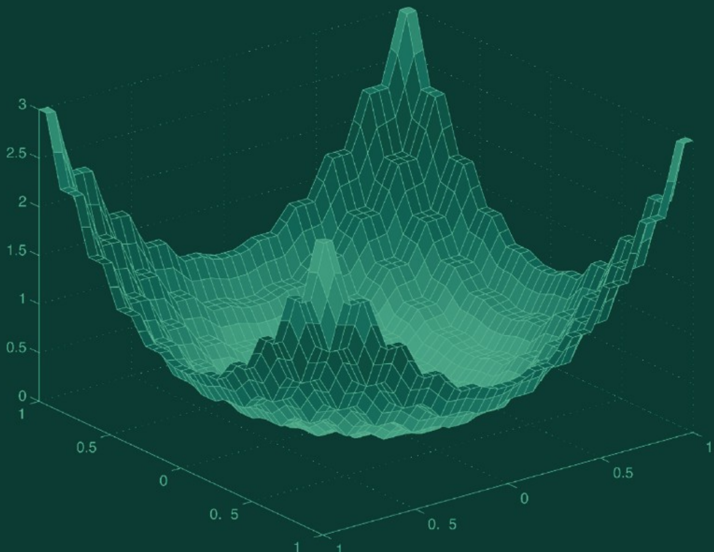
Sorin G. Gal

Global Smoothness and Shape Preserving Interpolation by Classical Operators



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Global Smoothness and Shape Preserving Interpolation by Classical Operators

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A tribute to Tiberiu Popoviciu

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Preface

This monograph is devoted to two interesting properties in the interpolation of functions: the first deals with the Global Smoothness Preservation Property (GSPP) for the well-known classical interpolation operators of Lagrange, Grünwald, Hermite–Fejér and Shepard type, while the second treats the Shape Preservation Property (SPP) for the same interpolation operators. The examination of the two properties involves both the univariate and bivariate cases. An exception is the GSPP of complex Hermite–Fejér interpolation polynomials based on the roots of unity.

An introduction to GSPP of some classical interpolation operators in the univariate case was given in a short chapter (chapter 6) of a monograph by Anastassiou–Gal [6]. In this work the univariate case is completed with many new results; an entirely new chapter is devoted to the bivariate case.

The study of SPP classical interpolation operators in both the univariate and bivariate cases appears here for the first time in book form.

This monograph consists of the author's research over the past five years with new concepts and results that have not been previously published. Many open problems suggested throughout may be of interest to different researchers. In general, the book may be used in various areas of mathematics — interpolation of functions, numerical analysis, approximation theory — as well as computer-aided geometric design, data fitting, fluid mechanics and engineering. Additionally, the work may be used as supplementary material in graduate courses as well.

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Oradea, September 2004 *Sorin G. Gal*

Introduction

The classical interpolation operators of Lagrange, Grünwald, Hermite–Fejér and Shepard on various systems of nodes, were introduced in mathematics mainly for the purpose of approximation of functions. As a consequence, most papers on these operators deal with their convergence properties.

However, in recent years it has been pointed out that many of these classical interpolation operators have other interesting properties too, like the (partial) preservation of the global smoothness and shape of the interpolated functions.

Systematic results in these directions have been obtained by the author of this monograph in a series of papers jointly written with other researchers.

The partial global smoothness preservation property can be described as follows. We say that the sequence of interpolation operators $L_n : C[a, b] \to C[a, b], n \in \mathbb{N}$, (in the sense that each $L_n(f)$ coincides with f on a system of given nodes) partially preserves the global smoothness of f, if for $f \in Lip_M\alpha$, $0 < \alpha \leq 1$, there exists $0 < \beta < \alpha$ independent of f and n, such that

$$\omega_1(L_n(f); h) \leq Ch^{\beta}, \quad \forall h \in [0, 1), n \in \mathbb{N},$$

(i.e., $L_n(f) \in Lip_C\beta, \forall n \in \mathbb{N}$) where $C > 0$ is independent of n but may depend on f. Here $\omega_1(f; \delta) = \sup\{|f(x + h) - f(x)|; 0 \leq h \leq \delta, x, x + h \in [a, b]\}$ is the uniform modulus of continuity, and of course it can be replaced by other kinds of moduli of continuity too.

It seems that in general (excepting, for example, some particular Shepard operators), the interpolation conditions do not permit a complete global smoothness preservation property, i.e., $\alpha = \beta$, as, for example, is the case of Bernstein-type approximation operators.

An introduction to the global smoothness preservation property (GSPP) in the univariate case was made in the recent book by Anastassiou–Gal [6].

In the present monograph, for the univariate case many new results are presented. For example, if $\tau(f; h) \leq Mh^{\alpha}, \forall h \in [0, 1)$, then there exists $0 < \beta \leq \alpha$ such that $\tau(L_n(f); h)_{L^1} \leq Ch^{\beta}, \forall n \in \mathbb{N}, h \in [0, 1)$, where $\tau(f; h)_{L^1}$ is the averaged

L^1-modulus of continuity and $L_n(f)$ is a certain interpolation operator. Also, new results on univariate Shepard operators on infinite intervals and on univariate Shepard–Lagrange operators on [0, 1] are included.

In addition, bivariate cases of these classical operators with respect to the following three kinds of moduli of continuity are examined:

- the bivariate uniform modulus of continuity given by

$$\omega_1(f;\delta,\eta)=\sup\{|f(x+h,y+k)-f(x,y)|;\quad 0\le h\le\delta, 0\le k\le\eta, (x,y),$$
$$(x+h,y+k)\in I\},$$

- the Bögel modulus of continuity given by

$$\omega^B(f;\delta,\eta)=\sup\{|\Delta_{h,k}f(x,y)|;\quad 0\le h\le\delta, 0\le k\le\eta, (x,y),$$
$$(x+h,y+k)\in I\},$$

  where $\Delta_{h,k}f(x,y)=f(x+h,y+k)-f(x+h,y)-f(x,y+k)+f(x,y)$,
- the bivariate Euclidean modulus of continuity given by

$$\omega^E(f;\rho)=\sup\{|f(x+h,y+k)-f(x,y)|;\quad 0\le h, 0\le k, (h^2+k^2)^{\frac{1}{2}}\le\rho,$$
$$(x,y),(x+h,y+k)\in I\},$$

  where $f:I\to\mathbb{R}$ with I a bidimensional interval.

The case of complex Hermite–Fejér interpolation polynomials on the roots of unity is studied as well.

On the other hand, the partial shape preservation property can be described as follows. Let $L:C[a,b]\to C[a,b]$ be an interpolation operator. We say that L partially preserves the shape of f, if for any monotone (convex) function $f(x)$ on $[a,b]$, there exist some points $\xi_i\in[a,b]$ (which may be independent of f) such that $L(f)(x)$ is of the same monotonicity (convexity, respectively) in some neighborhoods $V(\xi_i)$ of ξ_i.

The first results in this direction were obtained in 1960–1962 by T. Popoviciu, for the monotonicity case and Hermite–Fejér polynomials on some special Jacobi nodes. His results were of the qualitative type, i.e., only the existence of these neighborhoods $V(\xi_i)$ was proved, without any estimate of their lengths.

In this monograph, quantitative results of T. Popoviciu's results and new qualitative and quantitative results for other classical interpolation operators are presented.

Bivariate variants of these operators, with respect to various natural concepts of bivariate monotonicity and convexity, also are studied.

The results in this monograph, especially those in the bivariate case, have potential applications to data fitting, fluid dynamics, computer aided geometric design, curves, and surfaces. At the end of the book we present an Appendix with 20 graphs of various bivariate Shepard-type operators attached to some particular functions.

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September 2004

1

Global Smoothness Preservation, Univariate Case

In this chapter we present results concerning the global smoothness preservation by some classical interpolation operators.

1.1 Negative Results

Kratz–Stadtmüller [66] prove the property of global smoothness preservation for discrete sequences $\{L_n(f)\}_n$ of the form

$$L_n(f)(x) = \sum_{k=1}^{n} f(x_{k,n})p_{k,n}(x), \quad x \in [-1, 1], \quad f \in C[-1, 1], \tag{1.1}$$

satisfying the conditions

$$\sum_{k=1}^{n} p_{k,n}(x) \equiv s_n, \text{ which is independent of } x \in [-1, 1], \tag{1.2}$$

$$\sum_{k=1}^{n} |p_{k,n}(x)| \leq C_1, \quad x \in [-1, 1], \tag{1.3}$$

$$p_{k,n} \in C^1[-1, 1] \quad \text{and} \quad \sum_{k=1}^{n} |(x - x_{k,n})p'_{k,n}(x)| \leq C_2, \quad x \in [-1, 1], \tag{1.4}$$

for some constants C_1, C_2.

Since the Hermite–Fejér and the Lagrange polynomials are of type (1.1), it is natural to ask if for these polynomials (1.2)–(1.4) hold. Unfortunately, this is not so for Lagrange interpolation because of the following:

Theorem 1.1.1. *Let* $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbf{N}}$ ($\mathcal{M}_n = \{x_{k,n}\}_{k=1}^{n}$) *be an arbitrary triangular matrix of interpolation nodes in* $[-1, 1]$ *(i.e.,* $-1 \leq x_{n,n} < x_{n-1,n} < \cdots <$

$x_{1,n} \le 1$, $n \in \mathbf{N}$*) and* $l_{k,n}(x)$ *the fundamental polynomials of Lagrange interpolation on* $\mathcal{M}_n$*. Then for all* $n \ge 2$ *we have*

$$1 \le \liminf_{n\to\infty} \frac{\inf_{\mathcal{M}_n} \max_{|x|\le 1} \sum_{k=1}^{n} |x - x_{k,n}| \cdot |l'_{k,n}(x)|}{n}$$

$$\le \limsup_{n\to\infty} \frac{\inf_{\mathcal{M}_n} \max_{|x|\le 1} \sum_{k=1}^{n} |x - x_{k,n}| \cdot |l'_{k,n}(x)|}{n} \le 2.$$

Proof. Let us denote $x_{k,n} = x_k$, $k = 1, \ldots, n$ and

$$A_n(x; \mathcal{M}_n) = \sum_{k=1}^{n} |x - x_{k,n}| \cdot |l'_{k,n}(x)|,$$

where

$$l_{k,n}(x) = \frac{\omega_n(x)}{\omega'_n(x)(x - x_k)}, \quad k = 1, \ldots, n, \quad \omega_n(x) = \prod_{k=1}^{n}(x - x_k).$$

Consider an index $j \in \{1, 2, \ldots, n\}$ such that

$$|\omega'_n(x_j)| = \max_{1\le k\le n} |\omega'_n(x_k)|.$$

Then we get

$$A_n(x_j; \mathcal{M}_n) = |\omega'_n(x_j)| \sum_{\substack{k=1\\k\ne j}}^{n} \frac{1}{|\omega'_n(x_k)|} \ge n - 1,$$

which proves the first inequality in the statement of theorem.

To prove the second inequality, let us choose

$$\omega_n(x) = \frac{1}{2}[T_{n-2}(x) - T_n(x)] = \sin t \sin(n-1)t, \quad x = \cos t,$$

where $T_n(x) = \cos[n \arccos x]$ is the Chebyshev polynomials of degree n. Then $x_k = \cos t_k$, $t_k = \frac{(k-1)}{n-1}\pi$, $k = 1, \ldots, n$, and an easy calculation yields

$$|\omega'_n(x_k)| = \begin{cases} n-1, & \text{if } 2 \le k \le n-2 \\ 2n-1, & \text{if } k = 1, n, \end{cases}$$

and

$$\max_{|x|\le 1} |\omega_n(x)| \le 1, \quad \max_{|x|\le 1} |\omega'_n(x)| \le 2n - 2.$$

Thus, denoting an index j for which $|x - x_j| = \min_{1\le k\le n} |x - x_k|$, we obtain

$$A_n(x;\mathcal{M}_n) \le |\omega_n'(x)| \sum_{k=1}^{n} \frac{1}{|\omega_n'(x_k)|} + \sum_{k=1}^{n} \frac{|\omega_n(x)|}{|\omega_n'(x_k)|\cdot|x - x_k|}$$

$$\le (2n-2)\left(\frac{n-2}{n-1} + \frac{2}{2n-2}\right) + 2 + \sum_{\substack{k=1\\k\ne j}}^{n} \frac{\sin t}{(n-1)|\cos t - \cos t_k|}$$

$$\le 2n + \frac{1}{n-1}\sum_{\substack{k=1\\k\ne j}}^{n}\left(\frac{1}{\sin\frac{t-t_k}{2}} + \frac{1}{\sin\frac{t+t_k}{2}}\right)$$

$$\le 2n + \mathcal{O}\left(\sum_{\substack{k=1\\k\ne j}}^{n} \frac{1}{|j-k|}\right) = 2n + \mathcal{O}(\log n),$$

which completes the proof. □

Remarks. (1) If $\{L_n(f)\}_n$ are the classical Lagrange polynomials of any system of nodes, then it is known that (1.3) does not hold. As a conclusion, it seems that for classical interpolation polynomials all three conditions (1.2) to (1.4) cannot be verified.

(2) The shortcoming in Remark 1 above can be removed as follows. Let $L_n(f)(x)$ be of form (1.1) with $p_{k,n}(x)$ satisfying (1.4). Then $f \in \mathrm{Lip}_M(1;[-1,1])$ implies $L_n(f) \in \mathrm{Lip}_{C_2M}(1;[-1,1])$, for all $n \in \mathbf{N}$. Indeed, we have

$$|L_n(f)(x) - L_n(f)(y)| = \left|(x-y)\sum_{k=1}^{n} f(x_{k,n})p_{k,n}'(\xi_{x,y,n})\right|$$

$$= |x-y|\left|\sum_{k=1}^{n}[f(x_{k,n}) - f(\xi_{x,y,n})]p_{k,n}'(\xi_{x,y,n})\right|$$

$$\le |x-y|M\sum_{k=1}^{n}|x_{k,n} - \xi_{x,y,n}|\cdot|p_{k,n}'(\xi_{x,y,n})| \le C_2M|x-y|.$$

(3) If $\{L_n(f)\}_n$ are the Hermite–Fejér polynomials based on the Chebyshev nodes of first kind, then obviously (1.2) and (1.3) hold with $s_n \equiv 1$ and $C_1 = 1$.

But (1.4) cannot hold since then by Kratz–Stadtmüller [66] would follow that $\{L_n(f)\}_n$ have the property of global smoothness preservation and by Anastassiou–Cottin–Gonska [5] this does not hold.

(4) Let $x_{k,n} = -1 + \frac{2(k-1)}{n-1}$, $k = 1,\ldots,n$, be equidistant nodes in $[-1,1]$ and $H_{2n-1}(f)(x)$ the Hermite–Fejér interpolation polynomials on these nodes. Berman

[20] proved that for $f(x) = x, x \in [-1, 1]$, the sequence $\{H_{2n-1}(f)(x)\}_n$ is unboundedly divergent for any $0 < |x| < 1$. Hence this sequence has no the property of partial preservation of global smoothness. Indeed, if $f \in \text{Lip}_1(1; [-1, 1])$ and if we suppose that there exist $0 < \alpha < 1$ and $M > 0$ such that $H_{2n-1}(f) \in \text{Lip}_M(\alpha; [-1, 1])$ for all $n \in \mathbf{N}$, then

$$|H_{2n-1}(f)(x) - H_{2n-1}(f)(y)| \leq M|x - y|^{\alpha}, \quad \forall\, x, y \in [-1, 1], \quad n \in \mathbf{N}.$$

Taking $x = -1$ and $0 < |y| < 1$, letting $n \to \infty$ in the above inequality, we get a contradiction.

1.2 Global Smoothness Preservation by Some Lagrange, Hermite–Fejér, and Shepard-Type Operators

Let $x_k = \cos \frac{2k-1}{2n}\pi$, $k = 1, \ldots, n$, be the roots of the Chebyshev polynomial $T_n(x)$, and the Hermite–Fejér polynomial of an $f \in C[-1, 1]$ based on these roots,

$$H_n(f)(x) = \sum_{k=1}^{n} f(x_k)(1 - x x_k) \frac{T_n^2(x)}{n^2(x - x_k)^2}.$$

If we consider the Jackson interpolation trigonometric polynomials $J_n(f)(x)$ (see Section 1.4), then for $f \in C[-1, 1]$, denoting $F(t) = f(\cos t)$, it is known that (see, e.g., Szabados [95], p. 406)

$$H_n(f)(x) = J_{2n-1}(F)(t), \quad t = \arccos x.$$

Now, if $0 < \alpha < 1$ and $f \in \text{Lip}_M(\alpha; [-1, 1])$, then $J_{2n-1}(F) \in \text{Lip}_C \alpha$ (see Theorem 1.4.2), which can be written as ($x = \cos u$, $y = \cos v$)

$$|H_n(f)(x) - H_n(f)(y)| = |J_{2n-1}(F)(u) - J_{2n-1}(F)(v)| \leq C|u - v|^{\alpha}$$

$$= C|\arccos x - \arccos y|^{\alpha} \leq \frac{C\pi}{\sqrt{2}}|x - y|^{\alpha/2}, \quad \forall\, x, y \in [-1, 1],$$

since $\arccos x \in \text{Lip}_{\pi/\sqrt{2}}(1/2; [-1, 1])$, (see, e.g., Cheney [22], Problem 5, p. 88), which means that $H_n(f) \in \text{Lip}_{\frac{C\pi}{\sqrt{2}}}(\alpha/2; [-1, 1])$, for all $n \in \mathbf{N}$.

If $\alpha = 1$, in the same way we get

$$\omega_1(H_n(f); h) \leq C\left[h \log \frac{1}{h}\right]^{1/2}, \quad n \in \mathbf{N},\ h \in (0, 1).$$

As a conclusion, we can say that $\{H_n(f)\}_n$ has the property of partial preservation of global smoothness of f.

However, by a direct method we will improve the above considerations about $\{H_n(f)\}_n$.

Theorem 1.2.1. *For any $f \in C[-1,1]$, $h > 0$ and $n \in \mathbf{N}$ we have*

$$\omega_1(H_n(f);h)$$
$$= \min\left\{\mathcal{O}\left(hn\sum_{k=1}^{n}\omega_1\left(f;\frac{1}{k^2}\right)\right), \mathcal{O}\left(\frac{1}{n}\sum_{k=1}^{n}\omega_1\left(f;\frac{1}{k}\right)+\omega_1(f;h)\right)\right\},$$

where the constants in "$\mathcal{O}$" are independent of f, n and h.

Proof. First we obtain an upper estimate for $|H_n'(f)(x)|$. Let $x \in [-1,1]$ be fixed, the index j defined by $|x - x_j| = \min\{|x - x_j|;\ 1 \le k \le n\}$ and denote

$$A_k(x) = (1 - xx_k)\frac{T_n^2(x)}{n^2(x-x_k)^2}.$$

We have

$$A_k'(x) = -x_k\frac{T_n^2(x)}{(x-x_k)^2} + (1-xx_k)\frac{2T_n(x)T_n'(x)}{(x-x_k)^2} - \frac{2(1-xx_k)T_n^2(x)}{(x-x_k)^3},$$

$k = 1, \ldots, n$, which implies (by $1 - xx_k = 1 - x^2 + x(x - x_k)$)

$$|A_k'(x)| \le \frac{1}{(x-x_k)^2} + \frac{2(1-x^2)|T_n'(x)|}{(x-x_k)^2} + \frac{2|T_n'(x)|}{|x-x_k|}$$
$$+\frac{2(1-x^2)}{|x-x_k|^3} + \frac{2}{(x-x_k)^2}, \quad k = 1, \ldots, n.$$

For simplicity, all the constants (independent of f, n, and h) which appear will be denoted by C. Since

$$H_n'(f)(x) = \sum_{k=1}^{n} f(x_k)A_k'(x) = \sum_{k=1}^{n}[f(x_k) - f(x)]A_k'(x),$$

we obtain

$$|H_n'(f)(x)| \le \frac{C}{n^2}\sum_{\substack{k=1\\k\ne j}}^{n}\omega_1(f;|x-x_k|)\left[\frac{3}{(x-x_k)^2} + \frac{2(1-x^2)|T_n'(x)|}{(x-x_k)^2}\right.$$
$$\left.+\frac{2(1-x^2)}{|x-x_k|^3} + \frac{2|T_n'(x)|}{|x-x_k|}\right] + C\omega_1(f;|x-x_j|)|A_j'(x)|. \tag{1.5}$$

The following known relations (see, e.g., Szabados–Vértesi [100], p. 282) will be frequently used:

$$|x - x_j| \le \frac{Cj}{n^2}, \quad n\sqrt{1-x^2} \sim j, \quad |x - x_k| \sim \frac{|j^2 - k^2|}{n^2}, \quad k \ne j. \tag{1.6}$$

Now by (1.6) and by the combined Bernstein–Markov inequality, we get

$$|A_j'(x)| \leq \frac{n^2\|A_j\|}{n\sqrt{1-x^2}+1} \leq \frac{Cn^2}{j},$$

and

$$\omega_1(f;|x-x_j|)|A_j'(x)| \leq C\omega_1\left(f;\frac{j}{n^2}\right)\frac{n^2}{j} \leq C\omega_1\left(f;\frac{1}{n^2}\right)n^2.$$

Also, by (1.6) we obtain (using also the inequality $\omega_1(f;T)/T \leq 2\omega_1(f;t)/t$, for $t \leq T$)

$$\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)}{(x-x_k)^2} \leq \frac{n^2}{j}\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)}{|x-x_k|}$$

$$\leq \frac{Cn^4}{j}\sum_{k\neq j}\frac{\omega_1\left(f;\frac{|j^2-k^2|}{n^2}\right)}{|j^2-k^2|} \leq \frac{Cn^4}{j}\sum_{k=1}^{n}\frac{1}{k^2}\omega_1\left(f;\frac{k^2}{n^2}\right),$$

$$\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)(1-x^2)|T_n'(x)|}{(x-x_k)^2} \leq \sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)n\sqrt{1-x^2}}{(x-x_k)^2}$$

$$\leq Cj\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)}{(x-x_k)^2} \leq Cn^4\sum_{k=1}^{n}\frac{1}{k^2}\omega_1\left(f;\frac{k^2}{n^2}\right),$$

$$\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)(1-x^2)}{|x-x_k|^3} \leq C\sum_{k\neq j}\frac{\omega_1\left(f;\frac{|j^2-k^2|}{n^2}\right)\frac{j^2}{n^2}}{\frac{|j^2-k^2|^3}{n^6}}$$

$$\leq Cn^6\omega_1\left(f;\frac{1}{n^2}\right)\sum_{k\neq j}\frac{1}{|j^2-k^2|}\cdot\frac{j^2}{n^2} \leq \frac{Cn^6}{j^2}\omega_1\left(f;\frac{1}{n^2}\right)\frac{j^2}{n^2}$$

$$\leq Cn^4\omega_1\left(f;\frac{1}{n^2}\right),$$

$$\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)|T_n'(x)|}{|x-x_k|} \leq Cn^2\sum_{k\neq j}\frac{\omega_1(f;|x-x_k|)}{|x-x_k|}$$

$$\leq Cn^4\sum_{k=1}^{n}\frac{1}{k^2}\omega_1\left(f;\frac{k^2}{n^2}\right).$$

Collecting now all these estimates, by (1.5) we get

$$|H_n'(f)(x)| \leq Cn^2\sum_{k=1}^{n}\frac{1}{k^2}\omega_1\left(f;\frac{k^2}{n^2}\right).$$

Since

$$n^2 \sum_{k=1}^{n} \frac{1}{k^2} \omega_1\left(f; \frac{k^2}{n^2}\right) \sim n \int_{1/n}^{1} \frac{\omega_1(f; t^2)}{t^2} dt$$
$$= n \int_1^n \omega_1\left(f; \frac{1}{u^2}\right) du \sim n \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k^2}\right)$$

(we have used the equivalence between the Riemann integral sums and the integral itself, and made the substitution $t = 1/u$), the above estimate becomes

$$|H_n'(f)(x)| \leq Cn \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k^2}\right). \tag{1.7}$$

On the other hand, for $|x - y| \leq h$ and by, e.g., Szabados–Vértesi [100] , Theorem 5.1, p. 168, we get

$$|H_n(f)(x) - H_n(f)(y)| \leq |H_n(f)(x) - f(x)| + |f(x) - f(y)| + |f(y)$$
$$-H_n(f)(y)| \leq 2\|H_n(f) - f\| + \omega_1(f; h) \leq C \sum_{k=1}^{n} \frac{1}{k^2} \omega_1\left(f; \frac{k}{n}\right) + \omega_1(f; h).$$

But similarly to the above considerations,

$$\sum_{k=1}^{n} \frac{1}{k^2} \omega_1\left(f; \frac{k}{n}\right) \sim \frac{1}{n} \int_{1/n}^{1} \frac{\omega_1(f; t)}{t^2} dt = \frac{1}{n} \int_1^n \omega_1\left(f; \frac{1}{u}\right) du$$
$$\sim \frac{1}{n} \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k}\right),$$

and therefore we get

$$\omega_1(H_n(f); h) = \mathcal{O}\left[\frac{1}{n} \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k}\right)\right] + \omega_1(f; h). \tag{1.8}$$

Now, on the other hand, using (1.7), we obtain for $|x - y| \leq h$,

$$|H_n(f)(x) - H_n(f)(y)| \leq |H_n'(f)(\xi)|h \leq Cnh \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k^2}\right),$$

i.e.,

$$\omega_1(H_n(f); h) = \mathcal{O}\left[hn \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k^2}\right)\right],$$

which together with (1.8) proves the theorem. □

Corollary 1.2.1. *If* $f \in \mathrm{Lip}_M(\alpha; [-1, 1])$, $0 < \alpha \leq 1$, *then for all* $n \in \mathbf{N}$ *and* $0 < h < 1$ *we have*

$$\omega_1(H_n(f);h) = \begin{cases} \mathcal{O}\left(h^{\frac{\alpha}{\max(2-\alpha,1+\alpha)}}\right), & \textit{if} \;\; 0<\alpha<1/2 \;\; \textit{or} \;\; 1/2<\alpha<1, \\ \mathcal{O}\left(\left[h\log\frac{1}{h}\right]^{\frac{2\alpha+1}{6}}\right), & \textit{if} \;\; \alpha=1/2 \; \textit{or} \; 1. \end{cases}$$

Proof. The optimal choice in Theorem 1.2.1 is when $h = v_n$, where

$$v_n = \frac{\sum_{k=1}^{n}\omega_1\left(f;\frac{1}{k}\right)}{n^2\sum_{k=1}^{n}\omega_1\left(f;\frac{1}{k^2}\right)}.$$

When $h < v_n$, the minimum in Theorem 1.2.1 is the first term, and when $h > v_n$, it is the second term. By simple calculations we have

$$v_n = \begin{cases} \mathcal{O}(n^{\alpha-2}), & \text{if } 0<\alpha<1/2, \\ \mathcal{O}\left(\frac{1}{n^{3/2}\log n}\right), & \text{if } \alpha=1/2, \\ \mathcal{O}(n^{-1-\alpha}), & \text{if } 1/2<\alpha<1, \\ \mathcal{O}\left(\frac{\log n}{n^2}\right), & \text{if } \alpha=1. \end{cases}$$

Hence, by using Theorem 1.2.1 we arrive at the statement of the corollary. □

Remarks. (1) Let $0 < \alpha < 1$. The obvious inequalities

$$\frac{\alpha}{2} < \frac{\alpha}{\max(2-\alpha,1+\alpha)}, \qquad h^{1/4} > \left[h\log\frac{1}{h}\right]^{1/3}$$

mean that the preservation property given by Corollary 1.2.1 is better than that given for $H_n(f)(x)$ at the beginning of this section.

(2) It is an open question if the estimates of $\omega_1(H_n(f);h)$ in Theorem 1.2.1 and Corollary 1.2.1 are the best possible. However, if we choose, for example, $f_0(x) = x \in \mathrm{Lip}_1(1;[-1,1])$, then we can prove that $\omega_1(H_n(f_0);h) \sim \sqrt{h}$. Indeed, by

$$H_n(f_0)(x) = x - \frac{T_n(x)T_{n-1}(x)}{n} = x - \frac{T_{2n-1}(x)+T_1(x)}{2n}$$

(see, e.g., Anastassiou–Cottin–Gonska [5]) we get

$$|H_n'(f_0)(x)| = \left|1 - \frac{T_{2n-1}'(x)+1}{2n}\right| \geq \frac{C}{\sqrt{1-x^2}} \geq Cn,$$

for all $x \in \left[\frac{1+x_1}{2}, 1\right]$ and

$$\omega_1\left(H_n(f_0);\frac{1-x_1}{2}\right) \geq \left|H_n(f_0)(1) - H_n(f)\left(\frac{1+x_1}{2}\right)\right|$$

$$= |H_n'(f_0)(\xi)| \cdot \frac{1-x_1}{2} \geq Cn(1-x_1) = C\sqrt{\frac{1-x_1}{2}},$$

as claimed. Now we will prove that in fact $H_n(f_0) \in \mathrm{Lip}_M(1/2; [-1,1])$ for all $n \in \mathbf{N}$. Evidently, it suffices to prove that $\frac{T_{2n-1}(x)}{2n} \in \mathrm{Lip}_M(1/2; [-1,1])$, for all $n \in \mathbf{N}$. But by Cheney [22], Problem 5, p. 88

$$\frac{|T_{2n-1}(x) - T_{2n-1}(y)|}{2n} = \frac{\cos[(2n-1)\arccos x] - \cos[(2n-1)\arccos y]|}{2n}$$

$$\leq \frac{(2n-1)|\sin\xi| \cdot |\arccos x - \arccos y|}{2n} \leq M|\arccos x - \arccos y|$$

$$\leq \frac{M\pi}{\sqrt{2}}|x-y|^{1/2},$$

which was to be proved.

(3) It is well known (see, e.g., Szabados–Vértesi [101], relation (3.2)) that the mean convergence of $H_n(f)(x)$ to $f(x)$ is better than the uniform convergence, that is, we have

$$||f - H_n(f)||_{p,w} \leq C_p\omega_1\left(f; \frac{1}{n}\right), \quad \forall n \in \mathbb{N}, p > 0,$$

where $\omega_1\left(f; \frac{1}{n}\right)$ is the uniform modulus of continuity, $w(x) = \frac{1}{\sqrt{1-x^2}}$ and $||f||_{p,w} = (\int_{-1}^{1} |f(x)|^p dx)^{\frac{1}{p}}$.

This estimate suggests that by using a suitable chosen weighted L_w^1-modulus of continuity, $H_n(f)(x)$ may have a better global smoothness preservation property. In this sense, let us define

$$\omega_1(f;h)_{1,w} = \sup\{\int_{-1+t}^{1-t} w(x)|f(x+t) - f(x-t)|dx; 0 \leq t \leq h\},$$

$0 < h < 1$.

We present:

Theorem 1.2.2. *Let $H_n(f)$ be the Hermite–Fejér polynomial in Corollary 1.2.1. If $f \in Lip_M(\alpha; [-1,1])$, $0 < \alpha \leq 1$, then for all $n \in \mathbb{N}$ and $0 < h < 1$, it follows*

$$\omega_1(H_n(f);h)_{1,w} = \begin{cases} \mathcal{O}\left[h^{\frac{\alpha}{2-\alpha}}\right], \textit{if } 0 < \alpha < \frac{1}{2}, \\ \mathcal{O}\left[(h\log\frac{1}{h})^{1/3}\right], \textit{if } \alpha = \frac{1}{2} \\ \mathcal{O}\left[h^{\frac{\alpha}{1+\alpha}}\right], \textit{if } \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Proof. We have

$$w(x)|H_n(f)(x+t) - H_n(f)(x-t)|$$

$$\leq \frac{w(x)}{w(x+t)} w(x+t)|H_n(f)(x+t) - f(x+t)| + w(x)|f(x+t) - f(x-t)|$$

$$+ \frac{w(x)}{w(x-t)} w(x-t)|f(x-t) - H_n(f)(x-t)|.$$

Integrating the inequality from $-1+t$ to $1-t$ and applying to the first and third term the Hölder's inequality, we get

$$\int_{-1+t}^{1-t} w(x)|H_n(f)(x+t) - H_n(f)(x-t)|dx \leq \left(\int_{-1+t}^{1-t} [w^2(x)/w(x+t)]dx \right)^{1/2}$$

$$\left(\int_{-1+t}^{1-t} w(x+t)|H_n(f)(x+t) - f(x+t)|^2 dx \right)^{1/2}$$

$$+ \int_{-1+t}^{1-t} w(x)|f(x+t) - f(x-t)|dx$$

$$+ \left(\int_{-1+t}^{1-t} [w^2(x)/w(x-t)]dx \right)^{1/2} \left(\int_{-1+t}^{1-t} w(x-t)|H_n(x-t) - f(x-t)|^2 dx \right)^{1/2}$$

$$\leq CI_1(t)\omega_1(f; \frac{1}{n}) + \int_{-1+t}^{1-t} w(x)|f(x+t) - f(x-t)|dx + CI_2(t)\omega_1\left(f; \frac{1}{n}\right),$$

where

$$I_1(t) = \left(\int_{-1+t}^{1-t} [w^2(x)/w(x+t)]dx \right)^{1/2},$$

$$I_2(t) = \left(\int_{-1+t}^{1-t} [w^2(x)/w(x-t)]dx \right)^{1/2}.$$

Passing to supremum with $t \in [0, h]$ we get

$$\omega_1(H_n(f); h)_{1,w} \leq C\omega_1\left(f; \frac{1}{n}\right)[\sup\{I_1(t); t \in [0, h]\}$$

$$+ \sup\{I_2(t); t \in [0, h]\}] + \omega_1(f; h)_{1,w}.$$

In what follows we show that $\sup\{I_k(t); t \in [0, h]\} \leq C, \forall 0 < h < 1, k = 1, 2$, where $C > 0$ is independent of h.

We have

$$\int_{-1+t}^{1-t} \left[\frac{w^2(x)}{w(x+t)}\right] dx = \int_{-1+t}^{0} \frac{\sqrt{1-(x+t)^2}}{1-x^2} dx$$

$$+ \int_{0}^{1-t} \frac{\sqrt{1-(x+t)^2}}{1-x^2} dx \leq \int_{-1+t}^{0} \frac{dx}{\sqrt{1-x^2}}$$

$$+ \int_{-1+t}^{0} \frac{\sqrt{-2tx}}{1-x^2} dx + \int_{0}^{1-t} \frac{dx}{\sqrt{1-x^2}}$$

$$\leq C_1 + \int_{-1+t}^{0} \frac{\sqrt{-2tx}}{1-x^2}dx + C_2.$$

Also,

$$\int_{-1+t}^{0} \frac{\sqrt{-2tx}}{1-x^2}dx = \sqrt{2t}\int_{-1+t}^{0} \frac{\sqrt{-x}}{1-x^2}dx.$$

For $x \leq 0$ by some calculation we get

$$\int \frac{\sqrt{-x}}{1-x^2}dx = \text{arctg}\,(\sqrt{-x}) + \frac{1}{2}\log\frac{1-\sqrt{-x}}{1+\sqrt{-x}}.$$

Therefore, we easily obtain

$$\sqrt{2t}\int_{-1+t}^{0} \frac{\sqrt{-x}}{1-x^2}dx \leq c\sqrt{t}\log(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}},$$

and since by the l'Hôpital's rule it follows

$$\lim_{t\searrow 0}\sqrt{t}\log\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}} = 0,$$

as a consequence we immediately obtain

$$\sqrt{2t}\int_{-1+t}^{0} \frac{\sqrt{-x}}{1-x^2}dx \leq C_1, \forall t \in [0,1],$$

and $I_1(t) \leq C, \forall t \in [0,1]$.

Similarly, for $t \in [0,1]$ we have

$$\int_{-1+t}^{1-t}[w^2(x)/w(x-t)]dx = \int_{-1+t}^{0} \frac{\sqrt{1-(x-t)^2}}{1-x^2}dx$$

$$+\int_{0}^{1-t} \frac{\sqrt{1-(x-t)^2}}{1-x^2}dx \leq \int_{-1+t}^{0} \frac{dx}{\sqrt{1-x^2}}$$

$$+\int_{0}^{1-t} \frac{dx}{\sqrt{1-x^2}} + \int_{0}^{1-t} \frac{\sqrt{2tx}}{1-x^2}dx$$

$$\leq C + \sqrt{2t}\int_{0}^{1-t} \frac{\sqrt{x}}{1-x^2}dx.$$

Reasoning as in the case of $I_1(t)$, we easily obtain $I_2(t) \leq C$, for all $t \in [0,1]$.

As a first conclusion, we get the first estimate

$$\omega_1(H_n(f);h)_{1,w} \leq C\omega_1(f;1/n) + \omega_1(f;h)_{1,w}.$$

On the other hand, by the integral mean value theorem, for each fixed t, there exists $\xi \in (-1+t, 1-t)$ such that

$$\int_{-1+t}^{1-t} w(x)|H_n(f)(x+t) - H_n(f)(x-t)|dx$$
$$= |H_n(f)(\xi+t) - H_n(f)(\xi-t)| \int_{-1+t}^{1-t} \frac{dx}{\sqrt{1-x^2}}$$
$$\leq Ct|H_n'(f)(\eta)| \leq Ct||H_n'(f)|| \leq$$

(by the proof of Theorem 1.2.1) $\leq Cnt \sum_{k=1}^{n} \omega_1(f; 1/k^2)$. Passing to supremum with $t \in [0, h]$, it follows that

$$\omega_1(H_n(f); h)_{1,w} \leq Cnh \sum_{k=1}^{n} \omega_1(f; 1/k^2).$$

Let $f \in Lip_M(\alpha; [-1, 1])$. The last relation becomes

$$\omega_1(H_n(f); h)_{1,w} \leq \begin{cases} Chn^{2-2\alpha}, \text{ if } 0 < \alpha < \frac{1}{2}, \\ Chn\log(n), \text{ if } \alpha = \frac{1}{2} \\ Chn, \text{ if } \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Since $f \in Lip_M(\alpha; [-1, 1])$ implies $\omega_1(f; 1/n) \leq Mn^{-\alpha}$ and $\omega_1(f; h)_{1,w} \leq h^{\alpha}$, the first estimate becomes

$$\omega_1(H_n(f); 1/n)_{1,w} \leq Cn^{-\alpha} + Ch^{\alpha}.$$

For the three cases $0 < \alpha < 1/2, \alpha = 1/2, 1/2 < \alpha < 1$, by the standard technique, the optimal choices for n are from the equations $n^{2-2\alpha}h = n^{-\alpha}$, $hn\log(n) = n^{-1/2}$ and $nh = n^{-\alpha}$, respectively. Replacing in the last estimates we get the theorem. □

Remark. Comparing Theorem 1.2.2 with Corollary 1.2.1, we see that for all $\alpha \in (0, 1)$ we have the same global smoothness preservation property for both moduli $\omega_1(H_n(f); h)_{1,w}$ and $\omega_1(H_n(f); h)$, while for $\alpha = 1$, Theorem 1.2.2 gives a much better result than Corollary 1.2.1, since $\omega_1(H_n(f); h)_{1,w} \leq Ch^{1/2}$, while $\omega_1(H_n(f); h) \leq C(h\log(\frac{1}{h}))^{1/3}$.

Now, let us consider the Lagrange interpolation algebraic polynomial $L_n(f)$ based on the Chebyshev nodes of second kind plus the endpoints ± 1. It is known (Mastroianni–Szabados [72]) that

$$L_n(f)(x) = \sum_{k=1}^{n} f(x_k)l_k(x),$$

where $x_k = \cos t_k$, $t_k = \frac{k-1}{n-1}\pi$, $k = \overline{1, n}$, and

$$l_k(x) = \frac{(-1)^{k-1}\omega_n(x)}{(1+\delta_{k1}+\delta_{kn})(n-1)(x-x_k)}, \quad k = \overline{1, n},$$

with $\omega_n(x) = \sin t \sin(n-1)t$, $x = \cos t$ and δ_{kj} is the Kronecker's symbol.

Theorem 1.2.3. *For any* $f \in C[-1,1]$, $h > 0$ *and* $n \in \mathbf{N}$ *we have*

$$\omega_1(L_n(f); h) \le C \min\left\{hn \sum_{k=1}^{n} \omega_1\left(f; \frac{1}{k^2}\right), \omega_1\left(f; \frac{1}{n}\right) \log n + \omega_1(f; h)\right\},$$

where $C > 0$ *is independent of* f, n *and* h.

Proof. The method will follow the ideas in the proof of Theorem 1.2.1, taking also into account that the relations (1.6) hold in this case too. Therefore, let $x \in [0,1]$ be fixed (the proof in the case $x \in [-1,0]$ being similar), the index j defined by $|x - x_j| = \min\{|x - x_k|;\ 1 \le k \le n\}$ and let us denote $\omega_n(x) = U_n(x)(1 - x^2)$, where $U_n(x) = \frac{\sin(n-1)t}{\sin t}$, $x = \cos t$. By simple calculations we get

$$l_k'(x) = \frac{(-1)^{k-1}}{(1 + \delta_{k1} + \delta_{kn})(n-1)(x - x_k)} \times\left[\frac{U_n'(x)(1-x^2)}{x - x_k} - \frac{2xU_n(x)}{x - x_k} - \frac{U_n(x)(1-x^2)}{(x-x_k)^2}\right],$$

which, as in the proof of Theorem 1.2.1, implies

$$|L_n'(f)(x)| \le \frac{1}{n-1} \sum_{k \ne j} \omega_1(f; |x - x_k|)\left[\frac{|U_n'(x)|(1-x^2)}{|x - x_k|} + \frac{2|U_n(x)|}{|x - x_k|} + \frac{|U_n(x)|(1-x^2)}{(x-x_k)^2}\right] + \omega_1(f; |x - x_j|)|l_j'(x)|.$$

Now, the Bernstein–Markov inequality yields

$$|l_j'(x)| \le \frac{n^2}{n\sqrt{1-x^2} + 1}\|l_j\| \le \frac{Cn^2}{j}\|l_j\|$$

$$\le \frac{Cn^2}{j} \cdot \frac{1}{n-1} \cdot \frac{|U_n(x)|\frac{j^2}{n^2}}{\frac{1}{n^2}} \le \frac{Cn^2}{j} \cdot \frac{1}{n-1}(n-1) = \frac{Cn^2}{j}.$$

Therefore

$$\omega_1(f; |x - x_j|)|l_j'(x)| \le \omega_1\left(f; \frac{j}{n^2}\right)\frac{Cn^2}{j} \le n^2 \omega_1\left(f; \frac{1}{n^2}\right).$$

Now, we will use the obvious estimates

$$|U_n(x)|(1-x^2) \le \sqrt{1-x^2} \sim \frac{j}{n}, \quad |U_n'(x)|(1-x^2) \le 2(n-1).$$

Thus, we obtain

$$\frac{1}{n-1} \sum_{k \ne j} \omega_1(f; |x - x_k|)\frac{|U_n'(x)|(1-x^2)}{|x - x_k|} \le 2 \sum_{k \ne j} \frac{\omega_1(f; |x - x_k|)}{|x - x_k|}$$

$$\leq Cn^2 \sum_{k=1}^{n} \frac{1}{k^2} \omega_1 \left(f; \frac{k^2}{n^2} \right) \leq Cn \sum_{k=1}^{n} \omega_1 \left(f; \frac{1}{k^2} \right),$$

$$\frac{1}{n-1} \sum_{k \neq j} \omega_1(f; |x - x_k|) \frac{|U_n(x)|}{|x - x_k|} \leq \sum_{k \neq j} \frac{\omega_1(f; |x - x_k|)}{|x - x_k|}$$

$$\leq Cn \sum_{k=1}^{n} \omega_1 \left(f; \frac{1}{k^2} \right),$$

$$\frac{1}{n-1} \sum_{k \neq j} \omega_1(f; |x - x_k|) \frac{|U_n(x)|(1 - x^2)}{|x - x_k|} \leq \frac{Cj}{n^2} \sum_{k \neq j} \frac{\omega_1(f; |x - x_k|)}{(x - x_k)^2}$$

$$\leq \frac{Cj}{n^2} \sum_{k \neq j} \frac{\omega_1(f; |x - x_k|)}{\frac{j}{n^2} |x - x_k|} \leq C \sum_{k \neq j} \frac{\omega_1(f; |x - x_k|)}{|x - x_k|}$$

$$\leq Cn \sum_{k=1}^{n} \omega_1 \left(f; \frac{1}{k^2} \right).$$

Collecting all these estimates, we obtain

$$|L_n'(f)(x)| \leq Cn \sum_{k=1}^{n} \omega_1 \left(f; \frac{1}{k^2} \right),$$

which, by the same reasoning as in the proof of Theorem 1.2.1, yields

$$\omega_1(L_n(f); h) \leq Cnh \sum_{k=1}^{n} \omega_1 \left(f; \frac{1}{k^2} \right). \tag{1.9}$$

On the other hand, for $|x - y| \leq h$ we get

$$|L_n(f)(x) - L_n(f)(y)| \leq 2\|L_n(f) - f\| + \omega_1(f; h),$$

which implies

$$\omega_1(L_n(f); h) \leq 2\|L_n(f) - f\| + \omega_1(f; h).$$

Standard technique in interpolation theory (see Szabados–Vértesi [100]) gives

$$\|L_n(f) - f\| \leq C\omega_1 \left(f; \frac{1}{n} \right) \|\lambda_n\|,$$

where $\lambda_n(x) = \sum_{k=1}^{n} |l_k(x)|$, $x \in [-1, 1]$, is the Lebesgue function of interpolation. Here by (1.6)

$$\lambda_n(x) \leq \sum_{k=1}^{n} \left| \frac{U_n(x)(1 - x^2)}{(n-1)(x - x_k)} \right| \leq \sum_{k \neq j} \frac{|U_n(x)|(1 - x^2)}{(n-1)|x - x_k|} + |l_j(x)|$$

$$\le C\sum_{k\ne j}\frac{\frac{j}{n}}{(n-1)\frac{|j^2-k^2|}{n^2}}+|l_j(x)|\le C\sum_{k\ne j}\frac{k}{|k^2-j^2|}+|l_j(x)|$$
$$\le C\log n+|l_j(x)|.$$

Now,

$$|l_j(x)|=\left|\frac{U_n(x)(1-x^2)}{(n-1)(x-x_j)}\right|=|U_n'(\xi)|\frac{1-x^2}{n}.$$

Here, if $j=1$ or n, then $1-x^2\sim n^{-2}$, and by Markov's inequality $|U_n'(\xi)|\le n^2\|U_n\|=n^3$, i.e., by (1.6)

$$|U_n'(\xi)|\frac{1-x^2}{n}\le n^3\frac{\frac{C}{n^2}}{n}=C.$$

If $2\le j\le n-1$, then evidently $1-x^2\sim 1-\xi^2$ and

$$|l_j(x)|=\frac{|U_n(\xi)|(1-x^2)}{n}\le\frac{1-x^2}{1-\xi^2}=C,$$

whence in both cases $|l_j(x)|=\mathcal{O}(1)$, and $\lambda_n=\mathcal{O}(\log n)$. Thus

$$\omega_1(L_n(f);h)\le C\omega_1\left(f;\frac{1}{n}\right)\log n+\omega_1(f;h), \tag{1.10}$$

which together with (1.9) proves the theorem. □

Corollary 1.2.2. *(i) If* $f\in \mathrm{Lip}_M(\alpha;[-1,1])$, $0<\alpha\le 1$, *then for all* $n\in\mathbf{N}$ *and* $h\in(0,1)$ *we have*

$$\omega_1(L_n(f);h)=\begin{cases}\mathcal{O}\left[h^{\frac{\alpha}{2-\alpha}}\left(\log\frac{1}{h}\right)^{\frac{2-2\alpha}{2-\alpha}}\right], & \text{if } 0<\alpha<\frac{1}{2},\\ \mathcal{O}\left[h^{1/3}\log\frac{1}{h}\right], & \text{if } \alpha=\frac{1}{2}\\ \mathcal{O}\left[h^{\frac{\alpha}{1+\alpha}}\left(\log\frac{1}{h}\right)^{\frac{1}{1+\alpha}}\right], & \text{if } \frac{1}{2}<\alpha\le 1.\end{cases}$$

(ii) If $\omega_1(f;h)=\mathcal{O}\left(\frac{1}{\log^{\beta}\frac{1}{h}}\right)$, $\beta>1$, *then*

$$\omega_1(L_n(f);h)=\mathcal{O}\left(\frac{1}{\log^{\beta-1}\frac{1}{h}}\right).$$

(All the constants in $\mathcal{O}$ *are independent of n and h).*

Proof. (i) Let $f\in\mathrm{Lip}_M(\alpha;[-1,1])$, $0<\alpha\le 1$. Then (1.9) and (1.10) yield

$$\omega_1(L_n(f);h)=\begin{cases}\mathcal{O}(n^{2-2\alpha}h), & \text{if } 0<\alpha<\frac{1}{2},\\ \mathcal{O}(nh\log n), & \text{if } \alpha=\frac{1}{2}\\ \mathcal{O}(nh), & \text{if } \frac{1}{2}<\alpha\le 1,\end{cases} \tag{1.11}$$

and

$$\omega_1(L_n(f);h)=\mathcal{O}\left(\frac{\log n}{n^\alpha}+h^\alpha\right), \tag{1.12}$$

respectively. Now if n is smaller than

$$\left(\frac{1}{h}\log\frac{1}{h}\right)^{\frac{1}{2-\alpha}},\quad h^{-2/3},\quad \left(\frac{1}{h}\log\frac{1}{h}\right)^{\frac{1}{1+\alpha}},$$

in the cases $0<\alpha<1/2$, $\alpha=1/2$, $1/2<\alpha\le 1$, respectively, then we use the corresponding estimates in (1.11). Otherwise we use (1.12).

