentially regular at x.*

Consider now another concept of normal regularity introduced by Mordukhovich [192] in the finite dimensional setting and used latter by Mordukhovich and Shao [195–198] in Asplund–Banach spaces. Its definition is not in the same way like Definition 2.3. A subset S of an Asplund–Banach space X is said to be *Mordukhovich regular* (called *normally regular* in Mordukhovich [192]) at $x \in S$ provided that $N(S;x) = \widehat{N}(S;x)$. In the following theorem, we prove that Fréchet normal regularity and Mordukhovich regularity are equivalent in reflexive Banach spaces (see also [241] for the finite dimensional setting). We need the following relationship between the convexified (Clarke) normal cone and the basic normal cone. For its proof we refer the reader, for instance, to [89, 192]:

$$N^{\mathrm{C}}(S;\bar{x}) = \mathrm{cl}_{w^*} co\,[N(S;\bar{x})]\,. \tag{2.14}$$

Theorem 2.5. *Let S be a nonempty closed subset of a reflexive Banach space X with* $x \in S$. *Then the set S is Fréchet normally regular at x if and only if it is Mordukhovich regular at x.*

Proof. If S is Fréchet normally regular at x, then

$$N^{\mathrm{C}}(S;x) = \widehat{N}(S;x) \subset N(S;x) \subset N^{\mathrm{C}}(S;x)$$

and hence $\widehat{N}(S;x) = N(S;x)$, that is, S is Mordukhovich regular at x.

Assume now that S is Mordukhovich regular at x, i.e.,

$$\widehat{N}(S;x) = N(S;x). \tag{2.15}$$

By (2.14) and the convexity of the Fréchet normal cone we get

$$N^{\mathrm{C}}(S;x) = \mathrm{cl}_{w^*} co\,[N(S;x)] = \mathrm{cl}_{w^*}\left[\widehat{N}(S;x)\right]. \tag{2.16}$$

As $\widehat{N}(S;x)$ is strongly closed (see Part (3) in Proposition 1.8) and convex, it is a weak star closed convex set (since X is reflexive). So, the assumption (2.15) and the equality (2.16) ensure that $N^{\mathrm{C}}(S;x) = \widehat{N}(S;x)$, i.e., S is Fréchet normally regular at x. □

In a similar way, the concept of *Mordukhovich regularity* of a function $f : X \to \mathbf{R}$ can be defined as $\widehat{\partial} f(x) = \partial f(x)$ and the equivalence between the Mordukhovich regularity of the function d_S and its Fréchet subdifferential regularity can be established in reflexive Banach spaces. So, the arguments used in the proofs of the two above theorems give the following result.

Theorem 2.6. *Let X be a reflexive Banach space and S be a closed subset of X with $x \in S$. Then the following assertions are equivalent:*

(i) S is Mordukhovich regular at x;
(ii) d_S is Mordukhovich regular at x;
(iii) d_S is Fréchet subdifferentially regular at x.

2.7 Scalarization of Proximal Normal Regularity: [(PNR) ⇔ (PSR)]?

In this section, we will assume that X is a *reflexive Banach space*.

We have already seen that Fréchet normal regularity is not equivalent to proximal normal regularity. So, the present section is devoted to study some properties of proximal normal regularity, essentially we will give conditions under which this normal regularity can be characterized in terms of the distance function.

We establish first the following result on the relationship between the proximal normal cone and the proximal subdifferential of the distance function, which is the corresponding formula of (2.7) in the proximal case. It is due to Bounkhel and Thibault [61].

Theorem 2.7. *Let S be a nonempty closed subset of X and $x \in S$. Then*

$$\partial^P d_S(x) = N^P(S;x) \cap \mathbf{B}_*.$$

Proof. We begin by proving the inclusion

$$\partial^P d_S(x) \subset N^P(S;x) \cap \mathbf{B}_*.$$

Let $x^* \in \partial^P d_S(x)$. Then there exist $\sigma > 0$ and $\delta > 0$ such that for all $x' \in x + \delta \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2 + d_S(x') - d_S(x) = \sigma \|x' - x\|^2 + d_S(x')$$

and hence for all $x' \in S \cap (x + \delta \mathbf{B})$

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2$$

which ensures that $x^* \in N^P(S;x)$. Then, as one always has $\partial^P d_S(x) \subset \partial^C d_S(x) \subset \mathbf{B}_*$, then $x^* \in N^P(S;x) \cap \mathbf{B}_*$.

Now, we show the reverse inclusion

$$N^P(S;x) \cap \mathbf{B}_* \subset \partial^P d_S(x).$$

Fix $x^* \in N^P(S;x)$ with $\|x^*\| \le 1$. Then there exist $\sigma > 0$ and $\delta > 0$ such that

$$\langle x^*, x' - x\rangle \le \sigma\|x' - x\|^2 \quad \text{for all} \quad x' \in S \cap (x + \delta\mathbf{B}). \tag{2.17}$$

Fix now $\gamma = \min\{1, \frac{\delta}{3}\}$ and fix also any z in $x + \gamma\mathbf{B}$ and choose y_z in S such that

$$\|y_z - z\| \le d_S(z) + \|z - x\|^2. \tag{2.18}$$

Then $y_z \in x + \delta\mathbf{B}$, because (by (2.18)) $\|y_z - x\| \le \|y_z - z\| + \|z - x\| \le 3\|z - x\| \le 3\gamma \le \delta$, and hence

$$\begin{aligned}
\langle x^*, z - x\rangle &= \langle x^*, y_z - x\rangle + \langle x^*, z - y_z\rangle \\
&\le \sigma\|y_z - x\|^2 + \|y_z - z\| && \text{(by (2.17))} \\
&\le 9\sigma\|z - x\|^2 + d_S(z) + \|z - x\|^2 && \text{(by (2.18))} \\
&\le d_S(z) - d_S(x) + (9\sigma + 1)\|z - x\|^2.
\end{aligned}$$

This ensures that $x^* \in \partial^P d_S(x)$ and hence the proof is finished. □

Remark 2.7. Note that the idea of the proof of Theorem 2.7 can also be used to give another and different proof for Theorem 2.3. Note also that the proof shows that Theorem 2.7 holds for any normed vector space.

Theorem 2.8. *Let S be a nonempty closed subset of X and $x \in S$. Then S is proximally normally regular at x if and only if the function d_S is proximally subdifferentially regular at x.*

Proof.

1. Suppose that S is proximally normally regular at x, that is $N^P(S;x) = N^C(S;x)$. Then, by Theorem 2.7 one has

$$\partial^C d_S(x) \subset N^C(S;x) \cap \mathbf{B}_* = N^P(S;x) \cap \mathbf{B}_* = \partial^P d_S(x) \subset \partial^C d_S(x)$$

   which ensures the proximal subdifferential regularity of d_S at x.
2. Now, we assume that d_S is proximally subdifferentially regular (PSR) at x. Then the definition of the Fréchet subdifferential and the definition of (PSR) ensure that

$$\partial^P d_S(x) = \widehat{\partial} d_S(x) = \partial^C d_S(x).$$

So by Theorems 2.3 and 2.7 one has $N^P(S;x) = \widehat{N}(S;x)$. Moreover, as X is reflexive and $\widehat{N}(S;x)$ is convex and strongly closed in X^* (see Part (3) in Proposition 1.8), $\widehat{N}(S;x)$ is weak star closed and hence so is $N^P(S;x)$. Thus, the relation (2.10) yields

$$N^C(S;x) = \mathrm{cl}_{w^*}\left(\mathbf{R}_+ \partial^C d_S(x)\right)$$

$$= \mathrm{cl}_{w^*}\left(\mathbf{R}_+\partial^P d_S(x)\right)$$

$$= \mathrm{cl}_{w^*}\left(N^P(S;x)\right)$$

$$= N^P(S;x).$$

This completes the proof. □

Corollary 2.2. *Let S be a nonempty closed subset of $\mathbf{R}^N$ and let $x \in S$. Then S is proximal normally regular at x if and only if d_S is proximally subdifferentially regular at x.*

Remark 2.8. Note that in finite dimensional spaces (see Theorem 2.12 below), tangential regularity and Fréchet normal regularity are equivalent. Consequently, according to that equivalence, Corollaries 2.1 and 2.2, one concludes that even in finite dimensional spaces, tangential regularity and proximal normal regularity are not equivalent.

2.8 Weak Tangential Regularity of Sets

Another natural notion of regularity, in infinite dimensional setting, is the weak tangential regularity, that is, the Clarke tangent cone $T^{\mathrm{C}}(S;\bar{x})$ coincides with the *weak contingent cone* $K^{\mathrm{w}}(S;\bar{x})$, where

$K^{\mathrm{w}}(S;\bar{x})$ is the set of all $v \in X$ for which there exist a sequence of positive numbers $t_k \longrightarrow 0$ and a sequence $v_k \longrightarrow^w v$ such that $x + t_k v_k \in S$ for all k, where $\longrightarrow^w$ means the weak convergence.

Note that one always has $T^{\mathrm{C}}(S;x) \subset K(S;x) \subset K^{\mathrm{w}}(S;x)$.

We have proved in the previous section that in infinite dimensional case, Fréchet normal regularity is not equivalent to tangential regularity. One of our interests in this section is to prove (see Theorem 2.13) that Fréchet normal regularity can be characterized as weak tangential regularity, whenever the space X is reflexive. First, we prove the following theorem. Its proof is in the line of the proof of Fact 1 in Theorem 2.1.

Theorem 2.9. *Let S be a nonempty closed subset of a reflexive Banach space X and let $x \in S$. Then*

$$d_S^-(x;v) \geq d\left(K^{\mathrm{w}}(S;x);v\right), \quad \text{for all} \quad v \in X.$$

Proof. Fix any $v \in X$. Let (t_k) be a sequence of positive numbers converging to zero such that

$$d_S^-(x;v) = \lim_{k\to+\infty} t_k^{-1}\left[d_S(x+t_k v) - d_S(x)\right].$$

For each integer k choose $w_k \in t_k^{-1}(S-x)$ such that

$$d_{t_k^{-1}(S-x)}(v) \geq \|v - w_k\| - t_k.$$

Then,

$$\begin{aligned} d_S^-(x;v) &= \lim_{k\to+\infty} t_k^{-1} d_S(x+t_k v) \\ &= \lim_{k\to+\infty} d_{t_k^{-1}(S-x)}(v) \\ &\geq \limsup_{k\to+\infty} \|v - w_k\|. \end{aligned}$$

Since $d_S^-(x;v) \leq \|v\|$ (since d_S is 1-Lipschitz), the sequence (w_k) is bounded, and hence some subsequence converges weakly to $w \in K^{\mathrm{w}}(S;x)$. Hence,

$$\begin{aligned} d_S^-(x;v) &\geq \limsup_{k\to+\infty} \|v - w_k\| \\ &\geq \liminf_{k\to+\infty} \|v - w_k\| \\ &\geq \|v - w\| \\ &\geq d\big(K^{\mathrm{w}}(S;x);v\big), \end{aligned}$$

which completes the proof. □

In the following theorem we establish a relationship between the weak tangential regularity of a subset and the directional regularity of its distance function.

Theorem 2.10. *Let S be a nonempty closed subset of a reflexive Banach space and let $x \in S$. Assume that S is weakly tangentially regular at x. Then d_S is directionally regular at x.*

Proof. Assume that S is weakly tangentially regular at x, that is $K^{\mathrm{w}}(S;x) = T^{\mathrm{C}}(S;x)$. Then by Facts 1 and 2 in Theorem 2.1 and by Theorem 2.8 we obtain for all $v \in X$

$$\begin{aligned} d_S^-(x;v) &\leq d_S^0(x;v) \\ &\leq d\big(T^{\mathrm{C}}(S;x);v\big) \\ &= d\big(K^{\mathrm{w}}(S;x);v\big) \\ &\leq d_S^-(x;v). \end{aligned}$$

This finishes the proof. □

Now, we give the relationship between the tangential regularity of a nonempty closed subset $S \subset X$ at $x \in S$ and another type of normal regularity. First we need the definition of the lower Dini directional derivative for l.s.c functions (not necessarily locally Lipschitz continuous). Let $f : X \to \mathbf{R} \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} f$. The lower Dini directional derivative of f at $\bar{x}$ is given by

$$f^-(\bar{x};v) = \liminf_{\substack{v' \to v \\ t \downarrow 0}} t^{-1} f(\bar{x}+tv') - f(\bar{x}). \tag{2.19}$$

Using the same idea as in Sect. 1.3.2, we define the Dini subdifferential of f at $\bar{x}$, i.e.

$$\partial^- f(\bar{x}) = \{x^* \in X^* : \ \langle x^*, v\rangle \le f^-(\bar{x};v), \ \text{for all} \ v \in X\}.$$

Now, we define the new type of normal regularity of a closed set S at any point $\bar{x} \in S$ as follows: We will say that S is *Dini normally regular* provided that $N^C(S;x) = N^-(S;x)$, where $N^-(S;x)$ is the Dini normal cone to S at x defined as the Dini subdifferential of the indicator function ψ_S (note that ψ_S is l.s.c whenever S is closed and it is not Lipschitz) of S at x (see Chap. 3 for more details on the Dini subdifferential for lower semicontinuous functions). Note that one always has the inclusion $N^-(S;x) \subset N^C(S;x)$.

Theorem 2.11. *Let X be a normed vector space and let S be a nonempty closed subset of X with $x \in S$. Then S is tangentially regular (TR) at x if and only if it is Dini normally regular (DNR) at x.*

Proof. First, we prove the following relationship between the lower Dini derivative of the indicator function and the Bouligand tangent cone

$$\psi_S^-(x;v) = \psi_{K(S;x)}(v), \ \text{for all} \ v \in X. \tag{2.20}$$

Assume that $v \notin K(S;x)$, then by Part (2) in Proposition 1.6 one has $x + t_n v_n \notin S$ for all $t_n \downarrow 0$ and all $v_n \to v$. Then

$$\psi_S^-(x;v) = \liminf_{\substack{v' \to v \\ t \downarrow 0}} t^{-1} \psi_S(x+tv') = +\infty = \psi_{K(S;x)}(v).$$

Assume now that $v \in K(S;x)$. Then by Part (2) in Proposition 1.6 once again there exists $t_n \downarrow 0$ and $v_n \to v$ such that $x + t_n v_n \in S$ for all n. By the definition recalled above of the lower Dini derivative one has

$$0 \le \psi_S^-(x;v) = \liminf_{\substack{v' \to v \\ t \downarrow 0}} t^{-1} \psi_S(x+tv') \le \liminf_{n \to +\infty} t_n^{-1} \psi_S(x+t_n v_n) = 0 = \psi_{K(S;x)}(v)$$

and hence in both cases one has $\psi_S^-(x;v) = \psi_{K(S;x)}(v)$.

Now, we assume that S is tangentially regular at x and we prove that it is Dini normally regular at the same point x. By (2.20) and the definition of the Dini normal cone one gets

$$\begin{aligned} N^-(S;x) &= \{x^* \in X^* : \ \langle x^*, v\rangle \le \psi_S^-(x;v) = \psi_{K(S;x)}(v), \ \text{for all} \ v \in X\} \\ &= \{x^* \in X^* : \ \langle x^*, v\rangle \le 0, \ \text{for all} \ v \in K(S;x)\} = K^0(S;x), \end{aligned}$$

where $K^0(S;x)$ denotes the negative polar of $K(S;x)$. Therefore, by the tangential regularity of S at x one obtains

$$N^-(S;x) = K^0(S;x) = T^0(S;x) = N^{\mathrm{C}}(S;x).$$

For the converse, we assume that S is Dini normally regular at x, that is $N^-(S;x) = N^{\mathrm{C}}(S;x)$ and we show that S is tangential regularity at x. Then,

$$K(S;x) \subset (N^-(S;x))^0 = (N^{\mathrm{C}}(S;x))^0 = T^{\mathrm{C}}(S;x)$$

and hence $K(S;x) = T^{\mathrm{C}}(S;x)$ (because the reverse inclusion always holds). □

Now, we show that the Fréchet normal regularity of a nonempty closed subset $S \subset X$ at x implies the tangential regularity of S at x. If, in addition, X is assumed to be a finite dimensional space, then we have the equivalence.

Theorem 2.12. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Assume that S is Fréchet normally regular at x. Then, S is tangentially regular at x. If, in addition, X is a finite dimensional space, then one has the equivalence.*

Proof.

1. Assume that S is Fréchet normally regular at x, i.e., $N^{\mathrm{C}}(S;x) = \widehat{N}(S;x)$. As one always has $\widehat{N}(S;x) \subset N^-(S;x) \subset N^{\mathrm{C}}(S;x)$, the Dini normal regularity of S at x is ensured. Consequently, by Theorem 2.11, S is tangentially regular at x.
2. Now, we assume that X is a finite dimensional space and S is tangentially regular at x, i.e., $T^{\mathrm{C}}(S;x) = K(S;x)$. Let $x^* \in N^{\mathrm{C}}(S;x) = (T^{\mathrm{C}}(S;x))^0$. Then

$$\langle x^*, v\rangle \le 0 \quad \text{for all} \quad v \in T^{\mathrm{C}}(S;x) = K(S;x).$$

Consider a sequence (x_k) in S that converges to x with $x_k \neq x$ and such that

$$\limsup_{x' \to_S x} \left\langle x^*, \frac{x'-x}{\|x'-x\|} \right\rangle = \lim_{k\to+\infty} \left\langle x^*, \frac{x_k - x}{\|x_k - x\|} \right\rangle.$$

Extracting a subsequence if necessary we may suppose that

$$\frac{x_k - x}{\|x_k - x\|} \to v \in K(S;x) = T^{\mathrm{C}}(S;x).$$

Therefore, $\langle x^*, v\rangle \le 0$ and hence

$$\limsup_{x' \to_S x} \left\langle x^*, \frac{x'-x}{\|x'-x\|} \right\rangle \le 0$$

that is, $x^* \in \widehat{N}(S;x)$. So, $N^C(S;x) = \widehat{N}(S;x)$ and hence the proof is complete. □

Theorem 2.12, Corollary 2.1, and Theorem 2.6 give the following result. Note that the equivalence between (i), (ii), and (iii) of the corollary has been established (in a different way) in Corollary 6.29 of the book of Rockafellar and Wets [241].

Corollary 2.3. *Let S be a nonempty closed subset of $\mathbf{R}^N$ and let $x \in S$. Then the following assertions are equivalent:*

(i) S is Fréchet normally regular at x;
(ii) S is Mordukhovich regular at x;
(iii) S is tangentially regular at x;
(iv) d_S is Fréchet subdifferentially regular at x;
(v) d_S is directionally regular at x.

We have showed in the previous corollary that the equivalence between tangential regularity and Fréchet normal regularity is ensured whenever X is a finite dimensional space. We will see via a set constructed in Borwein and Fabian [36] that for any infinite dimensional space, this equivalence does not hold. For this purpose, we recall the following result due to Borwein and Strojwas (Proposition 3.1 in [37] see also Kruger [167, 175]) and we will characterize Fréchet normal regularity as weak tangential regularity in reflexive Banach spaces.

Proposition 2.2. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Then*

$$\widehat{N}(S;x) \subset \left(K^{\mathrm{w}}(S;x)\right)^0.$$

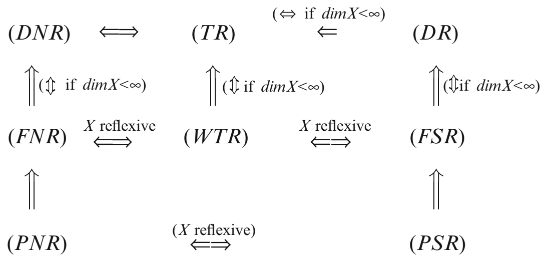
If, furthermore, X is a reflexive Banach space, then equality holds in the inclusion above.

Theorem 2.13. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Then,*

(i) S is weakly tangentially regular at x whenever it is Fréchet normally regular at x;
(ii) if, in addition, X is reflexive, then (i) becomes an equivalence.

Proof. 1. Assume that S is Fréchet normally regular at x, i.e., $\widehat{N}(S;x) = N^C(S;x) = (T^C(S;x))^0$. This ensures by the previous proposition that $(T^C(S;x))^0 \subset \left(K^{\mathrm{w}}(S;x)\right)^0$. Therefore, as $T^C(S;x)$ is a closed convex cone, we obtain $K^{\mathrm{w}}(S;x) \subset T^C(S;x)$ and hence $K^{\mathrm{w}}(S;x) = T^C(S;x)$ since the reverse inclusion always holds. So, S is weakly tangentially regular at x.

Fig. 2.4 Relationships between various concepts of regularity of sets



2. Now, assume that X is reflexive and S is weakly tangentially regular at x i.e. $T^{\mathrm{C}}(S;x) = K^{\mathrm{w}}(S;x)$. Then, one has $N^{\mathrm{C}}(S;x) = (T^{\mathrm{C}}(S;x))^0 = \big(K^{\mathrm{w}}(S;x)\big)^0$. On the other hand, we have (by Proposition 2.2) $\widehat{N}(S;x) = \big(K^{\mathrm{w}}(S;x)\big)^0$, and hence

$$\widehat{N}(S;x) = N^{\mathrm{C}}(S;x)$$

which ensures the Fréchet normal regularity of S at x. □

In Theorem 2.13, we have proved that when X is a reflexive Banach space one has **(1)** "$\widehat{N}(S;x) = N^{\mathrm{C}}(S;x) \Longleftrightarrow$ **(2)**" "S is w-tangentially regular at x." Moreover, we have proved in Theorem 2.10 that **(2)** $\Longrightarrow$ **(3)** "d_S is directionally regular at x." So **(1)** $\Longrightarrow$ **(3)**. On the other hand, in [36] Borwein and Fabian proved that we can find in any infinite dimensional Banach space a nonempty closed subset S of X and a point x in S for which **(4)** "S is tangentially regular at x" and d_S is not directionally regular at x. Consequently, in infinite dimensional Banach spaces **(4)** $\not\Longrightarrow$ **(3)**. This ensures that, for *infinite dimensional reflexive* Banach spaces, there is *no equivalence between tangential regularity and Fréchet normal regularity*.

Remark 2.9.

a. In the infinite dimensional case (even if X is assumed to be a reflexive Banach space), the inclusion $\widehat{N}(S;x) \subset N^{-}(S;x)$ is strict for the subset S constructed in Borwein and Fabian [36]. Indeed, according to the arguments above $\widehat{N}(S;x)$ is strictly included in $N^{\mathrm{C}}(S;x)$. On the other hand, as S is tangentially regular at x, Theorem 2.11 ensures that S is normally Dini regular at x, i.e., $N^{\mathrm{C}}(S;x) = N^{-}(S;x)$. Therefore, $\widehat{N}(S;x)$ is strictly included in $N^{-}(S;x)$ and in *infinite dimensional spaces Fréchet normal regularity and Dini normal regularity are not equivalent.*
b. The same subset S in [36] also shows (with the help of Theorem 2.11) that in *infinite dimensional spaces the Dini normal regularity of a set is not equivalent to the Dini subdifferential regularity of the distance function associated with this set.*

Conclusion. In the following diagram, we summarize all the relationships between all the various concepts of regularity of sets considered in this chapter (Fig. 2.4). Let X be a normed vector space, S be a nonempty closed subset of X, and x be some point in S. Then one has

However, there are important classes of sets (in finite and infinite dimensional spaces) for which all these types of regularity hold. It is obviously the case for convex sets. Another classes that appeared very recently are the one of prox-regular sets introduced by Poliquin and Rockafellar [229] and the one of proximally smooth sets introduced by Clarke et al. [89]. The next section is devoted to study this important class.

2.9 Uniform Prox-Regularity of Sets

First, we begin by recalling that, for a given $r \in (0,+\infty]$, a subset S is uniformly r-prox-regular (see [230]) (or equivalently r-proximally smooth see [89]) if and only if every nonzero proximal normal to S can be realized by an r-ball. This means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S;\bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ and we will just say in the sequel that S is uniformly r-prox-regular. Recall that for $r = +\infty$, the uniform r-prox-regularity of S is equivalent to the convexity of S. The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel of the book. For the proof of these results we refer the reader to [89, 230]. We use the notation $\text{proj}_S(x)$ instead of $\text{Proj}_S(x)$ whenever this set has a unique point.

Proposition 2.3. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $r > 0$. If the subset S is uniformly r-prox-regular, then the following hold:*

(i) For all $x \in \mathbf{H}$ with $d_S(x) < r$, $\text{proj}_S(x)$ exists;
(ii) For every $r' \in (0,r)$, the enlarged subset $S(r') := \{x \in \mathbf{H} : d_S(x) \leq r'\}$ is uniformly $(r - r')$-prox-regular;
(iii) The generalized gradient and the proximal subdifferential of d_S coincide at all points $x \in \mathbf{H}$ with $d_S(x) < r$.

The following proposition shows that in the inequality above characterizing the uniform prox-regularity one may use the proximal subdifferential of the distance function in place of the proximal normal cone. For a given subset S in $\mathbf{H}$ and a given $r > 0$ we will set

$$(P_r) \begin{cases} \text{for all } \bar{x} \in S \text{ and all } 0 \neq \xi \in N^P(S;\bar{x}) \text{ one has} \\ \left\langle \dfrac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \dfrac{1}{2r}\|x - \bar{x}\|^2 \quad \text{for all } x \in S \end{cases}$$

and

$$(P'_r) \qquad \begin{cases} \text{for all } \bar{x} \in S \text{ and all } \xi \in \partial^P d_S(\bar{x}) \text{ one has} \\ \langle \xi, x - \bar{x} \rangle \le \dfrac{1}{2r}\|x-\bar{x}\|^2 \quad \text{for all } x \in S. \end{cases}$$

Proposition 2.4. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $r > 0$. Then $(P_r) \Leftrightarrow (P'_r)$.*

Proof. $((P_r) \Rightarrow (P'_r))$. Assume that S satisfies (P_r). The property (P'_r) obviously holds for $\xi = 0$. Let $\bar{x} \in S$ and $0 \neq \xi \in \partial^P d_S(\bar{x}) \subset N^P(S;\bar{x})$. Then by (P_r) one has for all $x \in S$

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \le \frac{1}{2r}\|x-\bar{x}\|^2$$

and hence

$$\langle \xi, x - \bar{x} \rangle \le \frac{\|\xi\|}{2r}\|x-\bar{x}\|^2 \le \frac{1}{2r}\|x-\bar{x}\|^2$$

because $\|\xi\| \le 1$. The property (P'_r) then holds.

$((P'_r) \Rightarrow (P_r))$. Assume now that S satisfies (P'_r). Let $\bar{x} \in S$ and $0 \neq \xi \in N^P(S;\bar{x})$. Then by Theorem 2.7 one has $\dfrac{\xi}{\|\xi\|} \in \partial^P d_S(\bar{x})$ and hence one gets (by (P'_r))

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \le \frac{1}{2r}\|x-\bar{x}\|^2$$

for all $x \in S$. This completes the proof of the second implication and so the proof of the proposition is finished. □

The following lemma is proved in Sect. 2.11 in the context of a general normed vector space. It will be used in the proof of the next theorem. For the convenience of the reader, we show how the Hilbert norm allows us to give another simple proof. The reader will also note that the arguments work for any Kadec norm of a reflexive Banach space (see, e.g., [106–108] for the definition and properties of Kadec norms).

Lemma 2.1. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $r > 0$. Then for all $x \notin S(r)$ one has*

$$d_{S(r)}(x) = d_S(x) - r. \tag{2.21}$$

Proof. As the set $\{x \notin S(r) : \ \mathrm{Proj}_S(x) \neq \emptyset\}$ is dense in $X \setminus S(r)$ by [178], and as the functions d_S and $d_{S(r)}$ are continuous, it is enough to prove (2.21) only for points $x \notin S(r)$ satisfying $\mathrm{Proj}_S(x) \neq \emptyset$. Fix any such point x and fix also p in S such that $d_S(x) = \|x-p\|$. Set

$$u := p + \left(\frac{r}{\|x-p\|}\right)(x-p). \tag{2.22}$$

We observe that u is in $S(r)$ because (2.22) and the relation $p \in S$ ensure $d_S(u) \le \|u-p\| = r$.

Let us prove now that $u \in \mathrm{Proj}_{S(r)}(x)$. Consider any $y \in S(r)$, that is, $d_S(y) \leq r$, and fix any positive number ε. We may choose some $y_\varepsilon \in S$ satisfying

$$\|y - y_\varepsilon\| \leq d_S(y) + \varepsilon \leq r + \varepsilon.$$

Consequently

$$\|y - x\| \geq \|y_\varepsilon - x\| - \|y_\varepsilon - y\| \geq \|x - p\| - r - \varepsilon = \|x - u\| - \varepsilon$$

and this yields $d_{S(r)}(x) \geq \|x - u\| - \varepsilon$. As this holds for all $\varepsilon > 0$, we have $d_{S(r)}(x) \geq \|x - u\|$ and hence $d_{S(r)}(x) = \|x - u\|$ because u is in $S(r)$ as observed above. Writing by (2.22)

$$d_{S(r)}(x) = \|x - u\| = \|x - p\| - r = d_S(x),$$

the proof of the lemma is finished. □

We establish now the main result of this section from which some new characterizations of uniformly r-prox-regular sets will be derived. Here, the point where the proximal subdifferential of d_S is considered is not required to stay in S contrarily to Proposition 2.4.

Theorem 2.14. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $r > 0$. Assume that S is uniformly r-prox-regular. Then the following holds:*

$$(P_r'') \quad \begin{cases} \text{for all } x \in \mathbf{H}, \ \text{ with } d_S(x) < r, \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\[2ex] \langle \xi, x' - x \rangle \leq \dfrac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x), \\[2ex] \quad \text{for all } \ x' \in \mathbf{H} \ \text{ with } \ d_S(x') \leq r. \end{cases}$$

Proof.
Step 1. Firstly, we prove a stronger property for $x \in S$, more precisely we prove the following:

$$(P_r''') \quad \begin{cases} \text{for all } x \in S \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\[2ex] \langle \xi, x' - x \rangle \leq \dfrac{2}{r} \|x' - x\|^2 + d_S(x'), \\[2ex] \quad \text{for all } \ x' \in \mathbf{H} \ \text{ with } \ d_S(x') < r. \end{cases}$$

Fix any $x \in S$ and any $\xi \in \partial^P d_S(\bar{x})$. Fix also any $z \in \mathbf{H}$ satisfying $d_S(z) < r$. As S is uniformly r-prox-regular one can find some $y_z \in \mathrm{Proj}_S(z) \neq \emptyset$, that is, y_z is in S and

$$\|z - y_z\| = d_S(z). \tag{2.23}$$

Then,

$$\|y_z - x\| \le \|y_z - z\| + \|z - x\| \le 2\|z - x\|$$

and hence by (P'_r) and the inequality $\|\xi\| \le 1$, and also by the equality (2.23) one gets

$$\begin{aligned}\langle \xi, z - x\rangle &= \langle \xi, y_z - x\rangle + \langle \xi, z - y_z\rangle \\ &\le \frac{1}{2r}\|y_z - x\|^2 + \|\xi\|\|y_z - z\| \\ &\le \frac{2}{r}\|z - x\|^2 + d_S(z) - d_S(x).\end{aligned}$$

This completes the proof of (P'''_r).

Step 2. Note that (see Part (ii) in Proposition 2.3) for every $0 < r' < r$ the enlarged set $S(r')$ is uniformly $(r - r')$-prox-regular. Further, for any $u' \in \mathbf{H}$ it can be seen that the inequality $d_{S(r')}(u') < r - r'$ holds if and only if $d_S(u') < r$. Indeed, if we suppose that $d_{S(r')}(u') < r - r'$, then there exists some z in $\mathbf{H}$ with $d_S(z) \le r'$ and $\|u' - z\| < r - r'$, and hence

$$d_S(u') \le d_S(z) + \|u' - z\| < r.$$

Suppose now that $d_S(u') < r$. In the case $u' \in S(r')$, we can write $d_{S(r')}(u') = 0 < r - r'$. In the other case $u' \notin S(r')$, we have by Lemma 2.1

$$d_{S(r')}(u') = d_S(u') - r' < r - r'.$$

The equivalence then holds, and hence the property $(P'''_{(r-r')})$ may be written as

$$(P'''_{(r-r')}) \quad \begin{cases} \text{for all } u \in S(r'), \text{ and all } \zeta \in \partial^P d_{S(r')}(u) \text{ one has} \\ \langle \zeta, u' - u\rangle \le \dfrac{2}{(r - r')}\|u' - u\|^2 + d_{S(r')}(u'), \\ \qquad \text{for all } \; u' \in \mathbf{H} \; \text{ with } d_S(u') < r. \end{cases}$$

Now, fix any $x \in \mathbf{H}$ with $d_S(x) < r$ and any $\xi \in \partial^P d_S(x)$. We distinguish two cases:
Case 1. If $x \in S$, then by (P'''_r) one obtains for all $x' \in \mathbf{H}$ with $d_S(x') < r$

$$\langle \xi, x' - x\rangle \le \frac{2}{r}\|x' - x\|^2 + d_S(x') - d_S(x). \tag{2.24}$$

Case 2. If $x \notin S$, we put $r' := d_S(x) > 0$ in this case. Firstly, one observes that $\xi \in \partial^P d_{S(r')}(x)$. Indeed, one knows by Theorems 4.1 and 4.3 in Bounkhel and Thibault [61] (see also Theorem 3.2 in [89] for the equality in the following relation) that

$$\partial^P d_S(x) = N^P(S(r'),x) \cap \{\zeta : \|\zeta\| = 1\} \subset \partial^P d_{S(r')}(x)$$

and hence as ξ is fixed in $\partial^P d_S(x)$, one then gets $\xi \in \partial^P d_{S(r')}(x)$. Applying $(P'''_{(r-r')})$ in the form obtained above one gets for any $x' \in \mathbf{H}$ with $d_S(x') < r$

$$\langle \xi, x'-x \rangle \leq \frac{2}{r-r'}\|x'-x\|^2 + d_{S(r')}(x').$$

Consequently, for any $x' \in \mathbf{H}$ satisfying $d_S(x') < r$ and $x' \notin S(r')$ (that is, $r' < d_S(x') < r$) one gets according to Lemma 2.1

$$\langle \xi, x'-x \rangle \leq \frac{2}{r-r'}\|x'-x\|^2 + d_S(x') - d_S(x). \tag{2.25}$$

Now fix any $x' \in \mathbf{H}$ satisfying $d_S(x') < r$ and $x' \in S(r')$. We begin by noting that $(P'''_{(r-r')})$ ensures that the inequality

$$\langle \xi, y-x \rangle \leq \frac{2}{r-r'}\|y-x\|^2. \tag{2.26}$$

holds for all $y \in \mathbf{H}$ with $d_S(y) \leq r'$. Choose now, according to $\|\xi\| = 1$, some $u \in \mathbf{H}$ with $\|u\| = 1$ and such that $\langle \xi, u \rangle = 1$. Put $t := d_S(x) - d_S(x') \geq 0$. Then $x' + tu \in S(r')$, because $d_S(x'+tu) \leq d_S(x') + t = d_S(x) = r'$. Therefore, (2.26) allows us to write

$$\langle \xi, x'-x \rangle = \langle \xi, x'+tu-x \rangle - \langle \xi, tu \rangle \leq \frac{2}{r-r'}\|x'+tu-x\|^2 - t. \tag{2.27}$$

Observing that

$$\|x'+tu-x\| \leq \|x'-x\| + t \leq 2\|x'-x\|$$

we deduce from (2.27)

$$\langle \xi, x'-x \rangle \leq \frac{8}{r-r'}\|x'-x\|^2 + d_S(x') - d_S(x).$$

It then follows from (2.24), (2.25) and the last inequality that one has for all $x \in \mathbf{H}$ with $d_S(x) < r$ and all $\xi \in \partial^P d_S(x)$

$$\langle \xi, x'-x \rangle \leq \frac{8}{r-r'}\|x'-x\|^2 + d_S(x') - d_S(x) \quad \text{for all} \quad x' \in \mathbf{H} \quad \text{with} \quad d_S(x') < r.$$

Taking the continuity of both members of that inequality with respect to x' into account, we may replace the requirement $d_S(x') < r$ by $d_S(x) \leq r$. The proof of the theorem is then complete. □

The following corollary of the above theorem adds some further characterizations of uniformly prox-regular sets to the lists in Clarke et al. [89] and Poliquin et al. [230].

Corollary 2.4. *Let S be a nonempty closed subset of $\mathbf{H}$ and let $r > 0$. Then, the following assertions are equivalent:*

(a) S is uniformly r-prox-regular;
(b) the property (P_r'') holds for the proximal subdifferential of d_S;
(c) the property (P_r'') holds for the Fréchet subdifferential of d_S;
(d) the property (P_r'') holds for the basic (limiting) subdifferential of d_S;
(e) the property (P_r'') holds for the generalized gradient (Clarke subdifferential) of d_S.

Proof. The implication $(a) \Rightarrow (b)$ follows from Theorem 2.14 and $(b) \Rightarrow (c)$ holds because any $\xi \in \widehat{\partial} d_S(x)$ is the weak limit of a sequence $(\xi_n)_n$ such that $\xi_n \in \partial^P d_S(x_n)$ and $(x_n)_n$ converges to x. In the same way, the implication (c) implies (d) is true. The implication $(d) \Rightarrow (e)$ can be seen easily as a consequence of the definition of (P_r'') and of the formula characterizing the generalized gradient of a Lipschitz function as the closed convex hull of its basic (limiting) subdifferential. So, it remains to see $(e) \Rightarrow (a)$. We know that $\partial^C d_S(x)$ is nonempty at any x (see [88]). Supposing that (e) holds. This property tells us that any generalized subgradient is a proximal subgradient. Therefore, for any $x \in \mathbf{H}$ with $d_S(x) < r$ we have $\partial^P d_S(x) \neq \emptyset$. The implication is thus a consequence of corollary 4.3 in [230] (see also Theorem 4.1 in [89]). □

Observe that the assertion (e) in the corollary entails that the generalized gradient (Clarke subdifferential) and the proximal subdifferential (and hence also the Fréchet subdifferentials) of d_S coincide at all points $x \in \mathbf{H}$ with $d_S(x) < r$ provided that S is uniformly r-prox-regular. In fact, it is easily seen that this equality property of these subdifferentials characterizes the uniformly r-prox-regular sets.

2.10 Arc-Wise Essential Tangential Regularity

This section is devoted to study a different type of regularity for closed sets in Banach spaces. This concept has been introduced by Borwein and Moors in [34] in $\mathbf{R}^n$. Let us start with the following definition.

Definition 2.6. A closed subset C of a Banach space X is *arc-wise essentially tangentially regular* in X and we will write $C \in \mathscr{A}_{\mathrm{TR}}(X)$, if for each $x \in \mathscr{S}_e((0,1),X)$, the set

$$\{t \in (0,1) : x(t) \in C \ \text{ and } \ x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}$$

has null measure, where $\mathscr{S}_e((0,1),X)$ is the class of all locally Lipschitz mappings $x : (0,1) \to X$ which are strictly differentiable almost everywhere on $(0,1)$. The sets

$K(C;x)$ and $T^{\mathrm{C}}(C;x)$ denote the contingent cone and the Clarke tangent cone of C at x respectively (see Chap. 1).

Remark 2.10. As one always has $K(C;x) = T^{\mathrm{C}}(C;x) = X$, for each $x \in \operatorname{int} C$ (the topological interior of C), we can take x only in *bd C* (the boundary of C), in Definition 2.6, that is, C is arc-wise essentially tangentially regular if and only if for each $x \in \mathscr{S}_e((0,1),X)$ one has

$$\mu\big(\{t \in (0,1) : x(t) \in bd\, C \ \text{ and } \ x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}\big) = 0.$$

We proceed now to establish a characterization of the class $\mathscr{A}_{\mathrm{TR}}(X)$.

Proposition 2.5. *Let C be a nonempty closed subset of X. Then the set C is arc-wise essentially tangentially regular if and only if for each $x \in \mathscr{S}_e((0,1),X)$, both sets*

$$\{t \in (0,1) : x(t) \in C \ \text{ and } \ x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;(x(t))\}$$

and

$$\{t \in (0,1) : x(t) \in C \ \text{ and } \ -x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}$$

have null measure.

Proof. " $\Rightarrow$ " Assume that C is arc-wise essentially tangentially regular, that is for each $z \in \mathscr{S}_e((0,1),X)$, one has

$$\mu\big(\{t \in (0,1) : z(t) \in C \ \text{ and } \ z'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}\big) = 0.$$

Fix any $x \in \mathscr{S}_e((0,1),X)$ and put

$$E_x := \{t \in (0,1) : x(t) \in C \ \text{ and } \ x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;(x(t))\}$$

and

$$\tilde{E}_x := \{t \in (0,1) : x(t) \in C \ \text{ and } \ -x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;(x(t))\}.$$

We will show that $\mu(E_x \cup \tilde{E}_x) = 0$. Since by hypothesis, we have $\mu(E_x) = 0$, it suffices to show that $\mu(\tilde{E}_x) = 0$. To this end, define $h : (0,1) \longrightarrow (0,1)$ by $h(s) = 1 - s$ and $y : (0,1) \longrightarrow X$ by $y(s) = (x \circ h)(s) = x(1-s)$. Clearly, $y \in \mathscr{S}_e((0,1),X)$ and everywhere $y'(s) = -x'(h(s))$. For $s := 1 - t$ one has

$$\begin{aligned}
\tilde{E}_x &= \{h(s) \in (0,1) : x(h(s)) \in C \text{ and} -x'(h(s)) \in K(C;x(h(s))) \setminus T^{\mathrm{C}}(C;x(h(s)))\} \\
&= \{h(s) \in (0,1) : y(s) \in C \text{ and } \ y'(s) \in K(C;y(s)) \setminus T^{\mathrm{C}}(C;y(s))\} \\
&= h(E_y),
\end{aligned}$$

where $E_y := \{t \in (0,1) : y(t) \in C \text{ and } y'(t) \in K(C;y(t)) \setminus T^{C}(C;y(t))\}$. Since $y \in \mathscr{S}_e((0,1),X)$, the set E_y has null measure and hence $\mu(\tilde{E}_x) = 0$. Therefore, the proof of " $\Rightarrow$" is finished.

" $\Leftarrow$" This implication is obvious. □

Before proving the main theorem of this section which can be seen as a scalarization of arc-wise essential tangential regularity, we need the following different type of regularity for locally Lipschitz functions.

Definition 2.7. Let f be a locally Lipschitz function on a nonempty open subset Ω of X. We will say that f is arc-wise essentially strictly differentiable on Ω and we will write $f \in \mathscr{A}_{s,d}(\Omega)$ if for each $x \in \mathscr{AC}((0,1),\Omega)$, the set $\{t \in (0,1) : f$ is not strictly differentiable at $x(t)$ in the direction $x'(t)\}$ is Lebesgue-null in $\mathbf{R}$. Here, $\mathscr{AC}((0,1),\Omega)$ denotes the class of all absolutely continuous mappings defined from $(0,1)$ to Ω.

Also, we recall that a locally Lipschitz function f from X into $\mathbf{R}$ is directionally regular at x if $f'(x;v)$ exists for all $v \in X$ and $f^0(x;v) = f'(x;v)$ (See Chap. 1 for this definition and Chap. 3 for more details on directionally regular functions not necessarily locally Lipschitz).

Theorem 2.15. *Let C be a nonempty closed subset of a Banach space X.*

(i) C is arc-wise essentially tangentially regular whenever the associated distance function d_C is arc-wise essentially tangentially regular.
(ii) If the norm $\|.\|_X$ is uniformly Gâteaux differentiable, then C is arc-wise essentially tangentially regular if and only if d_C is arc-wise essentially strictly differentiable.

Proof.

1. Assume that $d_C \in \mathscr{A}_{s,d}(X,\mathbf{R})$, i.e., for each $x \in \mathscr{S}_e((0,1),X)$, the set

$$A := \{t \in (0,1) : d_C \text{ is not s.d. at } x(t) \text{ in the direction } x'(t)\}$$

is a Haar-null set. We will show that C is arc-wise essentially tangentially regular, i.e., $\mu(B) = 0$ where $B := \{t \in (0,1) : x(t) \in C \text{ and } x'(t) \in K(C;x(t)) \setminus T^{C}(C;(x(t))\})$. It is enough to prove that $B \subset A$. Let $t_0 \notin A$. Then d_C is s.d. at $x(t_0)$ in the direction $x'(t_0)$ and hence $d'_C(x(t_0);x'(t_0)) = d^-_C(x(t_0);x'(t_0)) = d^0_C(x(t_0);x'(t_0)) = 0$. So $x'(t_0) \in T^{C}(C;x(t_0))$ and hence $t_0 \notin B$. Consequently, each $t_0 \notin A$ does not lie in B. This completes the proof of the inclusion $B \subset A$.
2. Assume now that $\|.\|_X$ is a uniformly Gâteaux differentiable norm and assume that C is arc-wise essentially tangentially regular. Then, for each fixed x in $\mathscr{S}_e((0,1),X)$ by Proposition 2.5 we have

$$\mu(B_x) = 0,$$

where

$B_x = B_x^1 \cup B_x^2,$
$B_x^1 := \{t \in (0,1) : x(t) \in C, \text{ and } x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}$ and
$B_x^2 := \{t \in (0,1) : x(t) \in C \text{ and } -x'(t) \in K(C;x(t)) \setminus T^{\mathrm{C}}(C;x(t))\}.$

Put

$$A := \{t \in (0,1) : d_C \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t)\}.$$

It is not difficult to check that

$$A = \{t \in (0,1) : x(t) \in bd\, C, d_C \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t)\}.$$

Indeed, if $t \in (0,1)$ with $x(t) \in (X \setminus C) \cup \operatorname{int} C$ and d_C is not s.d. at $x(t)$ in the direction $x'(t)$, then $(-d_C)$ is not s.d. at $x(t)$ in the direction $x'(t)$ and so $(-d_C)$ is not directionally regular at $x(t)$ in the direction $x'(t)$, which is impossible, because $x(t) \in (X \setminus C) \cup \operatorname{int} C$, and Theorem 8 in [31]. Put now $D_{x'} := \{t \in (0,1) : x'(t) \text{ exists}\}$ hence

$$\mu(A \setminus D_{x'}) = 0 \tag{2.28}$$

and put also $I := I_r \cup I_l$ with I_r (resp. I_l) denotes the set of all isolated points in $A \cap D_{x'}$ relatively to the right topology (resp. the left topology). It is not difficult to check that I is countable and hence $\mu(I) = 0$. Fix $t_0 \in (A \cap D_{x'}) \setminus I$. Then there exist two sequences of real positive numbers $(\lambda_n)_n$ and $(\varepsilon_n)_n$ converging to zero such that for n sufficiently large $t_0 + \lambda_n$ and $t_0 - \varepsilon_n$ lie in $(A \cap D_{x'}) \setminus I$ and hence $x(t_0 + \lambda_n)$ and $x(t_0 - \varepsilon_n)$ lie in $bd\, C$, for n sufficiently large.

Put

$$v_n := \lambda_n^{-1}\left[x(t_0 + \lambda_n) - x(t_0)\right] \text{ and } w_n := \varepsilon_n^{-1}\left[x(t_0 - \varepsilon_n) - x(t_0)\right].$$

Clearly, $v_n \to x'(t_0)$ and $w_n \to -x'(t_0)$ and for n sufficiently large $x(t_0) + \lambda_n v_n$ and $x(t_0) + \varepsilon_n w_n$ lie in $bd\, C$. It follows by the sequential characterization of the contingent cone given Proposition 1.6, that $x'(t_0)$ and $-x'(t_0)$ lie in $K(C;x(t_0))$. Now, we distinguish two cases. Firstly, if $x'(t_0) \in K(C;x(t_0)) \setminus T^{\mathrm{C}}(C;x(t_0))$, then $t_0 \in B_x$. Secondly, if $x'(t_0) \in T^{\mathrm{C}}(C;x(t_0))$, then $-x'(t_0) \in K(C;x(t_0)) \setminus T^{\mathrm{C}}(C;x(t_0))$ (because, if $-x'(t_0) \in T^{\mathrm{C}}(C;x(t_0))$, we would have

$$d_C^0(x(t_0);x'(t_0)) = -d_C^0(x(t_0);-x'(t_0)) = 0,$$

so d_C would be s.d. at $x(t_0)$ in the direction $x'(t_0)$, which would contradict that $t_0 \in A$). Hence $t_0 \in B_x$. Thus $(D_{x'} \cap A) \setminus I \subset B_x$ and hence

$$\mu((D_{x'} \cap A) \setminus I) = 0. \tag{2.29}$$

Finally, according to (2.28) and (2.29), we obtain $\mu(A) = 0$. This ensures that $d_C \in \mathscr{A}_{s,d}(X, \mathbf{R})$ and hence the proof is finished. □

Remark 2.11. As observed by Borwein and Moors [34] all sets that are directionally tangentially regular except on a countable set are arc-wise essentially tangentially regular. Thus, all closed convex sets and smooth manifolds are arc-wise essentially tangentially regular.

2.11 More on the Regularity of Sets

In Sects. 2.5–2.7, we have scalarized some geometric notions of regularity of sets such as (TR), (FNR), and (PNR), via the distance function whenever the point is in the set. In this section we are interested in the following natural question: Given a closed nonempty set S and a point $\bar{x} \notin S$, is it possible to characterize the regularity of d_S at $\bar{x}$, which is well defined, in terms of some geometric notion regularity of the set S, or in other words is it possible to "geometrize" the regularity of d_S at points outside the set S. We restrict our study in this section to two concepts of regularity. We recall the following result from [71] which will be used in the present section.

Theorem 2.16. *Let S be a nonempty closed subset of a normed vector space X and let $x \notin S$ with d_S directionally regular at x and* $\mathrm{Proj}_S(x) \neq \emptyset$. *Then*

$$\partial^{\mathrm{C}} d_S(x) = N^{\mathrm{C}}(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

2.11.1 Fréchet Case

In this subsection, we will show that for every subset S of a reflexive Banach space X and every $x \notin S$, with $\mathrm{Proj}_S(x) \neq \emptyset$, the Fréchet subdifferential regularity of d_S at x implies the Fréchet normal regularity of the set $S(r)$ at x, where $r = d_S(x)$. In addition, if d_S is directionally regular at x, we have the equivalence.

In this subsection, we will put $r = d_S(x)$ for some point $x \notin S$. We begin this case with the following lemma which will be used in the proof of the next theorem. The lemma will be also used in the proximal case.

Lemma 2.2. *Let X be a normed vector space, S be a nonempty closed subset of X. Then for all $x' \notin S(r)$ one has*

$$d_{S(r)}(x') = d_S(x') - r.$$

Proof. Fix any $x' \notin S(r)$. Consider any $y \in S(r)$, that is, $d_S(y) \leq r$, and consider also any $\varepsilon > 0$. We may choose some $y_\varepsilon \in S$ satisfying

$$\|y - y_\varepsilon\| \leq d_S(y) + \varepsilon \leq r + \varepsilon.$$

Consequently

$$\|y-x'\| \geq \|y_\varepsilon - x'\| - \|y_\varepsilon - y\| \geq d_S(x') - \|y_\varepsilon - y\| \geq d_S(x') - r - \varepsilon.$$

As the inequality $\|y-x'\| \geq d_S(x') - r - \varepsilon$ holds for all $y \in S(r)$ and all $\varepsilon > 0$ we deduce

$$d_{S(r)}(x') \geq d_S(x') - r.$$

Let us prove the reverse inequality. Fix any $y \in S$ and consider the real-valued function h defined on $[0,+\infty)$ by $h(s) := d_S(sx' + (1-s)y)$. Observing that $h(0) = 0$ (because $y \in S$) and $h(1) > r$ (because $x' \notin S(r)$), we may apply the classical intermediate value theorem to get some $s_0 \in (0,1)$ such that $h(s_0) = r$. Putting $z := s_0 x' + (1-s_0)y$, we have $d_S(z) = r$ and

$$\|x'-y\| = \|x'-z\| + \|z-y\|.$$

Therefore, because $y \in S$ we obtain

$$\|x'-y\| \geq \|x'-z\| + d_S(z) = \|x'-z\| + r$$

and as $z \in S(r)$, it follows that

$$\|x'-y\| \geq d_{S(r)}(x') + r.$$

This yields the inequality $d_S(x') \geq d_{S(r)}(x') + r$ that completes the proof of the lemma. □

Now we establish the following result on the relationship between the Fréchet subdifferential of the distance function d_S at a point $x \notin S$ and the Fréchet normal cone of $S(r)$ at x. This result has been stated in [167] but the proof therein seems to need some further arguments. In the proof below, we use the previous lemma and strong ideas in the proof of Proposition 2.16 in Kruger [167].

Theorem 2.17. *Let X be a Banach space, S be a nonempty closed subset of X, and let $x \notin S$. Then,*

$$\widehat{\partial} d_S(x) \subset \widehat{N}(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \tag{2.30}$$

If, furthermore, X is a reflexive Banach space and $\|.\|$ denotes a Kadec equivalent norm on X, then equality holds in (2.30).

Proof. We begin by showing (2.30). Fix x^* in $\widehat{\partial} d_S(x)$ and fix also $\varepsilon > 0$. By definition there exists $\delta > 0$ such that

$$\langle x^*, x'-x \rangle \leq d_S(x') - d_S(x) + \varepsilon\|x'-x\|, \quad \text{for all} \quad x' \in x + \delta\mathbf{B}.$$

As $d_S(x') - d_S(x) \le 0$ for all $x' \in S(r)$, one obtains

$$\langle x^*, x' - x \rangle \le \varepsilon \|x' - x\|$$

for all $x' \in (x + \delta \mathbf{B}) \cap S(r)$, which ensures that $x^* \in \widehat{N}\big(S(r); x\big)$.

Now, we show that $\|x^*\| = 1$. Fix $\varepsilon > 0$. As $x^* \in \widehat{\partial} d_S(x)$, there exists $\delta > 0$ such that for all $x' \in x + \delta \mathbf{B}$

$$\langle x^*, x' - x \rangle \le d_S(x') - d_S(x) + \varepsilon \|x' - x\|. \tag{2.31}$$

Fix now, $\alpha := \min\{1, \varepsilon, \frac{\delta}{1 + d_S(x)}\}$ and choose x_α in S such that

$$\|x - x_\alpha\| \le d_S(x) + \alpha^2.$$

Put $x' := x + \alpha(x_\alpha - x)$. Since $\|x' - x\| \le \alpha \|x - x_\alpha\| \le \alpha d_S(x) + \alpha^2 \le \alpha(1 + d_S(x)) \le \delta$, one gets (by (2.31))

$$\begin{aligned}
\langle x^*, x' - x \rangle &\le \|x' - x_\alpha\| - \|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\| \\
&= (1 - \alpha)\|x - x_\alpha\| - \|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\| \\
&= -\alpha \|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle x^*, x_\alpha - x \rangle &\le -\|x - x_\alpha\| + \alpha + \varepsilon \|x - x_\alpha\| \\
&\le -\|x - x_\alpha\| + \varepsilon(1 + \|x - x_\alpha\|),
\end{aligned}$$

and hence

$$\frac{\langle x^*, x - x_\alpha \rangle}{\|x - x_\alpha\|} \ge 1 - \varepsilon \left(1 + \frac{1}{\|x - x_\alpha\|}\right) \ge 1 - \varepsilon \left(1 + \frac{1}{d_S(x)}\right).$$

This ensures that $\|x^*\| \ge 1$. Thus, as one always has $\widehat{\partial} d_S(x) \subset \mathbf{B}_*$, then $\|x^*\| \le 1$ and hence $\|x^*\| = 1$. This completes the proof of (2.30).

Now, we assume that X is a reflexive Banach space and that the norm $\|.\|$ of X is Kadec. Fix x^* in $\widehat{N}(S(r); x)$, with $\|x^*\| = 1$ and fix $\varepsilon > 0$. On the one hand, observe first that $x^* \in \widehat{\partial} d_{S(r)}(x)$ by Theorem 2.3. So, there exists $\delta_1 > 0$ such that for all $x' \in x + \delta_1 \mathbf{B}$

$$\langle x^*, x' - x \rangle \le d_{S(r)}(x') - d_{S(r)}(x) + \varepsilon \|x' - x\|.$$

By Lemma 3.2 one gets for any $x' \in (x + \delta_1 \mathbf{B}) \setminus S(r)$

$$\langle x^*, x' - x \rangle \le d_S(x') - d_S(x) + \varepsilon \|x' - x\|. \tag{2.32}$$

On the other hand, as $x^* \in \widehat{N}(S(r);x)$, there exists $\delta_2 > 0$ such that for all $x' \in (x+\delta_2\mathbf{B}) \cap S(r)$

$$\langle x^*, x' - x\rangle \le \frac{\varepsilon}{2}\|x' - x\|. \tag{2.33}$$

Since $\|x^*\| = 1$, we can choose $u \in X$, with $\|u\| = 1$, such that $\langle x^*, u\rangle = 1$.

Fix now $\delta_3 \in (0, \delta_2/2)$ and $x' \in (x+\delta_3\mathbf{B}) \cap S(r)$ and put $t_{x'} := d_S(x) - d_S(x') \ge 0$. Then, $x' + t_{x'}u \in S(r) \cap (x+\delta_2\mathbf{B})$ because

$$d_S(x' + t_{x'}u) \le d_S(x') + t_{x'} = d_S(x) = r$$

and

$$\|x' + t_{x'}u - x\| \le \|x' - x\| + t_{x'} \le 2\|x' - x\| \le 2\delta_3 \le \delta_2.$$

By (2.33) one gets

$$\langle x^*, x' + t_{x'}u - x\rangle \le \frac{\varepsilon}{2}\|x' + t_{x'}u - x\| \le \varepsilon\|x' - x\|,$$

and hence

$$\begin{aligned}\langle x^*, x' - x\rangle &= \langle x^*, x' + t_{x'}u - x\rangle - \langle x^*, t_{x'}u\rangle \\ &\le \varepsilon\|x' - x\| - t_{x'} \\ &\le \varepsilon\|x' - x\| + d_S(x') - d_S(x).\end{aligned} \tag{2.34}$$

According to (2.32) and (2.34), one obtains that for all $x' \in x + \delta\mathbf{B}$ with $\delta := \min\{\delta_1, \delta_3\}$ one has

$$\langle x^*, x' - x\rangle \le d_S(x') - d_S(x) + \varepsilon\|x' - x\|.$$

So $x^* \in \widehat{\partial} d_S(x)$ and hence the proof is complete. □

In order to establish the result on the Fréchet normal regularity of $S(r)$, we need to recall the following result of Borwein and Giles [31].

Theorem 2.18. *Let X be a reflexive Banach space, S be a nonempty closed subset of X and $x \notin S$. Let $\|.\|$ denote a Kadec equivalent norm on X. If $\widehat{\partial} d_S(x) \neq \emptyset$, then* $\mathrm{Proj}_S(x) \neq \emptyset$.

Now we are in position to state and prove the main result of this case.

Theorem 2.19. *Let X be a reflexive Banach space, S be a nonempty closed subset of X, and $x \notin S$. Let $\|.\|$ denote a Kadec equivalent norm on X.*

(i) If the function d_S is Fréchet subdifferentially regular at x, then $S(r)$ is Fréchet normally regular at x. Further, $\mathrm{Proj}_S(x) \neq \emptyset$ *and d_S is directional regular at x.*
(ii) Conversely, if $S(r)$ is Fréchet normally regular at x, d_S is directionally regular at x, and $\mathrm{Proj}_S(x) \neq \emptyset$, *then d_S is Fréchet subdifferentially regular at x.*

Proof. (i) Assume that d_S is Fréchet subdifferentially regular at x, i.e. $\widehat{\partial} d_S(x) = \partial^C d_S(x)$. As one always has $\partial^C d_S(x) \neq \emptyset$, one has $\mathrm{Proj}_S(x) \neq \emptyset$ by Theorem 2.18. Furthermore, d_S is directionally regular at x. Thus, by Theorem 2.16, we obtain

$$\partial^C d_S(x) = N^C\big(S(r);x\big) \cap \{x^* \in X^* : \|x^*\| = 1\}. \tag{2.35}$$

Now, fix any $x^* \in N^C\big(S(r);x\big)$ with $x^* \neq 0$. By (2.35) and Theorem 3.2, one has

$$\frac{x^*}{\|x^*\|} \in \partial^C d_S(x) = \widehat{\partial} d_S(x) \subset \widehat{N}\big(S(r);x\big) \cap \{u^* \in X^* : \|u^*\| = 1\},$$

which ensures that $x^* \in \widehat{N}\big(S(r);x\big)$ and hence $N^C\big(S(r);x\big) \subset \widehat{N}\big(S(r);x\big)$. Since the reverse inclusion always holds, the proof of (i) is complete.

(ii) Now, we assume $\widehat{N}(S(r);x) = N^C(S(r);x)$ as well as the other hypothesis in (ii) of the statement of the theorem. Thus, by Theorem 3.2 once again one has

$$\begin{aligned} \widehat{\partial} d_S(x) &= \widehat{N}(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\} \\ &= N^C(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \end{aligned}$$

Since d_S is directionally regular at x and $\mathrm{Proj}_S(x) \neq \emptyset$, one also has by Theorem 2.16

$$\partial^C d_S(x) = N^C(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Consequently, $\partial^C d_S(x) = \widehat{\partial} d_S(x)$ and hence the proof is complete. □

Remark 2.12. As X is reflexive, the Mordukhovich regularity of d_S at x could be used in the previous theorem in place of Fréchet subdifferential regularity.

2.11.2 Proximal Case

In this subsection, we will assume that X is a *reflexive Banach space*. We will also assume that the norm $\|.\|$ on X is *Kadec* and we will put $r = d_S(x)$ for some point $x \notin S$.

We have already seen that Fréchet normal regularity is not equivalent to proximal normal regularity. So, the present subsection is devoted to study similar properties of proximal normal regularity as in the previous subsection, essentially we will give conditions under which the proximal normal regularity can be characterized in terms of the distance function at points outside the set.

The following result has been established by Clarke, Stern, and Wolenski (see [89]) in the context of Hilbert spaces. Their proof is strongly based on the scalar product of the Hilbert space. Here we prove it in the general reflexive Banach space setting with a method in the line of the proof of Theorem 3.2.

Theorem 2.20. *Let S be a nonempty closed subset of X and $x \notin S$. Then,*

$$\partial^P d_S(x) = N^P(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Proof. We begin by showing the inclusion

$$\partial^P d_S(x) \subset N^P(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \tag{2.36}$$

Fix $x^* \in \partial^P d_S(x)$. Then there exist $\sigma > 0$ and $\delta > 0$ such that for all $x' \in x + \delta\mathbf{B}$

$$\langle x^*, x' - x\rangle \le d_S(x') - d_s(x) + \sigma\|x' - x\|^2.$$

As $d_S(x') - d_S(x) \le 0$ for all $x' \in S(r)$, one gets

$$\langle x^*, x' - x\rangle \le \sigma\|x' - x\|^2,$$

for all $x' \in (x + \delta\mathbf{B}) \cap S(r)$, which ensures that $x^* \in \widehat{N}(S(r);x)$. On the other hand, as in the proof of Theorem 3.2, we can check that $\|x^*\| = 1$. So, the proof of the inclusion (2.36) is complete.

Return now to the proof of the reverse inclusion

$$N^P(S(r);x) \cap \{x^* \in X^* : \|x^*\| = 1\} \subset \partial^P d_S(x).$$

Fix $x^* \in N^P(S(r);x)$ with $\|x^*\| = 1$. By Theorem 2.7, one has $x^* \in \partial^P d_{S(r)}(x)$ and hence there exist $\sigma_1 > 0$ and $\delta_1 > 0$ such that for all $x' \in x + \delta_1\mathbf{B}$

$$\langle x^*, x' - x\rangle \le d_{S(r)}(x') - d_{S(r)}(x) + \sigma_1\|x' - x\|^2.$$

By Lemma 3.2, one gets

$$\langle x^*, x' - x\rangle \le d_S(x') - d_S(x) + \sigma_1\|x' - x\|^2, \tag{2.37}$$

for any $x' \in (x + \delta\mathbf{B}) \setminus S(r)$. On the other hand, as $x^* \in N^P(S(r);x)$, there exist $\sigma_2 > 0$ and $\delta_2 > 0$ such that for all $x' \in (x + \delta_2\mathbf{B}) \cap S(r)$

$$\langle x^*, x' - x\rangle \le \sigma_2\|x' - x\|^2. \tag{2.38}$$

Since $\|x^*\| = 1$, we can choose $u \in X$, with $\|u\| = 1$, such that $\langle x^*, u\rangle = 1$.

Fix now $\delta := \min\{\delta_1, \delta_2/2\}$ and $x' \in (x + \delta\mathbf{B}) \cap S(r)$ and put $t_{x'} := d_S(x) - d_S(x') \ge 0$. Then $(x' + t_{x'}u) \in (x + \delta_2\mathbf{B}) \cap S(r)$(as in the proof of Theorem 3.2) and hence as $\|x' + t_{x'}u - x\| \le 2\|x' - x\|$ one gets by (2.38)

$$\begin{aligned}\langle x^*, x' - x\rangle &= \langle x^*, x' + t_{x'}u - x\rangle - \langle x^*, t_{x'}u\rangle \\ &\le \sigma_2\|x' + t_{x'}u - x\|^2 - t_{x'} \\ &\le 4\sigma_2\|x' - x\|^2 + d_S(x') - d_S(x).\end{aligned} \tag{2.39}$$

According to (2.37) and (2.39), one obtains that for all $x' \in x + \delta\mathbf{B}$ one has

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \sigma \|x' - x\|^2,$$

where $\sigma := max\{\sigma_1, 4\sigma_2\}$. So $x^* \in \partial^P d_S(x)$ and hence the proof is finished. □

Theorem 2.21. *Let S be a nonempty closed subset of X, $x \notin S$.*

1. *If the function d_S is proximally subdifferentially regular at x, then $S(r)$ is proximally normally regular at x. Further,* $\mathrm{Proj}_S(x) \neq \emptyset$ *and d_S is directionally regular at x.*
2. *Conversely, if $S(r)$ is proximally normally regular at x, d_S is directionally regular at x and* $\mathrm{Proj}_S(x) \neq \emptyset$, *then d_S is proximally subdifferentially regular at x.*

Proof.

1. Assume that d_S is proximally subdifferentially regular at x, i.e.,

$$\partial^P d_S(x) = \partial^C d_S(x). \tag{2.40}$$

As $\partial^C d_S(x)$ is always nonempty, then $\partial^P d_S(x) \neq \emptyset$. Thus, by Theorem 2.18, one has $\mathrm{Proj}_S(x) \neq \emptyset$. On the other hand, by (2.40) we have the directional regularity of d_S at x. Therefore, by Theorem 2.16, the following equality holds

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Thus (by (2.40) and Theorem 2.20),

$$N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} = N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Consequently,

$$N^P(S(r); x) = N^C(S(r); x),$$

which ensures that $S(r)$ is proximally normally regular at x.

2. Now, we assume that $N^P(S(r); x) = N^C(S(r); x)$. Thus, by Theorem 2.20 once again one has

$$\begin{aligned} \partial^P d_S(x) &= N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} \\ &= N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \end{aligned} \tag{2.41}$$

On the other hand, since d_S is directionally regular at x and $\mathrm{Proj}_S(x) \neq \emptyset$, one has by Theorem 2.16

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \tag{2.42}$$

Thus, by (2.41) and (2.42), we conclude that $\partial^{C} d_S(x) = \partial^{P} d_S(x)$ and hence the proof is complete. □

2.12 Commentary to Chap. 2

In this chapter, we present various concepts of regularity for sets and the possible relationships between them.

The results in Sect. 2.5 are due to Burke et al. [71]. Most results presented in Sects. 2.6–2.8 and 2.11 are taken from Bounkhel and Thibault [61]. Section 2.9 is devoted to the presentation and study of the very important concept of uniform prox-regularity in Hilbert spaces. The results stated in that section are proved in Bounkhel and Thibault [58]. Concerning this concept and its extensions to Banach spaces, we refer the reader to the following list of recent papers [22–24] and to the recent survey [96]. We have to point out, that a recent and very intersesting application of this concept to a real life phenomena is given in [186–189]. This application concerns the modeling of evacuation situations. The role of uniform prox-regularity in this application appears to be crucial. Indeed, the set of configurations in their model cannot be convex at all and it has been proved in [187–189] to be uniformly prox-regular.

In Sect. 2.10, we present a concept of regularity of sets introduced in finite dimensional settings in Borwein and Moors [33] and extended in Bounkhel [39] to Banach spaces. The results presented here are taken from [39]. The extension of this concept to set-valued mappings is introduced and studied recently in Bounkhel [41].

For interested readers on more regularity concepts for nonsmooth sets, the following list of references can be consulted [10, 22–24, 29, 36–39, 41, 44, 55, 61, 63, 71, 72, 88–91, 93, 94, 100–102, 104–106, 120, 133, 136, 137, 149, 160, 178, 190, 192, 193, 226–230, 235, 241, 258].

Chapter 3
Regularity of Functions

3.1 Introduction

It is well known that the smoothness of functions and mappings is preserved under some operations like addition, composition, etc., but contrarily to these operations there are many other operations of prime interest in analysis, like maximization and minimization that fail to preserve smoothness. So, it would be very natural to ask how the classical concept of differentiability can be enlarged in order to deal with such situations. As an answer to that question Clarke [86–88] introduced a general concept of generalized gradient (also known as Clarke subdifferential) for any extended-real-valued function defined on a finite dimensional space X. He defined first the generalized gradient of a locally Lipschitz function f at a point $\bar{x}$ with the help of Rademacher's theorem. This allowed him to consider a subset $S \subset X$ and for $\bar{x} \in S$ the normal cone $N^{\mathrm{C}}(S;\bar{x})$ to S at $\bar{x}$ as the closed cone generated by the generalized gradient at $\bar{x}$ of the distance function d_S to S. Then he defined the generalized gradient of any extended-real-valued function f at a point $\bar{x}$, where f is finite with the formula

$$\partial^{\mathrm{C}} f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N^{\mathrm{C}}(\operatorname{epi} f; \bar{x}, f(\bar{x}))\}. \qquad (\star)$$

Note that this approach also works for any normed vector space X. An important fact to record about this generalized gradient is that it enjoys general calculus rules (for instance, compositions, sums, maximum, etc.) that makes it applicable in mathematical programming, optimal control and several other mathematical fields. A prototype formula is given by

$$\partial^{\mathrm{C}}(f+g)(\bar{x}) \subset \partial^{\mathrm{C}} f(\bar{x}) + \partial^{\mathrm{C}} g(\bar{x}), \qquad (\star\star)$$

whenever f and g are locally Lipschitz.

Rockafellar in [236] showed the route to extend the definition for functions defined over any topological vector space X. He defined a generalized directional

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derivative $f^{\uparrow}(\bar{x};\cdot)$ for any extended-real-valued function f and showed that $f^{\uparrow}(\bar{x};\cdot)$ coincides with $f^0(\bar{x};\cdot)$ defined by Clarke, whenever f is locally Lipschitz. With this directional derivative he defined the generalized gradient of f at $\bar{x}$ as the set of all $x^* \in X^*$ such that $\langle x^*, h\rangle \leq f^{\uparrow}(\bar{x};h)$ for all $h \in X$ and showed that it is equal to the set given by $(\star)$ when X is a normed space. Rockafellar also extended formula $(\star\star)$ under a natural qualification condition to the case g is directionally Lipschitz. Several crucial properties of directionally Lipschitz functions are established in [145].

The equality in $(\star\star)$ (as those for composition, maximum, etc.) requires in Clarke [88] as well as in Rockafellar [236, 237] the functions to be directionally regular at $\bar{x}$ (i.e. $f^{\uparrow}(\bar{x};\cdot)$ coincides with the lower Dini directional derivative) and so makes clear the importance of this notion.

Another natural dual concept of regularity in Nonsmooth Analysis Theory is that of subdifferential regularity. This means that the generalized gradient (Clarke subdifferential) of f at $\bar{x}$ coincides with a prescribed subdifferential of f at $\bar{x}$. For example, the primal lower nice functions introduced by Poliquin [228] are regular in this sense with respect to the proximal subdifferential (we refer to [228–230] for the importance of these functions in Nonsmooth Analysis Theory). Our primary interest in this chapter is to study the relationships between these concepts of regularity for functions.

Let f be any extended-real-valued function on a real normed vector space X whose topological dual space is X^* and let $\bar{x}$ be any point where f is finite. The generalized directional derivative $f^{\uparrow}(\bar{x};\cdot)$ is defined by

$$\begin{aligned} f^{\uparrow}(\bar{x};h) &= \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t\downarrow 0}} \inf_{h'\to h} t^{-1}\left[f(x+th') - \alpha\right] \\ &:= \sup_{H\in\mathcal{N}(h)} \left[\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t\downarrow 0}} \left(\inf_{h'\in H} t^{-1}\left[f(x+th') - \alpha\right]\right)\right], \end{aligned}$$

where $(x,\alpha)\downarrow_f \bar{x}$ means $(x,\alpha)\in \operatorname{epi} f := \{(z,\beta)\in X\times\mathbf{R}; f(z)\leq\beta\}$ and $(x,\alpha)\longrightarrow (\bar{x}, f(\bar{x}))$ and $\mathcal{N}(h)$ denotes the filter of neighborhoods of h.

If f is lower semicontinuous (l.s.c.) at $\bar{x}$, the definition can be expressed in the following simpler form

$$f^{\uparrow}(\bar{x};h) = \limsup_{\substack{x\to_f \bar{x} \\ t\downarrow 0}} \inf_{h'\to h} t^{-1}\left[f(x+th') - f(x)\right],$$

where $x \to^f \bar{x}$ means $x\to\bar{x}$ and $f(x)\to f(\bar{x})$.

If f is Lipschitz around $\bar{x}$, then $f^{\uparrow}(\bar{x};h)$ coincides with the generalized directional derivative $f^0(\bar{x};\cdot)$ defined by

$$f^0(\bar{x};h) = \limsup_{\substack{x\to\bar{x} \\ t\downarrow 0}} t^{-1}\left[f(x+th) - f(x)\right].$$

Even if f is not necessarily Lipschitz around $\bar{x}$ we will put

$$f^0(\bar{x};h) := \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ t\downarrow 0}} t^{-1}\left[f(x+th)-\alpha\right].$$

The lower Dini directional derivative of f at $\bar{x}$ is defined by

$$f^-(\bar{x};h) = \liminf_{\substack{h'\to h \\ t\downarrow 0}} t^{-1}\left[f(\bar{x}+th')-f(\bar{x})\right],$$

and when f is Lipschitz around $\bar{x}$ one can check that this definition coincides with the definition of the lower Dini directional derivative given in Sect. 1.3, that is,

$$f^-(\bar{x};h) = \liminf_{t\downarrow 0} t^{-1}\left[f(\bar{x}+th)-f(\bar{x})\right],$$

that is one can take the direction fixed in the differential quotient in the definition of $f^-(\bar{x};\cdot)$.

Our study of regularity for functions will be given for a large class of non locally Lipschitz functions introduced in 1978 by Rockafellar [236]. This class of functions is large enough, it contains the indicator function of epi-Lipschitz subsets (which is never locally Lipschitz). Following Rockafellar [236], a function f is said to be directionally Lipschitz at $\bar{x}$ with respect to a vector h if

$$\limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ (t,h')\to(0^+,h)}} t^{-1}\left[f(x+th')-\alpha\right] < +\infty,$$

and this is reduced when f is l.s.c. at $\bar{x}$ to

$$\limsup_{\substack{(t,h')\to(0^+,h) \\ x\to_f \bar{x}}} t^{-1}\left[f(x+th')-f(x)\right] < +\infty.$$

If this relation holds for some h, one says that f is directionally Lipschitz at $\bar{x}$. Observe that f is Lipschitz around $\bar{x}$ if and only if it is directionally Lipschitz at $\bar{x}$ with respect to the vector zero (or equivalently with respect to every vector in X).

