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George A. Anastassiou
Department of Mathematical Sciences
University of Memphis

Sorin G. Gal

Shape-Preserving Approximation by Real and Complex Polynomials

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Sorin G. Gal
Department of Mathematics
and Computer Science
University of Oradea
410087 Oradea
Romania
galso@uoradea.ro

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To the memory of my parents
Gheorghe and Ana

Preface

In many problems arising in engineering and science one requires approximation methods to reproduce physical reality as well as possible. Very schematically, if the input data represents a complicated discrete/continuous quantity of information, of "shape" S (S could mean, for example, that we have a "monotone/convex" collection of data), then one desires to represent it by the less-complicated output information, that "approximates well" the input data and, in addition, has the same "shape" S.

This kind of approximation is called "shape-preserving approximation" and arises in computer-aided geometric design, robotics, chemistry, etc.

Typically, the input data is represented by a real or complex function (of one or several variables), and the output data is chosen to be in one of the classes polynomial, spline, or rational functions.

The present monograph deals in Chapters 1–4 with shape-preserving approximation by real or complex polynomials in one or several variables. Chapter 5 is an exception and is devoted to some related important but nonpolynomial and nonspline approximations preserving shape. The spline case is completely excluded in the present book, since on the one hand, many details concerning shape-preserving properties of splines can be found, for example, in the books of de Boor [49], Schumaker [344], Chui [69], DeVore–Lorentz [91], Kvasov [218] and in the surveys of Leviatan [229], Kocić–Milovanović [196], while on the other hand, we consider that shape-preserving approximation by splines deserves a complete study in a separate book.

The topic of shape-preserving approximation by real polynomials has a long history and probably begins with an earlier result of Pál [295] in 1925, which states that any convex function on an interval $[a, b]$ can be uniformly approximated on that interval by a sequence of convex polynomials.

The first constructive answer to the Pál's result seems to have been given by T. Popoviciu [317] in 1937, who proved that if f is convex (strictly convex) of order k on $[0, 1]$ (in the sense defined in Section 1.1), then the Bernstein polynomial $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$ is convex (strictly convex, respectively) of order k on $[0, 1]$, for all $n \in \mathbb{N}$.

Over time, much effort has been expended by many mathematicians to contribute to this topic. As good examples of surveys concerning shape-preserving approximation of univariate real functions by real polynomials, we can mention those of Leviatan [229], [230] in 1996 and 2000, that of Kocić–Milovanović [197] in 1997, and that of Hu–Yu [178] in 2000.

Also, a few aspects in the univariate real case are presented in the following books:

Lorentz–v. Golitschek–Makozov [249] in 1996, see Chapter 2, Section 3, titled Monotone Approximation (pp. 43–49), and page 82, with Problem 9.4 and Notes 10.1, 10.2,

Shevchuk [349] in 1992, referring to some results in monotone and convex approximation of univariate real functions by real polynomials,

Lorentz [247](see p. 23) in 1986, DeVore–Lorentz [91] in 1993, (see Chapter 10, Section 3, from page 307 to page 309), concerning some shape-preserving properties of real Bernstein polynomials, and

Gal [123] in 2005, concerning shape-preserving properties of classical Hermite–Fejér and Grünwald interpolation polynomials.

For the situation in the case of one complex variable, it is worth noting that two books concerning the study of complex polynomials have recently been published. The first is that of Sheil–Small [346] in 2002, which studies many geometric properties of complex polynomials and rational functions. But except for two small sections on the complex convolution polynomials through Cesàro and de la Vallée-Poussin trigonometric kernels (Sections 4.5 and 4.6, from page 156 to page 166), in fact that book does not deal with the preservation of geometric properties of analytic functions by approximating complex polynomials. The second book mentioned above is that of Rahman–Schmeisser [320] in 2002, which refers to the critical points, zeros, and extremal properties of complex polynomials, which are regarded as analytic functions of a special kind. Although some of its results improve classical inequalities of great importance in approximation theory (of Nikolskii, Bernstein, Markov, etc.), this book again does not deal with the preservation of geometric properties of analytic functions by approximating complex polynomials.

In the cases of two/several real or complex variables, there are no books at all treating the subject of shape-preserving approximation.

Therefore, we may conclude that despite the very large numbers of papers in the literature, at present, none of the books has been dedicated entirely to shape-preserving approximation by real and complex polynomials.

The present monograph seeks to fill this gap in the mathematical literature and is, to the best of our knowledge, the first book entirely dedicated to this topic. It attempts to assemble the main results from the great variety of contributions spread across a large number of journals all over the world.

This monograph contains the work of the main researchers in this area, as well as the research of the author over the past five years in these subjects and many new contributions that have not previously been published.

Chapter 1 mainly studies shape-preserving approximation and interpolation of real functions of one real variable by real polynomials. The "shapes" taken into consideration are convexity of order k (which includes the usual positivity, monotonicity, and convexity), some variations of positivity as almost positivity, strongly/weakly almost positivity, copositivity (with its variations almost copositivity, strongly/weakly copositivity), comonotonicity, and coconvexity. A variation of copositive approximation, called intertwining approximation (with its two variations almost and nearly intertwining), also is presented.

Chapter 2 deals with shape-preserving approximation of real functions of two/several real variables by bivariate/multivariate real polynomials. A main characteristic of this chapter is that to one concept of shape in univariate case, several concepts of shapes of a bivariate/multivariate function may be associated. For example, monotonicity has as variations *bivariate monotonicity, axial monotonicity, strong monotonicity*; convexity has the variations *axial-convexity, polyhedral convexity, strong convexity, and subharmonicity*, and so on.

In Chapter 3 we consider shape-preserving approximation of analytic functions of one complex variable by complex polynomials in the unit disk. The concepts of "shapes" preserved through approximation by polynomials are those in geometric function theory: *univalence, starlikeness, convexity, close-to-convexity, spiralikeness, growth of coefficients*, etc. The construction of such polynomials is mainly based on the Shisha-type method and on the convolution method.

Chapter 4 contains extensions of some results in Chapter 3 to shape-preserving approximation of analytic functions of several complex variables on the unit ball or the unit polydisk by polynomials of several complex variables.

It is worth noting that three constructive methods are "red lines" of the book, that is, they work for real univariate variables, real multivariate variables, complex univariate variables, and complex multivariate variables. These are the methods of Bernstein, producing Bernstein-type polynomials; the Shisha-type method; and the convolution-type method. As a consequence, Chapters 1–4 use these three methods. Also, although the error estimates produced by the *tensor product method* are not always the best possible, because of its simplicity we use it intensively in order to extend the results from the univariate to the bivariate/multivariate case.

Chapters 1–5 begin with an introductory section, in which we describe in detail the corresponding chapter and introduce the main concepts.

The book ends with Chapter 5, which is an appendix containing some related topics of great interest in shape-preserving approximation. Shape-preserving approximation by splines is not included in this chapter for the reasons mentioned at the beginning of this preface.

Let us mention that systematic results in Chapters 2–5 have been obtained by the author of this monograph in a series of papers, singly or jointly written with other researchers (as can be seen in the bibliography), and many new results appear for the first time here. Also, many open questions suggested at the end of Chapters 1–5 might be of interest for future research.

The book is intended for use in the fields of approximation of functions, mathematical analysis, numerical analysis, computer-aided geometric design, data fitting, fluid mechanics, and engineering, robotics, and chemistry. It is also suitable for graduate courses in the above domains.

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Sorin G. Gal
Department of Mathematics
University of Oradea
Romania

Contents

1

Shape-Preserving Approximation by Real Univariate Polynomials

In this chapter we present the main results concerning shape-preserving approximation by polynomials for real functions of one real variable, defined on compact subintervals of the real axis. There is a very rich literature dedicated to this topic that would suffice to write a separate book. Due to this fact, it was impossible for me to avoid the more pronounced survey-like character of this chapter. Also, for the proofs of some main results that are very technical and long, we will present here only their most important ideas and steps.

1.1 Introduction

In this section we will introduce the history of the subject, followed by very brief descriptions of the next sections in the chapter.

Probably one of the first results on the topic is an earlier result of Pál [295] in 1925, which states that any convex function on an interval $[a, b]$ can be uniformly approximated on that interval by a sequence of convex polynomials.

The first constructive solution to Pál's result seems to have been given by T. Popoviciu [317] in 1937, who proved that if f is convex (strictly convex) of order k on $[0, 1]$ (in the sense defined below in this section), then the *Bernstein* polynomial $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$ is convex (strictly convex, respectively) of order k on $[0, 1]$, for all $n \in \mathbb{N}$.

In the intervening years, a great deal of work has been done on this topic by many mathematicians. The aim of this chapter is to present this great effort in detail.

The topic of Chapter 1 might be divided into five main directions.

The first direction deals with the shape-preserving properties of interpolation polynomials, and this is the subject of Section 1.2. We mention here the contributions of (in alphabetical order) Deutch, Gal, Ivan, Kammerer, Kopotun, Lorentz, Morris, Nikolcheva, Passow, Popoviciu, Raymon, Roulier, Rubinstein, Szabados, Wolibner, Young, Zeller, and others.

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The second direction deals with the shape-preserving properties of the so-called Bernstein-type polynomial operators, (thus called because their constructions were suggested by the form of Bernstein's polynomials), representing the subjects of Section 1.3. We can mention here the contributions of (in alphabetical order) Berens, Butzer, Carnicer, Dahmen, Derrienic, DeVore, Gadzijev, Goodman, Ibikli, Ibragimov, Kocić, Lacković, Lupaş, Mastroianni, Micchelli, Munoz-Delgado, Müller, Nessel, Păltănea, Peña, Phillips, Ramirez-Gonzalez, Raşa, Sabloniére, Sauer, Stancu, Wood, and others.

Because of its close connection with the shape-preserving properties (see Section 5.1), the variation-diminishing property too is presented in Section 1.3.

The third direction deals with the so-called *Shisha*-type results, and it began with Shisha's paper of 1965. The method is, in general, based on polynomials of simultaneous approximation of a function and its derivatives, to which are added suitable polynomials (uniformly convergent to zero) in such a way that the new sum preserves some signs of the derivatives of the function. We mention here the contributions of (in chronological order) Shisha, Roulier, and Anastassiou–Shisha. It is contained in Section 1.4.

It is worth noting here the importance of Shisha's method, taking into account that because of its simplicity, it was extended to real functions of two real variables in Chapter 2, to complex functions of one complex variable in Chapter 3, and to complex functions of several complex variables in Chapter 4.

Note that the second direction of research produces rather weak degrees of approximation in terms of $\omega_k(f; \frac{1}{\sqrt{n}}), k = 1, 2$, while the third direction of research, although essentially improving the estimates of the second direction, has, however, the shortcoming that these estimates are given in terms of the moduli of smoothness of the derivatives of the function.

In order to obtain better estimates, that is, with respect to the moduli of smoothness (of various orders) of a function, one of the most used techniques (introduced for the first time in DeVore-Yu [86]) can be described as follows: first one approximates f by piecewise polynomials (splines) with the same shape as f, and then one replaces the piecewise polynomials by polynomials of the same shape. Estimates in terms of first- or higher-order moduli of smoothness in all the L^p-spaces, $0 < p \leq +\infty$, were found by (in alphabetical order) Beatson, DeVore, Ditzian, Dzyubenko, Hu, Iliev, Ivanov, Kopotun, Leviatan, Lorentz, Mhaskar, Newman, Operstein, Popov, Prymak, Shevchuk, Shvedov, Szabados, Wu, Yu, Zeller, Zhou, and others.

The main results are included in Sections 1.5, 1.6, 1.7 and are represented by the so-called *positive and copositive (with their variations like, almost, strongly/weakly, intertwining)* approximation, *monotone and comonotone* approximation (with the variation *nearly comonotone* approximation), and *convex and coconvex* approximation (with the variation *nearly coconvex* approximation), respectively. The above-mentioned variations of classical positive/copositive, comonotone, and coconvex approximations were introduced by the authors in order to improve the estimates, by requiring that the polynomials preserve the corresponding "shapes" in a major part of the interval, except for small neighborhoods of the endpoints and of the points where the approximated function changes the “shapes”.

The shape-preserving approximation results in Sections 1.5, 1.6, and 1.7 can also be classified with respect to the type of error estimate, as follows:

(i) approximation results with respect to the L^p-norm and in terms of best approximation quantities $E_n(f^{(i)})_p$, $i = 0, 1, 2$, and $0 < p \leq +\infty$.
(ii) approximation results with respect to the L^p-norm and in terms of the L^p-Ditzian–Totik moduli of smoothness, $0 < p \leq +\infty$.
Note that in both cases (i) and (ii), the uniform cases (i.e., $p = +\infty$) are richer in results than the cases $0 < p < +\infty$ and will be separately treated.
(iii) pointwise approximation on $[-1, 1]$ with DeVore-Telyakokovskii-Gopengauz-type estimates, in terms of the usual moduli of smoothness and with respect to the increments $\frac{1}{n^2} + \frac{1}{n}(1 - x^2)^{1/2}$ and $\frac{1}{n}(1 - x^2)^{1/2}$.
(iv) approximation in terms of higher moduli of smoothness of higher derivatives of functions.

Notice that while in monotone and convex approximation, the methods that produces estimates in terms of second-order moduli of smoothness are linear, the methods in convex approximation that produce the best possible order, i.e., in terms of the third-order moduli of smoothness, together with those in copositive, comonotone, and coconvex approximations, are nonlinear. It is not known whether there exist corresponding linear methods of approximation for these last three cases too.

Section 1.8 deals with the fifth direction of research, based on convolution-type polynomials and on the Boolean-sum method. This method produces good approximation errors of DeVore–Gopengauz type, but with respect to the previous ones has the advantage that the constructed polynomials preserve even higher-order convexities too. We mention here the contributions of Jia-Ding Cao and Gonska.

Section 1.9 presents a constructive example of a nonconvolution, positive linear polynomial operator that reproduces the linear functions, gives an error estimate of DeVore–Gopengauz-type in terms of second-order modulus of smoothness and preserves convexities of higher-order of the approximated function. The contributions belong to Jia-Ding Cao, Cottin, Gavrea, Gonska, Kacsó, Lupaş and Zhou.

In what follows, we introduce well-known concepts of shapes (monotonicities, convexities, etc.) necessary for the next sections of Chapter 1. Denote by $C[a, b]$ the space of all real functions defined and continuous on $[a, b]$.

Definition 1.1.1. (i) $f : [a, b] \to \mathbb{R}$ is called j-convex on $[a, b]$ (or convex of order j), if all the jth forward differences $\Delta_h^j f(t), 0 \leq h \leq (b - a)/j, t \in [a, b - jh]$ are non-negative (i.e., ≥ 0). Here $\Delta_h^j f(t) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(t + kh)$, for all $j = 0, 1, \ldots$. If there exists $f^{(j)}$ on $[a, b]$, a simple application of the mean value theorem shows that the

condition $f^{(j)}(x) \geq 0$, for all x $[a, b]$, implies that f is j-convex on $[a, b]$. Recall that the usual convexity (2-convexity in the above sense) can also be defined by the inequality $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$, for all $\lambda \in [0, 1]$ and $x, y \in [a, b]$.
Also, f is called j-concave on $[a, b]$ if all the jth forward differences $\Delta_h^j f(t), 0 \leq h \leq (b - a)/j, t \in [a, b - jh]$ are nonpositive (i.e., ≤ 0).

(ii) A function $f : [0, 1] \to \mathbb{R}$ is called starshaped on $[0, 1]$ if $f(\lambda x) \leq \lambda f(x)$, for all $\lambda \in [0, 1]$, $x \in [0, 1]$. If the above inequality is strict for all $\lambda \in (0, 1)$ then f is called strictly starshaped. Also, if there exists $f'(x)$ on $[0, 1]$, $f(0) = 0, f(x) \geq 0, x \in [0, 1]$, then the starshapedness (it is equivalent to) can be expressed by the differential inequality $xf'(x) - f(x) \geq 0$, for all $x \in (0, 1]$ (see, e.g., L. Lupaş [254]);
A function $f : [0, 1] \to \mathbb{R}$ is called α-star-convex on $[0, 1]$, where $\alpha \in [0, 1]$, if $f(\lambda x + (1 - \lambda)\alpha y) \leq \lambda f(x) + (1 - \lambda)\alpha f(y)$, for all $x, y \in [0, 1]$, $\lambda \in [0, 1]$ (see Toader [386]).

(iii) A function $f : [a, b] \to \mathbb{R}$, $f(x) > 0$, for all $x \in [a, b]$, is called logarithmic-convex on $[a, b]$, if $\log[f(x)]$ is a 2-convex function on $[a, b]$;

(iv) A function $f : [a, b] \to \mathbb{R}$ is called quasiconvex on $[a, b]$ if it satisfies the inequality $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. It is known that f is quasiconvex on $[a, b]$ if and only if for any $c \in \mathbb{R}$, $\{x \in [a, b]; f(x) \leq c\}$ is a convex set;
More generally, a function $f : [a, b] \to \mathbb{R}$ is called j-quasiconvex on $[a, b]$, $j \in \mathbb{N}$, if it satisfies the inequality

$$[x_2, \ldots, x_{j+1}; f] \leq \max\{[x_1, \ldots, x_j; f], [x_3, \ldots, x_{j+2}; f]\},$$

for every system of distinct points $x_1 < \cdots < x_{j+2}$ in $[a, b]$. Here

$$[x_1, \ldots, x_j; f] = \sum_{k=1}^{j} \frac{f(x_k)}{u_k(x_k)}$$

$\left(\text{with } u_k(x) = \frac{\Pi_{i=1}^j (x - x_i)}{x - x_k}\right)$ denotes the divided difference of f on the points $x_1, \ldots, x_j$, and $j = 1, 2, \ldots$. Note that for $j = 1$ we obtain again the usual quasi-convexity.

(v) Let $f, u \in C[a, b]$, $u(x) > 0$, for all $x \in [a, b]$. We say that f is u-monotone if $u(x_1)f(x_2) - u(x_2)f(x_1) \geq 0$, for all $a \leq x_1 < x_2 \leq b$.

(vi) For $(x_k)_{k=0}^n$, $0 \leq x_0 < x_1 < \cdots < x_n \leq 1$, let us denote by $S_{[0,1]}[f; (x_k)_k]$ the number of changes of sign in the finite sequence $f(x_0), f(x_1), \ldots, f(x_n)$, where zeros are disregarded. Also, define the number of changes of sign for f on $[0, 1]$ by $S_{[0,1]}[f] = \sup\{S_{[0,1]}[f; (x_k)_k]; (x_k)_{k=0}^n, n \in \mathbb{N}\}$. One says that the linear operator $L : C[0, 1] \to C[0, 1]$ is strongly variation-diminishing on $[0, 1]$, if $S_{[0,1]}[L(f)] \leq S_{[0,1]}[f]$, for all $f \in C[0, 1]$.

Remarks. (1) The concept of j-quasiconvexity belongs to E. Popoviciu (see, e.g., [311]) and the concept of u-monotonicity was introduced by Kocić–Lacković [195].

(2) The j-convexity introduced in Definition 1.1.1 (i) is sometimes called Jensen convexity of order j. A slightly more general concept of convexity, called Popoviciu convexity of order j, was introduced by Popoviciu [315] (in a slightly different denomination), as follows: one says that $f : [a, b] \to \mathbb{R}$ is Popoviciu convex of order j, if for all systems of distinct points (not necessarily equidistant) $a \leq x_0 < \cdots < x_j \leq b$, we have $[x_0, \ldots, x_j; f] \geq 0$. But, according to a result stated without proof by Popoviciu [318] in 1959, and completely proved in 1997 in, e.g., Ivan–Raşa [184], if f is continuous on $[a, b]$, then for any system of distinct points $a \leq x_0 < \cdots < x_j \leq b$, there are the points $c, c+h, \ldots, c+jh \in [a, b], h \geq 0$, such that $[x_0, \ldots, x_j; f] = \frac{1}{j!h^j} \Delta_h^j f(c)$.

This immediately implies that for continuous functions, the Jensen and Popoviciu convexities coincide and because in approximation, most of the time the functions considered are at least continuous, in those cases we will simply refer to j-convexity.

(3) The concept of an α-star-convex function $\alpha \in [0, 1]$ is an intermediate concept between the concept of usual convex and that of starshaped function. Indeed, in Definition 1.1.1 (ii), for $\alpha = 1$ we get the concept of usual convex function, while for $\alpha = 0$ we get the concept of starshaped function.

It is worthwhile to point out here the following main properties of an α-star-convex function $f : [0, 1] \to \mathbb{R}$, with $\alpha \in (0, 1]$ (see Mocanu–Şerb–Toader [274]): f is starshaped on $[0, 1]$ (for $f(0) \leq 0$), continuous on $(0, \alpha)$, bounded on $[0, 1]$, and Lipschitz in each compact subinterval of $(0, \alpha)$.

Also, we need the following.

Definition 1.1.2. (i) (see e.g. DeVore–Lorentz [91], p. 44) The modulus of smoothness of $f \in L^p[-1, 1]$, $0 < p \leq +\infty$, denoted by $\omega_k(f, t)_p$, $k \in \{0, 1, \ldots, \}$ is defined by $\omega_0(f, t)_p := \|f\|_{L^p[a,b]} := \|f\|_p$ and for $k \geq 1$ by

$$\omega_k(f, t)_p = \sup_{0 \leq h \leq t} \{\|\overline{\Delta}_h^k f(\cdot)\|_p\},$$

where $\overline{\Delta}_h^k f(x) = \Delta_h^k f(x)$ if $x, x + kh \in [-1, 1]$, $\overline{\Delta}_h^k f(x) = 0$; otherwise, $\Delta_h^k f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + ih)$. Here $L^\infty[-1, 1] = C[-1, 1]$, the space of all continuous functions on $[-1, 1]$.

(ii) (see Ditzian–Totik [98]) Set $\varphi(x) := \sqrt{1 - x^2}$ and define the kth symmetric difference

$$\Delta_{h\varphi}^k f(x) := \begin{cases} \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x + (i - \frac{k}{2}) h\varphi(x)), & x \pm \frac{k}{2} h\varphi(x) \in [-1, 1], \\ 0 & \text{otherwise}, \end{cases}$$

where $\Delta^0_{h\varphi} f(x) := f(x)$. Then the Ditzian–Totik modulus of smoothness of order k is given by

$$\omega_k^{\varphi}(f,t)_p := \sup_{0<h\le t} \|\Delta^k_{h\varphi} f\|_p.$$

(iii) (Sendov–Popov [345]) The kth averaged modulus of smoothness (called τ-modulus too) defined for a measurable bounded real function defined on $[a,b]$ is given by

$$\tau_k(f,t,[a,b])_p = \|\omega_k(f,\cdot,t)\|_{L^p[a,b]},$$

where $1 \le p \le \infty$, Δ^k_h is the kth symmetric difference from the above point (ii), and

$$\omega_k(f,x,t) = \sup\{|\Delta^k_h|; y \pm mh/2 \in [x - mt/2, x + mt/2] \cap [a,b]\}.$$

Remark. For $p = \infty$ one can modify these moduli by taking into account not only the position of x in the interval when setting $\Delta^k_{h\varphi} f$, but also how far the endpoints of the interval $[x - \frac{k}{2}h\varphi(x), x + \frac{k}{2}h\varphi(x)]$ are from the endpoints of $[-1,1]$. Thus, one can introduce the following.

Definition 1.1.3. (Shevchuk [349]) Let us define

$$\varphi_\delta(x) := \sqrt{(1 - x - \frac{\delta}{2}\varphi(x))(1 + x - \frac{\delta}{2}\varphi(x))}, \qquad x \pm \frac{\delta}{2}\varphi(x) \in [-1,1],$$

and by C^r_φ the set of functions $f \in C^r(-1,1) \cap C[-1,1]$, such that $\lim_{x\to\pm 1} \varphi^r(x) f^{(r)}(x) = 0$.

The modified Ditzian–Totik modulus of smoothness of order (k,r) is given by

$$\omega^{\varphi}_{k,r}(f^{(r)},t) := \sup_{0\le h\le t} \sup_x |\varphi^r_{kh}(x)\Delta^k_{h\varphi(x)} f^{(r)}(x)|, \quad t \ge 0,$$

where $\Delta^k_h f(x)$ denotes the kth symmetric difference and the inner supremum is taken over all x such that

$$x \pm \frac{k}{2}h\varphi(x) \in (-1,1).$$

Remarks. (1) For $k = 0$ we have

$$\omega^{\varphi}_{0,r}(f^{(r)},t) = \|\varphi^r f^{(r)}\|_\infty,$$

while for $r = 0$ we have

$$\omega^{\varphi}_{k,0}(f^{(0)},t) := \omega^{\varphi}_k(f,t).$$

The above condition guarantees that for $k \geq 1$, it follows that $\omega_{k,r}^{\varphi}(f^{(r)}, t) \to 0$, as $t \to 0$. Also, if $f \in C_{\varphi}^{r}$ and $0 \leq m < r$, then

$$\omega_{k+r-m,m}^{\varphi}(f^{(m)}, t) \leq C(k, r) t^{r-m} \omega_{k,r}^{\varphi}(f^{(r)}, t), \quad t \geq 0.$$

Conversely, if $f \in C[-1, 1]$, $m < \alpha < k$, and $\omega_{k}^{\varphi}(f, t) \leq t^{\alpha}$, then $f \in C_{\varphi}^{m}$ and

$$\omega_{k-m,m}^{\varphi}(f^{(m)}, t) \leq C(\alpha, k) t^{\alpha-m}, \quad t \geq 0.$$

(2) If $f \in C_{\varphi}^{m}$ and $\omega_{r-m,m}^{\varphi}(f^{(m)}, t) \leq t^{r-m}$, then

$$\|\varphi^{r} f^{(r)}\|_{\infty} \leq C(r).$$

If we denote the class of all functions satisfying this last inequality by $\mathbb{B}^{r}$, then the converse is valid too, that is, if $f \in \mathbb{B}^{r}$ and $0 \leq m < r$, then $f \in C_{\varphi}^{m}$ and

$$\omega_{r-m,m}^{\varphi}(f^{(m)}, t) \leq C(r) t^{r-m} \|\varphi^{r} f^{(r)}\|_{\infty}, \quad t \geq 0.$$

1.2 Shape-Preserving Interpolation by Polynomials

The existence of interpolating polynomials that are monotone with the interpolated data was established by Wolibner [399] and independently by Kammerer [189] and Young [404], as follows.

Theorem 1.2.1. (see Wolibner [399], Young [404], Kammerer [189]) *Let $(x_i, y_i), i = 1, \ldots, n$ be a set of data such that $x_1 < x_2 < \cdots < x_n$ and $y_i \neq y_{i+1}, i = 1, \ldots, n-1$, then there exists an algebraic polynomial p with the following properties:*

$$p(x_i) = y_i, i = 1, \ldots, n, \ \mathrm{sgn}[p'(x)] = \mathrm{sgn}[\Delta y_i], \ x \in [x_i, x_{i+1}], \ i = 1, \ldots, n-1,$$

where $\Delta y_i = y_{i+1} - y_i$.

Proof. We follow here the ideas in the proof of Wolibner [399]. Denote by $\phi(x)$ the continuous piecewise linear function defined on $[x_1, x_n]$ and passing through all the points (x_k, y_k). It is evident that we can define a twice differentiable function $f : [x_1, x_n] \to \mathbb{R}$ such that $f(x_k) = y_k, k = 1, \ldots, n$, f is comonotone with ϕ, (i.e., $f(x)$ is of the same monotonicity with $\phi(x)$ on each subinterval $[x_k, x_{k+1}]$), the monotonicity is given by the sign of the difference $f(x_{k+1}) - f(x_k)$), and, in addition f is strictly monotonic on each subinterval $[x_k, x_{k+1}]$.

It follows that f' can have only simple zeros. Denote by c_j, $j = 1, \ldots, m$, $m \leq n$, the x_j that are simple zeros. Then $F(x) = \frac{f'(x)}{\Pi_{k=1}^{m}(x - x_k)}$ cannot be zero on $[x_1, x_n]$, i.e., by the continuity of $F(x)$, we get that $F(x) > 0, \forall x \in [x_1, x_n]$ or $F(x) < 0, \forall x \in [x_1, x_n]$. In both cases, for any positive $\varepsilon > 0$, there exists an approximation polynomial P attached to F such that $\|F - P\|_{\infty} < \varepsilon$ and

P is strictly positive or strictly negative on $[x_1, x_n]$, as is F. Here $\|\cdot\|_\infty$ denotes the uniform norm on $[x_1, x_n]$.

Defining $Q(x) = f(x_1) + \int_{x_1}^{x} P(t)\Pi_{k=1}^{m}(t - x_k)dt$, it is easily seen that $Q(x)$ imitates the monotonicity of f on each subinterval $[x_k, x_{k+1}]$. Also, we get

$$\begin{aligned}|Q(x) - f(x)| &= \left| \int_{x_1}^{x} P(t)\Pi_{k=1}^{m}(t - x_k)dt - [f(x) - f(x_1)] \right| \\ &= \left| \int_{x_1}^{x} P(t)\Pi_{k=1}^{m}(t - x_k)dt - \int_{x_1}^{x} f'(t)dt \right| \\ &\le \varepsilon \int_{x_1}^{x_n} |\Pi_{k=1}^{m}(t - x_k)|dt \le (x_n - x_1)^{m+1}\varepsilon,\end{aligned}$$

for all $x \in [x_1, x_n]$. So for sufficiently small ε, we have that the $Q(x_k)$ are sufficiently close to $y_k = f(x_k),\ k = 1, \ldots, n$.

Now, for any $\varepsilon > 0$ and $s = 2, 3, \ldots, n$, choose $y_{k-1,\varepsilon}^{(s)} \ne y_{k,\varepsilon}^{(s)},\ k = 1, \ldots, n$, such that $|y_{k,\varepsilon}^{(s)}| < \varepsilon,\ k = 1, \ldots, s-1, |y_{k,\varepsilon}^{(s)} - 1| < \varepsilon, k = s, \ldots, n$, the points $x_1 < \cdots < x_n$ remaining the same. Also, the corresponding linear piecewise function passing through all the points $(x_k, y_{k,\varepsilon}^{(s)})$ is denoted by $\phi^{(s)}(x)$.

According to the above reasonings there exist the polynomials $Q_\varepsilon^{(s)}(x),\ s = 2, \ldots, n$, such that they are comonotone with $\phi^{(s)}(x)$ and satisfy

$$|Q_\varepsilon^{(s)}(x_k)| < \varepsilon,\ k = 1, \ldots, s-1,$$

and

$$|Q_\varepsilon^{(s)}(x_k) - 1| < \varepsilon,\ k = s, \ldots, n.$$

Also, by convention define $Q_\varepsilon^{(1)}(x) = 1$.

Denote by A_ε the value of the determinant $Q_\varepsilon^{(s)}(x_k),\ k, s = 1, \ldots, n$, and by $B_\varepsilon^{(s)}$ the value of the determinant obtained from the above one by replacing the sth column with $y_{k,\varepsilon}^{(s)},\ s = 1, \ldots, n$. Obviously, we have $\lim_{\varepsilon\to 0} A_\varepsilon = 1$ and $\lim_{\varepsilon\to 0} B_\varepsilon^{(s)} = y_s - y_{s-1}, s = 2, \ldots, n$. Therefore, for an ε_0 sufficiently small, we have $A_{\varepsilon_0} > 0$ and $\text{sign}(B_{\varepsilon_0}^{(s)}) = \text{sign}(y_s - y_{s-1}), s = 2, \ldots, n$.

Then the polynomial $W(x) = \sum_{s=1}^{n} \frac{B_{\varepsilon_0}^{(s)}}{A_{\varepsilon_0}} Q_{\varepsilon_0}^{(s)}(x)$ will satisfy the conditions in the statement. □

Remarks. (1) For generalizations of Wolibner's result see, e.g., Ivan [183].

(2) A direct consequence of the above theorem is the following result in Deutch–Morris [80], called SAIN (i.e., simultaneous approximation and interpolation-preserving norm)-type result: if $f \in C[a, b]$ and $x_0 < \cdots < x_n$ are distinct points in $[a, b]$, then for any $\varepsilon > 0$, there exists a polynomial p such that

$$p(x_i) = f(x_i), i = 0, \ldots, n, \|f - p\|_\infty < \varepsilon, \|p\|_\infty = \|f\|_\infty$$

(here $\|\cdot\|_\infty$ denotes the uniform norm on $C[a,b]$). In some particular cases, this result can also be considered to belong to the topic of approximation and interpolation by polynomials preserving positivity or positive bounds. Indeed, suppose $0 < f(x) \leq \|f\|_\infty$, for all $x \in [a,b]$ (obviously, the second inequality is always valid). From the continuity of f, there exists $c > 0$ such that $f(x) \geq c > 0$, for all $x \in [a,b]$, and therefore for any sufficiently small ε (more exactly for $0 < \varepsilon < c$), the approximating and interpolating polynomial p also satisfies $0 < p(x) \leq \|f\|_\infty$, for all $x \in [a,b]$.

(3) The Wolibner's theorem does not provide any information about the degree of the polynomial p. If we denote by s the smallest degree of p that still satisfies Theorem 1.2.1, then the first result concerning s was obtained by Rubinstein [330], but only for the particular case $n = 2$ and $y_0 < y_1 < y_2$. In Nikolcheva [285], for equidistant nodes in $[0,1]$ and for the hypothesis $\Delta y_i \geq cm^\alpha$, one obtains the best estimate, $s = O(\alpha \cdot \log(n))$. Similar results were obtained in Passow–Raymon [300] and Passow [299].

Another direction of research concerning the shape–preserving interpolation by polynomials was discovered by T. Popoviciu in a series of papers published between 1960 and 1962, see [312], [313], [314], and can be described as follows. First let us consider the following simple definition.

Definition 1.2.2. Let $f \in C[a,b]$ and $a \leq x_1 < x_2 < \cdots < x_n \leq b$ be fixed nodes. A linear operator $U : C[a,b] \to C[a,b]$ is said to be of interpolation type (on the nodes $x_i,\ i = 1, \ldots, n$) if for any $f \in C[a,b]$ we have

$$U(f)(x_i) = f(x_i),\ \forall\ i = 1, \ldots, n.$$

Remark. Important particular cases of U are of the form

$$U_n(f)(x) = \sum_{k=1}^{n} f(x_k) P_k(x), \quad n \in \mathbf{N},$$

where $P_k \in C[a,b]$ satisfy $P_k(x_i) = 0$ if $k \neq i$ and $P_k(x_i) = 1$ if $k = i$, and contain the classical Lagrange interpolation polynomials and Hermite–Fejér interpolation polynomials.

Now, if $f \in C[a,b]$ is, for example, monotone (or convex) on $[a,b]$, it is easy to note that because of the interpolation conditions, in general $U(f)$ cannot be monotone (or convex) on $[a,b]$.

However, it is a natural question whether $U(f)$ remains monotone (or convex) on neighborhoods of some points in $[a,b]$. In this sense, we can introduce the following definition.

Definition 1.2.3. Let $U : C[a,b] \to C[a,b]$ be a linear operator of interpolation type on the nodes $a \leq x_1 < \cdots < x_n \leq b$.

Let $y_0 \in (a,b)$. If for any $f \in C[a,b]$, nondecreasing on $[a,b]$, there exists a neighborhood of y_0, $V_f(y_0) = (y_0 - \varepsilon_f, y_0 + \varepsilon_f) \subset [a,b]$, $\varepsilon_f > 0$ (i.e., depending on f) such that $U(f)$ is nondecreasing on $V_f(y_0)$, then y_0 is called a point of weak preservation of partial monotonicity and correspondingly, U is said to have the property of weak preservation of partial monotonicity (about y_0).

If the above neighborhood $V(y_0)$ does not depend on f, then y_0 is called a point of strong preservation.

Similar definitions hold if monotonicity is replaced by, e.g., convexity (of any order).

For example, we present the following two results below concerning the Hermite–Fejér polynomials based on some special Jacobi nodes.

Theorem 1.2.4. (Gal–Szabados [139], Theorem 2.2; see also Gal [123], p. 46, Theorem 2.2.2) *For $n \in \mathbb{N}$, let $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$ be the classical Hermite–Fejér polynomial based on the roots $-1 < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < 1$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $\alpha, \beta \in (-1, 0]$ and*

$$h_{i,n}(x) = l_i^2(x)\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right],$$

$$l_i(x) = l(x)/[(x - x_{i,n})l'(x_{i,n})], \quad l(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

If $f : [-1, 1] \to \mathbb{R}$ is monotone on $[-1, 1]$, then for any root ξ of the polynomial $l'(x)$, there is a constant $c > 0$ (independent of n and of f) such that $H_n(f)(x)$ is of the same monotonicity with f in $\left(\xi - \frac{c_\xi}{n^{7+2\gamma}}, \xi + \frac{c_\xi}{n^{7+2\gamma}}\right) \subset (-1, 1)$, where $c_\xi = \frac{c}{(1-\xi^2)^{5/2+\delta}}$, $\gamma = \max\{\alpha, \beta\}$, and

$$\delta = \begin{cases} \alpha, \text{ if } 0 \le \xi < 1, \\ \beta, \text{ if } -1 < \xi \le 0. \end{cases}$$

Proof. Let us denote $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$, where

$$h_{i,n}(x) = l_i^2(x)\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right],$$

$$l_i(x) = l(x)/[(x - x_{i,n})l'(x_{i,n})], \quad l(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

By, e.g., Popoviciu [312] we have

$$h_{i,n}(0) = l^2(0)[2 - (1-\lambda)x_{i,n}^2]/[l'(x_{i,n})^2(1 - x_{i,n}^2)x_{i,n}^3],$$

for all $i = 1, \ldots, n$, and

$$H_n'(f)(x) = \sum_{i=1}^{n-1}[Q_i(x)][f(x_{i,n}) - f(x_{i+1,n})],$$

where $Q_i(x) = \sum_{j=1}^{i} h_{j,n}'(x), \; i = 1, \ldots, n-1.$

Reasoning as in the proof of Lemma 3 in Popoviciu [314], we get

$$Q_i(\xi) > \min\{h_{1,n}'(\xi), -h_{n,n}'(\xi)\} > 0, \text{ for all } i = 1, \ldots, n-1.$$

Let $a_n, b_n \in (0,1),\ a_n, b_n \searrow 0$ (when $n \to +\infty$) be such that $|h'_{1,n}(\xi)| \geq c_1 a_n,\ |h'_{n,n}(\xi)| \geq c_2 b_n$, and $s_n = \min\{a_n, b_n\}$.

It easily follows that $Q_i(\xi) \geq c_3 s_n, i = 1, \ldots, n-1$. By Szegö [383], Theorem 14.5, we have

$$\sum_{j=1}^{n} h_{j,n}(x) = 1,\ \forall\ x \in [-1,1],$$

where $h_{j,n}(x) \geq 0,\ \forall\ x \in [-1,1],\ j = 1, \ldots, n.$

Applying the Bernstein's inequality twice we obtain

$$Q_i(\xi) \leq c_1 |d_i - \xi| n^2/(1-\xi^2), \quad i = 1, \ldots, n-1,$$

where d_i is the nearest root of $Q_i(x)$ to ξ, and therefore

$$\max_{|x-\xi| \leq a_\xi \frac{s_n}{n^2}} Q_i(x) > 0, \quad i = 1, \ldots, n-1$$

with $a_\xi = c_2(1-\xi^2)$.

It remains to find a (lower) estimate for s_n. First we have

$$|P_n^{(\alpha,\beta)}(\xi)| \geq \frac{c_3 n^{-1/2}}{(1-\xi)^{\delta/2+1/4}},$$

(see Theorem 8.21.8 in Szegö [383]).

By Popoviciu [314], p. 79, relation (27),

$$h'_{1,n}(\xi) = \frac{l^2(\xi)}{(x_{1,n}-\xi)^3[l' x_{1,n})]^2}\left[2 + (x_{1,n} - \xi)\frac{l''(x_{1,n})}{l'(x_{1,n})}\right] > 0,$$
$$h'_{n,n}(\xi) = \frac{l^2(\xi)}{(x_{n,n}-\xi)^3[l' x_{n,n})]^2}\left[2 + (x_{n,n} - \xi)\frac{l''(x_{n,n})}{l'(x_{n,n})}\right] < 0.$$

By Szegö [383], Theorem 14.5, $2+(x_{i,n}-\xi)\dfrac{l''(x_{i,n})}{l'(x_{i,n})} \geq 1$ and by Szegö [383], (7.32.11),

$$h'_{1,n}(\xi) \geq \frac{l^2(\xi)}{(x_{1,n}-\xi)^3[l'(x_{1,n})]^2} = \frac{[P_n^{(\alpha,\beta)}(\xi)]^2}{(x_{1,n}-\xi)^3[P_n^{(\alpha,\beta)'}(x_{1,n})]^2}$$
$$\geq \frac{c_4[P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1-\xi)^3},$$

(where $q = \max\{2+\alpha, 2+\beta\}$).

Also,

$$-h'_{n,n}(\xi) = |h'_{n,n}(\xi)| \geq \frac{c_5[P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1+\xi)^3}.$$

Thus we obtain

$$Q_i(\xi) \geq \frac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}, \quad i=1,\dots,n-1.$$

Finally, taking $s_n = \dfrac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}$ we easily obtain the theorem. □

For $n \geq 3$ odd, let $H_n(f)(x)$ be the Hermite–Fejér interpolation polynomial based on the roots $x_{i,n} \in (-1,1),\ i=1,\dots,n$, of λ-ultraspherical polynomials of degree $n,\ \lambda > -1,\ \lambda \neq 0$. Also, consider the Côtes–Christoffel numbers of the Gauss–Jacobi quadrature given by

$$\lambda_{i,n} := 2^{2-\lambda}\pi \left[\Gamma\left(\frac{\lambda}{2}\right)\right]^{-2} \frac{\Gamma(n+\lambda)}{\Gamma(n+1)}(1-x_{i,n}^2)^{-1}[P_n^{(\lambda)'}(x_{i,n})]^{-2}, \quad i=1,\dots,n,$$

and define

$$\Delta_h^2 f(0) = f(h) - 2f(0) + f(-h).$$

We also have the following result:

Theorem 1.2.5. (Gal–Szabados [139], Theorem 2.3; see also Gal [123], p. 49, Theorem 2.2.3) *Let $f \in C[-1,1]$ satisfy*

$$\sum_{i=1}^{n} [\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)]/x_{i,n}^2 > 0$$

(if f is strictly convex on $[-1,1]$ then obviously it satisfies this condition). Then $H_n(f)(x)$ is strictly convex in $[-|d_n|, |d_n|]$, with

$$|d_n| \geq \frac{c(\lambda)\sum_{i=1}^{\frac{n-1}{2}} [\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)]/x_{i,n}^2}{n^2\left[\omega_1\left(f;\frac{1}{n}\right) + \|f - H_n(f)\|\right]_I},$$

where $c(\lambda) > 0$ is independent of f and n, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\omega_1\left(f;\frac{1}{n}\right)_{\left[-\frac{1}{2},\frac{1}{2}\right]}$ is the first-order modulus of continuity on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and $\|\cdot\|_{\left[-\frac{1}{2},\frac{1}{2}\right]}$ is the uniform norm on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof. Denote $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$, where

$$h_{i,n}''(x) = -4\frac{l''(x_{i,n})}{l'(x_{i,n})} l_i(x) l_i'(x) + 2[(l_i'(x))^2 + l_i(x) l_i''(x)] \left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right].$$

But $l_i(0) = 0$ and $l_i'(0) = -\dfrac{l'(0)}{x_{i,n}l'(x_{i,n})}$, for $i \neq (n+1)/2$ and

$$1 + x_{i,n}\frac{l''(x_{i,n})}{l'(x_{i,n})} = \frac{1+\lambda x_{i,n}^2}{1-x_{i,n}^2}, \quad i = 1, \ldots, n \quad \text{(see, e.g., \textit{Popoviciu} [312]).}$$

We obtain

$$h_{i,n}''(0) = \frac{2(l'(0))^2}{(l'(x_{i,n}))^2} \cdot \frac{1}{x_{i,n}^2}\left(\frac{1+\lambda x_{i,n}^2}{1-x_{i,n}^2}\right) > 0, \ \forall\ i \neq (n+1)/2.$$

Also, because $x_{i,n} = -x_{n+1-i,n}$, $i = 1, \ldots, n$, $l'(x_{i,n}) = l'(x_{n+1-i,n})$ (since n is odd) we easily get

$$h_{i,n}''(0) = h_{n+1-i,n}''(0).$$

But $\lambda_{i,n} = \dfrac{c_1\lambda\Gamma(n+\lambda)}{\Gamma(n+1)} \cdot \dfrac{1}{(l'(x_{i,n}))^2} \cdot \dfrac{1}{1-x_{i,n}^2}$ and $(l'(0))^2 \sim n^\lambda$, which together with the above inequality implies

$$h_{i,n}''(0) \geq c_2\lambda n\lambda_{i,n}/x_{i,n}^2, \text{ for all } i \neq (n+1)/2.$$

Therefore

$$H_n''(f)(0) = \sum_{i=1}^{(n-1)/(2)} h_{i,n}''(0)\Delta_{x_{i,n}}^2 f(0) \geq c_3\lambda n\sum_{i=1}^{n}\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)/x_{i,n}^2 > 0.$$

By this last relationship it follows that $H_n(f)$ is strictly convex in a neighborhood of 0. Let d_n be the nearest root of $H_n''(f)$ to 0. We may assume that $|d_n| \leq \dfrac{c}{n}$ (since otherwise there is nothing to prove, the interval of convexity cannot be larger than $\left[-\frac{c}{n}, \frac{c}{n}\right]$). Then by the mean value theorem, Bernstein's inequality and Stechkin's inequality (see, e.g., Szabados–Vértesi [381], p. 284) we get

$$\begin{aligned} H_n''(f)(0) &= |H_n''(f)(0) - H_n''(f)(d_n)| = |d_n| \cdot |H_n'''(f)(y)| \\ &\leq |d_n|c_4n^2\|H_n'(f)\|_J \leq c_5|d_n|n^3\omega_1\left(H_n(f); \frac{1}{n}\right)_I \\ &\leq c_5|d_n|n^3\left[\omega_1\left(f; \frac{1}{n}\right) + \omega_1\left(H_n(f) - f; \frac{1}{n}\right)\right]_I \\ &\leq c_5|d_n|n^3\left[\omega_1\left(f; \frac{1}{n}\right) + \|f - H_n(f)\|\right]_I, \end{aligned}$$

where $J = \left[-\frac{1}{4}, \frac{1}{4}\right]$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Combining the last inequality with the previous inequality satisfied by $H_n''(f)(0)$, the proof of the theorem is immediate. □

Remark. All the details concerning this direction of research can be found in Chapter 2 of the recent monograph Gal [123], where a deep and extensive study concerning shape-preserving interpolation by classical univariate interpolation polynomials (of Lagrange, Grünwald, or Hermite–Fejér type) is made.

For the error estimate in shape-preserving interpolation, we mention here the following four results.

The first two results show the existence of such polynomials with good approximation properties and can be stated as follows.

Theorem 1.2.6. (Ford–Roulier [120]) *Let $p \in \mathbb{N}$, $1 \le r_1 < r_2 < \cdots < r_s \le p$ with $r_i, i = 1, \ldots, s$, natural numbers, $\varepsilon_j = \pm 1, j = 1, \ldots, s$, and $a \le x_0 < \cdots x_m \le b$ interpolation nodes. For any $f \in C^p[a, b]$ satisfying*

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [a, b], \quad i = 1, \ldots, s,$$

there exists a sequence of polynomials $(P_n(x))_n$, degree$(P_n) \le n$, such that for sufficiently large n we have

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \forall x \in [a, b], \quad i = 1, \ldots, s, \text{ with } P_n(x_j) = f(x_j), \quad j = 0, \ldots, m,$$

and the estimate

$$\|f - P_n\|_\infty \le C n^{-p} \omega_1(f^{(p)}; 1/n)_\infty$$

holds, where $C > 0$ is independent of f and n. Here $\|\cdot\|_\infty$ denotes the uniform norm on $C[a, b]$.

Proof. Let us sketch the proof. According to a result in the doctoral thesis of Roulier [325], f can be extended to a function $F \in C^p[a - 1, b + 1]$ such that $\omega_1(F^{(p)}; h)_\infty \le \omega_1(f^{(p)}; h)_\infty$, for all $h \in [0, b - a]$. Denote by Q_n the polynomial of best approximation of degree $\le n$ attached to F on $[a-1, b+1]$. Jackson's theorem implies

$$\|Q_n - F\|_{C[a-1,b+1]} \le C n^{-p} \omega_1(F^{(p)}; 1/n)_\infty,$$

where ω_1 is the uniform modulus of continuity on $[a - 1, b + 1]$.

Now let L_m be the Lagrange's interpolation polynomial of degree $\le m$ satisfying $L_m(x_i) = \delta_i = F(x_i) - Q_n(x_i), i = 0, \ldots, m$.

Since $|\delta_i| \le C n^{-p} \omega_1(F^{(p)}; 1/n)_\infty$, for all $i = 0, \ldots, m$, it is easy to derive that $|L_m(x)| \le C_1 n^{-p} \omega_1(F^{(p)}; 1/n)_\infty$ for all $x \in [a - 1, b + 1]$, where C_1 depends only on m and the points x_i, $i = 0, \ldots, m$.

Setting $P_n(x) = Q_n(x) + L_m(x)$, it is easy to see that $P_n(x_i) = F(x_i) = f(x_i)$, $i = 0, \ldots, m$, and $\|P_n - f\|_{C[a-1,b+1]} \le C_2 n^{-p} \omega_1(f^{(p)}; 1/n)_\infty$, by the above mentioned result in Roulier [325] (here $\omega_1(f^{(p)}, 1/n)_\infty$ denotes the uniform modulus of continuity on $[a, b]$).

Also, according to another result in Roulier [325], $\|P_n - f\|_{C[a-1,b+1]} \le C_2 n^{-p}\omega_1(f^{(p)}; 1/n)_\infty$ implies $\|P_n^{(i)} - f^{(i)}\|_{C[a,b]} \le C_3 n^{i-p}\omega_1(f^{(p)}; 1/n)_\infty,\ i = 0, \dots, p$.

This means that $P_n^{(r_i)} \to f^{(r_i)}$, uniformly on $[a, b]$, for all $i = 1, \dots, s$, which because of the strict inequalities

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [a, b],\ i = 1, \dots, s,$$

immediately implies the conclusion in the statement. $\square$

Remark. A similar result to that of Theorem 1.2.6, but in the more general setting of nondifferentiable functions, has been considered by Szabados [379], who obtained estimates in terms of $\omega_1(f; \frac{\log n}{n})_\infty$.

Theorem 1.2.6 can be slightly refined by combining it with approximation by monotone sequences of polynomials, as follows. For simplicity, we consider the problem on $[0, 1]$.

Theorem 1.2.7. (Gal [130]) *Let $p \in \mathbb{N}$, $1 \le r_1 < r_2 < \dots < r_s \le p$ with $r_i, i = 1, \dots, s$ natural numbers, $\varepsilon_j = \pm 1, j = 1, \dots, s$ and $0 \le x_1 < \dots < x_m \le 1$ interpolation nodes. For any $f \in C^p[0, 1]$ satisfying*

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [0, 1],\ i = 1, \dots, s,$$

there exist sequences of polynomials $(P_n(x))_n, (Q_n(x))_n$, degree$(P_n) \le n$, degree$(Q_n) \le n$, such that for sufficiently large n, we have

$$\varepsilon_i P_n^{(r_i)}(x) > 0,\ \varepsilon_i Q_n^{(r_i)}(x) > 0,\ \forall x \in [0, 1],\ i = 1, \dots, s,$$
$$P_n(x_j) = Q_n(x_j) = f(x_j),\ j = 0, \dots, m,$$

the estimate

$$\|P_n - Q_n\|_\infty \le C n^{-p}\omega_1(f^{(p)}; 1/n)_\infty$$

holds, where $C > 0$ is independent of f and n, and in addition,

$$Q_n(x) \le Q_{n+1}(x) \le f(x) \le P_{n+1}(x) \le P_n(x),\ \forall x \in [0, 1],\ n \in \mathbb{N}.$$

Proof. From the proofs of the Theorem and Corollary 1 in Gal–Szabados [140], we distinguish two steps.

Step 1. We start with the polynomial sequence $(p_k)_k$, degree$(p_k) \le k$, satisfying Theorem 1.2.6, i.e., for sufficiently large n we have

$$\varepsilon_i p_k^{(r_i)}(x) > 0,\ \forall x \in [0, 1],\ i = 1, \dots, s,\ \text{where } p_k(x_j) = f(x_j),\ j = 0, \dots, m,$$

and the estimate

$$\|f - p_k\|_\infty \le C k^{-p}\omega_1(f^{(p)}; 1/k)_\infty$$

holds.

Step 2. With the aid of $(p_k)_k$, one construct the polynomials P_n and Q_n satisfying the relationships (5) and (8), respectively, in Gal–Szabados [140] (where P_n and Q_n are defined as special arithmetic means of p_k), replacing there $E_k(f)_\infty$ by the expression $Ck^{-p}\omega_1(f^{(k)}; 1/k)_\infty$.

By the mentioned proof, for all $n \geq 4$ we get

$$Q_n(x) \leq Q_{n+1}(x) \leq f(x) \leq P_{n+1}(x) \leq P_n(x), \ \forall x \in [0, 1], \ n \in \mathbb{N},$$

and

$$\|P_n - Q_n\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty.$$

Since the polynomials P_n and Q_n are arithmetic means of p_k, it is immediate that

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \quad \varepsilon_i Q_n^{(r_i)}(x) > 0, \quad \forall x \in [0, 1], i = 1, \ldots, s.$$

Now, in order to get the interpolation conditions too, let us redefine Q_n and P_n by $Q_n := Q_n + L_m^{(1)}$, $P_n := P_n + L_m^{(2)}$, where $L_m^{(1)}$ and $L_m^{(2)}$ are the Lagrange polynomials of degrees $\leq m$ satisfying the conditions $L_m^{(1)}(x_j) = f(x_j) - Q_n(x_j)$, $L_m^{(2)}(x_j) = f(x_j) - P_n(x_j)$, $j = 0, \ldots, m$.

Reasoning as in the proof of Theorem 1.2.6, for the redefined Q_n and P_n, we get

$$P_n(x_j) = Q_n(x_j) = f(x_j), \ j = 0, \ldots, m$$

and

$$\|Q_n - f\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty,$$
$$\|P_n - f\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty,$$

which by $\|Q_n - p_n\|_\infty \leq \|Q_n - f\|_\infty + \|f - P_n\|_\infty$, immediately implies

$$\|P_n - Q_n\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty.$$

Also, as in the proof of Theorem 1.2.6, we get the uniform convergence of $Q_n^{(r_i)}$ and $P_n^{(r_i)}$ to $f^{(r_i)}$, $i = 0, \ldots, s$, which for sufficiently large n also implies

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \quad \varepsilon_i Q_n^{(r_i)}(x) > 0, \quad \forall x \in [0, 1], \quad i = 1, \ldots, s.$$

Obviously, the monotonicity properties of the redefined sequences $(Q_n)_n$ and $(P_n)_n$ (with respect to n) become non-strict, because of interpolation conditions. The theorem is proved. □

Remark. Two recent results in Kopotun [202], Kopotun [203], give the approximation estimates necessarily verified by interpolation j-convex polynomials (interpolating a function which is not necessarily j-convex), in the case that the interpolation nodes are not close to the endpoints. These results remaining valid for j-convex functions too, it is clear that they can be considered to belong to the shape-preserving interpolation topic.

Now, since by the alternating Chebyshev theorem, the best approximation polynomial of degree $\leq n$ interpolates the function on at least $n+1$ points, in this section we also present three results concerning the preservation of j-convexity by the best approximation polynomials.

Theorem 1.2.8. (Roulier [327]) *Let $m \in \mathbb{N}$, $f \in C^{2m-1}[-1,1]$, $1 \leq i_1 < i_2 < \cdots < i_q < m$ be fixed integers and $\varepsilon_j, j = 1, \ldots, q$ be fixed signs. For any $n \in \{0, 1, \ldots, \}$, denote by Q_n the best approximation polynomial of degree $\leq n$ of f on $[-1,1]$. If $\varepsilon_j f^{(i_j)}(x) > 0$, for all $x \in [-1,1]$ and all $j = 1, \ldots, q$ and if $\sum_{k=1}^{+\infty} \frac{1}{k}\omega_1(f^{(2m-1)}; \frac{1}{k})_\infty < +\infty$, then for sufficiently large n, we have $\varepsilon_j Q_n^{(i_j)}(x) > 0$, for all $x \in [-1,1]$ and all $j = 1, \ldots, q$.*

Proof. We sketch here the proof using the ideas in Roulier [327]. In fact, it is based on two lemmas. The first one is well known (see, e.g., G.G. Lorentz's monograph [248]) and can be stated as follows.

Lemma (A). (Lorentz [248], p. 74) *There exist constants $M_p > 0$, $p = 1, 2, \ldots,$ such that if w is any modulus of continuity for which $\sum_{k=1}^{+\infty} \frac{1}{k} w(1/k) < +\infty$ and if for $f \in C[-1,1]$ and polynomials $q_n(x)$ of degree $\leq n$ we have the estimate*

$$|f(x) - q_n(x)| \leq C[\Delta_n(x)]^p w(\Delta_n(x)),$$

then f has continuous derivative $f^{(p)}$ and

$$|f^{(p)}(x) - q_n^{(p)}(x)| \leq M_p \sum_{k \geq [(\Delta_n(x))^{-1}]} \frac{1}{k} w(1/k), \forall x \in [-1,1].$$

Here $\Delta_n(x) = \max\{n^{-1}(1-x^2)^{1/2}, n^{1/2}\}$, $\Delta_0(x) = 1$.

Proof of Lemma A. Because of its importance in approximation theory, let us sketch its proof below. It is easy to see that we can write $f(x) = q_n(x) + \sum_{j=1}^{\infty}(q_{2^j n}(x) - q_{2^{j-1}n}(x))$, where by the hypothesis it follows that

$$|q_{2^j n}(x) - q_{2^{j-1}n}(x)| \leq 2[\Delta_{2^{j-1}n}(x)]^p w(\Delta_{2^{j-1}n}(x)).$$

This implies the uniform convergence of the series (on $[-1,1]$), that is, the differentiated series (of any order) is also uniformly convergent and we get

$$f^{(p)}(x) = q_n^{(p)}(x) + \sum_{j=1}^{\infty}(q_{2^j n}^{(p)}(x) - q_{2^{j-1}n}^{(p)}(x)).$$

Taking into account the elementary inequalities $\frac{1}{4}\Delta_n(y) \leq \Delta_{2n}(y) \leq \frac{1}{2}\Delta_n(y)$, valid for all $y \in [-1,1]$ and applying a well known Markov-type inequality in terms of the modulus of continuity (i.e., $|q_n(x)| \leq [\Delta_n(x)]^r w(\Delta_n(x))$, $|x| \leq 1$, implies $|q_n'(x)| \leq M_r[\Delta_n(x)]^{r-1} w(\Delta_n(x))$, $|x| \leq 1$, for its proof see, e.g., Theorem 3 in Lorentz [248], p. 71) p-times, we obtain

$$|q_{2^j n}^{(p)}(x) - q_{2^{j-1}n}^{(p)}(x)| \leq M_p w(\Delta_{2^j n}(x)).$$

Combining with Lemma 1 in Lorentz [248], pp. 58–59 and taking into account that $[\Delta_n(x)]^{-1} \ge n$, this implies

$$|f^{(p)}(x) - q_n^{(p)}(x)| \\ \le M_p \sum_{j=1}^{\infty} w(\Delta_{2^j n}(x)) \le M_p \sum_{k \ge [(\Delta_n(x))^{-1}]} \frac{1}{k} w(1/k), \forall x \in [-1,1],$$

and the existence of continuous $f^{(p)}$. □

From Lemma A, one deduces the following.

Lemma (B). (Roulier [327]) *If* $f \in C^{2m-1}[-1,1]$ *and*

$$\sum_{k=1}^{+\infty} \frac{1}{k} \omega_1 \left(f^{(2m-1)}; \frac{1}{k} \right)_{\infty} < +\infty,$$

then for the best-approximation polynomial Q_n*, there exists a constant* B_m *such that*

$$|f^{(i)}(x) - Q_n^{(i)}(x)| \le B_m \sum_{j=n}^{+\infty} \frac{1}{j} \omega_1 \left(f^{(2m-1)}; \frac{1}{j} \right)_{\infty},$$

for all $x \in [-1,1]$ *and* $1 \le i < m$.

Proof of Lemma B. From the well known Jackson's theorem we have

$$\|f - Q_n\|_{\infty} = E_n(f)_{\infty} \le \frac{C_m}{n^{2m-1}} \omega_1 \left(f^{(2m-1)}; \frac{1}{n} \right)_{\infty},$$

with $C_m > 0$ depending only on m.

Now, since $\frac{1}{n^2} \le \Delta_n(x) \le \frac{1}{n}$, for any $x \in [-1,1]$ and $k = 1, \dots, m-1$, by simple calculation we get

$$\frac{1}{n^{2m-1}} \omega_1 \left(f^{(2m-1)}; \frac{1}{n} \right)_{\infty} \le (\Delta_n(x))^k \omega_1 (f^{(2m-1)}; \Delta_n(x))_{\infty},$$

which implies

$$|Q_n(x) - f(x)| \le C_m (\Delta_n(x))^k \omega_1 (f^{(2m-1)}; \Delta_n(x))_{\infty}.$$

Applying Lemma A, we get the desired inequality in Lemma B.

Now, since $\sum_{j=n}^{+\infty} \frac{1}{j} \omega_1 (f^{(2m-1)}; \frac{1}{j})_{\infty} \to 0$ as $n \to +\infty$, the proof of Theorem 1.2.8 is immediate. □

As a negative-type result, we can mention the following.

Theorem 1.2.9. (see Passow–Roulier [301]) *Suppose that* $f \in C[-1,1]$ *has bounded* r*th-order divided differences and nonnegative* $(r+1)$*th-order divided* *differences on* $[-1,1]$. *Let* P_n *be the best approximation polynomial of* f *on* $[-1,1]$ *of degree* $\leq n$ *and assume that there is no* $C > 0$ *for which*

$$E_n(f)_\infty \leq \frac{C}{(n+1)^r}, \ n = 0, 1, \ldots.$$

Then there are infinitely many n *for which we do not have* $P_n^{r+1}(x) \geq 0$ *on* $[-1,1]$.

For other negative-type results see Passow–Roulier [301].

Remarks. Generalized results concerning the best comonotone approximation in $C[a,b]$ by elements from an n-dimensional extended Chebyshev subspace were obtained by Deutsch–Zhong [81].

1.3 Bernstein-Type Polynomials Preserving Shapes

Let $f : [0,1] \to \mathbb{R}$. The Bernstein polynomial on the interval $[0,1]$ given by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

is one of the most famous polynomials in approximation theory and was introduced in 1912 by S.N. Bernstein [45] in order to give the first constructive (and simple) proof to the Weierstrass approximation theorem. Note that the Bernstein polynomial attached to a function $f : [a,b] \to \mathbb{R}$ can be written by the formula $B_n(f;[a,b])(x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (x-a)^k (b-x)^{n-k} f(a + k\frac{b-a}{n})$.

The first approximation error of these polynomials was established by T. Popoviciu [316] in 1935, who proved the estimate

$$\|B_n(f) - f\|_\infty \leq \frac{3}{2}\omega_1\left(f; \frac{1}{\sqrt{n}}\right)_\infty,$$

for all $n \in \mathbb{N}$. The best constant in front of $\omega_1(f; \frac{1}{n})_\infty$ was found in 1961 by Sikkema [356], and it is $\frac{4306+837\sqrt{6}}{5832} = 1,089\ldots.$

Many decades later, the best order of approximation by Bernstein polynomials was found; namely Knoop–Zhou [194] and Totik [387] proved the following equivalence (with absolute constants C_1 and C_2):

$$C_1\omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right)_\infty \leq \|B_n(f) - f\|_\infty \leq C_2\omega_2^\varphi\left(f; \frac{1}{\sqrt{n}}\right)_\infty,$$

where we recall that

$$\omega_2^\varphi(f;\delta)_\infty$$
$$= \sup\{\sup\{\|f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))\|; x \in I_{2,h}\}, h \in [0,\delta]\}$$

(with $I_{2,h} = \left[-\frac{1-h^2}{1+h^2}, \frac{1-h^2}{1+h^2}\right]$, $\varphi(x) = \sqrt{x(1-x)}$, $\delta \leq 1$) denotes the second-order Ditzian–Totik modulus of smoothness.

Let us mention that earlier, the following pointwise estimate was proved:

$$|B_n(f)(x) - f(x)| \le C\omega_2\left(f; \sqrt{\frac{x(1-x)}{n}}\right)_\infty,$$

for all $x \in [0,1]$, $n \in \mathbb{N}$, $f \in C[0,1]$; see, e.g., Cao [54] (or Gonska–Meier [153] for $C = 5$).

Although the above estimates are rather weak with respect to the Jackson-type estimates attained by other approximation polynomials, the shape-preserving properties of Bernstein polynomials make them very important, with many applications in computer aided geometric design.

These properties can be summarized by the following theorem.

Theorem 1.3.1. *Let $f : [0,1] \to \mathbb{R}$.*

(i) (Popoviciu [317]) If we suppose that f is convex (strictly convex) of order $k \in \{0,1,2,\ldots,\}$ on $[0,1]$, then $B_n(f)$ is convex (strictly convex, respectively) of order k on $[0,1]$, for all $n \in \mathbb{N}$;

(ii) (Păltănea [296]) If f is quasiconvex of order $k \in \{1,2,\ldots,\}$ on $[0,1]$, then $B_n(f)$ is quasiconvex of order k on $[0,1]$, for all $n \in \mathbb{R}$;

(iii) (L. Lupaş [254]) If $f : [0,1] \to \mathbb{R}$ satisfy $f(0) = 0$, $f(x) \ge 0$, for all $x \in [0,1]$ and f is starshaped on $[0,1]$, then $B_n(f)(0) = 0$, $B_n(f)(x) \ge 0, x \in [0,1]$, and $B_n(f)$ is starshaped on $[0,1]$, for all $n \in \mathbb{N}$.

(Mocanu–Şerb–Toader [274]) For $f : [0,1] \to \mathbb{R}$ starshaped on $[0,1]$, define its order of star–convexity by $\alpha^*(f) = \sup\{\beta; f$ is β-star–convex$\}$. If, in addition, f is strictly starshaped with $f(0) < 0$ or f is strictly starshaped, $f \in C^2[0,1]$, $f(0) = 0, f''(0) \neq 0$, then

$$\lim_{n\to\infty} \alpha^*[B_n(f)] = \alpha^*(f).$$

(iv) (Goodman [154]) If f is logarithmic convex on $[0,1]$, then $B_n(f)$ is logarithmic convex on $[0,1]$, for all $n \in \mathbb{N}$;

(v) (Kocić–Lackcović [195]) If f is u-monotone, where $u(x) = x^\lambda$, for all $x \in [0,1]$ and $\lambda \in (0,1)$ is arbitrary and fixed, then $B_n(f)$ is u-monotone for all $n \in \mathbb{N}$.

(vi) (Pólya–Schoenberg [308]) $B_n(f)(x)$ are strongly variation-diminishing, that is $S_{[0,1]}[B_n(f)] \le S_{[0,1]}[f]$, for all $f \in C[0,1], n \in \mathbb{N}$.

Proof. (i) We may easily prove by mathematical induction with respect to k the formula,

$$B_n^{(k)}(f)(x) = n(n-1)\cdots(n-k+1)\sum_{j=0}^{n-k} \Delta_{1/n}^k f(j/n)p_{n-k,j}(x),$$

which immediately proves (i);

(ii) First we need two auxiliary results.

Lemma (A). (Păltănea [296]) *Let us define* $\Sigma = \{\sigma = (x_1 < \cdots < x_m); x_1, \ldots, x_m \in [0,1]\}$*, where* $m \geq k+1$*,* $k \in \mathbb{N}$*. For such a* $\sigma \in \Sigma$ *and for* $f : [0,1] \to \mathbb{R}$*,* $d_j = [x_j, \ldots, x_{j+k}; f], 1 \leq j \leq m-k$*, and let* $\mu(\sigma)$ *be the greatest index* p*,* $1 \leq j \leq m-k$*, such that* $d_p < 0$*. If there are no such indices, then we put* $\mu(\sigma) = 0$*.*

If $f : [0,1] \to \mathbb{R}$ *is* k*-quasiconvex (*$k \in \mathbb{N}$*) on* $[0,1]$*, then the following inequalities hold:*

$$d_j \leq 0, \quad 1 \leq \mu(\sigma) - 1, \quad d_{\mu(\sigma)} < 0,$$

and

$$d_j \geq 0, \quad \mu(\sigma) + 1 \leq j \leq m - k - 1.$$

Proof of Lemma A. Obviously it is enough to prove that $d_j \leq 0$, for all $j = 1, \ldots, \mu(\sigma)$. For this purpose, suppose that there is an index i, the case when we choose the greatest i satisfying $1 \leq i \leq \mu(\sigma) - 1$, such that $d_i > 0$. Therefore, we would have $d_j \leq 0$ for all j satisfying $i < j < \mu(\sigma)$. We will use the following properties of divided differences in Popoviciu [315]: for every system of strictly ordered points $x_1 < \cdots < x_m$, $m \geq 2$, and for every indice $1 = i_1 < \cdots < i_n = m$, $n \geq 1$, there exist real numbers $a_j \geq 0$, for all $1 \leq j \leq m - n + 1$, $a_1 > 0$, $a_{m-n+1} > 0$, such that

$$[x_{i_1}, \ldots, x_{i_n}; f] = \sum_{j=1}^{m-n+1} a_j \, [x_j, \ldots, x_{j+n-1}; f] \, .$$

Applying this relationship, there exist the numbers $a_j \geq 0$, for all $i+1 \leq j \leq \mu(\sigma)$, $a_{i+1} > 0$, $a_{\mu(\sigma)} > 0$, such that

$$[x_{i+1}, \ldots, x_{i+k+1}, x_{\mu(\sigma)+k+1}; f] = \sum_{j=i+1}^{\mu(\sigma)} a_j \cdot d_j \leq a_{\mu(\sigma)} \cdot d_{\mu(\sigma)} < 0,$$

a fact which obviously contradicts the k-quasiconvexity of f (on $x_i < x_{i+1} < \cdots < x_{i+k+1} < x_{\mu(\sigma)+k+1}$) in Definition 1.1.1, (iv). This proves the lemma. □

Lemma (B). (Păltănea [296]) *If* $f : [0,1] \to \mathbb{R}$ *is a polynomial and if there exist two subintervals* $[0,c]$ *and* $[c,1]$ *such that* f *is* j*-concave on* $[0,c]$ *and* j*-convex on* $[c,1]$*, then* f *is* j*-quasiconvex on* $[0,1]$*. Note that here* c *can be* 0 *or* 1 *too.*

Proof of Lemma B. By the hypothesis we have $f^{(k+1)}(x) \leq 0$, for all $x \in [0,c]$ and $f^{(k+1)}(x) \geq 0$, for all $x \in [c,1]$. First we claim that $f^{(k+2)}(c) \geq 0$ and there exists $\delta > 0$ such that $f^{(k+2)}(x) \geq 0$, for all $|x - c| < \delta$.

Indeed, the cases when the degree of f is not greater than $k+2$ or $f^{(k+2)}(c) > 0$ are obvious. In the opposite case we have $f^{(k+2)}(x) > 0$, for any x with $x \neq c$, $t_1 < x < t_2$, where t_1 is the greatest root of $f^{(k+2)}$, that is less than c (or $t_1 = -\infty$ if such a root does not exist), and t_2 is the least root of $f^{(k+2)}$, that is greater than c (or $t_2 = +\infty$ if such a root does not exist).

Now, let us consider the points $0 \leq x_1 < \cdots < x_{k+3} \leq 1$. To prove the k-quasiconvexity of f on $[0,1]$, it suffices to consider only the case when $x_1 < c < x_{k+3}$. Let us choose the points $0 \leq y_1 < \cdots < y_m \leq 1$, with the following properties: (1) there exist the indices $1 = i_1 < \cdots < i_{k+3} = m$, such that $y_{i_p} = x_p, 1 \leq p \leq k+3$ and (2) $y_{j+1} - y_j < \delta/(k+2), 1 \leq j \leq m-1$. We will denote $c_j = [y_j, \ldots, y_{j+k}; f], 1 \leq j \leq m-k$ and let $r \in \{2, \ldots, m\}$ be the least index such that $y_r \geq c$.

Also, in what follows will be useful the well-known mean-value theorem for divided differences $[t_1, \ldots, t_{p+1}; g] = g^{(p)}(\xi)/p!$, with $t_1 < \xi < t_{p+1}$.

With the above notations, we will prove the inequalities

$$c_{j+1} \leq \max\{c_j, c_{j+2}\}, 1 \leq j \leq m-k-2.$$

Indeed, first if $1 \leq j \leq r-k-2$, then $[y_j, \ldots, y_{j+k+1}; f] \leq 0$, which implies $c_j \geq c_{j+1}$. If $r-k-1 \leq j \leq \min\{r-1, m-k-2\}$, then $|y_p - c| < \delta$, $j \leq p \leq j+k+2$, and by the first claimed property in the proof and by the above mean-value theorem, it follows that $[y_j, \ldots, y_{j+k+2}; f] \geq 0$. Therefore, by the recurrence formula satisfied by the divided differences, it follows that $[y_{j+1}, \ldots, y_{j+k+2}; f] \geq [y_j, \ldots, y_{j+k+1}; f]$ and

$$c_{j+1} \leq \frac{p}{p+q} \cdot c_j + \frac{q}{p+q} c_{j+2},$$

where $p = y_{j+k+1} - y_j > 0$ and $q = y_{j+k+2} - y_{j+1} > 0$. This proves the inequality $c_{j+1} \leq \max\{c_j, c_{j+2}\}$ for these indices. Finally, if $r \leq r \leq m-k-2$, then it follows that $[y_{j+1}, \ldots, y_{j+k+2}; f] \geq 0$ and hence $c_{j+1} \leq c_{j+2}$ and therefore again we get $c_{j+1} \leq \max\{c_j, c_{j+2}\}$.

Now, take an index $1 < i < m$ such that $y_i \in \{x_1, \ldots, x_{k+3}\}$ and consider the points $z_j, j = 1, \ldots, m-1$, defined by $z_1 = y_1, \ldots, z_{i-1} = y_{i-1}, z_i = y_{i+1}, \ldots, z_{m-1} = y_m$. If we denote $d_j = [z_j, \ldots, z_{j+k}; f], 1 \leq j \leq m-k-1$, by again using the property of divided differences in Popoviciu [315] (stated in the proof of Lemma A also), we obtain $d_j = \lambda_j c_j + (1-\lambda_j) c_{j+1}$, with $\lambda_j \in [0,1]$, for any $1 \leq j \leq m-k-1$. Now the following property in Popoviciu [319] will be useful here: f is 0-quasiconvex in $[0,1]$, if and only if $[0,1]$ can be divided into two consecutive subintervals, such that f is nonincreasing on the first one and nondecreasing on the second one. Combining it with the inequality $c_{j+1} \leq \max\{c_j, c_{j+2}\}$, it implies

$$d_{j+1} \leq \max\{d_j, d_{j+2}\}, 1 \leq j \leq m-k-3.$$

Repeating this method of elimination of the points which differ from the points $x_1, \ldots, x_{k+3}$, finally after $m-k-3$ steps it follows that f is k-quasiconvex. □

Returning to the proof of Theorem 1.3.1, (ii), first we have the following formula which can easily be derived from the above point (i) (see also, e.g., Popoviciu [317]):

$$B_n^{(k)}(f)(x) = \frac{k!n!}{(n-k)!n^k} \sum_{i=0}^{n-k} \left[\frac{i}{n}, \dots, \frac{i+k}{n}; f\right] p_{n-k,i}(x).$$

Setting

$$a_i = \left[\frac{i}{n}, \dots, \frac{i+k}{n}; f\right] \frac{k!n!}{(n-k)!n^k} \binom{n-k}{i},$$

$y = \frac{x}{1-x}$, $x \in [0,1)$, we get

$$B_n^{(k)}(f)(x) = (1-x)^{n-k} \sum_{i=0}^{n-k} a_i y^i.$$

Because the theorem is obvious for $n < k$, let us suppose $n \geq k$. By the hypothesis on f and by Lemma A, we have only three possibilities: (1) $a_i \leq 0$, for all $i = 0, \dots, n-k$; (2) $a_i \geq 0$, for all $i = 0, \dots, n-k$; (3) there is p, $0 < p < n-k$, such that $a_i \leq 0$ for all $0 \leq i \leq p$, $\min_{0 \leq i \leq p} a_i < 0$, and $a_i \geq 0$, for all $p+1 \leq i \leq n-k$, $\max_{p+1 \leq i \leq n-k} a_i > 0$.

By the well-known Descartes's rule, the polynomial $P(y) = \sum_{i=0}^{n-k} a_i y^i$ has at most a positive root, y_0, and this is possible only in the case (3), when because of $\lim_{y \to +\infty} P(y) = +\infty$, we have $P(y) < 0$ for all $0 < y < y_0$ and $P(y) > 0$ for all $y_0 < y$.

As a consequence, we have only three possibilities: (1) $B_n^{(k)}(f)(x) \leq 0$, for all $x \in [0,1]$; (2) $B_n^{(k)}(f)(x) \geq 0$, for all $x \in [0,1]$; (3) there is $c \in (0,1)$ such that $B_n^{(k)}(f)(x) \leq 0$ for all $x \in [0,c]$ and $B_n^{(k)}(f)(x) \geq 0$ for all $x \in [c,1]$, which by Lemma B proves (ii).

(iii) First, it is obvious that $B_n(f)(0) = f(0) = 0$ and $B_n(f)(x) \geq 0$, for all $x \in [0,1]$. According to Definition 1.1.1 (ii), it suffices to prove that $xB_n'(f)(x) - B_n(f)(x) \geq 0$, for all $x \in (0,1]$.

We have

$$B_n'(f)(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} n\left[f((k+1)/n) - f(k/n)\right] p_{n-1,k}(x),$$

while

$$\frac{B_n(f)(x)}{x} = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n}{k+1} f((k+1)/n) p_{n-1,k}(x),$$

which implies

$$B_n'(f)(x) - \frac{B_n(f)(x)}{x} = \sum_{k=0}^{n-1} p_{n-1,k}(x) A_{n,k}(f),$$

where $A_{n,k}(f) = \frac{k}{k+1} f((k+1)/n) - f(k/n)$. Since f is starshaped, it follows that $A_{n,k}(f) \geq f\left(\frac{k}{k+1} \cdot \frac{k+1}{n}\right) - f(k/n) = 0$, and by $p_{n-1,k}(x) \geq 0$, for all $x \in (0,1]$, we get the desired inequality.

The proof of the second part in (iii) is more technical, and it was given in Mocanu–Şerb–Toader [274];

(iv) According to Goodman [154], the proof was suggested by C.A. Micchelli. Let us sketch it below. Since f is logarithmic convex on $[0,1]$, writing $y_i = f(\frac{i}{n})$, it easily follows that the polygonal arc with vertices $(\frac{i}{n}, \log[y_i]), i = 0, \ldots, n$, is convex, i.e., we may write $y_i^2 \geq y_{i-1}y_{i+1}, i = 1, \ldots, n-1$. We also can write $B_n(f)(t) = \sum_{i=1}^{n+1} y_i^{(n+1)} \binom{n+1}{i} t^i (1-t)^{n+1-i}$, where $y_i^{(n+1)} = \frac{i}{n+1} y_{i-1} + \frac{n+1-i}{n+1} y_i$.

But by simple calculations (see Goodman [154], p. 343) we obtain

$$(n+1)\left[(y_i^{(n+1)})^2 - y_{i-1}^{(n+1)} y_{i+1}^{(n+1)}\right] \geq 0,$$

taking into account the hypothesis $y_i^2 \geq y_{i-1}y_{i+1}$.

Repeating the reasoning, by induction one obtains that for all $m \geq n$ we have $B_n(f)(t) = \sum_{i=0}^{m} y_i^{(m)} \binom{m}{i} t^i (1-t)^{m-i}$, where $(y_i^{(m)})^2 \geq y_{i-1}^{(m)} y_{i+1}^{(m)}, i = 1, \ldots, m-1$.

From the uniform convergence on $[0,1]$ (as $m \to +\infty$) of the convex polygon with vertices $(i/m, \log[y_i^{(m)}]), i = 0, \ldots, m$, to the curve $(t, \log[B_n(f)(t)])$, it follows that $\log[B_n(f)(t)]$ is convex too, which proves (iv).

(v) Suppose $u(x) = x^\lambda$, with arbitrary fixed $\lambda \in [0,1]$. From the u-convexity, it easily follows that $g(x) = \frac{f(x)}{x^\lambda}$ is nondecreasing on $(0,1]$.

It suffices to prove that $[B_n(f)(x)/x^\lambda]' \geq 0,\ x \in (0,1]$. Simple calculations (see Kocić–Lackcović [195], p. 3) give us

$$\left[B_n(f)(x)/x^\lambda\right]' = \sum_{k=0}^{n-1} n x^\lambda p_{n-1,k}(x) Q_{n,k},$$

where $Q_{n,k} = f(\frac{k+1}{n})(1 - \frac{\lambda}{k+1}) - f(\frac{k}{n})$. Therefore, it is enough to prove that $Q_{n,k} \geq 0$.

But, the generalized Bernoulli's inequality $(1+t)^\lambda \leq 1 + \lambda t,\ \forall \lambda \in [0,1]$, $t \geq -1$, for $t = -\frac{1}{k+1}$ implies

$$\left(1 - \frac{1}{k+1}\right)^\lambda \leq 1 - \frac{\lambda}{k+1}.$$

It follows that

$$f(k/n) \leq f[(k+1)]/n](k/(k+1))^\lambda = f[(k+1)/n]\left(1 - \frac{1}{k+1}\right)^\lambda$$
$$\leq f[(k+1)/n]\left(1 - \frac{\lambda}{k+1}\right),$$

which means exactly that $Q_{n,k} \geq 0$ and ends the proof.

(vi) Denote by $Z_{(0,1)}[B_n(f)(x)]$ the number of real zeros of $B_n(f)$ in $(0, 1)$. Since it is evident that $S_{[0,1]}[B_n(f)] \leq Z_{(0,1)}[B_n(f)]$, it remains to prove that $Z_{(0,1)}[B_n(f)] \leq S_{[0,1]}[f]$.

We have

$$B_n(f)(x)/(1-x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} z^k,$$

where $z = x/(1-x)$. Recall the classical Descartes's rule of signs (in 1637!): if $p(x)$ is a polynomial of the form $\sum_{i=0}^{n} a_i \binom{n}{i} x^i$, then the number of times it changes sign on $(0, +\infty)$ is bounded by the number of changes of sign in the sequence $a_0, \ldots, a_n$ (the zeros are not counted). By Descartes's rule of signs, it follows that

$$\begin{aligned} Z_{(0,1)}[B_n(f)(x)] &= Z[B_n(f)(x)/(1-x)^n] \\ &= Z_{0<z<\infty} \left[\sum_{k=0}^{n} f(k/n) \binom{n}{k} z^k \right] \\ &\leq S_{[0,1]} \left[(f(k/n) \binom{n}{k})_k \right] \leq S_{[0,1]} \left[(f(k/n))_k \right] \leq S_{[0,1]}[f], \end{aligned}$$

which proves (vi) too. □

Remarks. (1) According to Theorem 1.3.1 (i), the Bernstein polynomials also represent a constructive answer to the earlier result of Pál [295] in 1925.

(2) Although it is not of interest in the preservation of shape, let us also mention the property $V_{[0,1]}(B_n(f)) \leq V_{[0,1]}(f)$, for all $n \in \mathbb{N}$, where $V_{[0,1]}(f)$ denotes the total variation of f (see Popoviciu [316] or Lorentz [247], p. 23).

The slow convergence of the Bernstein polynomials $B_n(f)$ to f is in fact a consequence of their shape-preserving properties, as is shown by the following.

Theorem 1.3.2. (Berens–DeVore [37]) *Let us denote by $\mathcal{T}_n$ the class of all operators $T_n : C[0, 1] \to C[0, 1]$ satisfying the following conditions: $T_n(f)$ is a polynomial of degree $\leq n$, $T_n(f) = f$ for f a linear function, and $T_n(f)^{(j)} \geq 0$ for $f^{(j)} \geq 0$, for all $j = 0, 1, \ldots, n$.*

For any $T_n \in \mathcal{T}_n$ we have

$$T_n \left[(\cdot - x)^2 \right](x) \geq B_n \left[\cdot - x)^2 \right](x) = \frac{x(1-x)}{n},$$

with equality if and only if $T_n = B_n$.

Proof. Here we will present the main ideas of the proof in Berens–DeVore [37]. The problem can be reformulated in terms of eigenvalues. Thus, if $T_n \in \mathcal{T}_n$, then the shape-preserving properties imply that $T_n[\Pi_j] = \Pi_j$, for all $j = 0, 1, \ldots, n$, where Π_j denotes the set of all (real) polynomials of degree $\leq j$. This immediately implies that for each j, the operator T_n has an eigenfunction $E_j \in \Pi_j$ of the form $E_j(x) = x^j + \cdots$. Denoting its corresponding eigenvalue by $\lambda_j(T_n)$, the preservation of linear functions easily implies that $\lambda_0(T_n) = \lambda_1(T_n) = 1$, while the shape-preserving properties imply in an easy way the inequalities $1 \geq \lambda_2(T_n) \geq \cdots \geq \lambda_n(T_n) \geq 0$ (see Lemma in Berens–DeVore [37]).

It is known that the eigenvalues of B_n are given by (see Călugăreanu [53]) $\lambda_j(B_n) = 1, j = 0, 1,\ \lambda_j(B_n) = (1 - 1/2) \cdots (1 - (j-1)/n), j = 2, \ldots, n.$

The inequality in the conclusion of the statement is equivalent to

$$\lambda_2(T_n) \leq \lambda_2(B_n) = 1 - 1/n, \forall T_n \in \mathcal{T}_n, n \in \mathbb{N}.$$

More exactly, the statement of Theorem 1.3.2 can be reformulated as follows: for any $T_n \in \mathcal{T}_n$, we have $\lambda_2(T_n) \leq \lambda_2(B_n) = 1 - 1/n$, or equivalently

$$T_n[(\cdot - x)^2](x) \geq B_n[\cdot - x)^2](x) = \frac{x(1-x)}{n},$$

with equality in both kinds of inequalities, if and only if $T_n = B_n$.

The proof is based on a special representation of each operator T_n. Indeed, since the polynomials $(p_{n,k}(x))_{k=0,\ldots,n}$ form a basis in Π_n, we can write

$$T_n(f)(x) = \sum_{k=0}^{n} a_k(f) p_{n,k}(x), \text{ with } a_k(f) = \int_0^1 f \cdot du_k,$$

where du_k is a Borel measure. Moreover, since $1 = \sum_{k=0}^{n} p_{n,k}(x)$ and $x = \sum_{k=0}^{n} p_{n,k}(x) k/n$, it easily follows that

$$\int_0^1 du_k = 1, \int_0^1 t du_k = k/n, k = 0, 1, \ldots, n.$$

Also, it is proved that all the Borel measures are positive measures, (i.e., have the property that $du_k \geq 0$, for all $k = 0, \ldots, n$), a fact which is used to prove that equality holds if and only if $T_n = B_n$.

For details, the interested reader can consult Berens–DeVore [37]. □

Remark. It is known that the degree of approximation $|f(x) - L_n(f)(x)|$ of a sequence of positive linear operators $(L_n)_n$ is controlled by the quantity $L_n(\cdot - x)^2)(x)$ (see, e.g., DeVore [82]). Therefore, Theorem 1.3.2 says that with respect to this quantity, the Bernstein polynomial $B_n(f)$ has the best rate of approximation among the operators in the class $\mathcal{T}_n$.

The form of the Bernstein polynomials gives suggestions for constructing many other approximation polynomials, called Bernstein-type polynomials. In what follows we present some well-known examples of Bernstein-type polynomials together with their approximation, shape-preserving, and strong variation-diminishing properties.

(1) The Stancu [362] polynomials are defined by

$$S_{n,\alpha}(f)(x) = \sum_{k=0}^{n} w_{n,k}^{\alpha}(x) f\left(\frac{k}{n}\right),$$

where $w_{n,k}^{\alpha}(x) = \binom{n}{k} \frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}$, $x^{(k,a)} = x[x-a] \cdots [x-(k-1)a]$, $\alpha \geq 0$.

(2) The Soardi [360] polynomials (also called Bernstein operators of the second kind) are defined by

$$\beta_n(f)(x) = \frac{2^{-n-1}}{(n+1)x} \sum_{k=0}^{[n/2]} \binom{n+1}{k} b_{n,k}(x) f\left(\frac{n-2k}{n}\right),$$

where $b_{n,k}(x) = (n+1-2k)[(1-x)^k(1+x)^{n+1-k} - (1-x)^{n+1-k}(1+x)^k]$.

(3) The q-Bernstein polynomials introduced by Phillips [303] (see also the book Phillips [304], p. 267) are defined by

$$B_{n,q}(f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) x^k \left[\binom{n}{k}\right] \prod_{s=0}^{n-k-1} (1-q^s x),$$

where $0 < q \leq 1$, the empty product is equal to 1, $[k] = \frac{(1-q)^k}{1-q}$ if $q \neq 1$, $[k] = k$ if $q = 1$, the q-factorial $[k]!$ is defined by $[k]! = [k][k-1]\cdots[1]$ if $k \in \mathbb{N}$, $[0]! = 1$, and the q-binomial coefficients are defined by $\left[\binom{n}{k}\right] = \frac{[n]!}{[k]![n-k]!}$.

(4) The Bernstein–Chlodowsky–Stancu polynomials introduced by Ibikli [179] are defined by

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} f\left(b_n \frac{k+\alpha}{n+\beta}\right),$$

where $0 \leq \alpha \leq \beta$, $b_n \to \infty$, $b_n = o(n)$ for $n \to \infty$, and $0 \leq x \leq b_n$.

(5) The Lupaş[2] polynomials introduced by Lupaş–Lupaş [261] are defined by

$$L_{n,(a_n)_n}(f)(x) = \sum_{k=0}^{n} s_{n,k}(a_n; x) f\left(\frac{k}{n}\right),$$

where $a_n > 0$, $\forall n = 1, 2, \ldots$, $s_{n,k}(a_n; x) = \frac{1}{(a_n)_n} \binom{n}{k} (a_n x)_k (a_n - a_n x)_{n-k}$, $(z)_k = z(z+1)\cdots(z+k-1)$, $(z)_0 = 1$.

(6) The Durrmeyer polynomials introduced by Durrmeyer [100] and studied by Derriennic [76], [77], are defined by

$$D_n(f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt,$$

for any integrable function on $[0, 1]$, where $p_{n,k}(x)$ is the Bernstein basis.

(7) The Durrmeyer polynomials based on the ultraspherical weight $t^\alpha(1-t)^\alpha$, $\alpha > -1$ (for $\alpha = 0$ one recapture the original Durrmeyer polynomials), introduced by Lupaş [255], are defined by

$$D_{n,\alpha}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{(2\alpha+2)_n}{(\alpha+1)_k(\alpha+1)_{n-k}} \int_0^1 \frac{t^{k+\alpha}(1-t)^{n-k+\alpha}}{B(\alpha+1, \alpha+1)} f(t) dt,$$

where $B(\alpha, \alpha)$ denotes the beta function.

(8) The Lazarević–Lupaş [221] polynomials are defined by

$$U_n(f)(x) = \sum_{k=0}^{n} l_{n,k}(x) \int_0^1 \left[\frac{t}{n+1}, \ldots, \frac{t+k}{n+1}; f \right] dt,$$

where $[x_0, \ldots, x_n; f]$ denotes the divided difference and

$$l_{n,k}(x) = \frac{n!}{n^k (n-k)!} \left(x - \frac{1}{2n+2} \right)^k .$$

(9) The generalized Bernstein-type polynomials introduced by Munoz–Delgado, Ramirez–González, and Sablonniére [278] are defined by

$$G_{n,k}(f)(x) = Q_{k-2}(f)(x) + \int_{x_0}^{x} \int_{x_1}^{t_{k-1}} \cdots \int_{x_{k-2}}^{t_2} B_{n-k+1}(f^{(k-1)})(t_1) dt_1 dt_2 \cdots dt_{k-1},$$

where $B_n(f)$ represents the Bernstein polynomials, $x_j = (1 - \varepsilon_j \varepsilon_{j+1})/2,\ j = 0, \ldots, k-2,\ \varepsilon_h = 1$ or $\varepsilon_h = -1$ for all $h \in \mathbb{N}$, and $Q_{k-2}(f)(x)$ is the unique polynomial of degree $\leq k-2$ that satisfies the interpolation conditions $Q_{k-2}^{(j)}(f)(x_k) = f^{(j)}(x_k)$, for all $j = 0, \ldots, k-2$.

(10) The modified Durrmeyer polynomials introduced by Păltănea [297] and independently by Berens–Xu [38] are defined by

$$D_n^{<\alpha,\beta>}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 t^{k+\alpha}(1-t)^{n-k+\beta} f(t) dt}{B(k+\alpha+1, n-k+\beta+1)}, \quad x \in [0,1],$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is the beta function.

(11) The Mache [264] polynomials defined by

$$P_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 t^{ck+a}(1-t)^{c(n-k)+b} f(t) dt}{B(ck+a+1, c(n-k)+b+1)}, \quad x \in [0,1],$$

where $a, b > -1$ and $c := c_n := [n^\alpha]$, $\alpha \geq 0$.

(12) The Stancu [363] polynomials defined by

$$L_{m,p}^{<\alpha,\beta,\gamma>}(f)(x) = \sum_{k=0}^{m+p} f\left(\frac{k+\beta}{m+\gamma} \right) \frac{\Pi_{\mu=0}^{k-1}(x+\mu\alpha) \cdot \Pi_{\mu=0}^{m+p-k-1}(1-x+\mu\alpha)}{\Pi_{\mu=0}^{m+p+1}(1+\mu\alpha)},$$

$x \in [0,1]$, where p is a natural number, $m \geq 1$, $\alpha \geq 0$, $0 \leq \beta \leq \gamma$, $f \in C[0, 1+p/m]$.

Concerning the above Bernstein-type polynomials, with respect to the approximation and shape-preserving properties, we can state the following.