(ii) In this case we get from (1.9) and (1.10)

$$\omega_1(L_n(f);h)=\mathcal{O}\left(\frac{n^2h}{\log n}\right)$$

and

$$\omega_1(L_n(f);h)=\mathcal{O}\left[\frac{1}{\log^{\beta-1}n}+\frac{1}{\log^\beta\frac{1}{h}}\right].$$

As before, we use these estimates according as n is smaller or bigger than $\frac{1}{\sqrt{h}}\left(\log\frac{1}{h}\right)^{\frac{1}{2}-\beta}$. □

Now, let us consider the case of the Shepard interpolation operator

$$S_{n,\lambda}(f)(x)=\frac{\sum_{k=0}^n f(k/n)\left|x-\frac{k}{n}\right|^{-\lambda}}{\sum_{k=0}^n\left|x-\frac{k}{n}\right|^{-\lambda}},\quad \lambda\ge 1,\ n\in\mathbf{N},$$

defined for an arbitrary $f\in C[0,1]$. Since by Szabados [98], Theorem 1 and Lemma 2, we have estimates for $\|S_{n,\lambda}-f\|$ and $\|S'_{n,\lambda}\|$, by applying the above method we immediately get

$$\begin{aligned}&\omega_1(S_{n,\lambda}(f);h)\\ &\le C\min\left\{hn^{2-\lambda}\int_{1/n}^1\frac{\omega_1(f;t)}{t^\lambda}dt,\ n^{1-\lambda}\int_{1/n}^1\frac{\omega_1(f;t)}{t^\lambda}dt+\omega_1(f;h)\right\}\\ &\le Ch^{\lambda-1}\int_h^1\frac{\omega_1(f;t)}{t^\lambda}dt,\quad 0<h<1<\lambda,\quad n\in\mathbf{N},\end{aligned}$$

where the constant C is independent of n. As an immediate consequence it follows the following.

Corollary 1.2.3. *(i) For* $1 < \lambda \leq 2$*,* $f \in \mathrm{Lip}_M(\alpha; [-1, 1])$*,* $0 < \alpha \leq 1$*, we get*

$$\omega_1(S_{n,\lambda}(f); h) = \begin{cases} \mathcal{O}(h^\alpha), & \text{if } \alpha < \lambda - 1, \\ \mathcal{O}\left(h^\alpha \log \frac{1}{h}\right), & \text{if } \alpha = \lambda - 1, \\ \mathcal{O}(h^{\lambda-1}), & \text{if } \lambda - 1 < \alpha. \end{cases}$$

(ii) If $\lambda > 2$ *then by*

$$\int_h^1 \frac{\omega_1(f; t)}{t^\lambda} dt \leq \frac{2\omega_1(f; t)}{h} \int_h^1 t^{1-\lambda} dt \leq C h^{1-\lambda} \omega_1(f; h),$$

we get the simpler relation

$$\omega_1(S_{n,\lambda}(f); h) \leq C\omega_1(f; h), \quad 0 < h < 1, \ \lambda > 2, \ n \in \mathbf{N}.$$

Remark. The above results show that if $\lambda > 2$, or $1 < \lambda \leq 2$ and $\alpha < \lambda - 1$, then the Shepard operators completely preserve the global smoothness of f. Also, the case $\lambda = 1$ remains unsolved, since in this case Lemma 2 in Szabados [98] does not give an estimate for $\|S'_{n,\lambda}\|$.

In what follows we consider the so-called Shepard–Lagrange operator studied in Coman–Trimbitas [26], defined here on the equidistant nodes $x_i = \frac{i}{n} \in [0, 1]$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \bigcup \{0\}$, $m < n$, $\lambda > 0$, as follows:

$$S_{n,\lambda,L_m}(f)(x) = \sum_{i=0}^{n} A_{i,\lambda}(x) L_{m,i}(f)(x),$$

where

$$A_{i,\lambda}(x) = \frac{|x - x_i|^{-\lambda}}{\sum_{k=0}^{n} |x - x_k|^{-\lambda}},$$

and

$$L_{m,i}(f)(x) = \sum_{j=0}^{m} \frac{u_i(x)}{(x - x_{i+j}) u'_i(x_{i+j})} f(x_{i+j}),$$

where $u_i(x) = (x - x_i) \cdots (x - x_{i+m})$ and $x_{n+\nu} = x_\nu$, $\nu = 1, \ldots, m$. Obviously the Shepard–Lagrange operator generalizes the classical Shepard operator, which is obtained for $m = 0$.

With respect to the classical Shepard operator in Corollary 1.2.3 which has the degree of exactness 0, the Shepard–Lagrange operator has the advantage that its degree of exactness is m (recall that an operator O is said to have the degree of exactness m, if $O(P) = P$, for any polynomial P of degree $\leq m$ and there exists a polynomial Q of degree $m + 1$, such that $O(Q) \neq Q$.)

Theorem 1.2.4. *Let us suppose that* $f : [0, 1] \to \mathbb{R}$*,* $f \in Lip_L \alpha$*, where* $0 < \alpha \leq 1$*. Then for all* $x \in [0, 1]$*,* $n \in \mathbb{N}$*,* $m \in \mathbb{N} \bigcup \{0\}$ *we have the estimates:*

(i)

$$|S_{n,\lambda,L_m}(f)(x) - f(x)| \leq C_m L E_\lambda^m(n),$$

where

$$E_\lambda^m(n) = \begin{cases} \frac{n^{2m+1-(m+1)\alpha}}{\log(n)}, \textit{ if } \lambda = 1, \\ n^{2m+2-(m+1)\alpha-\lambda}, \textit{ if } 1 < \lambda < m+2, \\ n^{m-(m+1)\alpha}\log(n), \textit{ if } \lambda = m+2, \\ n^{m-(m+1)\alpha}, \textit{ if } \lambda > m+2. \end{cases}$$

(ii)

$$|S'_{n,\lambda,L_m}(f)(x)| \le C\max\{nE_\lambda^m(n), n^m\},$$

for $\lambda \ge 2$*, even number.*

Proof. Everywhere in the proof we can consider $x \ne x_k$, $k = 0, \dots, n$, since in these cases $S_{n,\lambda,L_m}(f)(x_k) = f(x_k)$, $k = 0, \dots, n$ and the estimates are trivial.

(i) We have

$$|S_{n,\lambda,L_m}(f)(x) - f(x)| \le \sum_{i=0}^n A_i(x)|L_{m,i}(f)(x) - f(x)|$$

$$\le L\sum_{i=0}^n A_i(x)\left[\sum_{\nu=0}^m |x - x_i|\cdots|x - x_{i+\nu}|^\alpha\cdots|x - x_{i+m}|/|u_i'(x_{i+\nu})|\right].$$

It is easy to show that because of the equidistant nodes we have $|u_i'(x_{i+\nu})| \ge \frac{C_m}{n^m}$, $\forall i = 0, 1, \dots, n$, $\nu = 0, 1, \dots, m$ and from $|x - x_k| \le 1$, $\forall k$ it follows

$$|x - x_i|\cdots|x - x_{i+\nu}|^\alpha\cdots|x - x_{i+m}| \le |x - x_i|^\alpha\cdots|x - x_{i+\nu}|^\alpha\cdots|x - x_{i+m}|^\alpha.$$

Therefore,

$$|S_{n,\lambda,L_m}(f)(x) - f(x)| \le C_m L n^m \sum_{i=0}^n A_i(x)\frac{|x - x_i|\cdots|x - x_{i+m}|}{|x - x_i|^{1-\alpha}\cdots|x - x_{i+m}|^{1-\alpha}}.$$

Taking into account the relations (11) and (12) in Coman–Trimbitas [26], pp. 476–477 and denoting $\gamma(x, x_i) = |x - x_i|...|x - x_{i+m}|$, it follows $\frac{1}{|x-x_{i+l}|} \le \frac{1}{|2(j-l)-1|r}$ and

$$[1/\gamma(x, x_i)]^{1-\alpha} \le [1/r^{m+1}]^{1-\alpha}\left[1/\prod_{l=0}^m |2(j-l) - 1|\right]^{1-\alpha} \le [1/r^{m+1}]^{1-\alpha},$$

taking into account that $\prod_{l=0}^m |2(j-l) - 1| \ge 1$.

From Theorem 4 in Coman–Trimbitas [26], for the equidistant nodes we have there $M \le 3$, $r \sim \frac{1}{n}$ and reasoning as in the proof of the above mentioned Theorem 4, we get

$$|S_{n,\lambda,L_m}(f)(x) - f(x)| \le C_m L[1/r]^{m+(m+1)(1-\alpha)}\epsilon_\lambda^m(r),$$

where $\epsilon_\lambda^m(r)$ is given by the formula

$$\epsilon_{\lambda}^{m}(r) = \begin{cases} |\log(r)|^{-1}, \text{ if } \lambda = 1, \\ r^{\lambda-1}, \text{ if } 1 < \lambda < m+2, \\ r^{\lambda-1}|\log(r)|, \text{ if } \lambda = m+2, \\ r^{m+1}, \text{ if } \lambda > m+2. \end{cases}$$

All these prove (i).

(ii) We have

$$|S'_{n,\lambda,L_m}(f)(x)| \le |\sum_{i=0}^{n} A'(x)L_{m,i}(f)(x)|$$

$$+\left|\sum_{i=0}^{n} A_i(x)[L_{m,i}(f)(x)]'\right| := |E_1(x)| + |E_2(x)|,$$

where λ is considered even, i.e., $\lambda = 2p$, $p \in \mathbb{N}$.

Since

$$A'_i(x) = \frac{-2p(x-x_i)^{-2p-1}}{S} + \frac{2p(x-x_i)^{-2p}}{S} \frac{\sum_{k=0}^{n}(x-x_k)^{-2p-1}}{S},$$

where $S := \sum_{k=0}^{n}(x-x_k)^{-2p}$, we get

$$|A'_i(x)| \le 2p\frac{|x-x_i|^{-1}|x-x_i|^{-2p}}{S} + \frac{2p|x-x_i|^{-2p}}{S} \frac{\sum_{k=0}^{n}|x-x_k|^{-2p-1}}{S}.$$

For fixed $x \in [0,1]$, let x_d be the nearest point to x, i.e., $|x-x_d| := \min\{|x-x_i|; i = 0, 1, \ldots, n\} \le \frac{1}{n}$. It follows, for all $i \ne d$,

$$|A'_i(x)| \le 2pn\frac{|x-x_i|^{-2p}}{S}$$

and

$$|A'_d(x)| \le 2pn\frac{|x-x_d|^{-2p}}{S}.$$

Since $\sum_{k=0}^{n} A'_i(x) = 0$, we get

$$\sum_{i=0}^{n} A'_i(x)L_{m,i}(f)(x) = \sum_{i=0}^{n}[L_{m,i}(f)(x) - f(x)],$$

which implies

$$|E_1(x)| \le \sum_{i=0}^{n} |A'_i(x)||L_{m,i}(f)(x) - f(x)|$$

$$\leq Cn \sum_{i=0}^{n} |A_i(x)||L_{m,i}(f)(x) - f(x)| \leq CnE_\lambda^m(n),$$

(for the last inequality see (i)).

On the other hand, $[L_{m,i}(f)(x)]' \leq C_m n^m$, which implies $|E_2(x)| \leq C_m n^m$. As a conclusion,

$$|S'_{n,\lambda,L_m}(f)(x)| \leq C \max\{nE_\lambda^m(n), n^m\},$$

where C depends on f but it is independent of x and n. □

Remark. It is easy to see that the estimates in the expression of $E_\lambda^m(x)$ are not "good" for all the values of n, α and λ. For example, the worst situation seems to be in the $\lambda = 1$ case, when the estimate actually does not prove the convergence of $S_{n,\lambda,L_m}(f)(x)$ to $f(x)$.

Corollary 1.2.4 *Let us suppose that $f : [0,1] \to \mathbb{R}$, $f \in Lip_L\alpha$, where $0 < \alpha \leq 1$ and let us denote by $F_1(n,h)$ and $F_2(n,h)$, the estimates of $||S'_{n,\lambda,L_m}(f)||$ and $||S_{n,\lambda,L_m}(f) - f||$, respectively. We have:*

$$\omega_1(S_{n,\lambda,L_m}(f); h) \leq C_{m,f} \max\{hF_1(n,h), F_2(n,h) + \omega_1(f;h)\},$$

where n is the unique solution of equation $hF_1(n,h) = F_2(n,h)$. □

Proof. By the standard technique we obtain

$$\omega_1(S_{n,\lambda,L_m}(f); h)$$

$$\leq C_{m,f} \max\{h||S'_{n,\lambda,L_m}(f)(x)||, ||S_{n,\lambda,L_m}(f)(x) - f(x)|| + \omega_1(f;h)\},$$

for all $n \in \mathbb{N}$, $h \in (0,1)$. Combined with the estimates in Theorem 1.2.4, the standard method gives the optimal choice for n, as solution of the equation $hF_1(n,h) = F_2(n,h)$. □

Remark. Corollary 1.2.4 contains many global smoothness preservation properties, depending on the relations satisfied by λ, m and α. In some cases they are "good," in other cases are "bad." For example, let us suppose that $\lambda > m+2$, λ is even and $m > \frac{1-\alpha}{\alpha}$.

We get $nE_\lambda^m(n) = n^{(m+1)(1-\alpha)} < n^m$ and therefore we get the equation $hn^m = n^{m-(m+1)\alpha}$, which gives $n = h^{\frac{-1}{(m+1)\alpha}}$ and

$$\omega_1(S_{n,\lambda,L_m}(f); h)$$

$$\leq Ch \times (h^{\frac{-1}{(m+1)\alpha}})^m = Ch^{[(m+1)\alpha - m]/[(m+1)\alpha]}, \forall n \in \mathbb{N}, 0 < h < 1.$$

For a "good" property in this case, it is necessary to have $(m+1)\alpha - m > 0$, i.e., $m < \frac{\alpha}{1-\alpha}$. It follows the $\frac{\alpha}{1-\alpha} > \frac{1-\alpha}{\alpha}$, condition which necessarily implies $0 < \frac{1}{2} < \alpha < 1$. As a conclusion, if $\alpha < \frac{1}{2}$ then we don't have a "good" global smoothness preservation property.

As a concrete example, take $\alpha = \frac{4}{5}$. We obtain

$$\omega_1(S_{n,\lambda,L_m}(f); h) \leq Ch^{\frac{4-m}{4m+4}},$$

which for $m < 4$ and $\lambda > m + 2$, λ even, gives a "good" global smoothness preservation property.

In what follows we consider a sort of Shepard operator on the semi-axis $[0, +\infty)$, introduced and studied by the papers Della Vecchia–Mastroianni–Szabados [36]–[38].

Let us denote $C([0, +\infty]) = \{f : [0, +\infty) \to \mathbb{R}$; there exists $\lim_{x\to+\infty} f(x) = f(+\infty) \in \mathbb{R}\}$. If $f \in C([0, +\infty])$ then obviously the usual modulus of continuity $\omega(f; h) = \sup\{|f(x) - f(y)|; x, y \geq 0, |x - y| \leq h\}$ is finite and for the quantity $\epsilon_f(x) = \sup_{y\geq x} |f(+\infty) - f(y)|$ it follows that $\lim_{x\to+\infty} \epsilon_f(x) = 0$.

Putting $||f|| = \sup\{|f(x)|; x \geq 0\}$, let $\omega^{\Phi}(f; t) = \sup_{0<h\leq t} ||\Delta_{h\Phi}(f)||$ be the modulus of smoothness of f with step weight function $\Phi(x) = x^{1-\frac{1}{\gamma}}$, $\gamma \geq 1$.

Also, for the knots $x_k = \frac{k^\gamma}{n^{\gamma/2}}$, $k = 0, \ldots, n$, $\gamma \geq 1$, $s \geq 2$ and $f \in C([0, +\infty])$, let us consider the so-called Balázs–Shepard operator defined by

$$S_{n,s}(f)(x) = \frac{\sum_{k=0}^{n} |x - x_k|^{-s} f(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-s}}, x \geq 0.$$

The following estimates are known.

Theorem 1.2.5. (See Della Vecchia–Mastroianni–Szabados [37]) *Let us suppose that*

$$\limsup_{T>0} \left| \int_0^T [f(t) - f(+\infty)]/[\Phi(t)]dt \right| < +\infty$$

and

$$\epsilon_f(x) \leq \frac{C}{(1 + x)^{1/\gamma}}, \forall x \geq 0.$$

(i) *(See [37], Theorem 2.2) If* $s > 2$ *and* $0 < \alpha < 1$, *then* $\omega^{\Phi}(f; h) \leq Ch^\alpha$ *implies* $||S_{n,s}(f) - f|| \leq Cn^{-\alpha/2}$*;*

(ii) *(See [37], Theorem 2.4) Let* $s = 2$.

If $0 < \alpha < 1$, *then* $\omega^{\Phi}(f; h) \leq Ch^\alpha$ *implies* $||S_{n,s}(f) - f|| \leq Cn^{-\alpha/2}$ *and if* $\alpha = 1$ *then* $\omega^{\Phi}(f; h) \leq Ch$ *implies* $||S_{n,s}(f) - f|| \leq C\frac{\log(n)}{\sqrt{n}}$.

Theorem 1.2.6. (See Della Vecchia–Mastroianni–Szabados [37], Lemma 3.1, p. 446) *If* $s > 1$ *then* $||\Phi S'_{n,s}(f)|| \leq C\sqrt{n}||f||$.

Corollary 1.2.5. *Let us suppose that*

$$\limsup_{T>0} \left| \int_0^T [f(t) - f(+\infty)]/\Phi(t)dt \right| < +\infty$$

and

$$\epsilon_f(x) \leq \frac{C}{(1+x)^{1/\gamma}}, \forall x \geq 0.$$

(i) *If* $s > 2$ *and* $0 < \alpha < 1$, *then* $\omega^{\Phi}(f; h) \leq Ch^{\alpha}$ *implies*

$$\omega^{\Phi}(S_{n,s}(f); h) \leq Ch^{\alpha/(\alpha+1)},$$

for all $n \in \mathbb{N}, 0 < h < 1$, *where* $C > 0$ *is a constant independent of* n *and* h.

(ii) *If* $s = 2$ *and* $\omega^{\Phi}(f; h) \leq Ch^{\alpha}$ *then we have:*

$$\omega^{\Phi}(S_{n,s}(f); h) \leq Ch^{\alpha/(\alpha+1)} \text{ if } 0 < \alpha < 1$$

and

$$\omega^{\Phi}(S_{n,s}(f); h) \leq C[h \log(1/h)]^{1/2} \text{ if } \alpha = 1.$$

Both estimates take place for all $n \in \mathbb{N}, 0 < h < 1$.

Proof. (i) From the general inequality $\omega^{\Phi}(f; h) \leq Ch||\Phi f'||$, combined with Theorem 1.2.6, it follows $\omega^{\Phi}(S_{n,s}(f); h) \leq Ch\sqrt{n}||f||$. Combined now with the Theorem 1.2.5,(i), and by using the standard method, it immediately follows

$$\omega^{\Phi}(S_{n,s}(f); h) \leq C \max\{hn^{1/2}, n^{-\alpha/2} + h^{\alpha}\}.$$

From the equation $hn^{1/2} = n^{-\alpha/2}$, we get $n = h^{-2/(\alpha+1)}$ which replaced above gives the required estimate.

(ii) Let $s = 2$. If $0 < \alpha < 1$ then the proof follows similarly from Theorem 1.5.2,(ii). If $\alpha = 1$ then we get

$$\omega^{\Phi}(S_{n,s}(f); h) \leq C \max\{hn^{1/2}, \frac{\log(n)}{n^{1/2}} + h\},$$

which conducts to the equation $hn = \log(n)$. This implies $n = \frac{1}{h}\log(\frac{1}{h})$ and replacing it above, we get the desired estimate. The proof is complete. □

Unlike the results of this chapter which refer to real functions of one real variable, let us consider now the case of complex Hermite–Fejér interpolation polynomials on the roots of unity , attached to a complex function defined on the closed unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$.

Let us denote by $AC = \{f : \mathbb{D} \to \mathbb{C}; f$ is analytic on $\{|z| < 1\}$ and continuous on $\mathbb{D}\}$ and let $z_k = \exp^{2\pi ki/n}, k = 1, \ldots, n$ be the nth roots of unity. Then according to a result of Cavaretta–Sharma–Varga [21], there exists a uniquely determined complex polynomial $L_n(f)(z)$ of degree at most $2n - 1$, such that $L_n(f)(z_k) = f(z_k), k = 1, \ldots, n$ and $L_n'(f)(z_k) = 0, k = 1, \ldots, n$.

Denoting by $w(f; h) = \sup\{|f(\exp^{ix}) - f(\exp^{i(x+t)})|; x \in [0, 2\pi], 0 \leq t < h\}$, by Theorem 1 in Sharma–Szabados [89] the following estimate

$$||L_n(f) - f|| \leq C\left[w\left(f; \frac{1}{n}\right) + E_{[cn]}(f) \log(n)\right]$$

holds, where $||\cdot||$ represents the uniform norm on $\mathbb{D}$, C and c are independent of f and n and $E_n(f)$ denotes the error of best polynomial approximation.

Also, recall (see Rubel–Shields–Taylor [84]) that $w(f;h)$ is equivalent to the usual modulus of continuity on $\mathbb{D}$ defined by

$$\omega(f;h) = \sup\{|f(z_1) - f(z_2)|; z_1, z_2 \in \mathbb{D}, |z_1 - z_2| \leq h\}.$$

Therefore, we have the estimate

$$||L_n(f) - f|| \leq C\left[\omega\left(f; \frac{1}{n}\right) + E_{[cn]}(f)\log(n)\right].$$

We present

Theorem 1.2.7. *If $f \in AC$ and there exists $0 < \alpha \leq 1$ such that $\omega(f;h) \leq Ch^{\alpha}, \forall 0 \leq h < 1$ then*

$$\omega(L_n(f);h) \leq C_{f,\alpha}h^{\alpha/(1+\alpha)}\log\left(\frac{1}{h}\right), \quad \forall n \in \mathbb{N}, \quad 0 < h < 1,$$

where the constant $C_{f,\alpha} > 0$ is independent of n and h.

Proof. By the above error estimate we immediately obtain

$$||L_n(f) - f|| \leq C\left[\frac{1}{n^{\alpha}} + \frac{\log(n)}{n^{\alpha}}\right].$$

Now, let us prove that the standard method in the real case, works in the complex case too. Indeed, for $z_1, z_2 \in \mathbb{D}$, we obtain

$$|L_n(f)(z_1) - L_n(f)(z_2)| \leq |L_n(f)(z_1) - f(z_1)| + |f(z_1) - f(z_2)|$$
$$+|f(z_2) - L_n(f)(z_2)|,$$

which immediately implies

$$\omega(L_n(f);h) \leq 2||L_n(f) - f|| + \omega(f;h) \leq C_{\alpha}\left[\frac{\log(n)}{n^{\alpha}} + h^{\alpha}\right].$$

On the other hand, by the mean value theorem in the complex case, there exists ξ on the segment $[z_1, z_2]$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, such that

$$L_n(f)(z_1) - L_n(f)(z_2) = \lambda \cdot L_n'(f)(\xi)(z_1 - z_2),$$

which implies

$$|L_n(f)(z_1) - L_n(f)(z_2)| \leq |L_n'(f)(\xi)||z_1 - z_2|,$$

and therefore

$$\omega(L_n(f);h) \leq ||L_n'(f)|| \cdot h.$$

But by, e.g., Corollary 1.3 in DeVore–Lorentz [41], p. 98, we have $||L'_n(f)|| \leq Cn||L_n(f)||$. Combining it with relation (3.7), page 46 in Sharma–Szabados [89], we obtain $||L'_n(f)|| \leq Cn\log(n)||f||$.

As a conclusion, it follows $\omega(L_n(f); h) \leq C_f hn\log(n)$.

The best choice for n follows from the equation $hn\log(n) = \frac{\log(n)}{n^\alpha}$, which implies $n = h^{-1/(1+\alpha)}$ and finally the estimate $\omega(L_n(f); h) \leq C_{f,\alpha}h^{\alpha/(1+\alpha)}\log(\frac{1}{h})$, $\forall n \in \mathbb{N}$, $0 < h < 1$. The theorem is proved. □

At the end of this section, let us make some considerations on the interpolation of normed spaces valued mappings, of real variable. More exactly, let $(X, ||\cdot||)$ be a normed space over K, where $K = \mathbb{R}$ or $K = \mathbb{C}$ and $f : [a, b] \to X$, $[a, b] \subset \mathbb{R}$. First let us recall that in the recent paper [54], for $f : [0, 1] \to X$ continuous on $[0, 1]$ the following estimates hold:

(i)

$$C_1\omega_2^\phi\left(f; \frac{1}{\sqrt{n}}\right) \leq ||B_n(f) - f||_u \leq C_2\omega_2^\phi\left(f; \frac{1}{\sqrt{n}}\right),$$

where $||f||_u = \sup\{||f(x)||; x \in [0, 1]\}$ and $C_1, C_2 > 0$ are absolute constants.

(ii)

$$\begin{aligned} C_1\left[\omega_2^\phi\left(f; \frac{1}{\sqrt{n}}\right) + \omega_1\left(f; \frac{1}{n}\right)\right] &\leq ||K_n(f) - f||_u \\ &\leq C_2\left[\omega_2^\phi\left(f; \frac{1}{\sqrt{n}}\right) + \omega_1\left(f; \frac{1}{n}\right)\right]; \end{aligned}$$

(iii)

$$||B_n(f)(x) - f(x)|| \leq M\left[\frac{x(1-x)}{n}\right]^{\alpha/2} \forall x \in [0, 1], \text{ if and only if}$$

$$\omega_2(f; \delta) = O(\delta^\alpha),$$

where $\alpha \leq 2$.

(iv)

$$\omega_1(B_n(f); \delta) \leq 2\omega_1(f; \delta),$$

for all $n \in \mathbb{N}$ and all $\delta \leq 1$.

Here the usual first-order modulus of continuity $\omega_1(f; \delta)$, the usual second-order modulus of smoothness and the second Ditzian–Totik modulus of smoothness are defined by $\omega_1(f; \delta) = \sup\{||f(v) - f(w)||; v, w \in [0, 1], |v - w| \leq \delta\}$,

$$\omega_2(f; \delta) = \sup\{\sup\{||f(x+h) - 2f(x) + f(x-h)||; x - h, x, x + h \in [0, 1],$$

$$h \in [0, \delta]\},$$

where $\delta \leq \frac{1}{2}$ and

$$\omega_2^\phi(f; \delta) = \sup\{\sup\{||f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x))||; x \in I_{2,h}\},$$

$h \in [0, \delta]\}$,

respectively, with $I_{2,h} = \left[-\frac{1-h^2}{1+h^2}, \frac{1-h^2}{1+h^2}\right]$, $\phi(x) = \sqrt{x(1-x)}$, $\delta \leq 1$, the Bernstein and Bernstein–Kantorovich type operators attached to f are defined by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

and

$$K_n(f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt,$$

respectively, with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and the integral $\int_a^b f(t)dt$ is defined as the limit for $m \to \infty$ in the norm $||\cdot||$, of the all (classical defined) Riemann sums $\sum_{i=0}^{m} (x_{i+1} - x_i) f(\xi_i)$.

We want to show that these ideas can be used for interpolation too. Thus, for $f : [-1, 1] \to X$ we can attach to f the Lagrange polynomials $L_n(f)$ based on the Chebyshev knots of second kind plus the endpoints $-1, +1$ given by $L_n(f) : [-1, 1] \to X$,

$$L_n(f)(x) = \sum_{k=1}^{n} f(x_k) l_k(x),$$

where $x_k = cos t_k$, $t_k = \frac{(k-1)\pi}{n-1}$, $k = 1, \ldots, n$ and

$$l_k(x) = \frac{(-1)^{k-1} \omega_n(x)}{(1 + \delta_{k1} + \delta_{kn})(n-1)(x - x_k)}, \quad k = \overline{1, n},$$

with $\omega_n(x) = \sin t \sin(n-1)t$, $x = \cos t$ and δ_{kj} is the Kronecker's symbol.

We can prove the following.

Theorem 1.2.8. *Let* $f : [-1, 1] \to X$ *be continuous on* $[-1, 1]$. *We have:*
(i)

$$||L_n(f)(x) - f(x)|| \leq C\omega_1\left(f; \frac{1}{n}\right) \log n, \forall n \in \mathbb{N}, \quad x \in [-1, 1];$$

(ii) *if* $||f(v) - f(w)|| \leq M|v - w|^{\alpha}, \forall v, w \in [-1, 1]$, *where* $0 < \alpha < 1$, *then* $\omega_1(L_n(f); h)$ *satisfies the same estimates in the Corollary 1.2.2, (i).*

Proof. For $x^* \in X^* = \{x^* : X \to K; x^* \text{ is linear and continuous }\}$ with $|||x^*||| \leq 1$, define $g : [-1, 1] \to \mathbb{R}$ by $g(x) = x^*[f(x)]$, $x \in [-1, 1]$.

(i) By the relation (1.10) we have

$$||g - L_n(g)|| \leq C\omega_1\left(g; \frac{1}{n}\right) \log n, \quad \forall n \in \mathbb{N},$$

where $||g|| = \sup\{|g(x)|; x \in [-1, 1]\}$.

By the linearity of x^* we get $g(x) - L_n(g)(x) = x^*[f(x) - L_n(f)(x)]$ and therefore

$$|x^*[f(x) - L_n(f)(x)]| \le C\omega_1\left(g; \frac{1}{n}\right)\log n, \quad \forall x \in [-1, 1].$$

On the other hand, by $|g(v) - g(w)| = |x^*[f(v) - f(w)]| \le |||x^*||| \cdot ||f(v) - f(w)|| \le ||f(v) - f(w)||$, we easily get that $\omega_1(g; \frac{1}{n}) \le \omega_1(f; \frac{1}{n})$.

Combined with the above inequality, we get

$$|x^*[f(x) - L_n(f)(x)]| \le C\omega_1\left(f; \frac{1}{n}\right)\log n, \quad \forall x \in [-1, 1].$$

For fixed $x \in [-1, 1]$, passing to supremum with $|||x^*||| \le 1$ and taking into account the well-known classical equality

$$||x|| = \sup\{|x^*(x)|; x^* \in X^*, |||x^*||| \le 1\}, \quad \forall x \in X,$$

it follows

$$||L_n(f)(x) - f(x)|| \le C\omega_1\left(f; \frac{1}{n}\right)\log n, \quad \forall n \in \mathbb{N}, \quad x \in [-1, 1];$$

(ii) We have $|g(v) - g(w)| = |x^*[f(v) - f(w)]| \le |||x^*||| \cdot ||f(v) - f(w)|| \le M|v - w|^\alpha$, for all $v, w \in [-1, 1]$. Then by Corollary 1.2.2, (i), it follows that $\omega_1(L_n(g); h) \le CE_n(h, \alpha)$, where C and $E_n(h, \alpha)$ obviously do not depend on x^*.

Let $h > 0$ and $v, w \in [-1, 1]$ with $|v - w| \le h$, be fixed. We have

$$|x^*[L_n(f)(v) - L_n(f)(w)]| = |L_n(g)(v) - L_n(g)(w)|$$

$$\le \omega_1(L_n(g); h) \le CE_n(h, \alpha).$$

Passing to supremum with $x^* \in X^*$, $|||x^*||| \le 1$ we get as at the above point (i), $||L_n(f)(v) - L_n(f)(w)|| \le CE_n(h, \alpha)$. Passing now to supremum with $v, w \in [-1, 1]$, $|v - w| \le h$, we obtain $\omega_1(L_n(f); h) \le CE_n(h, \alpha)$, which proves the theorem. □

1.3 Algebraic Projection Operators and the Global Smoothness Preservation Property

Let Π_n be the set of algebraic polynomials of degree at most n. It is well known that the algebraic projection operators $L_n : C[-1, 1] \to \Pi_n$ are bounded linear operators having the properties:

(i) $f \in C[-1, 1]$ implies $L_n(f) \in \Pi_n$;

(ii) $f \in \Pi_n$ implies $L_n(f) \equiv f$.

The following approximation result for L_n is known.

Theorem 1.3.1. (See Szabados [97], Theorem 2.) *If* $f^{(s)} \in C[-1,1]$, $s \in \mathbf{N} \cup \{0\}$, *then*

$$\|f^{(s)} - L_n^{(s)}(f)\| \le C_s n^{-s} \omega_1(f^{(s)}; 1/n) \cdot |||L_n^{(s)}|||^*, \quad n \in \mathbf{N},$$

where $\|\cdot\|$ *represents the uniform norm on* $[-1,1]$ *and*

$$|||L_n^{(s)}|||^* = \sup\{\|L_n^{(s)}(f)\|;\ f(x)(1-x^2)^{-s/2} \in C[-1,1],$$
$$\|f(x)(1-x^2)^{-s/2}\| = 1\}.$$

Remark. For $s = 0$ it is easy to see that

$$|||L_n^{(s)}|||^* = |||L_n||| = \sup\{\|L_n(f)\|; \quad f \in C[-1,1], \quad |f\| = 1\}$$

and the above relation becomes

$$\|f - L_n(f)\| \le C\omega_1(f; 1/n) \cdot |||L_n|||, \quad n \in \mathbf{N},$$

where $|||L_n||| \ge C \log n$. (See, e.g., Szabados–Vértesi [100], pp. 266 and 268).

First we need the following simple

Lemma 1.3.1. *If* $f \in L^p[a,b]$, $1 \le p \le \infty$ *and* $\{L_n(f)\}_n$ *is a sequence of approximation operators such that* $L_n(f) \in L^p[a,b]$, $n \in \mathbf{N}$, *then for all* $n, r \in \mathbf{N}$, $h \in \left[0, \frac{b-a}{r}\right]$, *we have*

$$\omega_r(L_n(f); h)_p \le 2^r \|f - L_n(f)\|_p + \omega_r(f; h)_p,$$

where $\omega_r(f; h)_p$ *represents the usual modulus of smoothness,* $\|\cdot\|_p$ *is the classical* L^p*-norm,* $L^\infty[a,b] \equiv C[a,b]$, $\omega_r(f; \cdot)_\infty \equiv \omega_r(f; \cdot)$.

Proof. Let first $1 \le p < +\infty$. For $x \in [a, b - rt]$ we have

$$\Delta_t^r L_n(f)(x) = \Delta_t^r [L_n(f) - f](x) + \Delta_t^r f(x)$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} [L_n(f)(x+kt) - f(x+kt)] + \Delta_t^r f(x).$$

This implies

$$\left\{\int_a^{b-rt} |\Delta_t^r L_n(f)(x)|^p dx\right\}^{1/p}$$

$$\le \left\{\int_a^{b-rt} \left[\sum_{k=0}^{r} \binom{r}{k} |L_n(f)(x+kt) - f(x+kt)| + |\Delta_t^r f(x)|\right]^p dx\right\}^{1/p}$$

(by Minkowski's inequality)

$$\le 2^r \|L_n(f) - f\|_p + \left\{\int_a^{b-rt} |\Delta_t^r f(x)|^p dx\right\}^{1/p}.$$

Passing now to supremum with $0 \le t \le h$ we get the lemma.
For $p = +\infty$ the proof is obvious. □

The first main result of this section is the following:

Theorem 1.3.2. *If $f^{(s)} \in C[-1, 1]$, $s \in \mathbf{N} \cup \{0\}$, then for all $n \in \mathbf{N}$ and $h \in (0, 1)$ we have*

$$\omega_1(L_n^{(s)}(f); h)$$
$$\le C_s \min\{h\|f\| \cdot |||L_n^{(s+1)}|||, n^{-s}\omega_1(f^{(s)}; 1/n) \cdot |||L_n^{(s)}|||^* + \omega_1(f^{(s)}; h)\},$$

where $|||L_n||| = \sup\left\{\frac{\|L_n(f)\|}{\|f\|};\ 0 \not\equiv f \in C[-1, 1]\right\}$.

Proof. By Lemma 1.3.1 written for $p = +\infty$, $r = 1$, $f \equiv f^{(s)}$ and $L_n^{(s)}(f)$, and by Theorem 1.3.1 we get

$$\omega_1(L_n^{(s)}(f); h) \le 2\|L_n^{(s)}(f) - f^{(s)}\| + \omega_1(f^{(s)}; h)$$
$$\le 2C_s n^{-s}\omega_1(f^{(s)}; 1/n) \cdot |||L_n^{(s)}|||^* + \omega_1(f^{(s)}; h).$$

On the other hand, we obtain

$$\omega_1(L_n^{(s)}(f); h) \le h\|f\| \cdot |||L_n^{(s+1)}|||,$$

which completes the proof. □

Corollary 1.3.1. *Let us assume that $f^{(s)} \in \mathrm{Lip}_M(\alpha; [-1, 1])$, $s \in \mathbf{N} \cup \{0\}$. Then the best possible result concerning the partial preservation of global smoothness of f by $L_n^{(s)}(f)$ which can be derived by Theorem 1.3.2 is*

$$\omega_1(L_n^{(s)}(f); h) = \mathcal{O}\left[h^{\frac{\alpha}{2s+2+\alpha}}\left(\log\frac{1}{h}\right)^{\frac{2s+2}{2s+2+\alpha}}\right], \quad n \in \mathbf{N},\ h \in (0, 1),$$

attained if simultaneously we have

$$|||L_n^{(s)}|||^* = \mathcal{O}(n^s \log n) \quad \textit{and} \quad |||L_n^{(s+1)}||| = \mathcal{O}(n^{2(s+1)}).$$

(All the constants in "$\mathcal{O}$" are independent of n and h).

Proof. By Berman [18], the estimate

$$|||L_n^{(s)}||| = \mathcal{O}(n^{2s}), \quad s \in \mathbf{N},$$

is the best possible, and by Szabados–Vértesi [100], Theorem 8.1, p. 266, the estimate

$$|||L_n^{(s)}|||^* = \mathcal{O}(n^s \log n), \quad s \in \mathbf{N} \cup \{0\},$$

is the best possible.

Replacing in Theorem 1.3.2, we get

$$\omega_1(L_n^{(s)}(f); h) \le C_{s,\alpha,f} \min\{hn^{2(s+1)}, n^{-\alpha}\log n + h^\alpha\},\ s \in \mathbf{N} \cup \{0\}.$$

By $hn^{2(s+1)} = n^{-\alpha}\log n$, reasoning exactly as in the proof of Corollary 1.2.2,(i), we choose $n = \left(\frac{1}{h}\log\frac{1}{h}\right)^{\frac{1}{2s+2+\alpha}}$ and we obtain

$$\omega_1(L_n^{(s)}(f); h) = \mathcal{O}\left[h^{1-\frac{2s+2}{2s+2+\alpha}}\left(\log\frac{1}{h}\right)^{\frac{2s+2}{2s+2+\alpha}}\right]$$

$$= \mathcal{O}\left[h^{\frac{\alpha}{2s+2+\alpha}}\left(\log\frac{1}{h}\right)^{\frac{2s+2}{2s+2+\alpha}}\right],$$

which proves the theorem. □

Remark. In Szabados [97], Lagrange interpolation operators $\{L_n(f)\}_n$ are constructed, such that $|||L_n^{(s)}|||^* = \mathcal{O}(n^s \log n)$, $s \in \mathbf{N}$. But we do not know if simultaneously we have $|||L_n^{(s+1)}||| = \mathcal{O}(n^{2(s+1)})$.

Corollary 1.3.2. *Let us denote*

$$G_s = \{f \in C[-1,1];\ f(x)(1-x^2)^{-s/2} \in C[-1,1],\ \|f(x)(1-x^2)^{-s/2}\| = 1\},$$

$s \in \mathbf{N}$.

If $L_n(f)(x)$ represents the Lagrange interpolating polynomial based on the roots of the polynomial $\Omega_n(x)$ introduced in Szabados–Vértesi [99], then for all $f \in G_{s+1}$ with $f^{(s)} \in \mathrm{Lip}_M(\alpha; [-1,1])$, $s \in \mathbf{N} \cup \{0\}$, we have

$$\omega_1(L_n^{(s)}(f); h) = \mathcal{O}\left(h^{\frac{\alpha}{s+1+\alpha}}\log\frac{1}{h}\right),\quad 0 < h < 1,\ n \in \mathbf{N},$$

where the constant which appears in "$\mathcal{O}$" is independent of n and h.

Proof. We have

$$|L_n^{(s)}(f)(x) - L_n^{(s)}(f)(y)| = |L_n^{(s+1)}(f)(\xi)| \cdot |x-y|$$

$$\le |||L_n^{(s+1)}|||^* h,$$

for all $|x-y| \le h$ and $f \in G_{s+1}$, which immediately implies

$$\omega_1(L_n^{(s)}(f); h) \le h|||L_n^{(s+1)}|||^*,\quad 0 < h < 1,\ f \in G_{s+1}.$$

Then, reasoning as in the proof of Theorem 1.3.2., we get

$$\omega_1(L_n^{(s)}(f); h)$$

$$\le C\min\{h|||L_n^{(s+1)}|||^*, n^{-s}\omega_1(f^{(s)}; 1/n) \cdot |||L_n^{(s)}|||^* + \omega_1(f^{(s)}; h)\}.$$

By Theorem 1 in Szabados–Vértesi [99], we have

$$|||L_n^{(s)}|||^* = \mathcal{O}(n^s \log n), \quad |||L_n^{(s+1)}|||^* = \mathcal{O}(n^{s+1} \log n),$$

which replaced in the above inequality gives

$$\omega_1(L_n^{(s)}(f); h) \le C \min\{hn^{s+1} \log n, n^{-\alpha} \log n + h^\alpha\}.$$

The equation $hn^{s+1} \log n = n^{-\alpha} \log n$ gives the best choice, $h = n^{-s-1-\alpha}$, and the standard technique proves the corollary. □

1.4 Global Smoothness Preservation by Jackson Trigonometric Interpolation Polynomials

The goal of this section is to show that the sequence of Jackson interpolation trigonometric polynomials given by

$$J_n(f)(x) = \frac{2}{n+1} \sum_{k=0}^{n} f(t_k)\phi_n(x - t_k), \quad t_k = \frac{2\pi k}{n+1},$$

where $f \in C_{2\pi}$ and $\Phi(x) = \frac{1}{n+1} \frac{\sin^2((n+1)x/2)}{2 sin^2(x/2)}$ have the property of (partial) global smoothness preservation with respect to the uniform and average modulus of continuity.

First we need:

Definition 1.4.1. (Stechkin [94], pp. 219–220.) Let $k \in \mathbf{N}$. One says that the function φ is of N^k-class if it satisfies the following conditions:

(i) φ is defined on $[0, \pi]$;

(ii) φ is nondecreasing on $[0, \pi]$;

(iii) $\varphi(0) = 0$ and $\lim_{t \to 0} \varphi(t) = 0$;

(iv) $\varphi(t) \ge C_1 h^k$, for all $t > 0$.

(v) there exists a constant $C_2 > 0$ such that $0 < t < s \le \pi$ implies $s^{-k}\varphi(s) \le C_2 t^{-k} \varphi(t)$.

For $f \in C_{2\pi}$ and $\varphi \in N^k$ one says that f belongs to the $H_k(\varphi)$ class if

$$\omega_k(f; t) \le C_3 \varphi(t), \text{ for all } t \in [0, \pi],$$

where $\omega_k(f; t)$ represents the usual uniform modulus of smoothness of order k of f.

Remark. An obvious example of $\varphi \in N^k$ is $\varphi(t) = \omega_k(f; t)$, with fixed $f \in C_{2\pi}$.

Theorem 1.4.1. (Stechkin [94], Theorem 6, p. 230.) *Let $k \in \mathbf{N}$, $\varphi \in N^k$ and $f \in H_k(\varphi)$. If $\{T_n\}_n$ is a sequence of trigonometric polynomials with degree $T_n \le n$, which satisfy*

$$\|f - T_n\| \le C_4 \varphi(1/n), \text{ for all } n \in \mathbf{N},$$

then

$$\omega_k(T_n; h) \le C_5\varphi(h), \textit{ for all } h > 0, \; n \in \mathbf{N},$$

where $\|\cdot\|$ *represents the uniform norm and* C_4, C_5 *are absolute positive constants (independent of* f *and* n*).*

With respect to the uniform modulus of continuity, we present:

Theorem 1.4.2. *Let* $f \in C_{2\pi}$*. If* $f \in \text{Lip}_M \alpha$*,* $0 < \alpha < 1$*, (i.e.,* $\omega_1(f; h) \le Ch^\alpha$*) then*

$$\omega_1(J_n(f); h) \le Ch^\alpha, \quad h > 0, \; n \in \mathbf{N},$$

and if $f \in \text{Lip}_M 1$*, then*

$$\omega_1(J_n(f); h) \le Ch \log \frac{1}{h}, \quad h \in (0, 1), \; n \in \mathbf{N}.$$

Proof. By Szabados [95] the estimate

$$\|f - J_n(f)\| \le C[\omega_1(f; 1/n) + \omega(\tilde{f}; 1/n)], \quad n \in \mathbf{N},$$

holds, where $\tilde{f}$ represents the trigonometric conjugate of f. Let $f \in \text{Lip}_M \alpha$, $0 < \alpha \le 1$. If $\alpha < 1$, then it is known (see, e.g., Bari–Stechkin [9], p. 485) that this is equivalent to $\tilde{f} \in \text{Lip}_{\overline{M}} \alpha$, which, by the above estimate and by Theorem 1.4.1 (applied to $\varphi(h) = h^\alpha$) gives $\omega_1(J_n(f); h) \le Ch^\alpha$, $h > 0$, $n \in \mathbf{N}$. Also, if $\alpha = 1$, then by, e.g., Zygmund [110], p. 157 it follows that $\omega_1(\tilde{f}; h) \le Mh \log \frac{1}{h}$, which again with the above estimate and with Theorem 1.4.1 (applied to $\varphi(h) = h \log \frac{1}{h}$) yields

$$\omega(J_n(f); h) \le Ch \log \frac{1}{h}, \quad h \in (0, 1), \; n \in \mathbf{N}. \qquad \square$$

Now let us introduce the so-called average modulus of continuity by the following.

Definition 1.4.2. (Sendov [87]) The local modulus of continuity of a bounded 2π-periodic Riemann integrable function $f(x)$ is given by

$$\omega_1(f, x, h) = \sup\{|f(x') - f(x")|; x', x" \in [x - h/2, x + h/2]\},$$

while the average modulus of continuity in $L^p_{2\pi}$ is given by

$$\tau(f; h)_{L^p} = ||\omega(f, x, h)||_{L^p}, 1 \le p < \infty.$$

Remark. These concepts are very useful in some approximation theoretical problems where the ordinary L^p-modulus of continuity $\omega_1(f, h)_{L^p} = \sup\{||f(x + t) - f(x)||_{L^p}; |t| \le h\}$ is not applicable.

First we prove the following estimates.