The generalized gradient (respectively, the Dini, the Fréchet subdifferential) of f at x (with $f(x)$ finite) is defined by

$$\partial^C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^{\uparrow}(x;h), \quad \text{for all } h\in X\},$$

$\Bigg($respectively,

$$\partial^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^-(x;h), \quad \text{for all } h\in X\},$$

$$\widehat{\partial} f(x) = \left\{x^* \in X^* : \liminf_{x'\to x} \frac{f(x')-f(x)-\langle x^*, x'-x\rangle}{\|x'-x\|} \ge 0\right\}\Bigg).$$

It is not hard to check that these definitions coincide with those given in Sect. 1.3 for locally Lipschitz functions. We recall from Sect. 1.3 that the proximal subdifferential $\partial^P f(x)$ is the set of all $x^* \in X^*$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\mathbf{B}$

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \sigma\|x' - x\|^2.$$

By convention $\partial^C f(x) = \partial^- f(x) = \widehat{\partial} f(x) = \partial^P f(x) = \emptyset$ if $f(x)$ is not finite. Note that one always has $\partial^P f(x) \subset \widehat{\partial} f(x) \subset \partial^- f(x) \subset \partial^C f(x)$.

One says that f is *directionally regular* at $\bar{x}$ with respect to a vector $h \in X$ if one has $f^{\uparrow}(\bar{x}; h) = f^-(\bar{x}; h)$. When this holds for all $h \in X$ one says that f is directionally regular at $\bar{x}$.

Now, let us recall some definitions and results that will be used in all this chapter. A Banach space X is called an Asplund space if every continuous convex function defined on a convex open subset U of X is Fréchet differentiable on a dense G_δ subset of U (see [106]). Recall (see Diestel [108]) that the dual unit ball $\mathbf{B}_*$ is weak star sequentially compact, whenever the space X is Asplund or admits an equivalent norm that is Gâteaux differentiable away from the origin.

Definition 3.1. Let $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.c.s. and $\bar{x} \in \operatorname{dom} f$ and let $\partial^\# f$ be any subdifferential of f (for instance $\partial^C f$, $\partial^- f$, $\widehat{\partial} f$, ∂f, and $\partial^P f$). We will say that $\partial^\# f$ is *topologically closed* at $\bar{x}$ if, for every net $(x_j, x_j^*)_{j \in J}$ in $\partial^\# f$ such that $x_j^* \longrightarrow^{w^*} x^*$ and $x_j \longrightarrow^f \bar{x}$ one has $(\bar{x}, x^*) \in \partial^\# f$, where $\longrightarrow^{w^*}$ denotes the w^*-convergence in X^* and $(y, y^*) \in \partial^\# f$ means that $y^* \in \partial^\# f(y)$. When the set J is replaced by $\mathbf{N}$, we say that $\partial^\# f$ is sequentially closed at $\bar{x}$.

We will also say that the function f is *Dini subdifferentially regular* (respectively, *Fréchet, proximally*) at $\bar{x}$ whenever $\partial^- f(\bar{x})$ (respectively, $\widehat{\partial} f(\bar{x})$, $\partial^P f(\bar{x})$) coincides with $\partial^C f(\bar{x})$.

We finish this section by recalling the following results (by Zagrodny [262] and Rockafellar [236]). As it appears in [262] the following Zagrodny mean value theorem holds for any subdifferential concept although the proof in [262] was given for the generalized gradient (Clarke subdifferential).

Theorem 3.1. *Let X be a Banach space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. on X and $\partial^\# f$ be any subdifferential of f. Let $a, b \in \operatorname{dom} f$ (with $a \neq b$). Then there exist $x_n \longrightarrow^f c \in [a, b) := \{rb + (1-r)a : r \in [0,1)\}$ and $x_n^* \in \partial^\# f(x_n)$ such that*

$$f(b) - f(a) \leq \lim_n \langle x_n^*, b - a\rangle$$

and

$$\frac{\|b - c\|}{\|b - a\|}\big[f(b) - f(a)\big] \leq \lim_n \langle x_n^*, b - x_n\rangle.$$

Proposition 3.1. *Let f be any extended-real-valued function on X and $\bar{x}$ be any point where f is finite. If f is directionally Lipschitz at $\bar{x}$, then*

(i) for every $h \in \text{int dom } f^{\uparrow}(\bar{x};\cdot)$

$$f^{\uparrow}(\bar{x};h) = f^{0}(\bar{x};h) = \limsup_{\substack{(x,\alpha)\downarrow_f \bar{x} \\ (t,h')\to(0^+,h)}} t^{-1}\left[f(x+th') - \alpha\right];$$

(ii) $\text{int dom } f^{\uparrow}(\bar{x},.)$ *is the set of all vectors h with respect to which f is directionally Lipschitz and $f^{\uparrow}(\bar{x},.)$ is continuous over* $\text{int dom} f^{\uparrow}(\bar{x};\cdot)$*;*
(iii) $f^{\uparrow}(\bar{x};h) = \liminf_{h'\to h} f^{0}(\bar{x};h')$ *for every $h \in X$.*

3.2 Directional Regularity of Functions

In this section, we are going to prove (under some general assumptions) that the directional regularity is equivalent to the closedness of the Dini subdifferential. We begin by showing that the directional regularity of f at $\bar{x}$ coincides with the Dini subdifferential regularity of f at $\bar{x}$ whenever the generalized gradient of f at $\bar{x}$ is nonempty.

Proposition 3.2. *Let X be a real normed vector space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \text{dom} f$ with $\partial^{C} f(\bar{x}) \neq \emptyset$. Then the following assertions are equivalent*

(i) f is directionally regular at $\bar{x}$;
(ii) f is Dini subdifferentially regular at $\bar{x}$.

Proof. The implication $(i) \Longrightarrow (ii)$ is obvious. So, we will prove the reverse implication $(ii) \Longrightarrow (i)$. Fix any $h \in X$. Suppose that $\partial^{-} f(\bar{x}) = \partial^{C} f(\bar{x})$. As the inequality $f^{-}(\bar{x};h) \leq f^{\uparrow}(\bar{x};h)$ always holds, one has

$$\begin{aligned} f^{\uparrow}(\bar{x};h) &= \sup\{\langle x^*, h\rangle : x^* \in \partial^{C} f(\bar{x})\} \\ &= \sup\{\langle x^*, h\rangle : x^* \in \partial^{-} f(\bar{x})\} \\ &\leq f^{-}(\bar{x};h) \\ &\leq f^{\uparrow}(\bar{x};h). \end{aligned}$$

This ensures that f is directionally regular at $\bar{x}$ and proves the reverse implication. □

As a direct application of this proposition one obtains that the primal lower nice functions introduced by Poliquin [228] are directionally regular at all points of the domains of the subdifferentials. Indeed it is shown in [228] that all subdifferentials coincide for these functions.

Consider now the following lemma which has its own interest. It will allow us to prove the main propositions of this section.

Lemma 3.1. *Let X be a real normed vector space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \mathrm{dom} f$. Suppose that f is directionally Lipschitz at $\bar{x}$. Then, for all $h \in$ $\mathrm{int\,dom} f^{\uparrow}(\bar{x};\cdot)$ one has*

$$\limsup_{x \to_f \bar{x}} f^{\uparrow}(x;h) \leq f^{\uparrow}(\bar{x};h).$$

Proof. Fix any $h \in \mathrm{int\,dom} f^{\uparrow}(\bar{x};\cdot)$. As f is directionally Lipschitz at $\bar{x}$, one has (by (i) in Proposition 3.1)

$$f^{\uparrow}(\bar{x};h) = \limsup_{\substack{x \to_f \bar{x} \\ t \downarrow 0}} t^{-1}\big[f(x+th) - f(x)\big],$$

and there exists $\delta > 0$ such that f is directionally Lipschitz at each $x \in U(f,\bar{x},\delta) :=$ $\{x \in X : x \in \bar{x} + \delta \mathbf{B}$ and $|f(x) - f(\bar{x})| \leq \delta\}$. Let $\gamma > f^{\uparrow}(\bar{x};h)$. By definition of upper limit, there exists $0 < \delta' < \delta$ such that

$$t^{-1}\big[f(x+th) - f(x)\big] < \gamma,$$

for all $x \in U(f,\bar{x},\delta')$ and all $t \in (0,\delta')$. Fix any $x_0 \in U(f,\bar{x},\delta'/2)$. Then we have

$$t^{-1}\big[f(x+th) - f(x)\big] < \gamma,$$

for all $x \in U(f,x_0,\delta'/2)$ and all $t \in (0,\delta')$, and hence as f is directionally Lipschitz at each point $x \in U(f,\bar{x},\delta')$ one has

$$f^{\uparrow}(x_0;h) = \limsup_{\substack{x \to_f x_0 \\ t \downarrow 0}} t^{-1}\big[f(x+th) - f(x)\big] \leq \gamma.$$

Thus, since this inequality holds for all $x_0 \in U(f,\bar{x},\delta'/2)$ and all $\gamma > f^{\uparrow}(\bar{x};h)$, then taking upper limits as $x_0 \longrightarrow^{f} \bar{x}$ gives

$$\limsup_{x_0 \to_f \bar{x}} f^{\uparrow}(x_0;h) \leq f^{\uparrow}(\bar{x};h),$$

which completes the proof. □

Proposition 3.3. *Let X be a real normed vector space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \mathrm{dom} f$. If f is directionally Lipschitz at $\bar{x}$, then $\partial^{\mathrm{C}} f$ is topologically closed at $\bar{x}$.*

Proof. This follows immediately from Lemma 3.1 and the equality

$$\partial^{\mathrm{C}} f(\bar{x}) = \{x^* : \langle x^*, h\rangle \leq f^{\uparrow}(\bar{x};h) \quad \text{for all} \quad h \in \mathrm{int\,dom} f^{\uparrow}(\bar{x};\cdot)\ \},$$

because f is directionally Lipschitz at $\bar{x}$. □

Proposition 3.4. *Let X be a Banach space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. and $\bar{x} \in \mathrm{dom} f$. Suppose that f is directionally Lipschitz at $\bar{x}$. Then the following assertions are equivalent*

(i) *f is directionally regular at $\bar{x}$ with respect to any vector $h \in \mathrm{int}\ \mathrm{dom} f^{\uparrow}(\bar{x};\cdot)$;*
(ii) $\limsup_{x \to^f \bar{x}} f^{-}(x;h) \leq f^{-}(\bar{x};h)$, *for all* $h \in \mathrm{int}\ \mathrm{dom} f^{\uparrow}(\bar{x};\cdot)$.

Proof. The implication $(i) \Longrightarrow (ii)$ is immediate from Lemma 3.1 and the inequality $f^{-}(x;h) \leq f^{\uparrow}(x;h)$. Let us prove the reverse implication. Assume (ii) and fix any $\bar{h} \in \mathrm{int}\ \mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot)$. As we always have (see[136])

$$f^{\uparrow}(\bar{x};\bar{h}) \leq \sup_{\varepsilon>0} \left(\limsup_{x \to^f \bar{x}} \left(\inf_{h \in \bar{h}+\varepsilon \mathbf{B}} f^{-}(x;h) \right) \right),$$

then

$$\begin{aligned} f^{\uparrow}(\bar{x};\bar{h}) &\leq \limsup_{x \to^f \bar{x}} f^{-}(x;\bar{h}) \\ &\leq f^{-}(\bar{x};\bar{h}) (\text{by} \quad (ii)) \\ &\leq f^{\uparrow}(\bar{x};\bar{h}). \end{aligned}$$

This ensures the directional regularity of f at $\bar{x}$ with respect to $\bar{h}$ and hence the proof is complete. □

Throughout the following two lemmas we assume that $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous on X, $\bar{x} \in \mathrm{dom} f$ and $\partial^{\#} f$ is a subdifferential of f in an appropriate Banach space X and that satisfies $\partial^{\#} f \subset \partial^{C} f$ (for instance, we take $\widehat{\partial} f$ (respectively, $\partial^{-} f$, $\partial^{P} f$) whenever X is an Asplund space (respectively, a Banach space with a Gâteaux differentiable (away from the origin) renorm, a Hilbert space)). The following two lemmas will be also needed in the next section to establish characterizations of Fréchet and proximal subdifferential regularity.

Lemma 3.2. If f is directionally Lipschitz at $\bar{x}$ with respect to $\bar{h} \in X$, then there exists $\beta \in \mathbf{R}$ such that for every sequence $(x_n, x_n^*)_{n \in \mathbf{N}}$ in $\partial^{\#} f$, with $x_n \longrightarrow^f \bar{x}$, there are $n_0 \in \mathbf{N}$ and a neighborhood H of $\bar{h}$ such that for all $n \geq n_0$ one has

$$\langle x_n^*, h \rangle \leq \beta \qquad \text{for all} \quad h \in H.$$

Proof. As f is directionally Lipschitz at $\bar{x}$ with respect to $\bar{h}$, there exist $\beta \in \mathbf{R}$, $\delta > 0$ such that

$$t^{-1}\left[f(x+th) - f(x)\right] \leq \beta, \text{for all } t \in (0,\delta), h \in \bar{h} + \delta \mathbf{B}, x \in U(f,\bar{x},\delta). \tag{3.1}$$

Let $(x_n, x_n^*)_{n \in \mathbf{N}}$ in $\partial^{\#} f$, with $x_n \longrightarrow^f \bar{x}$. Then there exist $n_0 \in \mathbf{N}$ such that $x_n \in U(f,\bar{x},\delta)$, for all $n \geq n_0$. Fix any $h_0 \in \bar{h} + \frac{\delta}{2}\mathbf{B}$ and any $n \geq n_0$. Then by (3.1)

$$t^{-1}\left[f(x+th) - f(x)\right] \leq \beta, \text{for all } t \in (0,\delta), h \in h_0 + \frac{\delta}{2}\mathbf{B}, x \in U\left(f, x_n, \frac{\delta}{2}\right),$$

which ensures that

$$f^{\uparrow}(x_n, h_0) \le \beta, \quad \text{for all } n \ge n_0 \text{ and all } \quad h_0 \in \bar{h} + \frac{\delta}{2}\mathbf{B}.$$

Thus, as $x_n^* \in \partial^{\#} f(x_n) \subset \partial^{\mathrm{C}} f(x_n)$, one has

$$\langle x_n^*, h \rangle \le f^{\uparrow}(x_n, h) \le \beta, \quad \text{for all } n \ge n_0 \text{ and all } \quad h \in \bar{h} + \frac{\delta}{2}\mathbf{B},$$

which completes the proof. □

Lemma 3.3. *If f is directionally Lipschitz at $\bar{x}$ and $\partial^{\mathrm{C}} f(\bar{x}) \neq \emptyset$, then for each $h \in \mathit{int}\ \mathrm{dom} f^{\uparrow}(\bar{x}, .)$ there exist a sequence $u_n \longrightarrow^f \bar{x}$ and a bounded sequence $(u_n^*)_{n\in\mathbf{N}}$ in X^* such that*

(i) $u_n^* \in \partial^{\#} f(u_n)$ *for all* $n \in \mathbf{N}$;
(ii) $f^{\uparrow}(\bar{x}, h) \le \limsup_n \langle u_n^*, h \rangle$.

Proof. Let $\bar{h} \in \mathrm{int\ dom} f^{\uparrow}(\bar{x}, .)$. As $\partial^{\mathrm{C}} f(\bar{x}) \neq \emptyset$, one has

$$f^{\uparrow}(\bar{x}; \bar{h}) := \sup\{\langle x^*, \bar{h} \rangle; x^* \in \partial^{\mathrm{C}} f(\bar{x})\} > -\infty, \tag{3.2}$$

and by (i) in Proposition 3.1 one also has

$$f^{\uparrow}(\bar{x}; \bar{h}) = \limsup_{\substack{x \to_f \bar{x} \\ t \downarrow 0}} t^{-1} \left[f(x + t\bar{h}) - f(x) \right].$$

Let us consider sequences $x_n \longrightarrow^f \bar{x}$ and $t_n \longrightarrow 0^+$ such that

$$f^{\uparrow}(\bar{x}; \bar{h}) = \lim_n t_n^{-1} \left[f(x_n + t_n\bar{h}) - f(x_n) \right]. \tag{3.3}$$

For each $n \in \mathbf{N}$, we put $a_n := x_n$ and $b_n := x_n + t_n\bar{h}$. Note that, for n large enough, $f(x_n)$ is finite and that $f(b_n)$ is also finite because f is directionally Lipschitz at $\bar{x}$ with respect to $\bar{h}$. Then by Theorem 3.1, there exist a sequence $c_{k,n} \longrightarrow c_n \in [a_n, b_n)$ and a sequence $(x_{k,n}^*)_{k\in\mathbf{N}}$ in X^* that $x_{k,n}^* \in \partial^{\#} f(c_{k,n})$ for all $k \in \mathbf{N}$ and

$$t_n^{-1} \left[f(x_n + t_n\bar{h}) - f(x_n) \right] \le \lim_{k \to +\infty} \langle x_{k,n}^*, \bar{h} \rangle. \tag{3.4}$$

Thus, for each $n \in \mathbf{N}$, there exists $s(n) \in \mathbf{N}$ such that

$$\lim_k \langle x_{k,n}^*, \bar{h} \rangle \le \langle x_{s(n),n}^*, \bar{h} \rangle + \frac{1}{n+1}, \tag{3.5}$$

and

$$\|c_{s(n),n} - c_n\| \leq \frac{1}{n+1}.$$

Put $u_n^* := x_{s(n),n}^*$ and $u_n := c_{s(n),n}$. Then, $u_n^* \in \partial^\# f(u_n)$ for each $n \in \mathbf{N}$, which ensures (i).

As f is directionally Lipschitz at $\bar{x}$ with respect to $\bar{h}$ (see (ii) in Proposition 3.1), then by Lemma 3.2 above, there exist $\beta \in \mathbf{R}$ and $n_0 \in \mathbf{N}$ such that for all h around $\bar{h}$ and all $n \geq n_0$

$$\langle u_n^*, h\rangle \leq \beta.$$

By (3.2) and (3.3), there exist $n_1 \in \mathbf{N}$ and $\sigma \in \mathbf{R}$ such that for all $n \geq n_1$

$$\sigma \leq t_n^{-1}\left[f(u_n + t_n\bar{h}) - f(u_n)\right] \leq \langle u_n^*, \bar{h}\rangle + \frac{1}{n+1}, \quad \text{(by (3.4) and (3.5))}$$

and hence $\sigma - 1/(n_1+1) \leq \langle u_n^*, \bar{h}\rangle$ for all $n \geq n_1$. Put $N := \max\{n_0, n_1\}$ and $\sigma_1 := \sigma - \frac{1}{n_1+1}$. Choose $\delta > 0$ such that $\langle u_n^*, h\rangle \leq \beta$ for all $h \in \bar{h} + \delta\mathbf{B}$ and all $n \geq N$. Then, for all $n \geq N$ and all $b \in \mathbf{B}$, one has

$$\begin{aligned}\langle u_n^*, b\rangle &\leq \frac{1}{\delta}\left[\langle u_n^*, \bar{h} + \delta b\rangle - \langle u_n^*, \bar{h}\rangle\right]\\ &\leq \frac{1}{\delta}\left[\beta - \sigma_1\right].\end{aligned}$$

So for all $n \geq N$,

$$\|u_n^*\| \leq M := \frac{\beta - \sigma_1}{\delta},$$

which ensures that the sequence $(u_n^*)_{n\in\mathbf{N}}$ is bounded. Furthermore, (ii) is ensured by (3.3), (3.4) and (3.5), and hence the proof is finished. □

Now, we are ready to prove our main characterization of directional regularity for directionally Lipschitz functions.

Theorem 3.2. *Assume that X is a Banach space admitting an equivalent norm that is Gâteaux differentiable away from the origin. Let $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. on X and directionally Lipschitz at $\bar{x} \in \operatorname{dom} f$ with $\partial^C f(\bar{x}) \neq \emptyset$. Then the following assertions are equivalent*

(i) f is directionally regular at $\bar{x}$;
(ii) $\partial^- f$ is topologically closed at $\bar{x}$ and $\operatorname{dom} f^\uparrow(\bar{x};\cdot) = \operatorname{dom} f^-(\bar{x};\cdot)$;
(iii) $\partial^- f$ is sequentially closed at $\bar{x}$ and $\operatorname{dom} f^\uparrow(\bar{x};\cdot) = \operatorname{dom} f^-(\bar{x};\cdot)$.

Proof. Let us prove first the implication "$(i) \Longrightarrow (ii)$". The assumption (i) evidently implies

$$\mathrm{dom}\, f^{\uparrow}(\bar{x};\cdot) = \mathrm{dom}\, f^{-}(\bar{x};\cdot)$$

and $\partial^{-} f(\bar{x}) = \partial^{C} f(\bar{x})$. So Proposition 3.3 ensures that $\partial^{-} f$ is topologically closed at $\bar{x}$ and this finishes the proof of the first implication. Since the implication "$(ii) \Longrightarrow (iii)$" is obvious, it remains to show the third one "$(iii) \Longrightarrow (i)$". Fix any vector $\bar{h} \in \mathrm{int\,dom}\, f^{\uparrow}(\bar{x};\cdot)$. We apply Lemma 3.3 with the Dini subdifferential $\partial^{-} f$ and we obtain a sequence $x_n \longrightarrow^{f} \bar{x}$ and a bounded sequence $(x_n^*)_{n\in\mathbf{N}}$ in X^* such that

(a) $x_n^* \in \partial^{-} f(x_n)$ for all $n \in \mathbf{N}$;
(b) $f^{\uparrow}(\bar{x};\bar{h}) \le \limsup_{n\to+\infty} \langle x_n^*, \bar{h}\rangle$.

As the space X admits an equivalent norm that is Gâteaux differentiable away from the origin, the closed united ball of X^* is weak star sequentially compact and hence we may suppose that the sequence $(x_n^*)_{n\in\mathbf{N}}$ converges with respect to the weak star topology to some $x^* \in X^*$. Therefore, by the sequential closedness of $\partial^{-} f$ we have $x^* \in \partial^{-} f(\bar{x})$, and hence by (b) and Proposition 3.1

$$f^{0}(\bar{x};\bar{h}) = f^{\uparrow}(\bar{x};\bar{h}) \le \langle x^*, \bar{h}\rangle. \tag{3.6}$$

Note that the analysis above ensures that $\partial^{-} f(\bar{x}) \ne \emptyset$. Fix now $\bar{h} \in \mathrm{dom}\, f^{\uparrow}(\bar{x};\cdot)$, consider the lower semicontinuous convex function φ defined by

$$\varphi(h) := \sup\left\{\langle x^*, h\rangle : x^* \in \partial^{-} f(\bar{x})\right\},$$

and note that

$$\varphi(h) \le f^{-}(\bar{x};h) \le f^{\uparrow}(\bar{x};h) \quad \text{for all } h \in X. \tag{3.7}$$

Fix $v \in \mathrm{int\,dom}\, f^{\uparrow}(\bar{x};\cdot)$ and put $h_t := \bar{h} + t(v - \bar{h})$ for $t \in [0,1]$. By (3.7) we have $h_t \in \mathrm{dom}\,\varphi$ for all $t \in [0,1]$ and hence the function $t \mapsto \varphi(h_t)$ is continuous on $[0,1]$ (see Theorem 10.2 in [240]). Moreover for each $t \in \,]0,1]$ we have $h_t \in \mathrm{int\,dom}\, f^{\uparrow}(\bar{x};\cdot)$ and (3.6) ensures that there exists $x_t^* \in \partial^{-} f(\bar{x})$ with

$$f^{0}(\bar{x};h_t) = f^{\uparrow}(\bar{x};h_t) \le \langle x_t^*, h_t\rangle \le \varphi(h_t).$$

Therefore, by Propositions 3.1 and 3.4

$$\begin{aligned} f^{\uparrow}(\bar{x};\bar{h}) &= \liminf_{h\to\bar{h}} f^{0}(\bar{x};h) \\ &\le \liminf_{t\to 0^+} f^{0}(\bar{x};h_t) \\ &\le \lim_{t\to 0^+} \varphi(h_t) \\ &= \varphi(\bar{h}) \\ &\le f^{-}(\bar{x};\bar{h}). \end{aligned}$$

So $f^{\uparrow}(\bar{x};\bar{h}) = f^{-}(\bar{x};\bar{h})$ for all $\bar{h} \in \operatorname{dom} f^{\uparrow}(\bar{x};\cdot)$. As $\operatorname{dom} f^{\uparrow}(\bar{x};\cdot) = \operatorname{dom} f^{-}(\bar{x};\cdot)$ by assumption, we get $f^{\uparrow}(\bar{x};\bar{h}) = f^{-}(\bar{x};\bar{h})$ for all $\bar{h} \in X$ and the proof is complete. □

We close this section with the following result, which gives the equivalence between the various notions of Dini regularity given above and also the closedness of the Dini subdifferential, when the function is assumed be locally Lipschitz. Its proof follows directly from Propositions 3.2 and 3.4 and Theorem 3.2.

Corollary 3.1. *Assume that X is a Banach space admitting an equivalent norm that is Gâteaux differentiable away from the origin. Let $f : X \longrightarrow \mathbf{R}$ be a locally Lipschitz function around $\bar{x} \in \operatorname{dom} f$. Then the following assertions are equivalent*

(i) $f^{-}(\cdot;h)$ *is upper semicontinuous at $\bar{x}$ for all $h \in X$;*
(ii) f *is directionally regular at $\bar{x}$;*
(iii) f *is Dini subdifferentially regular at $\bar{x}$;*
(iv) $\partial^{-} f$ *is topologically closed at $\bar{x}$;*
(v) $\partial^{-} f$ *is sequentially closed at $\bar{x}$.*

3.3 Fréchet and Proximal Subdifferential Regularity of Functions

As in the previous section, we use the general results established in Lemmas 3.2 and 3.3, to prove the equivalence between the topological closedness of the Fréchet subdifferential (respectively, proximal subdifferential) and the Fréchet subdifferential regularity (respectively, proximal subdifferential regularity) of f whenever it is directionally Lipschitz.

Theorem 3.3. *Let X be an Asplund space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. on X with $\bar{x} \in \operatorname{dom} f$ and let $s(\bar{x};\cdot)$ be the support function of $\widehat{\partial} f(\bar{x})$, that is*

$$s(\bar{x};h) := \sup\{\langle y^*, h\rangle ; y^* \in \widehat{\partial} f(\bar{x})\}.$$

Suppose that f is directionally Lipschitz at $\bar{x}$. Then f is Fréchet subdifferentially regular at $\bar{x}$ if and only if $\widehat{\partial} f$ is topologically closed at $\bar{x}$ and $\operatorname{dom} f^{\uparrow}(\bar{x};\cdot) = \operatorname{dom} s(\bar{x};\cdot)$.

Proof. The implication "$\Longrightarrow$" follows from Proposition 3.3. So we will prove the reverse implication "$\Longleftarrow$". Fix any $\bar{h} \in \operatorname{int} \operatorname{dom} f^{\uparrow}(\bar{x};\cdot)$. As in the proof of Theorem 3.2 we apply Lemma 3.3 to obtain an element $x^* \in \widehat{\partial} f(\bar{x})$ that satisfies $f^{\uparrow}(\bar{x};\bar{h}) \le \langle x^*, \bar{h}\rangle$, and hence $\widehat{\partial} f(\bar{x}) \neq \emptyset$. Thus, for each $\bar{h} \in \operatorname{int} \operatorname{dom} f^{\uparrow}(\bar{x};\cdot)$ there exists $x^* \in \widehat{\partial} f(\bar{x})$ such that

$$f^{\uparrow}(\bar{x};\bar{h}) \le \langle x^*, \bar{h}\rangle \le s(\bar{x};\bar{h}) = \sup\left\{\langle y^*, \bar{h}\rangle ; y^* \in \widehat{\partial} f(\bar{x})\right\}. \tag{3.8}$$

Put $D := \mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot)$. It is obvious that the function $s(\bar{x};\cdot)$ is convex and lower semicontinuous. Fix $v \in \mathrm{int}\ D$ and $h \in D$ and put $h_t := h + t(v-h)$ for each $t \in [0,1]$. Observe that $h_t \in \mathrm{int}\, D$ for each $t \in (0,1]$ and that the function $t \mapsto s(\bar{x};h_t)$ is continuous on $[0,1]$ (see Theorem 10. in [240]). So, we have (see (iii) in Proposition 3.1)

$$\begin{aligned} f^{\uparrow}(\bar{x};h) &= \liminf_{h' \to h} f^0(\bar{x};h') \\ &\le \liminf_{t \to 0^+} f^0(\bar{x};h_t) \\ &\le \lim_{t \to 0^+} s(\bar{x};h_t)(\text{by}(8)) \\ &= s(\bar{x};h). \end{aligned}$$

Therefore, $f^{\uparrow}(\bar{x};h) \le s(\bar{x};h)$ for all $h \in \mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot)$. As $\mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot) = \mathrm{dom}\ s(\bar{x};\cdot)$, the inequality above holds for each $h \in X$, which ensures that the support function of $\partial^{C} f(\bar{x})$ is not greater than the support function of $\widehat{\partial} f(\bar{x})$. Since $\widehat{\partial} f(\bar{x})$ is weak-star closed (because of the topological closedness of $\widehat{\partial} f$ at $\bar{x}$) we get that $\partial^{C} f(\bar{x}) \subset \widehat{\partial} f(\bar{x})$ and hence $\partial^{C} f(\bar{x}) = \widehat{\partial} f(\bar{x})$ (the reverse inclusion being always true). So the proof is complete. □

In the following theorem, we prove that if X is assumed to be reflexive, the Fréchet subdifferential regularity of f at $\bar{x}$ is also equivalent to the sequential closedness of the Fréchet subdifferential. This equivalence is an open problem in the case of nonreflexive Banach spaces.

Theorem 3.4. *Let X be a reflexive Banach space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. on X with $\bar{x} \in \mathrm{dom}\ f$ and let $s(\bar{x};\cdot)$ be the support function of $\widehat{\partial} f(\bar{x})$, that is*

$$s(\bar{x};h) := \sup \left\{ \langle y^*, h \rangle ; y^* \in \widehat{\partial} f(\bar{x}) \right\}.$$

Suppose that f is directionally Lipschitz at $\bar{x}$. Then the following assertions are equivalent:

(i) f is Fréchet subdifferentially regular at $\bar{x}$;
(ii) $\widehat{\partial} f$ is topologically closed at $\bar{x}$ and $\mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot) = dom\ s(\bar{x};\cdot)$;
(iii) $\widehat{\partial} f$ is sequentially closed at $\bar{x}$ and $\mathrm{dom}\ f^{\uparrow}(\bar{x};\cdot) = dom\ s(\bar{x};\cdot)$.

Proof. We have only to prove the implications: "(ii) $\Longrightarrow$ (iii)" and "(ii) $\Longrightarrow$ (i)". As the implication "(ii) $\Longrightarrow$ (iii)" is obvious. So we will prove the implication "(iii)$\Longrightarrow$ (i)". To do that we follow the same idea used in the proof of Theorem 3.2 and we obtain

$$f^{\uparrow}(\bar{x};h) \le s(\bar{x};h) \quad \text{for all} \quad h \in X.$$

This ensures that the support function of $\partial^C f(\bar{x})$ is not greater than the support function of $\widehat{\partial} f(\bar{x})$. Since $\widehat{\partial} f(\bar{x})$ is weak-star closed (because of X is reflexive and the Fréchet subdifferential is convex strongly closed set in X^*) we get that $\partial^C f(\bar{x}) \subset \widehat{\partial} f(\bar{x})$ and hence $\partial^C f(\bar{x}) = \widehat{\partial} f(\bar{x})$ (the reverse inclusion being always true). So the proof of the implication " (iii)$\Longrightarrow$ (i)" is complete. □

We conclude the chapter with the theorem below, which concerns the proximal subdifferential regularity when X is assumed to be a Hilbert space. We omit the proof since it follows the arguments in Theorem 3.3.

Theorem 3.5. *Let X be a Hilbert space, $f : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be l.s.c. on X with $\bar{x} \in \mathrm{dom}\, f$ and let $s(\bar{x};\cdot)$ be the support function of $\partial^P f(\bar{x})$. Suppose that f is directionally Lipschitz at $\bar{x}$. Then f is proximally subdifferentially regular at $\bar{x}$ if and only if $\partial^P f$ is topologically closed at $\bar{x}$ and* $\mathrm{dom}\, f^{\uparrow}(\bar{x};\cdot) = dom\, s(\bar{x};\cdot)$.

Remark 3.1. Note that in contrast to Theorem 3.4 we cannot add in Theorem 3.5 the sequential closedness of the proximal subdifferential as a characterization of the proximal subdifferential regularity because the proximal subdifferential is not closed in general even in finite dimensional spaces.

3.4 Commentary to Chap. 3

In this chapter, we present various concepts of regularity of functions and we study the possible relationships between them. Most results are proved in Bounkhel and Thibault [60]. Other concepts of regularity for functions are introduced and studied by different authors. For more studies on the regularity of functions, we refer the reader to the following list of references: [22–25, 30–34, 36, 38, 39, 44, 46, 48, 55, 60, 61, 71, 72, 88, 89, 91, 94, 97, 98, 102, 104, 105, 121, 133, 134, 136, 137, 145, 147, 148, 183, 190, 192, 193, 226, 228–230, 236, 237, 239, 241, 242, 244, 251, 254–256, 258].

Chapter 4
Regularity of Set-Valued Mappings

4.1 Introduction

Let $M : E \rightrightarrows F$ be a set-valued mapping defined from a Hausdorff topological vector space E into a normed space F and let $(\bar{x}, \bar{y})$ be a point in $E \times F$. There are many ways to define the regularity for set-valued mappings. For instance, we can define the regularity of M at $(\bar{x}, \bar{y})$ in gph$M$ the graph of M in a geometrical point of view (via tangent and normal cones) or as the regularity of some scalar function associated with M.

The scalarization of any geometrical property of a set-valued mapping M requires the appropriate choice of the scalar function associated with M. As we have seen in the chapter reserved for sets, the appropriate scalar function associated with a subset S in a normed vector space was the distance function d_S. In our case here, the distance function to the graph of M has no sense because E is assumed to be a Hausdorff topological vector space. So, the distance function to the graph of M cannot be used for such study and the important results established in Chap. 1 cannot be applied. Another natural scalar function that can be associated with the set-valued mapping M is the distance function to images $\Delta_M : E \times F \to \mathbf{R} \cup \{+\infty\}$ defined by $\Delta_M(x,y) = d(y, M(x))$ for all $(x,y) \in E \times F$ (the distance between y and $M(x)$). This function is well defined because F is assumed to be a normed vector space. Moreover, it is generally more convenient to handle with Δ_M. For example, when M is a usual mapping f, one has $\Delta_f(x,y) = \|y - f(x)\|$.

The function Δ_M has been successfully used in optimization theory by a number of authors including Clarke [86–88], Castaing and Valadier [79], Rockafellar [238], Thibault [247], Ioffe [139], Bounkhel [44, 45] and Bounkhel and Thibault [59, 62, 63]. The aim of this chapter is to show how the function Δ_M recalled above allows to characterize the tangential regularity of the set-valued mapping M for general topological vector spaces E. To do that we need to prove in the next section some important results on the scalar function Δ_M.

M. Bounkhel, *Regularity Concepts in Nonsmooth Analysis: Theory and Applications*, Springer Optimization and Its Applications 59, DOI 10.1007/978-1-4614-1019-5_4,

4.2 On the Distance Function to Images Δ_M Around Points on the Graph

Let M be a set-valued mapping from a Hausdorff real topological vector space E into a real normed vector space F. The graph gph M (resp. the effective domain dom M) of $M : E \rightrightarrows F$ is the set gph $M := \{(x,y) \in E \times F : y \in M(x)\}$ (resp. dom $M := \{x \in E : M(x) \neq \emptyset\}$). The set-valued mapping M is said to be closed if its graph is closed. Consider the function Δ_M defined on $E \times F$ by

$$\Delta_M(x,y) = d(y, M(x)),$$

where the right term is $+\infty$ whenever $M(x) = \emptyset$, i.e., $x \notin \text{dom}\, M$.

If E is a normed space, it is easily seen that for all $(x,y) \in E \times F$

$$d_{\text{gph}\, M}(x,y) \leq \Delta_M(x,y) \leq \psi_M(x,y),$$

where $d_{\text{gph}\, M}(x,y) := d((x,y), \text{gph}\, M)$ and ψ_M denotes the indicator function of gph M i.e. $\psi_M(x,y) = 0$ if $(x,y) \in \text{gph}\, M$ and $+\infty$ otherwise.

Throughout this section E will be a Hausdorff *topological vector space*.

In [247], Thibault established a useful characterization of the generalized directional derivative of the function Δ_M at $(\bar{x},\bar{y})$ when E is a normed vector space. His characterization provides that we can replace $((x,y),\alpha) \downarrow_{\Delta_M} (\bar{x},\bar{y})$ in the definition of $\Delta_M^{\uparrow}(\bar{x},\bar{y};\cdot,\cdot)$ by $(x,y) \longrightarrow^M (\bar{x},\bar{y})$, i.e., we may take into account only the pairs (x,y) in the graph of M. In the following theorem, we show that the same characterization holds in the case when E is a topological vector space.

Theorem 4.1. *Let M be a set-valued mapping from E into F with $(\bar{x},\bar{y}) \in$ gph M. Then for all $(\bar{h},\bar{k}) \in E \times F$*

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) = \limsup_{\substack{(x,y)\to^M(\bar{x},\bar{y}) \\ t\downarrow 0}} \inf_{h\to\bar{h}} t^{-1}\Big[\Delta_M(x+th, y+t\bar{k})\Big]$$

and

$$\Delta_M^{0}(\bar{x},\bar{y};\bar{h},\bar{k}) = \limsup_{\substack{(x,y)\to^M(\bar{x},\bar{y}) \\ t\downarrow 0}} t^{-1}\Big[\Delta_M(x+t\bar{h}, y+t\bar{k})\Big].$$

Proof. Fix any $(\bar{h},\bar{k}) \in E \times F$. We only show the first equality, because the second one is obtained with $H = \{\bar{h}\}$ in the proof below. We denote by β the right-hand side of this equality and we begin by proving the following inequality

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) \leq \beta. \tag{4.1}$$

Obviously, we may suppose $\beta < +\infty$. Consider $H \times K \in \mathcal{N}(\bar{h},\bar{k})$ and $\varepsilon > 0$. By the definition of β, there exist $\delta \in (0,\varepsilon)$ and $X \times Y' \in \mathcal{N}(\bar{x},\bar{y})$ such that for all $t \in (0,\delta)$ and for all $(x,y) \in (X \times Y') \cap \operatorname{gph} M$ there is $h \in H$ with

$$t^{-1}\Delta_M(x+th, y+t\bar{k}) < \beta + \varepsilon. \tag{4.2}$$

Choose $\varepsilon' \in (0, \min(\varepsilon,\delta,1))$ and $Y \in \mathcal{N}(\bar{y})$ such that $Y + 2\varepsilon'\mathbf{B}_F \subset Y'$, and fix any $t \in (0,\varepsilon')$ and any $(x,y,\alpha) \in \big(X \times Y \times (-\varepsilon',\varepsilon')\big) \cap \operatorname{epi} \Delta_M$. As $\alpha \geq \Delta_M(x,y) = d(y, M(x))$ we have $M(x) \neq \emptyset$ and hence we can choose $y' \in M(x)$ satisfying

$$\|y - y'\| < t^2 + \Delta_M(x,y) < \varepsilon'^2 + \alpha < 2\varepsilon', \tag{4.3}$$

and hence $y' \in y + 2\varepsilon'\mathbf{B}_F \subset Y + 2\varepsilon'\mathbf{B}_F \subset Y'$, which ensures that $(x,y') \in (X \times Y') \cap \operatorname{gph} M$.

Therefore, by (4.2) there exists $h \in H$ such that

$$t^{-1}\Delta_M(x+th, y'+t\bar{k}) < \beta + \varepsilon,$$

and hence

$$\begin{aligned}
\beta + 2\varepsilon &> t^{-1}\Delta_M(x+th, y'+t\bar{k}) + \varepsilon \\
&> t^{-1}\big[\Delta_M(x+th, y'+t\bar{k}) + t^2\big] \\
&> t^{-1}\big[\Delta_M(x+th, y'+t\bar{k}) + \|y-y'\| - \Delta_M(x,y)\big] \quad \text{(by (4.3))} \\
&\geq t^{-1}\big[\Delta_M(x+th, y+t\bar{k}) - \alpha\big],
\end{aligned}$$

which proves (4.1).

Now, we prove the reverse inequality

$$\beta \leq \Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}). \tag{4.4}$$

We may suppose $\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) < +\infty$. Let $\varepsilon > 0$ and $H \in \mathcal{N}(\bar{h})$. Put $K = \bar{k} + \frac{\varepsilon}{2}\mathbf{B}_F$. There exists $X \times Y \in \mathcal{N}(\bar{x},\bar{y})$ and $\delta > 0$ such that for all $(x,y,\alpha) \in (X \times Y \times (-\delta,\delta)) \cap \operatorname{epi} \Delta_M$ and for all $t \in (0,\delta)$ there is $(h,k) \in H \times K$ with

$$t^{-1}\big[\Delta_M(x+th, y+tk) - \alpha\big] < \Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) + \frac{\varepsilon}{2}.$$

Fix any $(x,y) \in (X \times Y) \cap \operatorname{gph} M$. As $\Delta_M(x,y) = 0$ we have $(x,y,\Delta_M(x,y)) \in (X \times Y \times (-\delta,\delta)) \cap \operatorname{epi} \Delta_M$ and hence, by the last inequality above, there exist $h \in H$ and $k \in K$ such that

$$t^{-1}\big[\Delta_M(x+th, y+tk) - \Delta_M(x,y)\big] < \Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) + \frac{\varepsilon}{2}.$$

Therefore, as we always have

$$\Delta_M(x+th,y+t\bar{k}) \le \Delta_M(x+th,y+tk)+t\|k-\bar{k}\|,$$

then

$$t^{-1}\left[\Delta_M(x+th,y+t\bar{k})\right] < \Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k})+\|k-\bar{k}\|+\frac{\varepsilon}{2} \le \Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k})+\varepsilon.$$

This proves the inequality (4.4) and the proof is complete. □

The proof of the following theorem is straightforward and hence it is omitted.

Theorem 4.2. *Let M be a set-valued mapping from E into F with $(\bar{x},\bar{y}) \in \operatorname{gph} M$. Then for all $(\bar{h},\bar{k}) \in E \times F$*

$$\Delta_M^{-}(\bar{x},\bar{y};\bar{h},\bar{k}) = \liminf_{\substack{h\to\bar{h}\\ t\downarrow 0}} t^{-1}\left[\Delta_M(\bar{x}+th,\bar{y}+t\bar{k})\right].$$

These two above theorems allow us to describe the Clarke tangent cone $T^{\mathrm{C}}(\operatorname{gph} M;\bar{x},\bar{y})$ and the contingent cone $K(\operatorname{gph} M;\bar{x},\bar{y})$ to $\operatorname{gph} M$ in terms of the function Δ_M.

Theorem 4.3. *Let M be a set-valued mapping from E into F with $(\bar{x},\bar{y}) \in \operatorname{gph} M$. Then,*

$$T^{\mathrm{C}}(\operatorname{gph} M;\bar{x},\bar{y}) = \{(h,k) \in E\times F : \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)=0\}$$

and

$$K(\operatorname{gph} M;\bar{x},\bar{y}) = \{(h,k) \in E\times F : \ \Delta_M^{-}(\bar{x},\bar{y};h,k)=0\}.$$

Proof. We will prove the first equality (the proof of the second one being similar). As $\Delta_M \le \psi_M$ it follows from Theorem 4.1 that

$$0 \le \Delta_M^{\uparrow}(\bar{x},\bar{y};\cdot) \le \psi_M^{\uparrow}(\bar{x},\bar{y};\cdot)$$

and hence

$$T^{\mathrm{C}}(\operatorname{gph} M;\bar{x},\bar{y}) \subset \{(h,k) \in E\times F : \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)=0\}$$

since $(h,k) \in T^{\mathrm{C}}(\operatorname{gph} M;\bar{x},\bar{y})$ if and only if $\psi_M^{\uparrow}(\bar{x},\bar{y};h,k)=0$.
Now assume that $\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k})=0$. Consider any positive real number ε and any $H \in \mathcal{N}(\bar{h})$. By Theorem 4.1 once again we have

$$\limsup_{\substack{(x,y)\to_M(\bar{x},\bar{y})\\ t\downarrow 0}} \left[\inf_{h\in H} t^{-1}\Delta_M(x+th,y+t\bar{k})\right]=0.$$

So there exist $X \times Y \in \mathscr{N}(\bar{x}, \bar{y})$ and $\lambda > 0$ such that for all $(x,y) \in (X \times Y) \cap \text{gph } M$ and $t \in (0, \lambda)$ there exists $h \in H$ such that

$$t^{-1}\Delta_M(x+th, y+t\bar{k}) < \varepsilon \quad \text{i.e.} \quad d\big(y+t\bar{k}, M(x+th)\big) < t\varepsilon$$

and hence we may choose $k \in F$ with $y+tk \in M(x+th)$ and $\|(y+tk)-(y+t\bar{k})\| < t\varepsilon$. Therefore, for each $(x,y) \in (X \times Y) \cap \text{gph } M$ and $t \in (0,\lambda)$ there exists $(h,k) \in H \times (\bar{k} + \varepsilon \mathbf{B}_F)$ such that

$$(x,y) + t(h,k) \in \text{gph } M.$$

This shows the reverse inclusion and the proof is complete. □

Recall (see Chap. 1) that the convexified (Clarke) normal cone $N^{\mathrm{C}}(S;\bar{x})$ to a subset $S \subset E$ at $\bar{x} \in S$ is the negative polar cone of $T^{\mathrm{C}}(S;\bar{x})$, that is $N^{\mathrm{C}}(S;\bar{x}) = \{x^* \in E^*;\ \langle x^*, h\rangle \leq 0,\ \forall h \in T^{\mathrm{C}}(S;\bar{x})\}$.

The description of $T^{\mathrm{C}}(\text{gph } M; \bar{x}, \bar{y})$ in Theorem 4.3 ensures for $(\bar{x},\bar{y}) \in \text{gph } M$

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};0,0) = 0, \tag{4.5}$$

because the Clarke tangent cone always contains the zero vector. Note that this result can also be derived from the first equality in Theorem 4.1.

Corollary 4.1. *Assume that E is locally convex and M is a set-valued mapping from E into F and* $(\bar{x},\bar{y}) \in \text{gph } M$*. Then*

$$N^{\mathrm{C}}(\text{gph } M; \bar{x},\bar{y}) = \text{cl}_{w^*}\big(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\big).$$

Here, cl_{w^*} *denotes the closure with respect to the* w^**– topology in* $E^* \times F^*$.

Proof. The second member is obviously included in the first one. To prove the reverse inclusion we will prove that the polar cone of $\text{cl}_{w^*}\big(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\big)$ is included in the polar cone of $N^{\mathrm{C}}(\text{gph } M; \bar{x},\bar{y})$. Note first that

$$\Big(\text{cl}_{w^*}\big(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\big)\Big)^0 = \big(\partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\big)^0.$$

So, consider any $(h,k) \in \big(\partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\big)^0$. Then, we have

$$\big\langle (x^*,y^*),(h,k)\big\rangle \leq 0 \qquad \text{for all} \quad (x^*,y^*) \in \partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y}),$$

and hence (since $\partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y}) \neq \emptyset$)

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = \sup\{\big\langle (x^*,y^*),(h,k)\big\rangle; \quad (x^*,y^*) \in \partial^{\mathrm{C}} \Delta_M(\bar{x},\bar{y})\} \leq 0.$$

This ensures that $\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = 0$, and hence, by Theorem 4.2, we obtain

$$(h,k) \in T^{\mathrm{C}}(\mathrm{gph}\, M;\bar{x},\bar{y}) = \left(N^{\mathrm{C}}(\mathrm{gph}\, M;\bar{x},\bar{y})\right)^0.$$

□

Following Aubin [6], a geometric notion of derivative for set-valued mappings is associated with each notion of tangent cone.

Definition 4.1. Let $R(\mathrm{gph}\, M;\bar{x},\bar{y})$ be any tangent cone ($K(\mathrm{gph}\, M;\bar{x},\bar{y})$ or $T^{\mathrm{C}}(\mathrm{gph}\, M;\bar{x},\bar{y})$) to gph M at $(\bar{x},\bar{y}) \in$ gph M. The R-derivatives of M at $(\bar{x},\bar{y}) \in$ gph M is the set-valued mapping $D_R M(\bar{x},\bar{y})$ defined from E into F by

$$\mathrm{gph}\, (D_R M(\bar{x},\bar{y})) := R(\mathrm{gph}\, M;\bar{x},\bar{y}).$$

We will denote by $\Delta_{R,M}$ the distance function associated with the R-derivative of M at $(\bar{x},\bar{y})$ defined on $E \times F$ by

$$\Delta_{R,M}(x,y) := \Delta_{D_R M(\bar{x},\bar{y})}(x,y) = d\big(y, D_R M(\bar{x},\bar{y})(x)\big).$$

The next results of this section provide a relationship between the generalized directional derivative (resp. the lower Dini directional derivative) of Δ_M and the distance function $\Delta_{T^{\mathrm{C}},M}$ (resp. $\Delta_{K,M}$) associated with the T^{C}-derivative (respectively, K-derivative) of M at $(\bar{x},\bar{y})$.

Theorem 4.4. *Let $M : E \rightrightarrows F$ be a set-valued mapping with $(\bar{x},\bar{y}) \in$ gph M. Then for all $(\bar{h},\bar{k}) \in E \times F$*

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) \le \Delta_{T^{\mathrm{C}},M}(\bar{h},\bar{k}).$$

Proof. Fix $(\bar{h},\bar{k}) \in E \times F$. We may assume that $\Delta_{T^{\mathrm{C}},M}(\bar{h},\bar{k}) < +\infty$. Consider $\varepsilon > 0$. There exists $\bar{v} \in D_{T^{\mathrm{C}}} M(\bar{x},\bar{y})(\bar{h})$ (i.e. $(\bar{h},\bar{v}) \in T^{\mathrm{C}}(\mathrm{gph}\, M;\bar{x},\bar{y})$) such that

$$\|\bar{v} - \bar{k}\| \le \Delta_{T^{\mathrm{C}},M}(\bar{h},\bar{k}) + \frac{\varepsilon}{2}. \tag{4.6}$$

Let $H \in \mathscr{N}(\bar{h})$ and $V = \bar{v} + \frac{\varepsilon}{2}\mathbf{B}_F$. Then by the definition of the Clarke tangent cone, there exist $X \in \mathscr{N}(\bar{x})$, $Y \in \mathscr{N}(\bar{y})$ and $\delta > 0$ such that

$$\big((x,y) + t(H \times V)\big) \cap \mathrm{gph}\, M \ne \emptyset \quad \text{for all } (x,y) \in (X \times Y) \cap \mathrm{gph}\, M, \ \ t \in (0,\delta),$$

and hence, there exist $h \in H$ and $v \in V$ such that

$$(x,y) + t(h,v) \in \mathrm{gph} M, \text{ i.e., } v \in t^{-1}\big[M(x+th) - y\big]. \tag{4.7}$$

Moreover, we always have

$$t^{-1}\Delta_M(x+th, y+t\bar{k}) = t^{-1} d(y+t\bar{k}, M(x+th)) = d\left(\bar{k}, t^{-1}\left[M(x+th) - y\right]\right).$$

Then for every $(x,y) \in (X \times Y) \cap \text{gph } M$ and every $t \in (0,\delta)$ there is $(h,v) \in H \times V$ such that

$$\begin{aligned} t^{-1}\Delta_M(x+th, y+t\bar{k}) &\le \|\bar{k}-v\| (\text{by } (4.7)) \\ &\le \|\bar{k}-\bar{v}\| + \|\bar{v}-v\| \\ &\le \|\bar{k}-\bar{v}\| + \frac{\varepsilon}{2}, \end{aligned}$$

and hence

$$\sup_{H\in\mathcal{N}(\bar{h})} \inf_{\substack{X\times Y\in\mathcal{N}(\bar{x},\bar{y}) \\ \delta>0}} \sup_{\substack{(x,y)\in(X\times Y)\cap\text{gph } M \\ t\in(0,\delta)}} \inf_{h\in H} t^{-1}\Delta_M(x+th, y+t\bar{k}) \le \|\bar{k}-\bar{v}\| + \frac{\varepsilon}{2}.$$

According to (4.6) and Theorem 4.1 we conclude that

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};\bar{h},\bar{k}) \le \Delta_{T^C,M}(\bar{h},\bar{k}) + \varepsilon.$$

So the proof is complete. □

Theorem 4.5. *Let $M : E \rightrightarrows F$ be a set-valued mapping with $(\bar{x},\bar{y}) \in \text{gph } M$. Then for all $(\bar{h},\bar{k}) \in E \times F$*

$$\Delta_M^-(\bar{x},\bar{y};\bar{h},\bar{k}) \le \Delta_{K,M}(\bar{h},\bar{k}).$$

Proof. Let $(\bar{h},\bar{k}) \in E \times F$. Assume that $\Delta_{K,M}(\bar{h},\bar{k}) < +\infty$ (the other case is obvious) and fix $\varepsilon > 0$. There exists $\bar{v} \in D_K M(\bar{x},\bar{y})(\bar{h})$ (i.e. $(\bar{h},\bar{v}) \in K(\text{gph } M;\bar{x},\bar{y})$) such that

$$\|\bar{v}-\bar{k}\| \le d(\bar{k}, D_K M(\bar{x},\bar{y})(\bar{h})) + \frac{\varepsilon}{2} = \Delta_{K,M}(\bar{h},\bar{k}) + \frac{\varepsilon}{2}. \tag{4.8}$$

Let $H \in \mathcal{N}(\bar{h})$, $V = \bar{v} + \frac{\varepsilon}{2}\mathbf{B}_F$, and $\lambda > 0$. As $(\bar{h},\bar{v}) \in K(\text{gph } M;\bar{x},\bar{y})$, there exists $t \in (0,\lambda)$ such that

$$\left[(\bar{x},\bar{y}) + t(H \times V)\right] \cap \text{gph } M \ne \emptyset,$$

and hence, there are $h \in H$ and $v \in V$ with

$$v \in t^{-1}[M(\bar{x}+th) - \bar{y}]. \tag{4.9}$$

Moreover, for these h and v we always have

$$\begin{aligned} t^{-1}\Delta_M(\bar{x}+th, \bar{y}+t\bar{k}) &= d(\bar{k}, t^{-1}[M(\bar{x}+th) - \bar{y}]) \\ &\le \|\bar{k}-v\| \qquad (\text{by } (4.9)) \\ &\le \|\bar{k}-\bar{v}\| + \|\bar{v}-v\| \\ &\le \Delta_{K,M}(\bar{h},\bar{k}) + \varepsilon. \qquad (\text{by } (4.8)). \end{aligned}$$

So, for all $H \in \mathscr{N}(\bar{h})$ and $\lambda > 0$, there exists $t \in (0,\lambda)$ and $h \in H$ such that

$$t^{-1}\Delta_M(\bar{x}+th,\bar{y}+t\bar{k}) \leq \Delta_{K,M}(\bar{h},\bar{k})+\varepsilon,$$

and hence by Theorem 4.2

$$\Delta_M^-(\bar{x},\bar{y};\bar{h},\bar{k}) = \sup_{\substack{H\in\mathscr{N}(\bar{h})\\ \lambda>0}} \inf_{\substack{h\in H\\ t\in(0,\lambda)}} t^{-1}\Delta_M(\bar{x}+th,\bar{y}+t\bar{k}) \leq \Delta_{K,M}(\bar{h},\bar{k})+\varepsilon.$$

This completes the proof. □

Remark 4.1. Observe that the set-valued mapping $M : E \rightrightarrows F$ with $M(x) = S$ for any $x \in E$ (where S is a fixed subset of F) satisfies for all $h \in E$

$$D_K M(\bar{x},\bar{y})(h) = K(S;\bar{y}) \quad \text{and} \quad D_{T^{\mathrm{C}}} M(\bar{x},\bar{y})(h) = T^{\mathrm{C}}(S;\bar{y}),$$

where $\bar{y} \in S$.

Borwein and Fabian [36] showed that in any infinite dimensional Banach space F there exist a closed subset S and $\bar{y} \in S$ such that

$$d(\cdot,K(S;\bar{y})) \neq d_S^-(\bar{y};\cdot).$$

For this subset S and the set-valued mapping M associated as above, it is then easily seen (using the equalities above) that

$$\Delta_{K,M}(\cdot,\cdot) > \Delta_M^-(\bar{x},\bar{y};\cdot,\cdot).$$

So for infinite dimensional spaces F the inequality in the statement of Theorem 4.5 cannot be replaced by an equality.

4.3 Tangential Regularity of gph M and Directional Regularity of Δ_M

First, we are going to study some cases where the equality

$$\Delta_{K,M}(\cdot,\cdot) = \Delta_M^-(\bar{x},\bar{y};\cdot,\cdot) \tag{4.10}$$

holds. This will be a bridge between the tangential regularity of the graph of a set-valued mapping M at a point $(\bar{x},\bar{y}) \in \operatorname{gph} M$ and the directional regularity of the function Δ_M at the same point. It allows us to scalarize the tangential regularity of M at $(\bar{x},\bar{y}) \in \operatorname{gph} M$ with the help of the scalar function Δ_M.

In all the sequel of this section, we assume that E is a topological vector space. Let us recall that the contingent cone to a closed set $S \subset E$ at $\bar{u} \in S$ has also a characterization in terms of nets (a generalization of Proposition 1.6 part 1 for any topological vector space, see e.g. Penot [184]).

A vector $v \in K(S;\bar{u})$ if and only if there exist a net $(t_j)_{j\in J}$ of positive real numbers converging to zero and a net $(v_j)_{j\in J}$ in E converging to v such that

$$\bar{x} + t_j v_j \in S, \qquad \text{for each} \quad j \in J.$$

Let us begin with the case when F is a finite dimensional space.

The proof of the following theorem is adapted from the proof of Fact 1 in Theorem 2.1.

Theorem 4.6. *Let F be a finite dimensional vector space and let $M : E \rightrightarrows F$ be any set-valued mapping defined from E into F with $(\bar{x},\bar{y}) \in$ gph M. Then for all $(h,k) \in E \times F$*

$$\Delta_M^-(\bar{x},\bar{y};h,k) = \Delta_{K,M}(h,k).$$

Proof. Let $(h,k) \in E \times F$. The inequality

$$\Delta_M^-(\bar{x},\bar{y};h,k) \le \Delta_{K,M}(h,k)$$

always holds by Theorem 4.5. So, we will show the reverse inequality

$$\Delta_{K,M}(h,k) \le \Delta_M^-(\bar{x},\bar{y};h,k).$$

We may suppose that $\Delta_M^-(\bar{x},\bar{y};h,k) < \infty$. Let us consider (by Theorem 4.5) a net $(t_j,h_j)_{j\in J}$ in $(0,+\infty) \times E$ converging to $(0,h)$ and such that

$$\Delta_M^-(\bar{x},\bar{y};h,k) = \lim_{j\in J} d\left(k, t_j^{-1}\left[M(\bar{x}+t_jh_j) - \bar{y}\right]\right).$$

For each $j \in J$ we choose

$$v_j \in t_j^{-1}\left[M(\bar{x}+t_jh_j) - \bar{y}\right] \tag{4.11}$$

with

$$d\left(k, t_j^{-1}\left[M(\bar{x}+t_jh_j) - \bar{y}\right]\right) \ge \|k - v_j\| - t_j.$$

Then,

$$\Delta_M^-(\bar{x},\bar{y};h,k) \ge \limsup_{j\in J} \|k - v_j\|. \tag{4.12}$$

Since we have supposed that $\Delta_M^-(\bar{x},\bar{y};h,k) < +\infty$, there exist a real number $\beta > 0$ and j_0 such that

$$\|v_j\| \leq \beta, \quad \text{for all } j \geq j_0.$$

Thus, as F is finite dimensional, some subnet converges to some $v \in F$. Therefore, (4.11) ensures that $(h,v) \in K(\text{gph } M;\bar{x},\bar{y})$ i.e. $v \in D_K M(\bar{x},\bar{y})(h)$. So it follows from (4.12) that

$$\Delta_M^-(\bar{x},\bar{y};h,k) \geq \|k - v\| \geq d\big(k, D_K M(\bar{x},\bar{y})(h)\big) = \Delta_{K,M}(h,k),$$

which completes the proof. □

This theorem suggests to consider the following compatibility of the function Δ_M with respect to the contingent cone. We will say that a set-valued mapping $M : E \rightrightarrows F$ is *contingentially metrically stable* at $(\bar{x},\bar{y}) \in \text{gph } M$ if

$$\Delta_M^-(\bar{x},\bar{y};h,k) = \Delta_{K,M}(h,k) \qquad \text{for all} \quad (h,k) \in E \times F,$$

and we will denote by $\mathscr{K}(\bar{x},\bar{y})$ the collection of all such set-valued mappings.

By Theorem 4.6 all set-valued mappings M defined from a Hausdorff topological vector space E into a *finite* dimensional normed space F is in $\mathscr{K}(\bar{x},\bar{y})$, for each $(\bar{x},\bar{y}) \in \text{gph } M$.

Another general example of set-valued mappings in $\mathscr{K}(\bar{x},\bar{y})$ is given by Theorem 4.7 below concerning implicit set-valued mappings. Let E be a topological vector space and F and G be two Banach spaces. Let $f : E \times F \rightrightarrows G$ be a mapping that is strictly Fréchet differentiable at $(\bar{x},\bar{y})$ with respect to the second variable, that is, there exist a neighborhood X of $\bar{x}$, a continuous linear mapping $\nabla_2 f(\bar{x},\bar{y})$ from F into G and a mapping $r : X \times F \times F \longrightarrow G$ such that for all $(x,y,y') \in X \times F \times F$

$$f(x,y) - f(x,y') = \nabla_2 f(\bar{x},\bar{y})(y-y') + \|y-y'\| r(x,y,y')$$

and $\lim\limits_{(x,y,y') \to (\bar{x},\bar{y},\bar{y})} r(x,y,y') = 0$. We also assume that f is Gâteaux directionally differentiable at $(\bar{x},\bar{y})$ with respect to the first variable, that is, there exists a (not necessarily linear) continuous mapping $D_1 f(\bar{x},\bar{y};\cdot)$ from E into G such that for all $h \in E$

$$\lim_{h' \to h, t \to 0^+} t^{-1}\big[f(\bar{x}+th',\bar{y}) - f(\bar{x},\bar{y})\big] = D_1 f(\bar{x},\bar{y};h).$$

We are going to consider the implicit set-valued mapping $M : E \rightrightarrows F$ defined by

$$M(x) := \{y \in F : f(x,y) \in C\}, \tag{4.13}$$

where C is a closed convex subset of G with $\bar{z} := f(\bar{x},\bar{y}) \in C$. We suppose that the following Robinson qualification condition

$$0 \in \operatorname{core}\big[\operatorname{Im}\nabla_2 f(\bar{x},\bar{y}) - (C-\bar{z})\big] \tag{R.C.}$$

is satisfied. Recall that a point q is in the core of a subset S in G if for each $z \in G$ there exists some real number $\varepsilon > 0$ such that $\{q+t(z-q) : t \in [-\varepsilon,\varepsilon]\} \subset S$.

Theorem 4.7. *Under the assumptions above, the implicit set-valued mapping M given by (4.13) is in $\mathcal{K}(\bar{x},\bar{y})$.*

Proof. By Theorem 4.2 in Borwein [35] (note that Theorem 4.2 in [35] is presented for normed vector spaces E but its proof is actually valid for any topological vector spaces E using nets in place of sequences), the regularity condition (R.C.) implies that there exist two real positive numbers δ and γ and a neighborhood X of $\bar{x}$ such that for all $y \in \bar{y}+\delta\mathbf{B}$ and all $x \in X$

$$d\big(y,M(x)\big) \le \gamma d\big(f(x,y),C\big). \tag{4.14}$$

Consider any $(h,k) \in E\times F$ with $D_1 f(\bar{x},\bar{y};h) + \nabla_2 f(\bar{x},\bar{y})(k) \in K(C,\bar{z})$. Then for every positive number t small enough we have by (4.14)

$$t^{-1}\Delta_M(\bar{x}+th,\bar{y}+tk) \le \gamma t^{-1} d\big(f(\bar{x}+th,\bar{y}+tk),C\big). \tag{4.15}$$

Note that it is not difficult to see that

$$f(\bar{x}+th,\bar{y}+tk) = f(\bar{x},\bar{y}) + t\big(D_1 f(\bar{x},\bar{y};h) + \nabla_2 f(\bar{x},\bar{y})(k) + \rho(t)\big),$$

with $\lim_{t\longrightarrow 0^+} \rho(t) = 0$ and hence for $w := D_1 f(\bar{x},\bar{y};h) + \nabla_2 f(\bar{x},\bar{y})(k)$

$$\begin{aligned}\liminf_{t\longrightarrow 0^+} t^{-1} d\big(f(\bar{x}+th,\bar{y}+tk),C\big) &= \liminf_{t\longrightarrow 0^+} t^{-1} d\big(\bar{z}+t(w+\rho(t)),C\big)\\ &= \liminf_{t\longrightarrow 0^+} t^{-1} d\big(\bar{z}+tw,C\big) = 0,\end{aligned}$$

because $w \in K(C;\bar{z})$. So taking (4.15) and the definition of Δ_M^- into account, we get $\Delta_M^-(\bar{x},\bar{y};h,k) = 0$ and hence by Theorem 4.3 the vector (h,k) is in $K(\text{gph } M;\bar{x},\bar{y})$. As it is easily checked that the reverse inclusion always holds, we obtain

$$K(\text{gph } M;\bar{x},\bar{y}) = \{(h,k) \in E\times F : D_1 f(\bar{x},\bar{y};h) + \nabla_2 f(\bar{x},\bar{y})(k) \in K(C;\bar{z})\}. \tag{4.16}$$

Define now another set-valued mapping $Q : F \rightrightarrows G$ as follows

$$Q(v) = \nabla_2 f(\bar{x},\bar{y})(v) - K(C,\bar{z}) \qquad \text{for all} \quad v \in F.$$

This set-valued mapping has a closed convex graph and the regularity condition (R.C.) implies that $0 \in core\big[\mathrm{Im}Q\big]$. So, by Theorem 1 in Robinson [233] there exists $\eta > 0$ such that

$$\eta \mathbf{B}_G \subset Q(\mathbf{B}_F).$$

Since $0 \in Q(0)$, by Theorem 2 in Robinson [233], for all $z \in G$ with $\|z\| < \eta$ and all $y \in F$, one has

$$d\big(y, Q^{-1}(z)\big) \leq (\eta - \|z\|)^{-1}(1+\|y\|)d\big(z, Q(y)\big). \tag{4.17}$$

Now, we are ready to prove that $M \in \mathscr{K}(\bar{x},\bar{y})$ i.e. $\Delta_M^-(\bar{x},\bar{y};h,k) = \Delta_{K,M}(h,k)$ for all $(h,k) \in E \times F$. As in the proof of Theorem 4.6, it is enough to show the inequality

$$\Delta_{K,M}(h,k) \leq \Delta_M^-(\bar{x},\bar{y};h,k) \qquad \text{for all} \quad (h,k) \in E \times F. \tag{4.18}$$

As the functions $(h,k) \mapsto \Delta_{K,M}(h,k)$ and $(h,k) \mapsto \Delta_M^-(\bar{x},\bar{y};h,k)$ are positively homogeneous, we begin by proving (4.18) for all $h \in E$ with $\|D_1 f(\bar{x},\bar{y};h)\| \leq \frac{1}{2}\eta$ and all $k \in F$. Fix $(h,k) \in E \times F$ with $\|D_1 f(\bar{x},\bar{y};h)\| \leq \frac{1}{2}\eta$. Without loss of generality, we may suppose that $\Delta_M^-(\bar{x},\bar{y};h,k) < +\infty$. Let us consider a net $(t_j,h_j)_{j \in J}$ in $(0,+\infty) \times E$ converging to $(0,h)$ and such that

$$\Delta_M^-(\bar{x},\bar{y};h,k) = \lim_{j \in J} d\big(k, t_j^{-1}[M(\bar{x}+t_j h_j) - \bar{y}]\big).$$

For each $j \in J$ we choose v_j such that $(h_j, v_j) \in t_j^{-1}$ [gph $M - (\bar{x},\bar{y})$] and

$$\|k - v_j\| \leq d\big(k, t_j^{-1}\big[M(\bar{x}+t_j h_j) - \bar{y}\big]\big) + t_j. \tag{4.19}$$

As $\Delta_M^-(\bar{x},\bar{y};h,k) < +\infty$, there exist a real number $\beta > 0$ and $j_0 \in J$ such that

$$\|v_j\| \leq \beta \qquad \text{for all} \quad j \geq j_0.$$

On the other hand, we have

$$\Delta_{K,M}(h,k) = d\big(k, D_K M(\bar{x},\bar{y})(h)\big) \leq d\big(v_j, D_K M(\bar{x},\bar{y})(h)\big) + \|k - v_j\|. \tag{4.20}$$

Using (4.16), (4.17) and the definition of Q we can easily check that

$$\begin{aligned} D_K M(\bar{x},\bar{y})(h) &= \big(\nabla_2 f(\bar{x},\bar{y})\big)^{-1}\big(K(C;\bar{z}) - D_1 f(\bar{x},\bar{y};h)\big) \\ &= Q^{-1}\big(-D_1 f(\bar{x},\bar{y};h)\big). \end{aligned}$$

Thus, according to (4.17), we have for $j \geq j_0$

$$d\big(v_j, D_K M(\bar{x},\bar{y})(h)\big) \leq \frac{(1+\|v_j\|)}{\eta - \|D_1 f(\bar{x},\bar{y};h)\|} d\big(\nabla_2 f(\bar{x},\bar{y})(v_j), K(C;\bar{z}) - D_1 f(\bar{x},\bar{y};h)\big)$$
$$\leq 2\eta^{-1}(1+\beta) d\big(\nabla_2 f(\bar{x},\bar{y})(v_j), K(C;\bar{z}) - D_1 f(\bar{x},\bar{y};h)\big). \tag{4.21}$$

Because of our assumptions on the mapping f and since

$$(h_j, v_j) \in t_j^{-1}\, [\text{gph}\, M - (\bar{x},\bar{y})\,],$$

we have for some $\varepsilon_j \longrightarrow 0^+$

$$C \ni f(\bar{x}+t_j h_j, \bar{y}+t_j v_j) = f(\bar{x},\bar{y}) + t_j D_1 f(\bar{x},\bar{y};h) + t_j \nabla_2 f(\bar{x},\bar{y})(v_j) + t_j \varepsilon_j,$$

for all $j \geq j_0$ and hence

$$D_1 f(\bar{x},\bar{y};h) + \nabla_2 f(\bar{x},\bar{y})(v_j) + \varepsilon_j \in t_j^{-1}(C - \bar{z}) \subset K(C;\bar{z}).$$

Therefore, for all $j \geq j_0$

$$d\big(\nabla_2 f(\bar{x},\bar{y})(v_j), K(C;\bar{z}) - D_1 f(\bar{x},\bar{y};h)\big) \leq \|\varepsilon_j\|,$$

and hence using (4.21), we get that $d\big(v_j, D_K M(\bar{x},\bar{y})(h)\big)$ tends to zero. So by (4.19) and (4.20) we conclude that

$$\Delta_{K,M}(h,k) \leq \lim_{j\in J} d\big(v_j, D_K M(\bar{x},\bar{y})(h)\big) + \limsup_{j\in J} \|k - v_j\| \leq \Delta_M^-(\bar{x},\bar{y};h,k),$$

which finishes the proof of (4.18), for all $k \in F$ and all $h \in E$ with $\|D_1 f(\bar{x},\bar{y};h)\| \leq \frac{1}{2}\eta$. By positive homogeneity one easily obtains that

$$\Delta_{K,M}(h,k) \leq \Delta_M^-(\bar{x},\bar{y};h,k) \qquad \text{for all} \quad (h,k) \in E \times F,$$

and hence the proof of the theorem is complete. □

In the following theorem, we prove that for each set-valued mapping M defined from a Hausdorff topological vector space into a real normed vector space, the directional regularity of the function Δ_M at a point $(\bar{x},\bar{y}) \in \text{gph}\, M$ implies the tangential regularity of the graph gph M at the same point $(\bar{x},\bar{y})$. Moreover, the converse also holds whenever $M \in \mathscr{K}(\bar{x},\bar{y})$.

Theorem 4.8. *Let E be any Hausdorff topological vector space and F be any normed vector space and let $M : E \rightrightarrows F$ be any set-valued mapping defined from E into F with $(\bar{x},\bar{y}) \in$ gph M. Assume that Δ_M is directionally regular at $(\bar{x},\bar{y})$. Then* gph M *is tangentially regular at $(\bar{x},\bar{y})$. Moreover, if $M \in \mathscr{K}(\bar{x},\bar{y})$, then the converse is also true.*

Proof. Assume that Δ_M is directionally regular at $(\bar{x},\bar{y})$, i.e.,

$$\Delta_M^-(\bar{x},\bar{y};h,k) = \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \qquad \text{for all } \ (h,k) \in E \times F.$$

Then, by Theorem 4.3, one has

$$\begin{aligned} K(\text{gph } M;\bar{x},\bar{y}) &= \{(h,k) \in E \times F; \Delta_M^-(\bar{x},\bar{y};h,k) = 0\} \\ &= \{(h,k) \in E \times F; \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = 0\} \\ &= T^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y}), \end{aligned}$$

and hence gph M is tangentially regular at $(\bar{x},\bar{y})$.

Now, we assume that gph M is tangentially regular at $(\bar{x},\bar{y})$ and $M \in \mathscr{K}(\bar{x},\bar{y})$. Let $(h,k) \in E \times F$. As gph M is tangentially regular at $(\bar{x},\bar{y})$, one has

$$\Delta_{T,M}(h,k) = \Delta_{K,M}(h,k) \tag{4.22}$$

and as $M \in \mathscr{K}(\bar{x},\bar{y})$ one also has

$$\Delta_{K,M}(h,k) = \Delta_M^-(\bar{x},\bar{y};h,k). \tag{4.23}$$

On the other hand, one knows by Theorem 4.4 that

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le \Delta_{T^{\mathrm{C}},M}(h,k)$$

and hence according to (4.22) and (4.23) one gets

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le \Delta_{T,M}(h,k) = \Delta_{K,M}(h,k) = \Delta_M^-(\bar{x},\bar{y};h,k).$$

So

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le \Delta_M^-(\bar{x},\bar{y};h,k),$$

and since the reverse inequality is always true, the proof is complete. □

The following corollary is a direct consequence of Theorem 4.6 and Theorem 4.8.

Corollary 4.2. *Let E be any Hausdorff topological vector space and F be a finite dimensional vector space and let $M : E \rightrightarrows F$ be any set-valued mapping defined from E into F with $(\bar{x},\bar{y}) \in$ gph M. Then* gph M *is tangentially regular at $(\bar{x},\bar{y})$ if and only if the function Δ_M is directionally regular at $(\bar{x},\bar{y})$.*

Remark 4.2. Using Remark 4.1 it is easily seen that Theorem 4.8 and Corollary 4.2 do not hold for all set-valued mappings when F is infinite dimensional.