Theorem 1.3.3. *(i) The Stancu polynomial $S_{n,\alpha}(f)(x)$ satisfies (see Finta [117])*

$$\|S_{n,\alpha}(f) - f\|_\infty \le C\omega_2^\varphi(f; \sqrt{(1+n\alpha)/[n(1+\alpha)]})_\infty, \varphi^2(x) = x(1-x)$$

and preserves the convexities of any order of f (see Mastroianni [266]);

(ii) The Soardi polynomial $\beta_n(f)(x)$ satisfies (see Soardi [360])

$$\|\beta_n(f) - f\|_\infty \le C\omega_1\left(f; \frac{1}{\sqrt{n}}\right)_\infty.$$

In addition, if f is increasing on $[0,1]$, then so is $\beta_n(f)$ and if f is simultaneously increasing and convex on $[0,1]$, then so is $\beta_n(f)$ (see Raşa [321]);

(iii) The q-Bernstein polynomial $B_{n,q}(f)(x)$ satisfies (see Phillips [303])

$$\|B_{n,q}(f) - f\|_\infty \le \frac{3}{2}\omega_1\left(f; \frac{1}{[n]^{1/2}}\right)_\infty,$$

and, in addition, if f is increasing (convex) on $[0,1]$, then $B_{n,q}(f)$ is increasing (convex, respectively) on $[0,1]$ (see Goodman–Phillips [159] and the book Phillips [304], p. 287);

(iv) The Bernstein–Chlodowsky–Stancu $C_n(f)(x)$ preserves the convexities of any order of f on $[0, b_n]$ (see Ibikli [179]);

(v) The Lupaş[2] polynomial $L_{n,(a_n)_n}(f)(x)$ satisfies

$$\|L_{n,(a_n)_n}(f) - f\|_\infty \le 3\omega_1(f; \sqrt{1/n + 1/(2a_n)})_\infty$$

and preserves the convexities of any order of f (see L. Lupaş [260]);

(vi) The Durrmeyer polynomial $D_n(f)(x)$ satisfies the estimate (see Ditzian–Ivanov [96], Theorem 7.4)

$$\|D_n(f) - f\|_p \le C[\omega_2^\phi\left(f; \frac{1}{\sqrt{n}}\right)_p + n^{-1}\|f\|_p]$$

(where $\|\cdot\|_p$ denotes the $L^p[0,1]$ norm, $1 \le p \le +\infty$), and preserves the convexities of any order of f (see, e.g., Derriennic [76]);

(vii) The Durrmeyer polynomial $D_{n,\alpha}(f)(x)$ based on the ultraspherical weight $t^\alpha(1-t)^\alpha$, $\alpha > -1$ (for $\alpha = 0$ one recapture the original Durrmeyer polynomials), preserves the convexities of any order of f (see Lupaş [255]);

(viii) The Lazarević–Lupaş polynomial $U_n(f)(x)$, $n \ge [1/2\varepsilon]$, satisfies

$$\|U_n(f) - f\|_\infty \le \frac{19}{16}\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right)_\infty$$

and for any $\varepsilon \in (0, \frac{1}{2})$, if f is convex of order k on $[0,1]$, then $U_n(f)$ is convex of order k on $[\varepsilon, 1-\varepsilon]$, for all $n \ge [1/2\varepsilon]$ (see Lazarević–Lupaş [221]);

(ix) The generalized Bernstein polynomial $G_{n,k}(f)(x)$ satisfies the following properties: for any $2 \le k \le n$, if f is polynomial of degree k, then so is $G_{n,k}(f)$ and for all $i \le j$ and $j \ge k-1$

$$G_{n,k}(f)[C(i,j,\varepsilon)] \subset C(i,j,\varepsilon),$$

where $\varepsilon = (\varepsilon_k)_k$, with $\varepsilon_k = 1$ or -1 and $C(i,j,\varepsilon) := \{f \in C^n[0,1]; \varepsilon_k f^{(k)} \ge 0, k = i, i+1, \ldots, j\}$, $i \le j$ (see Munoz–Delgado, Ramirez–González and Sablonniére [278]).

Proof. Excepting the case of q-Bernstein polynomials, the method of proof for the shape-preserving properties of these polynomials is that in the case of classical Bernstein polynomials, by representing their derivative of a given order k as a sum of products between positive quantities and finite (or divided) differences of the same order k of f. In what follows, for some of them we will present the sketches of proofs for the shape-preserving properties only, while the quantitative estimates can be found in the corresponding mentioned papers.

(i) For the proof of shape-preserving properties see Mastroianni [266].

(ii) We can write

$$(\beta_n(f))'(x) = x^{-2}2^{-n-1} \sum_{k=0}^{[n/2]-1} r_{n,k}(x)[f((n-2k)/n) - f((n-2k-2)/n)],$$

where

$$r_{n,k}(x) = \binom{n}{k}(1-x^2)^k[(1+x)^{n-2k}((n-2k)x-1)+(1-x)^{n-2k}((n-2k)x+1)].$$

Since it is easy to prove that $r_{n,k}(x) \ge 0$, for all $x \in [0,1]$, it is immediate that $\beta_n(f)(x)$ preserves the monotonicity of f.

Then, denoting $m = [n/2]-1$, we have

$$\begin{aligned} x^3 2^{n+1}[\beta_n(f)]''(x) = &\sum_{k=0}^{m-1} [q_{n,k}(x) - q_{n,k}(-x)] \\ &\times \left[f\left(\frac{n-2k}{n}\right) - 2f\left(\frac{n-2k-2}{n}\right) + f\left(\frac{n-2k-4}{n}\right)\right] \\ &+ [q_{n,h}(x) - q_{n,h}(-x)]\left(f\left(\frac{n-2h}{n}\right) - f\left(\frac{n-2h-2}{n}\right)\right), \end{aligned}$$

where

$$\begin{aligned} q_{n,k}(x) = &\sum_{j=0}^{k} \binom{n}{j}[(n-2j)(1-x)^k(1+x)^{n-k-1} \\ &\times((n-2k-1)x^2 - 2x) + 2(1-x)^j(1+x)^{n-j}]. \end{aligned}$$

Since $q_{n,k}(x) - q_{n,k}(-x) \geq 0$, for all $0 \leq k \leq h$, $x \in [0,1]$ (see the proof in Raşa [321]), the conclusion is immediate.

(iii) First we prove that the operator $B_{n,q}$ is strongly variation-diminishing on $[0,1]$ (in the sense of Definition 1.1.1 (vi)). Indeed, for each $0 < q \leq 1$, denoting $P_{n,k,q}(x) = x^k \Pi_{s=0}^{n-k-1}(1-q^s x)$, we can write

$$B_{n,q}(f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\left[\binom{n}{k}\right] P_{n,k,q}(x)$$
$$= a_{0,q}P_{n,0,q}(x) + \cdots + a_{n,q}P_{n,n,q}(x).$$

Since each $\left[\binom{n}{k}\right]$ is a polynomial in q with positive integer coefficients, i.e., is positive, we immediately obtain that

$$S_{[0,1]}[B_{n,q}(f)] \leq S_{[0,1]}[f;([k]/[n])_k] \leq S_{[0,1]}(f).$$

Since $B_{n,q}$ reproduces any linear polynomial, this implies that for any $p \in \Pi_1$, we get

$$S_{[0,1]}[B_{n,q}(f) - p] = S_{[0,1]}[B_{n,q}(f-p)] \leq S_{[0,1]}(f-p).$$

In particular, for any constant c we have $S_{[0,1]}[B_{n,q}(f) - c] \leq S_{[0,1]}(f-c)$. Suppose that, for example, f is nondecreasing on $[0,1]$, it follows $S_{[0,1]}(f - c) \leq 1$, that is $S_{[0,1]}[B_{n,q}(f) - c] \leq 1$, which combined with the properties $B_{n,q}(f)(0) = f(0) \leq f(1) = B_{n,q}(f)(1)$, immediately implies that $B_{n,q}(f)$ must be nondecreasing on $[0,1]$.

Suppose now that f is convex on $[0,1]$, the graph of any $p \in \Pi_1$ can intersect that of f at no more than two points, which implies the $S_{[0,1]}(f - p) \leq 2$, and therefore $S_{[0,1]}(B_{n,q}(f) - p) \leq 2$. Suppose that the graph of p intersects that of $B_{n,q}(f)$ at u and v with $0 < u < v < 1$, that is $p(u) = B_{n,q}(f)(u), p(v) = B_{n,q}(f)(v)$. It easily follows that $B_{n,q}(f) - p$ cannot change its sign in (u,v). Varying u and v, it follows by a continuity argument that the sign of $B_{n,q}(f) - p$ remains the same. Since f is convex, for the limiting case $u = 0$ and $v = 1$, we get $0 \leq p(x) - f(x)$, for all $x \in [0,1]$, which implies that

$$0 \leq B_{n,q}(p-f)(x) = p(x) - B_{n,q}(f)(x), \forall x \in [0,1],$$

which is the convexity of $B_{n,q}(f)$ on $[0,1]$.

(iv) The preservation of convexity of any order m on the interval $[0, b_n]$, follows directly from the relationship (proved by mathematical induction)

$$[C_n(f)]^{(m)}(x) = \frac{n(n-1)\cdots(n-m+1)}{b_n^m}$$
$$\times \sum_{k=0}^{n-m} \binom{n-m}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-m-k} \Delta^m_{b_n/(n+\beta)} f((k+\alpha)b_n/(n+\beta)).$$

(v) The proof is very similar to that for Bernstein polynomials, see L. Lupaş [260].

(vi) For the proof, see, e.g., Derriennic [76].

(vii) The proof in Lupaş [255] is too technical to be reproduced here. Note that for $\alpha = 0$, one recapture the original Durrmeyer polynomials, so that in fact (vii) generalizes (vi).

(viii) The proof of the shape-preserving property is a direct consequence of the formula proved by mathematical induction

$$[U_n(f)]^{(j)}(x) = \frac{(n+1)^{n-j+1} n! j!}{n^n (n-j)!} \sum_{k=0}^{n-j} C_{k,j}(x) \times \int_{k/(n+1)}^{(k+1)/(n+1)} \left[t, t+\frac{1}{n+1}, \ldots, t+\frac{j}{n+1}; f\right] dt,$$

where

$$C_{k,j}(x) = \binom{n-j}{k} \left(x - \frac{1}{2n+2}\right)^k \left(\frac{2n+1}{2n+2} - x\right)^{n-j-k}.$$

(ix) Firstly, let $j \geq k-1$. For $C(j, j, \varepsilon)$ we get: if $f^{(j)} \geq 0$, then denoting $g = f^{(k-1)}$, it follows that $g^{(j-k+1)} \geq 0$ and $[G_{n,k}(f)]^{(j)} = [B_{n-k+1}(f)]^{j-k+1} \geq 0$. The general case is proved by mathematical induction on i. □

Remarks. (1) The approximation and shape-preserving properties of the q version of $D_n^{<\alpha,\beta>}(f)(x)$ polynomials, were studied in Derriennic [79].

(2) In 1930, Kantorovitch [191] introduced the polynomials

$$K_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt,$$

whose shape-preserving properties are an immediate consequence of the relationship $K_n(f)(x) = B'_{n+1}(F)(x)$ (see, e.g., Lorentz [247], p. 30), where $B_n(F)(x)$ denotes the Bernstein polynomial attached to $F(x) = \int_0^x f(t)dt$.

Concerning the relationship between the shape-preserving property and the strong variation-diminishing property, we present below the method of proof for some Bernstein-type polynomials.

Theorem 1.3.4. (Gavrea–Gonska–Kacsó [146]) *The modified Durrmeyer polynomials $D_n^{<\alpha,\beta>}(f)(x)$ (see Example 10 before Theorem 1.3.3) and Mache polynomials $P_n(f)(x)$ (see Example 11 before Theorem 1.3.3) preserve the convexity of orders $0, 1, \ldots, n$, the Stancu operator $L_{m,p}^{<\alpha,\beta,\gamma>}(f)(x)$ preserves the convexity of orders $0, \ldots, m+p$, and all have the strong variation-diminishing property.*

Proof. The proof follows the ideas in Gavrea–Gonska–Kacsó [146]. Thus, it suffices to prove that all the polynomials in the statement have the strong variation-diminishing property, which by Theorem 5.1.7 (ii) in Section 5.1 will imply the shape-preserving properties.

If we consider the beta-type operator

$$\mathcal{B}_n^{<\alpha,\beta>}(f)(x) = \frac{\int_0^1 t^{\alpha+nx}(1-t)^{n-nx+\beta} f(t)dt}{B(nx+\alpha+1, n-nx+\beta+1)},$$

then $D_n^{<\alpha,\beta>}(f)(x) = B_n[\mathcal{B}_n^{<\alpha,\beta>}(f)](x)$, where B_n is the classical Bernstein operator. We get

$$S_{[0,1]}[D_n^{<\alpha,\beta>}(f)] \le S_{[0,1]}[\mathcal{B}_n^{<\alpha,\beta>}(f)]$$
$$= S_{[0,1]}\left[\int_0^1 t^{nx}(1-t)^{n(t-x)} \cdot t^\alpha(1-t)^\beta f(t)dt\right].$$

But $S_{[0,1]}[\int_0^1 t^{nx}(1-t)^{n(t-x)} \cdot t^\alpha(1-t)^\beta f(t)dt] \le S_{[0,1]}[t^\alpha(1-t)^\beta f(t)] = S_{[0,1]}(f)$, which would prove $S_{[0,1]}[D_n^{<\alpha,\beta>}(f)] \le S_{[0,1]}[f]$.

Indeed, if we use now the substitution $u = \left(\frac{t}{1-t}\right)^n$, then the integral $\int_0^1 t^{nx}(1-t)^{n(1-x)} t^\alpha(1-t)^\beta f(t)dt$ becomes

$$\frac{1}{n}\int_0^{+\infty} u^x \frac{u^{1/n-1}}{(1+u^{1/n})^{n+2}} \frac{u^{\alpha/n}}{(1+u^{1/n})^{\alpha+\beta}} f\left(\frac{u^{1/n}}{1+u^{1/n}}\right) du.$$

Evidently, the number of sign changes of $f(t), t \in [0,1]$, equals the number of sign changes of the function $g(u) = \frac{u^{1/n}}{1+u^{1/n}}, u \in [0, +\infty)$. Applying now Theorem 5.1.7 (iii) in Section 5.1 for $A(g) = \int_0^{+\infty} g(u)du$ and putting $w(u) = \frac{u^{1/n-1}}{(1+u^{1/n})^{n+2}} \frac{u^{\alpha/n}}{(1+u^{1/n})^{\alpha+\beta}}$, we get

$$S_{[0,1]}\left[\int_0^1 t^{nx}(1-t)^{n(1-x)} t^\alpha(1-t)^\beta f(t)dt \le S_{[0,1]}[t^\alpha(1-t)^\beta f(t)\right]$$
$$= S_{[0,1]}[t^\alpha(1-t)^\beta f(t)] = S_{[0,1]}[f],$$

which implies

$$S_{[0,1]}[D_n^{<\alpha,\beta>}(f)] \le S_{[0,1]}[f].$$

For Mache's polynomial, we have $P_n(f) = B_n[\mathcal{B}_n^{<a,b,c>}(f)]$, where

$$\mathcal{B}_n^{<a,b,c>}(f)(x) = \frac{\int_0^1 t^{cnx+a}(1-t)^{cn(1-x)+b} f(t)dt}{B(cnx+a+1, cn(1-x)+b+1)}.$$

Reasoning exactly as in the case of the modified Durrmeyer polynomials, we arrive again at $S_{[0,1]}[P_n(f)] \le S_{[0,1]}[f]$.

In the case of Stancu's polynomials, by the following known recurrence formula for the beta function,

$$B(a+k, b+m+p-k)$$
$$= \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+m+p-k-1)}{(a+b)(a+b+1)\cdots(a+b+m+p-1)} B(a,b),$$

we obtain

$$\frac{\Pi_{\mu=0}^{k-1}(x+\mu\alpha)\cdot\Pi_{\mu=0}^{m+p-k-1}(1-x+\mu\alpha)}{\Pi_{mu=0}^{m+p-1}(1+\mu\alpha)}$$
$$=\frac{B(x/\alpha+k,(1-x)/\alpha+m+p-k)}{B(x/\alpha,(1-x)\alpha)}.$$

We therefore get

$$L_{m,p}^{<\alpha,\beta,\gamma>}(f)(x)$$
$$=\frac{1}{B(x/\alpha,(1-x)/\alpha)}\sum_{k=0}^{m+p}\binom{m+p}{k}B(x/\alpha+k,(1-x)/\alpha+m+p-k)f\left(\frac{k+\beta}{m+\gamma}\right)$$
$$=\frac{1}{B(x/\alpha,(1-x)/\alpha)}$$
$$\times\int_0^1\left[\sum_{k=0}^{m+p}\binom{m+p}{k}t^{x/\alpha+k+1}(1-t)^{(1-x)/\alpha+m+p-k-1}f\left(\frac{k+\beta}{m+\gamma}\right)\right]dt$$
$$=\frac{1}{B(x/\alpha,(1-x)/\alpha)}\int_0^1 t^{x/\alpha-1}(1-t)^{(1-x)/\alpha-1}B_{m,p}^{<\beta,\gamma>}(f)(t)dt,$$

where

$$B_{m,p}^{<\beta,\gamma>}(f)(x):=\sum_{k=0}^{m+p}p_{m+p,k}(x)f\left(\frac{k+\beta}{m+\gamma}\right).$$

Now, by Schoenberg [343], it is immediate that

$$S_{[0,1]}[L_{m,p}^{<\alpha,\beta,\gamma>}(f)]\le S_{[0,1+p/m]}[f].$$

We get

$$S_{[0,1]}[L_{m,p}^{<\alpha,\beta,\gamma>}(f)]=S_{[0,1]}\left[\int_0^1 t^{x/\alpha-1}(1-t)^{(1-x)/\alpha-1}B_{m,p}^{<\beta,\gamma>}(f)(t)dt\right],$$

and by the substitution $u=\left(\frac{t}{1-t}\right)^{1/\alpha}$, the above integral becomes

$$\alpha\int_0^\infty u^x\frac{1}{u(1+u^\alpha)^{1/\alpha}}B_{m,p}^{<\beta,\gamma>}(f)\left(\frac{u^\alpha}{1+u^\alpha}\right)du,$$

which by Theorem 5.1.7 (iii) in Section 5.1, implies

$$S_{[0,1]}[L_{m,p}^{<\alpha,\beta,\gamma>}(f)]\le S_{[0,1]}[B_{m,p}^{<\beta,\gamma>}(f)]\le S_{[0,1+p/m]}[f].$$

This proves the theorem. □

Remarks. As in the case of the modified Durrmeyer polynomials in Theorem 1.3.4, $D_n^{<\alpha,\beta>}(f)(x)$, it can be proved that the classical Durrmeyer polynomials $D_n(f)(x)$ in Theorem 1.3.3 (vi), have the strong variation-diminishing property, see Gavrea–Gonska–Kacsó [146].

1.4 Shisha-Type Results

Although the sequences of Bernstein-type approximation polynomials considered in Section 1.3 have very nice shape-preserving properties, their rates of approximation are rather weak, involving the quantities $\omega_k(f; \frac{1}{\sqrt{n}})_\infty, k = 1, 2$.

Interest in improving the estimates in shape-preserving approximation by polynomials began with the papers of Shisha [351] in 1965 and Lorentz–Zeller [250] in 1968.

Thus, Shisha [351] proved that if $p \geq 1$ and $k \in \{1, \ldots, p\}$, then for any $f \in C^p[a, b]$ satisfying $f^{(k)} \geq 0$ in $[a, b]$ and any $n > p$, there is an algebraic polynomial P_n of degree $\leq n$ satisfying $P_n^{(k)} \geq 0$ in $[a, b]$ and

$$\|f - P_n\|_\infty \leq \frac{C_{p,k}}{n^{p-k}} \omega_1(f^{(p)}; 1/n)_\infty.$$

Although this is a weaker Jackson-type estimate than those in the next sections of this chapter, since the idea of Shisha's method is simple and can easily be extended to real functions of two real variables and to complex variables, we present it here together with some generalizations and their proofs.

Instead of the proof of Shisha's first result in 1965, we present the proof of a generalization of it, as follows.

Theorem 1.4.1. (Anastassiou–Shisha [17]) *Let $f \in C^p[-1, 1]$, and let the integers $0 \leq h \leq k \leq p$ and the functions $a_j : [-1, 1] \to \mathbb{R}$, bounded on $[-1, 1]$ for all $j = h, \ldots, k$ be such that $a_h \geq c > 0$, for all $x \in [-1, 1]$ or $a_h(x) \leq d < 0$ for all $x \in [-1, 1]$. Define the differential operator $L(f)(x) = \sum_{j=h}^{k} a_j(x) f^{(j)}(x)$, $x \in [-1, 1]$, and suppose that $L(f)(x) \geq 0$, for all $x \in [-1, 1]$.*

Then for every $n \in \mathbb{N}$, there exists a real polynomial $P_n(x)$ of degree $\leq n$ such that

$$\|f - P_n\|_\infty \leq C n^{k-p} \omega_1 \left(f^{(p)}; \frac{1}{n} \right)_\infty, n \in \mathbb{N},$$

and in addition, $L(P_n)(x) \geq 0$ for all $x \in [-1, 1]$, $n \in \mathbb{N}$. Here C is independent of n and f and $\| \cdot \|_\infty$ denotes the uniform norm on $C[-1, 1]$.

Proof. The method is based on the simultaneous approximation result of Trigub [388]; namely, for any $g \in C^p[-1, 1]$, there exists a polynomial $p_n(x)$ of degree $\leq n$ with the property

$$\|g^{(j)} - p_n^{(j)}\|_\infty \leq R_p n^{j-p} \omega_1(g^{(p)}; 1/n)_\infty, \quad j = 0, 1, \ldots, p,$$

where R_p is independent of g and n.

Set $A_j = \|a_j / a_h\|_\infty$ and $\eta_n = R_p \omega_1(f^{(p)}; 1/n)_\infty \sum_{j=h}^{k} A_j n^{j-p}$.

Let us first suppose that $a_h(x) \geq c > 0$, for all $x \in [-1, 1]$. Writing $g(x) = f(x) + \eta_n x^h/(h!)$, let $P_n(x)$ be the polynomial of degree $\leq n$ satisfying

$$\begin{aligned}\|g^{(j)} - P_n^{(j)}\|_\infty &\leq R_p n^{j-p} \omega_1(g^{(p)}; 1/n)_\infty \\ &= R_p n^{j-p} \omega_1(f^{(p)}; 1/n)_\infty, \quad j = 0, 1, \ldots, p.\end{aligned}$$

We easily get

$$\|f - P_n\|_\infty \le \eta_n (h!)^{-1} + R_p n^{j-p} \omega_1(f^{(p)}; 1/n)_\infty$$
$$\le R_p/(1 + (h!)^{-1} \sum_{j=h}^{k} A_j) n^{k-p} \omega_1(f^{(p)}; 1/n)_\infty,$$

which implies the estimate in the theorem.

On the other hand, for all $-1 \le x \le 1$ we get

$$\frac{1}{a_h(x)} L(P_n)(x) = \frac{1}{a_h(x)} L(f)(x) + \eta_n$$
$$+ \sum_{j=h}^{k} \frac{1}{a_h(x)} a_j(x)[P_n(x) - f(x) - x^h \eta_n/(h!)]^{(j)}$$
$$\ge \eta_n - \sum_{j=h}^{k} A_j R_p n^{j-p} \omega_1(f^{(p)}; 1/n)_\infty = 0,$$

which proves $L(P_n)(x) \ge 0$ for all $x \in [-1, 1]$.

Now let us suppose that $a_h(x) \le d < 0$, for all $x \in [-1, 1]$ and for $g(x) = f(x) - \eta_n x^h/(h!)$, let $P_n(x)$ satisfy

$$\|g^{(j)} - P_n^{(j)}\|_\infty \le R_p n^{j-p} \omega_1(g^{(p)}; 1/n)_\infty$$
$$= R_p n^{j-p} \omega_1(f^{(p)}; 1/n)_\infty, \quad j = 0, 1, \ldots, p.$$

From here the proof is similar to that of the first case, which proves the theorem. □

Corollary 1.4.2. *Under the hypothesis and notations of Theorem 1.4.1, if $p \ge 0$ and $L(f)(x) \ge 0$, for all $x \in [-1, 1]$, then for every $n \in \mathbb{N}$, $n \ge p$, there exists a real polynomial $P_n(x)$ of degree $\le n$ such that*

$$\|f - P_n\|_\infty \le C_p n^{k-p} E_{n-p}(f^{(p)})_\infty,$$

and in addition, $L(P_n)(x) \ge 0$ for all $x \in [-1, 1]$, $n \in \mathbb{N}, n \ge p$.

Proof. Instead of the Trigub's result, we use the following improvement due to Leviatan [231], Theorem 2: for any $p \ge 0$, $g \in C^p[-1, 1]$ and $n \ge p$, there exists a polynomial $p_n(x)$ of degree $\le n$ with the properties

$$\|g^{(j)} - p_n^{(j)}\|_\infty \le R_p n^{j-p} E_{n-p}(f^{(p)})_\infty, \quad j = 0, 1, \ldots, p,$$

where R_p is independent of g and n and $E_r(g)_\infty = \inf_{p \in \Pi_r} \|g - p\|_\infty$. Then, repeating word for word the reasonings in the proof of Theorem 1.4.1, we get the corollary. □

Remarks. (1) If in the statement of Theorem 1.4.1 one takes $L(f)(x) = f^{(k)}(x)$, then one recovers the original result of Shisha [351]. Moreover, if in addition, one supposes that all $a_h, \ldots, a_k$ are continuous on $[-1, 1]$ and one considers the condition $L(f)(x) > 0$ for all $x \in [-1, 1]$, the conclusion of Theorem 1.4.1 remains true for n sufficiently large (see Anastassiou–Shisha [17]), recovering thus, in essence, for $L(f)(x) = f^{(k)}(x)$, the results in Roulier [326] too.

(2) Let us suppose that in Theorem 1.4.1, in addition to its hypothesis, all the functions a_j, $j = h, \ldots, k$, are continuous on $[-1, 1]$ and that $L(f)(x) > 0$ for all $x \in (-1, 1)$. By the continuity assumptions, it is immediate that $L(f)(x) \geq 0$ for all $x \in [-1, 1]$, and from the proof of the theorem, the conclusion $L(P_n)(x) > 0$ for all $x \in (-1, 1)$, $n \in \mathbb{N}$ follows easily.

This kind of remark will be very useful in Sections 3.2 and 4.2, where we will extend the method to complex functions of one or several complex variables.

(3) In Theorem 1.4.1, the hypothesis $a_h \geq c > 0$ for all $x \in [-1, 1]$ or $a_h(x) \leq d < 0$ for all $x \in [-1, 1]$ can be replaced, for example, by the hypothesis that a_h is continuous on $[-1, 1]$, which leads to the following "partial shape-preserving" approximation.

Corollary 1.4.3. *Suppose we are under the hypothesis and notation of Theorem 1.4.1, excepting that concerning a_h, which is supposed to be only continuous on $[-1, 1]$. Then for $f \in C^p[-1, 1]$ with $L(f)(x) \geq 0, \forall x \in [-1, 1]$, and $n \geq 1$, there exists a real polynomial P_n of degree $\leq n$ satisfying*

$$\|f - P_n\|_\infty \leq C_p n^{k-p} \omega_1(f^{(p)}; 1/n)_\infty$$

(or the better estimate $\|f - P_n\|_\infty \leq C_p n^{k-p} E_{n-p}(f^{(p)})_\infty$) such that for any $x_0 \in [-1, 1]$ with $a_h(x_0) \neq 0$, there exists a neighborhood of x_0, denoted by $V(x_0)$ and independent of f and n, such that $L(P_n)(x) \geq 0$ for all $x \in V(x_0)$.

Proof. Since $a_h(x_0) \neq 0$, by continuity it follows that there exists a neighborhood of x_0, denoted by $V(x_0)$, such that $a_h(x) > 0$ for all $x \in V(x_0)$, or $a_h(x) < 0$ for all $x \in V(x_0)$.

To make a choice, suppose that $a_h(x) > 0$ for all $x \in V(x_0)$. There exists $c > 0$, such that $a_h(x) \geq c > 0$, for all $x \in V(x_0)$. Repeating the reasoning to that of the proof of Theorem 1.4.1 with $A_j = \sup\{|a_j(x)|/|a_h(x)|; x \in V(x_0)\}$, it easily follows that $L(P_n)(x) \geq 0$ for all $x \in V(x_0)$. From the proof of Theorem 1.4.1, we observe that the approximation property of P_n is independent on the sign of a_h.

In the case when $a_h(x) < 0$ for all $x \in V(x_0)$, there exists $d < 0$ such that $a_h(x) \leq d < 0$ for all $x \in V(x_0)$, and we again repeat the proof of Theorem 1.4.1.

The case when instead of Trigub's approximation result, we use the improvement due to Leviatan [231], Theorem 2 (as in the proof of Corollary 1.4.2), which is completely similar and which proves the corollary. □

1.5 Positive and Copositive Polynomial Approximation

In this section we present the most important results in positive and copositive polynomial approximation. Since their proofs are, in general, very technical, we omit most of them.

For $f \in L^p[-1,1]$, let us define the unconstrained best approximation

$$E_n(f)_p := \inf_{p_n \in \Pi_n} \|f - p_n\|_p.$$

Denote by Δ^0 the set of all functions $f : [-1,1] \to \mathbb{R}$ such that $f \geq 0$ in $[-1,1]$, and for $f \in \Delta^0 \cap L^p[-1,1]$, the best positive approximation of f in the L^p-norm, $1 \leq p \leq \infty$, by algebraic polynomials of degree $\leq n$ will be denoted by

$$E_n^{(0)}(f)_p := \inf_{p_n \in \Pi_n \cap \Delta^0} \|f - p_n\|_p.$$

We first consider the case of positive uniform approximation.

Let us suppose that $f \in C[-1,1]$, $f \geq 0$. Then for $n \geq 0$, there exists $P_n \in \Pi_n$ such that,

$$\|f - P_n\|_\infty = E_n(f)_\infty.$$

It follows that

$$P_n(x) - f(x) \geq -E_n(f)_\infty,$$

which implies

$$R_n(x) := P_n(x) + E_n(f)_\infty \geq f(x) \geq 0.$$

Therefore R_n is nonnegative, and we have

$$\|f - R_n\|_\infty \leq 2E_n(f)_\infty,$$

which implies

$$E_n^{(0)}(f)_\infty \leq 2E_n(f)_\infty, \quad n \geq 0.$$

Therefore, the error estimate in the case of best positive uniform approximation in fact is equivalent to the error estimate in the case of best unconstrained uniform approximation.

But as we will see in what follows, the situation is completely different for pointwise estimates in approximation of nonnegative functions by nonnegative polynomials and for L^p-estimates of positive functions by positive approximation polynomials.

1.5.1 Pointwise Positive Approximation

The pointwise estimates in polynomial approximation of a nonnegative $f \in C^r[-1,1] \cap \Delta^0$ are of two kinds:

Timan–Brudnyi-type estimates of the form

$$|f(x) - p_n(x)| \leq C(r,k)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \quad -1 \leq x \leq 1, \quad n \geq N,$$

where $\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\varphi(x)$, $\varphi(x) = [1-x^2]^{1/2}$, $C(r,k)$ is a constant depending only on r and k (independent of f and n);

Telyakovskii–Gopengauz (or interpolatory) type estimates of the form

$$|f(x) - p_n(x)| \le C(r,k)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x))_\infty, \quad -1 \le x \le 1, \quad n \ge N,$$

where $\delta_n(x) := \frac{1}{n}\varphi(x)$.

Dzyubenko [103] proved that the above Timan–Brudnyi estimates are valid for positive approximation for all $n \ge N := r + k - 1$, while for the above Telyakovskii–Gopengauz estimates, we have the following.

Theorem 1.5.1. (Gonska–Leviatan–Shevchuk–Wenz [152]) *(i) Let either $r = 0$ and $k = 1, 2$, or $1 \le k \le r$. If $f \in C^r[-1,1] \cap \Delta^0$, then for any $n \ge N := 2[(r+k+1)/2]$, there exists a polynomial $p_n \in \Pi_n \cap \Delta^0$ with the property*

$$|f(x) - p_n(x)| \le C(r)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x))_\infty, \quad -1 \le x \le 1.$$

(ii) Let either $r = 0$ and $k > 2$, or $k > r \ge 1$. Then for each $n \ge 1$ and constant $A > 0$, there is a function $f = f_{k,r,n,A} \in C^r[-1,1] \cap \Delta^0$ such that for any polynomial $p_n \in \Pi_n \cap \Delta^0$, there exists a point $x \in [-1,1]$ such that

$$|f(x) - p_n(x)| > A\frac{(1-x)^{r/2}}{n^r}\omega_k\left(f^{(r)}, \frac{\sqrt{1-x}}{n}\right)_\infty$$

holds.

Remarks. (1) The case $r + k \le 2$ in Theorem 1.5.1, (i), is due to DeVore–Yu [92].

(2) Theorem 1.5.1, (i), may suggest the possibility to obtain some interpolatory estimates for copositive approximation. This question is completely open.

1.5.2 L^p-Positive Approximation, $0 < p < \infty$

The L^p-norm estimates for $0 < p < \infty$ in positive polynomial approximation are different from the case of positive uniform polynomial approximation. Thus, denote by $W_p^r[-1,1], 0 < p < \infty$, the Sobolev space of functions f such that $f^{(r-1)}$ is locally absolutely continuous in $(-1,1)$ and $f^{(r)} \in L^p[-1,1]$. First we present

Theorem 1.5.2. (Stojanova [372]) *(i) For any $f \in W_p^1[-1,1] \cap \Delta^0$, $1 \le p < \infty$, we have*

$$E_n^{(0)}(f)_p \le \frac{C}{n}E_{n-1}(f')_p \le \frac{C(k)}{n}\omega_k^\varphi\left(f', \frac{1}{n}\right)_p, \ n \ge 1,$$

where the constant $C(k)$ depends only on k and p;

(ii) For any $f \in \Delta^0 \cap L^p[-1,1]$, we have

$$E_n^{(0)}(f)_p \le C(k)\tau_k\left(f, \frac{1}{n}\right)_p, \quad n \ge 1,$$

where the $\tau_k(f,\cdot)_p$ denotes Sendov's averaged modulus of smoothness (see Definition 1.1.2 (iii)).

On the other hand, we have the following result.

Theorem 1.5.3. (Hu–Kopotun–Yu [172], Ivanov [185] for $1 \le p < \infty$) *For any $f \in \Delta^0 \cap L^p[-1,1]$ $0 < p < \infty$, there is a constant C such that*

$$E_n^{(0)}(f)_p \le C\omega_1^{\varphi}\left(f, \frac{1}{n}\right)_p,$$

where $C > 0$ is an absolute constant if $1 \le p < \infty$ and $C = C(p)$ if $0 < p < 1$.

Also for each $A > 0$, $n \ge 1$ and $0 < p < \infty$, there is a function $f := f_{A,n,p} \in \Delta^0 \cap L^p[-1,1]$, with the property

$$E_n^{(0)}(f)_p \ge A\omega_2(f,1)_p.$$

Proof. Here we will sketch the constructions by following the ideas in Hu–Kopotun–Yu [172], where the reader can find the complete proof. For the proof of the estimate, we first approximate f by a piecewise constant function $S_n(f)$ given by the formula

$$S_n(f,x) := s_n + \sum_{k=1}^{n-1}(s_k - s_{k+1})\chi_k(x),$$

where s_k is the best L^p-approximation constant function to f on the interval $[x_k, x_{k-1}]$, with $x_k = \cos(\frac{k\pi}{n}), k = 0,\dots,n$ and $\chi_k(x) = 0$ if $x \in [-1, x_k)$, $\chi(x) = 1$ if $x \in [x_k, 1]$. Obviously $S_n(x) = s_k$ on $[x_k, x_{k-1}]$, $S_n(x) \ge 0$ for all $x \in [-1,1]$, and by, e.g., DeVore–Lorentz [91] we have

$$\|f - s_k\|_{L^p[x_k,x_{k-1}]} \le C\omega_1(f, h_k, [x_k, x_{k-1}])_p,$$

where $h_k = x_{k-1} - x_k$ and $\omega_1(f, h_k, [x_k, x_{k-1}])_p$ denotes the L^p-modulus of continuity of f on the interval $[x_k, x_{k-1}]$. It follows that

$$\|f - S_n(f)\|_p^p = \sum_{k=1}^{n}\int_{x_k}^{x_{k-1}} |f(x) - s_k|^p dx$$
$$\le C^p \sum_{k=1}^{n}\omega_1(f, h_k, [x_k, x_{k-1}])_p \le C^p\omega_1^{\varphi}\left(f, \frac{1}{n+1}\right)_p^p.$$

Now define the polynomials

$$P_n(f,x) = s_n + \sum_{k=1}^{n-1}(s_k - s_{k+1})\left(\frac{\mathrm{sgn}(s_k - s_{k+1})+1}{2}T_k(x) + \frac{1-\mathrm{sgn}(s_k - s_{k+1})}{2}R_k(x)\right),$$

where $T_k(x)$ and $R_k(x)$ are suitable polynomials of degree $\leq k$ introduced in Lemma 3.1 in Hu–Kopotun–Yu [172].

We have

$$P_n(f,x) \geq s_n + \sum_{k=1}^{n-1}(s_k - s_{k+1})\chi_k(x) = S_n(f,x) \geq 0$$

for all $x \in [-1,1]$.

Also, the reasoning in Hu–Kopotun–Yu [172], pp. 330–331, imply

$$\begin{aligned}\|P_n - S_n(f)\|_p^p &\leq C^p \sum_{k=1}^{n-1} \|s_k - s_{k+1}\|^p_{L^p[x_k, x_{k-1}]} \\ &\leq C^p \sum_{k=1}^{n} \|f - s_k\|^p_{L^p([x_k,x_{k-1}]\cup[x_{k-1},x_{k-2}])} \\ &\leq C^p \sum_{k=1}^{n} \omega_1(f, h_k, [x_k, x_{k-1}] \cup [x_{k-1}, x_{k-2}])_p^p \leq C^p \omega_1^{\varphi}\left(f, \frac{1}{n+1}\right)_p^p.\end{aligned}$$

Combining with the estimate for $\|f - S_n(f)\|_p^p$, the desired estimate for $E_n^{(0)}(f)_p$ follows.

The counterexample function $f := f_{A,n,p} \in \Delta^0 \cap L^p[-1,1]$ is of the form $f(x) = b(1-x) - \log(1-x+e^{-b}) - \log(b)$, where $b \geq e^M$, with suitably chosen $M := M(n, \varepsilon, p, A)$. $\square$

1.5.3 Uniform and Pointwise Copositive Approximation

First we need some useful notation. For the integer $s \geq 0$, let $\mathbb{Y}_s$ be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points such that $y_{s+1} := -1 < y_s < \cdots < y_1 < 1 =: y_0$, where for $s = 0, Y_0 = \emptyset$. For $Y_s \in \mathbb{Y}_s$ we define

$$\Pi(x, Y_s) := \prod_{i=1}^{s}(x - y_i),$$

where the empty product $= 1$.

Let $\Delta^0(Y_s)$ be the set of all functions $f : [-1,1] \to \mathbb{R}$ that change their sign exactly at the points $y_i \in Y_s$ and that are nonnegative in $(y_1, 1)$. Obviously, $f \in \Delta^0(Y_s)$ is equivalent to

$$f(x)\Pi(x, Y_s) \geq 0, \qquad -1 < x < 1.$$

If $s = 0$, then this means that $Y_0 = \emptyset$, which implies $f \geq 0$ in $[-1, 1]$, and we will write as in the above subsections $f \in \Delta^0$.

One says that f and g are copositive on $[-1, 1]$ if $f(x)g(x) \geq 0$ for all $x \in [-1, 1]$.

For $f \in \Delta^0(Y_s) \cap L^p[-1, 1]$, we define

$$E_n^{(0)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^0(Y_s)} \|f - p_n\|_p,$$

the best copositive approximation of f by algebraic polynomials of degree $\leq n$. If $Y_0 = \emptyset$, then we write as in the above subsections $E_n^{(0)}(f)_p =: E_n^{(0)}(f, \emptyset)_p$, which represents the best positive approximation.

A natural extension of positive polynomial approximation is the so-called copositive polynomial approximation, i.e., the case of approximation of a function f by polynomials $(P_n)_n$, which changes their (same) signs with f at $Y_s \in \mathbb{Y}_s$, i.e., satisfying $f(x)P_n(x) \geq 0$, for all $x \in [-1, 1]$, $n \in \mathbb{N}$.

Remark. Since in the copositivity case we have $f(x)\Pi(x, Y_s) \geq 0, -1 < x < 1$, a first idea that might come to mind is that the results in positive approximation may be applicable to the positive function $F(x) = f(x)/\Pi(x, Y_s)$. But of course, first some hypothesis of smoothness on f at the points y_i would be necessary in order to ensure that F is continuous at all the points y_i, a fact that unfortunately would drastically reduce the generality on f. Also, even with the suitable smoothness hypothesis on f, if, for example, $q_n(F)$ would be the best positive approximation polynomial of degree $\leq n$ attached to F (i.e., satisfies $\|F - q_n(F)\|_\infty = \inf\{\|p - F\|_\infty; p_n \in \Pi_n, p_n \geq 0\} := E_n^{(0)}(F)_\infty$), then although the polynomial defined by $P_n(f)(x) = [q_n(F)(x) + E_n(F)_\infty]\Pi(x, Y_s)$ would be copositive with f, from the inequality

$$\|f - P_n(f)\|_\infty \leq \|q_n(F)\Pi(\cdot, Y_s) - f\|_\infty + CE_n^{(0)}(F)_\infty,$$

a very bad estimate follows, very far from any Jackson-type estimate in terms of the moduli of smoothness of f (since $E_n^{(0)}(F)_\infty$ and $\|q_n(F)\Pi(\cdot, Y_s) - f\|_\infty$ could be very bad with respect to $E_n(f)_\infty$ and $E_n^{(0)}(f, Y_s)_\infty$).

For uniform and pointwise approximation, the results can be summarized by the next theorem.

Theorem 1.5.4. *(Kopotun [207]) There exists a constant $C = C(Y_s)$ such that for any $f \in C[-1, 1] \cap \Delta^0(Y_s)$ we have*

$$E_n^{(0)}(f, Y_s)_\infty \leq C(Y_s)\omega_3^\varphi\left(f, \frac{1}{n}\right)_\infty, \quad n \geq 2.$$

Also, an immediate consequence of the above estimate and of the converse theorems in Ditzian–Totik [98] is the following: if $0 < \alpha < 3$ and $f \in C[-1, 1] \cap \Delta^0(Y_s)$, then

$$E_n(f)_\infty = O(n^{-\alpha}) \text{ iff } E_n^{(0)}(f, Y_s)_\infty = O(n^{-\alpha}).$$

(ii) (Hu–Kopotun–Yu [173]) There exists a constant $C = C(Y_s)$ *such that for any* $f \in C[-1,1] \cap \Delta^0(Y_s)$*, there exists a polynomial* $p_n \in \Pi_n \cap \Delta^0(Y_s)$ *satisfying*

$$|f(x) - p_n(x)| \le C(Y_s)\omega_3(f, \rho_n(x))_\infty, \quad n \ge 2,\ x \in [-1,1].$$

(iii) (Zhou [409]) Conversely, there is a function $f \in C^1[-1,1] \cap \Delta^0(\{0\})$ *such that*

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f,\{0\})_\infty}{\omega_4(f,\frac{1}{n})_\infty} = \infty.$$

(iv) (Hu–Kopotun–Yu [173]; see also Hu–Leviatan–Yu [176]) For any $f \in C^1[-1,1] \cap \Delta^0(Y_s)$ *we have*

$$E_n^{(0)}(f,Y_s)_\infty \le \frac{C(k,Y_s)}{n}\omega_k\left(f', \frac{1}{n}\right)_\infty, \quad n \ge k,$$

and there exists a polynomial $p_n \in \Pi_n \cap \Delta^0(Y_s)$ *such that*

$$|f(x) - p_n(x)| \le \frac{C(k,Y_s)}{n}\omega_k(f', \rho_n(x))_\infty, \quad n \ge k,\ x \in [-1,1].$$

Proof. We will prove here the estimate for $E_n^{(0)}(f,Y_s)_\infty$ in (iv). One reason why we chosen it is that it can be more easily extended to functions of two real variables, the extension proved in Section 2.6 (Theorem 2.6.6). The proof follows the ideas in Hu–Leviatan–Yu [176], pp. 213–217.

First we need two lemmas.

Lemma (A). (Hu–Leviatan–Yu [176]) *There exist absolute positive constants* A, B *and an odd and increasing (on* $[-1,1]$*) polynomial* $q_n(u)$ *of degree* $\le 2n$*, such that*

$$|q_n(u)| \le 1, \qquad \text{for} \quad |u| \le 1,$$
$$q_n'(u) \ge An, \quad \text{for} \quad |u| \le \frac{1}{n}, \quad \text{and } q_n'(u) \le \frac{B}{nu^2}, \quad \text{for} \quad \frac{1}{n} < |u| \le 1.$$

Proof of Lemma A. The polynomials are given by

$$q_n(u) := c_n \int_{-1}^{u} \frac{\sin^2((n/2)\arccos(1-t^2/2))}{\sin^2\frac{1}{2}\arccos(1-t^2/2)}dt - \frac{1}{2}, \quad -1 \le u \le 1,$$

where

$$c_n^{-1} := \int_{-1}^{1} \frac{\sin^2((n/2)\arccos(1-t^2/2))}{\sin^2\frac{1}{2}\arccos(1-t^2/2)}dt.$$

It is easy to see that $c_n \sim n^{-1}$, q_n is an odd and increasing (on $[-1,1]$) polynomial of degree $\le 2n$, satisfying the first estimate in the statement, and

$$q_n'(u) = c_n \frac{\sin^2((n/2)\arccos(1-u^2/2))}{\sin^2\frac{1}{2}\arccos(1-u^2/2)}, \qquad -1 \le u \le 1.$$

Now, for $|u| \le 1/n$, we get $(n/2)\arccos(1-u^2/2) \le \pi/2$, which by the well-known inequality $2t/\pi \le \sin t,\ 0 \le t \le \pi/2$, implies

$$\frac{\sin^2((n/2)\arccos(1-u^2/2))}{\sin^2 \frac{1}{2}\arccos(1-u^2/2)} \ge \left(\frac{(2/\pi)(n/2)\arccos(1-u^2/2)}{\frac{1}{2}\arccos(1-u^2/2)}\right)^2 = \frac{4}{\pi^2}n^2.$$

This shows that $q_n'(u) \ge An$.

For $1/n \le |u| \le 1$, it follows that

$$\frac{\sin^2((n/2)\arccos(1-u^2/2))}{\sin^2 \frac{1}{2}\arccos(1-u^2/2)} \le \frac{1}{((1/\pi)\arccos(1-u^2/2))^2} \le \frac{B'}{u^2},$$

i.e.,

$$q'(u) \le \frac{B}{nu^2},$$

which proves the lemma. □

Lemma (B). (see Hu–Leviatan–Yu [176]) *Let $f \in C^1[-1,1]$ and $k \ge 1$. For $y_0 := -1 < y_1 < \cdots < y_s < 1 =: y_{s+1}$, let us set $d = \min_{0\le i\le s}(y_{i+1}-y_i)$. For any $n \ge s$, a polynomial r_n of degree $\le n$ exists such that*

$$r_n(y_i) = f(y_i), \quad i = 1, 2, \ldots, s,$$

and

$$\|f - r_n\|_\infty \le Cn^{-1}\omega_k(f', 1/n)_\infty, \quad \|f' - r_n'\|_\infty \le C\omega_k(f', 1/n)_\infty,$$

where C depends on $s, k,$ and d.

Proof of Lemma B. By $f \in C^1[-1,1]$, a classical result of Gopengauz [161] says that there exists a polynomial $\tilde{r}_n$ of degree $\le n$ satisfying

$$\|f - \tilde{r}_n\|_\infty \le C_2(k)n^{-1}\omega_k(f', 1/n)_\infty, \quad \|f' - \tilde{r}_n'\|_\infty \le C_2(k)\omega_k(f', 1/n)_\infty.$$

Then $r_n(x) := \tilde{r}_n(x) + \tilde{h}_s(x)$, where $\tilde{h}_s$ is the polynomial of degree $s-1$ interpolating $f(x) - \tilde{r}_n(x)$ at $y_i, i = 1, 2, \ldots, s$, has the properties in the statement with $C > C_2(k)[1 + s(2/d)^{s-1}]$. □

Proof of Theorem 1.5.4, (iv). Fix $n > C_1 d^{-1}$, where C_1 is a constant that will be prescribed later. We will prove that the polynomial p_n of degree $\le 2sn$ given by

$$p_n(u) := r_n(u) + \varepsilon DCn^{-1}\omega_k(f', 1/n)_\infty \Pi_{i=1}^s q_n(u - y_i),$$

where q_n is given by Lemma A, $\varepsilon = \mathrm{sgn} f(u)$ for $u \in (y_s, 1)$ (C is the constant in Lemma B and D is a positive constant to be determined later), is copositive with f and satisfies the required estimate.

Note that the second term in the expression of $p_n(u)$ is copositive with f.

Because q_n is odd, increasing, and $q_n'(u) \geq An$ for $|u| \leq 1/n$, we get $|q_n(u)| \geq A, \quad |u| > 1/n$, and

$$|\Pi_{i=1}^{s} q_n(u - y_i)| \geq A^s, \quad u \notin \bigcup_{i=1}^{s} \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right].$$

If we take $D > A^{-s}$, then by the above estimate and by the estimate in Lemma B, we easily obtain

$$f(u)p_n(u) \geq 0, \quad u \notin \bigcup_{i=1}^{s} \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right]. \tag{1.0}$$

Then, from Lemma B it follows that $p_n(y_i) = f(y_i), \quad i = 1, 2, \ldots, s.$

Next, we prove that for $D > 2A^{-s}$, if f changes from $-$ to $+$ at y_i, then

$$f'(u) - p_n'(u) \leq 0, \quad u \in \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right] \tag{1.1}$$

and if f changes from $+$ to $-$ at y_i, then

$$f'(u) - p_n'(u) \geq 0, \quad u \in \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right]. \tag{1.2}$$

In this sense, we have

$$p_n'(u) = r_n'(u) + \varepsilon D C n^{-1} \omega_k(f', 1/n)_\infty \left(\Pi_{j=1}^{s} q_n(u - y_j) \right)'$$

and

$$\begin{aligned} \left(\Pi_{j=1}^{s} q_n(u - y_j) \right)' &= q_n'(u - y_i) \Pi^s{}_{j=1, j \neq i} q_n(u - y_j) \\ &\quad + q_n(u - y_i) \left[\Pi^s{}_{j=1, j \neq i} q_n(u - y_j) \right]' \\ &:= J_1(u) + J_2(u). \end{aligned}$$

By the second estimate in Lemma A and by $\left| \Pi_{i=1}^{s-1} q_n(u - y_i) \right| \geq A^{s-1}$, it follows that

$$|J_1(u)| \geq A n A^{s-1} = A^s n, \quad u \in \left[y_i - \frac{1}{n}, y_i + \frac{1}{n} \right].$$

Also, by Lemma A, for $u \in [y_i - 1/n, y_i + 1/n]$ and $n > 3/d$ we obtain

$$|J_2(u)| \leq \sum_{j=1, j \neq i}^{s} \frac{B}{n(u - y_j)^2} \leq \frac{B(s-1)}{n(2d/3)^2} \leq \frac{9Bs}{4nd^2}.$$

Defining $C_1 := \max\{3, 3\sqrt{Bs/2A^s}\}$, by $n > C_1 d^{-1}$ and by the above estimates for $|J_1(u)|$ and $|J_2(u)|$, we get

$$|J_2(u)| \leq \frac{9Bs}{4nC_1^2 n^{-2}} \leq \frac{1}{2}A^s n \quad \text{and} \quad |J_1(u)| - |J_2(u)| \geq \frac{1}{2}A^s n.$$

Therefore, if $n > C_1 d^{-1}$ and $D > 2A^{-s}$, then the second term in the expression of $p_n'(u)$, is in absolute value greater than $C\omega_k(f', 1/n)_\infty$. Thus by Lemma B, it follows that the sign of $f'(u) - p'(u)$ is exactly the sign of

$$-\varepsilon q_n'(u - y_i)\Pi^s{}_{j=1, j\neq i} q_n(u - y_j).$$

By

$$q_n'(u - y_i) \geq 0 \quad \text{and} \quad \mathrm{sgn}\left[\varepsilon \Pi_{j=1}^s q_n(u - y_j)\right] = \mathrm{sgn} f(u), u \in [-1, 1],$$

it follows that

$$\mathrm{sgn}\left[-\varepsilon q_n'(u - y_i)\Pi_{j=1, j\neq i}^s q_n(u - y_j)\right] = \mathrm{sgn}[-q_n(u - y_i)f(u)], u \in [-1, 1],$$

which implies (1.1) and (1.2).

Now, if $f(u) \leq 0$ for $u \in (y_i - 1/n, y_i)$ and $f(u) \geq 0$ for $u \in (y_i, y_i + 1/n)$, then by the mean value theorem, there is a number ξ between u and y_i such that

$$\begin{aligned} f(u) - p_n(u) &= [f(u) - p_n(u)] - [f(y_i) - p_n(y_i)] \\ &= (u - y_i)[f'(\xi) - p_n'(\xi)], \end{aligned} \tag{1.3}$$

which implies

$$f(u) - p_n(u) \geq 0, \quad u \in \left(y_i - \frac{1}{n}, y_i\right),$$

and

$$f(u) - p_n(u) \leq 0, \quad u \in \left(y_i, y_i + \frac{1}{n}\right).$$

Also, if $f(u) \geq 0$ for $u \in (y_i - 1/n, y_i)$ and $f(u) \leq 0$ for $u \in (y_i, y_i + 1/n)$, then by (1.2) and (1.3), we have

$$f(u) - p_n(u) \leq 0, \quad u \in \left(y_i - \frac{1}{n}, y_i\right),$$

and

$$f(u) - p_n(u) \geq 0, \quad u \in \left(y_i, y_i + \frac{1}{n}\right).$$

Hence, for $u \in \cup_{i=1}^s [y_i - 1/n, y_i + 1/n]$, we have either $p_n(u) \geq f(u) \geq 0$ or $p_n(u) \leq f(u) \leq 0$.

As a conclusion,

$$f(u)p_n(u) \geq 0, \quad u \in \cup_{i=1}^s [y_i - 1/n, y_i + 1/n],$$

which combined with (1.0) proves that p_n and f are copositive in $[-1, 1]$. □

Remarks. (1) Theorem 1.5.4, (i) in terms of the nonweighted modulus $\omega_3(f, 1/n)_\infty$ was proved by Hu–Yu [177].

(2) If in Theorem 1.5.4 (i) one replaces the third modulus of smoothness of f by the (first-order) modulus of continuity, then Leviatan [227] proved that the inequality holds with a constant $C = C(s)$ (so C does not depend on the points where the function changes sign, but depends on their number).

(3) The following result in trigonometric copositive approximation of 2π-periodic continuous functions (i.e., in the class denoted by $C_{2\pi}$) by trigonometric polynomials was proved in Pleshakov–Popov [305]: if $f \in C_{2\pi}$ changes sign at $Y_s : y_i \in [-\pi, \pi),\ i = 1, \ldots, 2s$, then for any $r \in \mathbb{N}$, there exists a trigonometric polynomial T_n of degree $\leq n$ that changes sign at the same points $y_i,\ i = 1, \ldots, 2s$ and satisfies $\|f - T_n\|_\infty \leq \frac{C(r,s)}{n^r}\omega_1(f^{(r)}; \pi/n)_\infty$, for all $n \geq N(Y_s, r)$, where $\| \cdot \|_\infty$ denotes the uniform norm in $C_{2\pi}$. Also, Pleshakov–Popov [306] proved the following zeros-preserving result: if $y_i \in [-\pi, \pi),\ i = 1, \ldots, 2s$ are distinct, then defining $y_i := y_{i+2s} + 2\pi, i \in \mathbb{Z}$, for any $f \in C_{2\pi}$ satisfying $f(y_i) = 0$, for all $i \in \mathbb{N}$, there exists a trigonometric polynomial T_n of degree $\leq n$, that has zeros at the points $y_i,\ i \in \mathbb{Z}$, and satisfies

$$\|f - T_n\|_\infty \leq C(s)\omega_1(f; \pi/n)_\infty \quad \text{for all } n \in \mathbb{N}.$$

1.5.4 L^p-Copositive Approximation, $0 < p < \infty$

For L^p-copositive polynomial approximation, $0 < p < \infty$, we present the following result.

Theorem 1.5.5. (Hu–Kopotun–Yu [172, 173]) *Let $0 < p < \infty$. For any $f \in L^p[-1,1] \cap \Delta^0(Y_s)$, we have*

$$E_n^{(0)}(f, Y_s)_p \leq C(Y_s)\omega_1^\varphi\left(f, \frac{1}{n}\right)_p, \quad n \geq 1,$$

where C depends on p too if $0 < p < 1$.

If $f \in W_p^1[-1,1] \cap \Delta^0(Y_s)$, with $1 \leq p < \infty$, then

$$E_n^{(0)}(f, Y_s)_p \leq \frac{C(Y_s)}{n}\omega_2^\varphi\left(f', \frac{1}{n}\right)_p, \quad n \geq 1.$$

In addition, if $f \in W_p^2[-1,1] \cap \Delta^0(Y_s)$, then

$$E_n^{(0)}(f, Y_s)_p \leq \frac{C(k, Y_s)}{n^2}\omega_k^\varphi\left(f'', \frac{1}{n}\right)_p, \quad n \geq k+1.$$

Conversely, for every $n \geq 1$, $0 < p < \infty$, and any $A > 0$ and $0 < \epsilon \leq 1$, there exists a function $f = f_{n,p,\epsilon,A} \in C^\infty[-1,1]$, satisfying $xf(x) \geq 0,\ -1 \leq x \leq 1$, such that for each $p_n \in \Pi_n$ with $p_n(0) \geq 0$,

$$\|f - p_n\|_{L^p[0,\varepsilon]} > A\omega_2(f, 1)_p.$$

Moreover, there exists a strictly increasing function $f = f_{n,p,\varepsilon,A} \in C^\infty[-1,1]$, *satisfying* $f(0) = 0$, *such that for each* $p_n \in \Pi_n$ *with* $p_n(0) = 0$, *and* $p_n(x) \geq 0$, $0 \leq x \leq \epsilon$, *we have*

$$\|f - p_n\|_{L^p[0,\epsilon]} > A\omega_3(f', 1)_p.$$

Proof. We will sketch the proof of the first estimate in the statement. Since for small n the theorem follows from the inequalities

$$\|f\|_p \leq C\omega_1^\varphi(f, 1)_p \leq C\omega_1\left(f, \frac{1}{n+1}\right)_p, n \leq C\delta^{-1},\ f \in \Delta^0(Y_s),$$

it suffices to prove it for sufficiently large n, let us say for $n \geq C\delta^{-1}$ (where $\delta = \min_{k=0,\dots,s}\{|y_k - y_{k+1}|\}$).

The proof is by induction on s, the number of sign changes, and follows the ideas in Hu–Kopotun–Yu [172], where the interested reader can find all the details. Thus, for $s = 0$ the result is true from Theorem 1.5.3. Assume now that the estimate is valid for $f \in L^p[-1,1] \cap \Delta^0(Y_{s-1})$, with $Y_{s-1} = \{y_1, \dots, y_{s-1}\}$. From Lemma 3.5, p. 328 in Hu–Kopotun–Yu [172], there exists a piecewise constant spline $R_n(f) \in \Delta^0(Y_s)$ satisfying the estimate $\|R_n(f) - f\|_p \leq C\omega_1^\varphi(f, \frac{1}{n})_p$. If we define now $S_n(f)(x) = R_n(f)(x)\text{sgn}(x - y_s)$, then $S_n(f) \in L^p[-1,1] \cap \Delta^0(Y_{s-1})$, and by the induction assumption, there exists a polynomial $Q_n(f) \in \Pi_n \cap \Delta^0(Y_{s-1})$ satisfying

$$\|S_n(f) - Q_n(f)\|_p \leq C\omega_1^\varphi\left(S_n(f), \frac{1}{n}\right)_p.$$

The desired polynomials are defined by $P_n(f)(x) = Q_n(f)(x)T_n(y_s)(x)$, where $T_n(y_s)(x)$ is the suitable increasing polynomial copositive with $\text{sgn}(x - y_s)$ given by Lemma 3.4, p. 327, in Hu–Kopotun–Yu [172]. In the rest of the proof it is shown that $\|f - P_n(f)\|_p \leq C\omega_1^\varphi(f, \frac{1}{n})_p$, which proves the first estimate in the statement. □

1.5.5 Copositive Approximation with Modified Weighted Moduli of Smoothness

First we recall some notations For $s \in \mathbb{N}$, let $\mathbb{Y}_s$ be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points, such that $-1 < y_s < \cdots < y_1 < 1$. For $Y_s \in \mathbb{Y}_s$ we define $\Pi(x, Y_s) := \prod_{i=1}^s (x - y_i)$.

Let $\Delta^0(Y_s)$ be the set of all functions $f : [-1,1] \to \mathbb{R}$ that change sign exactly at the points $y_i \in Y_s$ and that are nonnegative in $(y_1, 1)$. Obviously, $f \in \Delta^0(Y_s)$ is equivalent to $f(x)\Pi(x, Y_s) \geq 0,\ -1 < x < 1$.

Defining $\mathbf{Y} := \cup_{\mathbf{s}}\mathbb{Y}_{\mathbf{s}}$, we say that a collection $Y \in \mathbf{Y}$ is s-admissible for f if $Y \in \mathbb{Y}_s$ and $f \in \Delta^0(Y)$. Denote the set of all s-admissible collections by $A_s(f)$, and if $A_s(f)$ is nonempty, we write $f \in \Delta^{(0,s)}$.

For $Y \in \mathbf{Y}$ and $f \in C([-1,1])$ we define

$$E_n^{(0)}(f,Y)_\infty := \inf_{p_n \in \Pi_n \cap \Delta^0(Y)} \|f - p_n\|_\infty,$$

and for $f \in \Delta^{(0,s)}$, we put

$$E_n^{(0,s)}(f)_\infty := \sup\{E_n^{(0)}(f,Y); Y \in A_s(f)\}.$$

Taking into account also the notation in Definition 1.1.3 and its Remarks (1) and (2), we can state the main results as follows.

Theorem 1.5.6. (Smazhenko [359])

(i) If $f \in \Delta^{(0,s)} \cap \mathbb{B}^r$ and $(s,r) = (2,3)$ or $(s,r) = (1,2)$, or $r > 2s$, or $r = 1$, then

$$E_n^{(0,s)}(f)_\infty \le c(r,s) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1;$$

(ii) If $f \in \Delta^{(0,s)} \cap C_\varphi^r$ and either $r > 2s$, or $r + k = 2$ and $s = 1$, or $r = 0$ and $k = 1$, then

$$E_n^{(0,s)}(f)_\infty \le c(r,s,k)\omega_{k,r}^\varphi(f^{(r)}; 1/n), \quad n \ge k + r - 1;$$

(iii) If $\Delta^{(0,s)} \cap \mathbb{B}^r$ with $r \ge 1$ and $Y \in A_s(f)$, then

$$E_n^{(0)}(f,Y)_\infty \le c(r,Y) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1,$$

and

$$E_n^{(0)}(f,Y)_\infty \le c(r,s) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge N(r,Y);$$

(iv) If $f \in \Delta^{(0,s)} \cap C_\varphi^r$, $Y \in A_s(f)$, and either $r \ge 1$ or $r = 0$ and $k \le 3$, then

$$E_n^{(0)}(f,Y)_\infty \le c(r,k,Y)\omega_{k,r}^\varphi(f^{(r)}; 1/n), \quad n \ge k + r - 1,$$

and

$$E_n^{(0)}(f,Y)_\infty \le c(r,k,s)\omega_{k,r}^\varphi(f^{(r)}; 1/n), \quad n \ge N(k,r,Y);$$

(v) Writing $e_n^{(0,s)}(f)_\infty = \inf\{E_n^{(0)}(f,Y)_\infty; Y \in A_s(f)\}$, for every $A > 0$, $s \ge 2$ and $2 \le r \le 2s$ (excepting the case $(s,r) = (2,3)$), and any $n \in \mathbb{N}$, there exists $f = f_{s,r,n,A} \in \Delta^{(0,s)} \cap \mathbb{B}^r$ satisfying

$$E_n^{(0,s)}(f)_\infty \ge e^{(0,s)}(f)_\infty \ge A\|\varphi^r f^{(r)}\|_\infty > 0;$$

(vi) Supposing that $k, s \ge 1$ and $r \le 2s$ (excepting the cases $r = 0$, $k = 1$, and $r + k = 2$, $s = 1$), for any $A > 0$ and $n \in \mathbb{N}$, there is $f = f_{s,r,k,n,A} \in \Delta^{(0,s)} \cap C_\varphi^r$ satisfying

$$E_n^{(0,s)}(f)_\infty \ge e^{(0,s)}(f)_\infty \ge A\omega_{k,r}^\varphi(f^{(r)}; 1).$$

1.5.6 Generalizations

In this subsection we present some interesting generalizations of the concepts of positive and copositive polynomial approximation: almost positive, strong/weak almost positive, almost copositive, nearly copositive, strong/weak almost copositive, and almost/nearly intertwining polynomial approximation. Intertwining approximations are related to both copositive approximation and one-sided approximation and were introduced and studied in Hu–Kopotun–Yu [173, 174]. The main idea is to relax the preservation of the positivity/copositivity from the whole interval $[-1,1]$ to a major portion of that interval, but excepting small neighborhoods of some points.

We first introduce the following notation and concepts. Set $Y_s = \{y_j, j = 1,\dots,s\}$, where $y_{s+1} := -1 < y_s < \cdots < y_1 < 1 := y_0$, $\rho_n(x) = (1-x^2)^{1/2}/n + 1/n^2$, and for $\varepsilon \geq 0$, let $J_j(n,\varepsilon) = [y_j - \rho_n(y_j)n^{\varepsilon}, y_j + \rho_n(y_j)n^{\varepsilon}] \cap [-1,1]$, $j = 0,1,\dots,s+1$, $O_n(Y_s,\varepsilon) = \cup_{j=1}^{s} J_j(n,\varepsilon)$, $O_n^*(Y_s,\varepsilon) = \cup_{j=0}^{s+1} J_j(n,\varepsilon)$. If $\varepsilon = 0$ then we write $J_j = J_j(n,0)$, $O_n(Y_s) = O_n(Y_s,0)$, $O_n^*(Y_s) = O_n^*(Y_s,0)$. The L^p-norm is defined by $\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p dx\right)^{1/p}$ for $1 \leq p < \infty$, and $\|f\|_\infty := \|f\|$ is the uniform norm. Also, denote by Π_n the class of all real polynomials of degree $\leq n$.

Definition 1.5.7. (i) (Hu–Kopotun–Yu [173]) The best intertwining approximation by polynomials of degree $\leq n$, for $f \in L^p[-1,1]$, $1 \leq p \leq \infty$, with respect to the set Y_s is given by

$$\tilde{E}_n(f, Y_s)_p = \inf\{\|P - Q\|_p; P, Q \in \Pi_n, P - f \in \Delta^0(Y_s), f - Q \in \Delta^0(Y_s)\}.$$

We call (P, Q) an intertwining pair of polynomials for f with respect to Y_s if $P - f$ and $f - Q$ belong to $\Delta^0(Y_s)$.

For $s = 0$ we have $Y_0 = \emptyset$, and $\tilde{E}_n(f, Y_s)$ becomes the best one-sided approximation by polynomials of degree $\leq n$ for f given by

$$\tilde{E}_n(f)_p = \inf\{\|P - Q\|_p; P, Q \in \Pi_n, P(x) \geq f(x) \geq Q(x), \forall x \in [-1,1]\}.$$

(ii) (Hu–Kopotun–Yu [174]) With respect to Y_s, the functions f and g are called almost copositive on $[-1,1]$ if $f(x)g(x) \geq 0$, for all $x \in [-1,1] \setminus O_n^*(Y_s)$; *strongly (weakly) almost copositive* on $[-1,1]$ if $f(x)g(x) \geq 0$ for all $x \in [-1,1] \setminus O_n^*(Y_s,\varepsilon)$ with $\varepsilon < 0$ ($\varepsilon > 0$, respectively);

Note that for $\varepsilon = -\infty$, the strongly copositive functions are exactly copositive on $[-1,1]$;

Define the class of functions

$$(\varepsilon - \text{alm } \Delta)_n^0(Y_s) = \{f; (-1)^k f(x) \geq 0, x \in [-1,1] \setminus O_n^*(Y_s,\varepsilon), k = 0,\dots,s\}.$$

For $s = 0$, it becomes

$$\begin{aligned}&(\varepsilon - \text{ alm } \Delta)_n^0(Y_0)\\ &\quad := (\varepsilon - \text{ alm } \Delta)_n^0 = \{f; f(x) \geq 0, x \in [-1 + n^{-2+\varepsilon}, 1 - n^{-2+\varepsilon}]\},\end{aligned}$$

and it is called the set of all strongly (weakly) almost nonnegative functions on $[-1,1]$ if $\varepsilon < 0$ ($\varepsilon > 0$, respectively). If $\varepsilon = 0$, we omit the letter ε in the notation and use $(\text{alm } \Delta)_n^0(Y_s)$ and $(\text{alm } \Delta)_n^0$. The latter is exactly the set of almost nonnegative functions on $[-1,1]$, while for $\varepsilon = -\infty$, strongly almost nonnegative functions are exactly the nonnegative functions on $[-1,1]$.

The best almost positive approximation by polynomials of degree $\le n$, for $f \in L^p[-1,1]$, $1 \le p \le \infty$, is given by

$$E_n^{(0)}(f, \text{alm } Y_0)_p = \inf\{\|f - P\|_p; P \in \Pi_n \cap (\text{alm } \Delta)_n^0\}.$$

Similarly,

$$E_n^{(0)}(f, \varepsilon - \text{ alm } Y_0)_p = \inf\{\|f - P\|_p; P \in \Pi_n \cap (\varepsilon - \text{ alm } \Delta)_n^0\},$$

denotes the best strongly (weakly) almost positive approximation by polynomials of degree $\le n$ if $\varepsilon < 0$ ($\varepsilon > 0$, respectively).

The best almost copositive approximation by polynomials of degree $\le n$, for $f \in L^p[-1,1] \cap \Delta^0(Y_s)$, $1 \le p \le \infty$, is given by

$$E_n^{(0)}(f, \text{ alm } Y_s)_p = \inf\{\|f - P\|_p; P \in \Pi_n \cap (\text{alm } \Delta)_n^0(Y_s)\}.$$

Similarly,

$$E_n^{(0)}(f, \varepsilon - \text{ alm } Y_s)_p = \inf\{\|f - P\|_p; P \in \Pi_n \cap (\varepsilon - \text{ alm } \Delta)_n^0(Y_s)\}$$

denotes the best strongly (weakly) almost copositive approximation by polynomials of degree $\le n$ if $\varepsilon < 0$ ($\varepsilon > 0$, respectively).

The best almost intertwining approximation by polynomials of degree $\le n$, for $f \in L^p[-1,1]$, $1 \le p \le \infty$, with respect to the set Y_s is given by

$$\begin{aligned}&\tilde{E}_n(f, \text{ alm } Y_s)_p \\ &\quad = \inf\{\|P - f\|_p + \|f - Q\|_p; P, Q \in \Pi_n, (-1)^j(P(x) - f(x)) \ge 0 \text{ and} \\ &\qquad (-1)^j(f(x) - Q(x)) \ge 0 \text{ if } x \in [y_{j+1}, y_j] \setminus O_n(Y_s), j = 0, \dots, s\}.\end{aligned}$$

We call (P, Q) an almost intertwining pair of polynomials for f with respect to Y_s if P and Q satisfy the restrictions in the above definition.

The best nearly intertwining approximation by polynomials of degree $\le n$, for $f \in L^p[-1,1]$, $1 \le p \le \infty$, with respect to the set Y_s is given by

$$\begin{aligned}&\tilde{E}_n(f, \text{ nearly } Y_s)_p \\ &\quad = inf\{\|P - Q\|_p; P, Q \in \Pi_n, P - f \in \Delta^0(\tilde{Y}_s) \text{ and } f - Q \in \Delta^0(\tilde{Y}_s)\},\end{aligned}$$

where

$$\begin{aligned}\tilde{Y}_s = \{\tilde{y}_s, \dots, \tilde{y}_1; -1 = y_{s+1} < \tilde{y}_s < \dots < \tilde{y}_1 < y_0 = 1, \\ \text{and } |\tilde{y}_j - y_j| \le \rho_n(y_j), j = 1, \dots, s\}.\end{aligned}$$

We call (P, Q) a nearly intertwining pair of polynomials for f with respect to Y_s if $P - f$ and $f - Q$ belong to $\Delta^0(\tilde{Y}_s)$.

Remark. Between the above best-approximation quantities the following relationships hold:

$$\tilde{E}_n(f, \text{alm } Y_s)_p \leq \tilde{E}_n(f, \text{ nearly } Y_s)_p \leq \tilde{E}_n(f, Y_s)_p,$$

and for $f \in \Delta^0(Y_s)$,

$$\tilde{E}_n^{(0)}(f, \text{ alm } Y_s)_p \leq \tilde{E}_n(f, \text{ alm } Y_s)_p,$$
$$\tilde{E}_n^{(0)}(f, \text{ alm } Y_s)_p \leq \tilde{E}_n^{(0)}(f, Y_s)_p.$$

All the constrained best-approximation quantities in Definition 1.5.7 can be estimated in terms of various moduli of smoothness and for various additional smoothness properties for f. The next theorem summarizes a few results. For their proofs and for other estimates together with their proofs, the interested reader can consult the paper of Hu–Kopotun–Yu [174] (see also the survey Hu–Yu [178]).

Theorem 1.5.8. *(i) (see Hu–Kopotun–Yu [173]) (Intertwining approximation) If* $f \in W_p^1 = \{f; f$ *is absolutely continuous and* $f' \in L^p[-1,1]\}$, $1 \leq p < \infty$, *then* $\tilde{E}_n(f, Y_s)_p \leq Cn^{-1}\tau_k(f', n^{-1})_p$;
If $f \in C^1[-1,1]$ *then* $\tilde{E}_n(f, Y_s)_\infty \leq Cn^{-1}\omega_k^\varphi(f', n^{-1})_\infty$;

(ii) (Hu–Kopotun–Yu [174]) (Almost positive approximation) If we suppose $f \in L^p[-1,1] \cap \Delta^0$ *and* $1 \leq p < \infty$, *then*

$$E_n^{(0)}(f, \text{ alm } Y_0)_p \leq C\omega_2^\varphi(f, n^{-1})_p,$$

and ω_2^φ *cannot be replaced by* ω_3^φ;

(iii) (Hu–Kopotun–Yu [174]) (Strongly almost positive approximation, i.e., $\varepsilon < 0$*) If* $f \in L^p[-1,1] \cap \Delta^0$ *and* $1 \leq p < \infty$, *then*

$$E_n^{(0)}(f, \varepsilon - \text{alm } Y_0)_p \leq C\omega_1^\varphi(f, n^{-1})_p,$$

and $\omega_1^\varphi(f, 1/n)_p$ *cannot be replaced by* $\omega_2(f, 1/n)_p$;

(iv) (Hu–Kopotun–Yu [174]) (Weakly almost positive approximation, i.e., $0 < \varepsilon < 2$*) If* $f \in L^p[-1,1] \cap \Delta^0$ *and* $1 \leq p < \infty$, *then*

$$E_n^{(0)}(f, \varepsilon - \text{alm } Y_0)_p \leq C\omega_2^\varphi(f, n^{-1})_p,$$

and $\omega_2^\varphi(f, 1/n)_p$ *cannot be replaced by* $\omega_3(f, 1/n)_p$;

(v) (Hu–Kopotun–Yu [174]) (Almost copositive approximation) If $f \in L^p[-1,1] \cap \Delta^0(Y_s)$ *and* $1 \leq p < \infty$, *then*

$$E_n^{(0)}(f, \text{ alm } Y_s)_p \leq C\omega_2^\varphi(f, n^{-1})_p,$$

and $\omega_2^\varphi(f, 1/n)_p$ *cannot be replaced by* $\omega_3(f, 1/n)_p$ *if* $1 < p < \infty$ *and by* $\omega_4(f, 1/n)_p$ *if* $p = 1$;
If $f \in W_p^1 \cap \Delta^0(Y_s)$ *then*

$$E_n^{(0)}(f, \text{ alm } Y_s)_\infty \leq Cn^{-1}\omega_k^\varphi(f', n^{-1})_\infty;$$

(vi) (Hu–Kopotun–Yu [174]) (Almost intertwining approximation) If $f \in L^p[-1,1]$, $1 \le p < \infty$, *then*

$$\tilde{E}_n(f,\ \mathrm{alm}\ Y_s)_p \le C\tau_k(f, n^{-1})_p;$$

If $f \in W_p^1$ *then*

$$\tilde{E}_n(f,\ \mathrm{alm}\ Y_s)_\infty \le Cn^{-1}\omega_k^\varphi(f', n^{-1})_\infty;$$

(vii) (Hu–Kopotun–Yu [174]) (Nearly intertwining approximation) If $f \in W_p^1$, $1 \le p < \infty$, *then* $\tilde{E}_n(f,\ \textit{nearly}\ Y_s)_p \le Cn^{-1}\omega_k^\varphi(f', n^{-1})_p$.

Remark. From their proofs it follows that the above-presented methods in polynomial copositive approximation are nonlinear, such that even if two continuous functions $f, g \in C[-1,1]$ have the same points where the signs change, Y_s, and both are of the same sign on each subinterval, the copositive approximating polynomials P_n do not satisfy $P_n(f+g) = P_n(f) + P_n(g)$.

It is easy to show that $\Delta^0(Y_s)$ is a convex cone, i.e., $f, g \in \Delta^0(Y_s)$ and $\alpha \in \mathbb{R}_+$ implies $f + g \in \Delta^0(Y_s)$ and $\alpha f \in \Delta^0(Y_s)$.

Suggested by the proof of Theorem 1.2.1, we easily can construct a polynomial copositive with $f \in \Delta^0(Y_s)$ that is an additive and positive homogeneous operator on $\Delta^0(Y_s) \cap C^1[-1,1]$. In this sense we present our next result.

Theorem 1.5.9. *If* $f \in \Delta^0(Y_s)$ *is continuously differentiable in the interval* $[-1,1]$, *then a sequence of polynomials* $(P_n)_n$ *can be constructed such that degree*$(P_n) \le n+s$ *and for any* $\varepsilon > 0$, *there is* n_0 *with the properties*

$$\|f - P_n\|_\infty < \varepsilon, \qquad P_n(x)f(x) \ge 0, \quad \forall n \ge n_0, \quad x \in [-1,1],$$

where $P_n(\alpha g + \beta h) = \alpha P_n(g) + \beta P_n(h)$, *for all* $n \in \mathbb{N}$, $g, h \in \Delta^0(Y_s) \cap C^1[-1,1]$, $\alpha, \beta \ge 0$.

Recalling the notation $\Pi(x, Y_s) = \Pi_{i=1}^s (x - y_i)$, *the error estimate can be expressed by*

$$\|P_n(f) - f\|_\infty \le C(Y_s)\omega_2\left(\frac{f}{\Pi(\cdot, Y_s)}; \frac{1}{n}\right)_\infty,$$

for all $n \in \mathbb{N}$.

Proof. From the differentiability hypothesis, it easily follows that $F(x) = \frac{f(x)}{\Pi(x,Y_s)}$ is continuous (by extension) on $[-1,1]$. Also, by simple reasoning we get that $F(x) \ge 0$, for all $x \in [-1,1]$.

Now define $P_n(f)(x) := L_n(F)(x) \cdot \Pi(x, Y_s)$, where L_n, $n \in \mathbb{N}$, is a sequence of positive linear polynomial operators on $C[-1,1]$, satisfying degree $(L_n(f)) \le n$ and

$$\|L_n(f) - f\|_\infty \le C\omega_2(f; 1/n)_\infty,\ n = 1, 2, \ldots.$$

The conclusions in the statement are immediate. □

Remark. Obviously, the degree of $P_n(f)$ can be chosen $\leq n$, by considering that the degree of $L_n(f)$ is $\leq n-s$. But the shortcoming of the method in Theorem 1.5.9 is that it does not produce a Jackson-type estimate in terms of at least $\omega_1(f,1/n)_\infty$, or in terms of what we would expect, i.e., $\omega_2(f,1/n)_\infty$. The solution to this shortcoming remains an open question.

1.6 Monotone and Comonotone Polynomial Approximation

This section contains the most important results in monotone and comonotone polynomial approximation. As in the previous section, the proofs of the most results are omitted as being too technical. However, to give the reader a better look at the topic, for some proofs the main ideas will be given.

After Shisha's result in 1965, in the evolution of this topic the following result of Lorentz–Zeller in 1968 is an important step.

Theorem 1.6.1. (Lorentz–Zeller [250]) *If $f:[-1,1]\to\mathbb{R}$ is increasing on $[-1,1]$, then there exists a sequence of polynomials $(P_n(x))_{n\in\mathbb{N}}$ such that degree $(P_n)\leq n$, $P_n(x)$ is increasing on $[-1,1]$ for all $n\in\mathbb{N}$, and*

$$|f(x)-P_n(x)|\leq C\omega_1(f;\rho_n(x))_\infty,\ |x|\leq 1,\ n\in\mathbb{N},$$

where C is independent of n, f, and x, and $\rho_n(x)=(1-x^2)^{1/2}/n+1/n^2$.

To prove this theorem, first we need to prove the following trigonometric approximation lemma. Recall that a continuous 2π-periodic function on $[-\pi,\pi]$ is called *bell-shaped* if it is even and decreases on $[0,\pi]$.

Lemma 1.6.2. (Lorentz–Zeller [250]) *For any bell-shaped function f, there exists a bell-shaped trigonometric polynomial $T_n(f)$ such that*

$$|f(x)-T_n(f)(x)|\leq C\omega_1(f;1/n)_\infty,\forall x\in\mathbb{R},n\in\mathbb{N}.$$

Proof of Lemma 1.6.2. Let use denote the Jackson integral of f by

$$J_n(f)(x)=\int_{-\pi}^{\pi}K_n(x-t)f(t)dt,$$

where $K_n(t)=\lambda_n^{-1}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4$, $\int_{-\pi}^{\pi}K_n(t)dt=1$.

Let us define $L(f)(t):=L(t)=f\left(\frac{\pi k}{n}\right):=c_n$ for $\frac{\pi k}{n}\leq t<\frac{\pi(k+1)}{n}$, $k=0,1,\ldots,n$ and $L(f)(t)(t)=L(-t)$ for $t\in[-\pi,0]$. Obviously $L(f)(t)$ is a piecewise constant function, L is even, and $|f(t)-L(t)|\leq\omega_1(f;\pi/n)$. Writing $T_n(f)(t)=J_n(L)(t)$, obviously $T_n(f)$ is even and from the inequalities

$$\begin{aligned}|f(t)-T_n(f)(t)|&\leq|f(t)-J_n(f)(t)|+|J_n(f)(t)-J_n(L)(t)|\\&\leq C_1\omega_1(f;1/n)_\infty+\|f-L\|_\infty\leq C_2\omega_1(f;1/n)_\infty,\end{aligned}$$

it remains to show that $T_n(f)(x)$ is decreasing on $(0,\pi)$.

Define $b_k, k = 0, 1, \dots n-1$ by the recurrent relations $c_k = b_k + \dots + b_{n-1},\ k = 0, \dots, n-1$. We have $b_k \geq 0$ and

$$T_n(f)(x) = \sum_{k=0}^{n-1} b_k \int_{-\pi(k+1)/n}^{\pi(k+1)/n} K_n(x-t)dt$$
$$= \sum_{k=1}^{n} b_{k-1} \int_{-\pi k/n}^{\pi k/n} K_n(t)dt.$$

Therefore, it suffices to prove that the functions

$$\Psi_{k,n}(x) = \int_{-\pi k/n}^{\pi k/n} K_n(x-t)dt = \int_{x-\pi k/n}^{x+\pi k/n} K_n(t)dt$$

are decreasing on $(0, \pi)$.

We get

$$\Psi'_{k,n}(x) = K_n\left(x + \frac{\pi k}{n}\right) - K_n\left(x - \frac{\pi k}{n}\right)$$
$$= \lambda_n^{-1} \sin^4\left(\frac{\pi x + \pi k}{2}\right)\left[\frac{1}{\sin^4((x+\pi k/n)/2)} - \frac{1}{\sin^4((x-\pi k/n)/2)}\right] \leq 0,$$

which follows from the inequality $\sin(a+b) \geq |\sin(a-b)|$, for all $0 \leq a, b \leq \pi/2$, which is a consequence of $\sin(a+b) - \sin(a-b) = 2\sin(b)\cos(a) \geq 0$, and $\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b) \geq 0$.

Proof of Theorem 1.6.1. Defining $F(t) = f(\cos t)$, F is obviously bell-shaped. By Lorentz [248], p. 68, we have

$$|f(\cos t) - J_n(F)(t)| \leq C_1\omega_1(F; \alpha_n(t))_\infty \leq \omega_1(f; \alpha_n(t))_\infty,$$

where $\alpha_n(t) = \max\{\frac{|\sin t|}{n}, \frac{1}{n^2}\} \leq \rho_n(t)$. Set $T_n(F)(t) = J_n(F)(t)$.

Now, if we take $L(t) = L(F)(t)$, then as in the proof of Lemma 1.6.2 it follows that $J_n(L)(t)$ is bell-shaped and

$$|F(t) - L(t)| \leq \omega_1(F; h)_\infty \leq \omega_1(f; h)_\infty, h = \max|\cos(t_1) - \cos(t)|,$$

where (if, for example, $t > 0$), $t_1 = \pi k/n$, $t_1 \leq t < t_1 + \pi/n$ for some k.

Therefore,

$$h \leq 2\sin\left(\frac{k\pi}{2n}\right)\sin\left(\frac{k\pi}{2n}\right) \leq C\frac{1}{n}\sin(t) \leq C\alpha_n(t).$$

Finally, collecting all the above estimates, we obtain

$$|f(\cos t) - T_n(L)(t)| \leq |f(\cos t) - J_n(F)(t)| + |J_n(F)(t) - J_n(L)(t)|$$
$$\leq C_1\omega_1(f; \alpha_n(t))_\infty + \|F - L\|_\infty \leq C_2\omega_1(f; \alpha_n(t))_\infty.$$

Making now the substitution $x = \cos(t)$ and taking into account that $\alpha_n(t) \leq \rho_n(t)$, we arrive at the desired estimate. $\square$

After the publication of the above theorem of Lorentz–Zeller, many improvements between 1970 and 1980 have been made in monotone approximation first; for example, by Beatson, DeVore, Leviatan, Newman, Shvedov, Yu, and then, by many other researchers. In what follows we present the most important results in this topic.

As in the case of copositive approximation, first we need some useful notation. Let $\mathbb{Y}_s$ be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points, such that $y_{s+1} := -1 < y_s < \cdots < y_1 < 1 =: y_0$, where for $s = 0, Y_0 = \emptyset$. For $Y_s \in \mathbb{Y}_s$ we define

$$\Pi(x, Y_s) := \prod_{i=1}^{s} (x - y_i),$$

where the empty product is 1. Let $\Delta^1(Y_s)$ be the set of functions f that change monotonicity at the points $y_i \in Y_s$ and that are nondecreasing in $(y_1, 1)$ that is, f is nondecreasing in the intervals (y_{2j+1}, y_{2j}) and it is nonincreasing in (y_{2j}, y_{2j-1}). In particular, if $s = 0$, then f is nondecreasing in $[-1, 1]$ and we will write $f \in \Delta^1$. Moreover, if f is differentiable in $(-1, 1)$, then

$$f \in \Delta^1(Y_s) \quad \text{iff} \quad f'(x)\Pi(x, Y_s) \geq 0, \quad -1 < x < 1.$$

Now for $f \in \Delta^1(Y_s) \cap L^p[-1, 1]$, we denote by

$$E_n^{(1)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^1(Y_s)} \|f - p_n\|_p$$

the best comonotone approximation of f by polynomials of degree $\leq n$. If $Y_0 = \emptyset$, then we write $E_n^{(1)}(f)_p := E_n^{(1)}(f, \emptyset)_p$, and it will be called the best monotone approximation.

Remark. Clearly, f can belong to $\Delta^0(Y_{s_0}^0) \cap \Delta^1(Y_{s_1}^1)$, with $Y_{s_0}^0 \neq Y_{s_1}^1$ and $s_0 \neq s_1$. Thus, for such a function the quantity

$$e_n^{(\nu,s)}(f)_p := \inf E_n^{(\nu)}(f, Y_s)_p$$

is useful, where the infimum is taken over all sets Y_s of s points in which f changes its sign (corresponding to $\nu = 0$) or its monotonicity (corresponding to $\nu = 1$), respectively. This quantity is useful in negative results in comonotone approximation.

First let us make some simple observations on best monotone approximation. Supposing that $f \in \mathbb{C}^1[-1, 1]$ is nondecreasing, obviously $f' \geq 0$. By the considerations in the previous section, for any $n \geq 1$, a nonnegative $p_{n-1} \in \Pi_{n-1}$ exists satisfying

$$\|f' - p_{n-1}\|_\infty \leq 2E_{n-1}(f')_\infty.$$

Defining $P_n(x) := \int_0^x p_{n-1}(t)dt + f(0)$, we get that P_n is nondecreasing and

$$\|f - P_n\|_\infty \leq 2E_{n-1}(f')_\infty.$$

This implies

$$E_n^{(1)}(f)_\infty \le 2E_{n-1}(f')_\infty \quad n \ge 1.$$

It is clear that in this estimate we have a loss of order n with respect to the unconstrained approximation. Indeed, this follows by recalling that if $f \in W_p^1[-1,1],\ 1 \le p \le \infty$, then in unconstrained approximation we have

$$E_n(f)_p \le \frac{C}{n} E_{n-1}(f')_p, \quad n \ge 1,$$

where $C = C(p)$ is an absolute constant and $1 \le p$.

Some of this loss can be retrieved by proving Jackson-type estimates that are analogous to those in unconstrained approximation. However, Shevchuk [350] and Leviatan–Shevchuk [234] have proved that there exists a constant $C > 0$ such that for any $n \ge 1$, an $f = f_n \in \mathbb{C}^1[-1,1] \cap \Delta^1$ exists, satisfying

$$E_n^{(1)}(f)_\infty \ge C E_{n-1}(f')_\infty > 0,$$

which shows that the inequality $E_n^{(1)}(f)_\infty \le 2E_{n-1}(f')_\infty$ cannot be improved.

If $0 < p < \infty$, the situation is even more pronounced, since in this case Kopotun [208] proved that for any $n \ge 1$ and $A > 0$, there exists a function $f = f_{p,n,A} \in C^\infty[-1,1] \cap \Delta^1$ satisfying

$$E_n^{(1)}(f)_p \ge A E_{n-1}(f')_p.$$

1.6.1 L^p-Monotone Approximation, $0 < p \le \infty$

The following Jackson-type estimate recovers the previously mentioned loss.

Theorem 1.6.3. (see Shvedov [355], Yu [407], Leviatan [228], DeVore–Leviatan–Yu [90], DeVore–Leviatan [88]) *If $f \in L^p[-1,1] \cap \Delta^1$, $0 < p \le \infty$, then for every $n \ge 1$, we have*

$$E^{(1)}(f)_p \le C\omega_2^\varphi\left(f, \frac{1}{n}\right)_p,$$

where $C = C(p)$, the dependence on p being important only for $p \to 0$.

Conversely, if $k \ge 3$, then for every $A > 0$ and $n \ge 1$, there is an $f = f_{p,n,A} \in L^p[-1,1] \cap \Delta^1$ satisfying

$$E^{(1)}(f)_p \ge A\omega_k(f,1)_p > 0.$$

Proof. The sketch of proof for the first estimate will be presented in three separated cases: 1) $p = \infty$; 2) $0 < p < 1$; 3) $1 \le p < \infty$.

Case 1. (Leviatan [228]). The proof is based on the ideas of constructions in DeVore–Yu [86], i.e., first one approximates f by a piecewise linear function

$S_n(f)$ that interpolates f at a set of certain $2n+1$ points $-1 = \xi_{-n} < \xi_{-n+1} < \cdots < \xi_n = 1$, $n \in \mathbb{N}$, with the modifications in Leviatan [228], in the sense that one considers a subset of $n+1$ points. Consider the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin\, t/2} \right)^8, \qquad \int_{-\pi}^{\pi} J_n(t)dt = 1.$$

For $t_j = j\pi/n, \quad j = 0, \ldots, n$, and $x = \cos t$, let us define

$$T_j(t) = \int_{t-t_j}^{t+t_j} J_n(u)du, \quad \tau_j(x) = T_{n-j}(t), R_j(x) = \int_{-1}^{x} \tau_j(u)du, \quad j = 0, \ldots, n,$$

and the points ξ_j given by the equations $1 - \xi_j = R_j(1)$, $j = 0, \ldots, n$.

Since $T_{n-j} - T_{n-(j+1)} \geq 0$, we get $\tau_j - \tau_{j+1} \geq 0$ and that $R_j - R_{j+1}$ is increasing for $j = 0, 1, \ldots, n-1$, which implies $-1 = \xi_0 < \cdots < \xi_n = 1$.

The piecewise linear interpolant $S_n(f)$ is defined by

$$S_n(f)(x) = f(-1) + s_0(1+x) + \sum_{j=1}^{n-1} (s_j - s_{j-1})\varphi_j(x),$$

where

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \quad j = 0, \ldots, n-1,$$

and $\varphi_j(x) = (x - \xi_j)_+$.

In order to obtain the required polynomials, each $\varphi_j(x)$ is replaced by a sufficiently good approximation polynomial, i.e., by $R_j(x)$, which gives

$$\begin{aligned} P_n(f)(x) &= f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j(x) \\ &= f(-1) + \sum_{j=0}^{n-1} s_j (R_j(x) - R_{j+1}(x)). \end{aligned}$$

Note that $P_n(f) \in \Pi_{4n}$ is a linear operator with respect to f. Since each function $R_j - R_{j+1}$ is nondecreasing, it follows that $P_n(f)$ is nondecreasing if all $s_j \geq 0$, which is exactly the case when f is nondecreasing.

Also (reasoning as in the proof of Leviatan [232]), it follows that

$$\|f - P_n(f)\|_\infty \leq C\omega_2^{\varphi}(f, 1/n)_\infty.$$

Case 2. (DeVore–Leviatan–Yu [90], DeVore–Leviatan [88]). We sketch here the proof in DeVore–Leviatan [88]. First we choose the partition points $\xi_j,\ j = 0, \ldots, n$, as in the above Case 1, starting now from the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2} \right)^{2r}, \quad \int_{-\pi}^{\pi} J_n(t)dt = 1,$$

where r is a sufficiently large fixed natural number. For $x = \cos t$ and $t_j := j\pi/n, j = 0, \ldots, n$, let us define

$$T_j(t) = \int_{t-t_j}^{t+t_j} J_n(u)du, \tau_j(x) := T_{n-j}(t), \quad R_j(x) := \int_{-1}^{x} \tau_j(u)du, \; j = 0, \ldots, n,$$

and the partition points ξ_j given by the equations $1 - \xi_j = R_j(1), j = 0, \ldots, n$. Obviously R_j is a polynomial of degree $\leq nr$, and since $R_j(x) - R_{j+1}(x)$ is nondecreasing for all $j = 0, 1, \ldots, n-1$, we get $-1 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$.

Now we briefly describe the construction of a suitable piecewise linear continuous function, $S_n(f)$, attached to f and to the above partition. It is given by $S_n(f)(-1) = l_{j_0}(-1)$, $S_n(f)(1) = l_{j_1}(1)$, and for $j_0 \leq j \leq j_1$, $S_n(f)(\xi_j) = f(\xi_j) = l_j(\xi_j)$ and linear in between (i.e., $S_n(f)(x) = l_j(x)$, $x \in [\xi_{j-1}, \xi_j]$, $j = j_0, \ldots, j_1$, represent the linear functions that interpolates f at ξ_{j-1} and ξ_j), where j_0 and j_1 are indices chosen as follows. Writing $I_j = [\xi_{j-1}, \xi_j]$ and $\tilde{I}_j$ the interval with the same center as I_j and twice its length, j_0 will be the smallest index j such that $\tilde{I}_{j_0} \subseteq [-1, 1]$, and j_1 will be the largest index j such that $\tilde{I}_{j_1} \subseteq [-1, 1]$.

Without entering into details, we mention only that in DeVore–Leviatan [88] it is proved that l_j are near optimal L^p-approximation to f on the intervals I_j^*, where $I_j^* := \tilde{I}_j$ if $j_0 < j < j_1$, $I_{j_0}^* := \tilde{I}_{j_0} \cup [-1, \xi_{j_0}]$, and $I_{j_1}^* := \tilde{I}_{j_1} \cup [\xi_{j_1}, 1]$.