Theorem 1.4.3. *For the Jackson trigonometric interpolation polynomials* $J_n(f)(x)$ *defined above, it follows:*

(i)

$$||J_n'(f)||_{L^1} \leq C\sum_{k=1}^{n} \tau\left(f; \frac{1}{k}\right)_{L^1};$$

(ii)

$$\tau(J_n(f); h)_{L^1} \leq \frac{C}{n}\sum_{k=1}^{n} \tau\left(f; \frac{2}{k}+h\right)_{L^1} + \tau(f; h)_{L^1}.$$

Proof. (i) Reasoning as in the proof of Theorem 1 in Popov–Szabados [78] we easily get

$$|J_n'(f)(x)| \leq \frac{C}{n}\sum_{k=0}^{n} |f(t_k) - f(x)||\Phi_n'(x-t_k)|.$$

But

$$\Phi_n'(u) = \frac{1}{2(n+1)}\left[(n+1)\sin\left(\frac{(n+1)u}{2}\right)\cos\left(\frac{(n+1)u}{2}\right)/\sin^2\left(\frac{u}{2}\right)\right.$$
$$\left. - \cos\left(\frac{u}{2}\right)\sin^2\left(\frac{(n+1)u}{2}\right)/\sin^3\left(\frac{u}{2}\right)\right].$$

Denoting by x_j the nearest node to x and taking into account that $|x-t_k| \sim \frac{|j-k|}{n}$, $\forall k \neq j$, similar with the proof of Theorem 1 in Popov–Szabados [78] we obtain

$$|\Phi_n'(x-t_k)| \leq C[n^2|j-k|^{-2} + n^2|j-k|^{-3}] \leq Cn^2|j-k|^{-2},$$

and

$$|\Phi_n'(u)| = |\frac{1}{n+1}\sum_{k=1}^{n+1}(n+1-k)\sin(ku)| \leq \frac{1}{n+1}\sum_{k=1}^{n}(n+1-k)k \leq Cn^2.$$

It follows that

$$|J_n'(f)(x)| \leq \frac{C}{n}[cn^2\omega\left(f, x, \frac{1}{n}\right)$$

$$+n^2\sum_{k=0, k\neq j}^{n} |j-k|^{-2}\omega(f, x, |x-t_k|)] \leq Cn\sum_{k=1}^{n} k^{-2}\omega(f, x, k/n).$$

Integrating and reasoning exactly as in the proof of Theorem 1 in Popov–Szabados [78], we obtain

$$||J_n'(f)||_{L^1} \leq Cn\sum_{k=1}^{n} k^{-2}\tau\left(f; \frac{k}{n}\right)_{L^1} \leq C\sum_{k=1}^{n}\tau(f; \frac{1}{n})_{L^1}.$$

(ii) Let $x', x" \in [x-\frac{h}{2}, x+\frac{h}{2}]$. We have

$$|J_n(f)(x') - J_n(f)(x")| \leq |J_n(f)(x') - f(x')|$$

$$+|J_n(f)(x") - f(x")| + |f(x') - f(x")|$$

(see the proof of Theorem 1 in Popov–Szabados [78])

$$C\sum_{i=1}^{n} i^{-2}\omega(f, x', i/n) + C\sum_{i=1}^{n} i^{-2}\omega(f, x", i/n) + |f(x') - f(x")|.$$

Passing to supremum after $x', x" \in [x - \frac{h}{2}, x + \frac{h}{2}]$, it follows that

$$\omega(J_n(f), x, h) \leq C\sum_{i=1}^{n} i^{-2} \sup\left\{\omega(f, x', i/n); x' \in \left[x - \frac{h}{2}, x + \frac{h}{2}\right]\right\}$$

$$+C\sum_{i=1}^{n} i^{-2} \sup\left\{\omega(f, x", i/n); x" \in \left[x - \frac{h}{2}, x + \frac{h}{2}\right]\right\} + \omega(f, x, h).$$

But

$$\sup\left\{\omega(f, x', i/n); x' \in \left[x - \frac{h}{2}, x + \frac{h}{2}\right]\right\}$$

$$= \sup\left\{\sup\{|f(u') - f(u")|; u', u" \in \left[x' - i/n, x' + i/n\right]\}; x' \in \left[x - \frac{h}{2}, x + \frac{h}{2}\right]\right\}$$

$$\leq \omega\left(f, x, \frac{2i}{n} + h\right).$$

Integrating on $[0, 2\pi]$, it follows that

$$\tau(J_n(f); h)_{L^1} \leq C\sum_{i=1}^{n} i^{-2}\tau\left(f; \frac{2i}{n} + h\right)_{L^1} + \tau(f; h)_{L^1} \leq$$

(reasoning as in the proof of Theorem 1 in Popov–Szabados [78])

$$\frac{C}{n}\int_{2/n}^{2} u^{-2}\tau(f; u + h)_{L^1} du + \tau(f; h)_{L^1} \leq \frac{C}{n}\int_{1/2}^{n/2} \tau(f; 1/v + h)dv$$

$$+\tau(f; h)_{L^1} \leq \frac{C}{n}\sum_{i=1}^{n} \tau\left(f; \frac{2}{i} + h\right)_{L^1} + \tau(f; h)_{L^1},$$

which proves the theorem. □

Corollary 1.4.1. *Let us suppose that* $\tau(f; h)_{L^1} \leq Mh^{\alpha}$, *for all* $0 < h < 1$, *where* $\alpha \in (0, 1]$. *We have:*

$$\tau(J_n(f); h)_{L^1} \leq Ch^{\alpha}, \forall h \in (0, 1), n \in \mathbb{N}, \text{ if } 0 < \alpha < 1$$

and

$$\tau(J_n(f); h)_{L^1} \le Ch \log\left(\frac{1}{h}\right), \forall h \in (0,1), n \in \mathbb{N}, \ \text{if}\, \alpha = 1.$$

Proof. By Sendov–Popov [88], the property $\tau(f; h)_{L^p} \le Ch||f'||_{L^p}$, $\forall h \in (0,1)$, holds. This and Theorem 1.4.3,(i), imply

$$\tau(J_n(f); h)_{L^1} \le Ch||J_n'(f)||_{L^1} \le Ch \sum_{k=1}^{n} 1/k^{\alpha} \le Chn^{1-\alpha}, \quad \forall h \in (0,1).$$

That is, $\tau(J_n(f); h)_{L^1} \le Chn^{1-\alpha}$, for $\alpha \in (0,1)$ and $\tau(J_n(f); h)_{L^1} \le Ch \log(n)$, for $\alpha = 1$. Also, by Theorem 1.4.3,(ii), for $\alpha \in (0,1)$ we get $\tau(J_n(f); h)_{L^1} \le \frac{C}{n}[\sum_{k=1}^{n} \frac{1}{k^{\alpha}} + nh^{\alpha}] + h^{\alpha} \le Cn^{-\alpha} + Ch^{\alpha}$, while for $\alpha = 1$ we have $\tau(J_n(f); h)_{L^1} \le C\frac{\log(n)}{n} + Ch$. The standard method gives the optimal choices of $n = h^{-1}$ from the equations $n^{-\alpha} = hn^{1-\alpha}$ and $\frac{\log(n)}{n} = h\log(n)$, corresponding to the cases $0 < \alpha < 1$ and $\alpha = 1$, respectively. Replacing in the above estimates, we get the statement of the theorem. □

1.5 Trigonometric Projection Operators and the Global Smoothness Preservation Property

Let us denote by $\mathcal{T}_n$ the set of trigonometric polynomials of degree at most n. It is well known that the trigonometric projection operators $P_n : C_{2\pi} \to \mathcal{T}_n$ are bounded linear operators having the properties:

(i) $f \in C_{2\pi}$ implies $P_n(f) \in \mathcal{T}_n$

(ii) $f \in \mathcal{T}_n$ implies $P_n(f) \equiv f$.

The following approximation result for P_n is known.

Theorem 1.5.1 (Runck–Szabados–Vértesi [85], relation (19).) *Let* $s \in \mathbf{N} \cup \{0\}$. *If* $f^{(s)} \in C_{2\pi}$ *then*

$$\|f^{(s)} - P_n^{(s)}(f)\| \le C_s n^{-s} \omega_1(f^{(s)}; 1/n) \cdot |||P_n^{(s)}|||, \quad n \in \mathbf{N},$$

where $\|\cdot\|$ *represents the uniform norm on* $\mathbf{R}$ *and*

$$|||P_n^{(s)}||| = \sup\left\{\frac{\|P_n^{(s)}(f)\|}{\|f\|};\ 0 \not\equiv f \in C_{2\pi}\right\}.$$

Also, we need the following result.

Lemma 1.5.1. *If* $f \in L_{2\pi}^p$, $1 \le p \le +\infty$, *and* $\{T_n(f)\}_n$ *is a sequence of approximation operators such that* $T_n(f) \in L_{2\pi}^p$, $n \in \mathbf{N}$, *then for all* $n, r \in \mathbf{N}$, $h > 0$, *we have*

$$\omega_r(T_n(f); h)_p \le 2^r \|T_n(f) - f\|_p + \omega_r(f; h)_p,$$

where $\omega_r(f;h)_p$ represents the periodic L^p-modulus of smoothness of order r and $\|\cdot\|_p$ is the classical $L^p_{2\pi}$-norm ($L^\infty_{2\pi} \equiv C_{2\pi}$, $\omega_r(f;\cdot)_\infty \equiv \omega_r(f;\cdot)$).

Proof. For $0 \le t$ and $x \in \mathbf{R}$, we have

$$\Delta_t^r[T_n(f)](x) = \Delta_t^r[T_n(f) - f](x) + \Delta_t^r f(x)$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k}[T_n(f)(x+kt) - f(x+kt)] + \Delta_t^r f(x).$$

If $1 \le p < +\infty$, then we obtain

$$\left\{\int_0^{2\pi} |\Delta_t^r[T_n(f)](x)|^p dx\right\}^{1/p} \le \left\{\int_0^{2\pi}\left[\sum_{k=0}^{r}\binom{r}{k}|T_n(f)(x+kt) - f(x+kt)|\right.\right.$$

$$\left.\left. + |\Delta_t^r f(x)|\right]^p dx\right\}^{1/p} \le \text{(by Minkowski's inequality)}$$

$$\le \sum_{k=0}^{r}\binom{r}{k}\left\{\int_0^{2\pi} |T_n(f)(x+kt) - f(x+kt)|^p dx\right\}^{1/p}$$

$$+\left\{\int_0^{2\pi} |\Delta_t^r f(x)|^p dx\right\}^{1/p} = \sum_{k=0}^{r}\binom{r}{k}\left\{\int_{kt}^{2\pi+kt} |T_n(f)(u) - f(u)|^p du\right\}^{1/p}$$

$$+\left\{\int_0^{2\pi} |\Delta_t^r f(x)|^p dx\right\}^{1/p} = \sum_{k=0}^{r}\binom{r}{k}\left\{\int_0^{2\pi} |T_n(f)(u) - f(u)|^p du\right\}^{1/p}$$

$$+\left\{\int_0^{2\pi} |\Delta_t^r f(x)|^p dx\right\}^{1/p} = 2^r\|T_n(f) - f\|_p + \left\{\int_0^{2\pi} |\Delta_t^r f(x)|^p dx\right\}^{1/p}.$$

Passing to supremum with $0 \le t \le h$ we easily get the statement for $1 \le p < +\infty$. For $p = +\infty$ the proof is obvious. □

Now, concerning the (partial) global smoothness preservation by $P_n(f)$ we can prove the following results.

Theorem 1.5.2. *If $f^{(s)} \in C_{2\pi}$, $s \in \mathbf{N} \cup \{0\}$, then for all $n \in \mathbf{N}$ and $h \in (0, 1)$ we have*

$$\omega_1(P_n^{(s)}(f);h)$$
$$\le C_s \min\{h\|f\| \cdot |||P_n^{(s+1)}|||, n^{-s}\omega_1(f^{(s)};1/n) \cdot |||P_n^{(s)}||| + \omega_1(f^{(s)};h\}.$$

Proof. By Lemma 1.5.1, written for $p = +\infty$, $r = 1$, $f \equiv f^{(s)}$ and $T_n(f) \equiv P_n^{(s)}(f)$, and by Theorem 1.5.1, we get

$$\omega_1(P_n^{(s)}(f);h) \le 2\|P_n^{(s)}(f) - f^{(s)}\| + \omega_1(f^{(s)};h)$$

$$\leq 2C_s n^{-s}\omega_1(f^{(s)}; 1/n) \cdot |||P_n^{(s)}||| + \omega_1(f^{(s)}; h).$$

On the other hand, if $|x - y| \leq h$, $x, y \in \mathbf{R}$, we have

$$|P_n^{(s)}(f)(x) - P_n^{(s)}(f)(y)| \leq |x - y| \cdot \|P_n^{(s+1)}(f)\| \leq h\|P_n^{(s+1)}(f)\|$$

$$\leq h\|f\| \cdot |||P_n^{(s+1)}|||.$$

Passing to supremum, we get

$$\omega_1(P_n^{(s)}(f); h) \leq h\|f\| \cdot |||P_n^{(s+1)}|||.$$

Collecting the inequalities, we immediately obtain

$$\omega_1(P_n^{(s)}(f); h)$$

$$\leq C_s \min\{h\|f\| \cdot |||P_n^{(s+1)}|||, n^{-s}\omega_1(f^{(s)}; 1/n) \cdot |||P_n^{(s)}||| + \omega_1(f^{(s)}; h)\},$$

which proves the theorem. □

Corollary 1.5.1. *Let us assume that $f^{(s)} \in \mathrm{Lip}_M\alpha$, $s \in \mathbf{N} \cup \{0\}$, $0 < \alpha \leq 1$. Then the best possible result concerning the partial preservation of global smoothness of f by $P_n^{(s)}(f)$ which can be derived by Theorem 1.5.2 is*

$$\omega_1(P_n^{(s)}(f); h) \leq \overline{M} h^{\frac{\alpha}{s+1+\alpha}} \log \frac{1}{h}, \quad 0 < h < 1, \quad n \in \mathbf{N},$$

attained if simultaneously we have

$$|||P_n^{(s)}||| = \mathcal{O}(n^s \log n) \quad \textit{and} \quad |||P_n^{(s+1)}||| = \mathcal{O}(n^{s+1} \log n).$$

Proof. By Berman [19], the estimates

$$|||P_n^{(s)}||| = \mathcal{O}(n^s \log n), \quad |||P_n^{(s+1)}||| = \mathcal{O}(n^{s+1} \log n)$$

are the best possible.

Replacing in Theorem 1.5.2, we obtain

$$\omega_1(P_n^{(s)}(f); h) \leq C_{s,\alpha,f} \min\{hn^{s+1} \log n, n^{-\alpha} \log n + h^\alpha\},$$

for all $n \in \mathbf{N}$, $h \in (0, 1)$.

By the equation $n^{-\alpha} \log n = hn^{s+1} \log n$ we get $h = n^{-s-1-\alpha}$. This is the best choice for h, because when $h < n^{-s-1-\alpha}$, the minimum in the above inequality is $hn^{s+1} \log n$, and when $h > n^{-s-1-\alpha}$, it is $n^{-\alpha} \log n + h^\alpha$.

As a conclusion, replacing $n = h^{-\frac{1}{s+1+\alpha}}$, we immediately obtain the corollary. □

1.6 Bibliographical Remarks and Open Problems

The Theorems 1.1.1, 1.2.1, 1.2.3, Corollaries 1.2.1, 1.2.2, Theorem 1.4.2 and Corollary 1.2.3 are from Gal–Szabados [56]. Lemma 1.3.1, Theorem 1.3.2, Corollaries 1.3.1, 1.3.2, Lemma 1.5.1, Theorem 1.5.2 and Corollary 1.5.1 are from the book Anastassiou–Gal [6]. Completely new are the following results: Theorems 1.2.2, 1.2.4, 1.2.7, Corollaries 1.2.4, 1.2.5, Theorem 1.2.8, Theorem 1.4.3, and Corollary 1.4.1.

Also, below are described several open problems which might be of interest for future research.

Open Problem 1.6.1. Construct a sequence of algebraic projection operators for which in Corollary 1.3.1 we have $|||L_n^{(s)}|||^* = \mathcal{O}(n^s \log n)$ and $|||L_n^{(s+1)}||| = \mathcal{O}(n^{2(s+1)})$, $n \in \mathbf{N}$.

Open Problem 1.6.2. Prove global smoothness preservation properties for the Balász–Shepard operator on an infinite interval (semi-axis) from Della Vecchia–Mastroianni–Szabados [38], defined on the knots $x_k = \frac{k^\gamma}{n^{\gamma/2}}$, $k = 0, 1, \ldots, n$, $\gamma \geq 1$, with respect to the modulus of continuity $\omega^\Phi(f; t)_w$.

We recall that in [38], the convergence behavior is considered with respect to the weight $w(x) = (1+x)^{-\beta}$ and $\omega^\Phi(f; t)_w = \sup_{0 \leq h \leq t}\{||w(x)[f(x + h\Phi(x)) - f(x)]||\}$ -the modulus of continuity of f with the step function $\Phi(x) = x^{1-1/\gamma}$.

Open Problem 1.6.3. Construct a sequence of trigonometric projection operators $\{P_n(f)\}_n$ (for example, a sequence of interpolating trigonometric polynomials) simultaneously satisfying

$$|||P_n^{(s)}||| = \mathcal{O}(n^s \log n), \quad |||P_n^{(s+1)}||| = \mathcal{O}(n^{s+1} \log n), \quad n \in \mathbf{N},$$

for a fixed $s \in \mathbf{N} \cup \{0\}$.

It is possible to be useful the results in the algebraic case in Szabados–Vértesi [99].

Open Problem 1.6.4. Let us consider the so-called Grünwald interpolation polynomials given by $G_n(f)(x) = \sum_{i=1}^n l_i^2(x) f(x_i)$, where $l_i(x) = \frac{l(x)}{(x-x_i)l'(x_i)}$, $l(x) = \prod_{i=1}^n (x - x_i)$, $x_i = \cos(2i-1)\pi/[2n]$.

In Jiang Gongjian [64] the following estimate of convergence is proved: if $f \in \mathrm{Lip}\alpha(0 < \alpha \leq 1)$, then

$$|G_n[f)(x) - f(x)| \leq \frac{4\sqrt{2}}{(1+x)n}(1 + \log n)|f(x)| + g_n(x), (-1 < x \leq 1),$$

where

$$g_n(x) \leq \frac{8}{1+x}\left(2 + \frac{1}{1-\alpha}\right)\left(\frac{\pi}{n}\right)^\alpha, \quad (0 < \alpha < 1)$$

and

$$g_n(x) \leq \frac{4\sqrt{2}}{(1+x)n}(1 + \log n), \quad (\alpha = 1).$$

Then proving an estimate for $||G_n'(f)||$ and using the standard method in this chapter, find the global smoothness preservation property with respect to the usual uniform modulus of continuity, satisfied by the Grünwald interpolation polynomials.

More generally, let us consider the positive linear operators introduced by Criscuolo–Mastroianni [31] of the form

$$V_m(A;\,\Phi;\,f;\,x) = \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\Phi_m^2(x_{m,k})} f(x_{m,k}) \Big/ \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\Phi_m^2(x_{m,k})},$$

where $A = (x_{m,k})_{k=1}^{m}$ is a triangular matrix of nodes, $l_{m,k}(x)$ are the corresponding fundamental polynomials of Lagrange interpolation, and $(\Phi_m(x))_{m=1}^{\infty}$ is a sequence of functions such that $\Phi(x_{m,i}) \neq 0$, $i = 1, ..., m$. Obviously V_m are combined Shepard–Grünwald operators.

The question is to find global smoothness preservation properties of these operators with respect to the usual uniform modulus of continuity.

Open Problem 1.6.5. Let $X = (x_{k,n})_{k=1}^{n+1}$ be an interpolatory matrix on $[a, b]$. For any $f \in C[a, b]$ and $x \in [a, b]$, let us consider the Shepard operator

$$S_{n,s}(X;\,f;\,x) = \frac{\displaystyle\sum_{k=0}^{n+1} f(x_{k,n})|x - x_{n,k}|^{-s}}{\displaystyle\sum_{k=0}^{n+1} |x - x_{k,n}|^{-s}}, \quad s > 1.$$

Let $\phi(x)$ be a function defined on $[a, b]$ such that $\phi \sim 1$ locally and $\phi(x) \sim (x-a)^{\alpha}$, $x \to a+$ and $\phi(x) \sim (b-x)^{\beta}$, $x \to b-$, where α and β are non-negative numbers. The behavior of convergence in terms of the so-called Ditzian–Totik modulus of continuity $\omega_1^{\phi}(f;\,h)$ is estimated by Della Vecchia–Mastroianni–Vértesi [40].

The question is to find global smoothness preservation properties of $S_{n,s}(X;\,f;\,x)$ with respect to the Ditzian–Totik modulus of continuity $\omega_1^{\phi}(f;\,h)$.

Open Problem 1.6.6. Let $f \in C_{2\pi}$ and $I_n(f)$ be the trigonometric Lagrange interpolation polynomial of degree n on equidistant nodes in $[0, 2\pi)$. A classical theorem of Marcinkiewicz and Zygmund (see, e.g., Zygmund [111]) shows that

$$||I_n(f) - f||_{L_p} \leq C\omega\left(f;\,\frac{1}{n}\right)_{\infty}.$$

On the other hand, we can get

$$\omega(I_n(f);\,h)_{L_p} \leq C||I_n(f) - f||_{L_p} + \omega(f;\,h)_{L_p}$$

and

$$\omega(I_n(f);\,h)_{L_p} \leq Ch||I_n'(f)||_{L_p}.$$

The problem is to find an estimate of $||I_n'(f)||_{L_p}$ in terms of the $\omega(f;\,\frac{1}{n})_{\infty}$ modulus, or more generally in terms of the $\omega(f;\,\frac{1}{n})_{L_p}$ modulus, the fact of which would imply a global smoothness preservation property of $I_n(f)$ with respect to the usual L_p-modulus of continuity.

Open Problem 1.6.7. In Prestin–Xu [83] it is proved (see p. 118, Corollary 3.2 there) that if $f \in W_1^p$ and $m_1 > 1$, then

$$||F_n(f) - f||_{L_p} \le C_p n^{-1} \omega_{m_1-1}\left(f'; \frac{1}{n}\right)_{L_p}, \quad 1 < p < \infty,$$

where $F_n(f)$ represents the $(0, m_1)$ trigonometric interpolation polynomials on the nodes $2k\pi/n, k = 0, \dots, n-1$.

An estimate of $||F_n'(f)||_{L_p}$ in terms of the $\omega(f; \frac{1}{n})_{L_p}$ modulus, would imply by the standard method of this chapter a global smoothness preservation property of $F_n(f)$, with respect to the usual L_p modulus of continuity.

Open Problem 1.6.8. According to a result of Prasad and Varma (see, e.g., the survey Szabados–Vértesi [101] and its references), for the Hermite–Fejér polynomial $H_n(f)$ based on the knots of Chebyshev of first kind, the estimate

$$||H_n(f) - f||_{L_{p,w}} \le C\omega\left(f; \frac{1}{n}\right)_{\infty}$$

holds, where $w(x) = (1 - x^2)^{-1/2}$ and

$$||f||_{L_{p,w}} = \left(\int_{-1}^{1} w(x)|f(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty.$$

Defining the modulus

$$\omega(f; h)_{L_1,w} = \sup\left\{\int_{-1+t}^{1-t} w(x)|f(x+t) - f(x-t)|dx; 0 \le t \le h\right\}, \ 0 < h < 1,$$

and possibly taking into account the estimate for $||H_n'(f)||$ in the proof of Theorem 1.2.1 (see relation (1.7) there), find a global smoothness preservation result for $H_n(f)$ through the above weighted modulus of continuity.

Similar problem for the mean convergence of Lagrange interpolation polynomials.

Open Problem 1.6.9. Let $J_n(f)(x)$ be the Jackson interpolation trigonometric polynomials considered by Section 1.4. Using the estimate of $||f - J_n(f)||_{L^p}$, $1 < p < +\infty$, in terms of the L^p-average modulus of continuity $\tau(f; h)_{L^p}$ in Theorem 2 in Popov–Szabados [78] and the standard method in this chapter, find the global smoothness preservation properties of $J_n(f)(x)$ in terms of $\tau(f; h)_{L^p}$, $p > 1$. (The case $p = 1$ was solved by Corollary 1.4.1.)

Open Problem 1.6.10. Let $S_{n,\lambda,L_m}(f)(x)$ be the Shepard–Lagrange operator attached to $f \in C[-1, 1]$ and to the roots of the sequence of orthogonal polynomials in Trimbitas [104]. Using the estimate of $||f - S_{n,\lambda,L_m}(f)||$ in Theorem 2 in [104] and the standard method in this chapter, find the global smoothness preservation properties of $S_{n,\lambda,L_m}(f)$.

Open Problem 1.6.11. Let the Shepard–Taylor operator, given by

$$S_{n,\lambda,T_m}(f)(x) = \sum_{i=-n}^{n} A_{i,\lambda}(x)T_{m,i}(f)(x),$$

where

$$A_{i,\lambda}(x) = \frac{|x - x_i|^{-\lambda}}{\sum_{k=-n}^{n} |x - x_k|^{-\lambda}},$$

and

$$T_{m,i}(f)(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_i)(x - x_i)^j}{j!},$$

defined on the special matrix of nodes in Della Vecchia–Mastroianni [33]. Using the estimates of the approximation error in [33] and the standard method in this chapter, find the global smoothness preservation properties of $S_{n,\lambda,T_m}(f)(x)$.

Open Problem 1.6.12. Theorem 1.2.7 suggests the following problem: Find convergence and global smoothness preservation properties on $\mathbb{D}$, of the complex Shepard operators on the roots of unity given by

$$S_{n,2p}(f)(z) = \sum_{k=1}^{n} A_{k,2p}(z)(f)(z_k),$$

where

$$A_{k,2p}(z) = \frac{(z - z_k)^{-2p}}{\sum_{j=1}^{n}(z - z_j)^{-2p}},$$

$p \in \mathbb{N}, z_k = \exp^{2\pi ki/n}$, $f \in AC$ and $i = \sqrt{-1}$.

Open Problem 1.6.13. In order to reduce the large amount of computation required by the classical Shepard operator given by the formula

$$S_{n,p}(f)(x) = \sum_{k=1}^{n} s_{k,p}(x) f(x_k),$$

where

$$s_{k,p}(x) = \frac{(x - x_k)^{-p}}{\sum_{j=1}^{n}(x - x_j)^{-p}},$$

$p \in \mathbb{N}$, in, e.g., [50], [53], [106] is considered the so-called local variant of it, given by the formula

$$W_{n,p}(f)(x) = \sum_{k=1}^{n} w_{k,p}(x) f(x_k),$$

where

$$w_{k,p}(x) = \frac{B_{k,p}(x)}{\sum_{j=1}^{n} B_{j,p}(x)},$$

with $B_{k,p}(x) = \frac{(R-|x-x_k|)_+^p}{R^p|x-x_k|^p}$, $R > 0$ constant, $p \in \mathbb{N}$.

Also, replacing in the expression of $W_{n,p}(f)(x)$ the values $f(x_k)$ by the values of some interpolation operators (like Lagrange or Taylor), in, e.g., [105] the local variants of Shepard–Lagrange and Shepard–Taylor operators are considered.

Finally, an interesting problem would be to find global smoothness preservation properties for all these local variants of Shepard operators, too.

2

Partial Shape Preservation, Univariate Case

In this chapter we present results concerning the shape-preserving property of some classical interpolation operators.

2.1 Introduction

We begin with the following simple

Definition 2.1.1. Let $C[a, b] = \{f : [a, b] \to \mathbf{R};\ f$ continuous on $[a, b]\}$ and $a \leq x_1 < x_2 < \cdots < x_n \leq b$, be fixed knots. A linear operator $U : C[a, b] \to C[a, b]$ is called of interpolation-type (on the knots $x_i,\ i = 1, \ldots, n$) if for any $f \in C[a, b]$ we have

$$U(f)(x_i) = f(x_i),\ \forall\, i = 1, \ldots, n.$$

Remark. Important particular cases of U are of the form

$$U_n(f)(x) = \sum_{k=1}^{n} f(x_k) P_k(x), \quad n \in \mathbf{N},$$

where $P_k \in C[a, b]$ satisfy $P_k(x_i) = 0$ if $k \neq i,\ P_k(x_i) = 1$, if $k = i$, and contain the classical Lagrange interpolation polynomials and Hermite–Fejér interpolation polynomials.

Now, if $f \in C[a, b]$ is, for example, monotone (or convex) on $[a, b]$, it is easy to note that because of the interpolation conditions, in general $U(f)$ cannot be monotone (or convex) on $[a, b]$. However there is a natural question if $U(f)$ remains monotone (or convex) on neighborhoods of some points in $[a, b]$. In this sense, let us introduce the following

Definition 2.1.2. Let $U : C[a, b] \to C[a, b]$ be a linear operator of interpolation-type on the knots $a \leq x_1 < \cdots < x_n \leq b$.

Let $y_0 \in (a, b)$. If for any $f \in C[a, b]$, nondecreasing on $[a, b]$, there exists a neighborhood of y_0, $V_f(y_0) = (y_0 - \varepsilon_f, y_0 + \varepsilon_f) \subset [a, b]$, $\varepsilon_f > 0$ (i.e., depending on f) such that $U(f)$ is nondecreasing on $V_f(y_0)$, then y_0 is called a point of weak preservation of partial monotony and, correspondingly, U is said to have the property of weak preservation of partial monotony (about y_0).

If the above neighborhood $V(y_0)$ does not depend on f, then y_0 is called a point of strong preservation of partial monotony.

Similar definitions hold if monotony is replaced by, e.g., convexity (of any order).

In connection with Definition 2.1.2, in a series of papers, T. Popoviciu proved the following negative and positive results.

Theorem 2.1.1. (i) (Popoviciu [80], p. 328). *Let $a \leq x_0 < x_1 < \cdots < x_m \leq b$ be fixed. If $m \geq n + 3$, (where $n \in \{-1, 0, 1, 2, 3\}$), then there do not exist in (a, b) points of strong preservation of partial convexity of order n, for the Lagrange interpolation polynomial (of degree $\leq m$), $L_m(x_0, \ldots, x_m; f)(x)$. Here convexity of order n means that (the divided difference) $[z_1, \ldots, z_{n+2}; f] \geq 0$, for all distinct $z_i \in [a, b]$, $i = 1, \ldots, n + 2$. The monotony corresponds to $n = 0$ and the usual convexity to $n = 1$.*

(ii) (Popoviciu [81], p. 81, Theorem VII). *If we denote by $F_n(f)(x)$, $n \in \mathbf{N}$, the classical Hermite–Fejér polynomial based on the roots $x_{i,n} \in (-1, 1)$, $i = 1, \ldots, n$, of the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ of degree n with $-1 \leq \alpha \leq 1$, $-1 \leq \beta \leq 1$, then each root $x'_{i,n}$, $i = 1, \ldots, n-1$, of the polynomial $l'(x)$, where $l(x) = \prod_{i=1}^{n}(x-x_{i,n})$, is a point of strong preservation of partial monotony for $F_n(f)$.*

(iii) (Popoviciu [81], p. 82, Theorem VIII.) *There do not exist in $(-1, 1)$ points of strong preservation of partial (usual) convexity for $F_n(f)$, $n \in \mathbf{N}$.*

(iv) (Popoviciu [81], p. 82, Theorem IX.) *If, for example,*

$$G_n(f)(x) = \sum_{i=1}^{n} f(x_i) \left[\frac{l(x)}{(x - x_i)l'(x_i)} \right]^2,$$

$$l(x) = \prod_{i=1}^{n}(x - x_i), \quad x_i = \cos\frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, n,$$

are the Grünwald interpolation polynomials, then there do not exist in $(-1, 1)$ points of strong preservation of partial monotony for $G_n(f)$, $n \geq 2$.

In Section 2.2 first we obtain quantitative estimates of the lengths of neighborhoods $V(x'_{i,n})$, $i = 1, \ldots, n - 1$, in Theorem 2.1.1 (ii). In contrast with Theorem 2.1.1(iii), it is proved that in case when $n \geq 3$ is odd, 0 is a point of weak preservation of partial strict-convexity for $F_n(f)$. The Kryloff–Stayermann polynomials are studied as well.

A related result for Grünwald polynomials is obtained and quantitative estimates of the neighborhoods of shape preservation are proved. Also, taking into account the relationship between the Hermite–Fejér polynomials $F_n(f)$ based on the Chebyshev nodes of first kind and the trigonometric Jackson interpolation polynomials $J_n(f)$, the shape preserving properties of $J_n(f)$ are deduced.

Finally, qualitative and quantitative results in partial monotony (or convexity) preserving approximation by several Shepard-type operators are obtained.

2.2 Hermite–Fejér and Grünwald-Type Polynomials

First we obtain a quantitative estimate for a particular case of Theorem 2.1.1,(ii) (the qualitative version of the result is proved in Popoviciu [79]).

Theorem 2.2.1. *Let $n = 2m$ be even and let us denote by $F_n(x)$ the classical Hermite–Fejér polynomial based on the roots $x_{i,n} \in (-1, 1)$, $i = 1, \ldots, n$, of λ-ultraspherical polynomials of degree n, with $\lambda > -1$ (i.e., Jacobi polynomials of degree n, with $\alpha = \beta$ and $\lambda = \alpha + \beta + 1$, $-1 < \alpha, \ \beta \leq 1$),*

$$-1 < x_{n,n} < \cdots < x_{n+1-m,n} < 0 < x_{m,n} < \cdots < x_{1,n} < -1.$$

There exists a constant $c > 0$ (independent of f and n) such that if $f : [-1, 1] \to \mathbf{R}$ is monotone on $[-1, 1]$, then $F_n(f)(x)$ is monotone (of the same monotonicity) in $\left(-\frac{c}{n^4}, \frac{c}{n^4}\right) \subset (-1, 1)$.

Proof. Let us denote $F_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$, where

$$h_{i,n}(x) = l_i^2(x)\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right],$$

$$l_i(x) = l(x)/[(x - x_{i,n})l'(x_{i,n})], \quad l(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

By, e.g., Popoviciu [79] we have

$$h_{i,n}(0) = l^2(0)[2 - (1 - \lambda)x_{i,n}^2]/[l'(x_{i,n})^2(1 - x_{i,n}^2)x_{i,n}^3], \tag{2.1}$$

for all $i = 1, \ldots, n$, and

$$F_n'(f)(x) = \sum_{i=1}^{n-1}[Q_i(x)][f(x_{i,n}) - f(x_{i+1,n})],$$

where $Q_i(x) = \sum_{j=1}^{i} h'_{j,n}(x), \ i = 1, \ldots, n-1$.

First we will prove

$$Q_i(0) \geq h'_{1,n}(0) > 0, \ \text{for all } i = 1, \ldots, n-1. \tag{2.2}$$

Indeed, by (2.1) we get

$$\text{sign}\,\{h'_{i,n}(0)\} = +1, \ \text{for all } i = 1, \ldots, m,$$

which immediately implies

$$Q_i(0) \geq h'_{1,n}(0), \ \text{for all } i = 1, \ldots, m.$$

Now, let $m+1 \leq i < n$. Then by (see, e.g., Popoviciu [79], p. 245) $h'_{i,n}(0) = -h'_{n+1-i,n}(0)$, $i = 1, \ldots, n-1$, we again get

$$Q_i(0) = \sum_{j=1}^{i} h'_{j,n}(0) > h'_{1,n}(0), \quad i = m+1, \ldots, n-1,$$

which proves (2.2). On the other hand, simple calculations show

$$h'_{1,n}(0) \geq \frac{c_1}{n^2} \quad (c_1 > 0, \text{ independent of } n). \tag{2.3}$$

From Szegö [102], §14.6, we have $\max_{|x| \leq \frac{1}{2}} \sum_{j=1}^{n} |h_{j,n}(x)| \leq c_2$. Applying Bernstein's inequality, we obtain

$$\max_{|x| \leq \frac{1}{4}} \left| \sum_{j=1}^{i} h'_{j,n}(x) \right| = \max_{|x| \leq \frac{1}{4}} |Q_i(x)| \leq c_3 n, \quad i = 1, \ldots, n-1$$

and

$$\max_{|x| \leq \frac{1}{8}} |Q'_i(x)| \leq c_4 n^2, \quad i = 1, \ldots, n-1. \tag{2.4}$$

Let d_i be the nearest root of $Q_i(x)$ to zero. By (2.2), (2.3) and (2.4) it follows that

$$\frac{c_1}{n^2} \leq |Q_i(0)| = |Q_i(d_i) - Q_i(0)| = |d_i| \cdot |Q'_i(z)| \leq c_4 |d_i| n^2,$$

i.e.,

$$|d_i| \geq \frac{c}{n^4}, \text{ for all } i = 1, \ldots, n-1,$$

which proves the theorem. □

Also, we can prove the following

Theorem 2.2.2. *Let us denote by* $F_n(f)(x)$, $n \in \mathbf{N}$, *the classical Hermite–Fejér polynomial based on the roots* $-1 < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < 1$, *of the Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$, *with* $\alpha, \beta \in (-1, 0]$. *If* ξ *is any root of the polynomial* $l'(x)$, *then there exists a constant* $c > 0$ *(independent of* n *and of* f*) such that if* $f : [-1, 1] \to \mathbf{R}$ *is monotone on* $[-1, 1]$, *then* $F_n(f)(x)$ *is monotone (of the same monotony) in*

$$\left(\xi - \frac{c_\xi}{n^{7+2\gamma}}, \xi + \frac{c_\xi}{n^{7+2\gamma}}\right) \subset (-1, 1), \text{ where } c_\xi = \frac{c}{(1-\xi^2)^{5/2+\delta}}, \ \gamma = \max\{\alpha, \beta\},$$

and

$$\delta = \begin{cases} \alpha, & \text{if } 0 \leq \xi < 1 \\ \beta, & \text{if } -1 < \xi \leq 0. \end{cases}$$

Proof. Keeping the notations in the proof of Theorem 2.2.1, and reasoning as in the proof of Lemma 3 in Popoviciu [81] we get

$$Q_i(\xi) > \min\{h'_{1,n}(\xi), -h'_{n,n}(\xi)\} > 0, \text{ for all } i = 1, \dots, n-1.$$

Let $a_n, b_n \in (0,1)$, $a_n, b_n \searrow 0$ (when $n \to +\infty$) be such that $|h'_{1,n}(\xi)| \geq c_1 a_n$, $|h'_{n,n}(\xi)| \geq c_2 b_n$, and $s_n = \min\{a_n, b_n\}$.

It easily follows $Q_i(\xi) \geq c_3 s_n$, $\quad i = 1, \dots, n-1$. By Szegö [102], Theorem 14.5, we have

$$\sum_{j=1}^{n} h_{j,n}(x) = 1, \ \forall\, x \in [-1,1],$$

where $h_{j,n}(x) \geq 0$, $\forall\, x \in [-1,1]$, $j = 1, \dots, n$.

Applying twice the Bernstein's inequality and reasoning exactly as in the proof of Theorem 2.2.1 (with ξ instead of 0), we obtain

$$Q_i(\xi) \leq c_1 |d_i - \xi| n^2/(1-\xi^2), \quad i = 1, \dots, n-1,$$

where d_i is the nearest root of $Q_i(x)$ to ξ, and therefore

$$\max_{|x-\xi| \leq a_\xi \frac{s_n}{n^2}} Q_i(x) > 0, \quad i = 1, \dots, n-1,$$

with $a_\xi = c_2(1-\xi^2)$.

It remains to find a (lower) estimate for s_n. First we have

$$|P_n^{(\alpha,\beta)}(\xi)| \geq \frac{c_3 n^{-1/2}}{(1-\xi)^{\delta/2+1/4}},$$

(see Theorem 8.21.8 in Szegö [102]).

By Popoviciu [81], p. 79, relation (27),

$$h'_{1,n}(\xi) = \frac{l^2(\xi)}{(x_{1,n}-\xi)^3 [l' x_{1,n})]^2} \left[2 + (x_{1,n} - \xi) \frac{l''(x_{1,n})}{l'(x_{1,n})} \right] > 0,$$

$$h'_{n,n}(\xi) = \frac{l^2(\xi)}{(x_{n,n}-\xi)^3 [l' x_{n,n})]^2} \left[2 + (x_{n,n} - \xi) \frac{l''(x_{n,n})}{l'(x_{n,n})} \right] < 0.$$

By Szegö [102], Theorem 14.5, $2 + (x_{i,n} - \xi)\frac{l''(x_{i,n})}{l'(x_{i,n})} \geq 1$ and by Szegö [102], (7.32.11),

$$h'_{1,n}(\xi) \geq \frac{l^2(\xi)}{(x_{1,n}-\xi)^3 [l'(x_{1,n})]^2} = \frac{[P_n^{(\alpha,\beta)}(\xi)]^2}{(x_{1,n}-\xi)^3 [P_n^{(\alpha,\beta)'}(x_{1,n})]^2}$$

$$\geq \frac{c_4 [P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1-\xi)^3},$$

(where $q = \max\{2+\alpha, 2+\beta\}$).

Also,

$$-h'_{n,n}(\xi) = |h'_{n,n}(\xi)| \geq \frac{c_5 [P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1+\xi)^3}.$$

Thus we obtain

$$Q_i(\xi) \geq \frac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}, \quad i = 1, \dots, n-1.$$

Finally, taking $s_n = \frac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}$ we easily obtain the theorem. □

Remarks. (1) If ξ is near the endpoints, i.e., $1-\xi^2 \sim \frac{1}{n^2}$, then the interval of preservation of the monotony is

$$\left(\xi - \frac{c}{n^{2+2(\gamma-\delta)}}, \xi + \frac{c}{n^{2+2(\gamma-\delta)}}\right) \subset (-1, 1).$$

For $\alpha = \beta \in (-1, 0)$, i.e., in the ultraspherical case, we obtain the best possible interval

$$\left(\xi - \frac{c}{n^2}, \xi + \frac{c}{n^2}\right) \subset (-1, 1).$$

(2) Let us consider, for example, the case $\alpha = \beta = -\frac{1}{2}$, i.e., $x_{i,n} = \cos\frac{2i-1}{2n}\pi$, $i = 1, \dots, n$ and $f(x) = e_1(x) = x$, $\forall\, x \in [-1, 1]$. It is known that $F_n(f)(x) = x - \frac{T_{2n-1}(x)+x}{2n}$, $x \in [-1, 1]$, where $T_{2n-1}(x) = \cos[(2n-1)\arccos x]$. We have

$$F_n'(e_1)(x) = 1 - \frac{T_{2n-1}'(x)+1}{2n}, \quad F_n'(e_1)(-x) = F_n'(e_1)(x),$$

$$F_n''(e_1)(x) = -T_{2n-1}''(x)/(2n), \quad F_n''(0) = 0.$$

It follows that the roots of $F_n'(e_1)(x)$ are symmetric in respect with 0 and the equation $F_n'(e_1)(x) = 0$ is equivalent with $T_{2n-1}'(x) = 2n-1$. This last one reduces to

$$\sin[(2n-1)t] = \sin t, \quad t \in (0, \pi),$$

where $x = \cos t$.

Because of symmetry, we are interested only in the positive roots, i.e., when $t \in \left(0, \frac{\pi}{2}\right]$.

For n even, 0 is not a root of $F_n'(e_1)(x)$, while for n odd 0 is a double root.

Simple calculations show that the positive roots of $F_n'(e_1)(x)$ are given by

$$x_k^{(1)} = \cos\frac{2k\pi}{2n-2}, \quad k = 1, \dots, \left[\frac{n-1}{2}\right], \quad x_k^{(2)} = \cos\frac{(2k+1)\pi}{2n},$$

$$k = 0, 1, \dots, \left[\frac{n-1}{2}\right], \quad n \geq 2.$$

It is easy to show that for all k,

$$x_k^{(1)} > x_k^{(2)} > x_{k+1}^{(1)} > \cdots$$

Because $x_1^{(1)} - x_1^{(2)} \sim \frac{1}{n^2}$, near the endpoints we obtain the best possible estimates. Now, if for example n is even, then $\left[\frac{n-1}{2}\right] = \frac{n-2}{2}$ and for $k = \frac{n-2}{2}$ we easily get

$$x_k^{(1)} - x_k^{(2)} \sim \frac{1}{n},$$

which means that near 0, the estimate for the interval of preservation of monotonicity is much better than those given by Theorems 2.2.1 and 2.2.2.

For $n \geq 3$ odd, let us denote by $F_n(f)(x)$ the Hermite–Fejér interpolation polynomial based on the roots $x_{i,n} \in (-1, 1),\ i = 1, \dots, n$, of λ-ultraspherical polynomials of degree $n,\ \lambda > -1,\ \lambda \neq 0$. Also, let us consider the Côtes–Christoffel numbers of the Gauss–Jacobi quadrature

$$\lambda_{i,n} := 2^{2-\lambda}\pi \left[\Gamma\left(\frac{\lambda}{2}\right)\right]^{-2} \frac{\Gamma(n+\lambda)}{\Gamma(n+1)}(1 - x_{i,n}^2)^{-1}[P_n^{(\lambda)'}(x_{i,n})]^{-2}, i = 1, \dots, n$$

and denote

$$\Delta_h^2 f(0) = f(h) - 2f(0) + f(-h).$$

In contrast to the negative result in Theorem 2.1.1,(iii), we have the following:

Theorem 2.2.3. *If* $f \in C[-1, 1]$ *satisfies*

$$\sum_{i=1}^{n} \frac{[\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)]}{x_{i,n}^2} > 0, \tag{2.5}$$

then $F_n(f)(x)$ *is strictly convex in* $[-|d_n|, |d_n|]$*, with*

$$|d_n| \geq \frac{c(\lambda) \sum_{i=1}^{(n-1)/2} [\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)]/x_{i,n}^2}{n^2\left[\omega_1\left(f; \frac{1}{n}\right) + \|f - F_n(f)\|\right]_I},$$

where $c(\lambda) > 0$ *is independent of* f *and* n*,* $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$*,* $\omega_1\left(f; \frac{1}{n}\right)_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$ *is the modulus of continuity on* $\left[-\frac{1}{2}, \frac{1}{2}\right]$ *and* $\|\cdot\|_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$ *is the uniform norm on* $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof. Denoting $F_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x)f(x_{i,n})$, we have

$$h_{i,n}''(x) = -4\frac{l''(x_{i,n})}{l'(x_{i,n})}l_i(x)l_i'(x) + 2[(l_i'(x))^2 + l_i(x)l_i''(x)]$$

$$\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right].$$

But $l_i(0) = 0$ and $l_i'(0) = -\frac{l'(0)}{x_{i,n}l'(x_{i,n})}$, for $i \neq (n+1)/2$ and

$$1 + x_{i,n}\frac{l''(x_{i,n})}{l'(x_{i,n})} = \frac{1 + \lambda x_{i,n}^2}{1 - x_{i,n}^2}, \quad i = 1, \dots, n \quad \text{(see, e.g., Popoviciu [79])}.$$

We obtain

$$h''_{i,n}(0) = \frac{2(l'(0))^2}{(l'(x_{i,n}))^2} \cdot \frac{1}{x_{i,n}^2} \left(\frac{1 + \lambda x_{i,n}^2}{1 - x_{i,n}^2} \right) > 0, \ \forall\, i \neq (n+1)/2. \qquad (2.6)$$

Also, because $x_{i,n} = -x_{n+1-i,n},\ i = 1, \ldots, n,\ l'(x_{i,n}) = l'(x_{n+1-i,n})$ (since n is odd) we easily get

$$h''_{i,n}(0) = h''_{n+1-i,n}(0).$$

But

$$\lambda_{i,n} = \frac{c_1 \lambda \Gamma(n+\lambda)}{\Gamma(n+1)} \cdot \frac{1}{(l'(x_{i,n}))^2} \cdot \frac{1}{1 - x_{i,n}^2} \ \text{ and } \ (l'(0))^2 \sim n^{\lambda},$$

which together with (2.6) implies

$$h''_{i,n}(0) \geq c_2 \lambda n \lambda_{i,n} / x_{i,n}^2, \ \text{ for all } i \neq (n+1)/2.$$

Therefore

$$F''_n(f)(0) = \sum_{i=1}^{(n-1)/2} h''_{i,n}(0) \Delta^2_{x_{i,n}} f(0) \geq c_3 \lambda n \sum_{i=1}^{n} \frac{\lambda_{i,n} \Delta^2_{x_{i,n}} f(0)}{x_{i,n}^2} > 0. \qquad (2.7)$$

By (2.7) it follows that $F_n(f)$ is strictly convex in a neighborhood of 0. Let d_n be the nearest root of $F''_n(f)$ to 0. We may assume that $|d_n| \leq \frac{c}{n}$ (since otherwise there is nothing to prove, the interval of convexity cannot be larger than $\left[-\frac{c}{n}, \frac{c}{n}\right]$). Then by the mean value theorem, Bernstein's inequality and Stechkin's inequality (see, e.g., Szabados–Vértesi [100], p. 284) we get

$$\begin{aligned} F''_n(f)(0) &= |F''_n(f)(0) - F''_n(f)(d_n)| = |d_n| \cdot |F'''_n(f)(y)| \\ &\leq |d_n| c_4 n^2 \|F'_n(f)\|_J \leq c_5 |d_n| n^3 \omega_1 \left(F_n(f); \frac{1}{n} \right)_I \\ &\leq c_5 |d_n| n^3 \left[\omega_1 \left(f; \frac{1}{n} \right) + \omega_1 \left(F_n(f) - f; \frac{1}{n} \right) \right]_I \\ &\leq c_5 |d_n| n^3 \left[\omega_1 \left(f; \frac{1}{n} \right) + \|f - F_n(f)\| \right]_I, \end{aligned}$$

where $J = \left[-\frac{1}{4}, \frac{1}{4}\right]$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Combining the last inequality with (2.7), the proof of the theorem is immediate. □

Remarks. (1) It is obvious that if f is strictly convex on $[-1, 1]$, then (2.5) is satisfied (from $\lambda_{i,n} > 0,\ i = 1, \ldots, n$). Therefore, because the lower estimate for $|d_n|$ depends on f, we can say that 0 is a point of weak preservation of partial strict-convexity. Let us suppose that 0 would be point of strong preservation of partial strict-convexity. For any $\varepsilon > 0,\ f_1(x) = \varepsilon x^2 + x$ and $f_2(x) = \varepsilon x^2 - x$ are strictly convex on $[-1, 1]$. By hypothesis we obtain that there exists a neighborhood of 0,

$V_n(0)$, independent of $\varepsilon > 0$ (here $n \geq 3$ is odd fixed), such that $F_n''(f_1)(x) > 0$, $F_n''(f_2)(x) > 0$, $\forall\, x \in V_n(0)$. Passing with $\varepsilon \searrow 0$, we obtain $F_n''(e_1)(x) \geq 0$ and $F_n''(e_1)(x) \leq 0$, $\forall\, x \in V_n(0)$, i.e., $F_n''(e_1) = 0$, $\forall\, x \in V_n(0)$, where $e_1(x) = x$, $x \in [-1, 1]$. But this means that degree $[F_n(e_1)(x)] \leq 1$, $x \in V_n(0)$, which is a contradiction, because $F_n(e_1)(x) = x - \frac{T_{2n-1}(x)+x}{2n}$, $T_m(x) = \cos[m \arccos(x)]$.

As a conclusion, 0 cannot be a point of strong type for strict-convexity and $F_n(f)$.

Let us note that also the strict convexity of f on $[-1, 1]$ can be replaced by the weaker condition $\Delta_h^2 f(0) > 0$, for all $h \in (0, 1]$.

(2) More explicit estimate for $|d_n|$ in Theorem 2.2.3 can be obtained in the following particular case.

Let us suppose that f is strongly convex on $[-1, 1]$ (i.e., there exists $\gamma > 0$ such that $\eta f(x) + (1 - \eta) f(y) - f(\eta + (1 - \eta)y) \geq \eta(1 - \eta)\gamma(x - y)^2$, for all $\eta \in [0, 1]$, $x, y \in [-1, 1]$) and that $f \in \text{Lip}\,\alpha$, with $0 < \alpha \leq 1$.