Now, let us recall the Aubin property (also called local Lipschitz-like property, pseudo-Lipschitz property) for set-valued mappings (see Aubin [7] and Mordukhovich [192]).

Definition 4.2. Assume that E and F are normed vector spaces. A set-valued mapping $M : E \rightrightarrows F$ is said to have Aubin property with ratio l at $(\bar{x},\bar{y}) \in \text{gph}\, M$ (for a real number $l \geq 0$) if there exist neighborhoods $X \subset \text{dom}\, M$ of $\bar{x}$ and Y of $\bar{y}$ such that

$$Y \cap M(x') \subset M(x) + l\|x - x'\|\mathbf{B}_F, \quad \text{for all } \; x, x' \in X.$$

In [238], Rockafellar has proved that M has Aubin property with ratio l at $(\bar{x},\bar{y})$ if and only if there exist neighborhoods $X \subset \text{dom}\, M$ of $\bar{x}$ and Y of $\bar{y}$ such that

$$d(y,M(x)) \leq d(y',M(x')) + \|y - y'\| + l\|x - x'\|, \quad \text{for all } (x,y),(x',y') \in X \times Y.$$

We finish this section by giving a characterization of the generalized gradient (Clarke subdifferential) of the function Δ_M at $(\bar{x},\bar{y}) \in \text{gph}\, M$ when $M \in \mathscr{K}(\bar{x},\bar{y})$.

Theorem 4.9. *Let E be a Hausdorff topological vector space and F be a normed space, and let $M : E \rightrightarrows F$ be any set-valued mapping defined from E into F and $(\bar{x},\bar{y}) \in$ gph M. Suppose that $M \in \mathscr{K}(\bar{x},\bar{y})$ and gph M is tangentially regular at $(\bar{x},\bar{y})$. Then one has*

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = \Delta_{T^{\mathrm{C}},M}(h,k) \quad \text{for all } \; (h,k) \in E \times F \tag{4.24}$$

and

$$\partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y}) = (E^* \times \mathbf{B}_{F^*}) \cap N^{\mathrm{C}}(\text{gph}\, M;\bar{x},\bar{y}). \tag{4.25}$$

If, furthermore, E is a normed vector space and M has Aubin property with ratio l around $(\bar{x},\bar{y})$, then one has

$$\partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y}) = (l\mathbf{B}_{E^*}) \times \mathbf{B}_{F^*} \cap N^{\mathrm{C}}(\text{gph}\, M;\bar{x},\bar{y}). \tag{4.26}$$

Proof.

1. Let $(h,k) \in E \times F$. By definition of $\mathscr{K}(\bar{x},\bar{y})$ and by Theorem 4.8 one has

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = \Delta_M^{-}(\bar{x},\bar{y};h,k) = \Delta_{K,M}(h,k),$$

   and hence, since gph M is tangentially regular at $(\bar{x},\bar{y})$, one has $\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) = \Delta_{T,M}(h,k)$.
2. Let us prove the equality (4.25). Let $(x^*,y^*) \in (E^* \times \mathbf{B}_{F^*}) \cap N^{\mathrm{C}}(\text{gph}\, M;\bar{x},\bar{y})$. Put

$$\rho(h,k) = \|k\| + |\langle x^*,h\rangle| \quad \text{and} \quad \varphi(h,k) = \big\langle -(x^*,y^*),(h,k)\big\rangle,$$

for all $(h,k) \in E \times F$. It is easily seen that the inequality $\|y^*\| \le 1$ ensures that for all (h,k) and (h',k') in $E \times F$

$$\varphi(h',k') \le \varphi(h,k) + \rho(h-h',k-k').$$

Note that $\varphi(0,0) = 0$ and

$$\varphi(h',k') \ge 0 \quad \text{for all} \quad (h',k') \in T^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y})$$

because $(x^*,y^*) \in N^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y})$. Then for any $(h,k) \in E \times F$ and any $(h',k') \in T^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y})$ one has

$$\varphi(0,0) \le \varphi(h',k') \le \varphi(h,k) + \rho(h-h',k-k'),$$

and hence

$$0 = \varphi(0,0) \le \varphi(h,k) + \delta(h,k), \tag{4.27}$$

where

$$\delta(h,k) := \inf\{\rho(h-h',k-k') : (h',k') \in T^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y})\}.$$

Observe that for any $k' \in D_{T^{\mathrm{C}}}M(\bar{x},\bar{y})(h)$ $\big($i.e. $(h,k') \in T^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y})\big)$ one has

$$\delta(h,k) \le \rho(0,k-k') = \|k-k'\|,$$

and hence

$$\delta(h,k) \le \inf\{\|k-k'\| : k' \in D_{T^{\mathrm{C}}}M(\bar{x},\bar{y})(h)\} = \Delta_{T^{\mathrm{C}},M}(h,k).$$

So it follows from (4.27) and (4.24) that for all $(h,k) \in E \times F$

$$\big\langle (x^*,y^*),(h,k)\big\rangle \le \Delta_{T^{\mathrm{C}},M}(h,k) = \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k).$$

This ensures that $(x^*,y^*) \in \partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y})$, and hence

$$(E^* \times \mathbf{B}_{F^*}) \cap N^{\mathrm{C}}(\text{gph } M;\bar{x},\bar{y}) \subset \partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y}). \tag{4.28}$$

Consider now any $(x^*,y^*) \in \partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y})$. By Theorem 4.1 one has for all h in E and k,k' in F

$$\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k') + \|k-k'\|$$

and hence for $h=0$ and $k'=0$ (since $\Delta_M^{\uparrow}(\bar{x},\bar{y};0,0) = 0$ by (4.5))

$$\big\langle y^*,k\big\rangle \le \Delta_M^{\uparrow}(\bar{x},\bar{y};0,k) \le \|k\|.$$

This ensures that $\|y^*\| \leq 1$ and hence we get that

$$(x^*, y^*) \in (E^* \times \mathbf{B}_{F^*}) \cap N^{\mathrm{C}}(\operatorname{gph} M; \bar{x}, \bar{y}),$$

because $\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})$ is always included in $N^{\mathrm{C}}(\operatorname{gph} M; \bar{x}, \bar{y})$ by Corollary 4.1. So the converse inclusion to (4.28) is also established.

3. Assume further that E is a normed vector space and M has Aubin property with ratio l around $(\bar{x}, \bar{y})$. Then for each (x, y) and (x', y') around $(\bar{x}, \bar{y})$ with $(x, y) \in \operatorname{gph} M$ one has

$$\Delta_M(x', y') \leq \|y' - y\| + l\|x' - x\|,$$

which ensures by Theorem 4.1 that

$$\Delta_M^0(\bar{x}, \bar{y}; h, k) \leq \|k\| + l\|h\|, \quad \text{for all} \ \ (h, k) \in E \times F$$

and hence

$$\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}) \subset (l\mathbf{B}_{E^*}) \times \mathbf{B}_{F^*}.$$

So (4.26) follows from (4.28) and (4.25).

□

Remark 4.3. In this section, we have proved the equivalence between the directional regularity of Δ_M and the tangential regularity of gph M for a set-valued mapping M defined from a Hausdorff topological vector space E into a normed vector space F in two different cases. The first case is when F is assumed to be a finite dimensional space and M is any set-valued mapping. The second case is given by a general implicit set-valued mapping.

4.4 Tangential Regularity of Lipschitz Epigraphic Set-Valued Mappings

In the previous section, we have studied the equivalence between the tangential regularity of a set-valued mapping $M : E \rightrightarrows F$ and the directional regularity of the scalar function Δ_M associated with M. In this section, we will provide a general new important class of set-valued mappings for which the equivalence above holds in a weaker sense.

The class of set-valued mappings that we will explore (in this section) in connection with the tangential regularity is that of Lipschitz epigraphic set-valued mappings. We owe its consideration here to the part of the proof of Proposition 3.3 in Ioffe [141] establishing that the approximate and the geometric normal cone to an epi-Lipschitz set (of a Banach space) coincide.

Before defining this class, recall that a closed subset S of F is epi-Lipschitz around a point $\bar{x} \in S$ if it can be represented near $\bar{x}$ as the epigraph of a Lipschitz function. Rockafellar showed in [235] that S is epi-Lipschitz around $\bar{x}$ if and only if there exist some vector $\bar{h} \in F$, neighborhoods $H \in \mathscr{N}(\bar{h})$, $X \in \mathscr{N}(\bar{x})$ and a real number $\varepsilon > 0$ such that

$$S \cap X + tH \subset S,$$

for all $t \in [0, \varepsilon]$.

This geometrical characterization allows us to adapt the concept for set-valued mappings as follows.

Definition 4.3. Let $M : E \rightrightarrows F$ be a set-valued mapping with closed values that is lower semicontinuous at $\bar{x}$. We will say that M is *Lipschitz epigraphic* around $(\bar{x}, \bar{y}) \in \text{gph}\, M$ if there exist a vector $\bar{h} \in F$, neighborhoods $H \in \mathscr{N}(\bar{h})$, $X \in \mathscr{N}(\bar{x})$, and $Y \in \mathscr{N}(\bar{y})$ and a real number $\bar{\delta} > 0$ such that

$$M(x) \cap Y + tH \subset M(x),$$

for all $x \in X$ and $t \in [0, \bar{\delta}]$.

An adaptation of the proof of the geometrical characterization of epi-Lipschitz sets (Theorem 3 in [235]) by Rockafellar allows us to obtain a similar characterization for Lipschitz epigraphic set-valued mappings. We give it in the following proposition.

Proposition 4.1. *Let $M : E \rightrightarrows F$ be a set-valued mapping with closed values that is lower semicontinuous at $\bar{x}$. Then M is Lipschitz epigraphic around $(\bar{x}, \bar{y})$ in* gph M *if and only if, either $(\bar{x}, \bar{y}) \in \text{int gph}\, M$, or there exist a topological direct sum $F = G \oplus \mathbf{R}\bar{h}$ with $\bar{y} = \bar{z} + \bar{r}\bar{h}$ ($\bar{z} \in G$ and $\bar{r} \in \mathbf{R}$), a function $f : E \times G \rightrightarrows \mathbf{R}$, neighborhoods $X \times Z$ of $(\bar{x}, \bar{z})$ in $E \times G$ and I of $\bar{r}$ and a real positive number $l \geq 0$ such that*

(i) *f is upper semicontinuous at $(\bar{x}, \bar{z})$ and $f(\bar{x}, \bar{z}) = \bar{r}$;*
(ii) *$|f(x,z) - f(x,z')| \leq l\|z - z'\|$ for all $z, z' \in Z$ and $x \in X$ (here $\|.\|$ denotes the norm induced on G by the norm of F);*
(iii) *for $Y := Z + I\bar{h}$ and for all $x \in X$*

$$M(x) \cap Y = \{z + r\bar{h} : \quad (z,r) \in (Z \times I) \cap \text{epi}\, f(x, \cdot)\},$$

i.e.,

$$M(x) \cap Y = \{z + r\bar{h} : \quad (z,r) \in (Z \times I) \ \text{ and } \ f(x,z) \leq r\}.$$

Proof. *Necessity.* This implication can be checked in a straightforward way.

Sufficiency. Assume that $(\bar{x}, \bar{y}) \notin \text{int gph}\, M$. Let $\bar{h}$, X, Y and $\bar{\delta}$ as given by Definition 4.3. Choose a topological complement G of the one-dimensional subspace $\mathbf{R}\bar{h}$ of F

so that F appears as the topological direct sum $F = G \oplus \mathbf{R}\bar{h}$ and $\bar{y}$ may be written as $\bar{y} = \bar{z} + \bar{r}\bar{h}$ for some $\bar{z} \in G$ and $\bar{r} \in \mathbf{R}$. Choose a symmetric convex neighborhood V of zero in G endowed with the topology induced by that of F and a positive number $\delta < \min(\bar{\delta}, 1)$ such that $Y' := (\bar{z} + V) \oplus (\bar{r} - \delta, \bar{r} + \delta)\bar{h}$ is included in Y and $H' := V \oplus (1 - \delta, 1 + \delta)\bar{h}$ is included in H. So, Definition 4.3 says that for all $t \in [0, \delta]$ and $x \in X$

$$M(x) \cap Y' + tH' \subset M(x). \tag{4.29}$$

According to the lower semicontinuity of M at $(\bar{x}, \bar{y})$ there exists a neighborhood X' of $\bar{x}$ with $X' \subset X$ such that for all $x \in X'$

$$M(x) \cap \left[\left(\bar{z} + \frac{\delta}{8} V \right) \oplus \left(\bar{r} - \frac{\delta}{4}, \bar{r} + \frac{\delta}{4} \right) \bar{h} \right] \neq \emptyset. \tag{4.30}$$

Put

$$I := \left(\bar{r} - \frac{\delta}{2}, \bar{r} + \frac{\delta}{2} \right)$$

and consider the following function f defined on $E \times G$ by

$$f(x, y) := \inf\{ r \in I : \quad z + r\bar{h} \in M(x) \}. \tag{4.31}$$

First, we wish to show that for $Z := \bar{z} + \dfrac{\delta}{8} V$ and for every $(x, z) \in X' \times Z$ the set $\{ r \in I : \quad z + r\bar{h} \in M(x) \}$ is nonempty, which will ensure that f takes finite values over $X \times Z$. Fix then any $(x, z) \in X' \times Z$. There exists, by (4.30), some $z' \in \bar{z} + \dfrac{\delta}{8} V$ and some

$$r' \in \left(\bar{r} - \frac{\delta}{4}, \bar{r} + \frac{\delta}{4} \right)$$

such that $z' + r'\bar{h} \in M(x)$. Therefore,

$$\begin{aligned} z + \left(r' + \frac{\delta}{4} \right) \bar{h} = z' + r'\bar{h} + (z - z') + \frac{\delta}{4}\bar{h} &\in M(x) \cap Y' + \frac{\delta}{4} V + \frac{\delta}{4} \bar{h} \\ &= M(x) \cap Y' + \frac{\delta}{4} [V + \bar{h}] \\ &\subset M(x) \cap Y' + \frac{\delta}{4} H' \\ &\subset M(x). \qquad (\text{ by } (4.29)) \end{aligned}$$

This ensures that one has $\{r \in I : \quad z + r\bar{h} \in M(x)\} \neq \emptyset$ and this nonemptiness allows to write

$$f(x,y) = \inf\left\{r \in \mathbf{R} : \quad r > \bar{r} - \frac{\delta}{2} \quad \text{and} \quad z + r\bar{h} \in M(x)\right\}. \tag{4.32}$$

We turn now to prove (i), (ii) and (iii) of the statement of the theorem for the function f defined above. Fix any $(x,z,r) \in X' \times Z \times I$ with $(z,r) \in \operatorname{epi} f(x,\cdot)$. Then $f(x,z) \leq r$ and hence for any $s \in I$ with $s > r$ we can find (by (4.31)) $\rho \in I$ with $z + \rho\bar{h} \in M(x)$ and such that $f(x,z) \leq \rho < s$. Thus

$$z + s\bar{h} = z + \rho\bar{h} + (s - \rho)\bar{h} \in M(x) \cap Y' + (0,\delta)H' \subset M(x) \qquad \text{(by (4.29))}$$

and as $M(x)$ is closed we obtain $z + r\bar{h} \in M(x)$. So, we have proved that for $Y'' := Z \oplus I\bar{h}$

$$\{z + r\bar{h} : \quad (z,r) \in (Z \times I) \cap \operatorname{epi} f(x,\cdot)\} \subset M(x) \cap Y''.$$

As the reverse inclusion is obvious, the proof of (iii) is complete.

Now we show (ii). It is enough to prove the following

$$f(x, z + tv) \leq f(x,z) + t,$$

for all $x \in X'$, $z \in Z$, $v \in V$ and $t \in [0,\delta]$ with $z + tv \in Z$. Fix $(x,z) \in X' \times Z$ and fix also $r \in I$ with $z + r\bar{h} \in M(x)$. For all $v \in V$ and $t \in [0,\delta]$ with $z + tv \in Z$ one has by (4.29)

$$(z + tv) + (r + t)\bar{h} = (z + r\bar{h}) + t(v + \bar{h}) \in M(x) \cap Y' + tH' \subset M(x).$$

Since $r + t \geq r > \bar{r} - \dfrac{\delta}{2}$, one concludes (by (4.32)) that $f(x, z + tv) \leq r + t$ for all $r \in I$ with $z + r\bar{h} \in M(x)$. Hence, it follows from (4.31) that $f(x, z + tv) \leq f(x,z) + t$.

We finish the proof by showing (i). As $\bar{z} + \bar{r}\bar{h} \in M(\bar{x})$, one has $f(\bar{x},\bar{z}) \leq \bar{r}$. So, suppose by contradiction that $f(\bar{x},\bar{z}) < \bar{r}$. Choose (by (4.31)) $r' \in I$ and real numbers $\varepsilon > 0$ and $\eta \in (0,\delta/2)$ such that $\bar{z} + r'\bar{h} \in M(\bar{x})$, the closed ball Z_ε in G centered at $\bar{z}$ with radius ε is included in Z and $I_\varepsilon := (r' - \varepsilon, r' + \varepsilon) \subset I$ and also such that

$$f(\bar{x},\bar{z}) < r' + \varepsilon + 2l\varepsilon < \bar{r} - \eta.$$

Then, as M is lower semicontinuous at $\bar{x}$, there exists a neighborhood $X_\varepsilon \subset X'$ of $\bar{x}$ such that $(Z_\varepsilon + I_\varepsilon\bar{h}) \cap M(x) \neq \emptyset$ for all $x \in X_\varepsilon$. For every $(x,z) \in X_\varepsilon \times Z_\varepsilon$ one can choose $(z'' + r''\bar{h}) \in (Z_\varepsilon + I_\varepsilon\bar{h}) \cap M(x)$ and by (ii) and (4.31) one has

$$f(x,z) \leq f(x,z'') + l\|z - z''\| \leq r'' + 2l\varepsilon \leq r' + \varepsilon + 2l\varepsilon < \bar{r} - \eta$$

and hence, for $I_\eta := (\bar{r}-\eta, \bar{r}+\eta)$ one gets $X_\varepsilon \times Z_\varepsilon \times I_\eta \subset \text{epi}\, f$ which ensures by (iii) that $X_\varepsilon \times \left(Z_\varepsilon \oplus I_\eta \bar{h}\right) \subset \text{gph}\, M$. This contradicts that $(\bar{x}, \bar{z}+\bar{r}\bar{h}) = (\bar{x},\bar{y}) \notin$ int gph M. So $f(\bar{x},\bar{z}) = \bar{r}$. Using the same arguments as in what precedes, one can easily see that f is upper semicontinuous at $(\bar{x},\bar{z})$. So the proof of the theorem is complete. □

Theorem 4.10. *Let $M : E \rightrightarrows F$ be a set-valued mapping that is Lipschitz epigraphic around a point $(\bar{x},\bar{y}) \in \text{gph}\, M$ and lower semicontinuous at $\bar{x}$. Then M is tangentially regular at $(\bar{x},\bar{y})$ if and only if there exists some equivalent norm on F such that Δ_M (associated with that norm) is directionally regular at the same point $(\bar{x},\bar{y})$.*

Proof. We may assume that $(\bar{x},\bar{y})$ is not in the interior of gph M (because otherwise the result is trivial). By Proposition 4.1 we may also suppose that $F = G \times \mathbf{R}$ and $\bar{y} = (\bar{z},\bar{r})$ and that near $(\bar{x},\bar{y})$ the set-valued mapping M is given for all $x \in X$ by

$$M(x) \cap (Z \times I) = \{(z,r) \in Z \times I : \quad f(x,z) \le r\},$$

where X, Z, I and f are as in the statement of Proposition 4.1.

Step 1. We begin by proving the following inequality (which holds for any set-valued mapping defined from a Hausdorff topological vector space E into F endowed with any norm $\|\cdot\|$)

$$\Delta_M^{\uparrow}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) \le \Delta_{T^C,M}(\bar{h},\bar{k},\bar{s}) \quad \text{for all} \quad (\bar{h},\bar{k},\bar{s}) \in E \times G \times \mathbf{R}.$$

Here we use the notation

$$\Delta_{T^C,M}(h,k,s) := d\Big((k,s); D_{T^C}M(\bar{x},\bar{z},\bar{r})(h)\Big). \tag{4.33}$$

Fix $(\bar{h},\bar{k},\bar{s}) \in E \times G \times \mathbf{R}$. We may assume that $\Delta_{T^C,M}(\bar{h},\bar{k},\bar{s}) < +\infty$. Consider $\varepsilon > 0$. There exists $(\bar{v},\bar{\alpha}) \in D_{T^C}M(\bar{x},\bar{z},\bar{r})(\bar{h})$ (i.e. $(\bar{h},\bar{v},\bar{\alpha}) \in T^C(\text{gph}\, M;\bar{x},\bar{z},\bar{r})$) such that

$$\|(\bar{v},\bar{\alpha}) - (\bar{k},\bar{s})\| \le \Delta_{T^C,M}(\bar{h},\bar{k},\bar{s}) + \frac{\varepsilon}{2}. \tag{4.34}$$

Let $H \in \mathcal{N}(\bar{h})$, $V \in \mathcal{N}(\bar{v})$, and $J \in \mathcal{N}(\bar{\alpha})$ such that $\|(v,\alpha)-(\bar{v},\bar{\alpha})\| \le \frac{\varepsilon}{2}$ for all $(v,\alpha) \in V \times J$. Then, by the definition of the Clarke tangent cone, there exist $X \in \mathcal{N}(\bar{x})$, $Z \in \mathcal{N}(\bar{z})$, $\Lambda \in \mathcal{N}(\bar{r})$, and $\delta > 0$ such that

$$\big((x,z,s) + t(H \times V \times J)\big) \cap \text{gph}\, M \ne \emptyset$$

for all $(x,z,s) \in (X \times Z \times \Lambda) \cap \text{gph}\, M$, and all $t \in (0,\delta)$, and hence there exist $h \in H$, $v \in V$, and $\alpha \in J$ such that

$$(x,z,s) + t(h,v,\alpha) \in \text{gph}\, M, \quad \text{i.e.,} \quad (v,\alpha) \in t^{-1}\big[M(x+th) - (z,s)\big]. \tag{4.35}$$

Moreover, we always have

$$\begin{aligned} t^{-1}\Delta_M(x+th,z+t\bar{k},s+t\bar{s}) &= t^{-1}d\Big((z+t\bar{k},s+t\bar{s});M(x+th)\Big) \\ &= d\Big((\bar{k},\bar{s});t^{-1}\big[M(x+th)-(z,s)\big]\Big). \end{aligned}$$

Then for every $(x,z,s)\in(X\times Z\times\Lambda)\cap\operatorname{gph} M$ and every $t\in(0,\delta)$ there is by (4.35) some $(h,v,\alpha)\in H\times V\times J$ such that

$$\begin{aligned} t^{-1}\Delta_M(x+th,z+t\bar{k},s+t\bar{s}) &\le \|(v,\alpha)-(\bar{k},\bar{s})\| \\ &\le \|(v,\alpha)-(\bar{v},\bar{\alpha})\|+\|(\bar{v},\bar{\alpha})-(\bar{k},\bar{s})\| \\ &\le \|(\bar{v},\bar{\alpha})-(\bar{k},\bar{s})\|+\frac{\varepsilon}{2}, \end{aligned}$$

and hence we have by (4.34)

$$t^{-1}\Delta_M(x+th,z+t\bar{k},s+t\bar{s})\le \Delta_{T,M}(\bar{h},\bar{k},\bar{s})+\varepsilon.$$

Therefore, we deduce that for $\gamma:=\Delta_{T,M}(\bar{h},\bar{k},\bar{s})$

$$\sup_{H\in\mathscr{N}(\bar{h})}\ \inf_{\substack{X\times Z\in\mathscr{N}(\bar{x},\bar{z})\\ \Lambda\in\mathscr{N}(\bar{r}),\delta>0}}\ \sup_{\substack{(x,z,s)\in(X\times Z\times\Lambda)\cap\operatorname{gph} M\\ t\in(0,\delta)}}\ \inf_{h\in H} t^{-1}\Delta_M(x+th,z+t\bar{k},s+t\bar{s})\le\gamma+\varepsilon,$$

and according to Theorem 4.4 we conclude that

$$\Delta_M^{\uparrow}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s})\le\Delta_{T^C,M}(\bar{h},\bar{k},\bar{s}).$$

So the proof of Step 1 is complete.

Step 2. Let l be a Lipschitz constant of f as given by Proposition 4.1 and let $\|\cdot\|$ be the norm on $G\times\mathbf{R}$ defined by $\|(z,r)\|:=l\|z\|+|r|$ for all $(z,r)\in G\times\mathbf{R}$. In this second step, we will prove that, with respect to this norm $\|\cdot\|$, the following equality holds

$$\Delta_M^{-}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s})=\Delta_{K,M}(\bar{h},\bar{k},\bar{s})\quad\text{for all}\ \ (\bar{h},\bar{k},\bar{s})\in E\times G\times\mathbf{R},$$

where $\Delta_{K,M}(h,k,s)$ is defined in a similar way as in (2.5) with $D_TM(\bar{x},\bar{z},\bar{r})(h)$ in place of $D_KM(\bar{x},\bar{z},\bar{r})(h)$.

By the same techniques used in the previous step, we can prove (in fact with respect to any norm on $G\times\mathbf{R}$) the following inequality

$$\Delta_M^{-}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s})\le\Delta_{K,M}(\bar{h},\bar{k},\bar{s})\quad\text{for all}\ \ (\bar{h},\bar{k},\bar{s})\in E\times G\times\mathbf{R}. \tag{4.36}$$

So, we proceed to showing the reverse inequality

$$\Delta_{K,M}(\bar{h},\bar{k},\bar{s}) \le \Delta_M^-(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) \quad \text{for all} \quad (\bar{h},\bar{k},\bar{s}) \in E \times G \times \mathbf{R}. \tag{4.37}$$

We may assume that the second member is finite. Consider any real number $\rho > 0$ and any $(z,r) \in (\bar{z},\bar{r}) + \rho\mathbf{B}$. As the set-valued mapping M is lower semicontinuous at $(\bar{x},\bar{z},\bar{r}) \in \operatorname{gph} M$, there exists a neighborhood $U \in \mathscr{N}(\bar{x})$ such that for all $x \in U$ one has

$$M(x) \cap ((\bar{z},\bar{r}) + \rho\mathbf{B}) \neq \emptyset.$$

Let any $(z',r') \in M(x)$ with $\|(z',r') - (\bar{z},\bar{r})\| > 3\rho$. Choose $(z'',r'') \in M(x) \cap ((\bar{z},\bar{r}) + \rho\mathbf{B})$ and observe that

$$\begin{aligned}
\|(z',r') - (z,r)\| &\ge \|(z',r') - (\bar{z},\bar{r})\| - \|(z,r) - (\bar{z},\bar{r})\| \\
&> 3\rho - \rho = 2\rho \\
&\ge \|(z,r) - (\bar{z},\bar{r})\| + \|(z'',r'') - (\bar{z},\bar{r})\| \\
&\ge \|(z'',r'') - (z,r)\| \\
&\ge d\Big((z,r), M(x) \cap ((\bar{z},\bar{r}) + 3\rho\mathbf{B})\Big).
\end{aligned}$$

One deduces that one has for any $x \in U$

$$d\Big((z,r), M(x)\Big) \ge d\Big((z,r), M(x) \cap ((\bar{z},\bar{r}) + 3\rho\mathbf{B})\Big),$$

and hence (the reverse inequality being obvious)

$$d\Big((z,r), M(x)\Big) = d\Big((z,r), M(x) \cap ((\bar{z},\bar{r}) + 3\rho\mathbf{B})\Big). \tag{4.38}$$

Fix now any neighborhood $X \times Z \times I \in \mathscr{N}(\bar{x},\bar{z},\bar{r})$ as in Proposition 4.1 and fix also $\rho > 0$ such that $(\bar{z},\bar{r}) + 3\rho\mathbf{B} \subset Y \times I$. Choose $\delta > 0$ and a neighborhood $H \in \mathscr{N}(\bar{h})$ such that $\bar{x} + (0,\delta)H \subset X \cap U$ (where U is the neighborhood of $\bar{x}$ given in what precedes) and

$$(\bar{z} + (0,\delta)\bar{k}) \times (\bar{r} + (0,\delta)\bar{s}) \subset (\bar{z},\bar{r}) + \rho\mathbf{B}.$$

Fix $t \in (0,\delta)$ and $h \in H$. Then, for all $z \in Z$ and $r \ge 0$ with

$$(z, f(\bar{x} + th, z) + r) \in (\bar{z},\bar{r}) + 3\rho\mathbf{B}$$

one has

$$\begin{aligned}\|(\bar z+t\bar k,f(\bar x,\bar z)+t\bar s)-(z,f(\bar x+th,z)+r)\| &= l\|\bar z+t\bar k-z\|+|f(\bar x+th,z)-f(\bar x,\bar z)-t\bar s+r|\\ &\geq l\|\bar z+t\bar k-z\|+f(\bar x+th,z)-f(\bar x,\bar z)-t\bar s+r\\ &\geq (l\|\bar z+t\bar k-z\|+f(\bar x+th,z))-f(\bar x,\bar z)-t\bar s.\end{aligned}$$

From the Lipschitz property of f in (ii) of Proposition 4.1, we deduce that

$$\|(\bar z+t\bar k,f(\bar x,\bar z)+t\bar s)-(z,f(\bar x+th,z)+r)\|\geq f(\bar x+th,\bar z+t\bar k)-f(\bar x,\bar z)-t\bar s.$$

So, (4.38) and this inequality imply that

$$\begin{aligned}t^{-1}d\Big((\bar z+t\bar k,\bar r+t\bar s),M(\bar x+th)\Big) &= t^{-1}d\Big((\bar z+t\bar k,\bar r+t\bar s),(Z\times I)\cap \text{epi } f(\bar x+th,\cdot)\Big)\\ &\geq t^{-1}[f(\bar x+th,\bar z+t\bar k)-f(\bar x,\bar z)]-\bar s,\end{aligned}$$

which gives (taking (ii) of Proposition 4.1 and the definition of $f^-(\cdot;\cdot)$ into account)

$$\Delta_M^-(\bar x,\bar z,\bar r;\bar h,\bar k,\bar s)\geq f^-(\bar x,\bar z;\bar h,\bar k)-\bar s. \tag{4.39}$$

To finish this step, it is enough because of (4.39) to show the following inequality

$$f^-(\bar x,\bar z;\bar h,\bar k)-\bar s\geq \Delta_{K,M}(\bar h,\bar k,\bar s)\quad \text{for all}\quad (\bar h,\bar k,\bar s)\notin K(\text{gph } M;\bar x,\bar z,\bar r).$$

Indeed, if we take $(\bar h,\bar k,\bar s)\in K(\text{gph } M;\bar x,\bar z,\bar r)$, then $(\bar k,\bar s)\in D_KM(\bar x,\bar z,\bar r)(\bar h)$ and hence $\Delta_{K,M}(\bar h,\bar k,\bar s)=0$, which is always not greater than $\Delta_M^-(\bar x,\bar z,\bar r;\bar h,\bar k,\bar s)$. Fix now $(\bar h,\bar k,\bar s)\notin K(\text{gph}M;\bar x,\bar z,\bar r)$. As $(X\times Z\times I)\cap \text{gph } M=(X\times Z\times I)\cap \text{epi } f$ by Proposition 2.1, we have

$$K(\text{gph } M;\bar x,\bar z,\bar r)=K(\text{epi } f;\bar x,\bar z,f(\bar x,\bar z))=\text{epi } f^-(\bar x,\bar z;\cdot,\cdot). \tag{4.40}$$

So $(\bar h,\bar k,\bar s)\notin \text{epi } f^-(\bar x,\bar z;\cdot,\cdot)$, that is $f^-(\bar x,\bar z;\bar h,\bar k)>\bar s$. This implies in particular that $|f^-(\bar x,\bar z;\bar h,\bar k)|<\infty$ because the first member of (4.39) has been supposed to be finite. Furthermore, one has

$$f^-(\bar x,\bar z;\bar h,\bar k)-\bar s=|f^-(\bar x,\bar z;\bar h,\bar k)-\bar s|+l\|\bar k-\bar k\|=\|(\bar k,\bar s)-(\bar k,f^-(\bar x,\bar z;\bar h,\bar k))\|.$$

By (4.40), we know that $(\bar k,f^-(\bar x,\bar z;\bar h,\bar k))$ lies in $D_KM(\bar x,\bar z,f(\bar x,\bar z))(\bar h)$, which allows to conclude that

$$f^-(\bar x,\bar z;\bar h,\bar k)-\bar s\geq d\Big((\bar k,\bar s),D_KM(\bar x,\bar z,f(\bar x,\bar z))(\bar h)\Big)=\Delta_{K,M}(\bar h,\bar k,\bar s).$$

This completes the proof of the second step.

Step 3. We finish the proof of the theorem by proving the following : If M is tangentially regular at $(\bar{x},\bar{z},\bar{r}) \in \operatorname{gph} M$, then the scalar function Δ_M (associated with the norm $|||\cdot|||$ in Step 2) is directionally regular at the same point. It is sufficient to prove the assertion above because the converse of this assertion holds for any norm on $G \times \mathbf{R}$.

Assume that M is tangentially regular at $(\bar{x},\bar{z},\bar{r}) \in \operatorname{gph} M$. Let $(\bar{h},\bar{k},\bar{s}) \in E \times G \times \mathbf{R}$. Then one has

$$\Delta_{T^{C},M}(\bar{h},\bar{k},\bar{s}) = \Delta_{K,M}(\bar{h},\bar{k},\bar{s}),$$

and by Steps 1 and 2 one has

$$\Delta_M^{\uparrow}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) \leq \Delta_{T^{C},M}(\bar{h},\bar{k},\bar{s}),$$

and

$$\Delta_M^{-}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) = \Delta_{K,M}(\bar{h},\bar{k},\bar{s}).$$

Thus , one gets

$$\begin{aligned}\Delta_M^{\uparrow}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) &\leq \Delta_{T^{C},M}(\bar{h},\bar{k},\bar{s})\\ &= \Delta_{K,M}(\bar{h},\bar{k},\bar{s})\\ &= \Delta_M^{-}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}).\end{aligned}$$

So,

$$\Delta_M^{\uparrow}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s}) = \Delta_M^{-}(\bar{x},\bar{z},\bar{r};\bar{h},\bar{k},\bar{s})$$

because the reverse inequality is always true. This completes the proof of the theorem. □

Corollary 4.3. *Let S be a nonempty closed subset of a normed vector space F and $\bar{x} \in S$. Assume that S is epi-Lipschitz at $\bar{x}$. Then the two following assertions are equivalent:*

(i) S is tangentially regular at $\bar{x}$;
(ii) there exists an equivalent norm on F such that the associated distance function d_S is directionally regular at $\bar{x}$.

Proof. The proof of this corollary is a direct application of Theorem 4.10 with $M(x) = S$ for all $x \in E$. □

4.5 Tangential Regularity of Images

Throughout this section E will be a normed vector space. It is well known that any convex set-valued mapping $M : E \rightrightarrows F$ (i.e. gph M is convex) has convex image sets $M(x)$, but the converse is not true.

In this section we establish a similar result for the tangential regularity. We show that if the graph of a set-valued mapping M with Aubin property, is tangentially regular, then it has tangentially regular image sets. The converse is obviously not true in the general setting.

Theorem 4.11. *Let $M : E \rightrightarrows F$ be a set-valued mapping defined between two normed vector spaces E and F and let $(\bar{x}, \bar{y}) \in$ gph M. Assume that M has Aubin property at $(\bar{x}, \bar{y})$. If M is tangentially regular at $(\bar{x}, \bar{y})$, then $M(\bar{x})$ is tangentially regular at $\bar{y}$.*

Proof. It is not difficult to see that the following assertions ensure the conclusion of the theorem:

(1) $(0,k) \in K(\text{gph } M; \bar{x}, \bar{y})$ if and only if $k \in K(M(\bar{x}); \bar{y})$;
(2) If $(0,k) \in T^{\mathrm{C}}(\text{gph } M; \bar{x}, \bar{y})$, then $k \in T^{\mathrm{C}}(M(\bar{x}); \bar{y})$.

So, we begin by showing the first one.

1. Fix $(0,k) \in K(\text{gph } M; \bar{x}, \bar{y})$. There exist a sequence $(h_n, k_n) \longrightarrow (0,k)$ in $E \times F$ and a sequence $t_n \downarrow 0$ such that

$$k_n \in t_n^{-1}[M(\bar{x} + t_n h_n) - \bar{y}] \quad \text{for all} \;\; n \in \mathbf{N}.$$

As M has Aubin property at $(\bar{x}, \bar{y})$, for n sufficiently large we have

$$k_n \in t_n^{-1}[M(\bar{x}) + t_n l \|h_n\| \mathbf{B} - \bar{y}],$$

where l is a Lipschitz constant of M as in Definition 4.2. So, there exists a sequence $b_n \in \mathbf{B}$ such that for n sufficiently large

$$\bar{y} + t_n(k_n + l\|h_n\| b_n) \in M(\bar{x}).$$

By taking $w_n = k_n + l\|h_n\| b_n$, which converges to k, we conclude that $k \in K(M(\bar{x}); \bar{y})$.

Assume now that $k \in K(M(\bar{x}); \bar{y})$. Then there exist sequences $k_n \longrightarrow k$ and $t_n \downarrow 0$ such that

$$\bar{y} + t_n k_n \in M(\bar{x}) \;\; \text{i.e.} \;\; (\bar{x} + t_n h_n, \bar{y} + t_n k_n) \in \text{gph } M, \;\; \text{with } h_n = 0.$$

Thus, $(0,k) \in K(\text{gph } M; (\bar{x}, \bar{y}))$, which completes the proof of the first assertion.

2. Using the sequential characterization of the Clarke tangent cone in Proposition 1.6 and the same techniques in the proof of (1), it is not difficult to show the second assertion (2). □

4.6 On the Distance Function to Images Around Points Outside the Graph

For a set-valued mapping $M : E \rightrightarrows F$ and a given real number $r \geq 0$ we define the *r-enlargement* set-valued mapping $M_r : E \rightrightarrows F$ by $M_r(x) := \{y \in F : \ \Delta_M(x,y) \leq r\}$. It is obvious to see that, $\operatorname{gph} M_r = \{(x,y) \in E \times F : \Delta_M(x,y) \leq r\}$ and the 0-enlargement set-valued mapping M_0 coincides with M, whenever M is a closed valued mapping.

In this section, we present general results on the scalar function Δ_M. We establish some relationships between a set-valued mapping $M : E \rightrightarrows F$ and its associated r-enlargement set-valued mapping M_r, where $r := \Delta_M(\bar{x},\bar{y})$ and $(\bar{x},\bar{y}) \in E \times F$. An important result in this section is a characterization of the Clarke (resp. Bouligand) tangent cone to $\operatorname{gph} M_r$ in terms of the generalized directional derivative (resp. lower Dini directional derivative) of Δ_M for any closed set-valued mapping $M : E \rightrightarrows F$ defined from a Hausdorff topological vector space E into a normed vector space F. First, we state the following proposition needed in the sequel. Its proof is straightforward and hence it is omitted.

Proposition 4.2. *Let M be any set-valued mapping from E into F with $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in F$. Then*

$$\Delta_M^-(\bar{x},\bar{y};\bar{h},\bar{k}) = \liminf_{\substack{h \to \bar{h} \\ t \downarrow 0}} t^{-1} \Big[\Delta_M(\bar{x}+th,\bar{y}+t\bar{k}) - \Delta_M(\bar{x},\bar{y})\Big],$$

for all $(\bar{h},\bar{k}) \in E \times F$.

The following theorem provides:

- A relationship between the lower Dini directional derivatives of Δ_M and of Δ_{M_r} at $(\bar{x},\bar{y})$ and the distance function Δ_{K,M_r} associated with the K-derivative of M_r at the same point $(\bar{x},\bar{y})$.
- A description of the contingent cone to $\operatorname{gph} M_r$ in terms of the functions Δ_M and of Δ_{M_r}.

Theorem 4.12. *Let $M : E \rightrightarrows F$ be a set-valued mapping with $(\bar{x},\bar{y}) \in E \times F$. Then,*

1. for all $(\bar{h},\bar{k}) \in E \times F$

$$\Delta_M^-(\bar{x},\bar{y};\bar{h},\bar{k}) \leq \Delta_{M_r}^-(\bar{x},\bar{y};\bar{h},\bar{k}) \leq \Delta_{K,M_r}(\bar{h},\bar{k});$$

2.
$$\begin{aligned} K(\text{gph } M_r;\bar{x},\bar{y}) &= \{(h,k)\in E\times F:\ \ \Delta_M^-(\bar{x},\bar{y};h,k)\le 0\} \\ &= \{(h,k)\in E\times F:\ \ \Delta_{M_r}^-(\bar{x},\bar{y};h,k)=0\}. \end{aligned}$$

Proof.

1. By Theorem 4.5 one always has
$$\Delta_{M_r}^-(\bar{x},\bar{y};\bar{h},\bar{k})\le \Delta_{K,M_r}(\bar{h},\bar{k}),$$
we will show the first inequality, i.e.,
$$\Delta_M^-(\bar{x},\bar{y};\bar{h},\bar{k})\le \Delta_{M_r}^-(\bar{x},\bar{y};\bar{h},\bar{k}) \quad \text{for all} \quad (\bar{h},\bar{k})\in E\times F.$$
To this end, it is sufficient (by Proposition 4.2) to claim the following:
$$\Delta_M(x,y)-\Delta_M(\bar{x},\bar{y})\le \Delta_{M_r}(x,y)-\Delta_{M_r}(\bar{x},\bar{y}) \quad \text{for all} \quad (x,y)\in E\times F. \tag{4.41}$$
Fix any $(x,y)\in E\times F$, any $z\in M_r(x)$, and any $\varepsilon>0$. As $d(z,M(x))=\Delta_M(x,z)\le r$, one can find $z'\in M(x)$ such that $\|z-z'\|\le r+\varepsilon$. Thus,
$$\begin{aligned} \Delta_M(x,y)=d(y,M(x)) &\le d(z,M(x))+\|z-y\| \\ &\le \|z-z'\|+\|z-y\| \\ &\le \|z-y\|+r+\varepsilon=\|z-y\|+\Delta_M(\bar{x},\bar{y})+\varepsilon. \end{aligned}$$
Therefore, as z is taken arbitrary in $M_r(x)$, one gets
$$\Delta_M(x,y)-\Delta_M(\bar{x},\bar{y})\le d(y,M_r(x))+\varepsilon=\Delta_{M_r}(\bar{x},\bar{y})+\varepsilon,$$
for any $\varepsilon>0$. This ensures (4.41) and hence the proof of the first part of the theorem is complete.
2. By the first part, one has the following inclusions
$$\begin{aligned} K(\text{gph } M_r;\bar{x},\bar{y}) &\subset \{(h,k)\in E\times F:\ \ \Delta_{M_r}^-(\bar{x},\bar{y};h,k)=0\} \\ &\subset \{(h,k)\in E\times F:\ \ \Delta_M^-(\bar{x},\bar{y};h,k)\le 0\}. \end{aligned}$$
So, we will show the following one
$$\{(h,k)\in E\times F:\ \ \Delta_M^-(\bar{x},\bar{y};h,k)\le 0\}\subset K(\text{gph } M_r;\bar{x},\bar{y}). \tag{4.42}$$
Fix any (h,k) in $E\times F$ with $\Delta_M^-(\bar{x},\bar{y};h,k)\le 0$. Consider any real positive numbers ε and δ and any $H\in\mathcal{N}(h)$. Then, by Proposition 4.2, we have
$$\inf_{\substack{0<t<\delta\\ h'\in H}} t^{-1}\Big[\Delta_M(\bar{x}+th',\bar{y}+tk)-\Delta_M(\bar{x},\bar{y})\Big]\le \frac{\varepsilon}{2}.$$

So, there exist $h_0 \in H$ and $t_0 \in (0,\delta)$ such that

$$t_0^{-1}\Big[\Delta_M(\bar{x}+t_0h_0,\bar{y}+t_0k)-\Delta_M(\bar{x},\bar{y})\Big]<\varepsilon.$$

Therefore,

$$d(\bar{y}+t_0k, M(\bar{x}+t_0h_0)) < r+t_0\varepsilon,$$

and hence, we may choose some $k_0 \in F$ with $\bar{y}+t_0k_0 \in M(\bar{x}+t_0h_0)$ and $\|k_0 - k\| < \varepsilon + rt_0^{-1}$. Then there exists $b \in \mathbf{B}_F$ such that $k_0 = k+\varepsilon b + t_0^{-1}rb$. Therefore

$$\Delta_M(\bar{x}+t_0h_0,\bar{y}+t_0(k+\varepsilon b)) = \Delta_M(\bar{x}+t_0h_0,\bar{y}+t_0k_0-rb) \le r,$$

which ensures that

$$(\bar{x}+t_0h_0,\bar{y}+t_0(k+\varepsilon b)) = (\bar{x},\bar{y})+t_0(h_0,k+\varepsilon b) \in \operatorname{gph} M_r.$$

This shows that $(h,k) \in K(\operatorname{gph} M_r;\bar{x},\bar{y})$, which completes the proof of the inclusion (4.42). So, the proof of the theorem is finished. □

An important result can be deduced from the first part of the above theorem, by taking $M(x) = S$ for every $x \in E$. We give it in the following corollary generalizing Theorem 11 in [71]. Note that in Theorem 11 of [71] the authors only showed the direct inclusion of the first equality of this corollary.

Corollary 4.4. *Let S be a nonempty closed subset of F. Then for every $\bar{y} \notin S$, one has*

$$K(S(r);\bar{y}) = \{k \in F: \ d_S^-(\bar{y};k) \le 0\} = \Big\{k \in F: \ d_{S(r)}^-(\bar{y};k) = 0\Big\},$$

where $S(r) := \{y \in F: \ d_S(y) \le r\}$ and $r := d_S(\bar{y})$.

Now, we turn to establish similar results for the Clarke tangent cone to $\operatorname{gph} M_r$ at a given point $(\bar{x},\bar{y}) \in E \times F$ and the generalized directional derivative of Δ_M at the same point $(\bar{x},\bar{y})$.

Theorem 4.13. *Let $M: E \rightrightarrows F$ be any set-valued mapping defined from E into F and let $(\bar{x},\bar{y}) \in E \times F$. Then*

(a)

$$\{(h,k) \in E\times F: \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le 0\} \subset T^{\mathrm{C}}(\operatorname{gph} M_r;\bar{x},\bar{y}). \tag{4.43}$$

If Δ_M is directionally regular at $(\bar{x},\bar{y})$, then equality holds in (4.43) and M_r is tangentially regular at $(\bar{x},\bar{y})$.

(b) *Assume that E is locally convex and $\partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y}) \neq \emptyset$. Then one always has*

$$\Big(\{(h,k) \in E\times F: \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k) \le 0\}\Big)^0 = \mathrm{cl}_{w^*}\Big(\mathbf{R}_+\partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y})\Big).$$

Proof.

(a) Let any $(\bar{x},\bar{y})$ and any (h,k) in $E\times F$ with $\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)\le 0$. Consider any positive real number ε and any $H\in\mathcal{N}(\bar{h})$. By the definition of $\Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)$, there exist $X\times Y\in\mathcal{N}(\bar{x},\bar{y})$ and $\delta>0$ such that for all $(x,y,\alpha)\in X\times Y\times(r-\delta,r+\delta))\cap\mathrm{epi}\,\Delta_M$ and for all $t\in(0,\delta)$ there are $h\in H$ and $k\in\bar{k}+\frac{\varepsilon}{4}\mathbf{B}_F$

$$\Delta_M(x+th,y+tk)-\alpha\le t\frac{\varepsilon}{4}. \tag{4.44}$$

Fix any $(x,y)\in(X\times Y)\cap\mathrm{gph}\,M_r$, any $t\in(0,\delta)$. As $(x,y)\in\mathrm{gph}\,M_r$, i.e., $\Delta_M(x,y)\le r$, one has $(x,y,\Delta_M(\bar{x},\bar{y}))\in(X\times Y\times(r-\delta,r+\delta))\cap\mathrm{epi}\,\Delta_M$. So, by (4.44) there exist $h\in H$ and $k\in\bar{k}+\frac{\varepsilon}{4}\mathbf{B}_F$ such that

$$\Delta_M(x+th,y+tk)<t\frac{\varepsilon}{2}+r,\quad \text{i.e.,}\quad d(y+tk,M(x+th))<t\frac{\varepsilon}{2}+r.$$

Hence, we can take $k'\in F$ with $y+tk'\in M(x+th)$ and $\|k-k'\|\le\frac{\varepsilon}{2}+t^{-1}r$. Then, there exists $b\in\mathbf{B}_F$ such that $k'=k''+t^{-1}rb$ with $k''=k+\frac{\varepsilon}{2}b$, and hence

$$\Delta_M(x+th,y+tk'')\le\Delta_M(x+th,y+tk')+t\|k''-k'\|=t\|k''-k'\|\le r,$$

which ensures that $(x,y)+t(h,k'')\in\mathrm{gph}\,M_r$, with $\|k''-\bar{k}\|\le\varepsilon$. Therefore, for every $(x,y)\in(X\times Y)\cap\mathrm{gph}\,M_r$ and every $t\in(0,\delta)$ there exists $(h,k'')\in H\times(\bar{k}+\varepsilon\mathbf{B}_F)$ such that

$$(x,y)+t(h,k'')\in\mathrm{gph}\,M_r.$$

This ensures that $(\bar{h},\bar{k})\in T^{\mathrm{C}}(\mathrm{gph}\,M_r;\bar{x},\bar{y})$, which completes the proof of (4.43).

Now, assume that Δ_M is directionally regular at $(\bar{x},\bar{y})$, i.e., $\Delta_M^{-}(\bar{x},\bar{y};\cdot,\cdot)=\Delta_M^{\uparrow}(\bar{x},\bar{y};\cdot,\cdot)$. Then one has

$$\begin{aligned}K(\mathrm{gph}\,M_r;\bar{x},\bar{y})&=\{(h,k)\in E\times F:\ \ \Delta_M^{-}(\bar{x},\bar{y};h,k)\le 0\}\ \ (\text{by Theorem 4.12})\\&=\{(h,k)\in E\times F:\ \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)\le 0\}\ \ (\text{dir. reg.})\\&\subset T(\mathrm{gph}\,M_r;\bar{x},\bar{y})\ \ (\text{by (4.43)})\\&\subset K(\mathrm{gph}\,M_r;\bar{x},\bar{y}).\end{aligned}$$

This ensures the equality of all the sets above and the tangential regularity of M_r at $(\bar{x},\bar{y})$.

(b) We denote $T:=\{(h,k)\in E\times F:\ \ \Delta_M^{\uparrow}(\bar{x},\bar{y};h,k)\le 0\}$. First note that the second member is included in the first one. Indeed, consider (x^*,y^*) in $\partial^{\mathrm{C}}\Delta_M(\bar{x},\bar{y})\neq\emptyset$, then (by the definition) one has

$$\left\langle (x^*, y^*), (h,k) \right\rangle \le \Delta_M^{\uparrow}(\bar{x}, \bar{y}; h, k),$$

for all $(h,k) \in E \times F$. Therefore,

$$\left\langle (x^*, y^*), (h,k) \right\rangle \le 0, \quad \text{for all} \quad (h,k) \in T,$$

which ensures that $(x^*, y^*) \in T^0$.

Now, we return to prove the reverse inclusion. As the function $(h,k) \mapsto \Delta_M^{\uparrow}(\bar{x}, \bar{y}; h, k)$ is lower semi-continuous and sublinear, the set T is a closed convex cone and hence the bipolar $(T^0)^0$ coincides with T, i.e., $(T^0)^0 = T$. Therefore, we will prove that the polar cone of $\mathrm{cl}_{w^*}(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}))$ is included in T.

Note first that $\left(\mathrm{cl}_{w^*}(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}))\right)^0 = \left(\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})\right)^0$. So, consider any $(h,k) \in \left(\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})\right)^0$. Then we have

$$\left\langle (x^*, y^*), (h,k) \right\rangle \le 0, \quad \text{for all} \ (x^*, y^*) \in \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}),$$

and hence (since $\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}) \neq \emptyset$)

$$\Delta_M^{\uparrow}(\bar{x}, \bar{y}; h, k) = \sup\{\left\langle (x^*, y^*), (h,k) \right\rangle : \quad (x^*, y^*) \in \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})\} \le 0.$$

This ensures that $(h,k) \in T$ and the proof of the second part of the theorem is complete. □

Corollary 4.5. *Let M be any set-valued mapping defined from E into F. Assume that E is locally convex.*

(a) *If* $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$, *then*

$$T^{\mathrm{C}}(\mathrm{gph}\, M; \bar{x}, \bar{y}) = \{(h,k) \in E \times F : \ \Delta_M^{\uparrow}(\bar{x}, \bar{y}; h, k) = 0\},$$

and

$$N^{\mathrm{C}}(\mathrm{gph}\, M; \bar{x}, \bar{y}) = \mathrm{cl}_{w^*}(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})).$$

(b) *If* $(\bar{x}, \bar{y}) \notin \mathrm{gph}\, M$ *with* $\partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y}) \neq \emptyset$ *and* Δ_M *is directionally regular at* $(\bar{x}, \bar{y})$, *then*

$$T^{\mathrm{C}}(\mathrm{gph}\, M_r; \bar{x}, \bar{y}) = \{(h,k) \in E \times F : \ \Delta_M^{\uparrow}(\bar{x}, \bar{y}; h, k) \le 0\},$$

and

$$N^{\mathrm{C}}(\mathrm{gph}\, M_r; \bar{x}, \bar{y}) = \mathrm{cl}_{w^*}(\mathbf{R}_+ \partial^{\mathrm{C}} \Delta_M(\bar{x}, \bar{y})).$$

The next results in this section depend upon the non-emptiness of the set $\mathrm{Proj}_{M(x)}(y) := \{y' \in F : \ \Delta_M(x,y) = \|y-y'\|\}$. When F is a finite dimensional space, $\mathrm{Proj}_{M(x)}(y)$ is always nonempty as long as $M(x)$ is nonempty and closed. We begin with the following preliminary result. In this proposition, $\Delta_M'(\bar{x},\bar{y};h,k)$ will denote the directional derivative of Δ_M at $(\bar{x},\bar{y})$ in the direction (h,k), i.e., $\Delta_M'(\bar{x},\bar{y};h,k) = \lim_{t\downarrow 0} t^{-1}[\Delta_M(\bar{x}+th,\bar{y}+tk) - \Delta_M(\bar{x},\bar{y})]$, whenever this limit exists.

Proposition 4.3. *Let $M : E \rightrightarrows F$ be any set-valued mapping with $(\bar{x},\bar{y}) \notin \mathrm{gph}\, M$ and let $\tilde{y} \in \mathrm{Proj}_{M(\bar{x})}(\bar{y})$. Then,*

$$(1-t)\Delta_M(\bar{x},\bar{y}) = \Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y})), \tag{4.45}$$

for all $t \in [0,1]$. Moreover, $\Delta_M'(\bar{x},\bar{y};0,\tilde{y}-\bar{y})$ exists and equals $-\Delta_M(\bar{x},\bar{y})$ and we have

$$\partial^-\Delta_M(\bar{x},\bar{y}) \subset E^* \times \{y^* \in F^* : \ \|y^*\| = 1\}.$$

Proof. To see (4.45) simply observe that

$$\begin{aligned}
\Delta_M(\bar{x},\bar{y}) &= \Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y})-t(\tilde{y}-\bar{y}))\\
&\le \Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y}))+t\|\tilde{y}-\bar{y}\|\\
&= \Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y}))+t\Delta_M(\bar{x},\bar{y}),
\end{aligned}$$

and so $(1-t)\Delta_M(\bar{x},\bar{y}) \le \Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y}))$. Conversely,

$$\begin{aligned}
\Delta_M(\bar{x},\bar{y}+t(\tilde{y}-\bar{y})) &= \Delta_M(\bar{x},\tilde{y}+(1-t)(\bar{y}-\tilde{y}))\\
&\le \Delta_M(\bar{x},\tilde{y})+|1-t|\|\bar{y}-\tilde{y}\|\\
&= (1-t)\Delta_M(\bar{x},\bar{y}).
\end{aligned}$$

The fact that $\Delta_M'(\bar{x},\bar{y};0,\tilde{y}-\bar{y})$ exists and equals $-\Delta_M(\bar{x},\bar{y})$ follows immediately from (4.45). We turn now to show that for every $(x^*,y^*) \in \partial^-\Delta_M(\bar{x},\bar{y})$ one has $\|y^*\| = 1$. As the inequality $\|y^*\| \le 1$ always holds, we will prove the reverse inequality, i.e., $\|y^*\| \ge 1$. Let $t_n \downarrow 0$ be a sequence achieving the limit in the definition of $\Delta_M^-(\bar{x},\bar{y};0,\tilde{y}-\bar{y})$. Then,

$$\begin{aligned}
\Delta_M^-(\bar{x},\bar{y};0,\tilde{y}-\bar{y}) &= \lim_n t_n^{-1}[\Delta_M(\bar{x},\bar{y}+t_n(\tilde{y}-\bar{y}))-\Delta_M(\bar{x},\bar{y})]\\
&= \lim_n t_n^{-1}[(1-t_n)\Delta_M(\bar{x},\bar{y})-\Delta_M(\bar{x},\bar{y})] \quad (\text{by } (4.45))\\
&= -\Delta_M(\bar{x},\bar{y}).
\end{aligned}$$

By the definition of the Dini subdifferential, one gets

$$\left\langle (x^*, y^*), (0, \tilde{y} - \bar{y}) \right\rangle \le \Delta_M^-(\bar{x}, \bar{y}; 0, \tilde{y} - \bar{y}) = -\Delta_M(\bar{x}, \bar{y}) = -\|\tilde{y} - \bar{y}\|,$$

and so $\left\langle y^*, \dfrac{\bar{y} - \tilde{y}}{\|\tilde{y} - \bar{y}\|} \right\rangle \ge 1$, which ensures that $\|y^*\| \ge 1$ and hence the proof is finished. □

The following proposition is a direct consequence of the Part (b) in Corollary 4.5 and the above proposition. It extends Theorem 2.16 due to Burke et al. proved in [71] from sets to set-valued mapping.

Proposition 4.4. *Let $M : E \rightrightarrows F$ and $(\bar{x}, \bar{y}) \notin \mathrm{gph}\, M$. Assume that E is locally convex and that Δ_M is directionally regular at $(\bar{x}, \bar{y})$ and $\mathrm{Proj}_{M(\bar{x})}(\bar{y}) \neq \emptyset$. Then one has*

$$\partial^C \Delta_M(\bar{x}, \bar{y}) \subset N^C(\mathrm{gph}\, M_r; \bar{x}, \bar{y}) \cap \left(E^* \times \{y^* \in F^* : \quad \|y^*\| = 1\}\right).$$

We finish our results in this section with the following theorem.

Theorem 4.14. *Let $M : E \rightrightarrows F$ and $(\bar{x}, \bar{y}) \notin \mathrm{gph}\, M$. Assume that $\mathrm{Proj}_{M(\bar{x})}(\bar{y}) \neq \emptyset$. Then one has*

$$K(\mathrm{gph}\, M; \bar{x}, \tilde{y}) \subset K(\mathrm{gph}\, M_r; \bar{x}, \bar{y}), \quad \text{for every } \tilde{y} \in \mathrm{Proj}_{M(\bar{x})}(\bar{y}).$$

If, furthermore, M_r is tangentially regular at $(\bar{x}, \bar{y})$, then the inclusion holds for the Clarke tangent cone, that is,

$$T^C(\mathrm{gph}\, M; \bar{x}, \tilde{y}) \subset T^C(\mathrm{gph}\, M_r; \bar{x}, \bar{y}), \quad \text{for every } \tilde{y} \in \mathrm{Proj}_{M(\bar{x})}(\bar{y}).$$

Proof. We begin by showing the first inclusion. Fix any $(\bar{x}, \bar{y}) \notin \mathrm{gph}\, M$ with $\mathrm{Proj}_{M(\bar{x})}(\bar{y}) \neq \emptyset$ and any $\tilde{y} \in \mathrm{Proj}_{M(\bar{x})}(\bar{y})$. Fix also $(\bar{h}, \bar{k}) \in K(\mathrm{gph}\, M; \bar{x}, \tilde{y})$. Consider $V \times W \in \mathcal{N}(0,0)$ in $E \times F$ and $\lambda > 0$. There exists $t \in (0, \lambda)$ such that

$$\left[(\bar{x}, \tilde{y}) + t(V \times W + (\bar{h}, \bar{k}))\right] \cap \mathrm{gph}\, M \neq \emptyset.$$

Then there is $(v, w) \in V \times W$ such that $(\bar{x}, \tilde{y}) + t(v + \bar{h}, w + \bar{k}) \in \mathrm{gph}\, M$, i.e., $\tilde{y} + tw + t\bar{k} \in M(\bar{x} + tv + t\bar{h})$. Hence, $\bar{y} + tw + t\bar{k} \in M(\bar{x} + tv + t\bar{h}) + \bar{y} - \tilde{y}$.
Therefore,

$$\Delta_M(\bar{x} + tv + t\bar{h}, \bar{y} + tw + t\bar{k}) \le \|\bar{y} - \tilde{y}\| = \Delta_M(\bar{x}, \bar{y}),$$

which ensures that $(\bar{x} + tv + t\bar{h}, \bar{y} + tw + t\bar{k}) \in \mathrm{gph}\, M_r$ and hence

$$\left[(\bar{x}, \bar{y}) + t(V \times W + (\bar{h}, \bar{k}))\right] \cap \mathrm{gph}\, M_r \neq \emptyset.$$

This proves that $(\bar{h}, \bar{k}) \in K(\mathrm{gph}\, M_r; \bar{x}, \bar{y})$.

The second inclusion is an immediate consequence of the first one and the tangential regularity of M_r. □

Remark 4.4.

1. In Theorem 4.13, we have shown that for every $(\bar{x},\bar{y}) \in E \times F$, the set-valued mapping M_r (with $r := \Delta_M(\bar{x},\bar{y})$) is tangentially regular at $(\bar{x},\bar{y})$, whenever the scalar function Δ_M is directionally regular at $(\bar{x},\bar{y})$. The converse, in general, is not true. Indeed, we have showed in Sect. 4.3 that when $(\bar{x},\bar{y})$ lies in gph M and *dim* $F = +\infty$, the converse may fail. When $(\bar{x},\bar{y}) \notin$ gph M, the converse will not be valid even in the finite dimensional setting. Consider the constant set-valued mapping $M : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ defined by $M(x) := \{y \in \mathbf{R}^n : \quad \|y\| \geq 1\}$, for all $x \in \mathbf{R}^n$. It is not difficult to see that Δ_M is not directionally regular at $(\bar{x},0) \notin$ gph M (for all $\bar{x} \in \mathbf{R}^n$) while M_r (with $r := \Delta_M(\bar{x},0) = 1$) is tangentially regular at $(\bar{x},0)$.
2. The two inclusions in the statement of Theorem 4.14 may be strict, even in the finite dimensional case and under the directional regularity of Δ_M. To see this, it is enough to take M as in Example 4.1 below.
3. Note also that, in general, when $(\bar{x},\bar{y}) \notin$ gph M, the set-valued mapping M_r is better behaved than the set-valued mapping M, that is, M_r can be (even in the finite dimensional case) tangentially regular at $(\bar{x},\bar{y})$, but M is not tangentially regular at $(\bar{x},\tilde{y}) \in$ gph M, for every $\tilde{y} \in \text{Proj}_{M(\bar{x})}(\bar{y})$. In order to give the reader some insight into the problem, we use the Example 18 of [71].

Example 4.1 Let M be a constant set-valued mapping from any Hausdorff topological vector space E into $\mathbf{R}^2$ defined by $M(x) = S$ for any $x \in E$, where

$$S := \{(y_1,y_2) \in \mathbf{R}^2 : \quad y_2 = y_1 \sin(1/y_1), y_1 > 0\} \cup \{(0,0)\}.$$

Put $(\bar{x},\bar{y}) := (\bar{x},(-\bar{y}_1,0)) \in E \times \mathbf{R}^2$ with $\bar{y}_1 > 0$. Then one has $(\bar{x},\bar{y}) \notin$ gph $M = E \times S$ and

$$M_r(x) = S(r) := \{(y_1,y_2) \in \mathbf{R}^2 : \ \ \Delta_M(x,(y_1,y_2)) = d_S((y_1,y_2)) \leq r\},$$

with $r := \Delta_M(x,(-\bar{y}_1,0)) = d_S((-\bar{y}_1,0)) = \bar{y}_1$. It is not difficult to check that d_S is directionally regular at $\bar{y} = (-\bar{y}_1,0)$ and hence Δ_M is also directionally regular at $(\bar{x},\bar{y})$. This ensures by Theorem 4.13 that M_r is tangentially regular at $(\bar{x},\bar{y})$. On the other hand, it is easy to see that $\text{Proj}_{M(\bar{x})}(\bar{y}) = \{\tilde{y}\}$, with $\tilde{y} := (0,0)$ and M is not tangentially regular at $(\bar{x},\tilde{y})$.

4.7 Application of Δ_M: Calmness and Exact Penalization

Our primary goal in the present section is to make clear that the scalar function Δ_M can also be a powerful tool in the study of the calmness property of optimization problems. Recall that this concept was developed by [86, 88] (while suggested by

Rockafellar; see[86] page 172[1]). It was always required to establishing the existence of multipliers. Several variations of the original definition have been proposed and studied (see, for instance, [69, 70, 251, 260]). Here, we are interested in the one used by Burke [69, 70]. We will adapt his definition for a general perturbed problem with a constraint defined by a set-valued mapping and we will prove that it is equivalent to the existence of an exact penalization in terms of the distance function associated with the set-valued mapping defining the constraint of the problem. Like in [70], we compare this definition of calmness with the one given in Definition 6.4.1 in [88].

Consider the problem $(\mathscr{P})$, which consists in minimizing the function f over all $x \in E$ satisfying $0 \in M(x)$,

$$(\mathscr{P}) \qquad \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & 0 \in M(x), \end{cases}$$

where $M : E \rightrightarrows F$ is a closed set-valued mapping between two normed vector spaces E and F and $f : E \to \mathbf{R} \cup \{+\infty\}$ is an extended real-valued function. We begin with the definition of calmness.

Definition 4.4. Let f, M, E, and F be as in the statement of $(\mathscr{P})$ and consider the following perturbed problem

$$(\mathscr{P}_y) \qquad \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & y \in M(x). \end{cases}$$

Let $S : F \rightrightarrows E$ be the *feasible set-valued mapping* associated with $(\mathscr{P}_y)$, i.e.,

$$S(y) := \{x \in dom\ f : \quad y \in M(x)\}$$

and let $V : F \to \mathbf{R} \cup \{-\infty, +\infty\}$ be the value function for the family $(\mathscr{P}_y)$, i.e.,

$$V(y) := \inf\{f(x) : \quad y \in M(x)\}.$$

Let $(\bar{x}, \bar{y}) \in gph\ S$.

1. Following Burke [70], we will say that the problem $(\mathscr{P}_{\bar{y}})$ is *calm* at $\bar{x}$ if there are constants $\bar{\alpha} \geq 0$ and $\varepsilon > 0$ such that for every pair $(x, y) \in \text{gph } M$ with $x \in \bar{x} + \varepsilon \mathbf{B}_E$ we have
$$f(\bar{x}) \leq f(x) + \bar{\alpha}\|y - \bar{y}\|. \tag{4.46}$$
The constants $\bar{\alpha}$ and ε are called the *modulus of calmness* and *radius of calmness* for $(\mathscr{P}_{\bar{y}})$ at $\bar{x}$, respectively.
2. Following Clarke [88], the family of the perturbed problems $(\mathscr{P}_y)$ will be said to be *calm* if its value function satisfies $\liminf_{y \to 0} \dfrac{V(y) - V(0)}{\|y\|} > -\infty$.

[1]This remark has been communicated to us by B. Mordukhovich.

Remark 4.5.

1. Observe that if $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$, then $\bar{x}$ is necessarily a local solution to $(\mathscr{P}_{\bar{y}})$.
2. If the family $(\mathscr{P}_y)$ is calm, then $(\mathscr{P}_0)$ is calm at any point of its solution set.

For any problem $(\mathscr{P}_y)$ and any real number $\alpha \geq 0$ we will associate the function $P_{\alpha,y}$ defined by

$$P_{\alpha,y}(x) := f(x) + \alpha \Delta_M(x,y).$$

In the following theorem we state our main result in this section. It establishes a relationship between the calmness property and the existence of an exact penalization of the general perturbed problem $(\mathscr{P}_{\bar{y}})$ in terms of the distance function to images associated with the set-valued mapping M defining the constraint of the problem.

Theorem 4.15. *Let $(\bar{x},\bar{y}) \in gph\ S$. Then $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$ with modulus $\bar{\alpha}$ and radius ε if and only if $\bar{x}$ is a local minimum of radius ε for the function $P_{\alpha,\bar{y}}$, for all $\alpha \geq \bar{\alpha}$, that is, $P_{\alpha,\bar{y}}(\bar{x}) \leq P_{\alpha,\bar{y}}(x)$, for all $x \in \bar{x} + \varepsilon \mathbf{B}_E$ and all $\alpha \geq \bar{\alpha}$.*

Proof. $(\Rightarrow)$ Let $\delta > 0$. Given any $x \in \bar{x} + \varepsilon \mathbf{B}_E \cap \operatorname{dom} M \neq \emptyset$ such that $\|y - \bar{y}\| \leq d(\bar{y}, M(x)) + \delta$ with $y \in M(x)$. Thus, if $\alpha \geq \bar{\alpha}$, we obtain from the calmness hypothesis that

$$\begin{aligned} f(\bar{x}) &\leq f(x) + \alpha \|y - \bar{y}\| \\ &\leq f(x) + \alpha d(\bar{y}, M(x)) + \alpha \delta. \end{aligned}$$

Hence,

$$P_{\alpha,\bar{y}}(\bar{x}) = f(\bar{x}) \leq P_{\alpha,\bar{y}}(x) + \alpha \delta.$$

Since $\delta > 0$ was taken arbitrarily, the implication is established.

$(\Leftarrow)$ Let $(x,y) \in \operatorname{gph} M$ with $x \in \bar{x} + \varepsilon \mathbf{B}_E$. Then,

$$\begin{aligned} f(\bar{x}) = P_{\bar{\alpha},\bar{y}}(\bar{x}) &\leq P_{\bar{\alpha},\bar{y}}(x) \\ &= f(x) + \bar{\alpha} \Delta_M(x,\bar{y}) \\ &\leq f(x) + \bar{\alpha} \Delta_M(x,y) + \bar{\alpha} \|y - \bar{y}\| \\ &= f(x) + \bar{\alpha} \|y - \bar{y}\|. \end{aligned}$$

Hence, $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$ and the proof of the theorem is complete. □

The proof given above is an adaptation of the proof of Theorem 1.1 in [70]. In which, the author established the same result for a particular perturbed problem: $(\mathscr{P}_y)$ with $M(x) = g(x) - C$.

Recall now that in the definition of calmness used by Clarke [88], the variable y is required to satisfy $\|y-\bar{y}\| \le \varepsilon$ for inequality (4.46) to hold. In the following result we compare that definition to Definition 4.4.

Theorem 4.16. *Let $(\bar{x},\bar{y}) \in gph\ S$ and $\bar{\alpha}, \varepsilon > 0$. Assume that M is lower-Hausdorff semi-continuous at $(\bar{x},\bar{y})$ in the following sense, for any $\varepsilon' > 0$, there exists a neighborhood $X \times Y$ of $(\bar{x},\bar{y})$ such that $M(\bar{x}) \cap Y \subset M(x) + \varepsilon' \mathbf{B}_F$, for each $x \in X \cap \operatorname{dom} M$. If the problem $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$ with modulus $\bar{\alpha}$ and radius $\varepsilon > 0$ in the sense of Clarke [88], that is, for every pair $(x,y) \in \operatorname{gph} M$ with $x \in \bar{x} + \varepsilon \mathbf{B}_E$ and $y \in \bar{y} + \varepsilon \mathbf{B}_F$ we have*

$$f(\bar{x}) \le f(x) + \bar{\alpha}\|y-\bar{y}\|,$$

then there is $\bar{\varepsilon} \in (0,\varepsilon]$ such that $\bar{x}$ is a local minimum of radius $\bar{\varepsilon}$ for the function $P_{\bar{\alpha},\bar{y}}$, and consequently, $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$ with modulus $\bar{\alpha}$ and radius $\bar{\varepsilon} > 0$ in the sense of Definition 4.4.

Proof. Let $\delta \in (0, \frac{1}{2})$ and $\alpha \ge \bar{\alpha}$. Since M is lower Hausdorff semi-continuous at $(\bar{x},\bar{y})$, there exists $\bar{\varepsilon} \in (0,\varepsilon]$ such that, for $Y := \bar{y} + 3\bar{\varepsilon}\mathbf{B}_E$, one has

$$M(\bar{x}) \cap Y \subset M(x) + \frac{\varepsilon}{2}\mathbf{B}_F,$$

for every $x \in \bar{x} + \bar{\varepsilon}\mathbf{B}_E$. We can easily show that for all $x \in \bar{x} + \bar{\varepsilon}\mathbf{B}_E$ and all $y \in \bar{y} + \bar{\varepsilon}\mathbf{B}_F$ we have $\Delta_M(\bar{x},y) = d(y, M(\bar{x}) \cap Y)$. Consequently,

$$\Delta_M(\bar{x},y) \ge d\left(y, M(x) + \frac{\varepsilon}{2}\mathbf{B}_F\right) \ge d(y, M(x)) - \frac{\varepsilon}{2}.$$

Now, given $x \in (\bar{x} + \bar{\varepsilon}\mathbf{B}_E) \cap \operatorname{dom} M$ and choose $y_x \in M(x)$ such that

$$\|y_x - \bar{y}\| \le d(\bar{y}, M(x)) + \delta\bar{\varepsilon}.$$

Then, $(x, y_x) \in \operatorname{gph} M$ with $y_x \in \bar{y} + \varepsilon \mathbf{B}_F$. Indeed

$$\|y_x - \bar{y}\| \le d(\bar{y}, M(x)) + \delta\bar{\varepsilon} \le d(\bar{y}, M(\bar{x})) + \frac{\varepsilon}{2} + \delta\bar{\varepsilon} \le \varepsilon.$$

Thus,

$$\begin{aligned} f(\bar{x}) &\le f(x) + \bar{\alpha}\|y_x - \bar{y}\| \\ &\le f(x) + \alpha\|y_x - \bar{y}\| \\ &\le f(x) + \alpha d(\bar{y}, M(x)) + \alpha\delta\bar{\varepsilon}. \end{aligned}$$

Taking the limit as $\delta \downarrow 0$ one gets

$$P_{\alpha,\bar{y}}(\bar{x}) = f(\bar{x}) \le f(x) + \alpha \Delta_M(x,\bar{y}) = P_{\alpha,\bar{y}}(x),$$

for all $x \in (\bar{x} + \bar{\varepsilon}\mathbf{B}_E) \cap \mathrm{dom}\, M$. As $P_{\alpha,\bar{y}}(x) = +\infty$ whenever $x \notin \mathrm{dom}\, M$, then the last inequality still holds for all $x \in \bar{x} + \bar{\varepsilon}\mathbf{B}_E$, which means that $\bar{x}$ is a local minimum with radius $\bar{\varepsilon}$ for $P_{\alpha,\bar{y}}$. The fact that the problem $(\mathscr{P}_{\bar{y}})$ is calm at $\bar{x}$ with modulus $\bar{\alpha}$ and radius $\bar{\varepsilon}$ follows immediately from Theorem 4.15. □

4.8 Commentary to Chap. 4

Set-valued mappings are of special interest in variational analysis and optimization. The regularity concept of set-valued mappings is needed, in particular, to analyze the behavior of sets of feasible and optimal solutions to constraint and variational systems with respect to parameter perturbations. This chapter is devoted to the study of some concepts of regularity for set-valued mappings. First, we prove some important results on the distance to images around points on the graph (Sect. 4.2) and outside the graph (Sect. 4.6). These results are used (Sects. 4.3–4.5) to study the possible relationships between different types of regularity for set-valued mappings. The importance of the regularity of set-valued mappings is shown in Sect. 4.7 by giving an application to the study of calmness property to set-valued mappings.