We follow the ideas in the above case $p = \infty$ and take into account that we can write

$$\begin{aligned} S_n(f)(x) &= l_{j_0}(x) + \sum_{j=1}^{n-1} [a_j - a_{j-1}]\varphi_j(x) \\ &= l_{j_0}(x) - a_0(x+1) + \sum_{j=0}^{n-1} a_j[\varphi_j(x) - \varphi_{j+1}(x)], \end{aligned}$$

where $\varphi_j(x) = (x - \xi_j)_+$ and a_j is the slope of $l_j(x)$. If f is nondecreasing then $a_j \geq 0$ for all j, and therefore $S_n(f)$ is nondecreasing on $[-1, 1]$. Replacing φ_j by R_j, we get the nondecreasing polynomial

$$P_n(f)(x) = l_{j_0}(x) - a_0(x+1) + \sum_{j=0}^{n-1} a_j[R_j(x) - R_{j+1}(x)],$$

since $l_{j_0}(x) - a_0(x+1)$ is a constant and $R_j(x) - R_{j+1}(x)$ are nondecreasing.

Finally, in DeVore–Leviatan [88] it is proved that for large r we get

$$\|f - S_n(f)\|_p \leq C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p$$

and

$$\|S_n(f) - P_n(f)\|_p \leq C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p,$$

which proves the Case 2.

Case 3. (Yu [407], Leviatan–Yu [244], see Shvedov [355] for estimates in terms of $\omega_2(f, \frac{1}{n})_p$). Here we sketch the proof using Yu [407], Leviatan–Yu [244]. For that purpose, we keep the notation in the proof of Case 1 for $J_n(t)$, $T_j(t)$, t_j, $\tau_j(x)$, $R_j(x)$, ξ_j, $\varphi_j(x)$, and δ_j.

Let us set

$$\bar{f}(\xi_j) = \frac{1}{\eta_j} \int_{-\eta_j/2}^{\eta_j/2} f(\xi_j + t)dt,$$

where $\eta_j = \min\{\xi_j - \xi_{j-1}, \xi_{j+1} - \xi_j\},\ j = 1, \dots, n-1,$

$$\bar{S}_n(f) = \bar{f}(-1) + \sum_{j=1}^{n} \bar{s}_j(\varphi_{j-1} - \varphi_j)$$

is the piecewise linear function interpolating $\bar{f}(\xi_j)$ at the nodes $\xi_j, j = 1, \dots, n-1$, and extending linearly to the endpoints of $[-1, 1]$, where

$$\bar{s}_j = \frac{\bar{f}(\xi_j) - \bar{f}(\xi_{j-1})}{\xi_j - \xi_{j-1}}, j = 2, \dots, n-1,$$
$$\bar{s}_1 = \bar{s}_2, \quad \bar{s}_n = \bar{s}_{n-1},$$

and $\bar{f}(-1) = \bar{f}(\xi_1) - (1 + \xi_1)\bar{s}_1, \bar{f}(1) = f(\xi_{n-1}) + (1 - \xi_{n-1})\bar{s}_n$.

Defining

$$\bar{L}_n(f) = \bar{f}(-1) + \sum_{j=1}^{n} \bar{s}_j(R_{j-1} - R_j),$$

it is easy to see that $\bar{L}_n(f)$ is a polynomial of degree $\le 4n$ and is nondecreasing in $[-1, 1]$ whenever f is nondecreasing in $[-1, 1]$.

Finally, in Leviatan–Yu [244] (omitting here the details) the estimate

$$\|f - \bar{L}_n(f)\|_p \le C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p.$$

is proved. □

Remark. The estimate in Theorem 1.6.3, case $p = \infty$, can be refined, in the sense that for $f \in \Delta^1$ we can obtain polynomials $P_n(x) \in \Pi_n \cap \Delta^1$ satisfying an estimate of the form (see Ditzian–Jiang–Leviatan [97])

$$|f(x) - P_n(f)(x)| \le C(\lambda)\omega_2^{\varphi^\lambda}(f, n^{-1}\varphi(x)^{1-\lambda})_\infty,\ -1 \le x \le 1,$$

where $\lambda \in [0, 1]$.

If $1 \le p \le \infty$, then the estimate in Theorem 1.6.3 implies that for $f \in W_p^1[-1, 1] \cap \Delta^1$, we have

$$E^{(1)}(f)_p \le \frac{C}{n}\omega_1^{\varphi}\left(f', \frac{1}{n}\right)_p.$$

Thus, for smooth functions the question arises whether it would be possible to obtain estimates involving the moduli of smoothness of the derivatives. This is true for the uniform norm but not for the L^p-norms.

Theorem 1.6.4. *(i) (Shevchuk [347, 348]) Let $f \in C^1[-1,1] \cap \Delta^1$. For each $k \geq 1$, there exists a constant $C = C(k)$ with*

$$E^{(1)}(f)_\infty \leq \frac{C}{n}\omega_k\left(f', \frac{1}{n}\right)_\infty .$$

(ii) (Kopotun [208]) Let $0 < p < \infty$ and $k \geq 2$. Then for any $n \geq 1$, $\epsilon > 0$, and $A > 0$, there exists a function $f = f_{p,k,n,\epsilon,A} \in C^\infty[-1,1] \cap \Delta^1$ such that for all $p_n \in \Pi_n$ satisfying $p_n'(-1) \geq 0$, we have

$$\|f - p_n\|_{L^p[-1,-1+\epsilon]} > A\omega_k(f', 1)_p.$$

Note that if $k \geq 2$, then in Theorem 1.6.4, (i), one cannot replace the uniform norm by any of the L^p-norms, $0 < p < \infty$.

Now it is natural to ask whether the estimate in Theorem 1.6.3 still holds for ω_3^φ if we relax the requirement on the constant, by allowing that such a constant may depend on the function f (but not on n). Wu–Zhou [402] proved that this is impossible for $k = 4+[1/p]$. On the other hand, the following result closes the gap for monotonic continuous f.

Theorem 1.6.5. (Leviatan–Shevchuk [237]) *For any $f \in C[-1,1] \cap \Delta^1$, there exists a constant $C = C(f)$ such that*

$$E^{(1)}(f)_\infty \leq C\omega_3^\varphi\left(f, \frac{1}{n}\right)_\infty, \quad \forall n \geq 2.$$

Proof. Here we sketch the proof in Leviatan–Shevchuk [237]. For $n \geq 3$ and $x_j = -\cos(j\pi/n)$, $j = 0, \ldots, n$, denote by $\mathcal{S}_n$ the class of all continuous quadratic piecewise polynomials with the nodes at x_j, $j = 0, \ldots, n$.

For $f \in C[-1,1] \cap \Delta^{(1)}$ and $n \geq 3$, let l_1 and l_n be the linear functions interpolating f at the endpoints of the intervals $I_1 := [-1, x_1]$ and $I_n := [x_{n-1}, 1]$, respectively. By Lemma 1 in Leviatan–Shevchuk [236], there exists a piecewise quadratic polynomial $q_n \in \mathcal{S}_n \cap \Delta^{(1)}$ such that $q_n(x) = l_1(x)$, $x \in I_1$, $q_n(x) = l_n(x)$, $x \in I_n$, and

$$\|f - q_n\|_{[x_1, x_{n-1}]} \leq C\omega_3^\varphi(f, 1/n)_\infty.$$

The Burkill–Whitney inequality (see, e.g., (3.9) in Shevchuk [349]) implies

$$\|f - q_n\|(I_1)_\infty = \|f - l_1\|(I_1)_\infty \leq C\omega_2(f, 1/n^2)_\infty,$$
$$\|f - q_n\|(I_n)_\infty \leq C\omega_2(f, 1/n^2)_\infty,$$

taking into account that $|I_1|, |I_2| \leq Cn^{-2}$ (where $|I|$ denotes the length of the interval I).

On the other hand, by the proof of Marchaud's inequality (see, e.g., (3.6) in Shevchuk [349]), it follows that

$$\omega_2(f,\delta)_\infty \le C\delta^2 \int_\delta^1 \frac{\omega_3(f,u)_\infty}{u^3} du + C\delta^2 \omega_2(f,1)_\infty, \quad \delta \in (0,1),$$

which by the inequalities (see Ditzian–Totik [98])

$$\omega_3(f,\delta)_\infty \le C\omega_3^\varphi(f,\sqrt{\delta})_\infty \le Cn^3\delta^{3/2}\omega_3^\varphi(f,1/n)_\infty, \quad \delta \ge \frac{1}{n^2},$$

implies

$$\begin{aligned}\omega_2(f,1/n^2)_\infty &\le C\omega_3^\varphi(f,1/n)_\infty + \frac{C}{n^4}\omega_2(f,1)_\infty \\ &\le C\omega_3^\varphi(f,1/n)_\infty + \frac{C}{n^4}\omega_2^\varphi(f,1)_\infty.\end{aligned}$$

From the above inequalities, we obtain

$$\|f-q_n\|_\infty \le C\omega_3^\varphi(f,1/n)_\infty + Cn^{-4}\omega_2^\varphi(f,1)_\infty,$$

where

$$\omega_3^\varphi(q_n,1/n)_\infty \le \omega_3^\varphi(f,1/n)_\infty + 8\|f-q_n\|_\infty \le C\omega_3^\varphi(f,1/n)_\infty + Cn^{-4}\omega_2^\varphi(f,1)_\infty.$$

Hence, for $n > C$, since by Proposition 3 in Leviatan–Shevchuk [235], for $q_n \in \mathcal{S}_n \cap \Delta^1$, we have

$$E_{Cn}^{(1)}(q_n)_\infty \le C\omega_3^\varphi(s_n,1/n)_\infty,$$

the estimate in the statement follows. For $n \le C$ the estimate is immediate. □

1.6.2 Pointwise Monotone Approximation

Regarding pointwise estimates in monotone polynomial approximation, Dzyubenko [102] proved that for $f \in C^r[-1,1] \cap \Delta^1$ and $n \ge r+k-1$, there exists a polynomial $p_n \in \Delta^1$ for which the pointwise estimates of Timan–Brudnyi type are valid. However, as it was proved by DeVore–Yu [92], the interpolatory estimates of Telyakovskii–Gopengauz type are valid only when $r+k \le 2$.

Other positive and negative results can be summarized by the following.

Theorem 1.6.6. (Gonska–Leviatan–Shevchuk–Wenz [152])

(i) If $r > 2$, then for any n there is an $f = f_{r,n} \in W_\infty^r[-1,1] \cap \Delta^1$ such that for any polynomial $q_n \in \Pi_n \cap \Delta^1$, either

$$\limsup_{x\to -1} \frac{|f(x)-q_n(x)|}{\varphi^r(x)} = \infty$$

or

$$\limsup_{x\to 1} \frac{|f(x)-q_n(x)|}{\varphi^r(x)} = \infty.$$

(ii) Let $r+k>2$. Then for any n, there is an $f=f_{r,k,n}\in C^r[-1,1]\cap \Delta^1$, such that for any polynomial $q_n\in \Pi_n\cap\Delta^1$, either

$$\limsup_{x\to -1} \frac{|f(x)-q_n(x)|}{\varphi^r(x)\omega_k(f^{(r)},\varphi(x))_\infty} = \infty,$$

or

$$\limsup_{x\to 1} \frac{|f(x)-q_n(x)|}{\varphi^r(x)\omega_k(f^{(r)},\varphi(x))_\infty} = \infty.$$

(iii) For interpolation at only one of the end points, we have the following positive result. Suppose $k\le \max\{r,2\}$ and $f\in C^r[-1,1]\cap\Delta^1$. For any $n\ge N:=k+r-1$, there exists a polynomial $p_n\in\Pi_n\cap\Delta^1$ satisfying

$$|f(x)-p_n(x)|\le C(r)\rho_n^r(x)\omega_k(f^{(r)},\rho_n(x))_\infty,\quad x\in[-1,1],$$

and

$$|f(x)-p_n(x)|\le C(r)\frac{(1-x)^{r/2}}{n^r}\omega_k\left(f^{(r)},\frac{\sqrt{1-x}}{n}\right)_\infty,\quad x\in[-1,1].$$

(iv) Suppose $k>\max\{r,2\}$. Then for each $n\ge 1$ and every constant $A>0$, an $f=f_{r,k,n,A}\in C^r[-1,1]\cap\Delta^1$ exists such that for every polynomial $p_n\in\Pi_n\cap\Delta^1$, there is a point $x\in[-1,1]$ for which

$$|f(x)-p_n(x)|> A\frac{(1-x)^{r/2}}{n^r}\omega_k\left(f^{(r)},\frac{\sqrt{1-x}}{n}\right)_\infty$$

holds.

Remarks. (1) Note that for $f\in C^r[-1,1]\cap\Delta^1$, the first estimate in (iii), but in the uniform norm, was obtained by DeVore [84], who proved the existence of $p_n\in\Pi_n\cap\Delta^1$ satisfying

$$\|f-p_n\|_\infty\le C(r)n^{-r}\omega_1(f^{(r)},1/n)_\infty,$$

where $\|\cdot\|_\infty$ denotes the uniform norm in $C[-1,1]$.

(2) Concerning simultaneous pointwise estimates in monotone approximation, in Kopotun [204] it is proved that for $f\in C^1[-1,1]\cap\Delta^1$ and $n\ge 1$, there is a polynomial $p_n\in\Pi_n\cap\Delta^1$, satisfying

$$|f^{(i)}(x)-p_n^{(i)}(x)|\le C\omega_{2-i}(f^{(i)},\rho_n(x))_\infty,\qquad i=0,1,\qquad x\in[-1,1].$$

(3) Pointwise estimates in monotone approximation by polynomials with positive coefficients were obtained in Trigub [389].

1.6.3 L^p-Comonotone Approximation, $0 < p \leq \infty$

A natural extension of monotone polynomial approximation is the comonotone polynomial approximation, i.e., the approximation of a function $f \in L^p[-1,1], 0 < p \leq \infty$, by polynomials $(P_n)_n$ that change their (same) monotonicity with f at $Y_s = \{y_1, \ldots, y_s\} \in \mathbb{Y}_s$, i.e., if, for example, in addition to $f \in C^1[-1,1]$, then $f'(x)P_n'(x) \geq 0$ for all $x \in [-1,1]$, $n \in \mathbb{N}$. Set

$$d(Y_s) := \min_{0 \leq i \leq s} (y_i - y_{i+1}).$$

Theorem 1.6.7. *(i) (Beatson–Leviatan [34]) If f is continuously differentiable in $[-1,1]$ and changes monotonicity s times, $1 \leq s < \infty$, then for each $n \geq 1$ there is a polynomial p_n of degree $\leq n$, comonotone with f on $[-1,1]$ and satisfying*

$$\|f - p_n\|_\infty \leq \frac{C(s)}{n} \omega_1(f', 1/n)_\infty,$$

where $C(s)$ is a constant depending only on s.

(ii) (Kopotun–Leviatan [210]) If $f \in L^p[-1,1] \cap \Delta^1(Y_s)$, $0 < p \leq \infty$, then there is a constant $C = C(s)$ such that for any $n \geq C/d(Y_s)$, we have

$$E_n^{(1)}(f, Y_s)_p \leq C\omega_2^\varphi\left(f, \frac{1}{n}\right)_p.$$

Although it is weaker than the estimate in (ii), we present here only the proof of Theorem 1.6.7 (i), by following the ideas in Beatson–Leviatan [34], since they were very seminal for the next results and since it can more easily be extended to real functions of two real variables (see Section 2.6, Corollary 2.6.11).

In the proof of this result, C will denote a positive constant independent of f, n, and s but that can be different at each occurrence. Also, the method used is based on the so-called "flipped" function denoted by f_F and attached to f, with the property that it has one change fewer in monotonicity than f.

First we need a lemma on this "flipped" function.

Lemma (A). (Beatson–Leviatan [34]) *A constant $C > 0$ exists with the following property: if $f \in C[-1,1]$ changes its monotonicity at $s \geq 1$ points at $(-1,1)$ including 0, where it is supposed $f(0) = 0$, defining the "flipped" function*

$$f_F(x) = f(x), \quad x \geq 0,$$
$$f_F(x) = -f(x), \quad x < 0,$$

and supposing that for some $n \geq 1$ and $\delta > \omega_1(f_F', 1/n)_\infty$ there is a polynomial $p_n \in \Pi_n$ comonotone with f_F such that

$$\|f_F - p_n\|_\infty \leq C\delta/n, \quad \|f_F' - p_n'\|_\infty \leq C\delta,$$

then there is a polynomial $P_{2n} \in \Pi_{2n}$ *comonotone with* f *satisfying the estimates*

$$\|f - P_{2n}\|_\infty \le C\delta/n, \quad \|f' - P'_{2n}\|_\infty \le C\delta.$$

Proof of Lemma A. It is evident that f_F has one change of monotonicity fewer than f. Also for $0 < |x| < k/n$, where $k \ge 1$, we get

$$|f'_F(x)| \le \omega_1(f'_F, |x|)_\infty \le k\omega_1(f'_F, 1/n)_\infty.$$

By $f_F(0) = 0$ and by the mean value theorem, it follows that there is $\xi \in (0,1)$ such that

$$|f_F(x)| = |x| \cdot |f'_F(\xi x)| \le \frac{k^2}{n}\omega_1(f_F, 1/n)_\infty.$$

By DeVore [84], pp. 908–909, for each $n \ge 1$ we can approximate $\mathrm{sgn}(x)$ by the polynomials $q_n(x) = C_n \int_0^x (T_m(t)/t)^4 dt$, where m is the largest odd integer such that $q_n \in \Pi_n$ and C_n is chosen such that $q_n(1) = 1$. It is known that q_n is odd, monotone increasing, and

$$|\mathrm{sgn}(x) - q_n(x)| \le C|nx|^{-3}, \quad x \in [-1, 0) \cup (0, 1],$$
$$|\mathrm{sgn}(x) - q_n(x)| \le 1, \quad x \in [-1, 1].$$

Because $f_F(0) = 0$ we can suppose $p_n(0) = 0$, by replacing δ with 2δ in the inequality $\|f_F - p_n\|_\infty \le C\delta/n$.

Let us define now $P_{2n}(x) = \int_0^x p'_n(t)q_n(t)dt$. It is evident that P_{2n} is comonotone with f, and we get

$$\begin{aligned} f(x) - P_{2n}(x) &= \int_0^x [f'_F(t) - p'_n(t)]\mathrm{sgn}(t)dt + \int_0^x p'_n(t)[\mathrm{sgn}(t) - q_n(t)]dt \\ &= (f_F(x) - p_n(x))\mathrm{sgn}(x) + \int_0^x p'_n(t)[\mathrm{sgn}(t) - q_n(t)]dt \\ &:= J_1(x) + J_2(x). \end{aligned}$$

Write $\beta = \mathrm{sgn}(x)/n$. For $0 < |x| \le i/n,\ i \ge 1$, the above inequalities and $f'_F(0) = 0$ imply

$$\begin{aligned} |J_2(x)| = \left|\int_0^x p'_n(t)[\mathrm{sgn}(t) - q_n(t)]dt\right| &\le \sum_{k=0}^{i-1}\left|\int_{k\beta}^{(k+1)\beta} p'_n(t)[\mathrm{sgn}(t) - q_n(t)]dt\right| \\ \le \frac{1}{n}[\omega_1(f'_F, 1/n)_\infty + \delta] + \frac{1}{n}\sum_{k=1}^{i-1}&[(k+1)\omega_1(f'_F, 1/n)_\infty + \delta]Ck^{-3} \le C\delta/n. \end{aligned}$$

The estimate for $|J_1(x)|$ is similar, which proves the estimate $\|f - P_{2n}\|_\infty \le C\delta/n$.

The proof of the inequality $\|f' - P'_{2n}\|_\infty \le C\delta$ is similar. □

Proof of Theorem 1.6.7 (i). If $\alpha \in (-1,1)$ is a point where f changes its monotonicity, then $f'(\alpha)=0$.

For small values of n, let us say $n < M(s)$, the theorem is immediate since $f'(\alpha)=0$ implies

$$|f(x)-f(\alpha)| \leq |x-\alpha| \cdot |f'(\eta)| \leq 2\omega_1(f',2)_\infty \leq \frac{C(s)}{n}\omega_1(f',1/n)_\infty.$$

For large n, we prove the theorem by induction on the number of changes of monotonicity, s.

The theorem in true for $s=0$ by a construction in DeVore [83] (see Beatson–Leviatan [34], pp. 222–223 for details).

Note that without loss of generality, if $\alpha \in (-1,1)$ is a point where f changes its monotonicity, then we may assume that $f(\alpha)=0$ (otherwise, we subtract a constant from f and add it to the approximation polynomials).

Therefore, we have to prove the statement that there exist constants $C(s)$ and $M(s)$ such that for any function $f \in C^1[-1,1]$ with $s \geq 1$ changes of monotonicity (such that there is $\alpha \in (-1,1)$ with $f'(\alpha)=0$) and any $n \geq M(s)$, there is a $p_n \in \Pi_n$ comonotone with f satisfying

$$\|f-p_n\|_\infty \leq C(s)\frac{1}{n}\omega_1(f',1/n)_\infty, \quad \|f'-p_n'\|_\infty \leq C(s)\omega_1(f',1/n)_\infty.$$

Let us assume that the above statement is true for $s-1$ and let f with $s \geq 1$ changes of monotonicity. First we extend f linearly to $[-3,3]$, preserving the modulus of continuity of f'. Working with the interval I of length 4 centered at α (obviously $I \subseteq [-3,3]$) we deduce that a change of variable $y = \frac{1}{2}(x-\alpha)$ defines a function $g(y)=f(x)$ defined for $-1 \leq y \leq 1$ that satisfies $g'(0)=0$ and $\omega_1(g',\delta)_\infty \leq 4\omega_1(f',\delta)_\infty$.

Without loss of generality, we may assume that $g(0)=0$. Then the "flipped" function attached to g, i.e., g_F, has $s-1$ changes of monotonicity. Using the above Lemma A and the inductive hypothesis, there is a sequence $\{h_n\}_{n=2M(s-1)}^{\infty}$ of comonotone approximation polynomials to g. Now, inverting the change of variable, it follows that the sequence $\{p_n(x)\}, p_n(x)=h_n(y)$ satisfies the statement for s, which proves the theorem. □

Remarks. (1) As a negative result to Theorem 1.6.7 (ii) Zhou [410] proved that for any $0 < p \leq \infty$ and $s \geq 1$, there is a Y_s and an $f \in L^p[-1,1] \cap \Delta^1(Y_s)$ such that

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f)_p}{\omega_k(f,1/n)_p} = \infty,$$

with $k = 3 + [1/p]$. Therefore, if $p=\infty$ then Theorem 1.6.7 (ii) is not valid with any $k \geq 3$, even with $C = C(f)$ and $N = N(f)$.

(2) Let us mention that the estimate in Theorem 1.6.7 (i) was at the time (in 1983), an improvement with respect to the results in Iliev [181] and Newmann [283], who obtained an estimate of order $O(\omega_1(f,1/n)_\infty)$.

For $p = \infty$ we can say much more in comonotone approximation. The results can be summarized by the following.

Theorem 1.6.8. *(i) (Dzyubenko–Gilewicz–Shevchuk [106], Wu–Zhou [403]) If* $f \in C^r[-1,1] \cap \Delta^1(Y_s)$, *then the estimate*

$$E_n^{(1)}(f, Y_s)_\infty \le \frac{C}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right)_\infty, \quad n \ge N,$$

holds with $C = C(k,r,s)$ *and* $n = N(k,r,s)$ *only when either* $k = 1$ *or* $r > s$, *or in the case* $k = 2$ *and* $r = s$. *In addition, in these cases one can always take* $N = k + r - 1$. *If* $k = 2$ *and* $0 \le r < s$, *or* $k = 3$ *and* $1 \le r \le s$, *or* $k > 3$ *and* $2 \le r \le s$, *then the above estimate holds either with* $C = C(k,r,Y_s)$ *and* $N = k + r$, *or with* $C = C(k,r,s)$ *and* $N = N(k,r,Y_s)$, *and they fail to be valid with* $C = C(k,r,s)$ *and* $N = N(k,r,s)$. *Also, if either* $r = 0$ *or* $r = 1$, *then for every* $s \ge 1$, *there exist a* $Y_s \in \mathbb{Y}_s$ *and an* $f \in C^r[-1,1] \cap \Delta^1(Y_s)$ *such that*

$$\limsup_{n\to\infty} \frac{n^r E_n^{(1)}(f, Y_s)_\infty}{\omega_{3+r}(f^{(r)}, 1/n)_\infty} = \infty,$$

that is, the estimate does not hold even with constants that depend on f.

In particular, if $f \in W_\infty^r$ *then*

$$E_n^{(1)}(f, Y_s)_\infty \le C(r,s)\frac{\|f^{(r)}\|_\infty}{n^r}, \; n \ge r-1.$$

(ii) (Leviatan–Shevchuk [238]) For $s \ge 0$ *assume that* $f \in \mathbb{B}^r \cap \Delta^1(Y_s), r \ge 1$. *Then*

$$E_n^{(1)}(f, Y_s)_\infty \le C(r, Y_s)\frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1,$$

and

$$E_n^{(1)}(f, Y_s)_\infty \le C(r,s)\frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge N(r, Y_s).$$

In addition, if $f \in \mathbb{B}^r \cap \Delta^1(Y_s)$, *with either* $s = 0$, $r = 1$, *or* $r = 3$, $s = 1$, *or* $r > 2s + 2$, *then*

$$E_n^{(1)}(f, Y_s)_\infty \le C(r)\frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1.$$

(iii) (Leviatan–Shevchuk [235]) Given $s \ge 1$, *let* $A > 0$ *be arbitrary and* $2 \le r \le 2s + 2$, *excluding the case* $r = 3$, $s = 1$. *Then for every* n, *there is an* $f = f_{r,s,n} \in \mathbb{B}^r$, *that changes monotonicity* s *times in* $[-1,1]$, *such that*

$$e_n^{(1,s)}(f)_\infty \ge A\|\varphi^r f^{(r)}\|_\infty.$$

Remark. Pointwise estimates in comonotone approximation have new properties, described as follows. If $s = 1$, then when either $r \ge 2$ or in the three special cases the $k = 1$, $r = 0, 1$ and $k = 2$, $r = 1$, there is a polynomial $p_n \in \Pi_n \cap \Delta^1$ satisfying

$$|f(x) - p_n(x)| \le C(r)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \quad -1 \le x \le 1, \quad n \ge k + r - 1.$$

For two other pairs, $k = 2$, $r = 0$ and $k = 3$, $r = 1$, the above estimate holds with $C = C(Y_1) = C(y_1)$, while for the remaining pairs, i.e., $r = 0$, $k \ge 3$ and $r = 1$, $k \ge 4$, we have no such estimate. Thus, the situation is exactly that in Theorem 1.6.8 for $s = 1$.

On the other hand, if $s > 1$, then the situation for the above pointwise estimate can briefly be described as follows: it does not hold for $r = 0$, $k \ge 3$ and $r = 1$, $k \ge 4$, while for the rest it follows with $C = C(r, k, s)$ only for $n \ge N = N(r, k, Y_s)$ (for all these situations see, e.g., Leviatan [230]).

1.6.4 Comonotone Approximation with Modified Weighted Moduli of Smoothness

It is of interest to consider estimates in comonotone polynomial approximation with respect to the modified weighted moduli of smoothness, $\omega_{k,r}^\varphi$, introduced by Shevchuk [349] (see also Definition 1.1.3.). As a sample, we present the following.

Theorem 1.6.9. (Leviatan–Shevchuk [238], Dzyubenko–Listopad–Shevchuk [109] for $s = 0$) *Let $s \ge 0$. If $f \in C_\varphi^r \cap \Delta^1(Y_s)$, with $r > 2$, then*

$$E_n^{(1)}(f, Y_s)_\infty \le \frac{C(k, r, Y_s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \ge k + r - 1,$$

and

$$E_n^{(1)}(f, Y_s)_\infty \le \frac{C(k, r, s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \ge N(k, r, Y_s).$$

In addition, if $f \in C_\varphi^r \cap \Delta^1(Y_s)$ with $r > 2s + 2$, then

$$E_n^{(1)}(f, Y_s)_\infty \le \frac{C(k, r, s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \ge k + r - 1.$$

Remarks. (1) Concerning the dependence on the parameters of the constants C and N for the answer with respect to the estimates in Theorem 1.6.9, case $s = 0$, the following situations are possible (see Leviatan [230]):

If $r \ge 3$, then both N and C depend on k, r, s;

Let $r = 2$. If $k \ge 3$ then estimate holds, $k = 2$ is still an open question, if $k = 0$, then both constants C and N depend on k, r, s, if $k = 1$, then the first two estimates in Theorem 1.6.9 do not hold, but the third one holds with $C = C(f)$;

Let $r = 1$. If $k = 0$ or $k = 1$, then we have $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 2$, then the first two estimates in Theorem 1.6.9 do not hold but the third one holds with $C = C(f)$, $k = 3$ is still an open question, if $k > 3$, then no estimate holds.

Let $r = 0$. If $k = 1$ or $k = 2$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 3$, then the first two estimates in Theorem 1.6.9 do not hold but the third one holds with $C = C(f)$, if $k \geq 4$, then no estimate holds.

(2) Concerning the dependence on the parameters of the constants C and N, for the complete answer with respect to the estimates in Theorem 1.6.9, case $s = 1$, the following situations are possible (for all the cases see, e.g., Leviatan [230], except the cases where the references are specified):

If $r \geq 5$, then $C = C(k, r, s)$ and $N = N(k, r, s)$;

If $r = 4$, then one of the constants C or N depends on k, r and Y_s;

Let $r = 3$. If $k = 0$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k \geq 1$, then one of the constants C and N depends on k, r and Y_s;

Let $r = 2$. If $k = 0$ or $k = 1$, then one of the constants C and N depends on k, r and Y_s, if $k = 2$, then the estimates in Theorem 1.6.9 do not hold even for constants depending on the function (see Nesterenko–Petrova [282]), if $k \geq 3$, then no estimate holds.

Let $r = 1$. If $k = 0$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 1$ or $k = 2$, then one of the constants C or N depends on k, r and Y_s, if $k = 3$, then the estimates in Theorem 1.6.9 do not hold even for constants depending on the function (see Nesterenko–Petrova [282]), if $k \geq 4$, then no estimate holds.

Let $r = 0$. If $k = 1$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 2$, then one of the constants C or N depends on k, r and Y_s, if $k \geq 3$, then no estimate holds.

(3) Concerning the dependence on the parameters of the constants C and N, for the complete answer with respect to the estimates in Theorem 1.6.9, case $s > 1$, the following situations are possible (for all the cases see, e.g., Leviatan [230], excepting the cases for which references) are specified:

If $r \geq 2s + 3$, then $C = C(k, r, s)$ and $N = N(k, r, s)$;

If $3 \leq r \leq 2s + 2$, then one of the constants C and N depends on k, r, and Y_s;

Let $r = 2$. If $k = 0$ or $k = 1$, then one of the constants C and N depends on k, r and Y_s, if $k = 2$, then the estimates in Theorem 1.6.9 do not hold even for constants depending on the function (see Nesterenko–Petrova [282]), if $k \geq 3$, then no estimate holds.

Let $r = 1$. If $k = 0$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 1$ or $k = 2$, then one of the constants C and N depends on k, r and Y_s, if $k = 3$, then the estimates in Theorem 1.6.9 do not hold even for constants depending on the function (see Nesterenko–Petrova [282]), if $k \geq 4$, then no estimate holds.

Let $r = 0$. If $k = 1$, then $C = C(k, r, s)$ and $N = N(k, r, s)$, if $k = 2$, then one of the constants C and N depends on k, r and Y_s, if $k \geq 3$ then the estimates in Theorem 1.6.9 do not hold.

(4) When $s = 0$, we have $Y_0 = \emptyset$ in the first two estimates and hence the second estimate is just the third one. Also, in this case, the last estimate is valid for $0 \le r + k \le 2$.

(5) For other results in the comonotone approximation in terms of modified Ditzian–Totik moduli of smoothness, see Note 1.10.2.

1.6.5 Nearly Comonotone Approximation

By relaxing the condition of comonotonicity from the whole interval $[-1, 1]$ to only a major portion of that interval, one obtains the so-called nearly comonotone polynomial approximation.

This concept seems to appear for the first time in the paper of Newman–Passow–Raymon [284], as follows. They say that $f \in C[-1, 1]$ is piecewise monotone on $[-1, 1]$ if there exists $Y_s = \{y_j,\ j = 1, \dots, s\}$, with $-1 := y_{s+1} < y_s < \cdots < y_1 < 1 := y_0$, such that f is monotonic on each interval $(y_{i+1}, y_i), i = 0, \dots, s$. In particular, $f \in C[-1, 1]$ is called proper piecewise monotone on $[-1, 1]$ (with respect to Y_s) if for any $0 < \varepsilon < \frac{1}{2} \min_{i=0,\dots,s}\{(y_i - y_{i+1})\}$, there is $\delta > 0$ such that $|f(x) - f(y)|/|x - y| \ge \delta$, for all $x, y \in [y_{i+1} + \varepsilon, y_i - \varepsilon]$, $x \ne y$.

A sequence of algebraic polynomials $(P_n)_n$ is called nearly comonotone with f on $[-1, 1]$ if for any $0 < \varepsilon < \frac{1}{2} \min_{i=0,\dots,s}\{y_i - y_{i+1}\}$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \ge N_\varepsilon$, the polynomial P_n is of the same monotonicity as f on the intervals $(y_{i+1} + \varepsilon, y_i - \varepsilon)$, $i = 0, \dots, s$.

The first result on nearly comonotone approximation belongs to Newman–Passow–Raymon [284] and states that for any proper piecewise monotone $f \in C[-1, 1] \cap \mathrm{Lip}_1 M$, there exists a nearly comonotone sequence (of algebraic polynomials) with f, $(P_n)_n$, $P_n \in \Pi_n$, such that $\|f - P_n\|_\infty \le \frac{CM}{n}$, for all $n \in \mathbb{N}$. Later on, the restriction "proper" was removed by DeVore [83]. Roulier [328] generalized the Newman–Passow–Raymon [284] result by proving that for any piecewise monotone function $f \in C[-1, 1]$ and any sequence of positive numbers $d_n \to 0$ that satisfies $\lim_{n\to\infty} \frac{E_n(f)}{d_n} = 0$, there exists a nearly comonotone sequence of polynomials $(P_n)_n$, $P_n \in \Pi_n$, such that

$$\|f - P_n\|_\infty \le \omega_1(f, d_n)_\infty + E_n(f)_\infty,\ n \in \mathbb{N}.$$

where recall $E_n(f)_\infty = \inf\{\|f - q_n\|_\infty; q_n \in \Pi_n\}$.

Myers [281] proved that if $f \in C[-1, 1]$, then the above Roulier's estimates can be replaced by $\|f - P_n\| \le C\omega_1(f, 1/n)$, and if, in addition, $f \in C^1[-1, 1]$, then the estimate in nearly comonotone approximation can be improved to $\|f - P_n\|_\infty \le C_1 n^{-1}\omega_1(f', 1/n)_\infty$. (Here C and C_1 are absolute positive constants.)

All the above results can be improved. To be more precise, given the "comonotonicity points" $Y_s = \{y_1, \dots, y_s\}$, where $-1 := y_{s+1} < y_s < \cdots < y_1 < 1 =: y_0$, similar to the copositive case let us write

$$O(n, Y_s) = [-1, 1] \cap \{\cup_{i=1}^s (y_i - \rho_n(y_i), y_i + \rho_n(y_i))\}$$

and

$$O^*(n, Y_s) = O(n, Y_s) \cup [-1, -1 + 1/n^2] \cup [1 - 1/n^2, 1].$$

We have better results than those in comonotone approximation too, summarized by the following.

Theorem 1.6.10. *(i) (Leviatan–Shevchuk [240]) For any natural number M, there exists a constant $C = C(s, M)$ for which if $f \in C[-1, 1] \cap \Delta^1(Y_s)$, then for every $n \geq 2$, there exists a polynomial $P_n \in \Pi_n$ comonotone with f on $[-1, 1] \setminus O^*(Mn, Y_s)$ such that $\|f - P_n\|_\infty \leq C\omega_3^\varphi(f, 1/n)_\infty$. Also, ω_3 cannot be replaced by ω_4.*

(ii) (Leviatan–Shevchuk [240]) For each k and M natural numbers, there exists a constant $C = C(k, s, M)$ for which if $f \in \Delta^1(Y_s) \cap C^1[-1, 1]$, then for every $n \geq k$ there is a polynomial $P_n \in \Pi_n$ comonotone with f on $[-1, 1] \setminus O(Mn, Y_s)$ such that

$$\|f - P_n\|_\infty \leq \frac{C}{n}\omega_k^\varphi(f', 1/n)_\infty.$$

(iii) (Leviatan–Shevchuk [240]) For each k and M natural numbers, there exists a constant $C = C(k, s, M)$ for which if $f \in \Delta^1(Y_s) \cap C_\varphi^1$, then for every $n \geq k$ there is a polynomial $P_n \in \Pi_n$ comonotone with f on $[-1, 1] \setminus O^*(Mn, Y_s)$, such that

$$\|f - P_n\|_\infty \leq \frac{C}{n}\omega_{k,1}^\varphi(f', 1/n),$$

where $\omega_{k,1}^\varphi$ and C_φ^1 are defined in Definition 1.1.3.

(iv) (Leviatan–Shevchuk [236]) There exists a natural number $M = M(s)$ and a constant $C(s)$ such that if $f \in C^1[-1, 1] \cap \Delta^1(Y_s)$, then there exists a polynomial $P_n \in \Pi_n$, comonotone with f on $[-1, 1] \setminus O^*(Mn, Y_s)$ that for all $x \in [-1, 1]$ and $n \geq 2$ satisfies

$$|f(x) - P_n(x)| \leq C(s)\omega_3(f, \rho_n(x)).$$

(v) (Leviatan–Shevchuk [236]) There exists a natural number $M = M(s, k)$ and a constant $C(s, k)$ such that if $f \in C^1[-1, 1] \cap \Delta^1(Y_s)$, then there exists a polynomial $P_n \in \Pi_n$ comonotone with f on $[-1, 1] \setminus O(Mn, Y_s)$, that for all $x \in [-1, 1]$ and $n \geq k$ satisfies

$$|f(x) - P_n(x)| \leq C(s, k)\rho_n(x)\omega_k(f', \rho_n(x)).$$

Many other details, including the proofs in Theorem 1.6.10, can be found in the papers of Leviatan–Shevchuk [236, 240].

Remark. From their proofs it follows that the above-presented methods in polynomial comonotone approximation are nonlinear, such that even if two continuous functions $f, g \in C[-1, 1]$ have the same points Y_s where the monotonicity changes and both are of the same monotonicity on each subinterval, however the comonotone polynomials P_n do not satisfy $P_n(f + g) = P_n(f) + P_n(g)$.

It is easy to show that $\Delta^1(Y_s)$ is a convex cone, i.e., $f, g \in \Delta^1(Y_s)$ and $\alpha \in \mathbb{R}_+$ implies $f+g \in \Delta^1(Y_s)$ and $\alpha f \in \Delta^1(Y_s)$.

Suggested by the proof of Theorem 1.2.1, we easily can construct a polynomial comonotone with f, that is an additive and positive homogeneous operator on $\Delta^1(Y_s) \cap C^2[-1,1]$. In this sense, we present two very simple results.

Theorem 1.6.11. *If $f \in \Delta^1(Y_s)$ is twice continuously differentiable on $[-1,1]$, then a sequence of polynomials $(P_n)_n$ can be constructed such that degree$(P_n) \leq n+s+1$ and for any $\varepsilon > 0$, there is n_0 with the properties*

$$\|f - P_n\|_\infty < \varepsilon, \qquad P_n'(x) f'(x) \geq 0, \quad \forall n \geq n_0, \quad x \in [-1,1],$$

and $P_n(\alpha h + \beta g) = \alpha P_n(h) + \beta P_n(g)$, for all $n \in \mathbb{N}$, $h, g \in \Delta^1(Y_s) \cap C^2[-1,1]$, $\alpha, \beta \geq 0$.

Recalling the notation $\Pi(x, Y_s) = \Pi_{i=1}^s (x - y_i)$, the error estimate can be expressed by

$$\|P_n(f) - f\|_\infty \leq C(Y_s) \omega_2 \left(\frac{f'}{\Pi(\cdot, Y_s)}; \frac{1}{n} \right)_\infty,$$

for all $n \in \mathbb{N}$.

Proof. From the differentiability hypothesis, it easily follows that $F(x) = \frac{f'(x)}{\Pi(x,Y_s)}$ is continuous (by extension) on $[-1,1]$. Also, by simple reasoning we get that $F(x) \geq 0$, for all $x \in [-1,1]$. Note that without loss of generality, we may suppose $f(-1) = 0$.

Now define $P_n(f)(x) := \int_{-1}^x L_n(F)(t) \cdot \Pi(t, Y_s) dt$, where $L_n, n \in \mathbb{N}$, is a sequence of positive linear polynomial operators on $C[-1,1]$ satisfying degree$(L_n(f)) \leq n$ and

$$\|L_n(f) - f\|_\infty \leq C\omega_2(f; 1/n)_\infty, \; n = 1, 2, \ldots.$$

The conclusions in the statement are immediate. □

Remark. In the very particular case of a single point of change of monotonicity in the particular interval $(0,1)$, i.e., $Y_1 = \{y_1\}$, where $y_1 \in (0,1)$, we can obtain a different estimate in Theorem 1.6.11. In this case, $\Delta^1(Y_1)$ represents the space of all continuous functions on $[0,1]$, that are nonincreasing on $[0, y_1]$ and nondecreasing on $[y_1, 1]$.

Theorem 1.6.12. *If $f \in \Delta^1(Y_1)$ is continuously differentiable in $[0,1]$, $f'(x) \neq 0$, for all $x \neq y_1$, then for any $\varepsilon < y_1, \varepsilon < 1 - y_1$, there is $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $P_n(f)(x) = B_n(f)(x + x_n - y_1)$, with $B_n'(f)(x_n) = 0$, are polynomials comonotone with f on $[\varepsilon, 1-\varepsilon]$ (near comonotonicity) and*

$$\|P_n(f) - f\|_\infty \leq \frac{C}{n^{1/2}} \omega_1^\varphi(f; 1/n^{1/2})_\infty + \|f'\|_\infty \cdot |x_n - y_1|.$$

Here $B_n(f)(x)$ denotes the Bernstein polynomials on $[0,1]$.

Proof. Firstly, the estimate is immediate by the approximation property of Bernstein polynomials in e.g., Knoop–Zhou [194], Totik [387].

By E. Popoviciu [311], $f \in \Delta^1(Y_1)$ means that it is quasiconvex, which by Theorem 1.3.1 (ii) implies that all the Bernstein polynomials $B_n(f)$ are quasiconvex on $[0,1]$. Note that by hypothesis, it is easy to prove that for each fixed n, x_n is unique and that $\lim_{n\to\infty} x_n = y_1$, which implies the comonotonicity of $P_n(f)$ with f on $[\varepsilon, 1-\varepsilon]$, for sufficiently large n. □

Remark. An important shortcoming in Theorems 1.6.11 and 1.6.12 is that with respect to the Jackson-type estimates (i.e., in terms of the moduli $\omega_1(f,1/n)_\infty$ or $\omega_2(f,1/n)_\infty$), the approximation errors are weak. The solution to this shortcoming remains an open question.

1.7 Convex and Coconvex Polynomial Approximation

This section contains the main results in convex and coconvex polynomial approximation. Again, although we omit most of the proofs as being too technical, to have a look at the constructions, we sketch the proofs for a few of them.

It is worth noting that in recent years, the case of convex approximation in the uniform norm was completely solved. More exactly, for each r times differentiable convex function it is possible to say whether or not its degree of convex polynomial approximation in the uniform norm may be estimated by a Jackson-type estimate involving the weighted Ditzian–Totik kth modulus of smoothness, and how the constants in this estimate behave. We will see that for any pair (k,r), only one from the following three cases is possible: we have an estimate with constants depending only on these parameters, or we have an estimate but only with constants that depend on the function being approximated, or a Jackson-type estimate is not possible.

First we need some useful notation. Let $\mathbb{Y}_s$ be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points such that $y_{s+1} := -1 < y_s < \cdots < y_1 < 1 =: y_0$, where for $s=0, Y_0 = \emptyset$. For $Y_s \in \mathbb{Y}_s$ we define

$$\Pi(x, Y_s) := \prod_{i=1}^{s}(x - y_i),$$

where the empty product is 1. Let $\Delta^2(Y_s)$ be the set of functions f that change convexity at the points $y_i \in Y_s$ and that are convex in $(y_1, 1)$, (again denoted by Δ^2 if $Y_0 = \emptyset$), and for $f \in \Delta^2(Y_s) \cap L^p[-1,1]$, we denote by

$$E_n^{(2)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^2(Y_s)} \|f - p_n\|_p,$$

the best coconvex approximation of f by polynomials of degree $\leq n$. If $Y_0 = \emptyset$, then we write $E_n^{(2)}(f)_p := E_n^{(2)}(f, \emptyset)_p$, which represents the best convex approximation.

If $f, g \in C^2[-1,1]$ are coconvex, then this can be expressed by the condition $f''(x)g''(x) \geq 0$, for all $x \in [-1,1]$.

Now, reasoning as for the monotone case, if $f \in C^2[-1,1]$ is convex, i.e., $f'' \geq 0$, then we get

$$E_n^{(2)}(f)_\infty \leq E_{n-2}(f'')_\infty, \quad n \geq 2.$$

If $f \in W_p^2[-1,1]$, $1 \leq p \leq \infty$, then it is known that for unconstrained approximation we have

$$E_n(f)_p \leq \frac{C}{n^2} E_{n-2}(f'')_p, \quad n \geq 2,$$

where $C = C(p)$ is an absolute constant and $E_n(f)_p$ denotes the unconstrained best approximation.

Thus, for convex approximation we have a loss of order n^2. Some of this loss can be recaptured by proving Jackson-type estimates. However, we point out that Shevchuk [350] and Leviatan–Shevchuk [234] have proved that there is a constant $C > 0$ such that for any $n \geq 2$, an $f \in C^2[-1,1] \cap \Delta^2$ exists satisfying

$$E_n^{(2)}(f)_\infty \geq C E_{n-2}(f'')_\infty > 0,$$

that is, the above estimate in convex approximation cannot be improved.

The situation is more pronounced for $0 < p < \infty$. Kopotun [208] proved that if $0 < p < \infty$, then for each $n \geq 2$ and constant $A > 0$, there is function $f = f_{p,n,A} \in C^\infty[-1,1] \cap \Delta^2$ such that

$$E_n^{(2)}(f)_p \geq A E_{n-2}(f'')_p.$$

1.7.1 Linear Methods in Convex Approximation

Concerning Jackson-type estimates in convex approximation, the linear approximation methods give estimates involving second-order moduli of smoothness of various types. First we present the following theorem.

Theorem 1.7.1. (see Shvedov [355], Leviatan [228], DeVore–Leviatan [88]) *If $f \in L^p[-1,1] \cap \Delta^2$, $0 < p \leq \infty$, then for every $n \geq 1$, we have*

$$E^{(2)}(f)_p \leq C \omega_2^\varphi\left(f, \frac{1}{n}\right)_p,$$

where $C = C(p)$, the dependence on p being important only for $p \to 0$.

Proof. The sketch of proof for the estimate will be presented in three distinct cases: (1) $p = \infty$; (2) $0 < p < 1$; (3) $1 \leq p < \infty$.

Case 1. (Leviatan [228]). We use the same construction of polynomials as in the proof of Theorem 1.6.3, Case 1, i.e., (keeping the notation),

$$P_n(f)(x) = f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j(x).$$

We have to show that $P_n(f)$ is convex when f is convex. If f is convex, then $s_j - s_{j-1} \geq 0$ for $j = 1, \ldots, n-1$, and since $R_0(x) = 1 + x$, it follows by the above form of $P_n(f)(x)$ that for the convexity of $P_n(f)(x)$, it suffices to have $R_j'' \geq 0$ for $j = 1, \ldots, n-1$. But for $x = \cos t,\ 0 < t < \pi$, we get

$$\begin{aligned} R_j''(x) = \tau_j'(x) &= \frac{dt}{dx}\frac{d}{dt}\int_{t-t_{n-j}}^{t+t_{n-j}} J_n(u)du \\ &= \frac{-1}{\sin t}\left[J_n\left(t + \frac{(n-j)\pi}{n}\right) - J_n\left(t - \frac{(n-j)\pi}{n}\right)\right] \\ &= \frac{-\lambda_n}{\sin t}\sin^8 \frac{nt + (n-j)\pi}{2} \\ &\quad \times \left[\frac{1}{\sin^8 \frac{1}{2}(t + (n-j)\pi/n)} - \frac{1}{\sin^8 \frac{1}{2}(t - (n-j)\pi/n)}\right] \geq 0 \end{aligned}$$

due to the following inequality in Lorentz–Zeller [250] (see also the proof in Leviatan [228]):

$$\sin(\alpha + \beta) \geq |\sin(\alpha - \beta)|, \quad \text{for } 0 \leq \alpha, \beta \leq \pi/2.$$

Case 2. We follow here the ideas of proof in DeVore–Leviatan [88] and we use the same construction as in Case 2 in proof of Theorem 1.6.3, i.e.,

$$\begin{aligned} P_n(f)(x) &= l_{j_0}(x) - a_0(x+1) + \sum_{j=0}^{n-1} a_j[R_j(x) - R_{j+1}(x)] \\ &= l_{j_0}(x) + \sum_{j=1}^{n-1}(a_j - a_{j-1})R_j(x). \end{aligned}$$

Since the convexity of f implies $a_j - a_{j-1} \geq 0$ and since $R_j''(x) \geq 0,\ j = 0, \ldots, n$, $x \in [-1, 1]$, we immediately get that $P_n(f)(x)$ is a convex polynomial.

Case 3. (Leviatan–Yu [244], see Shvedov [355] for estimates in terms of $\omega_2(f, \frac{1}{n})_p$). We follow here the ideas in Yu [407], Leviatan–Yu [244], i.e., we use the same construction as in the proof of Theorem 1.6.3, Case 3,

$$\bar{L}_n(f) = \bar{f}(-1) + \sum_{j=1}^{n} \bar{s}_j (R_{j-1} - R_j),$$

which is convex when f is convex. □

Remark. The estimate in Theorem 1.7.1, case $p = \infty$, can be refined, in the sense that for $f \in \Delta^2$ we can obtain polynomials $P_n(x) \in \Pi_n \cap \Delta^2$ satisfying an estimate of the form (see Ditzian–Jiang–Leviatan [97])

$$|f(x) - P_n(f)(x)| \leq C(\lambda)\omega_2^{\varphi^\lambda}(f, n^{-1}\varphi(x)^{1-\lambda})_\infty, \ x \in [-1, 1],$$

where $\lambda \in [0, 1]$.

In what follows we show that the polynomials constructed in Theorem 1.6.3 (and Theorem 1.7.1), case $p = \infty$, have a new additional property, closely related to that of starshapedness introduced by Definition 1.1.1 (ii).

First we need some simple auxiliary results.

Lemma 1.7.2. *Let $F, G : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that F is (usual) convex on $[a, b]$ and*

$$F(a) = G(a), \ F(b) = G(b), \ G(x) \leq F(x), \ \forall x \in [a, b].$$

Let $a_0, a_1 \in [a, b]$ with $a_0 < a_1$. Then the straight line passing through the points $(a_0, F(a_0))$ and $(a_1, F(a_1))$ also cuts the graph of G in (at least) two distinct points of $[a, b]$.

Proof. The equation of the straight line is given by

$$Y(x) = F(a_1) + (x - a_1)\frac{F(a_1) - F(a_0)}{a_1 - a_0}, \ x \in [a, b].$$

By hypothesis it follows that $Y(a_0) = F(a_0) \geq G(a_0)$, and since F is convex on $[a, b]$, we get

$$\frac{F(a_1) - F(a)}{a_1 - a} \leq \frac{F(a_1) - F(a_0)}{a_1 - a_0}$$

and hence

$$F(a_1) + (a - a_1)\frac{F(a_1) - F(a_0)}{a_1 - a_0} \leq F(a) = G(a),$$

that is, $Y(a) \leq G(a)$.

Then since Y and G are continuous on $[a, b]$, there exists a point $\xi \in [a_0, a]$ such that $Y(\xi) = G(\xi)$.

Analogously, by the relationships in the hypothesis, we have $Y(a_1) = F(a_1) \geq G(a_1)$, and since F is convex on $[a, b]$, we get

$$\frac{F(a_1) - F(a_0)}{a_1 - a_0} \leq \frac{F(b) - F(a_1)}{b - a_1}.$$

This implies

$$Y(b) = F(a_1) + (b - a_1)\frac{F(a_1) - F(a_0)}{a_1 - a_0} \leq F(b) = G(b),$$

therefore there exists $\eta \in [a, b]$ such that $Y(\eta) = G(\eta)$, which proves the lemma. □

In what follows, keeping the notation and the constructions in the proof of Theorem 1.6.3, Case 1 (i.e., $p = \infty$), let

$$S_n(x) = f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j-1}) \varphi_j(x)$$

and

$$\begin{aligned} P_n(f)(x) &= f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j(x) \\ &= f(-1) + \sum_{j=0}^{n-1} s_j (R_j(x) - R_{j+1}(x)). \end{aligned}$$

Another auxiliary result is the following.

Lemma 1.7.3. *If $j = 0, 1, \ldots, n$ then $\varphi_j(x) \le R_j(x)$ for all $x \in [-1, 1]$.*

Proof. Since we have $\varphi_0(x) = R_0(x) = 1 + x$ and $\varphi_n(x) = R_n(x) = 0$ for all $x \in [-1, 1]$, we may suppose that $j \in \{1, \ldots, n-1\}$. For such a j, by $\xi_j \in (-1, 1)$ we obtain

$$\varphi_j(x) = 0,\ x \in [-1, \xi_j],\ \varphi_j(x) = x - \xi_j,\ x \in [\xi_j, 1].$$

On the other hand, we obviously have $\varphi_j(\pm 1) = R_j(\pm 1)$,

$$\varphi_j'(-1) = 0 \le \int_{\pi - t_{n-j}}^{\pi + t_{n-j}} J_n(u) du = R_j'(-1),$$

and

$$\begin{aligned} R_j'(1) &= T_{n-j}(\text{arccos} 1) = T_{n-j}(0) \\ &= \int_{-t_{n-j}}^{t_{n-j}} J_n(u) du \le \int_{-\pi}^{\pi} J_n(u) du = 1 = \varphi_j'(1). \end{aligned}$$

Now, let suppose that there exists $x_0 \in (-1, 1)$ such that $R_j(x_0) < \varphi_j(x_0)$. We have two possibilities: (i) $-1 < x_0 \le \xi_j$; (ii) $\xi_j < x_0 < 1$.

Case (i). The equation of the straight line passing through the points $(-1, 0)$ and $(x_0, R_j(x_0))$ is given by $y = (x+1)\frac{R_j(x_0)}{x_0+1},\ x \ge -1$, with slope

$$\frac{R_j(x_0)}{x_0 + 1} < \frac{\varphi_j(x_0)}{x_0 + 1} = 0.$$

On the other hand, since R_j is convex on $[-1, 1]$, we get

$$R_j(x) \le (x+1)\frac{R_j(x_0)}{x_0 + 1},\ \forall x \in [-1, 1],$$

which implies

$$R_j'(-1) = \lim_{x \searrow -1} \frac{R_j(x) - R_j(-1)}{x+1} \leq \frac{R_j(x_0)}{x_0+1} < 0,$$

thereby contradicting the condition proved above, $R_j'(-1) \geq 0$.

Case (ii). The equation of the straight line passing through the points $(1, R_j(1))$ and $(x_0, R_j(x_0))$ is given by $y = R_j(1) + (x-1)\frac{R_j(1)-R_j(x_0)}{1-x_0}$, with slope

$$\frac{R_j(1) - R_j(x_0)}{1-x_0} > \frac{\varphi_j(1) - \varphi_j(x_0)}{1-x_0} = 1.$$

On the other hand, since R_j is convex on $[-1,1]$, we get

$$R_j(x) \leq R_j(1) + (x-1)\frac{R_j(1) - R_j(x_0)}{1-x_0}, \ \forall x \in [x_0, 1].$$

Hence we obtain

$$R_j'(1) = \lim_{x \nearrow 1} \frac{R_j(x) - R_j(1)}{x-1} \geq \frac{R_j(1) - R_j(x_0)}{1-x_0} > 1 = \varphi_j'(1),$$

thereby contradicting the inequality $R'(1) \leq \varphi_j'(1)$ proved above. This proves the lemma. □

An immediate consequence is the following.

Corollary 1.7.4. *If $f \in C[-1,1]$ is convex on $[-1,1]$, then for the polynomials defined above we have $f(x) \leq P_n(f)(x)$, for all $x \in [-1,1]$.*

Proof. Since f is convex on $[-1,1]$, by simple geometric reasoning it obviously follows that we have $f(x) \leq S_n(x)$ for all $x \in [-1,1]$.

On the other hand, since f is convex, we have $s_j - s_{j-1} \geq 0$ for all $j = 1, \ldots, n-1$, and then by Lemma 1.7.3, we get $f(x) \leq S_n(x) \leq P_n(f)(x)$ for all $x \in [-1,1]$, which proves the corollary. □

Now, following Popoviciu [315], it is obvious that a function $f : [-1,1] \to \mathbb{R}$ is increasing on $[-1,1]$ if for any $a_0 < a_1$ in $[-1,1]$, the coefficient A of the Lagrange interpolation polynomial $Ax + B$ coinciding with f on the points a_0, a_1 satisfies $A \geq 0$.

By analogy, let us denote by $ST[-1,1]$ the class of all $f \in C[-1,1]$ such that for all $a_0 < a_1$ in $[-1,1]$, the coefficient B in the above Lagrange interpolation polynomial satisfies $B \geq 0$.

The first main result is the following.

Theorem 1.7.5. *If f is convex on $[-1,1]$ and $f \in ST[-1,1]$, then $P_n(f)$, $n \in \mathbb{N}$, are convex on $[-1,1]$ and $P_n(f) \in ST[-1,1]$ for all $n \in \mathbb{N}$.*

Proof. The fact that $P_n(f)$ is convex is proved in Leviatan [228] (see also the proof of Theorem 1.6.3). Let us suppose in addition that $f \in ST[-1,1]$, i.e., for any $a_0 < a_1$ in $[-1,1]$, the straight line passing through the points $(a_0, f(a_0))$, and $(a_1, f(a_1))$, cuts the y-axis at a point ≥ 0. Then, by Corollary 1.7.4 and applying Lemma 1.7.2 for $G(x) = f(x)$ and $F(x) = P_n(f)(x)$ (since $f(\pm 1) = P_n(f)(\pm 1)$), we easily obtain that $P_n(f) \in ST[-1,1]$. □

Corollary 1.7.6. *Let us suppose that $f \in C^1[-1,1]$ is convex on $[-1,1]$ and satisfies the differential inequality*

$$f(x) - xf'(x) \geq 0, \ \forall x \in [-1,1].$$

Then the polynomials $P_n(f)$ are convex on $[-1,1]$ and, in addition, satisfy the differential inequality

$$P_n(f)(x) - xP_n'(f)(x) \geq 0, \forall x \in [-1,1].$$

Proof. The fact that the P_n are convex follows from Leviatan [228]. Let $a_0 < a_1$ in $[-1,1]$ and let $Ax + B$ be the Lagrange interpolation polynomial coinciding with f on a_0 and a_1. By Pompeiu [309], it follows that for $0 \leq a_0 < a_1$ or $a_0 < a_1 \leq 0$, there is $\xi \in [a_0, a_1]$ such that $B = f(\xi) - \xi f'(\xi)$, i.e., by the differential inequality satisfied by f we have $B \geq 0$.

Now, if $a_0 < 0 < a_1$, then as above, for the Lagrange interpolation polynomials $A_1 x + B_1$ and $A_2 x + B_2$, coinciding with f at the points $\{a_0, 0\}$ and $\{0, a_1\}$, respectively, we have $B_1 \geq 0$ and $B_2 \geq 0$. But by simple geometric reasoning, it is easy to see that we always have $A_1 x + B_1 \leq Ax + B$ for all $x \in [a_0, 0]$, and $A_2 x + B_2 \leq Ax + B$ for all $x \in [0, a_1]$, which implies $B \geq 0$ in this case too.

As a conclusion, we get that $f \in ST[-1,1]$, which by Theorem 1.7.5 implies $P_n(f) \in ST[-1,1]$ too. This means that for all $a_0 < a_1$ in $[-1,1]$, the coefficient B_n of the Lagrange interpolation polynomial $A_n x + B_n$ coinciding with $P_n(f)$ on a_0 and a_1 satisfies $B_n \geq 0$. On the other hand, for all $a_0 < a_1$ we have

$$B_n = P_n(f)(a_0) - a_0 \frac{P_n(f)(a_1) - P_n(f)(a_0)}{a_1 - a_0} \geq 0.$$

Passing now to the limit with $a_0, a_1 \to x$, with arbitrary $x \in [-1,1]$, we get $P_n(f)(x) - xP_n'(f)(x) \geq 0$, which proves the theorem. □

Remark. Recall that by Definition 1.1.1 (ii) a function $f : [0,a] \to \mathbb{R}_+$ is called starshaped if $f(0) = 0$ and $f(\lambda x) \leq \lambda f(x)$ for all $x \in [0,a]$ and $\lambda \in [0,1]$, and if, in addition, it is continuously differentiable, then f is starshaped if and only if satisfies the differential inequality $xf'(x) - f(x) \geq 0$ for all $x \in (0,a]$. Comparing it with the differential inequality in Corollary 1.7.6, we see that $f \in ST[-1,1]$ implies that $-f$ is starshaped on $[0,1]$; therefore the class $ST[-1,1]$ is obviously related to the class of starshaped functions.

1.7.2 Nonlinear Methods in Convex Approximation

Concerning the Jackson-type estimates in convex approximation, recall that as a negative result, Shvedov [355] proved that it is impossible to get an estimate involving $\omega_4(f, 1/n)_p$ with an absolute constant (see also Wu–Zhou [403] for related results).

During the years 1994–1996, the gap between the affirmative estimates (in terms of $\omega_2^{\varphi}(f, 1/n)_p$) and the negative ones was closed in a series of papers by DeVore, Hu, Kopotun, Leviatan, and Yu, who proved, using nonlinear methods, the following.

Theorem 1.7.7. (Hu–Leviatan–Yu [175], Kopotun [204], DeVore–Hu–Leviatan [87]) *Let $f \in L^p[-1, 1] \cap \Delta^2$, $0 < p \leq \infty$. Then there is an absolute constant $C = C(p)$, such that for each $n \geq 2$ we have*

$$E_n^{(2)}(f)_p \leq C\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p.$$

Proof. Let $f \in L^p[-1, 1] \cap \Delta^2$ and $0 < p \leq \infty$. We sketch the proof using the nonlinear method in DeVore–Hu–Leviatan [87]: first one approximates f by a convex continuous piecewise quadratic q_n and then one approximates q_n by an algebraic polynomial.

Define $x_{n,j} := \cos[\pi(n-j)/n]$, $j = 0, \ldots, n$, $x_{n,j} := -1$, $j < 0$, and $x_{n,j} := 1$, $j > n$ and denote by q_n the convex continuous piecewise quadratic function for f and the points $\{x_{n,j}\}_{j=0}^n$. One can represent q_n as a sum of the truncated powers $(x - x_{n,j})_+$ and $(x - x_{n,j})_+^2, j = 1, \ldots, n$, as follows. First, one can classify the nodes $x_{n,j}$ according to four types depending on the second-order divided differences of q_n:

$$a_{n,j} := [x_{n,j-1}, x_{n,j}, x_{n,j+1}; q_n], \qquad j = 1, \ldots, n-1.$$

Also, we take $a_{n,0} = a_{n,n} := \infty$.

Let $1 \leq j \leq n-1$. Then $x_{n,j}$ are said to be of type I if

$$a_{n,j+1} < a_{n,j} \leq a_{n,j-1},$$

$x_{n,j}$ are type II if

$$a_{n,j-1} < a_{n,j} \leq a_{n,j+1},$$

$x_{n,j}$ are type III if

$$\max\{a_{n,j-1}, a_{n,j+1}\} < a_{n,j};$$

all the other $x_{n,j}$ are said to be of type IV.

Let us define

$$A_{n,j} := a_{n,j} - a_{n,j+1},\ j = 1, \ldots, n-2, \quad B_{n,j} := -A_{n,j-1},\ j = 2, \ldots, n-1,$$

and

$$A_{n,0} := [x_{n,0}, x_{n,1}; q_n] - [x_{n,1}, x_{n,2}; q_n] + [x_{n,0}, x_{n,2}; q_n].$$

If $x_{n,j}$ is of type II or III, then $A_{n,j} > 0$.

We obtain the following representation of $q_n(x)$ for $x \in [-1, 1]$:

$$\begin{aligned} q_n(x) = q_n(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ + \sum_{x_{n,j} \in I \cup III} A_{n,j}((x_{n,j+1} - x_{n,j})(x - x_{n,j})_+ - (x - x_{n,j})_+^2) \\ + \sum_{x_{n,j} \in II \cup III} B_{n,j}((x_{n,j} - x_{n,j-1})(x - x_{n,j})_+ + (x - x_{n,j})_+^2). \end{aligned}$$

Now, as a polynomial approximation to f, we take

$$\begin{aligned} P_n(x) := q_n(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ + \sum_{x_{n,j} \in I \cup III} A_{n,j}((x_{n,j+1} - x_{n,j})\sigma_{Mn,Mj}(x) - R_{Mn,Mj}(x)) \\ + \sum_{x_{n,j} \in II \cup III} B_{n,j}((x_{n,j} - x_{n,j-1})\sigma_{Mn,Mj}(x) + \bar{R}_{Mn,Mj}(x)), \end{aligned}$$

where $\sigma_{n,j}(x)$ is a good polynomial approximation to $(x - x_{n,j})_+$ and $R_{n,j}(x), \bar{R}_{n,j}(x)$ are good polynomial approximations to $(x - x_{n,j})_+^2$. Then P_n is a polynomial of degree at most $50Mn \max\{1, 1/p\}$ and satisfies

$$P_n''(x) \geq q_n''(x) \geq 0 \qquad \text{for all } x \in [-1, 1], \quad x \neq x_{n,j}, \quad 1 \leq i \leq n-1.$$

Thus P_n is convex on $[-1, 1]$. For the details, see DeVore–Hu–Leviatan [87]. □

Remark. However, notice that for $p = \infty$, surprisingly in Leviatan–Shevchuk [243] it was proved that $E_n^{(2)}(f)_\infty \leq C\omega_4^\varphi(f; 1/n)_\infty$, for $n \geq N(f)$ and $C > 0$ an absolute constant.

1.7.3 Pointwise Convex Approximation

Concerning pointwise convex polynomial approximation we mention the following result.

Theorem 1.7.8. *(i) (Kopotun [204]) Let $f \in C^r[-1, 1] \cap \Delta^2, 0 < r \leq 2$. For any $n \geq 2$, there is a polynomial $p_n \in \Pi_n \cap \Delta^2$ satisfying*

$$|f^{(i)}(x) - p_n^{(i)}(x)| \leq C\omega_{r-i}(f^{(i)}, \rho_n(x))_\infty, \quad 0 \leq i \leq r, \quad x \in [-1, 1].$$

(ii) (Manya; see Shevchuk [349], p. 148, Theorem 17.2) If $f \in C^r[-1, 1] \cap \Delta^2, r \geq 2$, then for any $n \geq r + k - 1$, there is a polynomial $p_n \in \Pi_n \cap \Delta^2$ satisfying

$$|f(x) - p_n(x)| \leq C\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \quad x \in [-1, 1],$$

where $C = C(r, k)$. *In particular,*

$$E_n^{(2)}(f)_\infty \leq Cn^{-r}\omega_k(f^{(r)}, 1/n)_\infty, \quad n \geq r + k - 1.$$

Remark. By virtue of Shvedov [355], for $f \in C[-1, 1] \cap \Delta^2$ one cannot, in general, achieve the pointwise estimates in Theorem 1.7.8 (ii), when the right-hand side is $\omega_4(f, \rho_n(x))_\infty$.

1.7.4 Convex Approximation with Modified Weighted Moduli of Smoothness

In this subsection, for each r times differentiable convex function, one studies whether or not its degree of convex polynomial approximation in the uniform norm may be estimated by a Jackson-type estimate involving the modified weighted moduli of smoothness of order (k, r) and how the constants in this estimate behave. We have the following possibilities: for some pairs (k, r), such an estimate with constants depending only on these parameters is valid. For other pairs the estimate is valid, but only with constants that depend on the approximated function, while there are pairs for which the Jackson-type estimate is invalid.

The following estimates on the degree of convex polynomial approximation of functions $f \in \mathbb{B}^r \cap \Delta^2$ were proved by Leviatan [228] ($r = 1$ and 2) and by Kopotun [209] ($r = 3$ and $r > 5$):

$$E_n^{(2)}(f)_\infty \leq \frac{c(r)}{n^r}\|\varphi^r f^{(r)}\|_\infty, \ n \geq r.$$

Here $\mathbb{B}^r$ was defined in Remark 2 after Definition 1.1.3.

Moreover, Kopotun [209] proved that the above estimate is invalid for $r = 4$. More exactly, for every $A > 0$ and $n \geq 1$, there is an $f = f_{n,A} \in \mathbb{B}^4 \cap \Delta^2$ such that

$$E_n^{(2)}(f)_\infty > A\|\varphi^4 f^{(4)}\|_\infty.$$

However, Leviatan and Shevchuk [243] have proved that for $f \in \mathbb{B}^4 \cap \Delta^2$, we have

$$E_n^{(2)}(f)_\infty \leq \frac{c}{n^4}\left(\|\varphi^4 f^{(4)}\|_\infty + \frac{1}{n^2}\|f\|_\infty\right), \ n \geq 1,$$

with an absolute constant c.

In fact, Leviatan [228] and Kopotun [205] have obtained more refined estimates, involving the Ditzian–Totik [98] moduli of smoothness and the modified Ditzian–Totik moduli of smoothness (introduced in Shevchuk [349]), see Definition 1.1.3. In particular, the following result follows (see, e.g., Theorem 1.1 in Kopotun–Leviatan–Shevchuk [213]).

Theorem 1.7.9. (Leviatan [228], Kopotun [209, 204, 205], Kopotun–Listopad [216], Leviatan–Shevchuk [243] and Kopotun–Leviatan–Shevchuk [213] (the case $\alpha = 4$)). *For $f \in \Delta^2$ and any $\alpha > 0$, we have*

$$E_n(f)_\infty = O(n^{-\alpha}), \quad n \to \infty \Leftrightarrow E_n^{(2)}(f)_\infty = O(n^{-\alpha}), \quad n \to \infty.$$

Proof. First, since obviously $E_n(f)_\infty \leq E_n^{(2)}(f)_\infty$, the implication from the right-hand side to the left-hand side is immediate. Conversely, take for example $0 < \alpha < 4$ and let us suppose that $E_n(f)_\infty = O(n^{-\alpha})$. According to a result of Ditzian–Totik in unconstrained approximation (see, e.g., Theorem 7.7, pp. 265 in DeVore–Lorentz [91]), this is equivalent to $\omega_4^\varphi(f;t)_\infty = O(t^\alpha)$. Then, by the remark after Theorem 1.7.7 we obtain $E_n^{(2)}(f)_\infty = O(n^{-\alpha})$. The case $\alpha \geq 4$ can be found in Kopotun–Leviatan–Shevchuk [213]. □

A problem of interest is to find the values of parameters k and r, for which the statement

$$\text{if } f \in \Delta^2 \cap \mathbb{C}_\varphi^r, \text{ then}$$
$$E_n^{(2)}(f)_\infty \leq \tfrac{C}{n^r}\omega_{k,r}^\varphi(f^{(r)}, 1/n), \quad n \geq N,$$

(where $C > 0$ and $N > 0$ are constants) is valid, and for which it is invalid. Here the class C_φ^r is defined by the Remark 2 of Definition 1.1.3.

As a sample concerning estimates in terms of these modified Ditzian–Totik moduli (defined by Definition 1.1.3) we present the following.

Theorem 1.7.10. (Kopotun [205]) *If $r, k \geq 0$, then for any $f \in C_\varphi^r \cap \Delta^2$ we have*

$$E_n^{(2)}(f)_\infty \leq Cn^{-r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq r + k - 1,$$

with $C = C(r,k)$, if and only if either $0 \leq r + k \leq 3$ or $r \geq 5$.

Remark. Notice that all the possibilities (concerning positive and negative results and how the constants depend on the parameters and on the function) in such of estimates, can be found in the paper Kopotun–Leviatan–Shevchuk [213] (for a more detailed discussion see Note 1.10.3).

1.7.5 Uniform Coconvex Approximation

A natural extension of convex polynomial approximation is coconvex polynomial approximation. In this subsection we present those coresponding to uniform approximation.

Theorem 1.7.11. (Kopotun–Leviatan–Shevchuk [212]) *If $f \in C[-1,1] \cap \Delta^2(Y_s)$, then there is a constant $C = C(s)$ such that for any $n \geq \frac{C}{d(Y_s)}$, we have*

$$E_n^{(2)}(f, Y_s)_\infty \leq C\omega_3^\varphi\left(f, \frac{1}{n}\right)_\infty,$$

where $d(Y_s)$ was defined in Section 1.6 (just before Theorem 1.6.7).

Concerning simultaneous comonotone and coconvex approximation, we have the following theorem.

Theorem 1.7.12. (Kopotun [206], Kopotun–Leviatan [211]) *Suppose* $f \in C^r[-1,1] \cap \Delta^r(Y_s)$, $1 \leq r \leq 2$. *Then there is a constant* $C = C(s)$ *with the property that for any* $n \geq \frac{C}{d(Y_s)}$, *there are polynomials* $P_n \in \Pi_n \cap \Delta^r(Y_s)$ *such that if* $r = 1$, *we simultaneously have*

$$\|f^{(i)} - P_n^{(i)}\|_\infty \leq \frac{C}{n^{1-i}} \omega_1^\varphi\left(f', \frac{1}{n}\right)_\infty, \quad 0 \leq i \leq 1;$$

and if $r = 2$, *we simultaneously have*

$$\|f^{(i)} - P_n^{(i)}\|_\infty \leq \frac{C}{n^{2-i}} \omega_1^\varphi\left(f'', \frac{1}{n}\right)_\infty, \quad 0 \leq i \leq 1,$$

and

$$\|f'' - P_n''\|_\infty \leq \frac{C}{d_0} \omega_1^\varphi\left(f'', \frac{1}{n}\right)_\infty,$$

where d_0 *is defined as in the result on simultaneous approximation in the Note 1.10.2.*

Proof. We sketch here the proof for $r = 2$ using Kopotun [206]. It is done by induction on s, the number of changes of convexity, using the so-called method of "flipped" functions, introduced in the case of comonotone approximation by Beatson–Leviatan [34] (see the proof of Theorem 1.6.7).

For $s = 0$, Theorem 1.7.12 becomes the result in convex approximation in Kopotun [204].

Let $s \geq 1$, $f \in C^2[-1,1]$, changing its convexity at the points of $Y_s = \{y_s < \cdots < y_1\}$, where $-1 =: y_{s+1} < y_s < \cdots < y_1 < 1 =: y_0$. Without loss of generality, we may assume that $f''(x) \geq 0$, for all $x \in [-1, y_s]$. Let us denote one fixed y_j by α, to have a choice set $y_s = \alpha$. Evidently, we have $f''(\alpha) = 0$, and we may assume $f(\alpha) = f'(\alpha) = 0$ too (otherwise, we subtract a linear function from f without affecting the convexity).

Following the ideas in Beatson–Leviatan [34] (see the proof of Theorem 1.6.7 too), we define the "flipped" function

$$f_F(x) := \begin{cases} f(x) & \text{if} \quad x \geq \alpha, \\ -f(x) & \text{if} \quad x < \alpha. \end{cases}$$

Then, it is easy to see that $f_F \in C^2[-1,1], f_F(\alpha) = f_F'(\alpha) = f_F''(\alpha) = 0$, and f_F has $s-1$ points its convexity changes, at $y_{s-1}, \ldots, y_1$, and as in, e.g., Leviatan [232], we have

$$\omega_1^\varphi(f_F'', t)_\infty \leq C\omega_1^\varphi(f'', t)_\infty, \ t > 0.$$

Define $d(s) := \min\{y_s + 1, \ldots, y_{s-1} - y_s, \ldots, 1 - y_1\}$.

By mathematical induction, there is a constant $A(s-1)$ such that for any $n > A(s-1)/d(s) \geq A(s-1)/d(s-1)$, there exists a polynomial $q_n \in \Pi_n$

with $f_F''(x)q_n''(x) \geq 0$, $x \in [-1,1]$ and the estimates in the statement (for $r = 2$) hold for both f_F and q_n. Notice that since $f_F(\alpha) = 0$, if we increase the constant in the estimate in the statement for $r = 2$ and $i = 0$, then we can suppose that $q_n(\alpha) = 0$.

Take $n > \max\{A(s-1)/d(s), 50/(y_2 - \alpha), 50(\alpha+1)\}$ and consider the decomposition $[-1,1] = \cup_{j=1}^{n}[x_j, x_{j-1}]$, where $x_j = \cos(j\pi/n)$. Defining j_0 such that $\alpha \in [x_{j_0}, x_{j_0-1}]$, it follows that $x_{j_0+3} \geq -1$ and $x_{j_0-4} \leq y_{s-1}$, that is, $[-1,\alpha]$ and $[\alpha, y_{s-1}]$ contain at least three intervals $[x_j, x_{j-1}]$ each. Therefore $\varphi(\alpha) \geq n^{-1}$ and $2\varphi(\alpha) \geq n\delta_n(\alpha)$, where $\delta_n(x) = \sqrt{1-x^2}/n + 1/n^2$.

Define the algebraic polynomial $P_n(x) := \int_\alpha^x P_n'(y)dy$ such that it satisfies

$$P_n'(x) = [q_n'(x) - q_n'(\alpha)] \cdot Q_n(x) + q_n'(\alpha) \cdot R_n(x),$$

where it remains to show that it is possible to choose the polynomials $Q_n(x)$ and $R_n(x)$ such that P_n is coconvex with f and the estimates in the statement are satisfied.

In Kopotun [206] it is proved that the following properties of Q_n and R_n are sufficient for the coconvexity of P_n with f:

$$\begin{aligned} Q_n(x)\mathrm{sgn}_\alpha(x) &\geq 0,\ x \in [-1,1],\ \mathrm{sgn}_\alpha(x) = \mathrm{sgn}(x-\alpha) \\ [q_n'(x) - q_n'(\alpha)]q_n''(x)Q_n'(x)\mathrm{sgn}_\alpha(x) &\geq 0,\ x \in [-1,1], \\ f''(x)R_n'(x)\mathrm{sgn}(q_n'(\alpha)) &\geq 0,\ x \in [-1,1]. \end{aligned}$$

Indeed, together with the inequality $f_F''(x)q_n''(x) \geq 0$, these above inequalities imply

$$\begin{aligned} \mathrm{sgn}\{P_n''(x)f''(x)\} &= \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))Q_n'(x)f''(x) + q_n''(x)Q_n(x)f''(x) \\ &\qquad + q_n'(\alpha)R_n'(x)f''(x)\} \\ &\geq \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))Q_n'(x)f''(x) + q_n''(x)Q_n(x)f''(x)\} \\ &= \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))Q_n'(x)q_n''(x)\mathrm{sgn}_\alpha(x) \\ &\quad + (q_n''(x))^2 Q_n(x)\mathrm{sgn}_\alpha(x)\} \geq 0. \end{aligned}$$

All the details can be found in Kopotun [206]. □

Remark. The following result in trigonometric coconvex approximation of 2π periodic continuous functions (i.e., in the class denoted by $C_{2\pi}$) by trigonometric polynomials was proved in Popov [310]: if $y_i, i = 1, \ldots, 2s$, are distinct points in $[-\pi, \pi)$, setting $y_i := y_{i+2s} + 2\pi$ and $Y = \{y_i\}_{i \in \mathbb{Z}}$, if $f \in C_{2\pi}$ changes its convexity at the points of Y, then for any $n \in \mathbb{N}$, there exists a trigonometric polynomial T_n of degree $\leq n$, coconvex with f, that satisfies $\|f - T_n\|_\infty \leq C(Y)\omega_2(f;\pi/n)_\infty$.

1.7.6 Coconvex Approximation with Modified Weighted Moduli of Smoothness

Similar to the case of convex polynomial approximation, there exists a complete characterization of the coconvex uniform polynomial approximation with respect to the modified Ditzian–Totik moduli of smoothness.

The first result presented is an analog in unconstrained polynomial approximation for coconvex polynomial approximation.

Theorem 1.7.13. (Kopotun–Leviatan–Shevchuk [214]) *For every* $s \geq 0$, $Y_s \in \mathbb{Y}_s$, $f \in \Delta^2(Y_s)$, *and* $\alpha > 0$, *we have*

$$E_n(f)_\infty = O(n^{-\alpha}), \quad n \to \infty \Leftrightarrow E_n^{(2)}(f, Y_s)_\infty = O(n^{-\alpha}),\ n \to \infty.$$

Proof. Since for any $s \geq 0$, we have $E_n(f)_\infty \leq E_n^{(2)}(f, Y_s)_\infty$, the implication from the right-hand side to the left-hand side is immediate. Conversely, suppose first, for example, $0 < \alpha < 3$ and that $E_n(f)_\infty = O(n^{-\alpha})$. By, e.g., Theorem 7.7, pp. 265 in DeVore–Lorentz [91], this is equivalent to $\omega_3^\varphi(f; t)_\infty = O(t^\alpha)$, which combined with Theorem 1.7.11, implies $E_n^{(2)}(f, Y_s)_\infty = O(n^{-\alpha})$. The general case ($\alpha \geq 3$) follows from the Jackson-type estimates involving the modified Ditzian–Totik moduli of smoothness and obtained in Kopotun–Leviatan–Shevchuk [214]. □

As in the case of convex approximation, we are interested in finding for which values of the parameters k, r and s the statement

$$\text{if } f \in \Delta^2(Y_s) \cap \mathbb{C}^r_\varphi, \text{then}$$
$$E_n^{(2)}(f, Y_s)_\infty \leq C n^{-r} \omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \geq N$$

(where $C > 0$ and $N > 0$ are constants), is valid, and for which it is invalid.

As a sample, we present here the following.

Theorem 1.7.14. (Kopotun–Leviatan–Shevchuk [214]) *If* $k \geq 1$, $r \geq 5$, $s \geq 1$, $Y_s \in \mathbb{Y}_s$ *and* $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$, *then*

$$E_n^{(2)}(f, Y_s)_\infty \leq c n^{-r} \omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \geq N(k, r, Y_s),$$

where $N(k, r, Y_s)$ *is a constant depending on* k, r *and* $Y(s)$.

Remark. A complete characterization (concerning positive and negative results and how the constants depend on the parameters and on the function) for such estimates, can be found in the paper Kopotun–Leviatan–Shevchuk [214] (for a more detailed discussion see Note 1.10.4).

1.7.7 Pointwise Coconvex Approximation

In this subsection we present results corresponding to pointwise approximation. They can be summarized by the following.

Theorem 1.7.15. *(i) (Dzyubenko–Gilewicz–Shevchuk [107]) Suppose that* $f \in C[-1, 1]$ *changes its convexity at the finite set* Y_s *of* s *distinct points in* $(-1, 1)$. *Then for each* $n > N(Y_s)$, *there is a polynomial of degree* $\leq n$, *coconvex with* f, *such that*

$$|f(x) - P_n(x)| \leq c\omega_2(f; \sqrt{1 - x^2}/n), \text{ for all } x \in [-1, 1],$$

where $c > 0$ *is an absolute constant and if* $s = 1$ *then* $N(Y_s) = 1$;

(ii) (Dzyubenko–Zalizko [110]) Suppose that $f \in C[-1,1]$ *changes its convexity at the finite set* Y_s *of* $s > 1$ *distinct points in* $(-1,1)$*. Then for each* $n \geq 2$*, there is a polynomial of degree* $\leq n$*, coconvex with* f*, such that*

$$|f(x) - P_n(x)| \leq c(Y_s)\omega_3(f; 1/n^2 + \sqrt{1-x^2}/n), \text{ for all } x \in [-1,1],$$

where $c(Y_s) > 0$ *depends only on the points of* Y_s*;*

(iii) (Dzyubenko–Zalizko [111]) Denote by W^r*,* $r \in \mathbb{N}$*, the set of functions continuous on* $[-1,1]$ *having absolutely continuous the* $(r-1)$*th derivative on* $[-1,1]$ *and satisfying* $|f^{(r)}(x)| \leq 1$*, for almost all* $x \in [-1,1]$*. If* $r \in \mathbb{N}$*,* $s \geq 2$ *and* $f \in W^r$ *changes its convexity at the finite set* Y_s *of* s *distinct points in* $(-1,1)$*, then for each* $n \geq r-1$*, there is a polynomial of degree* $\leq n$*, coconvex with* f*, such that*

$$|f(x) - P_n(x)| \leq C(Y_s, r)\rho_n^r(x), \text{ for all } x \in [-1,1],$$

where $C(Y_s, r) > 0$ *depends only on* r *and the points of* Y_s*.*

Remark. From the estimate (iii) in Theorem 1.7.15, obviously it follows that for $f \in W^r \cap \Delta^2(Y_s)$ (where we recall that $\Delta^2(Y_s)$ denotes the set of all functions that change their convexities on the points of Y_s such that on the last interval determined by the points of Y_s they are convex), we have

$$E^{(2)}(f, Y_s) \leq C(r, Y_s)\frac{\|f^{(r)}\|_\infty}{n^r}, \ n \geq r-1.$$

On the other hand, in the comonotone case, from Theorem 1.6.8 and its Remark, we get the estimate

$$E_n^{(1)}(f; Y_s) \leq C(r, s)\frac{\|f^{(r)}\|_\infty}{n^r}, n \geq N(r, Y_s),$$

and that for $s \geq 2$, $N(r, Y_s)$ cannot be replaced by $N(r, s)$. It is then natural to ask whether an analogue of the above estimate holds for coconvex approximation too, that is, the question is whether for $s \geq 2$ and $f \in W^r \cap \Delta^2(Y_s)$, then

$$E_n^{(2)}(f; Y_s) \leq C(r, s)\frac{\|f^{(r)}\|_\infty}{n^r}, \ n \geq N(r, Y_s),$$

holds. In the recent paper Dzyubenko–Gilewicz–Shevchuk [108] it was proved (by counterexample) that this is impossible for $r \geq 3$.

1.7.8 Nearly Coconvex Approximation

The first results on nearly coconvex polynomial approximation belong to Myers [281] and are particular cases of general results on nearly coconvex approximation of order $k \in \mathbb{N}$. Let us briefly recall these results. First, a function is called piecewise convex of order k on $[a,b]$ if there exists $Y_s = \{y_j, j = 1, \dots, s\}$, with $a := y_{s+1} < y_s < \cdots < y_1 < y_0 =: b$ such that f is k-convex or k-concave on each interval (y_{j+1}, y_j), $j = 0, \dots, s$. Here recall that f is called k-convex (concave) on an interval I, if $\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh) \geq 0$ ($\Delta_h^k f(x) \leq 0$, respectively) for all $x \in I$, $h \in \mathbb{R}$ with $x + kh \in I$. Similar to the comonotone case, the algebraic polynomials $(P_n)_n$ are said to form a nearly co-k-coconvex approximation sequence to the piecewise k-convex function f (with respect to Y_s) if for any $0 < \varepsilon < \frac{1}{2} \min_{j=0,\dots,s}\{y_j - y_{j+1}\}$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, the polynomials P_n are of the same k-convexity as f on the intervals $(y_{j+1} + \varepsilon, y_j - \varepsilon)$, $j = 0, \dots, s$. Note that for $k \geq 2$, the hypothesis that the piecewise convex function f, of order k, is continuous on $[a,b]$ does not imply that f necessarily changes its k-convexity at the points $y_j, j = 0, \dots, s$. The main result in Myers [281] states that if $k \in \mathbb{N}$ and $f \in C[a,b]$ is piecewise k-convex, then there exists a sequence of polynomials $(P_n)_n$, $P_n \in \Pi_n$, nearly co-k-convex to f, such that $\|f - P_n\|_\infty \leq C\omega_1(f, 1/n)_\infty$. If, in addition, $f \in C^1[a,b]$, then the estimate can be improved to $\|f - P_n\|_\infty \leq C_1 n^{-1} \omega_1(f', 1/n)_\infty$. (Here C and C_1 depend only on k.) For $k = 2$ we get results for nearly coconvex approximation.

All of Myers's results in [281] can be improved. To be more precise, take $[a,b] = [-1,1]$. Given the points $Y_s = \{y_j, j = 1, \dots, s\}$, where $-1 := y_{s+1} < y_s < \cdots < y_1 < 1 := y_0$, let us define

$$O(n, c, Y_s) = [-1,1] \cap \{\cup_{i=1}^{s} (y_i - c\rho_n(y_i), y_i + c\rho_n(y_i))$$

and let $\Delta_*^2(Y_s)$ be the class of all $f \in C[-1,1]$ that are convex on $[y_{i+1}, y_i]$ for i even and concave on $[y_{i+1}, y_i]$ for i odd.

Theorem 1.7.16. *(i) (Dzyubenko–Gilewicz [104] (resp. [105])) If $f \in \Delta^2(Y_s)$ (resp. $f \in \Delta_*^2(Y_s)$), then for each $n \geq 1$ (resp. $n \geq 3$), there exists a polynomial $P_n(x)$ of degree $\leq n$, coconvex with f on $[-1,1] \setminus O(n,c,Y_s)$ (resp. on $[-1 + c^2/n^2, 1 - c^2/n^2] \setminus O(n,c,Y_s)$), such that for $p = 3$ (resp. $p = 4$), we have*

$$|f(x) - P_n(x)| \leq C(s)\omega_p(f, \rho_n(x))_\infty, \; x \in [-1,1].$$

(ii) (Leviatan–Shevchuk [242]) Let $r = 0$ and $k \leq 4$, or $r = 1$ and $k \leq 3$, or $r \geq 2$ and $k \geq 1$. For any $f \in \Delta^2(Y_s) \cap C^r[-1,1]$ and $n \geq k + r - 1$, there exists a polynomial $P_n(x)$ of degree $\leq n$ such that

$$P_n''(x) \Pi_{i=1}^{s} (x - y_i) \geq 0$$

for all $x \in [-1,1] \setminus O(r,k,c^*,Y_s)$ and

$$\|f - P_n\|_\infty \leq \frac{C}{n^r} \omega_k^\varphi(f^{(r)}; 1/n)_\infty.$$

Here $c^* := c^*(k, r, s)$, $C := C(k, r, s)$, *and*

$$O(r, k, c^*, Y_s) \begin{cases} = \cup_{j=0}^{s+1}(y_j - c^*\rho_n(y_j), y_j + c^*\rho_n(y_j)) \text{ if } (r,k) = (0,4) \text{ or } (1,3), \\ = \cup_{i=1}^{s}(y_i - c^*\rho_n(y_i), y_i + c^*\rho_n(y_i)) \text{ otherwise.} \end{cases}$$

Remarks. (1) In Nissim–Yushchenko [289], it is proved that the estimate in Theorem 1.7.16 (ii) is invalid for $r = 1$ and $k > 3$, and for $r = 0$ and $k > 4$.

(2) From their proofs it follows that the methods in polynomial coconvex approximation are nonlinear, such that even if two continuous functions $f, g \in C[-1, 1]$ have the same points where the convexities change $Y_s = \{y_j, j = 1, \ldots, s\}$, and both are of the same convexity on each subinterval, the coconvex approximation polynomials P_n do not satisfy $P_n(f + g) = P_n(f) + P_n(g)$.

It is easy to show that $\Delta^2(Y_s)$ is a convex cone, i.e., $f, g \in \Delta^2(Y_s)$ and $\alpha \in \mathbb{R}_+$ implies $f + g \in \Delta^2(Y_s)$ and $\alpha f \in \Delta^2(Y_s)$.

Suggested by the proof of Theorem 1.2.1, we easily can construct a polynomial coconvex with f that is additive and a positive homogeneous operator on $\Delta^2(Y_s) \cap C^3[-1, 1]$. In this sense, we present the following result.

Theorem 1.7.17. *If* $f \in \Delta^2(Y_s)$ *is three times continuously differentiable in* $[-1, 1]$, *then for any* $n \in \mathbb{N}$, *a sequence of polynomials* $(P_n)_n$ *can be constructed such that degree*$(P_n) \le n + s + 2$ *and for any* $\varepsilon > 0$, *there is* n_0 *with the properties*

$$\|f - P_n\|_\infty < \varepsilon, \qquad P_n''(x) f''(x) \ge 0, \quad \forall n \ge n_0, \quad x \in [-1, 1],$$

and $P_n(\alpha h + \beta g) = \alpha P_n(h) + \beta P_n(g)$ *for all* $n \in \mathbb{N}$, $h, g \in \Delta^2(Y_s) \cap C^3[-1, 1]$, $\alpha, \beta \ge 0$.

Recalling the notation $\Pi(x, Y_s) = \Pi_{i=1}^{s}(x - y_i)$, *the error estimate can be expressed by*

$$\|P_n(f) - f\|_\infty \le C(Y_s)\omega_2 \left(\frac{f''}{\Pi(\cdot, Y_s)}; \frac{1}{n} \right)_\infty$$

for all $n \in \mathbb{N}$.

Proof. From the differentiability hypothesis, it easily follows that $F(x) = \frac{f''(x)}{\Pi(x, Y_s)}$ is continuous (by extension) on $[-1, 1]$. Also, by simple reasoning we get that $F(x) \ge 0$, for all $x \in [-1, 1]$. Note that without loss of generality, we may suppose $f(-1) = f'(-1) = 0$.

Now define $Q_n(f)(x) = \int_{-1}^{x} L_n(F)(t) \cdot \Pi(t, Y_s) dt$ and $P_n(f)(x) = \int_{-1}^{x} Q_n(f)(t) dt$, where $L_n, n \in \mathbb{N}$ is a sequence of positive linear polynomial operators on $C[-1, 1]$ satisfying degree$(L_n(f)) \le n$ and

$$\|L_n(f) - f\|_\infty \le C\omega_2(f; 1/n)_\infty, \ n = 1, 2, \ldots.$$

The conclusions in the statement are immediate. □

Remark. An important shortcoming of the estimate in Theorem 1.7.17 is that it is not satisfactory with respect to what we would expect, i.e., a Jackson-type estimate in terms of $\omega_2(f, 1/n)_\infty$. The solution to this shortcoming remains an open question.

1.8 Shape-Preserving Approximation by Convolution Polynomials

The convolution-type method used in this section is classical in approximation theory. It allows one to construct polynomials, linear as a function of the approximated function f, having good approximation properties in terms of second-order modulus of smoothness, and with respect to the methods in Sections 1.4–1.7 offering the advantage that in addition, they preserve higher order convexities too. In essence, it uses the convolution (with $\theta = \arccos x$)

$$\begin{aligned} G_{m(n)}(f, x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos t) \cdot S_{m(n)}[\arccos(\cos(\theta - t))] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos t) \cdot S_{m(n)}(\arccos x - t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f[\cos(\arccos x + t)] \cdot S_{m(n)}(t) dt, \end{aligned}$$

where $S_{m(n)}$ is an even trigonometric kernel of the form

$$S_{m(n)}(v) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot \cos kv,$$

$$S_{m(n)}(\arccos z) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot T_k(z),$$

and T_k denotes the kth Chebyshev polynomial, $S_{m(n)}(\arccos z)$ is an element of $\Pi_{m(n)}$, and for each f, $G_{m(n)}(f, \cdot)$ is also in $\Pi_{m(n)}$.

Using the proof of the classical Jackson's theorem, we obtain $S_{m(n)} = c_{n,2} \left(\frac{\sin(n\mu/2)}{\sin(\mu/2)} \right)^4$. In this section the higher-order Jackson kernels (also called Matsuoka kernels) given as follows (see, for example, DeVore [82], Matsuoka [269]), are important:

$$S_{sn-s}(\mu) := c_{n,s} \left(\frac{\sin(n\mu/2)}{\sin(\mu/2)} \right)^{2s},$$

for $s \in \mathbb{N}, s > 2$, where $c_{n,s}$ is chosen such that $\pi^{-1} \int_{-\pi}^{\pi} S_{sn-s}(\mu) d\mu = 1$. Thus we can write

$$S_{sn-s}(\mu) = \frac{1}{2} + \sum_{k=1}^{sn-s} \rho_{k,sn-s} \cos(k\mu).$$

The first result presented here regarding shape-preserving approximation by convolution polynomials is the following.