It easily follows that $\Delta_{x_{i,n}}^2 f(0)/x_{i,n}^2 \geq 2\gamma$, for all $i = 1, \ldots, (n-1)/2$. Also, by Szegö [102], relation (15.3.10), we have

$$\lambda_i \sim i^{\lambda}/n^{\lambda+1}, \text{ for all } i = 1, \ldots, n, \tag{2.8}$$

which immediately implies

$$\sum_{i=1}^{(n-1)/2} \lambda_{i,n} \geq c, \tag{2.9}$$

$c > 0$, constant independent of n.

For $\|f - F_n(f)\|_I$ we can use various estimates, (see, e.g., Szabados–Vértesi [100])

$$\|f - F_n(f)\|_I \leq c \sum_{k=0}^{n} \omega_1\left(f; \frac{k}{n}\right) k^{\lambda-2}, \quad \text{if } \lambda > 0.$$

Taking into account all the above estimates, by the estimate in Theorem 2.2.3 we obtain

$$|d_n| \geq c\lambda\gamma / \left\{ n^2 \left[\frac{1}{n^{\alpha}} + \sum_{i=0}^{n} \left(\frac{i}{n}\right)^{\alpha} i^{\lambda-2} \right] \right\}, \text{ if } \lambda > 0,$$

where the constant c (with $c\lambda > 0$) is independent of n but depends on f.

If for example $0 < \lambda + \alpha < 1$, by $\sum_{i=0}^{n} i^{\alpha+\lambda-2} \leq c$, we obtain

$$|d_n| \geq \frac{c\lambda\gamma}{n^{2-\alpha}}, \quad c, \lambda > 0.$$

(3) Two open questions appear in a natural way:

(i) If n is odd then find other points of weak preservation of partial strict-convexity for $F_n(f)$;

(ii) What happens if n is even?

(4) The dependence of $|d_n|$ of f in Theorem 2.2.3 can be dropped in a simple way for some subclasses of functions. Thus, let us define (for $a \in (0, 1]$)

$$B^*[-a, a] = \{f : [-1, 1] \to \mathbf{R};\ f \text{ is strictly convex on } [-a, a],$$

$$f(x) \geq 0,\ x \in [-a, a],\ f(0) = 0\}.$$

Let $n \geq 3$ be odd. Because $h_{i,n}(0) = h'_{i,n}(0) = 0$ and $h''_{i,n}(0) > 0$, for all $i \in \{1, \ldots, n\} \setminus \left\{\frac{n+1}{2}\right\}$, obviously there exists $\varepsilon_n > 0$ such that

$$h''_{i,n}(x) > 0,\ \text{for all } x \in [-\varepsilon_n, \varepsilon_n] \text{ and all } i \in \{1, \ldots, n\} \setminus \left\{\frac{n+1}{2}\right\}.$$

For all $x \in [-\varepsilon_n, \varepsilon_n]$ and all $f \in B^*[-1, 1]$ we get

$$F''_n(f)(x) = \sum_{i=1}^{n} h''_{i,n}(x) f(x_{i,n}) = \sum_{i=1 i \neq \frac{n+1}{2}}^{n} h''_{i,n}(x) f(x_{i,n})$$

$$\geq \sum_{j=1}^{(n-1)/2} h^*(x)[f(x_{j,n}) + f(x_{n+1-j,n})] > h^*(x)\frac{n-1}{2} 2f(0) = 0,$$

where $h^*(x) = \min\left\{h''_{i,n}(x);\ i \in \{1, \ldots, n\} \setminus \left\{\frac{n+1}{2}\right\}\right\} > 0,\ \forall\ x \in [-\varepsilon_n, \varepsilon_n]$, ε_n being independent of $f \in B^*[-1, 1]$.

On the other hand, it easily follows that $h_{i,n}(x) \geq 0$, for all $x \in [-\varepsilon_n, \varepsilon_n]$, i.e., $F_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n}) \geq 0$, for all $x \in [-\varepsilon_n, \varepsilon_n]$, $f \in B^*[-1, 1]$ and as a conclusion, $F_n(f) \in B^*[-\varepsilon_n, \varepsilon_n]$.

Now, let us consider the Hermite–Fejér type interpolation with quadruples nodes introduced by Kryloff–Stayermann [68] (see also, e.g., Gonska [59]), given by $K_n(f)(x) = \sum_{k=1}^{n} h_k(x) f(x_k)$, where $x_k = \cos \frac{2k-1}{2n}\pi$, $T_n(x) = \cos[n \arccos x]$ and $h_k(x) = \frac{p_k(x)}{(x-x_k)^4} T_n^4(x)$,

$$p_k(x) = \frac{1}{n^4}\left\{(1 - x x_k)^2 + \frac{1}{6}(x - x_k)^2[(4n^2 - 1)(1 - x x_k) - 3]\right\}.$$

It is known that $K_n(f)(x_k) = f(x_k)$, $K_n^{(i)}(f)(x_k) = 0, i = 1, 2, 3,\ k = 1, \ldots, n$, $\sum_{k=1}^{n} h_k(x) \equiv 1$. This implies $\sum_{k=1}^{n} h'_k(x) \equiv 0$ and reasoning exactly as in Popoviciu [79], p. 242, by $-1 < x_n < x_{n-1} < \cdots < x_1 < 1$ we obtain

$$K'_n(f)(x) = \sum_{i=1}^{n-1} [Q_i(x)][f(x_i) - f(x_{i+1})],$$

where $Q_i(x) = \sum_{j=1}^{i} h'_j(x)$.

First we present two results of qualitative type.

Theorem 2.2.4. *If n is even number (i.e., $n = 2m$), then there exists a neighborhood of 0 where $K_n(f)(x)$ preserves the monotonicity of f.*

Proof. We have $T'_n(0) = 0$ and

$$h'_k(x) = \frac{p_k(x)}{(x - x_k)^4} \cdot 4T_n^3(x)T'_n(x) + \frac{T_n^4(x)}{(x - x_k)^5}[p'_k(x)(x - x_k) - 4p_k(x)],$$

$$p_k'(x) = \frac{1}{n^4}\left\{-2x_k(1-xx_k) + \frac{1}{3}(x-x_k)[(4n^2-1)(1-xx_k)-3] + \frac{(x-x_k)^2}{6}(-x_k)(4n^2-1)\right\},$$

$$h_k'(0) = \frac{1}{x_k^5}[4p_k(0) + x_k p_k'(0)], \quad k = \overline{1,n},$$

$$p_k(0) = \frac{1}{n^4}\left[1 + \frac{2}{3}x_k^2(n^2-1)\right] > 0,$$

$$p_k'(0) = -\frac{1}{n^4}\left[2x_k + \frac{4}{3}x_k(n^2-1) + \frac{x_k^3}{6}(4n^2-1)\right], \quad k = \overline{1,n}.$$

By $x_{n+1-k} = -x_k,\ k = \overline{1,m}$, we get $p_{n+1-k}(0) = p_k(0)$, $p'_{n+1-k}(0) = -p_k'(0)$ and consequently $h'_{n+1-k}(0) = -h_k'(0),\ k = \overline{1,m}$.

By simple calculations

$$4p_k(0) + x_k p_k'(0) = \frac{1}{6n^4}[n^2(8x_k^2 - 4x_k^4) + 24 - 20x_k^2 + x_k^4] > 0,$$

which implies $h_k'(0) < 0$, if $x_k < 0$ and $h_k'(0) > 0$, if $x_k > 0$, i.e., $h_k'(0) > 0$ if $k = \overline{1,m}$ and $h_k'(0) < 0$, if $k = \overline{m+1,2m}$.

Reasoning now exactly as in Popoviciu [79], p. 245, we get $Q_i(0) > 0$, for all $i = \overline{1,2m-1}$. This implies that there exist neighborhoods of 0, $V_i(0)$, such that $Q_i(x) > 0,\ \forall\, x \in V_i(0),\ i = \overline{1,2m-1}$. Denoting $V(0) = \bigcap_{i=1}^{2m-1} V_i(0)$, we get $Q_i(x) > 0,\ \forall\, x \in V(0),\ i = \overline{1,2m-1}$, which proves the theorem. □

Theorem 2.2.5. *Let $n \geq 3$ be odd. Then there exists a neighborhood of 0 where $K_n(f)(x)$ preserves the strict-convexity of f.*

Proof. We can write

$$h_k'(x) = p_k(x)\frac{4T_n^3(x)T_n'(x)}{(x-x_k)^4} + T_n^4(x)E_1(x),$$

$$h_k''(x) = 12p_k(x)\frac{T_n^2(x)[T_n'(x)]^2}{(x-x_k)^4} + T_n^3(x)E_2(x),$$

$$h_k'''(x) = 24p_k(x)\frac{T_n(x)[T_n'(x)]^3}{(x-x_k)^4} + T_n^2(x)E_3(x),$$

$$h_k^{(4)}(x) = 24p_k(x)\frac{[T_n'(x)]^4}{(x-x_k)^4} + T_n(x)E_4(x).$$

For $n \geq 3$ odd and $k \neq \frac{n+1}{2}$ we obtain

$$h_k(0) = h_k'(0) = h_k''(0) = h_k'''(0) = 0 \text{ and } h_k^{(4)}(0)$$

$$= \frac{24}{x_k^4}\left[1 + \frac{2}{3}x_k^2(n^2-1)\right] > 0.$$

This implies $h_{n+1-k}^{(4)}(0) = h_k^{(4)}(0)$ and therefore

$$K_n^{(4)}(f)(0) = \sum_{i=1}^{n} h_k^{(4)}(0)f(x_k) = \sum_{j=1}^{(n-1)/2} [f(x_j) + f(x_{n+1-j})]h_j^{(4)}(0)$$

$$+f(0)h_{\frac{n+1}{2}}^{(4)}(0) > f(0)\sum_{j=1}^{(n-1)/2} 2h_j''(0) + f(0)h_{\frac{n+1}{2}}''(0) = f(0)\sum_{i=1}^{n} h_i''(0) = 0.$$

Because $K_n^{(i)}(f)(0) = 0,\ i =, 1, 2, 3$, it follows that 0 is minimum point for $K_n''(f)(x)$ and $K_n''(f)$ is strictly convex in a neighborhood of 0, $V(0)$. As a conclusion, $K_n''(f)(x) > 0,\ \forall\, x \in V(0)\setminus\{0\}$, which shows that $K_n(f)$ is strictly convex on $V(0)$.

Obviously here $V(0)$ depends on f (and of n of course). The theorem is proved. □

Remark. Theorem 2.2.5 remains valid if the strict-convexity of f on $[-1, 1]$ is replaced by the weaker condition

$$\Delta_h^2 f(0) = f(h) + f(-h) - 2f(0) > 0,\ \forall\, h \in (0, 1].$$

The quantitative version of Theorem 2.2.4 is the following.

Theorem 2.2.6. *Let n be even. There exists a constant* $c > 0$ *(independent of f and n) such that if* $f : [-1, 1] \to \mathbb{R}$ *is monotone on* $[-1, 1]$*, then* $K_n(f)$ *is of the same monotonicity in* $\left(-\frac{c}{n^4}, \frac{c}{n^4}\right) \subset (-1, 1)$.

Proof. By the proof of Theorem 2.2.4 we easily get

$$Q_i(0) \geq h_1'(0) > 0,\ \forall\, i = \overline{1, n-1}.$$

But $h_1'(0) = \frac{T_n^4(0)}{x_1^5 6n^4}[n^2(8x_1^2 - 4x_1^4) + 24 - 20x_1^2 + x_1^4] \geq \frac{4n^2x_1^2}{6n^4x_1^5} \geq \frac{2}{3n^2}$ (because $T_n^4(0) = 1,\ 8x_1^2 - 4x_1^4 > 4x_1^2$ and $24 - 20x_1^2 + x_1^4 > 0$).

On the other hand, by $0 \leq h_k(x),\ k = \overline{1, n},\ \sum_{k=1}^{n} h_k(x) = 1,\ \forall\, x \in [-1, 1]$ and by Bernstein's inequality we obtain (reasoning as in the proof of Theorem 2.2.1)

$$\max_{|x|\leq\frac{1}{4}} |Q_i(x)| \leq \max_{|x|\leq\frac{1}{4}} \frac{n\left\|\sum_{k=1}^{i} h_k\right\|}{\sqrt{1-x^2}} \leq c_1 n,\ i = \overline{1, n-1}$$

and

$$\max_{|x|\leq\frac{1}{8}} |Q_i'(x)| \leq c_2 n^2, \quad i = \overline{1, n-1},$$

where $c_1, c_2 > 0$ are independent of n.

Now, if d_i is the nearest root of $Q_i(x)$ to zero, we get

$$\frac{2}{3n^2} \le Q_i(0) = |Q_i(0)| = |Q_i(0) - Q_i(d_i)| = |d_i| \cdot |Q_i'(z)| \le c_2 |d_i| n^2,$$

i.e., $|d_i| \ge \frac{c}{n^4}$, for all $i = \overline{1, n-1}$, which proves the theorem. □

The quantitative version of Theorem 2.2.5 is the following.

Theorem 2.2.7. *Let $n \ge 3$ be odd. If $f : [-1, 1] \to \mathbb{R}$ satisfies*

$$\Delta_h^2 f(0) = f(h) + f(-h) - 2f(0) > 0, \ \forall\, h \in (0, 1],$$

then $K_n(f)(x)$ is strictly convex in $[-|d_n|, |d_n|]$, with

$$|d_n| \ge \frac{c \sum_{i=1}^{(n-1)/2} \Delta_{x_i}^2 f(0)/x_i^2}{n^3 \left[\omega_1\left(f; \frac{1}{n}\right) + \|f - K_n(f)\|\right]_I}, \quad I = \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Proof. By the proof of Theorem 2.2.5 we have

$$K_n^{(4)}(f)(0) = \sum_{i=1}^{(n-1)/2} h_i^{(4)}(0) \Delta_{x_i}^2 f(0),$$

where $h_i^{(4)}(0) = \frac{24}{x_i^4}\left[1 + \frac{2}{3} x_i^2 (n^2 - 1)\right] \ge \frac{n^2}{x_i^2}$, i.e.,

$$K_n^{(4)}(f)(0) \ge n^2 \sum_{i=1}^{(n-1)/2} \frac{\Delta_{x_i}^2 f(0)}{x_i^2} > 0.$$

Let d_n be the nearest root of $K_n^{(4)}(f)(x)$ to 0.

Reasoning as in the proof of Theorem 2.2.3, we have

$$\begin{aligned} K_n^{(4)}(f)(0) &= |K_n^{(4)}(f)(0) - K_n^{(4)}(f)(d_n)| = |d_n| \cdot |K_n^{(5)}(f)(y)| \\ &\le |d_n| c n^4 \|F_n'(f)\|_J \le c_1 |d_n| n^5 \omega_1\left(K_n(f); \frac{1}{n}\right)_I \\ &\le c_2 |d_n| n^5 \left[\omega_1\left(f; \frac{1}{n}\right) + \|f - K_n(f)\|\right]_I, \end{aligned}$$

where $J = \left[-\frac{1}{4}, \frac{1}{4}\right]$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Combining with the previous inequality, we obtain

$$|d_n| \ge \frac{c \sum_{i=1}^{(n-1)/2} \Delta_{x_i}^2 f(0)/x_i^2}{n^3 \left[\omega_1\left(f; \frac{1}{n}\right) + \|f - K_n(f)\|\right]_I}, \quad I = \left[-\frac{1}{2}, \frac{1}{2}\right],$$

where $c > 0$ is independent of n and f and $\omega_1(f;\cdot)_I$, $\|\cdot\|_I$ represents the corresponding concepts on I. □

Remarks. (1) For upper estimates of $\|f - K_n(f)\|$ can be used, for example, Gonska [59].

(2) Comparing Theorem 2.2.6 with Theorem 2.2.1, we see that although the polynomials $K_n(f)(x)$ satisfy higher-order Hermite–Fejér conditions, still the obtained estimate for the length of preservation of monotonicity is the same as that given by the classical Hermite–Fejér polynomials.

(3) As was pointed out in the case of classical Hermite–Fejér polynomials, there are also other points (different from 0) where the monotonicity of f is preserved. Therefore it is natural to look for other points of this type in the case of the polynomials $K_n(f)(x)$, too.

In contrast with the negative result in Theorem 2.1.1, (iv), in the case of convexity we can prove a shape-preserving property of Grünwald interpolation polynomials based on Chebyshev nodes of the first kind. In this sense, first we need the following simple result.

Lemma 2.2.1. *If* $x_k = \cos\frac{(2k-1)\pi}{2n}$, $k = 1, \dots, n$ *and* $l_k(x) = \frac{l(x)}{(x-x_k)l'_k(x_k)}$ *are the fundamental Lagrange interpolation polynomials (where* $l(x) = (x-x_1)\cdots(x-x_k)$*), then*

$$\left\| \sum_{k=1}^{n} |l_k(x)|^p \right\| \le C_p, \quad \forall p > 1,$$

and

$$\left\| \sum_{k=1}^{n} |l_k(x)|^p \right\| \le C_p n^{1-p}, \quad \forall 0 < p < 1,$$

where C_p *is a positive constant depending only on* p.

Proof. Let $x \in [-1, 1]$ be fixed and the index j defined by $|x - x_j| := \min\{|x - x_k|; 1 \le k \le n\}$. By the relations in Szabados–Vértesi [100], p. 282 (see also the relations (1.6) in the proof of Theorem 1.2.1), denoting $x = cos(t)$, $t_k = \frac{(2k-1)\pi}{2n}$ and $T_n(x) = \cos(n \arccos(x))$, for all $k \ne j$ we get

$$|l_k(x)| = \frac{|T_n(x)||\sin(t_k)|}{n|\cos(t) - \cos(t_k)|} \le \frac{\frac{k}{n}}{n\frac{|j^2-k^2|}{n^2}} \le \frac{C}{|j-k|}.$$

Also, $l_j(x_j) = 1$ and

$$|l_j(x)| = \frac{|T_n(x)||\sin(t_j)|}{n|\cos(t) - \cos(t_j)|} \le \frac{1}{n}\frac{\frac{Cj}{n}}{\frac{Cj}{n^2}} = C.$$

As a conclusion,

$$\sum_{k=1}^{n} |l_k(x)|^p \le C_p \sum_{k=1,k\ne j}^{n} \frac{1}{|k-j|^p} + C_p,$$

which immediately proves the lemma. □

Remark. In the case when $p = 2$ and the fundamental Lagrange interpolation polynomials are based on the roots of α-ultraspherical Jacobi polynomials of degree n, $\alpha \in (-1, 0)$, in Szegö [102], Problem 58, can be found the relation $|| \sum_{k=1}^{n} |l_k(x)|^2 || \leq \frac{1}{|\alpha|}$.

We present:

Theorem 2.2.8. *Let $n \geq 3$ be odd. If $\Delta_h^2 f(0) > 0$, $\forall\, h \in (0, 1]$, then the Grünwald polynomials $G_n(f)(x)$ on the Chebyshev nodes of first kind (given by Theorem 2.1.1 (iv)) are strictly convex on $[-|d_n|, |d_n|]$, where for $|d_n|$ we can have either of the following two lower estimates:*

(i)

$$|d_n| \geq C \frac{\sum_{k=1}^{(n-1)/2} \frac{2(1-x_k^2)}{x_k^2} \Delta_{x_k}^2 f(0)}{n^3 ||f||};$$

(ii)

$$|d_n| \geq C \frac{\sum_{k=1}^{(n-1)/2} \frac{2(1-x_k^2)}{x_k^2} \Delta_{x_k}^2 f(0)}{n^3 [\omega_1(f; \frac{1}{n}) + ||G_n(f) - f||]_I}.$$

Proof. It is known that

$$l_k(x) = (-1)^k \frac{\sqrt{1-x_k^2}}{x - x_k} \cdot \frac{T_n(x)}{n}.$$

Denoting $h_k(x) = l_k^2(x)$, we get $h_k'(0) = 2l_k(0)l_k'(0)$, $h_k''(0) = 2(l_k'(0))^2 + 2l_k(0)l_k''(0)$. Now, for $k \neq \frac{n+1}{2}$ we get $x_k \neq 0$ and $l_k(0) = 0$,

$$l_k'(x) = \frac{(-1)^k \sqrt{1-x_k^2}}{n} \cdot \frac{T_n'(x)(x - x_k) - T_n(x)}{(x - x_k)^2},$$

$$[l_k'(0)]^2 = \frac{1-x_k^2}{x_k^2}, \quad h_k'(0) = 0, \quad h_k''(0) = 2\frac{1-x_k^2}{x_k^2} = h_{n+1-k}''(0) > 0,$$

because $x_j = -x_{n+1-j}$.

Since $\sum_{k=1}^{n} h_k''(0) = 0$, it follows

$$h_{(n+1)/2}''(0) = -4 \sum_{k=1}^{(n-1)/2} \frac{1-x_k^2}{x_k^2}$$

and therefore

$$G_n''(f)(0) = \sum_{k=0}^{n} h_k''(0) f(x_k) = \sum_{k=1}^{(n-1)/2} \frac{2(1-x_k^2)}{x_k^2}[f(x_k) + f(x_{n+1-k})]$$

$$+ f(0) h''_{\frac{n+1}{2}}(0) = \sum_{k=1}^{(n-1)/2} \frac{2(1-x_k^2)}{x_k^2} \Delta_{x_k}^2 f(0).$$

On the other hand, if we denote d_n the nearest root to 0 of $G_n''(f)(x)$, then by reasonings similar to those in the proof of Theorem 2.2.3 and by the above Lemma 2.2.1, we get

$$G_n''(f)(0) = |G_n''(f)(0) - G_n''(f)(d_n)| = |d_n||G_n'''(\eta)|$$
$$\leq C|d_n|n^3||G_n|| \leq C|d_n|n^3||f||.$$

It immediately follows the estimate in (i).

Also, by reasonings similar to those in the proof of Theorem 2.2.3 (but without to use the Lemma 2.2.1), we get

$$G_n''(f)(0) = |G_n''(f)(0) - G_n''(f)(d_n)| = |d_n||G_n'''(\eta)|$$
$$\leq C|d_n|n^2||G_n'||_J \leq C|d_n|n^3\omega_1(G_n(f); \frac{1}{n})_I$$
$$\leq C|d_n|n^3[\omega_1(f; \frac{1}{n}) + \omega_1(G_n(f) - f; \frac{1}{n})]_I$$
$$\leq C|d_n|n^3 \left[\omega_1\left(f; \frac{1}{n}\right) + ||G_n(f) - f||\right]_I,$$

where $J = [-1/4, 1/4]$, $I = [-1/2, 1/2]$.

This proves the estimate (ii), too. □

Remarks. (1) Theorem 2.2.8 shows that 0 is a point of weak preservation of partial strict-convexity for the Grünwald polynomials. Because $G_n(e_1)(x),\ n \geq 3$ odd, is a polynomial of degree > 1, (here $e_1(x) = x,\ x \in [-1, 1]$), reasoning exactly as in Remark 1 of Theorem 2.2.3, we obtain that 0 cannot be point of strong preservation of partial strict-convexity for $G_n(f)$.

(2) For the estimate in Theorem 2.2.8 (ii), might be useful the degree of approximation in, e.g., Jiang Gongjian [64] and Sheng–Cheng [90].

We end the section with some remarks on the shape-preserving properties of trigonometric interpolation polynomials. They are based on the well-known relationship between the $J_n(f)(x)$ in Theorem 1.4.2 and the algebraic Hermite–Fejér polynomials $F_n(f)(x)$ on the Chebyshev knots of the first kind, given by (see, e.g., Szabados [95])

$$J_{2n-1}(f)(x) = F_n(g)(\cos(x)), x \in \mathbb{R},$$

where $f(t) = g(\cos(t))$.

This relation allows us to extend the properties of $F_n(f)$ in Theorems 2.2.1, 2.2.2 and 2.2.3, to $J_{2n-1}(f)$. First we need the following concept.

Definition 2.2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic. f is called periodically monotone if it is increasing in an interval (x_1, x_2) and decreasing in $(x_2, x_1 + 2\pi)$.

A particular periodically monotone function is the so-called bell-shaped function, which is 2π-periodic, even and decreasing on $[0, \pi]$ (i.e., $x_1 = \pi$ and $x_2 = 2\pi$).

Remark. The most natural bell-shaped function f can be generated by an increasing function $g : [-1, 1] \to \mathbb{R}$, defining $f(x) = g(\cos(x))$, $x \in \mathbb{R}$. Actually it is easy to show that g is increasing on $[-1, 1]$ if and only if f is bell-shaped.

The shape-preserving properties of $J_{2n-1}(f)(x)$ can be summarized by the following.

Theorem 2.2.9. *Let $F_n(g)(x)$ be the Hermite–Fejér polynomials based on the Chebyshev nodes of the first kind and let $J_n(f)(x)$ be the trigonometric Jackson interpolation polynomials, where $f(x) = g(\cos(x))$, $g : [-1, 1] \to \mathbb{R}$.*

(i) If f is bell-shaped and n is even, then $J_{2n-1}(f)$ is increasing in a neighborhood of $\frac{\pi}{2}$, of the form $(\frac{\pi}{2} - \frac{c}{n^2}, \frac{\pi}{2} + \frac{c}{n^2})$.

(ii) Let us suppose that f is bell-shaped. If ξ is any root of the polynomial $l'(x)$ (where $l(x)$ is given by $l(x) = \prod_{i=1}^{n}(x - x_{i,n})$, $x_{i,n} = \cos(2i-1)\pi/[2n]$), then $J_{2n-1}(f)$ is increasing in a neighborhood of $\eta = \arccos(\xi)$ of the form $(\eta - \frac{c}{n^3}, \eta + \frac{c}{n^3})$.

Proof. (i),(ii) It is immediate by taking $\alpha = \beta = -\frac{1}{2}$ (Chebyshev nodes of the first kind) in Theorems 2.2.1 and 2.2.2 and then taking into account that the function $h(x) = \arccos(x) \in Lip_{\frac{\pi}{\sqrt{2}}}(1/2)$. □

2.3 Shepard Operators

First, we present some results of qualitative type for a Shepard interpolation operator of the form

$$S_{n,p}(f)(x) = \sum_{k=-n}^{n} s_{k,n}(x) f(x_k), \quad n \in \mathbf{N}, \quad f : [-1, 1] \to \mathbf{R}, \tag{2.10}$$

$$-1 \le x_{-n} < x_{-n+1} < \cdots < x_1 < \cdots < x_n \le 1,$$

where $s_{k,n}(x) = (x - x_k)^{-2p} / \left(\sum_{i=-n}^{n}(x - x_i)^{-2p}\right)$, $p \in \mathbf{N}$, fixed.

It is easy to see that $S_{n,p}(f)(x_i) = f(x_i)$, $i = -n, \ldots, n$, $S_{n,p}(f)$ is a positive linear operator, $S_{n,p}^{(j)}(f)(x_i) = 0$, $i = -n, \ldots, n$, $j = 1, \ldots, 2p-1$ and $\sum_{k=-n}^{n} s_{k,n}(x) \equiv 1$.

Because $\sum_{k=-n}^{n} s'_{k,n}(x) \equiv 0$, reasoning exactly as in Popoviciu [81], p. 76, we obtain

$$S'_{n,p}(f)(x) = \sum_{i=-n}^{n-1} \left(- \sum_{j=-n}^{i} s'_{j,n}(x) \right) (f(x_{i+1}) - f(x_i)). \tag{2.11}$$

We also need the following basic result due to T. Popoviciu.

Lemma 2.3.1. (Popoviciu [81]) *Let* $F_m(f)(x) = \sum_{k=1}^{m} h_k(x) f(x_k)$, $x \in [a, b]$, $a \le x_1 < \cdots < x_m \le b$, $f : [a, b] \to \mathbf{R}$, $h_k \in C^1[a, b]$, $\sum_{k=1}^{m} h_k(x) \equiv 1$, $\forall\, x \in [a, b]$ *and* $x_0 \in [a, b]$ *be fixed.*

If $h_1'(x_0) < 0$, $h_m'(x_0) > 0$ *and the sequence*

$$h_1'(x_0), h_2'(x_0), \ldots, h_m'(x_0)$$

has exactly one variation of sign, then there exists a neighborhood of x_0, $V(x_0)$, *independent of* f *such that if* f *is monotone on* $[a, b]$, *then* $F_m(f)$ *is of the same monotonicity on* $V(x_0)$.

We have the following:

Theorem 2.3.1. *Let us denote* $l(x) = \sum_{i=-n}^{n} (x - x_i)^{-2p}$. *Any solution* $\xi \in (x_{-n}, x_n)$ *of the equation* $l'(x) = 0$ *is a point of strong preservation of partial monotony for* $S_{n,p}(f)(x)$.

Proof. Obviously $l'(\xi) = 0$ means

$$\sum_{i=-n}^{n} \frac{1}{(\xi - x_i)^{2p+1}} = 0, \quad \xi \ne x_i,\ i = -n, \ldots, n. \tag{2.12}$$

We have

$$s_{j,n}'(\xi) = \frac{-2p(\xi - x_j)^{-2p-1}}{l(\xi)},$$

where $l(\xi) > 0$.

Then,

$$s_{-n,n}'(\xi) = \frac{-2p(\xi - x_{-n})^{-2p-1}}{l(\xi)} < 0$$

and

$$s_{n,n}'(\xi) = \frac{-2p(\xi - x_n)^{-2p-1}}{l(\xi)} > 0.$$

Also, $\mathrm{sgn}[s_{j,n}'(\xi)] = \mathrm{sgn}(x_j - \xi)$, $j = -n, \ldots, n$, which implies that the sequence

$$s_{-n,n}'(\xi), s_{-n+1,n}'(\xi), \ldots, s_{1,n}'(\xi), \ldots, s_{n,n}'(\xi)$$

has exactly one variation of sign.

Applying now Lemma 2.3.1 and (2.11), we obtain the theorem. □

Remarks. (1) It is easy to see that (2.12) is equivalent to the polynomial equation

$$F_n(\xi) = \sum_{k=-n}^{n} \left[\prod_{i=-n\, i \ne k}^{n} (\xi - x_i) \right]^{2p+1} = 0.$$

Because simple calculations show that $F_n(x_j) F_n(x_{j+1}) < 0$, $j = -n, -n + 1, \ldots, n - 1$, it follows that in each interval (x_j, x_{j+1}) there exists a point ξ with $l'(\xi) = 0$.

(2) An open question is what happens in the case when

$$s_{j,n}(x) = |x - x_j|^{-(2p+1)} / \left[\sum_{i=-n}^{n} |x - x_i|^{-(2p+1)} \right], \quad p \in \mathbf{N}.$$

Concerning the convexity, we can prove the following:

Theorem 2.3.2. *If $S_{n,p}(f)(x)$ is of the form (2.10) with $x_{-k} = -x_k,\ x_0 = 0$, and $\Delta_h^2 f(0) > 0,\ \forall\, h \in (0, 1]$, then $S_{n,p}(f)(x)$ is strictly convex in a neighborhood of 0 (depending on f).*

Proof. We can write

$$s_{k,n}(x) = \frac{x^{2p}(x - x_k)^{-2p}}{1 + \sum_{i=-n i\neq k}^{n} x^{2p}(x - x_i)^{-2p}}, \quad k = -n, -n+1, \ldots, n,$$

and

$$s_{k,n}^{(i)}(0) = [x^{2p}(x - x_k)^{-2p}]_{x=0}^{(i)}, \ \forall\, i = 0, \ldots, 2p.$$

But from simple calculations, for all $k \neq 0$ we get

$$[x^{2p}(x - x_k)^{-2p}]_{x=0}^{(i)} = \begin{cases} 0, & i = 1, \ldots, 2p-1 \\ \dfrac{(2p)!}{x_k^{2p}}, & i = 2p. \end{cases}$$

As a conclusion

$$s_{k,n}^{(i)}(0) = \begin{cases} 0, & i = 1, \ldots, 2p-1 \\ \dfrac{(2p)!}{x_k^{2p}}, & i = 2p, \end{cases} \qquad k \neq 0,$$

$$S'_{n,p}(f)(0) = S''_{n,p}(f)(0) = \cdots = S_{n,p}^{(2p-1)}(f)(0) = 0$$

and

$$S_{n,p}^{(2p)}(f)(0) = \sum_{k=-n}^{n} s_{k,n}^{(2p)}(0) f(x_k) = \sum_{k=1}^{n} \frac{(2p)!}{x_k^{2p}} [f(x_k) + f(-x_k)]$$

$$+ f(0) s_{0,n}^{(2p)}(0) > f(0) \sum_{k=-n}^{n} s_{k,n}^{(2p)}(0) = 0.$$

It follows that for $S_{n,p}(f)(x), S''_{n,p}(f)(x), \ldots, S_{n,p}^{(2p-2)}(f)(x)$, 0 is a minimum point.

On the other hand, by $S_{n,p}^{(2p)}(f)(0) > 0$, there exists a neighborhood $V(0)$ of 0 (depending on f) such that $S_{n,p}^{(2p)}(f)(x) > 0$, for all $x \in V(0)$. It follows that $S_{n,p}^{(2p-2)}(f)(x)$ is strictly convex on $V(0)$, with $S_{n,p}^{(2p-2)}(f)(0) = 0$ and 0 minimum point for $S_{n,p}^{(2p-2)}(f)(x)$. This implies $S_{n,p}^{(2p-2)}(f)(x) > 0,\ \forall\, x \in V(0) \setminus \{0\}$. Reasoning by recurrence, at the end we obtain $S''_{n,p}(f)(x) > 0$, for all $x \in V(0) \setminus \{0\}$, i.e., $S_{n,p}(f)(x)$ is strictly convex in $V(0)$, which proves the theorem. □

Remarks. (1) Because for $f(x) \equiv x$, $S_{n,p}(f)(x)$ obviously is not a polynomial of degree ≤ 1 in any neighborhood, it follows that there do not exist points of strong preservation of partial convexity for $S_{n,p}(f)(x)$.

(2) Obviously Theorem 2.3.2 shows that 0 is a point of weak preservation of partial strict-convexity for $S_{n,p}(f)(x)$.

(3) Reasoning exactly as in Remark 1 of Theorem 2.2.3 and taking into account the above Remark 1, it follows that 0 cannot be a point of strong preservation of partial strict-convexity for $S_{n,p}(f)(x)$.

A quantitative version of Theorem 2.3.1 is the following (we use the notation of Theorem 2.3.1)

Theorem 2.3.3. *Let* $x_k = \frac{k}{n}$, $k = -n, \ldots, n$. *If* $\xi \in (x_{-n}, x_n)$ *is any solution of the equation* $l'(x) = 0$, *then there exists a constant* $c > 0$ *(independent of* f *and* n*) such that if* $f : [-1, 1] \to \mathbb{R}$ *is monotone on* $[-1, 1]$ *then* $S_{n,p}(f)(x)$ *is of the same monotonicity in* $\left(\xi - \frac{c}{n^{2p+3}}, \xi + \frac{c}{n^{2p+3}}\right) \subset (-1, 1)$.

Proof. Reasoning exactly as in the proof of Lemma 3 in Popoviciu [81] and taking into account the proof of Theorem 2.3.1, we get

$$Q_i(\xi) < \max\{s'_{-n,n}(\xi), -s'_{n,n}(\xi)\} < 0, \ \forall\, i = -n, \ldots, n-1,$$

where $Q_i(x) = \sum_{j=-n}^{i} s'_{j,n}(x)$, $s'_{-n,n}(\xi) = \frac{-2p(\xi - x_{-n})^{-2p-1}}{l(\xi)}$, $s'_{n,n}(\xi) = \frac{2p(x_n - \xi)^{-2p-1}}{l(\xi)}$, $l(\xi) > 0$. We have two possibilities.

Case 1: $x_{-n} < \xi \leq \frac{x_n + x_{-n}}{2} < x_n$. In this case,

$$\max\{s'_{-n,n}(\xi), -s'_{n,n}(\xi)\} = -s'_{n,n}(\xi).$$

Let j_0 be such that $|x_{j_0} - \xi| = \min\{|x_i - \xi|;\ i = \overline{-n, n}\}$. We obtain

$$s'_{n,n}(\xi) = \frac{2p}{(x_n - \xi)^{2p+1}} \cdot \frac{1}{\displaystyle\sum_{i=-n}^{n} \frac{1}{|\xi - x_i|^{2p}}} \geq \frac{c_1}{n^{2p}} > 0.$$

The last lower estimate is obtained by similar reasonings with those for $|E_1(x)|$ in the estimate of $|s''_{j,n}(x)|$ below in the proof.

Case 2: $x_{-n} < \frac{x_n + x_{-n}}{2} < \xi < x_n$. In this case,

$$\max\{s'_{-n,n}(\xi), -s'_{n,n}(\xi)\} = s'_{-n,n}(\xi)$$

and for x_{j_0} as in Case 1, we again obtain $|s'_{-n,n}(\xi) \geq \frac{c_2}{n^{2p}} > 0$. As a conclusion,

$$\frac{c}{n^{2p}} \leq |Q_i(\xi)|, \ \forall\, i = -n, \ldots, n-1.$$

On the other hand,

$$s_{j,n}(x) = \frac{1}{1 + (x - x_j)^{2p} \displaystyle\sum_{k=-n, k \neq j}^{n} \frac{1}{(x - x_k)^{2p}}},$$

$$s'_{j,n}(x) = \frac{2p(x-x_j)^{2p-1} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{x_k - x_j}{(x-x_k)^{2p+1}}}{\left[1 + (x-x_j)^{2p} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{1}{(x-x_k)^{2p}}\right]^2},$$

and

$$\begin{aligned} s''_{j,n}(x) &= \frac{2p(2p-1)(x-x_j)^{2p-2} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{x_k - x_j}{(x-x_k)^{2p+1}}}{\left[1 + (x-x_j)^{2p} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{1}{(x-x_k)^{2p}}\right]^2} \\ &\quad - \frac{2p(2p+1)(x-x_j)^{2p-1} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{x_k - x_j}{(x-x_k)^{2p+2}}}{\left[1 + (x-x_j)^{2p} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{1}{(x-x_k)^{2p}}\right]^2} \\ &\quad + \frac{4p(x-x_j)^{2p-1} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{x_k - x_j}{(x-x_k)^{2p+1}}}{\left[1 + (x-x_j)^{2p} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{1}{(x-x_k)^{2p}}\right]^2} \\ &\quad \cdot \frac{2p(x-x_j)^{2p-1} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{x_k - x_j}{(x-x_k)^{2p+1}}}{1 + (x-x_j)^{2p} \displaystyle\sum_{k=-n,k\neq j}^{n} \frac{1}{(x-x_k)^{2p}}} \\ &= E_1(x) - E_2(x) + E_3(x) \cdot E_4(x), \end{aligned}$$

where by $E_i(x)$, $i = \overline{1,4}$ we denote the expressions above in the order of their occurrence.

Obviously

$$|s''_{j,n}(x)| \leq |E_1(x)| + |E_2(x)| + |E_3(x)| \cdot |E_4(x)|.$$

Let us denote $|x - x_i| = \min\{|x - x_k|; -n \leq k \leq n\} \sim \frac{1}{n}$.

We have two cases.

Case 1: $i = j$. We obtain

$$|E_1(x)| \leq c_p \frac{1}{n^{2p-2}} \sum_{k=-n,k\neq j}^{n} \frac{|k-i|/n}{\left(\frac{|i-k|}{n}\right)^{2p+1}}$$
$$= c_p n^2 \sum_{k=-n,k\neq j}^{n} \frac{|k-i|}{|k-i|^{2p+1}} = c_p n^2 \sum_{k=-n,k\neq j}^{n} \frac{1}{|i-k|^{2p}} \leq c_p n^2.$$

It is easy to see that the same kinds of estimates applied to $E_2(x)$, $E_3(x)$ and $E_4(x)$ give us the estimates

$$|E_2(x)| \leq c_p \frac{1}{n^{2p-1}} \sum_{k=-n,k\neq j}^{n} \frac{|k-i|/n}{\left(\frac{|i-k|}{n}\right)^{2p+2}} \leq c_p n^2,$$
$$|E_3(x)| \leq c_p \frac{1}{n^{2p-1}} \sum_{k=-n,k\neq j}^{n} \frac{|k-i|/n}{\left(\frac{|i-k|}{n}\right)^{2p+1}} \leq c_p n,$$
$$|E_4(x)| \leq c_p n.$$

As a conclusion, in this case $|s''_{j,n}(x)| \leq c_p n^2$.

Case 2: $i \neq j$. We obtain

$$|E_1(x)| \leq c_{1,p} \frac{|x-x_j|^{2p-2} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}}}{\left(\frac{x-x_j}{x-x_i}\right)^{4p}}$$
$$\leq c_{1,p} \frac{(x-x_i)^{4p}}{(x-x_j)^{2p+2}} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}}$$
$$\leq c_{2,p} \frac{|x-x_i|^{2p-1}|x_i-x_j|}{|x-x_j|^{2p+2}}$$
$$+ c_{3,p} \frac{n^{-4p}}{\left(\frac{|i-j|}{n}\right)^{2p+2}} \sum_{k=-n,k\neq i,k\neq j}^{n} \frac{|k-j|/n}{\left(\frac{|i-k|}{n}\right)^{2p+1}}$$

$$\leq c_{4,p} \left\{ \frac{n^{1-2p}(|i-j|/n)}{\left(\frac{|i-j|}{n}\right)^{2p+2}} + \frac{n^2}{|i-j|^{2p+1}} \sum_{k=-n,k\neq i,k\neq j}^{n} \frac{|k-j|}{|i-k|^{2p+1}} \right\}$$
$$= c_{4,p} \left\{ \frac{n^2}{|i-j|^{2p+1}} + \frac{n^2}{|i-j|^{2p+1}} \left(\sum_{k=-n,k\neq i,k\neq j}^{n} \frac{1}{|i-k|^{2p}} + \frac{|i-j|}{|i-k|^{2p+1}} \right) \right\}$$
$$\leq c_p n^2.$$

Following exactly the same kinds of estimates, we easily get

$$|E_2(x)| \leq c_{1,p} \frac{|x-x_j|^{2p-1} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+2}}}{\left(\frac{x-x_j}{x-x_i}\right)^{4p}}$$

$$\leq c_{1,p} \frac{(x-x_i)^{4p}}{(x-x_j)^{2p+1}} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+2}} \leq cn^2,$$

$$|E_3(x)| \leq c_{1,p} \frac{|x-x_j|^{2p-1} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}}}{\left(\frac{x-x_j}{x-x_i}\right)^{4p}}$$

$$\leq c_{1,p} \frac{(x-x_i)^{4p}}{(x-x_j)^{2p+1}} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}} \leq cn,$$

$$|E_4(x)| \leq c_{1,p} \frac{|x-x_j|^{2p-1} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}}}{\left(\frac{x-x_j}{x-x_i}\right)^{2p}}$$

$$\leq c_{1,p} \frac{(x-x_i)^{2p}}{|x-x_j|} \sum_{k=-n,k\neq j}^{n} \frac{|k-j|/n}{|x-x_k|^{2p+1}}$$

$$\leq c_{2,p} \frac{|x_i-x_j|}{|x-x_j|\cdot|x-x_i|}$$

$$+ c_{3,p} \frac{n^{-2p}}{|i-j|/n} \sum_{k=-n,k\neq j,k\neq i}^{n} \frac{|k-j|/n}{\left(\frac{|i-k|}{n}\right)^{2p+1}}$$

$$\leq c_{4,p} \left\{ \frac{(|i-j|/n)}{\frac{|i-j|}{n}\cdot n^{-1}} + \frac{n}{|i-j|} \sum_{k=-n,k\neq j,k\neq i}^{n} \frac{|k-j|}{|i-k|^{2p+1}} \right\} \leq cn.$$

As a conclusion, in this case too we have $|s''_{j,n}(x)| \leq cn^2$. Let d_i be the nearest root of $Q_i(x)$ to ξ. It follows

$$\frac{c}{n^{2p}} \leq |Q_i(\xi)| = |Q_i(\xi) - Q_i(d_i)| = |d_i - \xi| \cdot |Q_i'(\eta)|$$

$$\leq |d_i - \xi| \cdot c_4 n^3, \quad \forall i = 1, \ldots, n-1,$$

and therefore

$$|d_i - \xi| \geq \frac{c}{n^{2p+3}}, \quad \forall i = 1, \ldots, n-1,$$

which proves the theorem. □

Remark. We would like to point out here an error appeared in the proof of Theorem 3.4 in the paper Gal–Szabados [57], where the estimate of the last line, page 241, and its analogue on page 242, line 3 from above (i.e., $s'_{n,n}(\xi) \geq \frac{c}{n}$, and $|s'_{-n,n}(\xi)| \geq \frac{c}{n}$) are wrong, having as a consequence a wrong estimate in Theorem 3.4. The correct proof and estimate are given by Theorem 2.3.3.

A quantitative version of Theorem 2.3.2 is the following.

Theorem 2.3.4. *If f satisfies $\Delta_h^2 f(0) > 0$, for all $h \in (0,1]$, then $S_{n,p}(f;x)$ of the form* (2.10) *with $x_k = \frac{k}{n}$, $k = -n, \ldots, n$ is strictly convex in $[-|d_n|, |d_n|]$, with*

$$|d_n| \geq \frac{c_p \sum_{k=1}^{n} [\Delta_{x_k}^2 f(0)]/x_k^{2p}}{n^{2p+1}||f||},$$

where c_p is a constant depending only on p.

Proof. By the proof of Theorem 2.3.2 we easily get

$$S_{n,p}^{(2p)}(f;0) = \sum_{k=1}^{n} s_{k,n}^{2p}(0)[\Delta_{x_k}^2 f(0)] = (2p)! \sum_{k=1}^{n} \frac{\Delta_{x_k}^2 f(0)}{x_k^{2p}}.$$

Let d_n be the nearest root of $S_{n,p}^{2p}(f;x)$ to 0. We have

$$S_{n,p}^{(2p)}(f;0) = |S_{n,p}^{(2p)}(f;0) - S_{n,p}^{(2p)}(f;d_n)| = |d_n||S_{n,p}^{(2p+1)}(f;y)|$$

$$\leq |d_n| \sum_{k=-n}^{n} |s_{k,n}^{(2p+1)}(x)||f(x_k)| \leq |d_n|||f|| \sum_{k=-n}^{n} |s_{k,n}^{(2p+1)}(x)|.$$

Taking into account the proof of Theorem 2.1 in Della Vecchia–Mastroianni [33], p. 149, we get

$$|s_{k,n}^{(2p+1)}(x)| \leq c_p s_{k,n}(x)(1/n)^{-(2p+1)} = c_p s_{k,n}(x) n^{2p+1}.$$

It follows

$$\sum_{k=1}^{n} [\Delta_{x_k}^2 f(0)]/x_k^{2p} = S_{n,p}^{(2p)}(f;0) \leq c_p |d_n|||f||n^{2p+1},$$

which immediately implies the inequality in the statement of theorem. □

Remarks. (1) If f is strictly convex on $[-1,1]$, i.e., there exists $\gamma > 0$ such that $\eta f(x) + (1-\eta) f(y) - f(\eta x + (1-\eta) y) \geq \eta(1-\eta)\gamma(x-y)^2$, for all $\eta \in [0,1]$ and $x, y \in [-1,1]$, then we easily get $\Delta_{x_k}^2 f(0)/x_k^2 \geq 2\gamma$, which implies

$$|d_n| \geq c_p 2\gamma \sum_{k=1}^{n} [1/x_k^{2p-2}]/[||f||n^{2p+1}] \geq 2\gamma c_p \sum_{k=1}^{n} [1/k^{2p-2}]/[||f||n^3],$$

which for $p = 1$ implies $|d_n| \geq 2\gamma c_p/[||f||n^2]$ and for $p \geq 2$ implies $|d_n| \geq 2\gamma c_p/[||f||n^3]$.

(2) Replacing $x_k = k/n$, we get $x_k^{2p} = k^{2p}/n^{2p}$ and the estimate in Theorem 2.3.4 becomes

$$|d_n| \geq \frac{c_p \sum_{k=1}^{n} \Delta^2_{x_k} f(0)/k^{2p}}{n||f||}.$$

(3) We don't know if the estimate of d_n in Theorem 2.3.4 is the best possible.

Concerning the Balász–Shepard operator defined on the semi-axis and whose global smoothness preservation properties were proved by Corollary 1.5.2, we have:

Theorem 2.3.5 *For the knots $x_k = \frac{k^\gamma}{n^{\gamma/2}}$, $k = 0, \ldots, n$, $n \in \mathbb{N}$, $\gamma \geq 1$, $s \geq 2$ and $f \in C([0, +\infty])$, let us consider the so-called Balázs–Shepard operator defined by*

$$S_{n,2p}(f)(x) = \frac{\sum_{k=0}^{n} |x - x_k|^{-2p} f(x_k)}{\sum_{k=0}^{n} |x - x_k|^{-2p}}, x \geq 0.$$

Let us denote $l(x) = \sum_{i=0}^{n} (x - x_i)^{-2p}$. Any solution $\xi \in (x_0, x_n) = (0, n^{\gamma/2})$ of the equation $l'(x) = 0$ is a point of strong preservation of partial monotony for $S_{n,2p}(f)(x)$.

Proof. Similar to the proof of Theorem 2.3.1. □

Remark. The Remarks 1 and 2 after the proof of Theorem 2.3.1 remain true in this case, too.