The results stated in this chapter are proved in [44, 45, 59, 63]. A different type of regularity for set-valued mapping is the so called *metric regularity*. This concept has been used successfully in nonlinear analysis and its applications, especially to optmization and related problems. This concept is completely different from the ones studied in our book. We refer the reader to the books [192, 193] of Mordukhovich for a complete study of that concept and its applications. For more studies on set-valued theory and its applications to parametric variational Analysis, the reader is referred to the books [8, 9, 38, 79, 103, 192, 193, 241, 244] and the references therein, as well as to the following list of papers: [6, 7, 28, 35, 41, 44, 59, 62, 63, 69–71, 111, 126, 133, 136, 137, 139, 142, 149–151, 168, 169, 173, 176, 177, 179, 181, 183, 184, 190, 195–200, 203–206, 226, 233, 238, 242, 247, 260].

Part III
Applications of Nonsmooth Analysis Theory

Chapter 5
First Order Differential Inclusions

5.1 Nonconvex Sweeping Processes and Nonconvex Differential Inclusions

5.1.1 Introduction

In this section, we study, nonconvex sweeping processes. We consider the following differential inclusion:

$$(P_1) \qquad \begin{cases} \dot{x}(t) \in -N^{\#}(C(t);x(t)), & \text{a.e.} \quad t \geq 0, \\ x(0) = x_0 \in C(0), \;\; x(t) \in C(t), & \forall \;\; t \geq 0, \end{cases}$$

where C is an absolutely set-valued mapping taking its values in Hilbert spaces, that is,

$$|d_{C(t)}(y) - d_{C(t')}(y)| \leq |v(t) - v(t')|, \tag{5.1}$$

where $v : \mathbf{R} \to \mathbf{R}$ is an absolutely continuous function, and $N^{\#}(C(t);x(t))$ denotes a prescribed normal cone to the set $C(t)$ at $x(t)$. The problem (P_1) is the so-called "*sweeping process problem*" (in French, rafle). It was introduced by Moreau in [207, 208] and studied intensively by himself in many papers (see for example [207–209]). This problem is related to the modelization of elasto-plastic materials (for more details see [210, 211]). The existence of solutions of (P_1) was resolved by Moreau in [209] for convex-valued mappings C taking their values in general Hilbert spaces. In [255, 256], Valadier proved for the first time the existence of solutions of (P_1) without convexity assumptions on C for some particular cases in the finite dimensional setting. Since, many authors have attacked the study of the existence of solutions for nonconvex sweeping processes (see, for instance, [16, 58, 61, 75, 92, 248] and the references therein). The next subsection is mainly concerned with the following problem: Under which conditions the solution set of (P_1) can be related to the solution set of the following convex compact differential inclusion (P_2)?

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$$(P_2) \qquad \begin{cases} \dot{x}(t) \in -|\dot{v}(t)|\partial^{\#} d_{C(t)}(x(t)), & \text{a.e. } t \geq 0, \\ x(0) = x_0 \in C(0), \end{cases}$$

where v is an absolutely continuous function and $\partial^{\#} d_{C(t)}(\cdot)$ stands for a prescribed subdifferential of the distance function $d_{C(t)}$ associated with the set $C(t)$.

This problem was considered by Thibault in [251] for convex-valued mappings C in the finite dimensional setting. His idea was to use the existence results for differential inclusions with convex compact values which is the case for (P_2) to prove existence results of the sweeping process (P_1). It is interesting to point out that his approach is new and different from those used by the authors who have studied the existence of solutions of the sweeping process (P_1).

5.1.2 Equivalence Between Nonconvex Sweeping Process and a Particular Nonconvex Differential Inclusion

Our main purpose of this subsection is to show, for a large class of set-valued mappings, that the solution set of the two following differential inclusions are the same:

$$(P_1) \qquad \begin{cases} \dot{x}(t) \in -N^{\mathrm{C}}(C(t);x(t)), & \text{a.e.} \quad t \geq 0, & (1) \\ x(0) = x_0 \in C(0), & & (2) \\ x(t) \in C(t), & \forall t \geq 0, & (3) \end{cases}$$

and

$$(P_2) \qquad \begin{cases} \dot{x}(t) \in -|\dot{v}(t)|\partial^{\mathrm{C}} d_{C(t)}(x(t)), & \text{a.e. } t \geq 0, & (4) \\ x(0) = x_0 \in C(0), & & (2) \end{cases}$$

i.e., a mapping $x(\cdot) : [0,+\infty) \to \mathbf{H}$ is a solution of (P_1) if and only if it is a solution of (P_2).

It is easy to see that one always has $(P_2)+(3) \Rightarrow (P_1)$. Indeed, let $x(\cdot) : [0,+\infty) \to \mathbf{H}$ be a solution of (P_2) satisfying (3). Then a.e. $t \geq 0$ we have

$$\dot{x}(t) \in -|\dot{v}(t)|\partial^{\mathrm{C}} d_{C(t)}(x(t)) \subset -N^{\mathrm{C}}(C(t);x(t)),$$

and hence $x(\cdot)$ is a solution of (P_1).

The use of (P_2) as an intermediate problem to prove existence results of the sweeping process (P_1) is due to Thibault [251]. His idea was to use the existence results for differential inclusions with compact convex values which is the case of the problem (P_2) to prove an existence result of the sweeping process (P_1). Note that all the authors (for example [16, 92]), who have studied the sweeping process

(P_1), have attacked it by direct methods for example by proving the convergence of the Moreau catching-up algorithm or by using some measurable arguments and new versions of the well known theorem of Scorza–Dragoni.

Recall that, Thibault [251] showed that, when C has closed convex values in a finite dimensional space $\mathbf{H}$, any solution of (P_2) is also a solution of (P_1) and as (P_2) has always at least one solution by Theorem VI. 13 in [79], then he obtained the existence of solutions of the convex sweeping process (P_1) in the finite dimensional setting. His idea is to show the viability of all solutions of (P_2), i.e., any solution of (P_2) satisfies (3) and so it is a solution of (P_1) by using the implication $(P_2)+(3) \Rightarrow (P_1)$. Recently, Thibault in [248] used the same idea to extend this result to the proximal smooth case.

In this section, we will follow this idea to extend his result in [251] to the nonconvex case by using powerful results by Borwein et al. [31] and recent results by Bounkhel and Thibault [58] (see also Sect. 2.6). We begin with the following theorem.

Theorem 5.1. *Any solution of (P_1) with the Fréchet normal cone satisfies the inequality $\|\dot{x}(t)\| \le |\dot{v}(t)|$ a.e. $t \ge 0$.*

Proof. Let $x(\cdot) : [0,+\infty) \to \mathbf{H}$ be an absolutely continuous solution of (P_1) with the Fréchet normal cone, that is, $-\dot{x}(t) \in \widehat{N}(C(t); x(t))$ a.e. $t \ge 0$, $x(0) = x_0 \in C(0)$, and $x(t) \in C(t) \quad \forall t \ge 0$. Fix any $t \ge 0$ for which $\dot{x}(t)$ and $\dot{v}(t)$ exist and fix also $\varepsilon > 0$. If $\dot{x}(t) = 0$, then we are done, so let suppose that $\dot{x}(t) \neq 0$. By the definition of the Fréchet normal cone, there exists $\delta := \delta(t,\varepsilon)$ such that

$$\big\langle -\dot{x}(t), x - x(t) \big\rangle \le \varepsilon \|x - x(t)\| \quad \forall x \in (x(t) + \delta\mathbf{B}) \cap C(t). \tag{5.2}$$

On the other hand there exists a mapping $\theta : \mathbf{R}_+ \to \mathbf{H}$ such that $\lim_{r \to 0^+} \theta(r) = 0$ and $x(t-r) = x(t) - r\dot{x}(t) - r\theta(r)$, for r small enough. Fix now $r > 0$ small enough such that $0 < r < \min\{1, \dfrac{\delta}{3\|\dot{x}(t)\|}\}$, $\|\theta(r)\| \le \delta/3$ and $|v(t-r) - v(t)| \le \delta/3$. By (5.1) and (3) one has $x(t-r) \in C(t-r) \subset C(t) + |v(t-r) - v(t)|\mathbf{B}$. So there exists $x_t \in C(t)$ and $b_t \in \mathbf{B}$ such that $x(t-r) = x_t - \xi_t$ where $\xi_t = |v(t-r) - v(t)|b_t$. Therefore $x_t = x(t-r) + \xi_t = x(t) - r\dot{x}(t) - r\theta(r) + \xi_t \in (x(t) + \delta\mathbf{B}) \cap C(t)$, since $\|x_t - x(t)\| = \| - r\dot{x}(t) - r\theta(r) + \xi_t\| \le \|\dot{x}(t)\| + \|\theta(r)\| + \|\xi_t\| \le \delta/3 + \dfrac{\delta}{3} + |v(t-r) - v(t)| \le \delta$. Thus, by (5.2)

$$\big\langle -\dot{x}(t), -r\dot{x}(t) - r\theta(r) + \xi_t \big\rangle \le \varepsilon \|r\dot{x}(t) + r\theta(r) - \xi_t\|$$

and hence

$$r\big\langle -\dot{x}(t), -\dot{x}(t) - \theta(r) + r^{-1}\xi_t \big\rangle \le \varepsilon r \Big[\|\dot{x}(t) + \theta(r)\| + r^{-1}|v(t-r) - v(t)| \Big]$$

and so

$$\begin{aligned}\langle \dot{x}(t), \dot{x}(t)\rangle &\leq \langle -\dot{x}(t), \theta(r) - r^{-1}\xi_t\rangle + \varepsilon\Big[\|\dot{x}(t)+\theta(r)\| + r^{-1}|v(t-r)-v(t)|\Big]\\ &\leq \|\dot{x}(t)\|\Big[\|\theta(r)\| + r^{-1}|v(t-r)-v(t)|\Big]\\ &\quad +\varepsilon\Big[\|\dot{x}(t)+\theta(r)\| + r^{-1}|v(t-r)-v(t)|\Big].\end{aligned}$$

By letting $\varepsilon, r \to 0^+$, one gets $\|\dot{x}(t)\|^2 \leq \|\dot{x}(t)\||\dot{v}(t)|$ and then $\|\dot{x}(t)\| \leq |\dot{v}(t)|$. This completes the proof. □

The following corollary generalizes Theorem 5.1 of Colombo et al. [92].

Corollary 5.1. *Assume that $C(t)$ is Fréchet normally regular for every $t \geq 0$. Then any solution of (P_1) satisfies the inequality $\|\dot{x}(t)\| \leq |\dot{v}(t)|$ a.e. $t \geq 0$.*

We recall from Chap. 1 the following notion of regularity of sets.

Definition 5.1. Let S be a nonempty closed subset of $\mathbf{H}$ and let x be a point in S. We will say that S is *Fréchet normally regular* at x if one has $\widehat{N}(S;x) = N^{\mathrm{C}}(S;x)$.

We summarize from Chap. 2, in the following proposition, some results needed in the rest of the present section.

Proposition 5.1. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $x \in S$. Then,*

(i) $\widehat{\partial} d_S(x) = \widehat{N}(S;x) \cap \mathbf{B}_*$;
(ii) If S is Fréchet normally regular at x, then it is tangentially regular at x. If, in addition, $\mathbf{H}$ is a finite dimensional space, then one has the equivalence.

Note that in the infinite dimensional setting, one can construct subsets that are tangentially regular but not Fréchet normally regular (see Chap. 1).

Now, we prove that, under the Fréchet normal regularity assumption, any solution of (P_1) must be a solution of (P_2).

Theorem 5.2. *Assume that $C(t)$ is Fréchet normally regular for every $t \geq 0$. Then any solution of (P_1) is also a solution of (P_2).*

Proof. Let $x(\cdot)$ be a solution of (P_1), that is, $x(0) = x_0 \in C(0)$, $x(t) \in C(t) \quad \forall t \geq 0$ and $-\dot{x}(t) \in N^{\mathrm{C}}(C(t);x(t))$ a.e. $t \geq 0$. Then, by the Fréchet normal regularity one has $-\dot{x}(t) \in N^{\mathrm{C}}(C(t);x(t)) = \widehat{N}(C(t);x(t))$ a.e. $t \geq 0$. By Theorem 5.1 one has $\|\dot{x}(t)\| \leq |\dot{v}(t)|$ a.e. $t \geq 0$. If $\dot{x}(t) = 0$, then $-\dot{x}(t) \in |\dot{v}(t)|\partial^{\mathrm{C}} d_{C(t)}(x(t))$, because $x(t) \in C(t)$. So we assume that $\dot{x}(t) \neq 0$ (and hence $\dot{v}(t) \neq 0$). Then, by Proposition 1.1 (i), one gets

$$\frac{-\dot{x}(t)}{|\dot{v}(t)|} \in \widehat{N}(C(t);x(t)) \cap \mathbf{B}_* = \widehat{\partial} d_{C(t)}(x(t)) \subset \partial^{\mathrm{C}} d_{C(t)}(x(t)).$$

Thus, $\dot{x}(t) \in -|\dot{v}(t)|\partial^{\mathrm{C}} d_{C(t)}(x(t))$, which ensures that $x(\cdot)$ is a solution of (P_2) and so the proof is finished. □

Now we proceed to prove the converse of Theorem 5.2, for a large class of set-valued mappings. We recall (see Chap. 1) the notion of Gâteaux directional differentiability. A locally Lipschitz function $f:\mathbf{H}\to\mathbf{R}$ is *directionally Gâteaux differentiable* at $\bar{x}\in\mathbf{H}$ in the direction $v\in\mathbf{H}$ if $\lim\limits_{t\to 0} t^{-1}[f(\bar{x}+tv)-f(\bar{x})]$ exists. We call such a limit the Gâteaux directional derivative of f at $\bar{x}$ in the direction v and we denote it by $\nabla_G f(\bar{x};v)$. When this limit exists for all $v\in\mathbf{H}$ and is linear in v we will say that f is Gâteaux differentiable at $\bar{x}$ and the Gâteaux derivative satisfies $\nabla_G f(\bar{x};v)=\langle\nabla_G f(\bar{x}),v\rangle$ for all $v\in\mathbf{H}$. We recall (see Chap. 1) that f is *directionally regular* at $\bar{x}$ in a direction $v\in\mathbf{H}$ provided that the generalized directional derivative $f^0(\bar{x};v)$ of f at $\bar{x}$ in the direction v coincides with $f^-(\bar{x};v)$ the lower Dini directional derivative of f at $\bar{x}$ in the same direction v.

Theorem 5.3. *Let $h:[0,+\infty)\to(0,+\infty)$ be a positive function. Assume that for every absolutely continuous mapping $x(\cdot):[0,+\infty)\to\mathbf{H}$ the following property (A) is satisfied: for a.e. $t\geq 0$ and for any $x(t)$ in the tube $U(h(t)):=\{u\in\mathbf{H}:\ \ 0<d_{C(t)}(u)<h(t)\}$ one has*

i) $\mathrm{Proj}_{C(t)}(x(t))\neq\emptyset$ *and $d_{C(t)}$ is directionally regular at $x(t)$ in both directions $\dot{x}(t)$ and $p(x(t))-x(t)$ for some $p(x(t))\in\mathrm{Proj}_{C(t)}(x(t))$.*

Then every solution z of (P_2) in $C(t)+h(t)\mathbf{B}$ for all $t\geq 0$ must lie in C(t) for all $t\geq 0$.

Before giving the proof of Theorem 5.3, we prove the following Lemmas.

Lemma 5.1. *Let S be a closed nonempty subset of* $\mathbf{H}$ *and u is any point outside S such that* $\mathrm{Proj}_S(u)\neq\emptyset$. *Assume that d_S is directionally regular at u in the direction $\bar{u}-u$, for some $\bar{u}\in\mathrm{Proj}_S(u)$. Then,* $\partial^C d_S(u)\subset\{\xi\in\mathbf{H}:\ \ \|\xi\|=1\}$.

Proof. Fix any $u\notin S$ with $\mathrm{Proj}_S(u)\neq\emptyset$ and any $\xi\in\partial^C d_S(u)$. As the inequality $\|\xi\|\leq 1$ always holds, we will prove the reverse inequality, i.e., $\|\xi\|\geq 1$. Firstly, we fix $\bar{u}\in\mathrm{Proj}_S(u)\neq\emptyset$ and we show that

$$(1-\delta)d_S(u)=d_S(u+\delta(\bar{u}-u)),\quad \text{for all}\quad \delta\in[0,1]. \tag{5.3}$$

Observe that one always has

$$d_S(u)\leq d_S(u+\delta(\bar{u}-u))+\delta\|\bar{u}-u\|=d_S(u+\delta(\bar{u}-u))+\delta d_S(u),$$

and so $(1-\delta)d_S(u)\leq d_S(u+\delta(\bar{u}-u))$. Conversely,

$$d_S(u+\delta(\bar{u}-u))=d_S(\bar{u}+(1-\delta)(u-\bar{u}))\leq(1-\delta)\|\bar{u}-u\|=(1-\delta)d_S(u).$$

Now, let δ_n be a sequence achieving the limit in the definition of $d_S^-(u;\bar{u}-u)$ the lower Dini directional derivative of d_S at u in the direction $\bar{u}-u$. Then, by (5.3), one gets

$$\begin{aligned}d_S^-(u;\bar{u}-u)&=\lim_n \delta_n^{-1}[d_S(u+\delta_n(\bar{u}-u))-d_S(u)]\\&=\lim_n \delta_n^{-1}[(1-\delta_n)d_S(u)-d_S(u)],\end{aligned}$$

and hence $d_S^-(u;\bar{u}-u) = -d_S(u)$. Finally, by the directional regularity of d_S at u in the direction $\bar{u}-u$ and by the definition of the generalized gradient one gets

$$\langle \xi, \bar{u}-u \rangle \leq d_S^0(u;\bar{u}-u) = d_S^-(u;\bar{u}-u) = -d_S(u) = -\|\bar{u}-u\|,$$

and so

$$\left\langle \xi, \frac{u-\bar{u}}{\|\bar{u}-u\|} \right\rangle \geq 1,$$

which ensures that $\|\xi\| \geq 1$. □

The following lemma is a direct consequence of Corollary 9 in [31]. We give its proof for the convenience of the reader.

Lemma 5.2. *Let S be a closed nonempty subset of* $\mathbf{H}$, *$u \notin S$ and $v \in \mathbf{H}$. Then the following are equivalent:*

1. $\langle \partial^{\mathrm{C}} d_S(u), v \rangle = \{d_S^0(u;v)\}$;
2. *d_S is directionally regular at u in the direction v;*
3. *d_S is Gâteaux differentiable at u in the direction v.*

Proof. The equivalence between (1) and (3) is given in [31]. The implication $(1) \Rightarrow (2)$ is obvious. So we proceed to proving the reverse one, i.e., $(2) \Rightarrow (1)$. By Theorem 8 in [31] one has $-d_S$ is directionally regular at u, hence $(-d_S)^0(u,v) = (-d_S)^-(u,v)$ and hence $d_S^0(u,-v) = -d_S^-(u,v)$. By (1) one has $d_S^0(u,v) = d_S^-(u,v)$. Therefore, one obtains $d_S^0(u,-v) = -d_S^0(u,v)$. Now, as we can easily check that $\langle \partial^{\mathrm{C}} d_S(u), v \rangle = [-d_S^0(u,-v), d_S^0(u,v)]$, then one gets $\langle \partial^{\mathrm{C}} d_S(u), v \rangle = \{d_S^0(u;v)\}$. This completes the proof of the lemma. □

Proof of Theorem 5.3. We prove the theorem for all $t \in [0,1]$ and we can extend the proof to $[0,+\infty)$ in the evident way by considering next the interval $[1,2]$, etc. We follow the proof of Proposition II.18 in Thibault [251]. Let z be a solution of (P_2) satisfying $z(t) \in C(t) + h(t)\mathbf{B}$ for all $t \in [0,1]$. Consider the real function f defined by $f(t) = d_{C(t)}(z(t))$. The function f is absolutely continuous because of (5.1). Put $\Omega := \{t \in [0,1] : z(t) \notin C(t)\}$. Ω is an open subset in $[0,1]$ because $\Omega = \{t \in [0,1] : f(t) > 0\}$. Assume by contradiction that $\Omega \neq \emptyset$. As $0 \notin \Omega$ there exists an interval $(\alpha,\beta) \subset \Omega$ such that $f(\alpha) = 0$ (it suffices to take (α,β) any connected component of $(0,1) \cap \Omega$). Since f, v and z are absolutely continuous, then their derivatives exist a.e. on $[0,1]$. Fix any $t \in (\alpha,\beta)$ such that $\dot{f}(t), \dot{v}(t)$ and $\dot{z}(t)$ exist. Observe that for such t and for every $\delta > 0$ we have

$$\begin{aligned}
\delta^{-1}[f(t+\delta) - f(t)] &= \delta^{-1}[d_{C(t+\delta)}(z(t+\delta)) - d_{C(t)}(z(t))] \\
&= \delta^{-1}[d_{C(t+\delta)}(z(t) + \delta\dot{z}(t) + \delta\varepsilon(\delta)) - d_{C(t+\delta)}(z(t) + \delta\dot{z}(t))] \\
&\quad + \delta^{-1}[d_{C(t+\delta)}(z(t) + \delta\dot{z}(t)) - d_{C(t)}(z(t) + \delta\dot{z}(t))] \\
&\quad + \delta^{-1}[d_{C(t)}(z(t) + \delta\dot{z}(t)) - d_{C(t)}(z(t))],
\end{aligned}$$

where $\varepsilon(\delta) \to 0^+$ as $\delta \to 0^+$ and hence

$$\delta^{-1}[f(t+\delta) - f(t)] \le \varepsilon(\delta) + \delta^{-1}|v(t+\delta) - v(t)| + \delta^{-1}[d_{C(t)}(z(t)+\delta\dot{z}(t)) - d_{C(t)}(z(t))].$$

Thus, for such t we have

$$\dot{f}(t) \le |\dot{v}(t)| + \limsup_{\delta\to 0^+} \delta^{-1}[d_{C(t)}(z(t)+\delta\dot{z}(t)) - d_{C(t)}(z(t))] \le |\dot{v}(t)| + d^0_{C(t)}(z(t);\dot{z}(t)).$$

Now, as z is a solution of (P_2) we have $\frac{-\dot{z}(t)}{|\dot{v}(t)|} \in \partial^{\mathrm{C}} d_{C(t)}(z(t))$ and hence

$$\left\langle \frac{-\dot{z}(t)}{|\dot{v}(t)|}, \dot{z}(t) \right\rangle \in \langle \partial^{\mathrm{C}} d_{C(t)}(z(t)), \dot{z}(t) \rangle = \{d^0_{C(t)}(z(t);\dot{z}(t))\} \quad \text{(by Lemma 5.2)}.$$

On the other hand as $z(t) \in U(h(t))$ and by hypothesis (A) and Lemma 5.1 one gets $\frac{\|-\dot{z}(t)\|}{|\dot{v}(t)|} = 1$ and hence $\|\dot{z}(t)\| = |\dot{v}(t)|$. Therefore,

$$d^0_{C(t)}(z(t);\dot{z}(t)) = -\left\langle \frac{\dot{z}(t)}{|\dot{v}(t)|}, \dot{z}(t) \right\rangle = -\frac{\|\dot{z}(t)\|^2}{|\dot{v}(t)|} = -|\dot{v}(t)|.$$

Now, for such $t \in (\alpha,\beta)$ we have $\dot{f}(t) \le 0$. So, as f is absolutely continuous we have $f(\theta) = f(\alpha) + \int_\alpha^\theta \dot{f}(t)\mathrm{d}t \le 0$ for every $\theta \in (\alpha,\beta)$. But by the definition of f we have $f(\theta) \ge 0$ for every θ. Thus, $f(\theta) = 0$ which contradicts that $(\alpha,\beta) \subset \Omega$. Hence, $\Omega = \emptyset$. This completes the proof. □

Now, we have the following corollary.

Corollary 5.2. *Put* $h(t) := 2\int_0^t |\dot{v}(s)|\mathrm{d}s$ *and assume that the hypothesis* (A) *holds. Then for every solution* z *of* (P_2)*, one has* $z(t) \in C(t)$ *for all* $t \ge 0$.

Proof. It is sufficient to show that every solution of (P_2) satisfies the hypothesis (A). Indeed, let z be a solution of (P_2). Then for a.e. $t \ge 0$ one has $\|\dot{z}(t)\| \le |\dot{v}(t)|$. So, by (5.1) one gets

$$d_{C(t)}(z(t)) \le \|z(t) - z(0)\| + |v(t) - v(0)| \le \int_0^t |\dot{v}(s)|\mathrm{d}s + \int_0^t \|\dot{z}(s)\|\mathrm{d}s \le h(t).$$

This ensures that $z(t) \in C(t) + h(t)\mathbf{B}$. □

Using Corollary 5.2 one gets the non-emptiness of the set of solutions of both problems (P_1) and (P_2) in the finite dimensional setting and that these two sets of solutions are the same. Note that this result is more strongly than the existence results of the problem (P_1) proved in [16, 92], because it is not necessary that a solution of (P_1) is in general a solution of (P_2). Note also that their existence results for the problem (P_1) have been obtained, respectively, for any Lipschitz set-valued mapping C taking its values in a finite dimensional space, and for any Lipschitz set-valued mapping C having locally compact graph and taking its values in a Hilbert space. Their proofs are strongly based on new versions of Scorza–Dragoni's theorem.

Theorem 5.4. *Assume that* $\dim \mathbf{H} < +\infty$ *and the hypothesis* (A) *holds with* $h(t) := 2\int_0^t |\dot{v}(s)|\mathrm{d}s$*. Then both problems* (P_1) *and* (P_2) *have the same set of solutions which is nonempty.*

Proof. By corollary 5.2 and the implication $(P_2) + (3) \Rightarrow (P_1)$, it is sufficient to show that (P_2) admits at least one solution. Indeed, we put $f_t(x) := -|\dot{v}(t)|d_{C(t)}(x)$ and we observe that this function satisfies all hypothesis of Lemma II.15 in Thibault [251] (we can apply directly Theorem VI.13 in Castaing and Valadier [79] as in the lemma I.15 in [251]). Then one gets by this lemma that (P_2) admits at least one solution. □

In order to give a concrete application of our abstract result in Theorem 5.3, we recall the definition of uniform prox-regularity for subsets which is a generalization of convex subsets as it was defined in Clarke [89] (see Chap. 1 for other characterization of uniform prox-regularity).

Definition 5.2. Let S be a closed nonempty subset in $\mathbf{H}$. Following Clarke et al. [89] we will say that S is uniformly r-prox-regular if d_S is continuously Gâteaux differentiable on the tube $U(r) := \{u \in \mathbf{H} : \quad 0 < d_S(u) < r\}$.

Corollary 5.3. *Put* $h(t) := 2\int_0^t |\dot{v}(s)|\mathrm{d}s$ *and assume that* $C(t)$ *is uniformly* $r(t)$*-prox-regular for all* $t \geq 0$ *with* $h(t) \leq r(t)$*. Then for every solution* z *of* (P_2)*, one has* $z(t) \in C(t)$ *for all* $t \geq 0$*.*

Proof. It is easily seen by Lemma 5.2 that under the uniform $r(t)$-prox-regularity of $C(t)$ for all $t \geq 0$ with $h(t) \leq r(t)$, the hypothesis (A) holds. So, we can directly apply Corollary 5.2. □

We close this section by establishing the following result. It is the combination of Theorem 5.2 and Corollary 5.2. It proves the equivalence between (P_1) and (P_2) for any set-valued mapping C satisfying the following hypothesis (A'):

$\ll$ Given a positive function $h : [0,+\infty) \to (0,+\infty)$. For every absolutely continuous mapping $x(\cdot) : [0,+\infty) \to \mathbf{H}$ and for a.e. $t \geq 0$ the two following assertions hold:

1. $C(t)$ is Fréchet normally regular at $x(t) \in C(t)$;
2. for every $x(t) \in U(h(t))$: $\mathrm{Proj}_{C(t)}(x(t)) \neq \emptyset$, $d_{C(t)}$ is directionally regular at $x(t)$ in both directions $\dot{x}(t)$ and $p(x(t)) - x(t)$, for some $p(x(t)) \in \mathrm{Proj}_{C(t)}(x(t))$. $\gg$

Theorem 5.5. *Assume that (A') holds with $h(t) := 2\int_0^t |\dot{v}(s)|\mathrm{d}s$. Then, (P_1) is equivalent to (P_2).*

Remark 5.1. Note that under the uniform $r(t)$-prox-regularity of $C(t)$ for all $t \geq 0$ with $h(t) \leq r(t)$, we can show (see Clarke et al. [89] for the part 1 in (A')) that the hypothesis (A') holds too. So we obtain the following result, also obtained in [248].

Theorem 5.6. *Put $h(t) := 2\int_0^t |\dot{v}(s)|\mathrm{d}s$ and assume that $C(t)$ is uniformly $r(t)$-prox-regular for all $t \geq 0$ with $h(t) \leq r(t)$. Then, (P_1) is equivalent to (P_2).*

5.1.3 Existence Results: Finite Dimensional Case

Throughout this subsection, $\mathbf{H}$ will be a finite dimensional space. Our aim here is to prove the existence of solutions to (P_1) and (P_2) by a new and a direct method and under another hypothesis which is incomparable in general with the hypothesis (A) given in the previous section.

We begin by recalling the following proposition (see e.g. [90])

Proposition 5.2. *Let X be a finite dimensional space. Let $F : X \rightrightarrows X$ be an upper semicontinuous set-valued mapping with compact convex images and let $S \subset \mathrm{dom}\, F$ be a closed subset in X. Then the two following assertions are equivalent:*

(i) $\forall x \in S$, $\forall p \in \mathrm{Proj}_S(x)$, $\sigma\Big(F(x), -p\Big) \geq 0$;
(ii) $\forall x_0 \in S$, $\exists$ *a solution* $x(\cdot) : [0, +\infty) \to \mathbf{H}$ *of the differential inclusion* $\dot{x}(t) \in F(x(t))$ *a.e.* $t \geq 0$ *such that* $x(0) = x_0$ *and* $x(t) \in S$ *for all* $t \geq 0$.

Here $\mathrm{Proj}_S(x)$ *denotes the set of all vectors* $\xi \in \mathbf{H}$ *such that* $d_S(x + \xi) = \|\xi\|$.

We prove the following result that is the key of the proof of Theorem 5.7.

Lemma 5.3. *Let $C : \mathbf{R}_+ \rightrightarrows \mathbf{H}$ be a set-valued mapping satisfying (5.1). For all $(t, x) \in \mathrm{gph}\, C$ and all $(q, p) \in \mathbf{R}_+ \widehat{\partial} \Delta_C(t, x)$ one has*

$$\sigma(F(t, x), -(q, p)) \geq 0,$$

for the set-valued mapping $F : \mathbf{R}_+ \times \mathbf{H} \rightrightarrows \mathbf{R}_+ \times \mathbf{H}$ defined by $F(t, x) := \{1\} \times \{-\beta(t)\partial^{\mathrm{C}} d_{C(t)}(x)\}$, where $\beta : \mathbf{R}_+ \to \mathbf{R}_+$ is any positive function satisfying $|\dot{v}(t)| \leq \beta(t)$ a.e. $t \geq 0$. Here $\Delta_C : \mathbf{R}_+ \times \mathbf{H} \to \mathbf{R}_+$ denotes the distance function to images associated with C and defined by $\Delta_C(t, x) := d_{C(t)}(x)$.

Proof. It is sufficient to show the inequality above for only $(q,p) \in \widehat{\partial}\Delta_C(t,x)$. Assume the contrary. There exist $(\bar{t},\bar{x}) \in \operatorname{gph} C$ and $(\bar{q},\bar{p})(\neq (0,0)) \in \widehat{\partial}\Delta_C(\bar{t},\bar{x})$ such that

$$\sigma(F(\bar{t},\bar{x}),-(\bar{q},\bar{p})) < 0. \tag{5.4}$$

Fix $\varepsilon > 0$. By the definition of the Fréchet subdifferential there exists $\eta > 0$ such that for all $|t-\bar{t}| \le \eta$, and all $\|x-\bar{x}\| \le \eta$ one has

$$\bar{q}(t-\bar{t}) + \langle \bar{p}, x-\bar{x}\rangle \le d_{C(t)}(x) + \varepsilon(|t-\bar{t}| + \|x-\bar{x}\|). \tag{5.5}$$

Taking $t=\bar{t}$ in (5.5) one obtains $\bar{p} \in \widehat{\partial} d_{C(\bar{t})}(\bar{x})$.

By (5.1) there exists for any $t \in \mathbf{R}_+$, some $x_t \in C(t)$ such that

$$\|x_t - \bar{x}\| \le |v(t) - v(\bar{t})|.$$

Taking now $x = x_t$ in (5.5) for all t sufficiently near to $\bar{t}$ one gets

$$\begin{aligned}\bar{q}(t-\bar{t}) &\le \langle -\bar{p}, x_t - \bar{x}\rangle + \varepsilon(|t-\bar{t}| + \|x_t - \bar{x}\|)\\ &\le \|\bar{p}\||v(t)-v(\bar{t})| + \varepsilon(|t-\bar{t}| + |v(t)-v(\bar{t})|),\end{aligned}$$

and hence

$$|\bar{q}| \le \|\bar{p}\||\dot{v}(t)| \le \|\bar{p}\|\beta(t). \tag{5.6}$$

If $\bar{p}=0$, then $\bar{q}=0$, which is impossible. Assume that $\bar{p}\neq 0$, then $\dfrac{\bar{p}}{\|\bar{p}\|} \in \widehat{\partial} d_{C(\bar{t})}(\bar{x})$, which ensures that

$$\left(1, -\beta(\bar{t})\frac{\bar{p}}{\|\bar{p}\|}\right) \in F(\bar{t},\bar{x}).$$

Thus, by (5.4) one gets

$$\left\langle \left(1, -\beta(\bar{t})\frac{\bar{p}}{\|\bar{p}\|}\right), -(\bar{q},\bar{p})\right\rangle < 0$$

and hence $\|\bar{p}\|\beta(\bar{t}) < |\bar{q}|$, which contradicts (5.6). This completes the proof. □

Now, we are ready to prove the main result of this section.

Theorem 5.7. *Assume that there is a continuous function $\beta : [0,+\infty) \to [0,+\infty)$ satisfying $|\dot{v}(t)| \le \beta(t)$ a.e. $t \ge 0$ and that the set-valued mapping $G : (t,x) \mapsto \widehat{\partial} d_{C(t)}(x)$ is u.s.c. on $\mathbf{R}\times\mathbf{H}$. Then there exists the same solution $x(\cdot) : [0,+\infty) \to \mathbf{H}$ for both problems (P_1) and (P_2), that is, (P_1) and (P_2) have the same nonempty set of solutions.*

Proof. Fix $x_0 \in C(0)$. Put $S := \operatorname{gph} C$ and $F(t,x) := \{1\} \times \{-\beta(t)\widehat{\partial} d_{C(t)}(x)\}$. It is well known that $\operatorname{Proj}_S(t,x)$ is always included in the Fréchet normal cone $\widehat{N}(S;(t,x))$

and hence by Proposition 5.2 part (i) one gets $\mathrm{Proj}_S(t,x) \subset \mathbf{R}_+\widehat{\partial}\Delta_C(t,x)$ for all $(t,x) \in S$. Therefore, Lemma 5.1 yields

$$\sigma(F(t,x), -(q,p)) \geq 0,$$

for all $(t,x) \in S$ and all $(q,p) \in \mathrm{Proj}_S(t,x)$. Now, as G is u.s.c. on $\mathbf{R}\times\mathbf{H}$ and β is continuous, then F is u.s.c. on $\mathbf{R}\times\mathbf{H}$ and hence it satisfies the hypothesis of Proposition 5.2 and then there exists a solution $(s(\cdot),x(\cdot)) : [0,+\infty) \to \mathbf{R}\times\mathbf{H}$ of the differential inclusion

$$\begin{cases} (\dot{s}(t),\dot{x}(t)) \in F(s(t),x(t)), & \text{a.e. } t \geq 0, \\ (s(0),x(0)) = (0,x_0) \in S, \\ (s(t),x(t)) \in S, & \forall t \geq 0. \end{cases}$$

Fix any $t \geq 0$ for which we have $x(t) \in C(s(t))$ and

$$(\dot{s}(t),\dot{x}(t)) \in F(s(t),x(t)) = \{1\} \times \{-\beta(s(t))\widehat{\partial}d_{C(s(t))}(x(t))\}.$$

Then,

$$\begin{cases} \dot{s}(t) = 1 \ \text{ and} \\ \dot{x}(t)) \in -\beta(s(t))\widehat{\partial}d_{C(s(t))}(x(t)). \end{cases}$$

Thus, as $s(0) = 0$ we get $s(t) = t$. Consequently, one concludes that $x(t) \in C(t)$ and $\dot{x}(t) \in -\beta(t)\widehat{\partial}d_{C(t)}(x(t)) \subset \widehat{N}(C(t),x(t))$. This ensures that $x(\cdot)$ is a solution of (P_1). To complete the proof we need by Theorem 5.5 to show that $C(t)$ is Fréchet normally regular for all $t \geq 0$. Indeed, consider any $\bar{t} \geq 0$ and any $\bar{x} \in C(\bar{t})$. Then the u.s.c. of G ensures that $\widehat{\partial}d_{C(\bar{t})}(\cdot)$ is closed at $\bar{x}$ in the following sense: for every $x_n \to \bar{x}$ and every $\xi_n \to \bar{\xi}$ with $\xi_n \in \widehat{\partial}d_{C(\bar{t})}(x_n)$ one has $\bar{\xi} \in \widehat{\partial}d_{C(\bar{t})}(\bar{x})$. Thus, by Theorem 5.1 in [61] and Corollary 3.1 in [60] one concludes that $C(t)$ is Fréchet normally regular. □

In order to make clear the importance of this result we give a concrete application. To this end, we need some new results by Bounkhel and Thibault [58] concerning uniformly prox-regular subsets (see Chap. 1).

Theorem 5.8. *Assume that C satisfies (5.1) and $C(t)$ is uniformly $r(t)$-prox-regular for all $t \geq 0$ with $r(t)$ bounded below by a positive number. Then the graph of G is closed and hence G is u.s.c. on $\mathbf{R}\times\mathbf{H}$.*

Now another existence result of solutions of uniformly prox-regular case in the finite dimensional setting of both problems (P_1) and (P_2) can be deduced from Theorem 5.7 and Theorem 5.8. We give it in the following theorem.

Theorem 5.9. *Under the hypothesis of Theorem 5.8, there exists a solution of both problems (P_1) and (P_2).*

5.2 Existence of Viable Solutions of Nonconvex First Order Differential Inclusions

5.2.1 Introduction

In this section, we consider the following class of differential inclusion (DI):

$$\dot{x}(t) \in G(x(t)) + F(t,x(t)), \ \text{ a.e. } \ [0,T],$$

where $T > 0$ is given, $F : [0,T] \times \mathbf{H} \rightrightarrows \mathbf{H}$ is a continuous set-valued mapping, $G : \mathbf{H} \rightrightarrows \mathbf{H}$ is an upper semicontinuous set-valued mapping such that $G(x) \subset \partial^{\mathrm{C}} g(x)$, with $g : \mathbf{H} \to \mathbf{R}$ is a locally Lipschitz function (not necessarily convex) and $\mathbf{H}$ is a finite dimensional space. Here, $\partial^{\mathrm{C}} g(x)$ denotes the generalized gradient (or Clarke subdifferential) of g at x (see the definition given in Chap. 1). By using some new concepts of regularity in Nonsmooth Analysis Theory, we prove under natural additional assumptions the existence of viable solutions for (DI), i.e., a solution x of (DI) such that $x(t) \in S$, for all $t \in [0,T]$, where S is a given closed subset in $\mathbf{H}$. Our main existence result in Theorem 5.11 is used to get existence results for a particular type of differential inclusions introduced by Henry [138] for the study of some economic problems.

5.2.2 Existence Criteria of Viable Solutions of Nonconvex Differential Inclusions

It is well known that the solution set of the following differential inclusion

$$\begin{cases} \dot{x}(t) \in G(x(t)), \ \text{a.e. } [0,T] \quad (T > 0), \\ x(0) = x_0 \in \mathbf{R}^d, \end{cases} \tag{5.7}$$

can be empty when the set-valued mapping G is upper semicontinuous with nonempty nonconvex values. In [67], the authors proved an existence result of (5.7), by assuming that the set-valued mapping G is included in the subdifferential of a convex lower semicontinuous (l.s.c.) function $g : R^d \to \mathbf{R}$. This result has been extended in many ways.

1. The first one was by [17], where the author replace the convexity assumption of g by its directional regularity in the finite dimensional setting. The infinite dimensional case with the directional regularity assumption on g and some other additional hypothesis has been proved by [17, 18].
2. The second extension was by [2]. An existence result has been obtained for the following nonconvex differential inclusion

$$\begin{cases} \dot{x}(t) \in G(x(t)) + f(t, x(t)), \text{ a.e. } [0,T] \quad (T > 0), \\ x(0) = x_0 \in \mathbf{R}^d, \end{cases} \tag{5.8}$$

under the assumption that G is an upper semicontinuous set-valued mapping with nonempty compact values contained in the subdifferential of a convex lower semicontinuous function, and f is a Caratheodory single-valued mapping.

3. The third way was to investigate the existence of a viable solution of (5.7) (i.e., a solution $x(\cdot)$ such that $x(t) \in S(t)$, where S is a set-valued mapping). The first existence result of viable solutions of (5.7) has been established by Rossi [243]. Later, Morchadi and Gautier [191] proved an existence result of viable solution of the inclusion (5.24).
4. The recent extension of (5.7) and (5.24) is given by [110]. The author proved an existence result of viable solutions for the following differential inclusion

$$(DI) \qquad \begin{cases} \dot{x}(t) \in G(x(t)) + F(t, x(t)), \text{ a.e. } [0,T] \quad (T > 0), \\ x(t) \in S, \end{cases}$$

when $F : [0,T] \times \mathbf{H} \rightrightarrows \mathbf{H}$ is a continuous set-valued mapping, $G : \mathbf{H} \rightrightarrows \mathbf{H}$ is an upper semicontinuous set-valued mapping such that $G(x) \subset \partial^{\text{conv}} g(x)$, where $g : \mathbf{H} \to \mathbf{R}$ is a convex continuous function and $S(t) \equiv S$ and the set S is locally compact in $\mathbf{H}$, with $\dim \mathbf{H} < +\infty$.

Our aim in this section is to establish an extension of the existence result of (5.7) that cover all the other extensions given in the finite dimensional setting, like the ones proved by [2, 17, 67, 110]. The infinite dimensional case has been proved recently in [51]. We will prove an existence result of viable solutions of (DI) when F is a continuous set-valued mapping, G is an u.s.c. set-valued mapping, g is a uniformly regular locally Lipschitz function over S (see Definition 5.3), and S is a closed subset in $\mathbf{H}$, with $dim\, \mathbf{H} < +\infty$.

In all the sequel, we will assume that $\mathbf{H}$ is a finite dimensional space.

We begin by recalling the following lemma proved in [110].

Lemma 5.4. *Assume that*

(i) *S is nonempty subset in $\mathbf{H}$, $x_0 \in S$, and $K_0 := S \cap (x_0 + \rho \mathbf{B})$ is a compact set for some $\rho > 0$;*
(ii) *$P : [0,T] \times \mathbf{H} \rightrightarrows \mathbf{H}$ is an u.s.c. set-valued mapping with nonempty compact values;*
(iii) *For any $(t,x) \in I \times S$ the following tangential condition holds:*

$$\liminf_{h \downarrow 0} h^{-1} d_S(x + hP(t,x)) = 0. \tag{5.9}$$

Let $a \in (0, \min\{T, \frac{\rho}{(M+1)}\})$, where $M := \sup\{\|P(t,x)\| : (t,x) \in [0,T] \times K_0\}$.

Then for any $\varepsilon \in (0,1)$*, any set* $N' = \{t'_i : t'_0 = 0 < \cdots < t'_{\nu'} = a\}$*, and any* $u_0 \in F(0,x_0)$*, There exist a set* $N = \{t_i : t_0 = 0 < \cdots < t_\nu = a\}$*, step functions* f*,* z*, and* x *defined on* $[0,a]$ *such that the following conditions hold for every* $i \in \{1,...,\nu\}$*:*

1. $\{t'_0,...,t'_{k(i)}\} \subset \{t_0,...,t_i\}$*, where* $k(i)$ *is the unique integer such that* $k(i) \in \{0,1,...,\nu'-1\}$ *and* $t'_{k(i)} \le t_i < t'_{k(i)+1}$*;*
2. $0 < t_{j+1} - t_j \le \alpha$*, for all* $j \in \{0,...,i-1\}$*, where*

$$\alpha := \varepsilon \min\{1, t'_1 - t'_0, ..., t'_{\nu'} - t'_{\nu'-1}\};$$

3. $f(0) = u_0$*,* $f(t) = f(\theta(t)) \in F(\theta(t), x(\theta(t)))$ *on* $[0,t_i]$ *where* $\theta(t) = t_j$ *if* $t \in [t_j, t_{j+1})$*, for all* $j \in \{0,1,...,i-1\}$ *and* $\theta(t_i) = t_i$*;*
4. $z(0) = 0$*,* $z(t) = z(t_{l+1})$ *if* $t \in (t_l, t_{l+1}]$*,* $l \le i-1$ *and* $\|z(t)\| \le 2\varepsilon(M+1)T$*;*
5. $x(t) = x_0 + \int_0^t f(s)\mathrm{d}s + z(t)$*, for all* $t \in [0,t_i]$*,* $x(t_j) = x_j \in K_0$ *and*

$$\|x_j - x_{j'}\| \le |t_j - t_{j'}|(M+1), \tag{5.10}$$

for $j, j' \in \{0,1,...,i\}$*.*

Now, we introduce our concept of regularity that will be used in this section.

Definition 5.3. Let $f : \mathbf{H} \to \mathbf{R} \cup \{+\infty\}$ be a l.s.c. function and let $O \subset \operatorname{dom} f$ be a nonempty open subset. We will say that f is *uniformly regular* over O if there exists a positive number $\beta \ge 0$ such that for all $x \in O$ and for all $\xi \in \partial^P f(x)$ one has

$$\langle \xi, x' - x \rangle \le f(x') - f(x) + \beta \|x' - x\|^2, \quad \text{for all } x' \in O.$$

We will say that f is uniformly regular over a closed set S if there exists an open set O containing S such that f is uniformly regular over O.

The class of functions that are uniformly regular over sets is so large. We state here some examples.

1. Any l.s.c. proper convex function f is uniformly regular over any nonempty subset of its domain with $\beta = 0$.
2. Any lower-C^2 function f is uniformly regular over any nonempty convex compact subset of its domain. Indeed, let f be a lower-C^2 function over a nonempty convex compact set $S \subset \operatorname{dom} f$. By Rockafellar's result (see for instance Theorem 10.33 in [241]) there exists a positive real number β such that $g := f + \frac{\beta}{2}\|\cdot\|^2$ is a convex function on S. Using the definition of the subdifferential of convex functions and the fact that $\partial^C f(x) = \partial g(x) - \beta x$ for any $x \in S$, we get the inequality in Definition 5.3 and so f is uniformly regular over S.

   One could think to deal with the class of lower-C^2 (see [241] for the definition of lower-C^2 property) instead of the class of uniformly regular functions. The inconvenience of the class of lower-C^2 functions is the need of the convexity and the compactness of the set S to satisfy the Definition 5.3 which is the exact property needed in our proofs. However, we can find many functions that are uniformly regular over nonconvex noncompact sets. To give an example we need to recall the following result by Bounkhel and Thibault [58] proved for Hilbert spaces $\mathbf{H}$ (see Chap. 1).

Theorem 5.10. *Let S be a nonempty closed subset in $\mathbf{H}$ and let $r > 0$. Then S is uniformly r-prox-regular if and only if the following holds*

$$(P_r) \quad \begin{cases} \text{for all } x \in \mathbf{H}, \text{ with } d_S(x) < r, \text{ and all } \xi \in \partial^P d_S(x): \\ \langle \xi, x' - x \rangle \le \dfrac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x), \\ \text{for all } x' \in \mathbf{H} \text{ with } d_S(x') \le r. \end{cases}$$

From Theorem 5.10 one deduces that for any uniformly r-prox-regular set S (not necessarily convex nor compact) the distance function d_S is uniformly regular over $S + (r - r')\mathbf{B} := \{x \in \mathbf{H} : \ d_S(x) \le r - r'\}$ for every $r' \in (0, r]$.

Some properties for uniformly regular locally Lipschitz functions over sets that will be needed in the next theorem can be stated in the following proposition.

Proposition 5.3. *Let $f : \mathbf{H} \to \mathbf{R}$ be a locally Lipschitz function and let $\emptyset \neq S \subset \operatorname{dom} f$. If f is uniformly regular over S, then the following hold:*

(i) the proximal subdifferential of f is closed over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ with $\xi_n \in \partial^P f(x_n)$ one has $\xi \in \partial^P f(x)$;
(ii) $\partial^C f(x) = \partial^P f(x)$ for all $x \in S$;
(iii) the proximal subdifferential of f is upper hemicontinuous over S, that is, the support function $x \mapsto \langle v, \partial^P f(x) \rangle$ is u.s.c. over S for every $v \in H$.

Proof.

(i) Let O be an open set containing S as in Definition 5.3. Let $x_n \to x \in S$ with $x_n \in S$ and let $\xi_n \to \xi$ with $\xi_n \in \partial^P f(x_n)$. Then by Definition 5.3 one has

$$\langle \xi_n, x' - x_n \rangle \le f(x') - f(x_n) + \beta \|x' - x_n\|^2, \quad \text{for all } x' \in O.$$

Letting n to $+\infty$ we get

$$\langle \xi, x' - x \rangle \le f(x') - f(x) + \beta \|x' - x\|^2, \quad \text{for all } x' \in O.$$

This ensures that $\xi \in \partial^P f(x)$ because O is a neighborhood of x.

(ii) Let x be any point in S. By the part (i) of the proposition we get $\partial^P f(x) = \partial f(x)$, where $\partial f(x)$ denotes the basic (or limiting or Mordukhovich) subdifferential of f at x (see Chap. 1). Now, as f is Lipschitz at x we get by Theorem 6.1 in [90] $\partial^C f(x) = \overline{co}\, \partial f(x) = \overline{co}\, \partial^P f(x) = \partial^P f(x)$. The part (iii) is a direct consequence of (i) and (ii) and so the proof is complete. □

Now we are in position to prove the main theorem in this section.

Theorem 5.11. *Let $S \subset H$ and let $g : H \to \mathbf{R}$ be a locally Lipschitz function that is uniformly regular over S with a constant $\beta \geq 0$. Assume that*

(i) S is a nonempty closed subset;
(ii) $G : H \rightrightarrows H$ is an u.s.c. set-valued mapping with compact values satisfying $G(x) \subset \partial^C g(x)$ for all $x \in S$;
(iii) $F : [0,T] \times H \rightrightarrows H$ is a continuous set-valued mapping with compact values;
(iv) For any $(t,x) \in I \times S$ the following tangential condition holds

$$\liminf_{h \downarrow 0} h^{-1} d_S(x + h(G(x) + F(t,x))) = 0. \tag{5.11}$$

Then, for any $x_0 \in S$ there exists $a \in (0,T)$ such that the differential inclusion (DI) has a viable solution on $[0,a]$.

Proof. Let $L > 0$ and ρ be two positives scalars such that g is Lipschitz with ratio L over $x_0 + \rho\mathbf{B}$. Put $K_0 := S \cap (x_0 + \rho\mathbf{B})$ is a compact set in H. Let M and a be two positives scalars such that $\|F(t,x)\| + \|G(x)\| \leq M$, for all $(t,x) \in [0,T] \times K_0$ and $a \in (0, \min\{T, \frac{\rho}{M+1}\})$. Let $N_0 = \{0,a\}$ and $\varepsilon_m = \frac{1}{2^m}$, for $m = 1,2,\ldots$.

First, the uniform continuity of F on the compact K_0 ensures the existence of $\delta_m > 0$ such that

$$\|(t,x) - (t',x')\| \leq (M+2)\delta_m \Longrightarrow \mathscr{H}(F(t,x), F(t',x')) \leq \varepsilon_m, \tag{5.12}$$

for $t,t' \in [0,a]$, $x,x' \in K_0$, where $\|(t,x)\| = |t| + \|x\|$. Here $\mathscr{H}(A,B)$ stands for the Hausdorff distance between A and B define by

$$\mathscr{H}(A,B) := \max\left\{ \sup_{a \in A} d_B(a), \sup_{b \in B} d_A(b) \right\}.$$

Now, applying Lemma 5.4 for the set-valued mapping $P := F + G$, the scalar ε_m, $m = 1,2,\ldots$, the set $N_0 = \{0,a\}$, and the set S, one obtains for every $m = 1,2,\ldots$ the existence of a set $N_m = \{t_i^m : t_0^m = 0 < \cdots < t_{\nu_m}^m = a\}$, step functions $y_m(\cdot)$, $f_m(\cdot)$, $z_m(\cdot)$, and $x_m(\cdot)$ defined on $[0,a]$ with the following properties:

(i) $N_m \subset N_{m+1}$, $m = 0,1,\ldots$;
(ii) $0 < t_{i+1}^m - t_i^m \leq \alpha_m$, for all $i \in \{0,\ldots,\nu_m - 1\}$, where

$$\alpha_m := \varepsilon_m \min\left\{1, \delta_m, t_1^{m-1} - t_0^{m-1}, \ldots, t_{\nu_{m-1}}^{m-1} - t_{\nu_{m-1}-1}^{m-1}\right\};$$

(iii) $f_m(t) = f_m(\theta_m(t)) \in F(\theta_m(t), x_m(\theta_m(t)))$ and $y_m(t) = y_m(\theta_m(t)) \in G(x_m(\theta_m(t)))$ on $[0,a]$ where $\theta_m(t) = t_i^m$ if $t \in [t_i^m, t_{i+1}^m)$, for all $i \in \{0,1,\ldots,\nu_m - 1\}$ and $\theta_m(a) = a$;
(iv) $z_m(0) = 0$, $z_m(t) = z_m(t_{i+1})$ if $t \in (t_i, t_{i+1}]$, $0 \leq i \leq \nu_m - 1$ and

$$\|z_m(t)\| \leq 2\varepsilon_m(M+1)T; \tag{5.13}$$

(v) $x_m(t) = x_0 + \int_0^t (y_m(s) + f_m(s))\mathrm{d}s + z_m(t)$ and $x_m(\theta_m(t)) \in K_0$, for all $t \in [0,a]$, and for $i, j \in \{0, 1, \ldots, \nu_m\}$

$$\|x_m(t_i^m) - x_m(t_j^m)\| \leq |t_i^m - t_j^m|(M+1). \tag{5.14}$$

Observe that (5.14) ensures that for $i, j \in \{0, 1, \ldots, \nu_m\}$

$$\|(t_i^m, x_m(t_i^m)) - (t_j^m, x_m(t_j^m))\| \leq |t_i^m - t_j^m|(M+2). \tag{5.15}$$

We will prove that the sequence $x_m(\cdot)$ converges to a viable solution of (DI). First, we note that the sequence f_m can be constructed with the relative compactness property in the space of bounded functions. We don't give the proof of this part here. It can be found in [109, 110, 123, 124]. Therefore, without loss of generality we can suppose that there is a bounded function f such that

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|f_m(t) - f(t)\| = 0. \tag{5.16}$$

Now, we use our characterizations of the uniform regularity proved in Proposition 5.3 and some techniques of [2, 58, 67] to prove that the approximate solutions $x_m(\cdot)$ converges to a function that is a viable solution of (DI).

Put $q_m(t) = x_0 + \int_0^t (y_m(s) + f_m(s))\mathrm{d}s$. By the property (iv), one has $\|\dot{z}_m(t)\| = 0$ a.e. on $[0,a]$. Then $\|\dot{q}_m(t)\| = \|\dot{x}_m(t)\| \leq M$ a.e. on $[0,a]$ and the sequence q_m is equicontinuous and the sequence of their derivatives $\dot{q}_m$ is equibounded. Hence, a subsequence of q_m may be extracted (without loss of generality we may suppose that this subsequence is q_m) that converges in the sup-norm topology to an absolutely continuous mapping $x : [0,a] \to H$ and such that the sequence of their derivatives $\dot{q}_m$ converges to $\dot{x}(\cdot)$ in the weak topology of $L^2([0,a], H)$. Since $\|q_m(t) - x_m(t)\| = \|z_m(t)\|$ and $\|\dot{z}_m(t)\| = 0$ a.e. on $[0,a]$ one gets by (5.29)

$$\begin{cases} \lim\limits_{m\to\infty} \max\limits_{t\in[0,a]} \|x_m(t) - x(t)\| = 0, \\ \dot{x}_m(\cdot) \rightharpoonup \dot{x}(\cdot) \ \text{ in the weak topology of } L^2([0,a], H). \end{cases} \tag{5.17}$$

Recall now that the sequence f_m converges pointwisely a.e. on $[0,a]$ to f. Then, the continuity of F and the closedness of $F(t, x(t))$ entail $f(t) \in F(t, x(t))$. Further, by the properties of the sequence x_m and the closedness of K_0, we get $x(t) \in K_0 \subset S$.

Put $y(t) = -f(t) + \dot{x}(t)$. It remains to prove that $y(t) \in G(x(t))$ a.e. $[0,a]$. By construction and the hypothesis on G and g we have $y_m(t) = \dot{x}_m(t) - f_m(t)$ and

$$y_m(t) \in G(x_m(\theta_m(t))) \subset \partial^{\mathrm{C}} g(x_m(\theta_m(t))) = \partial^P g(x_m(\theta_m(t))), \tag{5.18}$$

for a.e. on $[0,a]$, where the last equality follows from the uniform regularity of g over S and the part (ii) in Proposition 5.3.

The weak convergence (by (5.33)) in $L^2([0,a],H)$ of $\dot{x}_m(\cdot)$ to $\dot{x}(\cdot)$ and Mazur's Lemma entail

$$\dot{x}(t) \in \bigcap_m \overline{co}\{\dot{x}_k(t) : k \geq m\}, \quad \text{for a.e. on } [0,a].$$

Fix any such t and consider any $\xi \in H$. Then, the last relation above yields

$$\langle \xi, \dot{x}(t) \rangle \leq \inf_m \sup_{k \geq m} \langle \xi, \dot{x}_m(t) \rangle$$

and hence according to (5.34)

$$\begin{aligned}\langle \xi, \dot{x}(t) \rangle &\leq \limsup_m \sigma(\xi, \partial^P g(x_m(\theta_m(t))) + f_m(t)) \\ &\leq \sigma(\xi, \partial^P g(x(t)) + f(t)),\end{aligned}$$

where the second inequality follows from the upper hemicontinuity of the proximal subdifferential of uniformly regular functions (see part (ii) in Proposition 5.3) and the convergence pointwisely a.e. on $[0,a]$ of f_m to f, and the fact that $x_m(\theta_m(t)) \to x(t)$ in K_0 a.e. on $[0,a]$. Thus, by the convexity and the closedness of the proximal subdifferential of uniformly regular functions (part (ii) in Proposition 5.3) we obtain

$$y(t) = \dot{x}(t) - f(t) \in \partial^P g(x(t)). \tag{5.19}$$

To complete the proof we need to show that $y(t) \in G(x(t))$.

As $x(\cdot)$ is an absolutely continuous mapping and g is a uniformly regular locally Lipschitz function over S (hence directionally regular over S (see Chap. 1)), one gets by Theorem 2 in Valadier [255, 256] (see also [39]) for a.e. on $[0,a]$

$$\frac{\mathrm{d}}{\mathrm{d}t}(g \circ x)(t) = \langle \partial^P g(x(t)), \dot{x}(t) \rangle = \langle \dot{x}(t) - f(t), \dot{x}(t) \rangle = \|\dot{x}(t)\|^2 - \langle f(t), \dot{x}(t) \rangle.$$

Consequently,

$$g(x(a)) - g(x_0) = \int_0^a \|\dot{x}(s)\|^2 \mathrm{d}s - \int_0^a \langle f(s), \dot{x}(s) \rangle \mathrm{d}s. \tag{5.20}$$

On the other hand, we have by construction $\dot{x}_m(t) = y_i^m + f_i^m$ with $y_i^m \in G(x_i^m) \subset \partial^C g(x_i^m) = \partial^P g(x_i^m)$ for $t \in (t_i^m, t_{i+1})$, $i = 0, \ldots, \nu_m - 1$. Then, by Definition 5.3 one has

$$\begin{aligned}g(x_{i+1}^m) - g(x_i^m) &\geq \langle y_i^m, x_{i+1}^m - x_i^m \rangle - \beta \|x_{i+1}^m - x_i^m\|^2 \\ &= \left\langle \dot{x}_m(t) - f_m(t), \int_{t_i^m}^{t_{i+1}^m} \dot{x}_m(s) \mathrm{d}s \right\rangle - \beta \|x_{i+1}^m - x_i^m\|^2\end{aligned}$$

$$\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2 \mathrm{d}s - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s) \rangle \mathrm{d}s$$
$$- \beta(M+1)^2(t_{i+1}^m - t_i^m)^2$$
$$\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{x}_m(s)\|^2 \mathrm{d}s - \int_{t_i^m}^{t_{i+1}^m} \langle \dot{x}_m(s), f_m(s) \rangle \mathrm{d}s$$
$$- \beta(M+1)^2 \varepsilon_m (t_{i+1}^m - t_i^m).$$

By adding, we obtain

$$g(x_m(a)) - g(x_0) \geq \int_0^a \|\dot{x}_m(s)\|^2 \mathrm{d}s - \int_0^a \langle \dot{x}_m(s), f_m(s) \rangle \mathrm{d}s - \varepsilon_m (M+1)^2 a. \quad (5.21)$$

According to (5.32) and (5.33) one gets

$$\lim_m \int_0^a \langle \dot{x}_m(s), f_m(s) \rangle \mathrm{d}s = \int_0^a \langle \dot{x}(s), f(s) \rangle \mathrm{d}s.$$

Passing to the limit superior for $m \to \infty$ in (5.37) and the continuity of g yield

$$g(x(a)) - g(x_0) \geq \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 \mathrm{d}s - \int_0^a \langle \dot{x}(s), f(s) \rangle \mathrm{d}s,$$

and hence a comparison with (5.36) gives

$$\int_0^a \|\dot{x}(s)\|^2 \mathrm{d}s \geq \limsup_m \int_0^a \|\dot{x}_m(s)\|^2 \mathrm{d}s,$$

i.e.,

$$\|\dot{x}\|^2_{L^2([0,a],H)} \geq \limsup_m \|\dot{x}_m\|^2_{L^2([0,a],H)}.$$

On the other hand the weak lower semicontinuity of the norm ensures

$$\|\dot{x}\|_{L^2([0,a],H)} \leq \liminf_m \|\dot{x}\|_{L^2([0,a],H)}.$$

Consequently, we get

$$\|\dot{x}\|_{L^2([0,a],H)} = \lim_m \|\dot{x}_m\|_{L^2([0,a],H)}.$$

This means that the sequence $\dot{x}_m(\cdot)$ converges to $\dot{x}(\cdot)$ strongly in $L^2([0,a],H)$. Hence, there exists a subsequence of $\dot{x}_m(\cdot)$ still denoted $\dot{x}_m(\cdot)$ converges pointwisely a.e. on $[0,a]$ to $\dot{x}(\cdot)$. Finally, by the construction, one has $(x_m(t), \dot{x}_m(t) - f_m(t)) \in \operatorname{gph} G$ a.e. on $[0,a]$ and so the closedness of the graph ensures that $(x(t), \dot{x}(t) - f(t)) \in \operatorname{gph} G$ a.e. on $[0,a]$. This completes the proof of the theorem. □

Remark 5.2.

1. An inspection of the proof in Theorem 5.11 shows that the uniformity of the constant β was needed only over the set K_0 and so it was not necessary over all the set S. Indeed, it suffices to take the uniform regularity of g locally over S, that is, for every point $\bar{x} \in S$ there exist $\beta \geq 0$ and a neighborhood V of x_0 such that g is uniformly regular over $S \cap V$.
2. As we can see from the proof of Theorem 5.11, the assumption needed on the set S is the local compactness which holds in the finite dimensional setting for nonempty closed sets.
3. Under the assumptions (i)–(iv) of Theorem 5.11, if we assume that $F([0,T] \times S) + G(S)$ is bounded, then for any $a \in (0,T)$, the differential inclusion (DI) has a viable solution on $[0,a]$.

We close this section with two corollaries of the main result proved in Theorem 5.11.

Corollary 5.4. *Let $K \subset H$ be a nonempty uniformly prox-regular closed subset and $F : [0,T] \times H \rightrightarrows H$ be a continuous set-valued mapping with compact values. Then, for any $x_0 \in K$ there exists $a \in (0,T)$ such that the following differential inclusion*

$$\begin{cases} \dot{x}(t) \in -\partial^{\mathrm{C}} d_K(x(t)) + F(t,x(t)) \ \text{ a.e. on } \ [0,a] \\ x(0) = x_0 \in K, \end{cases}$$

has at least one absolutely continuous solution on $[0,a]$.

Proof. Theorem 5.10 shows that the function $g := d_K$ is uniformly regular over K and so it is uniformly regular over some neighborhood V of $x_0 \in K$. Thus, by Remark 5.3 part 1, we apply Theorem 5.12 with $S = H$ (hence, the tangential condition (5.27) is satisfied), $K_0 := V \cap S = V$, and the set-valued mapping $G := \partial^{\mathrm{C}} d_K$ which satisfies the hypothesis of Theorem 5.11. □

Our second corollary concerns the following differential inclusion

$$\begin{cases} \dot{x}(t) \in -N^{\mathrm{C}}(S;x(t)) + F(t,x(t)) \ \text{ a.e.} \\ x(t) \in S, \ \text{ for all } \ t, \ \text{ and } \ x(0) = x_0 \in S. \end{cases} \tag{5.22}$$

First, we recall that this type of differential inclusion has been introduced by Henry [138] for studying some economic problems. In the case when F is an u.s.c set-valued mapping and is autonomous (that is F is independent of t), he proved an existence result of (5.22) under the convexity assumption on the set S and on the images of the set-valued mapping F. In the autonomous case, this result has been extended by Cornet [100] by assuming the tangential regularity assumption on the set S and the convexity on the images of F with the u.s.c of F. In [248], Thibault proved in the non autonomous case, an existence result of (5.22) for any closed subset S (without any assumption on S), which also required the convexity of the images of F and the u.s.c. of F. The question arises whether we can drop the assumption of convexity of the images of F. Our corollary here establishes an existence result in this vein, but we will pay a heavy price for the absence of the convexity. We will assume that F is continuous, and above all, that the following tangential condition holds

$$\liminf_{h\downarrow 0} h^{-1} d_S(x+h(\partial^C d_S(x)+F(t,x))) = 0, \tag{5.23}$$

for any $(t,x) \in I \times S$.

Corollary 5.5. *Assume that*

(i) $F : [0,T] \times H \rightrightarrows H$ *is a continuous set-valued mapping with compact values;*
(ii) S *is a nonempty uniformly prox-regular closed subset in* H*;*
(iii) For any $(t,x) \in I \times S$ *the tangential condition (5.23) holds.*

Then, for any $x_0 \in S$*, there exists* $a \in (0,T)$ *such that the differential inclusion (5.22) has at lease one absolutely continuous solution on* $[0,a]$.

5.3 Existence Results for First Order Nonconvex Sweeping Processes: Infinite Dimensional Case

Our purpose, in this section, is to give an application (to the nonconvex sweeping process) of the results established in the first chapter on uniformly prox-regular sets.

We start with an important result of closedness of the proximal subdifferential of the distance function to images of set-valued mappings whose images are uniformly prox-regular. It has its own interest.

Proposition 5.4. *Let* $C : \mathbf{R} \rightrightarrows H$ *be a continuous set-valued mapping, that is, the relation (5.1) holds with* $v : \mathbf{R} \to \mathbf{R}$ *is a continuous function. Let* $r > 0$*. Assume that* $C(t)$ *is uniformly r-prox-regular for all t in some interval I of* $\mathbf{R}$*. For a given* $0 < \delta < r$*, the following closedness property of the proximal subdifferential of the distance function holds:*

"for any $\bar{t} \in I$*,* $\bar{x} \in C(\bar{t}) + (r-\delta)\mathbf{B}$*,* $x_n \to \bar{x}$*,* $t_n \to \bar{t}$ *with* $t_n \in I$*, (*x_n *is not necessarily in* $C(t_n)$*) and* $\xi_n \in \partial^P d_{C(t_n)}(x_n)$ *with* $\xi_n \to^w \bar{\xi}$*, one has* $\bar{\xi} \in \partial^P d_{C(\bar{t})}(\bar{x})$*." Here* $\to^w$ *means the weak convergence in* H.

Proof. Fix $\bar{t} \in I$ and $\bar{x} \in C(\bar{t}) + (r-\delta)\mathbf{B}$. As $x_n \to \bar{x}$ one gets for n sufficiently large $x_n \in \bar{x} + \frac{\delta}{4}\mathbf{B}$. On the other hand, since the subset $C(\bar{t})$ is uniformly r-prox-regular, one can choose a point $\bar{y} \in C(\bar{t})$ with $d_{C(\bar{t})}(\bar{x}) = \|\bar{y} - \bar{x}\|$. So, for every n large enough one can write by (5.1),

$$|d_{C(t_n)}(x_n) - d_{C(\bar{t})}(\bar{y}))| \le |v(t_n) - v(\bar{t})| + \|x_n - \bar{y}\|,$$

and hence the continuity of v yields for n large enough

$$d_{C(t_n)}(x_n) \leq \frac{\delta}{4} + \|x_n - \bar{x}\| + \|\bar{x} - \bar{y}\| \leq \frac{\delta}{4} + \frac{\delta}{4} + r - \delta = r - \frac{\delta}{2} < r.$$

Therefore, for any n large enough, we apply the property (P_r'') in Theorem 2.14 with $\xi_n \in \partial^P d_{C(t_n)}(x_n)$ to get

$$\langle \xi_n, u - x_n \rangle \leq \frac{8}{r - d_{C(t_n)}(x_n)} \|u - x_n\|^2 + d_{C(t_n)}(u) - d_{C(t_n)}(x_n), \tag{5.24}$$

for all $u \in H$ with $d_{C(t_n)}(u) < r$. This inequality still holds for all $u \in \bar{x} + \delta' \mathbf{B}$ with $0 < \delta' < \dfrac{\delta}{4}$ because for such u one has

$$d_{C(t_n)}(u) \leq \|u - \bar{x}\| + \|\bar{x} - x_n\| + d_{C(t_n)}(x_n) \leq \delta' + \frac{\delta}{4} + r - \frac{\delta}{2} < r.$$

Consequently, by the continuity (because of (5.1)) of the distance function with respect to (t,x), the inequality (5.24) gives, by letting $n \to +\infty$,

$$\langle \bar{\xi}, u - \bar{x} \rangle \leq \frac{8}{r - d_{C(\bar{t})}(\bar{x})} \|u - \bar{x}\|^2 + d_{C(\bar{t})}(u) - d_{C(\bar{t})}(\bar{x}),$$

for all $u \in \bar{x} + \delta' \mathbf{B}$. This ensures that $\bar{\xi} \in \partial^P d_{C(\bar{t})}(\bar{x})$ and so the proof of the proposition is complete. □

Remark 5.3. One obtains the same result if $C(t)$ is uniformly $r(t)$-prox-regular with either $r(t)$ is bounded below by a positive number $\alpha > 0$(i.e., $r(t) > \alpha > 0$, for all $t \in I$) or $r(\cdot)$ is a continuous positive function at $\bar{t}$.

The following existence theorem establishes the main result in this section. The result is proved by showing that the Moreau catching-up algorithm (introduced for convex sets in [209]) still converges for uniformly prox-regular sets.

Theorem 5.12. *Let H be a separable Hilbert space, $T > 0$, and $r > 0$. Assume that $C(t)$ is uniformly r-prox-regular for every $t \in I := [0,T]$ and that the assumption (5.1) holds with an absolutely continuous function v. Let $F : I \times H \to H$ be a set-valued mapping with convex compact values in H such that $F(t,\cdot)$ is u.s.c. on H for any fixed $t \in I$ and $F(\cdot,x)$ admits a measurable selection on I for any fixed $x \in H$. Assume that $F(t,x) \subset \mathcal{K}$ for all $(t,x) \in I \times H$, for some convex compact set $\mathcal{K} \subset H$. Then, for any $x_0 \in C(0)$, the sweeping process (SPP) with the perturbation F has at least one absolutely continuous solution, that is, there exists an absolutely continuous mapping $x : I \to H$ such that*

$$(SPP) \qquad \begin{cases} -\dot{x}(t) \in N^C\big(C(t); x(t)\big) + F(t, x(t)) \ \text{ a.e. } \ t \in I \\ \quad x(0) = x_0 \in C(0). \end{cases}$$

Proof. Step 1. We first assume that F is globally u.s.c. on $I \times H$ and we prove the conclusion of the theorem.