Theorem 1.8.1. (Beatson [31], Theorem 2) *Let j be positive integer. There exists a positive constant M_j such that for each $f \in C[-1, 1]$ and $n =$*

$0, 1, 2, \ldots,$ *there is a convolution type polynomial* $P_n \in \Pi_n$ *such that* P_n *is* i*-convex for any* $i \in [0, 1, \ldots\}$ *for which* f *is* i*-convex satisfying*

$$|f(x) - P_n(x)| \leq M_j \omega_1(f, \Delta_n(x))_\infty, \qquad |x| \leq 1.$$

Here $\Delta_0(x) := 1$ *and, for* $n \geq 1, \Delta_n(x) := \max(\sqrt{1-x^2}/n, 1/n^2)$.

Proof. Let us present the main lines of the proof. First we need two auxiliary results.

Lemma (A). *For the kernel* $k \in C[-1, 1]$, *define the convolution operator*

$$L(f)(x) = [f * k](x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos\, t) k(\cos(\theta - t))dt, x = \cos(\theta),$$

for all $f \in C[-1, 1]$. *Then, the cone of* j*-convex functions is invariant under* L*, if and only if* $k(x)$ *is* j*-convex (in the sense of Definition 1.1.1, (i)).*

Proof of Lemma A. First suppose that the cone of j-convex functions is invariant under L. Take the de la Vallée Poussin kernel $f_n(x) = c_n(1 + x)^n$, with c_n determined by the condition $\int_{-\pi}^{\pi} f_n[\cos(\theta)]d\theta = \pi$. Since each f_n is j-convex, by hypothesis it follows that $f_n * k$ is too. Taking into account that $f_n * k = k * f_n$ and that $k * f_n$ converges uniformly to k as $n \to \infty$, it follows that k is j-convex.

Conversely, supposing that k is j-convex, the key to the proof is a relationship between the convolution $*$ and the convolution structure of the ultraspherical polynomials (see the details in Beatson [31], proof of Theorem 1). □

Lemma (B). *Let* $(q_n)_n$ *be a sequence of algebraic polynomials, such that each* q_n *is* i*-convex, for* $i = 0, 1, \ldots, m$ *and* $\int_0^\pi q_n[\cos t]t^i dt \sim n^{-i}$, $i = 0, 1, \ldots, 2k$. *Then each* $h_{n+1}(x) = \int_{-1}^x q_n(s)ds$, $n \in \mathbb{N}$, *represents a* i*-convex polynomial for* $i = 0, 1, \ldots, m + 1$ *and* $\int_0^\pi h_n(\cos t)t^i dt \sim n^{-i-2}$, $i = 0, 1, \ldots, 2k - 2$.

Proof of Lemma B. It is immediate by integrating by parts and using the inequality $2x/\pi \leq \sin(x) \leq x$, for all $0 \leq x \leq \pi/2$. □

Proof of Theorem 1.8.1. From Lemma A and from well-known results in Lorentz [248], pp. 65–68, it is enough to construct a kernel $k_n \in \Pi_n$, i-convex for $i \in \{0, \ldots, j\}$, such that $\int_{-\pi}^{\pi} k_n(\cos t)dt = \pi$ and $\int_0^\pi k_n(\cos t)t^2 dt \sim n^{-2}$. For this purpose, we start with a sequence of non-negative polynomials $q_n \in \Pi_n$, $n = 0, 1, \ldots$, satisfying $\int_0^\pi q_n(\cos t)t^i dt \sim n^{-i}$, $i = 0, 1, \ldots, 2j + 2$ (for such sequences, see Lorentz [248], pp. 55–57). Then, applying j-times Lemma B, the theorem follows. □

In what follows, one investigates this matter by the Boolean sum method, systematically used by Jia-ding Cao and H.H. Gonska in a series of papers, obtaining thus even better estimates, i.e., in terms of $\omega_2\left(f, \sqrt{1-x^2}/n\right)_\infty$.

More exactly, first one constructs certain convolution operators $G_{n,s,j}$ based on modified Matsuoka kernels to generalize Beatson's Theorem 1.8.1 (see Theorem 1.8.2 below). Second, the Boolean sum modifications of $G_{n,s,j}$ are used in order to investigate the invariance of cones of j-convex functions in Telyakovskii- and Gopengauz-type estimates (Theorems 1.8.2 and 1.8.3). As special cases, the DeVore–Yu [92], Yu [406], and Leviatan [228] theorems are obtained. Finally, a type of Boolean sum modification is presented.

The results in this section belong to Cao–Gonska [55].

First we present the concept of Boolean sum of certain positive linear operators.

Let Lf be the linear function interpolating f at b, given by

$$L(f, x) = \frac{f(b)(x-a) + f(a)(b-x)}{b-a}, \qquad a \leq x \leq b.$$

For $A : C[a, b] \to C[a, b]$ linear, denote by A^+ the Boolean sum of L and A given by $A^+(f, x) := (L \oplus A)(f, x) = L(f; x) - (L \circ A)(f; x) = A(f, x) + \{(x-a)[f(b) - A(f, b)] + (b-x)[f(a) - A(f, a)]\}/(b-a)$.

The main aim of this section is to construct the Boolean sums approximating f in terms of the second-order modulus of smoothness and preserving the i-convexity of any order.

We first present the construction of certain useful kernels denoted by $\overline{Q}_{n,s,j}$.

For $\xi = \cos(\upsilon)$ one defines

$$S_{sn-n}(\arccos\xi) = C_{n,s}\left(\frac{\sin(n\arccos\xi/2)}{\sin(\arccos\xi/2)}\right)^{2s} = \frac{1}{2} + \sum_{k=1}^{sn-s} \rho_{k,sn-s} \cos(k\arccos\xi).$$

Obviously $S_{sn-s}(\arccos\xi) \in \Pi_{sn-s}$.

For $j \in \mathbb{N}$, let us define

$$Q_{n,s,j}(z) := \frac{1}{(j-1)!} \int_{-1}^{z} (z-\xi)^{j-1} S_{sn-s}(\arccos\xi) d\xi$$

to be a jth antiderivative of $S_{sn-s}(\arccos\xi)$, which means that $Q_{n,s,j} \in \Pi_{sn-s+j}$ and

$$Q_{n,s,j}(z) = \tau_{0,n,s,j} + \tau_{1,n,s,j} \cdot z + \cdots + \tau_{sn-s+j,n,s,j} \cdot z^{sn-s+j},$$

or

$$Q_{n,s,j}(\cos\upsilon) = \tau_{0,n,s,j} + \tau_{1,n,s,j} \cos\upsilon + \cdots + \tau_{sn-s+j,n,s,j} (\cos\upsilon)^{sn-s+j}.$$

Normalizing the kernel Q, we get

$$\overline{Q}_{n,s,j}(\upsilon) := \frac{\pi Q_{n,s,j}(\cos \upsilon)}{\int_{-\pi}^{\pi} Q_{n,s,j}(\cos t) dt},$$

so that

$$\frac{1}{\pi}\int_{\pi}^{\pi} \overline{Q}_{n,s,j}(v)dv = 1.$$

In addition, we can write

$$\overline{Q}_{n,s,j}(v) = \lambda_{0,n,s,j} + \lambda_{1,n,s,j}\cos v + \lambda_{2,n,s,j}\cos 2v + \cdots + \lambda_{sn-s+j,n,s,j}\cos(sn-s+j)v.$$

The next theorem summarizes the main results.

Theorem 1.8.2. (Cao–Gonska [55])

(i) For $j \in \mathbb{N}$ and $s \geq j+2$, there is $c_{j,s} > 0$ such that for each $f \in C[-1,1]$ and $n \geq 1$, the convolution polynomial $G_{n,s,j}(f,\cdot) \in \Pi_{sn-s+j}$ based on the kernel $\overline{Q}_{n,s,j}$ satisfies the inequality

$$|f(x) - G_{n,s,j}(f,x)| \leq c_{j,s}\omega_1\left(f, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)_{\infty}, \quad |x| \leq 1.$$

In addition, if the function f is i-convex, then so is $G_{n,s,j}(f,\cdot)$, for all $i \in \{0,1,\dots,j\}$.

(ii) If $j \in \mathbb{N}$ and $s \geq j+2$, then there exists a positive constant $\overline{c}_{j,s}$ such that for all $f \in C[-1,1]$, $n \geq 1$, we have $G^{+}_{n,s,j}(f,\cdot) \in \Pi_{sn-s+j}$ and

$$|f(x) - G^{+}_{n,s,j}(f,x)| \leq \overline{c}_{j,s}\omega_1\left(f, \sqrt{1-x^2}/n\right)_{\infty}.$$

In addition, if f is i-convex, then so is $G^{+}_{n,s,j}(f,\cdot)$ for all $i = 1,\dots,j$. Here $G^{+}_{n,s,j}(f,\cdot)$ denotes the Boolean sum of $G_{n,s,j}(f,\cdot)$ in (i), with $L(f,\cdot)$.

(iii) If $j \in \mathbb{N}$ and $s \geq j+3$, then there exists a positive constant $c_{j,s}$ such that for each $f \in C[-1,1]$ and $n \geq 2$ we have $G^{+}_{n,s,j}(f,\cdot) \in \Pi_{sn-s+j}$ and

$$|f(x) - G^{+}_{n,s,j}(f,x)| \leq c_{j,s}\omega_2\left(f, \sqrt{1-x^2}/n\right)_{\infty}, \qquad |x| \leq 1.$$

Also, if f is i-convex, then $G^{+}_{n,s,j}(f,\cdot)$ is i-convex, for all $i \in \{1,\dots,j\}$.

(iv) Let $j \in \mathbb{N}$ and $s \geq j+2$. Denoting the modified Boolean sum $G^{*}_{n,s,j} = G_{n,s,j} \bigoplus L = G_{n,s,j}(f;x) + L(f;x) - (G_{n,s,j} \circ L)(f;x) = G_{n,s,j}(f - Lf,x) + L(f,x)$, there exists a positive constant $c_{j,s}$ such that for all $f \in C[-1,1], n \geq 1$, we have $G^{*}_{n,s,j}(f,\cdot) \in \Pi_{sn-s+j}$ and

$$\|f - G^{*}_{n,s,j}(f,\cdot)\|_{\infty} \leq c_{j,s}\omega_2(f,1/n)_{\infty}.$$

In addition, if f is i-convex, then so is $G^{*}_{n,s,j}(f,\cdot)$ for any $i \in \{1,\dots,j\}$.

Remark. Note that above, (ii) is not a consequence of (iii). Details can be found in Cao–Gonska [55, 56, 57].

1.9 Positive Linear Polynomial Operators Preserving Shape

In this very short section we briefly present a constructive example of a positive linear polynomial operator of nonconvolution type, denoted by G_{m+2}, that reproduces the linear functions, produces DeVore–Gopengauz–type estimates in approximation, and in addition, preserves the convexities of higher order functions.

This polynomial operator was introduced by Gavrea in 1996, and it seems to be the first example of a positive linear polynomial operator that produces the following pointwise estimate of DeVore–Gopengauz type (see Gavrea [144]):

$$|(G_{m+2}f)(x) - f(x)| \leq c\omega_2\left(f; \frac{\sqrt{x(1-x)}}{m}\right)_\infty .$$

In more detail, Gavrea [144] first introduced the sequence of operators $H_m : C[0,1] \to \Pi_m,\ m \in \mathbb{N}$, given by

$$H_m(f)(x) \\ = f(0)(1-x)^m + x^m f(1) + (m-1)\sum_{k-1}^{m-1} p_{m,k}(x)\int_0^1 p_{m-2,k-1}(t)f(t)dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is the Bernstein basis, and then he defined $G_{m+2} : C[0,1] \to \Pi_{m+2}$ by

$$(G_{m+2}f)(x) = \sum_{k=0}^{m} \frac{a_k}{k+1}(H_{k+2}f)(x),$$

where the coefficients a_k are chosen so that the polynomial sequence $P_m \in \Pi_m, m \in \mathbb{N},\ P_m(x) = \sum_{k=0}^m a_k x^k$, attached to the above representation of $(G_{m+2}f), m \in \mathbb{N}$, and called the generator of $(G_{m+2}f),\ m \in \mathbb{N}$, satisfies the properties $P_m(x) \geq 0, P_m'(x) \geq 0$ for all $x \in [0,1]$, $m \in \mathbb{N}$, and $\int_0^1 P_m(x)dx = 1$ for all $m \in \mathbb{N}$.

Gavrea's original approximants are in Π_{2m+1}, but it was shown in, e.g., Gavrea–Gonska–Kacsó [145] that by a slight modification, their degree can be reduced to $m+2$.

The shape-preserving properties of G_{m+2} are given by the following.

Theorem 1.9.1. (Cottin–Gavrea–Gonska–Kacsó–Zhou [72]) *(i) The polynomial G_{m+2} preserves the monotonicity and the usual convexity (i.e., the 2-convexity in Definition 1.1.1, (i), in the introduction) of the function f.*

(ii) If $P_m(x) = \sum_{k=0}^m a_k x^k, m \in \mathbb{N}$ is the generator sequence for G_{m+2}, $m \in \mathbb{N}$, (i.e., satisfies the conditions $P_m(x) \geq 0, P_m'(x) \geq 0, \forall x \in [0,1]$, $\int_0^1 P_m(x)dx = 1$), and in addition, P_m is convex up to order r, then G_{m+2} preserves the convexity up to order $(r+1)$.

1.10 Notes

Note 1.10.1. Corollaries 1.4.2 and 1.4.3, Theorems 1.5.9, 1.6.11, 1.6.12, Lemmas 1.7.2, 1.7.3, Corollary 1.7.4, Theorem 1.7.5, Corollary 1.7.6, and Theorem 1.7.17 appear for the first time in this book.

Note 1.10.2. In comonotone approximation with respect to modified weighted moduli of smoothness, in addition to the results in subsection 1.6.4, we can present:

(Leviatan–Shevchuk [238]) For $s \ge 1$, let $0 \le r \le 2s+2$, excluding the three cases $r+k \le 1$. For any constant $A > 0$ and any $n \ge 1$, there exists an $f := f_{k,r,s,n,A} \in C^r_\varphi$ which changes monotonicity s times in $[-1,1]$, such that

$$e_n^{(1,s)}(f)_\infty > A\omega^\varphi_{k,r}(f^{(r)}, 1).$$

Here, $e_n^{(1,s)}(f)_\infty$ is defined in the remark after the proof of Theorem 1.6.1.

(Due to Leviatan–Shevchuk, see e.g., Leviatan [230].) Let $f \in \Delta^1$. Then there are constants $C = C(f)$, $N = N(f)$, and an absolute constant c, such that for all $0 \le k+r \le 3$, we have

$$E_n^{(1)}(f)_\infty \le C\omega^\varphi_{k,r}(f, \frac{1}{n}), \quad n \ge 2,$$

and

$$E_n^{(1)}(f)_\infty \le c\omega^\varphi_{k,r}(f, \frac{1}{n}), \quad n \ge N.$$

(Leviatan–Shevchuk [239]) For any $s \ge 0$, there is a $Y_s \in \mathbb{Y}_s$ and an $f \in C^2_\varphi \cap \Delta^1(Y_s)$, such that

$$\limsup_{n\to\infty} \frac{n^2 E_n^{(1)}(f, Y_s)_\infty}{\omega^\varphi_{3,2}(f'', 1/n)} = \infty.$$

Also, for simultaneous approximation in the comonotone case, with estimates in terms of usual Ditzian–Totik moduli of smoothness, we can present (Leviatan [232], Kopotun [206]) For $f \in C^1[-1,1] \cap \Delta^1(Y_s)$, there is a constant $C = C(s)$ such that for any $n \ge \frac{C}{d(Y_s)}$, there is a polynomial $p_n \in \Pi_n \cap \Delta^1(Y_s)$, satisfying

$$\|f - p_n\|_\infty \le \frac{C}{n}\omega_1^\varphi(f', \frac{1}{n})_\infty,$$

and

$$\|f' - p_n'\|_\infty \le \frac{C}{d_0}\omega_1^\varphi(f', \frac{1}{n})_\infty,$$

where $d_0 := \min\{\sqrt{1+y_s}, \sqrt{1-y_1}\}$. (See also Theorem 1.7.12).

Note 1.10.3. For convex approximation, in addition to the results presented in Subsection 1.7.4, in Kopotun–Leviatan–Shevchuk [213] the following estimates are proved with respect to the modified weighted Ditzian–Totik moduli of smoothness: if $f \in \mathbb{C}^2_\varphi \cap \Delta^2$, then $E^{(2)}_n(f)_\infty \le c(n^{-2}\omega^\varphi_{3,2}(f'', 1/n) + n^{-6}\|f''\|_{[-1/2,1/2]})$, $n \ge N$, where c and N are absolute constants (hence, $E^{(2)}_n(f)_\infty \le cn^{-2}\omega^\varphi_{3,2}(f'', 1/n), \quad n \ge N(f))$, while if $2 \le r \le 4$, $1 \le k \le 5-r$ and $f \in \mathbb{C}^r_\varphi \cap \Delta^2$, then

$$E^{(2)}_n(f)_\infty \le \frac{c}{n^r}\omega^\varphi_{k,r}(f^{(r)}, 1/n), n \ge N(f).$$

As negative results, in the same paper mentioned, it is proved that there exists an $f \in \mathbb{C}^4_\varphi \cap \Delta^2$, satisfying

$$\limsup_{n\to\infty} \frac{n^4 E^{(2)}_n(f)_\infty}{\omega^\varphi_{2,4}(f^{(4)}, 1/n)} = \infty,$$

while if $0 \le r \le 4$ and $k \ge 6-r$, then there exists an $f \in \mathbb{C}^4_\varphi \cap \Delta^2$, satisfying

$$\limsup_{n\to\infty} \frac{n^r E^{(2)}_n(f)_\infty}{\omega^\varphi_{k,r}(f^{(r)}, 1/n)} = \infty.$$

Note 1.10.4. For coconvex approximation, in addition to the results presented in Subsection 1.7.6, in Kopotun–Leviatan–Shevchuk [214] the following estimates are proved with respect to the modified weighted Ditzian–Totik moduli of smoothness: if $s \ge 2$, $2 \le r \le 4$, $1 \le k \le 5-r$, $Y_s \in \mathbb{Y}_s$ and $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$, then we have

$$E^{(2)}_n(f, Y_s)_\infty \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \ge N(Y_s);$$

if $s = 1$, $2 \le r \le 4$, $Y_1 \in \mathbb{Y}_1$ and $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_1)$, then we have

$$E^{(2)}_n(f, Y_1)_\infty \le cn^{-r}\omega^\varphi_{5-r,r}(f^{(r)}, 1/n), \quad n \ge N(f),$$

and for $1 \le k \le 4-r$,

$$E^{(2)}_n(f, Y_1)_\infty \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \ge N(Y_1);$$

if $k \ge 1$, $Y_1 \in \mathbb{Y}_1$ and $f \in \mathbb{C}^7_\varphi \cap \Delta^2(Y_1)$, then we have

$$E^{(2)}_n(f, Y_1)_\infty \le cn^{-7}\omega^\varphi_{k,7}(f^{(7)}, 1/n), \quad n \ge k+7;$$

if $k \ge 1$, $r \ge 7$, $Y_1 \in \mathbb{Y}_1$ and $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_1)$, then

$$E^{(2)}_n(f, Y_1)_\infty \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \ge k+r.$$

Also, in Kopotun–Leviatan–Shevchuk [212] for $r \le 3$ and Kopotun–Leviatan–Shevchuk [214] for $r \ge 4$, it is proved that if $r \ge 1$, $s \ge 1$, $Y_s \in \mathbb{Y}_s$ and $f \in \mathbb{B}^r \cap \Delta^2(Y_s)$, then

$$E_n^{(2)}(f,Y_s)_\infty \le cn^{-r}\|\varphi^r f^{(r)}\|_\infty, \quad n \ge N(r,Y_s),$$

where $N(r,Y_s)$ is a constant which may depend only on r and Y_s.

For $s=1$, if $r \le 2$ (see Leviatan–Shevchuk [241]) or $r \ge 7$ (see Kopotun–Leviatan–Shevchuk [214]), the above estimate is valid with $N = r$.

The paper Kopotun–Leviatan–Shevchuk [214], also contains negative results which show that these results cannot be improved.

Note 1.10.5. Concerning approximation-preserving shapes of higher order, we can present the following results. For $q > 1$ and $f \in \Delta^q \cap C[-1,1]$ (i.e., f is continuous and q-convex on $[-1,1]$), set $E_n^{(q)}(f)_\infty = \inf\{\|f-P\|_\infty; P \in \Pi_n \cap \Delta^q\}$ and consider the estimate

$$E_n^{(q)}(f)_\infty \le c(r,q)\frac{\|f^{(r)}\|_\infty}{n^r},\ n \ge M.$$

It holds for $r=1,\ q>1$ (see Beatson [33]), for $r=2,\ q>1$ (see Shvedov [355]), for $r=3,\ q=3$ (see Bondarenko [47]). It does not hold for $q \ge 4$ with $M = M(r)$, $r > 2$ (due to Konovalov–Leviatan [201]), and for $q \ge 4$, $r \ge 2$ with $M = M(f)$ and $c = c(f)$ (see Bondarenko–Primak [48]). The case $q = 3$ and $r \ge 4$ remains an open question. For $1 \le p < \infty$, the estimate $E_n^{(3)}(f)_p \le c\omega_3(f;1/n)_p$ does not hold with $c = c(f)$ (see Bondarenko–Primak [48]).

In the case $r = 3$, $q = 3$, is proved the estimate $E_n^{(3)}(f)_\infty \le c\omega_\varphi^3(f;1/n)_\infty,\ n > 1$, with $c > 0$ an absolute constant in Bondarenko [47].

2

Shape-Preserving Approximation by Real Multivariate Polynomials

Extending the results in the univariate case, in this chapter we prove approximation results preserving *multivariate shapes*, by *Bernstein*-type, *convolution*-type, and *tensor-product*-type polynomials. The *multivariate shapes* considered are *Popoviciu convexity*, *Schur convexity*, *axial convexity*, *polyhedral convexity*, *usual multivariate convexity*, *subharmonicity*, and in general, L-*convexity*, with L a bounded linear operator satisfying some suitable conditions.

For simplicity, most of the results in this chapter are presented for the bivariate case. However, in some cases, when the bivariate case is not representative for the multivariate case, the results are presented in three or several variables.

The approximation errors are given with respect to various bivariate moduli of continuity/smoothness or K-functionals, presented in the next section.

2.1 Introduction

In order to extend the results in monotone and convex approximation from the univariate case in Chapter 1 to the bivariate/multivariate case, we obviously need suitable bivariate/multivariate concepts of shapes (i.e., of monotonicities, convexities, harmonicity, subharmonicity), bivariate/multivariate moduli of smoothness, and bivariate/multivariate suitable polynomials.

First we present a few concepts of shapes in the bivariate case, which are natural extensions of the monotonicity and convexity in the univariate case, and some of them are obtained using the "tensor product" method.

Definition 2.1.1. Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$.

(i) We say that $f(x, y)$ is increasing (decreasing) with respect to x on $[-1, 1] \times [-1, 1]$ if

$$f(x + h, y) - f(x, y) \geq 0 \ (\leq 0), \ \forall\, y \in [-1, 1], \ \forall\, x, x + h \in [-1, 1], \ h \geq 0.$$

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(ii) We say that $f(x,y)$ is increasing (decreasing) with respect to y on $[-1,1]\times[-1,1]$ if

$$f(x,y+k)-f(x,y)\geq 0\ (\leq 0),\ \forall\, x\in[-1,1],\ \forall\, y,y+k\in[-1,1],\quad k\geq 0.$$

(iii) We say that $f(x,y)$ is upper (lower) bidimensional monotone on $[-1,1]\times[-1,1]$ (see, e.g., Marcus [265], p. 33) if

$$\Delta_2 f(x,y)=f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)\geq 0\ (\leq 0),$$

for all $x,x+h\in[-1,1],\quad y,y+k\in[-1,1],\quad h\geq 0,\quad k\geq 0.$

(iv) We say that $f(x,y)$ is totally upper (lower) monotone on $[-1,1]\times[-1,1]$ (see Nicolescu [286] or R.C. Young [405]) if (i), (ii), and (iii) hold, with all simultaneously ≥ 0 (or with all simultaneously ≤ 0).

(v) (Popoviciu [315], p. 78) The function $f:[-1,1]\times[-1,1]\to\mathbb{R}$ is called convex of order (n,m) in the Popoviciu sense (where $n,m\in\{0,1,\ldots\}$) if for any $n+1$ distinct points $x_1<x_2<\cdots<x_{n+1}$ and any $m+1$ distinct points $y_1<y_2<\cdots<y_{m+1}$ in $[-1,1]$, we have

$$\begin{bmatrix} x_1,x_2,\ldots,x_{n+1} & \\ & ;f \\ y_1,y_2,\ldots,y_{m+1} & \end{bmatrix}\geq 0,$$

where the symbol above represents the divided difference of a bivariate functions and it is defined iteratively (by means of the divided difference of univariate functions) as (see Popoviciu [315], pp. 64–65)

$$\begin{aligned}&[x_1,\ldots,x_{n+1};[y_1,\ldots,y_{m+1};f(x,\cdot)]_y]_x\\ &\quad=[y_1,\ldots,y_{m+1};[x_1,\ldots,x_{m+1};f(\cdot,y)]_x]_y.\end{aligned}$$

Here

$$[x_1,\ldots,x_p;g(\cdot)]=\sum_{i=1}^{p}\frac{g(x_i)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_p)}$$

represents the usual divided difference of a univariate function $g[g;x_1]=g(x_1)$.

To be in accordance with the definition of j-convexity in the univariate case (see Chapter 1, Definition 1.1.1 (i), and Remark (2) after it), note that the denomination in Popoviciu's original definition of the (n,m) convexity was slightly modified (in the original definition, the divided differences are taken on $n+2$ and $m+2$ points).

(vi) We also need to introduce the following concept, similar to that of total upper monotony: $f(x,y)$ will be called totally convex on $[-1,1]\times[-1,1]$ in the Popoviciu sense if $f(x,y)$ is simultaneously convex of orders $(0,2),(2,0),(1,2),(2,1)$, and $(2,2)$.

Remarks. (1) The most natural bivariate monotonicity seems to be that in Definition 2.1.1 (iv), because for such bivariate functions the set of discontinuity points is at most countable (see Nicolescu [286]).

(2) In the case that f has partial derivatives, the conditions (i)–(iv) in Definition 2.1.1 can be expressed as follows:

(i) by $\dfrac{\partial f(x,y)}{\partial x} \geq 0 \ (\leq 0), \ \forall \ x, y \in [-1,1]$,
(ii) by $\dfrac{\partial f(x,y)}{\partial y} \geq 0 \ (\leq 0), \ \forall \ x, y \in [-1,1]$,
(iii) by $\dfrac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0 \ (\leq 0), \ \forall \ x, y \in [-1,1]$ (see Nicolescu [286]),

while (iv) is represented by all conditions (i)–(iii).

(3) It is obvious that convexities of orders $(0,1)$ and $(1,0)$ in the Popoviciu sense mean in fact that $f(x,y)$ is increasing on $[-1,1]$ with respect to y (for any fixed $x \in [-1,1]$) and increasing with respect to x (for any fixed $y \in [-1,1]$), respectively. Also, one reduces convexity of order $(1,1)$ in the Popoviciu sense to upper bidimensional monotonicity introduced in Marcus [265], p. 33, simultaneous convexities of orders $(0,1), (1,0)$, and $(1,1)$ mean the total upper monotonicity in Nicolescu [286], convexity of order $(0,2)$ means in fact that $f(x,y)$ is convex on $[-1,1]$ with respect to y (for any fixed x), and so on.

(4) Suppose f is of class C^{n+m} on $[-1,1] \times [-1,1]$.

By the mean value theorem we get that if $\frac{\partial^{n+m} f(x,y)}{\partial x^n \partial y^m} \geq 0, \ \forall (x,y) \in [-1,1] \times [-1,1]$, then $f(x,y)$ is convex of order (n,m) in the Popoviciu sense on $[-1,1] \times [-1,1]$.

For the approximation errors, we will use the following kinds of bivariate/multuvariate moduli of smoothness and K-functionals.

Definition 2.1.2. Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$.
(i) For $\delta, \eta > 0$, we define

$$\omega_1^{(x)}(f;\delta) = \sup_{y \in [-1,1]} \sup\{|f(x+h,y) - f(x,y)|; \ x, x+h \in [-1,1], \ 0 \leq h \leq \delta\};$$

$$\omega_1^{(y)}(f;\eta) = \sup_{x \in [-1,1]} \sup\{|f(x,y+k) - f(x,y)|; \ y, y+k \in [-1,1], \ 0 \leq k \leq \eta\}$$

(i.e., the partial bivariate moduli of continuity; see, e.g., Timan [385]),

$$\omega_1(f;\delta,\eta) = \sup\{|f(x+h,y+k) - f(x,y)|; \ 0 \leq h \leq \delta, \ 0 \leq k \leq \eta,$$
$$x, x+h \in [-1,1], \ y, y+k \in [-1,1]\},$$
$$\omega_1(f;\alpha) = \sup\{|f(x+h,y+k) - f(x,y)|; \ h^2 + k^2 \leq \alpha^2,$$
$$x, x+h \in [-1,1], \ y, y+k \in [-1,1]\}$$

(if $f : [-1,1]^m \to \mathbb{R}$ then $\omega_1(f;h) = \{|f(x) - f(y)|; x, y \in [-1,1]^m, \|x - y\|_{\mathbb{R}^m} \le h\}$),

$$\omega^{(B)}(f;\delta,\eta) = \sup\{|\Delta_{h,k}f(x,y)|;\ 0 \le h \le \delta,\ 0 \le k \le \eta,\\ x, x+h \in [-1,1],\ y, y+k \in [-1,1]\}$$

(i.e., the Bögel modulus of continuity; see, e.g., Nicolescu [286]), where

$$\Delta_{h,k}f(x,y) = f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y).$$

By (Ditzian–Totik [98], Chapter 12) the (first order) Ditzian-Totik moduli of continuity are defined by

$$\omega_1^{\varphi}(f;\delta_1,\delta_2)\\ := \sup\{|\Delta_{h_1\varphi(x),h_2\varphi(y)}f(x,y)| : 0 \le h_i \le \delta_i, i = 1,2, x, y \in [-1,1]\},$$

and for $0 < p < +\infty$,

$$\omega_1^{\varphi}(f;\delta_1,\delta_2)_p := \sup_{0 \le h_i \le \delta_i, i=1,2} \left(\int_{-1}^{1} \int_{-1}^{1} |\Delta_{h_1\varphi(x),h_2\varphi(y)}f(x,y)|^p dx\, dy \right)^{1/p},$$

where $\varphi(t) = \sqrt{1-t^2}$,

$$\Delta_{h_1\varphi(x),h_2\varphi(y)}f(x,y) = f\left(x + \frac{h_1}{2}\varphi(x), y + \frac{h_2}{2}\varphi(y)\right)\\ -f\left(x - \frac{h_1}{2}\varphi(x), y - \frac{h_2}{2}\varphi(y)\right)$$

if $(x \pm \frac{h_1}{2}\varphi(x),\ y \pm \frac{h_2}{2}\varphi(y)) \in [-1,1] \times [-1,1]$, $\Delta_{h_1\varphi(x),h_2\varphi(y)}f(x,y) = 0$ elsewhere.

Also, the partial moduli of continuity are defined by

$$\omega_{1,x}^{\varphi}(f;\delta_1) := \sup\{|\Delta_{h_1\varphi(x),0}f(x,y)| : 0 \le h_1 \le \delta_1, x, y \in [-1,1]\},$$

$$\omega_{1,y}^{\varphi}(f;\delta_2) := \sup\{|\Delta_{0,h_2\varphi(y)}f(x,y)| : 0 \le h_2 \le \delta_2, x, y \in [-1,1]\},$$

and for $0 < p < +\infty$,

$$\omega_{1,x}^{\varphi}(f;\delta_1)_p := \sup_{0 \le h_1 \le \delta_1} \left(\int_{-1}^{1} \int_{-1}^{1} |\Delta_{h_1\varphi(x),0}f(x,y)|^p dx\, dy \right)^{1/p},$$

$$\omega_{1,y}^{\varphi}(f;\delta_2)_p := \sup_{0 \le h_2 \le \delta_2} \left(\int_{-1}^{1} \int_{-1}^{1} |\Delta_{0,h_2\varphi(y)}f(x,y)|^p dx\, dy \right)^{1/p}.$$

(ii) (Ditzian–Totik [98], Chapter 12) Besides the above first-order modulus ω_1^{φ}, the most used modulus in this chapter is the second-order Ditzian–Totik modulus of smoothness defined by

$$\omega_2^{\varphi}(f;\delta_1,\delta_2) \\ := \sup\{|\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y)| : 0 \le h_i \le \delta_i, i=1,2, x,y \in [-1,1]\},$$

and for $0 < p < +\infty$,

$$\omega_2^{\varphi}(f;\delta_1,\delta_2)_p := \sup\nolimits_{0\le h_i\le\delta_i, i=1,2} \left(\int_{-1}^{1}\int_{-1}^{1} |\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y)|^p dx\,dy\right)^{1/p},$$

where $\varphi(t) = \sqrt{1-t^2}$,

$$\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y) = \sum_{k=0}^{2}\binom{2}{k}(-1)^k f(x+(1-k)h_1\varphi(x), y+(1-k)h_2\varphi(y))$$

if $(x \pm h_1\varphi(x),\, y \pm h_2\varphi(y)) \in [-1,1]\times[-1,1]$, $\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y) = 0$ elsewhere.

(If $f : [-1,1]^m \to \mathbb{R}$, then the second-order Ditzian–Totik modulus of smoothness is defined by

$$\omega_2^{\varphi}(f;\delta_1,\ldots,\delta_m) = \sup\{|\Delta^2_{h_1\varphi(x_1),\ldots,h_m\varphi(x_m)}f(x_1,\ldots,x_m)|;$$

$$0 \le h_i \le \delta_i, \quad i = \overline{1,m}, \quad x_1,\ldots,x_m \in [-1,1]\}.)$$

Also, the second-order Ditzian–Totik partial moduli of smoothness are defined by

$$\omega_{2,x}^{\varphi}(f;\delta_1) := \sup\{|\Delta^2_{h_1\varphi(x),0}f(x,y)| : 0 \le h_1 \le \delta_1, x,y \in [-1,1]\},$$

$$\omega_{2,y}^{\varphi}(f;\delta_2) := \sup\{|\Delta^2_{0,h_2\varphi(y)}f(x,y)| : 0 \le h_2 \le \delta_2, x,y \in [-1,1]\},$$

and for $0 < p < +\infty$,

$$\omega_{2,x}^{\varphi}(f;\delta_1)_p := \sup\nolimits_{0\le h_1\le\delta_1} \left(\int_{-1}^{1}\int_{-1}^{1} |\Delta^2_{h_1\varphi(x),0}f(x,y)|^p dx\,dy\right)^{1/p},$$

$$\omega_{2,y}^{\varphi}(f;\delta_2)_p := \sup\nolimits_{0\le h_2\le\delta_2} \left(\int_{-1}^{1}\int_{-1}^{1} |\Delta^2_{0,h_2\varphi(y)}f(x,y)|^p dx\,dy\right)^{1/p}.$$

Note that in general, the Ditzian–Totik modulus of smoothness of order $r \ge 3$ is defined by

$$\omega_r^{\varphi}(f;\delta_1,\delta_2) \\ := \sup\{|\Delta^r_{h_1\varphi(x),h_2\varphi(y)}f(x,y)| : 0 \le h_i \le \delta_i, i=1,2, x,y \in [-1,1]\},$$

and for $0 < p < +\infty$,

$$\omega_r^{\varphi}(f;\delta_1,\delta_2)_p := \sup\nolimits_{0\le h_i\le\delta_i, i=1,2} \left(\int_{-1}^{1}\int_{-1}^{1} |\Delta^r_{h_1\varphi(x),h_2\varphi(y)}f(x,y)|^p dx\,dy\right)^{1/p},$$

where $\varphi(t) = \sqrt{1-t^2}$,

$$\Delta^r_{h_1\varphi(x),h_2\varphi(y)} f(x,y) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + (r/2-k)h_1\varphi(x), y + (r/2-k)h_2\varphi(y))$$

if $(x \pm rh_1\varphi(x)/2\ , y \pm rh_2\varphi(y)/2) \in [-1,1]\times[-1,1]$, $\Delta^r_{h_1\varphi(x),h_2\varphi(y)} f(x,y) = 0$ elsewhere.

The Ditzian–Totik partial moduli of smoothness of order r are defined accordingly.

Finally, another kind of (uniform) Ditzian–Totik modulus of smoothness of order $r \in \mathbb{N}$ can be defined by

$$\omega_r^{\varphi}(f;\delta) := \sup\{|\Delta^r_{h_1\varphi(x),h_2\varphi(y)} f(x,y)| : 0 \le h_1, 0 \le h_2, h_1^2 + h_2^2 \le \delta^2, x, y \in [-1,1]\}.$$

(iii) For $f \in C([a,b]\times[c,d])$ and $r \in \mathbb{N}$, we can consider other rth moduli of smoothness defined by

$$\omega_r(f;\delta_1,\delta_2) = \sup\left\{\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i} f(x+ih_1, y+ih_2)\right|; (x,y), (x+rh_1, y+rh_2) \in [a,b]\times[c,d], |h_1| \le \delta_1, |h_2| \le \delta_2\right\},$$

$$\omega_r(f;\alpha) = \sup\left\{\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i} f(x+ih_1, y+ih_2)\right|; (x,y), (x+rh_1, y+rh_2) \in [a,b]\times[c,d], h_1^2 + h_2^2 \le \alpha^2\right\}$$

(if $\Omega \subset \mathbb{R}^m$ and $f : \Omega \to \mathbb{R}$, then for any $\tau \ge 0$, we can define

$$\omega_r(f;\tau) = \sup_{\|t\|_{\mathbb{R}^m} \le \tau}\left\{\left|\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s} f(x+st)\right|; x+st \in \Omega, s = 0,\dots,r\right\},$$

where $\|t\|_{\mathbb{R}^m} = \sqrt{(t_1^2 + \cdots + t_m^2)}, t = (t_1, t_2, \dots, t_m)$),

$$\omega_r^*(f;\alpha) = \sup\left\{\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i} f(x+ih_1, y+ih_2)\right|; (x,y), (x \pm rh_1, y \pm rh_2) \in [a,b]\times[c,d], |h_1| \le \alpha, \quad |h_2| \le \alpha\right\}.$$

(iv) A kind of K_2-functional of two variables can be defined by

$$K_2(f;t,s) = \inf\left\{\|f-g\| + t^2\left\|\frac{\partial^2 g}{\partial x^2}\right\| + s^2\left\|\frac{\partial^2 g}{\partial y^2}\right\| + ts\left[\left\|\frac{\partial^2 g}{\partial x\partial y}\right\| + \left\|\frac{\partial^2 g}{\partial y\partial x}\right\|\right];\ g\in W^2([-1,1]\times[-1,1])\right\}.$$

Here $g \in W^2([-1,1]\times[-1,1])$ means that g is twice differentiable with its second-order partial derivatives bounded on $[-1,1]\times[-1,1]$, and $\|\cdot\|$ denotes the uniform norm.

Now, defining the norm of $g \in W^2([-1,1]\times[-1,1])$ by $\|g\|_{W^2} = \max\{\|\frac{\partial^2 g}{\partial x^2}\|, \|\frac{\partial^2 g}{\partial y^2}\|, \|\frac{\partial^2 g}{\partial x\partial y}\|, \|\frac{\partial^2 g}{\partial y\partial x}\|\}$, one can define another K_2-functional of one variable only, by

$$K_2(f;t) = \inf\{\|f-g\| + t\|g\|_{W^2}; g\in W^2([-1,1]\times[-1,1])\}$$

(see Johnen-Scherer [187]), which obviously is equivalent to a third kind of K_2-functional, defined by

$$K_2(f;t)^* = \inf\left\{\|f-g\| + t\left[\left\|\frac{\partial^2 g}{\partial x^2}\right\| + \left\|\frac{\partial^2 g}{\partial y^2}\right\| + \left\|\frac{\partial^2 g}{\partial x\partial y}\right\| + \left\|\frac{\partial^2 g}{\partial y\partial x}\right\|\right]\ g\in W^2([-1,1]\times[-1,1])\right\},$$

(see Dekel-Leviatan [75], DeVore [85]).

(v) The bivariate K_2^{φ}-functional is defined by

$$K_2^{\varphi}(f;t,s) = \inf\left\{\|f-g\| + t^2\left\|\varphi_x^2\frac{\partial^2 g}{\partial x^2}\right\| + s^2\left\|\varphi_y^2\frac{\partial^2 g}{\partial y^2}\right\| + ts\left[\left\|\varphi_x\varphi_y\frac{\partial^2 g}{\partial x\partial y}\right\| + \left\|\varphi_x\varphi_y\frac{\partial^2 g}{\partial y\partial x}\right\|\right];\ g\in W^{2,\varphi}([-1,1]\times[-1,1])\right\},$$

where $g\in W^{2,\varphi}([-1,1]\times[-1,1])$ means that

$$\left|\varphi_x^2\frac{\partial^2 g}{\partial x^2}(x,y)\right|,\ \left|\varphi_y^2\frac{\partial^2 g}{\partial y^2}(x,y)\right|,\ \left|\varphi_x\varphi_y\frac{\partial^2 g}{\partial x\partial y}(x,y)\right|,\ \left|\varphi_x\varphi_y\frac{\partial^2 g}{\partial y\partial x}(x,y)\right|$$

are all bounded on $[-1,1]\times[-1,1]$ and φ_u denotes $\sqrt{1-u^2}$.

Similarly, one can define

$$K_2^{\varphi}(f;t) = \inf\left\{\|f-g\| + t\left[\left\|\varphi_x^2\frac{\partial^2 g}{\partial x^2}\right\| + \left\|\varphi_y^2\frac{\partial^2 g}{\partial y^2}\right\| + \left\|\varphi_x\varphi_y\frac{\partial^2 g}{\partial x\partial y}\right\| + \left\|\varphi_x\varphi_y\frac{\partial^2 g}{\partial y\partial x}\right\|\right];\ g\in W^{2,\varphi}([-1,1]\times[-1,1])\right\}.$$

Some properties of these bivariate moduli of continuity/smoothness and K_2-functionals useful for the next results are given by the following.

Lemma 2.1.3. *Let* $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$.

(i) (see, e.g., Timan [385], pp. 111–114) $\omega_1(f; 0, 0) = 0$, $\omega_1(f; \delta, \eta)$ *is nondecreasing with respect to* δ *and* η,

$$\omega_1(f; \delta_1 + \delta_2, \eta_1 + \eta_2) \leq \omega_1(f; \delta_1, \eta_1) + \omega_1(f; \delta_2, \eta_2),$$
$$\omega_1(f; \delta, \eta) \leq \omega^{(x)}(f; \delta) + \omega^{(y)}(f; \eta) \leq 2\omega(f; \delta, \eta).$$

(ii) (see, e.g., Anastassiou–Gal [7], p. 81)

$$\omega^{(B)}(f; \delta, \eta) \leq \omega^{(x)}(f; \delta) + \omega^{(y)}(f; \eta).$$

(iii) (Johnen–Scherer [187]) $K_2(f; t^2) \sim \omega_2(f; t)$ *for all* $t \in (0, 1)$.
(iv) $K_2(f; t^2)^* \sim K_2(f; t^2) \sim K_2(f; t, t)$.
(v) $\omega_r^*(f; t) \sim \omega_r(f; t) \sim \omega_r(f; t, t)$ *and* $\omega_r^\varphi(f; t) \sim \omega_r^\varphi(f; t, t)$.
(vi) $K_2^\varphi(f; t, t) \sim K_2^\varphi(f; t^2)$.

(Recall that $a(t) \sim b(t)$ *means that there is* $t_0 > 0$ *and* M_1, M_2, *independent of* t, *such that* $M_1 a(t) \leq b(t) \leq M_2 a(t)$, *for all* $t \in (0, t_0)$.*)*

Proof. (i) The property that $\omega_1(f; 0, 0) = 0$ and the property that $\omega_1(f; \delta, \eta)$ is nondecreasing with respect to δ and η, follows directly from Definition 2.1.2. Also, the elementary inequalities:

$$|f(x + h_1 + h_2, y + k_1 + k_2) - f(x, y)| \leq |f(x + h_1 + h_2, y + k_1 + k_2) - f(x + h_1, y + k_1)| + |f(x + h_1, y + k_1) - f(x, y)|,$$

$$|f(x+h, y+k) - f(x, y)| \leq |f(x+h, y+k) - f(x+h, y)| + |f(x+h, y) - f(x, y)|,$$

immediately imply the other two inequalities in the statement of (i).

(ii) It is immediate by the inequality:

$$|\Delta_{h,k} f(x, y)| = |f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y)| \leq |f(x + h, y + k) - f(x + h, y)| + |f(x, y + k) - f(x, y)|.$$

(iii), (iv), (v), and (vi) are immediate from the corresponding definitions in Definition 2.1.2. □

In computer aided geometric Design (CAGD), the following concepts of Bernstein polynomials and Bézier surfaces represented in barycentric coordinates are important. First we present the case in $\mathbb{R}^2$, which has a simple and intuitive geometric interpretation, and then the general case in $\mathbb{R}^k$, $k > 2$.

Definition 2.1.4. (see Bézier [43], [44], Farin [113])

(i) Let P_1, P_2, P_3 be three noncollinear points in the plane and denote by T the interior together with the border of the triangle of vertices P_1, P_2, and P_3. In other words, T is the convex hull of the points P_1, P_2, and P_3, which means that for each point $P \in T$, there exist uniquely the numbers $u, v, w \geq 0$, $u+v+w=1$, called barycentric coordinates of P, such that $P = uP_1 + vP_2 + wP_3$, and we write $P = (u, v, w)$. Obviously $P_1 = (1, 0, 0), P_2 = (0, 1, 0), P_3 = (0, 0, 1)$.

If in Cartesian coordinates we have $P = (x, y)$, $P_k = (x_k, y_k)$, $k = 1, 2, 3$, then we easily get the relationships with the barycentric coordinates given by

$$x = ux_1 + vx_2 + wx_3, \quad y = uy_1 + vy_2 + wy_3.$$

If $f : T \to \mathbb{R}$, then the Bernstein polynomial of degree n in barycentric coordinates attached to f on the triangle T is defined by the formula

$$B_n^T(f)(u, v, w) = \sum_{i,j,k \geq 0, i+j+k=n} f(i/n, j/n, k/n) \frac{n!}{i!j!k!} u^i v^j w^k.$$

Here the expression $f(i/n, j/n, k/n)$ depends on the triangle T (actually on its vertices) as follows: if $f(x, y)$ is the function in Cartesian coordinates, then x and y are connected with u, v, w by the above relationships and $f(i/n, j/n, k/n)$ denotes in fact

$$f\left(x_1 \frac{i}{n} + x_2 \frac{j}{n} + x_3 \frac{k}{n}, y_1 \frac{i}{n} + y_2 \frac{j}{n} + y_3 \frac{k}{n}\right).$$

The above Bernstein polynomial rewritten in Cartesian coordinates (x, y) is called, in CAGD, a Bernstein–Bézier surface.

The Bézier net attached to $B_n^T(f)$ (and implicitly to f) is denoted by $\hat{f}_n$ and is defined as the function that satisfies the following conditions: $\hat{f}_n(i/n, j/n, k/n) = f(i/n, j/n, k/n)$, for all $i+j+k = n$, $i, j, k \geq 0$, and it is linear on each of the subtriangles $\{U_{i,j,k}, i+j+k = n-1, i, j, k \geq 0\}$ and $\{W_{i,j,k}, i+j+k = n-2, i, j, k \geq 0\}$, where $U_{i,j,k}$ has vertices (in barycentric coordinates)

$$\left(\frac{i+1}{n}, \frac{j}{n}, \frac{k}{n}\right), \ \left(\frac{i}{n}, \frac{j+1}{n}, \frac{k}{n}\right), \ \left(\frac{i}{n}, \frac{j}{n}, \frac{k+1}{n}\right),$$

and $W_{i,j,k}$ has vertices

$$\left(\frac{i}{n}, \frac{j+1}{n}, \frac{k+1}{n}\right), \ \left(\frac{i+1}{n}, \frac{j}{n}, \frac{k+1}{n}\right), \ \left(\frac{i+1}{n}, \frac{j+1}{n}, \frac{k}{n}\right).$$

Geometrically, $U_{i,j,k}$ and $W_{i,j,k}$ realize a division (triangulation) of the triangle T by smaller subtriangles all having their sides parallel with the sides of T, triangulation denoted by $\tau_n(T)$.

(ii) (see also Dinghas [93], Lorentz [247], p. 51) If $\Delta_k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k; x_1 + \cdots + x_k \leq 1, x_i \geq 0, i = 1, \ldots, k\}$ is the unit k-dimensional simplex and $f : \Delta_k \to \mathbb{R}$, then the Bernstein polynomial on the k-simplex Δ_k in Cartesian coordinates is defined by

$$B_n^{\Delta_k}(f)(x_1, \ldots, x_k) = \sum_{i_j \geq 0, i_1 + \cdots + i_k \leq n} f\left(\frac{i_1}{n_1}, \ldots, \frac{i_k}{n_k}\right) p_{n;i_1,\ldots,i_k}(x_1, \ldots, x_k),$$

where

$$p_{n;i_1,\ldots,i_k}(x_1, \ldots, x_k) = \binom{n}{i_1, \ldots, i_k} x_1^{i_1} \ldots x_k^{i_k} (1 - x_1 - \cdots - x_k)^{n - i_1 - \cdots - i_k},$$

$$\binom{n}{i_1, \ldots, i_k} = \frac{n!}{i_1! \ldots i_k! (n - i_1 - \cdots - i_k)!}.$$

To Δ_k corresponds the barycentric standard simplex $S_k = \{(u_1, \ldots, u_{k+1}) \in \mathbb{R}^{k+1}; u_j \geq 0, j = 1, \ldots, k+1, \sum_{j=1}^{k+1} u_j = 1\}$ and the above Bernstein polynomial can be rewritten in barycentric coordinates by

$$B_n^{S_k}(f)(u) = \sum_{|i|=n} f\left(\frac{i}{n}\right) B_i^n(u), \forall u = (u_1, \ldots, u_{k+1}) \in S_k,$$

where $i = (i_1, \ldots, i_{k+1})$, $|i| = i_1 + \cdots + i_{k+1}$, $\frac{i}{n} = (\frac{i_1}{n}, \ldots, \frac{i_{k+1}}{n})$, and

$$B_i^n(u) = \binom{n}{i} u^i := \frac{n!}{(i_1)! \cdots (i_{k+1})!} u_1^{i_1} \cdots u_{k+1}^{i_{k+1}}.$$

Given $k+1$ points $P_1, \ldots, P_{k+1} \in \mathbb{R}^k$ in general position (but preferably affine independent), any point $P \in \text{conv}\{P_1, \ldots, P_{k+1}\}$ (where the simplex $\text{D} = \text{conv}\{P_1, \ldots, P_{k+1}\}$ denotes the convex hull of the corresponding points) can be identified with its barycentric coordinates $u = (u_1, \ldots, u_{k+1})$, given by the relationships $P = \sum_{j=1}^{k+1} u_j P_j$, $\sum_{j=1}^{k+1} u_j = 1$. Due to this fact, any simplex $D = \text{conv}\{P_1, \ldots, P_{k+1}\}$ can be identified with the above barycentric standard simplex S_k.

As a consequence, for any k-dimensional simplex D and $f : D \to \mathbb{R}$, the Bernstein polynomial attached to f in barycentric coordinates is formally the same as the above $B_n^{S_k}(f)$ (excepting the values $f\left(\frac{|i|}{n}\right)$, which are recovered by the above relationships with the Cartesian coordinates of $P_1, \ldots, P_{k+1}$).

Therefore, without loss of generality and for simplicity, when one has to deal with functions and Bernstein polynomials defined on an arbitrary k-dimensional simplex, it suffices to consider functions and Bernstein polynomials defined on the barycentric standard simplex S_k.

(iii) The multivariate Bézier net (surface) $\hat{f}_n : S_k \to \mathbb{R}$, can be defined as follows: $\hat{f}_n(i/n) = f(i/n)$ for each multi-index i satisfying $|i| = n$, and piecewise linear on each subtriangle of the canonical triangulation constructed with the aid of an affine linear bijective map by Dahmen–Micchelli [74] (see also Sauer [339], p. 473). Note that because of the method of triangulation, the multivariate case is different from the bivariate case.

Remark. We can mention two interesting generalizations of the planar barycentric coordinates in Definition 2.1.4. One is the case of so-called barycentric coordinates associated with arcs, which leads to a theory of Bernstein–Bézier polynomials that parallels the familiar interval case and has close connections to trigonometric polynomials, see, e.g., Alfeld–Neamtu-Schumaker [3]. The other one is the case of barycentric coordinates on spheres or spherelike surfaces, as for example the spherical triangles, which leads to a theory of Bernstein–Bézier polynomials on spheres and spherelike surfaces, see, e.g., Alfeld–Neamtu–Schumaker [4].

Other concepts of shapes different from those in Definition 2.1.1 (excepting that in Definition 2.1.5 (i) below, which is an easy extension of the bivariate concept in Definition 2.1.1 (v)), are given by the following.

Definition 2.1.5. (i) Following the ideas in Popovicu [315], p. 78, we can say that the function $f : [-1,1]^m \to \mathbb{R}$, $m \in \mathbb{N}$, is called convex of order $(n_1, \dots, n_m)$, in the Popoviciu sense, where $n_i \in \{0,1,2,\dots\}$, $i = \overline{1,m}$ if for any $n_i + 1$ distinct points in $[-1,1]$, $x_1^{(i)} < x_2^{(i)} < \cdots < x_{n_i+1}^{(i)}$, $i = \overline{1,m}$, we have

$$\left[\begin{array}{cc} x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1+1}^{(1)} & \\ x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2+1}^{(2)} & \\ & ; f \\ \vdots & \\ x_1^{(m)}, x_2^{(m)}, \dots, x_{n_m+1}^{(m)} & \end{array}\right] \geq 0,$$

where the above symbol $[\cdot; f]$ means the divided difference of the function f and it is defined (by means of the divided difference of univariate functions) as $[x_1^{(1)}, \dots, x_{n_1+1}^{(1)}; \quad [x_1^{(2)}, \dots, x_{n_2+1}^{(2)}; \quad \dots [x_1^{(m)}, \dots, x_{n_m+1}^{(m)}; \quad f] \dots,]]$, (here each univariate divided difference $[x_1^{(i)}, \dots, x_{n_i+1}^{(i)}; \cdot]$ is considered with respect to the x_i variable, $\forall i = \overline{1,m}$).

(ii) $f : \Omega \to \mathbb{R}$ is called a convex function (on the convex set $\Omega \subset \mathbb{R}^m$) if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in [0,1], x, y \in \Omega$. Denote by $KO_1(\Omega)$ the class of all convex functions on Ω. The differential characterization of $f \in KO_1(\Omega)$ can be seen in Remark 3 after this definition.

(iii) It is a well-known concept that a twice-differentiable function $f : \Omega \to \mathbb{R}$ (where $\Omega \subset \mathbb{R}^m$ is a domain) is called a harmonic function on Ω if it satisfies $\Delta(f)(x) = 0$, for all $x \in \Omega$, where $\Delta(f)(x) = \sum_{k=1}^m \frac{\partial^2 f(x)}{\partial x_k^2}$ denotes the Laplacian of f.

By Gauss (for necessity) and Levi [226] (for sufficiency), if, e.g., $\Omega \subset \mathbb{R}^2$, then f is harmonic if and only if f is continuous in Ω and for all disks of center (x_0, y_0) and radius r included in Ω, it satisfies

$$f(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f[x_0 + r\cos(\theta), y_0 + r\sin(\theta)] d\theta,$$

for all $(x_0, y_0) \in \text{int}(\Omega)$. Denote the class of all harmonic functions on Ω by $H_1(\Omega)$.

As a generalization, f is called a polyharmonic function of order $p \in \mathbb{N}$ in Ω if it satisfies the iterated Laplace equation $\Delta^p(f)(x) = 0$ for all $x \in \Omega$, where $\Delta^p(f) = \Delta[\Delta^{p-1}(f)]$. Using some successive integral means (we do not reproduce them here), the above result of Gauss–Levi (i.e., the case $p = 1$) was extended by Nicolescu [287] to arbitrary $p \in \mathbb{N}$. Denote by $H_p(\Omega)$ the class of all polyharmonic functions of order p.

Also, it is well known that $f : \Omega \to \mathbb{R}$ (where $\Omega \subset \mathbb{R}^m$ is a domain) is called a subharmonic function on Ω if f is upper semicontinuous on Ω and for any sphere $B(x; r) \subset \Omega$, the value $f(x)$ is less than or equal to the mean value of f on the spherical surface $\partial B(x; r)$. For example, in the case of functions of two real variables, the inequality can be written as

$$f(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f[x_0 + r\cos(\theta), y_0 + r\sin(\theta)] d\theta,$$

for all $(x_0, y_0) \in \text{int}(\Omega)$ and all disks of center (x_0, y_0) and radius r included in Ω. Denote by $SH_1(\Omega)$ the class of all subharmonic functions on Ω.

In another paper, Nicolescu [288] introduced the concept of subharmonic function of order $p \in \mathbb{N}$, through the integral means used to represent a polyharmonic function of order p. Denote by $SH_p(\Omega)$ the class of all subharmonic functions of order p on Ω. For $p = 1$ one obtains the above concept of subharmonic function.

The differential characterization of $f \in SH_p(\Omega)$ can be seen in Remark 4 after this definition.

(iv) Let $S = (s_{i,k})_{i,k=1,\ldots,m}$ be a so-called double stochastic matrix, that is, satisfying

$$s_{i,k} \geq 0, \qquad \sum_{i=1}^{m} s_{i,k} = \sum_{k=1}^{m} s_{i,k} = 1,$$

for all $i, k = 1, \ldots, m$.

$D \subset \mathbb{R}^m$ is called an admissible domain if it has the following two properties:

(1) For any $x = (x_1, \ldots, x_m) \in D$, we have $(x_{p(1)}, \ldots, x_{p(m)}) \in D$ for all permutations p of the set $\{1, \ldots, m\}$.
(2) For any double-stochastic matrix $S = (s_{i,k})_{i,k=1,\ldots,m}$ and any $x = (x_1, \ldots, x_m) \in D$, we have $Sx \in D$.

Now, according to Ostrowski [293], a function $f : D \to \mathbb{R}$, $m \geq 2$, D an admissible domain, is called Schur convex if for any double-stochastic matrix S, we have $f(Sx) \leq f(x)$ for all $x \in D$.

(v) (Schmid [342] for the bivariate case), (Sauer [339], see also Dahmen–Micchelli [74], Dahmen [73] for the multivariate case) Let $k \in \mathbb{N}$ and $f \in C(S_k) = \{f : S_k \to \mathbb{R};\ \text{f continuous on } S_k\}$. We say that f is axially convex if

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v), \quad \forall \lambda \in [0,1],$$

whenever $u, v \in S_k$ lie on any line parallel to one of the edges/"axes" (sides in the $k = 2$ case) of S_k, i.e., if there exist an appropriate $c \neq 0$ and $0 \leq j < i$ with $u - v = c(e^j - e^i)$, where $e^j = (\delta_{j,s})_{s=1}^{k+1}$, ($\delta_{j,s}$ the Kronecker's symbol) defines the jth coordinate vector in $\mathbb{R}^{k+1}$.

(vi) (Lorente–Pardo, Sablonnière, Serrano–Perez [246]) Let T be a triangle in the plane with vertices P_1, P_2, P_3. One says that $f : T \to \mathbb{R}$ is w-subharmonic (weak subharmonic) if for all $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_2)$ with $|\alpha| = n$ (α_i positive integers) we have

$$0 \leq \sum_{i=1}^{3} \alpha_i \alpha_{i+1} \delta^2_{\gamma_{i-1}} \left[f\left(\frac{\alpha}{n}\right)\right],$$

where $\gamma_0 = \gamma_3, \gamma_4 = \alpha_1$, $\gamma_1 = \overrightarrow{P_2P_3}, \gamma_2 = \overrightarrow{P_3P_1}, \gamma_3 = \overrightarrow{P_1P_2}$,

$$\delta^2_{\gamma_i}\left[f\left(\frac{\alpha}{n}\right)\right] = f\left(\frac{\alpha - \gamma_i}{n}\right) - 2f\left(\frac{\alpha}{n}\right) + f\left(\frac{\alpha + \gamma_i}{n}\right),$$

for $\frac{\alpha}{n}$ belonging to the interior of the triangle T, and $\delta^2_{\gamma_i}[f\left(\frac{\alpha}{n}\right)] = 0$ otherwise.

Here $\frac{\alpha - \gamma i}{n}$ and $\frac{\alpha + \gamma i}{n}$ denote the points of intersection of the line parallel to the direction γ_i with the other two sides of the triangle T.

This is equivalent to a certain geometric property of all Bézier nets $\hat{f}_n$, $n \in \mathbb{N}$.

Note that w-subharmonicity implies the so-called weak axial convexity introduced in Beśka [40].

(vii) (Sauer [339]; see also Dahmen–Micchelli [74], Dahmen [73]) A function $f : S_k \to \mathbb{R}$, $k \geq 2$, is said to be polyhedral convex if all its Bézier nets $\hat{f}_n$, $n \in \mathbb{N}$, are convex.

(viii) (see Goodman–Sharma [160] for the bivariate case and Goodman–Peters [158] for the multivariate case) The continuous function $f : \Delta_2 \to \mathbb{R}$ is called strongly convex on Δ_2 if it satisfies the following three inequalities

$$f(x,y) + f(x+h,y) \leq f(x,y+h) + f(x+h,y-h),$$
$$f(x,y) + f(x,y+h) \leq f(x+h,y) + f(x-h,y+h),$$
$$f(x,y) + f(x+h,y-h) \leq f(x+h,y) + f(x,y-h),$$

for all the corresponding points belonging to Δ_2.

In the general case, given the affine independent $k+1$ points $P_1, \ldots, P_{k+1} \in \mathbb{R}^k$ and denoting the simplex D by $\text{conv}\{P_1, \ldots, P_{k+1}\}$, the function $f : D \to \mathbb{R}$ is called strongly convex on D if for any $h > 0$ and $0 \leq i < j \leq k+1$, we have

$$f(x + hP_i + hP_{j-1}) + f(x + hP_{i-1} + hP_j)$$
$$\geq f(x + hP_{i-1} + hP_{j-1}) + f(x + hP_i + hP_j)$$

for all x and h for which f is defined, where $P_{-1} := P_{k+1}$.

(ix) Let T be a triangle in the plane and $f : T \to \mathbb{R}$. One says that f is monotonically increasing with respect to the nonnull vector $d = (d_1, d_2) \in \mathbb{R}^2$ (or in the direction d) if for all $x = (x_1, x_2) \in T$ and all $t > 0$ with x, $x+td \in T$, we have $f(x + td) \geq f(x)$.

The directional derivative (in the direction d) at a point $x \in T$ is defined as the limit (supposed to exist) $D_d(f)(x) = \lim_{t \to 0} \frac{f(x+td)-f(x)}{t}$.

If, in addition, f is differentiable on t, then it follows that $D_d(f)(x) = d_1 \frac{\partial f}{\partial x_1}(x) + d_2 \frac{\partial f}{\partial x_1}(x)$, and f is monotonically increasing in the direction d if and only if $D_d(f)(x) \geq 0$ for all $x \in T$.

(x) Recall some well-known facts in multivariate analysis. A direction d in S_k is given by the difference of two points, i.e., $d = u - v$ for $u, v \in S_k$. Therefore, if $d = (d_1, \ldots, d_{k+1})$, the directional (Gâteaux) derivative of $f : S_k \to \mathbb{R}$ with respect to d is defined by $D_d(f)(u) = \lim_{t \to 0} \frac{f(u+td)-f(u)}{t}$, and if, in addition, $f \in C^1(S_k)$, then $D_d(f)(u) = \sum_{i=1}^{k+1} d_i \frac{\partial}{\partial u_i} f(u)$. The directional derivatives of higher order are defined by recurrence, for example $D^2_{d_1 d_2}(f) = D_{d_1}[D_{d_2}(f)]$, and so on.

For $\hat{e}_j = e^j - e^1$, $j = 1, \ldots, k+1$ (where e^j are the unit vectors in $\mathbb{R}^{k+1}$ defined at the above point (v)), we write $D_{i,j}(f) = D_{\hat{e}_i \hat{e}_j}(f)$.

Also, for $b_i \in \mathbb{R}^{k+1}$, where i is a multi-index, the multivariate forward difference operator can be inductively defined by $\Delta^0_{\hat{e}_j} b_i := b_i$,

$$\Delta^r_{\hat{e}_j} b_i = \Delta^{r-1}_{\hat{e}_j} b_{i+\hat{e}_j} - \Delta^{r-1}_{\hat{e}_j} b_i,$$
$$\Delta_{i,j} = \Delta_{\hat{e}_i} \Delta_{\hat{e}_j},$$

where $r = (r_1, \ldots, r_k)$ is a multi-index.

Remarks. (1) For $m = 2$ in Definition 2.1.5 (i), we get the concept in Popoviciu [315], p. 78.

(2) If f is of class $C^{n_1+\cdots+n_m}$ on $[-1, 1]^m$, then by the mean value theorem it follows that the condition

$$\frac{\partial^{n_1+\cdots+n_m} f(x_1, \ldots, x_m)}{\partial x_1^{n_1} \ldots \partial x_m^{n_m}} \geq 0 \quad \text{on } [-1, 1]^m$$

implies that f is convex of order $(n_1, \ldots, n_m)$.

(3) If f is twice continuously differentiable on Ω, then it is a classical result that $f \in KO_1(\Omega)$ if and only if for every $j = (j_1, \ldots, j_m) \in \mathbb{R}^m$ with $\|j\|_{\mathbb{R}^m} = 1$, we have

$$L_j(f)(x) = \sum_{1 \leq k,i \leq m} \frac{\partial^2 f(x)}{\partial x_i \partial x_k} j_i j_k \geq 0, \quad \forall x \in \Omega.$$

(4) If f is twice differentiable on Ω, according to a classical result due to Montel [276] (see also, e.g., Haymann–Kennedy [168], p. 41) we have $f \in SH_1(\Omega)$ if and only if $\Delta(f)(x) = \sum_{k=1}^{m} \frac{\partial^2 f(x)}{\partial x_k^2} \geq 0, \forall x \in \Omega$.

More generally, if f has all continuous partial derivatives of order $2p$ in Ω, then $f \in SH_p(\Omega)$ if and only if it satisfies the differential inequality $(-1)^p \Delta^p(f)(x) \leq 0$ for all $x \in \Omega$ (Nicolescu [288]).

(5) If $f \in C^2(S_k)$, then in Sauer [339], Proposition 2, it is proved that the axial convexity of f can be expressed in terms of some inequalities satisfied by the second-order directional derivatives (see also the proof of Theorem 2.2.3 (iii) below). For example, by taking for simplicity $k = 2$, if $T = S_2$ is a triangle in the plane with vertices P_1, P_2, P_3, and setting $\gamma_1 = \overrightarrow{P_2P_3}$, $\gamma_2 = \overrightarrow{P_3P_1}$, $\gamma_3 = \overrightarrow{P_1P_2}$, then f is axially convex if and only if we have $D^2_{\gamma_i}(f)(u) \geq 0$ for all $u \in T$ and $i = 1, 2, 3$, where $D^2_{\gamma_i}(f) = D_{\gamma_i}(f)[D_{\gamma_i}(f)]$ and the directional derivatives are those in Definition 2.1.5 (ix), (x).

(6) Similar to the above Remark 5, if $f \in C^2(S_k)$, then by Sauer [339], Proposition 10, the polyhedral convexity of f can be expressed in terms of some inequalities satisfied by the second-order directional derivatives. For example, keeping the notation in the bivariate case, f is polyhedral convex on the triangle T if and only if $D^2_{\gamma_i\gamma_j}(f)(u) \geq 0$ for all $u \in T$ and $i, j = 1, 2, 3$.

(7) If $D \subset \mathbb{R}^m$ is an admissible domain (as defined by Definition 2.1.5 (iv)), supposing D open and that $f : D \to \mathbb{R}$ has continuous partial derivatives of first order on D, then f is Schur convex on D if and only if

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_j}(x)\right) \geq 0,$$

for all $x = (x_1, \ldots, x_m) \in D$ and $i, j \in \{1, \ldots, m\}$ (see, e.g., Lupaş [259]).

(8) For bivariate continuous functions on triangles, it is known Sauer [339] that polyhedral convexity implies convexity, and convexity implies axial convexity and subharmonicity.

(9) The relationship between axial convexity and subharmonicity can depend on the geometry of the triangle. More specifically, we have (Lorente–Pardo–Sablonnière–Serrano–Pérez [246]):

(i) If T is not obtuse, i.e., $\cos(\theta_i) \geq 0$ for $i = 1, 2, 3$, then axial convexity implies subharmonicity.

(ii) If T is obtuse (there exists $i \in \{1, 2, 3\}$ with $\cos(\theta_i) < 0$), then there is no relation, a priori, between axial convexity and subharmonicity. It is enough to consider the functions $u_1 : T_1 \to \mathbb{R}$ and $u_2 : T_2 \to \mathbb{R}$ given by

$u_1(x,y) = -y(x+y)$ and $u_2(x,y) = 4x^2 - y^2$, respectively, where T_1 is the triangle with vertices $A = (0,0)$, $B = (1,0)$, and $C_1 = (-1,1)$, and T_2 is the triangle with vertices A, B, and $C_2 = (-1,-3)$. Then u_1 is axially convex but is not subharmonic. However, u_2 is subharmonic but is not axially convex.

(10) (Lorente–Pardo–Sablonnière–Serrano–Pérez [246]) Axial convexity implies w-subharmonicity on triangular domains.

2.2 Bernstein-Type Polynomials Preserving Shapes

Other multivariate Bernstein-type polynomials we consider in this section are given by the following.

Definition 2.2.1. (i) (Hildebrandt–Schoenberg [170]) If $f : [0,1]^k \to \mathbb{R}$, then the tensor product Bernstein polynomial (on the k-cube $[0,1]^k$) is defined by

$$B_{n_1,\dots,n_k}(f)(x_1,\dots,x_k) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} \left[\Pi_{j=1}^k p_{n_j,i_j}(x_j)\right] f\left(\frac{i_1}{n_1},\dots,\frac{i_k}{n_k}\right),$$

where $p_{n_j,i_j}(x_j) = \binom{n_j}{i_j} x_j^{i_j}(1-x_j)^{n_j-i_j}$ for all $j = 1,\dots,k$.

(ii) (Goodman–Sharma [160]) For $f : \Delta_k \to \mathbb{R}$, $k \in \mathbb{N}$, one defines the Bernstein-type polynomial $U_{n,k}(f)(x) = \sum_{|i|=n} p_{n,i}(x) \int_{\mathbb{R}^k} K_i(t)dt$, $n \in \mathbb{N}$, $x = (x_1,\dots,x_k) \in \Delta_k$, where $i = (i_1,\dots,i_k)$,

$$p_{n;i_1,\dots,i_k}(x_1,\dots,x_k) = \binom{n}{i_1,\dots,i_k} x_1^{i_1}\dots x_k^{i_k}(1-x_1-\cdots-x_k)^{n-i_1-\cdots-i_k},$$

$$\binom{n}{i_1,\dots,i_k} = \frac{n!}{i_1!\cdots i_k!(n-i_1-\cdots-i_k)!},$$

and

$$\int_{\mathbb{R}^k} K_i(t)dt = (|i|-1)! \int_{\Delta_{|i|-1}} f\left(\sum_{j=1}^k e^j \sum_{q=1}^{i_j} t_{(i_1+\cdots+i_{j-1})+q}\right) dt_1 \cdots dt_{|i|}.$$

Here $e^j = (0,\dots,1,\dots,0) \in \mathbb{R}^k$, with 1 in the jth position and $i_1+\cdots+i_{j-1} = 0$ for $j = 1$.

By Sauer [338], it follows that $U_{n,k}(f)$ is exactly the Bernstein–Durrmeyer–type polynomial that incorporates a particular Jacobi weight, which for $k = 1$ does not reduce to the usual multivariate Bernstein–Durrmeyer polynomial introduced in Derriennic [78], but one reduces to the modified Bernstein–Durrmeyer polynomial in Chen [68].

Remarks. (1) By similar reasoning to that of Popoviciu [316] in the case of one variable, we immediately obtain the estimate (for $k = 2$ see Feng [116])

$$|f(x_1, \dots, x_k) - B_{n_1,\dots,n_k}(f)(x_1, \dots, x_k)| \le C\omega_1(f; 1/\sqrt{n_1}, \dots, 1/\sqrt{n_k})$$

for all $x_j \in [0, 1]$, $j = 1, \dots, k$, where $C > 0$ is independent of f, $x_1, \dots, x_k$, $n_1, \dots, n_k$ and

$$\omega_1(f; \delta_1, \dots, \delta_k) \\ = \sup\{|f(x_1, \dots, x_k) - f(y_1, \dots, y_k)|; |x_j - y_j| \le \delta_j, j = 1, \dots, k\}.$$

(2) For the Bernstein polynomials defined on a simplex Δ_k, and $f \in C(\Delta_k) = \{f : \Delta_k \to \mathbb{R}; f \text{ is continuous on } \Delta_k\}$, we can recall the estimates

$$\|f - B_n^{\Delta_k}(f)\|_{C(\Delta_k)} \le C[K_\Phi(f; n^{-1}) + n^{-1}\|f\|_{C(\Delta_k)}]$$

(see Ditzian [94]) and

$$\|f - B_n^{\Delta_k}(f)\|_{C(\Delta_k)} \le Cn^{-1} \sum_{1 \le p \le \sqrt{n}} E_p(f)$$

(see Ditzian [95]), where $K_\Phi(f; t) = \inf_{g \in A}\{\|f - g\|_{C(\Delta_k)} + t\Phi(g)\}$, $g \in A$, means that $\partial g/\partial x_i \in AC_{\text{loc}}$ with respect to any variable x_j, $\partial^2 g/\partial x_i \partial x_j$ are continuous in the interior of Δ_k, $\Phi(g)$ is a suitably defined seminorm, and $E_p(f)$ is the best approximation of f on Δ_k (in the uniform norm $\|\cdot\|_{C(\Delta_k)}$) by polynomials (in the variables $x_1, \dots, x_k$) of total degree $\le p$.

(3) For the Bernstein–Durrmeyer-type polynomial $U_{n,2}(f)$, the following approximation error is proved in Goodman–Sharma [160]:

$$\|f - U_{n,k}(f)\| \le C_1\omega_1\left(f; \frac{1}{\sqrt{n}}\right) + \frac{C_2}{n}\|f\|,$$

where $\|\cdot\|$ is the uniform norm on Δ_k, $\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in \Delta_k, \|x - y\| \le \delta\}$, $\|x\| = x_1 + \cdots + x_k$, $x = (x_1, \dots, x_k) \in \Delta_k$.

(4) All the Bernstein-type polynomials in one variable introduced in Section 1.3 (see the 12 examples after Theorem 1.3.2) can be extended to several variables by the tensor product method or by the method used for Bernstein polynomials on a simplex. We can mention, for example, several bivariate/multivariate Bernstein–Stancu-type polynomials in Stancu [364]–[369], Stancu–Vernescu [370], Vlaic [392]–[395], Moldovan [275], the modified Bernstein–Durrmeyer-type polynomial on a simplex in Goodman–Sharma [160] (see also Sauer [338]), and the Bernstein–Durrmeyer polynomial on a simplex in Derriennic [78].

The history of applications of shape-preserving properties of Bernstein polynomials to computer aided geometric design begin with the pioneering work of the engineer and mathematician Pierre Bézier, who applied them to automobile design at the Renault company (for a history on this subject see, e.g., Laurent–Sablonnière [220]).

Concerning other types of Bernstein polynomials, an application of a certain Bernstein–Stancu-type base for defining generalized, more flexible (since they depend on some parameters) Bézier curves and surfaces was done in Gânsca–Coman–Ţâmbulea [147].

In Goodman–Sharma [160] (case k = 2), Goodman–Peters [158] (case $k > 2$), it is proved that the polynomials $U_{n,k}(f)$ preserve the strong convexity in Definition 2.1.4 (viii), while in Sauer [338], it is proved that $U_{n,k}(f)$ has several shape-preserving properties on a simplex.

In this section we present the main shape-preserving properties of the two classical Bernstein polynomials in Definition 2.1.4, (ii), Definition 2.2.1 (i), and those of modified Bernstein–Durrmeyer polynomial on a simplex.

For simplicity and without loss of generality, some results will be considered in two real variables.

In the case of a bivariate tensor product Bernstein polynomial, we present the following result.

Theorem 2.2.2. *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *and the tensor product Bernstein polynomial on* $[0,1] \times [0,1]$ *be defined by*

$$B_{n,m}(f)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{n,i}(x) p_{m,j}(y) f\left(\frac{i}{n}, \frac{j}{m}\right),$$

where $p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$ *for all* $i = 0, \ldots, n$, *and* $p_{m,j}(y)$ *in a similar manner is defined.*

If $f(x,y)$ *is* (r,s)*-convex in the Popoviciu sense, then so is* $B_{n,m}(f)(x,y)$. *In addition, if* f *is continuous on* $[0,1] \times [0,1]$, *then the following estimate holds:*

$$\|B_{n,m}(f) - f\| \leq C\omega_2^{\varphi}(f; 1/\sqrt{n}, 1/\sqrt{m}),$$

where $C > 0$ *is an absolute constant,* $\|\cdot\|$ *denotes the uniform norm on* $[0,1] \times [0,1]$, $\omega_2^{\varphi}(f; 1/\sqrt{n}, 1/\sqrt{m})$ *is the Ditzian–Totik modulus of smoothness in Definition 2.1.5 (i), and* $\varphi(x)^2 = x(1-x)$.

Proof. First we prove the estimate. It is known (see Knoop–Zhou [194] and Totik [387]) that for univariate $g : [0,1] \to \mathbb{R}$, we have

$$\|g - B_n(g)\| \leq C\omega_2^{\varphi}(g; \frac{1}{\sqrt{n}})_{\infty}.$$

But $\|B_n(g)\| \leq \|g\|$, which implies $\|\|B_n\|\| \leq 1, \forall n \in \mathbb{N}$ (here $B_n(g)$ denotes the univariate Bernstein polynomial).

Since $B_{n,m}(f)$ is the tensor product of B_n and B_m, applying Theorem 5 in Haussmann–Pottinger [167], we immediately get

$$\|f - B_{n,m}(f)\| \le C \left[\omega_{2,x}^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) + \omega_{2,y}^{\varphi}\left(f; \frac{1}{\sqrt{m}}\right)\right],$$

where $\omega_{2,x}^{\varphi}$ and $\omega_{2,y}^{\varphi}$ are the partial moduli defined in Ditzian-Totik [98], Chapter 2. Taking into account that obviously

$$\omega_{2,x}^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right) + \omega_{2,y}^{\varphi}\left(f; \frac{1}{\sqrt{m}}\right) \le 2\omega_{2}^{\varphi}\left(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}}\right),$$

we obtain the required estimate.

On the other hand, by the proof of Theorem 1.3.1 (i), (see also, e.g., Popoviciu [317]), for $g : [0,1] \to \mathbb{R}$, we have

$$B_n^{(r)}(g)(x) = \frac{r!n!}{(n-r)!n^r} \sum_{i=0}^{n-r} \left[\frac{i}{n}, \dots, \frac{i+r}{n}; g\right] p_{n-r,i}(x).$$

This immediately implies

$$\frac{\partial^{r+s} B_{n,m}(f)(x,y)}{\partial x^r \partial y^s} = \frac{r!n!}{(n-r)!n^r} \frac{s!m!}{(m-s)!m^s} \sum_{i=0}^{n-r} \sum_{j=0}^{m-s} p_{n-r,i}(x) p_{m-s,j}(y)$$
$$\times \left[\frac{i}{n}, \dots, \frac{i+r}{n}; \left[\frac{j}{m}, \dots, \frac{j+s}{m}; f(x,y)\right]_y\right]_x,$$

which by hypothesis on f implies $\frac{\partial^{r+s} B_{n,m}(f)(x,y)}{\partial x^r \partial y^s} \ge 0$, $\forall x, y \in [0,1]$, and proves the theorem. □

The Bernstein polynomials and the modified Bernstein–Durrmeyer polynomials on a simplex have interesting shape-preserving properties, summarized by the following.

Theorem 2.2.3. *(i) (Lupaş [259]) If $f : \Delta_2 \to \mathbb{R}$ is Schur convex on Δ_2, then so is $B_n^{\Delta_2}(f)$ for all $n \in \mathbb{N}$;*

(ii) (Chang–Davis [64], Theorem 5) If $f : \Delta_2 \to \mathbb{R}$ is strongly convex on Δ_2, then so is $B_n^{\Delta_2}(f)$.
(Goodman-Peters [158]) If $f : \Delta_k \to \mathbb{R}$, $k > 2$, is strongly convex on Δ_k, then so is $B_n^{\Delta_k}(f)$ too.

(iii) (Sauer [339]) If $f : S_k \to \mathbb{R}$ is continuous and axially convex on S_k, then $B_n^{S_k}(f)$ is axially convex for all $n \in \mathbb{R}$.

(iv) (Sauer [339]) If $f : S_k \to \mathbb{R}$ is continuous and polyhedrally convex on S_k, then $B_n^{S_k}(f)$ is polyhedrally convex for all $n \in \mathbb{R}$.

(v) (Goodman–Sharma [160]) If $f : \Delta_2 \to \mathbb{R}$ is strongly convex on Δ_2, then so is the polynomial $U_{n,2}(f)$.

(vi) (Sauer [338]) For any $n, k \in \mathbb{N}$ with $k \ge 2$, the polynomial $U_{n,k}(f)$ preserves any polynomial of degree $\le n$, the axial convexity, the polyhedral convexity, and the subharmonicity, but in general does not preserve the convexity.

Proof. (i) We follow the proof in Lupaş [259]. Notice together with Ostrowski [293] that if f is Schur convex on Δ_2 then it is symmetric too, i.e., $f(x,y) = f(y,x)$ for all $(x,y) \in \Delta_2$. Also, Lupaş [259] proved that the symmetry is preserved by $B_n^{\Delta_2}(f)$ for all $n \in \mathbb{N}$. Indeed, this is an immediate consequence of the general relationships $\sum_{k=0}^{n}\sum_{i=0}^{n-k} A_{i,k} = \sum_{k=0}^{n}\sum_{i=0}^{n-k} A_{k,i}$ and $\binom{n}{k}\binom{n-k}{i} = \binom{n}{i}\binom{n-i}{k}$.

In order to prove the Schur convexity of $B_n^{\Delta_2}(f)$, we will use Remark 7 of Definition 2.1.5. First we note that we can write $B_n^{\Delta_2}(f)(x,y) = \sum_{k=0}^{n}\sum_{i=0}^{n-k} p_{n,k,i}(x,y) f(k/n, i/n)$, where $p_{n,k,i}(x,y) = \binom{n}{k}\binom{n-k}{i} x^k y^i (1-x-y)^{n-k-i}$.

We have

$$\frac{\partial B_n^{\Delta_2}(f)}{\partial x} = n \sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} p_{n-1,k,i}[f((k+1)/n, i/n) - f(k/n, i/n)],$$

$$\frac{\partial B_n^{\Delta_2}(f)}{\partial y} = n \sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} p_{n-1,k,i}[f(k/n, (i+1)/n) - f(k/n, i/n)].$$

Write $F[B_n^{\Delta_2}(f)](x,y) = \frac{1}{n}(x-y)\left(\frac{\partial B_n^{\Delta_2}(f)}{\partial x} - \frac{\partial B_n^{\Delta_2}(f)}{\partial y}\right)$, $E(f)(a,x,y) = f(x,y) - f[ax + (1-a)y, (1-a)x + ay]$, and

$$q_{i,n}(x,y) = \binom{n-1}{i}\binom{n-1-i}{k-i} x^i y^k (1-x-y)^{n-k-1}(x-y)(x^{k-2i} - y^{k-2i}).$$

It is easy to see that f is Schur convex on Δ_2 if and only if $E(f)(a,x,y) \geq 0$ for all $(x,y) \in \Delta_2$, $a \in [0,1]$ and that $q_{i,n}(x,y) \geq 0$,

$$(x-y)[p_{n-1,2k-i,i}(x,y) - p_{n-1,i,2k-i}(x,y)] = q_{i,2k}(x,y),$$

for all $i = 0, \ldots, k-1$, $k = 1, \ldots, [(n-1)/2]$,

$$(x-y)[p_{n-1,2k+1-i,i}(x,y) - p_{n-1,i,2k+1-i}(x,y)] = q_{i,2k+1}(x,y),$$

for all $i = 0, \ldots, k$, $k = 0, 1, \ldots, [(n-2)/2]$.

By the general relationship

$$\sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} A_{i,k} = \sum_{i=0}^{[(n-1)/2]} A_{i,i} + \sum_{k=1}^{[(n-1)/2]}\sum_{i=0}^{k-1}(A_{i,2k-i} + A_{2k-i,i})$$
$$+ \sum_{k=0}^{[(n-2)/2]}\sum_{i=0}^{k}(A_{i,2k+1-i} + A_{2k+1-i,i}),$$

we obtain

$$\begin{aligned}
&F[B_n^{\Delta_2}(f)](x,y)\\
&=\sum_{k=0}^{n-1}\sum_{i=0}^{n-1-k} p_{n-1,k,i}(x,y)[f((k+1)/n, i/n) - f(k/n,(i+1)/n)]\\
&=\sum_{k=1}^{[(n-1)/2]}\sum_{i=0}^{k-1} q_{i,2k}(x,y)E(f)\left(\frac{2k-2i}{2k-2i+1}, \frac{2k-i+1}{n}, \frac{i}{n}\right)\\
&\quad+\sum_{k=0}^{[(n-2)/2]}\sum_{i=0}^{k} q_{i,2k+1}(x,y)E(f)\left(\frac{2k-2i+1}{2k-2i+2}, \frac{2k-i+2}{n}, \frac{i}{n}\right).
\end{aligned}$$

As a conclusion,

$$F[B_n^{\Delta_2}(f)](x,y) = \sum_{k=1}^{n-1}\sum_{i=0}^{[(k-1)/2]} q_{i,k}(x,y)E(f)\left(\frac{k-2i}{k-2i+1}, \frac{k-i+1}{n}, \frac{i}{n}\right),$$

which by the Schur convexity of f implies that of $B_n^{\Delta_2}(f)$ too.

(ii) Since the proofs are rather technical, we omit them. We mention only that for the case $k > 2$, the proof of (ii) is different from the case $k = 2$, is much based on the results in the paper Dahmen–Micchelli [74], and can be found in Goodman–Peters [158].

(iii) We sketch out the main lines of the proof. It is based on the following two simple/standard auxiliary results.

Lemma A. (Sauer [339]) *A continuous function $f : S_k \to \mathbb{R}$ is axially convex on S_k if and only if we have*

$$\Delta_{p,p}(f)(i/k) \geq 0, (\Delta_{j,j} + \Delta_{p,p} - 2\Delta_{j,p})f(i/k) \geq 0,$$

for all $|i| = k$ and $1 \leq j \leq p$.

Proof of Lemma A. We know that for continuous functions, convexity in one direction is in fact equivalent to the midpoint convexity in that direction. Now, convexity in the direction $e^p - e^1$ is equivalent to

$$\frac{1}{2}f\left(\frac{i+2\hat{e}_p}{k}\right) + \frac{1}{2}f\left(\frac{i}{k}\right) \geq f\left(\frac{i+\hat{e}_p}{k}\right),$$

which can easily be rewritten as the first inequality in statement.

Applying similar reasoning to the convexity in the direction $e^p - e^j$, we get that it is equivalent to the second inequality in the statement, which proves Lemma A.

Lemma B. (Sauer [339]) *A C^2-function $f : S_k \to \mathbb{R}$ is axially convex on S_k if and only if we have*

$$D_{p,p}(f)(u) \geq 0, \quad (D_{j,j} + D_{p,p} - D_{j,p})f(u) \geq 0,$$

for all $1 \leq j \leq p$ and $u \in S_k$.

Proof of Lemma B. Taking into account the integral representation of multivariate finite forward differences of functions of C^2-class in terms of directional/partial derivatives, it is immediate that the inequalities in Lemma A are equivalent to the inequalities in Lemma B.

Now, since we have

$$D_{j,j}B_n^{S_k}(f)(u) = \sum_{|i|=n-2} \Delta_{j,j} f\left(\frac{i+2e^1}{n}\right) B_i^{n-2}(u),$$

where $B_i^n(u)$ is given by Definition 2.1.4 (ii), we easily get that $D_{j,j}B_n^{S_k}(f) \geq 0$. In a similar manner we get $(D_{j,j}+D_{p,p}-D_{j,p})B_n^{S_k}(f)(u) \geq 0$, which combined with Lemma B proves the statement in (iii).

(iv) A detailed proof can be found in Sauer [339], p. 475, Theorem 11.

(v) We follow the ideas of the proof in Goodman–Sharma [160]. Let $f : \Delta_2 \to \mathbb{R}$ be strongly convex on Δ_2. First let us suppose that $f \in C^2(\Delta_k)$, i.e, $\frac{\partial^2 f}{\partial x^2} \geq \frac{\partial^2 f}{\partial x \partial y} \geq 0$ and $\frac{\partial^2 f}{\partial y^2} \geq \frac{\partial^2 f}{\partial x \partial y} \geq 0$. We have

$$\frac{\partial^2 U_{n,2}(f)(x,y)}{\partial x \partial y} = \sum_{i+j+k=n-2} p_{i,j,k}(x,y) \int_{\Delta_2} B_{i+1,j+1,k+2}(u,v) \frac{\partial^2 f}{\partial u \partial v}(u,v)du\, dv \geq 0,$$

by the hypothesis on f.

Similarly, from the hypothesis on f we can prove

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) U_{n,2}(f)(x,y) \geq 0$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) U_{n,2}(f)(x,y) \geq 0.$$

From Chang–Davis [64], Theorem 5, for an $f \in C(\Delta_2)$, it follows that $B_m^{\Delta_2}(f)$ is also strongly convex. Since $B_m^{\Delta_2}(f) \in C^2$, we get that $U_{n,2}(B_m^{\Delta_2}(f))$ is strongly convex with respect to Δ_2. By $B_m^{\Delta_2}(f) \to f$ as $m \to \infty$, we get $U_{n,2}[B_m^{\Delta_2}(f)] \to U_{n,2}(f)$, as $m \to \infty$. But the strong convexity means that $U_n(B_m^{\Delta_2}(f))$, $n, m \in \mathbb{N}$, satisfy the three inequalities in Definition 2.1.5 (vii). Passing to limit with $m \to \infty$ in these inequalities and taking into account that we easily have $\|U_{n,2}(f)\|_{\Delta_2} \leq \|f\|_{\Delta_2}$ (where $\|\cdot\|_{\Delta_2}$ denotes the uniform norm on Δ_2), we get that $\lim_{m\to\infty} U_{n,2}(B_m^{\Delta_2}(f)) = U_{n,2}(f)$, which immediately implies that $U_{n,2}(f)$ is strongly convex with respect to Δ_2.

(vi) We omit the proofs, which are long and technical. □

Remark. In Schmid [342] and Chang–Davis [64], pp. 12–13 it is proved (are given counterexamples) that if $f : \Delta_2 \to \mathbb{R}$ is convex on Δ_2 then $B_n^{\Delta_2}(f)$ is not necessarily convex on Δ_2; therefore $B_n^{\Delta_2}(f)$ does not preserve the convexity of f. Thus, in Schmid [342], it is shown that for the convex function $f(x,y) = |x-y|$ defined on the standard simplex $\Delta_2 \subset \mathbb{R}^2$, all the Bernstein polynomials $B_n^{\Delta_2}(f), n \in \mathbb{N}$, are not convex on Δ_2.

Another much-studied topic in CAGD concerns sufficient linear and nonlinear conditions for the (usual) convexity/monotonicity of bivariate/multivariate Bernstein polynomials.

The first sufficient conditions for convexity were derived for Bernstein polynomials on a triangle/simplex. Chang and Davis [64] showed that a sufficient condition for such a surface to be convex is that the Bézier net be convex and proved other sufficient linear conditions for convexity too. A weaker sufficient linear condition for convexity was found by Chang and Feng [66]. Also, Chang and Feng [65] derived another weaker but nonlinear condition, while Lai [219] derived a linear condition weaker than those in Chang–Davis [64]. In Carnicer–Floater–Peña [61] linear conditions that are weaker than all linear conditions mentioned above are introduced. These conditions have the advantage that they are symmetric with respect to the barycentric coordinates and moreover, can be interpreted geometrically.

On the other hand, sufficient conditions for convexity were derived for bivariate tensor-product Bernstein polynomials (called Bernstein–Bézier surfaces) too. First, Cavaretta and Sharma [63] showed that when the (bilinear) Bézier net of such a surface is convex, then so is the surface. Although the conditions are linear, they are very restrictive, since they imply that the Bernstein–Bézier surface $S(x,y)$ equals $f(x)+g(y)$, for some univariate functions f and g (in this case S is called translational). Then weaker but nonlinear conditions were obtained by Floater [118].

Starting from these last conditions, in Carnicer–Floater–Peña [61] linear conditions for convexity that do not require that S be translational were found. Even though these conditions are stronger than those in Floater [118], they have the advantage of being linear and moreover can be interpreted geometrically, in a similar way to the triangular case.

Also, several sufficient conditions for monotonicity were proved in Floater–Peña [119].

The next two theorems summarize the main results in this topic.

For Bernstein polynomials on a triangle/simplex, we present the following result.

Theorem 2.2.4. *Let us consider the Bernstein polynomial of degree n on the simplex $\Delta_2 = \{x \geq 0, y \geq 0, x+y \leq 1\}$,*

$$B_n^{\Delta_2}(f)(x,y) = \sum_{i+j+k=n} p_{i,j,k} B_{i,j,k}(x,y),$$

where $p_{i,j,k} := f(i/n, j/n, k/n)$ and

$$B_{i,j,k}(x,y) := \frac{(i+j+k)!}{i!j!k!} x^i y^j (1-x-y)^k.$$

Also, let us define the following difference operators:

$$\Delta_a p_{i,j,k} := p_{i+2,j,k} + p_{i,j+1,k+1} - p_{i+1,j+1,k} - p_{i+1,j,k+1},$$
$$\Delta_b p_{i,j,k} := p_{i,j+2,k} + p_{i+1,j,k+1} - p_{i+1,j+1,k} - p_{i,j+1,k+1},$$
$$\Delta_c p_{i,j,k} := p_{i,j,k+2} + p_{i+1,j+1,k} - p_{i+1,j,k+1} - p_{i,j+1,k+1},$$

for $i + j + k = n - 2$, *with* $i, j, k \geq 0$.

(i) *(Chang–Davis [64]) If the nth Bézier net (attached to* $B_n^{\Delta_2}(f)$*) is convex on* Δ_2*, then so is* $B_n^{\Delta_2}(f)$.
The result was extended to Δ_k, $k > 2$, *in Dahmen–Micchelli [74].*

(ii) *(Sauer [340]) If the Bézier net* $\hat{f}_\alpha$, $|\alpha| = n$, *is axially convex on* S_k*, then so is* $B_n^{S_k}(f)$.

(iii) *(Chang–Davis [64])* $B_n^{\Delta_2}(f)(x, y)$ *is convex, provided that*

$$\Delta_a p_{i,j,k} \geq 0, \quad \Delta_b p_{i,j,k} \geq 0, \quad \Delta_c p_{i,j,k} \geq 0,$$

for all $i + j + k = n - 2$, *with* $i, j, k \geq 0$.