At the end, we present some (negative) remarks on the shape preservation properties of the so-called Shepard–Lagrange operator, whose global smoothness preservation properties were studied by Corollary 1.2.4. These operators are defined on the equidistant nodes $x_i = \frac{i}{n} \in [-1, 1]$, $i = -n, \ldots, 0, \ldots, n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \bigcup \{0\}$, $m < n$, $\lambda > 0$, as follows:

$$S_{n,\lambda,L_m}(f)(x) = \sum_{i=-n}^{n} A_{i,\lambda}(x) L_{m,i}(f)(x),$$

where

$$A_{i,\lambda}(x) = \frac{|x - x_i|^{-\lambda}}{\sum_{k=-n}^{n} |x - x_k|^{-\lambda}},$$

and

$$L_{m,i}(f)(x) = \sum_{j=0}^{m} \frac{u_i(x)}{(x - x_{i+j}) u_i'(x_{i+j})} f(x_{i+j}),$$

where $u_i(x) = (x - x_i)...(x - x_{i+m})$ and $x_{n+\nu} = x_\nu$, $\nu = 1, \dots, m$.

Choose $\lambda = 2p$ even. Reasoning exactly as for the relation (2.11) in Section 2.3, we obtain

$$S'_{n,\lambda,L_m}(f)(x) = \sum_{i=-n}^{n} A'_{i,\lambda}(x)L_{m,i}(f)(x) + \sum_{i=-n}^{n} A_{i,\lambda}(x)L'_{m,i}(f)(x) =$$

$$\sum_{i=-n}^{n-1}[-\sum_{j=0}^{i} A'_{i,\lambda}(x)][L_{m,i+1}(f)(x) - L_{m,i}(f)(x)] + \sum_{i=-n}^{n} A_{i,\lambda}(x)L'_{m,i}(f)(x).$$

Suppose that f would be monotonically increasing on $[-1, 1]$. Unfortunately, by Popoviciu [80] (see Theorem 2.2.1 (i)) it follows that for $m \geq 3$, $L_{m,i}(f)(x)$ do not preserve the monotonicity (around certain points). Let us take the simplest case, i.e., $m = 1$. Then we get

$$S'_{n,2p,L_m}(f)(x) = \sum_{i=-n}^{n-1}\left[-\sum_{j=0}^{i} A'_{i,2p}(x)\right][L_{m,i+1}(f)(x) - L_{m,i}(f)(x)]$$

$$+ \sum_{i=-n}^{n} A_{i,2p}(x)m_i,$$

where all $m_i = L'_{1,i}(f)(x)$ are ≥ 0 (since if f is increasing on $[-1, 1]$ then it is easily seen that all the Lagrange polynomials of degree 1, $L_{1,i}(f)(x)$, are increasing).

Now, while the second sum above is obviously positive, the differences $L_{m,i+1}(f)(x) - L_{m,i}(f)(x)$ cannot be positive for all i at some point $\xi \in [-1, 1]$. As a conclusion, it seems that even in the simplest case $m = 1$, $S_{n,2p,L_1}(f)(x)$ cannot preserve the monotonicity around some points.

For the study of convexity, from the proof of Theorem 2.3.2 we get $S^{(k)}_{n,2p,L_m}(f)(0) = 0, \forall k = 1, \dots, 2p - 1$ and

$$S^{(2p)}_{n,2p,L_m}(f)(0) = \sum_{i=-n}^{n} A^{(2p)}_{i,2p}(0)L_{m,i}(f)(0)$$

$$= \sum_{i=-n,i\neq 0}^{n} \frac{(2p)!}{x_i^{2p}} L_{m,i}(f)(0) + A^{(2p)}_{0,2p}(0)L_{m,0}(f)(0).$$

Unfortunately, even in the simplest case $m = 1$, from this relation we cannot deduce any convexity-preserving property for $S_{n,2p,L_m}(f)(x)$, as we did in the proof of Theorem 2.3.2 for the usual Shepard operator.

Similar reasonings for the so-called Shepard–Taylor operator, given by

$$S_{n,\lambda,T_m}(f)(x) = \sum_{i=-n}^{n} A_{i,\lambda}(x)T_{m,i}(f)(x),$$

where

$$A_{i,\lambda}(x) = \frac{|x - x_i|^{-\lambda}}{\sum_{k=-n}^{n} |x - x_k|^{-\lambda}},$$

and

$$T_{m,i}(f)(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_i)(x - x_i)^j}{j!},$$

even for the simplest case $m = 1$, show that it does not have the shape-preservation property (around certain points).

2.4 Bibliographical Remarks and Open Problems

All the results in this chapter, except those where the authors are mentioned and Lemma 2.2.1, Theorems 2.2.4–2.2.9, 2.3.3–2.3.5, which are new, are from Gal–Szabados [57].

Open Problem 2.4.1. For the Kryloff–Stayermann polynomials, $K_n(f)(x)$, in Theorems 2.2.4 and 2.2.6, find other points (different from 0) of preservation for the monotonicity of f.

Open Problem 2.4.2. What happens if in the Theorems 2.2.1, 2.2.4, 2.2.6 n is odd and if in the Theorems 2.2.3, 2.2.5, 2.2.7 and 2.2.8 n is even?

Open Problem 2.4.3. What happens if in the statements of Theorems 2.3.1 and 2.3.2, the Shepard operators are of the form

$$\frac{\sum_{j=-n}^{n} |x - x_j|^{-(2p+1)} f(x_j)}{\sum_{k=-n}^{n} |x - x_k|^{-(2p+1)}} ?$$

Open Problem 2.4.4. For the Balász–Shepard operator defined on the semi-axis, prove a quantitative version of Theorem 2.3.5.

Also, another question for this operator is if there exist points such that in some neighborhoods of them, it preserves the strict-convexity of function.

Open Problem 2.4.5. For the general Shepard–Grünwald operators introduced by Criscuolo–Mastroianni [31] (considered in Open Problem 1.6.4, too), prove shape-preserving properties.

Open Problem 2.4.6. For the local variants of Shepard operators in Open Problem 1.6.13, prove shape-preserving properties.

3

Global Smoothness Preservation, Bivariate Case

Extending the results in the univariate case, we prove in this chapter that the bivariate interpolation polynomials of Hermite–Fejér based on the Chebyshev nodes of the first kind, those of Lagrange based on the Chebyshev nodes of second kind and ± 1, and those of bivariate Shepard operators, have the property of partial preservation of global smoothness with respect to various bivariate moduli of continuity.

3.1 Introduction

It is the aim of this chapter to extend the results of Chapter 1, with respect to various bivariate moduli of continuity.

In this sense, we will use the following kinds of bivariate moduli of continuity.

Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$. For $\delta, \eta > 0$, we define

$$\omega^{(x)}(f; \delta) = \sup_{y \in [-1,1]} \sup\{|f(x+h, y) - f(x, y)|;\ x, x+h \in [-1, 1],$$
$$0 \le h \le \delta\},$$
$$\omega^{(y)}(f; \eta) = \sup_{x \in [-1,1]} \sup\{|f(x, y+k) - f(x, y)|;\ y, y+k \in [-1, 1],$$
$$0 \le k \le \eta\},$$

(i.e., the partial bivariate moduli of continuity, see, e.g., Timan [103])

$$\omega(f; \delta, \eta) = \sup\{|f(x+h, y+k) - f(x, y)|;\ 0 \le h \le \delta,\ 0 \le k \le \eta,$$
$$x, x+h \in [-1, 1],\ y, y+k \in [-1, 1]\},$$
$$\omega^{(B)}(f; \delta, \eta) = \sup\{|\Delta_{h,k} f(x, y)|;\ 0 \le h \le \delta,\ 0 \le k \le \eta,$$
$$x, x+h \in [-1, 1],\ y, y+k \in [-1, 1]\},$$

(i.e., the Bögel modulus of continuity, see, e.g., Gonska–Jetter [61] or Nicolescu [76]) where

$$\Delta_{h,k} f(x, y) = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y).$$

The properties of these moduli of continuity useful in the next sections are given by the following

Lemma 3.1.1. (see, e.g., Timan [103], p. 111–114). (i) $\omega_1(f; 0, 0) = 0$, $\omega(f; \delta, \eta)$ *is nondecreasing with respect to* δ *and* η,

$$\omega(f; \delta_1 + \delta_2, \eta_1 + \eta_2) \le \omega(f; \delta_1, \eta_1) + \omega(f; \delta_2, \eta_2),$$

$$\omega(f; \delta, \eta) \le \omega^{(x)}(f; \delta) + \omega^{(y)}(f; \eta) \le 2\omega(f; \delta, \eta).$$

(ii) (see, e.g., Anastassiou–Gal [6], p. 81)

$$\omega^{(B)}(f; \delta, \eta) \le \omega^{(x)}(f; \delta) + \omega^{(y)}(f; \eta).$$

3.2 Bivariate Hermite–Fejér Polynomials

Let us define the bivariate Hermite–Fejér polynomial on the Chebyshev nodes of the first kind by

$$H_{n_1,n_2}(f)(x, y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h_{i,n_1}(x) h_{j,n_2}(y) f(x_{i,n_1}, x_{j,n_2}),$$

where $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$,

$$x_{i,n} = \cos \frac{2i - 1}{2n}\pi, \quad h_{i,n}(x) = \frac{T_n^2(x)(1 - x x_{i,n})}{n^2(x - x_{i,n})^2}, \qquad i = \overline{1, n}.$$

It is well known that we have

$$\sum_{i=1}^{n} h_{i,n}(x) = 1, \quad h_{i,n}(x) \ge 0, \qquad \forall\, x \in [-1, 1],\ \forall\, i = \overline{1, n}.$$

Let us denote by $H_n(f, x)$ the univariate Hermite–Fejér polynomials based on Chebyshev nodes of the first kind and

$$E(\alpha, \delta) = \begin{cases} O\left(\delta^{\frac{\alpha}{\max\{2-\alpha, 1+\alpha\}}}\right), & \text{if } 0 < \alpha < \frac{1}{2} \text{ or } \frac{1}{2} < \alpha < 1, \\ O\left(\left[\delta \log \frac{1}{\delta}\right]^{\frac{2\alpha+1}{6}}\right), & \text{if } \alpha = \frac{1}{2} \text{ or } 1, \end{cases}$$

Remark. In all subsequent results of this section, the constants involved in the signs O, will depend only on the functions considered.

The following result is known in the univariate case.

Theorem 3.2.1. (see Corollary 1.2.1). *If* $f \in \text{Lip}_M(\alpha; [-1, 1])$, $0 < \alpha \leq 1$, *then*

$$\omega(H_n(f); \delta) = E(\alpha, \delta).$$

We also need the following.

Lemma 3.2.1. *Let* $a, b \in (0, 1]$ *be fixed. Then,* $\omega(f; \delta, \eta) \leq C(\delta^a + \eta^b)$ *for all* $\delta, \eta \geq 0$ *if and only if* $\omega^{(x)}(f; \delta) \leq C\delta^a$, *and* $\omega^{(y)}(f; \eta) \leq C\eta^b$, *for all* $\delta, \eta \geq 0$. *Here* $C > 0$ *denotes an absolute constant, but which can be different at each occurrence.*

Proof. Indeed, by hypothesis and Lemma 3.1.1 (i), we get $\omega(f; \delta, \eta) \leq C(\delta^a + \eta^b)$.

Conversely, let us suppose that $\omega(f; \delta, \eta) \leq C(\delta^a + \eta^b)$ for all $\delta, \eta \geq 0$. From Lemma 3.1.1,(i) we get

$$\omega^{(x)}(f; \delta) + \omega^{(y)}(f; \eta) \leq 2C\delta^a + 2C\eta^b \quad \text{for all} \quad \delta, \eta \geq 0.$$

Now with $\eta = 0$ this implies

$$\omega^{(x)}(f; \delta) \leq 2C\delta^a \quad \text{for all} \quad \delta \geq 0$$

and

$$\omega^{(y)}(f; \eta) \leq 2C\eta^b \quad \text{for all} \quad \eta \geq 0,$$

respectively, which proves the lemma. □

The first main result is given by the following:

Theorem 3.2.2. *Let* $a, b \in (0, 1]$ *be fixed. If* $\omega(f; \delta, \eta) \leq C(\delta^a + \eta^b)$ *for all* $\delta, \eta \geq 0$, *then*

$$\omega(H_{n_1,n_2}(f); \delta, \eta) = E(a, \delta) + E(b, \eta) \quad \textit{for all} \quad \delta, \eta \geq 0.$$

Proof. For each fixed $y \in [-1, 1]$, let us denote

$$F_{n_2,y}(x) = \sum_{j=1}^{n_2} h_{j,n_2}(y) f(x, x_{j,n_2}).$$

By hypothesis and by Lemma 3.2.1 we get that $\omega^{(x)}(f; \delta) \leq C\delta^a$. This implies, for all $|x_1 - x_2| \leq \delta$ and y,

$$|F_{n_2,y}(x_1) - F_{n_2,y}(x_2)| \leq |\sum_{j=1}^{n_2} h_{j,n_2}(y)[f(x_1, x_{j,n_2}) - f(x_2, x_{j,n_2})]|$$

$$\leq \sum_{j=1}^{n_2} h_{j,n_2}(y)|f(x_1, x_{j,n_2}) - f(x_2, x_{j,n_2})| \leq \sum_{j=1}^{n_2} h_{j,n_2}(y)\omega^{(x)}(f; \delta) = \omega^{(x)}(f; \delta)$$

$$\leq C\delta^a,$$

where C is independent of n_2 and y. As a conclusion,

$$\omega^{(x)}(F_{n_2,y}(x); \delta) \leq C\delta^a \quad \text{for all} \quad y \in [-1, 1].$$

Now, it is easily seen that we can write

$$H_{n_1,n_2}(f)(x, y) = H_{n_1}[F_{n_2,y}(x)](x),$$

where the univariate H_{n_1} is applied to $F_{n_2,y}(x)$ as function of x (y is fixed arbitrary). According to Theorem 3.2.1, we immediately get that for every fixed y we have

$$\omega^{(x)}(H_{n_1,n_2}(f); \delta) = \omega^{(x)}(H_{n_1}[F_{n_2,y}(x)]; \delta) = E(a, \delta).$$

Similarly we obtain

$$\omega^{(y)}(H_{n_1,n_2}(f); \eta) = E(b, \eta).$$

Adding the last to relations we get the theorem. □

In what follows we deal with global smoothness preservation properties for the Hermite–Fejér polynomial through the Bögel modulus of continuity, but only for a special class of functions. In this sense, for $a, b \in (0, 1]$, let us define

$$D_{a,b} = \{G : [-1, 1] \times [-1, 1] \to \mathbb{R}; \quad G(x, y) = F[f(x)g(y)], \text{ where}$$

$$F : [-1, 1] \to \mathbb{R} \text{ satisfies } \sum_{i=1}^{\infty}(i^2/i!)|F^{(i)}(0)| < +\infty$$

and

$$f \in \mathrm{Lip}_{M_1}(a; [-1, 1]), g \in \mathrm{Lip}_{M_2}(b; [-1, 1]), ||f||, ||g|| \leq 1\}.$$

Remark. A simple example is $G(x, y) = \sin(xy) \in D_{1,1}$. For $a, b \in (0, 1]$, let us denote

$$\mathrm{Lip}^B(a, b; [-1, 1]) = \{G : [-1, 1] \times [-1, 1] \to \mathbb{R}; \quad \omega^{(B)}(G; a, b) \leq C\delta^a\eta^b\}.$$

Lemma 3.2.2. $D_{a,b} \subset \mathrm{Lip}^B(a, b; [-1, 1])$.

Proof. Let $G \in D_{a,b}$. Developing F in MacLaurin series, we get

$$G(x, y) = F(0) + \sum_{i=1}^{\infty} f^i(x)g^i(y)F^{(i)}(0)/i!.$$

This immediately implies

$$\Delta_{h,k}G(x, y) = \sum_{i=1}^{\infty} \Delta_h f^i(x)\Delta_k g^i(y)F^i(0)/i!.$$

On the other hand, by $f \in \mathrm{Lip}_{M_1}(a, [-1, 1])$ and $\|f\| \leq 1$

$$|f^i(x+h) - f^i(x)| \le iM_1h^a, \quad |g^i(y+k) - g^i(y)| \le iM_2k^b, i = 1, 2, \dots,$$

which implies

$$\omega^{(B)}(G;\delta,\eta) \le \sum_{i=1}^{\infty}(i^2/i!)|F^{(i)}(0)|\delta^a\eta^b \le C\delta^a\eta^b.$$

These prove the lemma. □

As a consequence we obtain

Corollary 3.2.1. *If* $G \in D_{a,b}$ *then*

$$\omega^{(B)}(H_{n_1,n_2}(G);\delta,\eta) = E(a,\delta)E(b,\eta).$$

Proof. We easily get

$$H_{n_1,n_2}(G)(x,y) = F(0) + \sum_{i=1}^{+\infty} H_{n_1}(f^i)(x)H_{n_2}(g^i)(y)F^{(i)}(0)/i!.$$

Then, as in the proof of Lemma 3.2.2, we have

$$\omega^{(B)}(H_{n_1,n_2}(G);\delta,\eta) \le \sum_{i=1}^{+\infty} \omega_1(H_{n_1}(f^i);\delta)\omega(H_{n_2}(g^i);\eta)F^{(i)}(0)/i!.$$

But for $f \in Lip_M(a;[-1,1])$, for all $i, n \in \mathbb{N}$ and all $\delta \ge 0$ we have $\omega_1(H_n(f^i);\delta) = iE(a,\delta)$. Indeed, from the univariate case (see Theorem 1.2.1) we get

$$\omega(H_n(f^i);\delta) \le Ci \min\left\{\delta n \sum_{k=1}^{n} 1/k^{2a}, n^{-1}\sum_{k=1}^{n} 1/k^a + \delta^a\right\},$$

where $C > 0$ depends only on the Lipschitz constant M of f. Then reasoning exactly as in the proof of Corollary 1.2.1, we get the required formula.

Taking now into account Theorem 3.2.1, we immediately obtain

$$\omega^{(B)}(H_{n_1,n_2}(G);\delta,\eta) \le E(a,\delta)E(b,\eta)\sum_{i=1}^{+\infty} i^2|F^{(i)}(0)|/i!,$$

which proves the corollary. □

3.3 Bivariate Shepard Operators

Let us first consider the bivariate Shepard operator as a tensor product by

$$S_{n,m}^{(\lambda,\mu)}(f)(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m}s_{i,\lambda}(x)s_{j,\mu}(y)f(i/n,j/m),\ \text{if } (x,y)\neq\left(\frac{i}{n},\frac{j}{m}\right),$$

$S_{n,m}^{(\lambda,\mu)}(f)(\frac{i}{n},\frac{j}{m})=f(\frac{i}{n},\frac{j}{m})$, where $1<\lambda,\mu$ and $f:[0,1]\times[0,1]\to\mathbb{R}$,

$$s_{i,\lambda}(x)=|x-i/n|^{-\lambda}/\left[\sum_{k=0}^{n}|x-k/n|^{-\lambda}\right],$$

$$s_{j,\mu}(y)=|y-j/m|^{-\mu}/\left[\sum_{k=0}^{m}|y-k/m|^{-\mu}\right].$$

Let $S_{n,\lambda}(f,x)$ denote the univariate Shepard operator (for univariate f) and let us define

$$E_{\lambda}(\alpha,\delta)=\begin{cases}O(\delta^{\alpha}), & \text{if } 0<\alpha<\lambda-1,\\ O\left(\delta^{\alpha}\log\frac{1}{\delta}\right), & \text{if } \alpha=\lambda-1\\ O\left(\delta^{\lambda-1}\right), & \text{if } \lambda-1<\alpha\leq 1,\end{cases}$$

for $1<\lambda\leq 2$.

The following result is known.

Theorem 3.3.1 (see Corollary 1.2.3). (i)*If $f\in Lip_M(\alpha;[0,1])$, $0<\alpha\leq 1$, $1<\lambda\leq 2$, then for all $\delta\geq 0$ and $n\in\mathbb{N}$ we have*

$$\omega(S_{n,\lambda}(f);\delta)=E_{\lambda}(\alpha,\delta).$$

(ii) *If $\lambda>2$ then for all $\delta\geq 0$ and $n\in\mathbb{N}$ we have*

$$\omega(S_{n,\lambda}(f);\delta)\leq C_{\lambda}\omega(f;\delta).$$

We present:

Theorem 3.3.2. *Let $f:[0,1]\times[0,1]\to\mathbb{R}$.*

(i) *Suppose $a,b\in(0,1]$, $1<\lambda,\mu\leq 2$ are fixed and $\omega(f;\delta,\eta)\leq C[\delta^{a}+\eta^{b}]$, for all $\delta,\eta\geq 0$. Then for all $\delta,\eta\geq 0$ and $n,m\in\mathbb{N}$ we have*

$$\omega(S_{n,m}^{(\lambda,\mu)}(f);\delta,\eta)=E_{\lambda}(a,\delta)+E_{\mu}(b,\eta).$$

(ii) *If $\lambda,\mu>2$ then*

$$\omega(S_{n,m}^{(\lambda,\mu)}(f);\delta,\eta)\leq C_{\lambda,\mu}\omega(f;\delta,\eta),$$

for all $\delta,\eta\geq 0$ and $n,m\in\mathbb{N}$.

Proof. (i) The reasonings are similar to those in the proof of Theorem 3.2.2, taking into account Theorem 3.3.1 (i) too.

(ii) By Theorem 3.3.1 (ii) and reasoning exactly as in the proof of Theorem 3.2.2, we get

$$\omega(S_{n,m}^{(\lambda,\mu)}(f);\delta,\eta)\leq C_{\lambda,\mu}[\omega^{(x)}(f;\delta)+\omega^{(y)}(f;\eta)],$$

which combined with Lemma 3.1.1 (i) proves the statement. □

Remark. It is of interest to know the approximation order by the tensor product Shepard operator $S_{n,m}^{(\lambda,\mu)}(f)(x, y)$. Since by, e.g., Szabados [98], Theorem 1, we easily obtain that for the univariate case we have

$$||S_{n,\lambda}(g)|| \leq ||S_{n,\lambda}(g) - g|| + ||g|| \leq C||g||,$$

then for bivariate functions f, the case $\lambda, \mu > 2$, by Haussmann–Pottinger [63], Theorem 5, it immediately follows that

$$||S_{n,m}^{(\lambda,\mu)}(f) - f|| \leq C_{\lambda,\mu}[\omega^{(x)}(f; 1/n) + \omega^{(y)}(f; 1/m)],$$

which combined with the second relation in Lemma 3.1.1,(i) implies

$$||S_{n,m}^{(\lambda,\mu)}(f) - f|| \leq C_{\lambda,\mu}\omega(f; 1/n, 1/m).$$

Here $||.||$ denotes the uniform norm.

For the case of Bögel modulus of continuity, firstly we need to define a new class of bivariate functions (which includes $D_{a,b}$) by

$$D^* = \{G : [0, 1] \times [0, 1] \to \mathbb{R}; \quad G(x, y) = F[f(x)g(y)], \text{ where}$$

$$F : [0, 1] \to \mathbb{R} \text{ satisfies}$$

$$\sum_{i=1}^{\infty}(i^2/i!)|F^{(i)}(0)| < +\infty \text{ and } ||f||, ||g|| \leq 1\}.$$

Reasoning exactly as in the previous section, we get the following.

Corollary 3.3.1. (i) *Suppose $a, b \in (0, 1]$, $1 < \lambda, \mu \leq 2$ are fixed. If $G \in D_{a,b}$, $G(x, y) = F(f(x)g(y))$ then for all $\delta, \eta \geq 0$ and $n, m \in \mathbb{N}$ we have*

$$\omega^{(B)}(S_{n,m}^{\lambda,\mu}(G); \delta, \eta) = E_{\lambda}(a, \delta)E_{\mu}(b, \eta).$$

(ii) *If $2 < \lambda, \mu$ and $G \in D^*$, $G(x, y) = F(f(x)g(y))$ then for all $\delta, \eta \geq 0$ and $n, m \in \mathbb{N}$ we have*

$$\omega^{(B)}(S_{n,m}^{\lambda,\mu}(G); \delta, \eta) \leq C_{\lambda,\mu}\omega(f; \delta)\omega(g; \eta).$$

Proof. As in the proofs of Lemma 3.2.2 and Corollary 3.2.1 we obtain

$$\omega^{(B)}(S_{n,m}^{\lambda,\mu}(G); \delta, \eta) \leq \sum_{i=1}^{+\infty} \omega(S_n^{\lambda}(f^i); \delta)\omega(S_m^{\mu}(g^i); \eta)|F^{(i)}(0)|/i!.$$

(i) The proof in this case follows the ideas of proof in Corollary 3.2.1, based on the formulas in the univariate case from the end of Section 1.2.

(ii) By mathematical induction we easily get $\omega(f^i;\delta)\le i\omega(f;\delta)$ and $\omega(g^i;\eta)\le i\omega(g;\eta)$.

Now, following the ideas in the proof of Corollary 3.2.1 and taking into account the above Theorem 3.3.1 too, we easily get the desired conclusion. □

It is well known that the original bivariate operator introduced by Shepard in [91], actually is not tensor product of the univariate case. More exactly, in [91] Shepard defines the bivariate operators by

$$S_{n,\mu}(f)(x,y)=\sum_{i=0}^{n} s_{n,i}^{(\mu)}(x,y)f(x_i,y_i),\ \text{if } (x,y)\neq(x_i,y_i),$$

$S_{n,\mu}(f)(x_i,y_i)=f(x_i,y_i)$, where $\mu>0$ is fixed, $f:D\to\mathbb{R}$, $D\subset\mathbb{R}^2$, $(x_i,y_i)\in D$, $i=0,\dots,n$, $x_0<x_1<\cdots<x_n$, $y_0<y_1<\cdots<y_n$,

$$s_{n,i}^{(\mu)}(x,y)=[(x-x_i)^2+(y-y_i)^2]^{-\mu/2}/l_n^{(\mu)}(x,y),$$

$$l_n^{(\mu)}(x,y)=\sum_{i=0}^{n}\left[(x-x_i)^2+(y-y_i)^2\right]^{-\mu/2}.$$

This kind of Shepard operator is useful in computer-aided geometric design (see, e.g., Barnhill–Dube–Little [13]).

In what follows, we consider another variant of the bivariate Shepard operator which is not a tensor product of the univariate case, has good approximation properties and implicitly better global smoothness preservation properties than that introduced in Shepard [91].

Thus, let us introduce

$$S_{n_1,n_2,\mu}(f;x,y)=\frac{T_{n_1,n_2,\mu}(f;x,y)}{T_{n_1,n_2,\mu}(1;x,y)},\ \text{if } (x,y)\neq(x_i,y_j),$$

$S_{n_1,n_2,\mu}(f;x_i,y_j)=f(x_i,y_j)$, where $\mu>0$ is fixed, $f:D\to\mathbb{R}$, $D=[0,1]\times[0,1]$, $x_i=i/n_1$, $i=0,\dots,n_1$; $y_j=j/n_2$, $j=0,1,\dots,n_2$ and

$$T_{n_1,n_2,\mu}(f;x,y)=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{f(x_i,y_j)}{[(x-x_i)^2+(y-y_j)^2]^{\mu}}.$$

The following result will be essential in establishing the smoothness-preserving properties of this operator.

Theorem 3.3.3. *For any $f\in C(D)$ and $\mu>3/2$ we have*

$$\|f-S_{n_1,n_2,\mu}(f)\|\le c\omega\left(f;\frac{1}{n_1},\frac{1}{n_2}\right)$$

and

$$\max\left[\left\|\frac{\partial S_{n_1,n_2,\mu}(f;x,y)}{\partial x}\right\|,\left\|\frac{\partial S_{n_1,n_2,\mu}(f;x,y)}{\partial y}\right\|\right]$$
$$\leq c(n_1+n_2)\omega\left(f;\frac{1}{n_1},\frac{1}{n_2}\right).$$

Proof. Let x_u and y_v denote the nodes closest to x and y, respectively, and let

$$K(x,y) := \{(i,j)|0\leq i\leq n_1,\ 0\leq j\leq n_2,\ (i,j)\neq(u,v)\}.$$

We prove that

$$\left\|\frac{\displaystyle\sum_{(i,j)\in K(x,y)}\frac{|x-x_i|^\lambda}{[(x-x_i)^2+(y-y_j)^2]^\nu}}{T_{n_1,n_2,\mu}(1;x,y)}\right\|\leq cn_1^{2\nu-2\mu-\lambda},\qquad \lambda<2\nu-2, \tag{3.1}$$

and

$$\left\|\frac{\displaystyle\sum_{(i,j)\in K(x,y)}\frac{|y-y_j|^\lambda}{[(x-x_i)^2+(y-y_j)^2]^\nu}}{T_{n_1,n_2,\mu}(1;x,y)}\right\|\leq cn_2^{2\nu-2\mu-\lambda},\qquad \lambda<2\nu-2. \tag{3.2}$$

Indeed,

$$\frac{\displaystyle\sum_{(i,j)\in K(x,y)}\frac{|x-x_i|^\lambda}{[(x-x_i)^2+(y-y_j)^2]^\nu}}{T_{n_1,n_2,\mu}(1;x,y)}\leq c\frac{n_1^{2\nu-\lambda}n_2^{2\nu}\sum_{i=1}^{n_1}\displaystyle\sum_{j=1}^{n_2}i^\lambda(n_2^2i^2+n_1^2j^2)^{-\nu}}{(n_1n_2)^{2\mu}\displaystyle\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(n_2^2i^2+n_1^2j^2)^{-\mu}}$$

$$\leq cn_1^{2\nu-2\mu-\lambda}n_2^{2\nu-2\mu}\frac{\displaystyle\sum_{i=1}^{n_1}\left(\sum_{j=1}^{[n_2i/n_1]}n_2^{-2\nu}i^{\lambda-2\nu}+i^\lambda\sum_{j=[n_2i/n_1]+1}^{n_2}n_1^{-2\nu}j^{-2\nu}\right)}{\displaystyle\sum_{i=1}^{n_1}\sum_{j=1}^{[n_2i/n_1]}(n_2i)^{-2\mu}}$$

$$\leq cn_1^{2\nu-2\mu-\lambda}n_2^{2\nu-2\mu}\frac{n_1^{-1}n_2^{1-2\nu}\displaystyle\sum_{i=1}^{n_1}i^{1+\lambda-2\nu}}{n_1^{-1}n_2^{1-2\mu}\displaystyle\sum_{i=1}^{n_1}i^{1-2\mu}}\leq cn_1^{2\nu-2\mu-\lambda}.$$

Equation (3.2) can be proved analogously. It is clear from the above argument that in the case $\lambda \geq 2\nu - 2\mu$ the summations in (3.1)–(3.2) can be extended to $0 \leq i \leq n_1,\ 0 \leq j \leq n_2$.

We shall also use the inequalities

$$|f(x_i, y_j) - f(x, y)| \leq \omega\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)(2 + n_1|x - x_i| + n_2|y - y_j|), \quad (3.3)$$

$$T_{n_1,n_2,\mu}(1; x, y) \geq c n_1^{2\mu-1} n_2, \quad (3.4)$$

$$\sum_{(i,j)\in K(x,y)} \frac{|x - x_i|^\lambda}{[(x - x_i)^2 + (y - y_j)^2]^\nu} \leq c n_1^{2\nu-\lambda-1} n_2, \qquad 0 \leq \lambda < 2\nu - 2 \quad (3.5)$$

and

$$\max\left[\left\|\frac{\partial T_{n_1,n_2,\mu}(1; x, y)/\partial x}{T_{n_1,n_2,\mu}(1; x, y)}\right\|, \left\|\frac{\partial T_{n_1,n_2,\mu}(1; x.y)/\partial y}{T_{n_1,n_2,\mu}(1; x, y)}\right\|\right] \leq c(n_1 + n_2). \quad (3.6)$$

Equation (3.3) easily follows from the properties of partial modulus of continuity. Equations (3.4)–(3.5) can be proved similarly to (3.1)–(3.2). Equation (3.6) follows from (3.1)–(3.2) with $\lambda = 1,\ \nu = \mu + 1$.

Next, using (3.1)–(3.2) with $\lambda = 0$ or 1 and $\nu = \mu > 3/2$, as well as the second relation in Lemma 3.1.1,(i) we obtain

$$\|f - S_{n_1,n_2,\mu}(f)\| \leq \omega\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)$$

$$+ \frac{\sum_{(i,j)\in K(x,y)} [\omega^{(x)}(f; |x-x_i|) + \omega^{(y)}(f; |y - y_j|)][(x - x_i)^2 + (y - y_j)^2]^{-\mu}}{T_{n_1,n_2,\mu}(1; x, y)}$$

$$\leq c\omega\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)\left[1 + \frac{\sum_{(i,j)\in K(x,y)} \frac{2 + n_1|x - x_i| + n_2|y - y_j|}{[(x - x_i)^2 + (y - y_j)^2]^\mu}}{T_{n_1,n_2,\mu}(1; x, y)}\right]$$

$$\leq c\omega\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right).$$

In order to prove the second relation in the theorem, besides (3.1)–(3.5) we use that for $f =$ const. we have $\frac{\partial S_{n_1,n_2,\mu}(f)}{\partial x} = \frac{\partial S_{n_1,n_2,\mu}(f)}{\partial y} \equiv 0$ for all $(x, y) \in D$, and get

$$\left|\frac{\partial S_{n_1,n_2,\mu}(f; x, y)}{\partial x}\right|$$

$$\leq c\omega\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)\left|\frac{\partial}{\partial x}\left(\frac{1}{[(x - x_u)^2 + (y - y_v)^2]^\mu T_{n_1,n_2,\mu}(1; x, y)}\right)\right|$$

$$+\frac{\sum_{(i,j)\in K(x,y)} \frac{|x-x_i|\cdot|f(x_i,y_j)-f(x,y)|}{[(x-x_i)^2+(y-y_j)^2]^{\mu+1}}}{T_{n_1,n_2,\mu}(1;x,y)}+$$

$$\frac{\sum_{(i,j)\in K(x,y)} \frac{|f(x_i,y_j)-f(x,y)|}{[(x-x_i)^2+(y-y_j)^2]^{\mu}}\cdot \sum_{(i,j)\in K(x,y)} \frac{|x-x_i|}{[(x-x_i)^2+(y-y_j)^2]^{\mu+1}}}{T_{n_1,n_2,\mu}(1;x,y)^2}$$

$$\leq c\omega\left(f;\frac{1}{n_1},\frac{1}{n_2}\right)$$

$$\left[|x-x_u|[(x-x_u)^2+(y-y_v)^2]^{\mu-1} \sum_{(i,j)\in K(x,y)} \frac{1}{[(x-x_i)^2+(y-y_j)^2]^{\mu}}\right.$$

$$+[(x-x_u)^2+(y-y_v)^2]^{\mu} \sum_{(i,j)\in K(x,y)} \frac{|x-x_i|}{[(x-x_i)^2+(y-y_j)^2]^{\mu+1}}$$

$$+\frac{\sum_{(i,j)\in K(x,y)} \frac{|x-x_i|[2+n_1|x-x_i|+n_2|y-y_j|]}{[(x-x_i)^2+(y-y_j)^2]^{\mu+1}}}{T_{n_1,n_2,\mu}(1;x,y)}$$

$$\left.+\frac{\sum_{(i,j)\in K(x,y)} \frac{2+n_1|x-x_i|+n_2|y-y_j|}{[(x-x_i)^2+(y-y_j)^2]^{\mu}}\cdot \sum_{(i,j)\in K(x,y)} \frac{|x-x_i|}{[(x-x_i)^2+(y-y_j)^2]^{\mu+1}}}{T_{n_1,n_2,\mu}(1;x,y)^2}\right]$$

$$\leq c\omega\left(f;\frac{1}{n_1},\frac{1}{n_2}\right)\left[\frac{1}{n_1}\left(\frac{1}{n_1^2}+\frac{1}{n_2^2}\right)^{\mu-1} n_1^{2\mu-1}n_2+\left(\frac{1}{n_1^2}+\frac{1}{n_2^2}\right)^{\mu} n_1^{2\mu}n_2\right.$$

$$\left.+\left[\frac{n_1^{2\mu}n_2^{2\mu}}{(n_1^2+n_2^2)^{\mu}}\cdot\frac{n_1}{n_1^{2\mu-1}n_2}+n_1+n_2\right]\leq c(n_1+n_2)\omega\left(f;\frac{1}{n_1},\frac{1}{n_2}\right).\right.$$

□

Now we are in the position to state our preservation results, unfortunately only in the special case $n_1 = n_2$. The case $n_1 \neq n_2$ remains open.

Theorem 3.3.4. *For* $\mu > 3/2$, $n \in \mathbb{N}$ *and* $h, k \geq 0$, *we have*

$$\omega(S_{n,n,\mu}(f); h, k) \leq c\omega(f; h+k, h+k).$$

Proof. We apply the standard technique. By the first estimate in Theorem 3.3.3 we get

$$\omega(S_{n,n,\mu}(f); h, k) \leq 2||S_{n,n,\mu}(f) - f|| + \omega(f; h, k)$$

$$\leq c\left[\omega\left(f; \frac{1}{n}, \frac{1}{n}\right) + \omega(f; h, k)\right].$$

Then, by the bivariate mean value theorem and by the second estimate in Theorem 3.3.3 we get

$$\omega(S_{n,n,\mu}(f); h, k) \leq h\left\|\frac{\partial S_{n,n,\mu}(f; x, y)}{\partial x}\right\| + k\left\|\frac{\partial S_{n,n,\mu}(f; x, y)}{\partial y}\right\|$$

$$\leq cn(h+k)\omega\left(f; \frac{1}{n}, \frac{1}{n}\right).$$

Hence by considering the cases $(h+k) \leq 1/n$ and $(h+k) > 1/n$ separately, we obtain the theorem. □

Taking into account the form of $S_{n,n,\mu}(f)(x, y)$, it is more natural to consider the so-called Euclidean bivariate modulus, defined by

$$\omega^{(E)}(f; \varrho) = \sup\{|f(x+h, y+k) - f(x, y)|;\ 0 \leq h, 0 \leq k, (h^2+k^2)^{1/2} \leq \varrho,$$

$$x, x+h \in [0, 1],\ \ y, y+k \in [0, 1]\},$$

(see, e.g., Anastassiou–Gal [6], p. 80, Definition 2.3.1).

With respect to this modulus we obtain the following:

Corollary 3.3.2. *For $\mu > 3/2$, $n \in \mathbb{N}$ and $\varrho \geq 0$, we have*

$$\omega^{(E)}(S_{n,n,\mu}(f); \varrho) \leq c\omega^{(E)}(f; \varrho).$$

Proof. Taking $h = k$ in Theorem 3.3.4, it follows that

$$\omega(S_{n,n,\mu}(f); h, h) \leq c\omega(f; h, h).$$

But by, e.g., Anastassiou–Gal [6], p. 81, we have

$$\omega^{(E)}(f; \varrho) \leq \omega(f; \varrho, \varrho) \leq \omega^{(E)}(f; \sqrt{2}\varrho) \leq 2\omega^{(E)}(f; \varrho),$$

which combined with the above inequality immediately proves the corollary. □

3.4 Bivariate Lagrange Polynomials

Let us define the bivariate Lagrange interpolation polynomials on the Chebyshev nodes of the second kind plus the endpoints ± 1, by

$$L_{n_1,n_2}(f)(x, y) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} h_{i,n_1}(x)h_{j,n_2}(y)f(x_{i,n_1}, y_{j,n_2}),$$

where

$$x_{i,n} = \cos\frac{i-1}{n-1}, \quad i = \overline{1,n},$$

and

$$h_{i,n}(x) = \frac{(-1)^{i-1}\omega_n(x)}{(1+\delta_{i1}+\delta_{in})(n-1)(x-x_{i,n})}, \quad \omega_n(x) = \sin t \sin(n-1)t.$$

Let us denote by $L_n(f,x)$ the univariate Lagrange polynomials based on Chebyshev nodes of the second kind plus the endpoints ± 1 and

$$E(\alpha,\delta) = \begin{cases} O\left[\delta^{\frac{\alpha}{2-\alpha}}\left(\ln\frac{1}{\delta}\right)^{\frac{2-2\alpha}{2-\alpha}}\right], & \text{if } 0 < \alpha < \frac{1}{2}, \\ O\left[\delta^{\frac{1}{3}}\ln\frac{1}{\delta}\right], & \text{if } \alpha = \frac{1}{2}, \\ O\left[\delta^{\frac{\alpha}{1+\alpha}}\left(\ln\frac{1}{\delta}\right)^{\frac{1}{1+\alpha}}\right], & \text{if } \frac{1}{2} < \alpha \le 1. \end{cases}$$

We need the following result in univariate case.

Theorem 3.4.1. (See Corollary 1.2.2) *If* $f \in \mathrm{Lip}_M(\alpha;[-1,1])$, $0 < \alpha \le 1$, *then*

$$\omega(L_n(f);\delta) = E(\alpha,\delta).$$

Firstly, we consider global smoothness preservation properties with respect to $\omega(f;\delta,\eta)$. The proofs of our main result require the following three lemmas.

Lemma 3.4.1. *For all* $(x,y) \in [-1,1]\times[-1,1]$ *we have:*

$$\left|\frac{\partial L_{n_1,n_2}(f)(x,y)}{\partial x}\right| \le cn_1 \sum_{i=1}^{n_1} \omega_1^{(x)}\left(f;\frac{1}{i^2}\right),$$

$$\left|\frac{\partial L_{n_1,n_2}(f)(x,y)}{\partial y}\right| \le cn_2 \sum_{j=1}^{n_2} \omega_1^{(y)}\left(f;\frac{1}{j^2}\right).$$

Here $c > 0$ *is an absolute constant (independent of* n_1, n_2, x, y *and* f*).*

Proof. Because $\sum_{i=1}^{n_1} h'_{i,n_1}(x) = 0$, we get

$$\left|\frac{\partial L_{n_1,n_2}(f)(x,y)}{\partial x}\right| = \left|\sum_{j=1}^{n_2} h_{j,n_2}(y)\left[\sum_{i=1}^{n_1} h'_{i,n_1}(x) f(x_i^{(1)},x_j^{(2)})\right]\right|$$

$$= \left|\sum_{j=1}^{n_2} h_{j,n_2}(y)\left\{\sum_{i=1}^{n_1} h'_{i,n_1}(x)[f(x_i^{(1)},x_j^{(2)}) - f(x,x_j^{(2)})]\right\}\right|$$

$$\leq \sum_{j=1}^{n_2} h_{j,n_2}(y) \sum_{i=1}^{n_1} |h'_{i,n_1}(x)| \omega_1^{(x)}(f; |x - x_i^{(1)}|)$$

(reasoning as in the univariate case in the proof of Theorem 1.2.2)

$$\leq \sum_{j=1}^{n_2} h_{j,n_2}(y) \left[nc_1 \sum_{i=1}^{n_1} \omega_1^{(x)} \left(f; \frac{1}{i^2} \right) \right] = cn_1 \sum_{i=1}^{n_1} \omega_1^{(x)} \left(f; \frac{1}{i^2} \right).$$

The estimate for $\left| \frac{\partial L_{n_1,n_2}(f)(x,y)}{\partial y} \right|$ is similar. □

Lemma 3.4.2. *We have the estimates:*

$$\omega^{(x)}(L_{n_1,n_2}(f); \delta) \leq \left\| \frac{\partial L_{n_1,n_2}(f)}{\partial x} \right\| \cdot \delta, \ \forall \, \delta \geq 0$$

and

$$\omega^{(y)}(L_{n_1,n_2}(f); \delta) \leq \left\| \frac{\partial L_{n_1,n_2}(f)}{\partial y} \right\| \cdot \delta, \ \forall \, \delta \geq 0$$

where $\| \cdot \|$ *represents the uniform norm on*

$$C[[-1, 1] \times [-1, 1]) = \{f : [-1, 1]^2 \to \mathbb{R}; \ f \text{ continuous on } [-1, 1]^2\}.$$

Proof. By the mean value theorem, we get

$$|L_{n_1,n_2}(f)(x + h, y) - L_{n_1,n_2}(f)(x, y)| = |h| \left| \frac{\partial L_{n_1,n_2}(f)(\xi, y)}{\partial x} \right|,$$

which immediately implies

$$\omega^{(x)}(L_{n_1,n_2}(f); \delta) \leq \delta \left\| \frac{\partial L_{n_1,n_2}(f)}{\partial x} \right\|.$$

The proof of second estimate is similar. □

Lemma 3.4.3. *We have the estimates:*

$$\omega^{(x)}(L_{n_1,n_2}(f); \delta) \leq 2\|L_{n_1,n_2}(f) - f\| + \omega^{(x)}(f; \delta),$$

and

$$\omega^{(y)}(L_{n_1,n_2}(f); \delta) \leq 2\|L_{n_1,n_2}(f) - f\| + \omega^{(y)}(f; \delta), \ \forall \, \delta \geq 0.$$

Proof. The first estimate is immediate by

$$|L_{n_1,n_2}(f)(x + h, y) - L_{n_1,n_2}(f)(x, y)| \leq |L_{n_1,n_2}(f)(x + h, y) - f(x + h, y)|$$

$$+|f(x + h, y) - f(x, y)| + |f(x, y) - L_{n_1,n_2}(f)(x, y)|.$$

We have a similar proof for the second estimate. □

We obtain the following consequences.

Corollary 3.4.1. *For any* $f \in C([-1,1]\times[-1,1])$, *any* $\delta > 0$ *and any* $n \in \mathbb{N}$, *we have:*

$$\omega^{(x)}(L_{n,n}(f);\delta) \le \min\left\{O\left(\delta n \sum_{i=1}^{n} \omega^{(x)}\left(f;\frac{1}{i^2}\right)\right),\right.$$

$$\left. O\left(\omega^{(x)}\left(f;\frac{1}{n}\right)\log n + \omega^{(y)}\left(f;\frac{1}{n}\right)\log n\right) + \omega^{(x)}(f;\delta)\right\}$$

and

$$\omega^{(y)}(L_{n,n}(f);\delta) \le \min\left\{O\left(\delta n \sum_{j=1}^{n} \omega^{(y)}\left(f;\frac{1}{j^2}\right)\right),\right.$$

$$\left. O\left[\omega^{(x)}\left(f;\frac{1}{n}\right)\log n + \omega^{(y)}\left(f;\frac{1}{n}\right)\log n\right] + \omega^{(y)}(f;\delta)\right\}.$$

Proof. It is immediate by Lemmas 3.4.1, 3.4.2, 3.4.3 for $n_1 = n_2 = n$ and by the fact that taking into account the technique in Shisha–Mond [92], p. 1275–1276 and the estimate in the univariate case, we have

$$\|L_{n_1,n_2}(f) - f\| \le c\left[\omega^{(x)}\left(f;\frac{1}{n_1}\right)\log n_1 + \omega^{(y)}\left(f;\frac{1}{n_2}\right)\log n_2\right].$$

□

Corollary 3.4.2. *Let* $f : [-1,1]\times[-1,1] \to \mathbb{R}$ *be Lipschitz of order* $\alpha \in (0,1]$ *with respect to x and y, respectively, i.e.,*

$$|f(x_1,y) - f(x_2,y)| \le L_1|x_1 - x_2|^{\alpha},\ \forall\, x_1, x_2, y \in [-1,1],$$

and

$$|f(x,y_1) - f(x,y_2)| \le L_2|y_1 - y_2|^{\alpha},\ \forall\, x, y_1, y_2 \in [-1,1].$$

Then, for all $n \in \mathbb{N}$, $\delta, \eta \in (0,1)$ *we have:*

$$\omega(L_{n,n}(f);\delta,\eta) \le C[\min\{\delta n, n^{-\alpha}\log n + \delta^{\alpha}\}$$

$$+\min\{\eta n, n^{-\alpha}\log n + \eta^{\alpha}\}],\ \text{if}\ \frac{1}{2} < \alpha \le 1;$$

$$\omega(L_{n,n}(f);\delta,\eta) \le C\left[\min\left\{\delta n\log n, \frac{\log n}{\sqrt{n}} + \sqrt{\delta}\right\}\right.$$

$$\left.+\min\left\{\eta n\log n, \frac{\log n}{\sqrt{n}} + \sqrt{\eta}\right\}\right],\ \text{if}\ \alpha = \frac{1}{2};$$

$$\omega(L_{n,n}(f);\delta,\eta) \le C[\min\{\delta n^{2-2\alpha}, n^{-\alpha}\log n + \delta^{\alpha}\}$$

$$+ \min\{\eta n^{2-2\alpha}, n^{-\alpha} \log n + \eta^{\alpha}\}], \text{ if } 0 < \alpha < \frac{1}{2}.$$

Proof. By Lemma 3.1.1,(i) we have

$$\omega(L_{n_1,n_2}(f); \delta, \eta) \le \omega^{(x)}(L_{n_1,n_2}(f); \delta) + \omega^{(y)}(L_{n_1,n_2}(f); \eta)$$

(actually $\omega(L_{n_1,n_2}(f); \delta, \eta) \sim [\omega^{(x)}(L_{n_1,n_2}(f); \delta) + \omega^{(y)}(L_{n_1,n_2}(f); \eta)]$ which justifies the method).

Because in general we have

$$\sum_{i=1}^{n} \frac{1}{i^{\alpha}} = \begin{cases} \mathcal{O}(n^{1-\alpha}), & \text{if } 0 < \alpha < 1 \\ \mathcal{O}(\log n), & \text{if } \alpha = 1 \\ C > 0, & \text{if } \alpha > 1 \end{cases}$$

it follows

$$n \sum_{i=1}^{n} \omega_1^{(z)}\left(f; \frac{1}{i^2}\right) \le C \begin{cases} n^{2-2\alpha}, & \text{if } 0 < \alpha < \frac{1}{2} \\ n \log n, & \text{if } \alpha = \frac{1}{2} \\ n, & \text{if } \frac{1}{2} < \alpha \le 1 \end{cases},$$

for $z := x$, $z := y$, $f \in Lip\alpha$ with respect to x and y.