Observing that (5.1) ensures for $t \le t'$

$$|d_{C(t')}(y) - d_{C(t)}(y)| \le \int_t^{t'} |\dot{v}(s)| \mathrm{d}s,$$

we may suppose (replacing $\dot{v}$ by $|\dot{v}|$ if necessary) that $\dot{v}(t) \ge 0$ for all $t \in I$. Consider for every $n \in \mathbf{N}$, the following partition of I:

$$t_{n,i} := \frac{iT}{2^n} \quad (0 \le i \le 2^n) \quad \text{and} \quad I_{n,i} := (t_{n,i}, t_{n,i+1}] \ \text{ if } \ 0 \le i \le 2^n - 1. \tag{5.25}$$

Put

$$\mu_n := \frac{T}{2^n}, \quad \varepsilon_{n,i} := \int_{t_{n,i}}^{t_{n,i+1}} \dot{v}(s)\mathrm{d}s, \quad \text{and} \quad \varepsilon_n := \max_{0 \le i < 2^n} \{\mu_n + \varepsilon_{n,i}\}. \tag{5.26}$$

As $\varepsilon_n \to 0$, we can fix $n_0 \ge 1$ satisfying for every $n \ge n_0$

$$2\mu_n < \frac{r}{(2l+1)} \quad \text{and} \quad 2\varepsilon_n < \min\left\{1, \frac{r}{(4l+3)}\right\}, \tag{5.27}$$

where l is a positive number satisfying $\mathscr{K} \subset l\mathbf{B}$ (because $\mathscr{K}$ is a compact set in H). For every $n \ge n_0$, we define by induction

$$u_{n,0} := x_0; \quad z_{n,0} \in F(t_{n,0}, u_{n,0});$$

$$z_{n,i} \in F(t_{n,i}, u_{n,i});$$

$$u_{n,i+1} := \mathrm{proj}_{C(t_{n,i+1})}(u_{n,i} - \mu_n z_{n,i}). \tag{5.28}$$

This last equality is well defined. Indeed, by (5.1) one has for all $t \in I$

$$d(u_{n,0} - \mu_n z_{n,0}, C(t)) \le l\mu_n + v(t) - v(t_{n,0}).$$

Then for $t := t_{n,1}$ one gets (by (5.26) and (5.27))

$$d(u_{n,0} - \mu_n z_{n,0}, C(t_{n,1})) \le l\mu_n + v(t_{n,1}) - v(t_{n,0}) \le (l+1)\varepsilon_n \le \frac{r}{2} < r$$

and hence as C has uniformly r-prox-regular values, one can choose a point $u_{n,1} := \mathrm{proj}_{C(t_{n,1})}(u_{n,0} - \mu_n z_{n,0})$. Similarly, we can define, by induction, the points $(u_{n,i})_{0 \le i \le 2^n}$ and $(z_{n,i})_{0 \le i \le 2^n}$. From (5.28) and (5.1) one deduces for every $0 \le i < 2^n$

$$\|u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}\| \le l\mu_n + \varepsilon_{n,i} \le (l+1)(\mu_n + \varepsilon_{n,i}). \tag{5.29}$$

For every $n \geq n_0$, these $(u_{n,i})_{0 \leq i \leq 2^n}$ and $(z_{n,i})_{0 \leq i \leq 2^n}$ are used to construct two mappings z_n and u_n from I to H by defining their restrictions to each interval $I_{n,i}$ as follows:

for $t = 0$, set $z_n(t) := z_{n,0}$ and $u_n(t) := u_{n,0} = x_0$,

for all $t \in I_{n,i}$ $(0 \leq i \leq 2^n)$, set $z_n(t) := z_{n,i}$ and

$$u_n(t) := u_{n,i} + \frac{a(t) - a(t_{n,i})}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - (t - t_{n,i})z_{n,i}, \tag{5.30}$$

where $a(t) := v(t) + t$ for all $t \in I$. Hence, for every t and t' in $I_{n,i}$ $(0 \leq i \leq 2^n)$ one has

$$u_n(t') - u_n(t) = \frac{a(t') - a(t)}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - (t' - t)z_{n,i}.$$

Thus, in view of (5.29), if $t, t' \in I_{n,i}$ $(0 \leq i < 2^n)$ with $t \leq t'$, one obtains

$$\|u_n(t') - u_n(t)\| \leq (2l + 1)(a(t') - a(t)), \tag{5.31}$$

and, by addition this also holds for all $t, t' \in I$ with $t \leq t'$. This inequality entails that u_n is absolutely continuous.

Coming back to the definition of u_n in (5.30), one observes that for $0 \leq i < 2^n$ $\dot{u}_n(t) = \dfrac{\dot{a}(t)}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - z_{n,i}$ for a. e. $t \in I_{n,i}$.

Then one obtains, in view of (5.29), for a. e. $t \in I$

$$\|\dot{u}_n(t) + z_n(t)\| \leq (l + 1)(\dot{v}(t) + 1). \tag{5.32}$$

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \rho_n(t) = t_{n,i} \quad \text{if} \quad t \in I_{n,i} \quad (0 \leq i < 2^n). \tag{5.33}$$

Then, by (5.28), the construction of u_n and z_n, and the properties of proximal normal cones to subsets, we have for a. e. $t \in I$

$$z_n(t) \in F(\rho_n(t), u_n(\rho_n(t))), \text{ and}$$

$$\dot{u}_n(t) + z_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))). \tag{5.34}$$

This last inclusion, relation (5.32) entail for a. e. $t \in I$

$$\dot{u}_n(t) + z_n(t) \in -(l + 1)\dot{a}(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))). \tag{5.35}$$

Let us show now that the sequence $(u_n)_n$ satisfies the Cauchy property in the space of continuous mappings $\mathscr{C}(I, H)$ endowed with the norm of uniform convergence.

Fix $m,n \in \mathbf{N}$ such that $m \geq n \geq n_0$ and fix also $t \in I$ with $t \neq t_{m,i}$ for $i = 0,...,2^m$ and $t \neq t_{n,j}$ for $j = 0,...,2^n$. Observe by (5.1), (5.26), and (5.31) that

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\leq v(\theta_n(t)) - v(\theta_m(t)) + \|u_m(\theta_m(t)) - u_m(t)\| \\ &\leq \int_{\theta_m(t)}^{\theta_n(t)} \dot{v}(s)\mathrm{d}s + (2l+1)\left[\int_t^{\theta_m(t)} \dot{v}(s)\mathrm{d}s + (\theta_m(t) - t)\right] \\ &\leq \varepsilon_n + (2l+1)\varepsilon_m \end{aligned} \tag{5.36}$$

and hence, by (5.27) $d_{C(\theta_n(t))}(u_m(t)) < r$. Set $\delta(t) := (l+1)\dot{a}(t)$. Then, (5.35) and $(P_r^{'''})$ in Theorem 2.14 and also (5.36) entail

$$\begin{aligned} &\left\langle \dot{u}_n(t) + z_n(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \\ &\leq \frac{2\delta(t)}{r}\|u_n(\theta_n(t)) - u_m(t)\|^2 + \delta(t) d_{C(\theta_n(t))}(u_m(t)) \\ &\leq \frac{2\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\Big]^2 \\ &\quad + \delta(t)(\varepsilon_n + (2l+1)\varepsilon_m), \end{aligned}$$

and this yields by (5.26) and (5.31)

$$\begin{aligned} \left\langle \dot{u}_n(t) + z_n(t), u_n(\theta_n(t)) - u_m(t) \right\rangle \leq\; & \frac{2\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + (2l+1)\varepsilon_n\Big]^2 \\ & + \delta(t)(2l+1)(\varepsilon_n + \varepsilon_m). \end{aligned} \tag{5.37}$$

Now, let us define $Z_n(t) := \int_0^t z_n(s)\mathrm{d}s$. Observe that for all $t \in I$ the set $\{Z_n(t) : n \geq n_0\}$ is contained in the strong compact set $T\mathscr{K}$ and so it is relatively strongly compact in H. Then by Arzela–Ascoli's theorem we get the relative strong compactness of set $\{Z_n : \; n \geq n_0\}$ with respect to the uniform convergence in $C(I,H)$ and so we may assume without loss of generality that (Z_n) converges uniformly to some mapping Z. As $\|z_n(t)\| \leq l$, we may suppose that (z_n) converges weakly in $L^1(I,H,\mathrm{d}t)$ to some mapping z. Then, for all $t \in I$,

$$Z(t) = \lim_n Z_n(t) = \lim_n \int_0^t z_n(s)\mathrm{d}s = \int_0^t z(s)\mathrm{d}s,$$

which gives that Z is absolutely continuous and $\dot{Z}(t) = z(t)$ for almost all $t \in I$.

Put now $w_n(t) := u_n(t) + Z_n(t)$ for all $n \geq n_0$ and all $t \in I$ and put $\eta_n := \max\{\varepsilon_n, \|Z_n - Z\|_\infty\}$. Then by (5.32) and (5.37) one gets

$$\begin{aligned}
&\left\langle \dot{w}_n(t), w_n(\theta_n(t)) - w_m(t) \right\rangle \\
&= \left\langle \dot{w}_n(t), u_n(\theta_n(t)) - u_m(t) \right\rangle + \left\langle \dot{w}_n(t), Z_n(\theta_n(t)) - Z_m(t) \right\rangle \\
&\leq \frac{2\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + \|Z_n(t) - Z_m(t)\| + (2l+1)\varepsilon_n \Big]^2 \\
&\quad + \delta(t)(2l+1)(\varepsilon_n + \varepsilon_m) + \delta(t)\|Z_n(\theta_n(t)) - Z_m(t)\| \\
&\leq \frac{2\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n \Big]^2 \\
&\quad + 2\delta(t)(2l+1)(\eta_n + \eta_m).
\end{aligned}$$

This last inequality ensures by (5.32)

$$\begin{aligned}
\left\langle \dot{w}_n(t), w_n(t) - w_m(t) \right\rangle &\leq \left\langle \dot{w}_n(t), w_n(t) - w_n(\theta_n(t)) \right\rangle + 2\delta(t)(2l+1)(\eta_n + \eta_m) \\
&\qquad + \frac{2\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n \Big]^2 \\
&\leq 4\delta(t)(2l+1)(\eta_n + \eta_m) \\
&\qquad + \frac{2\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n \Big]^2.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\left\langle \dot{w}_m(t), w_m(t) - w_n(t) \right\rangle &\leq 4\delta(t)(2l+1)(\eta_n + \eta_m) \\
&\qquad + \frac{2\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_m \Big]^2.
\end{aligned}$$

It then follows from both last inequalities that we have for some positive constant α independent of m, n, and t (note that $\|w_n(t)\| \leq lT + \|x_0\| + \int_0^T \dot{v}(s)\mathrm{d}s$)

$$2\left\langle \dot{w}_m(t) - \dot{w}_n(t), w_m(t) - w_n(t) \right\rangle \leq \alpha\delta(t)(\eta_n + \eta_m) + 8\frac{\delta(t)}{r}\|w_m(t) - w_n(t)\|^2,$$

and so, for some positive constants β and γ independent of m, n, and t

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_m(t) - w_n(t)\|^2 \right) \leq \beta\dot{a}(t)\|w_m(t) - w_n(t)\|^2 + \gamma\dot{a}(t)(\eta_n + \eta_m).$$

As $\|w_m(0) - w_n(0)\|^2 = 0$, the Gronwall inequality yields for all $t \in I$

$$\|w_m(t) - w_n(t)\|^2 \leq \gamma(\eta_n + \eta_m) \int_0^t \left[\dot{a}(s) \exp\left(\beta \int_s^t \dot{a}(u)\, \mathrm{d}u \right) \right] \mathrm{d}s$$

and hence for some positive constant K independent of m, n, and t we have

$$\|w_m(t) - w_n(t)\|^2 \leq K(\eta_n + \eta_m).$$

The Cauchy property in $\mathscr{C}(I,H)$ of the sequence $(w_n)_n = (u_n + Z_n)_n$ is thus established and hence this sequence converges uniformly to some mapping w. Therefore the sequence $(u_n)_n$ converges uniformly to $u := w - Z$. Furthermore, (5.32) ensures that a subsequence of $(\dot{u}_n)_n$ may be extracted that converges in the weak topology of $L^1(I,H,\mathrm{d}t)$. Without loss of generality, we may suppose that this subsequence is $(\dot{u}_n)_n$. Denote by p its weak limit in $L^1(I,H,\mathrm{d}t)$. Then, for all $t \in I$

$$u(t) = \lim_{n\to\infty} u_n(t) = x_0 + \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\mathrm{d}s = x_0 + \int_0^t p(s)\mathrm{d}s,$$

which gives that u is absolutely continuous and $\dot{u}(t) = p(t)$ for a. e. $t \in I$.

Moreover, for a.e. $t \in I$, by the definition (5.33) of $\theta_n(t)$ one has $|\theta_n(t) - t| \leq \dfrac{T}{2^n}$ and (by (5.31) and (5.26))

$$\begin{aligned} \|u_n(\theta_n(t)) - u(t)\| &\leq \|u_n(t) - u(t)\| + (2l+1)(a(\theta_n(t)) - a(t)) \\ &\leq \|u_n(t) - u(t)\| + (2l+1)\varepsilon_n. \end{aligned}$$

So,

$$\lim_{n\to\infty} \theta_n(t) = t \quad \text{and} \quad \lim_{n\to\infty} u_n(\theta_n(t)) = u(t). \tag{5.38}$$

As $u_n(\theta_n(t)) \in C(\theta_n(t))$, it follows from (5.1)

$$d_{C(t)}(u_n(\theta_n(t))) \leq v(\theta_n(t)) - v(t)$$

and hence, by (5.38), one obtains $u(t) \in C(t)$, because the set $C(t)$ is closed.

We proceed now to prove that $\dot{u}(t) + z(t) \in -N^{\mathrm{C}}(C(t); u(t))$ for almost all $t \in I$. We know by (5.35) that we have for almost all $t \in I$

$$\dot{u}_n(t) + z_n(t) \in -\delta(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))). \tag{5.39}$$

The weak convergence in $L^1(I,H,\mathrm{d}t)$ of $(\dot{u}_n)_n$ and $(z_n)_n$ to $\dot{u}$ and z respectively entail for almost all $t \in I$ (by Mazur's lemma)

$$\dot{u}(t) + z(t) \in \bigcap_n \overline{co}\{\dot{u}_k(t) + z_k(t) : k \geq n\}.$$

Here $\overline{co}$ denotes the closed convex hull. Fix any such $t \in I$ and consider any $\xi \in H$. The last relation above yields

$$\langle \xi, \dot{u}(t) + z(t) \rangle \le \inf_n \sup_{k \ge n} \langle \xi, \dot{u}_k(t) + z_k(t) \rangle,$$

and hence according to (5.39)

$$\begin{aligned} \langle \xi, \dot{u}(t) + z(t) \rangle &\le \limsup_n \sigma(-\delta(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))); \xi) \\ &\le \sigma(-\delta(t)\partial^P d_{C(t)}(u(t)); \xi), \end{aligned} \tag{5.40}$$

where the second inequality follows from the upper hemicontinuity property in Proposition 5.4 because (5.38) holds and $u(t) \in C(t)$.

As the set $\partial^P d_{C(t)}(u(t))$ is closed and convex, we obtain

$$\dot{u}(t) + z(t) \in -\delta(t)\partial^P d_{C(t)}(u(t)) \subset -N^P(C(t); u(t)).$$

By the global upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that $z(t) \in F(t, u(t))$ and so we get

$$-\dot{u}(t) \in N^P(C(t); u(t)) + F(t, u(t)),$$

which completes the proof of the first step. Note also by (5.32) that

$$\|\dot{u}(t)\| \le (2l+1)(\dot{v}(t)+1) \quad \text{for a. e.} \quad t \in I.$$

Step 2. Now, we assume that F satisfies the hypothesis in the statement of the theorem.

According to the proof of Theorem 2.1 in [75] (see [78] for more details concerning the existence of such approximation and their properties), there exists a sequence $(F_n)_n$ of globally u.s.c. set-valued mappings on $I \times H$ with convex compact values in H with $F_n(t,x) \subset T\mathscr{K}$ for all $(t,x) \in I \times H$ and satisfying: For any sequence (x_n) of Lebesgue measurable mappings from I to H which converges pointwise to a Lebesgue measurable mapping x and any sequence (z_n) converging weakly to z in $L^1(I, H, \mathrm{d}t)$ and such that $z_n(t) \in F_n(t, x_n(t))$ a.e. on I, one has

$$z(t) \in F(t, x(t)), \quad \text{a. e. on } I.$$

Since F_n satisfies the hypothesis of the first step, for every $n \ge 1$, there exists an absolutely continuous mapping $x_n : I \to H$ and a Lebesgue measurable mapping $z_n : I \to H$ satisfying $z_n(t) \in F_n(t, x_n(t)) \subset T\mathscr{K}$ for a.e. $t \in I$ and

$$\dot{x}_n(t) + z_n(t) \in -N^C(C(t); x_n(t)) \quad \text{a. e. on } I,$$

with $x_n(0) = x_0 \in C(0)$ and $\|\dot{x}_n(t)\| \leq (2lT+1)(\dot{v}(t)+1)$ for a.e. $t \in I$. Observe that (z_n) admits a subsequence (that we do not relabel) converging weakly in $L^1(I, H, \mathrm{d}t)$ to some mapping z. So, by the property of the sequence (F_n) stated above we conclude that $z(t) \in F(t, x(t))$ for a.e. $t \in I$. Now, with the same techniques as in the first step, we prove easily the uniform convergence of the sequence (x_n) to some absolutely continuous mapping x and that

$$\dot{x}(t) + z(t) \in -N^{\mathrm{C}}(C(t); x(t)) \quad \text{a. e. on } I.$$

Thus, we get $-\dot{x}(t) \in N^{\mathrm{C}}(C(t); x(t)) + F(t, x(t))$, for a.e. $t \in I$. This ends the proof of the theorem. □

The following corollary is a direct consequence of Theorem 5.12. A similar result is also established by Colombo and Goncharov [92] where the set-valued mapping C is assumed to be Lipschitz with ϕ-convex values.

Corollary 5.6. *Let H be any Hilbert space, $T > 0$, and $r > 0$. Assume that $C(t)$ is uniformly r-prox-regular for every $t \in I := [0, T]$ and that the assumption (5.1) holds with a nondecreasing absolutely continuous function v. Then the sweeping process (SP) associated with the set-valued mapping C has one and only one absolutely continuous solution.*

Proof. The existence follows from Theorem 5.12 since for $F = 0$ the separability of H is not needed as it is easily seen in the proof of the first step of Theorem 5.12. The uniqueness part follows from the proof of Corollary 5.1 in Thibault [248]. □

5.4 First Order Perturbed Nonconvex Sweeping Process with Delay

5.4.1 Introduction

In this section, we present some existence results for functional differential inclusions governed by nonconvex sweeping process of first order

$$\begin{cases} \dot{u}(t) \in -N(C(t); u(t)) + F(t, u_t), \text{ a.e. on } [0, T], \\ u(t) \in C(t), \text{ for all } t \in [0, T], \\ u(s) = T(0)u(s) = \varphi(s), \text{ for all } s \in [-\tau, 0], \end{cases} \tag{FOSPD}$$

where $\tau, T > 0$, $C : [0, T] \rightrightarrows H$ is a set-valued mapping taking values in a Hilbert space H, and $F : I \times \mathscr{C}_0 \rightrightarrows H$ is a set-valued mapping with convex compact values. Here $\mathscr{C}_0 := \mathscr{C}_0([-\tau, 0], H)$ is the Banach space of all continuous mapping from $[-\tau, 0]$ to H equipped with the norm of uniform convergence. For every $t \in I$, the function u_t is given by $u_t(s) = T(t)u(s) = u(t+s)$, for all $s \in [-\tau, 0]$. Such problems have been studied in several papers (see for example [76, 77, 109]). In [76], some topological properties of solutions set for (FOSPD) problem in the convex case are established, and in [77], the compactness of the solutions set is obtained in the nonconvex case when $H = \mathbf{R}^d$, using important properties of uniformly r-prox regular sets developed recently in [58, 61, 77, 230]. For more details on functional differential inclusions for sweeping process and related subjects, we refer the reader to [46, 74, 125] and the references therein.

Our main purpose in this section is to prove existence results for (FOSPD) when C has uniformly r-prox regular values and H is a separable Hilbert space. We start by proving the existence of approximate solutions for the (FOSPD) under the boundness of F. Under two different assumptions on F we prove the existence of absolutely continuous solutions of (FOSPD) by proving the convergence of the approximate solutions established in Theorem 5.13. Let $\varphi : X \rightrightarrows Y$ be a set-valued mapping defined between two topological vector spaces X and Y, we say that φ is upper semi-continuous (in short u.s.c.) at $x \in \mathrm{dom}(\varphi) := \{x' \in X : \varphi(x') \neq \emptyset\}$ if for any open O containing $\varphi(x)$ there exists a neighborhood V of x such that $\varphi(V) \subset O$.

We will deal with a finite delay $\tau > 0$. If $u : [-\tau, T] \to H$, then for every $t \in [0,T]$, we define the function $u_t(s) = u(t+s), s \in [-\tau, 0]$ and the Banach space $\mathscr{C}_T := \mathscr{C}_T([-\tau, T], H)$ (resp. $\mathscr{C}_0 := \mathscr{C}_0([-\tau, 0], H)$) of all continuous mapping from $[-\tau, T]$ (resp. $[-\tau, 0]$) to H with the norm given by

$$\|\varphi\|_{\mathscr{C}_T} := \max\{\|\varphi(s)\| : s \in [-\tau, T]\}$$

(respectively,

$$\|\varphi\|_{\mathscr{C}_0} := \max\{\|\varphi(s)\| : s \in [-\tau, 0]\}).$$

Clearly, if $u \in \mathscr{C}_T$, then $u_t \in \mathscr{C}_0$, and the mapping $u \to u_t$ is continuous in the sense of the uniform convergence.

Let $T > 0$, $I := [0,T]$, $r \in (0, +\infty]$, and $C : I \rightrightarrows H$ be an absolutely continuous set-valued mapping, that is, for any $y \in H$ and any $t, t' \in I$

$$|d_{C(t)}(y) - d_{C(t')}(y)| \leq |v(t) - v(t')|, \tag{5.41}$$

with $v : I \to \mathbf{R}$ is an absolutely continuous function. The following result provides an approximate solution for the (FOSPD) under consideration.

Theorem 5.13. *Assume that $C(t)$ is uniformly r-prox-regular for every $t \in I$. Let $F : I \times \mathscr{C}_0 \rightrightarrows H$ be a set-valued mapping with convex compact values in H such that $F(t, \cdot)$ is u.s.c. on $\mathscr{C}_0$ for any fixed $t \in I$ and $F(\cdot, \varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathscr{C}_0$. Assume that $F(t, \varphi) \subset l\mathbf{B}$ for all $(t, \varphi) \in I \times \mathscr{C}_0$, for some $l > 0$. Then, for any $\varphi \in \mathscr{C}_0$ with $\varphi(0) \in C(0)$ and for any n large enough there exists a continuous mapping $u_n : [-\tau, T] \to H$ which enjoys the following properties:*

1. $-\dot{u}_n(t) \in N^P(u_n(\theta_n(t)); C(\theta_n(t))) + F(\rho_n(t), T(\rho_n(t))u_n)$, *a.e.* $t \in I$, *where* $\theta_n, \rho_n : I \to I$ *with* $\theta_n(t) \to t$ *and* $\rho_n(t) \to t$ *for all* $t \in I$;
2. $\|\dot{u}_n(t)\| \leq (l+1)(\dot{v}(t)+1)$, *a.e.* $t \in I$.

Proof. We prove the conclusion of our theorem when F is globally u.s.c. on $I \times \mathscr{C}_0$ and then, we can proceed by approximation to prove it when $F(t,\cdot)$ is u.s.c. on $\mathscr{C}_0$ for any fixed $t \in I$ and $F(\cdot,\varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathscr{C}_0$.

First, observing that (5.41) ensures for $t \leq t'$

$$|d_{C(t')}(y) - d_{C(t)}(y)| \leq \int_t^{t'} |\dot{v}(s)| \mathrm{d}s, \tag{5.42}$$

we may suppose (replacing $\dot{v}$ by $|\dot{v}|$ if necessary) that $\dot{v}(t) \geq 0$ for all $t \in I$. We construct via discretization the sequence desired of continuous mappings $\{u_n\}_n$ in $\mathscr{C}_T$.

For every $n \in \mathbf{N}$, we consider the following partition of I:

$$t_{n,i} := \frac{iT}{2^n} \quad (0 \leq i \leq 2^n) \quad \text{and} \quad I_{n,i} := (t_{n,i}, t_{n,i+1}] \ \text{ if } \ 0 \leq i \leq 2^n - 1. \tag{5.43}$$

Put

$$\mu_n := \frac{T}{2^n}, \quad \varepsilon_{n,i} := \int_{t_{n,i}}^{t_{n,i+1}} \dot{v}(s) \mathrm{d}s, \quad \text{and} \quad \varepsilon_n := \max_{0 \leq i < 2^n} \{\mu_n + \varepsilon_{n,i}\}. \tag{5.44}$$

As $\varepsilon_n \to 0$, we can fix $n_0 \geq 1$ satisfying for every $n \geq n_0$

$$2\mu_n < \frac{r}{(2l+1)} \quad \text{and} \quad 2\varepsilon_n < \min\left\{1, \frac{r}{(4l+3)}\right\}. \tag{5.45}$$

First, we put

$$u_n(s) := \varphi(s), \ \text{for all } s \in [-\tau, 0] \text{ and for all } n \geq n_0. \tag{5.46}$$

For every $n \geq n_0$, we define by induction,

$$u_n(t_{i+1}^n) := u_{n,i+1} = \mathrm{proj}_{C(t_{n,i+1})}(u_{n,i} - \mu_n f_0(t_i^n, T(t_i^n)u_n)), \tag{5.47}$$

where $f_0(t_i^n, T(t_i^n)u_n)$ is the minimal norm element of $F(t_i^n, T(t_i^n)u_n)$, i.e.,

$$\|f_0(t_i^n, T(t_i^n)u_n)\| = \min\{\|y\| : \ y \in F(t_i^n, T(t_i^n)u_n)\} \leq l \tag{5.48}$$

and

$$T(t_i^n)u_n := (u_n)_{t_i^n}.$$

The above construction is possible although the nonconvexity of the images of C. Indeed, we can show that for every $n \geq n_0$ we have

$$d_{C(t_{n,i+1})}(u_{n,i} - \mu_n f_0(t_i^n, T(t_i^n)u_n)) \leq l\mu_n + v(t_{n,i+1}) - v(t_{n,i}) \leq (l+1)\varepsilon_n \leq \frac{r}{2}$$

and hence as C has uniformly r-prox-regular values, one can choose for all $n \geq n_0$ a point $u_{n,i+1} = \text{proj}_{C(t_{n,i+1})}(u_{n,i} - \mu_n f_0(t_i^n, T(t_i^n)u_n))$. Note that from (5.47) and (5.42) one deduces for every $0 \leq i < 2^n$

$$\|u_{n,i+1} - (u_{n,i} - \mu_n f_0(t_i^n, T(t_i^n)u_n))\| \leq l\mu_n + \varepsilon_{n,i} \leq (l+1)(\mu_n + \varepsilon_{n,i}). \tag{5.49}$$

By construction we have

$$u_i^n \in C(t_i^n), \text{ for all } 0 \leq i < 2^n. \tag{5.50}$$

For every $n \geq n_0$, these $(u_{n,i})_{0 \leq i \leq 2^n}$ and $(f_0(t_i^n, T(t_i^n)u_n)_{0 \leq i \leq 2^n}$ are used to construct two mappings u_n and f_n from I to H by defining their restrictions to each interval $I_{n,i}$ as follows:

for $t = 0$, set $f_n(t) := f_{n,0}$ and $u_n(t) := u_0^n = \varphi(0)$,
for all $t \in I_{n,i}$ $(0 \leq i \leq 2^n)$, set $f_n(t) := f_{n,i}$ and

$$u_n(t) := u_{n,i} + \frac{a(t) - a(t_{n,i})}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n f_{n,i}) - (t - t_{n,i})f_{n,i}, \tag{5.51}$$

where $f_{n,i} := f_0(t_i^n, T(t_i^n)u_n)$ and $a(t) := v(t) + t$ for all $t \in I$. Hence, for every t and t' in $I_{n,i}$ $(0 \leq i \leq 2^n)$ one has

$$u_n(t') - u_n(t) = \frac{a(t') - a(t)}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n f_{n,i}) - (t' - t)f_{n,i}.$$

Thus, in view of (5.49), if $t, t' \in I_{n,i}$ $(0 \leq i < 2^n)$ with $t \leq t'$, one obtains

$$\|u_n(t') - u_n(t)\| \leq (l+1)(a(t') - a(t)) + l(t' - t) \leq (2l+1)(a(t') - a(t)), \tag{5.52}$$

and, by addition this also holds for all $t, t' \in I$ with $t \leq t'$. This inequality entails that u_n is absolutely continuous.

Coming back to the definition of u_n in (5.51), one observes that for $0 \leq i < 2^n$
$\dot{u}_n(t) = \dfrac{\dot{a}(t)}{\varepsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n f_{n,i}) - f_{n,i}$ for a.e. $t \in I_{n,i}$.

Then one obtains, in view of (5.49), for a.e. $t \in I$

$$\|\dot{u}_n(t) + f_n(t)\| \leq (l+1)(\dot{v}(t) + 1), \tag{5.53}$$

which proves the part (2) of the theorem.

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \rho_n(t) = t_{n,i} \quad \text{if} \quad t \in I_{n,i} \quad (0 \leq i < 2^n). \tag{5.54}$$

Then, by (5.47), the construction of u_n and f_n, and the properties of proximal normal cones to subsets, we have for a.e. $t \in I$

$$f_n(t) \in F(\rho_n(t), T(\rho_n(t))u_n)$$

and

$$\dot{u}_n(t) + f_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))). \tag{5.55}$$

These last inclusions ensure the part (1) of the theorem and then the proof is complete. □

Now, we are able to state the first existence result for (FOSPD).

Theorem 5.14. *Assume that the assumptions of Theorem 5.13 are satisfied. Assume that $C(t)$ is strongly compact for every $t \in I$. Then for every $\varphi \in \mathscr{C}_0$ with $\varphi(0) \in C(0)$, there exists a continuous mapping $u : [-\tau, T] \to H$ such that u is absolutely continuous on I and satisfies:*

$$\begin{cases} \dot{u}(t) \in -N^P(C(t); u(t)) + F(t, u_t), & a.e. \text{ on } I, \\ u(t) \in C(t), & \forall t \in I, \\ u(s) = T(0)u(s) = \varphi(s), & \forall s \in [-\tau, 0], \end{cases} \tag{FOSPD}$$

and

$$\|\dot{u}(t)\| \leq (l+1)(\dot{v}(t) + 1), \quad a.e. \text{ on } I.$$

Proof. Let $\varphi \in \mathscr{C}_0$ with $\varphi(0) \in C(0)$. By Theorem 5.13 there exists a sequence of continuous mappings $\{u_n\}$ enjoys the properties (1) and (2) in Theorem 5.13. Let $n_0 \in \mathbf{N}$ satisfying (5.45). Then for any $n \geq n_0$ and any $t \in I$ we have

$$\begin{aligned} d(u_n(t), C(t)) &\leq \|u_n(t) - u_n(t_i^n)\| + \mathscr{H}(C(t_i^n), C(t)) \\ &\leq (2l+1)(a(t) - a(t_i^n)) + (v(t) - v(t_i^n)) \\ &\leq (2l+1)(\varepsilon_{n,i} + \mu_n) + \varepsilon_{n,i} \leq 2(l+1)\varepsilon_n. \end{aligned} \tag{5.56}$$

Since $C(t)$ is strongly compact and $\varepsilon_n \to 0$, (5.56) implies that the set $\{u_n(t) : \ n \geq n_0\}$ is relatively strongly compact in H for all $t \in I$. Thus, by Arzela–Ascoli's we can extract a subsequence of the sequence $\{u_n\}_n$ still denoted $\{u_n\}_n$, which converges uniformly on $[-\tau, T]$ to a continuous function u which clearly satisfies $u_0 = \varphi$. Now by letting $n \to +\infty$ we get for all $t \in I$

$$u(t) \in C(t). \tag{5.57}$$

On one hand, it follows from our construction in the proof of Theorem 5.13 that for all $t \in I$

$$\mathscr{H}(C(\theta_n(t)), C(t)) \leq |v(\theta_n(t)) - v(t)| \leq \varepsilon_n \to 0, \tag{5.58}$$

and by (5.52), (5.45), and the uniform convergence of $\{u_n\}_n$ to u over I we get

$$\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(\theta_n(t)) - u(\theta_n(t))\| + \|u(\theta_n(t)) - u(t)\| \to 0. \tag{5.59}$$

Now, using the same technique in [76] and the relations (5.45) and (5.52) we obtain

$$\lim_{n\to\infty} \|T(\rho_n(t))u_n - T(t)u_n\| = 0 \quad \text{in} \quad \mathscr{C}_0.$$

Therefore, as the uniform convergence of u_n to u in $[-\tau, T]$ implies that $T(t)u_n$ converges to $T(t)u$ uniformly on $[-\tau, 0]$, we conclude that

$$T(\rho_n(t))u_n \longrightarrow T(t)u = u_t \quad \text{in} \quad \mathscr{C}_0. \tag{5.60}$$

On the other hand, from $f_n(t) \in F(\rho_n(t), T(\rho_n(t))u_n)$ and (5.52), (f_n) and $(\dot{u}_n)$ are bounded sequences in $L^1(I, H, \mathrm{d}t)$, then by extracting subsequences we may suppose that f_n and $\dot{u}_n$ weakly converges in $L^1(I, H, \mathrm{d}t)$ to some mappings f and ω respectively. Then, for all $t \in I$ one has

$$u(t) = \lim_{n\to\infty} u_n(t) = \varphi(0) + \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\mathrm{d}s = x_0 + \int_0^t \omega(s)\mathrm{d}s,$$

which proves that u is absolutely continuous and $\dot{u}(t) = \omega(t)$ for a. e. $t \in I$.

Using now Mazur's lemma, we obtain

$$\dot{u}(t) + f(t) \in \bigcap_n \overline{co}\{\dot{u}_k(t) - f_k(t) : k \geq n\}, \qquad \text{a.e.} \quad t \in I.$$

Fix such t in I and any ξ in H, the last relation above yields

$$\langle \dot{u}(t) + f(t), \xi \rangle \leq \inf_n \sup_{k\geq n} \langle \dot{u}_k(t) + f_k(t), \xi \rangle.$$

By (5.53) and (5.55) we obtain for a.e. $t \in I$

$$\dot{u}_n(t) + f_n(t) \in N^P(C(\theta_n(t)); u_n(\theta_n(t))) \cap \delta(t)\mathbf{B}_* = \partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))),$$

where $\delta(t) := (l+1)(\dot{v}(t)+1)$. Hence, according to this last inclusion and Proposition 5.4 we get

$$\begin{aligned} \langle \dot{u}(t) + f(t), \xi \rangle &\leq \delta(t) \limsup_n \sigma(-\partial^P d_{C(\theta_n(t)}(u_n(\theta_n(t)); \xi) \\ &\leq \delta(t)\sigma(-\partial^P d_{C(t)}(u(t)); \xi). \end{aligned}$$

Since $\partial^P d_{C(t)}(u(t))$ is closed convex, we obtain

$$\dot{u}(t)+f(t)\in-\delta(t)\partial^P d_{C(t)}(u(t))\subset -N^P(C(t);u(t))$$

and then

$$-\dot{u}(t)\in N^P(C(t);u(t))+f(t),$$

because $u(t)\in C(t)$. Finally, from (5.60) and the global upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that

$$f(t)\in F(t,T(t)u)=F(t,u_t),\qquad \text{a.e.}\qquad t\in I.$$

Thus, the existence is proved. □

Under different assumptions another existence result for (FOSPD) is also proved in the following theorem.

Theorem 5.15. *Assume that the assumptions of Theorem 5.13 are satisfied. Assume also that $F(t,\varphi)\subset\mathscr{K}\subset l\mathbf{B}$ for every $(t,\varphi)\in I\times\mathscr{C}_0$, where $\mathscr{K}$ is a strongly compact set in H. Then for every $\varphi\in\mathscr{C}_0$ with $\varphi(0)\in C(0)$, there exists a continuous mapping $u:[-\tau,T]\to H$ such that u is absolutely continuous on I and satisfies*

$$\begin{cases}\dot{u}(t)\in -N^P(C(t);u(t))+F(t,u_t), & a.e.\quad on\quad I,\\ u(t)\in C(t), & \forall t\in I,\\ u(s)=T(0)u(s)=\varphi(s), & \forall s\in[-\tau,0],\end{cases}$$

and

$$\|\dot{u}(t)\|\le(l+1)(\dot{v}(t)+1),\qquad a.e.\ \ on\quad I.$$

Proof. Let $\varphi\in\mathscr{C}_0$ with $\varphi(0)\in C(0)$. By Theorem 5.13 there exists a sequence of continuous mappings $\{u_n\}$ enjoys the properties (1) and (2) in Theorem 5.13. Let $n_0\in\mathbf{N}$ satisfying (5.45). Let us show that the sequence $(u_n)_n$ satisfies the Cauchy property in the space of continuous mappings $\mathscr{C}(I,H)$ endowed with the norm of uniform convergence. Fix $m,n\in\mathbf{N}$ such that $m\ge n\ge n_0$ and fix also $t\in I$ with $t\neq t_{m,i}$ for $i=0,\dots,2^m$ and $t\neq t_{n,j}$ for $j=0,\dots,2^n$. Observe by (5.42), (5.44), and (5.52) that

$$\begin{aligned}d_{C(\theta_n(t))}(u_m(t))&=d_{C(\theta_n(t))}(u_m(t))-d_{C(\theta_m(t))}(u_m(\theta_m(t)))\\ &\le v(\theta_n(t))-v(\theta_m(t))+\|u_m(\theta_m(t))-u_m(t)\|\end{aligned}$$

$$\leq \int_{\theta_m(t)}^{\theta_n(t)} \dot{v}(s)\mathrm{d}s + (2l+1)\left[\int_t^{\theta_m(t)} \dot{v}(s)\mathrm{d}s + (\theta_m(t)-t)\right]$$
$$\leq \varepsilon_n + (2l+1)\varepsilon_m, \tag{5.61}$$

and hence, by (5.45) we get $d_{C(\theta_n(t))}(u_m(t)) < r$. Set $\delta(t) := (l+1)\dot{a}(t)$. Then, (5.55), (5.61), and Theorem 5.10 entail

$$\begin{aligned}
&\left\langle \dot{u}_n(t) + f_n(t), u_n(\theta_n(t)) - u_m(t)\right\rangle \\
&\leq \frac{2\delta(t)}{r}\|u_n(\theta_n(t)) - u_m(t)\|^2 + \delta(t) d_{C(\theta_n(t))}(u_m(t)) \\
&\leq \frac{2\delta(t)}{r}\left[\|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\right]^2 \\
&\quad + \delta(t)(\varepsilon_n + (2l+1)\varepsilon_m),
\end{aligned}$$

and this yields by (5.44) and (5.52)

$$\begin{aligned}
\left\langle \dot{u}_n(t) + f_n(t), u_n(\theta_n(t)) - u_m(t)\right\rangle &\leq \frac{2\delta(t)}{r}\left[\|u_n(t) - u_m(t)\| + (2l+1)\varepsilon_n\right]^2 \\
&\quad + \delta(t)(2l+1)(\varepsilon_n + \varepsilon_m).
\end{aligned} \tag{5.62}$$

Now, let us define $g_n(t) := \int_0^t f_n(s)\mathrm{d}s$ for all $t \in I$. Observe that for all $t \in I$ the set $\{g_n(t) : \ n \geq n_0\}$ is contained in the strong compact set $T\mathscr{K}$ and so it is relatively strongly compact in H. Then, as $\|f_n(t)\| \leq l$ a.e. on I, Arzela–Ascoli's theorem yields the relative strong compactness of the set $\{g_n : \ n \geq n_0\}$ with respect to the uniform convergence in $C(I,H)$ and so we may assume without loss of generality that (g_n) converges uniformly to some mapping g. Also, we may suppose that (f_n) weakly converges in $L^1(I,H,\mathrm{d}t)$ to some mapping f. Then, for all $t \in I$,

$$g(t) = \lim_n g_n(t) = \lim_n \int_0^t f_n(s)\mathrm{d}s = \int_0^t f(s)\mathrm{d}s,$$

which gives that g is absolutely continuous and $\dot{g} = f$ a.e. on I.
Put now $w_n(t) := u_n(t) + g_n(t)$ for all $n \geq n_0$ and all $t \in I$ and put $\eta_n := \max\{\varepsilon_n, \|g_n - g\|_\infty\}$. Then by (5.53) and (5.62) one gets

$$\begin{aligned}
&\left\langle \dot{w}_n(t), w_n(\theta_n(t)) - w_m(t)\right\rangle \\
&= \left\langle \dot{w}_n(t), u_n(\theta_n(t)) - u_m(t)\right\rangle + \left\langle \dot{w}_n(t), g_n(\theta_n(t)) - g_m(t)\right\rangle \\
&\leq \frac{2\delta(t)}{r}\left[\|w_n(t) - w_m(t)\| + \|g_n(t) - g_m(t)\| + (2l+1)\varepsilon_n\right]^2 \\
&\quad + \delta(t)(2l+1)(\varepsilon_n + \varepsilon_m) + \delta(t)\|g_n(\theta_n(t)) - g_m(t)\|
\end{aligned}$$

$$\leq \frac{2\delta(t)}{r}\Big[\|w_n(t)-w_m(t)\|+(\eta_n+\eta_m)+(2l+1)\eta_n\Big]^2$$
$$+2\delta(t)(2l+1)(\eta_n+\eta_m).$$

This last inequality ensures by (5.53)

$$\begin{aligned}\big\langle \dot{w}_n(t), w_n(t)-w_m(t)\big\rangle &\leq \big\langle \dot{w}_n(t), w_n(t)-w_n(\theta_n(t))\big\rangle + 2\delta(t)(2l+1)(\eta_n+\eta_m)\\ &\quad+\frac{2\delta(t)}{r}\Big[\|w_n(t)-w_m(t)\|+(\eta_n+\eta_m)+(2l+1)\eta_n\Big]^2\\ &\leq 4\delta(t)(2l+1)(\eta_n+\eta_m)\\ &\quad+\frac{2\delta(t)}{r}\Big[\|w_n(t)-w_m(t)\|+(\eta_n+\eta_m)+(2l+1)\eta_n\Big]^2.\end{aligned}$$

In the same way, we also have

$$\begin{aligned}\big\langle \dot{w}_m(t), w_m(t)-w_n(t)\big\rangle &\leq 4\delta(t)(2l+1)(\eta_n+\eta_m)\\ &\quad+\frac{2\delta(t)}{r}\Big[\|w_n(t)-w_m(t)\|+(\eta_n+\eta_m)+(2l+1)\eta_m\Big]^2.\end{aligned}$$

It then follows from both last inequalities that we have for some positive constant α independent of m, n, and t (note that $\|w_n(t)\| \leq lT + \|\varphi(0)\| + \int_0^T \dot{v}(s)\mathrm{d}s$)

$$2\big\langle \dot{w}_m(t)-\dot{w}_n(t), w_m(t)-w_n(t)\big\rangle \leq \alpha\delta(t)(\eta_n+\eta_m)+8\frac{\delta(t)}{r}\|w_m(t)-w_n(t)\|^2,$$

and so, for some positive constants β and γ independent of m, n, and t

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|w_m(t)-w_n(t)\|^2\Big) \leq \beta\dot{a}(t)\|w_m(t)-w_n(t)\|^2+\gamma\dot{a}(t)(\eta_n+\eta_m).$$

As $\|w_m(0)-w_n(0)\|^2 = 0$, the Gronwall inequality yields for all $t \in I$

$$\|w_m(t)-w_n(t)\|^2 \leq \gamma(\eta_n+\eta_m)\int_0^t \left[\dot{a}(s)\exp\left(\beta\int_s^t \dot{a}(u)\,\mathrm{d}u\right)\right]\mathrm{d}s$$

and hence for some positive constant K independent of m, n, and t we have

$$\|w_m(t)-w_n(t)\|^2 \leq K(\eta_n+\eta_m).$$

The Cauchy property in $\mathscr{C}(I,H)$ of the sequence $(w_n)_n = (u_n+g_n)_n$ is thus established and hence this sequence converges uniformly to some mapping w. Therefore, the sequence $(u_n)_n$ constructed in Theorem 5.13 converges uniformly to $u := w - g$. Following the same arguments in the proof of Theorem 5.14 we prove the conclusion of the theorem, i.e., the limit mapping u is continuous on $[-\tau, T]$ and absolutely continuous on I and satisfies

$$\begin{cases} \dot{u}(t) \in -N^P(C(t); u(t)) + F(t, u_t), & \text{a.e. on } I, \\ u(t) \in C(t), & \forall t \in I, \\ u(s) = T(0)u(s) = \varphi(s), & \forall s \in [-\tau, 0], \end{cases}$$

and

$$\|\dot{u}(t)\| \le (l+1)(\dot{v}(t)+1), \quad \text{a.e. on } I.$$

□

Remark 5.4. The results proved in this section generalizes many results given in [76, 77]. Theorem 5.14 extends the one given in [76] to the case of absolutely continuous set-valued mappings with nonconvex values, and Theorem 5.15 extends Theorem 2.1 in [77] given only in the finite dimensional setting. Note that the proof here is completely different of those given in [77] and it allows us to obtain the result in the infinite dimensional setting. It is interesting to point out that our assumptions on F are different to those supposed in Theorem 2.1 in [77]. They supposed that F has compact values and satisfies the linear growth condition and in our Theorem 5.15, F is supposed to be contained in a compact set.

5.5 Commentary to Chap. 5

Chapter 5 is devoted to problems described by first order differential inclusions under the regularity of the right hand side. Section 5.1 studies a special type of differential inclusions called sweeping process (SP), introduced and studied in the convex case by Moreau [207]. In Sect. 5.2, we present an existence result for first order differential inclusions, in finite dimensional setting, using a different concept of regularity introduced and studied in Bounkhel [42]. The existence of solution for (SP) in Hilbert spaces is presented in Sect. 5.3. The existence of solutions for (SP) with delay is presented in Sect. 5.4. The main results in Sects. 5.1 and 5.2 are proved in [42] while the main results in the last two sections are proved in [58] and [65,66], respectively. The continuation of the application of the uniform prox-regularity to the existence of solutions for differential inclusions has been the subject of a long list of recent works. We give here a bibliography on this subject for interested readers: [16, 26, 41, 42, 46, 47, 51, 52, 56, 58, 65, 66, 77, 83, 92, 95, 113–118, 130, 131, 161–165, 248, 249, 259].

Chapter 6
Second Order Differential Inclusions

6.1 Introduction

The existence of solutions for the second order differential inclusion

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t)) \qquad (SDI)$$

has been studied by many authors (see for example [3, 80, 81, 109, 185, 191, 246]). In [80], Castaing studied for the first time the existence problem for the following particular type of second order differential inclusions

$$\ddot{x}(t) \in -N^{\text{conv}}(K(x(t)); \dot{x}(t)) \quad \text{and} \quad \dot{x}(t) \in K(x(t)), \qquad (SSP)$$

where K is a convex set-valued mapping with compact values. Many papers (for example [80, 81, 185, 246]) studied since this particular problem. The general problem (SDI) has been treated in several ways. For instance, the authors in [3] solved the problem when G takes the following particular type: $G(t, x(t), \dot{x}(t)) = \gamma \dot{x}(t) + \partial^{\text{conv}} f(x(t))$, where $\gamma > 0$ and f is a lower semicontinuous convex function. Their motivations come from a mechanical problem that they called the heavy ball problem with friction. For more details we refer the reader to [3] and the references therein. In [53, 54], the authors studied the following particular problem of (SDI)

$$\ddot{x}(t) \in -N^{\text{conv}}(K(x(t)); \dot{x}(t)) + F(t, \dot{x}(t)). \qquad (SSPP1)$$

They proved several existence results when $K : \mathbf{H} \rightrightarrows \mathbf{H}$ is nonconvex set-valued mapping with compact values, $\mathbf{H}$ is a finite dimensional space, and the perturbation $F : [0, +\infty) \times \mathbf{H} \rightrightarrows \mathbf{H}$ is bounded with convex values. Their proofs are strongly based upon the fixed point theorems and some new existence results by [58] for first order sweeping processes. They also proved existence results for another particular problem of (SDI)

$$\ddot{x}(t) \in -N^{C}(K(x(t)); \dot{x}(t)) + F(t, x(t)), \qquad (SSPP2)$$

M. Bounkhel, *Regularity Concepts in Nonsmooth Analysis: Theory and Applications*, Springer Optimization and Its Applications 59, DOI 10.1007/978-1-4614-1019-5_6,

when K is a nonconvex set-valued mapping with compact values, $\mathbf{H}$ is a separable Hilbert space, and F is a nonconvex continuous set-valued mapping. Note that the problem $(SSPP2)$ with memory has been studied in [109] when K is a convex set-valued mapping with compact values.

Our aim in this chapter is to prove existence results for the following general problem

$$\ddot{x}(t) \in -N^{C}(K(x(t));\dot{x}(t)) + F(t,x(t),\dot{x}(t)) + G(t,x(t),\dot{x}(t)), \qquad (SSPMP)$$

where K is a nonconvex set-valued mapping with compact values, $\mathbf{H}$ is a separable Hilbert space, F is a scalarly upper semicontinuous convex set-valued mapping, and G is a nonconvex continuous set-valued mapping. This general problem covers all the problems studied before and mentioned above. We will call it the Second order Sweeping Process with Mixed Perturbations (in short $(SSPMP)$). The last section is devoted to study (SSPMP) with delay.

Throughout the chapter $\mathbf{H}$ will denote a *real separable Hilbert space*.

We close this section with the following theorem by Bounkhel and Thibault [58]. We give the proof here for the convenience of the reader. It proves a closedness property of the subdifferential of the distance function associated with a set-valued mapping. Note that the statement of this theorem in [58] is given with $X = \mathbf{R}$, but the same arguments of the proof still work for any normed vector space X because the proof is based on the uniform prox-regularity of the values of the set-valued mapping and it is independent from the structure of the space X. The key of the proof is the characterization of uniformly prox-regular subsets proved in Theorem 3.1 in [58].

Theorem 6.1. *Let $r \in (0,+\infty]$, Ω be an open subset in a normed vector space X, and $K : \Omega \rightrightarrows \mathbf{H}$ be a Hausdorff-continuous set-valued mapping with compact values. Assume that $K(z)$ is uniformly r-prox-regular for all z in Ω. Then for a given $0 < \delta < r$ the following holds:*

"for any $\bar{z} \in \Omega$, $\bar{x} \in K(\bar{z}) + (r-\delta)\mathbf{B}$, $x_n \to \bar{x}$, $z_n \to \bar{z}$ with $z_n \in \Omega$, (x_n is not necessarily in $K(z_n)$) and $\xi_n \in \partial^P d_{K(z_n)}(x_n)$ with $\xi_n \to^w \bar{\xi}$ one has $\bar{\xi} \in \partial^P d_{K(\bar{z})}(\bar{x})$." Here $\to^w$ means the weak convergence in $\mathbf{H}$.

Proof. Fix $\bar{z} \in \Omega$, and $\bar{x} \in K(\bar{z}) + (r-\delta)\mathbf{B}$. As $x_n \to \bar{x}$ one gets for n sufficiently large $x_n \in \bar{x} + \frac{\delta}{4}\mathbf{B}$. On the other hand, since the subset $K(\bar{z})$ is uniformly r-prox-regular one can choose a point $\bar{y} \in K(\bar{z})$ with $d_{K(\bar{z})}(\bar{x}) = \|\bar{y} - \bar{x}\|$. Hence one can write by the definition of the Hausdorff distance,

$$d_{K(z_n)}(x_n) \leq \mathscr{H}(K(z_n), K(\bar{z})) + \|x_n - \bar{y}\|,$$

and hence the Hausdorff-continuity of K yields for n large enough

$$d_{K(z_n)}(x_n) \leq \frac{\delta}{4} + \|x_n - \bar{x}\| + \|\bar{x} - \bar{y}\| \leq \frac{\delta}{4} + \frac{\delta}{4} + r - \delta = r - \frac{\delta}{2} < r.$$

Therefore, for any n large enough, we apply the property (P_r'') in Theorem 2.14 with $\xi_n \in \partial^P d_{K(z_n)}(x_n)$ to get

$$\langle \xi_n, u - x_n \rangle \leq \frac{8}{r - d_{K(z_n)}(x_n)} \|u - x_n\|^2 + d_{K(z_n)}(u) - d_{K(z_n)}(x_n), \tag{6.1}$$

for all $u \in \mathbf{H}$ with $d_{K(z_n)}(u) < r$. This inequality still holds for all $u \in \bar{x} + \delta' \mathbf{B}$ with $0 < \delta' < \frac{\delta}{4}$ because for such u one has

$$d_{K(z_n)}(u) \leq \|u - \bar{x}\| + \|\bar{x} - x_n\| + d_{K(z_n)}(x_n) \leq \delta' + \frac{\delta}{4} + r - \frac{\delta}{2} < r.$$

Consequently, by the continuity of the distance function with respect to (z, x), the inequality (6.3) gives, by letting $n \to +\infty$,

$$\langle \bar{\xi}, u - \bar{x} \rangle \leq \frac{8}{r - d_{K(\bar{z})}(\bar{x})} \|u - \bar{x}\|^2 + d_{K(\bar{z})}(u) - d_{K(\bar{z})}(\bar{x}) \quad \text{for all} \quad u \in \bar{x} + \delta' \mathbf{B}.$$

This ensures that $\bar{\xi} \in \partial^P d_{K(\bar{z})}(\bar{x})$ and so the proof of the theorem is complete. □

Remark 6.1. As a direct consequence of this theorem we have the upper semicontinuity of the set-valued mapping $(z, x) \mapsto \partial^P d_{K(z)}(x)$ from $T \times \mathbf{H}$ to $\mathbf{H}$ endowed with the weak topology, which is equivalent (see for example Proposition 1.4.1 and Theorem 1.4.2 in [9]) to the u.s.c. of the function $(z, x) \mapsto \sigma(\partial^P d_{K(z)}(x), p)$ for any $p \in \mathbf{H}$. Here, $\sigma(S, p)$ denotes the support function associated with S, i.e., $\sigma(S, p) := \sup_{s \in S} \langle s, p \rangle$.

6.2 Existence Theorems: Fixed Point Approach

Throughout this section $\mathbf{H}$ will be a *finite dimensional space*.

In this section, we prove several existence theorems for second order nonconvex sweeping processes with a perturbation. The method used is to follow an astute idea proposed in [246] that consists to use existence results for the first order sweeping process and standard methods of fixed point theorem. So, we begin by recalling an existence result for the first order nonconvex sweeping process proved in [58] (see Chap. 5).

Theorem 6.2. *Let $T > 0$ and $r \in (0, +\infty]$. Let $C : I := [0, T] \rightrightarrows \mathbf{H}$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in $\mathbf{H}$. Let $F : I \times \mathbf{H} \rightrightarrows \mathbf{H}$ be a set-valued mapping with convex closed values in $\mathbf{H}$ such that $F(t, \cdot)$ is u.s.c. on $\mathbf{H}$ for any fixed $t \in I$ and $F(\cdot, x)$ is Lebesgue* *measurable on I for any fixed $x \in \mathbf{H}$. Assume that F is bounded by $m \geq 0$. Then, for every $x_0 \in C(0)$, there exists at least one Lipschitz mapping $u : I \to \mathbf{H}$ satisfying*

$$\begin{cases} -\dot{u}(t) \in N^P(C(t);u(t)) + F(t,u(t)), & \text{a.e. on } I, \\ u(t) \in C(t), \ \ \forall t \in I, \\ u(0) = x_0, \end{cases}$$

and, $\| \dot{u}(t) \| \leq \lambda + m$ a.e. on I.

Now, we are in position to prove the first existence theorem in this section for the second order sweeping process with convex upper semicontinuous perturbation.

Theorem 6.3. *Let Ω be an open subset of $\mathbf{H}$, $r \in (0,+\infty]$, and $K : \Omega \rightrightarrows \mathbf{H}$ be a set-valued mapping with nonempty uniformly r-prox-regular values. Assume that K is Lipschitz with ratio $\lambda > 0$ and let $l := \sup_{x \in \Omega} | K(x) | < +\infty$. Let $F : \mathbf{R}_+ \times \mathbf{H} \rightrightarrows \mathbf{H}$ be a set-valued mapping with convex closed values in $\mathbf{H}$ such that $F(t,\cdot)$ is upper semicontinuous on $\mathbf{H}$ for any fixed $t \in \mathbf{R}_+$ and $F(\cdot,x)$ is Lebesgue measurable on $\mathbf{R}_+$ for any fixed $x \in \mathbf{H}$. Assume that F is bounded by $m \geq 0$. Then for all $x_0 \in \Omega$ and $u_0 \in K(x_0)$, there exist $T > 0$, two Lipschitz mappings $x : I := [0,T] \to \Omega$ and $u : I \to \mathbf{H}$ such that*

$$\begin{cases} x(t) = x_0 + \int_0^t u(s).\mathrm{d}s, \ \ \forall t \in I, \\ -\dot{u}(t) \in N^P(K(x(t));u(t)) + F(t,u(t)), \ \ \text{a.e. on } I, \\ u(t) \in K(x(t)), \ \ \forall t \in I, \ \ \text{and} \ \ u(0) = u_0, \end{cases} \qquad (SOSPP1)$$

with $\|\dot{x}(t)\| \leq l$ and $\|\dot{u}(t)\| \leq l\lambda + m$ a.e. on I. In other words, there is a Lipschitz solution $x : I \to \mathbf{H}$ to the Cauchy problem for the following second order nonconvex sweeping process with a convex perturbation:

$$\begin{cases} -\ddot{x}(t) \in N^P(K(x(t));\dot{x}(t)) + F(t,\dot{x}(t)), & \text{a. e. on } I, \\ \dot{x}(t) \in K(x(t)), \ \ \forall t \in I, \\ x(0) = x_0, \ \ \text{and} \ \ \dot{x}(0) = u_0. \end{cases}$$

Proof. Let $x_0 \in \Omega$, $u_0 \in K(x_0)$ and $T > 0$ such that $x_0 + lT\mathbf{B} \subset \Omega$.
Put $I := [0,T]$ and

$$\mathscr{X} := \left\{ x \in \mathscr{C}(I,\mathbf{H}) : x(t) = x_0 + \int_0^t \dot{x}(s)\mathrm{d}s, \forall t \in I \text{ and } \| \dot{x}(t) \| \leq l \text{ a.e. on } I \right\},$$

$$\mathscr{U} := \left\{ u \in \mathscr{C}(I,\mathbf{H}) : u(t) = u_0 + \int_0^t \dot{u}(s)\mathrm{d}s, \forall t \in I \text{ and } \| \dot{u}(t) \| \leq \lambda l + m \text{ a.e. on } I \right\}.$$

By Arzela–Ascoli's theorem $\mathscr{X}$ and $\mathscr{U}$ are convex compacts sets in $\mathscr{C}(I,\mathbf{H})$ and because of the choice of T one has $x(t) \in \Omega$ for all $x \in \mathscr{X}$ and all $t \in I$.

It is easily checked that for all $f \in \mathscr{X}$ the set-valued mapping $K \circ f$ is Lipschitz with ratio λl. Then for any $f \in \mathscr{X}$ there exists by Theorem 6.2, an integrable mapping $g \in L^1(I,\mathbf{H})$ and a Lipschitz mapping u_f satisfying

$$\begin{cases} -\dot{u}_f(t) \in N^P(K(f(t)); u_f(t)) + g(t), \ \text{ a. e. on } I, \\ g(t) \in F(t, u_f(t)), \ \text{ a. e. on } I, \\ u_f(t) \in K(f(t)), \ \forall t \in I, \ \text{ and } \ u_f(0) = u_0, \end{cases} \qquad (FOSPP1)$$

with $\|\dot{u}_f(t)\| \leq \lambda l + m$.

Let us consider the set-valued mapping $\Phi : \mathscr{X} \rightrightarrows \mathscr{U}$ such that

$$\Phi(f) = \{u_f \in \mathscr{U} : u_f \text{ is a solution of } (FOSPP1)\}.$$

First, we have to show that the graph $gph\,\Phi = \{(f, u_f) \in \mathscr{X} \times \mathscr{U} : u_f \in \Phi(f)\}$ of the set-valued mapping Φ is closed in $\mathscr{X} \times \mathscr{U}$. Let $(f_n, u_{f_n}) \in gph\,\Phi$ such that (f_n, u_{f_n}) converges uniformly to (f, w) in $\mathscr{X} \times \mathscr{U}$. We have to show that $(f, w) \in gph\,\Phi$. For each $n \in \mathbf{N}$, there exists an integrable mapping $g_n \in L^1(I,\mathbf{H})$ satisfying

$$\begin{cases} -\dot{u}_{f_n}(t) \in N^P(K(f_n(t)); u_{f_n}(t)) + g_n(t), \ \text{ a. e on } I, \\ g_n(t) \in F(t, u_{f_n}(t)), \ \text{ a. e on } I, \\ u_{f_n}(t) \in K(f_n(t)), \ \forall t \in I, \ \text{ and } \ u_{f_n}(0) = u_0, \end{cases} \tag{6.2}$$

with $\| \dot{u}_{f_n}(t) \| \leq \lambda l + m$.

Since g_n is bounded in $L^\infty(I,\mathbf{H})$, we may assume, by extracting a subsequence that g_n converges $*$-weakly to some mapping g in $L^\infty(I,\mathbf{H})$. On the other hand $g_n(t) \in F(t, u_{f_n}(t))$ and $u_{f_n} \longrightarrow w$, then by Theorem 1.4.1 in [9] one gets $g(t) \in F(t, w(t))$.

By (6.2) one has $u_{f_n}(t) \in K(f_n(t))$ for all $t \in I$. It follows from the Lipschitz property of K

$$d_{K(f(t))}(u_{f_n}(t)) \leq \lambda \|f(t) - f_n(t)\| \longrightarrow 0$$

and hence, one obtains $w(t) \in K(f(t))$, because the set $K(f(t))$ is closed.

By (6.2) once again one has

$$\dot{u}_{f_n}(t) + g_n(t) \in -N^P(K(f_n(t)); u_{f_n}(t)), \tag{6.3}$$

and

$$\| \dot{u}_{f_n}(t) + g_n(t) \| \leq \lambda l + 2m =: \alpha, \quad \text{i.e.,} \quad \dot{u}_{f_n}(t) + g_n(t) \in \alpha \mathbf{B}.$$

Therefore, we get

$$\dot{u}_{f_n}(t) + g_n(t) \in -\alpha \partial^P d_{K(f_n(t))}(u_{f_n}(t)) \ \text{ a.e } \ t \in I. \tag{6.4}$$

Now, as $(\dot{u}_{f_n}+g_n)$ converges weakly to $\dot{w}+g$ in $L^1(I,\mathbf{H})$, Mazur's lemma ensures that for a.e $t\in I$

$$\dot{w}(t)+g(t)\in\bigcap_n\overline{co}\{\dot{u}_{f_k}(t)+g_k(t):k\geq n\}.$$

Fix such t in I and any μ in $\mathbf{H}$, then the last relation gives

$$\langle\dot{w}(t)+g(t),\mu\rangle\leq\inf_n\sup_{k\geq n}\langle\dot{u}_{f_k}(t)+g_k(t),\mu\rangle$$

and hence according to (6.2)

$$\begin{aligned}\langle\dot{w}(t)+g(t),\mu\rangle&\leq\limsup_n\sigma(-\alpha\partial^P d_{K(f_n(t))}(u_{f_n}(t)),\mu)\\&\leq\sigma(-\alpha\partial^P d_{K(f(t))}(w(t)),\mu),\end{aligned}$$

where the second inequality follows from Remark 6.1 and Theorem 6.2.

As the set $\partial^P d_{K(f(t))}(w(t))$ is closed and convex, we obtain

$$\dot{w}(t)+g(t)\in-\alpha\partial^P d_{K(f(t)}(w(t))\subset-N^P(K(f(t));w(t)),$$

because $w(t)\in K(f(t))$.

This can be rephrased as

$$\begin{cases}-\dot{w}(t)\in N^P(K(f(t));w(t))+g(t), & \text{a. e. on } I,\\ g(t)\in F(t,w(t)), & \text{a. e. on } I,\\ w(t)\in K(f(t)), & \forall t\in I, \text{ and } w(0)=u_0.\end{cases}$$

In other words, w is of the form u_f with

$$\begin{cases}-\dot{u}_f(t)\in N^P(K(f(t));u_f(t))+F(t,u_f(t)), & \text{a. e. on } I,\\ u_f(t)\in K(f(t)), \quad \forall t\in I, & \text{and } u_f(0)=u_0.\end{cases}$$

Then $gph\,\boldsymbol{\Phi}$ is closed in $\mathscr{X}\times\mathscr{U}$.

Now, let us consider the set-valued mapping $A:\mathscr{X}\rightrightarrows\mathscr{C}(I,\mathbf{H})$ defined by

$$A(f)=\left\{x_f\in\mathscr{C}(I,\mathbf{H}):x_f(t)=x_0+\int_0^t u_f(s)\mathrm{d}s \text{ and } u_f\in\boldsymbol{\Phi}(f)\right\}.$$

Then A is a set-valued mapping with convex compact values in $\mathscr{C}(I,\mathbf{H})$. Observe that for any $x_f\in A(f)$ and for a.e. $t\in I$, one has $\dot{x}_f(t)=u_f(t)\in K(f(t))\subset l\mathbf{B}$. Then $x_f\in\mathscr{X}$ and so $A(f)\subset\mathscr{X}$. Moreover, the closedness of gph$(\boldsymbol{\Phi})$ in $\mathscr{X}\times\mathscr{U}$ ensures the closedness of the graph gph$A$ of A in $\mathscr{X} \times \mathscr{X}$. Consequently we get the upper semicontinuity of A and so by Kakutani–Ky Fan's theorem, the set-valued mapping A admits a fixed point, i.e., there exits $f \in \mathscr{X}$ such that $f \in A(f)$ and hence

$$\begin{cases} -\dot{u}_f(t) \in N^P(K(f(t)); u_f(t)) + F(t, u_f(t)), \text{ a. e. on } I, \\ u_f(t) \in K(f(t)), \ \forall t \in I, \text{ and } u_f(0) = u_0, \\ f(t) = x_0 + \int_0^t u_f(s)\mathrm{d}s, \ \forall t \in I. \end{cases}$$

Thus completing the proof. □

Observe that the set-valued mapping F in Theorem 6.3 is assumed to be bounded by some positive real number m. It would be interesting to have existence results for $(SOSPP1)$ in the case when F is an unbounded set-valued mapping. In what follows we will give a positive answer of this question for a particular type of unbounded set-valued mapping F. We begin first by proving the following existence result of a first order sweeping process with unbounded perturbation.

Theorem 6.4. *Let $T > 0$ and $r \in (0, +\infty]$. Let $C : I := [0, T] \rightrightarrows \mathbf{H}$ be an absolutely continuous set-valued mapping with nonempty closed uniformly r-prox-regular values, and $\gamma : I \to \mathbf{R}$ be an integrable function. Assume that $l := \sup\{\|C(t)\| : t \in I\} < \infty$. Then, for every $x_0 \in C(0)$, there exists one and only one absolutely continuous mapping $u : I \to \mathbf{H}$ satisfying*

$$\begin{cases} -\dot{u}(t) \in N^P(C(t); u(t)) + \gamma(t)u(t), \text{ a.e. on } I, \\ u(t) \in C(t), \text{ for all } t \in I, \\ u(0) = x_0. \end{cases} \qquad (FOSPP2)$$

and, $\| \dot{u}(t) \| \leq \kappa\dot{a}(t) + 2l|\gamma(t)|$ for almost every $t \in I$, with $\kappa := \beta \max\{\mathrm{e}^{-\int_0^s \gamma(z)\mathrm{d}z} : s \in I\}$, and $\beta = \max\{\mathrm{e}^{\int_0^s \gamma(z)\mathrm{d}z} : s \in I\}$.

Proof. We follow the idea of the proof of Theorem 4.1 in [75]. Let us define, for all $t \in I$:

$$\varphi(t) := \int_0^t \gamma(s)\mathrm{d}s, \ v(t) := \mathrm{e}^{\varphi(t)}u(t), \text{ and } D(t) := \mathrm{e}^{\varphi(t)}C(t).$$

Then it easily seen that φ is an absolutely continuous function from I to $\mathbf{R}$. Now, we prove that the new set-valued mapping D is absolutely continuous. Fix any $s, t \in I$ with $s < t$. Then, we have

$$\begin{aligned} \mathscr{H}(D(t), D(s)) &= \mathscr{H}(\mathrm{e}^{\varphi(t)}C(t), \mathrm{e}^{\varphi(s)}C(s)) \\ &\leq \mathscr{H}(\mathrm{e}^{\varphi(t)}C(t), \mathrm{e}^{\varphi(s)}C(t)) + \mathscr{H}(\mathrm{e}^{\varphi(s)}C(t), \mathrm{e}^{\varphi(s)}C(s)) \\ &\leq |\mathrm{e}^{\varphi(t)} - \mathrm{e}^{\varphi(s)}| \|C(t)\| + \mathrm{e}^{\varphi(s)}\mathscr{H}(C(t), C(s)). \end{aligned}$$

Since C is bounded one has $\|C(t)\| \leq l$ for all $t \in I$. Hence, we have

$$\mathscr{H}(D(t),D(s)) \leq l\left|\mathrm{e}^{\varphi(t)} - \mathrm{e}^{\varphi(s)}\right| + \mathrm{e}^{\varphi(s)}\left[a(t)-a(s)\right]$$
$$\leq l\int_s^t |\dot{\varphi}(\tau)|\mathrm{e}^{\varphi(\tau)}\mathrm{d}\tau + \beta\left[a(t)-a(s)\right].$$

Thus,

$$\mathscr{H}(D(t),D(s)) \leq g(t)-g(s), \quad \text{for all} \quad (0 \leq s \leq t \leq T),$$

where $g(t) := l\int_0^t |\dot{\varphi}(\tau)|\mathrm{e}^{\varphi(\tau)}\mathrm{d}\tau + \beta a(t)$ defines an absolutely continuous nondecreasing function on I.

Now, if u is absolutely continuous, then v is absolutely continuous with derivative

$$\dot{v}(t) = \mathrm{e}^{\varphi(t)}\left[\dot{u}(t) + \gamma(t)u(t)\right] \text{ a.e. on } I.$$

Observe that for any $\alpha > 0$ and any $S \subset \mathbf{H}$ one always has $d_{\alpha S}(\alpha x) = \alpha d_S(x)$ for all $x \in \mathbf{H}$ and $N^P(S;\bar{x}) = N^P(\alpha S;\alpha\bar{x})$ for all $\bar{x} \in S$. Thus, by using these two equalities and the fact that $\varphi(0) = 0$, it is easily seen that u solves $(FOSPP2)$ if and only if v solves

$$\begin{cases} -\dot{v}(t) \in N^P(D(t);v(t)), & \text{a.e. on } I, \\ v(t) \in D(t), & \text{for all } t \in I, \\ v(0) = x_0 \in D(0). \end{cases} \qquad (FOSP)$$

Now, using the characterization of the uniform prox-regularity proved in Theorem 3.1 in [58](see Chap. 2) one can check that for any $\alpha > 0$ one has a subset S is uniformly r-prox-regular if and only if αS is uniformly αr-prox-regular. Consequently, the new set-valued mapping D has prox-regular values. Thus all the hypothesis of Theorem 4.1 in [58] (see Chap. 5) are fulfilled and so there is a unique solution of $(FOSP)$ satisfying

$$\|\dot{v}(t)\| \leq \dot{g}(t) \text{ a.e. on } I.$$

Therefore, there is a unique solution of $(FOSPP2)$ satisfying for a.e. on I

$$\|\dot{u}(t)\| \leq |\gamma(t)|\|u(t)\| + \mathrm{e}^{-\varphi(t)}\|\dot{v}(t)\| \leq 2l|\gamma(t)| + \kappa\dot{a}(t). \text{ a.e. on } I.$$

This completes the proof. □

Now, we are ready to prove the second main theorem in this section.

Theorem 6.5. *Let Ω be an open subset of $\mathbf{H}$, $r \in (0,+\infty]$, and $K : \Omega \rightrightarrows \mathbf{H}$ be a set-valued mapping with nonempty uniformly r-prox-regular values. Assume that K is Lipschitz with ratio $\lambda > 0$ and $l := \sup_{x\in\Omega} \mid K(x) \mid < +\infty$. Let $\gamma \neq 0$. Then for all $x_0 \in \Omega$ and $u_0 \in K(x_0)$, there exist $T > 0$, two Lipschitz mappings, $x : I := [0,T] \to \Omega$ and $u : I \to \mathbf{H}$ such that*

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)\mathrm{d}s, \quad \forall t \in I, \\ -\dot{u}(t) \in N^P(K(x(t)); u(t)) + \gamma u(t), \quad \text{a.e. on } I, \\ u(t) \in K(x(t)), \quad \forall t \in I, \quad \text{and} \quad u(0) = u_0, \end{cases} \qquad (SOSPP2)$$

with $\|\dot{x}(t)\| \leq l$ and $\|\dot{u}(t)\| \leq l\lambda \mathrm{e}^{|\gamma|T} + 2l|\gamma|$ a.e. on I. In other words, there is a Lipschitz solution $x : [0,T] \to \mathbf{H}$ to the Cauchy problem for the following second order nonconvex sweeping process with an unbounded perturbation:

$$\begin{cases} -\ddot{x}(t) \in N^P(K(x(t)); \dot{x}(t)) + \gamma\dot{x}(t), \quad \text{a. e. on } I, \\ \dot{x}(t) \in K(x(t), \quad \forall t \in I, \\ x(0) = x_0, \quad \text{and} \quad \dot{x}(0) = u_0. \end{cases}$$

Proof. Let $x_0 \in \Omega$ and $T > 0$ such that $x_0 + lT\mathbf{B} \subset \Omega$. Put $I := [0,T]$ and

$$\mathscr{X} := \left\{ x \in \mathscr{C}(I, \mathbf{H}) : x(t) = x_0 + \int_0^t \dot{x}(s)\mathrm{d}s, \forall t \in I \text{ and } \parallel \dot{x}(t) \parallel \leq l \text{ a.e. on } I \right\}.$$

Then $\mathscr{X}$ is a convex compact set in $\mathscr{C}(I,\mathbf{H})$ and because of the choice of T one has $x(t) \in \Omega$ for all $x \in \mathscr{X}$ and all $t \in I$.

It is easily checked that for all $f \in \mathscr{X}$ the set-valued mapping $K \circ f$ is Lipschitz with ratio λl. Then for any $f \in \mathscr{X}$ there exists by Theorem 6.4, a unique Lipschitz mapping u_f satisfying

$$\begin{cases} -\dot{u}_f(t) \in N^P(K(f(t)); u_f(t)) + \gamma u_f(t), \quad \text{a. e. on } I, \\ u_f(t) \in K(f(t)), \quad \forall t \in I, \quad \text{and} \quad u_f(0) = u_0, \end{cases}$$

and $\parallel \dot{u}_f(t) \parallel \leq L := \mathrm{e}^{|\gamma|T}\lambda l + 2l|\gamma|$.

First, we prove that the mapping $f \in \mathscr{X} \mapsto u_f$ is continuous. Indeed, let (f_n) be a sequence of mappings of $\mathscr{X}$ converging uniformly to $f \in \mathscr{X}$. Then, for each $n \in \mathbf{N}$, there exists a unique Lipschitz mapping u_{f_n} satisfying

$$\begin{cases} -\dot{u}_{f_n}(t) \in N^P(K(f_n(t)); u_{f_n}(t)) + \gamma u_{f_n}(t), \quad \text{a. e on } I, \\ u_{f_n}(t) \in K(f_n(t)), \quad \forall t \in I, \quad \text{and} \quad u_{f_n}(0) = u_0, \end{cases} \qquad (6.5)$$

and $\parallel \dot{u}_{f_n}(t) \parallel \leq L$. Thus, since $u_{f_n}(t) \in l\mathbf{B}$ for all $n \in \mathbf{N}$ and all $t \in I$, by Theorem 0.3.4 in [9] there exists a subsequence of (u_{f_n}) again denoted (u_{f_n}) such that

(i) (u_{f_n}) converges uniformly to a Lipschitz mapping $w \in \mathscr{C}(I,\mathbf{H})$,
(ii) $(\dot{u}_{f_n})$ converges weakly in $L^1(I,\mathbf{H})$ to $\dot{w}$, and
(iii) $w(t) = u_0 + \int_0^t \dot{w}(s)\mathrm{d}s$.

Now, we wish to prove that w satisfies $(SOSPP2)$, i.e., $w = u_f$.

By (6.5) one has for a.e. $t \in I$

$$\dot{u}_{f_n}(t) + \gamma u_{f_n}(t) \in -N^P(K(f_n(t))); u_{f_n}(t)). \tag{6.6}$$

By (6.5) once again one has $u_{f_n}(t) \in K(f_n(t))$ for all $t \in I$. It follows from the Lipschitz property of K

$$d_{K(f(t))}(u_{f_n}(t)) \leq \lambda \| f(t) - f_n(t) \| \longrightarrow 0,$$

and hence, one obtains $w(t) \in K(f(t))$, because the set $K(f(t))$ is closed.