(iv) *(Chang–Feng [66])* $B_n^{\Delta_2}(f)(x, y)$ *is convex, provided that the matrices*

$$A = \begin{pmatrix} (\Delta_a + \Delta_c)p_{i,j,k} & \Delta_c p_{i,j,k} \\ \Delta_c p_{i,j,k} & (\Delta_b + \Delta_c)p_{i,j,k} \end{pmatrix}$$

are positive semidefinite for all $i, j, k \geq 0$ *with* $i + j + k = n - 2$.

(v) *(Chang–Feng [65]) The nonlinear conditions*

$$(\Delta_a + \Delta_b)p_{i,j,k} \geq 0, \quad (\Delta_b + \Delta_c)p_{i,j,k} \geq 0, \quad (\Delta_c + \Delta_a)p_{i,j,k} \geq 0,$$
$$\Delta_b p_{i,j,k}\Delta_c p_{i,j,k} + \Delta_c p_{i,j,k}\Delta_a p_{i,j,k} + \Delta_a p_{i,j,k}\Delta_b p_{i,j,k} \geq 0,$$

for all $i, j, k \geq 0$, *with* $i + j + k = n - 2$, *imply the convexity of* $B_n^{\Delta_2}(f)(x, y)$.

(vi) *(Lai [219]) The linear conditions*

$$\Delta_a p_{i,j,k} \geq 0, \quad \Delta_b p_{i,j,k} \geq 0, \quad \Delta_a p_{i,j,k} + 2\Delta_c p_{i,j,k} \geq 0,$$

$$\Delta_b p_{i,j,k} + 2\Delta_c p_{i,j,k} \geq 0,$$

for all $i + j + k = n - 2$, *with* $i, j, k \geq 0$, *imply the convexity of* $B_n^{\Delta_2}(f)(x, y)$.

(vii) *(Carnicer–Floater–Peña [61]) The linear conditions*

$$\Delta_a p_{i,j,k} + 2\Delta_c p_{i,j,k} \geq 0,\ \Delta_b p_{i,j,k} + 2\Delta_c p_{i,j,k} \geq 0,\ \Delta_c p_{i,j,k} + 2\Delta_a p_{i,j,k} \geq 0,$$
$$\Delta_c p_{i,j,k} + 2\Delta_b p_{i,j,k} \geq 0,\ \Delta_b p_{i,j,k} + 2\Delta_a p_{i,j,k} \geq 0,\ \Delta_a p_{i,j,k} + 2\Delta_b p_{i,j,k} \geq 0,$$

for all $i, j, k \geq 0$, *with* $i + j + k = n - 2$, *imply the convexity of* $B_n^{\Delta_2}(f)(x, y)$.

(viii) *(Floater–Peña [119]) If the nth Bézier net (attached to* $B_n^{\Delta_2}(f)$*) is monotonically increasing with respect to all nonnull vectors* $d \in \mathbb{R}^2$*, then so is* $B_n^{\Delta_2}(f)(x, y)$.

(ix) (Floater–Peña [119]) Let us denote by P_1, P_2, P_3 *the vertices of the triangle* Δ_2*. If the nth Bézier net is monotonically increasing with respect to the vectors* $\overrightarrow{P_2P_1}$, $\overrightarrow{P_3P_2}$, $\overrightarrow{P_1P_3}$*, then so is* $B_n^{\Delta_2}(f)(x,y)$*.*

(x) (Floater–Peña [119]) Let P_1, P_2, P_3 *be the vertices of triangle* Δ_2*. The condition* $E_d c_p \geq 0$, $\forall |p| = n-1$*, implies that* $B_n^{\Delta_2}(f)(x,y)$ *is monotonically increasing in the direction d, where* $E_d c_p = \alpha_1 c_{p+e^1} + \alpha_2 c_{p+e^2} + \alpha_3 c_{p+e^3}$*, with* $\alpha_1, \alpha_2, \alpha_3$ *solutions of the equations*

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 = d, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0,$$

$e^1 = (1,0,0)$, $e^2 = (0,1,0)$, $e^3 = (0,0,1)$*, and, e.g.,* $c_{p+e^1} = f\left(\frac{p_1+1}{n-1}, \frac{p_2}{n-1}, \frac{p_3}{n-1}\right)$*, for* $p = (p_1, p_2, p_3)$ *(the others* c_{p+e^2} *and* c_{p+e^3} *are analogously defined).*

For the tensor-product Bernstein polynomial, we present the following sufficient conditions for convexity.

Theorem 2.2.5. *Let us consider the tensor-product Bernstein polynomial of degree* (m,n) *given by*

$$B_{n,m}(f)(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{i,j} B_{i,n}(x) B_{j,m}(y),$$

where $p_{i,j} = f(i/n, j/m)$, $B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $i = 0, \ldots, n$*, and we assume that* $n \geq 2$, $m \geq 2$*.*

Let us define the finite difference operator

$$\Delta_{k,i} p_{i,j} = \sum_{i=0}^{n}\sum_{s=0}^{m} (-1)^{k+l-r-s} \binom{k}{r}\binom{l}{s} p_{i+r,j+s}.$$

By analogy with the Bézier net on triangles, a Bézier net (attached to the above tensor-product Bernstein polynomial) can be defined by

$$\begin{aligned}\hat{f}_{n,m}(x,y) = p_{i,j}(i+1-nx)(j+1-my) + p_{i+1,j}(nx-i)(j+1-my)\\ + p_{i,j+1}(i+1-nx)(my-j) + p_{i+1,j+1}(nx-i)(my-j),\end{aligned}$$

for $\frac{i}{n} \leq x \leq \frac{i+1}{n}$, $\frac{j}{m} \leq y \leq \frac{j+1}{m}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$*.*

(i) (Cavaretta–Sharma [63]) If we suppose that the Bézier net of the polynomial $B_{n,m}(f)(x,y)$ *is convex, then so is* $B_{n,m}(f)(x,y)$*.*

(ii) (Floater [118]) If

$$\begin{aligned}\Delta_{2,0} p_{i,j} \geq 0 \text{ for } i = 0, \ldots, n-2, \quad j = 0, \ldots, m,\\ \Delta_{0,2} p_{ij} \geq 0 \text{ for } i = 0, \ldots, n, \quad j = 0, \ldots, m-2,\end{aligned}$$

and

$$\Delta_{2,0}p_{i,j+l+s}\Delta_{0,2}p_{i+k+r,j} - k(\Delta_{1,1}p_{i+k,j+l})^2 \geq 0,$$

for all $i = 0, \ldots, n-2$, $j = 0, \ldots, m-2$, *and* $k, l, r, s \in \{0,1\}$, *where* $k = mn/(m-1)(n-1)$, *then* $B_{n,m}(f)(x,y)$ *is convex.*

(iii) (Carnicer–Floater–Peña [61]) The linear conditions

$$\Delta_{2,0}p_{i,j+s} \geq 2|\Delta_{1,1}p_{i+k,j}|$$

for all $i = 0, \ldots, n-2$, $j = 0, \ldots, m-1$, *and* $k, s \in \{0,1\}$,

$$\Delta_{0,2}p_{i+r,j} \geq 2|\Delta_{1,1}p_{i,j+l}|,$$

for all $i = 0, \ldots, n-1$, $j = 0, \ldots, m-2$, *and* $l, r \in \{0,1\}$, *imply that* $B_{n,m}(f)(x,y)$ *is convex.*

Remark. In the paper Jüttler [188], a general construction of linear sufficient conditions for convexity of multivariate tensor-product Bernstein polynomials is obtained. These conditions can be made as weak as desired, and the conditions in Theorem 2.2.4 (iii) and Theorem 2.2.5 (iii) are special cases of this general construction.

The connection between the shape-preserving properties and the strongly variation diminishing property in the univariate case is well known (see Chapter 5, Section 5.1, Theorem 5.1.7). At the end of this section, we briefly present some concepts of total variation together with corresponding properties for bivariate Bernstein polynomials defined on triangles and show a connection with the preservation of convexity.

Definition 2.2.6. (i) (Goodman [155]) For $f : T \to \mathbb{R}$, $f \in C^2(T)$, where T is a triangle in the plane, one defines two kinds of variations of f on T by

$$V(f;T) = \int_T \left[\left(\frac{\partial f(x,y))}{\partial x} \right)^2 + \left(\frac{\partial f(x,y)}{\partial y} \right)^2 \right]^{1/2} dx\, dy,$$

and

$$V_1(f;T) = \int_T \left[\left(\frac{\partial^2 f(x,y)}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2} \right)^2 \right]^{1/2} dx\, dy.$$

(ii) (Chang–Hoschek [67]) For $f \in C^2(T)$, T a triangle in the plane, one defines the variation of f on T by

$$V_1^*(f;T) = \int_T \left| \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \right| dx\, dy.$$

(iii) (Goodman [156]) For any $f \in C^2(T)$, T a triangle in the plane, and S a seminorm in $\mathbb{R}^3$, one defines the generalized variation of f on T by

$$V_S(f;T) = \int_T S\left(\frac{\partial^2 f(x,y)}{\partial x^2}, \frac{\partial^2 f(x,y)}{\partial x \partial y}, \frac{\partial^2 f(x,y)}{\partial y^2} \right) dx\, dy.$$

(iv) (Cavaretta-Sharma [63]) For $f : [0,1] \times [0,1] \to \mathbb{R}$, one defines the variation

$$V_1([0,1]^2)(f) = \int_0^1 \int_0^1 \left[\left(\frac{\partial^2 f(x,y)}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f(x,y)}{\partial y^2} \right)^2 \right]^{1/2} dx\, dy.$$

We present without proof the following.

Theorem 2.2.7. *(i) (Goodman [155]) Let $f \in C^2(T)$ and denote by $\hat{f}_n$ the nth Bézier net attached to f. We have $V_1(B_n^T(f);T) \leq V_1(\hat{f}_n;T)$ and $V_1(B_n^T(f);T) \leq C \cdot V_1(f;T)$, where $B_n^T(f)$ denotes the Bernstein polynomial attached to f on the triangle T and $C > 0$ is a constant independent of f and n, depends only on T, and can explicitly be given in terms of the angles of T (for example, if T is equilateral, then $C = \frac{\sqrt{34}}{3}$). (Since $\hat{f}_n$ is not of $C^2(T)$-class, a specific formula is given for $V_1(\hat{f}_n;T)$).*
Also,

$$V(B_n^T(f);T) \leq \frac{2n}{n+1} V(\hat{f}_n;T), \ \forall n \in \mathbb{N}, \ f \in C^2(T),$$

but there is no constant $C > 0$ independent of f such that $V(B_n^T(f);T) \leq CV(f;T), n \in \mathbb{N}$.

(ii) (Chang–Hoschek [67]) Let $f \in C^2(T)$. We have

$$V_1^*(B_n^T(f);T) \leq V_1^*(\hat{f}_n;T) = V_1(\hat{f}_n;T), \ \forall n \in \mathbb{N},$$

with equality if and only if the Bézier net $\hat{f}_n$ is either convex or concave over T.

(iii) (Goodman [156]) For any $f \in C^2(T)$, we have

$$V_S(B_n^T(f);T) \leq C(S,T) V_S(f;T), \ \forall n \in \mathbb{N},$$

where the constant $C(S,T)$ can explicitly be calculated and depends only on S and T, but is independent of f and n. The result generalizes (i) and (ii).

(iv) (Cavaretta-Sharma [63]) Keeping the notation in Theorem 2.2.5 and Definition 2.2.6 (iv), we have $V_1([0,1]^2)(B_{n,m}(f)) \leq V_1([0,1]^2)(\hat{f}_{n,m})$, where a specific formula is given for $V_1([0,1]^2)(\hat{f}_{n,m})$ (since $\hat{f}_{n,m}$ is not twice differentiable).

Remarks. (1) Note that the concepts in Definition 2.2.6 (i) are naturally suggested by the total variation in the univariate case (see Goodman [155], pp. 111–112).

(2) The above equality property in (ii) connects, in the case of bivariate Bernstein polynomials over triangles, the preservation of convexity with the preservation of total variation. Indeed, if equality holds and $\hat{f}_n$ is convex over

the triangle Δ_2, then according to Theorem 2.2.4 (i), the Bernstein polynomial $B_n^{\Delta_2}(f)$ is convex on Δ_2.

(3) Since the approximation error in shape-preserving approximation by bivariate Bernstein polynomials is rather weak, it is natural to search for Jackson-type estimates in shape-preserving approximation by bivariate polynomials. This question will be solved in the next sections of this chapter.

Mainly, the results will be obtained in using the following methods:

(a) the Shisha-type method in the univariate case adapted to the bivariate/multivariate case and its generalization in vector spaces, called L-positive approximation and introduced by Anastassiou–Ganzburg;

(b) the tensor-product method;

(c) the method consisting in approximating first $f(x, y)$ by piecewise bivariate linear functions $L_n(f)(x, y)$ with the same shape as $f(x, y)$, and then replacing $L_n(f)(x, y)$ by suitable bivariate polynomials $P_n(x, y)$.

2.3 Shisha-Type Methods and Generalizations

The ideas of Shisha and Anastassiou–Shisha's methods used in the univariate case for the proof of Theorem 1.4.1 were extended for the first time to the bivariate case, by Anastassiou [5] in 1991. Unfortunately, since it is based on the simultaneous approximation by bivariate Bernstein polynomials, the estimate is rather weak, involving the modulus of continuity $\omega_1(f; \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{m}})$.

Using a different method, similar results were obtained by Xu-guang Lu [251], [252] in 1988 and 1992, respectively. Note that in these cases too, the error estimates are rather weak, in terms of $\omega_1(f; \frac{1}{\sqrt{n}}) = \sup\{|f(x, y) - f(u, v)|; \sqrt{(x-u)^2 + (y-v)^2} \le \frac{1}{\sqrt{n}}\}$.

2.3.1 Shisha-Type Approximation

In this subsection we present new essential improvements of the above-mentioned estimates. For simplicity, only the bivariate case will be presented, but the results can easily be extended to functions of several real variables.

The first main tool used in this sense is a recent result of Beutel–Gonska [41], [42] concerning simultaneous approximation by tensor-product operators.

For $f : [a, b] \times [c, d] \to \mathbb{R}$, let us write $f^{(k,l)} = D^{(k,l)} f = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}$, $C^{p,q}([a, b] \times [c, d]) = \{f : [a, b] \times [c, d] \to \mathbb{R}; D^{(k,l)} f$ is continuous on $[a, b] \times [c, d], \forall (0, 0) \le (k, l) \le (p, q)\}$, where $(0, 0) \le (k, l) \le (p, q)$ means $0 \le k \le p$, $0 \le l \le q$.

For $f \in C^{0,0}([a, b] \times [c, d])$ and $r \ge 1$, consider the modulus of smoothness, $\omega_r(f; \delta_1, \delta_2)$, defined by Definition 2.1.2 (iii).

Concerning the degree of simultaneous approximation by tensor product operators, we recall here the following result, which will be used later in our proofs.

Theorem 2.3.1. (Beutel–Gonska [41], Theorem 1) *Let* $L : C^p(I) \to C^{p'}(I')$, $M : C^q(J) \to C^{q'}(J')$ *be two linear continuous operators, where for each* g *in the space* $C^s(I)$ *the norm is* $\|g\| := \max\{\|g^{(i)}\|_\infty; i = 0, \ldots, s\}$, $\|\cdot\|_\infty$ *is the uniform norm on* $C(I)$, $I' \subset I$, $J' \subset J$ *are compact subintervals of the real axis, and* $(0,0) \leq (p', q') \leq (p, q)$. *Let us suppose that the operators* L *and* M *satisfy the conditions*

$$|[g - L(g)]^{(k)}(x)| \leq \sum_{r \in T} K_{r,k,L}(x) \omega_r(g^{(p)}; \gamma_{r,L}(x))$$

for all $0 \leq k \leq p' \leq p$, $x \in I'$, $g \in C^p(I)$, *and*

$$|[h - M(h)]^{(l)}(y)| \leq \sum_{s \in S} K_{s,l,M}(y) \omega_s(h^{(q)}; \gamma_{s,M}(y)),$$

for all $0 \leq l \leq q' \leq q$, $y \in J'$, $h \in C^q(J)$, *where* T, S *are finite (nonempty) subsets of* $\mathbb{N} \cup \{0\}$ *and* K, γ *are bounded functions on their domains of definition.*

Then, for all $(x, y) \in I' \times J'$, $f \in C^{p,q}(I \times J)$, $(0,0) \leq (k, l) \leq (p', q') \leq (p, q)$, *we have the estimates*

$$|[f - ({}_xL \circ_y M)(f)]^{k,l}(x,y)| \leq \sum_{r \in T} K_{r,k,L}(x) \omega_r(f^{(p,l)}; \gamma_{r,L}(x), 0)$$
$$+ \|D^k \circ L\| \cdot \sup_{0 \leq i \leq p} \sum_{s \in S} K_{s,l,M}(y) \cdot \omega_s(f^{(i,q)}; 0, \gamma_{s,M}(y))$$

and

$$|[f - ({}_xL \circ_y M)(f)]^{k,l}(x,y)| \leq \sum_{s \in S} K_{s,l,M}(y) \omega_s(f^{(k,q)}; 0, \gamma_{s,M}(x))$$
$$+ \|D^l \circ M\| \cdot \sup_{0 \leq j \leq q} \sum_{r \in T} K_{r,k,L}(y) \cdot \omega_r(f^{(p,j)}; \gamma_{r,L}(x), 0),$$

where by definition,

$$\|L\| = \sup\{\max_{0 \leq k \leq p'} \max_{x \in J} |[L(f)]^{(k)}(x)|; f \in C^p(I), \|f\| \leq 1\}.$$

An application of Theorem 2.3.1 is a bivariate version of the Brudnyi–Gopengauz pointwise-type result in the univariate case. We need only its immediate uniform (i.e., in the $\|\cdot\|_\infty$-norm) consequence below.

Theorem 2.3.2. *(Beutel–Gonska [42], Section 3.1, p. 10) For any* $f \in C^{p,q}([-1,1]^2)$ *and for* $(0,0) \leq (k,l) \leq (p', q')$, *there exists a sequence of bivariate polynomials* $Q_{n,m}(f)(x,y)$ *of degree* $\leq n$ *in* x *and of degree* $\leq m$ *in* y *for all* $n \geq \max\{4(p+1), p+r\}$, $m \geq \max\{4(q+1), q+s\}$, *such that*

$$\|[f - Q_{n,m}]^{(k,l)}\|_\infty \leq c_{p,r}\left(\frac{1}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p,l)}; \frac{1}{n}, 0\right)$$
$$+ c_{p,q,r,s}\left(\frac{1}{m}\right)^{q-l} \cdot \sup_{0\leq i\leq p} \omega_s\left(f^{(i,q)}; 0, \frac{1}{m}\right).$$

Here $p' = \min\{p, p - r + 2\} \leq p$ *and* $q' = \min\{q - s + 2, q\} \leq q$.

Based on Theorem 2.3.2, we can prove the following new bivariate Anastassiou–Shisha-type result.

Theorem 2.3.3. *Let* h_1, h_2, p, q, r, s *be positive integers,* $0 \leq h_1 \leq p' \leq p$, $0 \leq h_2 \leq q' \leq q$, *with* $p' = \min\{p, p - r + 2\} \leq p$ *and* $q' = \min\{q - s + 2, q\} \leq q$, *and let* $f \in C^{p,q}([-1, 1]^2)$. *Consider the bounded functions* $\alpha_{i,j} : [-1, 1] \times [-1, 1] \to \mathbb{R}$, $i = h_1, \ldots, p'$, $j = h_2, \ldots, q'$, *assume that* a_{h_1,h_2} *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *on* $[-1, 1]^2$ *and define the differential operator*

$$L = \sum_{i=h_1}^{p'} \sum_{j=h_2}^{q'} \alpha_{i,j}(x, y) \frac{\partial^{i+j}}{\partial x^i \partial y^j}.$$

Also, define

$$M_{n,m}^{k,l}(f) = c_{p,r}\left(\frac{1}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p,l)}; \frac{1}{n}, 0\right)$$
$$+c_{p,q,r,s}\left(\frac{1}{m}\right)^{q-l} \cdot \sup_{0\leq i\leq p} \omega_s\left(f^{(i,q)}; 0, \frac{1}{m}\right),$$
$$P_{n,m}(L; f) = \sum_{i=h_1}^{p'} \sum_{j=h_2}^{q'} l_{i,j} M_{n,m}^{i,j},$$

and $l_{i,j} = \sup_{x,y\in[-1,1]}\{|\alpha_{h_1,h_2}^{-1}(x, y) \cdot \alpha_{i,j}(x, y)|\} < \infty$. *(Here* a^{-1} *means* $\frac{1}{a}$.*)*

If $L(f)(x, y) \geq 0, \forall x, y \in [-1, 1]$, *then for all* n, m *integers with* $n \geq \max\{4(p + 1), p + r\}$, $m \geq \max\{4(q + 1), q + s\}$, *there exists a bivariate polynomial* $Q_{n,m}(f)(x, y)$ *of degree* $\leq n$ *in* x *and* $\leq m$ *in* y, *satisfying* $L[Q_{n,m}](f)(x, y) \geq 0, \forall x, y \in [-1, 1]$ *and*

$$\|f - Q_{n,m}\|_\infty \leq \frac{P_{n,m}(L; f)}{(h_1)!(h_2)!} + c_{p,r}\frac{1}{n^p}\omega_r\left(f^{(p,0)}; \frac{1}{n}, 0\right)$$
$$+c_{p,q,r,s}\frac{1}{m^q} \sup_{0\leq i\leq p} \omega_s\left(f^{(i,q)}; \frac{1}{m}\right).$$

Proof. Case 1. Assume first $\alpha_{h_1,h_2}(x, y) \geq \alpha > 0$ for all $x, y \in [-1, 1]$. From Theorem 2.3.2 there exists the bivariate polynomial $Q_{n,m}(x, y)$ such that we have

$$\left\|\left[f + P_{n,m}(L; f)\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right]^{(k,l)} - Q_{n,m}^{(k,l)}\right\|_\infty \leq M_{n,m}^{k,l},$$

which implies

$$\|f^{(k,l)} - Q_{n,m}^{(k,l)}\|_\infty \le \frac{P_{n,m}(L;f)}{(h_1-k)!(h_2-l)!} + M_{n,m}^{k,l}(f),$$

for all $(0,0) \le (k,l) \le (h_1,h_2)$.

Taking now $k = l = 0$, it follows that

$$\begin{aligned}\|f - Q_{n,m}\|_\infty &\le \frac{P_{n,m}(L;f)}{(h_1)!(h_2)!} + M_{n,m}^{0,0}(f)\\ &= \frac{P_{n,m}(L;f)}{(h_1)!(h_2)!} + c_{p,r}\frac{1}{n^p}\omega_r\left(f^{(p,0)}; \frac{1}{n}, 0\right)\\ &\quad + c_{p,q,r,s}\frac{1}{m^q}\sup_{0\le i\le p}\omega_s\left(f^{(i,q)}; \frac{1}{m}\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\alpha_{h_1,h_2}^{-1}(x,y)L(Q_{n,m})(x,y) = \alpha_{h_1,h_2}^{-1}(x,y)L(f)(x,y)\\ &\quad + P_{n,m}(L;f) + \sum_{i=h_1}^{p'}\sum_{j=h_2}^{q'}\alpha_{h_1,h_2}^{-1}(x,y)\alpha_{i,j}(x,y)\\ &\quad \times \left[Q_{n,m}(x,y) - f(x,y) - P_{n,m}(L;f)\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right]^{(i,j)}\\ &\ge \alpha_{h_1,h_2}^{-1}(x,y)L(f)(x,y) + P_{n,m}(L;f) - \sum_{i=h_1}^{p'}\sum_{j=h_2}^{q'} l_{i,j}M_{n,m}^{i,j}\\ &= \alpha_{h_1,h_2}^{-1}(x,y)L(f)(x,y) \ge 0,\end{aligned}$$

by hypothesis.

The case $\alpha_{h_1,h_2} \le \beta < 0$ is similar, which proves the theorem. □

Remarks. (1) Theorem 2.3.3 improves the estimates of Theorem 2 in Anastassiou [5].

(2) Let us suppose in Theorem 2.3.3 that in addition to the present hypothesis, all the functions $a_{i,j}(x,y), i = h_1, \ldots, p', j = h_2, \ldots, q'$ are continuous on $[-1,1] \times [-1,1]$ and that $L(f)(x,y) > 0$, for all $x, y \in (-1,1)$. By the continuity assumptions, it is immediate that $L(f)(x,y) \ge 0$, for all $x, y \in [-1,1]$, and from the proof of theorem, the conclusion $L(Q_{n,m})(x,y) > 0$, for all $x, y \in (-1,1)$, $n, m \in \mathbb{N}$, $n \ge \max\{4(p+1), p+r\}$, $m \ge \max\{4(q+1), q+s\}$ follows easily.

2.3.2 *L*-Positive Approximation

A generalization of the above method was introduced in Anastassiou–Ganzburg [16] and is called L-positive approximation. The main idea is that if a parametric family of linear bounded operators on a vector space $(L_\gamma)_{\gamma\in\Gamma}$ satisfies some suitable conditions, then the quantity of best approximation by elements in a finite-dimensional subspace, constrained by the positivity through $(L_\gamma)_{\gamma\in\Gamma}$ (called L-positivity), can be estimated by a constant times the best unconstrained approximation quantity.

Thus, the known estimates for unconstrained best approximation will become estimates for the best L-positive approximation too.

In what follows, this generalization is presented in greater detail.

Everywhere in this subsection we use the general notation and hypothesis in the following.

Definition 2.3.4. *For $(X, \|\cdot\|_X)$ a normed space, S a linear subspace of X, $M \subset X$ with $S\cap M \neq \emptyset$, and $f \in X$, the best approximations of f by elements from S and from $S \cap M$ are defined by*

$$E_S(f : X) = \inf_{g\in S}\{\|f - g\|_X\}$$

and

$$E_{S,M}(f; X) = \inf_{g\in S\cap M}\{\|f - g\|_X\},$$

respectively.

Remarks. (1) It is obvious that $E_S(f; X) \leq E_{S,M}(f; X)$, for all $f \in X$.

(2) If, in addition, S is finite-dimensional, then by a classical result in functional analysis (see, e.g., Singer [357], p. 91, Corollary 2.2), it follows that there exists $g^* \in S$ such that $E_S(f; X) = \|f - g^*\|_X$. On the other hand, $S\cap M$ is obviously a finite-dimensional set, so it is closed, and since obviously $\text{span}\{S\cap M\} \subset S$, due to an old result of Ascoli [28] (see, e.g., Muntean [279], p. 126), it follows that $S\cap M$ is proximinal, i.e., there exists $g^*_M \in S\cap M$ such that $E_{S,M}(f; X) = \|f - g^*_M\|_X$.

Let us denote by $L_\infty(\Omega) = \{f : \Omega \to \mathbb{R}; \text{esssup}_{x\in\Omega}|f(x)| < +\infty\}$, where $\Omega \subset \mathbb{R}^m$, and let $L : X \to L_\infty(\Omega)$ be a linear bounded operator, i.e., $|||L||| = \sup\{\|L(x)\|_{L_\infty(\Omega)}; \|x\|_X \leq 1\} < \infty$. Also, let $M(L) = \{f \in X; L(f)(x) \geq 0,\ \text{a.e. on } \Omega\}$.

The goal is under some conditions on L and $S \subset X$ to obtain an estimate of the form

$$E_{S,M(L)}(f; X) \leq CE_S(f; X), \forall f \in X,$$

where C is a constant independent of f and S.

Obviously that the above kind of inequality is not always valid.

The first main result is the following.

Theorem 2.3.5. *(Anastassiou–Ganzburg [16]) For $(X, \|\cdot\|_X)$ a normed space and $S \subset X$ a linear subspace, let us consider a family of linear bounded operators $L_j : X \to L_\infty(\Omega_j)$, $\Omega_j \subset \mathbb{R}^m$, $j \in J$ (J an arbitrary set), satisfying the conditions*

(i) $\sup_{j\in J} |||L_j||| < \infty$;
(ii) there is an element $e \in S$ *such that for any* $j \in J$ *we have*

$$L_j(e)(x) \geq 1, \text{ a.e. on } \Omega_j.$$

Then, for every $f \in X$ *and* $P \in S$, *there exist* $Q_i \in S$, $i = 1, 2$, *such that*

$$(-1)^{i+1} L_j(Q_i - f)(x) \geq 0, \; x \in \Omega_j, \; j \in J,$$

and the estimate

$$\|f - Q_i\|_X \leq (1 + \|e\|_X \sup_{j\in J} |||L_j|||) \cdot \|f - P\|_X, \, i = 1, 2,$$

holds.

Proof. Setting $Q_i = P + (-1)^{i+1}\lambda e$, where $\lambda = \sup_{j\in J}\{|||L_j|||\} \cdot \|f - P\|_X$ for any $i = 1, 2$, we obtain

$$\begin{aligned}(-1)^{i+1} L_j(Q_i - f)(x) &= (-1)^{i+1} L_j(P - f)(x) + \lambda L(e)(x) \\ &\geq \lambda - \sup_{j\in J}\{|||L_j||| \cdot \|f - P\|_X = 0.\end{aligned}$$

In addition,

$$\|f - Q_i\|_X \leq \|f - P\|_X + \lambda\|e\|_X = (1 + \|e\|_X \sup_{j\in J} |||L_j|||)\|f - P\|_X, \quad i = 1, 2,$$

which proves the theorem. □

Also, we have the following.

Theorem 2.3.6. (Anastassiou–Ganzburg [16]) *Let us consider a family of linear bounded operators* $L_j : X \to L_\infty(\Omega_j)$, $\Omega_j \subset \mathbb{R}^m$, $j \in J$ *(J an arbitrary set), satisfying conditions (i) and (ii) (with S a linear subspace of X) in the statement of Theorem 2.3.5, and for* $j \in J$ *let us define*

$$\begin{aligned} M^+(L_j) &= \{f \in X : L_j(f)(x) \geq 0, \text{ a.e. on } \Omega_j\}, \\ M^-(L_j) &= \{f \in X : L_j(f)(x) \leq 0, \text{ a.e. on } \Omega_j\}, \\ M^\pm &= \bigcap_{j\in J} M^\pm(L_j). \end{aligned}$$

Then for any $f \in M^\pm$, *we have*

$$E_{S,M^\pm}(f; X) \leq (1 + \|e\|_X \sup_{j\in J} |||L_j|||) E_S(f; X).$$

Proof. We prove the case $f \in M^+$, since the case $f \in M^-$ is similar. By Theorem 2.3.5, there exists $Q_1 \in S$ such that $L_j(Q_1)(x) \geq L_j(f)(x) \geq 0$, a.e. $x \in \Omega_j$, for all $j \in J$. Thus $Q_1 \in M^+$ and the conclusion immediately follows from the estimate in Theorem 2.3.5. □

Corollary 2.3.7. *(Anastassiou–Ganzburg [16]) Let* $L : X \to L_\infty(\Omega)$, $\Omega \subset \mathbb{R}^m$, *be a linear bounded operator such that these exists* $e \in S$ *(*S *a linear subspace of* X*) with* $L(e) \geq 1$, *a.e. on* Ω. *If we set* $M(L) = \{f \in X : L(f)(x) \geq 0,\ on\ \Omega\}$, *then for any* $f \in M(L)$ *we have*

$$E_{S,M(L)}(f;X) \leq (1 + |||L||| \cdot \|e\|_X)E_S(f;X).$$

Proof. It is immediate by taking $L_j = L$, for all $j \in J$ in Theorem 2.3.6. □

Remarks. (1) The constant $(1 + |||L||| \cdot \|e\|_X)$ can obviously be improved by replacing $\|e\|_X$ with $c = \inf\{\|e\|_X; e \in S, L(e)(x) \geq 1,\ \text{a.e. on}\ \Omega\}$.

(2) Among many applications of the above results for particular spaces X, S, and L_j in Anastassiou–Ganzburg [16], some of them refer to convex and monotone approximation, the univariate case. Since that topic in fact was studied, by different methods but in detail, in Sections 1.6 and 1.7, we omit them here. Other applications refer to multivariate convex and subharmonic approximation and will be presented in the next section.

At the end of this subsection we present new results that are refinements of Theorems 2.3.5, 2.3.6 and Corollary 2.3.7, in the sense that the L-positivity (i.e. ≥ 0) can be replaced by strict L-positivity (i.e., > 0).

Corollary 2.3.8. *In the hypothesis of Theorem 2.3.5, for every* $f \in X$ *and* $P \in S$, $P \neq f$, *there exist* $Q_i \in S$, $i = 1, 2$, *such that*

$$(-1)^{i+1}L_j(Q_i - f)(x) > 0,\ x \in \Omega_j,\ j \in J$$

and the estimate

$$\|f - Q_i\|_X \leq (1 + \|e\|_X + \|e\|_X \sup_{j \in J} |||L_j|||) \cdot \|f - P\|_X,\ i = 1, 2,$$

holds.

Proof. Indeed, this conclusion easily follows if in the proof of Theorem 2.3.5 we take $\lambda = (1 + \sup_{j \in J} |||L_j|||) \cdot \|f - P\|_X$. □

Corollary 2.3.9. *In the hypothesis of Theorem 2.3.6 but introducing the notation*

$$\begin{aligned} M_0^+(L_j) &= \{f \in X : L_j(f)(x) > 0,\ a.e.\ on\ \Omega_j\}, \\ M_0^-(L_j) &= \{f \in X : L_j(f)(x) < 0,\ a.e.\ on\ \Omega_j\}, \\ M_0^\pm &= \bigcap_{j \in J} M_0^\pm(L_j), \end{aligned}$$

for any $f \in M_0^\pm$, *we have*

$$E_{S,M_0^\pm}(f;X) \leq (1 + \|e\|_X + \|e\|_X \sup_{j \in J} |||L_j|||)E_S(f;X).$$

Proof. We prove the case $f \in M_0^+$, since the case $f \in M_0^-$ is similar. By Corollary 2.3.8, for any $f \in X$ and $P \in S$, $P \neq f$, there exists $Q_1 \in S$ such that $L_j(Q_1)(x) > L_j(f)(x) > 0$, a.e. $x \in \Omega_j$, for all $j \in J$. Thus $Q_1 \in M_0^+$, and from the estimate in Corollary 2.3.8 we get

$$\|f - Q_1\|_X \leq (1 + \|e\|_X + \|e\|_X \sup_{j \in J} \|\|L_j\|\|) \cdot \|f - P\|_X,\ i = 1, 2,$$

which immediately implies the estimate in Corollary 2.3.9. □

Corollary 2.3.10. *In the hypothesis of Corollary 2.3.7, writing* $M_0(L) = \{f \in X : L(f)(x) > 0,\ \text{on } \Omega\}$, *then for any* $f \in M_0(L)$ *we have*

$$E_{S,M_0(L)}(f;X) \leq (1 + \|e\|_X + \|\|L\|\| \cdot \|e\|_X)E_S(f;X).$$

Proof. The proof is immediate by taking $L_j = L$ for all $j \in J$ in Corollary 2.3.9. □

Remark. Other slightly different variants of the above results hold. For example, in Corollary 2.3.10 let us suppose that $L : X \to C(\Omega)$, with $\Omega \subset \mathbb{R}^m$ compact, is a linear bounded operator such that these exists $e \in S$ (S a linear subspace of X) with $L(e) \geq 1$ on Ω. If we define $M_0(L) = \{f \in X : L(f)(x)>0,\ \text{on int}\{\Omega\}\}$, then for any $f \in M_0(L)$ we have

$$E_{S,M_0(L)}(f;X) \leq (1 + \|\|L\|\| \cdot \|e\|_X)E_S(f;X).$$

The estimate is immediate from the proof of Corollary 2.3.7.

Note that by $L(f) \in C(\Omega)$, we easily get $L(f)(x) \geq 0$ for all $x \in \Omega$, i.e., actually $f \in M(L) = \{f \in X : L(f)(x) \geq 0,\ \text{on } \Omega\}$.

2.4 Approximation Preserving Three Classical Shapes

This section presents approximation by polynomials preserving three classical "shapes" of functions of two or several variables: harmonicity, subharmonicity, and convexity.

2.4.1 Harmonic Polynomial Approximation

The following Weierstrass-type result in approximation of harmonic functions by harmonic polynomials is well known:

Theorem 2.4.1. *(Walsh [397], see also Keldysh–Lavrent'ev [193]) If* $A \subset \mathbb{R}^m$ *is simply connected and* $f : A \to \mathbb{R}$ *is harmonic on* A *(i.e.,* $f \in H_1(A)$ *according to Definition 2.1.5 (iii)), then for any compact set* $E \subset A$ *and any* $\varepsilon > 0$, *there exists a polynomial* $P \in H_1(\mathbb{R}^m)$ *such that* $\|f - P\|_{C(E;\mathbb{R})} < \varepsilon$, *where* $\|\cdot\|_{C(E;\mathbb{R})}$ *denotes the uniform norm on* $C(E;\mathbb{R}) = \{f : E \to \mathbb{R}; f \text{ is continuous on } E\}$.

Since it is well known that any harmonic function of two real variables on a domain in $\mathbb{R}^2$ is (excepting an additive constant) the real part of an analytic (holomorphic) function of one complex variable, we immediately obtain the following quantitative result for the unit disk.

Theorem 2.4.2. *Denote by $\mathbb{D} \subset \mathbb{C}$ the open unit disk and by $S_1 = \{(x, y) \in \mathbb{R}^2; x^2+y^2 < 1\}$ its geometric image in the plane. If $f : S_1 \to \mathbb{R}$ is harmonic in S_1 (i.e. $f \in H_1(S_1)$), continuous in $\overline{S_1}$ and $p \in \mathbb{N}$, then there exists a sequence of harmonic polynomials $(H_n(x, y))_{n\in\mathbb{N}}$ (i.e., $H_n(x, y) \in H_1(S_1)$, $n \in \mathbb{N}$) of total degree $(H_n(x, y)) \leq n$ such that*

$$\|f - H_n\|_{C(\overline{S_1};\mathbb{R})} \leq C_p[\omega_p(f; 1/n)_{\infty,\partial S_1} + \omega_p(g; 1/n)_{\infty,\partial S_1}],\ n = 1, 2, \dots,$$

where g is harmonic conjugate to f and

$$\omega_p(f; 1/n)_{\infty,\partial S_1} = \sup\{|\Delta_u^p f(\cos v, \sin v)|; |u| \leq 1/n, |v| \leq \pi\}.$$

Proof. Indeed, let

$$F \in A(\mathbb{D}) = \{h : \overline{D} \to \mathbb{C}; h \text{ is analytic in } \mathbb{D} \text{ and continuous in } \overline{\mathbb{D}}\},$$

with $\mathrm{Re}[F] = f$ in $\mathbb{D}$.

By Gaier [121], p. 4 (see also Gaier [122], p. 53), there exists a sequence of complex polynomials $(P_n(z))_n$, degree$(P_n) \leq n$, such that

$$\|F - P_n\| \leq C(p)\omega_p(F; 1/n)_{\infty,\partial\mathbb{D}},\ n = 1, 2, \dots,$$

where $\|\cdot\|$ denotes the uniform norm in $C(\overline{\mathbb{D}}; \mathbb{C})$ and

$$\omega_p(F; \delta)_{\infty,\partial\mathbb{D}} = \sup\{|\Delta_u^p F(e^{iv})|; |v| \leq \pi, |u| \leq \delta\},$$

$i^2 = -1$, $\Delta_u^p F(e^{iv}) = \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} F(e^{i(v+ju)})$.

Writing $H_n = \mathrm{Re}[P_n]$, we have degree$(H_n(x, y)) \leq n$, and it follows that

$$\begin{aligned}\|f - H_n\|_{C(\overline{S_1};\mathbb{R})} &\leq \|F - P_n\| \\ &\leq C(p)\omega_p(f + ig; 1/n)_{\infty,\partial\mathbb{D}} \leq C_p[\omega_p(f; 1/n)_{\infty,\partial S_1} + \omega_p(g; 1/n)_{\infty,\partial S_1}].\end{aligned}$$

□

Results concerning the approximation of harmonic functions (of two or several variables) by harmonic polynomials on sets more general than the unit disk were obtained by, e.g., Andrievskii [21, 22, 23, 24], Andrievskii–Belyi–Dzjadyk [25], Chapter 5, Section 4, pp. 172–183, Bergman [39], Holmes [171], Kamzolov [190], Lenkhorova [224], Lenkhorova–Maymeskul [225], Paramonov [298], Pavlov [302], and others.

For example, the following quantitative estimates were proved.

Theorem 2.4.3. *(i) (Andrievskii [23]) Let M be a bounded continuum in $\mathbb{C}$ such that $A := \overline{\mathbb{C}} \setminus M$ is simply connected and M is the closure of a quasidisk. Denote by L_R the level curve of the Green function for A, i.e., $L_R := \{z; G(z) = \log(R)\}$, $R > 1$, and $\rho_R(z) := \inf_{u \in L_R}\{|u - z|\}$, $z \in \mathbb{C}$. If ω is a modulus of continuity, then for any $f \in H_1(\text{int}(M))$ continuous on M, with the property*

$$|f(z_1) - f(z_2)| \leq C\omega(|z_1 - z_2|), \forall z_1, z_2 \in M,$$

and any $n \in \mathbb{N}$, there exists a polynomial $H_n \in H_1(\text{int}(M))$ of total degree $\leq n$ such that

$$|f(z) - H_n(z)| \leq C\omega(\rho_{1+1/n}(z)), \; z \in \partial(M).$$

(ii) (Andrievskii [23]) Let $f \in H_1(\text{int}(M))$ be continuous in the compact set $M \subset \mathbb{R}^3$ supposed to have the additional property that each point of the set $A := \mathbb{R}^3 \setminus M$ is the vertex of a circular cone of fixed solid angle that lies in A. Define $\gamma(t) := \sup_{0<s<t}\{\frac{s}{1-s}\log(t/s)\}, t \in (0,1)$. Then for any $\varepsilon > 0$, there exists a sufficiently large constant $C(\varepsilon)$ such that

$$E_n^H(f) \leq C(\varepsilon)\omega_1(f; n^{-\gamma(\sin\alpha)+\varepsilon}), \quad n \in \mathbb{N},$$

where α is the solid angle of the cone and $E_n^H(f)$ denotes the best uniform approximation of f on M by harmonic polynomials of total degree $\leq n$, i.e.,

$$E_n^H(f) := \inf\{\|f - P\|_M; P \in H_1(\mathbb{R}^3), \text{degree}(P) \leq n\}.$$

Remark. Extensions of Theorems 2.4.1 and 2.4.3 to polyharmonic functions of order $r \in \mathbb{N}$, $H_r(\Omega)$, are not known yet, being the subject of Open Problem 2.7.5. However, because of the fact that polyharmonic functions can be considered a multivariate analogue to the univariate linear functions, we can mention here the following result due to Kounchev [217] that generalizes in a sense the Weierstrass-type Theorem 2.4.1: if f has all the partial derivatives of orders $2r$ continuous on $\overline{\Omega}$ (but f is not necessarily polyharmonic), then for any $s \geq r+1$, there exists a polyharmonic function $T_s(f)$ of order s, i.e., $T_s(f) \in H_s(\Omega)$, such that

$$|f(x) - T_s(f)(x)| \leq \frac{C}{s^{2r}}\omega_1^H(\Delta^r(F_r); 1/s), x \in \overline{\Omega},$$

where $C > 0$ depends only on Ω and r, $F_r(x) := f(x) - h(f)(x), x \in \overline{\Omega}$, $h(f)(x)$ is the solution of the boundary value problem

$$\Delta^{r+1}(h)(x) = 0, \quad x \in \Omega, \quad \Delta^j(h)(x) = \Delta^j(f)(x), \quad j = 0, \ldots, r, \quad x \in \partial D,$$

$\Delta^j(f)$ denotes the jth iteration of the Laplace operator $\Delta(f)$, and $\omega_1^H(G; 1/s)$ denotes the so-called harmonicity modulus of continuity of G defined in Kounchev [217]. In other words, this result gives a quantitative estimate in approximation of nonpolyharmonic functions by polyharmonic ones, with respect to the orders of polyharmonic functions.

2.4.2 Subharmonic Polynomial Approximation

First we recall the following Weierstrass-type result.

Theorem 2.4.4. (Shvedov [353]) *If Ω is a simply connected subset of $\mathbb{R}^m$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous subharmonic function, then for any compact $E \subset \Omega$ and any $\varepsilon > 0$, there exists a subharmonic polynomial p on $\mathbb{R}^m$ such that*

$$\|f - p\|_{C(E;\mathbb{R})} < \varepsilon,$$

where $\|f\|_{C(E;\mathbb{R})} = \sup\{|f(x)|; x \in E\}$.

As a quantitative result we mention the following.

Theorem 2.4.5. (Lu [252]) *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. If $f : \Omega \to \mathbb{R}$ is continuous on $\overline{\Omega}$ and subharmonic on Ω (i.e., $f \in SH_1(\Omega)$ according to Definition 2.1.5 (iii)), then there exists a sequence of polynomials $P_n \in SH_1(\Omega), n \in \mathbb{N}$, with (total) degree$(P_n) \leq n$ such that*

$$\|f - P_n\|_{C(\overline{\Omega};\mathbb{R})} \leq C\omega_1(f; n^{-1/2})_{\infty,\overline{\Omega}}, n \in \mathbb{N}.$$

Here $C > 0$ is an absolute constant,

$$\omega_1(f;\delta)_{\infty,\overline{\Omega}} = \sup\{|f(x) - f(y)|; x, y \in \overline{\Omega}, \|x - y\| \leq \delta\},$$

and $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^m$.

Unfortunately, the above estimate is rather weak. However, by applying the L-positive approximation results in Section 2.3.2, we will show that under some additional differentiability conditions, it can essentially be improved. In this sense, first we need the following definition.

Definition 2.4.6. For $\Omega \subset \mathbb{R}^m$ compact, recall that the Sobolev space $W^2_\infty(\Omega)$ is the space of twice-differentiable functions on Ω, $f : \Omega \to \mathbb{R}$, with bounded partial derivatives, endowed with the norm $\|f\|_{W^2_\infty(\Omega)} = \sum_{0 \leq |a| \leq 2} \|D^a(f)\|_{L_\infty(\Omega)} < \infty$, where $a = (a_1, \ldots, a_m), a_i \geq 0, i = 1, \ldots, m$, $|a| = \sum_{i=1}^m a_i$, $D^a f(x) = \frac{\partial^{|a|} f(x)}{\partial x_1^{a_1} \ldots \partial x_m^{a_m}}, x = (x_1, \ldots, x_m)$.

Denoting by $\mathcal{P}_n^m$ the class of algebraic polynomials in m variables and of total degree $\leq n$, we present the following result.

Theorem 2.4.7. (Anastassiou–Ganzburg [16]) *If $f \in SH_1(\Omega) \cap W^2_\infty(\Omega)$ (see Definition 2.1.5 (iii) and Remark 4 after Definition 2.1.5), then for any $n \in \mathbb{N}$, $n \geq 2$, there exists $Q_n^* \in \mathcal{P}_n^m \cap SH_1(\Omega)$ such that*

$$\|f - Q_n^*\|_{W^2_\infty} \leq C(m) \cdot E_{\mathcal{P}_n^m}(f; W^2_\infty(\Omega)),$$

where $C_m > 0$ is a constant independent of f and n.

Proof. Since it requires some rather technical details, for the proof see Anastassiou-Ganzburg [16], pp. 482–483, Corollary 3.4. □

Now, taking into account the Jackson-type results for simultaneous approximation by unconstrained polynomials in Anastassiou–Ganzburg [16], Section 4, we immediately get the following.

Corollary 2.4.8. (Anastassiou–Ganzburg [16]) *For any* $f \in SH_1(\Omega) \cap W^2_\infty(\Omega)$, $k \geq 1$, $n > k+2$, *there exists* $Q^*_n \in \mathcal{P}^m_n \cap SH_1(\Omega)$ *such that*

$$\|f - Q^*_n\|_{W^2_\infty(\Omega)} \leq C_m n^{-2} \max_{|b|=2} \omega_k(D^b(f); Cn^{-1}).$$

Here, $C_m, C > 0$ *are constants independent of* f *and* n *and for any* $h \geq 0$, $\omega_k(f;h) = \sup\{|\sum_{s=0}^k (-1)^{k-s}\binom{k}{s} f(x+st)| : x, \ldots, x+kt \in \Omega, \|t\|_{R^m} \leq h\}$ *is of the kind introduced by Definition 2.1.2 (iii).*

Proof. It is an immediate consequence of Theorem 2.4.7 and of the following Jackson-type estimates (see a particular case of Theorem 4.1, p. 485 in Anastassiou–Ganzburg [16]): for any $k \geq 1$, $n > k+2$, $f \in W^2_\infty(\Omega)$, there exists a polynomial $R_n \in \mathcal{P}^m_n$ such that for every multi-index $a = (a_1, \ldots, a_m)$ with $0 \leq |a| \leq 2$, we have

$$\|D^a[f - R_n]\|_{L_\infty(\Omega)} \leq C_1 n^{|a|-2} \max_{|b|=2} \omega_k(D^b(f); Cn^{-1}),$$

where $C_1, C > 0$ are constants independent of f and n.

Indeed, since by definition we have

$$\|f - R_n\|_{W^2_\infty(\Omega)} = \sum_{0\leq|a|\leq 2} \|D^a[f - R_n]\|_{L_\infty(\Omega)},$$

from the above Jackson-type estimate for $|a| = 0$, it easily follows that

$$\|f - R_n\|_{W^2_\infty(\Omega)} \leq C_m n^{-2} \max_{|b|=2} \omega_k(D^b(f); Cn^{-1}).$$

Since

$$E_{\mathcal{P}^m_n}(f; W^2_\infty(\Omega)) \leq \|f - R_n\|_{W^2_\infty(\Omega)},$$

combined with Theorem 2.4.7, we easily get the desired conclusion. □

Remarks. (1) For other Jackson-type results in the multivariate case that are potentially applicable above, see, e.g., Ganzburg [142], [143].

(2) Despite its generality, the above L-positive method seems to have a shortcoming, namely in the cases of subharmonic functions, it seems not to be applicable to nondifferentiable functions. Indeed, the condition of the type $L(e)(x) \geq 1$, a.e. $x \in \Omega$, for a suitable chosen subharmonic polynomial e is too strong in the nondifferentiability case. For example, in the case $m = 2$, the corresponding family of linear bounded operators would be $L_r(f)(x,y) =$

$\frac{1}{2\pi}\int_0^{2\pi} f[x+r\cos(\theta), y+r\sin(\theta)]d\theta - f(x,y)$, with $0 < r < r_0$ and r_0 sufficiently small. It is easy to show that for any $f \in SH_1(\Omega)$, for r sufficiently small, the quantity $|L_r(f)(x,y)|$ can be made as small we want.

Theorem 2.4.7 and Corollary 2.4.8 can be extended to approximation of subharmonic functions of order $p \in \mathbb{N}$. Defining for $\Omega \subset \mathbb{R}^m$ a bounded domain the class $W_\infty^{2p}(\Omega)$ of all $2p$-times-differentiable functions on Ω, $f : \Omega \to \mathbb{R}$, with bounded partial derivatives, endowed with the norm $\|f\|_{W_\infty^{2p}(\Omega)} = \sum_{0\le|a|\le 2p}\|D^a(f)\|_{L_\infty(\Omega)} < \infty$, where $a = (a_1, \ldots, a_m), a_i \ge 0, i = 1, \ldots, m$, $|a| = \sum_{i=1}^m a_i$, $D^a f(x) = \frac{\partial^{|a|}f(x)}{\partial x_1^{a_1}\cdots\partial x_m^{a_m}}$, $x = (x_1, \ldots, x_m)$, we present

Theorem 2.4.9. *Let $f \in SH_p(\Omega) \cap W_\infty^{2p}(\Omega)$ (i.e., according to Remark 4 after Definition 2.1.5 we have $(-1)^p\Delta^p(f)(x) \le 0$, for all $x \in \Omega$).*

(i) For any $n \in \mathbb{N}$, $n \ge 2p$, there exists $Q_n^* \in \mathcal{P}_n^m \cap SH_p(\Omega)$ such that

$$\|f - Q_n^*\|_{W_\infty^{2p}} \le C(m,p)\cdot E_{\mathcal{P}_n^m}(f; W_\infty^{2p}(\Omega)),$$

where $C(m,p) > 0$ is a constant independent of f and n.

(ii) For any $k \ge 1$, $n > \max\{k+2, 2p\}$, there exists $Q_n^* \in \mathcal{P}_n^m \cap SH_p(\Omega)$ such that

$$\|f - Q_n^*\|_{W_\infty^{2p}(\Omega)} \le C_{m,p}n^{-2p}\max_{|b|=2p}\omega_k(D^b(f); Cn^{-1}).$$

Here $C_{m,p}, C > 0$ are constants independent of f and n.

Proof. Denoting the differential operator by $L(f)(x) = (-1)^{p+1}\Delta^p(f)(x)$, obviously L is linear, bounded on W_∞^{2p}, and is a sum of partial derivatives of f up to order $2p$. Also, the function $\rho(x) = \frac{(p-2)!}{m(2p!)}\sum_{k=1}^m x_k^{2p}$, $x = (x_1, \ldots, x_m) \in \Omega$, satisfies $L(\rho)(x) = 1$ for all $x \in \Omega$.

(i) It is easy to see that this $L(f)(x)$ satisfies all the required conditions in the proof of Corollary 3.1 (and Remark 3.1) in Anastassiou–Ganzburg [16], pp. 480–481. Therefore our estimate is a direct consequence of Remark 3.1, p. 481 in Anastassiou–Ganzburg [16].

(ii) It is an immediate consequence of the above point (i) and of the general Jackson-type estimate in Anastassiou–Ganzburg [16], p. 485, Theorem 4.1. □

2.4.3 Convex Polynomial Approximation

We have the following Weierstrass-type result, which in fact is a consequence of a quantitative-type (Jackson-type) result.

Theorem 2.4.10. (Shvedov [354], Theorem 2) *If $\Omega \subset \mathbb{R}^m$ is a compact convex set and f is convex on Ω (i.e., $f \in KO_1(\Omega)$), then for any $\varepsilon > 0$, there exists a convex polynomial p on $\mathbb{R}^m$ such that $\|f - p\|_{C(\Omega;\mathbb{R}^m)} < \varepsilon$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\omega_1(f; \frac{1}{n+1}; \Omega) \to 0$ as $n \to \infty$, choose $n \in \mathbb{N}$ with $\omega_1(f; \frac{1}{n+1}; \Omega) < \varepsilon / C_m$ and apply Theorem 1 in Shvedov [354] (that is Theorem 2.4.11 below). □

In what follows we prove the Jackson-type result.

Theorem 2.4.11. (Shvedov [354], Theorem 1) *If $f : \Omega \to \mathbb{R}$ is convex on $\Omega \subset \mathbb{R}^m$, where Ω is supposed to be compact convex, then for any $n \in \mathbb{N}$, there exists a convex polynomial $P_n \in \mathcal{P}_n^m$ such that*

$$\|f - P_n\|_{C(\Omega;\mathbb{R})} \leq C_m \omega_1 \left(f; \frac{1}{n+1}; \Omega \right),$$

where $C_m > 0$ is independent of f and n (but depends on m and increases with m), $\omega_1(f; \delta; \Omega) = \{|f(x) - f(y)|; x, y \in \Omega, |x - y|_\Omega \leq \delta\}$, $|x|_\Omega := \inf\{\mu \geq 0; x \in \mu\Omega_0\}$ and $\Omega_0 = \Omega \setminus O(\Omega)\}$, $O(\Omega)$ the center of gravity of Ω.

Proof. We follow the ideas of proof in Shvedov [354]. First we remark that if, for example, $m = 2$ and $\Omega = [-1, 1] \times [-1, 1]$, then the norm $|\cdot|_\Omega$ is given by $|M|_\Omega = \max\{|x|, |y|\}$ for all $M = (x, y) \in \mathbb{R}^2$. Also, since $|\cdot|_\Omega$ is equivalent to the Euclidean norm on $\mathbb{R}^2$, denoted by $\|\cdot\|_{\mathbb{R}^2}$, it follows that the modulus of continuity in the statement is in fact equivalent to that in Definition 2.1.2 (i), given by $\omega_1(f; \delta) = \{|f(x) - f(y)|; x, y \in [-1, 1], \|x - y\|_{\mathbb{R}^2} \leq \delta\}$.

In what follows we will describe the main steps in the proof.

Step 1. In Lemma 3 in Shvedov [354] one constructs a sequence $(g_a)_{a \in (0,1]}$ of uniformly continuous convex piecewise affine functions on $\mathbb{R}^m$ such that

$$\|f - g_a\|_{C(\Omega;\mathbb{R})} \leq C(m) \omega_1(f; a; \Omega),$$

and $\omega_1(g_a; a; \Omega) \leq C_m \omega_1(f; a; \Omega)$.

For example, in the particular cases $m = 1$ and $m = 2$, this construction has simple geometrical interpretations, if we suppose in addition that f is differentiable.

Thus, for $m = 1$, if, for example, $\Omega = [-1, 1]$, then for any $a \in (0, 1]$, one takes a division $-1 = x_0 < x_1 < \cdots < x_{k-1} < x_k = 1$ such that $\max_{i=0,\ldots,k-1}\{x_{i+1} - x_i\} \leq a$. Suppose f is differentiable and convex on $[-1, 1]$. Then for each i, at the middle of the subinterval (x_i, x_{i+1}), i.e., at $m_i = \frac{x_i + x_{i+1}}{2}$, one takes the tangent at the graphic of f (of the equation $y = P_i(f)(x)$) and at the middle of the subinterval (x_{i+1}, x_{i+2}), i.e., at $m_{i+1} = \frac{x_{i+1} + x_{i+2}}{2}$ one takes the tangent at the graphic of f (of equation $y = P_{i+1}(f)(x)$). These two tangents intersect at a point between m_i and m_{i+1}, and so on, finally, $g_a(x)$ will be the continuous polygonal line, denoted by $P(f)(x)$, circumscribed about the graphic of f and tangent to it at the points $m_i, i = 0, \ldots, k-1$. Since f is convex on $[-1, 1]$, this continuous polygonal line, is in fact given by the formula $P(f)(x) = \max_i\{P_i(f)(x)\}$ for all $x \in [-1, 1]$.

For $m = 2$, if for example $\Omega = [-1, 1] \times [-1, 1]$, then for any $a \in (0, 1]$, one takes two divisions of $[-1, 1]$: one is on the OX-axis denoted by $-1 = x_0 < x_1 < \cdots < x_{k-1} < x_k = 1$ and the other one is on the OY axis denoted by $-1 = y_0 < y_1 < \cdots < y_{s-1} < y_s = 1$, such that $\max_{i=0,\ldots,k-1}\{x_{i+1} - x_i\} \leq a$ and $\max_{j=0,\ldots,s-1}\{y_{j+1} - y_j\} \leq a$. In this way, $\Omega = [-1, 1] \times [-1, 1]$ is decomposed into $k \cdot s$ rectangles $T_{i,j}$, $i = 1, \ldots, k$, $j = 1, \ldots, s$. Take the "center" $M_{i,j}$ of each rectangle $T_{i,j}$, i.e., $M_{i,j}$ is the intersection of diagonals of $T_{i,j}$. At each point $M_{i,j}$, consider the tangent plane to the convex surface $z = f(x, y)$ and define as $g_a(x, y)$ the maximum (at (x, y)) of all these tangent planes. In other words, $g_a(x, y)$ is the uniformly continuous affine convex function (on $\mathbb{R}^2$) circumscribed about the convex surface $z = f(x, y)$ and tangent to it at the points $M_{i,j}$.

Step 2. For any g a uniformly continuous convex function on $\mathbb{R}^m$ and $n \in \mathbb{N}$, by Lemma 5 in Shvedov [354] one constructs a polynomial of total degree $\leq n$, convex on $2Y$, that satisfies the estimate

$$\|g - P_n\|_{C(Y;\mathbb{R})} \leq C(m)\omega_1(f; 1/(n + 1); Y)_{\mathbb{R}^m},$$

where $Y = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m; |x_i| < a_i/2, |i = 1, \ldots, m\}$, with fixed $a_i > 0, |i = 1, \ldots, m$ and $\omega_1(f; \delta; Y)_{\mathbb{R}^m} = \{|f(x) - f(y)|; x, y \in \mathbb{R}^m, |x - y|_\Omega \leq \delta\}$. This construction is given as follows.

First one considers a multivariate algebraic polynomial $J_{n-1}(t)$, $t = (t_1, \ldots, t_m)$, as

$$J_{n-1}(t) = \Pi_{i=1}^m K_{n-1}^{(i)}(t_i),$$

where $K_{n-1}^{(i)}(t_i) := \frac{1}{5a_i} K_{n-1}(\frac{t_i}{5a_i})$, and $K_{n-1}(u)$ is the even univariate algebraic polynomial of degree $\leq n - 1$ introduced in Dzjadyk [101], pp. 136–138, having the properties

$$K_{n-1}(u) \geq 0, \quad u \in [-2, 2], \quad \int_{-1}^1 K_{n-1}(u)du = 1,$$

$$\int_{-2}^2 u^j K_{n-1}(u)du \leq \frac{A}{n^j}, \quad j = 1, 2, 3.$$

Second, one defines $h(x) = g(x')$, where x' is the point of $4Y$ closest to x in the Euclidean distance (note that $h(x) = g(x)$ for all $x \in 4Y$). For $n \geq 2m$ and $\delta = 1/n$, define

$$f_\delta(x) = \frac{1}{\delta^{2m}\psi^2} \int_{\delta Y} \int_{\delta Y} h(x + u + v)du\, dv,$$

where $\psi = V_m(Y)$ represents the m-dimensional volume of the set Y. It is proved that

$$\|f_\delta - h\|_{C(\mathbb{R}^m;\mathbb{R})} \leq A\omega_1(h; \delta; Y)_{\mathbb{R}^m}.$$

Then one chooses the polynomial

$$\pi_n(x) := \Pi_n^{(1)}(x) + \Pi_n^{(2)}(x) + \lambda(x),$$

where

$$\Pi_n^{(1)}(x) = \int_{10Y\setminus Y} f_\delta(x+t)J_{n-1}(t)dt,$$
$$\Pi_n^{(2)}(x) = \int_Y f_\delta(x+t)J_{n-1}(t)dt,$$

and $\lambda(x)$ is of the form

$$\lambda(x) = A_0\omega_1(h,\delta;Y)_{\mathbb{R}^m}\sum_{i=1}^m \frac{x_i^2}{a_i^2}, \quad x=(x_1,\dots,x_m).$$

The final result follows from the above Step 1 and Step 2. For all the details, the interested reader can consult Shvedov [354]. □

Remark. A much weaker estimate, of order $O(\omega_1(f;n^{-1/2}))$, in approximation of convex bivariate functions by convex bivariate polynomials was obtained in Lu [251]. Also, let us mention the following negative-type result due to Lu [253]: if a sequence of multivariate linear and positive operators preserves the usual convexity and the affine functions, then it is necessarily a sequence of affine functions (i.e., trivial), a fact that is in contrast to what happens in the univariate case.

By applying the L-positive approximation results in Section 2.3.2, we will show that under some additional differentiability conditions, the estimate in Theorem 2.4.11 can essentially be improved. Keeping the notation, we present the following theorem.

Theorem 2.4.12. (Anastassiou–Ganzburg [16]) *Let $\Omega \subset \mathbb{R}^m$ be convex and compact. If $f \in KO_1(\Omega)\cap W_\infty^2(\Omega)$ (see Definition 2.1.5 (ii) and Remark 3 after Definition 2.1.5) and $n \geq 2$, then there exists $P_n^* \in \mathcal{P}_n^m \cap KO_1(\Omega)$ such that*

$$\|f - P_n^*\|_{W_\infty^2(\Omega)} \leq C(m)\cdot E_{\mathcal{P}_n^m}(f;W_\infty^2(\Omega)).$$

Proof. Let $J = \{j=(j_1,\dots,j_m); \|j\|_{\mathbb{R}^m}=1\}, X = W_\infty^2(\Omega)$ and consider

$$L_j(f)(x) = \sum_{1\leq k,i\leq m}\frac{\partial^2 f(x)}{\partial x_i\partial x_k}j_ij_k,\ \forall x\in\Omega,\ j\in J.$$

By Remark 3 after Definition 2.1.5, $f \in KO_1(\Omega)\cap W_\infty^2(\Omega)$ if and only if $L_j(f)(x)\geq 0$ for all $j\in J$, $x\in\Omega$. It is immediate that $\sup_{j\in J}|||L_j|||\leq 1$, and choosing $e(x) = \frac{1}{2}\sum_{k=1}^m x_k^2 \in \mathcal{P}_n^m\cap KO_1(\Omega), \forall n\geq 2$, we easily get $L_j(e)(x) = 1, \forall j\in J$, and the result is an immediate consequence of Theorem 2.3.6 and of Remark 2 after Definition 2.3.4. □

Corollary 2.4.13. (Anastassiou–Ganzburg [16]) *Let $\Omega \subset \mathbb{R}^m$ be convex and compact. For $f \in KO_1(\Omega) \cap W^2_\infty(\Omega)$, $k \geq 1$, $n > k+2$, there exists $Q^*_n \in \mathcal{P}^m_n \cap KO_1(\Omega)$ such that*

$$\|f - Q^*_n\|_{W^2_\infty(\Omega)} \leq C_m n^{-2} \max_{|b|=2} \omega_k(D^b(f); Cn^{-1}).$$

Here $C_m, C > 0$ are constants independent of f and n, and for any $h \geq 0$, $\omega_k(f;h) = \sup\{\| \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} f(x+st)\| : x, \ldots, x+kt \in \Omega, \|t\|_{R^m} \leq h\}$ is of the kind introduced in Definition 2.1.2, (iii).

Proof. It is an immediate consequence of Theorem 2.4.12 and of the following Jackson-type estimates (see a particular case of Theorem 4.1, p. 485 in Anastassiou–Ganzburg [16]): for any $k \geq 1$, $n > k+2$, $f \in W^2_\infty(\Omega)$, there exists a polynomial $R_n \in \mathcal{P}^m_n$ such that for every multi-index $a = (a_1, \ldots, a_m)$ with $0 \leq |a| \leq 2$, we have

$$\|D^a[f - R_n]\|_{L_\infty(\Omega)} \leq C_1 n^{|a|-2} \max_{|b|=2} \omega_k(D^b(f); Cn^{-1}),$$

where $C_1, C > 0$ are constants independent of f and n.

The details are those in the proof of Corollary 2.4.8. □

Remarks. (1) For other Jackson-type results in the multivariate case that are potentially applicable above, see, e.g., Ganzburg [142], [143].

(2) Despite its generality, the above L-positive method seems to have a shortcoming, namely in the cases of convex functions, it seems to be not applicable to nondifferentiable functions. Indeed, the condition of the type $L(e)(x) \geq 1$, a.e. $x \in \Omega$, for a suitable chosen convex polynomial e is too strong in the nondifferentiability case. For example, in the case $m = 1$, the corresponding family of linear bounded operators would be $L_j(f)(s,t) = jf(s) + (1-j)f(t) - f[js + (1-j)t], j \in (0,1), s,t \in [a,b]$. It is easy to show that if $j \to 0$ or if $s \to t$, $s \neq t$, then $L_j(f)(s,t)$ tends to zero, so it cannot be made ≥ 1 for any convex function $f : [a,b] \to \mathbb{R}$.

Theorem 2.4.12 and Corollary 2.4.13 can be extended to approximation of the so-called classes of polyconvex functions of order $p \in \mathbb{N}$ defined in what follows.

For $\Omega \subset \mathbb{R}^m$ convex and compact, let $W^{2p}_\infty(\Omega)$ be the class defined before the statement of Theorem 2.4.9.

Definition 2.4.14. Let $f \in W^{2p}_\infty(\Omega)$. By Remark 3 after Definition 2.1.5, the class of convex functions (of order 1) is denoted by $KO_1(\Omega)$ and consists of all the functions that for every $j = (j_1, \ldots, j_m) \in \mathbb{R}^m$ with $\|j\|_{\mathbb{R}^m} = 1$, satisfy the condition

$$L_j(f)(x) = \sum_{1 \leq k,i \leq m} \frac{\partial^2 f(x)}{\partial x_i \partial x_k} j_i j_k \geq 0, \forall x \in \Omega.$$

The class of polyconvex functions of order $p \in \mathbb{N}$ is denoted by $KO_p(\Omega)$ and is the class of all functions that for every $j = (j_1, \ldots, j_m) \in \mathbb{R}^m$ with $\|j\|_{\mathbb{R}^m} = 1$, satisfy the condition

$$L_j^p(f)(x) \geq 0, \ \forall x \in \Omega,$$

where $L_j^p(f) := L_j[L_j^{p-1}(f)], L_j^0(f) = f$.

Remark. Suppose, for example, that $m = 2$, i.e., $\Omega \subset \mathbb{R}^2$. Then it is easy to show by induction that $f \in KO_p(\Omega)$ if and only if we have (formally written)

$$\left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}\right)^{2p} (f)(x) \geq 0$$

for all $(x, y) \in \Omega$ and $\alpha^2 + \beta^2 = 1$.

For example, for the function $f(x, y) = x^2 y^2$ it is easy to check that $f \in KO_2(\Omega)$ but $f \notin KO_1(\Omega)$.

For general $m > 2$, it is easy to show that $f \in KO_p(\Omega)$ if and only if we have (formally written)

$$\left[\sum_{i=1}^{m} \gamma_i \frac{\partial}{\partial x_i}\right]^{2p} (f)(x) \geq 0,$$

for all $x = (x_1, \ldots, x_m) \in \Omega$, $\gamma = (\gamma_1, \ldots, \gamma_m)$ with $\|\gamma\|_{\mathbb{R}^m} = 1$.

For simplicity we consider below the bivariate case.

Theorem 2.4.15. *Let $\Omega \subset \mathbb{R}^2$ be compact convex and $f \in KO_p(\Omega)$, $p > 1$.*

(i) For any $n \in \mathbb{N}$, $n \geq 2p$, there exists $Q_n^* \in \mathcal{P}_n^2 \cap KO_p(\Omega)$ such that

$$\|f - Q_n^*\|_{W_\infty^{2p}} \leq C(p) \cdot E_{\mathcal{P}_n^2}(f; W_\infty^{2p}(\Omega)),$$

where $C(p) > 0$ is a constant independent of f and n.

(ii) For any $k \geq 1$, $n > \max\{k + 2, 2p\}$, there exists $Q_n^* \in \mathcal{P}_n^2 \cap KO_p(\Omega)$ such that

$$\|f - Q_n^*\|_{W_\infty^{2p}(\Omega)} \leq C_p n^{-2p} \max_{|b|=2p} \omega_k(D^b(f); Cn^{-1}).$$

Here, $C_p, C > 0$ are constants independent of f and n.

Proof. Fix $p > 1$. Obviously, for each j, $L_j^p(f)$ is a linear bounded operator on the space $KO_p(\Omega)$ endowed with the norm in $W_\infty^{2p}(\Omega)$. Also, from $|||L_j||| \leq 1$, it follows that $|||L_j^p||| \leq 1$, for all $j \in \mathbb{R}^2$ with $\|j\| = 1$.

On the other hand, for $e(x, y) = 2^{p-1}[x^{2p} + y^{2p}]$ we have

$$L_j^p(e)(x, y) \geq 1,$$

for all $(x, y) \in \Omega$ and all $j = (j_1, j_2) \in \mathbb{R}^2$ with $\|j\| = 1$. Indeed, by simple calculation we get $L_j^p(e)(x, y) = 2^{p-1}(2p!)[j_1^{2p} + j_2^{2p}]$, with $j_1^2 + j_2^2 = 1$. Making the substitutions $j_1^2 = u$ we get $L_j^p(x, y) = 2^{p-1}(2p!)[u^p + (1-u)^p], u \in [0, 1]$. As a function of u, it is easy to show that it is a convex function attaining its minimum value at $u = 1/2$, i.e., $L_j(e)(x, y) \geq 2^{p-1}(2p!)/2^{p-1} = (2p!) \geq 1$ for all $(x, y) \in \Omega$.

(i) Taking into account the above considerations, it is an immediate consequence of Theorems 2.3.5 and 2.3.6 and of Remark 2 after Definition 2.3.4.

(ii) It is an immediate consequence of the above point (i) and of the general Jackson-type estimate in Anastassiou–Ganzburg [16], p. 485, Theorem 4.1. □

In what follows, for bivariate convex approximation we will obtain good estimates of Jackson type in terms of ω_2 moduli of smoothness and K_2-functionals, under the continuity hypothesis on f only (without differentiability conditions on f).

Looking more closely into the proof of the above Shvedov's Theorem 2.4.11 (which is Theorem 1 in [354]), we will show that it can be modified, such that the estimate in Theorem 2.4.11 can be improved in terms of the ω_2 moduli of smoothness. Since the estimate in Theorem 1 in Shvedov [354] is a direct consequence of the estimates (in terms of the ω_1 modulus) in Lemmas 3 and 5 there, in what follows we will re-prove these lemmas, obtaining their estimates in terms of the ω_2 modulus of smoothness.

For simplicity, we consider below the case $\Omega = [-1, 1] \times [-1, 1]$ and $m = 2$. The proof for arbitrary m is similar. Thus first, the construction in Lemma 3 in Shvedov [354] will be replaced by the following one.

We divide the square $[-1, 1] \times [-1, 1]$ into rectangles, by the points $-1 = x_0 < \cdots < x_{n_1} = 1$ and $-1 = y_0 < \cdots < y_{n_2} = 1$, then we take the diagonals of the generated rectangles as follows:

- For the four rectangles with $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ as one of their vertices, the corresponding diagonals pass through these vertices.
- For all the other rectangles, the diagonals are taken in the same directions with the diagonals of the two rectangles having $(-1, 1)$ and $(1, -1)$ as vertices.

In this way, we get a division of the square $[-1, 1] \times [-1, 1]$ by triangles (i.e., a triangulation of the square $[-1, 1] \times [-1, 1]$).

Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be continuous and convex, i.e., the surface $z = f(x, y)$ is convex. For each triangle of the above division we consider the unique interpolatory bivariate linear piecewise function of the form $Ax + By + C$ that passes through the three vertices of the triangle. In this way, we obtain a continuous bivariate linear spline, $S_{n_1,n_2}(f)(x, y)$, inscribed in the convex surface $z = f(x, y)$. Simple geometric reasoning shows us that this continuous bivariate linear spline is also a convex surface, and in addition, $f(x, y) \leq S_{n_1,n_2}(f)(x, y)$, for all $(x, y) \in [-1, 1] \times [-1, 1]$. Also, note that while this construction is inscribed in the convex surface $z = f(x, y)$, the construction in Lemma 3 in Shvedov [354] is circumscribed about the same surface.

For $\|f - S_{n_1,n_2}(f)\|$, we can prove the following kinds of estimates.

Theorem 2.4.16. (Gal [136]) *Let* $f \in C([-1,1]\times[-1,1])$ *and the distinct nodes* $-1 = x_0 < x_i < x_{i+1} < x_{n_1} = 1$, $i = 1,\dots,n_1-2$, $-1 = y_0 < y_j < y_{j+1} < y_{n_2} = 1$, $j = 1,\dots,n_2-2$.

(i) If the nodes are equidistant, i.e., $x_{i+1} - x_i = \frac{1}{n_1}$, $i = 0,\dots,n_1-1$ *and* $y_{j+1} - y_j = \frac{1}{n_2}$; $j = 0,\dots,n_2-1$, *then we have the estimate*

$$\|f - S_{n_1,n_2}(f)\| \le 6K_2\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right),$$

where the bivariate $K_2(f;t,s)$*-functional is defined by Definition 2.1.2 (iv).*

(ii) If the nodes $x_0,\dots,x_{n_1}$ *and* $y_0,\dots,y_{n_2}$ *are chosen as in Leviatan [228], p. 473, Lemma A, then*

$$\|f - S_{n_1,n_2}(f)\| \le CK_2^{\varphi}\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right),$$

where the bivariate $K_2^{\varphi}(f;t,s)$*-functional is defined in Definition 2.1.2 (v).*

Proof. By construction, the interpolation operator $S_{n_1,n_2}(f)(x,y)$ is defined as follows.

Case (1). If $(x,y) \in [x_i,x_{i+1}]\times[y_j,y_{j+1}]$, with $(x_{i+1},y_{j+1}) \ne (1,1)$ and $(x_i,y_j) \ne (-1,-1)$, then

$$S_{n_1,n_2}(f)(x,y) = f(x_i,y_{j+1}) + (x-x_i)\frac{f(x_{i+1},y_j) - f(x_i,y_j)}{x_{i+1}-x_i} + (y-y_{j+1})\frac{f(x_i,y_{j+1}) - f(x_i,y_j)}{y_{j+1}-y_j}$$

for all $(x,y) \in [x_i,x_{i+1}]\times[y_j,y_{j+1}]$ satisfying $F_{i,j}(x,y) < 0$, and

$$S_{n_1,n_2}(f)(x,y) = f(x_i,y_{j+1}) + (x-x_i)\frac{f(x_{i+1},y_{j+1}) - f(x_i,y_{j+1})}{x_{i+1}-x_i} + (y-y_{j+1})\frac{f(x_{i+1},y_j) - f(x_{i+1},y_{j+1})}{y_j-y_{j+1}}$$

for all $(x,y) \in [x_i,x_{i+1}]\times[y_j,y_{j+1}]$ satisfying $F_{i,j}(f)(x,y) \ge 0$, where $F_{i,j}(x,y) = (y-y_{i+1}) - (x-x_i)\frac{y_j-y_{j+1}}{x_{i+1}-x_i}$.

Case (2). If $(x,y) \in [-1,x_1]\times[-1,y_1]$, then

$$S_{n_1,n_2}(f)(x,y) = f(-1,-1) + (x+1)\frac{f(x_1,-1) - f(-1,-1)}{x_1+1} + (y+1)\frac{f(x_1,y_1) - f(x_1,-1)}{y_1+1}$$

for all $(x,y) \in [-1,x_1] \times [-1,y_1]$ satisfying $F_{0,0}(x,y) < 0$, and

$$S_{n_1,n_2}(f)(x,y) = f(-1,-1) + (x+1)\frac{f(x_1,y_1) - f(-1,y_1)}{x_1+1} + (y+1)\frac{f(-1,y_1) - f(-1,-1)}{y_1+1}$$

for all $(x,y) \in [-1,x_1] \times [-1,y_1]$ satisfying $F_{0,0}(x,y) \geq 0$, where $F_{0,0}(x,y) = (y+1) - (x+1)\frac{y_1+1}{x_1+1}$.

Case (3). If $(x,y) \in [x_{n_1-1},1] \times [y_{n_2-1},1]$, then

$$\begin{aligned} &S_{n_1,n_2}(f)(x,y) \\ &\quad = f(x_{n_1-1},y_{n_2-1}) + (x - x_{n_1-1})\frac{f(1,y_{n_2-1}) - f(x_{n_1-1},y_{n_2-1})}{1 - x_{n_1+1}} \\ &\quad + (y - y_{n_2-1})\frac{f(1,1) - f(1,y_{n_2-1})}{1 - y_{n_2-1}} \end{aligned}$$

for all $(x,y) \in [x_{n_1-1},1] \times [y_{n_2-1},1]$ satisfying $F_{n_1-1,n_2-1}(x,y) < 0$, and

$$S_{n_1,n_2}(f)(x,y) = f(x_{n_1-1},y_{n_2-1}) + (x - x_{n_1-1})\frac{f(1,1) - f(x_{n_1-1},1)}{1 - x_{n_1-1}} + (y - y_{n_2-1})\frac{f(x_{n_1-1},1) - f(x_{n_1-1},y_{n_2-1})}{1 - y_{n_1-1}}$$

for all $(x,y) \in [x_{n_1-1},1] \times [y_{n_2-1},1]$ satisfying $F_{n_1-1,n_2-1}(x,y) \geq 0$, where $F_{n_1-1,n_2-1}(x,y) = (y-1) - (x-1)\frac{1-y_{n_2-1}}{1-x_{n_1-1}}$.

(i) We will estimate the difference $|g(x,y) - S_{n_1,n_2}(g)(x,y)|$ for $g \in W^2([-1,1] \times [-1,1])$. If $(x,y) \in [-1,+1] \times [-1,+1]$, then there exist i and j such that $(x,y) \in [x_i,x_{i+1}] \times [y_j,y_{j+1}]$. To make a choice, let us suppose that we are in Case (1) above, with $F_{i,j}(x,y) < 0$. (The proofs for all the other cases are similar). By a simple calculation we have

$$g(x,y) - S_{n_1,n_2}(g)(x,y) = (y - y_{j+1})E + (x - x_i)F,$$

where

$$\begin{aligned} E &= \frac{g(x,y) - g(x,y_{j+1})}{y - y_{j+1}} - \frac{g(x_i,y_{j+1}) - g(x_i,y_j)}{y_{j+1} - y_j}, \\ F &= \frac{g(x,y_{j+1}) - g(x_i,y_{j+1})}{x - x_i} - \frac{g(x_{i+1},y_j) - g(x_i,y_j)}{x_{i+1} - y_i}. \end{aligned}$$

Then, using the notation for the divided difference of bivariate functions in Popoviciu [315], we easily can write

$$\begin{aligned} E &= [y,y_{j+1};[x;g(\cdot,y)]] - [y_j,y_{j+1};[x_i;g(\cdot,y)]] \\ &= [y_j,y,y_{j+1};[x;g(\cdot,y)]](y_{j+1} - y_j) + [y_j,y;[x,x_i;g(\cdot,y)]](x - x_i) \\ &\quad - [y_{j+1},y_j,y;[x_i;g(\cdot,y)]](y_{j+1} - y) \end{aligned}$$

and

$$\begin{aligned} F &= [x, x_i; [y_{j+1}; g(x, \cdot)]] - [x_i, x_{i+1}; [y_j; g(x, \cdot)]] \\ &= [x_{i+1}, x_i, x; [y_{j+1}; g(x, \cdot)]](x - x_{i+1}) + [x_i, x_{i+1}; [y_{j+1}, y; g(x, \cdot)]](y_{j+1} - y) \\ &\quad + [x_i, x_{i+1}; [y, y_j; g(x, \cdot)]](y - y_j). \end{aligned}$$

By the mean value theorem for divided differences (on distinct nodes), we have $|[a, b, c; [x; g(\cdot, y)]]| \le \|\frac{\partial^2 g}{\partial y^2}\|$, $[a, b, c; [y; g(x, \cdot)]] \le \|\frac{\partial^2 g}{\partial x^2}\|$, $[a, b; [c, d; g(\cdot, y)]] \le \|\frac{\partial^2 g}{\partial y \partial x}\|$, $[a, b; [c, d; g(x, \cdot)]] \le \|\frac{\partial^2 g}{\partial x \partial y}\|$, which implies

$$\begin{aligned} \|E\| &\le \left\|\frac{\partial^2 g}{\partial y^2}\right\| \frac{1}{n_2} + \left\|\frac{\partial^2 g}{\partial x \partial y}\right\| \frac{1}{n_1} + \left\|\frac{\partial^2 g}{\partial y^2}\right\| \frac{1}{n_2}, \\ \|F\| &\le \left\|\frac{\partial^2 g}{\partial x^2}\right\| \frac{1}{n_1} + \left\|\frac{\partial^2 g}{\partial y \partial x}\right\| \frac{1}{n_2} + \left\|\frac{\partial^2 g}{\partial y \partial x}\right\| \frac{1}{n_2}, \end{aligned}$$

and

$$\begin{aligned} \|g - S_{n_1,n_2}(g)\| &\le \frac{1}{n_2}\|E\| + \frac{1}{n_1}\|F\| \\ &\le \frac{2}{n_2^2}\|\frac{\partial^2 g}{\partial y^2}\| + \frac{1}{n_1^2}\|\frac{\partial^2 g}{\partial x^2}\| + \frac{2}{n_1 n_2}\left[\|\frac{\partial^2 g}{\partial x \partial y}\| + \|\frac{\partial^2 g}{\partial y \partial x}\|\right] \\ &\le 2\left[\frac{1}{n_2^2}\|\frac{\partial^2 g}{\partial y^2}\| + \frac{1}{n_1^2}\|\frac{\partial^2 g}{\partial x^2}\| + \frac{1}{n_1 n_2}\left(\|\frac{\partial^2 g}{\partial x \partial y}\| + \|\frac{\partial^2 g}{\partial y \partial x}\|\right)\right]. \end{aligned}$$

By the linearity of S_{n_1,n_2}, for any $g \in W^2([-1, 1] \times [-1, 1])$ we obtain

$$\|f - S_{n_1,n_2}(f)\| \le \|f - g\| + \|g - S_{n_1,n_2}(g)\| + \|\|S_{n_1,n_2}\|\| \cdot \|f - g\|,$$

where $\|\|S_{n_1,n_2}\|\| := \sup\{\|S_{n_1,n_2}(f)\|; \|f\| \le 1\}$. But in Case (1), with $F_{i,j}(x, y) < 0$, we have

$$\begin{aligned} |S_{n_1,n_2}(f)(x, y)| &\le |f(x_i, y_{j+1})| + |x - x_i| \frac{|f(x_{i+1}, y_j)| + |f(x_i, y_j)|}{|x_{i+1} - x_i|} \\ &\quad + |y - y_{j+1}| \frac{|f(x_i, y_{j+1})| + |f(x_i, y_j)|}{|y_{j+1} - y_j|} \le 5\|f\|, \end{aligned}$$

which implies that $\|\|S_{n_1,n_2}\|\| \le 5$. It is easy to see that this inequality is valid in all the cases of $(x, y) \in [-1, +1] \times [-1, +1]$.

As a consequence, for any $g \in W^2([-1, 1] \times [-1, 1])$ it follows that

$$\begin{aligned} &\|f - S_{n_1,n_2}(f)\| \le 6\|f - g\| \\ &+ 2\left[\frac{1}{n_2^2}\left\|\frac{\partial^2 g}{\partial y^2}\right\| + \frac{1}{n_1^2}\left\|\frac{\partial^2 g}{\partial x^2}\right\| + \frac{1}{n_1 n_2}\left(\left\|\frac{\partial^2 g}{\partial x \partial y}\right\| + \left\|\frac{\partial^2 g}{\partial y \partial x}\right\|\right)\right], \end{aligned}$$

i.e., passing to the infimum with g,

$$\|f - S_{n_1,n_2}(f)\| \le 6K_2\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right),$$

which proves (i).

(ii) Let $g \in W^{2,\varphi}([-1,1]\times[-1,1])$ and $(a,b) \in [-1,1]\times[-1,1]$ be arbitrary, i.e., there exist i and j such that $x_i \le a \le x_{i+1}$, $y_j \le b \le y_{j+1}$. From the estimates at the above point (ii), we easily get

$$\begin{aligned} g(a,b) - S_{n_1,n_2}(g)(a,b) &= (b - y_{j+1})((b-y_j)[y_j, b, y_{j+1}; g(a,y)]_y \\ &\quad + (b - y_{j+1})(a - x_i)[y_j, b; [a, x_i; g(x,y)]_x]_y \\ &\quad + (a - x_i)(a - x_{i+1})[x_{i+1}, x_i, a; g(x, y_{j+1})]_x \\ &\quad + (a - x_i)(y_{j+1} - b)[x_i, x_{i+1}; [y_{j+1}, b; g(x,y)]_y]_x \\ &\quad + (a - x_i)(b - y_j)[x_i, x_{i+1}; [b, y_j; g(x,y)]_y]_x \\ &:= E_1 + E_2 + E_3 + E_4 + E_5. \end{aligned}$$

Now, reasoning exactly as in the paper Leviatan [232], at pages 7–8, we get

$$\begin{aligned} E_2 &= (b - y_{j+1})[y_j, b; \int_{x_i}^{a} \frac{\partial g}{\partial x}(u,y)du]_y \\ &= \frac{b - y_{j+1}}{b - y_j} \int_{y_j}^{b} \int_{x_i}^{a} \frac{\partial^2 g}{\partial x \partial y}(u,v)dudv \\ &= \frac{1}{b - y_j} \frac{1}{a - x_i} \int_{y_j}^{b} \int_{x_i}^{a} (b - y_{j+1})(a - x_i) \frac{\partial^2 g}{\partial x \partial y}(u,v)dudv. \end{aligned}$$

But by Lemma A (iii) in Leviatan [228], since $x_i \le u \le a \le x_{i+1}$ and $y_j \le v \le b \le y_{j+1}$, we get $a - x_i \le x_{i+1} - x_i \le c\frac{\sqrt{1-u^2}}{n_1}$, $y_{j+1} - b \le y_{j+1} - y_j \le c\frac{\sqrt{1-v^2}}{n_2}$, which implies

$$\begin{aligned} |E_2| &\le \frac{C}{n_1 n_2} \frac{1}{b - y_j} \frac{1}{a - x_i} \int_{y_j}^{b} \int_{x_i}^{a} \varphi_u \varphi_v \left| \frac{\partial^2 g}{\partial x \partial y}(u,v) \right| dudv \\ &\le \frac{C}{n_1 n_2} \left\| \varphi_u \varphi_v \frac{\partial^2 g}{\partial x \partial y}(u,v) \right\|. \end{aligned}$$

Similarly, we get

$$\begin{aligned} |E_1| &\le \frac{1}{y_{j+1} - y_j} \int_{y_j}^{y_{j+1}} (v - y_j)(y_{j+1} - v) \left| \frac{\partial g^2}{\partial y^2}(a,v) \right| du \\ &\le \frac{C}{n_2^2} \left\| \varphi_v^2 \frac{\partial^2 g}{y^2}(u,v) \right\|, \\ |E_3| &\le \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} (u - x_i)(x_{i+1} - u) \left| \frac{\partial g^2}{\partial x^2}(u, y_{j+1}) \right| \\ &\le \frac{C}{n_1^2} \left\| \varphi_u^2 \frac{\partial^2 g}{x^2}(u,v) \right\|. \end{aligned}$$

Then

$$E_4 = (a - x_i)\left[x_i, x_{i+1}; \int_b^{y_{j+1}} \frac{\partial g}{\partial y}(x, v)dv\right]_x$$
$$= \frac{1}{x_{i+1} - x_i}\frac{1}{y_{j+1} - b}\int_{x_i}^{x_{i+1}}\int_b^{y_{j+1}}(a - x_i)(y_{j+1} - b)\frac{\partial g^2}{\partial y \partial x}(u, v)du\, dv,$$

which by similar reasoning gives

$$|E_4| \leq \frac{C}{n_1 n_2}\left\|\varphi_u \varphi_v \frac{\partial^2 g}{\partial y \partial x}(u, v)\right\|.$$

Finally, by similar reasoning for E_5, we obtain

$$|E_5| \leq \frac{C}{n_1 n_2}\left\|\varphi_u \varphi_v \frac{\partial^2 g}{\partial y \partial x}(u, v)\right\|.$$

Reasoning exactly as for the K_2-functional from the above point (i), we easily arrive at the desired estimate. The theorem is proved. □

Corollary 2.4.17. (Gal [136]) *Let* $f \in C([-1, 1] \times [-1, 1])$ *be continuous and convex on* $[-1, 1] \times [-1, 1]$ *and* $a \in (0, 1]$.

(i) Denoting by $z = H_{i,j}(x, y)$ *the equations of the linear plane pieces (inscribed in the convex surface* $z = f(x, y)$*) corresponding to the (finite number of) triangulations in the division of* $[-1, 1] \times [-1, 1]$ *in the proof of Theorem 2.4.16, we have* $S_{n_1,n_2}(f)(x, y) = \max_{i,j}\{H_{i,j}(x, y)\}$, *for all* $x, y \in [-1, 1]$. *Moreover, prolonging* $S_{n_1,n_2}(f)$ *to* $S^*_{n_1,n_2}(f)(x, y) = \max_{i,j}\{H_{i,j}(x, y)\}$ *for all* $(x, y) \in \mathbb{R}^2$, *it follows that* $S^*_{n_1,n_2}(f)$ *is convex and uniformly continuous on* $\mathbb{R}^2$.

(ii) For all $\frac{1}{n_1} \leq a$, $\frac{1}{n_2} \leq a$, *we have*

$$\|f - S_{n_1,n_2}(f)\| \leq C\omega_2(f; a).$$

Also,

$$\omega_2(S^*_{n_1,n_2}(f); a) \leq C\omega_2(f; a),$$

where $\omega_2(f; a)$ *is given by Definition 2.1.2 (ii). Note that in this inequality,* $\omega_2(S^*_{n_1,n_2}(f); a)$ *is considered on the whole space* $\mathbb{R}^2$, *while* $\omega_2(f; a)$ *is considered on* $[-1, 1] \times [-1, 1]$.

Proof. (i) First, since f is convex, it is immediate that $S_{n_1,n_2}(f)(x, y) = \max_{i,j}\{H_{i,j}(x, y)\}$, for all $x, y \in [-1, 1]$. Also, since each $H_{i,j}(x, y)$ is of the form $H_{i,j}(x, y) = A_{i,j}x + B_{i,j}y + C_{i,j}$ for all $(x, y) \in \mathbb{R}^2$ and since $H_{i,j}, |H_{i,j}|$ are uniformly continuous on $\mathbb{R}^2$, we immediately get that $S^*_{n_1,n_2}(f)(x, y)$ is uniformly continuous as a finite sum of uniformly continuous functions on $\mathbb{R}^2$, and it is convex as the maximum of convex functions on $\mathbb{R}^2$.

(ii) By Theorem 2.4.16 we get

$$\|f - S_{n_1,n_2}(f)\| \leq 6K_2\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right) \leq 6K_2(f; a, a),$$

which combined with Lemma 2.1.3 (iii), (iv), implies $K_2(f; a, a) \sim \omega_2(f; a)$ and proves the first estimate.

On the other hand,

$$\begin{aligned}|\Delta_h^2 S^*_{n_1,n_2}(f)(M)| &\leq |\Delta_h^2 S^*_{n_1,n_2}(f)(M) - \Delta_h^2 f(M)| + |\Delta_h^2 f(M)| \\ &\leq 2\|S_{n_1,n_2}(f) - f\| + \omega_2(f; a) \leq C\omega_2(f; a),\end{aligned}$$

for all $M, M \pm h \in [-1, 1] \times [-1, 1]$, with $|h_1| \leq a$, $|h_2| \leq a$, where $M = (x, y)$, $h = (h_1, h_2)$ and $\Delta_h^2(f)(M) = f(M + h) - 2f(M) + f(M - h)$, $M = (x, y)$, $h = (h_1, h_2)$.

We prove that the function $F(M, h) = \Delta_h^2 S^*_{n_1,n_2}(f)(M)$, $M \in \mathbb{R}^2$, $h = (h_1, h_2) \in [-1, 1] \times [-1, 1]$, is uniformly continuous on $\mathbb{R}^2 \times [-1, 1]^2$.

We first have

$$\begin{aligned}&|\Delta_h^2 S^*_{n_1,n_2}(f)(M) - \Delta_{h'}^2 S^*_{n_1,n_2}(f)(M')| \\ &\quad \leq 2|S^*_{n_1,n_2}(f)(M) - S^*_{n_1,n_2}(f)(M')| \\ &\qquad + |S^*_{n_1,n_2}(f)(M + h) - S^*_{n_1,n_2}(f)(M' + h')| \\ &\qquad + |S^*_{n_1,n_2}(f)(M - h) - S^*_{n_1,n_2}(f)(M' - h')| \\ &\quad \leq \frac{\varepsilon}{3} + |S^*_{n_1,n_2}(f)(M + h) - S^*_{n_1,n_2}(f)(M' + h')| \\ &\qquad + |S^*_{n_1,n_2}(f)(M - h) - S^*_{n_1,n_2}(f)(M' - h')|,\end{aligned}$$

for all $M, M' \in \mathbb{R}^2$ and all $h, h' \in [-1, 1] \times [-1, 1]$.