Taking now $n = n_1 = n_2$ in Lemma 3.1.1, by Corollary 3.4.1 we get the statement in corollary. □

Now, we are in position to prove the following global smoothness preservation property.

Corollary 3.4.3. *Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be Lipschitz of order $\alpha \in (0, 1]$ with respect to x and y, respectively, which obviously is equivalent to*

$$\omega(f; \delta, \eta) \le C(\delta^{\alpha} + \eta^{\alpha}), \ \forall\, \delta, \eta > 0.$$

For all $n \in \mathbb{N}$, $\delta, \eta \in (0, 1)$, we have:

$$\omega(L_{n,n}(f); \delta, \eta) = E(\alpha, \delta) + E(\alpha, \eta).$$

Proof. By Corollary 3.4.2 and reasoning exactly as in the univariate (for example, if $1/2 < \alpha \le 1$, then we consider separately the cases $\delta n \le n^{-\alpha} \log n$, $\delta n > n^{-\alpha} \log n$, $\eta n \le n^{-\alpha} \log n$, $\eta n > n^{-\alpha} \log n$), we immediately obtain

$$\omega(L_{n,n}(f); \delta, \eta) = E(\alpha, \delta) + E(\alpha, \eta).$$ □

Remark. The method applied for Hermite–Fejér interpolation polynomials in Section 3.2, unfortunately does not work for Lagrange polynomials. As a consequence, the case $n_1 \neq n_2$ still remains open for Lagrange polynomials.

For the case of Bögel modulus of continuity, reasoning exactly as in the previous Section 3.2, we immediately get the following.

Corollary 3.4.4. *If* $G \in D_{a,b}$, $G(x, y) = F(f(x)g(y))$ *then*

$$\omega^{(B)}(L_{n_1,n_2}(G); \delta, \eta) = E(a, \delta)E(b, \eta).$$

Remark. All the above results can easily be extended for m variables, $m > 2$.

3.5 Bibliographical Remarks and Open Problems

All the results in this chapter except those where the authors are mentioned belong to Gal–Szabados [58].

Open Problem 3.5.1. Prove global smoothness preservation properties for the Shepard operators $S_{n_1,n_2,\mu}(f)(x, y)$ in Theorem 3.3.3 and for the Lagrange polynomials $L_{n_1,n_2}(f)(x, y)$ in Lemmas 3.4.1–3.4.3, for the general case $n_1 \neq n_2$. (The particular case $n_1 = n_2$ is solved by Theorem 3.3.4, Corollary 3.3.2 and Corollary 3.4.3.)

Open Problem 3.5.2. For the bivariate tensor product Shepard operators on the semi-axis, generated by the univariate case in Della Vecchia–Mastroianni–Szabados [36], prove global smoothness preservation properties.

Open Problem 3.5.3. For the kinds of univariate Shepard operators considered by Chapters 1 and 2, various bivariate combinations different from those considered by Chapter 3 can be considered as follows.

We suppose that all the kinds of Shepard operators are defined on equidistant nodes in the interval $[-1, 1] \times [-1, 1]$, i.e., are of the form $x_k = \frac{k}{n}$, $k = -n, \dots, 0, \dots, n$ and $y_j = \frac{j}{m}$, $j = -m, \dots, 0, \dots, m$.

Type 1. "Original Shepard–Lagrange operator,"

$$S_{n,p,L_m}(f)(x, y) = \sum_{i=-n}^{n} s_{i,n,p}(x, y)L_m^i(f)(x, y),$$

where

$$s_{i,n,p}(x, y) = \frac{[(x - x_i)^2 + (y - y_i)^2]^{-p}}{\sum_{k=-n}^{n}[(x - x_k)^2 + (y - y_k)^2]^{-p}}, \quad p > 2, p \in \mathbb{N},$$

and $L_m^i(f)(x, y)$ is the polynomial defined as in, e.g., Coman–Trimbitas [27], on page 43.

Type 2. "Tensor product Shepard–Lagrange operator"

$$S_{n,m,2p,2q,n_1,n_2}(f)(x, y)$$
$$= \sum_{i=-n}^{n} \sum_{j=-m}^{m} s_{i,n,2p}(x)s_{j,m,2q}(y)L_{n_1,n_2}^{i,j}(f)(x, y), \; p, q \geq 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x - x_i)^{-2p}}{\sum_{k=-n}^{n} (x - x_k)^{-2p}},$$

$$s_{j,m,2q}(y) = \frac{(y - y_j)^{-2q}}{\sum_{k=-m}^{m} (y - y_k)^{-2q}},$$

and

$$L^{i,j}_{n_1,n_2}(f)(x, y)$$

$$= \sum_{\nu=0}^{n_1} \sum_{\mu=0}^{n_2} \frac{u_i(x)}{(x - x_{i+\nu})u_i'(x_{i+\nu})} \frac{v_j(y)}{(y - y_{j+\mu})v_j'(y_{j+\mu})} f(x_{i+\nu}, y_{j+\mu}),$$

with $u_i(x) = (x - x_i) \cdots (x - x_{i+n_1})$, $v_j(y) = (y - y_j) \cdots (y - y_{j+n_2})$ and $x_{n_1+\nu} = x_\nu$, $y_{n_2+\mu} = y_\mu$.

Type 3. "Tensor product Shepard–Lagrange-Taylor operator"

$$S_{n,m,2p,2q,n_1,n_2}(f)(x, y)$$

$$= \sum_{i=-n}^{n} \sum_{j=-m}^{m} s_{i,n,2p}(x) s_{j,m,2q}(y) LT^{i,j}_{n_1,n_2}(f)(x, y), \quad p, q \geq 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x - x_i)^{-2p}}{\sum_{k=-n}^{n} (x - x_k)^{-2p}},$$

$$s_{j,m,2q}(x) = \frac{(y - y_j)^{-2q}}{\sum_{k=-m}^{m} (y - y_k)^{-2q}},$$

and

$$LT^{i,j}_{n_1,n_2}(f)(x, y) = \sum_{\nu=0}^{n_1} \sum_{\mu=0}^{n_2} \frac{u_i(x)}{(x - x_{i+\nu})u_i'(x_{i+\nu})} \frac{(y - y_j)^\mu}{(\mu)!} \frac{\partial^\mu f(x_{i+\nu}, y_j)}{\partial y^\mu}.$$

Obviously, if we change the place of x and y, then we get the "tensor product Shepard–Taylor–Lagrange" type operator, which is similar.

Type 4. "Original Shepard–Taylor operator"

$$S_{n,p,T_{m_{-n},\ldots,m_n}}(f)(x, y) = \sum_{i=-n}^{n} s_{i,n,p}(x, y) T^i_{m_i}(f)(x, y),$$

where

$$s_{i,n,p}(x,y) = \frac{[(x-x_i)^2 + (y-y_i)^2]^{-p}}{\sum_{k=-n}^{n} [(x-x_k)^2 + (y-y_k)^2]^{-p}}, \quad p > 2, p \in \mathbb{N},$$

and

$$T_{m_i}^i(f)(x,y) = \sum_{r+s \le m_i} \frac{(x-x_i)^r}{r!} \frac{(y-y_j)^s}{s!} \frac{\partial^{r+s} f(x_i, y_i)}{\partial x^r \partial y^s}.$$

Type 5. "Tensor product Shepard–Taylor operator"

$$S_{n,m,2p,2q,n_{-n},\ldots,n_n,m_{-m},\ldots,m_m}(f)(x,y)$$

$$= \sum_{i=-n}^{n} \sum_{j=-m}^{m} s_{i,n,2p}(x) s_{j,m,2q}(y) TT_{n_i,m_j}^{i,j}(f)(x,y), \quad p, q \ge 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x-x_i)^{-2p}}{\sum_{k=-n}^{n} (x-x_k)^{-2p}},$$

$$s_{j,m,2q}(x) = \frac{(y-y_j)^{-2q}}{\sum_{k=-m}^{m} (y-y_k)^{-2q}},$$

and

$$TT_{n_i,m_j}^{i,j}(f)(x,y) = \sum_{\nu=0}^{n_i} \sum_{\mu=0}^{m_j} \frac{(x-x_i)^\nu}{(\nu)!} \frac{(y-y_j)^\mu}{(\mu)!} \frac{\partial^{\nu+\mu} f(x_i, y_j)}{\partial x^\nu \partial y^\mu}.$$

Type 6. "Shepard–Gal–Szabados–Taylor operator"

$$S_{n,m,p,n_{-n},\ldots,n_n,m_{-m},\ldots,m_m}(f)(x,y)$$

$$= \frac{\sum_{i=-n}^{n} \sum_{j=-m}^{m} [(x-x_i)^2 + (y-y_j)^2]^{-p} TT_{n_i,m_j}^{i,j}(f)(x,y)}{\sum_{i=-n}^{n} \sum_{j=-m}^{m} [(x-x_i)^2 + (y-y_j)^2]^{-p}},$$

where

$$TT_{n_i,m_j}^{i,j}(f)(x,y) = \sum_{\nu=0}^{n_i} \sum_{\mu=0}^{m_j} \frac{(x-x_i)^\nu}{(\nu)!} \frac{(y-y_j)^\mu}{(\mu)!} \frac{\partial^{\nu+\mu} f(x_i, y_j)}{\partial x^\nu \partial y^\mu}.$$

It would be interesting to find global smoothness preservation properties (with respect to various bivariate moduli of continuity) of the above six types of bivariate Shepard operators.

Open Problem 3.5.4. Starting from the local variants of Shepard operators in the univariate case defined in Open Problem 1.6.13 (Chapter 1), local bivariate tensor products can be defined as in the above Open Problem 3.5.3.

The problem is to find global smoothness preservation properties (with respect to various bivariate moduli of continuity) of these local bivariate Shepard operators.

4

Partial Shape Preservation, Bivariate Case

In this chapter we extend the results of Chapter 2 to the bivariate case.

4.1 Introduction

As it was pointed out in Chapter 2, it is evident that because of the interpolation conditions, the interpolating operators do not completely preserve the shape of an univariate function f, on the whole interval that contains the points of interpolation. A key result used in the univariate case for the proofs of qualitative-type results, is the following simple one

Lemma 4.1.1. (Popoviciu [81]). *Let $f : [a, b] \to \mathbb{R}$, $a \leq x_1 < x_2 < \cdots < x_n \leq b$ and $F_n(f)(x) = \sum_{i=1}^{n} h_i(x) f(x_i)$, where $h_i \in C^1[a, b]$ and $\sum_{i=1}^{n} h_i(x) = 1$, $\forall x \in [a, b]$.*

(i) *We have*

$$F_n'(f)(x) = \sum_{i=1}^{n-1} \left(-\sum_{j=1}^{i} h_j'(x) \right) [f(x_{i+1}) - f(x_i)].$$

(ii) *If there exists $x_0 \in (a, b)$ such that $h_1'(x_0) < 0$, $h_n'(x_0) > 0$ and the sequence $h_1'(x_0), h_2'(x_0), \ldots, h_n'(x_0)$ has a unique variation of sign, then*

$$-\sum_{j=0}^{i} h_j'(x_0) > 0, \quad \textit{for all } i = \overline{1, n-1},$$

and consequently by (i) there exists a neighborhood $V(x_0)$ of x_0, where the monotonicity of f assumed on the whole $[a, b]$ is preserved.

In this chapter qualitative and quantitative results for bivariate Hermite–Fejér polynomial and Shepard operators are obtained. New aspects appear because of various possible natural concepts for bivariate monotonicity and convexity. Also, three different kinds of bivariate Shepard operators are studied.

4.2 Bivariate Hermite–Fejér Polynomials

If $g: [-1,1] \to \mathbb{R}$ and $-1 < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < 1$ are the roots of Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$, then it is well known that (see, e.g., Fejér [43] or Popoviciu [81]) the (univariate) Hermite–Fejér polynomials based on the roots above are given by $F_n(g)(x) = \sum_{i=1}^n h_{i,n}(x)g(x_{i,n})$, where

$$h_{i,n}(x) = \ell_{i,n}^2(x) \cdot \left[1 - \frac{\ell_n''(x_{i,n})}{\ell_n'(x_{i,n})}(x - x_{i,n})\right],$$

$$\ell_{i,n}(x) = \frac{\ell_n(x)}{(x - x_{i,n})\ell_n'(x_{i,n})}, \quad \ell_n(x) = \prod_{i=1}^n (x - x_{i,n}).$$

We have $\sum_{i=1}^n h_{i,n}(x) = 1$, for all $x \in [-1,1]$.

Now, if $f: [-1,1] \times [-1,1] \to \mathbb{R}$, then according to, e.g., Shisha–Mond [92], the bivariate Hermite–Fejér polynomial is defined by

$$F_{n_1,n_2}(f)(x,y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h_{i,n_1}^{(1)}(x) h_{j,n_2}^{(2)}(y) f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) \tag{4.1}$$

where $h_{i,n_1}^{(1)}(x), x_{i,n_1}^{(1)}, i = \overline{1,n_1}$ and $h_{j,n_2}^{(2)}(y)$ and $x_{j,n_2}^{(2)}, j = \overline{1,n_2}$ are defined as in the univariate case above, $n_1, n_2 \in \mathbb{N}$.

We easily see that

$$F_{n_1,n_2}(f)(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) = f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}), \quad \forall i = \overline{1,n_1}, j = \overline{1,n_2}.$$

The key result of this section is

Theorem 4.2.1. *With the notations above, we have*

$$\begin{aligned}\frac{\partial^2 F_{n_1,n_2}(f)(x,y)}{\partial x \partial y} = \sum_{i=1}^{n_1-1} \Bigg[\left(\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x) \right) \cdot \Bigg\{ \sum_{j=1}^{n_2-1} \left(\sum_{q=1}^{j} h_{q,m}^{(2)'}(y) \right) \\ \cdot (f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i,n_1}^{(1)}, x_{j+1,n_2}^{(2)}) \\ - f(x_{i+1,n_1}^{(1)}, x_{j,n_2}^{(2)}) + f(x_{i+1,n_1}^{(1)}, x_{j+1,n_2}^{(2)})) \Bigg\} \Bigg].\end{aligned}$$

Proof. We observe

$$\frac{\partial F_{n_1,n_2}(f)(x,y)}{\partial x} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h_{i,n_1}^{(1)'}(x) h_{j,n_2}^{(2)}(y) f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)})$$

$$= \sum_{j=1}^{n_2} \left(\sum_{i=1}^{n_1} h_{i,n_1}^{(1)'}(x) f(x_{i,n_1}, x_{j,n_2}) \right) h_{j,n_2}^{(2)}(y) (\text{by Lemma 4.1.1 (i) })$$

$$= \sum_{j=1}^{n_2}\Bigg[\sum_{i=1}^{n_1-1}\left(\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x)\right)(f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i+1,n_1}^{(1)}, x_{j,n_2}^{(2)}))\Bigg] \cdot h_{j,n_2}^{(2)}(y)$$

$$= \sum_{i=1}^{n_1-1}\left(\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x)\right) \cdot \left\{\sum_{j=1}^{n_2} h_{j,n_2}^{(2)}(y)(f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i+1,n_1}^{(1)}, x_{j,n_2}^{(2)}))\right\}.$$

It follows that

$$\frac{\partial^2 F_{n_1,n_2}(f)(x,y)}{\partial x \partial y} = \sum_{i=1}^{n_1-1}\Bigg[\left(\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x)\right) \cdot \Bigg\{\sum_{j=1}^{n_2} h_{j,n_2}^{(2)}(y)$$

$$\cdot(f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i+1,n_1}^{(1)}, x_{j,n_2}^{(2)}))\Bigg\}\Bigg]_y' \text{ (by Lemma 4.1.1 (i))}$$

$$= \sum_{i=1}^{n_1-1}\left(\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x)\right) \cdot \sum_{j=1}^{n_2-1}\left(\sum_{q=1}^{j} h_{q,n_2}^{(2)'}(y)\right)$$

$$\cdot(f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i+1,n_1}^{(1)}, x_{j,n_2}^{(2)}) - f(x_{i,n_1}^{(1)}, x_{j+1,n_2}^{(2)}) + f(x_{i+1,n_1}^{(1)}, x_{j+1,n_2}^{(2)})),$$

which proves the theorem. □

Also, we need the following:

Definition 4.2.1. (see, e.g., Marcus [71], p. 33). We say that $f: [a,b]\times[c,d] \to \mathbb{R}$ is bidimensional or hyperbolical upper (lower) monotone on $[a,b]\times[c,d]$, if

$$\Delta_2(f)(x,y;\alpha,\beta) = f(x+\alpha, y+\beta) - f(x, y+\beta) - f(x+\alpha, y) + f(x,y) \geq 0$$

(≤ 0, respectively), for all $\alpha, \beta \geq 0$ and $(x,y) \in [a,b]\times[c,d]$ such that $(x+\alpha, y+\beta) \in [a,b]\times[c,d]$.

Remark. If $f \in C^2([a,b]\times[c,d])$ and $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$, for all $(x,y) \in [a,b]\times[c,d]$, then f is bidimensional upper monotone on $[a,b]\times[c,d]$ (see, e.g., Marcus [71]).

Corollary 4.2.1. *Let $n_1 = 2p_1$, $n_2 = 2p_2$ be even numbers and let us consider the bivariate Hermite–Fejér polynomials $F_{n_1,n_2}(f)(x,y)$ given by (4.1), based on the roots $x_{i,n_1}^{(1)}$, $i = \overline{1,n_1}$ of λ_1-ultraspherical polynomials of degree n_1 with $\lambda_1 > -1$ (i.e., the Jacobi polynomials $J_{n_1}^{(\alpha_1,\beta_1)}$ with $\alpha_1 = \beta_1$, $\lambda_1 = \alpha_1+\beta_1+1$, $-1 < \alpha_1, \beta_1 \leq 1$) and on the roots $x_{j,n_2}^{(2)}$, $j = \overline{1,n_2}$ of λ_2-ultraspherical polynomials of degree n_2, $J_{n_2}^{(\alpha_2,\beta_2)}$, $\lambda_2 > -1$ (i.e., $\alpha_2 = \beta_2$, $\lambda_2 = \alpha_2+\beta_2+1$, $-1 < \alpha_2, \beta_2 \leq 1$). There exists a constant $c > 0$ (independent of f and n_1, n_2) such that if $f: [-1,1]\times[-1,1] \to \mathbb{R}$ is bidimensional monotone on $[-1,1]\times[-1,1]$, then $F_{n_1,n_2}(f)(x,y)$ is bidimensional monotone (of the same monotonicity) on $\left(-\frac{c}{n_1^4}, \frac{c}{n_1^4}\right) \times \left(-\frac{c}{n_2^4}, \frac{c}{n_2^4}\right)$.*

Proof. By the proof of Theorem 2.2.1 (see relation (2.2) and the last relation there), we have

$$\sum_{p=1}^{i} h_{p,n_1}^{(1)'}(x) > 0, \ \sum_{q=1}^{j} h_{q,n_2}^{(2)'}(y) > 0, \quad \forall i = \overline{1, n_1 - 1}, \, j = \overline{1, n_2 - 1},$$

$$\forall x \in \left(-\frac{c}{n_1^4}, \frac{c}{n_1^4}\right), \quad \forall y \in \left(-\frac{c}{n_2^4}, \frac{c}{n_2^4}\right).$$

Taking into account Theorem 4.2.1, we obtain

$$\frac{\partial^2 F_{n_1,n_2}(f)(x,y)}{\partial x \partial y} \geq 0, \quad \forall (x,y) \in \left(-\frac{c}{n_1^4}, \frac{c}{n_1^4}\right) \times \left(-\frac{c}{n_2^4}, \frac{c}{n_2^4}\right),$$

which by the Remark after Definition 4.2.1 proves the theorem. □

Corollary 4.2.2. *Let us consider $F_{n_1,n_2}(f)(x,y)$ given by (4.1), based on the roots of Jacobi polynomials $J_{n_1}^{(\alpha_1,\beta_1)}$, $J_{n_2}^{(\alpha_2,\beta_2)}$, of degree n_1 and n_2, respectively, with $\alpha_i, \beta_i \in (-1, 0]$, $i = 1, 2$. If ξ is any root of the polynomial $\ell_{n_1}^{(1)'}(x)$ and η is any root of the polynomial $\ell_{n_2}^{(2)'}(y)$ (here $\ell_{n_1}^{(1)}(x) = \prod_{i=1}^{n_1}(x - x_{i,n_1}^{(1)})$, $\ell_{n_2}^{(2)}(y) = \prod_{j=1}^{n_2}(y - x_{j,n_2}^{(2)})$), then there exists a constant $c > 0$ (independent of n_1, n_2 and f) such that if f is bidimensional monotone on $[-1, 1] \times [-1, 1]$, then $F_{n_1,n_2}(f)(x,y)$ is bidimensional monotone (of the same monotonicity) on*

$$\left(\xi - \frac{c_\xi}{n_1^{7+2\gamma_1}}, \xi + \frac{c_\xi}{n_1^{7+2\gamma_1}}\right) \times \left(\eta - \frac{c_\eta}{n_2^{7+2\gamma_2}}, \eta + \frac{c_\eta}{n_2^{7+2\gamma_2}}\right) \subset (-1, 1) \times (-1, 1),$$

where

$$c_\xi = \frac{c}{(1-\xi^2)^{5/2+\delta_1}}, \ c_\eta = \frac{c}{(1-\eta^2)^{5/2+\delta_2}}, \ \gamma_i = \max\{\alpha_i, \beta_i\}, i = 1, 2$$

and

$$\delta_1 = \begin{cases} \alpha_1, & \text{if } 0 \leq \xi < 1 \\ \beta_1, & \text{if } -1 < \xi \leq 0, \end{cases}$$

$$\delta_2 = \begin{cases} \alpha_2, & \text{if } 0 \leq \eta < 1 \\ \beta_2, & \text{if } -1 < \eta \leq 0. \end{cases}$$

Proof. An immediate consequence of Theorem 4.2.1 above and of Theorem 2.2.2. □

Remarks. (1) Because $\ell_{n_1}^{(1)'}$ and $\ell_{n_2}^{(2)'}$ have exactly $n_1 - 1$ and $n_2 - 1$ roots in $(-1, 1)$, respectively, it follows that in $(-1, 1) \times (-1, 1)$ there exists a grid of $(n_1 - 1)(n_2 - 1)$ points (ξ, η) from Corollary 4.2.2.

(2) From Remark 1, after Theorem 2.2.2, it follows that if ξ and η are near the endpoints in the ultra-spherical case, for example, (i.e., $\alpha_i = \beta_i \in (-1, 0)$, $i = 1, 2$) then the best possible bidimensional interval of preservation of bidimensional monotonicity is $\left(\xi - \frac{c}{n_1^2}, \xi - \frac{c}{n_1^2}\right) \times \left(\eta - \frac{c}{n_2^2}, \eta - \frac{c}{n_2^2}\right)$.

In what follows we will extend the convexity problem from the univariate case. In this sense, we need the following.

Definition 4.2.2. (see, e.g., Nicolescu [76]). We say that $f : [-1,1]\times[-1,1] \to \mathbb{R}$ is strictly double convex on $[-1,1]\times[-1,1]$, if $\Delta_{h_1}^{2,y}[\Delta_{h_2}^{2,x} f(a,b)] > 0$, for all $h_1, h_2 > 0$, $(a,b) \in [-1,1]\times[-1,1]$, with $a \pm h_2, b \pm h_1 \in [-1,1]$, where

$$\Delta_{h_2}^{2,x} f(\alpha,\beta) = f(\alpha+h_2,\beta) - 2f(\alpha,\beta) + f(\alpha-h_2,\beta)$$

and

$$\Delta_{h_1}^{2,y} f(\alpha,\beta) = f(\alpha,\beta+h_1) - 2f(\alpha,\beta) + f(\alpha,\beta-h_1).$$

Remark. By the mean value theorem it is easy to see that if $\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2}(x,y) > 0$, for all $(x,y) \in [-1,1]^2$, then f is strictly double convex on $[-1,1]^2$.

Now, let $n_1, n_2 \geq 3$ be odd and let us consider as $F_{n_1,n_2}(f)(x,y)$ the Hermite–Fejér polynomial given by (4.1), based on the roots $x_{i,n}^{(1)}$, $i = \overline{1,n_1}$ and $x_{j,n_2}^{(2)}$, $j = \overline{1,n_2}$ of the λ_1-ultraspherical polynomials $p_{n_1}^{(\lambda_1)}$ of degree n_1 and λ_2-ultraspherical polynomials $P_{n_2}^{(\lambda_2)}$ of degree n_2, respectively, $\lambda_1, \lambda_2 \in [0,1]$, and the Côtes–Christoffel numbers of the Gauss–Jacobi quadrature

$$\begin{aligned}\lambda_{i,n_1}^{(1)} &:= 2^{2-\lambda_1}\pi\left[\Gamma\left(\frac{\lambda_1}{2}\right)\right]^{-2}\frac{\Gamma(n_1+\lambda_1)}{\Gamma(n_1+1)}[1-(x_{i,n_1}^{(1)})^2]^{-1}\\ &\quad\cdot[P_{n_1}^{(\lambda_1)'}(x_{i,n_1}^{(1)})]^{-2}, \quad i = \overline{1,n_1},\\ \lambda_{j,n_2}^{(2)} &:= 2^{2-\lambda_2}\pi\left[\Gamma\left(\frac{\lambda_2}{2}\right)\right]^{-2}\frac{\Gamma(n_2+\lambda_2)}{\Gamma(n_2+1)}[1-(x_{j,n_2}^{(2)})^2]^{-1}\\ &\quad\cdot[P_{n_2}^{(\lambda_2)'}(x_{j,n_2})]^{-2}, \quad j = \overline{1,n_2}.\end{aligned}$$

Theorem 4.2.2. *If $f \in C([-1,1]\times[-1,1])$ satisfies*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\lambda_{i,n_1}^{(1)}\lambda_{j,n_2}^{(2)}\Delta_{x_{j,n_2}^{(2)}}^{2,y}\frac{\Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0,0)}{(x_{i,n_1}^{(1)}x_{j,n_2}^{(2)})^2} > 0, \tag{4.2}$$

then $F_{n_1,n_2}(f)(x,y)$ is strictly double convex in $V(0,0) = \{(x,y); x^2+y^2 < d_{n_1,n_2}^2\}$, with

$$|d_{n_1,n_2}|$$

$$\geq c_{f,\lambda_1,\lambda_2}\frac{n_1 n_2 \sum_{i=1}^{\frac{n_1-1}{2}}\sum_{j=1}^{\frac{n_2-1}{2}}\lambda_{i,n_1}^{(1)}\lambda_{j,n_2}^{(2)}\Delta_{x_{j,n_2}^{(2)}}^{2,y}[\Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0,0)]/(x_{i,n_1}^{(1)}x_{j,n_2}^{(2)})^2}{(n_1+n_2)^5},$$

where $c_{f_1\lambda_1,\lambda_2} > 0$ is independent of n_1 and n_2.

Proof. We observe

$$\frac{\partial^4 F_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j,n_2}^{(2)''}(y) \left(\sum_{i=1}^{n_1} h_{i,n_1}^{(1)''}(x) f(x_{i,n_1}^{(1)}, x_{j,n_2}^{(2)}) \right)$$

and reasoning as in the proof of Theorem 2.2.3 in (see relation (2.7) there) we obtain

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j,n_2}^{(2)''}(0) \left[\sum_{i=1}^{\frac{n_1-1}{2}} h_{i,n_1}^{(1)''}(0) \Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0, x_{j,n_2}^{(2)}) \right].$$

Denoting $G(y) = \sum_{i=1}^{\frac{n_1-1}{2}} h_{i,n_1}^{(1)''}(0) \cdot \Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0,y)$, we get

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_2} h_{j,n_2}^{(2)''}(0) G(x_{j,n_2}^{(2)}) = \sum_{j=1}^{\frac{n_2-1}{2}} h_{j,n_2}^{(2)''}(0) \Delta_{x_{j,n_2}^{(2)}}^{2,y} G(0)$$

$$= \sum_{j=1}^{\frac{n_2-1}{2}} \sum_{i=1}^{\frac{n_1-1}{2}} h_{j,n_2}^{(2)''}(0) h_{i,n_1}^{(1)''}(0) \Delta_{x_{j,n_2}^{(2)}}^{2,y} [\Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0,0)].$$

Therefore, again by relation (2.7) and by hypothesis we obtain

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} \geq c_3 \lambda_1 \lambda_2 n_1 n_2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda_{i,n_1}^{(1)} \lambda_{j,n_2}^{(2)} \Delta_{x_{j,n_2}^{(2)}}^{2,y}$$

$$[\Delta_{x_{i,n_1}^{(1)}}^{2,x} f(0,0)] / (x_{j,n_1}^{(2)} x_{i,n_2}^{(1)})^2 > 0. \tag{4.3}$$

So it follows that $F_{n_1,n_2}(f)(x,y)$ is strictly double convex in a neighborhood of $(0,0)$. Let $(\alpha_{n_1,n_2}, \beta_{n_1,n_2})$ be the nearest root of $\frac{\partial^4 F_{n_1,n_2}(f)}{\partial x^2 \partial y^2}$ to $(0,0)$, in the sense that the distance $d_{n_1,n_2} = \sqrt{\alpha_{n_1,n_2}^2 + \beta_{n_1,n_2}^2}$ is minimum for all the roots of $\frac{\partial^4 F_{n_1,n_2}(f)}{\partial x^2 \partial y^2}$. Then, for all $(x,y) \in V(0,0) = \{(x,y) \in \mathbb{R}^2; \sqrt{x^2+y^2} < d_{n_1,n_2}\}$ we necessarily have $\frac{\partial^4 F_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} > 0$. By the mean value theorem for bivariate functions we get

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} = \left| \frac{\partial F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} - \frac{\partial^4 F_{n_1,n_2}(f)(\alpha_{n_1,n_2}, \beta_{n_1,n_2})}{\partial x^2 \partial y^2} \right|$$

$$\leq |\alpha_{n_1,n_2}| \cdot \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| + |\beta_{n_1,n_2}| \cdot \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3} \right|$$

$$\leq |d_{n_1,n_2}| \cdot \left[\left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2} \right| + \left| \frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3} \right| \right].$$

Because degree $(F_{n_1,n_2}(f)) \leq 2n_1 - 1 + 2n_2 - 1 = 2n_1 + 2n_2 - 2$, we have degree $\left(\frac{\partial^5 F_{n_1,n_2}(f)}{\partial x^3 \partial y^2} \right) \leq 2n_1 + 2n_2 - 7$, degree $\left(\frac{\partial^5 F_{n_1,n_2}(f)}{\partial x^2 \partial y^3} \right) \leq 2n_1 + 2n_2 - 7$. As in the proof of Theorem 2.2.3, we can assume that the interval of convexity cannot be larger than $\left[-\frac{c_1}{n_1}, \frac{c_1}{n_1}\right] \times \left[-\frac{c_2}{n_2}, \frac{c_2}{n_2}\right]$. Consequently, we may assume that $|d_{n_1,n_2}| \leq \frac{c}{\min\{n_1,n_2\}}$.

Now, by the Bernstein theorem in Kroó–Révész [67], p. 136, relation (8), we obtain

$$\left|\frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2}\right| \leq c(2n_1+2n_2-7)(2n_1+2n_2-6)(2n_1+2n_2-5)$$

$$\cdot(2n_1+2n_2-4)(2n_1+2n_2-3) \cdot \|F_{n_1,n_2}\|_{C([-1,1]\times[-1,1])}$$
$$\leq c(n_1+n_2)^5 \cdot \|F_{n_1,n_2}\|_{C([-1,1]\times[-1,1])}.$$

But because by Fejér [43], the fundamental interpolation polynomials $h^{(1)}_{i,n_1}(x)$ and $h^{(2)}_{j,n_2}(y)$ are ≥ 0, $\forall i = \overline{1,n_1}$, $\forall j = \overline{1,n_2}$, $\forall (x,y) \in [-1,1]\times[-1,1]$, denoting $M_f = \|f\|_{C([-1,1]\times[-1,1])}$, it follows that

$$\left|\frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^3 \partial y^2}\right| \leq c(n_1+n_2)^5 M_f, \quad \left|\frac{\partial^5 F_{n_1,n_2}(f)(\xi,\eta)}{\partial x^2 \partial y^3}\right| \leq c(n_1+n_2)^5 M_f$$

and consequently

$$\frac{\partial^4 F_{n_1,n_2}(f)(0,0)}{\partial x^2 \partial y^2} \leq c_f(n_1+n_2)^5 d_{n_1,n_2},$$

where $c_f > 0$ is independent of n_1 and n_2 (but dependent on f).

Combining this estimate with (4.3), we easily get the lower estimate for $|d_{n_1,n_2}|$ in the statement of Theorem 4.2.2. □

Remarks. (1) As in the univariate case, the neighborhood $V(0,0)$ of preservation of strict convexity depends on f too.

(2) The estimate of $|d_{n_1,n_2}|$ in the bivariate case seems to be weaker, in a sense, than that of the univariate case, because it was not proved yet to be a Stechkin-type inequality for bivariate polynomials. That would be useful for a better estimate.

(3) If $f : [-1,1]\times[-1,1] \to \mathbb{R}$ is strictly double convex on $[-1,1]\times[-1,1]$, then the condition (4.2) is obviously satisfied and consequently $F_{n_1,n_2}(f)(x,y)$ preserve the strictly double convexity in a disc centered at (0, 0), having for its ray $|d_{n_1,n_2}|$ the lower estimate of Theorem 4.2.2.

(4) Let $F_{n_1,n_2}(f)(x,y)$ be given by (4.1), based on the roots $x^{(1)}_{i,n_1}$, $i = \overline{1,n_1}$ and $x^{(2)}_{j,n_2}$, $j = \overline{1,n_2}$ of the λ_1-ultraspherical polynomials $P^{(\lambda_1)}_{n_1}$ and λ_2-ultraspherical polynomials $P^{(\lambda_2)}_{n_2}$, respectively, where $\lambda_1, \lambda_2 \in [0,1]$. Because by Fejér [43], the polynomials $h^{(1)}_{i,n_1}(x)$, $h^{(2)}_{j,n_2}(y) \geq 0$, $\forall i = \overline{1,n_1}$, $\forall j = \overline{1,n_2}$, $\forall (x,y) \in [-1,1]\times[-1,1]$, by the formulas

$$\frac{\partial^p F_{n_1,n_2}(f)(x,y)}{\partial y^p} = \sum_{i=1}^{n_1} h^{(1)}_{i,n_1}(x)\left[\sum_{j=1}^{n_2} \frac{\partial^p h^{(2)}_{j,n_2}(y)}{\partial y^p} f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2})\right],$$

$p = 1, 2$, from the univariate case in Section 2.2, the following results are immediate: If $f(x, y)$ is nondecreasing with respect to $y \in [-1, 1]$ (for all fixed $x \in [-1, 1]$), then for $n_1, n_2 \in \mathbb{N}$ and η root of $P_{n_2}^{(\lambda_2)'}(y)$, $F_{n_1,n_2}(f)(x, y)$ is nondecreasing with respect to

$$y \in \left(\eta - \frac{c_\eta}{n_2^{7+2\gamma_2}}, \eta + \frac{c_\eta}{n_2^{7+2\gamma_2}}\right),$$

for all fixed $x \in [-1, 1]$ (here c_η and γ_2 are given by Corollary 4.2.2).

If $f(x, y)$ is strictly convex with respect to $y \in [-1, 1]$ (for all fixed $x \in [-1, 1]$), then for all $n_1 \in \mathbb{N}$, arbitrary and $n_2 \in \mathbb{N}$, $n_2 \geq 3$, n_2 odd number, there exists a neighborhood $V(0)$ of 0, such that for all fixed $x \in [-1, 1]$, $F_{n_1,n_2}(f)(x, y)$ is strictly convex with respect to $y \in V(0)$.

Similar results hold if we consider $\frac{\partial^p F_{n_1,n_2}(f)(x,y)}{\partial x^p}$, $p = 1, 2$.

(5) All the results above can easily be extended for n variables, $n > 2$.

4.3 Bivariate Shepard Operators

For the three types of bivariate Shepard operators considered in Section 3 of Chapter 3, we prove here that preserve natural kinds of bivariate monotonicity and convexity in the neighborhoods of some points and quantitative estimates of the lengths of these neighborhoods are obtained.

Let us first consider the bivariate Shepard operator defined as a tensor product by

$$S_{n,m}^{(\lambda,\mu)}(f; x, y) = \sum_{i=-n}^{n} \sum_{j=-m}^{m} s_{i,\lambda}(x) s_{j,\mu}(y) f(x_i, y_j), \text{ if } (x, y) \neq (x_i, y_j),$$

$S_{n,m}^{(\lambda,\mu)}(f; x_i, y_j) = f(x_i, y_j)$, where $1 < \lambda, \mu$, $-1 \leq x_{-n} < \cdots < x_n \leq 1$, $-1 \leq y_{-m} < \cdots < y_m \leq 1$ and $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$,

$$s_{i,\lambda}(x) = |x - x_i|^{-\lambda} / \left[\sum_{k=-n}^{n} |x - x_k|^{-\lambda}\right],$$

$$s_{j,\mu}(y) = |y - y_j|^{-\mu} / \left[\sum_{k=-m}^{m} |y - y_k|^{-\mu}\right].$$

The global smoothness preservation properties and convergence properties of these operators were studied in Section 3 of Chapter 3. In this section we consider their properties of preservation of shape.

In this sense, a key result is the following.

Theorem 4.3.1. *With the notations above, we have:*

$$\frac{\partial^2 S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x \partial y} = \sum_{i=-n}^{n-1} \left(-\sum_{p=-n}^{i} s'_{p,\lambda}(x)\right) \sum_{j=-m}^{m-1} \left(-\sum_{q=-m}^{j} s'_{q,\mu}(y)\right)$$

$$\times(f(x_i, y_j) - f(x_i, y_{j+1}) - f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})).$$

Proof. We observe

$$\frac{\partial S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x} = \sum_{i=-n}^{n} \sum_{j=-m}^{m} s'_{i,\lambda}(x) s_{j,\mu}(y) f(x_i, y_j)$$

$$= \sum_{j=-m}^{m} \left(\sum_{i=-n}^{n} s'_{i,\lambda}(x) f(x_i, y_j) \right) s_{j,\mu}(y) \text{ (by Lemma 4.1.1)}$$

$$= \sum_{j=-m}^{m} \left[\sum_{i=-n}^{n-1} \left(- \sum_{p=-n}^{i} s'_{p,\lambda}(x) \right) (f(x_{i+1}, y_j) - f(x_i, y_j)) \right] s_{j,\mu}(y)$$

$$= \sum_{i=-n}^{n-1} \left(- \sum_{p=-n}^{i} s'_{p,\lambda}(x) \right) \sum_{j=-m}^{m} s_{j,\mu}(y) \times (f(x_{i+1}, y_j) - f(x_i, y_j)).$$

It follows

$$\frac{\partial^2 S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x \partial y} = \sum_{i=-n}^{n-1} \left(- \sum_{p=-n}^{i} s'_{p,\lambda}(x) \right) \sum_{j=-m}^{m} s'_{j,\mu}(y)$$

$$\times(f(x_{i+1}, y_j) - f(x_i, y_j)),$$

which by the above Theorem 4.1.1, immediately implies the formula in the statement. □

Corollary 4.3.1. *Let* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, $\lambda = 2p$, $\mu = 2q$, $p, q \in \mathbb{N}$, $x_i = i/n, i \in \{-n, \ldots, n\}$, $y_j = j/m$, $j \in \{-m, \ldots, m\}$, $l_{n,p}(x) = \sum_{i=-n}^{n} (x - x_i)^{-2p}$, $l_{m,q}(y) = \sum_{j=-m}^{m} (y - y_j)^{-2q}$. *If* ξ *is any solution of the equation* $l'_{n,p}(x) = 0$ *and* η *is any solution of the equation* $l'_{m,q}(y) = 0$, *then there exists a constant* $c > 0$ *independent of* n, m *and* f, *such that if* $f(x, y)$ *is bidimensional monotone on* $[-1, 1] \times [-1, 1]$, *then* $S_{n,m}^{(\lambda,\mu)}(f; x, y)$ *is bidimensional monotone (of the same monotonicity) on*

$$(\xi - c/n^{2p+3}, \xi + c/n^{2p+3}) \times (\eta - c/m^{2q+3}, \eta + c/m^{2q+3}) \subset (-1, 1) \times (-1, 1).$$

Proof. It is immediate by Theorem 4.3.1 and Theorem 2.3.3. □

Remark. Because by Remark 1 after Theorem 2.3.1, each equation $l'_{n,p}(x) = 0$, $l'_{m,q}(y) = 0$ has $2n$, respectively, $2m$ solutions, it follows that in Corollary 4.3.1 there exists a grid of $4mn$ points (ξ, η).

In what follows, we will extend the convexity problem from the univariate case.

Corollary 4.3.2. *Let* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, $\lambda = 2p$, $\mu = 2q$, $p, q \in \mathbb{N}$, $x_i = i/n$, $i \in \{-n, \ldots, n\}$, $y_j = j/m$, $j \in \{-m, \ldots, m\}$. *If* $\Delta_h^{2,y}[\Delta_k^{2,x} f(0, 0)] > 0$, *for all* $h, k \in (0, 1]$, *then* $S_{n,m}^{(\lambda,\mu)}(f; x, y)$ *is strictly double convex in a bivariate neighborhood* $V(0, 0)$ *of* $(0, 0)$.

Proof. We observe

$$\frac{\partial^4 S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^2 \partial y^2} = \sum_{j=-m}^{m} s''_{j,\mu}(y) \left(\sum_{i=-n}^{n} s''_{i,\lambda}(x) f(x_i, y_j) \right).$$

Reasoning as in the proof of univariate case of Theorem 2.3.2 , we obtain

$$\frac{\partial^{r+s} S_{n,m}^{(\lambda,\mu)}(f; 0, y)}{\partial x^r \partial y^s} = \sum_{j=-m}^{m} s_{j,\mu}^{(s)}(y) \left[\sum_{i=-n}^{n} s_{i,\lambda}^{(r)}(0) f(x_i, y_j) \right]$$

$$= \frac{\partial^{r+s} S_{n,m}^{(\lambda,\mu)}(f; x, 0)}{\partial x^r \partial y^s} = \sum_{j=-m}^{m} s_{j,\mu}^{(s)}(0) \left[\sum_{i=-n}^{n} s_{i,\lambda}^{(r)}(x) f(x_i, y_j) \right] = 0,$$

for all $r \in \{1, \ldots, 2p - 1\}$, $s \in \{1, \ldots, 2q - 1\}$, $(x, y) \in [-1, 1] \times [-1, 1]$, and

$$\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f; 0, 0)}{\partial x^{2p} \partial y^{2q}} = \sum_{j=-m}^{m} s_{j,\mu}^{(2q)}(0) \left[\sum_{i=-n}^{n} s_{i,\lambda}^{(2p)}(0) f(x_i, y_j) \right]$$

$$= \sum_{j=-m}^{m} s_{j,\mu}^{(2q)}(0) \sum_{i=1}^{n} s_{i,\lambda}^{(2p)}(0) \Delta_{x_i}^{2,x} f(0, y_j) := A.$$

Denoting

$$G(y) = \sum_{i=1}^{n} s_{i,\lambda}^{(2p)}(0) \Delta_{x_i}^{2,x} f(0, y)$$

we get

$$A = \sum_{j=-m}^{m} s_{j,\mu}^{(2q)}(0) G(y_j) = \sum_{j=1}^{m} s_{j,\mu}^{(2q)}(0) \Delta_{y_j}^{2,y} G(0)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} s_{j,\mu}^{(2q)}(0) s_{i,\lambda}^{(2p)}(0) \Delta_{y_j}^{2,y} [\Delta_{x_i}^{2,x} f(0, 0)] > 0,$$

by hypothesis.

As a conclusion,

$$\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f; 0, 0)}{\partial x^{2p} \partial y^{2q}} > 0,$$

which implies that there exists a neighborhood $V(0, 0)$ of $(0, 0)$ such that

$$\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^{2p} \partial y^{2q}} > 0,$$

for all $(x, y) \in V(0, 0)$.

Denote by V_y the projection of $V(0, 0)$ on the OY-axis and by V_x the projection of $V(0, 0)$ on the OX-axis. Firstly let $y \in V_y$ be fixed. Reasoning as in the univariate case, from the above relations it follows that $(0, y)$ is a minimum point for

$$\frac{\partial^{2p+2q-2} S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^{2p-2} \partial y^{2q}},$$

and since by the last relation $\frac{\partial^{2p+2q-2} S_{n,m}^{(\lambda,\mu)}(f;x,y)}{\partial x^{2p-2} \partial y^{2q}}$ is convex with respect to $x \in V_x$, it follows that

$$\frac{\partial^{2p+2q-2} S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^{2p-2} \partial y^{2q}} > 0,$$

for all $(x, y) \in V(0, 0)$, with $x \neq 0$.

Now, let $x \in V_x$, $x \neq 0$, be fixed. Reasoning for $\frac{\partial^{2p+2q-2} S_{n,m}^{(\lambda,\mu)}(f;x,y)}{\partial x^{2p-2} \partial y^{2q}}$ as above, similarly we get

$$\frac{\partial^{2p-2+2q-2} S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^{2p-2} \partial y^{2q-2}} > 0,$$

for all $(x, y) \in V(0, 0)$, with $x \neq 0$ and $y \neq 0$.

Reasoning by induction, finally we arrive at

$$\frac{\partial^{4} S_{n,m}^{(\lambda,\mu)}(f; x, y)}{\partial x^{2} \partial y^{2}} > 0,$$

for all $(x, y) \in V(0, 0)$, with $x \neq 0$ and $y \neq 0$, which proves the theorem. □

As an immediate consequence of the univariate result, for the bivariate tensor product Shepard operator, we obtain the following quantitative version of Corollary 4.3.2.

Corollary 4.3.3. *Let* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, $\lambda = 2p$, $\mu = 2q$, $p, q \in \mathbb{N}$, $x_i = i/n$, $i \in \{-n, \dots, n\}$, $y_j = j/m$, $j \in \{-m, \dots, m\}$. *If* $\Delta_h^{2,y}[\Delta_k^{2,x} f(0, 0)] > 0$, *for all* $h, k \in (0, 1]$, *then* $S_{n,m}^{(\lambda,\mu)}(f; x, y)$ *is strictly double convex in the bivariate neighborhood of* $(0, 0)$, $V(0, 0) = \{(x, y); x^2 + y^2 < d_{n,m}^2\}$, *with*

$$d_{n,m} \geq \frac{c_{p,q} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_{x_i}^{2,x}[\Delta_{y_j}^{2,y} f(0, 0)]/(x_i^{2p} y_j^{2p})}{[n^{2p+1} + m^{2q+1}]||f||},$$

where $c_{p,q}$ *is a constant depending only on* p, q.

Proof. From the proof of Corollary 4.3.2 and from the univariate case we get

$$\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f; 0, 0)}{\partial x^{2p} \partial y^{2q}} = \sum_{j=1}^{m} \sum_{i=1}^{n} s_{j,\mu}^{(2q)}(0) s_{i,\lambda}^{(2p)}(0) \Delta_{y_j}^{2,y}[\Delta_{x_i}^{2,x} f(0, 0)]$$

$$\geq c_{p,q} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_{x_i}^{2,x}[\Delta_{y_j}^{2,y} f(0, 0)]/(x_k^{2p} y_j^{2p}) > 0.$$

Let $(\alpha_{n,m}, \beta_{n,m})$ be the nearest root to $(0,0)$ of $\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;x,y)}{\partial x^{2p}\partial y^{2q}}$, in the sense that the distance $d_{n,m} = (\alpha_{n,m}^2 + \beta_{n,m}^2)^{1/2}$ is minimum for all the roots of $\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;x,y)}{\partial x^{2p}\partial y^{2q}}$. Then for all $(x,y) \in \{(x,y); (x^2+y^2)^{1/2} < d_{n,m}\}$, we necessarily have $\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;x,y)}{\partial x^{2p}\partial y^{2q}} > 0$. By the mean value theorem for bivariate functions, we get

$$\begin{aligned}
\frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;0,0)}{\partial x^{2p}\partial y^{2q}} &= \frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;0,0)}{\partial x^{2p}\partial y^{2q}} - \frac{\partial^{2p+2q} S_{n,m}^{(\lambda,\mu)}(f;\alpha_{n,m},\beta_{n,m})}{\partial x^{2p}\partial y^{2q}} \\
&\le |\alpha_{n,m}| \left| \frac{\partial^{2p+2q+1} S_{n,m}^{(\lambda,\mu)}(f;\xi,\eta)}{\partial x^{2p+1}\partial y^{2q}} \right| + |\beta_{n,m}| \left| \frac{\partial^{2p+2q+1} S_{n,m}^{(\lambda,\mu)}(f;\xi,\eta)}{\partial x^{2p}\partial y^{2q+1}} \right| \\
&\le |d_{n,m}| \left[\left| \frac{\partial^{2p+2q+1} S_{n,m}^{(\lambda,\mu)}(f;\xi,\eta)}{\partial x^{2p+1}\partial y^{2q}} \right| + \left| \frac{\partial^{2p+2q+1} S_{n,m}^{(\lambda,\mu)}(f;\xi,\eta)}{\partial x^{2p}\partial y^{2q+1}} \right| \right] \\
&\le |d_{n,m}| [n^{2p+1} + m^{2q+1}] \|f\|,
\end{aligned}$$

taking into account the proof of Theorem 2.1 in Della Vecchia–Mastroianni [33], p. 149, too (see also the proof of Theorem 2.3.4). □

In what follows, let us consider the original bivariate Shepard operator introduced in Shepard [91] by

$$S_{n,2p}(f)(x,y) = \sum_{i=1}^{n} s_{n,i}^{(2p)}(x,y) f(x_i,y_i), \text{ if } (x,y) \neq (x_i,y_i),$$

$S_{n,2p}(f)(x_i,y_i) = f(x_i,y_i)$, where $p \in \mathbb{N}$ is fixed, $f : D \to \mathbb{R}$, $D = [a,b] \times [c,d]$, $(x_i,y_i) \in D$, $i = 1,\dots,n$, $x_1 < \dots < x_n$, $y_1 < \dots < y_n$,

$$s_{n,i}^{(2p)}(x,y) = [(x-x_i)^2 + (y-y_i)^2]^{-p} / l_n^{(2p)}(x,y),$$

$$l_n^{(2p)}(x,y) = \sum_{i=1}^{n} \left[(x-x_i)^2 + (y-y_i)^2 \right]^{-p}.$$

Convergence properties of this kind of operators can be found in, e.g., Gonska [60], Farwig [42], Allasia [3], [4], while global smoothness preservation properties have not yet been proved.