On the other hand we have

$$\| \dot{u}_{f_n}(t) + \gamma u_{f_n}(t) \| \leq L + l|\gamma|, \quad \text{i.e.,} \quad \dot{u}_{f_n}(t) + \gamma u_{f_n}(t) \in (L + l|\gamma|)\mathbf{B},$$

then by (6.6), we obtain

$$\dot{u}_{f_n}(t) + \gamma u_{f_n}(t) \in -(L + l|\gamma|)\partial^P d_{K(f_n(t))}(u_{f_n}(t)) \ \text{ a.e } \ \in I. \tag{6.7}$$

Now, as $(\dot{u}_{f_n} + \gamma u_{f_n})$ converges weakly to $\dot{w} + \gamma w$ in $L^1(I, \mathbf{H})$, Mazur's lemma ensures that for a.e $t \in I$

$$\dot{w}(t) + \gamma w(t) \in \bigcap_n \overline{co}\{\dot{u}_{f_k}(t) + \gamma u_{f_k}(t) : k \geq n\}.$$

Fix such t in I and any μ in $\mathbf{H}$, then the last relation gives

$$\langle \dot{w}(t) + \gamma w(t), \mu \rangle \leq \inf_n \sup_{k \geq n} \langle \dot{u}_{f_k}(t) + \gamma u_{f_k}(t), \mu \rangle,$$

and hence according to (6.7) (for $\alpha := L + l|\gamma|$)

$$\begin{aligned} \langle \dot{w}(t) + \gamma w(t), \mu \rangle &\leq \limsup_n \sigma(-\alpha \partial^P d_{K(f_n(t))}(u_{f_n}(t)), \mu) \\ &\leq \sigma(-\alpha \partial^P d_{K(f(t))}(w(t)), \mu) \end{aligned}$$

where the second inequality follows from Remark 6.1 and Theorem 6.2.

As the set $\partial^P d_{K(f(t))}(w(t))$ is closed and convex (for uniformly prox-regular sets), we obtain

$$\dot{w}(t) + \gamma w(t) \in -(L + l|\gamma|)\partial^P d_{K(f(t))} w((t))$$

and then

$$\dot{w}(t) + \gamma w(t) \in -N^P(K(f(t)); w(t)),$$

because $w(t) \in K(f(t))$.

This can be rephrased as

$$\begin{cases} -\dot{w}(t) \in N^P(K(f(t)); w(t)) + \gamma w(t), \ \text{a. e. on } I, \\ w(t) \in K(f(t)), \ \forall t \in I, \ \text{and } w(0) = u_0. \end{cases}$$

In other words, w is of the form u_f with

$$\begin{cases} -\dot{u}_f(t) \in N^P(K(f(t)); u_f(t)) + \gamma u_f(t), \ \text{a. e. on } I, \\ u_f(t) \in K(f(t)), \ \forall t \in I, \ \text{and } u_f(0) = u_0. \end{cases}$$

Let us consider the mapping $h : \mathscr{X} \to \mathscr{C}(I, \mathbf{H})$ defined by $h(f)(t) = x_0 + \int_0^t u_f(s)\mathrm{d}s$ for all $f \in \mathscr{X}$ and all $t \in I$. Then, by what precedes the mapping h is continuous from $\mathscr{X}$ to $\mathscr{C}(I, \mathbf{H})$. By our hypothesis one has

$$u_f(t) \in K(f(t)) \subset l\mathbf{B},$$

for all $f \in \mathscr{X}$ and all $t \in I$. It follows then that $h(f) \in \mathscr{X}$ for all $f \in \mathscr{X}$. By Schauder's theorem, the mapping h admits a fixed point, i.e., there exists $f \in \mathscr{X}$ such that $h(f) = f$ and hence

$$\begin{cases} -\dot{u}_f(t) \in N^P(K(f(t)); u_f(t)) + \gamma u_f(t), \ \text{a. e. on } I, \\ u_f(t) \in K(f(t)), \ \forall t \in I, \ \text{and } u_f(0) = u_0, \\ f(t) = x_0 + \int_0^t u_f(s)\mathrm{d}s, \ \forall t \in I. \end{cases}$$

Thus, the proof of the theorem is complete. □

We close this section with an existence result of the first order nonconvex sweeping process with unbounded perturbation when the uniform prox-regularity assumption on the values of the set-valued mapping K are replaced by the local compactness of the graph of K. Using the same techniques in our proof of Theorem 6.4 and an existence result in Hilbert spaces of the first order nonconvex sweeping process without perturbation in Theorem 4.2 in [92] (see also [16]), we can prove the following result. Note that the local compactness of the graph of K needed to apply Theorem 4.2 in [92], is satisfied in our case since $\mathbf{H}$ is a finite dimensional space.

Theorem 6.6. *Let $T > 0$, $C : I := [0,T] \rightrightarrows \mathbf{H}$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ and with nonempty closed values, and $\gamma : I \to \mathbf{R}$ be a continuous function. Then, for every $x_0 \in C(0)$, there exists at least one Lipschitz mapping $u : I \to \mathbf{H}$ satisfying*

$$\begin{cases} -\dot{u}(t) \in N^P(C(t); u(t)) + \gamma(t)u(t) \ \text{a.e. } t \in I, \\ u(t) \in C(t), \ \forall t \in I, \ \text{and } u(0) = x_0, \end{cases}$$

and $\| \dot{u}(t) \| \leq \kappa\lambda + 2|\gamma(t)|.\max\{\|C(t)\|, t \in I\}$ *for almost every* $t \in I$ *with* $\kappa = \beta \max\{e^{-\int_0^t \gamma(s)\mathrm{d}s} : t \in I\}$ *and* $\beta = \max\{e^{\int_0^t \gamma(s)\mathrm{d}s} : t \in I\}$.

Proof. It is omitted. □

6.3 Existence Theorems: Direct Approach

In the present and next section, let $r \in (0, +\infty]$, $x_0 \in \mathbf{H}$, $u_0 \in K(x_0)$, $\mathscr{V}_0$ be an open neighborhood of x_0 in $\mathbf{H}$, and $K : \mathrm{cl}\,(\mathscr{V}_0) \rightrightarrows \mathbf{H}$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in $\mathbf{H}$. Our aim in this section is to prove the local existence of $(SSPMP)$ on $\mathrm{cl}\,(\mathscr{V}_0)$, that is, there exists $T > 0$, Lipschitz mappings $x : [0,T] \to \mathrm{cl}\,(\mathscr{V}_0)$ and $u : [0,T] \to \mathbf{H}$ such that

$$\begin{cases} u(0) = u_0, \ u(t) \in K(x(t)), \quad \text{for all } t \in [0,T], \\ x(t) = x_0 + \displaystyle\int_0^t u(s)\mathrm{d}s, \quad \text{for all } t \in [0,T], \\ \dot{u}(t) \in -N^P(K(x(t)); u(t)) + F(t, x(t), u(t)) + G(t, x(t), u(t)), \quad \text{a. e. } [0,T]. \end{cases}$$

We begin by recalling the following lemma proved in [123, 124].

Lemma 6.1. *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces and let* $h : X \to Y$ *be a uniformly continuous mapping. Then for every sequence* $(\varepsilon_n)_{n\geq 1}$ *of positive numbers there exists a strictly decreasing sequence of positive numbers* $(e_n)_{n\geq 1}$ *converging to* 0 *such that*

1. *for any* $n \geq 2$, $\frac{1}{e_{n-1}}$ *and* $\frac{e_{n-1}}{e_n}$ *are integers* ≥ 2*;*
2. *for any* $n \geq 1$*, and any* $x_1, x_2 \in X$*, one has*

$$d_X(x_1, x_2) \leq e_n \Longrightarrow d_Y(h(x_1), h(x_2)) \leq \varepsilon_n.$$

We prove the first main theorem in this section.

Theorem 6.7. *Let* $G, F : [0, +\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ *be two set-valued mappings and let* $\varsigma > 0$ *such that* $x_0 + \varsigma\mathbf{B} \subset \mathscr{V}_0$*. Assume that the following assumptions are satisfied:*

(i) For all $x \in \mathrm{cl}(\mathscr{V}_0)$*,* $K(x) \subset \mathscr{K}_1 \subset l\mathbf{B}$*, for some convex compact set* $\mathscr{K}_1$ *in* $\mathbf{H}$ *and some* $l > 0$*;*
(ii) F *is scalarly u.s.c. on* $[0, \varsigma/l] \times \mathrm{gph}K$ *with nonempty convex weakly compact values;*
(iii) G *is uniformly continuous on* $[0, \varsigma/l] \times \alpha\mathbf{B} \times l\mathbf{B}$ *into nonempty compact subsets of* $\mathbf{H}$*, for* $\alpha := \|x_0\| + \varsigma$*;*
(iv) F *and* G *satisfy the linear growth condition, that is,*

$$F(t, x, u) \subset \rho_1(1 + \|x\| + \|u\|)\mathbf{B} \ \text{ and } \ G(t, x, u) \subset \rho_2(1 + \|x\| + \|u\|)\mathbf{B},$$

for all $(t,x,u) \in [0,\varsigma/l] \times \mathrm{gph}K$ *for some* $\rho_1,\rho_2 \geq 0$. *Then for every* $T \in (0,\varsigma/l]$ *there exist Lipschitz mappings* $x : [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ *and* $u : [0,T] \to \mathbf{H}$ *such that*

$$\begin{cases} u(0) = u_0,\, u(t) \in K(x(t)),\, x(t) = x_0 + \displaystyle\int_0^t u(s)\mathrm{d}s, \text{for all } t \in [0,T], \\ \dot{u}(t) \in -N^P(K(x(t));u(t)) + F(t,x(t),u(t)) + G(t,x(t),u(t)), \text{a.e. } [0,T], \end{cases} \tag{6.8}$$

with $\|\dot{x}(t)\| \leq l$ *and* $\|\dot{u}(t)\| \leq l\lambda + 2(1+\alpha+l)(\rho_1+\rho_2)$ *a.e. on* $[0,T]$. *In other words, there is a Lipschitz solution* $x : [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ *to the Cauchy problem for the second order differential inclusion:*

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t));\dot{x}(t)) + F(t,x(t),\dot{x}(t)) + G(t,x(t),\dot{x}(t)), \text{ a. e. } [0,T], \\ x(0) = x_0, \dot{x}(0) = u_0, \dot{x}(t) \in K(x(t)), \text{ for all } t \in [0,T], \end{cases}$$

with $\|\dot{x}(t)\| \leq l$ *and* $\|\ddot{x}(t)\| \leq l\lambda + 2(\rho_1+\rho_2)(1+\alpha+l)$ *a.e. on* $[0,T]$.

Proof. We give the proof in four steps.

Step 1. Construction of the approximants.

Let $T \in (0,\varsigma/l]$ and put $I := [0,T]$ and $\mathscr{K} := I \times \alpha\mathbf{B} \times l\mathbf{B}$. Then by the assumption (iv) we have

$$\|F(t,x,u)\| \leq \rho_1(1+\|x\|+\|u\|) \leq \rho_1(1+\alpha+l) =: \zeta_1, \tag{6.9}$$

and

$$\|G(t,x,u)\| \leq \rho_2(1+\|x\|+\|u\|) \leq \rho_2(1+\alpha+l) =: \zeta_2, \tag{6.10}$$

for all $(t,x,u) \in \mathscr{K} \cap (I \times \mathrm{gph}K)$. Note that $\mathscr{K} \cap (I \times \mathrm{gph}K) \neq \emptyset$ because $(x_0,u_0) \in (\alpha\mathbf{B} \times l\mathbf{B}) \cap \mathrm{gph}K$.

Let $\varepsilon_n = \frac{1}{2^n}$, $(n = 1,2,....)$. Then by the uniform continuity of G on the set $\mathscr{K}$ and Lemma 6.1, there is a strictly decreasing sequence of positive numbers (e_n) converging to 0 such that $e_n \leq 1$, and $\frac{T}{e_{n-1}}$ and $\frac{e_{n-1}}{e_n}$ are integers ≥ 2 and the following implication holds:

$$\|(t,x,u)-(t',x',u')\| \leq \eta e_n \Longrightarrow \mathscr{H}(G(t,x,u),G(t',x',u')) \leq \varepsilon_n, \tag{6.11}$$

for every $(t,x,u),(t',x',u') \in \mathscr{K}$ where $\|(t,x,u)\| = |t|+\|x\|+\|u\|$ and $\eta = (1+3l+l\lambda+2(\zeta_1+\zeta_2))$.

As the sequence $e_n \longrightarrow 0^+$, one can fix a positive integer n_0 such that

$$(\lambda l + \zeta_1 + \zeta_2)e_{n_0} \leq \frac{r}{2}. \tag{6.12}$$

For each $n \geq n_0$, we consider the partition of I given by

$$P_n = \left\{ t_{n,i} = ie_n : i = 0,1,...,\mu_n = \frac{T}{e_n} \right\}. \tag{6.13}$$

We recall (see [123]) some important properties of the sequence of partitions $(P_n)_n$ needed in the sequel.

(Pr_1) $P_n \subset P_{n+1}$, for all $n \geq n_0$;

(Pr_2) For every $n \geq n_0$ and for every $t_{n,i} \in P_n \setminus P_1$ there exists a unique couple (m, j) of positive integers depending on $t_{n,i}$, such that $n_0 \leq m < n$, $t_{n,i} \notin P_s$ for every $s \leq m$, $t_{n,i} \in P_s$ for every $s > m$, $0 \leq j < \mu_m$ and $t_{m,j} < t_{n,i} < t_{m,j+1}$.

Put $I_{n,i} := [t_{n,i}, t_{n,i+1})$, for all $i = 0, \ldots, \mu_n - 1$ and $I_{n,\mu_n} := \{T\}$. For every $n \geq n_0$ we define the following approximating mappings on each interval $I_{n,i}$ as

$$\begin{cases} u_n(t) := u_{n,i}, \\ x_n(t) = x_0 + \displaystyle\int_0^t u_n(s)\mathrm{d}s, \\ f_n(t) := f_{n,i} \in F(t_{n,i}, x_n(t_{n,i}), u_{n,i}), \text{ and} \\ g_n(t) := g_{n,i} \in G(t_{n,i}, x_n(t_{n,i}), u_{n,i}), \end{cases} \tag{6.14}$$

where $u_{n,0} = u_0$ and for all $i = 0, \ldots, \mu_n - 1$, the point $u_{n,i+1}$ is given by

$$u_{n,i+1} \in \mathrm{Proj}(u_{n,i} + e_n(f_{n,i} + g_{n,i}), K(x_n(t_{n,i+1}))). \tag{6.15}$$

Although the absence of the convexity of the images of K, we have the last equality is well defined. Indeed, as

$$x_n(t_{n,1}) = x_0 + \int_0^{t_{n,1}} u_n(s)\mathrm{d}s \in x_0 + t_{n,1} l\mathbf{B} \subset x_0 + \varsigma\mathbf{B} \subset \mathscr{V}_0,$$

then by the Lipschitz property of K and the relations (i), (6.9), (6.10), (6.14), and (6.15) we get for $x := x_n(t_{n,1})$

$$\begin{aligned} d_{K(x_n(t_{n,1}))}(u_{n,0} + e_n(f_{n,0} + g_{n,0})) &\leq \mathscr{H}(K(x_n(t_{n,0})), K(x_n(t_{n,1}))) + e_n \|f_{n,0} + g_{n,0}\| \\ &\leq \lambda \parallel x_n(t_{n,0}) - x_n(t_{n,1}) \parallel + e_n(\zeta_1 + \zeta_2) \\ &\leq \lambda (t_{n,1} - t_{n,0}) \parallel u_{n,0} \parallel + e_n(\zeta_1 + \zeta_2) \\ &\leq (l\lambda + \zeta_1 + \zeta_2) e_{n_0} \leq \frac{r}{2} < r. \end{aligned} \tag{6.16}$$

and hence as K has uniformly r-prox-regular values, one can choose a point $u_{n,1} \in \mathrm{Proj}(u_{n,0} + e_n(f_{n,0} + g_{n,0}), K(x_n(t_{n,1})))$. Similarly, we can define, by induction, the points $(u_{n,i})_{(0 \leq i \leq \mu_n)}$, $(f_{n,i})_{(0 \leq i \leq \mu_n)}$, and $(g_{n,i})_{(0 \leq i \leq \mu_n)}$.

Let us define $\theta_n(t) := t_{n,i}$, if $t \in I_{n,i}$. Then, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ and the assumption (i) yield for all $t \in I$,

$$u_n(t) \in K(x_n(\theta_n(t))) \subset \mathscr{K}_1 \subset l\mathbf{B}. \tag{6.17}$$

So, all the mappings $x_n(\cdot)$ are Lipschitz with ratio l and they are also equibounded, with $\|x_n\|_\infty \leq \|x_0\| + lT$. Here and thereby $\|x\|_\infty := \sup_{t\in I} \|x(t)\|$. Observe also that for all $n \geq n_0$ and all $t \in I$ one has

$$x_n(t) \in \alpha \mathbf{B} \cap \mathscr{V}_0. \tag{6.18}$$

Indeed, the definition of $x_n(\cdot)$ and $u_n(\cdot)$ ensure that, for all $t \in I$,

$$x_n(t) = x_0 + \int_0^t u_n(s)\mathrm{d}s \in x_0 + tl\mathbf{B} \subset x_0 + \varsigma \mathbf{B} \subset \alpha \mathbf{B} \cap \mathscr{V}_0,$$

and hence $K(x_n(t))$ is well defined for all $t \in I$.

Now we define the piecewise affine approximants

$$v_n(t) := u_{n,i} + e_n^{-1}(t - t_{n,i})(u_{n,i+1} - u_{n,i}), \quad \text{if} \quad t \in I_{n,i}. \tag{6.19}$$

Observe that $v_n(\theta_n(t)) = u_{n,i}$, for all $i = 0, \ldots, \mu_n$ and so by (6.15), (6.18), and the assumption (ii), one has $v_n(\theta_n(t)) \in K(x_n(t_{n,i})) = K(x_n(\theta_n(t))) \subset l\mathbf{B}$. Then by (6.9),(6.10), (6.14), (6.18), and the last relation we obtain for all $t \in I$ and all $n \geq n_0$

$$\begin{cases} f_n(t) \in F(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))) \cap \zeta_1 \mathbf{B} \quad \text{and} \\ \\ g_n(t) \in G(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))) \cap \zeta_2 \mathbf{B}. \end{cases} \tag{6.20}$$

Now we check that the mappings v_n are equi-Lipschitz with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$. Indeed, by (6.15) and the Lipschitz property of K one has

$$\begin{aligned} \| u_{n,i+1} - u_{n,i} \| &\leq \| u_{n,i+1} - (u_{n,i} + e_n(f_{n,i} + g_{n,i})) \| + e_n \| f_{n,i} + g_{n,i} \| \\ &\leq d_{K(x_n(t_{n,i+1}))}(u_{n,i} + e_n(f_{n,i} + g_{n,i})) + (\zeta_1 + \zeta_2)e_n \\ &\leq \mathscr{H}(K(x_n(t_{n,i})), K(x_n(t_{n,i+1}))) + 2(\zeta_1 + \zeta_2)e_n \\ &\leq (l\lambda + 2(\zeta_1 + \zeta_2)e_n, \end{aligned} \tag{6.21}$$

and hence,

$$\| v_n(t) - v_n(s) \| = e_n^{-1} \mid t - s \| \| u_{n,i+1} - u_{n,i} \| \leq (l\lambda + 2(\zeta_1 + \zeta_2)) \mid t - s \mid.$$

It is also clear, by the definitions of $u_n(\cdot)$ and $v_n(\cdot)$, that

$$\| v_n(t) - u_n(t) \| \leq e_n^{-1} \mid t - t_{n,i} \| \| u_{n,i+1} - u_{n,i} \| \leq (l\lambda + 2(\zeta_1 + \zeta_2))e_n, \tag{6.22}$$

and hence

$$\| v_n - u_n \|_\infty \longrightarrow 0.$$

Let us define, $\nu_n(t) := t_{n,i+1}$ if $t \in I_{n,i}$ and $i = 0, \ldots, \mu_n - 1$. The definition of $v_n(\cdot)$ given by (6.19) and the relation (6.15) yield

$$v_n(\nu_n(t)) \in K(x_n(\nu_n(t))), \quad \text{for all} \quad t \in I_{n,i} \ (i = 0, \ldots, \mu_n - 1), \tag{6.23}$$

and for all $t \in I \setminus \{t_{n,i} : i = 0, .., \mu_n\}$ one has

$$\dot{v}_n(t) = e_n^{-1}(u_{n,i+1} - u_{n,i}). \tag{6.24}$$

So, we get for all $t \in I \setminus \{t_{n,i} : i = 0, .., \mu_n\}$

$$e_n(\dot{v}_n(t) - (f_n(t) + g_n(t))) = u_{n,i+1} - (u_{n,i} + e_n(f_{n,i} + g_{n,i}))$$
$$\in \text{Proj}(u_{n,i} + e_n(f_{n,i} + g_{n,i}), K(x_n(t_{n,i+1}))) - (u_{n,i} + e_n(f_{n,i} + g_{n,i})).$$

Then, the properties of the proximal normal cone to subsets, ensure that we have for all $t \in I \setminus \{t_{n,i} : i = 0, .., \mu_n\}$

$$\begin{aligned} \dot{v}_n(t) - (f_n(t) + g_n(t)) &\in -N^P(K(x_n(t_{n,i+1})); u_{n,i+1}) \\ &= -N^P(K(x_n(\nu_n(t))); v_n(\nu_n(t))). \end{aligned} \tag{6.25}$$

On the other hand, by (6.21) and (6.24), it is clear that

$$\|\dot{v}_n(t)\| \leq (l\lambda + 2(\zeta_1 + \zeta_2)). \tag{6.26}$$

Put $\delta := (l\lambda + 3(\zeta_1 + \zeta_2))$. Therefore, the relations (6.20), (6.25), and (6.26), and Theorem 4.1 in [58] entail for all $t \in I \setminus \{t_{n,i} : i = 0, .., \mu_n\}$

$$\dot{v}_n(t) - (f_n(t) + g_n(t)) \in -\delta \partial^P d_{K(x_n(\nu_n(t)))}(v_n(\nu_n(t))). \tag{6.27}$$

Step 2. Uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$.

Since $e_n^{-1}(t - t_{n,i}) \leq 1$, for all $t \in I_{n,i}$ and $u_{n,i}, u_{n,i+1} \in \mathscr{K}_1$, and $\mathscr{K}_1$ is a convex set in $\mathbf{H}$ one gets for all $t \in I$,

$$v_n(t) = u_{n,i} + e_n^{-1}(t - t_{n,i})(u_{n,i+1} - u_{n,i}) \in \mathscr{K}_1.$$

Thus, for every $t \in I$, the set $\{v_n(t) : \ n \geq n_0\}$ is relatively strongly compact in $\mathbf{H}$. Therefore, the estimate (6.26) and Theorem 0.4.4 in [9] ensure that there exists a Lipschitz mapping $u : I \to \mathbf{H}$ with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$ such that
(v_n) converges uniformly to u on I;
$(\dot{v}_n)$ weakly converges to $\dot{u}$ in $L^1(I, \mathbf{H})$.

Now we define the Lipschitz mapping $x : I \to \mathbf{H}$ as

$$x(t) = x_0 + \int_0^t u(s)\mathrm{d}s, \quad \text{for all} \quad t \in I. \tag{6.28}$$

Then by the definition of x_n one obtains for all $t \in I$,

$$\|x_n(t) - x(t)\| = \|\int_0^t (u_n(s) - u(s))\mathrm{d}s\| \leq T\|u_n - u\|_\infty$$

and so by (6.22) we get

$$\|x_n - x\|_\infty \leq T\|u_n - v_n\|_\infty + T\|v_n - u\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{6.29}$$

This completes the second step.

Step 3. Relative strong compactness of (g_n).

The points $(g_{n,i})_{i=,0...,\mu_n}$ defining the step function $g_n(\cdot)$ was chosen arbitrarily in our construction. Nevertheless, by using the uniform continuity of the set-valued mapping G over $\mathscr{K}$ and the techniques of [123] (see also [109, 110]), the sequence $g_n(\cdot)$ can be constructed *relatively strongly compact for the uniform convergence* in the space of bounded functions. The construction of the sequence $g_n(\cdot)$ is similar to the one presented in [109, 110]. We give it here for the completeness and for the reader's convenience.

To prove the relative strong compactness for the uniform convergence in the space of bounded functions we will use a very useful compactness criterion proved in Theorem 0.4.5 in [9]. First, we need to prove that for all $t \in I$, the set $\{g_n(t) : \quad n \geq n_0\}$ is relatively strongly compact in $\mathbf{H}$. By the definition of $\theta_n(\cdot)$ we have for all $t \in I$ and all $n \geq n_0$ $|\theta_n(t) - t| \leq e_n$. Then, $(x_n \circ \theta_n)$ and $(v_n \circ \theta_n)$ converge uniformly on I to x and u respectively. Now, by (6.20) and the continuity of G on $I \times \mathrm{gph} K$ one has

$$\begin{aligned} d_{G(t,x(t),u(t))}(g_n(t)) \leq \mathscr{H}(G(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))),& \\ G(t,x(t),u(t))) \to 0 \text{ as } n \to \infty.& \end{aligned}$$

This implies the relative strong compactness of the set $\{g_n(t) : \quad n \geq n_0\}$ in $\mathbf{H}$ for all $t \in I$ because $G(t,x(t),u(t))$ is a strongly compact set in $\mathbf{H}$. Now, we have to show that the sequence is *an equioscillating* family of bounded functions in the sense of [9]. Recall that a family $\mathscr{F}$ of bounded mappings $x : I \to \mathbf{H}$ is equioscillating if for every $\varepsilon > 0$, there exists a finite partition of I into subintervals J_j $(j = 0, \ldots, m)$ such that for all $x \in \mathscr{F}$ and all $j = 0, \ldots, m$ one has $\omega_{J_j}(x) \leq \varepsilon$, where $\omega_J(x)$ denotes the oscillation of x in J defined by

$$\omega_J(x) := \sup\{\|x(s) - x(t)\| : \quad s,t \in J\}. \tag{6.30}$$

Fix any $\varepsilon > 0$ and let $m_0 \geq n_0$ such that $4\varepsilon_{m_0} \leq \varepsilon$. Consider the finite partition $J_j := [t_{m_0,j}, t_{m_0,j+1})$ $(j = 0, \ldots, \mu_{m_0} - 1)$ of I. We shall prove that

$$\omega_{J_j}(g_n) \leq \varepsilon, \text{ for all } n \geq n_0 \text{ and all } j = 0, \ldots, \mu_{m_0} - 1. \tag{6.31}$$

For that purpose, we have to choose $g_{n,i}$ in (6.14) in such way that the following condition holds for every $n \geq n_0$ and $i = 0,\ldots,\mu_{m_0} - 1$:

$$\begin{aligned} \|g_n(t_{n,i}) - g_n(t_{n,i-1})\| &\leq \varepsilon_n, \text{ if } t_{n,i} \in P_1, \\ \|g_n(t_{n,i}) - g_n(t_{m,p})\| &\leq \varepsilon_m, \text{ if } t_{n,i} \notin P_1, \end{aligned} \tag{6.32}$$

where (m,p) is the unique pair of integers assigned to $t_{n,i}$ such that $m < n$, $t_{n,i} \notin P_j$ for $j \leq m$, $t_{n,i} \in P_j$ for $j > m$ and $t_{m,p} < t_{n,i} < t_{m,p+1}$. For $i = 0$, we take $g_{n,0} \in G(0,x_0,u_0)$. By induction we assume that $g_{n,j} \in G(t_{n,j},x_n(t_{n,j}),u_{n,j})$ have been defined for all $j \in \{0,\ldots,i-1\}$.

If $t_{n,i} \in P_1$, it suffices to take $g_{n,i} \in G(t_{n,i},x_n(t_{n,i}),u_{n,i})$ such that:

$$\|g_{n,i} - g_{n,i-1}\| \leq \mathscr{H}(G(t_{n,i},x_n(t_{n,i}),u_{n,i})),G(t_{n,i-1},x_n(t_{n,i-1}),u_{n,i-1})).$$

Indeed, by virtue of (6.17), (6.21), and (6.26) we have

$$\|(t_{n,i},x_n(t_{n,i}),u_{n,i}) - (t_{n,i-1},x_n(t_{n,i-1}),u_{n,i-1})\| \leq (1+l+l\lambda+2(\zeta_1+\zeta_2))e_n \leq \eta e_n,$$

which in combining with (6.12) gives

$$\|g_n(t_{n,i}) - g_n(t_{n,i-1})\| = \|g_{n,i} - g_{n,i-1}\| \leq \varepsilon_n.$$

If $t_{n,i} \notin P_1$, then $t_{m,p} \in P_n$ (because $m < n$) and so there is a unique integer $q < i$ such that $t_{m,p} = t_{n,q}$. Hence, $t_{n,i} - t_{n,q} = t_{n,i} - t_{m,p} < t_{m,p+1} - t_{m,p} \leq e_m$. This with (6.17) and (6.26) imply

$$\|(t_{n,i},x_n(t_{n,i}),u_{n,i})) - (t_{n,q},x_n(t_{n,q}),u_{n,q}))\| \leq (1+l+l\lambda+2(\zeta_1+\zeta_2))e_m \leq \eta e_m,$$

which together with (6.12) yield

$$\mathscr{H}(G(t_{n,i},x_n(t_{n,i}),u_{n,i})),G(t_{n,q},x_n(t_{n,q}),u_{n,q}))) \leq \varepsilon_m.$$

Since $g_n(t_{m,p}) = g_n(t_{n,q}) = g_{n,q} \in G(t_{n,q},x_n(t_{n,q}),u_{n,q}))$, we may choose $g_{n,i} \in G(t_{n,i},x_n(t_{n,i}),u_{n,i}))$ such that

$$\|g_n(t_{n,i}) - g_n(t_{m,p})\| = \|g_{n,i} - g_{n,q}\| \leq \varepsilon_m,$$

which is the second inequality in (6.32).

Next, we prove that (6.31) holds.

If $n \leq m_0$, then $\frac{e_n}{e_{m_0}}$ is an integer and every J_j is contained in some interval $[t_{n,k},t_{n,k+1})$ in which g_n is constant. Thus, (6.31) is trivial in this case:

$$\omega_{J_j}(g_n) = 0, \text{ for all } j = 0,\ldots,\mu_{m_0} \text{ and all } n \leq m_0.$$

Let $n > m_0$. As $\frac{e_{m_0}}{e_n}$ is an integer, then $2e_n \leq e_{m_0}$. By property (Pr_1), it follows that $t_{m_0,j}, t_{m_0,j+1} \in P_n$. Thus, there exist ρ, ϑ such that $0 \leq \vartheta < \rho$, $t_{m_0,j} = t_{n,\vartheta}$ and $t_{m_0,j+1} = t_{n,\rho}$. The values of the mapping g_n on $J_j = [t_{m_0,j}, t_{m_0,j+1}) = [t_{n,\vartheta}, t_{n,\rho})$ are $g_n(t_{n,s}) = g_{n,s}$, with $\vartheta < s < \rho$. So we shall prove that, for all $\vartheta < s < \rho$,

$$\|g_n(t_{n,s}) - g_n(t_{m_0,j})\| \leq 2\varepsilon_{m_0}, \tag{6.33}$$

and so $\|g_n(t) - g_n(t_{m_0,j})\| \leq 2\varepsilon_{m_0}$, for all $t \in J_j$ and all $n > m_0$. Then it will follow that for all t and s in J_j,

$$\|g_n(t) - g_n(s)\| \leq \|g_n(t) - g_n(t_{m_0,j})\| + \|g_n(t_{m_0,j}) - g_n(s)\| \leq 4\varepsilon_{m_0} \leq \varepsilon.$$

Hence, $\omega_{J_j}(g_n) \leq \varepsilon$, and (6.31) holds.

Let $t_{n,s} \in P_n$ such that $\vartheta < s < \rho$. Then $t_{n,s} \notin P_{m_0}$ and consequently $t_{n,s} \notin P_1$. Now by property Pr_2, there exists a unique couple (m_1, p_1) such that $m_1 < n$, $t_{n,s} \in P_{m_1+1} \setminus P_{m_1}$ and $t_{m_1,p_1} < t_{n,s} < t_{m_1,p_1+1}$, with $p_1 < \mu_{m_1}$. By virtue of the second inequality in (6.32), we obtain that

$$\|g_n(t_{n,s}) - g_n(t_{m_1,p_1})\| \leq \varepsilon_{m_1}. \tag{6.34}$$

Using the same techniques in [109, 110, 123, 124] we can show that $t_{m_0,j} \leq t_{m_1,p_1}$.

If $t_{m_0,j} = t_{m_1,p_1}$, then (6.33) is true, by (6.34) and the fact that $m_1 \geq m_0$ implies $\varepsilon_{m_1} \leq \varepsilon_{m_0}$.

If $t_{m_0,j} < t_{m_1,p_1}$, then since $t_{m_1,p_1} < t_{m_0,j+1}$ it follows that $t_{m_1,p_1} \notin P_{m_0}$ and so $t_{m_1,p_1} \notin P_1$. Then, by Pr_2, there is a unique couple (m_2, p_2) such that $m_2 < m_1$, $t_{n,s} \in P_{m_2+1} \setminus P_{m_2}$ and $t_{m_2,p_2} < t_{m_1,p_1} < t_{m_2,p_2+1}$, with $p_2 < \mu_{m_2}$. Again by virtue of the second inequality in (6.32), we obtain that

$$\|g_n(t_{m_2,p_2}) - g_n(t_{m_1,p_1})\| \leq \varepsilon_{m_2}, \tag{6.35}$$

because $t_{m_1,p_1} \in P_n$ ($m_1 < n$ implies $P_{m_1} \subset P_n$). As mentioned above for the couple (m_1, p_1), it is not hard to check that $t_{m_0,j} \leq t_{m_2,p_2}$. If $t_{m_0,j} = t_{m_2,p_2}$, then (6.33) follows by summing (6.34) and (6.35), since $\varepsilon_{m_1} + \varepsilon_{m_2} \leq \varepsilon_{m_0}$ (because $m_1, m_2 \geq m_0$). The case If $t_{m_0,j} < t_{m_2,p_2}$ is treated as above.

The inductive procedure is now clear: There exists a finite sequence $\{(m_i, p_i)\}$, $i = 0, \ldots, k$ such that $m_0 \leq m_k < m_{k-1} < \cdots < m_1 < n$, $t_{m_k,p_k} = t_{m_0,j}$, $t_{m_i,p_i} \in P_{m_i} \subset P_n$ for all i and

$$\|g_n(t_{m_i,p_i}) - g_n(t_{m_{i+1},p_{i+1}})\| \leq \varepsilon_{m_{i+1}}, \quad \text{for} \quad i = 0, \ldots, k-1.$$

Consequently, by applying these inequalities, (6.34), and the triangle inequality, we obtain

$$\|g_n(t_{n,s}) - g_n(t_{m_1,p_1})\| \leq \varepsilon_{m_1} + \varepsilon_{m_2} + \cdots + \varepsilon_{m_k} \leq 2\varepsilon_{m_0}.$$

Thus, completing the proof of (6.33) and so we get the relative strong compactness for the uniform convergence in the space of bounded mappings of the sequence $g_n(\cdot)$. Therefore, there exists a bounded mapping $g(\cdot) : I \to \mathbf{H}$ such that $\|g_n - g\|_\infty \longrightarrow 0$.

Step 4. Existence of a solution.

Since $(x_n \circ \theta_n)$ and $(v_n \circ \theta_n)$ converge uniformly on I to x and u respectively, then by the continuity of G on $I \times \alpha\mathbf{B} \times l\mathbf{B}$, the closedness of the set $G(t,x(t),u(t))$, and the fact that $g_n(t) \in G(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t)))$ a.e. on I (by (6.20)), we obtain $g(t) \in G(t,x(t),u(t))$ a. e. on I.

Recall that $v_n(\theta_n(t)) \in K(x_n(\theta_n(t)))$, for all $t \in I$ and all $n \geq n_0$. It follows then by the closedness and the continuity of K that $u(t) \in K(x(t))$, for all $t \in I$ and hence the first part of (6.8) holds.

By (6.20) one can assume without loss of generality that the sequence f_n converges weakly in $L^1(I,\mathbf{H})$ to some mapping f. Therefore, from (6.20) once again, we can classically (see Theorem V-14 in [79]) conclude that $f(t) \in F(t,x(t),u(t))$ a.e. on I, because by hypothesis F is scalarly u.s.c. with convex weakly compact values. The weak convergence of $(\dot{v}_n - (f_n + g_n))$ to $\dot{u} - (f+g)$ in $L^1(I,\mathbf{H})$ (by what precedes and Step 2) entails (Mazur's lemma) that for a.e. $t \in I$

$$\dot{u}(t) - f(t) - g(t) \in \bigcap_n \overline{co}[\dot{v}_k(t) - f_k(t) - g_k(t),\ k \geq n].$$

Fix such t in I an any $\xi \in \mathbf{H}$. Then the last relation gives

$$\langle \dot{u}(t) - f(t) - g(t), \xi \rangle \leq \inf_n \sup_{k \geq n} \langle \dot{v}_n(t) - f_n(t) - g_n(t), \xi \rangle.$$

Hence, by (6.27), one obtains

$$\langle \dot{u}(t) - f(t) - g(t), \xi \rangle \leq \limsup_n\ \sigma(-\delta\partial^P d_{K(x_n(\nu_n(t)))}(v_n(\nu_n(t)), \xi).$$

Since $|\nu_n(t) - t| \leq e_n$ on $[0,T)$, then $\nu_n(t) \longrightarrow t$ uniformly on $[0,T)$. It follows then by Remark 6.1 and Theorem 6.2 that for a.e. $t \in I$ and any $\xi \in \mathbf{H}$,

$$\langle \dot{u}(t) - f(t) - g(t), \xi \rangle \leq \sigma(-\delta\partial^P d_{K(x(t))}(u(t)), \xi).$$

Since $\partial^P d_{K(x(t))}(u(t))$ is a convex closed set, then the last inequality entails

$$\dot{u}(t) - f(t) - g(t) \in -\delta\partial^P d_{K(x(t))}(u(t)) \subset -N^P(K(x(t)); u(t)),$$

because $u(t) \in K(x(t))$. Thus,

$$\begin{aligned} \dot{u}(t) &\in -N^P(K(x(t)); u(t)) + f(t) + g(t) \\ &\subset -N^P(K(x(t)); u(t)) + F(t,x(t)), u(t)) + G(t,x(t)), u(t)), \end{aligned} \tag{6.36}$$

and so the second part of (6.8) holds and the proof of the theorem is complete. □

It would be interesting in the infinite dimensional setting to ask whether the compactness assumption on K, i.e., $K(x) \subset \mathscr{K}_1 \subset l\mathbf{B}$, can be replaced by, $K(x) \subset l\mathbf{B}$, the boundness of the set-valued mapping K. Here, we give a positive answer when K is anti-monotone, G satisfies the strong linear growth condition, i.e.,

$$G(t,x,u) \subset (1+\|x\|+\|u\|)\kappa_2 \subset \rho_2(1+\|x\|+\|u\|)\mathbf{B},$$

for all $(t,x,u) \in [0,\varsigma/l] \times \mathrm{gph}K$, where κ_2 is a convex compact subset in $\mathbf{H}$ and $\rho_2 \geq 0$, and F satisfies one of the two following assumptions:

1. The monotonicity with respect to the third variable on $[0,\varsigma/l] \times \mathrm{gph}K$, that is, for any $(t_i,x_i,u_i) \in [0,\varsigma/l] \times \mathrm{gph}K$ and any $z_i \in F(t_i,x_i,u_i)$ $(i=1,2)$ one has
$$\left\langle z_1 - z_2, u_1 - u_2 \right\rangle \geq 0;$$
2. The strong linear growth condition, that is,
$$F(t,x,u) \subset (1+\|x\|+\|u\|)\kappa_1 \subset \rho_1(1+\|x\|+\|u\|)\mathbf{B},$$
for all $(t,x,u) \in [0,\varsigma/l] \times \mathrm{gph}K$, where κ_1 is a convex compact subset in $\mathbf{H}$ and $\rho_1 \geq 0$.

We need the definition of anti-monotone set-valued mappings. We will say that K is anti-monotone if the set-valued mapping $-K$ is monotone in the usual sense, that is, for any $(x_i,u_i) \in \mathrm{gph}K (i=1,2)$ one has

$$\left\langle u_1 - u_2, x_1 - x_2 \right\rangle \leq 0.$$

In the following theorem, we prove the first case when F is monotone with respect to the third variable.

Theorem 6.8. *Let $F,G : [0,+\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be two set-valued mappings and $\varsigma > 0$ such that $x_0 + \varsigma\mathbf{B} \subset \mathscr{V}_0$. Assume that the following assumptions are satisfied:*

(i) *K is anti-monotone and for all $x \in \mathrm{cl}(\mathscr{V}_0)$, $K(x) \subset l\mathbf{B}$, for some $l > 0$;*
(ii) *F is scalarly u.s.c. on $[0, \frac{\varsigma}{l}] \times \mathrm{gph}K$ with nonempty convex weakly compact values;*
(iii) *G satisfies the strong linear growth condition and it is uniformly continuous on $[0,\varsigma/l] \times \alpha\mathbf{B} \times l\mathbf{B}$ into nonempty compact subsets of $\mathbf{H}$, for $\alpha := \|x_0\| + \varsigma$;*
(iv) *F satisfies the linear growth condition, that is,*
$$F(t,x,u) \subset \rho_1(1+\|x\|+\|u\|)\mathbf{B},$$
for all $(t,x,u) \in [0,\varsigma/l] \times \mathrm{gph}K$ for some $\rho_1 \geq 0$;
(v) *F is monotone with respect to the third variable on $[0,\varsigma/l] \times \mathrm{gph}K$. Then for every $T \in (0,\varsigma/l]$ there is a Lipschitz solution $x : I := [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ of (SSPMP) satisfying $\|\dot{x}(t)\| \leq l$ and $\|\ddot{x}(t)\| \leq l\lambda + 2(\rho_1+\rho_2)(1+\alpha+l)$ a.e. on I.*

Proof. An inspection of the proof of Theorem 6.7 shows that the compactness assumption on K was used in Steps 2 and 3 to get the uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$ and the relative strong compactness of $g_n(\cdot)$. Then we have to prove Steps 2 and 3. First, we need, for technical reasons, to fix n_0 satisfying $((4\sqrt{T}+3)l\lambda+2(\zeta_1+\zeta_2))\sqrt{e}_{n_0} \leq \frac{r}{2}$.

Observe by (6.20) and the strong linear growth of G that for every $t \in I$ and every $n \geq n_0$

$$g_n(t) \in G(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t))) \subset (1+\alpha+l)\kappa_2.$$

Then the set $\{g_n(t) : n \geq n_0\}$ is relatively strongly compact in $\mathbf{H}$ for all $t \in I$. On the other hand as $g_n(\cdot)$ is equioscillating by the same arguments in Step 3 in the proof of Theorem 6.7, then we get the relative strong compactness of $g_n(\cdot)$ for the uniform convergence in the space of bounded mappings. Consequently, we may assume without loss of generality that $g_n(\cdot)$ converges uniformly to a bounded mapping g, i.e.,

$$\|g_n - g\|_\infty \to 0 \quad \text{as} \quad n \to +\infty. \tag{6.37}$$

Now we prove the uniform convergence of $x_n(\cdot)$. Put for all positive integers m and $n \geq n_0$

$$w_{m,n}(t) := \frac{1}{2}\|x_n(t) - x_m(t)\|^2.$$

Then,

$$\frac{\mathrm{d}^+ w_{m,n}}{\mathrm{d}t}(t) = \langle x_m(t) - x_n(t), u_m(t) - u_n(t)\rangle, \quad \text{for all} \quad t \in [0,T),$$

since u_n is the right-derivative of x_n. Observe that for any $t \in [0,T)$, there exist positive integers i and j such that $t \in I_{n,i} \cap I_{m,j}$. Then, $u_m(t) = u_{m,j}$ belongs to $K(x_m(t_{m,j}))$ and $u_n(t) = u_{n,i}$ belongs to $K(x_n(t_{n,i}))$. It follows by the anti-monotony of K that

$$\langle x_m(t_{m,j}) - x_n(t_{n,i}), u_m(t) - u_n(t)\rangle \leq 0,$$

and thus

$$\frac{\mathrm{d}^+ w_{m,n}}{\mathrm{d}t}(t) \leq \langle x_m(t) - x_m(t_{m,j}), u_m(t) - u_n(t)\rangle + \langle x_n(t_{n,i}) - x_n(t), u_m(t) - u_n(t)\rangle.$$

Since (6.17) holds and since all the mappings x_n have the same Lipschitz constant l, we have

$$\frac{\mathrm{d}^+ w_{m,n}}{\mathrm{d}t}(t) \leq 2l^2|t - t_{m,j}| + 2l^2|t - t_{n,i}| \leq 2l^2(e_m + e_n).$$

Moreover, $w_{m,n}(0) = 0$. Hence, $w_{m,n}(t) \leq 2l^2(e_m + e_n)t$, and so

$$\|x_m - x_n\|_\infty \leq 2l\sqrt{T}(\sqrt{e}_m + \sqrt{e}_n), \tag{6.38}$$

which ensures that $x_n(\cdot)$ is a Cauchy sequence for the uniform convergence; hence, it converges uniformly to a Lipschitz mapping $x(\cdot)$ with ratio l. So, we have

$$\|x_n - x\|_\infty \to 0 \ \text{ as } \ n \to +\infty.$$

Now, we proceed to prove the Cauchy property of the sequence $v_n(\cdot)$ for the uniform convergence in the space of continuous mappings $\mathscr{C}(I, \mathbf{H})$. We will follow the idea used in [58].

Fix $m, n \geq n_0$ and fix also $t \in I$ with $t \neq t_{m,j}$ for $j = 0, \ldots, \mu_m - 1$ and $t \neq t_{n,i}$ for $i = 0, \ldots, \mu_n - 1$. Observe by the Lipschitz property of K and the relations (6.23), (6.26), and (6.38) that

$$\begin{aligned} d_{K(x_n(\nu_n(t)))}(v_m(t)) &\leq \mathscr{H}(K(x_n(\nu_n(t))), K(x_m(\nu_m(t)))) + \|v_m(\nu_m(t)) - v_m(t)\| \\ &\leq \lambda \|x_n(\nu_n(t)) - x_m(\nu_m(t))\| + \|v_m(\nu_m(t)) - v_m(t)\| \\ &\leq \lambda \ [\|x_n(\nu_n(t)) - x_m(\nu_n(t))\| + \|x_m(\nu_m(t)) - x_m(\nu_n(t))\| \\ &\quad + (l\lambda + 2(\zeta_1 + \zeta_2))|\nu_m(t) - t| \\ &\leq \lambda [2l\sqrt{T}(\sqrt{e}_n + \sqrt{e}_m) + l|\nu_m(t) - \nu_n(t)|] + (l\lambda + 2(\zeta_1 + \zeta_2))e_m \\ &\leq (2\sqrt{T} + 1)l\lambda\sqrt{e}_n + (2(\sqrt{T} + 1)l\lambda + 2(\zeta_1 + \zeta_2))\sqrt{e}_m \\ &\leq [(4\sqrt{T} + 3)l\lambda + 2(\zeta_1 + \zeta_2)]\sqrt{e}_{n_0} \leq \frac{r}{2} < r. \end{aligned}$$

Put $\alpha_1 := 2(\sqrt{T} + 1)$ and $\widetilde{e}_n := \max\{\sqrt{e}_n, \|g_n - g\|_\infty\}$, for all $n \geq n_0$. Then, (6.27) entails

$$\begin{aligned} \left\langle \dot{v}_n(t) - (f_n(t) + g_n(t)), v_n(\nu_n(t)) - v_m(t) \right\rangle &\leq \frac{2\delta}{r} \|v_n(\nu_n(t)) - v_m(t)\|^2 \\ &\quad + \delta d_{K(x_n(\nu_n(t)))}(v_m(t)) \\ &\leq \frac{2\delta}{r} \Big[\|v_n(\nu_n(t)) - v_n(t)\| \\ &\qquad + \|v_n(t) - v_m(t)\| \Big]^2 \\ &\quad + \delta[\alpha_1 l\lambda\widetilde{e}_n + (\alpha_1 l\lambda + 2(\zeta_1 + \zeta_2))\widetilde{e}_m] \\ &\leq \frac{2\delta}{r} \Big[\delta\widetilde{e}_n + \|v_n(t) - v_m(t)\| \Big]^2 \\ &\quad + \delta(\alpha_1 l\lambda + 2(\zeta_1 + \zeta_2))(\widetilde{e}_n + \widetilde{e}_m). \end{aligned}$$

This last inequality and (6.26) yield

$$\begin{aligned}\big\langle \dot v_n(t), v_n(t)-v_m(t)\big\rangle &\le \big\langle f_n(t)+g_n(t), v_m(t)-v_n(\nu_n(t))\big\rangle\\ &\quad+\big\langle \dot v_n(t), v_n(t)-v_n(\nu_n(t))\big\rangle\\ &\quad+\frac{2\delta}{r}\Big[\delta\widetilde e_n+\|v_n(t)-v_m(t)\|\Big]^2\\ &\quad+\delta(\alpha_1 l\lambda+2(\zeta_1+\zeta_2))(\widetilde e_n+\widetilde e_m)\\ &\le \big\langle f_n(t)+g_n(t), v_m(t)-v_n(\nu_n(t))\big\rangle\\ &\quad+\frac{2\delta}{r}\Big[\delta\widetilde e_n+\|v_n(t)-v_m(t)\|\Big]^2\\ &\quad+\delta(\alpha_1 l\lambda+2(\zeta_1+\zeta_2))(\widetilde e_n+\widetilde e_m)+\delta^2\widetilde e_n.\end{aligned}$$

On the other hand by (6.20) and (6.26) we have

$$\begin{aligned}\big\langle f_n(t)+g_n(t), v_m(t)-v_n(\nu_n(t))\big\rangle &= \big\langle f_n(t)+g_n(t), v_m(\theta_m(t))-v_n(\theta_n(t))\big\rangle\\ &\quad+\big\langle f_n(t)+g_n(t), v_m(t)-v_m(\theta_m(t))\big\rangle\\ &\quad+\big\langle f_n(t)+g_n(t), v_n(\theta_n(t))-v_n(\nu_n(t))\big\rangle\\ &\le \big\langle f_n(t)+g_n(t), v_m(\theta_m(t))-v_n(\theta_n(t))\big\rangle\\ &\quad+\delta^2(\widetilde e_m+\widetilde e_n).\end{aligned}$$

Therefore, we get for some positive constant α_2 independent of m,n, and t

$$\begin{aligned}\big\langle \dot v_n(t), v_n(t)-v_m(t)\big\rangle &\le \big\langle f_n(t)+g_n(t), v_m(\theta_m(t))-v_n(\theta_n(t))\big\rangle\\ &\quad+\alpha_2(\widetilde e_m+\widetilde e_n)]+\frac{2\delta}{r}\Big[\delta\widetilde e_n+\|v_n(t)-v_m(t)\|\Big]^2.\end{aligned}$$

In the same way, we also have

$$\begin{aligned}\big\langle \dot v_m(t), v_m(t)-v_n(t)\big\rangle &\le \big\langle f_m(t)+g_m(t), v_n(\theta_n(t))-v_m(\theta_m(t))\big\rangle\\ &\quad+\alpha_2(\widetilde e_m+\widetilde e_n)+\frac{2\delta}{r}\Big[\delta\widetilde e_m+\|v_n(t)-v_m(t)\|\Big]^2.\end{aligned}$$

It then follows from both last inequalities (note that $\|v_n(t)\| \le \|u_0\|+\delta T$) that we have for some positive constant β_1 independent of m,n, and t

$$\begin{aligned}\left\langle \dot{v}_m(t)-\dot{v}_n(t), v_m(t)-v_n(t)\right\rangle &\leq \left\langle f_m(t)-f_n(t), v_n(\theta_n(t))-v_m(\theta_m(t))\right\rangle \\ &+\left\langle g_m(t)-g_n(t), v_n(\theta_n(t))-v_m(\theta_m(t))\right\rangle \\ &+\frac{2\delta}{r}\|v_n(t)-v_m(t)\|^2+\frac{\beta_1}{2}(\widetilde{e}_m+\widetilde{e}_n).\end{aligned}$$

By (6.20) one has for all $t \in I$

$$(x_n(\theta_n(t)), v_n(\theta_n(t))) \in \operatorname{gph} K \text{ and } f_n(t) \in F(\theta_n(t), x_n(\theta_n(t)), v_n(\theta_n(t)))$$

and hence by the monotonicity of F with respect to the third variable on $I \times \operatorname{gph} K$ we get

$$\left\langle f_m(t)-f_n(t), v_n(\theta_n(t))-v_m(\theta_m(t))\right\rangle \leq 0.$$

On the other hand, one has for some $\beta_2 > 0$ (because $\|v_n(t)\| \leq \|u_0\| + \delta T$)

$$\left\langle g_m(t)-g_n(t), v_n(\theta_n(t))-v_m(\theta_m(t))\right\rangle \leq \frac{\beta_2}{2}\|g_m-g_n\|_\infty \leq \frac{\beta_2}{2}(\widetilde{e}_n+\widetilde{e}_m).$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v_m(t)-v_n(t)\|^2\right) \leq (\beta_1+\beta_2)(\widetilde{e}_m+\widetilde{e}_n)+\frac{4\delta}{r}\|v_m(t)-v_n(t)\|^2.$$

As $\|v_m(0)-v_n(0)\|^2 = 0$, Gronwall's inequality yields for all $t \in I$

$$\|v_m(t)-v_n(t)\|^2 \leq \int_0^t \left[(\beta_1+\beta_2)(\widetilde{e}_m+\widetilde{e}_n)\exp\int_s^t \left(\frac{4\delta}{r}\mathrm{d}\tau\right)\right]\mathrm{d}s,$$

and hence for some positive constant β independent of m, n, and t we have

$$\|v_m(t)-v_n(t)\|^2 \leq \beta(\widetilde{e}_m+\widetilde{e}_n).$$

The Cauchy property in $\mathscr{C}(I, \mathbf{H})$ of the sequence $(v_n)_n$ is thus established and hence this sequence converges uniformly to some Lipschitz mapping u with ratio $l\lambda + 2(\zeta_1+\zeta_2)$. Thus, the proof of the theorem is complete. □

Now, we prove the case when F satisfies the strong linear growth.

Theorem 6.9. *Let $F, G : [0, +\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be two set-valued mappings and $\varsigma > 0$ such that $x_0 + \varsigma\mathbf{B} \subset \mathscr{V}_0$. Assume that the following assumptions are satisfied:*

(*i*) *K is anti-monotone and for all $x \in \operatorname{cl}(\mathscr{V}_0)$, $K(x) \subset l\mathbf{B}$, for some $l > 0$;*
(*ii*) *F is scalarly u.s.c. on $[0, \varsigma/l] \times \operatorname{gph} K$ with nonempty convex weakly compact values;*

(*iii*) *G is uniformly continuous on* $[0,\varsigma/l]\times\alpha\mathbf{B}\times l\mathbf{B}$ *into nonempty compact subsets of* $\mathbf{H}$*, for* $\alpha := \|x_0\|+\varsigma$*;*
(*iv*) *F and G satisfy the strong linear growth condition.*

Then for every $T\in(0,\varsigma/l]$ *there is a Lipschitz solution* $x : I := [0,T]\to \mathrm{cl}(\mathscr{V}_0)$ *of (SSPMP) satisfying* $\|\dot{x}(t)\|\le l$ *and* $\|\ddot{x}(t)\|\le l\lambda+2(\rho_1+\rho_2)(1+\alpha+l)$ *a.e. on I.*

Proof. As in the proof of Theorem 6.8 we have to prove Steps 2 and 3 in Theorem 6.7, i.e., the uniform convergence of both sequences $x_n(\cdot)$ and $v_n(\cdot)$ and the relative strong compactness of $g_n(\cdot)$. Using the anti-monotonicity of K we can show as in the proof of Theorem 6.8 the uniform convergence of $x_n(\cdot)$ and so we may assume that (6.38) holds. Also, the relative strong compactness of $g_n(\cdot)$ can be proved as in the proof of Theorem 6.8 by using the strong linear growth condition of G. So we may assume that (6.37) holds. Thus, it remains only to prove the uniform convergence of $v_n(\cdot)$. To do that we need, for technical reasons, to fix n_0 as in the proof of Theorem 6.8, i.e., satisfying $((4\sqrt{T}+3)l\lambda+2(\zeta_1+\zeta_2))\widetilde{e}_{n_0}\le\frac{r}{2}$.

Put $h_n(t) := \int_0^t f_n(s)\mathrm{d}s$ and $w_n(t) := v_n(t)-h_n(t)$ for all $t\in I$. By the strong linear growth condition of F and our construction in Theorem 6.7 we have

$$f_n(t)\in(1+\alpha+l)\kappa_1 \text{ and } h_n(t)\in T(1+\alpha+l)\kappa_1 \text{ for all } t\in I. \tag{6.39}$$

Then Arzela–Ascoli's theorem ensures that we may extract a subsequence of h_n that converges uniformly to a mapping h with $h(t)=\int_0^t f(s)\mathrm{d}s$ and f is the weak limit of a subsequence of f_n in $L^1(I,\mathbf{H})$. Put for all $n\ge n_0$

$$\widehat{e}_n := \max\{\delta^{-1}\|h_n-h\|_\infty,\widetilde{e}_n\},\quad \text{for all}\quad n\ge n_0. \tag{6.40}$$

Now, we proceed to prove the Cauchy property of the sequence $v_n(\cdot)$ for the uniform convergence in the space of continuous mappings $\mathscr{C}(I,\mathbf{H})$.

Fix $m,n\ge n_0$ and fix also $t\in I$ with $t\ne t_{m,j}$ for $j=0,...,\mu_m-1$ and $t\ne t_{n,i}$ for $i=0,...,\mu_n-1$. As in the proof of Theorem 6.8 we get for almost every $t\in I$

$$\begin{aligned}\left\langle \dot{w}_n(t)-g_n(t),v_n(\nu_n(t))-v_m(t)\right\rangle \le{}& \frac{2\delta}{r}\Big[\delta\widetilde{e}_n+\|v_n(t)-v_m(t)\|\Big]^2\\ &+\delta(\alpha_1 l\lambda+2(\zeta_1+\zeta_2))(\widetilde{e}_n+\widetilde{e}_m).\end{aligned}$$

Then by (6.26), (6.20), (6.39), and (6.40) one gets

$$\begin{aligned}\left\langle \dot{w}_n(t),w_n(t)-w_m(t)\right\rangle \le{}& \left\langle \dot{w}_n(t),w_n(t)-w_n(\nu_n(t))\right\rangle+\left\langle g_n(t),v_n(\nu_n(t))-v_m(t)\right\rangle\\ &+\delta(\alpha_1 l\lambda+2(\zeta_1+\zeta_2))(\widetilde{e}_n+\widetilde{e}_m)+\left\langle \dot{w}_n(t),h_m(t)-h_n(\nu_n(t))\right\rangle\\ &+\frac{2\delta}{r}\Big[\delta\widetilde{e}_n+\|w_n(t)-w_m(t)\|+\|h_n(t)-h_m(t)\|\Big]^2\end{aligned}$$

$$\leq \delta^2 e_n + \zeta_2 \delta e_n + \langle g_n(t), v_n(t) - v_m(t) \rangle$$
$$+\delta(\alpha_1 l\lambda + 2(\zeta_1+\zeta_2))(\widetilde{e}_n + \widetilde{e}_m) + \delta(\delta\widehat{e}_n + \delta\widehat{e}_m + \zeta_1 e_n)$$
$$+\frac{2\delta}{r}\Big[2\delta(\widehat{e}_n+\widehat{e}_m) + \|w_n(t) - w_m(t)\|\Big]^2.$$

Therefore, we get for some $\beta_1 > 0$ (independent of m, n, and t)

$$\langle \dot{w}_n(t), w_n(t) - w_m(t) \rangle \leq \langle g_n(t), v_n(t) - v_m(t) \rangle + \frac{\beta_1}{2}(\widehat{e}_n + \widehat{e}_m)$$
$$+\frac{2\delta}{r}\Big[2\delta(\widehat{e}_n+\widehat{e}_m) + \|w_n(t) - w_m(t)\|\Big]^2.$$

In the same way, we also have

$$\langle \dot{w}_m(t), w_m(t) - w_n(t) \rangle \leq \langle g_m(t), v_m(t) - v_n(t) \rangle + \frac{\beta_1}{2}(\widehat{e}_n + \widehat{e}_m)$$
$$+\frac{2\delta}{r}\Big[2\delta(\widehat{e}_n+\widehat{e}_m) + \|w_n(t) - w_m(t)\|\Big]^2.$$

It then follows from both last inequalities, the relation (6.40), the definition of $\widetilde{e}_n$ and the equiboundedness of v_n and w_n, that for some $\beta_2 > 0$ independent of m, n, and t one has

$$\langle \dot{w}_m(t) - \dot{w}_n(t), w_m(t) - w_n(t) \rangle \leq \frac{2\delta}{r}\|w_n(t) - w_m(t)\|^2 + \frac{\beta_2}{2}(\widehat{e}_m + \widehat{e}_n).$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|w_m(t) - w_n(t)\|^2\Big) \leq \beta_2(\widehat{e}_m + \widehat{e}_n) + \frac{4\delta}{r}\|w_m(t) - w_n(t)\|^2.$$

As $\|w_m(0) - w_n(0)\|^2 = 0$, Gronwall's inequality yields for some $\beta > 0$ independent of m, n, and t

$$\|w_m(t) - w_n(t)\|^2 \leq \beta^2(\widehat{e}_m + \widehat{e}_n),$$

for all $t \in I$. Finally, by (6.40) one obtains

$$\|v_m(t) - v_n(t)\| \leq \beta(\widehat{e}_m + \widehat{e}_n)^{1/2} + \delta(\widehat{e}_m + \widehat{e}_n) \leq (\beta + \delta)(\widehat{e}_m + \widehat{e}_n)^{1/2}.$$

The Cauchy property in $\mathscr{C}(I, \mathbf{H})$ of the sequence $(v_n)_n$ is thus established and hence this sequence converges uniformly to some Lipschitz mapping u with ratio $l\lambda + 2(\zeta_1 + \zeta_2)$. Thus the proof of the theorem is complete. □

Remark 6.2. Observe that in the proof of Theorems 6.7–6.9, the constant of Lipschitz of $\dot{x}$ (the derivative of the solution x) as well as the construction of the sequences and their convergence depend upon the initial point x_0, the neighborhood $\mathscr{V}_0$, and the constant T. Nevertheless, an inspection of the proof of Theorem 6.7 shows that if we take $\mathscr{V}_0 = \mathbf{H}$ and if we replace the linear growth condition of F and G by the following bounded-linear growth condition (bounded in x and linear growth in u)

$$(BLGC) \qquad F(t,x,u) \subset \rho_1(1+\|u\|)\mathbf{B} \text{ and } G(t,x,u) \subset \rho_2(1+\|u\|)\mathbf{B}$$

for all $(t,x,u) \in [0,+\infty) \times \text{gph} K$ for some $\rho_1, \rho_2 \geq 0$, then for every $T > 0$ there exists a solution $x : [0,T] \to \mathbf{H}$ independently upon the constant T. Consequently, by extending in the evident way the solution x to $[0,+\infty)$ by considering the interval $[0,1]$ and next the interval $[1,2]$, etc, we obtain the following global existence result:

Theorem 6.10. *Let $x_0 \in \mathbf{H}$, $u_0 \in K(x_0)$, and $G, F : [0,+\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be two set-valued mappings. Assume that the following assumptions are satisfied:*

(i) For all $x \in \mathbf{H}$, $K(x) \subset \mathscr{K}_1 \subset l\mathbf{B}$, for some convex compact set $\mathscr{K}_1$ in $\mathbf{H}$ and some $l > 0$;
(ii) F is scalarly u.s.c. on $[0,+\infty) \times \text{gph} K$ with nonempty convex weakly compact values;
(iii) For any $\alpha > 0$, G is uniformly continuous on $[0,+\infty) \times \alpha\mathbf{B} \times l\mathbf{B}$ into nonempty compact subsets of $\mathbf{H}$;
(iv) F and G satisfy the bounded-linear growth condition (LGC).

Then there is a Lipschitz solution $x : [0,+\infty) \to \mathbf{H}$ to

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t)); \dot{x}(t)) + F(t,x(t),\dot{x}(t)) + G(t,x(t),\dot{x}(t)), & \text{a. e. } [0,+\infty), \\ \dot{x}(t) \in K(x(t)), \quad \text{for all } t \in [0,+\infty), \\ x(0) = x_0, \quad \text{and} \quad \dot{x}(0) = u_0. \end{cases}$$

Remark 6.3. As in Remark 6.1, global existence results can be obtained in Theorems 6.8 and 6.9 when we take $\mathscr{V}_0 = \mathbf{H}$ and we replace in Theorems 6.7 (resp. Theorems 6.9) the linear growth for F and the strong linear growth for G (resp. the strong linear growth for both F and G) by the bounded-linear growth (BLGC) for F and the strong bounded-linear growth for G (resp. the strong bounded-linear growth for both F and G, i.e.,

$$F(t,x,u) \subset (1+\|u\|)\kappa_1 \text{ and } G(t,x,u) \subset (1+\|u\|)\kappa_2,$$

for all $(t,x,u) \in [0,\infty) \times \text{gph} K$, where κ_1 an κ_2 are two convex compact sets in $\mathbf{H}$.)

In the previous theorems we have proved many existence results for the problem $(SSPMP)$ when the perturbation F is assumed to be globally scalarly u.s.c. Our aim in the next part of the present section is to prove that for the problem $(SSPCP)$ (the Second Order Sweeping Process with a Convex Perturbation F, i.e., the case when

$G = \{0\}$), the global scalarly upper semicontinuity of F on $[0, \frac{\varsigma}{l}] \times \mathrm{gph}\, K$ can be replaced by the following weaker assumptions:

(A_1) For any $t \in [0, \frac{\varsigma}{l}]$, the set-valued mapping $F(t;\cdot,\cdot)$ is scalarly u.s.c. on $\mathrm{gph}\, K$;
(A_2) F is scalarly measurable with respect to the σ-field of $[0, \frac{\varsigma}{l}] \times \mathrm{gph}\, K$ generated by the Lebesgue sets in $[0, \frac{\varsigma}{l}]$ and the Borel sets in the space $\mathbf{H}$.

The proof here is based on an approximation method. The idea is to approximate a set-valued mapping F that satisfies (A_1) and (A_2) by a sequence of globally scalarly u.s.c. set-valued mappings F_n and study the convergence of the solutions x_n of $(SSPCP)_n$ associated with each F_n (the existence of such solutions is ensured by our results in Theorems 6.7–6.9). We will use a special approximation F_n of F defined by

$$F_n(t,x,u) := \frac{1}{\eta_n} \int_{I_{t,\eta_n}} F(s,x,u)\mathrm{d}s$$

for all $(t,x,u) \in I \times \mathbf{H} \times \mathbf{H}$, where I is some compact interval, η_n is a sequence of strictly positive numbers converging to zero and $I_{t,\eta_n} := I \cap [t, t+\eta_n]$. For more details concerning this approximation we refer the reader to [78, 246] and the references therein. We need the two following lemmas. For their proofs we refer to [78, 246].

Lemma 6.2. *Let $T > 0$, S be a Suslin metrizable space, and $F : [0,T] \times S$ be a set-valued mapping with nonempty convex weakly compact values. Assume that F satisfies the following assumptions:*

(a) For any $t \in [0,T]$, $F(t;\cdot)$ is scalarly u.s.c. on S;
(b) F is scalarly measurable w.r.t. the σ-field of $[0,T] \times S$ generated by the Lebesgue sets in $[0,T]$ and the Borel sets in the topological space S;
(c) $F(t,y) \subset \rho(1+\|y\|)\mathbf{B}$, for all $(t,y) \in [0,T] \times S$ and for some $\rho > 0$.

Then F_n is a globally scalarly u.s.c. set-valued mapping on $[0,T] \times S$ with nonempty convex compact values satisfying

$$F_n(t,y) \subset \rho T(1+\|y\|)\mathbf{B},$$

for all $(t,y) \in [0,T] \times S$ and all n.

Lemma 6.3. *Let $T > 0$, S be a Suslin metrizable space and $F : [0,T] \times S \rightrightarrows \mathbf{H}$ be a set-valued mapping with nonempty convex weakly compact values. Assume that F is bounded on $[0,T] \times S$ and that satisfies the hypothesis (a), (b), and (c) in Lemma 6.2. Then for any sequence y_n of Lebesgue measurable mappings from $[0,T]$ to S which converges pointwisely to a Lebesgue measurable mapping y, any sequence z_n in $L^1([0,T],\mathbf{H})$ weakly converging to z in $L^1([0,T],\mathbf{H})$ and satisfying $z_n(t) \in F_n(t,y_n(t))$ a.e. on I one has*

$$z(t) \in F(t,y(t)) \text{ a. e. on } [0,T].$$

Now we are able to prove the following result.

Theorem 6.11. *Let $F : [0,+\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be a set-valued mapping and $\varsigma > 0$ such that $x_0 + \varsigma \mathbf{B} \subset \mathscr{V}_0$. Assume that the hypothesis (i), (iv) in Theorem 6.7 are satisfied and assume that F satisfies (A_1) and (A_2). Then for every $T \in (0, \frac{\varsigma}{l}]$ there exists a Lipschitz solution $x : [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ of $(SSPCP)$ satisfying $\|\dot{x}(t)\| \le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2T\rho_1(1+\alpha+l)$ a.e. on $[0,T]$.*

Proof. Let $T \in (0, \frac{\varsigma}{l}]$ and put $I := [0,T]$ and $S := \alpha \mathbf{B} \times l\mathbf{B}$. Clearly S is a Suslin metrizable space. Let η_n be a sequence of strictly positive numbers that converges to zero. For each $n \ge 1$ we put

$$F_n(t,x,u) := \frac{1}{\eta_n} \int_{I_{t,\eta_n}} F(s,x,u)\mathrm{d}s$$

for all $(t,x,u) \in I \times \mathbf{H} \times \mathbf{H}$. By Lemma 6.2 the set-valued mappings F_n are scalarly u.s.c. on $I \times S$ with nonempty convex compact values and satisfies

$$F_n(t,x,u) \subset T\rho_1(1+\|x\|+\|u\|)\mathbf{B} \subset T\rho_1(1+\alpha+l)\mathbf{B} =: T\zeta_1\mathbf{B},$$

for any $(t,x,u) \in I \times S$ and all $n \ge 1$. So that we can apply the result of Theorem 6.7. For each $n \ge 1$, there exists a Lipschitz mapping $x_n : I \to \mathrm{cl}(\mathscr{V}_0)$ satisfying

$$(SSPCP)_n \quad \begin{cases} \ddot{x}_n(t) \in -N^P(K(x_n(t)); \dot{x}_n(t)) + F_n(t, x_n(t), \dot{x}_n(t)), & \text{a. e. on } I, \\ \dot{x}_n(t) \in K(x_n(t)), \quad \text{for all } t \in I, \\ x_n(0) = x_0, \text{ and } \dot{x}_n(0) = u_0, \end{cases}$$

with $\|\dot{x}_n(t)\| \le l$ and $\|\ddot{x}_n(t)\| \le l\lambda + 2T\zeta_1$ a.e. on I and for all $n \ge 1$.

Since $\dot{x}_n(t) \in K(x_n(t)) \subset \mathscr{K}_1$ for all $n \ge 1$ and all $t \in I$, then we get the relative strong compactness of the set $\{\dot{x}_n(t) : n \ge 1\}$ in $\mathbf{H}$ for all $t \in I$. Therefore, by Arzela–Ascoli's theorem we may extract from $\dot{x}_n$ a subsequence that converges uniformly to some Lipschitz mapping $\dot{x}$. By integrating, we get the uniform convergence of the sequence x_n to x because they have the same initial value $x_n(0) = x_0$, for all $n \ge 1$. Now, by $(SSPCP)_n$ there is for any $n \ge 1$ a Lebesgue measurable mapping $f_n : I \to \mathbf{H}$ such that

$$f_n(t) \in F_n(t, x_n(t), \dot{x}_n(t)) \subset T\rho_1(1+\|x_n(t)\|+\|\dot{x}_n(t)\|)\mathbf{B} \subset T\zeta_1\mathbf{B}. \tag{6.41}$$

and

$$f_n(t) - \ddot{x}_n(t) \in N^P(K(x_n(t)); \dot{x}_n(t)) \cap \delta\mathbf{B} = \delta\partial^P d_{K(x_n(t))}(\dot{x}_n(t)), \tag{6.42}$$

for a.e. $t \in I$, where $\delta := l\lambda + 3T\zeta_1$. Observe by (6.41) and $(SSPCP)_n$ that f_n and $\ddot{x}_n(\cdot)$ are equibounded in $L^1(I,\mathbf{H})$ and so subsequences may be extracted that converge in the weak topology of $L^1(I,\mathbf{H})$. Without loss of generality, we may suppose that these subsequences are f_n and $(\ddot{x}_n)_n$, respectively. Denote by f and w their weak limits respectively. Then, for each $t \in I$

$$u_0 + \int_0^t \ddot{x}(s)\mathrm{d}s = \dot{x}(t) = \lim_{n\to\infty} \dot{x}_n(t) = u_0 + \lim_{n\to\infty} \int_0^t \ddot{x}_n(s)\mathrm{d}s = u_0 + \int_0^t w(s)\mathrm{d}s,$$

which gives the equality $\ddot{x}(t) = w(t)$ for almost all $t \in I$, that is, $(\ddot{x}_n)_n$ converges weakly in $L^1(I,\mathbf{H})$ to $\ddot{x}$.

It follows then from $(SSPCP)_n$ and the Lipschitz property of K that

$$d_{K(x(t))}(\dot{x}_n(t)) \le \mathscr{H}(K(x(t)), K(x_n(t)) \le \|x_n(t) - x(t)\| \to 0,$$

and hence one obtains $\dot{x}(t) \in K(x(t))$, because the set $K(x(t))$ is closed.

Now, the weak convergence in $L^1(I,\mathbf{H})$ of $(\ddot{x}_n)_n$ and $(f_n)_n$ to $\ddot{x}$ and f, respectively, entail for almost all $t \in I$ (by Mazur's lemma)

$$f(t) - \ddot{x}(t) \in \bigcap_n \overline{co}\{f_k(t) - \ddot{x}_k(t) : k \ge n\}.$$

Fix any such $t \in I$ and consider any $\xi \in \mathbf{H}$. The last relation ensures

$$\langle \xi, f(t) - \ddot{x}(t)\rangle \le \inf_n \sup_{k\ge n} \langle \xi, f_k(t) - \ddot{x}_k(t)\rangle,$$

and hence according to (6.42) and Theorem 6.2 we get

$$\langle \xi, f(t) - \ddot{x}(t)\rangle \le \limsup_n \sigma(\delta\partial^P d_{K(x_n(t))}(\dot{x}_n(t)), \xi) \le \sigma(\delta\partial^P d_{K(x(t))}(\dot{x}(t)), \xi).$$

As the set $\partial^P d_{K(t)}(u(t))$ is closed and convex (for uniformly prox-regular sets), we obtain

$$f(t) - \ddot{x}(t) \in \delta\partial^P d_{K(x(t))}(\dot{x}(t)) \subset N^P(K(x(t)); \dot{x}(t)), \tag{6.43}$$

because $\dot{x}(t) \in K(x(t))$. Now we check that $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. on I. Since F is bounded on $I \times S$, f_n converges weakly to f in $L^1(I,\mathbf{H})$, and $(x_n, \dot{x}_n)$ is a sequence of Lebesgue measurable mappings from I to S (because $\dot{x}_n(t) \in K(x_n(t)) \subset l\mathbf{B}$ and $\|x_n(t)\| \le \alpha$ for all $t \in I$) converging uniformly to $(x, \dot{x})$, it follows then from Lemma 6.3 that $f(t) \in F(t, x(t), \dot{x}(t))$ for a.e. on I . Consequently, we obtain by (6.43) and so

$$\ddot{x}(t) \in -N^P(K(x(t)); \dot{x}(t)) + F(t, x(t), \dot{x}(t)).$$

Thus, completing the proof of the theorem. □

Now we prove the following main theorem.