Now, let $\varepsilon > 0$ be arbitrary, fixed. Since $S^*_{n_1,n_2}(f)$ is uniformly continuous on $\mathbb{R}^2$, there exists $\delta > 0$ such that for all $M = (x, y)$, $M' = (x', y') \in \mathbb{R}^2$ with $\|M - M'\|_{\mathbb{R}^2} < \delta$, we have

$$|S^*_{n_1,n_2}(f)(M) - S^*_{n_1,n_2}(f)(M')| < \frac{\varepsilon}{6}.$$

Suppose now $\|M - M'\|_{\mathbb{R}^2} < \delta/2$ and $\|h - h'\|_{\mathbb{R}^2} < \delta/2$. This immediately implies $\|(M \pm h) - (M' \pm h')\|_{\mathbb{R}^2} \leq \delta$, and therefore by the uniform continuity of $S^*_{n_1,n_2}(f)$, it follows that $|S^*_{n_1,n_2}(f)(M \pm h) - S^*_{n_1,n_2}(f)(M' \pm h')| < \frac{\varepsilon}{6}$, which finally leads to

$$|\Delta_h^2 S^*_{n_1,n_2}(f)(M) - \Delta_{h'}^2 S^*_{n_1,n_2}(f)(M')| < \varepsilon.$$

As a conclusion, $F(M, h)$ is uniformly continuous on $\mathbb{R}^2 \times [-1, 1]^2$.

Let $M - h, M, M + h \in \mathbb{R}^2$ be with $h = (h_1, h_2)$, $|h_1| \leq a$, $|h_2| \leq a$, fixed, where M does not necessarily belong to $[-1, 1] \times [-1, 1]$. Obviously, there exist i, j and a plane linear piece $K_{i,j}$ of equation $z = H_{i,j}(x, y)$ such that $M \in K_{i,j}$.

From the uniform continuity of $F(M,h)$, for $\varepsilon = \omega_2(f;a)$, there exists $\delta > 0$ such that

$$|\Delta_h^2 S^*_{n_1,n_2}(f)(M) - \Delta_{h'}^2 S^*_{n_1,n_2}(f)(M')| < \omega_2(f;a),$$

for all $\|M - M'\|_{\mathbb{R}^2} < \delta$ and $\|h - h'\|_{\mathbb{R}^2} < \delta$.

But we can choose M' sufficiently close to M and h' sufficiently small that $M', M' - h', M' + h' \in K_{i,j}$, which immediately implies $\Delta_{h'}^2 S^*_{n_1,n_2}(f)(M')=0$. As a consequence, we get $|\Delta_h^2 S^*_{n_1,n_2}(f)(M) < \omega_2(f;a)$, which implies $\omega_2(S^*_{n_1,n_2}(f);a) \le C\omega_2(f;a)$ and proves the corollary. □

Remark. According to Lemma 2.1.3 (v), the modulus $\omega_2(f;a)$ in Corollary 2.4.17 is equivalent to $\omega_2^*(f;a)$ (introduced by Definition 2.1.2 (iii)), which, in fact, represents the modulus used in the proof of Theorem 2.4.18 below (but with a different notation).

We are now in position to present the main result.

Theorem 2.4.18. (Gal [136]) *If $f : [-1,1]\times[-1,1] \to \mathbb{R}$ is convex on $[-1,1]\times[-1,1]$, then for any $n \in \mathbb{N}$, there exists a convex polynomial $P_n \in \mathcal{P}_n^2$ such that*

$$\|f - P_n\| \le C\omega_2\left(f; \frac{1}{n+1}\right),$$

where $\omega_2(f;\delta)$ is that in Definition 2.1.2 (iii), and $C > 0$ is independent of f and n.

Proof. Everywhere we will recall and use some notation in the paper of Shvedov [354] (see also the notation in the proofs of Theorem 2.4.11 and Theorem 2.4.16). The estimate in Theorem 1 in Shvedov [354] is a direct consequence of the estimates (in terms of the ω_1 modulus) in Lemmas 3 and 5 there. Since Lemma 3 there was already replaced by Corollary 2.4.17, in what follows we show that the statement of Lemma 5 in Shvedov [354] is valid by replacing ω_1 with ω_2 there.

For that, keeping the notation in the above-mentioned Lemma 5, we remark that since (in general dimensions m) it is easy to see that

$$f_\delta(x) = \frac{1}{\delta^{2m}\psi^2}\int_{\delta Y}\int_{\delta Y} h(x+u+v)du\,dv = \frac{1}{\delta^{2m}\psi^2}\int_{\delta Y}\int_{\delta Y} h(x-u-v)du\;dv$$

for all $x \in \mathbb{R}^m$, we can rewrite f_δ as

$$f_\delta(x) = \frac{1}{\delta^{2m}\psi^2}\int_{\delta Y}\int_{\delta Y} \frac{h(x+u+v) + h(x-u-v)}{2}du\;dv,$$

which immediately implies

$$\begin{aligned}|f_\delta(x) - h(x)| &\le \frac{1}{\delta^{2m}\psi^2}\int_{\delta Y}\int_{\delta Y}\left|\frac{h(x+u+v) + h(x-u-v)}{2} - h(x)\right| du\;dv \\ &\le A\omega_2(h;a;Y)_{\mathbb{R}^m},\end{aligned}$$

i.e., the estimate (16) in the proof of Lemma 5 in Shvedov [354] can be written in terms of the ω_2 modulus of smoothness too.

Here, according to the remark after the proof of Corollary 2.4.17, $\omega_2(h;a;Y)_{\mathbb{R}^m}$ denotes the second-order modulus of smoothness of h on $\mathbb{R}^2$ denoted by $\omega_2^*(h;a)$, which is equivalent to $\omega_2(h;a)$ (both of the form in Definition 2.1.2 (iii), but for h defined on the whole space $\mathbb{R}^2$ and not only on the bidimensional interval $[a,b]\times[c,d]$). Also, $\omega_2(f;a,M)$ (with $M=[-1,1]\times[-1,1]$) in the proof of Theorem 1 in Shvedov [354], p. 524, denotes in fact $\omega_2^*(f;a)$.

Then (16) implies (17) in terms of ω_2 too (for (17) in terms of ω_1 see Shvedov [354], p. 521). Here (16) and (17) refer to formulas in Shvedov [354].

Taking into account the formula for $\frac{\partial^2 f_\delta}{\partial x_1^2}(x_1,x_2)$ in Shvedov [354], p. 521, i.e. (considering here for simplicity $m=2$),

$$\begin{aligned}\frac{\partial^2 f_\delta}{\partial x_1^2}&(x_1,x_2)\\ &=\frac{1}{\delta^4\psi^2}\int_{-\delta a_1/2}^{\delta a_1/2}\int_{-\delta a_2/2}^{\delta a_2/2}\Big[h(x_1+\delta a_1,x_2+u_2+v_2)\\ &\qquad\qquad -2h(x_1,x_2+u_2+v_2)\\ &\qquad\qquad +h(x_1-\delta a_1,x_2+u_2+v_2)\Big]du_2\,dv_2,\end{aligned}$$

it immediately follows that

$$\left\|\frac{\partial^2 f_\delta}{\partial x_1^2}\right\|_{C(\mathbb{R}^2;\mathbb{R})}\le\frac{A\omega_2(h;a;Y)_{\mathbb{R}^2}}{\delta^2 a_1^2}.$$

A similar estimate in terms of ω_2 satisfies $\|\frac{\partial^2 f_\delta}{\partial x_2^2}\|_{C(\mathbb{R}^2;\mathbb{R})}$.

Then, by the formula in Shvedov [354], p. 521, we have

$$\begin{aligned}\frac{\partial^2 f_\delta}{\partial x_1\partial x_2}&(x_1,x_2)\\ &=\frac{1}{\delta^4\psi^2}\int_{-\delta a_1/2}^{\delta a_1/2}\int_{-\delta a_2/2}^{\delta a_2/2}\Big[h(x_1+\delta a_1/2+v_1,x_2+\delta a_2/2+v_2)\\ &\qquad\qquad -h(x_1+\delta a_1/2+v_1,x_2-\delta a_2/2+v_2)\Big]\\ &+\Big[-h(x_1-\delta a_1/2+v_1,x_2+\delta a_2/2+v_2)\\ &\qquad +h(x_1-\delta a_1/2+v_1,x_2-\delta a_2/2+v_2)\Big]dv_1\,dv_2.\end{aligned}$$

But it is easy to show that

$$h(x_1+\delta a_1/2+v_1, x_2+\delta a_2/2+v_2) - h(x_1+\delta a_1/2+v_1, x_2-\delta a_2/2+v_2)$$
$$- h(x_1-\delta a_1/2+v_1, x_2+\delta a_2/2+v_2) + h(x_1-\delta a_1/2+v_1, x_2-\delta a_2/2+v_2)$$
$$= \Delta^2_{(\delta a_1/2, \delta a_2/2)} f(x+v) - \Delta^2_{(\delta a_1/2, -\delta a_2/2)} f(x+v),$$

where $x=(x_1,x_2)$, $v=(v_1,v_2)$, $\Delta^2_{\alpha} f(x) = f(x+\alpha) - 2f(x) + f(x-\alpha)$.

This immediately implies

$$\left\| \frac{\partial^2 f_\delta}{\partial x_1 \partial x_2}(x_1,x_2) \right\|_{C(\mathbb{R}^2;\mathbb{R})} \le \frac{A\omega_2(h;a;Y)_{\mathbb{R}^2}}{\delta^2 a_1 a_2},$$

i.e., the estimates (18) in the proof of Lemma 5 in Shvedov [354], p. 521, hold in terms of ω_2 moduli of smoothness too.

Keeping the notation for the polynomials $\Pi_n^{(1)}$, $\Pi_n^{(2)}$ in Shvedov [354], pp. 521–522 (see also the proof of Theorem 2.4.11 above) as a first immediate consequence, it follows that estimate (19) there holds in terms of ω_2 too.

Also, writing $\lambda(x_1,x_2) = 2A_0\omega_2(h;\delta;Y)_{\mathbb{R}^2}\left[\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}\right]$, the polynomials in Lemma 5 in Shvedov [354] are now given by $\pi_n(x) = \Pi_n^{(1)}(x) + \Pi_n^{(2)}(x) + \lambda(x)$, with the new above form for $\lambda(x)$.

Now, taking into account that the multivariate algebraic polynomial $J_{n-1}(t), t=(t_1,\ldots,t_m)$ (introduced in Shvedov [354], p. 519) is even, we immediately get

$$\int_{10Y} f_\delta(x+t)J_{n-1}(t)dt = \int_{10Y} f_\delta(x-t)J_{n-1}(t)dt,$$

which implies

$$f_\delta(x) - \Pi_n^{(1)}(x) - \Pi_n^{(2)}(x) = \int_{10Y}\left[f_\delta(x) - \frac{f_\delta(x+t)+f_\delta(x-t)}{2}\right]J_{n-1}(t)dt,$$

and therefore

$$\|f_\delta(x) - \Pi_n^{(1)}(x) - \Pi_n^{(2)}(x)\|_{C(Y;\mathbb{R})} \le \int_{10Y} \omega_2(f_\delta;|t|_Y,Y)_{\mathbb{R}^m} J_{n-1}(t)dt$$
$$\le \omega_2(f_\delta;\delta,Y)_{\mathbb{R}^m}\int_{10Y}[n|t|_Y+1]^2 J_{n-1}(t)dt,$$

where $|t|_Y = \max\{\frac{2}{a_1}|t_1|,\ldots,\frac{2}{a_m}|t_m|\}$, which by the estimates (13) in Shvedov [354], p. 519 (see also the estimates satisfied by $K_{n-1}(t_i)$ written in the proof of Theorem 2.4.11), immediately implies the estimate (21) in Shvedov [354], p. 522, in terms of ω_2, which is exactly the estimate in Lemma 5 in Shvedov [354], in terms of ω_2.

Finally, following the lines in the proof of Theorem 1 in Shvedov [354] at page 524 and the above considerations, it is easily seen that its estimate holds in terms of ω_2 too.

The theorem is proved. □

Remark. If in Theorem 2.4.18, f is of C^2-class, then the estimate in the statement of the theorem becomes $\|f - P_n\| \leq C/n^2$, and one reobtains the result of Budnik [50].

2.5 Bivariate Monotone Approximation by Convolution Polynomials

In this section, we construct bivariate polynomials of convolution type, attached to a bivariate function, that approximate in the uniform and L^p norms with Jackson-type rate (by using suitable bivariate moduli of smoothness) and preserve some bivariate kinds of monotonicity of function.

Recalling that in the univariate case, the first Jackson-type estimate (without using derivatives) presented for monotone approximation in the uniform norm, was Theorem 1.6.1 in Section 1.6, a bivariate analogue of Theorem 1.6.1 is the following.

Theorem 2.5.1. (Gal [125]) *If $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is continuous on $[-1,1] \times [-1,1]$, then there exists a sequence of bivariate polynomials $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2 \in \mathbb{N}}$, of degree n_1 with respect to x and n_2 with respect to y, such that*

$$|f(x,y) - P_{n_1,n_2}(f)(x,y)| \leq C\omega_1\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right),$$

$\forall\, n_1, n_2 \in \mathbb{N},\ \forall\, x, y \in [-1,1],\ C > 0$ *independent of f, n_1, n_2, x, and y (here $\omega_1(f;t,s)$ is defined by Definition 2.1.2 (i)) satisfying moreover the following shape-preserving properties (for their definitions see the introduction to this chapter):*

(i) If $f(x,y)$ is increasing (decreasing) with respect to x on $[-1,1] \times [-1,1]$, then so is $P_{n_1,n_2}(f)(x,y)$;
(ii) If $f(x,y)$ is increasing (decreasing) with respect to y on $[-1,1] \times [-1,1]$, then so is $P_{n_1,n_2}(f)(x,y)$;
(iii) If $f(x,y)$ is upper (lower) bidimensional monotone on $[-1,1] \times [-1,1]$, then so is $P_{n_1,n_2}(f)(x,y)$;
(iv) If $f(x,y)$ is totally upper (lower) monotone on $[-1,1] \times [-1,1]$, then so is $P_{n_1,n_2}(f)(x,y)$.

First we need the following lemma.

Lemma 2.5.2. (Gal [125]) *Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$ be continuous on $[-1,1] \times [-1,1]$ and $F(t,s) = f(\cos t, \cos s)$, $t, s \in \mathbb{R}$.*

(i) We have

$$\omega_1(F; \alpha, \beta) \leq \omega_1(f; \alpha, \beta),\ \forall\, \alpha, \beta \geq 0,$$

where

$$\omega_1(F;\alpha,\beta) \\ = \sup\{|F(t_1,s_1)-F(t_2,s_2)|; t_1,t_2,s_1,s_2 \in \mathbb{R},\ |t_1-t_2| \le \alpha,\ |s_1-s_2| \le \beta\}.$$

(ii) If we define $g(F)(t,s) : [-\pi,\pi] \times [-\pi,\pi] \to \mathbb{R}$ *by*

$$g(F)(t,s) = F\left(\frac{\pi k}{n_1}, \frac{\pi j}{n_2}\right) := c_{k,j},$$

if $t \in \left[\frac{\pi k}{n_1}, \frac{\pi(k+1)}{n_1}\right)$ *and* $s \in \left[\frac{\pi j}{n_2}, \frac{\pi(j+1)}{n_2}\right)$, $k = \overline{0,n_1}$, $j = \overline{0,n_2}$,

$$g(F)(t,s) := g(-t,s),\ \text{if } t \in [-\pi,0] \text{ and } s \in [0,\pi],$$
$$g(F)(t,s) := g(t,-s),\ \text{if } t \in [0,\pi] \text{ and } s \in [-\pi,0],$$
$$g(F)(-t,-s) := g(-t,-s),\ \text{if } t \in [-\pi,0] \text{ and } s \in [-\pi,0],$$

then $g(F)(\cdot,s), g(F)(t,\cdot), g(F)(\cdot,\cdot)$ *are even and*

$$|F(t,s) - g(F)(t,s)| \le \omega_1\left(F; \frac{\pi}{n_1}, \frac{\pi}{n_2}\right),\ \forall (t,s) \in [-\pi,\pi] \times [-\pi,\pi].$$

Proof. (i) Let $|t_1 - t_2| \le \alpha$, $|s_1 - s_2| \le \beta$, $t_1,t_2,s_1,s_2 \in \mathbb{R}$. We get

$$|F(t_1,s_1) - F(t_2,s_2)| = |f(\cos t_1, \cos s_1) - f(\cos t_2, \cos s_2)| \\ \le \omega_1(f; |\cos t_1 - \cos t_2|, |\cos s_1 - \cos s_2|) \le \omega_1(f;\alpha,\beta),$$

reasoning as in the univariate case.

(ii) Reasoning as in univariate case (see the proof of Lemma 1.6.2 in Chapter 1), the conclusion is immediate. □

Proof of Theorem 2.5.1. For $f : [-1,1] \times [-1,1] \to \mathbb{R}$ define F and $g(F)$ as in Lemma 2.5.2.

Let us consider the Jackson kernel with

$$m' = \left[\frac{m}{2}\right] + 1, \quad K_m(t) = \lambda_{m'}^{-1} \left(\frac{\sin \frac{m't}{2}}{\sin \frac{t}{2}}\right)^4, \quad \int_{-\pi}^{\pi} K_m(t)dt = 1,$$

and the Jackson double integral

$$J_{n_1,n_2}(g(F))(t,s) := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_{n_1}(t-u)K_{n_2}(s-v)g(F)(u,v)du\,dv.$$

Define

$$P_{n_1,n_2}(f)(x,y) := J_{n_1,n_2}(g(F))(\arccos x, \arccos y), \quad x,y \in [-1,1].$$

Obviously $P_{n_1,n_2}(f)(x,y)$ is an algebraic polynomial of degree $\le n_1+n_2$ (i.e., of degree $\le n_1$ in x and of degree $\le n_2$ in y).

First we will deduce the estimate in Theorem 2.5.1.

Using Lemma 2.5.2 (i), (ii), we get

$$\begin{aligned}|F(t,s)-J_{n_1,n_2}(g(F))(t,s)| &\le |F(t,s)-J_{n_1,n_2}(F)(t,s)| \\ &\quad + |J_{n_1,n_2}(F)(t,s)-J_{n_1,n_2}(g(F))(t,s)| \\ &\le |F(t,s)-J_{n_1,n_2}(F)(t,s)| + \|F-g(F)\| \\ &\le |F(t,s)-J_{n_1,n_2}(F)(t,s)| + \omega_1\left(F;\frac{\pi}{n_1},\frac{\pi}{n_2}\right) \\ &\le |F(t,s)-J_{n_1,n_2}(F)(t,s)| + \omega_1\left(f;\frac{\pi}{n_1},\frac{\pi}{n_2}\right).\end{aligned}$$

Here $\|G\| := \sup\{|G(t,s)|;\ t,s\in[-\pi,\pi]\}$.

On the other hand, as in the univariate case (see Lemma 1.6.2 in Chapter 1), we obtain

$$\begin{aligned}&|F(t,s)-J_{n_1,n_2}(F)(t,s)| \\ &\quad\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|F(t,s)-F(t+u,s+v)|K_{n_1}(u)K_{n_2}(v)du\,dv \\ &\quad\le C\omega_1\left(F;\frac{\pi}{n_1},\frac{\pi}{n_2}\right)\le C\omega_1\left(f;\frac{\pi}{n_1},\frac{\pi}{n_2}\right).\end{aligned}$$

So the conclusion is

$$|F(t,s)-J_{n_1,n_2}(g(F))(t,s)|\le C\omega_1\left(f;\frac{\pi}{n_1},\frac{\pi}{n_2}\right),$$

and making the substitutions $t=\arccos x$, $s=\arccos y$, we have obtained the desired estimate.

In what follows, we will prove the shape-preserving properties. Writing

$$H_{n_1,n_2}(t,s,u,v)=K_{n_1}(t-u)K_{n_2}(s-v)g(F)(u,v)$$

and using the above notation, we have

$$\begin{aligned}J_{n_1,n_2}(g(F))(t,s) &= \int_{-\pi}^{0}\int_{-\pi}^{0}H_{n_1,n_2}(t,s,u,v)dudv \\ &\quad+\int_{-\pi}^{0}\int_{0}^{\pi}H_{n_1,n_2}(t,s,u,v)dudv \\ &\quad+\int_{0}^{\pi}\int_{-\pi}^{0}H_{n_1,n_2}(t,s,u,v)dudv \\ &\quad+\int_{0}^{\pi}\int_{0}^{\pi}H_{n_1,n_2}(t,s,u,v)dudv = I_1+I_2+I_3+I_4.\end{aligned}$$

Let us suppose, first, that $f(x,y)$ is increasing on $[-1,1]$ as a function of x, for any fixed $y \in [-1,1]$. We have

$$I_1 = \left(\sum_{k=0}^{n_1-1} \int_{-\pi(k+1)/n_1}^{-\pi k/n_1} K_{n_1}(t-u)du \right) A_k(s),$$
$$I_3 = \left(\sum_{k=0}^{n_1-1} \int_{\pi k/n_1}^{\pi(k+1)/n_1} K_{n_1}(t-u)du \right) A_k(s),$$
$$I_2 = \left(\sum_{k=0}^{n_1-1} \int_{-\pi(k+1)/n_1}^{-\pi k/n_1} K_{n_1}(t-u)du \right) B_k(s),$$
$$I_4 = \left(\sum_{k=0}^{n_1-1} \int_{\pi k/n_1}^{\pi(k+1)/n_1} K_{n_1}(t-u)du \right) B_k(s),$$

where

$$A_k(s) = \sum_{j=0}^{n_2-1} c_{k,j} \int_{-\pi(j+1)/n_2}^{-\pi j/n_2} K_{n_2}(s-v)dv,$$
$$B_k(s) = \sum_{j=0}^{n_2-1} c_{k,j} \int_{\pi j/n_2}^{\pi(j+1)/n_2} K_{n_2}(s-v)dv.$$

We obtain (as in the univariate case, see the proof of Lemma 1.6.2 in Chapter 1)

$$I_1 + I_3 = \sum_{k=1}^{n_1} a_{k-1}(s) \int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du,$$
$$I_2 + I_4 = \sum_{k=1}^{n_1} b_{k-1}(s) \int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du,$$

where $a_k(s)$ and $b_k(s)$ are defined by the relations

$$A_k(s) = a_k(s) + a_{k+1}(s) + \cdots + a_{n-1}(s), \text{ i.e., } a_k(s) = A_k(s) - A_{k+1}(s)$$

and

$$B_k(s) = b_k(s) + b_{k+1}(s) + \cdots + b_{n-1}(s), \text{ i.e., } b_k(s) = B_k(s) - B_{k+1}(s).$$

But

$$a_k(s) = A_k(s) - A_{k+1}(s) = \sum_{j=0}^{n_2-1} (c_{k,j} - c_{k+1,j}) \int_{-\pi(j+1)/n_2}^{-\pi j/n_2} K_{n_2}(s-v)dv \geq 0,$$

because by hypothesis f is increasing with respect to x, that is F, is decreasing with respect to x, that implies $c_{k,j} - c_{k+1,j} \geq 0, \ \forall \ k = \overline{0, n_1 - 1}$.

Similarly,

$$b_k(s) = B_k(s) - B_{k+1}(s) = \sum_{j=0}^{n_2-1} (c_{k,j} - c_{k,j+1}) \int_{\pi j/n_2}^{\pi(j+1)/n_2} K_{n_2}(s-v)dv \geq 0.$$

As a conclusion,

$$\begin{aligned}\frac{\partial J_{n_1,n_2}(g(F))(t,s)}{\partial t} &= \frac{\partial}{\partial t}(I_1 + I_3) + \frac{\partial}{\partial t}(I_2 + I_4) \\ &= \sum_{k=1}^{n_1} a_{k-1}(s) \frac{\partial}{\partial t}\left(\int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du\right) \\ &\quad + \sum_{k=1}^{n_1} b_{k-1}(s) \frac{\partial}{\partial t}\left(\int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du\right) \leq 0,\end{aligned}$$

taking into account the reasoning at the end of proof of Lemma 1.6.2.

This immediately implies that $P_{n_1,n_2}(f)(x,y)$ is increasing with respect to x.

Now, if we suppose that $f(x,y)$ is increasing on $[-1,1]$ with respect to y, we rewrite I_1, I_2, I_3, I_4 with K_{n_2} first and then we group similarly $I_1 + I_3$ and $I_2 + I_4$.

Finally, let us suppose that $f(x,y)$ is upper bidimensional monotone on $[-1,1] \times [-1,1]$.

Applying successively two times the decomposition in the univariate case in the proof of Lemma 1.6.2, we get

$$\begin{aligned}&J_{n_1,n_2}(g(F))(t,s) \\ &= \sum_{k=1}^{n_1} \sum_{j=0}^{n_2-1} (c_{k,j+1} - c_{k,j} - c_{k-1,j+1} + c_{k-1,j}) \\ &\quad \times \int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du \int_{-\pi(j+1)/n_2}^{\pi(j+1)/n_2} K_{n_2}(s-v)dv.\end{aligned}$$

Indeed,

$$\begin{aligned}J_{n_1,n_2}(g(F))(t,s) &= \sum_{k=1}^{n_1} (a_{k-1}(y) + b_{k-1}(y)) \int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du \\ &= \sum_{k=1}^{n_1} [(A_{k-1}(y) - A_k(y)) + (B_{k-1}(y) - B_k(y))] \\ &\quad \times \int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du\end{aligned}$$

$$= \sum_{k=1}^{n_1} \left(\int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du \right) \left\{ \sum_{j=0}^{n_2-1} (c_{k-1,j} - c_{k,j}) \right.$$

$$\left. \times \left[\int_{-\pi(j+1)/n_2}^{-\pi j/n_2} K_{n_2}(s-v)dv + \int_{\pi j/n_2}^{\pi(j+1)/n_2} K_{n_2}(s-v)dv \right] \right\}$$

$$= \sum_{k=1}^{n_1} \left(\int_{-\pi k/n_1}^{\pi k/n_1} K_{n_1}(t-u)du \right) D_k,$$

where applying the reasoning in the proof Lemma 1.6.2, we get

$$D_k = \sum_{j=0}^{n_2-1} [(c_{k-1,j} - c_{k,j}) - (c_{k-1,j+1} - c_{k,j+1})] \int_{-\pi(j+1)/n_2}^{\pi(j+1)/n_2} K_{n_2}(s-v)dv.$$

As a consequence we obtain

$$\frac{\partial^2 J_{n_1,n_2}(g(F))(t,s)}{\partial t \partial s} \geq 0 \quad \left(\text{which implies } \frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x \partial y} \geq 0 \right),$$

by the proof of Lemma 1.6.2 and by the inequalities

$$c_{k,j+1} - c_{k,j} - c_{k-1,j+1} + c_{k-1,j} \geq 0, \ \forall\, k = \overline{1, n_1}, \ \forall\, j = \overline{0, n_2 - 1}.$$

The last inequalities (concerning $c_{k,j}$) hold from the fact that $f(x,y)$ upper bidimensional monotone on $[-1,1] \times [-1,1]$ implies that $F(t,s)$ is upper bidimensional monotone on $[0,\pi] \times [0,\pi]$.

Indeed, let us first suppose that f is of C^2-class.

We easily get

$$\frac{\partial^2 F(t,s)}{\partial t \partial s} = \sin t \sin s \frac{\partial^2 f(\cos t, \cos s)}{\partial t \partial s} \geq 0, \ \forall\, t,s \in [0,\pi].$$

Now, if f is only continuous on $[-1,1] \times [-1,1]$, then by standard procedures we easily get that the Bernstein polynomials

$$B_{m,n}(f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x^i (1-x)^{m-i} y^j (1-y)^{n-j}$$

are uniformly convergent to $f(x,y)$ (with $m,n \to +\infty$) and are upper bidimensional monotone on $[-1,1]$.

Because obviously $B_{m,n}(f)(x,y)$ are of C^2-class, reasoning as above it follows that $B_{m,n}(f)(\cos t, \cos s)$ is upper bidimensional monotone on $[0,\pi] \times [0,\pi]$, that is, for all $m,n \in \mathbb{N}$, all $t,s \in [0,\pi]$, and all $\alpha, \beta \geq 0$ we get

$$B_{m,n}(f)(\cos(t+\alpha), \cos(s+\beta)) - B_{m,n}(f)(\cos(t+\alpha), \cos s)$$

$$-B_{m,n}(f)(\cos t, \cos(s+\beta)) + B_{m,n}(\cos t, \cos t) \geq 0.$$

Passing to limit with $m,n \to +\infty$, we obtain that $F(t,s) = f(\cos t, \cos s)$ is upper bidimensional monotone on $[0,\pi] \times [0,\pi]$.

This completes the proof. □

2.6 Tensor Product Polynomials Preserving Popoviciu's Convexities

In order to extend the results in the real univariate case to the real bivariate/multivariate case, in this section we intensively use the tensor product method. Although the error estimates obtained are not always the best possible, this method is very accessible because it allows us to use the results in the univariate cases.

2.6.1 Bivariate/Multivariate Monotone and Convex Approximation

The next result extends that in the univariate case of Leviatan [228], [232] and improves, in most cases that in Theorem 2.5.1 (excepting the cases of usual monotonicities with respect to each variable, when the other one is fixed, i.e., Theorem 2.5.1 (i) and (ii)).

Theorem 2.6.1. *For any continuous function* $f : [-1,1] \times [-1,1] \to \mathbb{R}$*, a sequence of bivariate polynomials* $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2 \in \mathbb{N}}$ *exists, with degree* $(P_{n_1,n_2}(f)(x,y)) \leq n_1 + n_2$*, such that (see Gal [126])*

$$\|f - P_{n_1,n_2}\| \leq C\omega_2^{\varphi}\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right), \forall n_1, n_2 \in \mathbb{N},$$

where $C > 0$ *is independent of* f*,* n_1*, and* n_2*, (*ω_2^{φ} *is in Definition 2.1.2 (ii)), satisfying, in addition, the following shape-preserving properties :*

(i) (Gal [126]) If f *is convex of order* $(1,1)$ *on* $[-1,1]\times[-1,1]$ *(i.e., according to Remark 3 of Definition 2.1.1,* f *is upper bidimensional monotone), then so is* $P_{n_1,n_2}(f)$*.*

(ii) (Gal [126]) If f *is simultaneously convex of orders* $(0,1), (1,0)$*, and* $(1,1)$ *(i.e., according to Remark 3 of Definition 2.1.1,* $f(x,y)$ *is totally upper monotone) then so is* $P_{n_1,n_2}(f)$*.*

(iii) (Gal [126]) if f *is convex of order* $(2,2)$ *on* $[-1,1] \times [-1,1]$*, then so is* $P_{n_1,n_2}(f)$*.*

(iv) (Gal [126]) if f *is totally convex on* $[-1,1] \times [-1,1]$*, then so is* $P_{n_1,n_2}(f)$*.*

(v) Let us denote by $L_{1,1}(f)(x,y)$ *the tensor product of the Lagrange interpolation polynomials on the nodes* $a_0 < a_1$*,* $a_0, a_1 \in [-1,1]$ *and* $b_0 < b_1$*,* $b_0, b_1 \in [-1,1]$*, respectively, i.e., we have* $L_{1,1}(f)(x,y) = Axy + Bx + Cy + D$*, where* A, B, C, D *are given by the interpolation conditions* $L_{1,1}(f)(a_i, b_j) = f(a_i, b_j)$*,* $i = 0, 1$*,* $j = 0, 1$ *and*

$$D = \frac{a_1 b_1 f(a_0, b_0) - a_1 b_0 f(a_0, b_1) - a_0 b_1 f(a_1, b_0) + a_0 b_0 f(a_1, b_1)}{(a_1 - a_0)(b_1 - b_0)}.$$

Let us denote by $ST([-1,1]\times[-1,1])$ *the class of all functions satisfying the condition* $D \geq 0$ *for all* $-1 \leq a_0 < a_1 \leq 1$, $-1 \leq b_0 < b_1 \leq 1$.

If f *is totally convex and* $f \in ST([-1,1]\times[-1,1])$, *then* $P_{n_1,n_2}(f)$ *is totally convex and satisfies the differential inequality*

$$P(x,y) - x\frac{\partial P}{\partial x}(x,y) - y\frac{\partial P}{\partial y}(x,y) + xy\frac{\partial^2 P}{\partial x \partial y}(x,y) \geq 0$$

for all $x, y \in [-1,0]$ *(see Open Problem 2.7.9).*

Proof. For $h : [-1,1] \to R$, according to Leviatan [228], relation (5) (see also the proof of Theorem 1.6.3, Case 1), the approximation polynomials are given by

$$P_n(h)(x) = h(-1) + \sum_{j=0}^{n-1} s_{j,n}(R_{j,n}(x) - (R_{j+1,n}(x)),$$

where $s_{j,n} = \frac{h(\xi_{j+1,n})-h(\xi_{j,n})}{\xi_{j+1,n}-\xi_{j,n}}$, $\{\xi_{j,n}, j = \overline{0,n}\}$ are suitable nodes in $[-1,1]$, and $R_{j,n}(x)$ are suitable chosen polynomials of degree $\leq n$.

Also, by Leviatan [228], Theorem 1 (see also the proof of Theorem 1.6.3, Case 1), we have

$$\|h - P_n(h)\| \leq C\omega_2^{\varphi}\left(h; \frac{1}{n}\right)_{\infty}, \forall n \in \mathbb{N},$$

where $\omega_2^{\varphi}(h;\delta)_{\infty}$ is the usual Ditzian–Totik uniform modulus of smoothness.

We will construct the polynomials $P_{n_1,n_2}(f)(x,y)$ using the classical tensor product method (see, e.g., Nürnberger [290], pp. 195–196).

We get

$$\begin{aligned}
P_{n_1,n_2}(f)(x,y) &= f(-1,-1) \\
&+ \sum_{i=0}^{n_2-1} \frac{f(-1,\eta_{i+1,n_2}) - f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2} - \eta_{i,n_2}}[R_{i,n_2}(y) - R_{i+1,n_2}(y)] \\
&+ \sum_{j=0}^{n_1-1} \frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}}[R_{j,n_1}(x) - R_{j+1,n_1}(x)] \\
&+ \sum_{j=0}^{n_1-1}\sum_{i=0}^{n_2-1} S^*_{i,j}[R_{i,n_2}(y) - R_{i+1,n_2}(y)][R_{j,n_1}(x) - R_{j+1,n_1}(x)],
\end{aligned}$$

where

$$\begin{aligned}
&S^*_{i,j} \\
&= \frac{f(\xi_{j+1,n_1},\eta_{i+1,n_2}) - f(\xi_{j,n_1},\eta_{i+1,n_2}) - f(\xi_{j+1,n_1},\eta_{i,n_2}) + f(\xi_{j,n_1},\eta_{i,n_2})}{(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \xi_{i,n_2})} \\
&= \begin{bmatrix} \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ; f \\ \eta_{i,n_2}, \eta_{i+1,n_2} & \end{bmatrix},
\end{aligned}$$

and $\{\xi_{j,n_1}\}$, $R_{j,n_1}(x)$, $j = \overline{0, n_1}$, $\{\eta_{i,n_2}\}$, $R_{i,n_2}(y)$, $i = \overline{0, n_2}$, are constructed as in the univariate case in Leviatan [228] (see also the proof of Theorem 1.6.3, Case 1).

Obviously, degree $(P_{n_1,n_2}(f)) \leq n_1 + n_2$.

First we deduce the estimate in Theorem 2.6.1. Thus, for any univariate function h, we have

$$\|P_n(h)\| \leq \|h\| + \|P_n(h) - h\| \leq \|h\| + C\omega_2^{\varphi}\left(h; \frac{1}{n}\right)_{\infty} \leq (1 + 4C)\|h\|,$$

that is, passing to sup with $\|h\| \leq 1$, we obtain (for the linear operator P_n)

$$|||P_n||| \leq (1 + 4C), \quad \forall n \in \mathbb{N},$$

with $C > 0$ independent of n.

Then by Theorem 5 in Haussmann–Pottinger [167], we immediately obtain

$$\|f - P_{n_1,n_2}(f)\| \leq C\left[\omega_{2,x}^{\varphi}\left(f; \frac{1}{n_1}\right) + \omega_{2,y}^{\varphi}\left(f; \frac{1}{n_2}\right)\right],$$

where $\omega_{2,x}^{\varphi}$ and $\omega_{2,y}^{\varphi}$ are the partial moduli defined in Ditzian–Totik [98], Chapter 12. Since obviously we have

$$\omega_{2,x}^{\varphi}\left(f; \frac{1}{n_2}\right) + \omega_{2,y}^{\varphi}\left(f; \frac{1}{n_2}\right) \leq 2\omega_2^{\varphi}\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right),$$

we obtain the desired estimate.

Now, we will prove the shape-preserving properties.

(i) Let f be convex of order $(1, 1)$. We will prove that

$$\frac{\partial^2 P_{n_1,n_2}(f)(x, y)}{\partial x \partial y} \geq 0, \quad \forall x, y \in [-1, 1].$$

Indeed, we get

$$\frac{\partial^2 P_{n_1,n_2}(f)(x, y)}{\partial x \partial y} = \sum_{j=0}^{n_1-1} \sum_{i=0}^{n_2-1} S_{i,j}^*(R'_{i,n_2}(y) - R'_{i+1,n_2}(y))(R'_{j,n_1}(x) - R'_{j+1,n_1}(x)) \geq 0,$$

since from univariate case, each of $(R_{i,n_2}(y) - R_{i+1,n_2}(y))$ and $(R_{j,n_1}(x) - R_{j+1,n_1}(x)$ is increasing with respect to y and x, respectively, and $S_{i,j}^* \geq 0$, $\forall i = \overline{0, n_2}, \quad j = \overline{0, n_1}$.

(ii) Let f be simultaneously convex of orders $(0,1),(1,0)$, and $(1,1)$. This implies

$$\frac{f(-1,\eta_{i+1,n_2})-f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2}-\eta_{i,n_2}}\geq 0,$$
$$\frac{f(\xi_{j+1,n_1},-1)-f(\xi_{j,n_1},-1)}{\xi_{j+1,n_2}-\xi_{j,n_2}}\geq 0,$$

and $S^*_{i,j}\geq 0$, $\forall i=\overline{0,n_2}$, $j=\overline{0,n_1}$.

Also, $R_{j,n_1}(-1)=R_{i,n_2}(-1)=0$, $\forall j=\overline{0,n_1}$, $i=\overline{0,n_2}$,

$$R_{j,n_1}(x)-R_{j+1,n_1}(x)\geq 0,\quad R_{i,n_2}(y)-R_{i+1,n_2}(y)\geq 0,$$
$$\forall j=\overline{0,n_1}, i=\overline{0,n_2}.$$

Therefore, we get

$$\frac{\partial P_{n_1,n_2}(f)(x,y)}{\partial x}$$
$$=\sum_{j=0}^{n_1-1}\frac{f(\xi_{j+1,n_1},-1)-f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1}-\xi_{j,n_1}}(R'_{j,n_1}(x)-R'_{j+1,n_1}(x))$$
$$+\sum_{j=0}^{n_1-1}\sum_{i=0}^{n_2-1}\begin{bmatrix}\xi_{j+1,n_1},\xi_{j,n_1} & \\ & ;f\\ \eta_{i+1,n_2},\eta_{i,n_2} & \end{bmatrix}$$
$$\times(R'_{j,n_1}(x)-R'_{j+1,n_1}(x))(R_{i,n_2}(y)-R_{i+1,n_2}(y))$$
$$\geq 0,\forall x,y\in[-1,1]\times[-1,1].$$

Similarly, one obtains

$$\frac{\partial P_{n_1,n_2}(f)(x,y)}{\partial y}\geq 0,\quad \forall x,y\in[-1,1],$$

and finally from the previous point (i), we get

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x\partial y}\geq 0,\quad \forall x,y\in[-1,1].$$

(iii) By relation (5) in Leviatan [228], p. 473 (see also the proof of Theorem 1.6.3, Case 1), we obtain the form

$$
\begin{aligned}
P_{n_1,n_2}(f)(x,y) = f(-1,-1) &+ \frac{f(-1,\eta_{1,n_2}) - f(-1,-1)}{\eta_{1,n_2}+1}(1+y) \\
&+ \sum_{i=1}^{n_2-1} \left[\frac{f(-1,\eta_{i+1,n_2}) - f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2} - \eta_{i,n_2}} \right. \\
&\qquad \left. - \frac{f(-1,\eta_{i,n_2}) - f(-1,\eta_{i-1,n_2})}{\eta_{i,n_2} - \eta_{i-1,n_2}} \right] R_{i,n_2}(y) \\
&+ \frac{f(\xi_{1,n_1},-1) - f(-1,-1)}{\xi_{1,n_2}+1}(1+x) \\
&+ \sum_{j=1}^{n_1-1} \left[\frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}} \right. \\
&\qquad \left. - \frac{f(\xi_{j,n_1},-1) - f(\xi_{j-1,n_1},-1)}{\xi_{j,n_1} - \xi_{j-1,n_1}} \right] R_{j,n_1}(x) \\
&+ (1+x)(1+y)S^*_{0,0} + (1+x) \sum_{i=1}^{n_2-1} (S^*_{i,0} - S^*_{i-1,0}) R_{i,n_2}(y) \\
&+ (1+y) \sum_{j=1}^{n_1-1} (S^*_{0,j} - S^*_{0,j-1}) R_{j,n_1}(x) \\
&+ \sum_{j=1}^{n_1-1} \sum_{i=1}^{n_2-1} (S^*_{i,j} - S^*_{i,j-1} - S^*_{i-1,j} + S^*_{i-1,j-1}) R_{i,n_2}(y) R_{j,n_1}(x),
\end{aligned}
$$

where

$$
S^*_{i,j} = \frac{f(\xi_{j+1,n_1},\eta_{i+1,n_2}) - f(\xi_{j,n_1},\eta_{i+1,n_2}) - f(\xi_{j+1,n_1},\eta_{i,n_2}) + f(\xi_{j,n_1},\eta_{i,n_2})}{(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})}.
$$

Note that

$$
\frac{f(-1,\eta_{i+1,n_2}) - f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2} - \eta_{i,n_2}} - \frac{f(-1,\eta_{i,n_2}) - f(-1,\eta_{i-1,n_2})}{\eta_{i,n_2} - \eta_{i-1,n_2}}
$$

$$
= (\eta_{i+1,n_2} - \eta_{i-1,n_2}) \left[\begin{matrix} -1 \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} \end{matrix} ; f \right];
$$

$$
\frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}} - \frac{f(\xi_{j,n_1},-1) - f(\xi_{j-1,n_1} - 1)}{\xi_{j,n_1} - \xi_{j-1,n_1}}
$$

$$
= (\xi_{j+1,n_1} - \xi_{j-1,n_1}) \left[\begin{matrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ -1 \end{matrix} ; f \right];
$$

$$S^*_{i,0} - S^*_{i-1,0} = \left[\begin{matrix} -1, \xi_{1,n_1} \\ \eta_{i,n_2}, \eta_{i+1,n_2} \end{matrix} ; f\right] - \left[\begin{matrix} -1, \xi_{1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2} \end{matrix} ; f\right]$$
$$= (\eta_{i+1,n_2} - \eta_{i-1,n_2}) \left[\begin{matrix} -1, \xi_{1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} \end{matrix} ; f\right];$$
$$S^*_{0,j} - S^*_{0,j-1} = (\xi_{j+1,n_1} - \xi_{j-1,n_1}) \left[\begin{matrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ -1, \eta_{1,n_2} \end{matrix} ; f\right]$$

$$S^*_{i,j} - S^*_{i,j-1} - S^*_{i-1,j} + S^*_{i-1,j-1} = S^*_{i,j} - S^*_{i,j-1} - (S^*_{i-1,j} - S^*_{i-1,j-1})$$
$$= (\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})$$
$$\times \left[\begin{matrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} \end{matrix} ; f\right].$$

By the hypothesis on f and by $R''_{i,n_2}(y) \geq 0$, $R''_{j,n_1}(x) \geq 0$, $\forall x, y \in [-1, 1]$ (see the proof of Theorem 1.7.1, Case 1), we get

$$\frac{\partial^4 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_1-1} \sum_{i=1}^{n_2-1} (\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})$$
$$\times \left[\begin{matrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ \eta_{i-1,n_2}, \eta_{i+1,n_2}, \eta_{i,n_2} \end{matrix} ; f\right] R''_{i,n_2}(y) R''_{j,n_1}(x) \geq 0.$$

(iv) By construction we have (see Leviatan [228] or the proofs of Theorems 1.6.3 and 1.7.1, Case 1)

$$R'_{j,n_1} \geq 0, R_{j,n_1}(x) \geq 0, R''_{j,n_1}(x) \geq 0, \forall x \in [-1, 1], j = \overline{0, n_1},$$
$$R'_{i,n_2}(y) \geq 0, R_{i,n_2}(y) \geq 0, R''_{i,n_2}(y) \geq 0, \ \forall y \in [-1, 1], i = \overline{0, n_2}.$$

These inequalities combined with the hypothesis on f immediately give

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x^2} \geq 0, \frac{\partial^3 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y} \geq 0, \frac{\partial^4 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} \geq 0,$$
$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial y^2} \geq 0, \frac{\partial^3 P_{n_1,n_2}(f)(x,y)}{\partial y^2 \partial x} \geq 0, \forall x, y \in [-1, 1].$$

(v) First we need to prove that the polynomials $R_j(x)$, $j = 0, 1, \ldots,$ satisfy the differential inequality

$$R_j(x) - x R'_j(x) \geq 0, \ \forall x \in [-1, 0].$$

Indeed, writing $F(x) = R_j(x) - xR_j'(x)$ we have $F'(x) = -xR_j''$. Since $R_j''(x) \geq 0$ for al $x \in [-1,1]$, we get that F is increasing on $[-1,0]$ and decreasing on $[0,1]$.

Now, it suffices to prove that $F_j(-1) \geq 0$. We have

$$F_j(-1) = R_j(-1) + R_j'(-1) = R_j'(-1) = T_{n-j}(\arccos(-1)) = T_{n-j}(\pi) \geq 0.$$

Note here that in order to have $R_j(x) - xR_j'(x) \geq 0$, $\forall x \in [-1,1]$, (a fact suggested by intuition), it would be necessary and sufficient to prove that $F_j(1) = R_j(1) - R_j'(1) = 1 - \xi_j - R_j'(1) \geq 0$. For those j with $\xi_j \leq 0$, the above condition is clearly satisfied, because we easily see that by definition we have $0 \leq R_j'(1) \leq 1$. But in the present proof, the cases of j satisfying $\xi_j > 0$ remain unsettled (see Open Problem 2.7.9).

Continuing the proof, the polynomials $P_{n_1,n_2}(f)(x,y)$ can be written in the form

$$\begin{aligned}
P_{n_1,n_2}(f)(x,y) = f(-1,-1) &+ B(1+y) + \sum_{i=1}^{n_2-1} B_i R_{i,n_2}(y) \\
&+ A(1+x) + \sum_{j=1}^{n_1-1} A_j R_{j,n_1}(x) + (1+x)(1+y)S_{0,0}^* \\
&+ (1+x)\sum_{i=1}^{n_2-1}(S_{i,0}^* - S_{i-1,0}^*)R_{i,n_2}(y) + (1+y) \\
&\times \sum_{j=1}^{n_1-1}(S_{0,j}^* - S_{0,j-1}^*)R_{j,n_1}(x) \\
&+ \sum_{j=1}^{n_1-1}\sum_{i=1}^{n_2-1}(S_{i,j}^* - S_{i,j-1}^* - S_{i-1,j}^* + S_{i-1,j-1}^*)R_{i,n_2}(y) \\
&\times R_{j,n_1}(x),
\end{aligned}$$

where

$$\begin{aligned}
S_{i,j}^* &= \begin{bmatrix} \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ; f \\ \eta_{i,n_2}, \eta_{i+1,n_2} & \end{bmatrix}, \\
A &= \begin{bmatrix} -1 & \\ & ; f \\ -1, \eta_{1,n_2} & \end{bmatrix} B = \begin{bmatrix} -1, \xi_{1,n_1} & \\ & ; f \\ -1 & \end{bmatrix}, \\
B_i &= (\eta_{i+1,n_2} - \eta_{i-1,n_2}) \begin{bmatrix} -1 & \\ & ; f \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i,+1,n_2} & \end{bmatrix},
\end{aligned}$$

$$A_j = (\xi_{j+1,n_1} - \xi_{j-1,n_1}) \begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ; f \\ -1 & \end{bmatrix},$$

$$S^*_{i,0} - S^*_{i-1,0} = (\eta_{i+1,n_2} - \eta_{i-1,n_2}) \begin{bmatrix} -1, \xi_{1,n_1} & \\ & ; f \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} & \end{bmatrix},$$

$$S^*_{0,j} - S^*_{0,j-1} = (\xi_{j+1,n_1} - \xi_{j-1,n_1}) \begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ; f \\ -1, \eta_{1,n_2} & \end{bmatrix},$$

$$S^*_{i,j} - S^*_{i,j-1} - S^*_{i-1,j} + S^*_{i-1,j-1}$$
$$= (\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2}) \begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ; f \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} & \end{bmatrix}.$$

By a simple calculation we get

$$P_{n_1,n_2}(f)(x,y) - x\frac{\partial P_{n_1,n_2}(f)}{\partial x}(x,y) - y\frac{\partial P_{n_1,n_2}(f)}{\partial y}(x,y) + xy\frac{\partial^2 P_{n_1,n_2}(f)}{\partial x \partial y}(x,y)$$

$$\begin{aligned}
&= f(-1,-1) + S^*_{0,0} \\
&\quad + \begin{bmatrix} -1 & \\ & ; f \\ -1, \eta_{1,n_2} & \end{bmatrix} + \begin{bmatrix} -1, \xi_{1,n_1} & \\ & ; f \\ -1 & \end{bmatrix} \\
&\quad + \sum_{i=1}^{n_2-1} B_i [R_{i,n_2}(y) - yR'_{i,n_2}(y)] \\
&\quad + \sum_{j=1}^{n_1-1} A_j [R_{j,n_1}(x) - xR'_{j,n_1}(x)] \\
&\quad + \sum_{i=1}^{n_2-1} (S^*_{i,0} - S^*_{i-1,0}) [R_{i,n_2}(y) - yR'_{i,n_2}(y)] \\
&\quad + \sum_{j=1}^{n_1-1} (S^*_{0,j} - S^*_{0,j-1}) [R_{j,n_1}(x) - xR'_{j,n_1}(x)] \\
&\quad + \sum_{j=1}^{n_1-1} \sum_{i=1}^{n_2-1} (S^*_{i,j} - S^*_{i,j-1} - S^*_{i-1,j} + S^*_{i-1,j-1}) \\
&\quad \times [R_{i,n_2}(y) - yR'_{i,n_2}(y)][R_{j,n_1}(x) - xR'_{j,n_1}(x)] \geq 0,
\end{aligned}$$

by the condition that f be totally convex and since $f \in ST([-1,1] \times [-1,1])$, implies

$$f(-1,-1) + S^*_{0,0} + \begin{bmatrix} -1 & \\ & ; f \\ -1, \eta_{1,n_2} & \end{bmatrix} + \begin{bmatrix} -1, \xi_{1,n_1} & \\ & ; f \\ -1 & \end{bmatrix}$$
$$= \frac{\xi_{1,n_1}\eta_{1,n_2} f(-1,-1) + \xi_{1,n_1} f(-1,\eta_{1,n_2}) + \eta_{1,n_2} f(\xi_{1,n_1},-1) + f(\xi_{1,n_1},\eta_{1,n_2})}{(\xi_{1,n_1}+1)(\eta_{1,n_2}+1)},$$

an expression that is ≥ 0 by replacing $a_0 = b_0 = -1$ and $a_1 = \xi_{1,n_1}, b_1 = \eta_{1,n_2}$ in the definition of the class $ST([-1,1] \times [-1,1])$. The theorem is proved. □

Remarks. (1) The polynomials constructed by Theorem 2.6.1 do not preserve the usual monotonicity (as Theorem 2.5.1 (i), (ii) does), but still preserve the total monotonicity, which seems to be the most natural concept of bivariate monotonicity, because by Nicolescu [286], a totally monotone function has at most a countable numbers of points of discontinuity.

(2) The bivariate differential operator on the left-hand side of the differential inequality in Theorem 2.6.1 (v) in fact represents the "tensor product" of the univariate differential operator in Corollary 1.7.6. Indeed, applying it first with respect to x, we get the expression $F(x,y) = P(x,y) - x\frac{\partial P}{\partial x}$ and then applying it to F with respect to y we obtain

$$P(x,y) - x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y} + xy\frac{\partial^2 P}{\partial x \partial y}.$$

In what follows, we will restate Theorem 2.6.1 (i)–(iv) for the multivariate case, with a proof for the case of functions of three real variables, because the $m = 2$ case seems to be not representative for the general case $m \in \mathbb{N}$ (as, for example, would be the $m = 3$ case). Also, we will see that the proof of the general result requires much more intricate calculation than in the case of Theorem 2.6.1.

Theorem 2.6.2. (Gal–Gal [137]) *Suppose that the function $f : [-1,1]^m \to \mathbb{R}$, $m \geq 2$, is continuous. Then there exists a sequence of multivariate polynomials $\{P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)); n_1,\ldots,n_m \in \mathbb{N}\}$, where degree $(P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)) \leq n_k$ with respect to the kth variable, $k = \overline{1,m}$, such that*

$$\|f - P_{n_1,\ldots,n_m}(f)\| \leq C_m \omega_2^{\varphi}\left(f; \frac{1}{n_1}, \ldots, \frac{1}{n_m}\right), \quad \forall n_1,\ldots,n_m \in \mathbb{N},$$

where $C_m > 0$ is independent of f and n_i, $i = \overline{1,m}$ (here $\omega_2^{\varphi}(f;\delta_1,\ldots,\delta_m)$ is in Definition 2.1.2, (ii)) satisfying moreover the following shape-preserving properties:

(i) if f is convex of order $(1,\ldots,1)$ on $[-1,1]^m$, then so is $P_{n_1,\ldots,n_m}(f)$.

(ii) if f is simultaneously convex of orders $(s_1, \ldots, s_m) \in \{(s_1, \ldots, s_m);\ \ s_i \in \{0,1\},\ \ \forall i = \overline{1,m}$, *and* $\exists k$ *with* $s_k = 1\}$, *then so is* $P_{n_1,\ldots,n_m}(f)$.

(iii) if f is convex of order $(2, \ldots, 2)$ *on* $[-1,1]^m$ *then so is* $P_{n_1,\ldots,n_m}(f)$.

(iv) if f is simultaneously convex of orders $(s_1, \ldots, s_m) \in \{(s_1, \ldots, s_m);\ \ s_i \in \{0,1,2\},\ \ \forall i = \overline{1,m}$, *and* $\exists k$ *with* $s_k = 2\}$, *then so is* $P_{n_1,\ldots,n_m}(f)$.

Proof. If $h : [-1,1] \to \mathbb{R}$, then according to Leviatan [228], relation (5) (see also the proof of Theorem 1.6.3, Case 1), the approximation polynomials are given by

$$P_n(h)(x) = h(-1) + \sum_{j=0}^{n-1} s_{j,n}(R_{j,n}(x) - R_{j+1,n}(x)),$$

where $s_{j,n} = \frac{h(\xi_{j+1,n}) - h(\xi_{j,n})}{\xi_{j+1,n} - \xi_{j,n}}$, $\xi_{j,n}$, $j = \overline{0,n}$, are suitable nodes in $[-1,1]$ and $R_{j,n}(x)$ are suitable chosen polynomials of degree $\leq n$.

According to Leviatan [228], Theorem 1 (see also the proof of Theorem 1.6.3, Case 1), we have

$$\|h - P_n(h)\| \leq C\omega_2^{\varphi}\left(h; \frac{1}{n}\right)_{\infty}, \quad \forall n \in \mathbb{N},$$

where $C > 0$ is independent of h and n.

We will construct the polynomials $P_{n_1,\ldots,n_m}(f)(x_1, \ldots, x_m)$ by applying the tensor product method (see e.g. Nürnberger [290], p. 195–296). By mathematical induction we get

$$\begin{aligned}
&P_{n_1,\ldots,n_m}(f)(x_1, \ldots, x_m) \\
&\quad = f(-1, \ldots, -1) + \sum_{k=1}^{m} \left\{ \sum_{i_k=0}^{n_k-1} [\cdot; f]_k \cdot [R_{i_k,n_k}(x_k)(x_k) - R_{i_k+1,n_k}(x_k)] \right\} \\
&\qquad + \sum_{\substack{k,j=1 \\ k<j}}^{m} \left\{ \sum_{i_k=0}^{n_k-1} \sum_{i_j=0}^{n_j-1} [\cdot; f]_{j,k} \cdot [R_{i_k,n_k}(x_k) - R_{i_k+1,n_k}(x_k)] \right. \\
&\qquad\qquad \left. \times [R_{i_j,n_j}(x_j) - R_{i_j+1,n_j}(x_j)] \right\} + \cdots \\
&\qquad + \sum_{\substack{p_1,\ldots,p_{m-1}=1 \\ p_1<\cdots<p_{m-1}}}^{m} \left\{ \sum_{i_{p_1}=0}^{n_{p_1}-1} \cdots \sum_{i_{p_{m-1}}=0}^{n_{p_{m-1}}-1} [\cdot; f]_{p_1,\ldots,p_{m-1}} \right. \\
&\qquad\qquad \left. \times \prod_{s=1}^{m-1} [R_{i_{p_s},n_{p_s}}(x_{p_s}) - R_{i_{p_s}+1,n_{p_s}}(x_{p_s})] \right\} \\
&\qquad + \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_m=0}^{n_m-1} [\cdot; f]_{1,\ldots,m} \cdot \prod_{k=1}^{m} [R_{i_k,n_k}(x_k) - R_{i_k+1,n_k}(x_k)],
\end{aligned}$$

where $\xi^{(k)}_{i_k,n_k}$, $R_{i_k,n_k}(x_k)$, $i_k = \overline{0, n_k}$, $k = 1, 2, \ldots, m$, are constructed as in the univariate case in Leviatan [228] (see also the proof of Theorem 1.6.3, Case 1), the value of f at the point $(-1, \ldots, -1)$ by definition can be written as a divided difference by

$$f(-1,\ldots,-1) = \begin{bmatrix} -1 & \\ & ; f \\ \vdots & \\ -1 & \end{bmatrix},$$

where the notation in the right-hand side denotes the divided difference of f with m lines (see Definition 2.1) with -1 on each line, then $[\cdot; f]_k$ denotes the divided difference obtained from the above, by replacing the line k (which has only one node, -1), with another one composed by the two nodes $\xi^{(k)}_{i_k,n_k}$ and $\xi^{(k)}_{i_k+1,n_k}$ (the rest of the lines remaining unchanged), then $[\cdot; f]_{k,j}$, $k \neq j$ denotes the divided difference obtained (from the same divided difference that represents $f(-1, \ldots, -1)$) by replacing the lines k and j (which have only the node -1) with lines composed by the pairs of nodes $\xi^{(k)}_{i_k,n_k}, \xi^{(k)}_{i_k+1,n_k}$ and $\xi^{(j)}_{i_j,n_j}, \xi^{(j)}_{i_j+1,n_j}$, respectively, and so on.

Note that finally,

$$[\cdot; f]_{1,\ldots,m} = \begin{bmatrix} \xi^{(1)}_{i_1,n_1}, \xi^{(1)}_{i_1+1,n_1} & \\ & ; f \\ \vdots & \\ \xi^{(m)}_{i_m,n_m}, \xi^{(m)}_{i_m+1,n_m} & \end{bmatrix},$$

i.e., it is a divided difference with m lines, having two nodes on each line.

Obviously, degree $(P_{n_1,\ldots,n_m}(f)) \leq n_k$ with respect to the kth variable, $k = \overline{1, m}$.

First we prove the estimate in the theorem. For any univariate function h, we have (as in the proof of Theorem 2.6.1)

$$\|P_n(h)\| \leq \|h\| + \|P_n(h) - h\| \leq \|h\| + C\omega_2^{\varphi}\left(h; \frac{1}{n}\right)_{\infty} \leq (1 + 2C)\|h\|,$$

that is, passing to the supremum with $\|h\| \leq 1$, for the linear operator P_n we obtain

$$|||P_n||| \leq (1 + 2C), \quad \forall n \in \mathbb{N},$$

where $c > 0$ is independent of n.

Applying now Theorem 5 in Haussmann–Pottinger [167], we immediately get

$$\|f - P_{n_1,\ldots,n_m}(f)\| \leq C \sum_{i=1}^{m} \omega^{\varphi}_{2,x_i}\left(f; \frac{1}{n_i}\right),$$

where $\omega_{2,x_i}^{\varphi}(f;\delta_i)$, $i=\overline{1,m}$, are the partial moduli of smoothness defined by

$$\omega_{2,x_i}^{\varphi}(f;\delta_i)=\sup\{|\Delta_{h_i\varphi(x_i)}^{2,x_i}f(x_1,\ldots,x_m)|;0\le h_i\le\delta_i,x_i,\ldots,x_m\in[-1,1]\},$$

where $\varphi(t)=\sqrt{1-t^2}$,

$$\Delta_{h_i\varphi(x_i)}^{2,x_i}f(x_1,\ldots,x_m)$$
$$=\sum_{k=0}^{2}\binom{2}{k}(-1)^k f(x_1,\ldots,x_{i-1},x_i+(1-k)h_i\varphi(x_i),x_{i+1},\ldots,x_m)$$

if $x_1,\ldots,x_{i-1},x_i\pm h_i\varphi(x_i),x_{i+1},\ldots,x_m\in[-1,1]$,

$$\Delta_{h_i\varphi(x_i)}^{2,x_i}f(x_1,\ldots,x_m)=0,\ \text{elsewhere.}$$

Taking into account that obviously

$$\sum_{i=1}^{m}\omega_{2,x_i}^{\varphi}\left(f;\frac{1}{n_i}\right)\le m\omega_2^{\varphi}\left(f;\frac{1}{n_1},\ldots,\frac{1}{n_m}\right),$$

we obtain the desired estimate.

In what follows we will prove the shape-preserving properties.

(i) Suppose f is convex of order $(1,\ldots,1)$. We have to prove that

$$\frac{\partial^m P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)}{\partial x_1\ldots\partial x_m}\ge 0 \text{ on } [-1,1]^m.$$

We get

$$\frac{\partial^m P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)}{\partial x_1\ldots\partial x_m}$$
$$=\sum_{i_1=0}^{n_1-1}\cdots\sum_{i_m=0}^{n_m-1}[\cdot;f]_{1,\ldots,m}\prod_{k=1}^{m}\left[\frac{\partial R_{i_k,n_k}(x_k)}{\partial x_k}-\frac{\partial R_{i_k+1,n_k}(x_k)}{\partial x_k}\right]\ge 0,$$

because $[\cdot;f]_{1,\ldots,m}\ge 0$ (f is convex of order $(0,\ldots,0)$) and from the univariate case each $R_{i_k,n_k}(x_k)-R_{i_k+1,n_k}(x_k), k=\overline{1,m}$ is increasing as a function of $x_k\in[-1,1]$.

(ii) By the hypothesis on f, it follows that all the quantities

$$[\cdot,f]_k,\ [\cdot,f]_{k,j},\ldots,[\cdot;f]_{p_1,\ldots,p_{m-1}},\ [\cdot;f]_{1,\ldots,m}$$

in the expression of $P_{n_1,\ldots,n_m}(f)$ are ≥ 0.

By $R_{i_k,n_k}(-1)=0$, $\forall i_k=\overline{0,n_k}$, $k=\overline{1,m}$, we immediately get

$$R_{i_k,n_k}(x_k)-R_{i_k+1,n_k}(x_k)\ge 0,\ \forall i_k=\overline{0,n_k-1},\ x_k\in[-1,1],\ k=\overline{1,m}.$$

Also, from the univariate case, we have

$$R'_{i_k,n_k}(x_k) - R'_{i_k+1,n_k}(x_k) \geq 0,\ \forall i_k = \overline{0, n_k - 1},\ x_k \in [-1,1],\ k = \overline{1,m}.$$

Let $(s_i, \ldots, s_m) \in \{(s_1, \ldots, s_m);\ s_i \in \{0,1\},\ \forall i=\overline{1,m},\quad \exists\quad k \text{ with } s_k=1\}$. The above hypothesis and simple calculations (similar to those in the bivariate case, see the proof of Theorem 2.6.1 (ii)), immediately implies that $P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)$ is convex of order $(s_1,\ldots,s_m)$, which proves (ii).

(iii) We have to prove that

$$\frac{\partial^{2m} P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)}{\partial x_1^2 \ldots \partial x_m^2} \geq 0.$$

Applying with respect to each variable the relation in the univariate case (see relation (5) in Leviatan [228], p. 473, or also the proof of Theorem 1.6.3, Case 1), i.e.,

$$P_n(g)(x) = g(-1) + \sum_{j=0}^{n-1} s_{j,n}(R_{j,n}(x) - R_{j+1,n}(x)),$$

where

$$s_{j,n} = \frac{g(\xi_{j+1,n}) - g(\xi_{j,n})}{\xi_{j+1,n} - \xi_{j,n}},$$

we get

$$\begin{aligned} &P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m) = f(-1,\ldots,-1) \\ &\quad + \sum_{k=1}^{m}(1+x_k)C_k + F(x_1,\ldots,x_m) + E(x_1,\ldots,x_m) \\ &\quad + \sum_{i_1=1}^{n_1-1} \cdots \sum_{i_m=1}^{n_m-1} \left(\xi^{(1)}_{i_1+1,n_1} - \xi^{(1)}_{i_1,n_1}\right) \cdots \left(\xi^{(m)}_{i_m+1,n_m} - \xi^{(m)}_{i_m,n_m}\right) \left(\prod_{k=1}^{m} R_{i_k,n_k}(x_k)\right) \cdot \\ &\quad \times \begin{bmatrix} \xi^{(1)}_{i_1-1,n_1}, \xi^{(1)}_{i_1,n_1}, \xi^{(1)}_{i_1+1,n_1} & \\ & ; f \\ \vdots & \\ \xi^{(m)}_{i_m-1,n_m}, \xi^{(m)}_{i_m,n_m}, \xi^{(m)}_{i_m+1,n_m} & \end{bmatrix}, \end{aligned}$$

where $F(x_1,\ldots,x_m)$ is a sum of several expressions of the form $(1+x_{i_1})\cdots(1+x_{i_k})C$ with distinct indexes $i_1,\ldots i_k$, $k < m$, C a real constant (which can be different at each occurrence) and $E(x_1,\ldots,x_m)$ is a sum of several expressions, each expression being a simple or multiple sum of terms, where each term is represented by the product between:

(a) various $R_{i_k,n_k}(x_k)$, or product of distinct $R_{i_k,n_k}(x_k)$ (having at most $m-1$ terms in that product),
and

(b) a divided difference of f on one, two, or three nodes with respect to each variable x_k, such that at least with respect to one variable the divided difference is taken on three nodes,
and

(c) a positive quantity of the form: $\left(\xi^{(k)}_{i_k+1,n_k} - \xi^{(k)}_{i_k,n_k}\right)$ or $\left(\xi^{(k)}_{i_k+1,n_k} - \xi^{(k)}_{i_k-1,n_k}\right)$ or product of such distinct quantities.
Moreover, the above mentioned expressions in $E(x_1, \ldots, x_m)$, which depend on the variables $x_k, k = \overline{1,m}$ through $R_{i_k,n_k}(x_k)$, are of two kinds:

(d) expressions that do not depend on all variables $x_k, k = \overline{1,m}$;

(e) expressions that depend on all variables $x_k, k = \overline{1,m}$, but at least one $R_{i_k,n_k}(x_k) = 1 + x_k$.

Let us exemplify the passing from $m = 2$ to $m = 3$. Therefore, let f be a function of three variables, i.e., $f = f(x_1, x_2, x_3)$.

Applying the formula in the univariate case (specified at the beginning of (iii)) with respect to the variables x_1 and x_2, by the formulas in Gal [126], pp. 31–32 (see also the proof of Theorem 2.6.1), we immediately get

$$\begin{aligned}
P_{n_1,n_2}(f)(x_1,x_2,x_3) &= f(-1,-1,x_3) \\
&\quad + (1+x_2)\begin{bmatrix} -1 & \\ & ;f \\ \xi^{(2)}_{1,n_2}, \xi^{(2)}_{0,n_2} & \end{bmatrix} \\
&\quad + (1+x_1)\begin{bmatrix} \xi^{(1)}_{1,n_1}, \xi^{(1)}_{0,n_1} & \\ & ;f \\ -1 & \end{bmatrix} \\
+ \sum_{i_2=1}^{n_2-1} \left(\xi^{(2)}_{i_2+1,n_2} - \xi^{(2)}_{i_2-1,n_2}\right) &\begin{bmatrix} -1 & \\ & ;f \\ \xi^{(2)}_{i_2-1,n_2}, \xi^{(2)}_{i_2,n_2}, \xi^{(2)}_{i_2+1,n_2} & \end{bmatrix} R_{i_2,n_2}(x_2) \\
+ \sum_{i_1=1}^{n_1-1} \left(\xi^{(1)}_{i_1+1,n_1} - \xi^{(1)}_{i_1-1,n_1}\right) &\begin{bmatrix} \xi^{(1)}_{i_1-1,n_1}, \xi^{(1)}_{i_1,n_1}, \xi^{(1)}_{i_1+1,n_1} & \\ & ;f \\ -1 & \end{bmatrix} R_{i_1,n_1}(x_1) \\
&(1+x_1)(1+x_2)S^*_{0,0}(f)(x_3) \\
+ (1+x_1)\sum_{i_2=1}^{n_2-1} \left(\xi^{(2)}_{i_2+1,n_2} - \xi^{(2)}_{i_2-1,n_2}\right) &\begin{bmatrix} -1, \xi^{(1)}_{1,n_1} & \\ & ;f \\ \xi^{(2)}_{i_2-1,n_2}, \xi^{(2)}_{i_2,n_2}, \xi^{(2)}_{i_2+1,n_2} & \end{bmatrix} R_{i_2,n_2}(x_2) \\
+ (1+x_2)\sum_{i_1=1}^{n_1-1} \left(\xi^{(1)}_{i_1+1,n_1} - \xi^{(1)}_{i_1-1,n_1}\right) &\begin{bmatrix} \xi^{(1)}_{i_1-1,n_1}, \xi^{(1)}_{i_1,n_1}, \xi^{(1)}_{i_1+1,n_1} & \\ & ;f \\ -1, \xi^{(2)}_{1,n_2} & \end{bmatrix} R_{i_1,n_1}(x_1)
\end{aligned}$$

$$+\sum_{i_1=1}^{n_1-1}\sum_{i_2=1}^{n_2-1}\left(\xi^{(1)}_{i_1+1,n_1}-\xi^{(1)}_{i_1,n_1}\right)\left(\xi^{(2)}_{i_2+1,n_2}-\xi^{(2)}_{i_2,n_2}\right)\left[\begin{matrix}\xi^{(1)}_{i_1-1,n_1},\xi^{(1)}_{i_1,n_1},\xi^{(1)}_{i_1+1,n_1} & \\ & ;f \\ \xi^{(2)}_{i_2-1,n_2},\xi^{(2)}_{i_2,n_2},\xi^{(2)}_{i_2+1,n_2} & \end{matrix}\right]$$
$$\times R_{i_1,n_1}(x_1)R_{i_2,n_2}(x_2),$$

where all the divided differences are considered with respect to the variables x_1, x_2, and x_3 is fixed, arbitrary. Also, recall that the formula for $S^*_{0,0}(f)(x_3)$ is given by Gal [126], p. 31 (see also the proof of Theorem 2.6.1), and depends on the values of $f(.,.,x_3)$ on some nodes, where f is considered as a function of the variables x_1 and x_2.

Now, applying the formula in the univariate case with respect to x_3 to $P_{n_1,n_2}(f)(x_1,x_2,x_3)$, i.e., to each term of it and taking into account the recurrence formula satisfied by the divided differences, we immediately obtain $P_{n_1,n_2,n_3}(f)(x_1,x_2,x_3)$ of the claimed form.

As a conclusion, all these immediately imply

$$\frac{\partial^{2m}P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)}{\partial x_1^2\ldots\partial x_m^2}$$
$$=\sum_{i_1=1}^{n_1-1}\cdots\sum_{i_m=1}^{n_m-1}\left(\prod_{k=1}^{m}\left(\xi^{(k)}_{i_k+1,n_k}-\xi^{(k)}_{i_k,n_k}\right)\right)\left(\prod_{k=1}^{m}R''_{i_k,n_k}(x_k)\right)$$
$$\times\left[\begin{matrix}\xi^{(1)}_{i_1-1,n_1},\xi^{(1)}_{i_1,n_1},\xi^{(1)}_{i_1+1,n_1} & \\ & ;f \\ \vdots & \\ \xi^{(m)}_{i_m-1,n_m},\xi^{(m)}_{i_m,n_m},\xi^{(m)}_{i_m+1,n_m} & \end{matrix}\right]\geq 0,$$

by the hypothesis on f and by the conditions $R''_{i_k,n_k}(x_k)\geq 0,\quad \forall i_k=\overline{0,n_k},\quad x_k\in[-1,1],\quad k=\overline{1,m}$ (see Leviatan [228], or also the proof of Theorem 1.7.1, Case 1).

(iv) First, let us recall that by construction we have (see Leviatan [228], or also the proof of Theorem 1.7.1, Case 1)

$$R_{i_k,n_k}(x_k)\geq 0,\quad R'_{i_k,n_k}(x_k)\geq,\quad R''_{i_k,n_k}(x_k)\geq 0,$$
$$\forall i_k=\overline{0,n_k},\quad x_k\in[-1,1],\quad k=\overline{1,m}.$$

We have to check the inequalities

$$\frac{\partial^{r}P_{n_1,\ldots,n_m}(f)(x_1,\ldots,x_m)}{\partial x_{i_1}^{r_1}\ldots\partial x_{i_p}^{r_p}}\geq 0 \text{ on } [-1,1]^m,$$

for all $r\in\{2,\ldots,m\},\quad p\in\{1,\ldots,m\},\quad i_k\neq i_j$ if $i\neq j,\quad r=r_1+\ldots+r_p$, where at least one r_l is equal to 2 and $r_k\in\{0,1,2\},\quad k=\overline{1,p}$.

By hypothesis, the divided differences of f that contains, at least on a line, three distinct points all are ≥ 0. Then, taking into account the forms of $F(x_1,\ldots,x_m)$ and $E(x_1,\ldots,x_m)$ described at the previous point (iii), we immediately obtain the required conclusion. □

Remarks. (1) For $m = 2$ we recapture Theorem 2.6.1 (see also Theorem 3.1 in Gal [126]).

(2) Since in the univariate case (i.e., $m = 1$), the property in Theorem 2.6.2 (i) reduces to the usual increasing monotonicity and in this case, according to Shvedov [355] (see Theorem 1.6.3 in Chapter 1) we know that $\omega_2^{\varphi}(f;.)$ cannot be replaced by higher-order moduli of smoothness $\omega_k^{\varphi}(f;.)$ with $k \geq 3$ (and with an universal constant), it follows that for arbitrary $m \geq 2$ the same phenomenon is expected, i.e., the estimate in Theorem 2.6.2 cannot be improved by higher-order moduli of smoothness with a constant C_m depending only on m.

With respect to the L^p-norm, $1 \leq p < +\infty$, we can prove, for example, the following.

Corollary 2.6.3. *Let us denote by $C^P([-1,1] \times [-1,1])$ the class of all bounded functions $f : [-1,1] \times [-1,1] \to \mathbb{R}$ such that f is (partially) continuous on $[-1,1]$ with respect to each variable x and y. Obviously*

$$
\begin{aligned}
&C^P([-1,1] \times [-1,1]) \subset L^p([-1,1] \times [-1,1]) \\
&= \{f; \|f\|_p := \left(\int_{-1}^{+1} \int_{-1}^{+1} |f(x,y)|^p dx\, dy \right)^{1/p} < +\infty\}, \quad 1 \leq p < +\infty
\end{aligned}
$$

(here the letter P is only a notation and is not to be confused with the number p). If $f \in C^P([-1,1] \times [-1,1])$, then there exists a sequence of bivariate polynomials $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2 \in \mathbb{N}}$, of degrees $\leq n_1$ with respect to x and $\leq n_2$ with respect to y such that

$$\|f - P_{n_1,n_2}(f)\|_p \leq C\omega_2^{\varphi}\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)_p,$$

$\forall\, n_1, n_2 \in \mathbb{N}$, $C > 0$ independent of f, n_1 and n_2 (here $\omega_2^{\varphi}(f;t,s)_p$ is defined by Definition 2.1.2, (ii)), preserving the convexity of order $(1,1)$ in the sense of Popoviciu.

Proof. We will use the tensor product method for the construction in the univariate case in Leviatan–Yu [244] (see also Yu [407]), related to that in the uniform approximation case in the proof of Theorem 2.6.1. Keeping the notation for ξ_k and $R_k(x)$ in the univariate case in Leviatan [228] and for $g : [-1,1] \to \mathbb{R}$ denoting

$$
\begin{aligned}
&\overline{S}_j(g) := \frac{\overline{g}(\xi_{j+1}) - \overline{g}(\xi_j)}{\xi_{j+1} - \xi_j}, \quad j = 1, \ldots, n-2, \\
&\overline{S}_0(g) := \overline{S}_1(g), \quad \overline{S}_{n-1}(g) := \overline{S}_{n-2}(g), \\
&\overline{g}(\xi_j) := \frac{1}{\alpha_j} \int_{-\alpha_j/2}^{\alpha_j/2} g(\xi_j + t)dt, \alpha_j = \min\{\xi_j - \xi_{j-1}, \xi_{j+1} - \xi_j\}, \\
&\qquad j = 1, \ldots, n-1, \\
&\overline{g}(-1) := \overline{g}(\xi_1) - (1 + \xi_1)\overline{S}_1(g), \quad \overline{g}(1) := \overline{g}(\xi_{n-1}) + (1 - \xi_{n-1})\overline{S}_{n-2}(g),
\end{aligned}
$$

recall that the polynomials in Leviatan–Yu [244] are given by

$$P_n(g)(x) = \overline{g}(-1) + \sum_{j=0}^{n-1} \overline{S}_j(g)[R_j(x) - R_{j+1}(x)]$$
$$= \overline{g}(-1) + \overline{S}_0(g)R_0(x) + \sum_{j=1}^{n-1}[\overline{S}_j(g) - \overline{S}_{j-1}(g)]R_j(x)$$
$$= \overline{g}(-1) + \overline{S}_0(g)R_0(x) + \sum_{j=2}^{n-2}[\overline{S}_j(g) - \overline{S}_{j-1}(g)]R_j(x).$$

Also, recall that for all j, $R_j(x) - R_{j+1}(x)$ are increasing, $R_j \geq 0$, $R_j' \geq 0$ on $[-1, 1]$, and $R_j'' \geq 0$ on $[-1, 1]$ (see Yu [407] and Leviatan [228]).

First let us consider that $f \in C^P([-1, 1] \times [-1, 1])$ is convex of order $(1, 1)$ in the Popoviciu sense. For the construction of the bivariate tensor product polynomial we use the first form of the above univariate polynomial P_n. Since for the tensor product polynomial denoted by $P_{n_1,n_2}(f)(x, y)$, we have to check that $\frac{\partial^2 P_{n_1,n_2}(f)}{\partial x \partial y} \geq 0$ on $[-1, 1] \times [-1, 1]$, we see that for this purpose, the term $\overline{g}(-1)$ in the univariate case can be omitted from the tensor product, so that if $\{\xi_{j,n_1}\}$, $R_{j,n_1}(x)$, $j = \overline{0, n_1}$, $\{\eta_{i,n_2}\}$, $R_{i,n_2}(y)$, $i = \overline{0, n_2}$, are constructed as in the univariate case in Leviatan [228], then reasoning as in the proof of Theorem 2.6.1 (i), we get

$$\frac{\partial^2 P_{n_1,n_2}(f)(x, y)}{\partial x \partial y}$$
$$= \sum_{j=0}^{n_1-1} \sum_{i=0}^{n_2-1} \overline{S}_{i,j}[R_{i,n_2}'(y) - R_{i+1,n_2}'(y)][R_{j,n_1}'(x) - R_{j+1,n_1}'(x)],$$

where

$$\overline{S}_{i,j} = \frac{\overline{f}(\xi_{j+1,n_1}, \eta_{i+1,n_2}) - \overline{f}(\xi_{j,n_1}, \eta_{i+1,n_2}) - \overline{f}(\xi_{j+1,n_1}, \eta_{i,n_2}) + \overline{f}(\xi_{j,n_1}, \eta_{i,n_2})}{(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})},$$

and

$$\overline{f}(\xi_{j,n_1}, \eta_{i,n_2}) = \frac{1}{\alpha_{j,n_1}\beta_{i,n_2}} \int_{-\alpha_{j,n_1}/2}^{\alpha_{j,n_1}/2} \int_{-\beta_{i,n_2}/2}^{\beta_{i,n_2}/2} f(\xi_{j,n_1} + t, \eta_{i,n_2} + s) dt\, ds,$$

$\alpha_{j,n_1} = \min\{\xi_{j,n_1} - \xi_{j-1,n_1},\ \ \xi_{j+1,n_1} - \xi_{j,n_1}\}$, $j = 1, \ldots, n_1 - 1$, $\beta_{i,n_2} = \min\{\eta_{i,n_2} - \eta_{i-1,n_2},\ \ \eta_{i+1,n_2} - \eta_{i,n_2}\}$, $i = 1, \ldots, n_2 - 1$. Now suppose that for the arbitrary intervals denoted by $[A, B]$, $[C, D]$, $[a, b]$, $[c, d]$, the intersections $[A, B] \cap [a, b]$ and $[C, D] \cap [c, d]$ are empty or have at most one element. We will prove the following auxiliary result : there exist $\xi_M \in [A, B]$, $\xi_m \in [a, b]$, $\eta_M \in [C, D]$, $\eta_m \in [c, d]$, such that

$$\begin{aligned} I := &\frac{1}{(B-A)(D-C)} \int_A^B \int_C^D f(x,y)dx\,dy \\ &- \frac{1}{(b-a)(D-C)} \int_a^b \int_C^D f(x,y)dx\,dy \\ &- \frac{1}{(B-A)(d-c)} \int_A^B \int_c^d f(x,y)dx\,dy \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dx\,dy \\ = &f(\xi_M, \eta_M) - f(\xi_m, \eta_M) - f(\xi_M, \eta_m) + f(\xi_m, \eta_m). \end{aligned}$$

Indeed, we can write

$$I = \frac{1}{D-C} \int_C^D F(y)dy - \frac{1}{d-c} \int_c^d F(y)dy = F(\eta_M) - F(\eta_m),$$

where $F(y) = \frac{1}{B-A} \int_A^B f(x,y)dx - \frac{1}{b-a} \int_a^b f(x,y)dx$ is continuous with respect to $y \in [-1,1]$ and the points $\eta_M \in [C,D]$, $\eta_m \in [c,d]$ are given by the integral mean value theorem.

Therefore, applying once again the integral mean value theorem for continuous functions (with respect to x), there exist $\xi_M \in [A,B]$, $\xi_m \in [a,b]$, such that

$$\begin{aligned} I = &\left[\frac{1}{B-A} \int_A^B f(x,\eta_M)dx - \frac{1}{b-a} \int_a^b f(x,\eta_M)dx \right] \\ &- \left[\frac{1}{B-A} \int_A^B f(x,\eta_m)dx - \frac{1}{b-a} \int_a^b f(x,\eta_m)dx \right] \\ = &\frac{1}{B-A} \int_A^B [f(x,\eta_M) - f(x,\eta_m)]dx - \frac{1}{b-a} \int_a^b [f(x,\eta_M) - f(x,\eta_m)]dx \\ = &f(\xi_M, \eta_M) - f(\xi_m, \eta_M) - f(\xi_M, \eta_m) + f(\xi_m, \eta_m). \end{aligned}$$

Applying now this auxiliary result to the numerator in the expression of $\overline{S}_{i,j}$, there exist $\xi'_{j,n_1} < \xi'_{j+1,n_1}$, $\eta'_{i,n_2} < \eta'_{i+1,n_1}$ such that

$$\begin{aligned} &\overline{f}(\xi_{j+1,n_1}, \eta_{i+1,n_2}) - \overline{f}(\xi_{j,n_1}, \eta_{i+1,n_2}) - \overline{f}(\xi_{j+1,n_1}, \eta_{i,n_2}) + \overline{f}(\xi_{j,n_1}, \eta_{i,n_2}) \\ &= f(\xi'_{j+1,n_1}, \eta'_{i+1,n_2}) - f(\xi'_{j,n_1}, \eta'_{i+1,n_2}) - f(\xi'_{j+1,n_1}, \eta'_{i,n_2}) + f(\xi'_{j,n_1}, \eta'_{i,n_2}) \geq 0, \end{aligned}$$

from the hypothesis on f.

As a conclusion, it follows that $\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x \partial y} \geq 0$ on $[-1,1] \times [-1,1]$.

Taking into account the estimate in the univariate case in Leviatan–Yu [244] and reasoning as in the proof of Theorem 2.6.1, i.e., applying Theorem 5 in Haussmann–Pottinger [167], we immediately get

$$\|f - P_{n_1,n_2}(f)\|_p \leq C \left[\omega_{2,x}^{\varphi}\left(f; \frac{1}{n_1}\right)_p + \omega_{2,y}^{\varphi}\left(f; \frac{1}{n_2}\right)_p \right],$$

where $\omega_{2,x}^{\varphi}(f;\delta)_p$ and $\omega_{2,y}^{\varphi}(f;\delta)_p$ are the partial moduli defined in Ditzian–Totik [98], Chapter 12. Since we obviously have

$$\omega_{2,x}^{\varphi}\left(f; \frac{1}{n_2}\right)_p + \omega_{2,y}^{\varphi}\left(f; \frac{1}{n_2}\right)_p \leq 2\omega_2^{\varphi}\left(f; \frac{1}{n_1}, \frac{1}{n_2}\right)_p,$$

we obtain the desired estimate, which proves the corollary. □

Remarks. (1) The convexity of order $(1,1)$ of f does not necessarily imply that we have $f \in C^P([-1,1] \times [-1,1])$. Indeed, for example, according to Definition 2.1.1 (v), the $(1,1)$ convexity is equivalent to

$$[x_1, x_2; [y_1, y_2; f]_y]_x \geq 0$$

for all $-1 \leq x_1 < x_2 \leq 1$, $-1 \leq y_1 < y_2 \leq 1$. It is easy to see that if we define $F(x,y) = f(x,y) + \varphi(x) + \psi(y)$, where the functions $\varphi(x)$ and $\psi(y)$ are discontinuous at each point in $[-1,1]$, then F is discontinuous at each point in $[-1,1]$ but

$$[x_1, x_2; [y_1, y_2; F]_y]_x \geq 0$$

for all $-1 \leq x_1 < x_2 \leq 1$, $-1 \leq y_1 < y_2 \leq 1$.

(2) Since in the univariate case, the property in Corollary 2.6.3 reduces to the usual increasing monotonicity, and in this case according to Shvedov [355] (see Theorem 1.6.3 in Chapter 1), we know that $\omega_2^{\varphi}(f;.)_p$ cannot be replaced by higher-order moduli of smoothness $\omega_k^{\varphi}(f;.)_p$ with $k \geq 3$ (and with a constant in front of ω_k^{φ} independent of f), then in Corollary 2.6.3 is the same phenomenon expected.

2.6.2 Concepts in Bivariate Coshape Approximation

The tensor-product method can be applied in order to obtain new results in bivariate copositive, comonotone, and coconvex approximations. For that purpose, first we need suitable concepts of copositivity, comonotonicity, and coconvexity on grids for bivariate functions, which can be defined as follows.

Definition 2.6.4. Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$.

(i) We say that f is bivariate copositive with g if $f(x,y)g(x,y) \geq 0$ for all $(x,y) \in [a,b] \times [c,d]$.

Let $f \in C([a,b] \times [c,d])$, $a < x_1 < \ldots < x_k < b$ and $c < y_1 < \ldots < y_s < d$. One says that f changes sign on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i,\ i \in \{1,\ldots,k\}, y = y_j,\ j \in \{1,\ldots,s\}$, if

$$f(x,y) \cdot \Pi_{i=1}^{k}(x - x_i) \cdot \Pi_{j=1}^{s}(y - y_j) \geq 0,\ \forall (x,y) \in (a,b) \times (c,d)$$

or

$$f(x,y)\Pi_{i=1}^{k}(x-x_i)\Pi_{j=1}^{s}(y-y_j) \le 0,\ \forall (x,y) \in (a,b) \times (c,d).$$

One says that f changes sign on the degenerated grid in $(a,b) \times (c,d)$, determined by the segments parallel to the OY axis, $x = x_i, i = 1, \dots, k$, if

$$f(x,y) \cdot \Pi_{i=1}^{k}(x-x_i) \ge 0, \forall (x,y) \in (a,b) \times (c,d)$$

or

$$f(x,y)\Pi_{i=1}^{k}(x-x_i) \le 0, \forall (x,y) \in (a,b) \times (c,d).$$

Also, one says that f changes its sign on the degenerated grid in $(a,b)\times(c,d)$, determined by the segments parallel to the OX axis, $y = y_j, j = 1, \dots, s$, if

$$f(x,y) \cdot \Pi_{j=1}^{s}(y-y_j) \ge 0,\ \forall (x,y) \in (a,b) \times (c,d)$$

or

$$f(x,y)\Pi_{j=1}^{s}(y-y_j) \le 0,\ \forall (x,y) \in (a,b) \times (c,d);$$

(ii) We say that f is upper bidimensional comonotone with g if for all points $a \le x_1 < x_2 \le b$ and $c \le y_1 < y_2 \le d$, we have

$$\begin{bmatrix} x_1, x_2 & \\ & ; f \\ y_1, y_2 & \end{bmatrix} \cdot \begin{bmatrix} x_1, x_2 & \\ & ; g \\ y_1, y_2 & \end{bmatrix} \ge 0.$$

Equivalently, if f and g are continuous, then we say that f is upper bidimensional comonotone with g if $\Delta_{h,k}^{1,1}(f)(x,y) \cdot \Delta_{h,k}^{1,1}(g)(x,y) \ge 0$ for all $x, x+h \in [a,b]$, $y, y+k \in [c,d]$, $h,k \ge 0$, where $\Delta_{h,k}^{1,1}(f)(x,y) = f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y)$.

If, in addition, f and g are twice continuously differentiable on $[a,b]\times[c,d]$, then the upper bidimensional comonotonicity can be written by the condition $\frac{\partial^2 f}{\partial x \partial y}(x,y) \cdot \frac{\partial^2 g}{\partial x \partial y}(x,y) \ge 0$ for all $(x,y) \in [a,b] \times [c,d]$.

If f is twice continuously differentiable on $[a,b] \times [c,d]$, then we say that f changes the upper bidimensional monotonicity on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i,\ i \in \{1, \dots, k\},\ y{=}y_j$, $j \in \{1, \dots, s\}$, if

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) \cdot \Pi_{i=1}^{k}(x-x_i) \cdot \Pi_{j=1}^{s}(y-y_j) \ge 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) \cdot \Pi_{i=1}^{k}(x-x_i) \cdot \Pi_{j=1}^{s}(y-y_j) \le 0, \quad \forall (x,y) \in (a,b) \times (c,d).$$

Also, if f is twice continuously differentiable on $[a,b] \times [c,d]$, then one says that f changes the upper bidimensional monotonicity on the degenerate grid in $(a,b)\times(c,d)$ determined by the segments parallel to the OY axis $x=x_i, i=1,\dots,k$ (or determined by the segments parallel to the OX axis, $y=y_j,\ j=1,\dots,s$, respectively) if

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\cdot \Pi_{i=1}^{k}(x-x_i)\geq 0,\quad \forall(x,y)\in(a,b)\times(c,d),$$

or

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\cdot \Pi_{i=1}^{k}(x-x_i)\leq 0,\quad \forall(x,y)\in(a,b)\times(c,d)$$

(or respectively

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\Pi_{j=1}^{s}(y-y_j)\geq 0,\quad \forall(x,y)\in(a,b)\times(c,d),$$

or

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\Pi_{j=1}^{s}(y-y_j)\leq 0,\quad \forall(x,y)\in(a,b)\times(c,d)).$$

(iii) We say that f is totally upper comonotone with g if for all points $a\leq x_1<x_2\leq b$ and $c\leq y_1<y_2\leq d$, we have

$$\begin{bmatrix} x_1,x_2 & \\ & ;f \\ y_1,y_2 & \end{bmatrix}\cdot\begin{bmatrix} x_1,x_2 & \\ & ;g \\ y_1,y_2 & \end{bmatrix}\geq 0,$$

$$\begin{bmatrix} x_1 & \\ & ;f \\ y_1,y_2 & \end{bmatrix}\cdot\begin{bmatrix} x_1 & \\ & ;g \\ y_1,y_2 & \end{bmatrix}\geq 0,$$

and

$$\begin{bmatrix} x_1,x_2 & \\ & ;f \\ y_1 & \end{bmatrix}\cdot\begin{bmatrix} x_1,x_2 & \\ & ;g \\ y_1 & \end{bmatrix}\geq 0.$$

If f and g are twice differentiable on $[a,b]\times[c,d]$, then the above conditions can be replaced by

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)\cdot\frac{\partial^2 g}{\partial x\partial y}(x,y)\geq 0,\quad \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}(x,y)\geq 0,\quad \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}(x,y)\geq 0,$$

for all $(x,y)\in[a,b]\times[c,d]$.

(iv) We say that f is coconvex of order $(2,2)$ (in the Popoviciu sense) with g if for all $a\leq x_1<x_2<x_3\leq b$ and $c\leq y_1<y_2<y_3\leq d$, we have

$$\begin{bmatrix} x_1, x_2, x_3 & \\ & ; f \\ y_1, y_2, y_3 & \end{bmatrix} \cdot \begin{bmatrix} x_1, x_2, x_3 & \\ & ; g \\ y_1, y_2, y_3 & \end{bmatrix} \geq 0.$$

Equivalently, for continuous f and g, one says that f is coconvex of order $(2,2)$ with g if $\Delta_{h,k}^{1,1}[\Delta_{h,k}^{1,1}(f)(x,y)] \cdot \Delta_{h,k}^{1,1}[\Delta_{h,k}^{1,1}(g)(x,y)] \geq 0$, for all $h, k \geq 0$ and $(x,y) \in][a,b] \times [c,d]$ such that $\Delta_{h,k}^{1,1}[\Delta_{h,k}^{1,1}(f)(x,y)]$ is defined.

If f and g have all the partial derivatives of order 4, then the above condition can be written by

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \frac{\partial^4 g}{\partial x^2 \partial y^2}(x,y) \geq 0$$

for all $(x,y) \in [a,b] \times [c,d]$.

Let f have all its partial derivatives of order 4 continuous on $[a,b] \times [c,d]$.

One says that f changes its convexity of order $(2,2)$ on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i,\ i \in \{1,\ldots,k\},\ y = y_j,\ j \in \{1,\ldots,s\}$ if

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{i=1}^{k}(x - x_i) \cdot \Pi_{j=1}^{s}(y - y_j) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{i=1}^{k}(x - x_i) \cdot \Pi_{j=1}^{s}(y - y_j) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d).$$

Also, one says that f changes its convexity of order $(2,2)$ on the degenerate grid in $(a,b) \times (c,d)$ determined by the segments parallel to the OY axis, $x = x_i,\ i = 1,\ldots,k$ (or determined by the segments parallel to the OX axis, $y = y_j,\ j = 1,\ldots,s$, respectively) if

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{i=1}^{k}(x - x_i) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{i=1}^{k}(x - x_i) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d)$$

(or respectively

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{j=1}^{s}(y - y_j) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \Pi_{j=1}^{s}(y - y_j) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d)).$$

(v) If f and g have all the partial derivatives of order 4, then we say that f is totally coconvex of order $(2,2)$ with g if

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \cdot \frac{\partial^4 g}{\partial x^2 \partial y^2}(x,y) \geq 0, \quad \frac{\partial^3 f}{\partial x^2 \partial y}(x,y) \cdot \frac{\partial^3 g}{\partial x^2 \partial y}(x,y) \geq 0,$$
$$\frac{\partial^3 f}{\partial x \partial y^2}(x,y) \cdot \frac{\partial^3 g}{\partial x \partial y^2}(x,y) \geq 0,$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 g}{\partial x^2}(x,y) \geq 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y) \cdot \frac{\partial^2 g}{\partial y^2}(x,y) \geq 0,$$

for all $(x,y) \in [a,b] \times [c,d]$.