On the other hand, concerning the partial shape-preserving property we first can prove the following qualitative results.

Theorem 4.3.2. *If $f : D \to \mathbb{R}$, $D = [a,b] \times [c,d]$ is such that $f(x,y)$ is nondecreasing as a function of x (for each fixed y) and nondecreasing as a function of y (for each fixed x), then for any point $(\xi,\eta) \in (a,b) \times (c,d)$ that is a solution of the system of equations*

$$\frac{\partial l_n^{(2p)}(x,y)}{\partial x}=0,\ \frac{\partial l_n^{(2p)}(x,y)}{\partial y}=0,$$

there exists a neighborhood $V(\xi,\eta)$ of it (depending on n and p but independent of f) such that $S_{n,2p}(f)(x,y)$ is nondecreasing as a function of x and as a function of y on $V(\xi,\eta)$.

Proof. By using Lemma 4.1.1, (i), we get

$$\begin{aligned}\frac{\partial S_{n,2p}(f)(x,y)}{\partial x}&=\sum_{i=1}^{n-1}\left[-\sum_{j=1}^{i}\frac{\partial s_{n,j}^{(2p)}(x,y)}{\partial x}\right]\\&\quad\cdot[f(x_{i+1},y_{i+1})-f(x_i,y_i)],\\ \frac{\partial S_{n,2p}(f)(x,y)}{\partial y}&=\sum_{i=1}^{n-1}\left[-\sum_{j=1}^{i}\frac{\partial s_{n,j}^{(2p)}(x,y)}{\partial y}\right]\\&\quad\cdot[f(x_{i+1},y_{i+1})-f(x_i,y_i)].\end{aligned}$$

By hypothesis,

$$\begin{aligned}f(x_{i+1},y_{i+1})-f(x_i,y_i)&=f(x_{i+1},y_{i+1})-f(x_{i+1},y_i)+f(x_{i+1},y_i)\\&\quad-f(x_i,y_i)\geq 0,\quad \text{for all } i=\overline{1,n-1}.\end{aligned}$$

Let $(\xi,\eta)\in(a,b)\times(c,d)$ be a solution of the system in the statement. Because

$$\begin{aligned}\frac{\partial s_{n,j}^{(2p)}(f)(x,y)}{\partial x}&=\frac{-2p(x-x_j)[(x-x_j)^2+(y-y_j)^2]^{-p-1}}{\ell_n^{(2p)}(x,y)}\\&\quad-\frac{\frac{\partial \ell_n^{(2p)}(x,y)}{\partial x}[(x-x_j)^2+(y-y_j)^2]^{-p}}{[\ell_n^{(2p)}(x,y)]^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial s_{n,j}^{(2p)}(f)(x,y)}{\partial y}&=\frac{-2p(y-y_j)[(x-x_j)^2+(y-y_j)^2]^{-p-1}}{\ell_n^{(2p)}(x,y)}\\&\quad-\frac{\frac{\partial \ell_n^{(2p)}(x,y)}{\partial y}[(x-x_j)^2+(y-y_j)^2]^{-p}}{[\ell_n^{(2p)}(x,y)]^2},\end{aligned}$$

we immediately get

$$\frac{\partial s_{n,1}^{(2p)}(\xi,\eta)}{\partial x}<0,\quad \frac{\partial s_{n,n}^{(2p)}(\xi,\eta)}{\partial x}>0,$$

$$\operatorname{sgn}\left[\frac{\partial s_{n,j}^{(2p)}(\xi,\eta)}{\partial x}\right]=\operatorname{sgn}(x_j-\xi)$$

and

$$\frac{\partial s_{n,1}^{(2p)}(\xi,\eta)}{\partial y} < 0, \quad \frac{\partial s_{n,n}^{(2p)}(\xi,\eta)}{\partial y} > 0,$$

$$\text{sgn}\left[\frac{\partial s_{n,j}^{(2p)}(\xi,\eta)}{\partial y}\right] = \text{sgn}(y_j - \eta).$$

By Lemma 4.1.1,(ii), we easily obtain

$$-\sum_{j=1}^{i} \frac{\partial s_{n,j}^{(2p)}(\xi,\eta)}{\partial x} > 0, \; -\sum_{j=1}^{i} \frac{\partial s_{n,j}^{(2p)}(\xi,\eta)}{\partial y} > 0, \quad \forall i = \overline{1, n-1}$$

and consequently

$$\frac{\partial S_{n,2p}(f)(x,y)}{\partial x} > 0, \; \frac{\partial S_{n,2p}(f)(x,y)}{\partial y} > 0, \quad \forall (x,y) \in V(\xi,\eta)$$

which is a neighborhood of (ξ, η).

The theorem is proved. □

Remark. A natural question is whether or not the system in the statement has solutions in $(a,b) \times (c,d)$. For some particular choices of the nodes $x_k, y_k, k = \overline{1,n}$, a positive answer can easily be derived.

Thus, let us first consider the case $[a,b] = [c,d]$ and $x_k = y_k, k = \overline{1,n}$.

An easy calculation shows that it is equivalent to the system

$$\sum_{k=1}^{n} \frac{x - x_k}{[(x-x_k)^2 + (y-y_k)^2]^{p+1}} = 0,$$

$$\sum_{k=1}^{n} \frac{y - y_k}{[(x-x_k)^2 + (y-y_k)^2]^{p+1}} = 0.$$

Taking now $x_k = y_k, k = \overline{1,n}$ and subtracting the equations, it necessarily follows $x = y$.

Replacing in the first equation of the above system, we obtain $\sum_{k=-n}^{n} \frac{1}{(x-x_k)^{2p+1}} = 0$, which is exactly equation (2.12) in the proof of Theorem 2.3.1. But according to Remark 1 after the proof of Theorem 2.3.1, equation (2.12) has $2n$ solutions. So, the above system has in this case $2n$ solutions of the form (ξ, ξ). Another particular choice would be $[a,b] = [c,d] = [-1,1]$ and $x_k = -y_k, k = \overline{1,n}$.

In this second case, by adding both equations we necessarily obtain $x = -y$, that is, replacing in the first equation we easily obtain the equation

$$\sum_{k=1}^{n} \frac{x - x_k}{(x^2 + x_k^2)^{p+1}} = 0.$$

Denoting $F(x) = \sum_{k=1}^{n} \frac{x-x_k}{(x^2+x_k^2)^{p+1}}$, we get $F(x_1) < 0$ and $F(x_n) > 0$, that is, there exists $\xi \in (x_1, x_n)$ with $F(\xi) = 0$, and as a conclusion, this (ξ, ξ) will be a solution of the system in the statement. Finally, notice that if $n = 2$ and $p \in \mathbb{N}$, $(x_k, y_k) \in [a, b] \times [c, d]$, $k = \overline{1, n}$, then (ξ, η) with $\xi = \frac{x_1+x_2}{2}$ and $\eta = \frac{y_1+y_2}{2}$ is a solution.

Now, let us discuss some properties of qualitative kind of $S_{n,2p}(f)(x, y)$ related to the convexity. In this sense, we will consider $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, the equidistant interpolation knots $x_{-i} = -x_i$, $y_{-i} = -y_i$, $i = \overline{1, n}$, $x_0 = y_0 = 0$ and the Shepard operator given by

$$S_{n,2p}(f)(x, y) = \sum_{k=-n}^{n} s_{n,k}^{(2p)}(x, y) f(x_k, y_k), \tag{4.4}$$

where $s_{n,k}^{(2p)}(x, y)$ can be written in the form

$$s_{n,k}^{(2p)}(x, y) = \frac{(x^2 + y^2)^p[(x - x_k)^2 + (y - y_k)^2]^{-p}}{1 + \sum\limits_{\substack{j=-n \\ j\neq 0}}^{n} (x^2 + y^2)^p[(x - x_j)^2 + (y - y_j)^2]^{-p}}. \tag{4.5}$$

We also need the following

Definition 4.3.1. The function $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ is called strictly convex on $[-1, 1] \times [-1, 1]$ if for any $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2) \in [-1, 1] \times [-1, 1]$ and any $\lambda \in (0, 1)$, we have

$$f(\lambda P_1 + (1 - \lambda)P_2) < \lambda f(P_1) + (1 - \lambda) f(P_2),$$

where $\lambda P_1 + (1 - \lambda)P_2 = (\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in (-1, 1) \times (-1, 1)$.

Remark. It is well known (see, e.g., Fleming [44], p. 114]) that if $f \in C^2([-1, 1] \times [-1, 1])$ and

$$\frac{\partial^2 f(x, y)}{\partial x^2} > 0, \quad \frac{\partial^2 f(x, y)}{\partial y^2} > 0, \quad \frac{\partial^2 f(x, y)}{\partial x^2} \cdot \frac{\partial^2 f(x, y)}{\partial y^2} > \left(\frac{\partial^2 f(x, y)}{\partial x \partial y}\right)^2,$$

$\forall (x, y) \in [-1, 1] \times [-1, 1]$, then f is strictly convex on $[-1, 1] \times [-1, 1]$. Moreover, if the two strict inequalities above are valid for all $(x, y) \in [-1, 1] \times [-1, 1] \setminus \{(0, 0)\}$ and $\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0$, then f is strictly convex on $[-1, 1] \times [-1, 1]$ and $(0, 0)$ is its global minimum point.

We present the following:

Theorem 4.3.3. *Let $S_{n,2p}(f)(x, y)$ be given by (4.4), (4.5), with $p = 1$.*

If $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ is strictly convex on $[-1, 1] \times [-1, 1]$, then there exists a neighborhood of $(0, 0)$, denoted by $V(0, 0)$ (depending on f and n), such that $S_{n,2p}(f)(x, y)$ is strictly convex in $V(0, 0)$.

Proof. By (4.5) and by simple calculations, for all $k \neq 0$ we get

$$\frac{\partial^i s_{n,k}^{(2p)}(0,0)}{\partial x^i} = \frac{\partial^i s_{n,k}^{(2p)}(0,0)}{\partial y^i} = \begin{cases} 0, & \text{if } k = \overline{1, 2p-1} \\ \dfrac{(2p)!}{(x_k^2 + y_k^2)^p}, & \text{if } k = 2p. \end{cases} \tag{4.6}$$

We have

$$\frac{\partial^2 S_{n,2}(f)(0,0)}{\partial x^2} = \sum_{k=-n}^{n} \frac{\partial^2 s_{n,k}^{(2)}(0,0)}{\partial x^2} f(x_k, y_k)$$

$$= \sum_{k=-n}^{n} \frac{2!}{(x_k^2 + y_k^2)^2} f(x_k, y_k) = \sum_{k=1}^{n} \frac{2}{(x_k^2 + y_k^2)^2} [f(x_k, y_k) + f(-x_k, -y_k)]$$

$$+ f(0,0) \cdot \frac{\partial^2 s_{n,0}^{(2)}(0,0)}{\partial x^2} > f(0,0) \cdot \sum_{k=-n}^{n} \frac{\partial^2 s_{n,k}^{(2)}(0,0)}{\partial x^2} = 0,$$

taking into account that the strict convexity of f implies $f(x_k, y_k) + f(-x_k, -y_{-k}) > 2f(0,0)$ and that the identity $\sum_{k=-n}^{n} s_{n,k}^{(2p)}(x, y) = 1$, implies $\sum_{k=-n}^{n} \frac{\partial^2 s_{n,k}^{(2p)}(0,0)}{\partial x^2} = 0$. Similarly, $\frac{\partial^2 S_{n,2}(f)(0,0)}{\partial y^2} > 0$. On the other hand, by simple calculations we get $\frac{\partial^2 s_{n,k}^{(2)}(0,0)}{\partial x \partial y} = 0, \forall k = \overline{-n, n}$, which implies

$$\frac{\partial^2 S_{n,2}(f)(0,0)}{\partial x \partial y} = 0.$$

So, it easily follows

$$\frac{\partial^2 S_{n,2}(f)(0,0)}{\partial x^2} \cdot \frac{\partial^2 S_{n,2}(f)(0,0)}{\partial y^2} > \left(\frac{\partial^2 S_{n,2}(f)(0,0)}{\partial x \partial y} \right) = 0.$$

As a conclusion, there exists a neighborhood of $(0,0)$, denoted by $V(0,0)$ (obviously depending on f and n) such that for all $(x, y) \in V(0,0)$ we have $\frac{\partial^2 S_{n,2}(f)(x,y)}{\partial x^2} > 0$, and

$$\frac{\partial^2 S_{n,2}(f)(x,y)}{\partial x^2} \cdot \frac{\partial^2 S_{n,2}(f)(x,y)}{\partial y^2} > \left(\frac{\partial^2 S_{n,2}(f)(x,y)}{\partial x \partial y} \right)^2,$$

that is $S_{n,2}(f)(x, y)$ is strictly convex in $V(0,0)$. □

Remark. Let $p \geq 2$. According to the Remark after Definition 4.3.1, it will be enough if we will prove that there exists a neighborhood $V(0,0)$ of $(0,0)$, such that

$$\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x^2} > 0, \quad \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial y^2} > 0,$$

$$\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x^2} \cdot \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial y^2} > \left(\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x \partial y} \right)^2, \tag{4.7}$$

for all $(x, y) \in V(0,0) \setminus \{(0,0)\}$ because by relations (4.6) we obviously have

$$\frac{\partial S_{n,2p}(f)(0,0)}{\partial x} = \frac{\partial S_{n,2p}(f)(0,0)}{\partial y} = 0.$$

Because of the same relations (4.6) we have

$$\frac{\partial^2 S_{n,2p}(f)(0,0)}{\partial x^2} = \frac{\partial^2 S_{n,2p}(f)(0,0)}{\partial y^2} = 0,$$

the idea is to prove that the functions

$$F(x,y) = \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x^2}, \quad G(x,y) = \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial y^2}$$

and

$$H(x,y) = F(x,y) \cdot G(x,y) - \left(\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x \partial y} \right)^2$$

are strictly convex on a neighborhood of $(0,0)$, having as global minimum point $(0,0)$, which would imply the required relations (4.7).

In order to prove the "qualitative result," Theorem 4.3.3, for all $p \in \mathbb{N}$, $p \geq 2$, the following three lemmas are necessary.

Lemma 4.3.1. *If $p \in \mathbb{N}$, $p \geq 2$ and $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is strictly convex on $[-1,1] \times [-1,1]$, then there exists a neighborhood $V(0,0)$ of $(0,0)$, such that*

$$\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x^2} > 0, \ \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial y^2} > 0, \ \forall (x,y) \in V(0,0) - \{(0,0)\},$$

where $S_{n,2p}(f)(x,y)$ is given by (4.4) and (4.5).

Proof. Denoting

$$E(x,y) = \frac{[(x - x_k)^2 + (y - y_k)^2]^{-p}}{1 + (x^2 + y^2)^p \sum_{\substack{j=-n \\ j \neq 0}}^{n} [(x - x_j)^2 + (y - y_j)^2]^{-p}}, \quad k \neq 0,$$

we have $s_{n,k}^{(2p)}(x,y) = P(x,y) \cdot E(x,y)$, where

$$\begin{aligned} P(x,y) &= (x^2 + y^2)^p = \sum_{i=0}^{p} \binom{p}{i} x^{2i} y^{2p-2i} \\ &= y^{2p} + \binom{p}{1} x^2 y^{2p-2} + \binom{p}{2} x^4 y^{2p-4} + \binom{p}{3} x^6 y^{2p-6} \\ &\quad + \cdots + \binom{p}{p-2} x^{2p-4} y^4 + \binom{p}{p-1} x^{2p-2} y^2 + x^{2p}, \end{aligned}$$

and $E(x,y)$ has at $(0,0)$ partial derivatives of any order. Denote

$$R_k(x, y) = \frac{\partial^{2p-2} s_{n,k}^{(2p)}(x, y)}{\partial x^{2p-2}}, \quad k \neq 0.$$

Firstly, by (4.6) we get

$$\frac{\partial^2 R_k(0,0)}{\partial x^2} = \frac{\partial^{2p} s_{n,k}^{(2p)}(0,0)}{\partial x^{2p}} = \frac{(2p)!}{(x_k^2 + y_k^2)^p} > 0.$$

Then

$$\begin{aligned}
\frac{\partial^2 R_k(x, y)}{\partial y^2} &= \frac{\partial}{\partial y^2}\left[\sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial^i P(x, y)}{\partial x^i}\cdot\frac{\partial^{2p-2-i}E(x, y)}{\partial x^{2p-2-i}}\right] \\
&= \frac{\partial}{\partial y}\left[\sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial^{i+1} P(x, y)}{\partial x^i \partial y}\cdot\frac{\partial^{2p-2-i}E(x, y)}{\partial x^{2p-2-i}}\right. \\
&\quad \left. + \sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial^i P(x, y)}{\partial x^i}\cdot\frac{\partial^{2p-1-i}E(x, y)}{\partial x^{2p-2-i}\partial y}\right] \\
&= \sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial^2}{y^2}\left[\frac{\partial^i P(x, y)}{\partial x^i}\right]\cdot\frac{\partial^{2p-2-i}E(x, y)}{\partial x^{2p-2-i}} \\
&\quad +2\sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial}{\partial y}\left[\frac{\partial^i P(x, y)}{\partial x^i}\right]\cdot\frac{\partial^{2p-1-i}E(x, y)}{\partial x^{2p-2-i}\partial y} \\
&\quad + \sum_{i=0}^{2p-2}\binom{2p-2}{i}\frac{\partial^i P(x, y)}{\partial x^i}\cdot\frac{\partial^{2p+i}E(x, y)}{\partial x^{2p-2-i}\partial y^2}.
\end{aligned}$$

If we take $x = y = 0$ in these sums, then all the terms that contain x or (and) y will become zero, so taking into account the form of polynomial $P(x, y)$, we obtain

$$\frac{\partial^2 R_k(0,0)}{\partial y^2} = 2p\binom{2p-2}{2p-2}(2p-2)!\frac{1}{(x_k^2 + y_k^2)^p} = \frac{2p(2p-2)!}{(x_k^2 + y_k^2)^p} > 0.$$

Reasoning for $S_{n,2p}(f)(x, y)$ exactly as in the case $p = 1$ (see the proof of Theorem 4.3.3), we easily obtain

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial^{2p-2} S_{n,2p}(f)(0,0)}{\partial x^{2p-2}}\right] = \frac{\partial^{2p} S_{n,2p}(f)(0,0)}{\partial x^{2p}} > 0,$$
$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^{2p-2} S_{n,2p}(f)(0,0)}{\partial x^{2p-2}}\right] = \frac{\partial^{2p} S_{n,2p}(f)(0,0)}{\partial x^{2p-2}\partial y^2} > 0.$$

So, there exists a neighborhood $V_1(0,0)$ of $(0,0)$, such that for all $(x, y) \in V_1(0,0)$ we have

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial^{2p-2}S_{n,2p}(f)(x,y)}{\partial x^{2p-2}}\right] > 0, \quad \frac{\partial^2}{\partial y^2}\left[\frac{\partial^{2p-2}S_{n,2p}(f)(x,y)}{\partial x^{2p-2}}\right] > 0.$$

On the other hand, reasoning as above, we immediately get

$$\frac{\partial^2}{\partial y \partial x}\left[\frac{\partial^{2p-2}s_{n,k}^{(2p)}(0,0)}{\partial x^{2p-2}}\right] = \frac{\partial^{2p}s_{n,k}^{(2p)}(0,0)}{\partial x^{2p-1}\partial y} = 0.$$

As a first conclusion, $\frac{\partial^{2p-2}S_{n,2p}(f)(x,y)}{\partial x^{2p-2}}$ is strictly convex in a neighborhood $V_1(0,0)$ of $(0,0)$, and because

$$\frac{\partial^{2p-2}S_{n,2p}(f)(0,0)}{\partial x^{2p-2}} = \frac{\partial^{2p-1}S_{n,2p}(f)(0,0)}{\partial x^{2p-1}} = \frac{\partial^{2p-1}S_{n,2p}(f)(0,0)}{\partial x^{2p-2}\partial y} = 0,$$

it follows that $\frac{\partial^{2p-2}S_{n,2p}(f)(x,y)}{\partial x^{2p-2}} > 0, \forall (x,y) \in V_1(0,0) \setminus \{(0,0)\}$.

By symmetry, we get

$$\frac{\partial^{2p-2}S_{n,2p}(f)(x,y)}{\partial y^{2p-2}} > 0, \quad \forall (x,y) \in V_2(0,0).$$

Now, if $p = 2$ then we exactly obtain the statement of the lemma.

If $p > 2$, then by similar reasonings as above, we obtain that $\frac{\partial^{2p-4}S_{n,2p}(f)(x,y)}{\partial x^{2p-4}}$ and $\frac{\partial^{2p-4}S_{n,2p}(f)(x,y)}{\partial y^{2p-4}}$ are strictly convex on a neighborhood $U(0,0)$ of $(0,0)$, etc., and as a conclusion

$$\frac{\partial^{2p-4}S_{n,2p}(f)(x,y)}{\partial x^{2p-4}} > 0, \quad \frac{\partial^{2p-4}S_{n,2p}(f)(x,y)}{\partial y^{2p-4}} > 0,$$

$$\forall (x,y) \in U(0,0) \setminus \{(0,0)\}.$$

We can continue in this way until we arrive at

$$\frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial x^2} > 0, \quad \frac{\partial^2 S_{n,2p}(f)(x,y)}{\partial y^2} > 0, \quad \forall (x,y) \in V(0,0) \setminus \{(0,0)\}. \quad \square$$

Lemma 4.3.2. *Let $p \in \mathbb{N}$, $p \geq 2$. Then we have*:

$$\frac{\partial^r S_{n,2p}(f)(0,0)}{\partial x^i \partial y^j} = \begin{cases} = 0, & \textit{if } r < 2p \textit{ or } r > 2p. \\ = 0, & \textit{if } r = 2p \textit{ and both } i, j \textit{ are odd} \\ > 0, & \textit{if } r = 2p \textit{ and both } i, j \textit{ are even.} \end{cases}$$

Proof. Let $P(x,y) = (x^2 + y^2)^p$. It is easy to check that

$$\frac{\partial^r P(0,0)}{\partial x^i \partial y^j} = \begin{cases} = 0, & \textit{if } r < 2p \textit{ or } r > 2p. \\ = 0, & \textit{if } r = 2p \textit{ and both } i, j \textit{ are odd} \\ > 0, & \textit{if } r = 2p \textit{ and both } i, j \textit{ are even.} \end{cases}$$

We have $s_{n,k}^{(2p)}(x, y) = P(x, y) \cdot E(x, y), \forall k \neq 0$, where $E(x, y)$ is given in the proof of Lemma 4.3.1, and

$$\frac{\partial^r s_{n,k}^{(2p)}(x, y)}{\partial x^i \partial y^j} = \frac{\partial^i}{\partial x^i}\left[\sum_{q=0}^{j}\binom{j}{q}\frac{\partial^2 P(x, y)}{\partial y^2} \cdot \frac{\partial^{j-q} E(x, y)}{\partial y^{j-q}}\right],$$

which combined with the above properties of $\frac{\partial^r P(0,0)}{\partial x^i \partial y^j}$ and with the method of proof in Lemma 4.3.1 (because

$$\left(\frac{\partial^r S_{n,p}(f)(0, 0)}{\partial x^i \partial y^j} = \sum_{k=-n}^{n}\frac{\partial^r s_{n,k}^{(2p)}(0, 0)}{\partial x^i \partial y^j} \cdot f(x_k, y_k)\right)$$

proves the lemma. □

Lemma 4.3.3. *Let $p \in \mathbb{N}$, $p \geq 2$. We have*

$$\frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x^2} \cdot \frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial y^2} - \left(\frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x \partial y}\right)^2 > 0,$$

for all $(x, y) \in V(0, 0) \setminus \{(0, 0)\}$, where $V(0, 0)$ is a neighborhood of $(0, 0)$ (depending on f, n and p).

Proof. Denote

$$H(x, y) = \frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x^2} \cdot \frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial y^2} - \left(\frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x \partial y}\right)^2.$$

According to the remark after the proof of Theorem 4.3.3, we have to prove that

$$\frac{\partial^2 H(x, y)}{\partial x^2} > 0, \quad \frac{\partial^2 H(x, y)}{\partial y^2} > 0, \quad \frac{\partial^2 H(x, y)}{\partial x^2} \cdot \frac{\partial^2 H(x, y)}{\partial y^2} - \left(\frac{\partial^2 H(x, y)}{\partial x \partial y}\right)^2 > 0,$$

for all $(x, y) \in V(0, 0) \setminus \{(0, 0)\}$, and that $\frac{\partial H(0,0)}{\partial x} = \frac{\partial H(0,0)}{\partial y} = 0$ (because by Lemma 4.3.2 we have $H(0, 0) = 0$).

Now, by Lemma 4.3.2 we easily get that $\frac{\partial^r H(0,0)}{\partial x^i \partial y^i} > 0$ only if $r = 2p$ and both i, j are even, all the other partial derivatives of H at $(0, 0)$ being 0, so we obtain

$$\frac{\partial^{2p} H(0, 0)}{\partial x^{2p}} > 0, \quad \frac{\partial^{2p} H(0, 0)}{\partial x^{2p-2} \partial y^2} > 0, \quad \frac{\partial^{2p} H(0, 0)}{\partial x^{2p-1} \partial y} = 0,$$

that is, $\frac{\partial^{2p-2} H(x,y)}{\partial x^{2p-2}}$ is strictly convex in a neighborhood of zero. Because $\frac{\partial^{2p-2} H(0,0)}{\partial x^{2p-2}} = 0$ and $\frac{\partial^{2p-1} H(0,0)}{\partial x^{2p-1}} = \frac{\partial^{2p-1} H(0,0)}{\partial x^{2p-2} \partial y} = 0$, it follows that $(0, 0)$ is a global minimum point, so

$$\frac{\partial^{2p-2} H(x, y)}{\partial x^{2p-2}} > 0, \quad \forall (x, y) \in V_1(0, 0) \setminus \{(0, 0)\}.$$

Reasoning by symmetry, we get

$$\frac{\partial^{2p-2}H(x,y)}{\partial y^{2p-2}} > 0, \quad \forall (x,y) \in V_2(0,0) \setminus \{(0,0\}.$$

Similarly, by Lemma 4.3.2 we obtain

$$\frac{\partial^{2p-2}H(x,y)}{\partial x^{2p-4}\partial y^2} > 0, \quad \forall (x,y) \in V_3(0,0) \setminus \{(0,0)\}$$

and consequently

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^{2p-4}H(x,y)}{\partial x^{2p-4}}\right) > 0, \; \frac{\partial^2}{\partial y^2}\left(\frac{\partial^{2p-4}H(x,y)}{\partial x^{2p-4}}\right) > 0,$$
$$\forall (x,y) \in V_3(0,0) \setminus \{(0,0)\}.$$

Let us denote

$$H_1(x,y) = \frac{\partial^{2p-2}H(x,y)}{\partial x^{2p-2}} \cdot \frac{\partial^{2p-2}H(x,y)}{\partial x^{2p-4}\partial y^2} - \left(\frac{\partial^{2p-2}H(x,y)}{\partial x^{2p-3}\partial y}\right)^2.$$

By the same Lemma 4.3.2, we obtain

$$\frac{\partial^2 H_1(0,0)}{\partial x^2} > 0, \quad \frac{\partial^2 H_1(0,0)}{\partial y^2} > 0, \quad \frac{\partial^2 H_1(0,0)}{\partial x \partial y} = 0,$$

that is $H_1(x,y)$ is strictly convex in a neighborhood of $(0,0)$. But $H_1(0,0) = 0$ and $\frac{\partial H_1(0,0)}{\partial x} = \frac{\partial H_1(0,0)}{\partial y} = 0$, so it follows that $H_1(x,y) > 0, \forall (x,y) \in V_4(0,0) \setminus \{(0,0)\}$.

As a conclusion, it follows that $\frac{\partial^{2p-4}H(x,y)}{\partial x^{2p-4}}$ is strictly convex in a neighborhood of zero, and reasoning as above, we get

$$\frac{\partial^{2p-4}H(x,y)}{\partial x^{2p-4}} > 0, \quad \forall (x,y) \in V_5(0,0) \setminus \{(0,0)\}.$$

Continuing this process by recurrence, finally we will arrive at

$$\frac{\partial^2 H(x,y)}{\partial x^2} > 0, \quad \forall (x,y) \in V_7(0,0) \setminus \{(0,0)\},$$

and by reason of symmetry at

$$\frac{\partial^2 H(x,y)}{\partial y^2} > 0, \quad \forall (x,y) \in V_8(0,0) \setminus \{(0,0)\},$$

and then get that

$$\frac{\partial^2 H(x,y)}{\partial x^2} \cdot \frac{\partial^2 H(x,y)}{\partial y^2} - \left(\frac{\partial^2 H(x,y)}{\partial x \partial y}\right)^2 > 0, \;\; \forall (x,y) \in V_9(0,0) \setminus \{(0,0)\},$$

which proves the lemma. □

Corollary 4.3.4. *Let* $S_{n,2p}(f)(x, y)$ *be given by (4.4), (4.5), with* $p \in \mathbb{N}$, $p \geq 2$. *If* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ *is strictly convex on* $[-1, 1] \times [-1, 1]$, *then there exists a neighborhood* $V(0, 0)$ *of* $(0, 0)$ *(depending on* f, n *and* p*) such that* $S_{n,2p}(f)(x, y)$ *is strictly convex in* $V(0, 0)$.

Proof. Immediate by the Lemmas 4.3.1, 4.3.2 and 4.3.3. □

Remarks. (1) The idea of the proof of Corollary 4.3.4 is in fact that in the univariate case.

(2) The results can be extended to n variables, $n > 2$.

In what follows we present some quantitative versions for the above results, on the bidimensional interval $[-1, 1] \times [-1, 1]$ and for $x_i = y_i = i/n, i = -n, \ldots, 0, \ldots, n$, i.e.,

$$S_{n,2p}(f)(x, y) = \sum_{i=-n}^{n} s_{n,i}^{(2p)}(x, y) f(i/n, i/n),$$

where

$$s_{n,i}^{(2p)}(x, y) = [(x - i/n)^2 + (y - i/n)^2]^{-p} / l_n^{(2p)}(x, y),$$

$$l_n^{(2p)}(x, y) = \sum_{i=-n}^{n} \left[(x - i/n)^2 + (y - i/n)^2 \right]^{-p}.$$

Theorem 4.3.4. *If* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ *is such that* $f(x, y)$ *is nondecreasing as a function of* x *(for each fixed* y*) and nondecreasing as a function of* y *(for each fixed* x*), then for any point* $(\xi, \xi) \in (-1, 1) \times (-1, 1)$ *which is solution of the system of equations*

$$\frac{\partial l_n^{(2p)}(x, y)}{\partial x} = 0, \quad \frac{\partial l_n^{(2p)}(x, y)}{\partial y} = 0,$$

there exists a constant $c > 0$ *(independent of* f *and* n*) such that* $S_{n,2p}(f)(x, y)$ *is nondecreasing as a function of* x *and as a function of* y *in* $(\xi - c/n^{2p+3}, \xi + c/n^{2p+3}) \times (\xi - c/n^{2p+3}, \xi + c/n^{2p+3})$.

Proof. By the proof of Theorem 4.3.2 and by the hypothesis on the points (ξ, ξ), we immediately get

$$\frac{\partial s_{n,i}^{(2p)}(\xi, \xi)}{\partial x} = \frac{\partial s_{n,i}^{(2p)}(\xi, \xi)}{\partial y} = \frac{-p(\xi - i/n)^{-2p-1}}{\sum_{k=-n}^{n} (\xi - k/n)^{-2p}},$$

which for $i = -n$ and $i = n$ give the same expressions (except the constant 1/2) with those in the univariate case in the proof of Theorem 2.3.3. Then, combining the ideas in the bivariate case in the proof of Theorem 4.3.2 with those in the univariate case in the proof of Theorem 2.3.3, we immediately obtain

$$|Q_j(\xi, \xi)| \geq c/(n^{2p}), |P_j(\xi, \xi)| \geq c/(n^{2p}), j = -n, \ldots, 0, \ldots, n - 1,$$

where

$$Q_j(x,x) = \sum_{i=-n}^{j} \frac{\partial s_{n,i}^{(2p)}(x,x)}{\partial x}, \quad P_j(y,y) = \sum_{i=-n}^{j} \frac{\partial s_{n,i}^{(2p)}(y,y)}{\partial y}.$$

By taking $x = y$ in the proof of Theorem 4.3.2, we immediately get that $\frac{\partial s_{n,i}^{(2p)}(x,x)}{\partial x}$ actually is $1/2$ times the first derivative of the fundamental univariate rational functions $s'_{j,n}(x)$ in the proof of Theorem 2.3.3.

It follows that our $Q_j(x,x)$ is exactly $1/2$ times the function $Q_j(x)$ which appears in univariate case, in the proof of Theorem 2.3.3.

Let α_j be the nearest root of $Q_j(x,x)$ to ξ and β_j the nearest root of $P_j(y,y)$ to ξ.

Reasoning exactly as in the proof of Theorem 2.3.3, we get

$$|\alpha_j - \xi| \geq \frac{c}{n^{2p+3}}, |\beta_j - \xi| \geq \frac{c}{n^{2p+3}},$$

which proves the theorem. □

For the convexity result, first we need the following bivariate analogous of the estimate in univariate case in Della Vecchia–Mastroianni [33].

Lemma 4.3.4. *For all $l, k \geq 0$ with $l + k = q$ and for*

$$s_{n,i}^{(2p)}(x,y) = \frac{[(x - i/n)^2 + (y - i/n)^2]^{-p}}{l_n^{(2p)}(x,y)},$$

where

$$l_n^{(2p)}(x,y) = \sum_{i=-n}^{n} \left[(x - i/n)^2 + (y - i/n)^2\right]^{-p},$$

we have

$$\left| \frac{\partial^q s_{n,i}^{(2p)}(x,y)}{\partial x^l \partial y^k} \right| \leq C n^q s_{n,i}^{(2p)}(x,y),$$

and

$$\left| \frac{\partial^q S_{n,2p}(f)(x,y)}{\partial x^l \partial y^k} \right| \leq C n^q ||f||,$$

where $||f||$ is the uniform norm.

Proof. First let us denote

$$F_{k,p}(x,y) = \frac{\sum_{i=-n}^{n} (x - x_i)^k [(x - x_i)^2 + (y - y_i)^2]^{-p-k}}{\sum_{i=-n}^{n} [(x - x_i)^2 + (y - y_i)^2]^{-p}},$$

and x_u (respectively, y_v), the closest point x_i (respectively, y_j) to x (respectively, y). We have

$$|F_{k,p}(x,y)| \le |x-x_u|^{-k}.$$

Indeed, by $|x-x_u| \le |x-x_i|$, for all i, we get

$$\frac{|x-x_i|^k}{[(x-x_i)^2+(y-y_i)^2]^k} \le \frac{|x-x_i|^k}{(x-x_i)^{2k}} = 1/|x-x_k|^k \le |x-x_u|^{-k}.$$

Similarly, if we denote

$$G_{k,p}(x,y) = \frac{\sum_{i=-n}^{n} (y-y_i)^k [(x-x_i)^2+(y-y_i)^2]^{-p-k}}{\sum_{i=-n}^{n} [(x-x_i)^2+(y-y_i)^2]^{-p}},$$

we get

$$|G_{k,p}(x,y)| \le |y-y_v|^{-k}.$$

Now, if we denote

$$H_{k,l,p}(x,y) = \frac{\sum_{i=-n}^{n} (x-x_i)^k (y-y_i)^l [(x-x_i)^2+(y-y_i)^2]^{-p-k-l}}{\sum_{i=-n_1}^{n_1} [(x-x_i)^2+(y-y_i)^2]^{-p}},$$

we have

$$|H_{k,l,p}(x,y)| \le |x-x_u|^{-k}|y-y_v|^{-l}, k,l \ge 0.$$

Then

$$\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial x} = \frac{-2p(x-x_i)}{(x-x_i)^2+(y-y_i)^2} s_{n,i}^{(2p)}(x,y) + 2ps_{n,i}^{(2p)}(x,y)F_{1,p}(x,y),$$

which immediately implies

$$\left|\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial x}\right| \le C|x-x_u|^{-1} s_{n,i}^{(2p)}(x,y) \le Cns_{n,i}^{(2p)}(x,y).$$

Similarly,

$$\left|\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial y}\right| \le Cns_{n,i}^{(2p)}(x,y).$$

Then

$$\frac{\partial^2 s_{n,i}^{(2p)}(x,y)}{\partial x^2}$$

$$= -2ps_{n,i}^{(2p)}(x,y)\left[\frac{1}{(x-x_i)^2+(y-y_i)^2} - \frac{2(x-x_i)^2}{[(x-x_i)^2+(y-y_i)^2]^2}\right]$$

$$-\frac{2p(x-x_i)}{(x-x_i)^2+(y-y_i)^2}\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial x} + 2p\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial x}F_{1,p}(x,y)$$

$$+2ps_{n,i}^{(2p)}(x,y)\frac{\partial F_{1,p}(x,y)}{\partial x},$$

where

$$\frac{\partial F_{1,p}(x,y)}{\partial x} = \frac{\sum_{i=-n}^{n}[(x-x_i)^2+(y-y_i)^2]^{-p-1}}{\sum_{i=-n}^{n}[(x-x_i)^2+(y-y_i)^2]^{-p}} - 2(p+1)F_{2,p}(x,y)$$

$$+2p[F_{1,p}(x,y)]^2.$$

We immediately get

$$\left|\frac{\partial^2 s_{n,i}^{(2p)}(x,y)}{\partial x^2}\right| \le C|x-x_u|^{-2}s_{n,i}^{(2p)}(x,y) \le Cn^2 s_{n,i}^{(2p)}(x,y).$$

Also

$$\frac{\partial^2 s_{n,i}^{(2p)}(x,y)}{\partial x\partial y} = \frac{4p(x-x_i)(y-y_i)}{[(x-x_i)^2+(y-y_i)^2]^2}s_{n,i}^{(2p)}(x,y)$$

$$-\frac{2p(x-x_i)}{(x-x_i)^2+(y-y_i)^2}\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial y} + 2p\frac{\partial s_{n,i}^{(2p)}(x,y)}{\partial y}F_{1,p}(x,y)$$

$$+2ps_{n,i}^{(2p)}(x,y)\frac{\partial F_{1,p}(x,y)}{\partial y},$$

where

$$\frac{\partial F_{1,p}(x,y)}{\partial y} = -2(p+1)H_{1,1,p}(x,y) + 2pF_{1,p}(x,y)G_{1,p}(x,y).$$

It immediately follows that

$$\left|\frac{\partial^2 s_{n,i}^{(2p)}(x,y)}{\partial x\partial y}\right| \le Cn^2 s_{n,i}^{(2p)}(x,y).$$

Reasoning in this way, we obtain

$$\left|\frac{\partial^3 s_{n,i}^{(2p)}(x,y)}{\partial x^3}\right| \le Cn^3 s_{n,i}^{(2p)}(x,y),$$

$$\left|\frac{\partial^3 s_{n,i}^{(2p)}(x, y)}{\partial x^2 \partial y}\right| \le Cn^3 s_{n,i}^{(2p)}(x, y),$$

and so on

$$\left|\frac{\partial^q s_{n,i}^{(2p)}(x, y)}{\partial x^l 2 \partial y^k}\right| \le Cn^q s_{n,i}^{(2p)}(x, y),$$

which proves the first estimate in the lemma. The second one is immediate by

$$S_{n,2p}(f)(x, y) = \sum_{i=-n}^{n} s_{n,i}^{(2p)}(x, y) f(i/n, i/n). \qquad \square$$

Theorem 4.3.5. *Let $p = 1$ and $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be strictly convex on $[-1, 1]\times[-1, 1]$. Then $S_{n,2p}(f)(x, y)$ is strictly convex in $V(0, 0) = \{x^2+y^2 < d_n^2\}$, where*

$$|d_n| \ge c\left[\sum_{k=1}^{n} x_k^{-4}[\Delta_{x_k}^2 F(0)]\right]^2 /n^5,$$

with $F(x) = f(x, x)$, for all $x \in [-1, 1] \times [-1, 1]$.

Proof. Let us denote

$$H_n(x, y) = \frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x^2} \frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial y^2} - \left[\frac{\partial^2 S_{n,2p}(f)(x, y)}{\partial x \partial y}\right]^2.$$

By the proof of Theorem 4.3.3 we have

$$H_n(0, 0) = \left[\sum_{k=1}^{n} \frac{x_k^{-4}}{2}[\Delta_{x_k}^2 F(0)]\right]^2 > 0.$$

Let (α_n, β_n) be the nearest root to $(0, 0)$ (in the sense of Euclidean distance in $\mathbb{R}^2$) of $H_n(x, y)$. Denoting $d_n = [\alpha_n^2 + \beta_n^2]^{1/2}$, by the mean value theorem we get

$$0 < H_n(0, 0) = |H_n(0, 0) - H_n(\alpha_n, \beta_n)| \le |\alpha_n| \left|\frac{\partial H_n(\xi, \eta)}{\partial x}\right|$$

$$+|\beta_n| \left|\frac{\partial H_n(\xi, \eta)}{\partial y}\right| \le |d_n| \left[\left|\frac{\partial H_n(\xi, \eta)}{\partial x}\right| + \left|\frac{\partial H_n(\xi, \eta)}{\partial y}\right|\right].$$

By simple calculation and by the above Lemma 4.3.4 we immediately get

$$\left|\frac{\partial H_n(\xi, \eta)}{\partial x}\right| \le cn^5, \left|\frac{\partial H_n(\xi, \eta)}{\partial y}\right| \le cn^5,$$

for all (ξ, η), which immediately implies

$$|d_n| \ge cH_n(0, 0)/n^5,$$

and the proof is complete. □

Remark. Because of the complicated proof of the qualitative result (see the proofs of Lemmas 4.3.1, 4.3.2 and 4.3.3), a variant of the above Theorem 4.3.5 for $p \geq 2$ still remains an open question.

Now, let us consider the third kind of Shepard operator (which is not a tensor product), defined by

$$S_{n_1,n_2,\mu}(f; x, y) = \frac{T_{n_1,n_2,\mu}(f; x, y)}{T_{n_1,n_2,\mu}(1; x, y)}, \text{ if } (x, y) \neq (x_i, y_j),$$

$S_{n_1,n_2,\mu}(f; x_i, y_j) = f(x_i, y_j)$, where $\mu > 0$ is fixed, $f : D \to \mathbb{R}, D \subset \mathbb{R}^2$, $D = [-1, 1] \times [-1, 1], x_i = i/n_1,\ i = -n_1, \ldots, n_1;\ y_j = j/n_2,\ j = -n_2, \ldots, n_2$ and

$$T_{n_1,n_2,\mu}(f; x, y) = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \frac{f(x_i, y_j)}{[(x - x_i)^2 + (y - y_j)^2]^{\mu}}.$$

The global smoothness preservation properties and convergence properties of these operators were studied in Section 3.3. In what follows we consider their properties of preservation of shape.

Remark. Let us note that with respect to preserving monotonicity, it is unfortunate that this does not seem to be a useful method for dealing with this kind of Shepard operator in the univariate case — as it had been in dealing with the original Shepard operator (see the proof of Theorem 4.3.2) or with the tensor product Shepard operator. This seems to happen since there is not a way to put the knots $(x_i, y_j), i \in \{-n_1, \ldots, n_1\}, j \in \{-n_2, \ldots, n_2\}$ in a sort of "increasing" sequence. That is why we consider here only some properties related to the usual bivariate convexity.

For simplicity, first we consider this Shepard operator for $\mu = 1$.

Theorem 4.3.6. *If* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ *is strictly convex on* $[-1, 1] \times [-1, 1]$*, then there exists a neighborhood* $V(0, 0)$ *of* $(0, 0)$ *(depending on* f *and* n_1, n_2*) such that* $S_{n_1,n_2,1}(f; x, y)$ *is strictly convex in* $V(0, 0)$.

Proof. We observe that we can write

$$S_{n_1,n_2,\mu}(f; x, y) = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} f(x_i, y_j) h_{i,j,\mu}(x, y),$$

with

$$h_{i,j,\mu}(x, y) = \frac{(x^2 + y^2)^{\mu}[(x - x_i)^2 + (y - y_j)^2]^{-\mu}}{1 + \sum_{i=-n_1}^{n_1} \sum^{*n_2}_{j=-n_2} (x^2 + y^2)^{\mu}[(x - x_i)^2 + (y - y_j)^2]^{-\mu}},$$

where $\sum \sum^*$ means that the index (i, j) of double sum is different from $(0, 0)$.

By simple calculation, for $\mu \in \mathbb{N}$, $i \neq 0$ and $j \neq 0$ we get

$$\frac{\partial^r h_{i,j,\mu}(0, 0)}{\partial x^r} = \frac{\partial^s h_{i,j,\mu}(0, 0)}{\partial y^s} = 0, \tag{4.8}$$

if $r, s = 1, \dots, 2\mu - 1$ and

$$\frac{\partial^{2\mu} h_{i,j,\mu}(0,0)}{\partial x^{2\mu}} = \frac{\partial^{2\mu} h_{i,j,\mu}(0,0)}{\partial y^{2\mu}} = (2\mu)!/[(x_i^2 + y_j^2)^\mu]. \tag{4.9}$$

By (4.9) and by the obvious relation $\frac{\partial^{2\mu} h_{0,0,\mu}(0,0)}{\partial x^{2\mu}} = \frac{\partial^{2\mu} h_{0,0,\mu}(0,0)}{\partial y^{2\mu}} = 0$, we get

$$\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x^2} = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \frac{\partial^2 h_{i,j,1}(0,0)}{\partial x^2} f(x_i, y_j)$$

$$= \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2}{}^{*} (2!) f(x_i, y_j)/(x_i^2 + y_j^2) + f(0,0) \frac{\partial^2 h_{0,0,1}(0,0)}{\partial x^2}$$

$$= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*} [2/(x_i^2 + y_j^2)][f(x_i, y_j) + f(-x_i, -y_j) + f(-x_i, y_j) + f(x_i, -y_j)]$$

$$+ f(0,0) \frac{\partial^2 h_{0,0,1}(0,0)}{\partial x^2} > f(0,0) \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \frac{\partial^2 h_{i,j,1}(0,0)}{\partial x^2} = 0,$$

taking into account that the strict convexity of f implies $f(x_i, y_j) + f(-x_i, -y_j) > 2f(0,0)$, $f(-x_i, y_j) + f(x_i, -y_j) > 2f(0,0)$ and that we have the identity $\sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} h_{i,j,1}(x,y) = 1$.

Similarly, $\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial y^2} > 0$. On the other hand, by simple calculation we get $\frac{\partial^2 h_{i,j,1}(0,0)}{\partial x \partial y} = 0$, for all $i = -n_1, \dots, n_1$, $j = -n_2, \dots, n_2$, which implies $\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x \partial y} = 0$.

So it is immediate

$$\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x^2} \frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial y^2} > \left(\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x \partial y} \right)^2.$$

As a conclusion, there exists a neighborhood $V(0,0)$ of $(0,0)$ (depending obviously on f and n) such that

$$\frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x^2} \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial y^2} > \left(\frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x \partial y} \right)^2,$$

for all $(x,y) \in V(0,0)$, which proves the theorem. □

Remark. For the cases $\mu \in \mathbb{N}$, $\mu \geq 2$, it is enough if we prove that there exists a neighborhood $V(0,0)$ of $(0,0)$, such that

$$\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2} > 0, \quad \frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2} > 0,$$

$$\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2}\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2} > \left(\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x \partial y}\right)^2, \qquad (4.10)$$

for all $(x, y) \in V(0,0) \setminus (0,0)$, because by the relations (4.8)–(4.9) we obviously have

$$\frac{\partial S_{n_1,n_2,\mu}(f;0,0)}{\partial x} = \frac{\partial S_{n_1,n_2,\mu}(f;0,0)}{\partial y} = 0.$$

Taking into account that for $\mu \geq 2$ relations (4.8)–(4.9) imply

$$\frac{\partial^2 S_{n_1,n_2,\mu}(f;0,0)}{\partial x^2} = \frac{\partial^2 S_{n_1,n_2,\mu}(f;0,0)}{\partial y^2} = 0,$$

the idea of proof (for the $\mu \geq 2$ case) will be to prove that the functions

$$F(x,y) = \frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2}, \quad G(x,y) = \frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2},$$

and

$$H(x,y) = F(x,y)G(x,y) - \left(\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x \partial y}\right)^2$$

are strictly convex on a neighborhood of $(0, 0)$, having as global minimum point $(0, 0)$, which would imply the required relations (4.10).