Theorem 6.12. *Let $F : [0,+\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be a set-valued mapping and $\varsigma > 0$ such that $x_0 + \varsigma\mathbf{B} \subset \mathscr{V}_0$. Assume that the hypothesis (i) and (iv) in Theorem 6.8 and (A_1) and (A_2) are satisfied. Then for every $T \in (0, \frac{\varsigma}{l}]$ there exists a Lipschitz solution $x : [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ of $(SSPCP)$ satisfying $\|\dot{x}(t)\| \le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2T\rho_1(1 + \alpha + l)$ a.e. on $[0,T]$.*

Proof. We do as in the proof of Theorem 6.11 to get, for all $n \geq 1$, a Lipschitz solution x_n of $(SSPCP)_n$ with the estimates with $\|\dot{x}_n(t)\| \leq l$ and $\|\ddot{x}_n(t)\| \leq l\lambda + 2T\zeta_1$ a.e. on I. Then, we prove the uniform convergence of the sequences $x_n(\cdot)$ and $\dot{x}_n(\cdot)$. For this end, we denote $w_{m,n}(t) := \frac{1}{2}\|x_n(t) - x_m(t)\|^2$, for all $t \in I$ and for every $m,n \geq 1$. Then,

$$\frac{\mathrm{d}^+ w_{m,n}}{\mathrm{d}t}(t) = \langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t) \rangle, \text{ for all } t \in [0,T).$$

Therefore, by $(SSPP)_n$ and the anti-monotonicity of K we get

$$\frac{\mathrm{d}^+ w_{m,n}}{\mathrm{d}t}(t) \leq 0,$$

for all $t \in [0,T)$. Moreover, by $(SSPP)_n$ one has $w_{m,n}(0) = \frac{1}{2}\|x_n(0) - x_m(0)\|^2 = 0$. Hence, $w_{m,n}(t) = 0$ for all $t \in I$ and then $x_n(\cdot)$ is a constant sequence. Let x be its limit. Then $(\dot{x}_n)$ and $(\ddot{x}_n)$ converge uniformly to $\dot{x}$ and $\ddot{x}$, respectively.

Now, by $(SSPP)_n$ there is for any $n \geq 1$ a Lebesgue measurable mapping $f_n : I \to \mathbf{H}$ such that

$$f_n(t) \in F_n(t, x(t), \dot{x}(t)) \subset T(1 + \|x(t)\| + \|\dot{x}(t)\|)\mathscr{K} \subset T\zeta_1 \mathbf{B} \tag{6.44}$$

and

$$f_n(t) - \ddot{x}(t) \in N^P(K(x(t)); \dot{x}(t)) \cap \delta\mathbf{B} = \delta\partial^P d_{K(x(t))}(\dot{x}(t)), \tag{6.45}$$

for a.e. $t \in I$, where $\delta := l\lambda + 3T\zeta_1$. Observe by (6.44) that f_n is equibounded in $L^1(I, \mathbf{H})$ and so a subsequence may be extracted that converges in the weak topology of $L^1(I, \mathbf{H})$. Without loss of generality, we may suppose that this subsequence is f_n. Denote by f its weak limit. Then, by using Mazur's lemma and the properties of the subdifferential of the distance function for uniformly prox-regular sets, it is easy to conclude that for almost every $t \in I$

$$f(t) \in \delta\partial^P d_{K(x(t))}(\dot{x}(t)) + \ddot{x}(t) \subset N^P(K(x(t)); \dot{x}(t)) + \ddot{x}(t). \tag{6.46}$$

Finally, with the same arguments, as in the proof of Theorem 4.1, we can check that $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. on I and so we obtain by (6.46)

$$\ddot{x}(t) \in -N^P(K(x(t)); \dot{x}(t)) + F(t, x(t), \dot{x}(t)).$$

Thus, completing the proof. □

Remark 6.4. The generalization of Theorem 6.8, in the same way as like in Theorems 6.11–6.12, to the case of set-valued mappings F satisfying the assumptions (A_1) and (A_2), depends on the monotonicity of the approximation F_n which is the key of Theorem 6.8. Since one cannot be sure that the monotonicity of F whether implies or not the monotonicity of F_n, then it is not clear for us the generalization of

Theorem 6.8. Thus, the question will be what are the assumptions on F implying the monotonicity of F_n? under such assumptions the both proofs in Theorems 6.11–6.12 still work to obtain a generalization of Theorem 6.8.

6.4 Properties of Solution Sets

Throughout this section let $r \in (0,+\infty]$, Ω be an open subset in $\mathbf{H}$, $F : [0,+\infty) \times \mathbf{H} \times \mathbf{H} \rightrightarrows \mathbf{H}$ be a set-valued mapping, and $K : \mathrm{cl}(\Omega) \rightrightarrows \mathbf{H}$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in $\mathbf{H}$. In this section we are interested by some topological properties of the solution set of the problem $(SSPCP)$. Let $x_0 \in \Omega$, $u_0 \in K(x_0)$, and $T > 0$ such that $x_0 + Tl\mathbf{B} \subset \Omega$. We denote by $\mathscr{S}_F(x_0,u_0)$ the set of all continuous mappings $(x,u) : [0,T] \to \mathrm{cl}(\Omega) \times \mathbf{H}$ such that

$$(SSPCP) \qquad \begin{cases} u(0) = u_0, \\ x(t) = x_0 + \displaystyle\int_0^t u(s)\mathrm{d}s, \quad \text{for all } t \in [0,T], \\ u(t) \in K(x(t)), \quad \text{for all } t \in [0,T], \\ \dot{u}(t) \in -N^P(K(x(t));u(t)) + F(t,x(t),u(t)), \quad \text{a. e. on } [0,T]. \end{cases}$$

Proposition 6.1. *Assume that the hypothesis of one of the Theorems 6.7–6.9 are satisfied and that* $\mathrm{gph}\,K$ *is strongly compact in* $\mathrm{cl}(\Omega) \times l\mathbf{B}$. *Then the set* $\mathscr{S}_F(x_0,u_0)$ *is relatively strongly compact in* $\mathscr{C}([0,T],\mathbf{H}\times\mathbf{H})$.

Proof. By Theorem 6.7–6.9 the set of solution (x,u) of $(SSPCP)$ are equi-Lipschitz and for any $t \in [0,T]$ one has $\{(x(t),u(t)) : (x,u) \in \mathscr{S}_F(x_0,u_0)\}$ is relatively strongly compact in $\mathbf{H}\times\mathbf{H}$ because it is contained in the strong compact set $\mathrm{gph}\,K$. Then Arzela–Ascoli's theorem gives the relative strong compactness of the set $\mathscr{S}_F(x_0,u_0)$ in $\mathscr{C}(I,\mathbf{H}\times\mathbf{H})$. □

Remark 6.5. Assume that $\Omega = \mathbf{H}$ and let T be any strictly positive number. Put

$$\mathscr{S}_F(\mathrm{gph}\,K) := \bigcup_{(x_0,u_0)\in\mathrm{gph}\,K} \mathscr{S}_F(x_0,u_0)$$

With the same arguments, as in the proof of Proposition 6.1, we can show that under the same hypothesis in Proposition 6.1 the set $\mathscr{S}_F(\mathrm{gph}\,K)$ is relatively strongly compact in $\mathscr{C}([0,T],\mathbf{H}\times\mathbf{H})$.

Now we wish to prove the closedness of the set-valued mapping $\mathscr{S}_F$.

Proposition 6.2. *Assume that the hypothesis of one of the Theorems 6.7–6.9 are satisfied. Then the set-valued mapping* $\mathscr{S}_F$ *has a closed graph in* $\Omega \times K(\Omega) \times \mathscr{C}([0,T],\mathbf{H}\times\mathbf{H})$.

Proof. Let $((x_0^n,u_0^n))_n \in \Omega \times K(\Omega)$ and $((x^n,u^n))_n \in \mathscr{C}([0,T],\mathbf{H}\times\mathbf{H})$ with $(x^n,u^n) \in \mathscr{S}_F((x_0^n,u_0^n))$ such that $(x_0^n,u_0^n) \to (x_0,u_0) \in \Omega \times K(\Omega)$ uniformly, and $(x^n,u^n) \to (x,u) \in \mathscr{C}([0,T],\mathbf{H}\times\mathbf{H})$ uniformly. We have to show that $(x,u) \in \mathscr{S}_F(x_0,u_0)$. First observe that for n sufficiently large $x_0^n \in x_0 + lT\mathbf{B}$. Now, it is not difficult to check that the closedness of gph$K$ and the uniform convergence of both sequences $((x_0^n,u_0^n))_n$ and $((x^n,u^n))_n$ imply that $(x(0),u(0)) = (x_0,u_0)$ and that $u(t) \in K(x(t))$ for all $t \in [0,T]$. On the other hand one has for all $t \in [0,T]$

$$x(t) = \lim_n x^n(t) = x_0 + \lim_n \int_0^t u^n(s)\mathrm{d}s = x_0 + \int_0^t u(s)\mathrm{d}s.$$

It remains then to show that

$$\dot{u}(t) \in -N^P(K(x(t));u(t)) + F(t,x(t),u(t)), \text{ a. e. on } [0,T].$$

For every n, one has

$$\dot{u}^n(t) \in -N^P(K(x^n(t));u^n(t)) + F(t,x^n(t),u^n(t)), \text{ a. e. on } [0,T].$$

Then for every n there exists a measurable selection f^n such that

$$f^n(t) \in F(t,x^n(t),u^n(t)) \;\text{ and }\; -\dot{u}^n(t) + f^n(t) \in N^P(K(x^n(t));u^n(t)), \tag{6.47}$$

for a. e. $t \in [0,T]$. By Theorems 6.7–6.9 one has for n sufficiently large

$$\|\dot{u}^n(t)\| \le l\lambda + 2\rho_1(1 + \|x_0^n\| + Tl + l) \le l\lambda + 2\rho_1(1 + \|x_0\| + 2Tl + l). \tag{6.48}$$

By (iv) in Theorems 6.7–6.9 and the fact that $u^n(t) \in K(x^n(t))$ one gets

$$\|f^n(t)\| \le \rho_1(1 + \|x_0\| + Tl + l). \tag{6.49}$$

Therefore, we may suppose without loss of generality that $\dot{u}^n \to \dot{u}$ and $f^n \to f$ weakly in $L^1([0,T],\mathbf{H})$. Since $F(t,\cdot,\cdot)$ is scalarly upper semicontinuous with convex compact values, then we get easily that $f(t) \in F(t,x(t),u(t))$ a.e. $t \in [0,T]$. Now by (6.47)–(6.49) and Theorem 4.1 in [61] (see also Chap. 2) we have for $\delta := l\lambda + 3\rho_1(1 + \|x_0\| + 2Tl + l)$

$$-\dot{u}^n(t) + f^n(t) \in \delta\partial^P d_{K(x^n(t))}(u^n(t)) \;\text{ a.e. }\; t \in [0,T].$$

Then by using Mazur's lemma and the same techniques used in the proof of the previous theorems, it is easy to conclude that for a. e. $t \in [0,T]$

$$f(t) - \dot{u}(t) \in \delta\partial^P d_{K(x(t))}(u(t)) \subset N^P(K(x(t));u(t)).$$

Thus we get for a.e. $t \in [0,T]$

$$\dot{u}(t) \in -N^P(K(x(t));u(t)) + F(t,x(t),u(t)),$$

which completes the proof of the proposition. □

Remark 6.6. The proof of Proposition 6.1 shows that the solution set $\mathscr{S}_F(x_0,u_0)$ associated to the problem (*SSPMP*) is relatively strongly compact in $\mathscr{C}([0,T], \mathbf{H}\times\mathbf{H})$ whenever the graph gph$K$ is strongly compact in $\mathbf{H}$. Contrarily, our proof in Proposition 6.2 cannot provide the closedness of the graph of the set-valued mapping $\mathscr{S}_F$ associated to the problem (*SSPMP*). The difficulty that prevents to conclude is the absence of the convexity of G.

6.5 Particular Case

In this section, let $\mathbf{H}$ be a finite dimensional space and let us focus our attention to the special case when F is defined by

$$F(t,x,u) = -\partial^C f_t(x) + \gamma u,$$

where $\gamma \in \mathbf{R}$, $f_t := f(t,\cdot)$, $f : [0,T]\times \mathrm{cl}\,(\mathscr{V}_0) \to \mathbf{R}$ is a globally measurable function and β-equi-Lispchitz w.r.t. the second variable, $\mathscr{V}_0$ is an open neighborhood of x_0, and $T > 0$ satisfies $x_0 + Tl\mathbf{B} \subset \mathscr{V}_0$. Here $\partial^C f_t(x)$ denotes the Clarke subdifferential of f_t at x given by

$$\partial^C f_t(x) = \{\xi \in \mathbf{H} : \ \langle \xi, h\rangle \le f_t^0(x;h), \ \text{for all} \ h \in \mathbf{H}\},$$

where $f_t^0(x;h)$ is the Clarke directional derivative of f_t at x in the direction h, that is,

$$f_t^0(x;h) := \limsup_{\substack{\delta\downarrow 0\\ x'\to x}} \delta^{-1}[f(t,x'+\delta h) - f(t,x')].$$

It is not difficult to see that the set-valued mapping F satisfies the hypothesis (A_1), (A_2), and (iv) in Theorem 6.11. Indeed, for the hypothesis (A_1), (A_2) it suffices to observe that the support function associated with F is given by

$$\sigma(F(t,x,u),h) = \sigma(-\partial^C f_t(x),h) + \gamma\langle u,h\rangle = (-f_t)^0(x;h) + \gamma\langle u,h\rangle,$$

for all $h \in \mathbf{H}$. Then the measurability and the scalar u.s.c. of F follow easily from the hypothesis on f and the properties of the Clarke directional derivative. Since f_t is β-equi-Lispchitz w.r.t. the second variable we get

$$F(t,x,u) = -\partial^C f_t(x) + \gamma u \subset \beta\mathbf{B} + \gamma u \subset \rho_1(1+\|u\|)\mathbf{B},$$

with $\rho_1 := \max\{\beta, |\gamma|\}$ and so the hypothesis (iv) is satisfied. Now applying Theorem 6.11 we get the following result.

Theorem 6.13. *For every $u_0 \in K(x_0)$ there is a Lipschitz solution $x : [0,T] \to \mathrm{cl}\,(\mathscr{V}_0)$ to the Cauchy problem for the second order differential inclusion:*

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t));\dot{x}(t)) - \partial^{\mathrm{C}} f_t(x(t)) + \gamma\dot{x}(t), \ \textit{a. e. on } [0,T], \\ \dot{x}(t) \in K(x(t)), \ \textit{for all } t \in [0,T], \\ x(0) = x_0, \ \textit{and } \dot{x}(0) = u_0, \end{cases}$$

with $\|\dot{x}(t)\| \le l$ *and* $\|\ddot{x}(t)\| \le l\lambda + 2T\rho_1(1+\alpha+l)$.

It would be interesting to ask whether the result in Theorem 6.13 remains true if we take f_t is not necessarily Lipschitz? Such problem is till now open and in our opinion is so hard to attacked it in a direct manner. Nevertheless, in what follows we give a positive answer for a special case when f_t is the indicator function associated to some set-valued mapping C. To this aim we use the result stated in Theorem 6.1 for the distance function which satisfies all the hypothesis of that theorem and then we prove the viability of the solution x, i.e., $x(t) \in C(t)$ for all $t \in I$. So applying Theorem 6.13 for $f_t = d_{C(t)}$ we get a Lipschitz mapping $x : [0,T] \to \mathrm{cl}\,(\mathscr{V}_0)$ such that

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t));\dot{x}(t)) - \partial^{\mathrm{C}} d_{C(t)}(x(t)) + \gamma\dot{x}(t), \quad \text{a. e. on } [0,T], \\ \dot{x}(t) \in K(x(t)), \quad \text{for all } t \in [0,T], \\ x(0) = x_0, \ \text{and} \quad \dot{x}(0) = u_0, \end{cases}$$

with $\|\dot{x}(t)\| \le l$ and $\|\ddot{x}(t)\| \le l\lambda + 2T\rho_1(1+\alpha+l)$, where $\rho_1 := \max\{|\gamma|, 1\}$. Now we come back to our construction in Theorem 6.7. Observe that the solution x is always bounded by $\|x_0\| + lT$. Thus, if we assume that for all $t \in [0,T]$ the set $C(t)$ contains the ball $M\mathbf{B}$ where $M := \|x_0\| + lT$, then we get $x(t) \in C(t)$ and consequently the solution would satisfy $\partial^{\mathrm{C}} d_{C(t)}(x(t)) \subset N^{\mathrm{C}}(C(t);x(t)) = \partial^{\mathrm{C}} \psi_{C(t)}(x(t))$ and so

$$\ddot{x}(t) \in -N^P(K(x(t));\dot{x}(t)) - N^{\mathrm{C}}(C(t);x(t)) + \gamma\dot{x}(t).$$

Therefore, we obtain the following result.

Theorem 6.14. *Let $C : [0,T] \rightrightarrows \mathbf{H}$ be any set-valued mapping such that its associated distance function to images $(t,x) \mapsto d_{C(t)}(x)$ is globally measurable. Assume that $l, T,$ and x_0 satisfy $(\|x_0\| + lT)\mathbf{B} \subset C(t)$ for all $t \in [0,T]$. Then, for every $u_0 \in K(x_0)$ there is a Lipschitz mapping $x : [0,T] \to \mathrm{cl}\,(\mathscr{V}_0)$ satisfying*

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t));\dot{x}(t)) - N^{\mathrm{C}}(C(t);x(t)) + \gamma\dot{x}(t), \quad \text{a. e. on } [0,T], \\ \dot{x}(t) \in K(x(t)), \quad \text{for all } t \in [0,T], \\ x(0) = x_0, \ \text{and} \quad \dot{x}(0) = u_0, \end{cases}$$

with $\|\dot{x}(t)\| \le l$ *and* $\|\ddot{x}(t)\| \le l\lambda + 2\rho_1(1+\alpha+l)$, *where* $\rho_1 := \max\{|\gamma|, 1\}$.

6.6 Second Order Perturbed Sweeping Process with Delay

In all this section we let $r \in (0,\infty]$, $x_0 \in \mathbf{H}$, $u_0 \in K(x_0)$, $\mathscr{V}_0$ be an open neighborhood of x_0 in $\mathbf{H}$, and $\zeta > 0$ such that $x_0 + \zeta\mathscr{B} \subset \mathscr{V}_0$, and let $K : \mathrm{cl}(\mathscr{V}_0) \to \mathbf{H}$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in $\mathbf{H}$.

First we state the following result used in our main proofs. It is a direct consequence of Theorem 6.7 in the previous section by taking $G(t,x,u) = \{0\}$.

Theorem 6.15. *Assume that*

(i) $\forall x \in \mathrm{cl}(\mathscr{V}_0), K(x) \subset \mathscr{K}_1 \subset l\mathbf{B}, \mathscr{K}_1$ *is a convex compact set in* $\mathbf{H}$, *and* $l > 0$;
(ii) $F : [0,\infty) \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ *is scalarly upper semi-continuous on* $[0, \frac{\zeta}{l}] \times \mathrm{gph}(K)$ *with nonempty convex weakly compact values;*
(iii) $F(t,x,u) \subset \rho(1 + \|x\| + \|u\|)\mathbf{B}, \forall (t,x,u) \in [0, \frac{\zeta}{l}] \times \mathrm{gph}(K)$.

Then for any $T \in (0, \frac{\zeta}{l}]$, *there exists a Lipschitz mapping* $x : I = [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ *such that*

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t)); \dot{x}(t)) + F(t, x(t), \dot{x}(t)), & \text{a.e.} \quad \text{on} \quad I, \\ \dot{x}(t) \in K(x(t)), \quad \forall t \in I, \\ x(0) = x_0, \dot{x}(0) = u_0, \end{cases}$$

with $\|\dot{x}(t)\| \leq l, \|\ddot{x}(t)\| \leq l\lambda + 2(1+\alpha+l)\rho$ *a.e on I.*

Remark 6.7. We point out that the solution mapping x obtained in Theorem 6.15 is differentiable everywhere on I.

6.6.1 Existence Theorems

Now let us state the existence result for the second order perturbed sweeping process with delay (SOSPD).

Theorem 6.16. *Assume that (i) and the following conditions hold:*

(ii)′ $F : [0,+\infty) \times \mathscr{C}_0 \times \mathscr{C}_0 \rightrightarrows \mathbf{H}$ *is scalarly upper semi-continuous on* $[0, \frac{\zeta}{l}] \times \mathscr{C}_0 \times \mathscr{C}_0$, *taking convex weakly compact values in* $\mathbf{H}$, *and*
(iii)′ $F(t,\varphi,\phi) \subset \rho(1 + \|\varphi(0)\| + \|\phi(0)\|)\mathbf{B}, \forall (t,\varphi,\phi) \in \left[0, \frac{\zeta}{l}\right] \times \mathscr{C}_0 \times \mathscr{C}_0$.

Then for every $T \in (0, \frac{\zeta}{l}]$ *and for every* $\phi \in \mathscr{C}_0$ *verifying* $\phi(0) = u_0$, *there exists a Lipschitz mapping* $x : [0,T] \to \mathrm{cl}(\mathscr{V}_0)$ *such that:*

$$\begin{cases} \ddot{x}(t) \in -N^P(K(x(t)); \dot{x}(t)) + F(t, T(t)x, T(t)\dot{x}), & \text{a.e.} \quad \text{on} \quad [0,T], \\ \dot{x}(t) \in K(x(t)), \quad \forall t \in [0,T], \\ T(0)x = \varphi \quad \text{and} \quad T(0)\dot{x} = \phi \quad \textit{on} \quad [-\tau, 0], \end{cases}$$

with $\varphi(t) = x_0 + \int_0^t \phi(s)\mathrm{d}s$*, for all* $t \in [-\tau, 0]$*, and* $\|\dot{x}(t)\| \le l$ *and* $\|\ddot{x}(t)\| \le l\lambda + 2(1+\alpha+l)\rho$ *a.e on* $[0,T]$.

Proof. Without loss of generality, we may take $T = 1$. Step 1. Let $\phi \in \mathscr{C}_0$ satisfying $\phi(0) = u_0$, and put $\varphi(t) := x_0 + \int_0^t \phi(s)\mathrm{d}s$ for all $t \in [-\tau, 0]$. Let $(\mathscr{P}_n)$ be a subdivision of $[0,1]$ defined by the points : $t_i^n := \frac{i}{n}, (i = 0, 1, ..., n)$. For every $(t,x,u) \in [-\tau, t_1^n] \times \mathrm{gph}(K)$, we define $f_0^n : [-\tau, t_1^n] \times \mathrm{cl}(\mathscr{V}_0) \to \mathbf{H}$ and $g_0^n : [-\tau, t_1^n] \times K(\mathrm{cl}(\mathscr{V}_0)) \to \mathbf{H}$ by

$$f_0^n(t,x) = \begin{cases} \varphi(t) & \forall t \in [-\tau, 0], \\ \varphi(0) + nt(x - \varphi(0)) & \forall t \in [0, t_1^n], \end{cases}$$

and

$$g_0^n(t,u) = \begin{cases} \phi(t) & \forall t \in [-\tau, 0], \\ \phi(0) + nt(u - \phi(0)) & \forall t \in [0, t_1^n]. \end{cases}$$

We have $f_0^n(\frac{1}{n}, x) = x$ and $g_0^n(\frac{1}{n}, u) = u$ for all $(x,u) \in \mathrm{gph}(K)$. Observe that the mapping $(x,u) \mapsto (T(t_1^n)f_0^n(.,x), T(t_1^n)g_0^n(.,u))$ from $\mathrm{gph}(K)$ to $\mathscr{C}_0 \times \mathscr{C}_0$ is nonexpansive. Indeed, we have for all $(x,y) \in \mathbf{H} \times \mathbf{H}$

$$\begin{aligned} \|T(1/n)f_0^n(.,x) - T(1/n)f_0^n(.,y)\|_{\mathscr{C}_0} &= \sup_{s\in[-\tau,0]} \|f_0^n(s+(1/n),x) - f_0^n(s+(1/n),y)\| \\ &= \sup_{s\in[-\tau+\frac{1}{n},\frac{1}{n}]} \|f_0^n(s,x) - f_0^n(s,y)\| \\ &= \sup_{0\le s\le \frac{1}{n}} \|ns(x-\varphi(0)) - ns(y-\varphi(0))\| \\ &= \sup_{0\le s\le \frac{1}{n}} \|ns(x-y)\| \\ &= \|x-y\|. \end{aligned}$$

In the same way, we get for all $(u,v) \in \mathbf{H} \times \mathbf{H}$

$$\|T(1/n)g_0^n(.,u) - T(1/n)g_0^n(.,v)\|_{\mathscr{C}_0} = \|u-v\|.$$

Hence, the mapping $(x,u) \mapsto (T(t_1^n)f_0^n(.,x), T(t_1^n)g_0^n(.,u))$ from $\mathrm{gph}(K)$ to $\mathscr{C}_0 \times \mathscr{C}_0$ is nonexpansive and so the set-valued mapping $F_0^n : [0, \frac{1}{n}] \times \mathrm{gph}(K) \rightrightarrows \mathbf{H}$ defined by: $F_0^n(t,x,u) = F(t, T(\frac{1}{n})f_0^n(.,x), T(\frac{1}{n})g_0^n(.,u))$ is scalarly upper semi-continuous on $[0, \frac{1}{n}] \times \mathrm{gph}(K)$ because F it is on $[0, \frac{1}{n}] \times \mathscr{C}_0 \times \mathscr{C}_0$, with nonempty convex weakly compact values in $\mathbf{H}$ and satisfies

$$F_0^n(t,x,u) = F(t, T(1/n)f_0^n(.,x), T(1/n)g_0^n(.,u)) \subset \rho(1 + \|x\| + \|u\|),$$

for all $(t,x,u) \in [0,(1/n)] \times \text{gph}(K)$ because $T(\frac{1}{n})f_0^n(0,x) = x$ and $T(\frac{1}{n})g_0^n(0,u) = u$. Hence, F_0^n verifies conditions of Theorem 6.7, provides a Lipschitz differentiable solution $y_0^n : [0,\frac{1}{n}] \to \text{cl}(\mathscr{V}_0)$ to the problem

$$\begin{cases} \ddot{y}_0^n(t) \in -N^P(K(y_0^n(t)); \dot{y}_0^n(t)) + F(t, T(\frac{1}{n})f_0^n(.,y_0^n(t)), T(\frac{1}{n})f_1^n(.,\dot{y}_0^n(t))), \\ \hfill \text{a.e on}[0,\frac{1}{n}], \\ \dot{y}_0^n(t) \in K(y_0^n(t)), \qquad \forall t \in [0,\frac{1}{n}], \\ y_0^n(0) = x_0 = \varphi(0), \quad \dot{y}_0^n(0) = u_0 = \phi(0). \end{cases}$$

Further, we have $\|\dot{y}_0^n(t)\| \le l$; and $\|\ddot{y}_0^n(t)\| \le l\lambda + 2(1+\alpha+l)\rho$.

Set

$$y_n(t) = \begin{cases} \varphi(t) & \forall t \in [-\tau,0], \\ y_0^n(t) & \forall t \in [0,\frac{1}{n}]. \end{cases}$$

Then, y_n is well defined on $[-\tau,\frac{1}{n}]$, with $y_n = \varphi$ on $[-\tau,0]$ and

$$\dot{y}_n(t) = \begin{cases} \phi(t) & \forall t \in [-\tau,0], \\ \dot{y}_0^n(t) & \forall t \in (0,\frac{1}{n}), \end{cases}$$

and

$$\begin{cases} \ddot{y}_n(t) \in -N^P(K(y_n(t)); \dot{y}_n(t)) + F(t, T(\frac{1}{n})f_0^n(\cdot,y_n(t)), T(\frac{1}{n})g_0^n(\cdot,\dot{y}_n(t))), \\ \hfill \text{a.e on } [0,\frac{1}{n}], \\ \dot{y}_n(t) \in K(y_n(t)), \qquad \forall t \in [0,\frac{1}{n}], \\ y_n(0) = x_0 = \varphi(0), \quad \dot{y}_n(0) = u_0, \end{cases}$$

with $\|\dot{y}_n(t)\| \le l$ and $\|\ddot{y}_n(t)\| \le l\lambda + 2(1+\alpha+l)\rho$ a. e. $t \in [0,\frac{1}{n}]$.

Suppose that y_n is defined on $[-\tau,\frac{k}{n}]$ $(k \ge 1)$ with $y_n = \varphi$ on $[-\tau,0]$ and satisfies

$$y_n(t) = \begin{cases} y_0^n(t) = x_0 + \int_0^t \dot{y}_n(s)\text{d}s, & \forall t \in \left[0,\frac{1}{n}\right], \\ y_1^n(t) := y_n(\frac{1}{n}) + \int_{\frac{1}{n}}^t \dot{y}_n(s)\text{d}s, & \forall t \in \left[\frac{1}{n},\frac{2}{n}\right], \\ . \quad . \quad . \quad . \quad . \\ y_{k-1}^n(t) := y_n(\frac{k-1}{n}) + \int_{\frac{k-1}{n}}^t \dot{y}_n(s)\text{d}s, & \forall t \in \left[\frac{k-1}{n},\frac{k}{n}\right], \end{cases}$$

and y_n is a Lipschitz solution of

$$\begin{cases} y_n(t) = y_{k-1}^n(t) := y_n(\frac{k-1}{n}) + \int_{\frac{k-1}{n}}^t \dot{y}_n(s)\mathrm{d}s, & \forall t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \\ \ddot{y}_n(t) \in -N^P(K(y_n(t)); \dot{y}_n(t)) + F(t, T(\frac{k}{n})f_{k-1}^n(.,y_n(t)), T(\frac{k}{n})g_{k-1}^n(.,\dot{y}_n(t)), & \text{a.e. } [\frac{k-1}{n}, \frac{k}{n}]; \\ \dot{y}_n(t) \in K(y_n(t)), & \forall t \in [\frac{k-1}{n}, \frac{k}{n}], \end{cases}$$

where f_{k-1}^n and g_{k-1}^n are defined for any $(x,u) \in \mathrm{gph}\,(K)$ as follows

$$f_{k-1}^n(t,x) = \begin{cases} y_n(t), & \forall t \in \left[-\tau, \frac{k-1}{n}\right], \\ y_n(\frac{k-1}{n}) + n(t - \frac{k-1}{n})(x - y_n(\frac{k-1}{n})), & \forall t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \end{cases}$$

and

$$g_{k-1}^n(t,u) = \begin{cases} \dot{y}_n(t), & \forall t \in \left[-\tau, \frac{k-1}{n}\right], \\ \dot{y}_n(\frac{k-1}{n}) + n(t - \frac{k-1}{n})(u - \dot{y}_n(\frac{k-1}{n})), & \forall t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \end{cases}$$

Similarly, we can define $f_k^n, g_k^n : \left[-\tau, \frac{k+1}{n}\right] \times \mathbf{H} \to \mathbf{H}$ as

$$f_k^n(t,x) = \begin{cases} y_n(t), & \forall t \in \left[-\tau, \frac{k}{n}\right], \\ y_n(\frac{k}{n}) + n(t - \frac{k}{n})(x - y_n(\frac{k}{n})), & \forall t \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \end{cases}$$

and

$$g_k^n(t,u) = \begin{cases} \dot{y}_n(t), & \forall t \in \left[-\tau, \frac{k}{n}\right], \\ \dot{y}_n(\frac{k}{n}) + n(t - \frac{k}{n})(u - \dot{y}_n(\frac{k}{n})), & \forall t \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \end{cases}$$

for any $(x,u) \in \mathrm{gph}\,(K)$. Note that $T(\frac{k+1}{n})f_k^n(0,x) = f_k^n(\frac{k+1}{n}, x) = x$ and $T(\frac{k+1}{n})g_k^n(0,u) = g_k^n(\frac{k+1}{n}, u) = u$, for all $(x,u) \in \mathrm{gph}(K)$.

Note also that, for all $(x,u),(y,v) \in \mathrm{gph}(K)$, we have

$$\begin{aligned} \left\| T\left(\frac{k+1}{n}\right) f_k^n(.,x) - T\left(\frac{k+1}{n}\right) f_k^n(.,y) \right\|_{\mathscr{C}_0} &= \sup_{s \in [-\tau,0]} \left\| f_k^n\left(s + \frac{k+1}{n}, x\right) \right. \\ &\qquad \left. - f_k^n\left(s + \frac{k+1}{n}, y\right) \right\| \\ &= \sup_{s \in \left[-\tau + \frac{k+1}{n}, \frac{k+1}{n}\right]} \| f_k^n(s,x) - f_k^n(s,y) \|, \end{aligned}$$

and

$$\left\|T\left(\frac{k+1}{n}\right)g_k^n(.,u)-T\left(\frac{k+1}{n}\right)g_k^n(.,v)\right\|_{\mathscr{C}_0} = \sup_{s\in[-\tau,0]}\left\|g_k^n\left(s+\frac{k+1}{n},u\right) - g_k^n\left(s+\frac{k+1}{n},v\right)\right\|$$
$$= \sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]}\|g_k^n(s,u)-g_k^n(s,v)\|.$$

We distinguish two cases

(1) If $-\tau+\frac{k+1}{n}<\frac{k}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]}\|f_k^n(s,x)-f_k^n(s,y)\| = \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}\|f_k^n(s,x)-f_k^n(s,y)\|$$
$$= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}}\|n(s-(k/n))(x-y)\| = \|x-y\|$$

and

$$\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]}\|g_k^n(s,v)-g_k^n(s,v)\| = \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}\|g_k^n(s,u)-g_k^n(s,v)\|$$
$$= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}}\|n(s-(k/n))(u-v)\| = \|u-v\|.$$

(2) If $\frac{k}{n}\le-\tau+\frac{k+1}{n}\le\frac{k+1}{n}$, we have

$$\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]}\|f_k^n(s,x)-f_k^n(s,y)\| \le \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}\|f_k^n(s,x)-f_k^n(s,y)\|$$
$$= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}}\|n(s-(k/n))(x-y)\| = \|x-y\|$$

and

$$\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]}\|g_k^n(s,v)-g_k^n(s,v)\| \le \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}\|g_k^n(s,u)-g_k^n(s,v)\|$$
$$= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}}\|n(s-(k/n))(u-v)\| = \|u-v\|.$$

So the mapping $(x,u)\mapsto(T(\frac{k+1}{n})f_k^n(.,x),T(\frac{k+1}{n})g_k^n(.,u))$ from $\mathrm{gph}\,(K)$ to $\mathscr{C}_0\times\mathscr{C}_0$ is nonexpansive. Hence, the set-valued mapping $F_k^n:[0,1]\times\mathrm{gph}\,(K)\rightrightarrows\mathbf{H}$ defined by

$$F_k^n(t,x,u):=F\left(t,T\left(\frac{k+1}{n}\right)f_k^n(.,x),T\left(\frac{k+1}{n}\right)g_k^n(.,u)\right)$$

is scalarly upper semi-continuous on $[0,1]\times \mathrm{gph}(K)$ with nonempty convex weakly compact values. As above we can easily check that F_k^n satisfies the growth condition:

$$F_k^n(t,x,u)\subset \rho(1+\|x\|+\|u\|),\quad \forall (t,x,u)\in [0,1]\times \mathrm{gph}(K).$$

Applying Theorem 6.7 gives a Lipschitz solution $y_k^n : \left[\frac{k}{n},\frac{k+1}{n}\right]\to \mathrm{cl}(\mathscr{V}_0)$ to the problem

$$\begin{cases} \ddot{y}_k^n(t)\in -N^P(K(y_k^n(t));\dot{y}_k^n(t))+F_k^n(t,y_k^n(t),\dot{y}_k^n(t)), & \text{a.e. } \left[\frac{k}{n},\frac{k+1}{n}\right],\\ y_k^n(\frac{k}{n})=y_n(\frac{k}{n}), & \\ \dot{y}_k^n(t)\in K(y_k^n(t), & \forall t\in \left[\frac{k}{n},\frac{k+1}{n}\right], \end{cases}$$

with $\|\dot{y}_k^n(t)\|\le l; \|\ddot{y}_k^n(t)\|\le l\lambda+2(1+\alpha+l)\rho$.

Consequently, there exists $h_k^n\in L^1([\frac{k}{n},\frac{k+1}{n}],\mathbf{H})$ such that

$$\begin{cases} h_k^n(t)\in F(t,T(\frac{k+1}{n})f_k^n(.,y_k^n(t)),T(\frac{k+1}{n})g_k^n(.,\dot{y}_k^n(t))), \ \text{a.e. } \left[\frac{k}{n},\frac{k+1}{n}\right],\\ \ddot{y}_k^n(t)\in -N^P(K(y_k^n(t));\dot{y}_k^n(t))+h_k^n(t), \ \text{a.e. } \left[\frac{k}{n},\frac{k+1}{n}\right],\\ y_k^n(\frac{k}{n})=y_n(\frac{k}{n}),\\ \dot{y}_k^n(t)\in K(y_k^n(t)), \qquad \forall t\in \left[\frac{k}{n},\frac{k+1}{n}\right]. \end{cases}$$

Thus, by induction, we can construct a continuous function $y_n : [-\tau,1]\to \mathrm{cl}(\mathscr{V}_0)$ with $y_n=\varphi$ on $[-\tau,0]$ such that its restriction on each interval $[\frac{k}{n},\frac{k+1}{n}]$ is a solution to

$$\begin{cases} \ddot{x}(t)\in -N^P(K(x(t));\dot{x}(t))+F(t,T(\frac{k+1}{n})f_k^n(.,x(t)),T(\frac{k+1}{n})g_k^n(.,\dot{x}(t))),\\ \hfill \text{a.e. } \left[\frac{k}{n},\frac{k+1}{n}\right],\\ x(\frac{k}{n})=y_n(\frac{k}{n}),\\ \dot{x}(t)\in K(x(t)), \qquad \forall t\in \left[\frac{k}{n},\frac{k+1}{n}\right]. \end{cases}$$

Indeed, set

$$y_n(t):=\begin{cases} \varphi(t), & \forall t\in[-\tau,0],\\ y_0^n(t), & \forall t\in[0,\frac{1}{n}],\\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot\\ y_k^n(t), & \forall t\in[\frac{k}{n},\frac{k+1}{n}], \end{cases}$$

$$\dot{y}_n(t):=\begin{cases} \phi(t), & \forall t\in[-\tau,0],\\ \dot{y}_0^n(t), & \forall t\in(0,\frac{1}{n}),\\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot\\ \dot{y}_k^n(t), & \forall t\in(\frac{k}{n},\frac{k+1}{n}), \end{cases}$$

and

$$h_n(t) := h_k^n(t) \qquad \text{on} \quad \left(\frac{k}{n}, \frac{k+1}{n}\right].$$

Also, for notational convenience, we set $\theta_n(t) := \frac{k+1}{n}$ and $\delta_n(t) := \frac{k}{n}$, for $t \in (\frac{k}{n}, \frac{k+1}{n}]$. Then, we get

$$\begin{aligned}
h_n(t) &\in F(t, T(\theta_n(t))f^n_{n\theta_n(t)-1}(.,y_n(t)), T(\theta_n(t))g^n_{n\theta_n(t)-1}(.,\dot{y}_n(t))), \text{ a.e. } (0,1],\\
\ddot{y}_n(t) &\in -N^P(K(y_n(t)); \dot{y}_n(t)) + h_n(t), \text{ a.e. } (0,1],\\
y_n(0) &= x_0 = \varphi(0), \quad \dot{y}_n(0) = u_0 \in K(x_0),\\
\dot{y}_n(t) &\in K(y_n(t), \qquad \forall t \in [0,1],
\end{aligned} \tag{6.50}$$

with

$$\|\dot{y}_n(t)\| \le l; \quad \|\ddot{y}_n(t)\| \le l\lambda + 2(1+\alpha+l)\rho \text{ a.e. } (0,1],$$

and for a.e. $t \in [0,1]$,

$$\begin{aligned}
&F(t, T(\theta_n(t))f^n_{n\theta_n(t)-1}(.,y_n(t)), T(\theta_n(t))g^n_{n\theta_n(t)-1}(.,\dot{y}_n(t)))\\
&\quad \subset \rho(1 + \|y_n(t)\| + \|\dot{y}_n(t)\|).
\end{aligned}$$

Step 2. uniformconvergence of (y_n):

Let (y_n) and h_n as in (6.50), we have

$$\|h_n(t)\| \le \rho(1 + \|y_n(t)\| + \|\dot{y}_n(t)\|) \le \rho(1 + \|x_0\| + Tl + l),$$

a.e. $t \in [0,1], \forall n$. So the sequence (h_n) is bounded in $\mathscr{L}^1([0,1], \mathbf{H})$. By extracting a subsequence, we may assume that (h_n) converges weakly to some $h \in L^1([0,1], \mathbf{H})$. Further, $y_n)$ is relatively compact in $\mathscr{C}([0,1], \mathbf{H})$, so we may suppose that (y_n) converges in $\mathscr{C}([0,1], \mathbf{H})$ to some $z \in \mathscr{C}([0,1], \mathbf{H})$ with

$$z(t) = \varphi(0) + \int_0^t \dot{z}(s)\mathrm{d}s, \qquad \forall t \in [0,1].$$

For each $t \in [-\tau, 0]$, we set

$$y(t) = \begin{cases} \varphi(t), & \forall t \in [-\tau, 0],\\ z(t), & \forall t \in [0,1]. \end{cases}$$

Then, $y \in \mathscr{C}_1$ and y_n converges to y in $\mathscr{C}_1$.

Step 3. We claim that $T(\theta_n(t))f^n_{n\theta_n(t)-1}(.,y_n(t))$ and $T(\theta_n(t))g^n_{n\theta_n(t)-1}(.,\dot{y}_n(t))$ converge pointwisely on $(0,1]$ to $T(t)y$ and $T(t)\dot{y}$ respectively in the Banach space $\mathscr{C}_0$. The proof is similar to the one given in Theorem 2.1 in [75].

Step 4. Existence of solutions:

We have for a.e. $t \in [0,1]$

$$h_n(t) \in F(t, T(\theta_n(t)) f^n_{n\theta_n(t)-1}(., y_n(t)), T(\theta_n(t)) g^n_{n\theta_n(t)-1}(., \dot{y}_n(t))).$$

As h_n converges weakly to h in $L^1([0,1], \mathbf{H})$ and for all $t \in (0,1]$

$$\|T(\theta_n(t)) f^n_{n\theta_n(t)-1}(., y_n(t)) - T(t)y\|_{\mathscr{C}_0} \to 0,$$

and

$$\|T(\theta_n(t)) g^n_{n\theta_n(t)-1}(., \dot{y}_n(t)) - T(t)\dot{y}\|_{\mathscr{C}_0} \to 0,$$

and as the multifunction F is scalarly upper semi-continuous with convex weakly compact values, by a classical closure result (see, for instance [79], we get $h(t) \in F(t, T(t)y, T(t)\dot{y})$.

Further, as $\|\dot{y}_n(t)\| \le l$, we may assume that $\dot{y}_n$ converges weakly to $\dot{y}$ and similarly $\ddot{y}_n$ converges weakly to $\ddot{y}$. By what precedes one has $(\ddot{y}_n - h_n)$ weakly converges to $\ddot{y} - h$ in $L^1([0,1], \mathbf{H})$ and so Mazur's lemma ensures that for almost every $t \in [0,1]$

$$\ddot{y}(t) - h(t) \in \bigcap_n \overline{co}\{\ddot{y}_k(t) - h_k(t) : k \ge n\}.$$

Fix such t in I and any μ in $\mathbf{H}$, then the last relation gives

$$\langle \ddot{y}(t) - h(t), \mu \rangle \le \inf_n \sup_{k \ge n} \langle \ddot{y}_n(t) - h_n(t), \mu \rangle$$

and hence according to (6.50) one has

$$\ddot{y}_n(t) - h_n(t) \in -N^P(K(y_n(t)); \dot{y}_n(t)) \qquad \text{a.e. } t \in (0,1],$$

with $\|\ddot{y}_n(t) - h_n(t)\| \le l\lambda + 2(1 + \alpha + l) + \rho(1 + \|x_0\| + Tl + l) = \delta$. Then, for a.e $t \in (0,1]$

$$\ddot{y}_n(t) - h_n(t) \in -\delta \partial^P d_{K(y_n(t))} \dot{y}_n(t),$$

hence, one obtains

$$\langle \ddot{y}(t) - h(t), \mu \rangle \le \limsup_n \sigma(-\delta \partial^P d_{K(y_n(t))n}(\dot{y}_n(t)), \mu) \le \sigma(-\delta \partial^P d_{K(y(t))}(\dot{y}(t)), \mu).$$

As the set $\partial^P d_{K(y(t))}(\dot{y}(t))$ is closed convex (for uniformly prox-regular sets), we obtain

$$\ddot{y}(t) - h(t) \in -\delta \partial^P d_{K(y(t))}(\dot{y}(t)),$$

and then

$$\ddot{y}(t) - h(t) \in -N^P(K(y(t)); \dot{y}(t)),$$

since $\dot{y}(t) \in K(x(t))$. Thus,

$$\ddot{y}(t) \in -N^P(K(y(t)); \dot{y}(t)) + F(t, T(t)y, T(t)\dot{y}).$$

The proof then is complete. □

Remark 6.8.

1. Note that some existence results have been given when the perturbation F depends only on the variables t and $\dot{x}_t$, see e.g., [109]. So, the result in Theorem 6.14 is more general than those proved in [109] because our perturbation F depends on all the variables t, x_t, and $\dot{x}_t$.
2. In [109], the set-valued mapping K is assumed with convex values and the perturbation F is assumed to be uniformly continuous and depending only on t and $\dot{x}_t$, and with nonconvex values, which is another variant of existence results for (SOSPD), because the perturbation F in Theorem 6.14 has convex values and upper semi-continuous. The techniques used here can be also used to adapt the proof of the result in [109] to obtain a generalization when K has nonconvex values and under the same assumptions on F.

6.7 Commentary to Chap. 6

Chapter 6 is devoted to problems described by a special type of second order differential inclusions, called second order sweeping processes (SOSP), under the regularity of the right hand side. We were interested in the application of uniform prox-regularity concept to the existence of solutions for (SOSP) and to the study of some topological properties of the solution sets. Section 6.2 studies the existence of solutions for (SOPS) in finite dimensional spaces by using a fixed point approach and the existence of solutions for first order sweeping processes in Chap. 5. Theses results are taken from Bounkhel [53]. In Sect. 6.3, we present an existence result for (SOSP), in Hilbert space settings, using a direct approach. The study of some topological properties of solution sets is treated in Sect. 6.4. A very interesting case is presented and studied in Sect. 6.5. The existence of solutions for (SOSP) with delay is considered in Sect. 6.6. All the previous studies have been done under the uniform prox-regularity assumption on the moving set. The main results in Sects. 6.2–6.6 are taken from [46, 53, 54, 65, 66]. A long list of works study the existence of solutions for second order differential inclusions in the nonconvex case and under some regularity concepts. We give here the following recent references which may be of interest to the readers: [11, 12, 43, 46, 50, 53, 54, 56, 65, 66, 109].

Chapter 7
Quasi-Variational Inequalities

7.1 Introduction

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connections with main areas of pure and applied sciences have been made, see for example [13, 20, 21, 112, 127, 128, 132, 135] and the references cited therein.

One of the typical formulations of the variational inequality problem found in the literature is the following

$$\text{Find } x^* \in C \text{ and } y^* \in F(x^*) \text{ s.t. } \langle y^*, x - x^* \rangle \geq 0, \text{ for all } x \in C, \tag{VI}$$

where C is a convex subset of a Hilbert space $\mathbf{H}$ and $F : \mathbf{H} \rightrightarrows \mathbf{H}$ is a set-valued mapping. A tremendous amount of research has been done in the case where C is convex, both on the existence of solutions of (VI) and the construction of solutions, see for example [68, 135, 217, 245]. Only the existence of solutions of (VI) has been considered in the case where C is nonconvex, see for instance [55]. To the best of our knowledge, nothing has been done concerning the construction of solutions in this case.

We intend in this chapter to first generalize problem (VI) to take into account the nonconvexity of the set C and then construct a suitable algorithm to solve the generalized (VI). Note that (VI) is usually a reformulation of some minimization problem of some functional over convex sets. For this reason, it does not make sense to generalize (VI) by just replacing the convex sets by nonconvex ones. Also, a straightforward generalization to the nonconvex case of the techniques used when set C is convex cannot be done. This is because these techniques are strongly based on properties of the projection operator over convex sets and these properties do not hold in general when C is nonconvex. Based on the above two arguments, and to

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take advantage of the techniques used in the convex case, we propose to reformulate problem (VI) when C is convex as the following equivalent problem

$$\text{Find a point } \; x^* \in C: \quad F(x^*) \cap -N^{\text{conv}}(C;x^*) \neq \emptyset, \tag{VP}$$

where $N^{\text{conv}}(C;x)$ denotes the normal cone of C at x in the sense of convex analysis. Equivalence of problems (VI) and (VP) will be proved in Proposition 7.2 below. The corresponding problem when C is not convex will be denoted (NVP). This reformulation allows us to consider the resolution of problem (NVP) as the desired suitable generalization of the problem (VI). We point out that the resolution of (VI) with C nonconvex is not, at least from our point of view, a good way for such generalization. Our idea of the generalization is inspired from [55] where the authors studied the existence of generalized equilibriums.

In the present chapter we make use of some recent techniques and ideas from Nonsmooth Analysis Theory [55, 58] to overcome the difficulties that arise from the nonconvexity of the set C. Specifically, we will be considering the class of uniformly prox-regular sets which is sufficiently large to include the class of convex sets, p-convex sets (see [72]), $C^{1,1}$ submanifolds (possibly with boundary) of $\mathbf{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets.

$$\text{Find a point } \; x^* \in C(x^*): \quad F(x^*) \cap -N^P(C(x^*);x^*) \neq \emptyset. \tag{SNVP}$$

We present an algorithm to solve problem (NVP). The algorithm is an adaptation of the standard projection algorithm that we reproduce below for completeness (for more details concerning this type of projection and convergence analysis in the convex case we refer the reader to [135] and the references therein).

Algorithm 7.1

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1. *Select* $x^0 \in \mathbf{H}, \;\; y^0 \in F(x^0)$, *and* $\rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - \rho y^n$ *and select:* $x^{n+1} \in Proj_C(z^{n+1}), \;\; y^{n+1} \in F(x^{n+1})$.

It is well known that the projection algorithm above has been introduced in the convex case ([135]) and its convergence has been proved is this case. Observe that Algorithm 7.1 is well defined provided the projection on C is not empty. The convexity assumption on C made by all researchers considering Algorithm 7.1 was not required for its well definedness because it may be well defined, even in the nonconvex case (for example when C is a closed subset of a finite dimensional space, or when C is a compact subset of a Hilbert space, etc.). Rather, convexity was required for its convergence analysis. Our adaptation of the projection algorithm is based on the following two observations:

1. The sequence of points $\{z^n\}_n$ that it generates must be sufficiently close to C.
2. The projection operator $Proj_C(\cdot)$ must be Lipschitz on an open set containing the sequence of points $\{z^n\}_n$.

Recently, a new class of nonconvex sets, called *uniformly prox-regular sets* (for the definition of this class and its important properties, we refer to Sect. 2.9 Chap. 2), has been introduced and studied in [89]. It has been successfully used in many nonconvex applications such as optimization, economic models, dynamical systems, differential inclusions, etc. For such applications see [39, 42, 53, 55, 58]. This class seems particularly well suited to overcome the difficulties arised by the nonconvexity assumption on C.

The following proposition summarizes from Chap. 2, some important consequences of the uniform prox-regularity needed in our proofs in the present chapter.

Proposition 7.1. *Let C be a nonempty closed subset in $\mathbf{H}$ and let $r \in (0,+\infty]$. If the subset C is uniformly r-prox-regular then the following hold:*

(i) For all $x \in \mathbf{H}$ with $d_C(x) < r$, one has $Proj_C(x) \neq \emptyset$;
(ii) Let $r' \in (0,r)$. The operator $Proj_C$ is Lipschitz with rank $\frac{r}{r-r'}$ on $C_{r'}$;
(iii) The proximal normal cone is closed as a set-valued mapping.
(iv) For all $x \in C$ and all $0 \neq \xi \in N^P(C;x)$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x' - x \right\rangle \leq \frac{2}{r}\|x' - x\|^2 + d_C(x'),$$

for all $x' \in \mathbf{H}$ with $d_C(x') < r$.

The following proposition establishes the relationship between (VI) and (VP) in the convex case.

Proposition 7.2. *If C is convex, then* (VI) $\Longleftrightarrow$ (VP).

Proof. It follows directly from the definition of $N^{\text{conv}}(C;x)$. □

The next proposition shows that the nonconvex variational problem (NVP) can be rewritten as the following nonconvex variational inequality:

$$\text{Find } x^* \in C \;\; y^* \in F(x^*) \text{ s.t. } \langle y^*, x - x^* \rangle + \frac{\|y^*\|}{2r}\|x - x^*\|^2 \geq 0, \tag{NVI}$$

for all $x \in C$.

Proposition 7.3. *If C is uniformly r-prox-regular, then* (NVI) $\Longleftrightarrow$ (NVP).

Proof. ($\Longrightarrow$) Let $x^* \in C$ be a solution of (NVI), i.e., there exists $y^* \in F(x^*)$ such that

$$\langle y^*, x - x^* \rangle + \frac{\|y^*\|}{2r}\|x - x^*\|^2 \geq 0, \quad \text{for all } \; x \in C.$$

If $y^* = 0$, then we are done because the vector zero always belongs to any normal cone. If $y^* \neq 0$, then, for all $x \in C$, one has

$$\left\langle \frac{-y^*}{\|y^*\|}, x - x^* \right\rangle \leq \frac{1}{2r}\|x - x^*\|^2.$$

Therefore, $\frac{-y^*}{\|y^*\|} \in N^P(C;x^*)$ and so $-y^* \in N^P(C;x^*)$, which completes the proof of the necessity part.

($\Longleftarrow$) It follows directly from the definition of uniformly prox-regular sets. □

In what follows we will let C be a uniformly r'-prox-regular subset of $\mathbf{H}$ with $r' > 0$ and we will let $r \in (0, r')$. Now, we are ready to present our adaptation of Algorithm 7.1 to the uniform prox-regular case.

7.2 Main Theorems

Our first algorithm Algorithm 7.2 below is proposed to solve problem (NVP).

Algorithm 7.2

1. *Select* $x^0 \in C, y^0 \in F(x^0)$, *and* $\rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - \rho y^n$ *and select:* $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

In our analysis we need the following assumptions on F:

Assumptions $\mathscr{A}_1$**.**

1. $F : \mathbf{H} \rightrightarrows \mathbf{H}$ is strongly monotone on C with constant $\alpha > 0$, i.e., th ere exists $\alpha > 0$ such that $\forall x, x' \in C$

$$\langle y - y', x - x' \rangle \geq \alpha \|x - x'\|^2, \ \ \forall y \in F(x), \ y' \in F(x').$$

2. F has nonempty compact values in $\mathbf{H}$ and is Hausdorff Lipschitz on C with constant $\beta > 0$, i.e., there exists $\beta > 0$ such that $\forall x, x' \in C$

$$\mathscr{H}(F(x), F(x')) \leq \beta \|x - x'\|.$$

Here $\mathscr{H}$ stands for the Hausdorff distance relative to the norm associated with the Hilbert space $\mathbf{H}$.

3. The constants α and β satisfy the following inequality

$$\alpha\zeta > \beta\sqrt{\zeta^2 - 1},$$

where $\zeta = \frac{r'}{r' - r}$.

Theorem 7.1. *Assume that $\mathscr{A}_1$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha}{\beta^2} - \varepsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \varepsilon, \frac{r}{\|y^n\| + 1}\right\},$$

where $\varepsilon = \frac{\sqrt{(\alpha\zeta)^2 - \beta^2(\zeta^2-1)}}{\zeta\beta^2}$, then the sequences $\{z^n\}_n$, $\{x^n\}_n$, and $\{y^n\}_n$ generated by algorithm Algorithm 7.2 converge strongly to some z^*, x^*, and y^* respectively, and x^* is a solution of (NVP).

Proof. From Algorithm 7.2, we have

$$\begin{aligned}\|z^{n+1} - z^n\| &= \|\,(x^n - \rho y^n) - \left(x^{n-1} - \rho y^{n-1}\right)\| \\ &= \|x^n - x^{n-1} - \rho(y^n - y^{n-1})\|.\end{aligned}$$

As the elements $\{x^n\}_n$ belong to C by construction and by using the fact that F is strongly monotone and Hausdorff Lipschitz on C, we have

$$\left\langle y^n - y^{n-1}, x^n - x^{n-1}\right\rangle \geq \alpha\|x^n - x^{n-1}\|^2$$

and

$$\|y^n - y^{n-1}\| \leq \mathscr{H}(F(x^n), F(x^{n-1})) \leq \beta\|x^n - x^{n-1}\|,$$

respectively. Note that

$$\begin{aligned}\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\|^2 = \|x^n - x^{n-1}\|^2 - 2\rho\left\langle y^n - y^{n-1}, x^n - x^{n-1}\right\rangle \\ + \rho^2\|y^n - y^{n-1}\|^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\|^2 \leq \|x^n - x^{n-1}\|^2 - 2\rho\alpha\|x^n - x^{n-1}\|^2 \\ + \rho^2\beta^2\|x^n - x^{n-1}\|^2,\end{aligned}$$

i.e.,

$$\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\|^2 \leq (1 - 2\rho\alpha + \rho^2\beta^2)\|x^n - x^{n-1}\|^2.$$

So,

$$\|x^n - x^{n-1} - \rho(y^n - y^{n-1})\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|x^n - x^{n-1}\|.$$

Finally, we deduce directly that

$$\|z^{n+1} - z^n\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|x^n - x^{n-1}\|.$$

Now, by the choice of ρ in the statement of the theorem, $\rho < \frac{r}{\|y^n\| + 1}$, we can easily check that the sequence of points $\{z^n\}_n$ belongs to $C_r := \{x \in \mathbf{H} : d_C(x) < r\}$. Consequently, the Lipschitz property of the projection operator on C_r mentioned in Proposition 7.1, yields

$$\begin{aligned}\|x^{n+1} - x^n\| &= \|\mathrm{Proj}_C(z^{n+1}) - \mathrm{Proj}_C(z^n)\| \leq \zeta\|z^{n+1} - z^n\| \\ &\leq \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|x^n - x^{n-1}\|.\end{aligned}$$

Let $\xi = \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$. Our assumption (3) in $\mathscr{A}_1$ and the choice of ρ in the statement of the theorem yield $\xi < 1$. Therefore, the sequence $\{x^n\}_n$ is a Cauchy sequence and hence it converges strongly to some point $x^* \in \mathbf{H}$. By using the continuity of the operator F, the strong convergence of the sequences $\{y^n\}_n$ and $\{z^n\}_n$ follows directly from the strong convergence of $\{x^n\}_n$.

Let y^* and z^* be the limits of the sequences $\{y^n\}_n$ and $\{z^n\}_n$ respectively. It is obvious to see that $z^* = x^* - \rho y^*$ with $x^* \in C$, $y^* \in F(x^*)$. We wish to show that x^* is the solution of our problem (NVP).

By construction we have for all $n \geq 0$

$$x^{n+1} \in \mathrm{Proj}_C(z^{n+1}) = \mathrm{Proj}_C(x^n - \rho y^n),$$

which gives by the definition of the proximal normal cone

$$(x^n - x^{n+1}) - \rho y^n \in N^P(C; x^{n+1}).$$

Using the closedness property of the proximal normal cone in (iii) of Proposition 7.1 and by letting $n \to \infty$ we get

$$\rho y^* \in -N^P(C; x^*).$$

Finally, as $y^* \in F(x^*)$ we conclude that $-N^P(C; x^*) \cap F(x^*) \neq \emptyset$ with $x^* \in C$. This completes the proof. □

Remark 7.1. If C is given in an explicit form, then we select for starting point x^0 in C. However, if we do not know the explicit form of C, then the choice of $x^0 \in C$ may not be possible. Assume that we know instead an explicit form of a δ-neighborhood of C, with $\delta < r/2$. So, we start with a point x^0 in the δ-neighborhood and instead of Algorithm 7.2, we use Algorithm 7.3 below. The convergence analysis of Algorithm 7.2 can be conducted along the same lines under the following choice of ρ:

$$\frac{\alpha}{\beta^2} - \varepsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \varepsilon, \frac{\delta}{\|y^n\| + 1}\right\}.$$

Indeed, if $x^0 \in \delta$-neighborhood of C, then $z^1 := x^0 - \rho y^0$ and so $d(z^1, C) \leq d(x^0, C) + \rho\|y^0\| < \delta + \frac{\delta}{\|y^0\|+1}\|y^0\| < \delta + \delta = 2\delta < r$. Therefore, we can project z^1 on C to get $x^1 \in C$, and then all subsequent points of the sequence x^n will be in C.

Algorithm 7.3

1. *Select* $x^0 \in C + \delta\mathbf{B}$, *with* $0 < 2\delta < r$, $y^0 \in F(x^0)$, *and* $\rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - \rho y^n$ *and select:* $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F(x^{n+1})$.

Remark 7.2. An inspection of our proof of Theorem 7.1 shows that the sequence $\{y^n\}_n$ is bounded. We state two sufficient conditions ensuring the boundedness of the sequence $\{y^n\}_n$:

1. The set-valued mapping F is bounded on C.
2. The set C is bounded and the set-valued mapping F has the linear growth property on C, that is,

$$F(x) \subset \alpha_1(1 + \|x\|)\mathbf{B},$$

   for some α_1 and for all $x \in C$.

We end this section by noting that our result in Theorem 7.1 can be extended (see Theorem 7.2 below) to the case $F = F_1 + F_2$ where F_1 is a Hausdorff Lipschitz set-valued mapping strongly monotone on C and F_2 is only Hausdorff Lipschitz set-valued mapping on C not necessarily monotone. It is interesting to point out that in this case F is not necessarily strongly monotone on C and so the following result cannot be covered by our previous result. In this case, Algorithm 7.2 becomes

Algorithm 7.4

1. *Select* $x^0 \in C$, $y^0 \in F_1(x^0)$, $w^0 \in F_2(x^0)$ *and* $\rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - \rho(y^n + w^n)$ *and select:* $x^{n+1} \in Proj_C(z^{n+1})$, $y^{n+1} \in F_1(x^{n+1})$, $w^{n+1} \in F_2(x^{n+1})$.

The following assumptions on F_1 and F_2 are needed for the proof of the convergence of Algorithm 7.4.

Assumptions $\mathscr{A}_2$.

1. F_1 is strongly monotone on C with constant $\alpha > 0$.
2. F_1 and F_2 have nonempty compact values in $\mathbf{H}$ and are Hausdorff Lipschitz on C with constant $\beta > 0$ and $\eta > 0$, respectively.
3. The constants α, ζ, η, and β satisfy the following inequality:

$$\alpha\zeta > \eta + \sqrt{(\beta^2 - \eta^2)(\zeta^2 - 1)}.$$

Theorem 7.2. *Assume that $\mathscr{A}_2$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha\zeta-\eta}{\zeta(\beta^2-\eta^2)}-\varepsilon<\rho<\min\left\{\frac{\alpha\zeta-\eta}{\zeta(\beta^2-\eta^2)}+\varepsilon,\frac{1}{\eta\zeta},\frac{r}{\|y^n+w^n\|+1}\right\},$$

where $\varepsilon=\frac{\sqrt{(\alpha\zeta-\eta)^2-(\beta^2-\eta^2)(\zeta^2-1)}}{\zeta(\beta^2-\eta^2)}$, *then the sequences* $\{z^n\}_n$, $\{x^n\}_n$, *and* $\{y^n\}_n$ *generated by algorithm Algorithm 7.4 converge strongly to some* z^*, x^*, *and* y^*, *respectively, and* x^* *is a solution of (NVP) associated to the set-valued mapping* $F=F_1+F_2$.

Proof. It follows the same lines as the proof of Theorem 7.1 with slight modifications. From Algorithm 7.4, we have

$$\begin{aligned}\|z^{n+1}-z^n\|&=\|\left[x^n-\rho(y^n+w^n)\right]-\left[x^{n-1}-\rho(y^{n-1}+w^{n-1}\right]\|\\&\leq\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\|+\rho\|w^n-w^{n-1}\|.\end{aligned}$$

As the elements $\{x^n\}_n$ belong to C by construction and by using the fact that F_1 is strongly monotone and Hausdorff Lipschitz continuous on C, we have

$$\left\langle y^n-y^{n-1},x^n-x^{n-1}\right\rangle\geq\alpha\|x^n-x^{n-1}\|^2$$

and

$$\|y^n-y^{n-1}\|\leq\mathscr{H}(F_1(x^n),F_1(x^{n-1}))\leq\beta\|x^n-x^{n-1}\|.$$

Note that

$$\begin{aligned}\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\|^2=\|x^n-x^{n-1}\|^2-2\rho\left\langle y^n-y^{n-1},x^n-x^{n-1}\right\rangle\\+\rho^2\|y^n-y^{n-1}\|^2.\end{aligned}$$

Thus, a simple computation yields

$$\|x^n-x^{n-1}-\rho(y^n-y^{n-1})\|^2\leq(1-2\rho\alpha+\rho^2\beta^2)\|x^n-x^{n-1}\|^2.$$

On the other hand, since F_2 is Hausdorff Lipschitz on C, we have

$$\|w^n-w^{n-1}\|\leq\mathscr{H}(F_2(x^n),F_2(x^{n-1}))\leq\eta\|x^n-x^{n-1}\|.$$

Finally,

$$\|z^{n+1}-z^n\|\leq\sqrt{1-2\rho\alpha+\rho^2\beta^2}\|x^n-x^{n-1}\|+\rho\eta\|x^n-x^{n-1}\|.$$

Now, by the choice of ρ in the statement of the theorem and the Lipschitz property of the projection operator on C_r mentioned in Proposition 7.1, we have

$$\begin{aligned}\|x^{n+1}-x^n\| &= \|\mathrm{Proj}_C(z^{n+1})-\mathrm{Proj}_C(z^n)\| \le \zeta\|z^{n+1}-z^n\| \\ &\le \zeta\left(\sqrt{1-2\rho\alpha+\rho^2\beta^2}+\rho\eta\right)\|x^n-x^{n-1}\|.\end{aligned}$$

Let $\xi = \zeta\left(\sqrt{1-2\rho\alpha+\rho^2\beta^2}+\rho\eta\right)$. Our assumption (3) in $\mathscr{A}_2$ and the choice of ρ in the statement of the theorem yield $\xi < 1$. Therefore, the proof of Theorem 7.1 completes the proof. □

Remark 7.3.

1. Theorem 7.2 generalizes the main result in [217] to the case where C is nonconvex.
2. As we have observed in Remark 7.1, Algorithm 7.4 may also be adapted to the case where the starting point x^0 is selected in a δ-neighborhood of the set C with $0 < 2\delta < r$.

7.3 Extensions

In this section, we are interested in extending the results obtained so far to the case where the set C, instead of being fixed, is a set-valued mapping. Besides being a more general case, it also has many applications, see for example [20]. The problem that will be considered is the following:

$$\text{Find a point } x^* \in C(x^*): \quad F(x^*)\cap -N^P(C(x^*);x^*) \neq \emptyset. \tag{SNVP}$$

This problem will be called Set-valued Nonconvex Variational Problem (SNVP). We need the following proposition which is an adaptation of Theorem 4.1 in [58] (see also Sect. 5.3) to our problem. We recall the following concept of Lipschitz property for set-valued mappings: A set-valued mapping C is said Lipschitz if there exists $\kappa > 0$ such that

$$|d(y,C(x)) - d(y',C(x'))| \le \|y-y'\| + \kappa\|x-x'\|,$$

for all $x,x',y,y' \in \mathbf{H}$. In such a case we also say that C is Lipschitz with constant κ. It is easy to see that for set-valued mappings the above concept of Lipschitz property is weaker than the Hausdorff Lipschitz property.

Proposition 7.4. *Let $r \in (0,+\infty]$ and let $C : \mathbf{H} \rightrightarrows \mathbf{H}$ be a Lipschitz set-valued mapping with uniformly r-prox-regular values. Then, the following closedness property holds: "For any $x^n \to x^*, y^n \to y^*$, and $u^n \to u^*$ with $y^n \in C(x^n)$ and $u^n \in N^P(C(x^n);y^n)$, one has $u^* \in N^P(C(x^*);y^*)$".*

Proof. Let $x^n \to x^*, y^n \to y^*$, and $u^n \to u^*$ with $y^n \in C(x^n)$ and $u^n \in N^P(C(x^n); y^n)$. If $u^* = 0$, then we are done. Assume that $u^* \neq 0$ (hence $u^n \neq 0$ for n large enough). Observe first that $y^* \in C(x^*)$ because C is Lipschitz. As $y^n \to y^*$ one gets for n sufficiently large $y^n \in y^* + \frac{r}{2}\mathbf{B}$. Therefore, the uniform r-prox-regularity of the images of C and Proposition 7.1 (iv) give

$$\left\langle \frac{u^n}{\|u^n\|}, z - y^n \right\rangle \leq \frac{2}{r}\|z - y^n\|^2 + d_{C(x^n)}(z),$$

for all $z \in \mathbf{H}$ with $d_{C(x^n)}(z) < r$. This inequality still holds for n sufficiently large and for all $z \in y^* + \delta\mathbf{B}$ with $0 < \delta < \frac{r}{2}$, because for such z one has

$$d_{C(x^n)}(z) \leq \|z - y^*\| + \|y^* - y^n\| \leq \delta + \frac{r}{2} < r.$$

Consequently, the continuity of the distance function with respect to both variables (because C is Lipschitz) and the above inequality give, by letting $n \to +\infty$,

$$\left\langle \frac{u^*}{\|u^*\|}, z - y^* \right\rangle \leq \frac{2}{r}\|z - y^*\|^2 + d_{C(x^*)}(z) \quad \text{for all } z \in y^* + \delta\mathbf{B}.$$

Hence,

$$\left\langle \frac{u^*}{\|u^*\|}, z - y^* \right\rangle \leq \frac{2}{r}\|z - y^*\|^2 \quad \text{for all } z \in (y^* + \delta\mathbf{B}) \cap C(x^*).$$

This ensures, by the definition of the proximal normal cone, that $\frac{u^*}{\|u^*\|} \in N^P(C(x^*); y^*)$ and so $u^* \in N^P(C(x^*); y^*)$. Thus, completing the proof of the proposition. □

In all what follows, C will be a set-valued mapping with nonempty closed uniformly r'-prox-regular values for some $r' > 0$. We will also let $r \in (0, r')$ and $\zeta = \frac{r'}{r' - r}$.

The next algorithm, Algorithm 7.5, solves problem (SNVP).

Algorithm 7.5

---

1. *Select* $x^0 \in C(x^0)$*,* $y^0 \in F(x^0)$*, and* $\rho > 0$.
2. *For* $n \geq 0$*, compute:* $z^{n+1} = x^n - \rho y^n$ *and select:* $x^{n+1} \in Proj_{C(x^n)}(z^{n+1})$*,* $y^{n+1} \in F(x^{n+1})$.

We make the following assumptions on the set-valued mappings F and C:

Assumptions $\mathscr{A}_3$.

1. F has nonempty compact values and is strongly monotone with constant $\alpha > 0$.
2. F is Hausdorff Lipschitz and C is Lipschitz with constants $\beta > 0$ and $0 < \kappa < 1$, respectively.

3. For some constant $0 < k < 1$, the operator $\mathrm{Proj}_{C(\cdot)}(\cdot)$ satisfies the condition

$$\|\mathrm{Proj}_{C(x)}(z) - \mathrm{Proj}_{C(y)}(z)\| \leq k\|x-y\|, \quad \text{for all } \ x,y,z \in \mathbf{H}.$$

4. Let λ be a sufficiently small positive constant such that $0 < \lambda < \dfrac{r(1-\kappa)}{1+3\kappa}$.
5. The constants α, β, ζ and k satisfy:

$$\alpha\zeta > \beta\sqrt{\zeta^2 - (1-k)^2}.$$

Theorem 7.3. *Assume that $\mathscr{A}_3$ holds and that for each iteration the parameter ρ satisfies the inequalities*

$$\frac{\alpha}{\beta^2} - \varepsilon < \rho < \min\left\{\frac{\alpha}{\beta^2} + \varepsilon, \frac{\lambda}{\|y_n\|+1}\right\},$$

where

$$\varepsilon = \frac{\sqrt{(\alpha\zeta)^2 - \beta^2[\zeta^2 - (1-k)^2])}}{\zeta\beta^2},$$

then the sequences $\{z^n\}_n$, $\{x^n\}_n$, and $\{y^n\}_n$ generated by Algorithm 7.5 converge strongly to some z^*, x^*, and y^* respectively, and x^* is a solution of (SNVP).

We prove the following lemma needed in the proof of Theorem 7.3. It has its own interest.