(vi) According to Popoviciu [315] (see p. 84, the expression above relationship (92)), it follows that $f : [a,b] \times [c,d] \to \mathbb{R}$ is usually convex if for all points $a \leq x_1 < x_2 < x_3 \leq b$ and $c \leq y_1 < y_2 < y_3 \leq d$, we have $D(f) \geq 0$ and $E(f) \geq 0$, where

$$D(f) = \left[\begin{matrix} x_1, x_2, x_3 & \\ & ; f \\ y_1 & \end{matrix}\right]$$

and

$$E(f) = 4 \cdot \left[\begin{matrix} x_1, x_2, x_3 & \\ & ; f \\ y_1 & \end{matrix}\right] \cdot \left[\begin{matrix} x_3 & \\ & ; f \\ y_1, y_2, y_3 & \end{matrix}\right] - \left[\begin{matrix} x_2, x_3 & \\ & ; f \\ y_1, y_2 & \end{matrix}\right] \cdot \left[\begin{matrix} x_2, x_3 & \\ & ; f \\ y_1, y_2 & \end{matrix}\right].$$

Then f and g are called usually coconvex if $D(f) \cdot D(g) \geq 0$ and $E(f) \geq 0$, $E(g) \geq 0$, for all points $a \leq x_1 < x_2 < x_3 \leq b$ and $c \leq y_1 < y_2 < y_3 \leq d$.

In the presence of corresponding partial derivatives, the above conditions can be replaced by $F(f)(x,y)F(g)(x,y) \geq 0$ and $G(f)(x,y) \geq 0$, $G(g)(x,y) \geq 0$ for all $(x,y) \in [a,b] \times [c,d]$, where

$$F(f)(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y)$$

and

$$G(f)(x,y) := \frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2.$$

If f has all the partial derivatives of order two continuous on $[a,b] \times [c,d]$, then one says that f changes its convexity on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i,\ i \in \{1, \ldots, k\},\ y = y_j,\ j \in \{1, \ldots, s\}$ if

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 \geq 0$$

for all $(x,y) \in [a,b] \times [c,d]$ and

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{i=1}^k (x - x_i) \cdot \Pi_{j=1}^s (y - y_j) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{i=1}^k (x - x_i) \cdot \Pi_{j=1}^s (y - y_j) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d).$$

Also, if

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 \geq 0$$

for all $(x,y) \in [a,b] \times [c,d]$, then one says that f changes its convexity on the degenerate grid in $(a,b) \times (c,d)$ determined by the segments parallel to the OY axis, $x = x_i,\ i = 1, \ldots, k$ (or determined by the segments parallel to the OX axis, $y = y_j,\ j = 1, \ldots, s$, respectively) if

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{i=1}^k (x - x_i) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{i=1}^k (x - x_i) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d)$$

(or respectively

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{j=1}^s (y - y_j) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$\frac{\partial^2 f}{\partial x^2}(x,y) \cdot \Pi_{j=1}^s (y - y_j) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d)).$$

(vii) Let $f \in C([a,b] \times [c,d])$. The concepts in the above points (i)-(vi), can easily be generalized by replacing the grids with finite systems of continuous arcs of curves or continuous closed curves of the Cartesian equations $g_i(x,y)=0, i = 1, \ldots, k$, contained in the rectangle $[a,b] \times [c,d]$. Let us first consider the case (i), of change of sign.

We say that f changes sign on the system of continuous arcs (curves) $g_i,\ i = 1, \ldots, k$, if

$$f(x,y) \cdot \Pi_{i=1}^k g_i(x,y) \geq 0, \quad \forall (x,y) \in (a,b) \times (c,d),$$

or

$$f(x,y) \Pi_{i=1}^k g_i(x,y) \leq 0, \quad \forall (x,y) \in (a,b) \times (c,d).$$

For the other cases, we have the following concepts. Everywhere we suppose that the corresponding partial derivatives of f exist and are continuous on $[a, b] \times [c, d]$.

We say that f changes its upper bidimensional monotonicity on the system of continuous arcs (curves) $g_i i = 1, \ldots, k$, if

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) \cdot \Pi_{i=1}^k g_i(x, y) \geq 0, \quad \forall (x, y) \in (a, b) \times (c, d),$$

or

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) \Pi_{i=1}^k g_i(x, y) \leq 0, \quad \forall (x, y) \in (a, b) \times (c, d).$$

We say that f changes its convexity of order $(2, 2)$ on the system of continuous arcs (curves) $g_i i = 1, \ldots, k$, if

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y) \cdot \Pi_{i=1}^k g_i(x, y) \geq 0, \quad \forall (x, y) \in (a, b) \times (c, d),$$

or

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y) \Pi_{i=1}^k g_i(x, y) \leq 0, \quad \forall (x, y) \in (a, b) \times (c, d).$$

One says that f changes its usual convexity on the system of continuous arcs (curves) $g_i i = 1, \ldots, k$, if

$$\frac{\partial^2 f}{\partial x^2}(x, y) \cdot \frac{\partial^2 f}{\partial y^2}(x, y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x, y) \right)^2 \geq 0$$

for all $(x, y) \in [a, b] \times [c, d]$ and

$$\frac{\partial^2 f}{\partial x^2}(x, y) \cdot \Pi_{i=1}^k g_i(x, y) \geq 0, \quad \forall (x, y) \in (a, b) \times (c, d),$$

or

$$\frac{\partial^2 f}{\partial x^2}(x, y) \cdot \Pi_{i=1}^k g_i(x, y) \leq 0, \quad \forall (x, y) \in (a, b) \times (c, d).$$

Remarks. (1) The continuity of f immediately implies that if f changes sign on the proper rectangular grid in $(a, b) \times (c, d)$ determined by the segments $x{=}x_i, i \in \{1, \ldots, k\}, y = y_j,\ j \in \{1, \ldots, s\}$, then we have $f(x_i, y) = f(x, y_j) = 0$ for all $(x, y) \in [a, b] \times [c, d]$, $i = 1, \ldots, k$ and $j = 1, \ldots, s$.

Simple examples of such f are $f(x, y) = \Pi_{i=1}^k (x - x_i) \cdot \Pi_{j=1}^s (y - y_j) g(x, y)$ and $f(x, y) = \Pi_{i=1}^k (e^x - e^{x_i}) \cdot \Pi_{j=1}^s (e^y - e^{y_j}) g(x, y)$, where $g \geq 0$ on $[a, b] \times [c, d]$ or $g \leq 0$ on $[a, b] \times [c, d]$.

Also, if, for example, f changes sign on the degenerate grid in $(a, b) \times (c, d)$ determined by the segments parallel to the OY axis, $x = x_i,\ i = 1, \ldots, k$ (or determined by the segments parallel to the OX axis, $y = y_j,\ j = 1, \ldots, s$, respectively), then thecontinuity of f immediately implies $f(x_i, y) = 0$ for all $y \in (c,d)$, $i = 1,\ldots,k$ (or $f(x,y_j) = 0$, for all $x \in (a,b)$, $j = 1,\ldots,s$, respectively). Simple examples are $f(x,y) = \Pi_{i=1}^k (x - x_i) \cdot g(x,y)$, $f(x,y) = \Pi_{i=1}^k (e^x - e^{x_i}) \cdot g(x,y)$, where $g \geq 0$ on $[a,b] \times [c,d]$ or $g \leq 0$ on $[a,b] \times [c,d]$ (or respectively $f(x,y) = \Pi_{j=1}^s (y - y_j) g(x,y)$, $f(x,y) = \Pi_{j=1}^s (e^y - e^{y_j}) g(x,y)$, where $g \geq 0$ on $[a,b] \times [c,d]$ or $g \leq 0$ on $[a,b] \times [c,d]$).

(2) The continuity of the partial derivative $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ in Definition 2.6.4 (ii) implies that if f changes its upper bidimensional monotonicity on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i$, $i \in \{1,\ldots,k\}, y = y_j, j \in \{1,\ldots,s\}$, then $\frac{\partial^2 f}{\partial x \partial y}(x_i,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y_j) = 0$ for all $(x,y) \in [a,b] \times [c,d]$, $i = 1,\ldots,k$ and $j = 1,\ldots,s$.

Simple examples of such f are given by

$$f(x,y) = \int_a^x \int_c^y \Pi_{i=1}^k (u - x_i) \cdot \Pi_{j=1}^s (v - y_j) g(u,v) du\, dv,$$
$$f(x,y) = \int_a^x \int_c^y \Pi_{i=1}^k (e^u - e^{x_i}) \cdot \Pi_{j=1}^s (e^v - e^{y_j}) g(u,v) du\, dv,$$

with g continuous and satisfying $g \geq 0$ on $[a,b] \times [c,d]$ or $g \leq 0$ on $[a,b] \times [c,d]$.

If, for example, f changes its upper bidimensional monotonicity on the degenerate grid in $(a,b) \times (c,d)$ determined by the segments parallel to the OY axis, $x = x_i, i = 1,\ldots,k$, then $\frac{\partial^2 f}{\partial x \partial y}(x_i,y) = 0$ for all $y \in [c,d]$, $i = 1,\ldots,k$ and simple examples are

$$f(x,y) = \int_a^x \int_c^y \Pi_{i=1}^k (u - x_i) g(u,y) du,$$
$$f(x,y) = \int_a^x \int_c^y \Pi_{i=1}^k (e^u - e^{x_i}) g(u,y) du,$$

with g continuous and satisfying $g \geq 0$ on $[a,b] \times [c,d]$ or $g \leq 0$ on $[a,b] \times [c,d]$.

(3) The continuity of the partial derivative $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)$ in Definition 2.6.4 (iv) implies that if f changes its convexity of order $(2,2)$ on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i$, $i \in \{1,\ldots,k\}\, y = y_j, j \in \{1,\ldots,s\}$, then $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x_i,y) = \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y_j) = 0$, for all $(x,y) \in [a,b] \times [c,d]$, $i = 1,\ldots,k$ and $j = 1,\ldots,s$.

Similarly, simple examples of such f are obtained by integrating $\Pi_{i=1}^k (u - x_i) \cdot \Pi_{j=1}^s (v - y_j) g(u,v)$ or $\Pi_{i=1}^k (e^u - e^{x_i}) \cdot \Pi_{j=1}^s (e^v - e^{y_j}) g(u,v)$ (where g keeps the same sign), first twice with respect to u from a to x, and then twice with respect to v from c to y.

(4) The continuity of the second-order partial derivatives of f in Definition 2.6.4 (vi) implies that if f changes its usual convexity on the proper rectangular grid in $(a,b) \times (c,d)$ determined by the segments $x = x_i$, $i \in \{1,\ldots,k\}$, $y = y_j, j \in \{1,\ldots,s\}$, then $\frac{\partial^2 f}{\partial x^2}(x_i,y) = \frac{\partial^2 f}{\partial x^2}(x,y_j) = 0$, for all $(x,y) \in [a,b] \times [c,d]$ and $i = 1,\ldots,k$, $j = 1,\ldots,s$.

Again, simple examples are obtained by integrating from a to x the expression $\Pi_{i=1}^{k}(u-x_i)\cdot\Pi_{j=1}^{s}(v-y_j)g(u,v)$ or $\Pi_{i=1}^{k}(e^u-e^{x_i})\cdot\Pi_{j=1}^{s}(e^v-e^{y_j})g(u,v)$ (where g keeps the same sign), twice with respect to u.

(5) Let us suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is continuous on $[a,b]\times[c,d]$ and that f changes sign (or bidimensional monotonicity or convexity of order $(2,2)$ or usual convexity, respectively) on the system of continuous curves of implicit equations $g_i(x,y)=0$, $i=1,\dots,k$, contained in the rectangle $[a,b]\times[c,d]$, as in Definition 2.6.4 (vii). If, in addition, each equation $g_i(x,y)=0$ can be explicitly written as $y=h_i(x)$, then the continuity of f (or of the corresponding partial derivatives of f, respectively) obviously implies that we necessarily have $f(x,h_i(x))=0$ (or $\frac{\partial^2 f}{\partial x\partial y}(x,h_i(x))=0$, or $\frac{\partial^4 f}{\partial x^2\partial y^2}(x,h_i(x))=0$, or $\frac{\partial^2 f}{\partial x^2}(x,h_i(x))=0$, respectively) for all $x\in(a,b)$ and $i=1,\dots,k$.

For example, if $f:[0,1]\times[0,1]\to\mathbb{R}$, $g_1(x,y)=x-y$, $g_2(x,y)=x-y^2$ and we suppose that f changes sign (or bidimensional monotonicity, or convexity of order $(2,2)$, or usual convexity, respectively) on the first bisector in $(0,1)\times(0,1)$ of equation $g_1(x,y)=0$ and on the arc of the parabola in $(0,1)\times(0,1)$ of equation $g_2(x,y)=0$, then f necessarily satisfies $f(u,u)=f(u^2,u)=0$ (or $\frac{\partial^2 f}{\partial x\partial y}(u,u)=\frac{\partial^2 f}{\partial x\partial y}(u^2,u)=0$, or $\frac{\partial^4 f}{\partial x^2\partial y^2}(u,u)=\frac{\partial^4 f}{\partial x^2\partial y^2}(u^2,u)=0$, or $\frac{\partial^2 f}{\partial x^2}(u,u)=\frac{\partial^2 f}{\partial x^2}(u^2,u)=0$, respectively) for all $u\in(0,1)$.

Concrete simple functions f with respect to the above concrete system of curves are:

(i) $f(x,y)=(x-y)(x-y^2)g(x,y)$, $f(x,y)=(e^x-e^y)(x-y^2)g(x,y)$ with $g\geq 0$ on $[0,1]\times[0,1]$ or $g\leq 0$ on $[0,1]\times[0,1]$, change their signs in this system;

(ii) $f(x,y)=\int_0^x\int_0^y(u-v)(e^{u^2}-e^v)g(u,v)dudv$, with $g\geq 0$ on $[0,1]\times[0,1]$ or $g\leq 0$ on $[0,1]\times[0,1]$ changes its bidimensional monotonicity in this system.

Similarly, having as models the above Remarks 3 and 4, we easily can produce simple example of functions f that change their convexity of order $(2,2)$ and usual convexity on the mentioned system of curves.

2.6.3 Bivariate Copositive Approximation

In this subsection, in order to obtain results in bivariate copositive approximation, we will use the results in the univariate case through the tensor product method.

For this purpose, we need the following.

Corollary 2.6.5. (Beutel–Gonska [42]) *For any $f:[0,1]\times[0,1]\to\mathbb{R}$, with $\frac{\partial^2 f}{\partial x\partial y}$ continuous on $[0,1]\times[0,1]$, $n,m\geq 8$, there exists a sequence of bivariate polynomials $\tilde{p}_{n,m}(x,y)$, of degrees $\leq n$ in x and $\leq m$ in y such that*

$$\left\|\tilde{p}_{n,m} - f\right\| \leq C\left[\frac{a}{n} + \frac{b}{m}\right], \quad \left\|\frac{\partial \tilde{p}_{n,m}}{\partial x} - \frac{\partial f}{\partial x}\right\| \leq C\left[a + \frac{b}{m}\right],$$
$$\left\|\frac{\partial \tilde{p}_{n,m}}{\partial y} - \frac{\partial f}{\partial y}\right\| \leq C\left[\frac{a}{n} + b\right], \quad \left\|\frac{\partial^2 \tilde{p}_{n,m}}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right\| \leq C[a+b],$$

where $C > 0$ is independent of f, n, m, also $\|\cdot\|$ denotes the uniform norm and

$$a = \omega_2(\frac{\partial f}{\partial x}; \frac{1}{n}, 0) + \omega_2\left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, 0\right),$$
$$b = \omega_2(\frac{\partial f}{\partial y}; 0, \frac{1}{m}) + \omega_2\left(\frac{\partial^2 f}{\partial x \partial y}; 0, \frac{1}{m}\right).$$

Proof. If we take $p = q = 1$ and $r = s = 2$ in Theorem 2.3.2, then we get $p' = q' = \min\{1, 1\} = 1$ and taking all the possible values $(0,0), (0,1), (1,0)$ and $(1,1)$ for (k,l), the estimates in Theorem 2.3.2 become exactly the estimates in Corollary 2.6.5. □

We are in a position to prove (keeping the notation in Corollary 2.6.5) the following result.

Theorem 2.6.6. *(Gal [132]) If $f : [0,1] \times [0,1] \to \mathbb{R}$ has the partial derivative $\frac{\partial^2 f}{\partial x \partial y}$ continuous on $[0,1] \times [0,1]$ and changes its sign on the proper rectangular grid in $(0,1) \times (0,1)$, determined by the distinct segments $x = x_i, i \in \{1,\ldots,k\}, y = y_j, j \in \{1,\ldots,s\}$, then for all $n \geq n_0$ and $m \geq m_0$ (with n_0 and m_0 depending only on k, s, α, β, where $\alpha = \min_{0 \leq i \leq k}(x_{i+1} - x_i), \beta = \min_{0 \leq j \leq s}(y_{j+1} - y_j)$, $0 = x_0 = y_0, 1 = x_{k+1} = y_{s+1}$), there exists a polynomial $P_{n,m}(x,y)$ of degrees $\leq n$ in x and $\leq m$ in y that satisfies*

$$\|f - P_{n,m}\| \leq C\left[\frac{a}{n} + \frac{b}{m}\right], \text{ where } C = C(k,s,\alpha,\beta) > 0,$$

and is copositive with f on $[0,1]^2 \setminus \{A \cup B\}$, where

$$A = \{(x,y) \in [0,1]^2; x \in \cup_{i=1}^k [x_i - 1/n, x_i + 1/n],$$
$$y \notin \cup_{j=1}^s [y_j - 1/m, y_j + 1/m], \Pi_{j=1}^s (y - y_j) < 0\},$$
$$B = \{(x,y) \in [0,1]^2; y \in \cup_{j=1}^s [y_j - 1/m, y_j + 1/m],$$
$$x \notin \cup_{i=1}^k [x_i - 1/n, x_i + 1/n], \Pi_{i=1}^k (x - x_i) < 0\}.$$

Proof. By Remark 1 of Definition 2.6.4, we have $f(x_i, y) = f(x, y_j) = 0$, for all $(x,y) \in [0,1] \times [0,1]$, $i = 1,\ldots,k$ and $j = 1,\ldots,s$.

For $\tilde{p}_{n,m}$ in Corollary 2.6.5, let us define

$$p_{n,m}(x,y) = \tilde{p}_{n,m}(x,y) + \sum_{i=1}^k \sum_{j=1}^s l_i(x) l_j(y)[f(x_i, y_j) - \tilde{p}_{n,m}(x_i, y_j)],$$

where $l_i(x)$ and $l_j(y)$ represent the fundamental Lagrange interpolation polynomials based on the nodes $x_1 < \cdots < x_k$ and $y_1 < \cdots < y_s$, respectively.

We define now

$$Q_{n,m}(x, y) = p_{n,m}(x, y) + \sum_{i=1}^{k} l_i(x)[f(x_i, y) - p_{n,m}(x_i, y)] + \sum_{j=1}^{s} l_j(y)[f(x, y_j) - p_{n,m}(x, y_j)].$$

Note that $Q_{n,m}(x_i, y) = Q_{n,m}(x, y_j) = f(x_i, y) = f(x, y_j) = 0$ for all $x, y \in (0,1)$, $i = 1, \ldots, k$ and $j = 1, \ldots, s$. Also, $Q_{n,m}$ gives the same orders for the approximation errors as $p_{n,m}$, i.e., as $\tilde{p}_{n,m}$, with a constant (independent of f, n, m, x, y) that also will be denoted by C without loss of generality.

Let us consider

$$P_{n,m}(x, y) = Q_{n,m}(x, y) + E_{n,m}(x, y),$$

where $E_{n,m}(x, y) = \varepsilon DC \left[\frac{a}{n} + \frac{b}{m}\right] \Pi_{i=1}^{k} q_n(x - x_i) \Pi_{j=1}^{s} q_m(y - y_j)$, $q_n(x)$, $q_m(y)$ are the polynomials with the properties in Lemma 1 in Hu–Leviatan–Yu [176] (see also the proof of Theorem 1.5.4 (iv) in the univariate case in Chapter 1), a, b are given by the statement of Corollary 2.6.5, $\varepsilon = \text{sgn} f(x, y)$, $(x, y) \in (x_k, 1) \times (y_s, 1)$, C is as above, and $D > 0$ will be determined later.

It is easy to show that $P_{n,m}(x_i, y) = P_{n,m}(x, y_j) = 0$, for all $x, y \in [0,1]$, $i = 1, \ldots, k, j = 1, \ldots, s$, and that $E_{n,m}(x, y) f(x, y) \geq 0$ for all $x, y \in [0,1]$.

We have four possibilities :

(i) $x \notin \cup_{i=1}^{k} [x_i - 1/n, x_i + 1/n]$ and $y \notin \cup_{j=1}^{r} [y_j - 1/m, y_j + 1/m]$;
(ii) there is i such that $x \in [x_i - 1/n, x_i + 1/n]$ and $y \notin \cup_{j=1}^{r} [y_j - 1/m, y_j + 1/m]$;
(iii) there is j such that $y \in [y_j - 1/m, y_j + 1/m]$ and $x \notin \cup_{i=1}^{k} [x_i - 1/n, x_i + 1/n]$;
(iv) there is i such that $x \in [x_i - 1/n, x_i + 1/n]$ and there is j such that $y \in [y_j - 1/m, y_j + 1/m]$.

Case (i). Reasoning as in the proof of the univariate case Theorem 1.5.4 (iv) (see also Theorem 2 in Hu–Leviatan–Yu [176]) and choosing $D > A^{-(k+s)}$, we get $|E_{n,m}(x, y)| > C \left[\frac{a}{n} + \frac{b}{m}\right]$ and $f(x, y) P_{n,m}(x, y) \geq 0$.

Case (ii). Reasoning as in the univariate case (see Theorem 1.5.4, (iv) in Chapter 1 or Hu–Leviatan–Yu [176]) we obtain $\text{sgn} \left[\frac{\partial f}{\partial x} - \frac{\partial P_{n,m}}{\partial x}\right](x, y) = \text{sgn}\{-q_n(x - x_i) f(x, y)\}$, if $\Pi_{j=1}^{s} q_m(y - y_j) \geq 0$ (i.e., if $\Pi_{j=1}^{s} (y - y_j) \geq 0$).

By the mean value theorem, we can write $f(x, y) - P_{n,m}(x, y) = f(x, y) - P_{n,m}(x, y) - [f(x_i, y) - P_{n,m}(x_i, y)] = (x - x_i) \left(\frac{\partial f}{\partial x} - \frac{\partial P_{n,m}}{\partial x}\right)(\xi, y)$, with ξ between x and x_i. Reasoning now with respect to x (y fixed) as in the univariate case, we similarly get $f(x, y) P_{n,m}(x, y) \geq 0$, for all $n \geq n_0$ and $m \geq 8$.

Case (iii). Reasoning as in the univariate case (Theorem 1.5.4., (iv)) we get $\text{sgn}\left[\frac{\partial f}{\partial y} - \frac{\partial P_{n,m}}{\partial y}\right](x,y) = \text{sgn}\{-q_m(y-y_j)f(x,y)\}$ if $\Pi_{i=1}^k q_n(x-x_i) \geq 0$.

Using the mean value theorem, we can write $f(x,y) - P_{n,m}(x,y) = f(x,y) - P_{n,m}(x,y) - [f(x,y_j) - P_{n,m}(x,y_j)] = (y-y_j)\left(\frac{\partial f}{\partial y} - \frac{\partial P_{n,m}}{\partial y}\right)(x,\eta)$, with η between y and y_j. Reasoning now with respect to y (x fixed) as in the univariate case, we again get $f(x,y)P_{n,m}(x,y) \geq 0$, for all $n \geq 8$ and $m \geq m_0$.

Case (iv). Reasoning as in the univariate case (Theorem 1.5.4 (iv)), it follows that

$$\text{sgn}\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 P_{n,m}}{\partial x \partial y}\right](x,y) = \text{sgn}\{-q_n(x-x_i)q_m(y-y_j)f(x,y)\}.$$

We can write $f(x,y) - P_{n,m}(x,y) = f(x,y) - P_{n,m}(x,y) - [f(x_i,y) - P_{n,m}(x_i,y)] - [f(x,y_j) - P_{n,m}(x,y_j)] + [f(x_i,y_j) - P_{n,m}(x_i,y_j)] = F(x,y) - F(x_i,y) - F(x,y_j) + F(x_i,y_j)$, with $F(x,y) = f(x,y) - P_{n,m}(x,y)$. Applying a known mean value theorem (see, e.g., Nicolescu [286]), there exists (ξ,η), ξ between x and x_i and η between y and y_j, such that

$$\begin{aligned} f(x,y) - P_{n,m}(x,y) &= F(x,y) - F(x_i,y) - F(x,y_j) + F(x_i,y_j) \\ &= (x-x_i)(y-y_j)\frac{\partial^2 F}{\partial x \partial y}(\xi,\eta). \end{aligned}$$

Repeating now the type of reasoning in the univariate case with respect to both x and y, similarly we obtain $f(x,y)P_{n,m}(x,y) \geq 0$, for all $n \geq n_0$, $m \geq m_0$. Finally, $P_{n,m}(x,y)$ obviously gives the estimate in the statement.

Note that from the proof, it follows in fact that $P_{n,m}(x,y)$ is of degrees $\leq 2kn$ in x and $\leq 2sm$ in y. But by a standard procedure, we may reduce it to degrees $\leq n$ in x and $\leq m$ in y, which proves the theorem. □

Corollary 2.6.7. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be supposed to have the partial derivative $\frac{\partial^2 f}{\partial x \partial y}$ continuous on $[0,1] \times [0,1]$.*

(i) (Gal [132]) If f changes sign on the degenerate grid in $(0,1) \times (0,1)$ determined by the distinct segments parallel to the OY axis, $x = x_i, i = 1,\ldots,k$, then for all $n \geq n_0$ and $m \geq 8$ (n_0 depends only on k,α, where $\alpha = \min_{0 \leq i \leq k}(x_{i+1} - x_i)$, $0 = x_0, 1 = x_{k+1}$), there is a polynomial $P_{n,m}(x,y)$ of degrees $\leq n$ in x and $\leq m$ in y that is copositive with f on $[0,1]^2$ and satisfies

$$\begin{aligned} \|f - P_{n,m}\| \leq C\Bigg[&\frac{1}{n}\left(\omega_2(\frac{\partial f}{\partial x};\frac{1}{n},0) + \omega_2(\frac{\partial^2 f}{\partial x \partial y};\frac{1}{n},0)\right) \\ &+ \frac{1}{m}\left(\omega_2(\frac{\partial f}{\partial y};0,\frac{1}{m}) + \omega_2(\frac{\partial^2 f}{\partial x \partial y};0,\frac{1}{m})\right)\Bigg] \end{aligned}$$

with $C > 0$ depending only on k,α.

(ii) (Gal [132]) If f changes sign on the degenerate grid in $(0,1) \times (0,1)$ *determined by the distinct segments parallel to the OX axis,* $y = y_j, j = 1, \dots, s$, *then for all* $n \geq 8$ *and* $m \geq m_0$ *(with* m_0 *depending only on* s, β, *where* $\beta = \min_{0 \leq j \leq s}(y_{j+1} - y_j)$, $0 = y_0, 1 = y_{s+1}$*), there exists a polynomial* $P_{n,m}(x,y)$ *of degrees* $\leq n$ *in* x *and* $\leq m$ *in* y *that is copositive with* f *on* $[0,1]^2$ *and* $\|f - P_{n,m}\|$ *satisfies the estimate from the above point (i), with* $C > 0$ *depending only on* s, β.

(iii) If f *changes sign on the first bisector in* $[0,1] \times [0,1]$ *of the equation* $x - y = 0$ *(which necessarily implies* $f(u,u) = 0$ *for all* $u \in [0,1]$*), then for all* $0 < \eta < 1/2$, $n \geq n_0$, *and* $m \geq 8$ *(with* $n_0 = n_0(\eta)$ *independent of* f*), there exists a polynomial* $P_{n,m}(x,y)$ *of degrees* $\leq n$ *in* x *and* $\leq \max\{n, [n/2] + m\}$ *in* y *that is copositive with* f *on* $[0,1] \times [\eta, 1-\eta]$ *and* $\|f - P_{n,m}\|$ *satisfies the estimate from the point (i), with* $C > 0$ *depending only on* η.

(iv) If f *changes its sign on the arc of the parabola in* $[0,1] \times [0,1]$ *of equation* $x - y^2 = 0$ *(which necessarily implies* $f(u^2, u) = 0$, *for all* $u \in [0,1]$*), then for all* $0 < \eta < 1/2$, $n \geq n_0$, *and* $m \geq 8$ *(with* $n_0 = n_0(\eta)$ *independent of* f*), there exists a polynomial* $P_{n,m}(x,y)$ *of degrees* $\leq n$ *in* x *and* $\leq \max\{2n, n+m\}$ *in* y *that is copositive with* f *on* $[0,1] \times [\eta, 1-\eta]$ *and* $\|f - P_{n,m}\|$ *satisfies the estimate from the above (i), with* $C > 0$ *depending only on* η.

(v) Let us suppose that f *changes sign on the system of curves composed by the first bisector and the arc of the parabola* $x - y^2 = 0$ *in* $[0,1] \times [0,1]$ *(which necessarily implies* $f(u,u) = 0$ *and* $f(u^2,u) = 0$, *for all* $u \in [0,1]$*) and let* $p_{n,m}(x,y)$ *be a bivariate polynomial of degree* $\leq n$ *in* x *and* $\leq m$ *in* y *such that* $\|f - p_{n,m}\| \leq C\Phi(f)_{n,m}$, $\|\frac{\partial f}{\partial x} - \frac{\partial p_{n,m}}{\partial x}\| \leq C\Psi(f)_{n,m}$, *where* $\Psi(f)_{n,m} \leq n\Phi(f)_{n,m}$, *for all* $n, m \in \mathbb{N}$, *with* $C > 0$ *independent of* f, n, m, *and* $p_{n,m}(y,y) = p_{n,m}(y^2,y) = 0$ *for all* $y \in [0,1]$. *For all* $0 < \eta < 1/2$, $n \geq n_0$, $m \in \mathbb{N}$ *(with* $n_0 = n_0(\eta)$ *independent of* f*), there is a polynomial* $P_{n,m}(x,y)$ *of degrees* $\leq 2n$ *in* x *and* $\leq \max\{3n, m\}$ *in* y, *copositive with* f *on* $[0,1] \times [\eta, 1-\eta]$ *and* $\|f - P_{n,m}\| \leq C_1\Phi(f)_{n,m}$, *with* $C_1 > 0$ *a absolute constant.*

Proof. (i) By Remark 1 of Definition 2.6.4, we have $f(x_i, y) = 0$ for all $y \in (0,1)$, $i = 1, \dots, k$.

For $\tilde{p}_{n,m}$ in Corollary 2.6.5, let us define

$$Q_{n,m}(x,y) = \tilde{p}_{n,m}(x,y) + \sum_{i=1}^{k} l_i(x)[f(x_i,y) - \tilde{p}_{n,m}(x_i,y)],$$

where $l_i(x)$, $i = 1, \dots, k$ represent the fundamental Lagrange interpolation polynomials based on the nodes $x_1 < \dots < x_k$.

It is easy to see that the approximation errors by $Q_{n,m}$ are the same as those given by $\tilde{p}_{n,m}$ in Corollary 2.6.5, with a constant (independent of f, n, m) that will also be denoted by C without loss of generality. Also, we have $Q_{n,m}(x_i, y) = f(x_i, y) = 0$ for all $y \in (0,1)$, $i = 1, \dots, k$.

Let us define

$$P_{n,m}(x,y)=Q_{n,m}(x,y)+E_{n,m}(x),$$

where $E_{n,m}(x)=\varepsilon DC\left[\frac{a}{n}+\frac{b}{m}\right]\Pi_{i=1}^{k}q_n(x-x_i)$, $q_n(x)$ are the polynomials with the properties in Lemma 1 in Hu–Leviatan–Yu [176] (see also the proof of Theorem 1.5.4 (iv) in Chapter 1), a,b, are given by the statement of Corollary 2.6.5, $\varepsilon=\text{sgn}f(x,y)$, $(x,y)\in(x_k,1)\times(0,1)$, $C>0$ is as above and $D>0$ will be properly chosen later.

It is easy to show that $P_{n,m}(x_i,y)=0$, for all $y\in[0,1],\ i=1,\dots,k$, and that $E_{n,m}(x)f(x,y)\ge 0$, for all $x,y\in[0,1]$.

For arbitrary fixed $y\in(0,1)$, we have two possibilities:

(a) $x\notin\cup_{i=1}^{k}[x_i-1/n,x_i+1/n]$;

(b) there is i such that $x\in[x_i-1/n,x_i+1/n]$;

Case (a). Reasoning as in the proof of the univariate case Theorem 1.5.4 (iv) (see also Theorem 2 in Hu–Leviatan–Yu [176]) and choosing $D>A^{-k}$, we get $|E_{n,m}(x)|>C\left[\frac{a}{n}+\frac{b}{m}\right]$ and $f(x,y)P_{n,m}(x,y)\ge 0$.

Case (b). Reasoning as in the univariate case (see Theorem 1.5.4 (iv) or Hu–Leviatan–Yu [176]), we obtain $\text{sgn}\left[\frac{\partial f}{\partial x}-\frac{\partial P_{n,m}}{\partial x}\right](x,y)=\text{sgn}\{-q_n(x-x_i)f(x,y)\}$.

By the mean value theorem, we can write $f(x,y)-P_{n,m}(x,y)=f(x,y)-P_{n,m}(x,y)-[f(x_i,y)-P_{n,m}(x_i,y)]=(x-x_i)\left(\frac{\partial f}{\partial x}-\frac{\partial P_{n,m}}{\partial x}\right)(\xi,y)$, with ξ between x and x_i. Reasoning now with respect to x (y fixed) as in the univariate case, we similarly get $f(x,y)P_{n,m}(x,y)\ge 0$, for all $n\ge n_0$ and $m\ge 8$. Finally, $P_{n,m}(x,y)$ obviously gives the estimate in the statement.

Note that from the proof, in fact it follows that $P_{n,m}(x,y)$ is of degrees $\le 2kn$ in x and $\le m$ in y. But by a standard procedure we may reduce it to the degrees $\le n$ in x and $\le m$ in y, which proves (i).

(ii) The proof is similar to that of (i).

(iii) Defining $A_1=\{(x,y)\in[0,1]^2;x-y<0\}$, $A_2=\{(x,y)\in[0,1]^2;x-y>0\}$, and $A_3=\{(x,y)\in[0,1]^2;x-y=0\}$, it is obvious that $[0,1]^2=A_1\cup A_2\cup A_3$, $A_i\cap A_j=\emptyset$ for all $i\ne j$, and by hypothesis we get $f(x,x)=0$ for all $x\in[0,1]$. Fix $y\in(\eta,1-\eta)$, arbitrary.

For $\tilde{p}_{n,m}$ in Corollary 2.6.5, let us define $p_{n,m}(x,y)=\tilde{p}_{n,m}(x,y)+[f(y,y)-\tilde{p}_{n,m}(y,y)]$. It is easy to see that the approximation errors by $p_{n,m}(x,y)$ (and by $\frac{\partial p_{n,m}}{\partial x}$) are the same as those given by $\tilde{p}_{n,m}(x,y)$ (and by $\frac{\partial\tilde{p}_{n,m}}{\partial x}$), with a constant (independent of f,n,m,y but depending on η) that will also be denoted by C without loss of generality. Also, $p_{n,m}(y,y)=f(y,y)=0$.

Define now

$$P_{n,m}(x,y)=p_{n,m}(x,y)+E_{n,m}(x,y),$$

where $E_{n,m}(x,y)=\varepsilon DC[a/n+b/m]q_n(x-y)$, $q_n(u)$ is in Lemma 1 in Hu–Leviatan–Yu [176], $\varepsilon=\text{sgn}\{f(x,y),(x,y)\in A_2\}$, a,b are given by Corollary 2.6.5, $C>0$ is as above and $D>0$ will be determined below.

First, it is easy to check that $f(x,y)E_{n,m}(x,y)\ge 0$, for all $(x,y)\in[0,1]^2$.

Now fix $y \in (\eta, 1-\eta)$ arbitrary and let $x \in [0,1]$ be variable. We have two possibilities :

(1) $|x-y| > \frac{1}{n}$

and

(2) $|x-y| \leq \frac{1}{n}$.

Case (1). Choosing $D > A^{-1}$, we get

$$f(x,y)P_{n,m}(x,y) = f(x,y)[p_{n,m}(x,y) - f(x,y)] + f^2(x,y) + f(x,y)E_{n,m}(x,y),$$

and since

$$f(x,y)E_{n,m}(x,y) = |f(x,y)E_{n,m}(x,y)| \geq |f(x,y)|C[a/n + b/m],$$

it follows that $f(x,y)P_{n,m}(x,y) \geq 0$.

Case (2). First we easily get $P_{n,m}(y,y) = p_{n,m}(y,y) = 0 = f(y,y)$. Defining $F(x) = f(x,y), x \in [0,1]$, $R_{n,m}(x) = P_{n,m}(x,y)$, we obtain $F(y) = R_{n,m}(y) = 0$, and reasoning exactly as in the proof of Theorem 2 in the univariate case in Hu–Leviatan–Yu [176], pp. 215-217 (since $|x-y| \leq \frac{1}{n}$), for $D > 2A^{-1}$ we get $F(x)R_{n,m}(x) \geq 0$ for all $n \geq n_0 = n_0(\eta)$ and $m \geq 8$.

From the continuity of f and $P_{n,m}(x,y)$, we obtain $f(x,y)P_{n,m}(x,y) \geq 0$, for all $x \in [0,1]$ and all $y \in [\eta, 1-\eta]$.

Finally, $P_{n,m}(x,y)$ obviously gives the estimate in the statement. Note that from the proof, it follows in fact that $P_{n,m}(x,y)$ is of degrees $\leq 2n$ in x and $\leq \max\{n+m, 2n\}$ in y. But by a standard procedure we may reduce it to the degrees $\leq n$ in x and $\leq \max\{[n/2]+m, n\}$ in y, which proves (iii).

(iv) The proof is identical to that in (iii), using the constructions

$$p_{n,m}(x,y) = \tilde{p}_{n,m}(x,y) + [f(y^2,y) - \tilde{p}_{n,m}(y^2,y)],$$
$$P_{n,m}(x,y) = p_{n,m}(x,y) + E_{n,m}(x,y),$$

where $E_{n,m}(x,y) = \varepsilon DC[a/n + b/m]q_n(x-y^2)$, $\varepsilon = \text{sgn}\{f(x,y); x - y^2 > 0\}$.

(v) Fix $y \in (\eta, 1-\eta)$ and take $x \in [0,1]$ variable. Our required polynomials will be defined by $P_{n,m}(x,y) = p_{n,m}(x,y) + E_{n,m}(x,y)$, where

$$E_{n,m}(x,y) = \varepsilon DC\Phi(f)_{n,m}q_n(x-y)q_n(x-y^2),$$

$\varepsilon = \text{sgn}\{f(x,y); (x,y) \in A_3\}$, $A_3 = \{(x,y) \in [0,1]^2 : y < x\}$, with $D > 0$ to be chosen later. Setting $A_2 = \{(x,y) \in [0,1]^2; y^2 < x < y\}$, $A_1 = \{(x,y) \in [0,1]^2; x \leq y^2\}$, we obviously have $[0,1]^2 = A_1 \cup A_2 \cup A_3$, with the $A_j \cap A_i = \emptyset$ for $j \neq i$.

First we easily get $f(x,y)E_{n,m}(x,y) \geq 0$ for all $x, y \in [0,1]$.

Since $y \in (\eta, 1-\eta)$, we have $y^2 < y$. In what follows, reasoning exactly as in the proof of Theorem 2 in the univariate case in Hu–Leviatan–Yu [176], pp. 215–217 (with respect to x and n in the expression of $f(x,y)$ and $P_{n,m}(x,y)$ and setting $y_1 = y^2$, $y_2 = y$), we get $f(x,y)P_{n,m}(x,y) \geq 0$, for all $x \in [0,1]$, $y \in (\eta, 1-\eta)$, $n \geq n_0 = n_0(\eta)$, $m \in \mathbb{N}$.

From the continuity of f and $P_{n,m}(x,y)$, again we immediately obtain that $f(x,y)P_{n,m}(x,y) \geq 0$ for all $x \in [0,1]$ and all $y \in [\eta, 1-\eta]$, $n \geq n_0 = n_0(\eta)$, $m \in \mathbb{N}$.

Note that from the proof, it follows in fact that $P_{n,m}(x,y)$ is of degrees $\leq 4n$ in x and $\leq \max\{m, 6n\}$ in y. But by a standard procedure we may reduce it to the degrees $\leq 2n$ in x and $\leq \max\{m, 3n\}$ in y. □

Corollary 2.6.8. (Gal [132]) *In terms of the modulus of smoothness of the second kind introduced in Definition 2.1.2 (iii), the estimate in Corollary 2.6.7 (i)–(iv), can be expressed by*

$$\|f - P_{n,m}\| \leq C \left[\frac{1}{n}\left(\omega_2(\frac{\partial f}{\partial x};\frac{1}{n}) + \omega_2(\frac{\partial^2 f}{\partial x \partial y};\frac{1}{n})\right)\right.$$
$$\left. + \frac{1}{m}\left(\omega_2(\frac{\partial f}{\partial y};\frac{1}{m}) + \omega_2(\frac{\partial^2 f}{\partial x \partial y};\frac{1}{m})\right)\right].$$

Proof. It is immediate by the obvious inequalities

$$\omega_2(f;\delta,0) \leq \omega_2(f;\delta), \quad \omega_2(f;0,\delta) \leq \omega_2(f;\delta). \quad \square$$

Remarks. (1) Theorem 2.6.6 is in fact a result of "almost nearly" copositive approximation and corrects Theorem 2.1 in Gal [132]. Also, although the estimates in Theorem 2.6.6 and Corollaries 2.6.7, 2.6.8 are not the best possible, at least they may be considered as the first Weierstrass-type results in bivariate copositive approximation.

(2) If we consider the general concept in Definition 2.6.4 (vii), it is obvious that if the system of arcs or curves on which f changes sign are not algebraic curves, then the copositive approximation by bivariate polynomials might no longer be possible. For example, suppose that the continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ changes sign on the arc of the curve $y = e^x$ in $[0,1]$. Then we necessarily have $f(x, e^x) = 0$ for all $x \in [0,1]$, and any bivariate polynomial $P_{n,m}(x,y)$ that is copositive with f necessarily satisfies $P_{n,m}(x, e^x) = 0$ for all $x \in [0,1]$, which is impossible.

(3) In the proof of Corollary 2.6.7 (v), we cannot use completely the method in the univariate case, as we did in cases (i)–(iv), fact that has as a consequence a different type of estimate. Indeed, the problem is starting from a polynomial, let us say $\tilde{p}_{n,m}$ in Corollary 2.6.5, to define another polynomial $p_{n,m}(x,y)$ having the same error approximation as $\tilde{p}_{n,m}$ has and satisfying, in addition, the interpolation conditions $p_{n,m}(y,y) = 0$ $(= f(y,y))$ and $p_{n,m}(y^2,y) = 0 (= f(y^2,y))$.

The standard Lagrange interpolation method produces

$$p_{n,m}(x,y) = \tilde{p}_{n,m}(x,y) + \frac{x-y}{y^2-y}[f(y^2,y) - \tilde{p}_{n,m}(y^2,y)]$$
$$+ \frac{x-y^2}{y-y^2}[f(y,y) - \tilde{p}_{n,m}(y,y)],$$

which unfortunately it is not a polynomial in x and y (it is a bivariate rational function), although the approximation error by $p_{n,m}(x,y)$ is the same with that given by $\tilde{p}_{n,m}(x,y)$, with a constant $C>0$ (independent of f,n,m) and $p_{n,m}(y,y)=f(y,y)=0$, $p_{n,m}(y^2,y)=f(y^2,y)=0$.

On the other hand, given f with the properties $f(y,y)=f(y^2,y)=0$ for all $y\in[0,1]$, polynomials satisfying $p_{n,m}(y,y)=p_{n,m}(y^2,y)=0$ for all $y\in[0,1]$ can easily be constructed in the form $p_{n,m}(x,y)=(x-y)(x-y^2)g(x,y)$, with $g(x,y)$ an arbitrary bivariate polynomial. The problem is to find such of $p_{n,m}(x,y)$ which, in addition, approximate well the function f. In this sense, we may proceed as follows. For all $n\geq 2$ and $m\geq 3$, consider

$$B_{n,m}(f)=\inf\{\|f-p\|;p\in\mathcal{P}_{n,m}\cap G\},$$

where $\mathcal{P}_{n,m}$ denotes the set of all bivariate polynomials of degrees $\leq n$ in x and $\leq m$ in y, $G=\{g;g(x,y)=(x-y)(x-y^2)h(x,y);h(x,y)\geq 0(\text{ or } h(x,y)\leq 0),\forall x,y\in[0,1]\}$, and $\|\cdot\|$ is the uniform norm in $C([0,1]\times[0,1])$. According to Remark 2 after Definition 2.3.4, there exists $p_{n,m}\in\mathcal{P}_{n,m}\cap G$ such that $B_{n,m}(f)=\|f-p_{n,m}\|$. Starting from this $p_{n,m}(x,y)$ and reasoning exactly as in the proof of Corollary 2.6.7, (v), we get the polynomials $P_{n,m}(x,y)$ which are copositive with f on $[0,1]\times[\eta,1-\eta]$ and satisfy the error estimate

$$\|f-P_{n,m}\|\leq C\cdot B_{n,m}(f),$$

with the constant $C>0$ independent of f,n,m. The question is to find good estimates in terms of moduli of smoothness for the quantity $B_{n,m}(f)$.

2.6.4 Bivariate Comonotone Approximation

We apply now some ideas in the univariate case to deduce a result in bivariate comonotone approximation. This is possible because we use the concept of change of bidimensional monotonicity on a proper grid.

For that purpose, we need the following.

Theorem 2.6.9. (Gal [126], Theorem 3.1) *For any $f\in C([-1,1]\times[-1,1])$ with the continuous partial derivative $\frac{\partial^2 f}{\partial x\partial y}\geq 0$ on $[-1,1]\times[-1,1]$, there exists a sequence of bivariate polynomials $P_{n,m}(f)(x,y)$, of degrees $\leq n$ in x and $\leq m$ in y, such that*

$$\|P_{n,m}-f\|\leq C\omega_2^{\varphi}\left(f;\frac{1}{n},\frac{1}{m}\right),$$

and $\frac{\partial^2 P_{n,m}}{\partial x\partial y}\geq 0$ on $[-1,1]\times[-1,1]$, where $C>0$ is independent of f, n, m, and $\|\cdot\|$ denotes the uniform norm.

Proof. Since f has continuous partial derivative $\frac{\partial^2 f}{\partial x\partial y}\geq 0$ on $[-1,1]\times[-1,1]$, according to Remark 2 (iii) after Definition 2.1.1, f is upper bidimensional monotone in $[-1,1]\times[-1,1]$. Then Theorem 2.6.9 follows directly from Theorem 2.6.1 (i). □

We will extend Theorem 1.6.7 (i), to bivariate approximation. The first main result is the following.

Theorem 2.6.10. *If* $f : [-1,1] \times [-1,1] \to \mathbb{R}$ *has the partial derivative* $g = \frac{\partial^2 f}{\partial x \partial y}$ *continuous on* $[-1,1] \times [-1,1]$ *and changes its upper bidimensional monotonicity on the proper rectangular grid in* $(-1,1) \times (-1,1)$*, determined by the distinct segments* $x = x_i,\ i \in \{1,\dots,k\},\ y = y_i,\ i \in \{1,\dots,k\}$*, then for all* $n \geq 1$ *and* $m \geq 1$*, there exists a polynomial* $P_{n,m}(x,y)$ *of degrees* $\leq n$ *in* x *and* $\leq m$ *in* y *that is comonotone with* f *on* $[-1,1]^2$ *and satisfies*

$$\|f - P_{n,m}\| \leq C(k) \left[\left(\frac{n}{m} + \frac{m}{n} \right) \omega_1^{\varphi} \left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, \frac{1}{m} \right) + \omega_2^{\varphi} \left(f; \frac{1}{n}, \frac{1}{m} \right) \right],$$

with $C(k) > 0$ *depending only on* k.

Before to prove it, let us consider the following important

Remark. It is evident that the estimates produced by Theorem 2.6.10 can become very bad (tends to ∞ with $n \to \infty$) and actually unuseful if, for example, we take $m = n^s$, with $s \geq 2$ (a similar phenomenon can be found in bivariate global smoothness preservation by interpolation, see Gal-Szabados [141]). To completely eliminate this shortcoming, we will reformulate Theorem 2.6.10 in the following more suitable form in Corollary 2.6.11, such that the error is expressed with respect to the global degree of the polynomials. Note that although the estimate in the next corollary is not the best possible, at least this corollary could be considered as the first Weierstrass-type result in bivariate comonotone approximation with respect to the total degree of the polynomials.

Corollary 2.6.11. *(i) If* $f : [-1,1] \times [-1,1] \to \mathbb{R}$ *has the partial derivative* $g = \frac{\partial^2 f}{\partial x \partial y}$ *continuous on* $[-1,1] \times [-1,1]$ *and changes its upper bidimensional monotonicity on the proper rectangular grid in* $(-1,1) \times (-1,1)$ *determined by the distinct segments* $x = x_i$, $i \in \{1,\dots,k\}$, $y = y_i$, $i \in \{1,\dots,k\}$*, then for any* $r \geq 2$*, there exists a polynomial* $P_r(x,y)$ *of total degree* $\leq r$*, that is comonotone with* f *on* $[-1,1]^2$ *and satisfies*

$$\|f - P_r\| \leq C(k) \left[\omega_1^{\varphi} \left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{r}, \frac{1}{r} \right) + \omega_2^{\varphi} \left(f; \frac{1}{r}, \frac{1}{r} \right) \right],$$

with $C(k) > 0$ *depending only on* k.

(ii) In terms of the Ditzian–Totik moduli of smoothness introduced at the end of Definition 2.1.2 (ii), the above estimate at the point (i) can be expressed by

$$\|f - P_r\| \leq C(k) \left[\omega_1^{\varphi} \left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{r} \right) + \omega_2^{\varphi} \left(f; \frac{1}{r} \right) \right].$$

Proof. (i) For even r, take $m = n = r/2$ in Theorem 2.6.10. There exists a polynomial $P_{r/2}(x, y)$ of degrees $\leq r/2 \leq r$ in x and $\leq r/2 \leq r$ in y, that is comonotone with f and satisfies

$$\|f - P_{n,m}\| \leq C(k) \left[\omega_1^{\varphi} \left(\frac{\partial^2 f}{\partial x \partial y}; \frac{2}{r}, \frac{2}{r} \right) + \omega_2^{\varphi} \left(f; \frac{2}{r}, \frac{2}{r} \right) \right],$$

with $C(k) > 0$ depending only on k, which proves the statement.

For odd r, take $n = [r/2] \leq r$ and $m = [r/2] + 1 \leq r$ in Theorem 2.6.10. Since $n/m + m/n \leq 3$, again we easily obtain the statement.

(ii) It is immediate by Lemma 2.1.3 (v). □

For the proof of Theorem 2.6.10 some auxiliary constructions and results are necessary, as follows.

First we prove an additional approximation property of the polynomials in Theorem 2.6.9.

Theorem 2.6.12. *For any $f \in C([0, 1] \times [0, 1])$ with the continuous partial derivative $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ on $[-1, 1] \times [-1, 1]$, there exists a sequence of bivariate polynomials $P_{n,m}(f)(x, y)$, of degrees $\leq n$ in x and $\leq m$ in y, such that*

$$\|P_{n,m} - f\| \leq C\omega_2^{\varphi} \left(f; \frac{1}{n}, \frac{1}{m} \right),$$

$$\left\| \frac{\partial^2 P_{n,m}}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} \right\| \leq C\omega_1^{\varphi} \left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, \frac{1}{m} \right),$$

and $\frac{\partial^2 P_{n,m}}{\partial x \partial y} \geq 0$ on $[-1, 1] \times [-1, 1]$, where $C > 0$ is independent of f, n, m, and $\| \cdot \|$ denotes the uniform norm.

Proof. Let us consider the nodes $\xi_{i,n}$, $\eta_{j,m}$ in $[-1, 1]$ and $R_{i,n}(x), R_{j,m}(y)$ defined as in the proof of Theorem 2.6.1 (see also Gal [126]), suggested by the univariate case in Leviatan [228], and the tensor product of the polynomials in the proof of Proposition 8, p. 11, in Leviatan [232] (with the corresponding simplifications of nodes in Leviatan [228]), given by

$$Q_{n,m}(f)(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{1}{(\xi_{i+1,n} - \xi_{i,n})(\eta_{j+1,m} - \eta_{j,m})} \times \int_{\xi_{i,n}}^{\xi_{i+1,n}} \int_{\eta_{j,m}}^{\eta_{j+1,m}} f(u, v) du\, dv [R'_{i,n}(x) - R'_{i+1,n}(x)] \times [R'_{j,m}(y) - R'_{j+1,m}(y)].$$

By the estimate in the univariate case in Leviatan [232], Proposition 8, and by Haussmann–Pottinger [167], Theorem 5, we immediately get

$$\|Q_{n,m}(f) - f\| \leq C[\omega_{1,x}^{\varphi}(f; 1/n) + \omega_{1,y}^{\varphi}(f; 1/m)] \leq 2C\omega_1^{\varphi}(f; 1/n, 1/m).$$

Replacing f by $\frac{\partial^2 f}{\partial x \partial y}$ we obtain

$$\left\| Q_{n,m}\left(\frac{\partial^2 f}{\partial x \partial y}\right) - \frac{\partial^2 f}{\partial x \partial y}\right\| \le C\omega_1^{\varphi}\left(\frac{\partial^2 f}{\partial x \partial y}; 1/n, 1/m\right).$$

Let us define

$$\begin{aligned} L_{n,m}(f)(x,y) &= \int_{-1}^{x}\int_{-1}^{y} Q_{n,m}\left(\frac{\partial^2 f}{\partial u \partial v}\right)(w,z)dw\,dz \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} S^*_{i,j}[R_{i,n}(x) - R_{i+1,n}(x)][R_{j,m}(y) - R_{j+1,m}(y)], \end{aligned}$$

where $S^*_{i,j}$ has the meaning in the proof of Theorem 2.6.1.

Setting now

$$P_{n,m}(f)(x,y) = L_{n,m}(f)(x,y) + G_{m,n}(f)(x,y),$$

where

$$\begin{aligned} &G_{n,m}(f)(x,y) \\ &\quad = f(-1,-1) + \sum_{j=0}^{m-1} \frac{f(-1,\eta_{j+1,m}) - f(-1,\eta_{j,n})}{\eta_{j+1,m} - \eta_{j,m}}[R_{j,m}(y) - R_{j+1,m}(y)] \\ &\quad\quad + \sum_{i=0}^{n-1} \frac{f(\xi_{i+1,n},-1) - f(\xi_{i,n},-1)}{\xi_{i+1,n} - \xi_{i,n}}[R_{i,n}(x) - R_{i+1,n}(x)], \end{aligned}$$

it is easy to see that we reobtain exactly the polynomials in Theorem 2.6.1 (see also Gal [126], p. 29), i.e., we have

$$\|P_{n,m}(f) - f\| \le C\omega_2^{\varphi}(f; 1/n, 1/m) \text{ and } \frac{\partial^2 P_{n,m}}{\partial x \partial y} \ge 0 \text{ on } [-1,1]\times[-1,1].$$

On the other hand, since $\frac{\partial^2 P_{n,m}}{\partial x \partial y} = \frac{\partial^2 L_{n,m}}{\partial x \partial y} = Q_{n,m}(\frac{\partial^2 f}{\partial x \partial y})$, we get

$$\left\|\frac{\partial^2 P_{n,m}}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right\| \le C\omega_1^{\varphi}\left(\frac{\partial^2 f}{\partial x \partial y}; 1/n, 1/m\right). \square$$

In what follows we need to define a certain suitable extension of functions to larger bidimensional intervals and to introduce the so-called bivariate "flipped" function.

Writing $g(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y), x,y \in [-1,1]$, let us consider its extension $G(g) := G$ to $[-3,3]\times[-3,3]$, as follows: $G(x,y) = g(x,y)$ if $x,y \in [-1,1]$; $G(x,y) = g(1,y)$ if $x \in [1,3], y \in [-1,1]$; $G(x,y) = g(-1,y)$ if $x \in [-3,-1], y \in [-1,1]$; $G(x,y) = g(x,1)$ if $x \in [-1,1]$, $y \in [1,3]$; $G(x,y) = g(x,-1)$ if $x \in [-1,1], y \in [-3,-1]$; $G(x,y) = g(1,1)$ if $x,y \in [1,3]$; $G(x,y) = g(-1,1)$ if $x \in [-3,-1], y \in [1,3]$; $G(x,y) = g(-1,-1)$ if $x,y \in [-3,-1]$; $G(x,y) = g(1,-1)$ if $x \in [1,3]$, $y \in [-3,-1]$.

It is easy to prove the following.

Lemma 2.6.13. *(i) If g is continuous on $[-1,1]\times[-1,1]$ then so is $G(g)$ on $[-3,3]\times[-3,3]$.*

(ii) The functions g and $G(g)$ change signs on the same proper rectangular grid, i.e.,

$$\text{if } \Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,y) \geq 0,\ \forall x,y \in [-1,1],\ \text{ or}$$
$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,y) \leq 0,\ \forall x,y \in [-1,1],$$
$$\text{then } \Pi_{i=1}^{k}(x-x_i)(y-y_i)G(g)(x,y) \geq 0,\ \forall x,y \in [-3,3],\ \text{ or}$$
$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)G(g)(x,y) \leq 0,\ \forall x,y \in [-3,3].$$

(iii) If we set $\delta_0 = \min_{i=0,..,k}\{x_{i+1}-x_i, y_{i+1}-y_i\}$ (where $x_0 = y_0 = -1, x_{k+1} = y_{k+1} = 1$), then for all $0 \leq \delta \leq \delta_0$, $0 \leq \eta \leq \delta_0$ we have $\omega_1^{\varphi}(g;\delta,\eta)_{[-1,1]\times[-1,1]} = \omega_1^{\varphi}(G(g);\delta,\eta)_{[-3,3]\times[-3,3]}$.

Proof. (i) By definition, we easily see that $G(g)$ prolongs g by continuity, from $[-1,1]^2$ to $[-3,3]^2$, which implies the continuity of $G(g)$.

(ii) To make a choice, suppose for example that

$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,y) \geq 0, \forall x,y \in [-1,1].$$

We have nine possibilities : (1) $(x,y) \in [-1,1]\times[-1,1]$; (2) $(x,y) \in [-1,1]\times[-3,-1]$; (3) $(x,y) \in [-1,1]\times[1,3]$; (4) $(x,y) \in [-3,-1]\times[-1,1]$; (5) $(x,y) \in [1,3]\times[-1,1]$; (6) $(x,y) \in [1,3]\times[-3,-1]$; (7) $(x,y) \in [1,3]\times[1,3]$; (8) $(x,y) \in [-3,-1]\times[1,3]$; (9) $(x,y) \in [-3,-1]\times[-3,-1]$.

Case (1). Since $G(g) = g$, we get

$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,y) \geq 0,\ \forall x,y \in [-1,1].$$

Case (2). By definition we have $G(g)(x,y) = g(x,-1)$, that is

$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)G(g)(x,y) = \Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,-1)$$
$$= \frac{\Pi_{i=1}^{k}(y-y_i)}{\Pi_{i=1}^{k}(-1-y_i)}\Pi_{i=1}^{k}(x-x_i)(-1-y_i)g(x,-1) \geq 0.$$

Case (3). By definition we have $G(g)(x,y) = g(x,1)$, that is

$$\Pi_{i=1}^{k}(x-x_i)(y-y_i)G(g)(x,y) = \Pi_{i=1}^{k}(x-x_i)(y-y_i)g(x,1)$$
$$= \frac{\Pi_{i=1}^{k}(y-y_i)}{\Pi_{i=1}^{k}(1-y_i)}\Pi_{i=1}^{k}(x-x_i)(1-y_i)g(x,1) \geq 0.$$

The proofs for the other Cases (4)–(9), are similar.

(iii) The proof is very simple. First, because $G(g)$ prolongs g, it is obvious the inequality $\omega_1^{\varphi}(g;\delta,\eta)_{[-1,1]^2} \leq \omega_1^{\varphi}(G(g);\delta,\eta)_{[-3,3]^2}$. For the converse inequality, we easily observe that any difference $G(g)(u,v) - G(g)(x,y)$ with $u,v,x,y \in [-3,3]$, can be expressed in terms of a similar difference of g, that is there exist points $a,b,c,d \in [-1,1]$, such that $G(g)(u,v) - G(g)(x,y) = g(a,b) - g(c,d)$. □

Definition 2.6.14. Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$ be with partial derivative $g = \frac{\partial^2 f}{\partial x \partial y}$ continuous on $[-1,1]\times[-1,1]$ such that f changes its upper bidimensional monotonicity on the proper rectangular grid in $(-1,1) \times (-1,1)$ determined by the distinct segments $x = x_i$, $i \in \{1, \ldots, k\}$, $y = y_i$, $i \in \{1, \ldots, k\}$, such that there exists j with $x_j = y_j = 0$.

Define the "flipped" function attached to f, denoted by $F(f)$, as follows: first define $\tilde{g}(x,y) = g(x,y)$ if $xy \geq 0$, $\tilde{g}(x,y) = -g(x,y)$ if $xy < 0$ (which remains continuous on $[-1,1] \times [-1,1]$ since by the Remark 2 after Definition 2.6.4, we necessarily have $g(x_i, y) = g(x, y_i) = 0$ for all $x, y \in [-1,1]$, $i = 1, \ldots, k$). Then one takes $F(f)(x,y) = \int_0^x \int_0^y \tilde{g}(u,v) du\, dv$ for all $x, y \in [-1,1]$.

Remarks. (1) Obviously we have $\frac{\partial^2 F(f)}{\partial x \partial y} = \tilde{g}$, continuous on $[-1,1] \times [-1,1]$.

(2) Since by its definition, it easily follows that $\tilde{g}$ changes its sign on the proper rectangular grid in $(-1,1) \times (-1,1)$ determined by the distinct segments $x = x_i, i \in \{1, \ldots, k\} \setminus \{j\}$, $y = y_i, i \in \{1, \ldots, k\} \setminus \{j\}$, so on a proper rectangular grid in $(-1,1)\times(-1,1)$, determined by a number of segments with two fewer than that for g, it means that $F(f)$ changes its upper bidimensional monotonicity on a proper rectangular grid in $(-1,1) \times (-1,1)$, determined by a number of segments two less than that for f.

(3) We have $\tilde{g}(x,y) = \text{sgn}\{xy\} g(x,y)$, i.e., $\tilde{g}(x,y) = \text{sgn}\{xy\} \frac{\partial^2 f}{\partial x \partial y}(x,y)$, which immediately implies $F(f)(x,y) = \text{sgn}\{xy\}[f(x,y) - f(x,0) - f(0,y) + f(0,0)]$.

(4) In the next proofs, $C > 0$ denotes a constant that is independent of f, n, m, k and can be different at each occurrence.

We also need the following lemma.

Lemma 2.6.15. *Let $f : [-1,1]\times[-1,1] \to \mathbb{R}$ be with continuous partial derivative $g := \frac{\partial^2 f}{\partial x \partial y}$ continuous on $[-1,1] \times [-1,1]$ such that f changes its upper bidimensional monotonicity on the proper rectangular grid in $(-1,1)\times(-1,1)$, determined by the distinct segments $x = x_i, i \in \{1, \ldots, k\}, y = y_i$, $i \in \{1, \ldots, k\}$ and there exists j with $x_j = y_j = 0$ and $f(0,0) = 0$. Denoting by $F(f)$ the "flipped" function in Definition 2.6.14, let us suppose that for some $n, m \geq 1$ and some $\varepsilon > \omega_1^{\varphi}(\frac{\partial^2 F(f)}{\partial x \partial y}; 1/n, 1/m)$ there exists a bivariate polynomial $p_{n,m}(x,y)$ of degrees $\leq n$ in x and $\leq m$ in y, which is upper bidimensional comonotone with $F(f)$ on $[-1,1]^2$ and satisfies the estimates*

$$\|F(f) - p_{n,m}\| \leq \varepsilon \left(\frac{n}{m} + \frac{m}{n}\right) + C_1 \omega_2^{\varphi}\left(f; \frac{1}{n}, \frac{1}{m}\right),$$

$$\left\| \frac{\partial^2 F(f)}{\partial x \partial y} - \frac{\partial^2 p_{n,m}}{\partial x \partial y} \right\| \leq \varepsilon,$$

where $\|\cdot\|$ denotes the uniform norm on $C([-1,1] \times [-1,1])$ and $C_1 > 0$ is independent of f, n, and m.

Then there exists a polynomial $P_{2n,2m}(x,y)$ of degrees $\leq 2n$ in x and $\leq 2m$ in y, upper bidimensional comonotone with f on $[-1,1]^2$, satisfying

$$\|f - P_{n,m}\| \leq C \left[\varepsilon \left(\frac{n}{m} + \frac{m}{n} \right) + \omega_2^{\varphi} \left(f; \frac{1}{n}, \frac{1}{m} \right) \right],$$

$$\left\| \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 P_{2n,2m}}{\partial x \partial y} \right\| \leq C\varepsilon,$$

with $C > 0$ independent of f, n, and m.

Proof. First we show that in essence, the polynomial $p_{n,m}(x,y)$ in the hypothesis can be redefined to satisfy the conditions $p_{n,m}(0,0) = 0, p_{n,m}(x,0) = p_{n,m}(0,y) = 0$, for all $x, y \in [-1,1]$. Indeed, writing $\overline{p}_{n,m}(x,y) = p_{n,m}(x,y) - p_{n,m}(x,0) - p_{n,m}(0,y) + p_{n,m}(0,0)$, we have $\overline{p}_{n,m}(0,0) = p_{n,m}(x,0) = p_{n,m}(0,y) = 0$, and since we also have $F(f)(0,0) = F(f)(x,0) = F(f)(0,y) = 0$, it follows that $\|F(f) - \overline{p}_{n,m}\| \leq 4\|F(f) - p_{n,m}\|$, while $\overline{p}_{n,m}(x,y)$ has the same degrees as $p_{n,m}(x,y)$ and $\frac{\partial^2 \overline{p}_{n,m}}{\partial x \partial y} = \frac{\partial^2 p_{n,m}}{\partial x \partial y}$.

By $f(0,0) = 0$ we get

$$f(x,y) = \int_0^x \int_0^y \frac{\partial^2 f}{\partial u \partial v}(u,v) du\, dv + f(x,0) + f(0,y).$$

There exist (see, e.g., Ditzian–Totik [98]) $r_n(x)$ and $s_m(y)$ polynomials of degrees $\leq n$ in x and $\leq m$ in y, respectively, such that

$$|f(u,0) - r_n(u)| \leq C\omega_2^{\varphi}(f(\cdot,0), 1/n) \leq C\omega_{2,x}^{\varphi}(f; 1/n),\ \forall u \in [-1,1]$$

and

$$|f(0,v) - s_m(v)| \leq C\omega_2^{\varphi}(f(0,\cdot), 1/m) \leq C\omega_{2,y}^{\varphi}(f; 1/m),\ \forall v \in [-1,1].$$

Define

$$R_{2n,2m}(x,y) = \int_0^x \int_0^y \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v) q_n(u) q_m(v) du\, dv,$$

where $q_n(u)$ and $q_m(v)$ represent the–good approximation polynomials of $\text{sgn}(u)$ and $\text{sgn}(v)$, of degrees $\leq n$ in u and $\leq m$ in v, respectively, given as in the proof of the Lemma in Beatson–Leviatan [34], p. 221 (see also the proof of Lemma A in the proof of Theorem 1.6.7). Therefore $R_{n,m}(x,y)$ is a polynomial of degrees $\leq 2n$ in x and $\leq 2m$ in y.

Note that $R_{2n,2m}(x,y)$ is upper bidimensional comonotone with respect to $F(f)(x,y)\text{sgn}\{xy\} = [f(x,y) - f(x,0) - f(0,y)]$ (see Remark 3 after Definition 2.6.14). Since $f(x,y) - f(x,0) - f(0,y)$ is obviously upper bidimensional comonotone with $f(x,y)$, it follows that $R_{n,m}(x,y)$ is upper bidimensional comonotone with $f(x,y)$.

If we define now $P_{2n,2m}(x,y) = R_{2n,2m}(x,y) + r_n(x) + s_m(y)$, then obviously that $P_{2n,2m}(x,y)$ remains upper bidimensional comonotone with $f(x,y)$ and it is of the same degrees as $R_{2n,2m}(x,y)$.

Also, we have

$$\begin{aligned} &f(x,y) - P_{2n,2m}(x,y) \\ &\quad = \left[\int_0^x \int_0^y \frac{\partial^2 f}{\partial u \partial v}(u,v) du dv - R_{2n,2m}(x,y)\right] + [f(x,0) - r_n(x)] \\ &\quad\quad + [f(0,y) - s_m(y)], \end{aligned}$$

where

$$\begin{aligned} &\int_0^x \int_0^y \frac{\partial^2 f}{\partial u \partial v}(u,v) du\, dv - R_{2n,2m}(x,y) \\ &= \int_0^x \int_0^y g(u,v) du\, dv - \int_0^x \int_0^y \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v) q_n(u) q_m(v) du\, dv \\ &= \int_0^x \int_0^y [\tilde{g}(u,v) - \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v)] \mathrm{sgn}\{u\} \mathrm{sgn}\{v\} du\, dv \\ &\quad + \int_0^x \int_0^y \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v) [\mathrm{sgn}\{u\}\mathrm{sgn}\{v\} - q_n(u) q_m(v)] du\, dv := I + J. \end{aligned}$$

For the integral I, taking into account Remark 1 of Definition 2.6.14 and the above possibility to choose $p_{n,m}$, we easily get

$$\begin{aligned} I &= \int_0^x \int_0^y [\tilde{g}(u,v) - \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v)] \mathrm{sgn}\{u\} \mathrm{sgn}\{v\} du\, dv \\ &= \int_0^x \int_0^y [\frac{\partial^2 F(f)}{\partial u \partial v}(u,v) - \frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v)] \mathrm{sgn}\{u\} \mathrm{sgn}\{v\} du\, dv \\ &= \mathrm{sgn}\{x\} \mathrm{sgn}\{y\} [F(f)(x,y) - p_{n,m}(x,y)]. \end{aligned}$$

Therefore, $\|I\| \le \|F(f) - p_{n,m}\|$, which implies

$$\begin{aligned} |f(x,y) - P_{2n,2m}(x,y)| &\le \|I\| + \|J\| + C\omega_{2,x}^{\varphi}(f;1/n) + C\omega_{2,y}^{\varphi}(f;1/m) \\ &\le \|F(f) - p_{n,m}\| + 2C\omega_2^{\varphi}(f;1/n,1/m) + \|J\|. \end{aligned}$$

We used above the obvious inequality

$$\omega_{2,x}^{\varphi}(f;1/n) + \omega_{2,y}^{\varphi}(f;1/m) \le 2\omega_2^{\varphi}(f;1/n,1/m).$$

In what follows we estimate the integral J. Setting $\eta = \mathrm{sgn}\{x\}/n$, $\xi = \mathrm{sgn}\{y\}/m$, for $0 < |x| \le i/n, i = 1, \dots, n$, and $0 < |y| \le j/m, j = 1, \dots, m$, we get

$$|J| \leq \sum_{a=0}^{i-1}\sum_{b=0}^{j-1}\int_{a\eta}^{(a+1)\eta}\int_{b\xi}^{(b+1)\xi}\left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v)\right| \cdot |\text{sgn}\{u\}\text{sgn}\{v\} - q_n(u)q_m(v)| du\, dv$$
$$\leq 2\sum_{a=0}^{i-1}\sum_{b=0}^{j-1}\int_{a\eta}^{(a+1)\eta}\int_{b\xi}^{(b+1)\xi}\left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v)\right| \cdot |\text{sgn}\{u\} - q_n(u)| du\, dv$$
$$+ 2\sum_{a=0}^{i-1}\sum_{b=0}^{j-1}\int_{a\eta}^{(a+1)\eta}\int_{b\xi}^{(b+1)\xi}\left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v)\right| \cdot |\text{sgn}\{v\} - q_m(v)| du\, dv := J_1 + J_2.$$

We used above the easy-to prove inequality (taking into account that by Beatson–Leviatan [34], formula (6), we have $\|q_n\| \leq 2$)

$$|\text{sgn}\{u\}\text{sgn}\{v\} - q_n(u)q_m(v)| \leq 2|\text{sgn}\{u\} - q_n(u)| + 2|\text{sgn}\{v\} - q_m(v)|.$$

Now we estimate J_1. First, keeping the notation in Definition 2.6.14, we have

$$\left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v)\right| \leq \left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v) - \tilde{g}(u,v)\right| + |\tilde{g}(u,v)|,$$

where recall that by Remark 1 after Definition 2.6.14, we have $\frac{\partial^2 F(f)}{\partial x\partial y} = \tilde{g}$, the properties of g implies $\tilde{g}(u,0) = \tilde{g}(0,v) = \tilde{g}(0,0) = 0$, for all $u, v \in [-1,1]$, and therefore, if $|u| \leq i/n$ and $|v| \leq j/m$, then we get

$$|\tilde{g}(u,v)| \leq |\tilde{g}(u,v) - \tilde{g}(u,0)| + |\tilde{g}(0,v) - \tilde{g}(0,0)|$$
$$\leq \omega_{1,v}^{\varphi}(\tilde{g}; 2|v|) + \omega_{1,u}^{\varphi}(\tilde{g}; 2|u|)$$
$$\leq Cj\omega_{1,v}^{\varphi}(\tilde{g}; 1/m) + Ci\omega_{1,u}^{\varphi}(\tilde{g}; 1/n) \leq C(i+j)\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m).$$

The above estimates easily follow from the fact that given, for example, $u \in [-1,1]$, there are $w \in [-1,1]$ and $h > 0$, with $h \leq 2|u|$, satisfying the system $w + \frac{h}{2}\varphi(w) = u, w - \frac{h}{2}\varphi(w) = 0$, which implies that for any function H we have

$$|H(u) - H(0)| = \left|H\left(w + \frac{h}{2}\varphi(w)\right) - H\left(w - \frac{h}{2}\varphi(w)\right)\right| \leq \omega_1^{\varphi}(H; h).$$

Indeed, we choose $w = u/2$ which implies h given by the equation $h^2 = \frac{4u^2}{4-u^2}$.

As a conclusion, we get

$$\left|\frac{\partial^2 p_{n,m}}{\partial u\partial v}(u,v)\right| \leq \left\|\frac{\partial^2 p_{n,m}}{\partial u\partial v} - \tilde{g}\right\| + C(i+j)\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)$$
$$\leq \varepsilon + C(i+j)\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m),$$

for all $|u| \leq i/n$ and $|v| \leq j/m$

We can write

$$\begin{aligned} J_1 :&= \sum_{a=0}^{i-1}\sum_{b=0}^{j-1} c_{a,b} \\ &= 2\int_0^{\eta}\int_0^{\xi}\left|\frac{\partial^2 p_{n,m}}{\partial u \partial v}(u,v)\right| \cdot |\text{sgn}\{u\} - q_n(u)| du\, dv + \sum_{b=1}^{j-1} c_{0,b} + \sum_{a=1}^{i-1} c_{a,0} \\ &\quad + \sum_{a=1}^{i-1}\sum_{b=1}^{j-1} c_{a,b} := S_1 + S_2 + S_3 + S_4. \end{aligned}$$

Taking into account the above inequality, the properties of $q_n(x), q_m(y)$, and reasoning similar to the case of one real variable in Beatson-Leviatan [34], proof of lemma, pp. 221–222 (see also Lemma A in the proof of Theorem 1.6.7), we get

$$S_1 \le \frac{C}{nm}[\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m) + \varepsilon],$$

$$S_2 \le \frac{C}{nm}\sum_{b=1}^{j-1}[\varepsilon + (b+2)\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)] \le \frac{C\varepsilon}{n} + C\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)\frac{m}{n},$$

$$S_3 \le \frac{C}{nm}\sum_{a=1}^{i-1} a^{-3}[(a+2)\omega_1(\tilde{g}; 1/n, 1/m) + \varepsilon] \le \frac{C\varepsilon}{mn} + \omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)\frac{C}{nm},$$

$$\begin{aligned} S_4 &\le \frac{C}{nm}\sum_{a=1}^{i-1}\sum_{b=1}^{j-1} a^{-3}[(a+b+2)\omega_1(\tilde{g}; 1/n, 1/m) + \varepsilon] \\ &\le \frac{C\varepsilon}{n} + C\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)\frac{m}{n}. \end{aligned}$$

This implies

$$J_1 = S_1 + S_2 + S_3 + S_4 \le \frac{C\varepsilon}{n} + C\omega_1^{\varphi}(\tilde{g}; 1/n, 1/m)\frac{m}{n},$$

which for $\varepsilon > \omega_1^{\varphi}(\frac{\partial^2 F(f)}{\partial x \partial y}; 1/n, 1/m)$, gives $J_1 \le C\varepsilon\frac{m}{n}$.

Similarly, by symmetry it follows that $J_2 \le C\varepsilon\frac{n}{m}$.

As a general conclusion, we obtain $\|J\| \le C\varepsilon[m/n + n/m]$ and

$$\begin{aligned} \|f - P_{2n,2m}\| &\le \|F(f) - p_{n,m}\| + 2\omega_2^{\varphi}(f; 1/n, 1/m) + \|J\| \\ &\le C[\varepsilon(\frac{m}{n} + \frac{n}{m}) + \omega_2^{\varphi}(f; 1/n, 1/m)]. \end{aligned}$$

Reasoning as in the univariate case (see Beatson–Leviatan [34]), we get

$$\left|\frac{\partial^2 f}{\partial x\partial y}(x,y)-\frac{\partial^2 P_{2n,2m}}{\partial x\partial y}(x,y)\right| = \left|\frac{\partial^2 f}{\partial x\partial y}(x,y)-\frac{\partial^2 p_{n,m}}{\partial x\partial y}(x,y)q_n(x)q_m(y)\right|$$
$$= \left|\frac{\partial^2 F(f)}{\partial x\partial y}(x,y)-\frac{\partial^2 p_{n,m}}{\partial x\partial y}(x,y)\text{sgn}\{x\}q_n(x)\text{sgn}\{y\}q_m(y)\right|$$
$$\leq C\left\|\frac{\partial^2 F(f)}{\partial x\partial y}-\frac{\partial^2 p_{n,m}}{\partial x\partial y}\right\| \leq C\varepsilon,$$

which proves the lemma. □

Remark. It is easy to see that the polynomials $P_{2n,2m}$ of degrees $\leq 2n$ in x and $\leq 2m$ in y can in fact be modified to become of degrees $\leq n$ in x and $\leq m$ in y.

Proof of Theorem 2.6.10. Let (α,β) be a point such that the segments $(x=\alpha, y\in(-1,1))$ and $(y=\beta, x\in(-1,1))$ belong to the proper rectangular grid of changes for the upper bivariate monotonicity of f in the statement of Theorem 2.6.10, which obviously implies $\frac{\partial^2 f}{\partial x\partial y}(\alpha,\beta)=0$.

Let $h_i(x)$ and $g_j(y)$ be polynomials of degrees $\leq i$ in x and $\leq j$ in y, respectively, such that

$$|f(x,\beta)-h_i(x)| \leq C\omega_1^{\varphi}(f(\cdot,\beta);1/i) \leq C\omega_{1,x}^{\varphi}(f;1/i,0), \forall x\in[-1,1],$$

and

$$|f(\alpha,y)-g_j(y)| \leq C\omega_1^{\varphi}(f(\alpha,\cdot);1/j) \leq C\omega_{1,y}^{\varphi}(f;0,1/j), \forall y\in[-1,1].$$

For small n and m, e.g., $n,m<N(k)$ the theorem is trivial. Indeed, we obtain

$$|f(x,y)-h_n(x)-g_m(y)+f(\alpha,\beta)|$$
$$\leq |f(x,y)-f(x,\beta)-f(\alpha,y)+f(\alpha,\beta)|+|f(x,\beta)-h_n(x)|+|f(\alpha,y)-g_m(y)|$$
$$\leq |x-\alpha|\cdot|y-\beta|\left|\frac{\partial^2 f}{\partial x\partial y}(\xi,\eta)\right| + C[\omega_{1,x}^{\varphi}(f;1/n,0)+\omega_{1,y}^{\varphi}(f;0,1/m)]$$
$$\leq 2C\omega_1^{\varphi}(f;1/n,1/m)+2\left|\frac{\partial^2 f}{\partial x\partial y}(\xi,\eta)-\frac{\partial^2 f}{\partial x\partial y}(\alpha,\beta)\right|$$
$$\leq C\omega_1^{\varphi}(f;1/n,1/m)+2C\omega_1^{\varphi}(\frac{\partial^2 f}{\partial x\partial y};2,2)$$
$$\leq C\omega_1^{\varphi}(f;1/n,1/m)+C(k)\omega_2^{\varphi}(\frac{\partial^2 f}{\partial x\partial y};1/n,1/m)$$
$$\leq C(k)\left[(\frac{n}{m}+\frac{m}{n})\omega_1^{\varphi}(\frac{\partial^2 f}{\partial x\partial y};\frac{1}{n},\frac{1}{m})+\omega_2^{\varphi}(f;\frac{1}{n},\frac{1}{m})\right].$$

Therefore, we will prove the theorem for large n and m, by induction on k, with k representing the number of distinct segments parallel to the OY axis and also the number of distinct segments parallel to the OX axis, such that on the grid determined by these segments, f changes its upper bidimensional monotonicity.

The theorem is obviously valid for $k = 0$ by Theorem 2.6.12.

Assuming now that the theorem is true for $k - 1$ with $k \geq 1$, we prove it for k.

By Lemma 2.6.13 we can continuously extend g to $G(g)$ on $[-3, 3] \times [-3, 3]$, with $G(g)$ having exactly the same proper grid of changes for upper bidimensional monotonicity as g and preserving the modulus ω_1^{φ} of g.

Without loss of generality, we can suppose that $f(0, 0) = 0$ (otherwise subtract a constant from f and add it to approximation polynomials). Since $\int_0^x \int_0^y g(u, v)dudv = f(x, y) - f(x, 0) - f(0, y) + f(0, 0) = f(x, y) - f(x, 0) - f(0, y)$, let $h_1(x)$ and $h_2(y)$ be two continuously two-times differentiable extensions on $[-3, 3]$ of $f(x, 0)$ and $f(0, y)$, respectively. Then the two-times differentiable extension of $f(x, y)$ to $[-3, 3] \times [-3, 3]$ will be

$$H(f)(x, y) = \int_0^x \int_0^y G(g)(u, v)du\, dv + h_1(x) + h_2(y).$$

Also, it is evident that

$$\omega_1^{\varphi}\left(\frac{\partial^2 f}{\partial x \partial y}; \delta_1, \delta_2\right)_{[-1,1]\times[-1,1]} = \omega_1^{\varphi}\left(\frac{\partial^2 H(f)}{\partial x \partial y}; \delta_1, \delta_2\right)_{[-3,3]\times[-3,3]},$$

and that $H(f)(x, y)$ changes its upper bidimensional monotonicity exactly on the same proper grid as f.

Let $x = \alpha,\ y = \beta$ with $-1 < \alpha < 1$, $-1 < \beta < 1$ belonging to a proper grid of changing for upper bidimensional monotonicity of f, i.e., according to the Remark 2 of Definition 2.6.4, we have $\frac{\partial^2 f}{\partial x \partial y}(\alpha, \beta) = \frac{\partial^2 f}{\partial x \partial y}(\alpha, y) = \frac{\partial^2 f}{\partial x \partial y}(x, \beta) = 0$, for all $x, y \in [-1, 1]$. Let us consider the bidimensional interval centered at (α, β), $I = [\alpha - 2, \alpha + 2] \times [\beta - 2, \beta + 2] \subset [-3, 3] \times [-3, 3]$. By the change of variables $u = (x - \alpha)/2$, $v = (y - \beta)$, the function $h(u, v) = H(f)(x, y)$ is defined for $(u, v) \in [-1, 1] \times [-1, 1]$, the segments $(x = 0, y \in (-1, 1))$, $(y = 0, x \in (-1, 1))$, belong to a rectangular proper grid such that h changes its upper bidimensional monotonicity on that grid and $\omega_1^{\varphi}(\frac{\partial^2 h}{\partial x \partial y}; \delta_1, \delta_2)_{[-1,1]\times[-1,1]} \leq C\omega_1^{\varphi}(\frac{\partial^2 H(f)}{\partial x \partial y}; \delta_1, \delta_2)_{[-3,3]\times[-3,3]}$.

Now, without loss of generality, we may suppose that $h(0, 0) = 0$. Apply Lemma 2.6.15 to the "flipped" function $F(h)$. It follows that $F(h)$ changes its upper bidimensional monotonicity on a grid determined by $k - 1$ segments parallel to the OY axis and by $k - 1$ segments parallel to the OX axis, and moreover

$$\omega_1^{\varphi}\left(\frac{\partial^2 F(h)}{\partial x \partial y}; \delta_1, \delta_2\right)_{[-1,1]\times[-1,1]} \leq C\omega_1^{\varphi}\left(\frac{\partial^2 h}{\partial x \partial y}; \delta_1, \delta_2\right)_{[-1,1]\times[-1,1]}.$$

Using Lemma 2.6.15 and the inductive hypothesis, there exists a sequence of polynomials $h_{n,m}(x, y)$, $n, m \in \mathbb{N}$, upper bidimensional comonotone with h

and satisfying the estimate in its statement. Then inverting the linear changes of variables in this sequence, we get the desired approximation polynomial sequence for f, which proves the theorem. □

Remark. It is a natural question to ask whether the L-positive approximation method presented in Section 2.3 could be used to obtain approximation results in, e.g., bivariate copositive approximation, bivariate comonotone approximation, and bivariate coconvex of order $(2,2)$ approximation. The corresponding L linear bounded operators would be in these cases of the form $L(f)(x,y) = f(x,y)\Pi_{j=1}^{s}(x-x_j)(y-y_j)$, $L(f)(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)\Pi_{j=1}^{s}(x-x_j)(y-y_j)$ and $L(f)(x,y) = \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\Pi_{j=1}^{s}(x-x_j)(y-y_j)$, respectively, where x_j, y_j, $j=1,\dots,s$ are fixed. Unfortunately, the answer is negative, since no bivariate polynomial (actually no any function) e exists satisfying $L(e)(x,y) \geq 1$, for all $x, y \in [-1,1]$. Indeed, we see that for x close to any point x_j or for y close to any point y_j, these $L(e)(x,y)$ become arbitrarily close to zero.

However, in the case of bivariate bidimensional monotone approximation or $(2,2)$-convex approximation, as applications of the L-positive method, we get the following.

Corollary 2.6.16. *(i) Let $f \in W_{\infty}^{2}([-1,1] \times [-1,1])$, satisfying the relation $\frac{\partial^2 f}{\partial x \partial y}(x,y) \geq 0$ for all $x, y \in [-1,1]$.*

(i_1) For any $n \in \mathbb{N}$, $n \geq 2$, there exists $Q_n^* \in \mathcal{P}_n^2$ such that $\frac{\partial^2 Q_n^*}{\partial x \partial y}(x,y) \geq 0$ for all $x, y \in [-1,1]$ and

$$\|f - Q_n^*\|_{W_{\infty}^{2p}([-1,1]\times[-1,1])} \leq C \cdot E_{\mathcal{P}_n^2}(f; W_{\infty}^{2}([-1,1] \times [-1,1])),$$

where $C > 0$ is a constant independent of f and n.

(i_2) Also, for any $k \geq 1$, $n > k+2$, there exists $Q_n^* \in \mathcal{P}_n^2$ such that $\frac{\partial^2 Q_n^*}{\partial x \partial y}(x,y) \geq 0$, for all $x, y \in [-1,1]$ and

$$\|f - Q_n^*\|_{W_{\infty}^{2}(\Omega)} \leq C n^{-2} \max_{|b|=2} \omega_k(D^b(f); C n^{-1}).$$

(ii) Let $f \in W_{\infty}^{4}([-1,1] \times [-1,1])$, satisfying $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \geq 0$, for all $x, y \in [-1,1]$.

(ii_1) For any $n \in \mathbb{N}$, $n \geq 4$, there exists $Q_n^* \in \mathcal{P}_n^2$ such that $\frac{\partial^4 Q_n^*}{\partial x^2 \partial y^2}(x,y) \geq 0$, for all $x, y \in [-1,1]$ and

$$\|f - Q_n^*\|_{W_{\infty}^{4}([-1,1]\times[-1,1])} \leq C \cdot E_{\mathcal{P}_n^2}(f; W_{\infty}^{4}([-1,1] \times [-1,1])),$$

where $C > 0$ is a constant independent of f and n.

(ii_2) Also, for any $k \geq 1$, $n > \max\{k+2, 4\}$, there exists $Q_n^* \in \mathcal{P}_n^2$ such that $\frac{\partial^4 Q_n^*}{\partial x^2 \partial y^2}(x,y) \geq 0$, for all $x, y \in [-1,1]$ and

$$\|f - Q_n^*\|_{W_{\infty}^{4}([-1,1]\times[-1,1])} \leq C n^{-4} \max_{|b|=4} \omega_k(D^b(f); C n^{-1}).$$

Proof. (i) Defining $L(f)(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$, it is a bounded linear operator on $W^2_\infty([-1,1] \times [-1,1])$ and $e(x,y) = xy$ satisfies $L(e)(x,y) = 1$, for all $x, y \in [-1,1]$. Then, (i_1) is a direct consequence of Corollary 3.1 (and Remark 3.1) in Anastassiou–Ganzburg [16], pp. 480–481 and (i_2) is an immediate consequence of (i_1) and of the general Jackson-type estimate in Anastassiou–Ganzburg [16], p. 485, Theorem 4.1.

(ii) Defining now $L(f)(x,y) = \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)$, it is a bounded linear operator on $W^4_\infty([-1,1] \times [-1,1])$ and $e(x,y) = \frac{x^2 y^2}{4}$ satisfies $L(f)(e)(x,y) = 1$, for all $x, y \in [-1,1]$. The proof is as above. □

2.6.5 Bivariate Shape-Preserving Interpolation

We present the following two results concerning the preservation of shape by tensor product interpolation polynomials.

If $g\colon [-1,1] \to \mathbb{R}$ and $-1 < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < 1$ are the roots of Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$, then it is well known that (see, e.g., Fejér [115] or Popoviciu [314]) that the (univariate) Hermite–Fejér polynomials based on the roots above are given by $F_n(g)(x) = \sum_{i=1}^{n} h_{i,n}(x) g(x_{i,n})$, where

$$h_{i,n}(x) = \ell^2_{i,n}(x) \cdot \left[1 - \frac{\ell''_n(x_{i,n})}{\ell'_n(x_{i,n})}(x - x_{i,n})\right],$$

$$\ell_{i,n}(x) = \ell_n(x)/[(x - x_{i,n})\ell'_n(x_{i,n})], \quad \ell_n(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

Now, if $f\colon [-1,1] \times [-1,1] \to \mathbb{R}$, then according to, e.g., Shisha–Mond [352], the tensor product bivariate Hermite–Fejér polynomial is defined by

$$F_{n_1,n_2}(f)(x,y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h^{(1)}_{i,n_1}(x) h^{(2)}_{j,n_2}(y) f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}),$$

where $h^{(1)}_{i,n_1}(x)$, $x^{(1)}_{i,n_1}$, $i = \overline{1,n_1}$ and $h^{(2)}_{j,n_2}(y)$ and $x^{(2)}_{j,n_2}$, $j = \overline{1,n_2}$, are defined as in the univariate case above, $n_1, n_2 \in \mathbb{N}$.

We easily see that

$$F_{n_1,n_2}(f)(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}) = f(x^{(1)}_{i,n_1}, x^{(2)}_{j,n_2}), \quad \forall i = \overline{1,n_1},\ j = \overline{1,n_2}.$$

An extension of Theorem 1.2.4 to the bivariate case is the following.

Theorem 2.6.17. *(Anastassiou-Gal [8], see also Gal [123], p. 94, Corollary 4.2.2) Let us consider $F_{n_1,n_2}(f)(x,y)$ given as above, based on the roots of Jacobi polynomials $J_{n_1}^{(\alpha_1,\beta_1)}$, $J_{n_2}^{(\alpha_2,\beta_2)}$, of degree n_1 and n_2, respectively, with $\alpha_i, \beta_i \in (-1,0]$, $i = 1,2$. If ξ is any root of the polynomial $\ell^{(1)'}_{n_1}(x)$ and η is* *any root of the polynomial* $\ell_{n_2}^{(2)'}(y)$ *(here* $\ell_{n_1}^{(1)}(x) = \prod_{i=1}^{n_1}(x - x_{i,n_1}^{(1)})$, $\ell_{n_2}^{(2)}(y) = \prod_{j=1}^{n_2}(y - x_{j,n_2}^{(2)})$*), then there exists a constant* $c > 0$ *(independent of* n_1, n_2*, and* f*) such that if* f *is bidimensional monotone on* $[-1,1] \times [-1,1]$*, then* $F_{n_1,n_2}(f)(x,y)$ *is bidimensional monotone (of the same monotonicity) on*

$$\left(\xi - \frac{c_\xi}{n_1^{7+2\gamma_1}}, \xi + \frac{c_\xi}{n_1^{7+2\gamma_1}}\right) \times \left(\eta - \frac{c_\eta}{n_2^{7+2\gamma_2}}, \eta + \frac{c_\eta}{n_2^{7+2\gamma_2}}\right) \subset (-1,1) \times (-1,1),$$

where

$$c_\xi = \frac{c}{(1-\xi^2)^{5/2+\delta_1}}, \ c_\eta = \frac{c}{(1-\eta^2)^{5/2+\delta_2}}, \ \gamma_i = \max\{\alpha_i, \beta_i\}, \ i = 1, 2,$$

and

$$\delta_1 = \begin{cases} \alpha_1, & \text{if } 0 \le \xi < 1, \\ \beta_1, & \text{if } -1 < \xi \le 0, \end{cases}$$

$$\delta_2 = \begin{cases} \alpha_2, & \text{if } 0 \le \eta < 1, \\ \beta_2, & \text{if } -1 < \eta \le 0. \end{cases}$$

In what follows we present an extension of Theorem 1.2.5 to the bivariate case. In this sense, it is known that (see, e.g., Nicolescu [286]) $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is strictly doubly convex on $[-1,1] \times [-1,1]$, if $\Delta_{h_1}^{2,y}[\Delta_{h_2}^{2,x} f(a,b)] > 0$, for all $h_1, h_2 > 0$, $(a,b) \in [-1,1] \times [-1,1]$, with $a \pm h_2$, $b \pm h_1 \in [-1,1]$, where

$$\Delta_{h_2}^{2,x} f(\alpha, \beta) = f(\alpha + h_2, \beta) - 2f(\alpha, \beta) + f(\alpha - h_2, \beta)$$

and

$$\Delta_{h_1}^{2,y} f(\alpha, \beta) = f(\alpha, \beta + h_1) - 2f(\alpha, \beta) + f(\alpha, \beta - h_1).$$

Remark. By the mean value theorem it is easy to see that from the inequality $\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2}(x,y) > 0$ for all $(x,y) \in [-1,1]^2$, it follows that f is strictly doubly convex on $[-1,1]^2$. In fact, the doubly convex bivariate functions represent the convex functions of order $(2,2)$ in the Popoviciu sense in Definition 2.1.1 (v).