But the cases $\mu \geq 2$ also require the following three lemmas.

Lemma 4.3.5. *Let $\mu \in \mathbb{N}$, $\mu \geq 2$. If $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is strictly convex on $[-1,1] \times [-1,1]$, then there exists a neighborhood $V(0,0)$ of $(0,0)$ such that*

$$\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2} > 0, \quad \frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2} > 0,$$

for all $(x, y) \in V(0,0) \setminus (0,0)$.

Proof. Denoting for $(i, j) \neq (0, 0)$

$$E(x,y) = \frac{[(x-x_i)^2 + (y-y_j)^2]^{-\mu}}{1 + \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{*n_2} (x^2+y^2)^{\mu} [(x-x_i)^2 + (y-y_j)^2]^{-\mu}},$$

we have $h_{i,j,\mu}(x, y) = P(x, y)E(x, y)$, where

$$P(x,y) = (x^2+y^2)^{\mu} = \sum_{k=0}^{\mu} \binom{\mu}{k} x^{2k} y^{2\mu-2k} = y^{2\mu}$$

$$+\binom{\mu}{1} x^2 y^{2\mu-2k} + \binom{\mu}{2} x^4 y^{2\mu-4} + \cdots + \binom{\mu}{\mu-1} x^{2\mu-2} y^2 + x^{2\mu}.$$

Let us denote

$$R_{i,j}(x,y) = \frac{\partial^{2\mu-2} h_{i,j,\mu}(x,y)}{\partial x^{2\mu-2}}.$$

Firstly, by (4.8)–(4.9) we get

$$\frac{\partial^2 R_{i,j}(x, y)}{\partial x^2} = \frac{(2\mu)!}{[x_i{}^2 + y_j{}^2]^\mu} > 0.$$

Then

$$\frac{\partial^2 R_{i,j}(x, y)}{\partial y^2} = \sum_{k=0}^{2\mu-2} \binom{2\mu-2}{k} \frac{\partial^2}{\partial y^2} \left[\frac{\partial^k P(x, y)}{\partial x^k} \right] \frac{\partial^{2\mu-2-k} E(x, y)}{\partial x^{2\mu-2-k}}$$

$$+2 \sum_{k=0}^{2\mu-2} \binom{2\mu-2}{k} \frac{\partial}{\partial y} \left[\frac{\partial^k P(x, y)}{\partial x^k} \right] \frac{\partial^{2\mu-1-k} E(x, y)}{\partial x^{2\mu-2-k} \partial y}$$

$$+ \sum_{k=0}^{2\mu-2} \binom{2\mu-2}{k} \frac{\partial^k P(x, y)}{\partial x^k} \frac{\partial^{2\mu-k} E(x, y)}{\partial x^{2\mu-2-k} \partial y^2}.$$

If we take $x = y = 0$ in these sums, then all the terms that contain x or (and) y will become zero, so taking into account the form of the polynomial $P(x, y)$, we obtain

$$\frac{\partial^2 R_{i,j}(0, 0)}{\partial y^2} = 2\mu \binom{2\mu-2}{2\mu-2} \frac{(2\mu-2)!}{[x_i{}^2 + y_j{}^2]^\mu} > 0.$$

Reasoning for $S_{n_1,n_2,\mu}(f; x, y)$ exactly as in the case $\mu = 1$ (see the proof of Theorem 4.3.6) we easily obtain

$$\frac{\partial^2}{\partial x^2} \left[\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu-2}} \right] = \frac{\partial^{2\mu} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu}} > 0,$$

$$\frac{\partial^2}{\partial y^2} \left[\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu-2}} \right] = \frac{\partial^{2\mu} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu-2} \partial y^2} > 0.$$

Therefore, there exists a neighborhood $V_1(0, 0)$ of $(0, 0)$ such that for all $(x, y) \in V_1(0, 0)$, we have

$$\frac{\partial^2}{\partial x^2} \left[\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f; x, y)}{\partial x^{2\mu-2}} \right] > 0, \quad \frac{\partial^2}{\partial y^2} \left[\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f; x, y)}{\partial x^{2\mu-2}} \right] > 0.$$

On the other hand, reasoning as above we immediately get

$$\frac{\partial^2}{\partial x \partial y} \left[\frac{\partial^{2\mu-2} h_{i,j,\mu}(0, 0)}{\partial x^{2\mu-2}} \right] = \frac{\partial^{2\mu} h_{i,j,\mu}(0, 0)}{\partial x^{2\mu-1} \partial y} = 0.$$

As a first conclusion, $\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f;x,y)}{\partial x^{2\mu-2}}$ is strictly convex on the neighborhood $V_1(0, 0)$ and because

$$\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu-2}} = \frac{\partial^{2\mu-1} S_{n_1,n_2,\mu}(f; 0, 0)}{\partial x^{2\mu-1}}$$

$$= \frac{\partial^{2\mu-1} S_{n_1,n_2,\mu}(f;0,0)}{\partial x^{2\mu-2}\partial y} = 0,$$

it follows that $\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f;x,y)}{\partial x^{2\mu-2}} > 0$, for all $(x, y) \in V_1(0,0) \setminus (0,0)$.

By symmetry, we get $\frac{\partial^{2\mu-2} S_{n_1,n_2,\mu}(f;x,y)}{\partial y^{2\mu-2}} > 0$, for all $(x, y) \in V_2(0,0) \setminus (0,0)$.

Now, if $\mu = 2$ then we exactly obtain the statement of the lemma.

If $\mu > 2$ then by similar reasonings as above, we obtain that $\frac{\partial^{2\mu-4} S_{n_1,n_2,\mu}(f;x,y)}{\partial x^{2\mu-4}}$ and $\frac{\partial^{2\mu-4} S_{n_1,n_2,\mu}(f;x,y)}{\partial y^{2\mu-4}}$ are strictly convex in a neighborhood $U(0,0)$ of $(0,0)$, and as a conclusion

$$\frac{\partial^{2\mu-4} S_{n_1,n_2,\mu}(f;x,y)}{\partial x^{2\mu-4}} > 0, \quad \frac{\partial^{2\mu-4} S_{n_1,n_2,\mu}(f;x,y)}{\partial y^{2\mu-4}} > 0,$$

for all $(x, y) \in U(0,0) \setminus (0,0)$.

We can continue in this way until we arrive at

$$\frac{\partial^{2} S_{n_1,n_2,\mu}(f;x,y)}{\partial x^{2}} > 0, \quad \frac{\partial^{2} S_{n_1,n_2,\mu}(f;x,y)}{\partial y^{2}} > 0,$$

for all $(x, y) \in V(0,0) \setminus (0,0)$, which proves the lemma. □

Lemma 4.3.6. *Let $\mu \in \mathbb{N}$, $\mu \geq 2$, and let us denote*

$$A := \frac{\partial^{r} S_{n_1,n_2,\mu}(f;0,0)}{\partial x^{i}\partial y^{j}}.$$

We have: $A = 0$ if $r < 2\mu$ or $r > 2\mu$, $A = 0$ if $r = 2\mu$ and both i, j are odd, and $A > 0$ if $r = 2\mu$ and both i, j are even.

Proof. Let $P(x, y) = (x^2 + y^2)^{\mu}$. Denoting $B := \frac{\partial^r P(0,0)}{\partial x^i \partial y^j}$, it is easy to check that we have $B = 0$ if $r < 2\mu$ or $r > 2\mu$, $B = 0$ if $r = 2\mu$ and both i, j are odd, and $B > 0$ if $r = 2\mu$ and both i, j are even. But $h_{i,j,\mu}(x, y) = P(x, y)E(x, y)$, where $E(x, y)$ is given in the proof of Lemma 4.3.5 and

$$\frac{\partial^{r} h_{i,j,\mu}(x,y)}{\partial x^{i}\partial y^{j}} = \sum_{q=0}^{j} \binom{j}{q} \frac{\partial^{i}}{\partial x^{i}} \left[\frac{\partial^{2} P(x,y)}{\partial y^{2}} \frac{\partial^{j-q} E(x,y)}{\partial y^{j-q}} \right],$$

which combined with the above properties of $\frac{\partial^r P(0,0)}{\partial x^i \partial y^j}$ and with the method of proof in Lemma 4.3.5, taking into account that

$$\frac{\partial^{r} S_{n_1,n_2,\mu}(f;0,0)}{\partial x^{i}\partial y^{j}} = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \frac{\partial^{r} h_{i,j,\mu}(0,0)}{\partial x^{i}\partial y^{j}} f(x_i, y_j),$$

proves the lemma. □

Lemma 4.3.7. *For $\mu \in \mathbb{N}$, $\mu \geq 2$, we have*

$$\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2}\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2} - \left(\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x \partial y}\right)^2 > 0,$$

for all $(x, y) \in V(0,0) \setminus (0,0)$, where $V(0,0)$ is a neighborhood of $(0,0)$ (depending on f, n_1, n_2 and μ).

Proof. Denote

$$H(x,y) = \frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x^2}\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial y^2} - \left(\frac{\partial^2 S_{n_1,n_2,\mu}(f;x,y)}{\partial x \partial y}\right)^2.$$

According to the Remark after the proof of Theorem 4.3.6, we have to prove that

$$\frac{\partial^2 H(x,y)}{\partial x^2} > 0, \frac{\partial^2 H(x,y)}{\partial y^2} > 0,$$

$$\frac{\partial^2 H(x,y)}{\partial x^2}\frac{\partial^2 H(x,y)}{\partial y^2} - \left(\frac{\partial^2 H(x,y)}{\partial x \partial y}\right)^2 > 0,$$

for all $(x, y) \in V(0,0) \setminus (0,0)$, and that $\frac{\partial H(0,0)}{\partial x} = \frac{\partial H(0,0)}{\partial y} = 0$ (because by Lemma 4.3.6 we have $H(0,0) = 0$).

Now, by Lemma 4.3.6 we easily get that $\frac{\partial^r H(0,0)}{\partial x^i \partial j} > 0$, only if $r = 2\mu$ and both i, j are even, all the other partial derivatives of H on $(0,0)$ being 0, so we get

$$\frac{\partial^{2\mu} H(0,0)}{\partial x^{2\mu}} > 0, \quad \frac{\partial^{2\mu} H(0,0)}{\partial x^{2\mu-2}\partial y^2} > 0, \quad \frac{\partial^{2\mu} H(0,0)}{\partial x^{2\mu-1}\partial y} = 0,$$

that is, $\frac{\partial^{2\mu-2} H(x,y)}{\partial x^{2\mu-2}}$ is strictly convex in a neighborhood of zero. Because $\frac{\partial^{2\mu-2} H(0,0)}{\partial x^{2\mu-2}} = 0$ and

$$\frac{\partial^{2\mu-1} H(0,0)}{\partial x^{2\mu-1}} = \frac{\partial^{2\mu-1} H(0,0)}{\partial x^{2\mu-2}\partial y} = 0,$$

it follows that $(0,0)$ is a global minimum point, so

$$\frac{\partial^{2\mu-2} H(x,y)}{\partial x^{2\mu-2}} > 0,$$

for all $(x, y) \in V_1(0,0) \setminus (0,0)$.

Reasoning by symmetry, we get

$$\frac{\partial^{2\mu-2} H(x,y)}{\partial y^{2\mu-2}} > 0,$$

for all $(x, y) \in V_2(0,0) \setminus (0,0)$.

Similarly, by Lemma 4.3.6, we obtain

$$\frac{\partial^{2\mu-2} H(x,y)}{\partial x^{2\mu-4}\partial y^2} > 0,$$

for all $(x, y) \in V_3(0,0) \setminus (0,0)$, and consequently

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^{2\mu-4}H(x,y)}{\partial x^{2\mu-4}}\right) > 0, \quad \frac{\partial^2}{\partial y^2}\left(\frac{\partial^{2\mu-4}H(x,y)}{\partial x^{2\mu-4}}\right) > 0,$$

for all $(x, y) \in V_3(0,0) \setminus (0,0)$.

Let us denote

$$H_1(x,y) = \frac{\partial^{2\mu-2}H(x,y)}{\partial x^{2\mu-2}} \frac{\partial^{2\mu-2}H(x,y)}{\partial x^{2\mu-4}\partial y^2} - \left(\frac{\partial^{2\mu-2}H(x,y)}{\partial x^{2\mu-3}\partial y}\right)^2.$$

By Lemma 4.3.6 we obtain

$$\frac{\partial^2 H_1(0,0)}{\partial x^2} > 0, \quad \frac{\partial^2 H_1(0,0)}{\partial y^2} > 0, \quad \frac{\partial^2 H_1(0,0)}{\partial x \partial y} = 0,$$

that is, $H_1(x, y)$ is strictly convex in a neighborhood of $(0, 0)$. But $H_1(0, 0) = 0$ and $\frac{\partial H_1(0,0)}{\partial x} = \frac{\partial H_1(0,0)}{\partial y} = 0$, so it follows that $H_1(x, y) > 0$, for all $(x, y) \in V_4(0,0) \setminus (0,0)$.

As a conclusion, it follows that $\frac{\partial^{2\mu-4}H(x,y)}{\partial x^{2\mu-4}}$ is strictly convex in a neighborhood of zero, and reasoning as above we get

$$\frac{\partial^{2\mu-4}H(x,y)}{\partial x^{2\mu-4}} > 0,$$

for all $(x, y) \in V_5(0,0) \setminus (0,0)$.

Continuing this process by recurrence, finally we arrive to $\frac{\partial^2 H(x,y)}{\partial x^2} > 0$, for all $(x, y) \in V_6(0,0) \setminus (0,0)$, by reason of symmetry to $\frac{\partial^2 H(x,y)}{\partial y^2} > 0$, for all $(x, y) \in V_7(0,0) \setminus (0,0)$, and then

$$\frac{\partial^2 H(x,y)}{\partial x^2}\frac{\partial^2 H(x,y)}{\partial y^2} - \left(\frac{\partial^2 H(x,y)}{\partial x \partial y}\right)^2 > 0,$$

for all $(x, y) \in V_8(0,0) \setminus (0,0)$, which proves the lemma. □

Corollary 4.3.5. *Let* $\mu \in \mathbb{N}$, $\mu \geq 2$. *If* $f : [-1,1]\times[-1,1] \to \mathbb{R}$ *is strictly convex on* $[-1,1]\times[-1,1]$, *then there exists a neighborhood* $V(0,0)$ *of* $(0,0)$ *(depending on* f, μ *and* n_1, n_2*) such that* $S_{n_1,n_2,\mu}(f; x, y)$ *is strictly convex in* $V(0,0)$.

Proof. Immediate by Lemmas 4.3.5–4.3.7.

Remark. For a quantitative estimate of the length of convexity neighborhood $V(0,0)$ in Corollary 4.3.5, we need the following.

Lemma 4.3.8. *Denoting* $n = \max\{n_1, n_2\}$ *and*

$$A_{i,j}(x,y) = \frac{[(x-x_i)^2 + (y-y_j)^2]^{-p}}{T_{n_1,n_2,p}(1;x,y)},$$

we have

$$\left|\frac{\partial^q A_{i,j}(x,y)}{\partial x^l \partial y^k}\right| \leq Cn^q A_{i,j}(x,y).$$

Proof. Firstly, let us denote

$$F_{k,p}(x,y) = \frac{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}(x-x_i)^k[(x-x_i)^2+(y-y_j)^2]^{-p-k}}{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}[(x-x_i)^2+(y-y_j)^2]^{-p}},$$

and x_u (respectively, y_v), the closest point x_i (respectively, y_j) to x (respectively, y). We have

$$|F_{k,p}(x,y)| \leq |x-x_u|^{-k}.$$

Indeed, by $|x-x_u| \leq |x-x_i|$, for all i, we get

$$\frac{|x-x_i|^k}{[(x-x_i)^2+(y-y_j)^2]^k} \leq \frac{|x-x_i|^k}{(x-x_i)^{2k}} = 1/|x-x_k|^k \leq |x-x_u|^{-k}.$$

Similarly, if we denote

$$G_{k,p}(x,y) = \frac{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}(y-y_j)^k[(x-x_i)^2+(y-y_j)^2]^{-p-k}}{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}[(x-x_i)^2+(y-y_j)^2]^{-p}},$$

we get

$$|G_{k,p}(x,y)| \leq |y-y_v|^{-k}.$$

Now, if we denote

$$H_{k,l,p}(x,y) = \frac{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}(x-x_i)^k(y-y_j)^l[(x-x_i)^2+(y-y_j)^2]^{-p-k-l}}{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}[(x-x_i)^2+(y-y_j)^2]^{-p}},$$

then we have

$$|H_{k,l,p}(x,y)| \leq |x-x_u|^{-k}|y-y_v|^{-l}, k,l \geq 0.$$

Then

$$\frac{\partial A_{i,j}(x,y)}{\partial x} = \frac{-2p(x-x_i)}{(x-x_i)^2+(y-y_j)^2}A_{i,j}(x,y) + 2pA_{i,j}(x,y)F_{1,p}(x,y),$$

which immediately implies

$$\left|\frac{\partial A_{i,j}(x,y)}{\partial x}\right| \le C|x-x_u|^{-1}A_{i,j}(x,y) \le CnA_{i,j}(x,y).$$

Similarly,

$$\left|\frac{\partial A_{i,j}(x,y)}{\partial y}\right| \le CnA_{i,j}(x,y).$$

Then

$$\frac{\partial^2 A_{i,j}(x,y)}{\partial x^2}$$

$$= -2pA_{i,j}(x,y)\left[\frac{1}{(x-x_i)^2+(y-y_j)^2} - \frac{2(x-x_i)^2}{[(x-x_i)^2+(y-y_j)^2]^2}\right]$$

$$-\frac{2p(x-x_i)}{(x-x_i)^2+(y-y_j)^2}\frac{\partial A_{i,j}(x,y)}{\partial x} + 2p\frac{\partial A_{i,j}(x,y)}{\partial x}F_{1,p}(x,y)$$

$$+2pA_{i,j}(x,y)\frac{\partial F_{1,p}(x,y)}{\partial x},$$

where

$$\frac{\partial F_{1,p}(x,y)}{\partial x} = \frac{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}[(x-x_i)^2+(y-y_j)^2]^{-p-1}}{\sum_{i=-n_1}^{n_1}\sum_{j=-n_2}^{n_2}[(x-x_i)^2+(y-y_j)^2]^{-p}}$$

$$-2(p+1)F_{2,p}(x,y) + 2p[F_{1,p}(x,y)]^2.$$

We immediately get

$$\left|\frac{\partial^2 A_{i,j}(x,y)}{\partial x^2}\right| \le C|x-x_u|^{-2}A_{i,j}(x,y) \le Cn^2A_{i,j}(x,y).$$

Also

$$\frac{\partial^2 A_{i,j}(x,y)}{\partial x\partial y} = \frac{4p(x-x_i)(y-y_j)}{[(x-x_i)^2+(y-y_j)^2]^2}A_{i,j}(x,y)$$

$$-\frac{2p(x-x_i)}{(x-x_i)^2+(y-y_j)^2}\frac{\partial A_{i,j}(x,y)}{\partial y} + 2p\frac{\partial A_{i,j}(x,y)}{\partial y}F_{1,p}(x,y)$$

$$+2pA_{i,j}(x,y)\frac{\partial F_{1,p}(x,y)}{\partial y},$$

where

$$\frac{\partial F_{1,p}(x,y)}{\partial y} = -2(p+1)H_{1,1,p}(x,y) + 2pF_{1,p}(x,y)G_{1,p}(x,y).$$

It immediately follows

$$\left|\frac{\partial^2 A_{i,j}(x,y)}{\partial x \partial y}\right| \le Cn^2 A_{i,j}(x,y).$$

Reasoning in this way, we obtain

$$\left|\frac{\partial^3 A_{i,j}(x,y)}{\partial x^3}\right| \le Cn^3 A_{i,j}(x,y), \quad \left|\frac{\partial^3 A_{i,j}(x,y)}{\partial x^2 \partial y}\right| \le Cn^3 A_{i,j}(x,y),$$

and so on

$$\left|\frac{\partial^q A_{i,j}(x,y)}{\partial x^l \partial y^k}\right| \le Cn^q A_{i,j}(x,y),$$

which proves the lemma. □

As an immediate consequence, we get

Corollary 4.3.6. *For* $n = \max\{n_1, n_2\}$ *and*

$$S_{n_1,n_2,p}(x,y) = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} A_{i,j}(x,y) f(x_i, y_j),$$

we have

$$\left|\frac{\partial^q S_{n_1,n_2,p}(x,y)}{\partial x^l \partial y^k}\right| \le Cn^q ||f||,$$

where $||f||$ *is the uniform norm.*

The quantitative estimate of convexity in Theorem 4.3.6 is given by the following.

Theorem 4.3.7. *Let* $f : [-1,1] \times [-1,1] \to \mathbb{R}$ *supposed to be strictly convex on* $[-1,1] \times [-1,1]$. *If* $n = \max\{n_1, n_2\}$ *then* $S_{n_1,n_2,1}(f)(x,y)$ *is strictly convex in* $V(0,0) = \{x^2 + y^2 < d^2_{n_1,n_2}\}$, *where*

$$|d_{n_1,n_2}| \ge C \left[\sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*} [2/(x_i^2 + y_j^2)] E(x_i, y_j) \right]^2 / n^5,$$

with $E(x_i, y_j) = [(f(x_i, y_j) + f(-x_i, -y_j) - 2f(0,0)) + (f(-x_i, y_j) + f(x_i, -y_j) - 2f(0,0))]$ *and the sum* $\sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*}$ *means that the index* (i, j) *of double sum is different from* $(0, 0)$.

Proof. From the proof of Theorem 4.3.6, we immediately obtain

$$\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x^2} = \frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial y^2}$$

$$= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*} [2/(x_i^2 + y_j^2)][f(x_i, y_j) + f(-x_i, -y_j) + f(-x_i, y_j) + f(x_i, -y_j)]$$

$$= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*} [2/(x_i^2 + y_j^2)] E(x_i, y_j) > 0,$$

where $E(x_i, y_j) = (f(x_i, y_j) + f(-x_i, -y_j) - 2f(0,0)) + (f(-x_i, y_j) + f(x_i, -y_j) - 2f(0,0)) > 0$, from the convexity of f.

Also, we have $\frac{\partial^2 S_{n_1,n_2,1}(f;0,0)}{\partial x \partial y} = 0$.

Let us denote

$$H(f)(x,y) = \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x^2} \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial y^2} - \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x \partial y}.$$

We have $H(f)(0,0) > 0$. Let $(\alpha_{n_1,n_2}, \beta_{n_1,n_2})$ be the nearest root to $(0,0)$ of $H(f)(x,y)$, in the sense that the distance $d_{n_1,n_2} = ((\alpha_{n_1,n_2})^2 + (\beta_{n_1,n_2})^2)^2)^{1/2}$ is minimum for all the roots of $H(f)(x,y)$. Then, for all $(x,y) \in \{(x,y); (x^2+y^2)^{1/2}\}$, we necessarily have $H(f)(x,y) > 0$. By the mean value theorem for bivariate functions, we get

$$H(f)(0,0) = |H(f)(0,0)| = |H(f)(0,0) - H(f)(\alpha_{n_1,n_2}, \beta_{n_1,n_2})|$$

$$\leq |d_{n_1,n_2}| \left[\left| \frac{\partial H(f)(\xi,\eta)}{\partial x} \right| + \left| \frac{\partial H(f)(\xi,\eta)}{\partial y} \right| \right].$$

But

$$\frac{\partial H(f)(x,y)}{\partial x} = \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial x^3} \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial y^2}$$

$$+ \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial y^2 \partial x} \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x^2} - \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial x^2 \partial y},$$

which by Corollary 4.3.6 implies

$$\left| \frac{\partial H(f)(x,y)}{\partial x} \right| \leq \left| \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial x^3} \right| \left| \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial y^2} \right|$$

$$+ \left| \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial y^2 \partial x} \right| \left| \frac{\partial^2 S_{n_1,n_2,1}(f;x,y)}{\partial x^2} \right| + \left| \frac{\partial^3 S_{n_1,n_2,1}(f;x,y)}{\partial x^2 \partial y} \right| \leq Cn^5,$$

with C depending on f, but independent of n.

Analogously, $\left| \frac{\partial H(f)(x,y)}{\partial y} \right| \leq Cn^5$. We get $0 < H(f)(0,0) \leq C|d_{n_1,n_2}|n^5$, i.e.,

$$|d_{n_1,n_2}| \geq C \left[\sum_{i=0}^{n_1} \sum_{j=0}^{n_2}{}^{*} [2/(x_i^2 + y_j^2)] E(x_i, y_j) \right]^2 / n^5,$$

which proves the theorem. □

Remark. A version of Theorem 4.3.7 for arbitrary $p \geq 2$, i.e., for $S_{n_1,n_2,p}(f)(x,y)$, still remains an open question, because of the complicated reasonings in the proofs of Lemmas 4.3.5, 4.3.6 and 4.3.7.

4.4 Bibliographical Remarks and Open Problems

Theorem 4.2.1, Corollaries 4.2.1, 4.2.2, Theorem 4.2.2 are in Anastassiou–Gal [7], Theorems 4.3.2, 4.3.3, Lemmas 4.3.1–4.3.3 and Corollary 4.3.4 are in Anastassiou–Gal [8]. All the other results of this chapter (except those where the authors are mentioned), are from Gal–Gal [55].

Open Problem 4.4.1. Prove Theorems 4.3.5 and 4.3.7 for all $p \geq 2$.

Open Problem 4.4.2. For the bivariate tensor product operator generated by the univariate Balász–Shepard operator defined on the semi-axis considered by Theorem 2.3.5, prove qualitative and quantitative shape-preserving properties.

Open Problem 4.4.3. For the bivariate tensor product operator generated by the univariate Shepard–Grünwald operator introduced by Criscuolo–Mastroianni [31] (see Open Problem 1.6.4, too), prove qualitative and quantitative shape-preserving properties.

Open Problem 4.4.4. For the bivariate Shepard kind operator defined by

$$S_{n_1,n_2,\mu}(f; x, y) = \frac{T_{n_1,n_2,\mu}(f; x, y)}{T_{n_1,n_2,\mu}(1; x, y)}, \text{ if } (x, y) \neq (x_i, y_j),$$

$S_{n_1,n_2,\mu}(f; x_i, y_j) = f(x_i, y_j)$, where $\mu > 0$ is fixed, $f : D \to \mathbb{R}, D \subset \mathbb{R}^2$, $D = [-1, 1] \times [-1, 1], x_i = i/n_1,\ i = -n_1, \ldots, n_1;\ y_j = j/n_2,\ j = -n_2, \ldots, n_2$ and

$$T_{n_1,n_2,\mu}(f; x, y) = \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \frac{f(x_i, y_j)}{[(x - x_i)^2 + (y - y_j)^2]^{\mu}},$$

prove properties concerning the preservation of monotonicity of f, in neighborhoods of some points.

Open Problem 4.4.5. For the local bivariate tensor products, Shepard operators mentioned in Open Problem 3.5.4 (previous chapter) prove the shape-preserving properties.

A

Appendix: Graphs of Shepard Surfaces

Due to the usefulness of their properties in approximation theory, data fitting, CAGD, fluid dynamics, curves and surfaces, the Shepard operators (univariate and bivariate variants) have been the object of much work. Let us mention here the following papers: [1]–[4], [7], [8], [10]–[17], [23]–[40], [42], [45]–[53], [55]–[58], [60], [62], [65], [69], [70]–[75], [77], [86], [91], [93], [96], [98], [104]–[109].

In this appendix we present some pictures for various kinds of bivariate Shepard operators, which illustrate the shape-preserving property of them.

I thank very much Professor Radu Trimbitas from “Babeş-Bolyai” University, Faculty of Mathematics and Informatics, Cluj-Napoca, Romania, who made all the graphs in this section.

Except type 4, when the original Shepard–Lagrange operator is defined on 100 random knots uniformly distributed into $[-1,1]\times[-1,1]$ (which were then sorted in ascending order with respect to their relative distance), all the other types of bivariate Shepard operators in this section are defined on equidistant knots in the bidimensional interval $[-1,1]\times[-1,1]$, i.e., are of the form $x_k=\frac{k}{n}$, $k=-n,\ldots,0,\ldots,n$ and $y_j=\frac{j}{m}$, $j=-m,\ldots,0,\ldots,m$, with $n=m=5$.

Recall that by the previous sections, we have considered nine main types of Shepard operators, given by the following formulas.

Type 1. Original Shepard operator:

$$S_{n,p}(f)(x,y)=\sum_{i=-n}^{n}s_{i,n,p}(x,y)f(x_i,y_i),$$

where

$$s_{i,n,p}(x,y)=\frac{[(x-x_i)^2+(y-y_i)^2]^{-p}}{\sum_{k=-n}^{n}[(x-x_k)^2+(y-y_k)^2]^{-p}},\quad p>2,\ p\in\mathbb{N}.$$

Type 2. Tensor product Shepard operator:

$$S_{n,m,2p,2q}(f)(x,y)=\sum_{i=-n}^{n}\sum_{j=-m}^{m}s_{i,n,2p}(x)s_{j,m,2q}(y)f(x_i,y_j),\quad p,q\geq 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x - x_i)^{-2p}}{\sum_{k=-n}^{n}(x - x_k)^{-2p}}$$

and

$$s_{j,m,2q}(x) = \frac{(y - y_j)^{-2q}}{\sum_{k=-m}^{m}(y - y_k)^{-2q}}.$$

Type 3. Shepard–Gal–Szabados operator

$$S_{n,m,p}(f)(x, y) = \frac{\sum_{i=-n}^{n}\sum_{j=-m}^{m}[(x - x_i)^2 + (y - y_j)^2]^{-p} f(x_i, y_j)}{\sum_{i=-n}^{n}\sum_{j=-m}^{m}[(x - x_i)^2 + (y - y_j)^2]^{-p}},$$

where $p > 2$, $p \in \mathbb{N}$.

Type 4. Original Shepard–Lagrange operator:

$$S_{n,p,L_m}(f)(x, y) = \sum_{i=-n}^{n} s_{i,n,p}(x, y) L_m^i(f)(x, y),$$

where

$$s_{i,n,p}(x, y) = \frac{[(x - x_i)^2 + (y - y_i)^2]^{-p}}{\sum_{k=-n}^{n}[(x - x_k)^2 + (y - y_k)^2]^{-p}}, \; p > 2, \; p \in \mathbb{N},$$

and $L_m^i(f)(x, y)$ is the Lagrange interpolation bivariate polynomial of degree m (i.e., of the form $\sum_{j+k\le m} a_{j,k}x^j y^k$, where $a_{j,k} \in \mathbb{R}$), uniquely defined by the conditions $L_m(f)(x_{i+\mu}, y_{i+\mu}) = f(x_{i+\mu}, y_{i+\mu})$, $\mu = 1, \dots, m-1$, $m = (n+1)(n+2)/2$, $m < n$, $(x_{n+\mu}, y_{n+\mu}) = (x_\mu, y_\mu)$ (see, e.g., Coman–Trimbitas [27], page 9).

Type 5. Tensor product Shepard–Lagrange: operator

$$S_{n,m,2p,2q,n_1,n_2}(f)(x, y)$$

$$= \sum_{i=-n}^{n}\sum_{j=-m}^{m} s_{i,n,2p}(x) s_{j,m,2q}(y) L_{n_1,n_2}^{i,j}(f)(x, y), \; p, q \ge 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x - x_i)^{-2p}}{\sum_{k=-n}^{n}(x - x_k)^{-2p}},$$

$$s_{j,m,2q}(y) = \frac{(y-y_j)^{-2q}}{\sum\limits_{k=-m}^{m}(y-y_k)^{-2q}},$$

and

$$L^{i,j}_{n_1,n_2}(f)(x,y)$$
$$= \sum_{\nu=0}^{n_1}\sum_{\mu=0}^{n_2} \frac{u_i(x)}{(x-x_{i+\nu})u_i'(x_{i+\nu})}\frac{v_j(y)}{(y-y_{j+\mu})v_j'(y_{j+\mu})} f(x_{i+\nu}, y_{j+\mu}),$$

with $u_i(x) = (x-x_i)\cdots(x-x_{i+n_1})$, $v_j(y) = (y-y_j)\cdots(y-y_{j+n_2})$.

Type 6. Tensor product Shepard–Lagrange–Taylor operator:

$$S_{n,m,2p,2q,n_1,n_2}(f)(x,y)$$
$$= \sum_{i=-n}^{n}\sum_{j=-m}^{m} s_{i,n,2p}(x)s_{j,m,2q}(y)LT^{i,j}_{n_1,n_2}(f)(x,y),\ p,q \geq 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x-x_i)^{-2p}}{\sum\limits_{k=-n}^{n}(x-x_k)^{-2p}},$$

$$s_{j,m,2q}(x) = \frac{(y-y_j)^{-2q}}{\sum\limits_{k=-m}^{m}(y-y_k)^{-2q}},$$

and

$$LT^{i,j}_{n_1,n_2}(f)(x,y) = \sum_{\nu=0}^{n_1}\sum_{\mu=0}^{n_2} \frac{u_i(x)}{(x-x_{i+\nu})u_i'(x_{i+\nu})}\frac{(y-y_j)^{\mu}}{(\mu)!}\frac{\partial^{\mu} f(x_{i+\nu}, y_j)}{\partial y^{\mu}}.$$

Remark. Evidently, by changing the variable x with y we get the "Tensor product Shepard–Taylor–Lagrange" kind of operator.

Type 7. Original Shepard–Taylor operator:

$$S_{n,p,T_{m_{-n},\ldots,m_n}}(f)(x,y) = \sum_{i=-n}^{n} s_{i,n,p}(x,y)T^{i}_{m_i}(f)(x,y),$$

where

$$s_{i,n,p}(x,y) = \frac{[(x-x_i)^2+(y-y_i)^2]^{-p}}{\sum\limits_{k=-n}^{n}[(x-x_k)^2+(y-y_k)^2]^{-p}}, \quad p > 2,\ p \in \mathbb{N},$$

and

$$T^i_{m_i}(f)(x,y) = \sum_{r+s\le m_i} \frac{(x-x_i)^r}{r!}\frac{(y-y_j)^s}{s!}\frac{\partial^{r+s} f(x_i,y_i)}{\partial x^r \partial y^s}.$$

Type 8. Tensor product Shepard–Taylor operator:

$$S_{n,m,2p,2q,n_{-n},\ldots,n_n,m_{-m},\ldots,m_m}(f)(x,y)$$

$$= \sum_{i=-n}^{n}\sum_{j=-m}^{m} s_{i,n,2p}(x) s_{j,m,2q}(y) TT^{i,j}_{n_i,m_j}(f)(x,y), \quad p,q \ge 2,$$

where

$$s_{i,n,2p}(x) = \frac{(x-x_i)^{-2p}}{\sum_{k=-n}^{n}(x-x_k)^{-2p}},$$

$$s_{j,m,2q}(x) = \frac{(y-y_j)^{-2q}}{\sum_{k=-m}^{m}(y-y_k)^{-2q}},$$

and

$$TT^{i,j}_{n_i,m_j}(f)(x,y) = \sum_{\nu=0}^{n_i}\sum_{\mu=0}^{m_j} \frac{(x-x_i)^\nu}{(\nu)!}\frac{(y-y_j)^\mu}{(\mu)!}\frac{\partial^{\nu+\mu} f(x_i,y_j)}{\partial x^\nu \partial y^\mu}.$$

Type 9. Shepard–Gal–Szabados–Taylor operator:

$$S_{n,m,p}(f)(x,y)$$

$$= \frac{\sum_{i=-n}^{n}\sum_{j=-m}^{m}[(x-x_i)^2+(y-y_j)^2]^{-p} TT^{i,j}_{n_i,m_j}(f)(x,y)}{\sum_{i=-n}^{n}\sum_{j=-m}^{m}[(x-x_i)^2+(y-y_j)^2]^{-p}},$$

where

$$TT^{i,j}_{n_i,m_j}(f)(x,y) = \sum_{\nu=0}^{n_i}\sum_{\mu=0}^{m_j} \frac{(x-x_i)^\nu}{(\nu)!}\frac{(y-y_j)^\mu}{(\mu)!}\frac{\partial^{\nu+\mu} f(x_i,y_j)}{\partial x^\nu \partial y^\mu}.$$

The Shepard surfaces generated by the Shepard operators of the types 2, 3, 4, 6, 7 and 8, will be illustrated for the following five examples of functions $f(x,y)$, called in the sequel "test functions."

Example 1. Let $f_1(x,y) = xy$, $x,y \in [-1,1]$. It is a bidimensional (upper) monotone function, i.e., it satisfies the condition:

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0, \quad \forall x, y \in [-1, 1].$$

Another example of this kind is $f_4(x, y) = x^3 y^3$.

Example 2. Let $f_2(x, y) = x^2 y^2$. It is a "strictly double convex" function, i.e., it satisfies the condition:

$$\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} > 0, \quad \forall x, y \in [-1, 1].$$

Example 3. Let $f_3(x, y) = x^2 y^2 + x^2 + y^2$. It is an ordinary strictly convex bivariate function, i.e., satisfies the conditions:

$$\frac{\partial^2 f(x,y)}{\partial x^2} > 0, \quad \frac{\partial^2 f(x,y)}{\partial y^2} > 0,$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} \frac{\partial^2 f(x,y)}{\partial y^2} > \left[\frac{\partial^2 f(x,y)}{\partial x \partial y} \right]^2,$$

$\forall x, y \in (-1, 1)$.

Example 4. Let $f_5(x, y) = 9 \exp(x+y) - xy$. Simple calculations show that f_5 is simultaneously bidimensional (upper) monotone, strictly double convex and ordinary strictly convex.

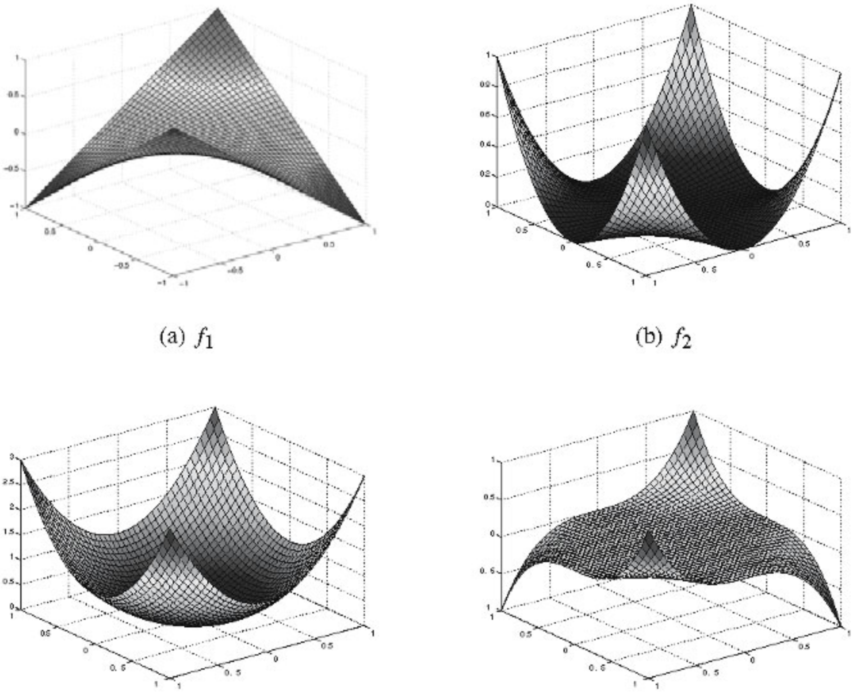
The graphs of the first four functions, f_1, f_2, f_3, f_4, are given in Figure A.1. Also, Figures 5.2, 5.3 and 5.4 contain the graphs of various Shepard operators corresponding to these four test functions.

Figure A.2 shows the graphs for Shepard–Gal–Szabados operators, type 3, $p = 3$.

Figure A.3 gives the graphs for tensor product Shepard–Lagrange–Taylor operators (type 6), $p = 2$, $q = 2$.

In Figure A.4 appear the graphs for original Shepard–Taylor operators (type 7) for $p = 4$.

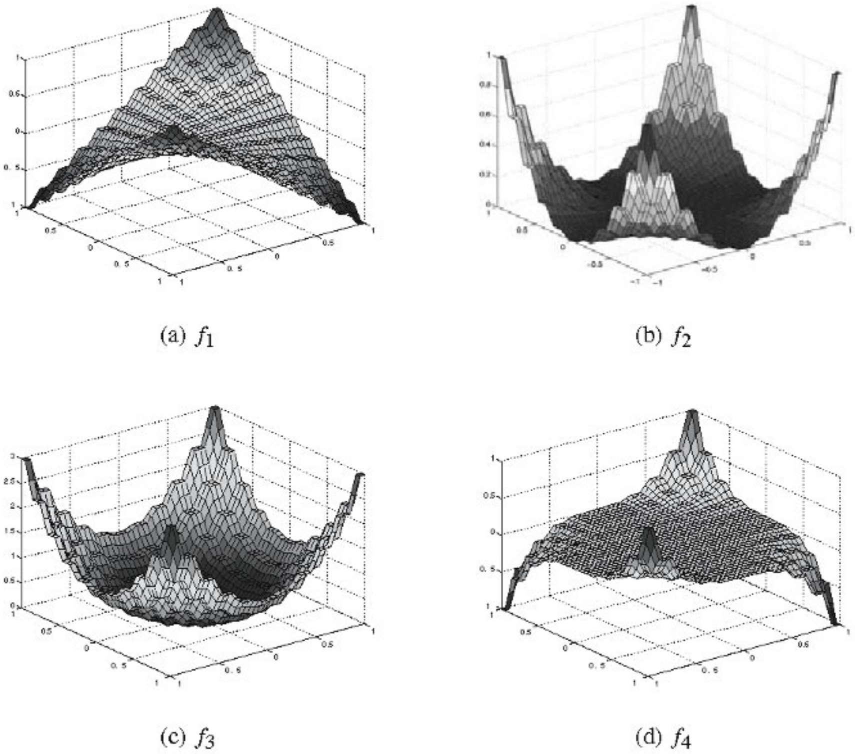
Finally, in Figure A.5 we give the graphs for Shepard operators of types 2, 4 and 8, corresponding to the function $f_5(x, y) = 9 \exp(x + y) - xy$. The graph of this function is given in Figure A.5(a).



(a) f_1 (b) f_2

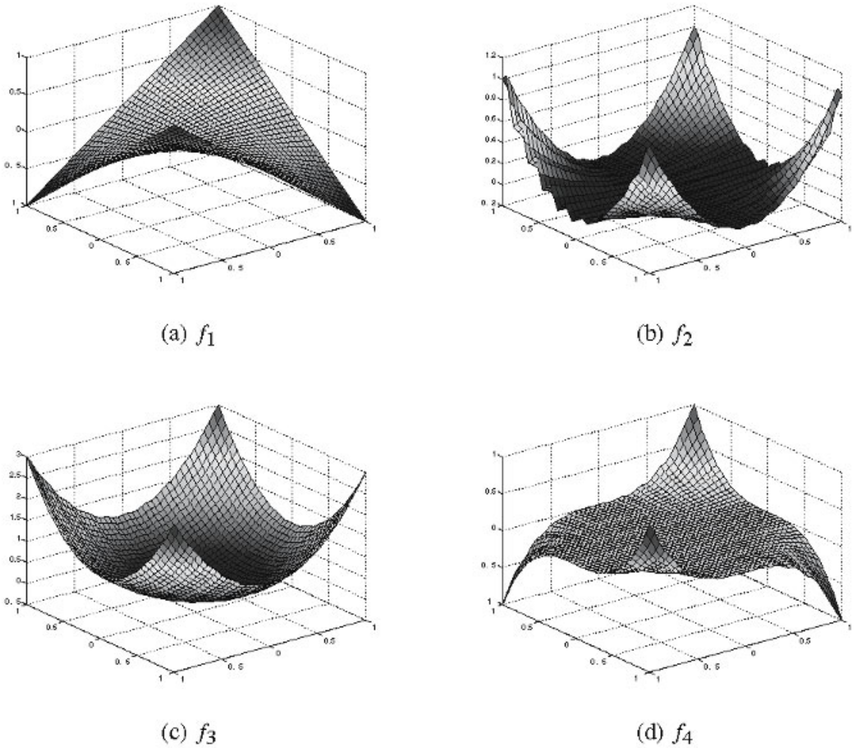
(c) f_3 (d) f_4

Fig. A.1. Graphs of the test functions, f_1, f_2, f_3, f_4



(a) f_1 (b) f_2 (c) f_3 (d) f_4

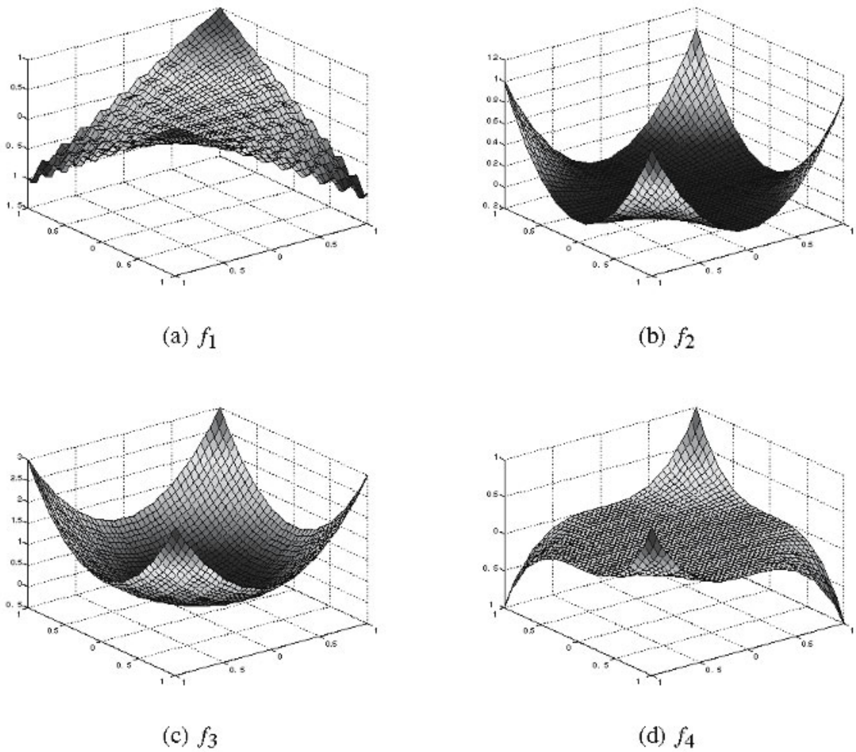
Fig. A.2. Graphs for Shepard–Gal–Szabados operators, Type 3, $p = 3$



(a) f_1 (b) f_2

(c) f_3 (d) f_4

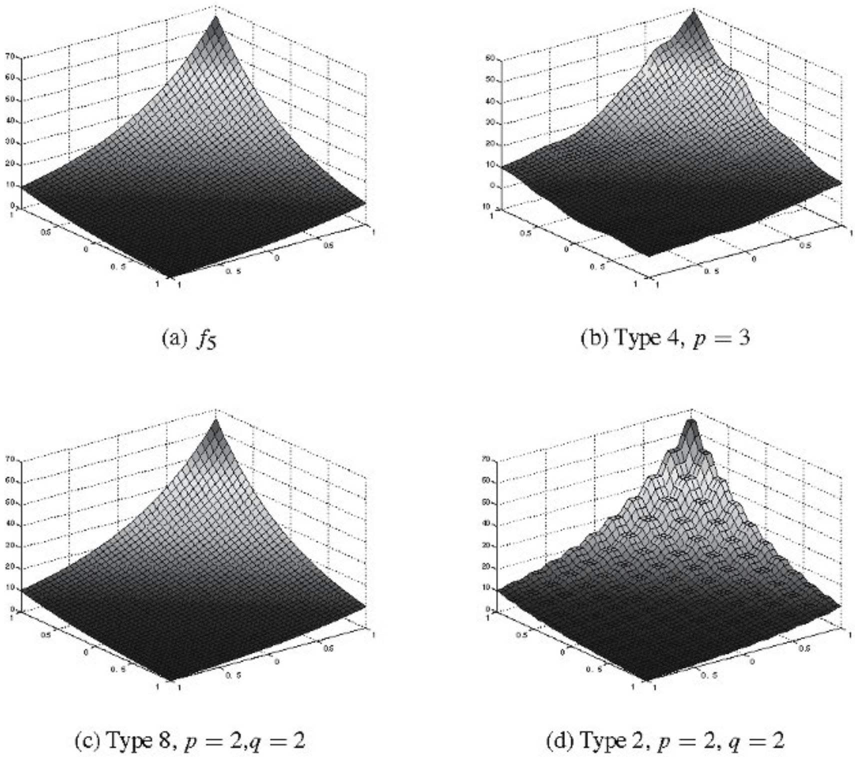
Fig. A.3. Graphs for tensor product Shepard–Lagrange–Taylor operators, type 6, $p = 2, q = 2$



(a) f_1 (b) f_2

(c) f_3 (d) f_4

Fig. A.4. Graphs for original Shepard–Taylor operators, type 7, $p = 4$



(a) f_5

(b) Type 4, $p = 3$

(c) Type 8, $p = 2, q = 2$

(d) Type 2, $p = 2$, $q = 2$

Fig. A.5. Graphs for f_5 and Shepard operators of types 2, 4 and 8

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