Lemma 7.1. *Under the hypothesis of Theorem 7.3, the sequences of points $\{x^n\}_n$ and $\{z^n\}_n$ generated by Algorithm 7.5 are such that:*

$$z^n \text{ and } z^{n+1} \in [C(x^n)]_r := \{y \in \mathbf{H} : d_{C(x^n)}(y) < r\}, \quad \text{for all } n \geq 1.$$

Proof. Observe that by the definition of the algorithm

$$d(z^1, C(x^0)) = d(x^0 - \rho y^0, C(x^0)) \leq d(x^0, C(x^0)) + \rho\|y^0\| \leq \lambda.$$

For $n = 1$, we have by (ii)–(iv) of $\mathscr{A}_3$

$$\begin{aligned} d(z^2, C(x^1)) = d(x^1 - \rho y^1, C(x^1)) &\leq d(x^1, C(x^1)) \\ &- d(x^1, C(x^0)) + \rho\|y^1\| \leq \kappa\|x^1 - x^0\| + \lambda, \end{aligned}$$

and by the Lipschitz property of C once again and the first inequality of this proof we get

$$\begin{aligned} d(z^1, C(x^1)) \leq d(z^1, C(x^0)) + \kappa\|x^1 - x^0\| &= d(x^0 - \rho y^0, C(x^0)) \\ &+ \kappa\|x^1 - x^0\| \leq \lambda + \kappa\|x^1 - x^0\|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x^1 - x^0\| \le \|x^1 - z^1\| + \|z^1 - x^0\| &= d(z^1, C(x^0)) + \|z^1 - x^0\| \\ &= d(x^0 - \rho y^0, C(x^0)) + \rho\|y^0\| < 2\lambda.\end{aligned}$$

Thus, we get both $d(z^2, C(x^1))$ and $d(z^1, C(x^1))$ are less than $2\kappa\lambda + \lambda$ which is strictly less than r. Similarly, we have for general n

$$d(z^{n+1}, C(x^n)) \le d(x^n, C(x^n)) + \rho\|y^n\| \le \kappa\|x^n - x^{n-1}\| + \lambda$$

and

$$\begin{aligned}d(z^n, C(x^n)) &\le d(z^n, C(x^{n-1})) + \kappa\|x^n - x^{n-1}\| \\ &\le \kappa\|x^{n-1} - x^{n-2}\| + \lambda + \kappa\|x^n - x^{n-1}\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|x^n - x^{n-1}\| &\le \|x^n - z^n\| + \|z^n - x^{n-1}\| \\ &\le d(z^n, C(x^{n-1})) + \lambda \\ &\le d(x^{n-1}, C(x^{n-1})) - d(x^{n-1}, C(x^{n-2})) + 2\lambda \\ &\le \kappa\|x^{n-1} - x^{n-2}\| + 2\lambda.\end{aligned}$$

Hence, using that $\|x^1 - x^0\| < 2\lambda$, we get

$$\|x^n - x^{n-1}\| \le \frac{2\lambda(1-\kappa^n)}{1-\kappa}.$$

Therefore,

$$\begin{aligned}d(z^{n+1}, C(x^n)) &\le \frac{2\kappa\lambda(1-\kappa^n)}{1-\kappa} + \lambda \\ &\le \lambda\frac{1+\kappa-2\kappa^{n+1}}{1-\kappa} < \frac{\lambda(1+3\kappa)}{1-\kappa} < r\end{aligned}$$

and

$$\begin{aligned}d(z^n, C(x^n)) &\le \kappa\|x^{n-1} - x^{n-2}\| + \lambda + \kappa\|x^n - x^{n-1}\| \\ &\le (\kappa^2 + \kappa)\|x^{n-1} - x^{n-2}\| + 2\lambda\kappa + \lambda \\ &\le (\kappa^2 + \kappa)\frac{2\lambda(1-\kappa^{n-1})}{1-\kappa} + 2\lambda\kappa + \lambda \\ &\le \frac{\lambda(1+3\kappa)}{1-\kappa} < r.\end{aligned}$$

This completes the proof. □

Proof of Theorem 7.3. Following the proof of Theorem 7.1 and using the fact that F is strongly monotone and Hausdorff Lipschitz, we get from Algorithm 7.5,

$$\|z^{n+1} - z^n\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|x^n - x^{n-1}\|.$$

On the other hand, by Lemma 7.1, we have z^n and $z^{n+1} \in [C(x^n)]_r$ and so Proposition 7.1 yields that $\mathrm{Proj}_{C(x^n)}(z^n)$ and $\mathrm{Proj}_{C(x^n)}(z^{n+1})$ are not empty, and the operator $\mathrm{Proj}_{C(x^n)}(\cdot)$ is ζ-Lipschitz on $[C(x^n)]_r$. Then, by the assumption (iii) in $\mathscr{A}_3$,

$$\begin{aligned}\|x^{n+1} - x^n\| &= \|\mathrm{Proj}_{C(x^n)}(z^{n+1}) - \mathrm{Proj}_{C(x^{n-1})}(z^n)\| \\ &\leq \|\mathrm{Proj}_{C(x^n)}(z^{n+1}) - \mathrm{Proj}_{C(x^n)}(z^n)\| \\ &\quad + \|\mathrm{Proj}_{C(x^n)}(z^n) - \mathrm{Proj}_{C(x^{n-1})}(z^n)\| \\ &\leq \zeta\|z^{n+1} - z^n\| + k\|x^n - x^{n-1}\| \\ &\leq \left[\zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k\right]\|x^n - x^{n-1}\|.\end{aligned}$$

Let $\xi = \zeta\sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k$. Our assumptions (iv) and (v) in $\mathscr{A}_3$ and the choice of ρ in the statement of the theorem yield $\xi < 1$. As in the proof of Theorem 7.1, we can prove that the sequences $\{x^n\}_n, \{y^n\}_n$, and $\{z^n\}_n$ strongly converge to some $x^*, y^*, z^* \in \mathbf{H}$, respectively. It is obvious to see that $z^* = x^* - \rho y^*$ with $x^* \in C(x^*)$, $y^* \in F(x^*)$. We wish to show that x^* is the solution of our problem (SNVP).

By construction we have for all $n \geq 0$

$$x^{n+1} \in \mathrm{Proj}_{C(x^n)}(z^{n+1}) = \mathrm{Proj}_{C(x^n)}(x^n - \rho y^n),$$

which gives by the definition of the proximal normal cone

$$(x^n - x^{n+1}) - \rho y^n \in N^P(C(x^n); x^{n+1}).$$

Using the closedness property of the proximal normal cone in Proposition 7.4 and by letting $n \to \infty$ we get

$$\rho y^* \in -N^P(C(x^*); x^*).$$

Finally, as $y^* \in F(x^*)$ we conclude that $-N^P(C(x^*); x^*) \cap F(x^*) \neq \emptyset$ with $x^* \in C(x^*)$. This completes the proof. □

We extend Theorem 7.3 to the case $F = F_1 + F_2$ where F_1 is a Hausdorff Lipschitz set-valued mapping strongly monotone and F_2 is only Hausdorff Lipschitz set-valued mapping. In this case, Algorithm 7.5 becomes

Algorithm 7.6

1. *Select* $x^0 \in C(x^0)$, $y^0 \in F_1(x^0)$, $w^0 \in F_2(x^0)$ *and* $\rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - \rho(y^n + w^n)$ *and select:* $x^{n+1} \in Proj_{C(x^n)}(z^{n+1})$, $y^{n+1} \in F_1(x^{n+1})$, $w^{n+1} \in F_2(x^{n+1})$.

The following assumptions on F_1 and F_2 are needed for the proof of the convergence of Algorithm 7.6.

Assumptions $\mathscr{A}_4$**.**

1. The assumptions on the set-valued mapping C are as in $\mathscr{A}_3$.
2. F_1 is strongly monotone with constant $\alpha > 0$.
3. F_1 and F_2 have nonempty compact values and are Hausdorff Lipschitz with constant $\beta > 0$ and $\eta > 0$, respectively.
4. The constants $\alpha, \beta, \eta, \zeta$, and k satisfy the following inequality:

$$\alpha\zeta > (1-k)\eta + \sqrt{(\beta^2 - \eta^2)[\zeta^2 - (1-k)^2]}.$$

Theorem 7.4. *Assume that* $\mathscr{A}_4$ *holds and that for each iteration the parameter* ρ *satisfies the inequalities*

$$\frac{\alpha\zeta - (1-k)\eta}{\zeta(\beta^2 - \eta^2)} - \varepsilon < \rho < \min\left\{\frac{\alpha\zeta - (1-k)\eta}{\zeta(\beta^2 - \eta^2)} + \varepsilon, \frac{1-k}{\zeta\eta}, \frac{r}{\|y^n + w^n\| + 1}\right\},$$

where

$$\varepsilon = \frac{\sqrt{[\alpha\zeta - (1-k)\eta]^2 - (\beta^2 - \eta^2)[\zeta^2 - (1-k)^2]}}{\zeta(\beta^2 - \eta^2)},$$

then the sequences $\{z^n\}_n$, $\{x^n\}_n$, *and* $\{y^n\}_n$ *generated by algorithm Algorithm 7.6 converge strongly to some* z^*, x^*, *and* y^*, *respectively, and* x^* *is a solution of (SNVP) associated to the set-valued mapping* $F = F_1 + F_2$.

Proof. As we adapted the proof of Theorem 7.1 to prove Theorem 7.2, we can adapt, in a similar way, the proof of Theorem 7.3 to prove Theorem 7.4. □

Remark 7.4.

1. Theorem 7.4 generalizes Theorem 7.2 in [216] to the case where C is nonconvex.
2. As we have observed in Remark 7.1, algorithms Algorithm 7.5 and Algorithm 7.6 may be also adapted to the case where the starting point x^0 is selected in a δ-neighborhood of the set $C(x^0)$ with $0 < 2\delta < r$.

Example 7.1. In many applications (see for example [20]), the set-valued mapping C has the form $C(x) = S + f(x)$, where S is a fixed closed subset in $\mathbf{H}$ and f is a point-to-point mapping from **H** to **H**. In this case, assumption (iii) on C in $\mathscr{A}_3$ and the Lipschitz property of C are satisfied provided the mapping f is Lipschitz. Indeed, it is not hard (using the relation below) to show that, if f is γ-Lipschitz then the set-valued mapping C is γ-Lipschitz and it satisfies the assumption (iii) in $\mathscr{A}_3$ with $k = 2\gamma$. Using the well known relation

$$\bar{x} \in \mathrm{Proj}_{S+v}(\bar{u}) \Longleftrightarrow \bar{x} - v \in \mathrm{Proj}_S(\bar{u} - v),$$

the Algorithms 7.5 and 7.6 can be rewritten in simpler forms. For example Algorithm 7.6 becomes

Algorithm 7.7

1. *Select* $x^0 \in (I - f)^{-1}(S)$, $y^0 \in F_1(x^0)$, $w^0 \in F_2(x^0)$ $textit and \rho > 0$.
2. *For* $n \geq 0$, *compute:* $z^{n+1} = x^n - f(x^n) - \rho(y^n + w^n)$ *and select:* $x^{n+1} \in Proj_S(z^{n+1}) + f(x^n)$, $y^{n+1} \in F_1(x^{n+1})$, $w^{n+1} \in F_2(x^{n+1})$.

Here I is the Identity operator from **H** to **H**.

Conclusion. Our Algorithms proposed here can be extended to solve the following general variational problem:

Find $x^* \in \mathbf{H}$ with $g(x^*) \in C(x^*)$ such that

$$F(x^*) \cap -N^P(C(x^*); g(x^*)) \neq \emptyset, \qquad (g\text{-SNVP})$$

where $g : \mathbf{H} \to \mathbf{H}$ is a point-to-point mapping. It is obvious that (g-SNVP) coincides with (SNVP) when $g = I$. An important reason for considering this general variational problem (g-SNVP) is to extend all (or almost all) the types of variational inequalities existing in the literature in the convex case to the nonconvex case by the same way presented in this chapter. For instance, when the set-valued mapping C is assumed to have convex values the general variational problem (g-SNVP) coincides with the so-called *generalized multivalued quasi-variational inequality* introduced by Noor [218] and studied by himself and many other authors.

7.4 Commentary to Chap. 7

Our main goal in Chap. 7 is to show how the regularity concept can be used to study nonconvex variational (or quasi-variational) inequalities. The results presented in this chapter have been demonstrated in Bounkhel [64]. Various other works studied variational and quasi-variational inequalities under some other regularity concepts. We refer the interested readers to the following recent references: [49,212–214,219–223].

Chapter 8
Economic Problems and Equilibrium Theory

8.1 Introduction

The distance function d_S for a closed subset S of a Hilbert space $\mathbf{H}$ is a very important tool in many fields of mathematics, such as, optimization, differential inclusions, control theory, geometry of Banach spaces, economic models, etc. Many papers [31, 44, 45, 58, 61, 88–91, 120, 230, 255, 258] studied and characterized some important properties of this function. For example, in [89, 230] the authors established in several interesting ways, important characterization of the differentiability of the distance function d_S for closed sets S. In [230] the authors showed that the class of sets S for which d_S is continuously differentiable outside of S on some neighborhood of a point $x \in S$, is equivalent to the class of uniformly prox-regular sets, which is a condition on normal vectors. They also gave connections to proximally smooth sets introduced in [89]. This class is so large, it englobes all closed convex sets, smooth submanifolds and many other nonconvex sets (for more details see Sect. 2.9 in Chap. 2). Later, in [58] the authors established new characterizations of *uniformly* prox-regular sets and they gave a very important application to the existence of solutions of nonconvex differential inclusions. In the present chapter, we prove some properties of the class of uniformly prox-regular sets in terms of the distance function, in order to give some applications in economies (Sect. 5) and equilibrium theory (Sect. 6). The chapter is organized as follow: In Sect. 2, we introduce the concept of *uniform* lower-C^2 property for functions and we study the stability of this class under some operations. We study the connection between this property and the uniform prox-regularity. We prove for example the uniform prox-regularity of level sets defined by uniform lower-C^2 functions. Section 3 is devoted to the study of both the subdifferential and the co-normal stability of uniformly prox-regular sets. In Sects. 4 and 5, we present two different applications of our results obtained in the previous sections. In Sect. 5, we prove, using the subdifferential stability result in Sect. 4, the stability of the quasi-equilibrium prices for nonconvex nontransitive economies. The case of economies defined by nonconvex utility functions is also considered. The application in Sect. 6

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concerns the existence of generalized equilibrium of set-valued mappings over a given nonconvex set. A new existence result in the infinite dimensional setting is established.

8.2 Uniform Prox-Regularity of Level Sets and Uniform Lower-C^2 Property

In [235], Rockafellar introduced in the finite dimensional setting an important class of nonsmooth functions which he called "lower-C^2". He showed that such class has favorable properties in optimization. We recall that a function $f : O \to \mathbf{R}$ is said to be lower-C^2 on an open subset O of $\mathbf{R}^n$ if on some neighborhood V of each $\bar{x} \in O$ there is a representation $f(x) = \max_{t\in T} f_t(x)$ in which the functions f_t are of C^2 on V and the index set T is a compact space such that $f_t(x)$ and $\nabla f_t(x)$ depend continuously not on just on x but jointly on $(t,x) \in T \times V$. A particular example of lower-C^2 functions one has $f(x) = \max\{f_1(x),\ldots,f_m(x)\}$, when f_i is of class C^2.

Rockafellar [235] proved an important characterization of lower C^2 functions in finite dimensional setting. He showed that a function f is lower-C^2 on an open set $O \subset \mathbf{R}^n$ if and only if, relative to some neighborhood of each point of O, there is an expression $f = g - \frac{\rho}{2}\|\cdot\|^2$, in which g is finite convex function and $\rho \geq 0$. Therefore, a natural way to generalize the concept of lower C^2 property to infinite dimensional setting is to take this local characterization as a definition and one gets.

Definition 8.1. Let $\mathbf{H}$ be a Hilbert space. A function $f : \mathbf{H} \to \mathbf{R}$ is said to be lower C^2 around a point $\bar{x} \in \mathbf{H}$ if there exist an open convex neighborhood O and $\rho \geq 0$ such that $f + \frac{\rho}{2}\|\cdot\|^2$ is convex over O. We will say that f is lower C^2 around a subset $S \subset \mathbf{H}$ if it is lower C^2 around each point of S.

In this section, some uniformity of this characterization will be needed. So this definition is not appropriate for our purpose. Therefore, we will take the following definition.

Definition 8.2. Let Ω be a convex set of a Hilbert space $\mathbf{H}$. We will say that a function $f : \mathrm{cl}(\Omega) \to \mathbf{R}$ is *uniformly lower* C^2 over Ω if there exists $\rho \geq 0$ such that $f + \frac{\rho}{2}\|\cdot\|^2$ is convex over Ω. In such case, we will say that f is uniformly ρ-lower C^2 over Ω.

The class of functions that are uniformly lower C^2 over sets is so large. We state here some examples.

1. Any lower semicontinuous proper convex function f is uniformly regular over any nonempty subset of its domain with $\rho = 0$.
2. Any lower-C^2 function f over the closure of a nonempty open convex set Ω in the sense of Definition 8.1 is uniformly lower C^2 over the closure of Ω whenever Ω is relatively strongly compact in $\mathbf{H}$ as the following proposition shows it.

Proposition 8.1. *Let Ω be an open convex relatively strongly compact of* $\mathbf{H}$. *If f is a lower-C^2 function on* $\mathrm{cl}(\Omega)$, *then f is uniformly lower C^2 over* $\mathrm{cl}(\Omega)$.

Proof. Assume that f is lower-C^2 on $\mathrm{cl}(\Omega)$. For every $x \in \mathrm{cl}(\Omega)$ there exist, by Definition 8.1, an open convex neighborhood O_x of x and $\rho_x \geq 0$ such that $f + \frac{\rho_x}{2}\|\cdot\|^2$ is finite convex over O_x. The family of open sets $\{O_x\}_{x\in\mathrm{cl}(\Omega)}$ covers $\mathrm{cl}(\Omega)$, so by the strong compactness of $\mathrm{cl}(\Omega)$ there exist points $x_1, x_2, \ldots, x_n$ in $\mathrm{cl}(\Omega)$ such that $\mathrm{cl}(\Omega) \subset \cup_{k=1}^{k=n} O_{x_k}$, and $\rho > 0$ sufficiently large such that $f + \frac{\rho}{2}\|\cdot\|^2$ is finite convex over $\cup_{k=1}^{k=n} O_{x_k}$ and so over the convex set $\mathrm{cl}(\Omega)$. This ensures the uniform lower C^2 property of f over $\mathrm{cl}(\Omega)$ and hence the proof is complete. □

One could think to deal with the class of lower-C^2 functions in the sense of Definition 8.1 instead of our class of uniformly lower-C^2 functions. The inconvenience of the class of lower-C^2 functions is the need of the relative strong compactness of the set to get the uniform concept in Definition 8.2 which is the exact property needed in our proofs. However, we can find many functions that are uniformly lower-C^2 over noncompact sets see Theorem 2.14.

Note that it is easy to see that for a function f that is uniformly lower-C^2 over a convex set Ω all the classical subdifferentials included in the generalized gradient, coincide at each point x in Ω.

Now, we prove a characterization of uniformly lower-C^2 functions in terms of their subdifferentials. It will be our main tool to study the stability of the class of uniform lower-C^2 functions. Also, it will be used to establish a connection between the uniform lower-C^2 property of a function f and the uniform prox-regularity of its associated level set $[f \leq 0]$. Note that a similar result has been proved in [89].

Proposition 8.2. *Let $\rho \geq 0$, Ω be an open convex relatively strongly compact subset of* $\mathbf{H}$, *and let f be a Lipschitz function on* $\mathrm{cl}(\Omega)$. *Then the following assertions are equivalent:*

(i) *f is uniformly ρ-lower-C^2 on* $\mathrm{cl}(\Omega)$;
(ii) *For each $\bar{x} \in \mathrm{cl}(\Omega)$ and $\xi \in \partial^P f(\bar{x})$ one has*

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2 \quad \forall x \in \mathrm{cl}(\Omega); \tag{8.1}$$

(iii) *For each $\bar{x} \in \Omega$ and $\xi \in \partial^P f(\bar{x})$ one has*

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2 \quad \forall x \in \Omega; \tag{8.2}$$

(iv) *For each $\bar{x} \in \Omega$ and $\xi \in \partial^C f(\bar{x})$ one has*

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2 \quad \forall x \in \Omega. \tag{8.3}$$

Proof.

(i) $\Rightarrow (ii)$: Let $\bar{x} \in \mathrm{cl}(\Omega)$ and $\xi \in \partial^P f(\bar{x})$. Then one has $\xi + \rho\bar{x} \in \partial^P(f + \frac{\rho}{2}\|\cdot\|^2)(\bar{x})$. By Definition 8.2 the function $f + \frac{\rho}{2}\|\cdot\|^2$ is finite and convex on $\mathrm{cl}(\Omega)$ and so for all $x \in \mathrm{cl}(\Omega)$ one has

$$\langle \xi + \rho\bar{x}, x - \bar{x}\rangle \leq \left(f + \frac{\rho}{2}\|\cdot\|^2\right)(x) - \left(f + \frac{\rho}{2}\|\cdot\|^2\right)(\bar{x}).$$

This ensures

$$\begin{aligned}\langle \xi, x - \bar{x}\rangle &\leq f(x) - f(\bar{x}) + \frac{\rho}{2}\left[\|x\|^2 - \|\bar{x}\|^2 - 2\langle \bar{x}, x - \bar{x}\rangle\right] \\ &= f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2,\end{aligned}$$

for all $x \in \mathrm{cl}(\Omega)$ and then the proof of (8.1) is complete.

(ii) $\Rightarrow (iii)$: It is obvious.

(iii) $\Rightarrow (iv)$: It follows directly from the formula $\partial^C f(x) = \overline{co}\left[\partial^P f(x)\right]$ proved in [91].

(iv) $\Rightarrow (i)$: By (8.3) we have for each $x \in \Omega$

$$f(x) \geq f(\bar{x}) + \langle \xi, x - \bar{x}\rangle - \frac{\rho}{2}\|x - \bar{x}\|^2 \ \text{ for all } \ \bar{x} \in \Omega \ \text{ and } \ \xi \in \partial^C f(\bar{x}).$$

So, we have

$$f(x) = \max_{(\bar{x},\xi)\in \mathrm{cl}(\Omega)\times\mathscr{E}} \left\{ f(\bar{x}) + \langle \xi, x - \bar{x}\rangle - \frac{\rho}{2}\|x - \bar{x}\|^2 \right\},$$

where $\mathscr{E} := \bigcup_{x\in \mathrm{cl}(\Omega)} \partial^C f(x)$ which is a weakly compact subset in $\mathbf{H}$ by Theorem II-25 in [79]. It follows then that $f = g - \frac{\rho}{2}\|\cdot\|^2$ on $\mathrm{cl}(\Omega)$ with

$$g(x) = \max_{(\bar{x},\xi)\in \mathrm{cl}(\Omega)\times\mathscr{E}} \left\{ f(\bar{x}) + \langle \xi, x - \bar{x}\rangle - \frac{\rho}{2}\|\bar{x}\|^2 + \rho < x, \bar{x} > \right\},$$

which is a finite convex function on $\mathrm{cl}(\Omega)$. Thus, f is uniformly ρ-lower C^2 over $\mathrm{cl}(\Omega)$ and so the proof of this implication is complete. □

The study of the stability under some operations of the class of uniform lower-C^2 functions is very important for applications. Noting that it follows directly from Definition 8.2 that the addition of uniform lower-C^2 functions on an open convex subset Ω of $\mathbf{H}$ is uniform lower-C^2 on Ω. In what follows we prove the stability of this class under the following operations: pointwise maximum, composition, and integral.

Proposition 8.3. *Pointwise maximum of uniformly lower-C^2 functions Let Ω be an open convex relatively strongly compact set of* $\mathbf{H}$. *The pointwise maximum of Lipschitz uniformly lower-C^2 functions over* $\mathrm{cl}(\Omega)$ *is Lipschitz uniformly lower-C^2 on* $\mathrm{cl}(\Omega)$.

Proof. Assume that $f(x) := \max_{1\le i\le m} f_i(x)$, for all $x \in \mathrm{cl}(\Omega)$, where f_i, $1 \le i \le m$ is Lipschitz uniformly ρ_i-lower-C^2 on $\mathrm{cl}(\Omega)$. The Lipschitz property of f is obvious. So we have to show its uniform lower-C^2 property over $\mathrm{cl}(\Omega)$. Taking $\rho := \max_{1\le i\le m} \rho_i$, we get $(f_i)_{1\le i\le m}$ are Lipschitz uniformly ρ-lower-C^2 on $\mathrm{cl}(\Omega)$. Fix now any $x \in \mathrm{cl}(\Omega)$ and any $\xi \in \partial^{\mathrm{C}} f(x)$. By subdifferential calculus there exist $\xi_j \in \partial^{\mathrm{C}} f_j(x)$ and $\alpha_j \ge 0$, $j \in I(x) := \{i \in \{1,\ldots,m\} : \; f_i(x) = f(x)\}$ such that $\xi = \sum_{j\in I(x)} \alpha_j \xi_j$ and $\sum_{j\in I(x)} \alpha_j = 1$. By Proposition 8.2 we obtain for all $j \in I(x)$ and all $x' \in \mathrm{cl}(\Omega)$

$$\langle \xi_j, x'-x\rangle \le f_j(x') - f_j(x) + \frac{\rho}{2}\|x'-x\|^2.$$

This yields, for all $x' \in \mathrm{cl}(\Omega)$

$$\begin{aligned}\langle \xi, x'-x\rangle &= \sum_{j\in I(x)} \alpha_j \langle \xi_j, x'-x\rangle \le \sum_{j\in I(x)} \alpha_j \left[f_j(x') - f_j(x) + \frac{\rho}{2}\|x'-x\|^2\right]\\ &\le \sum_{j\in I(x)} \alpha_j \left[f(x') - f(x) + \frac{\rho}{2}\|x'-x\|^2\right] \le f(x') - f(x) + \frac{\rho}{2}\|x'-x\|^2\end{aligned}$$

and so by Proposition 8.2, the function f is uniformly ρ-lower-C^2 on $\mathrm{cl}(\Omega)$. □

Proposition 8.4. *Let Ω be an open convex relatively strongly compact subset in* $\mathbf{H}$. *Let $F : \mathbf{H} \to \mathbf{H}'$ ($\mathbf{H}'$ is another Hilbert space) be a C^2 mapping and let h be a Lipschitz uniformly ρ-lower-C^2 function over $F(\mathrm{cl}(\Omega))$. Then the function $f = h \circ F$ is uniformly ρ'-lower-C^2 on $\mathrm{cl}(\Omega)$ for some $\rho' > 0$.*

Proof. The fact that f is Lipschitz is straightforward. Let $\alpha := \sup_{x\in \mathrm{cl}(\Omega)} \|\nabla F(x)\|$ and $\beta := \sup_{x\in \mathrm{cl}(\Omega)} \|\nabla^2 F(x)\|$, and let $\lambda > 0$ be the Lipschitz constant of h over $\mathrm{cl}(\Omega)$. Fix now any $x \in \mathrm{cl}(\Omega)$ and any $\xi \in \partial^{\mathrm{C}} f(x)$. By subdifferential calculus there exists $\zeta \in \partial^{\mathrm{C}} h(F(x))$ such that $\xi = \nabla F(x)^* \zeta$. Using Proposition 8.2 and the fact that h is uniformly ρ-lower-C^2 over $F(\mathrm{cl}(\Omega))$, we get for all $x' \in \mathrm{cl}(\Omega)$,

$$\begin{aligned}\langle \zeta, F(x') - F(x)\rangle &\le f(x') - f(x) + \frac{\rho}{2}\|F(x') - F(x)\|^2\\ &\le f(x') - f(x) + \frac{\alpha^2\rho}{2}\|x'-x\|^2.\end{aligned}$$

On the other hand, as F is C^2 we have

$$F(x') = F(x) + \nabla F(x)(x'-x) + \frac{1}{2}\langle \nabla^2 F(c)(x'-x), x'-x\rangle,$$

with $c = x + \theta(x'-x) \in \mathrm{cl}(\Omega)$ for some $\theta \in [0,1]$. Therefore, this equality and the inequality before give

$$\begin{aligned}\langle \xi, x'-x\rangle &= \langle \zeta, \nabla F(x)(x'-x)\rangle \\ &= \langle \zeta, F(x') - F(x)\rangle + \frac{1}{2}\langle \zeta, \langle \nabla^2 F(c)(x'-x), x'-x\rangle\rangle \\ &\le f(x') - f(x) + \frac{\alpha^2\rho}{2}\|x'-x\|^2 + \frac{1}{2}\|\zeta\|\|\nabla^2 F(c)\|\|x'-x\|^2 \\ &\le f(x') - f(x) + \frac{\rho'}{2}\|x'-x\|^2,\end{aligned}$$

where $\rho' := \alpha^2\rho + \beta\lambda$. The proof then is complete by (iv) in Proposition 8.2. □

Now we are going to study the stability under the integral operation. Let $I := [0,T]$ with $T > 0$ and let us consider the functional I_f defined from $L^p(I,\mathbf{H})$, with $p \in [2,+\infty)$ to $(-\infty,+\infty]$ by

$$I_f(u) := \int_0^T f(t,u(t))\mathrm{d}t, \quad \text{for all} \quad u \in L^p(I,\mathbf{H}),$$

where f is a function from $I \times \mathbf{H}$ to $(-\infty,+\infty]$.

Before proving our stability result we need the following lemma. Its proof is standard.

Lemma 8.1. *For any $p \in [2,+\infty)$ the function $h : L^p(I,\mathbf{H}) \to \mathbf{R}$ defined by*

$$h(u) := \frac{1}{2}\int_0^T \|u(t)\|^2\mathrm{d}t, \quad \text{for all} \quad u \in L^p(I,\mathbf{H})$$

is continuously Fréchet differentiable on $L^p(I,\mathbf{H})$ and its Fréchet derivative is given by $\nabla h(u) = u$.

Let us consider the following assumptions:

(A_1) $f : I \times \mathbf{H} \to \mathbf{R}$ is measurable with respect to the σ-field of subsets of $I \times \mathbf{H}$ generated by the Lebesgue sets in I and the Borel sets in $\mathbf{H}$.

(A_2) there exist $a \in L^q(I,\mathbf{H})$, $b \in L^1(I,\mathbf{R})$, and $c \in L^p(I,\mathbf{H})$ such that

$$f(t,c) \in L^1(I,\mathbf{R}) \quad \text{and} \quad f(t,x) \ge \langle a(t), x\rangle + b(t),$$

for all $t \in I$ and all $x \in \mathbf{H}$. Here q satisfies $1/p + 1/q = 1$.

Theorem 8.1. *Let Ω be an open convex relatively strongly compact subset of* $\mathbf{H}$ *and $\rho \geq 0$. Let f be a continuous function from $I \times \mathbf{H}$ to $\mathbf{R}$. Assume that f satisfies the assumptions (A_1) and (A_2) and for all $t \in I$ the function $f(t,\cdot)$ is uniformly ρ-lower-C^2 on $\mathrm{cl}(\Omega)$. Then the functional I_f is uniformly ρ-lower-C^2 on the set $\mathscr{K} := \{u \in L^p(I,\mathbf{H}) : u(t) \in \mathrm{cl}(\Omega)$ for all $t \in I\}$. Furthermore, for each $u \in \mathscr{K}$ one has*

$$\partial^P I_f(u) = \int_0^T \partial^P f(t,u)\mathrm{d}t := \left\{\xi \in L^q(I,\mathbf{H}) : \;\; \xi(t) \in \partial^P f(t,u(t)) \;\text{ a. e. on }\; I\right\}.$$

Proof. By Definition 8.2 one has $g(t,\cdot) := f(t,\cdot) + \frac{\rho}{2}\|\cdot\|^2$ is finite convex on $\mathrm{cl}(\Omega)$ for all $t \in I$. Then for any $x \in \mathrm{cl}(\Omega)$ and any $t \in I$ one has

$$\partial^P f(t,x) = \partial^{\mathrm{conv}} g(t,x) - \rho x,$$

because the norm $\|\cdot\|$ of $\mathbf{H}$ is smooth. Now, since it is easy to see that g also satisfies the assumptions (A_1) and (A_2), then Proposition 2.8 in [15] and the convexity of $g(t,\cdot)$ ensure that the functional I_g is finite convex on $\mathscr{K}$ and its subdifferential at any $u \in \mathscr{K}$ is given by

$$\begin{aligned}\partial^{\mathrm{conv}} I_g(u) &= \int_0^T \partial^{\mathrm{conv}} g(t,u)\mathrm{d}t \\ &:= \left\{\zeta \in L^q(T,\mathbf{H}) : \zeta(t) \in \partial^{\mathrm{conv}} g(t,u(t)) \text{ a.e. on } I\right\}.\end{aligned} \tag{8.4}$$

Fix now any $u \in \mathscr{K}$. Then the calculus rules for subdifferentials and Lemma 8.1 yield

$$\partial^P I_f(u) = \partial^P(I_g - \rho h)(u) = \partial^{\mathrm{conv}} I_g(u) - \rho \nabla h(u) = \partial^{\mathrm{conv}} I_g(u) - \rho u.$$

Therefore, for any $\xi \in \partial^P I_f(u)$ one has $\xi + \rho u \in \partial^{\mathrm{conv}} I_g(u)$ and so by (8.4) we get $\xi(t) \in \partial^{\mathrm{conv}} g(t,u(t)) - \rho u(t) = \partial^P f(t,u(t))$ a.e. on I, as claimed. Conversely, it is easy to see that every $\xi \in L^q(I,\mathbf{H})$ satisfying the latter belongs to $\partial^P I_f(u)$. This completes the proof. $\square$

As a direct application of the subdifferential formula in Theorem 8.1, we give necessary conditions of optimality for the following nonconvex variational problem:

$$(\mathscr{P}) \qquad \text{minimize} \int_0^T f(t,u(t))\mathrm{d}t$$

over $\mathscr{K} = \{u \in L^p(I,\mathbf{H}) : \;\; u(t) \in \mathrm{cl}(\Omega)$ for all $t \in I\}$, where Ω is an open convex relatively strongly compact subset of $\mathbf{H}$ and f satisfies the hypothesis in Theorem 8.1. We note that $\mathscr{K}$ is a closed convex set in $L^p(I,\mathbf{H})$ not necessarily compact.

Theorem 8.2. *If u solves the problem $(\mathscr{P})$ then*

$$f(t,u(t)) = \min_{v \in \mathrm{cl}(\Omega)} \left\{ f(t,v) + \frac{\rho}{2}\|u(t)-v\|^2 \right\}, \text{ for a. e. } t \in I.$$

Proof. Let u be a solution of $(\mathscr{P})$. Then $0 \in \partial^P I_f(u) + N^{\mathrm{conv}}(\mathscr{K};u)$ and so there exists $\xi \in \partial^P I_f(u)$ with $-\xi \in N^{\mathrm{conv}}(\mathscr{K};u)$. Using Theorem 8.1 we get for a.e. $t \in I$

$$\xi(t) \in \partial^P f(t,u(t)) = \partial^{\mathrm{conv}} g(t,u(t)) - \rho u(t),$$

where g is as in Theorem 8.1 (a convex finite function on $\mathrm{cl}(\Omega)$). Then for every $x \in \mathrm{cl}(\Omega)$ and for a.e. $t \in I$ one gets

$$\big\langle \xi(t) + \rho u(t), x - u(t) \big\rangle \leq g(t,x) - g(t,u(t)).$$

Now as $-\xi \in N^{\mathrm{conv}}(\mathscr{K};u)$ it follows easily that

$$\big\langle -\xi(t), x - u(t) \big\rangle \leq 0,$$

for every $x \in \mathrm{cl}(\Omega)$ and for a.e. $t \in I$. Therefore, from both last inequalities we get

$$\begin{aligned} f(t,u(t)) - f(t,x) &= g(t,u(t)) - g(t,x) - \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) \\ &\leq \langle -\xi(t) - \rho u(t), x - u(t) \rangle - \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) \\ &\leq \rho \big\langle u(t), u(t) - x \big\rangle + \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) \\ &= \frac{\rho}{2}\|u(t) - x\|^2, \end{aligned}$$

for every $x \in \mathrm{cl}(\Omega)$ and for a.e. $t \in I$. Thus, completing the proof. □

In [89], the authors proved that a function f is lower-C^2 in the sense of Rockafellar [239] if and only if its epigraph $\mathrm{epi}\, f$ is uniformly prox-regular. In the next theorem, we establish a sufficient condition of uniform prox-regularity for level sets.

Theorem 8.3. *Let Ω be an open convex relatively strongly compact subset in $\mathbf{H}$ and let f be a Lipschitz uniformly ρ-lower-C^2 function on $\mathrm{cl}(\Omega)$. Let $m := \inf\{\|\xi\| : \xi \in \partial^P f(x) \text{ with } x \in \mathrm{cl}(\Omega)\}$. Then either $m = 0$ or the level set $S := \{x \in \mathrm{cl}(\Omega) : f(x) \leq 0\}$ is uniformly r-prox-regular for $r := \frac{1}{m\rho}$.*

Proof. Put $\mathscr{E} := \bigcup_{x\in \mathrm{cl}(\Omega)} \partial^P f(x)$. Since $x \mapsto \partial^P f(x)$ is weak compact-valued and upper hemicontinuous and as $\mathrm{cl}(\Omega)$ is strongly compact then $\mathscr{E}$ is a weak compact set in $\mathbf{H}$. Hence, $m < +\infty$. Assume that $m \neq 0$. Fix any $\bar{x} \in S$ and any $0 \neq \xi \in N^P(S;\bar{x})$. Without loss of generality, we may assume that $f(\bar{x}) = 0$, because in the other case, i.e., $f(\bar{x}) < 0$ one has $N^P(S;\bar{x}) = \{0\}$. By Theorem 2.4.7 in [88] there exists $\lambda_\xi > 0$ such that $\frac{\xi}{\lambda_\xi} \in \partial^C f(\bar{x}) = \partial^P f(\bar{x})$. Further, $\|\frac{\xi}{\lambda_\xi}\| \geq m > 0$. On the other hand, by Proposition 8.2, we have

$$f(x) \geq -\frac{\rho}{2}\|x-\bar{x}\|^2 + \left\langle \frac{\xi}{\lambda_\xi}, x-\bar{x} \right\rangle + f(\bar{x}) \ \ \forall x \in \mathrm{cl}(\Omega).$$

Hence,

$$\langle \xi, x-\bar{x} \rangle \leq \frac{\lambda_\xi \rho}{2}\|x-\bar{x}\|^2 \ \ \forall x \in S,$$

and hence

$$\left\langle \frac{\xi}{\|\xi\|}, x-\bar{x} \right\rangle \leq \frac{\rho \lambda_\xi}{2\|\xi\|}\|x-\bar{x}\|^2 \leq \frac{\rho}{2m}\|x-\bar{x}\|^2,$$

for all $x \in S$. This ensures that S is uniformly r-prox-regular with $r = \frac{m}{\rho}$. The proof then is complete. □

Using this theorem and the stability results of the class of uniformly lower-C^2 functions proved above we prove in the following corollary the uniform prox-regularity of some special level sets.

Corollary 8.1. *Let Ω be an open convex relatively strongly compact subset in $\mathbf{H}$.*

1. *If f_1 and f_2 are two Lipschitz uniformly lower-C^2 functions on $\mathrm{cl}(\Omega)$ satisfying $\partial^P f_1(x) \cap \{-\partial^P f_1(x)\} = \emptyset$, for all $x \in \mathrm{cl}(\Omega)$, then the set $\{x \in \mathrm{cl}(\Omega) : \ f_1(x) + f_2(x) \leq 0\}$ is uniformly prox-regular.*
2. *If f_i, $i = 1, \ldots, N$ are Lipschitz uniformly lower-C^2 functions on $\mathrm{cl}(\Omega)$ satisfying $0 \notin \partial^P f_i(x)$, for all $x \in \mathrm{cl}(\Omega)$ and all $i = 1, \ldots, N$, then the set $\{x \in \mathrm{cl}(\Omega) : f_i(x) \leq 0, \text{for all } i = 1, \ldots, N\}$ is uniformly prox-regular.*
3. *If $F : \mathbf{H} \to \mathbf{H}'$ ($\mathbf{H}'$ is another Hilbert space) is a C^2 mapping and h is a Lipschitz uniformly lower-C^2 function on $F(\mathrm{cl}(\Omega))$ satisfying $0 \notin \partial^P h(y)$, for all $y \in F(\mathrm{cl}(\Omega))$, then the set $\{x \in\in \mathrm{cl}(\Omega) : \ h \circ F(x) \leq 0\}$ is uniformly prox-regular.*
4. *Let f be a continuous function from $I \times \mathbf{H}$ to $\mathbf{R}$ satisfying the assumptions (A_1) and (A_2). Assume that for all $t \in I$ the function $f(\cdot,t)$ is Lipschitz uniformly ρ-lower-C^2 on $\mathrm{cl}(\Omega)$ and $0 \notin \partial^P f(t,x)$, for all $x \in \mathrm{cl}(\Omega)$ and all $t \in I$. Then the set $\{u \in L^p(I,\mathbf{H}) : u(t) \in \mathrm{cl}(\Omega) \ \text{and} \ I_f(u) \leq 0\}$ is uniformly prox-regular.*

8.3 Subdifferential and Co-normal Stability

Our purpose in this section is to study the stability of normal cones and of the subdifferential of the distance functions to uniformly prox-regular sets. That property is very useful for applications. Our motivations come from some applications in economies and equilibrium theory (see the next sections). We start with the following definitions.

Definition 8.3. Let $\{S_k\}_k$ be any sequence of nonempty closed sets in $\mathbf{H}$. We will say that a nonempty closed set S is *the Painlevé–Kuratowski PK-lower limit* (resp. PK-upper limit) of S_k provided that

$$S \subset \liminf_k S_k := \{x \in \mathbf{H} : \exists \ \ x_k \to x \ \text{ such } \ \text{that}\ x_k \in S_k\},$$

$$(\text{ resp.}\{x \in \mathbf{H} : \exists \ \ x_k \to x \quad \text{such that}\ x_k \in S_{s(k)}\} =: \limsup_k S_k \subset S.)$$

Here $S_{s(k)}$ is a subsequence of S_k.

We will say that S_k PK-converges to S or S is the PK-limit of S_k provided that S is both the PK-upper limit and the PK-lower limit of S_k.

Definition 8.4. Let $\{S_k\}_k$ be a sequence of nonempty closed sets in $\mathbf{H}$ that converges in some sense to a closed set S in $\mathbf{H}$. We will say that the sequence $\{S_k\}_k$ is *subdifferentially stable* if one has

$$\limsup_{x_k \longrightarrow \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

that is, for any sequence x_k (not necessarily in S_k) and such that $x_k \to x$ and any $\xi \in \partial^P d_{S_k}(x_k)$ weakly converging to some $\xi \in \mathbf{H}$, one has $\xi \in \partial^P d_S(\bar{x})$. In the same way, we will say that $\{S_k\}_k$ is *co-normally stable* provided that

$$\limsup_k N^P(S_k; x_k) \subset N^P(S; x),$$

that is, for any sequence x_k such that $x_k \in S_k$ and $x_k \to x$ and any $\xi \in N^P(S_k; x_k)$ weakly converging to some $\xi \in \mathbf{H}$, one has $\xi \in N^P(S; x)$.

We recall the following lemma needed in the proof of the next results. It gives a characterization of the PK-lower limit in terms of the distance function to sets. For its proof we refer the reader to [241]. Note that the proof in [241] is given in the finite dimensional setting but it can be extended in the evident way to the Hilbert case.

Lemma 8.2. *Let $\{S_k\}_k$ and S be nonempty closed sets in $\mathbf{H}$. Then S is the PK-lower limit of the sequence $\{S_k\}_k$ if and only if there exists for each $\rho > 0$ and $\varepsilon > 0$ an integer $k_0 \in \mathbf{N}$ such that for all $x \in \rho\mathbf{B}$ and all $k \geq k_0$ one has*

$$d(x, S_k) \leq d(x, S) + \varepsilon.$$

Now we are ready to prove the subdifferential and co-normal stability for uniformly prox-regular sets under an additional hypothesis on their distance functions.

Theorem 8.4. *Let $\{S_k\}_{k\in\mathbf{N}}$ be a sequence of nonempty closed subsets in $\mathbf{H}$ and let S be a nonempty closed in $\mathbf{H}$. Let $r>0$ and $\bar{x}\in S$. Assume that S is the PK-lower limit of $\{S_k\}_{k\in\mathbf{N}}$ and that all the subsets $\{S_k\}_{k\in\mathbf{N}}$ are uniformly r-prox-regular. Then*

(i) the sequence $\{S_k\}_k$ is subdifferentially stable, that is,

$$\limsup_{x_k \xrightarrow{S_k} \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

where $x_k \xrightarrow{S_k} \bar{x}$ means that x_k converging to $\bar{x}$ and $x_k\in S_k$ for all $k\in\mathbf{N}$.

(ii) If, in addition, $\mathbf{H}$ is a finite dimensional space, then the sequence $\{S_k\}_k$ is co-normally stable.

Proof. (i) Let $x_k \xrightarrow{S_k} \bar{x}$ and $\{\xi_k\}_{k\in\mathbf{N}}$ be any sequence weakly converging to some $\bar{\xi}$ with $\xi_k\in\partial^P d_{S_k}(x_k)$ for all $k\in\mathbf{N}$. As the subsets $\{S_k\}_{k\in\mathbf{N}}$ are uniformly r-prox-regular, Theorem 2.14 ensures that for all $k\in\mathbf{N}$ one has

$$\begin{cases} \langle \xi_k, x-x_k\rangle \le \dfrac{2}{r}\|x-x_k\|^2 + d_{S_k}(x), \\ \text{for all } \; x\in\mathbf{H} \; \text{ with } \; d_{S_k}(x)\le r. \end{cases} \tag{8.5}$$

Fix any $y\in\bar{x}+\frac{r}{2}\mathbf{B}$. Then, by Lemma 8.2 there exists $k_0\in\mathbf{N}$ such that

$$d_{S_k}(y) \le d_S(y) + \frac{1}{1+k} \quad \text{for all} \quad k\ge k_0. \tag{8.6}$$

One may choose k_0 large enough so that

$$\frac{1}{1+k} \le \frac{r}{2}$$

for all $k\ge k_0$. Thus, one gets

$$d_{S_k}(y) \le d_S(y) + \frac{r}{2} \le \|y-\bar{x}\| + \frac{r}{2} \le \frac{r}{2} + \frac{r}{2} \le r$$

and so one can apply (8.5) with $x=y$ to get for all $k\ge k_0$

$$\langle \xi_k, y-x_k\rangle \le \frac{2}{r}\|y-x_k\|^2 + d_{S_k}(y),$$

and by (8.6) one obtains for all $k\ge k_0$

$$\langle \xi_k, y-x_k\rangle \le \frac{2}{r}\|y-x_k\|^2 + d_S(y) + \frac{1}{1+k}.$$

By letting $k \to +\infty$ in the last inequality one gets

$$\langle \bar{\xi}, y - \bar{x} \rangle \leq \frac{2}{r}\|y - \bar{x}\|^2 + d_S(y) - d_S(\bar{x}),$$

for all $y \in \bar{x} + \frac{r}{4}\mathbf{B}$. This ensures that $\bar{\xi} \in \partial^P d_S(\bar{x})$.

Assume that $dim\,\mathbf{H} < +\infty$. Let $x_k \xrightarrow{S_k} \bar{x}$ and $\xi_k \longrightarrow \bar{\xi}$ with $\xi_k \in N^P(S_k; x_k)$ for all $k \in \mathbf{N}$. Put

$$\zeta_k := \frac{\xi_k}{1 + \|\xi_k\|}.$$

Then, $\zeta_k \in N^P(S_k; x_k) \cap \mathbf{B}$ and hence by Theorem 2.7 one gets $\zeta_k \in \partial^P d_{S_k}(x_k)$. Now as $\dim \mathbf{H} < +\infty$ the sequence ζ_k converges to

$$\frac{\bar{\xi}}{1 + \|\bar{\xi}\|}$$

and so we get by (i) that $\frac{\bar{\xi}}{1+\|\bar{\xi}\|} \in \partial^P d_S(\bar{x}) \subset N^P(S; \bar{x})$, which ensures that

$$\bar{\xi} \in N^P(S, \bar{x}).$$

This completes the proof. □

Now we proceed to prove a similar result for level sets. First, we recall the following definition

Definition 8.5. Let $\{f_k\}_k$ be any sequence of functions on $\mathbf{H}$ and let $x \in \mathbf{H}$. We will say that f_k *upper-epi-converges* to some function f at x provided that

$$\text{epi-}\limsup_n f_n(x) \leq f(x),$$

or equivalently there exists $x_k \to x$ such that

$$\limsup_k f_k(x_k) \leq f(x).$$

Recall now the following lemma needed in the proof. Its proof can be found in [241] in the finite dimensional setting which can be extended evidently to the Hilbert case.

Lemma 8.3. *Let $\{f_k\}_k$ be a sequence of functions on $\mathbf{H}$. Assume that $\{f_k\}_k$ upper-epi-converges to some function f over $\mathbf{H}$. Then one has*

$$[f \leq 0] \subset \liminf_k \{[f_k \leq \alpha_k]\},$$

for some sequence $\alpha_k \downarrow 0$. In other words the level set $[f \leq 0]$ is the PK-lower limit of the sequence of the level sets $\{[f_k \leq \alpha_k]\}_{k \in \mathbf{N}}$.

Now we are able to prove a co-normal stability result for level sets.

Theorem 8.5. *Let Ω be an open convex relatively strongly compact subset of* $\mathbf{H}$ *and let $\{f_k\}_k$ be a sequence of functions on* $\mathbf{H}$ *that upper-epi-converges to some function f over* $\mathbf{H}$. *Let $\sigma > 0$ and $\bar{x} \in \mathrm{cl}(\Omega)$ with $f(\bar{x}) = 0$. Assume that $\{f_k\}_k$ are uniformly σ-lower-C^2 on $\mathrm{cl}(\Omega)$ with $\inf\{\|\xi\| : \ \xi \in \partial^P f_k(x)$ for all $x \in \mathrm{cl}(\Omega)$ and all $k\} \geq m$ for some $m > 0$. Then there exists $\alpha_k \downarrow 0$ such that the sequence $\{S_k\}_k$ is subdifferentially stable, that is,*

$$\limsup_{x_k \xrightarrow{S_k} \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

where $S_k := [f_k \leq \alpha_k]$ and $S := [f \leq 0]$. If, in addition, $\mathbf{H}$ *is a finite dimensional space, then the sequence $\{S_k\}_k$ is co-normally stable.*

Proof. By Lemma 8.3 there exists $\alpha_k \downarrow 0$ such that S is the PK-lower limit of the sequence $\{S_k\}_k$. Put $\tilde{f}_k(\cdot) = f_k(\cdot) - \alpha_k$ and $m_k := \inf\{\|\xi\| : \ \xi \in \partial^P \tilde{f}_k(x)$ for all $x \in \mathrm{cl}(\Omega)\}$ for any k. Observe that $m_k = \inf\{\|\xi\| : \ \xi \in \partial^P f_k(x)$ for all $x \in \mathrm{cl}(\Omega)\}$ and $S_k = [\tilde{f}_k \leq 0]$. By our hypothesis we have $m_k \geq m > 0$ for all k and hence we get by Theorem 8.3 that the subsets S_k are uniformly r_k-prox-regular for $r_k = \frac{1}{m_k \sigma}$. Since $m_k \geq m > 0$ we get by the properties of uniform prox-regularity that S_k are uniformly r-prox-regular for $r = \frac{1}{m\sigma}$. Therefore, all the hypothesis of Theorem 8.4 are satisfied and so the subdifferential stability of d_{S_k} and the co-normal stability of S_k follow from Theorem 8.4, which completes the proof. □

8.4 Nonconvex Nontransitive Economies

In this section, we consider the following economic model established by Arrow and Debreu (1959). In this model, there are an infinite number of goods and a finite number of consumers m and producers n. Each consumer has a preference set-valued mapping $P_i : \prod_k X_k \rightrightarrows X_i$, where $X_i \subset \mathbf{H}$ is a set of consumptions for the consumer i. For a given $(x_1, \ldots, x_m) \in \prod_k X_k$, the set $\mathrm{cl}P_{i_0}(x_1, \ldots, x_m)$ (resp. $P_{i_0}(x_1, \ldots, x_m)$) represents all those elements in X_{i_0} that are preferred (resp. strictly preferred) to $(x_1, \ldots, x_m)$ for the consumer i_0. Each producer j has a production set $Y_j \subset \mathbf{H}$. Thus, an economy $\mathscr{E}$ is defined as $\mathscr{E} = ((X_i), (P_i), (Y_j), e)$, where $e \in \mathbf{H}$ is the total initial endowment for the economy, that is, $e = \sum_{i=1}^m e_i$ with e_i is the initial endowment for the consumer i. A fundamental result of this theory is the second welfare theorem which gives a price decentralization of a Pareto optimum allocation. An extension of this welfare's theorem to general nonconvex economies was proved in [144]. Before stating it we need to recall the definitions of feasible allocation, Pareto optimum, and the *Asymptotic Included Condition (A.I.C)* for the economy $\mathscr{E}$.

Definition 8.6.

1. We will say that $((x_i^*), (y_j^*)) \in \prod X_i \times \prod Y_j$ is a *feasible allocation* for the economy $\mathscr{E}$ if the following conditions are satisfied:

(a) for each $i = 1, \dots, m$ and $j = 1, \dots, n$, $x_i^* \in X_i$, and $y_j^* \in Y_j$;
(b) $\sum_i x_i^* - \sum_j y_j^* = e$.

2. A feasible allocation $((x_i^*),(y_j^*)) \in \prod X_i \times \prod Y_j$ is a *Pareto optimum* for the economy $\mathscr{E}$ if there is no feasible allocation $((x_i'),(y_j')) \in \prod X_i \times \prod Y_j$ such that

(i) for each $i \in \{1, \dots, m\}$, $x_i' \in \mathrm{cl}P_i(x_1^*, \dots, x_m^*)$;
(ii) for some $i_0 \in \{1, \dots, m\}$, $x_{i_0}' \in P_{i_0}(x_1^*, \dots, x_m^*)$.

3. We will say that $\mathscr{E}$ satisfies the A.I.C. at a point $((x_i^*),(y_j^*)) \in \prod X_i \times \prod Y_j$ if there exists $i_0 \in \{1, \dots, m\}$, $\varepsilon > 0$, and a sequence $h_k \to 0$ such that for k sufficiently large we have

$$h_k + \sum_i [\mathrm{cl}P_i^* \cap B(x_i^*, \varepsilon)] - \sum_j [Y_j \cap B(y_j^*, \varepsilon)] \subseteq P_{i_0}^* + \sum_{i \neq i_0} [\mathrm{cl}P_i^*] - \sum_j Y_j,$$

where $P_i^* := P_i(x_1^*, \dots, x_m^*)$ and $B(x, \varepsilon) := x + \varepsilon \mathbf{B}$.

Remark 8.1. Note that A.I.C. is satisfied whenever $P_{i_0}^*$ for some $i_0 \in \{1, \dots, m\}$ is epi-Lipschitz at $x_{i_0}^*$ in the sense of Rockafellar [237].

We recall (see [37]) that a set S is called compactly epi-Lipschitz at $\bar{x} \in S$ if there exist $\varepsilon, \alpha > 0$, and a compact set K satisfying $S \cap (\bar{x} + \varepsilon \mathbf{B}) + t\mathbf{B} \subset S + tK$ for all $t \in [0, \alpha]$. We also recall the following characterization of compactly epi-Lipschitz sets by Ioffe [142], needed in the proof of Theorem 8.7.

Lemma 8.4. *Let S be a closed subset of $\mathbf{H}$ and $\bar{x} \in S$. The following properties are equivalent:*

1. *S is compactly epi-lipschitz at $\bar{x}$;*
2. *For any bounded sequence ξ_n such that $\xi \in N^{\mathrm{C}}(S; x_n)$ with $x_n \in S$ and $x_n \to \bar{x}$, we have that $\xi_n \xrightarrow{w} 0$ (ξ_n weakly converges to 0) implies $\xi_n \xrightarrow{\|\cdot\|} 0$ (its norm convergence to zero) too.*

The following theorem is taken from [144] and will be needed in the proof of Theorem 8.7.

Theorem 8.6. *Let $((x_i^*),(y_j^*)) \in \prod X_i \times \prod Y_j$ be a Pareto optimum point for the economy $\mathscr{E} = ((X_i),(P_i),(Y_j),e)$ which satisfies the A.I.C. on it. If for each $i \in \{1, \dots, m\}$, $x_i^* \in \mathrm{cl}P_i^*$, and $\mathrm{cl}P_{i_0}^*$ or Y_{i_0} is compactly epi-Lipschitz at $x_{i_0}^*$ or $y_{j_0}^*$ for some i_0 and j_0 respectively, then there exists a price vector $p^* \in \mathbf{H}$ such that*

$$\|p^*\| \geq \frac{1}{n+m}, \quad p^* \in \bigcap_j \partial d_{Y_j}(y_j^*), \quad \textit{and} \; -p^* \in \bigcap_i \partial d_{\mathrm{cl}P_i^*}(x_i^*).$$

The price vector p^* is called a quasi-equilibrium price for the economy $\mathscr{E}$. In [143] the authors studied, in the finite dimensional case (when the economy $\mathscr{E}$ is defined with a finite number of goods $l > 0$, i.e., $\mathbf{H} = \mathbf{R}^l$), the stability of the quasi-equilibrium prices, i.e., if we are given a sequence e^k converging to some $e \in \mathbf{H}$ and we assume that each economy $\mathscr{E}_k := ((X_i),(P_i),(Y_j),e_k)$ satisfies A.I.C. at a Pareto optimum point $((x^*_{i,k}),(y^*_{j,k}))$ and that this Pareto optimum sequence converges to some $((x^*_i),(y^*_j))$, is it possible to get the conclusion of Theorem 8.6 for the limit economy $\mathscr{E} := ((X_i),(P_i),(Y_j),e)$ at $((x^*_i),(y^*_j))$? They gave a positive answer under some hypothesis on the subdifferential of the distance function to the producers and preferences sets. Our main result in this section is in this vein. We will use the abstract results proved in the previous sections to prove that stability for general nonconvex economies with an infinite number of goods.

Let $\mathscr{E}^k := ((X_{i,k}),(P_{i,k}),(Y_{j,k}),e_k)$ be a sequence of nonconvex economies, with $e_k \to e \in \mathbf{H}$, $(X_{i,k})$ and $(Y_{j,k})$ PK-lower-converge to (X_i) and (Y_j) respectively in $\mathbf{H}$, and the sequence of set-valued mappings $(P_{i,k})$ admits a PK-lower limit set-valued mappings (P_i) in the following sense: for each $i \in \{1,\ldots,m\}$ and for any $(x_{1,k},\ldots,x_{m,k}) \to (x_1,\ldots,x_m)$ one has

(A_1)

$$\liminf_k \operatorname{cl} P_{i,k}(x_{1,k},\ldots,x_{m,k}) \subset \operatorname{cl} P_i(x_1,\ldots,x_m).$$

Assume that each economy $\mathscr{E}_k$ satisfies A.I.C. at some Pareto optimum point $((x^*_{i,k}),(y^*_{j,k}))$ and that this Pareto optimum sequence strongly converges to some $((x^*_i),(y^*_j))$. Let us consider the following two conditions:

(A_2) there exists $j_0 \in \{1,\ldots,n\}$ such that (Y_{j_0}) is compactly epi-Lipschitz at $y^*_{j_0}$ and $(Y_{j_0,k}) = (Y_{j_0}), \forall k$;

(A_3) there exists $i_0 \in \{1,\ldots,m\}$ such that $(\mathrm{cl}P^*_{i_0})$ is compactly epi-Lipschitz at $x^*_{i_0}$ and $(\mathrm{cl}P^*_{i_0,k}) = (\mathrm{cl}P^*_{i_0}), \forall k$.

Then we can state the main result.

Theorem 8.7. *Assume that* (A_1) *is satisfied and either* (A_2) *or* (A_3) *is satisfied, and for each* $i \in \{1,\ldots,m\}$ *and for all* k, $x^*_{i,k} \in \mathrm{cl}P^*_{i,k}$, *and that both sequences* $(Y_{j,k})$ *and* $(\mathrm{cl}P_{i,k})$ *are r-uniformly prox-regular, for some* $r > 0$. *Then there exists a price vector* $p^* \in \mathbf{H}$ *with* $p^* \neq 0$ *such that*

$$p^* \in \bigcap_j \partial^P d_{Y_j}(y^*_j) \quad \text{and} \quad -p^* \in \bigcap_i \partial^P d_{\mathrm{cl}P^*_i}(x^*_i).$$

Proof. By Theorem 8.6 there exists a sequence of prices $p^*_k \in \mathbf{H}$ with $1 \geq \|p^*_k\| \geq \frac{1}{n+m} > 0$, satisfying

$$p^*_k \in \bigcap_j \partial^P d_{Y_{j,k}}(y^*_{j,k}) \quad \text{and} \quad -p^*_k \in \bigcap_i \partial^P d_{\mathrm{cl}P^*_{i,k}}(x^*_{i,k}),$$

where $\mathrm{cl}P^*_{i,k} := \mathrm{cl}P_{i,k}(x^*_{i,k})$. By our assumption (A_1) the set $\mathrm{cl}P^*_i := \mathrm{cl}P_i(x^*_i)$ is the PK-lower limit of the sequence $\mathrm{cl}P^*_{i,k}$. Now as all the sets $\mathrm{cl}P^*_{i,k}$ are uniformly prox-regular we get by Theorem 8.4 that the weak limit p^* of p^*_k will belong to $-\bigcap_i \partial^P d_{\mathrm{cl}P^*_i}(x^*_i)$. Applying Theorem 8.4 once again with the sequences $Y_{j,k}$ and their PK-lower limits Y_j we get $p^* \in \bigcap_i \partial^P d_{Y_j}(y^*_j)$. Now we check that $p^* \neq 0$.

Assume first that the assumption (A_2) is satisfied. Then there exists $j_0 \in \{1,\ldots,n\}$ such that $(Y_{j_0,k}) = Y_{j_0}$ for all k with Y_{j_0} is a compactly epi-Lipschitz set at $y^*_{j_0}$. Therefore,

$$p^*_k \in \partial^P d_{Y_{j_0,k}}(y^*_{j,k}) = \partial^P d_{Y_{j_0}}(y^*_{j,k}) \subset N^P(Y_{j_0}; y^*_{j,k}).$$

Assume by contradiction that $p^* = 0$. Then p^*_k is a bounded sequence in the normal cone $N^P(Y_{j_0}; y^*_{j,k})$ and it converges weakly to $p^* = 0$. Thus, by Lemma 8.4 the sequence would converge in norm to 0 which contradicts the fact that $\|p^*_k\| \geq \frac{1}{m+n} > 0$. Consequently $p^* \neq 0$. The same argument concludes the proof when the assumption (A_3) is satisfied. □

Many corollaries can be obtained directly from this theorem. We give only the two followings. First we begin with the case when the preference $(P_{i,k})$ defining the economic $\mathscr{E}_k$ is not perturbed, that is, $\mathscr{E}_k = ((X_i),(P_i),(Y_{j,k}),e_k)$. In this case, the assumption (A_1) becomes the lower semi-continuity property of the set-valued mapping $\mathrm{cl}\,P_i$ at (x^*_i), i.e., for each $i = 1,\ldots,m$ and for any $(x_{i,k}) \to (x^*_i)$ one has

$$\liminf_k \mathrm{cl}\,P_i(x_{1,k},\ldots,x_{m,k}) \subset \mathrm{cl}\,P_i(x^*_1,\ldots,x_m).$$

Corollary 8.2. *Assume that the set-valued mapping $\mathrm{cl}P_i$ has uniformly r-prox-regular values for some $r > 0$ and that it is l.s.c. at (x^*_i). Assume that for each $i \in \{1,\ldots,m\}$, $x^*_i \in \mathrm{cl}P^*_i$, and that the sequence $(Y_{j,k})$ is uniformly r-prox-regular for some $r_2 > 0$. If either (A_2) is satisfied or $(\mathrm{cl}P^*_{i_0})$ is compactly epi-Lipschitz at $x^*_{i_0}$ for some $i_0 \in \{1,\ldots,m\}$, then there exists a price vector $p^* \in \mathbf{H}$ with $p^* \neq 0$ such that*

$$p^* \in \bigcap_j \partial^P d_{Y_j}(y^*_j) \quad \text{and} \quad -p^* \in \bigcap_i \partial^P d_{\mathrm{cl}P^*_i}(x^*_i).$$

Now, we assume that the preferences $P_{i,k}$ are defined by Lipschitz utility functions $f_{i,k} : X_i \to \mathbf{R}$, that is,

$$P_{i,k}(x_1,\ldots,x_m) = \{x \in X_i : \quad f_{i,k}(x) > f_{i,k}(x_{i,k})\}.$$

Corollary 8.3. *Assume that the following assumptions are satisfied:*

(i) *$X_{i,k}$ are convex strongly compact in $\mathbf{H}$ and $x^*_{i,k} \in \mathrm{cl}P^*_{i,k}$ for all k;*
(ii) *$(Y_{j,k})$ are uniformly r-prox-regular sets for some $r > 0$ and for some $j_0 \in \{1,\ldots,n\}$ the set $(Y_{j,k})$ is compactly epi-Lipschitz at $y^*_{j_0}$.*
(iii) *$-f_{i,k}$ are uniformly σ-lower-C^2 on $X_{i,k}$ for some $\sigma > 0$ and upper-epi-converges to some function f_i over $\mathbf{H}$ with*

$$\inf\{\|\xi\| : \ \xi \in \partial^P f_{i,k}(x^*_{i,k}) \text{ for all } i,k\} > c,$$

for some $c > 0$. Then there exists a price vector $p^* \in \mathbf{H}$ with $p^* \neq 0$ such that

$$p^* \in \bigcap_j \partial^P d_{Y_j}(y^*_j) \quad \text{and} \quad -p^* \in \bigcap_i \partial^P d_{\mathrm{cl}P^*_i}(x^*_i),$$

where P_i is the limit preference defined by the limit utility function f_i.

8.5 Existence of Nonconvex Equilibrium

In this last section, we are going to give an application of our co-normal and subdifferential stability results to the equilibrium theory for nonconvex sets in the infinite dimensional setting. We start with the following definition of generalized equilibriums.

Definition 8.7. For a closed set $S \subset \mathbf{H}$ and a set-valued mapping $F : S \rightrightarrows \mathbf{H}$, we will say that $\bar{x} \in S$ is a *generalized equilibrium* of F over S if one has

$$0 \in F(\bar{x}) - N^{\#}(S;\bar{x}),$$

where $N^{\#}(S;x)$ is a prescribed normal cone.

This concept of equilibrium has been considered in [241] and studied later by [101] in the finite dimensional setting. We recall now the classical definition of equilibrium.

Definition 8.8. For a closed set $S \subset \mathbf{H}$ and a set-valued mapping $F : S \rightrightarrows \mathbf{H}$, we will say that $\bar{x} \in S$ is an *equilibrium* of F over S if one has $0 \in F(\bar{x})$.

The existence of equilibrium has been the subject of many works in the finite (see for example [90, 101]) and infinite dimensional setting (see e.g. [19, 90] and the references therein). The best known equilibrium result in the Hilbert (infinite dimensional) setting is the following theorem by Ben-El-Mechaiekh and Kryszewski [19].

Theorem 8.8. *Let S be a compact $\mathscr{L}$-retract in $\mathbf{H}$ with $\chi(S) \neq 0$. If $F : S \rightrightarrows \mathbf{H}$ is an upper hemicontinuous map with closed convex values satisfying* for all $x \in S$ and all $p \in ret^{-1}(x)$*:*

$$\inf_{y \in F(x)} \langle p - x, y \rangle \leq 0,$$

then F has an equilibrium over S.

Here, $\chi(S)$ stands the Euler characteristic of S. Recall that (see [19]) a closed subset $S \subset \mathbf{H}$ is said to be $\mathscr{L}$-retract if there exist an open neighborhood O of S, a continuous retraction $\text{ret} : O \to S$, and a constant $L \geq 0$ such that

$$\|x - \text{ret}(x)\| \leq L d_S(x), \quad \text{for all } x \in O.$$

This definition was introduced by [19] for metric spaces. To prove our main theorem in this section we need to prove some preliminary results.

Lemma 8.5. *Every uniformly prox-regular set is $\mathscr{L}$-retract.*

Proof. Let $r > 0$ be the constant of the uniform prox-regularity of S and let $r' \in (0,r)$. Put $U(r') := \{x \in \mathbf{H} : \ 0 < d_S(x) < r'\}$ and $S(r') := \{x \in \mathbf{H} : \ 0 \leq d_S(x) < r'\}$. It suffices to take $ret := \text{Proj}_S$, $O := S(r')$, and $L := 1$. Indeed, by Theorem 4.2 in

[89], the projection Proj_S is single-valued Lipschitz mapping of rank $\frac{r}{r-r'}$ on $U(r')$. In particular, it is continuous on the open set $S(r')$. Finally, as $\|x - \mathrm{Proj}_S(x)\| = d_S(x)$ for all $x \in S(r')$, the proof then is complete. □

Remark 8.2. Note that in Proposition 5.1 in [101] the authors proved in the finite dimensional setting that every uniformly prox-regular (more general every proximal nondegenerate (see [101] for the definition)) and compact set is $\mathscr{L}$-retract. In Lemma 8.5, we don't need the compactness of S. So, it generalizes Proposition 5.1 in [101] to uniformly prox-regular sets not necessarily compact and to the Hilbert space setting.

Lemma 8.6. *Let S be a uniformly r-prox-regular subset in $\mathbf{H}$ for some $r > 0$ and let $F : S \rightrightarrows \mathbf{H}$ be any set-valued mapping. Then the following assertions are equivalent:*

1. *for all $x \in S$ and all $p \in \mathrm{Proj}_S^{-1}(x)$ one has*
$$\inf_{\xi \in F(x) - \partial^P d_S(x)} \langle p - x, \xi \rangle \leq 0;$$
2. *for all $x \in S$ and all $p \in \mathrm{Proj}_S^{-1}(x)$ one has*
$$\inf_{\xi \in F(x)} \langle p - x, \xi \rangle \leq \|p - x\|;$$
3. *for all $x \in S$ and all $p \in \mathrm{Proj}_S^{-1}(x)$ with $p \neq x$ one has*
$$\inf_{\xi \in F(x)} \left\langle \frac{p - x}{\|p - x\|}, \xi \right\rangle \leq 1;$$

Proof. Assume that (1) holds. Then for any $x \in S$ and $p \in \mathrm{Proj}_S^{-1}(x)$, there exists $\xi_1 \in F(x)$ and $\xi_2 \in \partial^P d_S(x)$ such that $\langle p - x, \xi_1 \rangle \leq \langle p - x, \xi_2 \rangle$. So $\langle p - x, \xi_1 \rangle \leq \|\xi_2\| \|p - x\| \leq \|p - x\|$, because one always has $\partial^P d_S(x) \subset \mathbf{B}$. Therefore, (2) holds.

As the equivalence between (2) and (3) is obvious, we have to show (3) $\Rightarrow$ (1). Assume that (3) holds. Fix any $x \in S$ and $p \in \mathrm{Proj}_S^{-1}(x)$ with $p \neq x$. Then by (3), there exists $\xi \in F(x)$ such that

$$\left\langle \frac{p - x}{\|p - x\|}, \xi \right\rangle \leq 1. \tag{8.7}$$

As $p \in \mathrm{Proj}_S^{-1}(x)$, we have $\frac{p-x}{\|p-x\|} \in \partial^P d_S(x)$. Put $\widetilde{\xi} := \xi - \frac{p-x}{\|p-x\|} \in F(x) - \partial^P d_S(x)$. Then (8.7) yields

$$\langle p - x, \widetilde{\xi} \rangle = \langle p - x, \xi \rangle - \left\langle p - x, \frac{p - x}{\|p - x\|} \right\rangle = \langle p - x, \xi \rangle - \|p - x\| \leq 0.$$

Thus, (1) holds and so the proof is complete. □

Now we are in position to prove the main result of this section.

Theorem 8.9. *Let S_k be a sequence of compact uniformly r-prox-regular subsets in* $\mathbf{H}$ *with* $\chi(S_k) \neq 0$ *and let* $F : S_k \rightrightarrows \mathbf{H}$ *be an upper hemicontinuous map with closed convex values. Assume that S_k PK-converges to some compact subset S. Assume also that for all* $x_k \in S_k$ *and all* $p_k \in \mathrm{Proj}_{S_k}^{-1}(x_k)$ *one has*

$$\inf_{\xi_k \in F(x_k)} \langle p_k - x_k, \xi_k \rangle \leq \|p_k - x_k\|, \tag{8.8}$$

then F has a generalized equilibrium over S with respect to the proximal normal cone, i.e., there exists $\bar{x} \in S$ *such that* $0 \in F(\bar{x}) - N^P(S;\bar{x})$.

Proof. For every $k \geq 1$ we define the set-valued mapping $\widetilde{F_k} := F - \partial^P d_{S_k}$. By Lemma 8.6 our hypothesis (8.8) is equivalent to

$$\inf_{\xi_k \in \widetilde{F_k}(x)} \langle p_k - x_k, \xi_k \rangle \leq 0;$$

for all $x_k \in S_k$ and all $p_k \in \mathrm{Proj}_{S_k}^{-1}(x_k)$. On the other hand, by Lemma 8.5 the set S_k is $\mathscr{L}$-retract with $ret := \mathrm{Proj}_{S_k}$. Then as it is easily to see that the set-valued mapping $\widetilde{F_k}$ is upper hemicontinuous with closed convex values, we can apply Theorem 8.8. to get an equilibrium of $\widetilde{F_k}$ over S_k, i.e., there exists $\bar{x}_k \in S_k$ such that

$$0 \in \widetilde{F_k}(\bar{x}_k) = F(\bar{x}_k) - \partial^P d_{S_k}(\bar{x}_k). \tag{8.9}$$

Now, using the fact that S_k *PK*-converges to S, we get that $d_S(\bar{x}_k) \to 0$ as $k \to \infty$, which ensures the relative compactness of the sequence $\bar{x}_k$ because S is a compact set in $\mathbf{H}$. There exists then some subsequence of $\bar{x}_k$ that converges to some point $\bar{x} \in S$. On the other hand, by the relation (8.9) there exists $\xi_k \in \partial^P d_{S_k}(\bar{x}_k) \cap F(\bar{x}_k) \subset \mathbf{B}$. Then, a subsequence of ξ_k may be extracted converging weakly to some $\bar{\xi}$. Finally, by the subdifferential stability result in Theorem 8.4, we conclude that $\bar{\xi} \in \partial^P d_S(\bar{x})$ and by the upper hemicontinuity of F we also have $\bar{\xi} \in F(\bar{x})$. Therefore,

$$0 \in F(\bar{x}) - \partial^P d_S(\bar{x}) \subset F(\bar{x}) - N^P(S;\bar{x}).$$

This ends the proof.

Remark 8.3.

1. In the statement of Theorem 8.9, we specify the normal cone of S with which we work, because the limit set S is not necessarily uniformly prox-regular and so the classical subdifferentials a priori do not coincide with the proximal one. Therefore, our result in Theorem 8.9 proves the existence of generalized equilibrium for nonconvex sets that a priori are not necessarily uniformly prox-regular.

2. From the Part (1) of Remark 8.3, the result in Theorem 8.9 cannot be covered by Corollary 3.4 in [101] even in the finite dimensional setting, because the limit set S in Theorem 8.9 is not necessarily proximally nondegenerate in the sense of [101].
3. Another approach, but with less importance relatively to our Theorem 8.9, that can be used to prove the existence of generalized equilibrium for uniformly prox-regular sets not necessarily convex, is to approximate a set S with uniformly prox-regular sets S_k satisfying (8.8) and all the other hypothesis of Theorem 8.9. Then we use the subdifferential stability result in Theorem 8.4 to get the condition (8.8) for the set S and then we follow the same argument in the proof of Theorem 8.9 to obtain a generalized equilibrium of the set-valued mapping F over S.

8.6 Commentary to Chap. 8

Our main objectiveof this chapter is to present an application of the regularity concept to some economic problems and equilibrium theory. Sections 9.2 and 9.3 are devoted to prove some additional results for uniformly prox-regular sets that will be of a great importance in the proofs of the main theorems in this chapter. Sections 9.4 and 9.5 present two applications of uniform prox-regularity to nonstransitive economies and equilibrium problems respectively. The main results stated in this chapter are taken from [55].

It is very interesting to point out to a detailed and very well presented section in Mordukhovich [193] on Competitive Equilibria and Pareto Optimality in Welfare Economies. We refer the reader, to Chap. 8 in [193], for more details, more results, and more references, and especially to the section Commentary to Chap. 8. This section contains an excellent survey on economic problems and equilibrium theory. For the completeness of our work we give here some references on this subject: [1, 4, 5, 14, 27, 99, 101, 119, 122, 129, 138, 143, 144, 147, 152–158, 181, 182, 201, 202, 204, 231, 232].

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Index

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