Now let $n_1, n_2 \ge 3$ be odd and let us consider as $F_{n_1,n_2}(f)(x,y)$ the tensor product Hermite–Fejér polynomial given as above, based on the roots $x_{i,n}^{(1)}$, $i = \overline{1, n_1}$, and $x_{j,n_2}^{(2)}$, $j = \overline{1, n_2}$, of the λ_1-ultraspherical polynomials $p_{n_1}^{(\lambda_1)}$ of degree n_1 and λ_2-ultraspherical polynomials $P_{n_2}^{(\lambda_2)}$ of degree n_2, respectively, $\lambda_1, \lambda_2 \in [0,1]$, and the Côtes–Christoffel numbers of the Gauss–Jacobi quadrature

$$\lambda_{i,n_1}^{(1)} := 2^{2-\lambda_1}\pi\left[\Gamma\left(\frac{\lambda_1}{2}\right)\right]^{-2}\frac{\Gamma(n_1+\lambda_1)}{\Gamma(n_1+1)}[1-(x_{i,n_1}^{(1)})^2]^{-1}\cdot[P_{n_1}^{(\lambda_1)'}(x_{i,n_1}^{(1)})]^{-2},$$
$$i=\overline{1,n_1},$$
$$\lambda_{j,n_2}^{(2)} := 2^{2-\lambda_2}\pi\left[\Gamma\left(\frac{\lambda_2}{2}\right)\right]^{-2}\frac{\Gamma(n_2+\lambda_2)}{\Gamma(n_2+1)}[1-(x_{j,n_2}^{(2)})^2]^{-1}\cdot[P_{n_2}^{(\lambda_2)'}(x_{j,n_2})]^{-2},$$
$$j=\overline{1,n_2}.$$

Theorem 2.6.18. *(Anastassiou–Gal [8], see also Gal [123], p. 95, Theorem 4.2.2) If $f\in C([-1,1]\times[-1,1])$ satisfies*

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\lambda_{i,n_1}^{(1)}\lambda_{j,n_2}^{(2)}\Delta^{2,y}_{x_{j,n_2}^{(2)}}[\Delta^{2,x}_{x_{i,n_1}^{(1)}}f(0,0)]/(x_{i,n_1}^{(1)}x_{j,n_2}^{(2)})^2>0,$$

then $F_{n_1,n_2}(f)(x,y)$ is strictly doubly convex in $V(0,0)=\{(x,y);\, x^2+y^2<d_{n_1,n_2}^2\}$, with

$$|d_{n_1,n_2}|\geq c_{f,\lambda_1,\lambda_2}\frac{n_1n_2\sum_{i=1}^{\frac{n_1-1}{2}}\sum_{j=1}^{\frac{n_2-1}{2}}\lambda_{i,n_1}^{(1)}\lambda_{j,n_2}^{(2)}\Delta^{2,y}_{x_{j,n_2}^{(2)}}[\Delta^{2,x}_{x_{i,n_1}^{(1)}}f(0,0)]/(x_{i,n_1}^{(1)}x_{j,n_2}^{(2)})^2}{(n_1+n_2)^5},$$

where $c_{f_1,\lambda_1,\lambda_2}>0$ is independent of n_1 and n_2.

Remark. For details in bivariate shape-preserving interpolation, the interested reader may consult Chapter 4 in the book Gal [123].

2.7 Bibliographical Notes and Open Problems

Theorems 2.2.2, 2.3.3, Corollaries 2.3.8–2.3.10, Theorem 2.4.2, Theorem 2.4.9, Definition 2.4.14, Theorem 2.4.15, Theorem 2.6.1 (v), Corollary 2.6.3, Definition 2.6.4, Corollary 2.6.7 (iii–v), Theorem 2.6.10, Corollary 2.6.11, Theorem 2.6.12, Lemmas 2.6.13, 2.6.15, Corollary 2.6.16 are new results.

Open Problem 2.7.1. Find shape-preserving properties of the bivariate/multivariate Bernstein–Stancu polynomials mentioned in Remark 4 after Definition 2.2.1, with respect to convexity, axial convexity, polyhedral convexity, subharmonicity, and w-subharmonicity.

Open Problem 2.7.2. For each concept of "shape" introduced in CAGD, as for e.g. axial monotonicity, axial convexity, polyhedral convexity, and so on (see Definition 2.1.5 (iv)–(x)), construct a sequence of bivariate polynomials $(P_n)_n$ on a triangle, preserving that shape, but having better approximation properties than those of the Bernstein polynomials (i.e., satisfying Jackson-type estimates in terms of a suitable modulus of smoothness $\omega_p\left(f;\frac{1}{n}\right)$, with $p\geq 1$).

Open Problem 2.7.3. It is an open question whether the estimate in Theorem 2.4.18 could be obtained in terms of the Ditzian–Totik modulus ω_2^{φ} and then in terms of ω_3^{φ}.

Open Problem 2.7.4. For $f \in C^P([-1,1] \times [-1,1])$ convex of order $(2,2)$ and $P_{n_1,n_2}(f)(x,y)$, the polynomials constructed in the proof of Corollary 2.6.3 (which give the approximation order in terms of $\omega_2^{\varphi}(f;1/n_1,1/n_2)_p, \quad 1 \le p < +\infty$), it is an open problem whether they preserve the $(2,2)$ convexity of f. Below we present some possible hints to solve this problem.

In fact, we have to check that $\frac{\partial^4 P_{n_1,n_2}(f)}{\partial x^2 \partial y^2} \ge 0$ on $[-1,1] \times [-1,1]$, which from the last form of the univariate case mentioned in the proof of Corollary 2.6.3, implies that only the term $Q_n(g)(x) = \sum_{j=2}^{n-2}[\overline{S}_j(g) - \overline{S}_{j-1}(g)]R_j(x)$ matters.

In other words, reasoning exactly as in the proof of Theorem 2.6.1 (iii), the tensor product of the above univariate polynomials $Q_n(f)(x)$ will become

$$Q_{n_1,n_2}(f)(x,y) = \sum_{j=2}^{n_1-2}\sum_{i=2}^{n_2-2}(\overline{S}_{i,j} - \overline{S}_{i,j-1} - \overline{S}_{i-1,j} + \overline{S}_{i-1,j-1})R_{i,n_2}(y)R_{j,n_1}(x),$$

where $\overline{S}_{i,j}$ and $\overline{f}(\xi_{j,n_1}, \eta_{i,n_2})$ are defined as in the proof of Theorem 2.6.3.

But as in the proof of Theorem 2.6.1 (iii), we get

$$\begin{aligned}&\overline{S}_{i,j} - \overline{S}_{i,j-1} - \overline{S}_{i-1,j} + \overline{S}_{i-1,j-1}\\ &\quad = (\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})\left[\begin{array}{cc} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ;\overline{f} \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} & \end{array}\right],\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial^4 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} &= \frac{\partial^4 Q_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2}\\ &= \sum_{j=2}^{n_1-2}\sum_{i=2}^{n_2-2}(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})\\ &\quad \times \left[\begin{array}{cc} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ;\overline{f} \\ \eta_{i-1,n_2}, \eta_{i+1,n_2}, \eta_{i,n_2} & \end{array}\right] R''_{i,n_2}(y)R''_{j,n_1}(x).\end{aligned}$$

Therefore, it remains to show that if f is convex of order $(2,2)$ then

$$\left[\begin{array}{cc} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} & \\ & ;\overline{f} \\ \eta_{i-1,n_2}, \eta_{i+1,n_2}, \eta_{i,n_2} & \end{array}\right] \ge 0,$$

for all the points ξ_{j,n_1} and η_{i,n_2}, where $\overline{f}$ is defined in the proof of Corollary 2.6.3.

Probably it could be proved that there exist the points $\xi'_{j-1,n_1} < \xi'_{j,n_1} < \xi'_{j+1,n_1}$ and $\eta_{i-1,n_2} < \eta_{i+1,n_2} < \eta_{i,n_2}$ such that

$$\left[\begin{array}{c} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ ; \overline{f} \\ \eta_{i-1,n_2}, \eta_{i+1,n_2}, \eta_{i,n_2} \end{array}\right] = C_{i,j} \left[\begin{array}{c} \xi'_{j-1,n_1}, \xi'_{j,n_1}, \xi'_{j+1,n_1} \\ ; f \\ \eta'_{i-1,n_2}, \eta'_{i+1,n_2}, \eta'_{i,n_2} \end{array}\right],$$

with $C_{i,j} > 0$ depending on the points ξ, η, ξ', and η'.

On the other hand, in Kopotun [204] for uniform aproximation, i.e., $p = +\infty$, and then in DeVore–Hu–Leviatan [87] for L^p approximation, $0 < p < +\infty$ (see Theorem 1.7.7 in Chapter 1), for continuous univariate convex functions on $[-1, 1]$, a sequence (in essence the same for all p) of convex polynomials $(P_n(f))_n$ of degree $(P_n(f)) \leq n$ is constructed such that $\|f - P_n(f)\|_p \leq C_p \omega_3^{\varphi}(f; 1/n)_p$. This sequence $(P_n(f))_n$, is nonlinear with respect to f. Because of its definition, it seems to be difficult to define the tensor product of this polynomial in the way we did in the proof of Theorem 2.6.1. Moreover, the absence of linearity does not allow us to use the error estimate for tensor products of linear approximation processes in Haussmann–Pottinger [167].

As a consequence, it remains an open question whether the error estimates in terms of $\omega_2^{\varphi}(f; 1/n_1, 1/n_2)$ in Theorem 2.6.1 (iii) and (iv) (i.e., in the cases of convexity of order $(2, 2)$ and totally convexity of order $(2, 2)$) can be reobtained in terms of $\omega_3^{\varphi}(f; 1/n_1, 1/n_2)$.

A similar open question arises in the case of L^p-approximation, $0 < p < +\infty$, for bivariate convexities of orders $(1, 1)$ and $(2, 2)$, i.e., whether the error estimate can be obtained in terms of $\omega_3^{\varphi}(f; 1/n_1, 1/n_2)_p$.

Because of the difficult matter, however, it would be satisfactory to obtain extensions of the whole of Theorem 2.6.1 to L^p-spaces, $0 < p < +\infty$, (note that Corollary 2.6.3 extends only the case of bidimensional monotonicity and $1 \leq p < +\infty$) i.e., to obtain error estimates in terms of only $\omega_2^{\varphi}(f; 1/n_1, n_2)_p$.

These last-mentioned problems would represent extensions to the bivariate case of the results in the univariate case in Yu [407], [408], Leviatan–Yu [244], DeVore–Leviatan–Yu [90] and DeVore–Leviatan [88].

Open Problem 2.7.5. Extend the results in Theorem 2.4.3 (i) and (ii) to approximation of polyharmonic functions of arbitrary order $p \in \mathbb{N}$, $f \in H_p(\text{int}(M))$, by polyharmonic polynomials in $H_p(\text{int}(M))$, so as to recapture Theorem 2.4.3 for $p = 1$.

Open Problem 2.7.6. Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be together with all partial derivatives of order 4 continuous, such that it changes the tensor product convexity of Popoviciu type of order $(2, 2)$ on a proper or degenerate rectangular grid in $[-1, 1] \times [-1, 1]$ according to the Definition 2.6.4 (iv). Construct a sequence of bivariate polynomials $(P_{n,m}(x, y))_{n,m\in\mathbb{N}}$ of degrees $\leq n$ in x and $\leq m$ in y such that each $P_{n,m}$ is coconvex of order $(2, 2)$ with f on $[-1, 1] \times [-1, 1]$ (as in the same Definition 2.6.4 (iv)), having good approximation error expressed in terms of (at least) the moduli of smoothness $\omega_1(\frac{\partial^4 f}{\partial x^i \partial y^j}; 1/n, 1/m)$ or $\omega_1^{\varphi}(\frac{\partial^4 f}{\partial x^i \partial y^j}; 1/n, 1/m)$, with $i + j = 4$.

A possible idea would be to consider the tensor product of the polynomials in coconvex approximation in the univariate case (note that this idea worked in bivariate copositive approximation and bivariate comonotone approximation, see Theorem 2.6.6 and Corollary 2.6.11, respectively).

Open Problem 2.7.7. Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ be together with all partial derivatives of order 2 continuous such that it changes the (usual) convexity on a proper or degenerate rectangular grid in $[-1, 1] \times [-1, 1]$ according to Definition 2.6.4 (vi). Construct a sequence of bivariate polynomials $(P_{n,m}(x, y))_{n,m\in\mathbb{N}}$, of degrees $\leq n$ in x and $\leq m$ in y, such that each $P_{n,m}$ is coconvex with f on $[-1, 1] \times [-1, 1]$ as in Definition 2.6.4 (vi), having good approximation error expressed in terms of (at least) the moduli of smoothness $\omega_1(\frac{\partial^2 f}{\partial x^i \partial y^j}; 1/n, 1/m)$ or $\omega_1^{\varphi}(\frac{\partial^2 f}{\partial x^i \partial y^j}; 1/n, 1/m)$, with $i + j = 2$.

A possible idea of proof would be by mathematical induction on the number of segments that determine the grid, taking into account the results in convex approximation obtained by Corollary 2.4.13 and Theorem 2.4.18.

Open Problem 2.7.8. Suppose that $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$ changes sign (or bidimensional monotonicity, or convexity of order $(2, 2)$ or usual convexity, respectively) on a finite general system of algebraic arcs and curves (contained in $[-1, 1] \times [-1, 1]$ (see Definition 2.6.4 (vii), Remark 5 after this definition, and Remark 3 after Corollary 2.6.8).

It is then an open question how to solve the problem of copositive (or upper bidimensional comonotone, or coconvex of order $(2, 2)$, or usual coconvex, respectively) approximation by algebraic bivariate polynomials in this more general setting.

A way would be to start the study with some particular simpler algebraic arcs and closed curves, as we did in the copositive case by Corollary 2.6.7, (iii), (iv), (v).

For other examples, we may start with the hypothesis that f changes its bidimensional monotonicity (or convexity of order $(2, 2)$ or usual convexity, respectively) on the first bisector of the equation $x - y = 0$ contained in $[-1, 1] \times [-1, 1]$, or on the arc of the parabola in $[-1, 1] \times [-1, 1]$ of equation $x - y^2 = 0$, or on the circle of equation $x^2 + y^2 - \frac{1}{4} = 0$, or on any combination of these three curves. Then, the question would be how well we can approximate the surface $z = f(x, y)$ by a sequence of algebraic surfaces $z = P_{n,m}(x, y)$ (i.e., $P_{n,m}(x, y)$ are polynomials of degree $\leq n$ in x and $\leq m$ in y), that are upper bidimensional comonotone (or coconvex of order $(2, 2)$, or usually coconvex, respectively) with it.

Open Problem 2.7.9. In order to have Theorem 2.6.1 (v) as a natural extension of the univariate results (in Theorem 1.7.5 and Corollary 1.7.6), the bivariate differential inequality must be satisfied for all $x, y \in [-1, 1]$, and not only for all $x, y \in [-1, 0]$. According to the proof of Theorem 2.6.1 (v) and keeping the notation as there, it would remain to prove that $R_j'(1) \leq R_j(1)$, for all $j = 1, \ldots, n - 1$.

Open Problem 2.7.10. Since any real polynomial of one real variable degree $\le n$, is obviously $(n+1)$-convex (concave) in the sense of Definition 1.1.1 (i), an interesting subject in shape-preserving approximation is how the Bernstein–Markov inequalities are preserved to higher order convex functions. In this sense, the first results were found by Popoviciu [315], who proved that if f is $(n+1)$-convex (concave) on $[-1,1]$, then for all $x \in (-1+\lambda, 1-\lambda)$, $|f'(x)| < 4(2+\sqrt{3})n^2\|f\|_\infty/[3(2-\sqrt{3})]$ and $|f'(x)| < 2(2+\sqrt{3})n\|f\|_\infty/[(2-\sqrt{3})\sqrt{1-x^2}]$, where $\lambda = 2(1-\cos(\pi/2n))/(1+\cos(\pi/2n))$.

These results were extended by Mastroianni–Szabados [267] to the whole interval $(-1,1)$, who proved, among others, that if f is $(n+1)$-convex (concave) on $[-1,1]$, then for all $n \ge 2$, $|x| < 1$, $1 \le r \le n-1$ we have $|f^{(r)}(x)| \le c_r[n/\sqrt{1-x^2} + 1/(1-x^2)]^r\|f\|_\infty$.

Popoviciu's and Mastroianni–Szabados' results were extended by Gal [133] to bivariate convex functions of higher order, by proving e.g., that if the bivariate function f is convex (concave) of order n in the sense of Popoviciu [315], p. 84 (for $n=1$ this means the usual convexity (concavity) in Definition 2.1.5, (ii)) then $|\text{grad}(f)(x,y)|_2 \le 4(2+\sqrt{3})n\|f\|/[(2-\sqrt{3})\sqrt{1-\varphi_K^2(x,y)}]$, for all $x, y \in (-1+\lambda, 1-\lambda)$, where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^2$ and $\varphi_K(x,y)$ is the Minkowski functional attached to $K = [-1,1] \times [-1,1]$. Also, similar results were obtained for convex (concave) functions of order $(n+1, m+1)$ in the sense of Definition 2.1.5 (i).

An interesting open question would be to extend the results in Gal [133] to bivariate convex functions of higher order defined on more general subsets $K \subset \mathbb{R}^2$, e.g., on symmetric or nonsymmetric convex subsets.

3

Shape-Preserving Approximation by Complex Univariate Polynomials

In this chapter we present results concerning approximation of analytic functions in the unit disk by univariate complex polynomials preserving properties in geometric function theory.

3.1 Introduction

A central concept in geometric function theory is that of univalence. Also, many sufficient conditions of geometric kind that imply univalence are important, such as starlikeness, convexity, close-to-convexity, α-convexity, spirallikeness, bounded turn, and so on.

All these geometric sufficient conditions for univalence are mainly studied for analytic functions, because in this case they can easily be expressed by nice (and simple) differential inequalities. Also, because of the *Riemann mapping theorem*, in general it suffices to study these properties on the open unit disk.

First, let us recall some classical definitions in geometric function theory. Everywhere in this chapter we denote the open unit disk by $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.

Definition 3.1.1. (see, e.g., the monograph Mocanu–Bulboacă–Sălăgean [273], Chapter 4, for (i) to (ix))

(i) A function $f : E \to \mathbb{C}$, where $E \subset \mathbb{C}$ is a domain, is called univalent in E if for all $u, v \in E$ with $u \neq v$, we have $f(u) \neq f(v)$.

(ii) Let $f : \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D}$ with $f(0) = 0$. It is called starlike in $\mathbb{D}$ with respect to the origin (or simply starlike) if it is univalent on $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain (i.e., for any $w \in f(\mathbb{D})$, the segment joining the origin with w is entirely contained in $f(\mathbb{D})$).

If, in addition, $f'(0) \neq 0$ (for example, if it is normalized, i.e., $f(0) = f'(0) - 1 = 0$), then it is well known that f is starlike in $\mathbb{D}$ if and only if $\text{Re}\left[z\frac{f'(z)}{f(z)}\right] > 0$ for all $z \in \mathbb{D}$.

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As a generalization, f is called starlike of order $\alpha \geq 0$ if f is normalized and $\text{Re}\left[z\frac{f'(z)}{f(z)}\right] > \alpha$ for all $z \in \mathbb{D}$.

We set

$$S_\alpha^*(\mathbb{D}) = \Bigg\{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic on } \mathbb{D}, f(0) = f'(0) - 1 = 0,$$
$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \text{ for all } z \in \mathbb{D}\Bigg\},$$

and $S_0^*(\mathbb{D})$ by $S^*(\mathbb{D})$.

(iii) The analytic function $f : \mathbb{D} \to \mathbb{C}$ is called convex in $\mathbb{D}$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a convex domain, i.e., for any $w_1, w_2 \in f(\mathbb{D})$, the segment joining w_1 and w_2 is entirely contained in $f(\mathbb{D})$.

If, in addition, $f'(0) \neq 0$ (e.g., if it is normalized), then it is well known that f is convex in $\mathbb{D}$ if and only if $\text{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1 > 0$ for all $z \in \mathbb{D}$.

As a generalization, f is called convex of order $\alpha \geq 0$ if f is normalized and $\text{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1 > \alpha$ for all $z \in \mathbb{D}$.

We set

$$K_\alpha(\mathbb{D}) = \Bigg\{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic on } \mathbb{D}, f(0) = f'(0) - 1 = 0,$$
$$\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha, \text{ for all } z \in \mathbb{D}\Bigg\},$$

and denote $K_0(\mathbb{D})$ by $K(\mathbb{D})$.

(iv) The analytic function $f : \mathbb{D} \to \mathbb{C}$ is called close-to-convex in $\mathbb{D}$ (with respect to φ) if there exists an analytic convex function φ in $\mathbb{D}$ such that $\text{Re}\left[\frac{f'(z)}{\varphi'(z)}\right] > 0$ for all $z \in \mathbb{D}$.

Equivalently, an analytic normalized function f is close-to-convex if there exists an analytic normalized starlike function h such that $\text{Re}\left[\frac{zf'(z)}{h(z)}\right] > 0$ for all $z \in \mathbb{D}$.

It is well known that the close-to-convexity of f implies its univalence in $\mathbb{D}$.

As a generalization, f is called close-to-convex of order $\alpha \geq 0$ and type β if there exists a convex function of order α and $\beta \in \mathbb{R}$ such that

$$\text{Re}\left[e^{i\beta}\frac{f'(z)}{g'(z)}\right] > 0, \ \forall z \in \mathbb{D}.$$

(v) Let $f : \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D}$ with $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Then f is called α-convex in $\mathbb{D}$ if

$$\text{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0$$

for all $z \in \mathbb{D}$.

The α-convex functions have a natural geometrical interpretation and for $\alpha \in [0,1]$ realize a continuous connection between starlike and convex functions (it is something in between).

Also, it is well known that for any $\alpha \in \mathbb{R}$, the class of normalized α-convex functions is included in the class of starlike normalized functions and that in fact, the condition $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \mathbb{D}$ is not necessary in the definition.

(vi) An analytic function $f : \mathbb{D} \to \mathbb{C}$ is called convex in the direction of the imaginary axis if the intersection of any parallel to the imaginary axis with $f(\mathbb{D})$ is a segment or does not intersect it. Analytically, this property is expressed by the differential inequality $\mathrm{Re}[(1-z^2)f'(z)] > 0$ for all $z \in \mathbb{D}$.

(vii) The analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ is called spirallike of type γ (where $|\gamma| < \frac{\pi}{2}$) if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a spirallike domain of type γ. We recall that a domain E that contains the origin is called spirallike of type γ, with $|\gamma| < \frac{\pi}{2}$, if for any point $w_0 \in E \setminus 0$, the γ-spiral arc joining w_0 with the origin is entirely contained in E (the equation of the γ-spiral arc is given by $w(t) = w_0 e^{-(\cos\gamma + i\sin\gamma)t}, t \in \mathbb{R}$).

It is well known that if in addition, $f'(0) \neq 0$ and $f(z) \neq 0$ for all $z \in \mathbb{D}$, then f is spirallike of type γ if and only if

$$\mathrm{Re}\left[e^{-i\gamma}\frac{zf'(z)}{f(z)}\right] > 0$$

for all $z \in \mathbb{D}$.

As a generalization, f is called spirallike of type γ and order $\alpha \geq 0$ if f is normalized and

$$\mathrm{Re}\left[e^{-i\gamma}\frac{zf'(z)}{f(z)}\right] > \alpha$$

for all $z \in \mathbb{D}$.

We set

$$S_\gamma^\alpha(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic on } \mathbb{D}, f(0) = f'(0) - 1 = 0, \right.$$
$$\left. \mathrm{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > \alpha, \text{ for all } z \in \mathbb{D} \right\}$$

and denote $S_\gamma^0(\mathbb{D})$ by $S_\gamma(\mathbb{D})$.

(viii) The analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = f'(0) - 1 = 0$ is called of bounded turn if $\arg[f'(z)] < \frac{\pi}{2}$ for all $z \in \mathbb{D}$.

It is well known that f is of bounded turn if and only if $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$.

(ix) If f, g are two analytic functions on $\mathbb{D}$, recall that g is subordinated to f (and we write $g \prec f$) if there exists an analytic function Φ on $\mathbb{D}$ with $|\Phi(z)| \leq |z|$ and $f = g \circ \Phi$.

It is well known that if f is univalent on $\mathbb{D}$, then $g \prec f$ if and only if $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$.

(x) (Sălăgean [337]) A normalized analytic function f is called n-starlike in the unit disk ($n \in \{0, 1, 2, \dots,\}$) if it satisfies $\text{Re}\left[\frac{D^{n+1}(f)(z)}{D^n(f)(z)}\right] > 0$ for all $z \in \mathbb{D}$, where $D^0(f)(z) = f(z), D(f)(z) := D^1(f)(z) = zf'(z), \dots, D^{n+1}(f)(z) = D[D^n(f)](z)$. For $n = 0$ we recapture the starlike functions, while for $n = 1$ we get the convex functions.

It is proved in Sălăgean [337] that for any $n \in \{0, 1, 2, \dots,\}$, if f is n-starlike then f is univalent in $\mathbb{D}$.

(xi) (Padmanabhan [294]) The function $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, 1 \leq q \leq p$, with $p, q \in \mathbb{N}$, analytic in $\mathbb{D}$, is called p-valently starlike in $|z| < 1$ if there exists $\rho \in (0, 1)$ such that for all $r \in (\rho, 1)$ we have

$$H(r, \theta) = \text{Re}\{re^{i\theta} f'(re^{i\theta})/f(re^{i\theta})\} > 0, \ 0 \leq \theta \leq 2\pi, \ \int_0^{2\pi} H(r, \theta) d\theta = 2\pi p.$$

The function $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \ 1 \leq q \leq p$, with $p, q \in \mathbb{N}$, analytic in $\mathbb{D}$, is called p-valently convex in $|z| < 1$ if there exists $\rho \in (0, 1)$ such that for all $r \in (\rho, 1)$ we have

$$G(r, \theta) = \text{Re}\{1 + re^{i\theta} f''(re^{i\theta})/f'(re^{i\theta})\} > 0, \ 0 \leq \theta \leq 2\pi, \int_0^{2\pi} G(r, \theta) d\theta = 2\pi p.$$

(xii) (Umezawa [391]) The function $f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \ 1 \leq q \leq p$, with $p, q \in \mathbb{N}$, analytic in $\mathbb{D}$, is called p-valently close-to-convex in $|z| < 1$ with respect to φ if there exists a p-valently convex function $\varphi(z)$ in $|z| < 1$ such that $\text{Re}\{f'(z)/\varphi'(z)\} > 0$ for all $|z| < 1$.

Remark. There are many other sufficient conditions of univalence known in geometric function theory. Some of them will be defined and used directly in some theorems of the next sections.

Returning now to the topic of this chapter, the following problem is well motivated: how well can an analytic function be approximated having a given property in geometric function theory by polynomials keeping the same property?

In what follows, first we will briefly review, without proofs, the history of this problem. To the best of our knowledge, there are three main directions of research:

(1) Approximation-preserving geometric properties by the partial sums of the Taylor expansion;
(2) Approximation-preserving geometric properties by Cesàro means and convolution polynomials;
(3) Approximation of univalent functions by subordinate polynomials in the unit disk.

First we review Direction 1.

Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. This implies that we have the expansion $f(z) = \sum_{k=0}^{+\infty} a_k z^k, \ z \in \mathbb{D}$, where $a_k \in \mathbb{C}$. So the simplest way to approximate f by polynomials would be to consider the partial sums of this expansion, i.e., $S_n(f)(z) = \sum_{k=0}^{n} a_k z^k,\ n = 1, 2, \dots$.

Now, supposing in addition that f is univalent (or starlike or convex, and so on), it is natural to ask whether $S_n(f)(z),\ n \in \mathbb{N}$, keep this property.

In this sense, we mention the following results.

Theorem 3.1.2. *(i) (Szegö [384]) If f is univalent on $\mathbb{D}$, then all the partial sums $S_n(f)(z),\ n = 1, 2, \dots$, are univalent in the disk $\mathbb{D}_{1/4} = \{z \in \mathbb{D}; |z| < \frac{1}{4}\}$.*

(ii) (Alexander [2]) Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function having the expansion $f(z) = \sum_{k=0}^{+\infty} a_k z^k,\ z \in \mathbb{D}$. If $a_k \in \mathbb{R},\ a_k \geq 0,\ \forall k \in \mathbb{N}$, and the sequence $(ka_k)_k$ is decreasing, then f together with all its partial sums $S_n(f)$ are univalent in $\mathbb{D}$. A concrete example is the univalent function $f(z) = log[\frac{z}{1-z}] = \sum_{k=1}^{\infty} \frac{1}{n} z^k$.

(iii) (Szegö [384]) If f is a normalized (i.e., of the form $f(z) = z + a_2 z^2 + \cdots +$) starlike function in $\mathbb{D}$, then all the partial sums $S_n(f),\ n \in \mathbb{N}$, are starlike in $\mathbb{D}_{1/4}$. In both cases (i) and (iii), the radius $\frac{1}{4}$, in general, cannot be improved. The function $f(z) = \frac{z}{1-z}$ provides a counterexample. Similarly, if instead of starlikeness we consider the convexity of f, then all the partial sums $S_n(f),\ n \in \mathbb{N}$, are convex in $\mathbb{D}_{1/4}$.

(iv) (Ruscheweyh [331], Ruscheweyh–Wirths [336]) If f is convex (close-to-convex, respectively) in $\mathbb{D}$, then the partial sum $S_n(f),\ n \in \mathbb{N}$, is convex (close-to-convex, respectively) in the disk $\mathbb{D}_{r_1} = \{z; |z| < r_1\}$, where r_1 is the radius of convexity of the particular polynomial $P_n(z) = z + z^2 + \cdots + z^n$ (the radius of convexity is, by definition, the greatest value r for which $P_n(z)$ is convex in $\mathbb{D}_r$). Also, if r_2 is the radius of close-to-convexity of $P_n(z)$, then $S_n(f)(z)$ is close-to-convex in $\mathbb{D}_{r_2}$ for every convex f. These results are sharp and the values r_1, r_2 are computed.

(v) (Suffridge [373], p. 236) If we denote by S the class of all analytic normalized univalent functions in $\mathbb{D}$, then the family of univalent polynomials of the form $T_n(z) = z + a_2 z^2 + \cdots + \frac{1}{n} z^n,\ n = 1, 2, \dots$, is dense in the class S (with respect to the uniform convergence on compact subsets in $\mathbb{D}$).

(vi) (Suffridge [374]) For the convex function in $\mathbb{D}$, $f(z) = e^{1+z} = \sum_{k=0}^{\infty} \frac{(1+z)^k}{k!},\ z \in \mathbb{D}$, all its partial sums $S_n(f)(z) = \sum_{k=0}^{n} \frac{(1+z)^k}{k!},\ n = 1, 2, \dots$, are convex in $\mathbb{D}$.

(vii) (MacGregor [263]) If f is univalent in $\mathbb{D}$, then there exists a sequence of polynomials $(P_n)_n$ such that $P_n \to f$ uniformly in any compact subset of $\mathbb{D}$, degree$(P_n) = n$, each P_n is univalent in $\mathbb{D}$ and $P_1 \prec P_2 \prec \cdots \prec P_n \prec \cdots \prec f$, where $\prec$ means subordination and it was defined in Definition 3.1.1 (ix).

(viii) (Padmanabhan [294]) If $f(z) = z^2 + \sum_{k=3}^{\infty} a_k z^k$ is 2-valently starlike in $|z| < 1$, then any partial sum $S_n(z) = z^2 + \sum_{k=3}^{n} a_k z^k$ is 2-valently starlike in $|z| < 1/6$ and the result is sharp.

If $f(z) = z^2 + \sum_{k=3}^{\infty} a_k z^k$ is 2-valently close-to-convex in $|z| < 1$ with respect to $\varphi(z) = z^2 + \sum_{k=3}^{\infty} A_k z^k$, then for any $n \geq 3$, the partial sum $S_n(z) = z^2 + \sum_{k=3}^{n} a_k z^k$ is 2-valently close-to-convex in $|z| < 1/6$, with respect to the partial sum $F_n(z) = z^2 + \sum_{k=3}^{n} a_k z^k$ and the result is sharp.

Remark. It is interesting to note that the univalence of polynomials in the unit disk generates some nonnegative trigonometric sums and conversely, so that the proof of univalence can be reduced to the proof of nonnegativity of some trigonometric sums. Also, for example, the vertical-convexity can be connected with the monotonicity of some trigonometric sums. For details concerning these connections, see, e.g., Gluchoff–Hartmann [149] and the references cited there and some sections in Chapter 4 in the book of Sheil–Small [346].

In the case of Direction 2, we present the following.

Theorem 3.1.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function on $\mathbb{D}$ having the Taylor expansion $f(z) = \sum_{k=1}^{+\infty} a_k z^k, \ z \in \mathbb{D}$, with $a_k \in \mathbb{C}, \ k = 1, 2, \ldots$.*

The nth Cesàro mean of order $\alpha \geq 0$ of f is defined by

$$S_n^{\alpha}(f)(z) = \frac{1}{\binom{n+\alpha-1}{n-1}} \sum_{k=1}^{n} \binom{n+\alpha-k}{n-k} a_k z^k,$$

and the de la Vallée Poussin mean of order n is defined by

$$V_n(f)(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k.$$

(i) (Fejér [115]) If f is convex with respect to a direction parallel to the imaginary axis (i.e., it is verticaly convex), then all $S_n^3(f), n = 1, 2, \ldots,$ are vertically convex.

(ii) (Robertson [322]) Denote by G the class of univalent functions on $\mathbb{D}$ of the form (i.e., normalized) $f(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \ z \in \mathbb{D}$, with the property that all the partial sums $S_n(f)(z) = z + \sum_{k=2}^{n} a_k z^k, \ n = 2, 3, \ldots,$ are also univalent in $\mathbb{D}$. For any f in the above class, also all the Cesàro means $S_n^1(f)(z), \ n \in \mathbb{N}$, are univalent in $\mathbb{D}$.

(iii) (Bustoz [51]) If $f \in G$ (with G defined above) and $k \in \mathbb{N}$, then all the Cesàro means $S_n^k(f)(z), \ n \in \mathbb{N}$, are univalent in $\mathbb{D}$.

(iv) (Lewis [245]) If f is a normalized convex function in $\mathbb{D}$, then for any $\alpha \geq 1$, all the Cesàro means $S_n^{\alpha}(f)(z), n \in \mathbb{N}$, are close-to-convex in $\mathbb{D}$.

(v) (Egerváry [112]) If f is a normalized convex function in $\mathbb{D}$, then all the Cesàro means $S_n^3(f)(z), \ n \in \mathbb{N}$, are convex in $\mathbb{D}$.

(vi) (Ruscheweyh [332]) If f is a normalized convex function in $\mathbb{D}$, then for any $\alpha > 3$, all the Cesàro means $S_n^{\alpha}(f)(z), n \in \mathbb{N}$, are convex in $\mathbb{D}$.

(vii) (Ruscheweyh [333]) If f is a normalized convex function of order $\frac{1}{2}$ in $\mathbb{D}$, then all $S_n^0(f)(z), \ n \in \mathbb{N}$; are close-to-convex of order $\frac{1}{2}$.

(viii) (Pólya–Schoenberg [308]) If f is an analytic (normalized) function in $\mathbb{D}$, then f is convex (starlike, respectively) in $\mathbb{D}$ if and only if all the de la Vallée-Poussin means $V_n(f)(z), \ n \in \mathbb{N}$, are convex (starlike, respectively) in $\mathbb{D}$. (Note that the variation-diminishing property in the case of univariate

real functions in Chapter 1, see Section in 1.1, Definition 1.1.1 (vi), is the main tool in proving these results.)

(ix) (Ruscheweyh–Sheil–Small [335]) If the (normalized) analytic function f is close-to-convex in $\mathbb{D}$, then all $V_n(f)(z)$, $n = 1, 2, \ldots$, are close-to-convex in $\mathbb{D}$.

(x) (Gal [129]) If f is an analytic normalized m-starlike function in $\mathbb{D}$ for a fixed $m \in \{0, 1, 2, \ldots, \}$, then the de la Vallée Poussin means $V_n(f)$ are m-starlike for all $n \in \mathbb{N}$.

Remark. Pólya–Schoenberg [308] observed that we can write

$$V_n(f)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ze^{it}) \Omega_n(t) dt,$$

where $\Omega_n(t) = \frac{(n!)^2}{(2n)!} (2\cos[t/2])^{2n}$ is the trigonometric polynomial called the de la Vallée Poussin kernel. (Because of their nice shape-preserving properties in Theorem 3.1.3 (viii), the de la Vallée Poussin kernels can be considered as the trigonometric counterpart of the algebraic Bernstein polynomials.) Also, Ruscheweyh [332] remarked that we can write

$$S_n^{\alpha}(f)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ze^{it}) K_n^{\alpha}(t) dt,$$

where $K_n^{\alpha}(t)$ is the nth Cesàro kernel of order α and it is a trigonometric polynomial given by

$$K_n^{\alpha}(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} D_k(t),$$

with $D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin(t/2)}$. These integral-convolution forms are more useful in approximation theory, because they easily allow one to deduce the rate of approximation. Unfortunately, in the case of the de la Vallée Poussin polynomials, the rate of approximation is rather weak (of order $\omega_1(f; \frac{1}{\sqrt{n}})$), while in the case of the Cesàro's kernel, the rate of approximation is even worse (see Gal [127]).

This shortcoming stimulated the author of the present book to search in a series of papers Gal [127, 128, 129] for convolution polynomials preserving properties in geometric function theory and having good approximation properties too.

A very nice generalization of the Pólya–Schoenberg result [308], is due to Ruscheweyh–Salinas [334]. For that purpose, first we need the following definition.

Definition 3.1.4. *(see Ruscheweyh–Salinas [334]) If f is a real 2π-periodic function, then its cyclic variation is defined by*

$$\mu_c(f) = \sup\{\mu(f(x_1), \ldots, f(x_m), f(x_1))\},$$

where $\mu(f(x_1),\ldots,f(x_m),f(x_1))$ *denotes the number of sign changes* (0 *is disregarded) in the sequence* $f(x_1),\ldots,f(x_m),f(x_1)$ *and the supremum is considered for all the finite sequences* $x_1 < x_2 < \cdots < x_m < x_1 + 2\pi,\ m \in \mathbb{N}$.

The function f is called periodically monotone (we write $f \in PM$) if $\mu_c(f-a) \le 2, \forall a \in \mathbb{R}$ (in other words, f increases in an interval (x_1,x_2) and decreases in $(x_2, x_1+2\pi)$). We denote by PM_b the functions $f \in PM$ that are bounded.

A 2π-periodic function $g : \mathbb{R} \to \mathbb{R}$, with $\int_0^{2\pi} |g(x)|dx < +\infty$ (we write $g \in L$), is called periodic monotonicity preserving (we write $g \in PMP$) if $f \star g \in PM$ for all $f \in PM_b$, where $(f\star g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)g(x-y)dy, \forall x \in \mathbb{R}$.

Theorem 3.1.5. (Ruscheweyh–Salinas [334]) *If* $g \in L$ *then* $g \in PMP$ *if and only if there exists* h *such that* $g = h$ *a.e., where* $h \in PM_b$ *satisfies the following conditions:*

(i) h *is continuous except for at most two points in a period. Also, if we set* $S = \sup\{h(x); x \in \mathbb{R}\}$, $I = \inf\{h(x); x \in \mathbb{R}\}$ *and* h *is not continuous at* t_0, *then* $|h(t_0+0) - h(t_0-0)| = S - I$.

(ii) h *is continuously differentiable in each interval where* h *neither assumes nor approaches* S *or* I. *Furthermore,* $\log|h'|$ *is concave in those intervals.*

This result is very important, because Schoenberg [343] observed that the PMP problem is identical to the characterization of integral "kernels" K that preserve the convexity of f in the sense that if $f : \mathbb{D} \to \mathbb{C}$ is an analytic convex function in $\mathbb{D}$, then $f \star K$ is analytic convex in $\mathbb{D}$. If we take as $K := K_n$ some trigonometric kernel and if K_n satisfies the conditions of g in Theorem 3.1.5, then the convolution $f \star K_n$ will produce complex polynomials preserving the convexity of f in $\mathbb{D}$.

Remark. At the end of Direction 2 of research, let us briefly describe a related result obtained by Goodman–Lee [157] for kernels K that are piecewise smooth, 2π-periodic nonnegative functions.

For f, g two 2π-periodic piecewise smooth functions and the parametric form $\gamma(t) = (f(t), g(t)),\ t \in [0, 2\pi]$, of a closed curve in $\mathbb{R}^2$, the convolution transform

$$\Gamma(x) = (\Gamma_1(x), \Gamma_2(x)) = \left(\int_0^{2\pi} K(x-t)f(t)dt, \int_0^{2\pi} K(x-t)g(t)dt\right)$$

also is a closed curve in $\mathbb{R}^2$.

First recall that the curve γ is called convex if any straight line does not intersect γ more than twice. Also, γ is called locally convex if its Wronskian is ≥ 0 on $[0, 2\pi]$, i.e.,

$$W(f', g')(x) = f'(x)g''(x) - f''(x)g'(x) \ge 0$$

for all $x \in [0, 2\pi]$. Geometrically, local convexity means that the tangent to the curve turns in the same direction as the point moves along the curve (in the positive sense).

The main result is the following.

Theorem 3.1.6. (Goodman–Lee [157]) *A necessary and sufficient condition for the convolution transform $\Gamma(x)$ to map a positively oriented convex curve γ onto the positively oriented locally convex curve Γ is that the curve $K^*(x) = (K'(x), K(x)), x \in [0, 2\pi]$, be positively oriented convex.*

Remarks. (1) In fact, in Goodman–Lee [157] is proved only the sufficient condition in Theorem 3.1.6, because the necessity part is attributed to Loewner by Schoenberg [343].

(2) Considerations on the convolution polynomials through de la Vallée Poussin and Cesàro means can also be found in Chapter 4 in the book of Sheil–Small [346].

Let us now describe Direction 3 of research.

Definition 3.1.7. (i) (see, e.g., Córdova–Ruscheweyh [71]) Let $\Omega \subset \mathbb{C}$ be a simply connected domain, with $0 \in \Omega$. Denote by $\mathcal{P}_n$, $n \in \mathbb{N}$, the set of all complex polynomials (with complex coeficients) of degree $\leq n$, $\mathcal{P}_n(\Omega) = \{p \in \mathcal{P}_n; p(0) = 0, p(\mathbb{D}) \subset \Omega\}$.

The set $\Omega_n = \bigcup_{p \in \mathcal{P}_n} p(\mathbb{D})$ is called the maximal range of these polynomials.

(ii) (Andrievskii–Ruscheweyh [26]) For $f : \mathbb{D} \to \mathbb{C}$ and a parameter $0 < s < 1$, define $f_s(z) = f[(1-s)z]$ for all $z \in \mathbb{D}$.

The results contained in Direction 3 of research can be summarized by the following.

Theorem 3.1.8. *(i) (Andrievskii–Ruscheweyh [26]) A universal constant $c > 0$ exists such that for any univalent function $f : \mathbb{D} \to \Omega$ with $f(0) = 0$, for any $n \geq 2c$, there exists a polynomial of degree n, P_n, univalent in $\mathbb{D}$, with $P_n(0) = 0$, such that*

$$f_{c/n} \prec P_n \prec f.$$

In particular,

$$f\left[\left(1 - \frac{c}{n}\right)\mathbb{D}\right] \subset \Omega_n \subset \Omega.$$

(ii) (Greiner [163]) For the above constant c, we have $\pi \leq c < 73$, where π is sharp (attained for $f(z) = \frac{z}{(1-z)^2}$).

(iii) (Greiner–Ruscheweyh [164]) If, in addition, f is convex in $\mathbb{D}$, then for any $\alpha \geq 1$, all the Cesàro means $S_n^\alpha(f)(z)$, $n \in \mathbb{N}$, are univalent in $\mathbb{D}$,

$$f_{n/(n+\alpha+1)} \prec S_n^\alpha(f) \prec f,$$

and the universal constant is given by $c = 2$, for all convex functions f.

(iv) (Greiner–Ruscheweyh [164]) If f is convex in $\mathbb{D}$, then for any $n \in \mathbb{N}$, there exists a convex univalent polynomial P_n of degree $\leq n$ such hat

$$f_{1-4/n} \prec P_n \prec f.$$

(v) (MacGregor [263]) If f is univalent in $\mathbb{D}$, then there exists a sequence of polynomials $(P_n)_n$ such that $P_n \to f$ uniformly in any compact subset of $\mathbb{D}$, degree$(P_n) = n$, each P_n is univalent in $\mathbb{D}$, and $P_1 \prec P_2 \prec \cdots \prec P_n \prec \cdots \prec f$, where $\prec$ means subordination and it was defined in Definition 3.1.1 (ix).

Remark. The main result is Theorem 3.1.8 (i) which and has a constructive proof based on an approach due to Dzyadyk [101]. More exactly, in Andrievskii–Ruscheweyh [26], the polynomial P_n is constructed by the formula

$$P_n(z) := T_{m,6}\left[\left(1 - \frac{c}{m}\right) z\right],$$

where $m = [n/7]$, $n \geq 14$, and $T_{m,k}(z)$ are algebraic polynomials of degree $\leq (k+1)(m-1) - 1$, given by the formula

$$T_{m,k}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_{m,k}(t) \frac{1}{2\pi i}\int_{|\xi|=1-1/m} \frac{f(\xi)}{\xi - z}\left[1 - \left(1 - \frac{\xi - z}{\xi e^{-it} - z}\right)^{k-1}\right] d\xi \, dt,$$

with the normalized kernel $J_{m,k}(t) = a_{m,k}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2(k+1)}$.

Briefly, the contents of the next sections in Chapter 3 can be described as follows. In Section 3.2 we use an analogous method to that in Section 1.4 (in the proof of Theorem 1.4.1) and obtain Shisha-type results for complex approximation. Then one generalize these results in an abstract setting to the so-called Re$[L]$-positive approximation.

Section 3.3 presents the results in Gal [129], obtained in Direction 2 of research, i.e., using convolution polynomials.

Section 3.4 deals with some geometric properties of the complex Bernstein polynomials in the unit disk.

Section 3.5 refers to some bibliographical remarks and open questions.

3.2 Shisha-Type Methods and Generalizations

In this section we extend some methods in the real univariate case to the case of one complex variable in the unit disk. The basic tool is represented by the simultaneous approximation results expressed by Theorem 3.2.1 (i), (ii), (iii) below, stated here for the particular case of the unit disk only.

Notice that Theorem 3.2.1 (i) below, in fact, was proved by Vorob'ev [396] for the so-called domains of type A in the complex plane (including the unit disk), while Theorem 3.2.1 (ii) was proved by Andrievskii–Pritsker–Varga [27] for general continua in the complex plane (also including the unit disk).

Unfortunately, the constants appearing in the estimates of Theorem 3.2.1 (i) and (ii) are claimed as independent of n and z only, without independence of f being mentioned too. For this reason, in the case of the unit disk, in Theorem 3.2.1 (iii) we present here a new simple proof that clearly shows that the constant can be chosen independent of n, z, and f too.

Set $A(\mathbb{D}) = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f$ is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}\}$ and for $p \in \mathbb{N}$, $A^p(\mathbb{D})$ denotes the space of p-times continuously differentiable functions on $\overline{\mathbb{D}}$.

Theorem 3.2.1. *(i) (Vorob'ev [396]) Let $p \in \mathbb{N}$. For any $f \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial P_n of degree $\leq n$ such that for all $j = 0, \ldots, p$ we have*

$$|f^{(j)}(z) - P_n^{(j)}(z)| \leq An^{j-p}\omega_1\left(f^{(p)}; \frac{1}{n}\right), \quad \forall z \in \partial\mathbb{D},$$

where A is independent of n and z. Here $\omega_1(g;\delta) = \sup\{|f(u) - f(v)|; u, v \in \overline{\mathbb{D}}, |u - v| \leq \delta\}$.

(ii) (Andrievskii–Pritsker–Varga [27]) Let us suppose that $p, q, r \in \mathbb{N}$, $f \in A^p(\mathbb{D})$, and consider the distinct points $|z_l| = 1,\ l = 1, \ldots, q$. Then, for any $n \in \mathbb{N}$, $n \geq qp + r$, there exists a polynomial P_n of degree $\leq n$ such that for all $j = 0, \ldots, p$ we have

$$|f^{(j)}(z) - P_n^{(j)}(z)| \leq cn^{j-p}\omega_r^*\left(f^{(p)}; \frac{1}{n}\right), \quad \forall z \in \partial\mathbb{D},$$

and

$$P_n^{(j)}(z_l) = f^{(j)}(z_l),\ l = 1, \ldots, q,$$

where c is independent of n and z. Here

$$\omega_r^*(g;\delta) := \sup_{z\in\overline{\mathbb{D}}}\{E_{r-1}(g; \overline{\mathbb{D}} \cap B(z;\delta))\},$$
$$B(z;\delta) = \{\xi \in \mathbb{C}; |\xi - z| \leq \delta\},$$

$$E_m(g;M) := \inf\{\|g - P\|_M; P \text{ complex polynomial of degree } \leq m\},$$

and $\|\cdot\|_M$ is the uniform norm on the set M.

(iii) (Gal [131]) Let $p \in \mathbb{N}$ and $f \in A^p(\mathbb{D})$. For any $n \geq p$, a polynomial P_n of degree $\leq n$ exists such that

$$\|f^{(k)} - P_n^{(k)}\| \leq Cn^{k-p}E_{n-p}(f^{(p)}), \text{ for all } k = 0, \ldots, p,$$

where $C > 0$ depends on p but it is independent of n and f. Here $\|\cdot\|$ denotes the uniform norm in $C(\overline{\mathbb{D}})$ and

$$E_n(f^{(p)}) = inf\{\|f^{(p)} - P\|; P \text{ is complex polynomial of degree} \leq n\}.$$

Proof. (iii) According to, e.g., Stechkin [371], p. 61, relation (0.3), the de la Vallée Poussin sums attached to a continuous 2π-periodic function g are defined by

$$\sigma_{n,m}(g)(x) = \frac{1}{m+1} \sum_{j=n-m}^{n} s_j(g)(x),$$

where $0 \leq m \leq n$, $n = 0, 1, \ldots,$ and $s_j(g)(x)$ denotes the jth Fourier partial sum attached to g. We also have (see, e.g., Stechkin [371], p. 63)

$$\sigma_{n,m}(g)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x+t) V_{n,m}(t) dt,$$

where

$$V_{n,m}(t) = \frac{1}{m+1} \sum_{k=n-m}^{n} D_k(t),$$

and $D_k(t) = \frac{1}{2} + \sum_{j=1}^{k} \cos(jt)$ represents the Dirichlet kernel of order k.

For $f \in A^p(\mathbb{D})$ and $0 \leq m \leq n$, now let us define (using the same notation $\sigma_{n,m}$)

$$\sigma_{n,m}(f)(z) = \frac{1}{m+1} \sum_{k=n-m}^{n} T_k(f)(z),$$

where $T_k(f)(z) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} z^j$ represents the kth Taylor partial sum of f.

Reasoning as in, e.g., the proof of Lemma 1, pp. 881–882 in Mujica [277], we easily get the formula

$$\sigma_{n,m}(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) V_{n,m}(t) dt \text{ for all } z \in \overline{\mathbb{D}}.$$

Let us reproduce below the proof of this integral representation. Indeed, writing $z = re^{it}$ and $f(z) = f(ze^{it}) = \sum_{k=0}^{\infty} c_k r^k e^{ikt}$, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{-ikt} dt = c_k r^k \text{ for all } k = 0, 1, \ldots.$$

Now for any $k = 1, 2, \ldots,$ we get

$$\begin{aligned} c_k r^k e^{ik\theta} &= \frac{e^{ik\theta}}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{-ikt} dt + \frac{e^{-ik\theta}}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) e^{ikt} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it}) [e^{ik(\theta - t)} + e^{ik(t-\theta)}] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(re^{it}) \cos[k(t-\theta)] dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(re^{i(t+\theta)}) \cos(kt) dt. \end{aligned}$$

This implies

$$T_n(f)(re^{i\theta}) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i(t+\theta)})\left(1+\sum_{k=1}^{n}\cos(kt)\right)dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{i(t+\theta)})D_n(t)dt,$$

which gives the required integral representation.

It is evident that $\sigma_{2n,n-p}(f)(z)$ is a complex polynomial of degree $\leq 2n$, while from the obvious property $T'_k(f)(z) = T_{k-1}(f')(z)$, we easily get

$$\sigma^{(k)}_{2n,n-p}(f)(z) = \sigma_{2n-k,n-p}(f^{(k)})(z),$$

for all $k = 0, \ldots, p$.

After a careful examination of the proof of Theorem 4, p. 69 in Stechkin [371], concerning the approximation property of $\sigma_{n,m}(f)(z)$, it is not difficult to see that we get exactly the same kind of estimate as that for the trigonometric case, similar to relationship (2.1) in the statement of Theorem 4, p. 69 in Stechkin [371]), that is,

$$\|f - \sigma_{n,m}(f)\| \leq A\sum_{j=0}^{n}\frac{E_{n-m+j}(f)}{m+j+1}, \text{ for all } 0 \leq m \leq n,$$

where $\|\cdot\|$ denotes the uniform norm in $C(\overline{\mathbb{D}})$ and $A > 0$ is an absolute constant independent of f, n, and m.

Let us briefly describe how we can use the reasoning in the trigonometric case in Stechkin [371] for the complex setting too. Indeed, the proof of Theorem 4 in Stechkin [371] is based on the previous Lemma 1, Lemma 2, and Theorem 1 there. But Lemma 1 remains valid because one refers only to the trigonometric kernels $V_{n,m}(t)$. Also, for the proof of estimate (1.5) in the statement of Lemma 2, by the maximum modulus principle we can take $|z| = 1$, that is, $z = e^{i\alpha}$ and $f(e^{i\alpha}) = G[\cos(\alpha), \sin(\alpha)] + iH[\cos(\alpha), \sin(\alpha)]$. Then, because any complex polynomial $P_{n-m}(z)$ of degree $\leq n - m$ can be written as $P_{n-m}(z) = S_{n-m}(\alpha) + iQ_{n-m}(\alpha)$, where $z = re^{i\alpha}$ and $S_{n-m}(\alpha)$, $Q_{n-m}(\alpha)$ are trigonometric polynomials of degrees $\leq n-m$ in α, we can apply the same reasoning to get the inequality of Lemma 2 for the complex setting. Furthermore, the proof of Theorem 1, pp. 66–68, is based on Lemma 2 and on some inequalities obtained by standard reasoning and that remain valid in the complex setting too. Finally, in the proof of Theorem 4, pp. 69–71, the constructions quoted in the relationships (2.2), (2.3), (2.4) obviously remain valid in the complex setting too, and also, Theorem 1 and some standard metric kind inequalities are used that remain valid in the complex setting too.

Consequently, for all $k = 0, \dots, p$ and $n \geq p$, we get

$$\begin{aligned}
&\|f - \sigma_{2n-k,n-p}(f)\| \\
&\quad \leq A \sum_{j=0}^{2n-k} \frac{E_{n+p-k+j}(f)}{n-p+j+1} \leq AE_{n+p-k}(f) \sum_{j=0}^{2n-k} \frac{1}{n-p+j+1} \\
&\quad \leq AE_{n+p-k}(f) \frac{2n-k+1}{n-p+1} \leq AE_{n+p-k}(f) \frac{2n+1}{n-p+1} \\
&\quad = AE_{n+p-k}(f)[2 + (2p-1)/(n-p+1)] \leq A(2p+1)E_{n+p-k}(f).
\end{aligned}$$

Let $P_n(z)$ denote the best-approximation polynomial of degree $\leq n$, that is, $E_n(f) = \|f - P_n\|$ (or any near-best approximation polynomial of degree $\leq n$, that is, one that satisfies $\|f - P_n\| \leq CE_n(f)$, with $C > 0$ independent of n and f).

Taking into account the above estimate, Bernstein's inequality for complex polynomials, and the well known inequality $E_n(f) \leq Cn^{-p}E_{n-p}(f^{(p)})$ for any $n \geq p$ and $k = 0, \dots, p$, we obtain (the constants C_p below can be different at each occurrence, but are independent of f and n)

$$\begin{aligned}
\|f^{(k)} - P_n^{(k)}\| &\leq \|f^{(k)} - \sigma_{2n,n-p}^{(k)}(f)\| + \|\sigma_{2n,n-p}^{(k)}(f) - P_n^{(k)}\| \\
&= \|f^{(k)} - \sigma_{2n-k,n-p}(f^{(k)})\| + \|\sigma_{2n,n-p}^{(k)}(f) - P_n^{(k)}\| \\
&\leq C_pE_{n+p-k}(f^{(k)}) + \|(\sigma_{2n,n-p}(f) - P_n)^{(k)}\| \\
&\leq C_pE_{n+p-k}(f^{(k)}) + (2n)^k\|\sigma_{2n,n-p}(f) - P_n\| \\
&\leq C_pE_{n+p-k}(f^{(k)}) + C_pn^k[\|\sigma_{2n,n-p}(f) - f\| + \|f - P_n\|] \\
&\leq C_pE_{n+p-k}(f^{(k)}) + C_pn^k[E_{n+p}(f) + E_n(f)] \\
&\leq C_pE_{n+p-k}(f^{(k)}) + C_pn^kE_n(f) \\
&\leq C_pE_{n+p-k}(f^{(k)}) + C_pn^kn^{-p}E_{n-p}(f^{(p)}) \\
&\leq C_p(n+p-k)^{-p+k}E_n(f^{(p)}) + C_pn^{-p+k}E_{n-p}(f^{(p)}) \\
&\leq C_pn^{-p+k}E_{n-p}(f^{(p)}),
\end{aligned}$$

which proves (iii). □

Remarks. (1) From the proof of Theorem 3.2.1 (iii), it easily follows that in fact we have

$$\begin{aligned}
\|f^{(k)} - \sigma_{2n,n-p}^{(k)}(f)\| &= \|f^{(k)} - \sigma_{2n-k,n-p}(f^{(k)})\| \leq C_pE_{n+p-k}(f^{(k)}) \\
&\leq C_p(n+p-k)^{-p+k}E_n(f^{(p)}) \leq C_pn^{-p+k}E_n(f^{(p)}),
\end{aligned}$$

for all $k = 0, \dots, p$ and $n \geq p$.

(2) It is natural to ask whether instead of the de la Vallée Poussin sums, we could use directly the Taylor polynomials attached to f, defined by $T_n(f)(z) =$

$\sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} z^j$. Indeed, these polynomials reproduce any polynomial of degree $\leq n$ and satisfy $T_n^{(k)}(f) = T_{n-k}(f^{(k)})$. Unfortunately, as linear operators on $A(\mathbb{D})$, the family $\{T_n, n \in \mathbb{N}\}$ is not uniformly bounded on $A(\mathbb{D})$. This shortcoming could be solved by imposing a stronger hypothesis on f, as for example that $f \in A^p(\mathbb{D}_R)$ with $R > 1$, where $\mathbb{D}_R = \{|z| < R\}$. In this case, taking into account Cauchy's estimates for the coefficients, for any fixed $1 < r < R$ we have $\frac{|f^{(j)}(0)|}{j!} \leq \frac{\|f\|_r}{r^j}$, where $\|f\|_r$ denotes the uniform norm in $C(\overline{\mathbb{D}_r})$ (for simplicity, here $\|\cdot\|_1$ is denoted simply by $\|\cdot\|$) and

$$\|T_n(f)\| \leq \frac{r}{r-1}\|f\|_r, \quad \forall n \in \mathbb{N},$$

which shows that $T_n : A(\overline{\mathbb{D}_r}) \to A(\mathbb{D}),\ n \in \mathbb{N}$, is a family of bounded linear operators. Reasoning now as in the proof of Theorem 3.2.1 (iii), let P_n be the polynomial of best approximation of degree $\leq n$ of f and let Q_{n-k} be the polynomial of best approximation of degree $\leq n-k$ of $f^{(k)}$, both on $\overline{\mathbb{D}_r}$. We obtain

$$\begin{aligned}
\|f^{(k)} - P_n^{(k)}\| &\leq \|f^{(k)} - T_{n-k}(f^{(k)})\| + \|T_n^{(k)}(f) - P_n^{(k)}\| \\
&\leq \|f^{(k)} - Q_{n-k}\| + \|Q_{n-k} - T_{n-k}(f^{(k)})\| + \|[T_n(f) - P_n]^{(k)}\| \\
&\leq E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + \|\|T_{n-k}\|\| \cdot \|f^{(k)} - Q_{n-k}\|_r + n^k \|T_n(f) - P_n\| \\
&\leq E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + \frac{r}{r-1} E_{n-k}(f^{(k)}; \overline{\mathbb{D}_r}) + n^k \|T_n[f - P_n]\| \\
&\leq C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) + \|\|T_n\|\| \cdot n^k \|f - P_n\|_r \\
&\leq C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) + C_{r,p,k} n^{-p+k} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}) \\
&\leq \frac{C_{r,p,k}}{n^{p-k}} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}).
\end{aligned}$$

Thus, for any $f \in A^p(\mathbb{D}_R)$, $p \geq 1$ (with $R > 1$), any fixed $1 < r < R$, and any $n > p$, there is a sequence of polynomials $P_n(f)$ of degree$(P_n(f)) \leq n$, $n \in \mathbb{N}$, such that for any $k = 0, 1, \ldots, p$, we have

$$\|f^{(k)} - P_n^{(k)}(f)\| \leq \frac{C_{r,p}}{n^{p-k}} E_{n-p}(f^{(p)}; \overline{\mathbb{D}_r}),\ k = 0, 1, \ldots, p,$$

where $C_{r,p} > 0$ is a constant independent of f and n. This inequality is very similar to that in the statement of Theorem 3.2.1 (iii). Here $E_n(F; \overline{\mathbb{D}_r})$ denotes the best approximation of F on $\overline{\mathbb{D}_r}$ by polynomials of degree $\leq n$.

(3) Although it is not directly connected with the topic covered by the title of this book, let us remark that the reasoning about the de la Vallée Poussin sums in the proof of Theorem 3.2.1 (iii) suggests that two interesting trigonometric inequalities of Leindler [222] and Leindler–Meir [223], generalizing the Steckin's result in [371], might hold in the complex setting too. More exactly, writing, for all nonnegative integers $0 \leq m \leq n$ and real $p \geq 1$,

$$G_{n,m}(f)(z) = \frac{1}{m+1} \sum_{k=n-m}^{n} |T_k(f)(z) - f(z)|$$

and

$$G_{n,m,p}(f)(z) = \left(\frac{1}{m+1} \sum_{k=n-m}^{n} |T_k(f)(z) - f(z)|^p \right)^{1/p},$$

we conjecture that for any $f \in A(\mathbb{D})$, we have

$$\|G_{n,m}(f)\| \leq K \sum_{j=0}^{n} \frac{E_{n-m+j}(f)}{m+j+1}$$

and

$$\|G_{n,m,p}(f)\| \leq C[log(n)]^{1-1/p} \left(\sum_{k=0}^{n} \frac{E^p_{n-m+k}(f)}{m+k+1} \right)^{1/p},$$

where $C > 0$ is an absolute constant. The complete proofs of these two inequalities remain open questions.

3.2.1 Shisha-Type Approximation

The first main result is the following.

Theorem 3.2.2. (Gal [131]) *Let us consider the integers $h, k, p \in \mathbb{N}$, $0 \leq h \leq k \leq p$, the functions $a_j : \overline{\mathbb{D}} \to \mathbb{C}$, continuous on $\overline{\mathbb{D}}$ for all $j = h, \ldots, k$ such that $a_h(z) = 1$, for all $z \in \overline{\mathbb{D}}$, the distinct interpolation points $|z_i| \leq 1$, $i = 1, \ldots, h$ (by convention, if $h = 0$, then we do not have any interpolation point), and the complex differential operator on $A^p(\mathbb{D})$ defined by $L(f)(z) = \sum_{j=h}^{k} a_j(z) f^{(j)}(z)$, $z \in \mathbb{D}$.*

For any $f \in A^p(\mathbb{D})$ satisfying $\mathrm{Re}[L(f)(z)] \geq 0$ for all $z \in \mathbb{D}$, and any $n \in \mathbb{N}$, $n \geq p$, there exists a complex polynomial $P_n(z)$ of degree $\leq n$ satisfying

$$\|f - P_n\| \leq C n^{k-p} E_{n-p}(f^{(p)})$$

(with C independent of n, f), $\mathrm{Re}[L(P_n)(z)] \geq 0$ for all $z \in \mathbb{D}$ and $P_n(z_i) = f(z_i)$, $i = 1, \ldots, h$ (if $h = 0$ then we don't have interpolative conditions). Here recall that $\| \cdot \|$ denotes the uniform norm in $C(\overline{\mathbb{D}})$.

Proof. By Theorem 3.2.1 (iii), for any $F \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial $p_n(z)$ of degree $\leq n$ such that

$$\|F^{(j)} - p_n^{(j)}\| \leq C n^{j-p} E_{n-p}(F^{(p)}),$$

$j = 0, 1, \ldots, p$, with $C > 0$ independent of n and F.

Defining $q_n(z) = p_n(z) + Q(F - p_n)(z)$, where $Q(F - p_n)(z)$ represents the Lagrange interpolation polynomial attached to $F - p_n$ at the points $z_1, \dots, z_h$, we easily obtain $q_n(z_i) = F(z_i),\ i = 1, \dots, h$ and

$$\|q_n - F\| \le \|p_n - F\| + \|Q(F - p_n)\| \le c_1 n^{-p} E_{n-p}(F^{(p)}),$$

with c_1 independent of n and F.

From the continuity of each $a_j(z)$ on $\overline{\mathbb{D}}$, setting $A_j = \|a_j\|$, $j = h, \dots, k$, we get $A_h = 1$ and there exists $M > 0$ with $A_j \le M,\ j = h+1, \dots, k$. Since $c_1 E_{n-p}(f^{(p)}) \sum_{j=h}^{k} A_j n^{j-p} \le c_1 \cdot \max\{1, M\}(k-h+1) n^{k-p} E_{n-p}(f^{(p)}) =: \eta_n$, taking $F(z) = f(z) + \eta_n[(z - z_1) \cdots (z - z_h)]/(h!)$ (if $h = 0$ then $F(z) = f(z) + \eta_n$), let $P_n(z)$ be the polynomial of degree $\le n$ satisfying $P_n(z_i) = F(z_i),\ i = 1, \dots, h$ and

$$\|F^{(j)} - P_n^{(j)}\| \le c_1 n^{j-p} E_{n-p}(F^{(p)}) = c_1 n^{j-p} E_{n-p}(f^{(p)}),\ j = 0, 1, \dots, p.$$

(Here c_1 is independent of n and F, therefore independent of f too.)

First, it is clear that $P_n(z_i) = F(z_i) = f(z_i),\ i = 1, \dots, h$.

Also, we obtain

$$\begin{aligned} \|f - P_n\| &\le 2^h \eta_n (h!)^{-1} + c_1 n^{-p} E_{n-p}(f^{(p)}) \\ &\le C n^{k-p} E_{n-p}(f^{(p)}), \end{aligned}$$

with C independent of n and f, which implies the estimate in the theorem.

On the other hand, if $z \in \mathbb{D}$ (keeping the convention in the $h = 0$ case), it is easy to see that

$$\begin{aligned} L(P_n)(z) &= L(f)(z) + \eta_n \\ &\quad + \sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z - z_1) \cdots (z - z_h)]\eta_n/(h!)\}^{(j)}, \end{aligned}$$

and we obtain

$$\begin{aligned} \mathrm{Re}[L(P_n)(z)] &= \mathrm{Re}[L(f)(z)] + \eta_n \\ &\quad + \mathrm{Re}\left\{ \sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) \right. \\ &\quad \left. - [(z - z_1) \cdots (z - z_h)]\eta_n/(h!)\}^{(j)} \right\}. \end{aligned}$$

Now, by

$$\begin{aligned} &\left| \mathrm{Re}\left\{ \sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z - z_1) \cdots (z - z_h)]\eta_n/(h!)\}^{(j)} \right\} \right| \\ &\le \left| \sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z - z_1) \cdots (z - z_h)]\eta_n/(h!)\}^{(j)} \right| \\ &\le \max\{1, M\} c_1 \cdot (k - h + 1) n^{k-p} E_{n-p}(f^{(p)}) = \eta_n, \end{aligned}$$

we get

$$\eta_n + \mathrm{Re}\left\{ \sum_{j=h}^{k} a_j(z)\{P_n(z) - f(z) - [(z - z_1)\cdots(z - z_h)]\eta_n/(h!)\}^{(j)} \right\} \geq 0,$$

and since $\mathrm{Re}[L(f)(z)] \geq 0$, we finally obtain $\mathrm{Re}[L(P_n)(z)] \geq 0$ for all $z \in \mathbb{D}$. □

Remarks. (1) It is easy to see that the statement of Theorem 3.2.2 remains valid if we replace the real part "Re" of the corresponding quantities with the imaginary part "Im".

(2) If in Theorem 3.2.2 we take $\mathrm{Re}[L(f)(z)] > 0$ for all $z \in \mathbb{D}$, then from the proof it easily follows that $\mathrm{Re}[L(P_n)(z)] > 0$ for all $z \in \mathbb{D}$, $n \geq p$.

Another consequence of Theorem 3.2.1 (iii) is the following.

Corollary 3.2.3. (Gal [131]) *Let us consider the integers* $h, k, p \in \mathbb{N}$, $0 \leq h \leq k \leq p$, *the functions* $a_j : \overline{\mathbb{D}} \to \mathbb{C}$ *continuous on* $\overline{\mathbb{D}}$ *for all* $j = h, \ldots, k$ *such that* $a_h(z) = 1$ *for all* $z \in \overline{\mathbb{D}}$, *the point* $|z_0| \leq 1$, *and the complex differential operator on* $A^p(\mathbb{D})$ *defined by* $L(f)(z) = \sum_{j=h}^{k} a_j(z) f^{(j)}(z)$, $z \in \mathbb{D}$.

For any $f \in A^p(\mathbb{D})$ *satisfying* $\mathrm{Re}[L(f)(z)] \geq 0$ *for all* $z \in \mathbb{D}$, *and for every* $n \in \mathbb{N}$, $n \geq p$, *there exists a complex polynomial* $P_n(z)$ *of degree,* $\leq n$, *satisfying*

$$\|f - P_n\| \leq C n^{k-p} E_{n-p}(f^{(p)})$$

(with C *independent of* n, f*),* $\mathrm{Re}[L(P_n)(z)] \geq 0$ *for all* $z \in \mathbb{D}$, *and* $P_n^{(i)}(z_0) = f^{(i)}(z_0)$, $i = 0, \ldots, h$.

Proof. As in the proof of Theorem 3.2.2, for any $F \in A^p(\mathbb{D})$ and $n \geq p$, there exists a polynomial $p_n(z)$ of degree $\leq n$ such that

$$\|F^{(j)} - p_n^{(j)}\| \leq c n^{j-p} E_{n-p}(F^{(p)}),$$

$j = 0, 1, \ldots, p$.

Define now $q_n(z) = p_n(z) + T_h(z)$, where $T_h(z)$ denotes the Taylor polynomial of degree h attached to the point z_0 and to $F - p_n$, that is, $T_h(z) = \sum_{j=0}^{h} \frac{(z-z_0)^j}{j!}[F - p_n]^{(j)}(z_0)$.

We easily get $q_n^{(j)}(z_0) = F^{(j)}(z_0)$, $j = 0, \ldots, h$, and

$$\|q_n - F\| \leq c_1 \sum_{j=0}^{h} \|F^{(j)} - p_n^{(j)}\| \leq c_1 n^{k-p} E_n(F^{(p)}),$$

with $c_1 > 0$ independent of n and f, since $h \leq k$.

Defining $\eta_n = c_1 \max\{1, M\}(k - h + 1) n^{k-p} E_{n-p}(f^{(p)})$ (where M is given in the proof of Theorem 3.2.2) and taking $F(z) = f(z) + \eta_n (z - z_0)^h/(h!)$, let $P_n(z)$ be the polynomial of degree $\le n$ satisfying $P_n^{(i)}(z_0) = F^{(i)}(z_0),\ i = 0, \dots, h$, and

$$\|F^{(j)} - P_n^{(j)}\| \le c_1 n^{j-p} E_{n-p}(F^{(p)}) = c_1 n^{j-p} E_{n-p}(f^{(p)}) \le C n^{k-p} E_{n-p}(f^{(p)}),\ j = 0, 1, \dots, k.$$

From here the proof is identical to that of Theorem 3.2.2. □

We present some applications of Corollary 3.2.3.

Theorem 3.2.4. (Gal [131]) *(i) Let $p \in \mathbb{N}$. For any $f \in A^p(\mathbb{D})$ normalized in $\mathbb{D}$ (that is, $f(0) = f'(0) - 1 = 0$) satisfying $\mathrm{Re}[f'(z)] > 0$ for all $z \in \mathbb{D}$ and any $n \ge p$, there exists a polynomial P_n of degree $\le n$ such that $P_n(0) = f(0)$, $P_n'(0) = f'(0)$, $\mathrm{Re}[P_n'(z)] > 0$ for all $z \in \mathbb{D}$, and*

$$\|f - P_n\| \le C \frac{1}{n^{p-1}} E_{n-p}(f^{(p)}),$$

where C is independent of n and f.

(ii) Let $p \in \mathbb{N}$, $p \ge 2$. For any $f \in A^p(\mathbb{D})$ normalized in $\mathbb{D}$ and any $n \in \mathbb{N}$, $n \ge p$, there exists a polynomial P_n of degree $\le n$ such that $P_n(0) = f(0), P_n'(0) = f'(0)$,

$$\|f - P_n\| \le C \frac{1}{n^{p-2}} E_{n-p}(f^{(p)}),$$

with C independent of n, f, that in addition has the following properties (the choice of $P_n(z)$ depends on the property):

(a) If $\mathrm{Re}[f'(z) + zf''(z)] > 0$ for all $z \in \mathbb{D}$, then $\mathrm{Re}[P_n'(z) + zP_n''(z)] > 0$ $\forall z \in \mathbb{D}$.

(b) If $\mathrm{Re}[f'(z) + \frac{1}{2}zf''(z)] > 0$ for all $z \in \mathbb{D}$, then $\mathrm{Re}[P_n'(z) + \frac{1}{2}zP_n''(z)] > 0$ for all $z \in \mathbb{D}$.

(c) If $\mathrm{Re}[f'(z) + \frac{1}{\gamma}zf''(z)] > 0$ for all $z \in \mathbb{D}$, where $-1 < \gamma \le \gamma_0 = 1.869\dots$, then $\mathrm{Re}[P_n'(z) + \frac{1}{\gamma}zP_n''(z)] > 0$ $\forall z \in \mathbb{D}$.

(iii) Let $p \in \mathbb{N}$, $p \ge 2$. For any $g \in A^p(\mathbb{D})$ satisfying $g(0) = a$, with $\mathrm{Re}[a] > 0$ and $\mathrm{Re}[g(z) + zg'(z) + z^2 g''(z)] > 0$ for all $z \in \mathbb{D}$, and any $n \in \mathbb{N}$, $n \ge p$, there exists a polynomial P_n of degree $\le n$ such that $P_n(0) = g(0)$, $\mathrm{Re}[P_n(z) + zP_n'(z) + z^2 P_n''(z)] > 0$ for all $z \in \mathbb{D}$, and

$$\|g - P_n\| \le C \frac{1}{n^{p-2}} E_{n-p}(g^{(p)}),$$

where C is independent of n and f.

(iv) Let $p \in \mathbb{N}$. For any $g \in A^p(\mathbb{D})$ satisfying $g(0) = a$, with $\mathrm{Re}[a] > 0$ and $\mathrm{Re}[g(z) + zB(z)g'(z)] > 0$, with $\mathrm{Re}[B(z)] > 0$ for all $z \in \mathbb{D}$, and any $n \in \mathbb{N}$, $n \ge p$, there exists a polynomial P_n of degree $\le n$ such that $P_n(0) = g(0)$, $\mathrm{Re}[P_n(z) + zB(z)P_n'(z)] > 0$ for all $z \in \mathbb{D}$, and

$$\|g - P_n\| \le C \frac{1}{n^{p-1}} E_{n-p}(g^{(p)}),$$

where C is independent of n and f.

Proof. (i) Take $L(f)(z) = f'(z)$ and apply Remark 2 of Theorem 3.2.2 and Corollary 3.2.3 for $z_0 = 0$, $h = k = 1$.

(ii), (a), (b), and (c) Take $L(f)(z) = f'(z) + zf''(z)$ (or $L(f)(z) = f'(z) + \frac{z}{2}f''(z)$, respectively) and apply Remark 2 of Theorem 3.2.2 and Corollary 3.2.3 for $z_0 = 0$, $h = 1$, $k = 2$.

(iii) Apply Remark 2 of Theorem 3.2.2 and Corollary 3.2.3 for $z_0 = 0$, $h = 0$, and $k = 2$.

(iv) Apply Remark 2 of Theorem 3.2.2 and Corollary 3.2.3 for $z_0 = 0$, $h = 0$, and $k = 1$. □

Remarks. (1) By, e.g., Mocanu–Bulboacă–Sălăgean [273], p. 78, it is well known that if f is normalized and satisfies the condition $\text{Re}[f'(z)] > 0$ for all $z \in \mathbb{D}$, then f is univalent and of bounded turn in $\mathbb{D}$ (i.e., $|\arg[f'(z)]| < \frac{\pi}{2}$, for all $z \in \mathbb{D}$). Consequently, the approximation polynomials P_n, $n \geq p$, are univalent and of bounded turn on $\mathbb{D}$.

(2) By, e.g., Singh–Singh [358], Mocanu [272], and Mocanu–Bulboacă–Salagean [273], p. 358, respectively, the fact that f is normalized together with any of the three conditions (a), (b), and (c) in Theorem 3.2.4 (ii) implies the starlikeness of f (and as a consequence the starlikeness of P_n too) in $\mathbb{D}$ (for the above sufficient conditions of starlikeness (a) and (b), see also Mocanu–Bulboacă–Sălăgean [273], p. 363).

(3) By, e.g., Mocanu–Bulboacă–Sălăgean [273], Problem 9.6.5 (ii), p. 221, the conditions satisfied by g in Theorem 3.2.4 (iii) imply $\text{Re}[g(z)] > 0$ for all $z \in \mathbb{D}$.

(4) By, e.g., Mocanu–Bulboacă–Sălăgean [273], p. 192, the conditions in Theorem 3.2.4 (iv) imply $\text{Re}[g(z)] > 0$ for all $z \in \mathbb{D}$.

Similar results are given by the following.

Theorem 3.2.5. (Gal [131]) *(i) If $f \in A(\mathbb{D})$ satisfies $\text{Re}[f(z)] > 0$ for all $z \in \mathbb{D}$ and $E_n(f) \neq 0$ for all $n \in N$, then for each $n \in \mathbb{N}$, there exists P_n, a complex polynomial of degree $\leq n$, satisfying $\|P_n - f\| \leq 2E_n(f)$ and $\text{Re}[P_n(z)] > 0$ for all $z \in \mathbb{D}$.*

(ii) Let $f \in A^1(\mathbb{D})$ be such that there exists $\gamma \in \mathbb{R}$, with $\text{Re}[e^{i\gamma}f'(z)] > 0$ for all $z \in \mathbb{D}$ and suppose that $E_n(f'; 1/n) \neq 0$ for all $n \in N$. Then for any $n \geq 1$, there exists a polynomial $Q_n(f)(z)$ of degree $\leq n$ satisfying $\text{Re}[e^{i\gamma}Q_n'(z)] > 0$ for all $z \in \mathbb{D}$ and $\|Q_n(f) - f\| \leq cE_{n-1}(f')$, where c is independent of n and f.

Proof. (i) Let P_n^* be the polynomial of degree $\leq n$ satisfying $\|f - P_n^*\| = E_n(f)$. Since $|\text{Re}[P_n^*(z) - f(z)]| \leq \|f - P_n^*\| = E_n(f) < 2E_n(f)$ and $\text{Re}[P_n(z) - f(z)] = \text{Re}[P_n^*(z) - f(z)] + 2E_n(f) > 0$, it is easy to see that $P_n(z) = P_n^*(z) + 2E_n(f)$ satisfies the required conditions.

(ii) From Theorem 3.2.1 (iii), there exists $P_n(z)$ satisfying $\|P_n - f\| \leq c\frac{1}{n}E_{n-1}(f') =: \alpha_n$, and $\|P_n' - f'\| \leq cE_{n-1}(f') =: \beta_n$.

Setting $Q_n(z) = P_n(z) + z\frac{2\beta_n}{\cos(\gamma)}$, we get $\|Q_n - f\| \le \|P_n - f\| + 2\beta_n \le \alpha_n + 2\beta_n$, which proves the approximation estimate.

Also,

$$\begin{aligned}\mathrm{Re}[e^{i\gamma}(Q_n'(z) - f'(z))] &= \mathrm{Re}[e^{i\gamma}(P_n'(z) - f'(z))] + 2\beta_n \mathrm{Re}\left[\frac{e^{i\gamma}}{cos(\gamma)}\right] \\ &= \mathrm{Re}[e^{i\gamma}(P_n'(z) - f'(z))] + 2\beta_n > 0,\end{aligned}$$

since $|\mathrm{Re}[e^{i\gamma}(P_n'(z) - f'(z))]| \le |e^{i\gamma}[P_n'(z) - f'(z)]| \le \|P_n' - f'\| \le \beta_n < 2\beta_n$, which proves the theorem. □

Remark. It is well known that $\mathrm{Re}[e^{i\gamma} f'(z)] > 0$ for all $z \in \mathbb{D}$ is the Noshiro, Warschawski, Wolff's sufficient condition of univalence for f (see also, e.g., Mocanu–Bulboacă–Sălăgean [273], p. 78).

3.2.2 Re[*L*]-Positive Approximation

In this subsection we generalize the above Shisha-type methods and results to an abstract setting by extending the abstract method of L-positive approximation in Section 2.3 to a complex variable.

Let $(X, \|\cdot\|_X)$ be a complex normed space, S a linear subspace of X, $M \subset X$ with $S \cap M \ne \emptyset$, and $f \in X$. Then the best approximation of f by elements from S or from $S \cap M$ are defined by

$$E_S(f : X) = \inf_{g \in S}\{\|f - g\|_X\}$$

and

$$E_{S,M}(f; X) = \inf_{g \in S \cap M}\{\|f - g\|_X\},$$

respectively.

We have $E_S(f; X) \le E_{S,M}(f; X)$ for all $f \in X$, and if in addition, S is finite-dimensional, we recall (see Section 2.3, Remark 2 after Definition 2.3.4) that then there exist $g^* \in S$ and $g_M^* \in S \cap M$ such that $E_S(f; X) = \|f - g^*\|_X$ and $E_{S,M}(f; X) = \|f - g_M^*\|_X$.

For $\Omega \subset \mathbb{C}$ let us denote by $(\mathcal{F}(\Omega), \|\cdot\|_{\mathcal{F}})$ a normed space of complex-valued functions defined on Ω and let $L : X \to \mathcal{F}(\Omega)$ be a linear bounded operator, i.e., $|||L||| = \sup\{\|L(x)\|_{\mathcal{F}}; \|x\|_X \le 1\} < \infty$. Also, set $M[Re(L)] = \{f \in X; \mathrm{Re}[L(f)(z)] \ge 0, \text{ for all } z \in \Omega\}$ and $M[\mathrm{Im}(L)] = \{f \in X; \mathrm{Im}[L(f)(z)] \ge 0,$ for all $z \in \Omega\}$.

The goal is under some conditions on L and $S \subset X$ to obtain estimates of the form

$$E_{S,M[\mathrm{Re}(L)]}(f; X) \le CE_S(f; X),\ \forall f \in X,$$

and

$$E_{S,M[\mathrm{Im}(L)]}(f;X) \leq CE_S(f;X),\ \forall f \in X,$$

where C is a constant independent of f and S.

The first main result is the following.

Theorem 3.2.6. *For $(X, \|\cdot\|_X)$ a complex normed space and $S \subset X$ a linear subspace, let us consider a family of linear bounded operators $L_j : X \to \mathcal{F}_j(\Omega_j)$, $\Omega_j \subset \mathbb{C}$, where $(\mathcal{F}_j(\Omega_j), \|\cdot\|_{\mathcal{F}_j})$ are normed spaces of complex-valued functions defined on Ω_j for all $j \in J$, respectively (J an arbitrary set), satisfying the following conditions:*

(i) $\sup_{j\in J} |||L_j||| < \infty$;

(ii) there is an element $e \in S$ such that for any $j \in J$ we have

$$\mathrm{Re}[L_j(e)(z)] \geq 1,\ \textit{for all } z \in \Omega_j.$$

Then, for every $f \in X$ and $P \in S$, there exist $Q_i \in S$, $i = 1, 2$, such that

$$(-1)^{i+1}\mathrm{Re}[L_j(Q_i - f)(z)] \geq 0,\ z \in \Omega_j,\ j \in J,$$

and the estimate

$$\|f - Q_i\|_X \leq (1 + \|e\|_X sup_{j\in J} |||L_j|||) \cdot \|f - P\|_X,\ i = 1, 2,$$

holds.

Proof. Writing $Q_i = P + (-1)^{i+1}\lambda e$, where $\lambda = \sup_{j\in J}\{|||L_j||| \cdot \|f - P\|_X$, for any $i = 1, 2$, we obtain

$$(-1)^{i+1}L_j(Q_i - f)(z) = (-1)^{i+1}L_j(P - f)(z) + \lambda L(e)(z).$$

Taking the real part, it follows, that

$$\begin{aligned}(-1)^{i+1}\mathrm{Re}[L_j(Q_i - f)(z)] &= (-1)^{i+1}\mathrm{Re}[L_j(P - f)(z)] + \lambda\mathrm{Re}[L(e)(z)]\\ &\geq (-1)^{i+1}\mathrm{Re}[L_j(P - f)(z)] + \lambda \geq 0,\end{aligned}$$

since $|(-1)^{i+1}\mathrm{Re}[L_j(P - f)(z)]| \leq |L_j(P - f)(z)| \leq |||L_j||| \cdot \|P - f\|_X \leq \lambda$.

In addition,

$$\|f - Q_i\|_X \leq \|f - P\|_X + \lambda\|e\|_X = (1 + \|e\|_X \sup_{j\in J} |||L_j|||)\|f - P\|_X, \quad i = 1, 2,$$

which proves the theorem. □

Replacing above the real part with the imaginary part, we immediately get the following.

Corollary 3.2.7. *For $(X, \|\cdot\|_X)$ a complex normed space and $S \subset X$ a linear subspace, let us consider a family of linear bounded operators $L_j : X \to \mathcal{F}_j(\Omega_j)$, $\Omega_j \subset \mathbb{C}$, $j \in J$ (J an arbitrary set), satisfying the following conditions:*

(i) $\sup_{j\in J} |||L_j||| < \infty$;
(ii) there is an element $e \in S$ *such that for any* $j \in J$ *we have*

$$\mathrm{Im}[L_j(e)(z)] \geq 1, \text{ for all } z \in \Omega_j.$$

Then, for every $f \in X$ *and* $P \in S$*, there exist* $Q_i \in S, i = 1, 2$*, such that*

$$(-1)^{i+1} Im[L_j(Q_i - f)(z)] \geq 0, z \in \Omega_j, j \in J,$$

and the estimate

$$\|f - Q_i\|_X \leq (1 + \|e\|_X \sup_{j\in J} |||L_j|||) \cdot \|f - P\|_X, \ i = 1, 2,$$

holds.

Another consequence of Theorem 3.2.6 is the next result.

Theorem 3.2.8. *Let* $(X, \|\cdot\|_X)$ *be a complex normed space with* $S \subset X$ *a linear subspace. Let us consider a family of linear bounded operators* $L_j : X \to \mathcal{F}_j(\Omega_j)$, $\Omega_j \subset \mathbb{C}$*, where* $(\mathcal{F}_j(\Omega_j), \|\cdot\|_{\mathcal{F}_j})$ *are normed spaces of complex-valued functions defined on* Ω_j*, for all* $j \in J$*, respectively (*J *an arbitrary set), satisfying the conditions (i) and (ii) in the statement of Theorem 2.3.5, and for* $j \in J$ *let us define*

$$M^+(\mathrm{Re}[L_j]) = \{f \in X : \mathrm{Re}[L_j(f)(z)] \geq 0, \text{ for all } z \in \Omega_j\},$$
$$M^-(\mathrm{Re}[L_j]) = \{f \in X : \mathrm{Re}[L_j(f)(z)] \leq 0, \text{ for all } z \in \Omega_j\},$$
$$M^{\pm}[\mathrm{Re}] = \bigcap_{j\in J} M^{\pm}(\mathrm{Re}[L_j]).$$

Then, for any $f \in M^{\pm}[\mathrm{Re}]$*, we have*

$$E_{S,M^{\pm}[\mathrm{Re}]}(f; X) \leq (1 + \|e\|_X \sup_{j\in J} |||L_j|||) E_S(f; X).$$

Proof. We prove the case $f \in M^+(\mathrm{Re}[L_j])$, since the case $f \in M^-(\mathrm{Re}[L_j])$ is similar. By Theorem 3.2.6, there exists $Q_1 \in S$ such that $\mathrm{Re}[L_j(Q_1)(z)] \geq \mathrm{Re}[L_j(f)(z)] \geq 0$, for all $z \in \Omega_j$ for all $j \in J$. Thus $Q_1 \in M^+(\mathrm{Re}[L_j])$ and the conclusion immediately follows from the estimate in Theorem 3.2.6. □

Remark. Theorem 3.2.8 remains valid if in the statement we replace Re with Im.

Corollary 3.2.9. *For* $(X, \|\cdot\|_X)$ *a complex normed space and* $S \subset X$ *a linear subspace, let* $L : X \to \mathcal{F}(\Omega)$*, where* $\Omega \subset \mathbb{C}$ *and* $(\mathcal{F}(\Omega), \|\cdot\|_{\mathcal{F}})$ *is a certain normed space of complex-valued functions defined on* Ω*. If* L *is a linear bounded operator such that there exists* $e \in S$ *with* $\mathrm{Re}[L(e)](z) \geq 1$ *for all* $z \in \Omega$*, and if we set* $M(\mathrm{Re}[L]) = \{f \in X : \mathrm{Re}[L(f)(z)] \geq 0$ *for all* $z \in \Omega\}$*, then for any* $f \in M(\mathrm{Re}[L])$ *we have*

$$E_{S,M(\mathrm{Re}[L])}(f; X) \leq (1 + |||L||| \cdot \|e\|_X) E_S(f; X).$$

The proof is an immediate consequence of Theorem 3.2.8. □

Remarks. (1) Corollary 3.2.9 remains valid by replacing Re with Im.

(2) The constant $(1+|||L|||\cdot\|e\|_X)$ can obviously be improved, by replacing $\|e\|_X$ with $c=\inf\{\|e\|_X; e\in S,\ \mathrm{Re}[L(e)(z)]\geq 1,\ \text{for all } z\in\Omega\}$.

At the end of this section we present the following refinements of Theorems 3.2.6, 3.2.8 and Corollary 3.2.9, in the sense that the positivity condition $\mathrm{Re}[L]\geq 0$ can be replaced by the strict positivity condition $\mathrm{Re}[L]>0$.

Corollary 3.2.10. *In the hypothesis of Theorem 3.2.6, for every $f\in X$ and $P\in S$, $P\neq f$, there exist $Q_i\in S$, $i=1,2$, such that*

$$(-1)^{i+1}\mathrm{Re}[L_j(Q_i-f)(z)]>0,\ z\in\Omega_j,\ j\in J,$$

and the estimate

$$\|f-Q_i\|_X\leq(1+\|e\|_X+\|e\|_X\sup_{j\in J}|||L_j|||)\cdot\|f-P\|_X \quad i=1,2,$$

holds.

Proof. Indeed, this conclusion easily follows if in the proof of Theorem 3.2.6 we take $\lambda=(1+\sup_{j\in J}|||L_j|||)\cdot\|f-P\|_X$ and we follow the reasoning of that proof. □

Corollary 3.2.11. *In the hypothesis of Theorem 3.2.8 but introducing the notations*

$$\begin{aligned} M_0^+(L_j)=&\{f\in X:\mathrm{Re}[L_j(f)(z)]>0,\ \textit{a.e. on } \Omega_j\},\\ M_0^-(L_j)=&\{f\in X:\mathrm{Re}[L_j(f)(z)]<0,\ \textit{a.e. on } \Omega_j\},\\ M_0^{\pm}=&\bigcap_{j\in J}M_0^{\pm}(L_j), \end{aligned}$$

for any $f\in M_0^{\pm}$, we have

$$E_{S,M_0^{\pm}}(f;X)\leq(1+\|e\|_X+\|e\|_X\sup_{j\in J}|||L_j|||)E_S(f;X).$$

Proof. We prove the case $f\in M_0^+$, since the case $f\in M_0^-$ is similar. By Corollary 3.2.10, for any $f\in X$ and $P\in S$, $P\neq f$, there exists $Q_1\in S$ such that $\mathrm{Re}[L_j(Q_1)(z)]>\mathrm{Re}[L_j(f)(z)]>0$, a.e. $z\in\Omega_j$, for all $j\in J$. Thus $Q_1\in M_0^+$ and from the estimate in Corollary 3.2.10, we get

$$\|f-Q_1\|_X\leq(1+\|e\|_X+\|e\|_X\sup_{j\in J}|||L_j|||)\cdot\|f-P\|_X,\ i=1,2,$$

which immediately implies the estimate in Corollary 3.2.11. □

Corollary 3.2.12. *In the hypothesis of Corollary 3.2.9, setting $M_0(L)=\{f\in X:\mathrm{Re}[L(f)(z)]>0,\ \text{on } \Omega\}$, then for any $f\in M_0(L)$ we have*

$$E_{S,M_0(L)}(f;X)\leq(1+\|e\|_X+|||L|||\cdot\|e\|_X)E_S(f;X).$$

The proof is an immediate consequence of Corollary 3.2.11. □

Remarks. (1) Other slightly different variants of the above results hold. For example, in Corollary 3.2.12 let us suppose that $L : X \to C(\overline{\mathbb{D}})$, with $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, is a linear bounded operator such that there exists $e \in S$ with $\text{Re}[L(e)(z)] \geq 1$ on $\overline{\mathbb{D}}$. If we set $M_0(L) = \{f \in X : \text{Re}[L(f)(z)] > 0, \text{ on } \mathbb{D}\}$, then for any $f \in M_0(L)$ we have

$$E_{S,M_0(L)}(f;X) \leq (1 + |||L||| \cdot \|e\|_X) E_S(f;X).$$

The estimate is immediate from the proof of Theorem 3.2.8.

Note that by $L(f) \in C(\overline{\mathbb{D}})$, we easily get $\text{Re}[L(f)(z)] \geq 0$ for all $x \in \overline{\mathbb{D}}$, i.e., actually $f \in M(L) = \{f \in X : \text{Re}[L(f)(z)] \geq 0, \text{ on } \overline{\mathbb{D}}\}$.

(2) In order to get an application, let us particularize the spaces X, S, and the operator L.

If we take $X = A^1(\mathbb{D})$, the space of continuously differentiable functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ on the closed unit disk $\overline{\mathbb{D}}$ (endowed with the norm $\|f\|_{A^1} = \max\{\|f\|, \|f'\|\}$, where $\|\cdot\|$ denotes the uniform norm on $C(\overline{\mathbb{D}})$), and $L : A^1(\mathbb{D}) \to C(\overline{\mathbb{D}})$ defined by $L(f)(z) = f'(z)$, then obviously L is linear and bounded. Also, $e(z) = z, \forall z \in \overline{\mathbb{D}}$, satisfies $\text{Re}[L(e)(z)] = 1$ for all $z \in \overline{\mathbb{D}}$. Therefore, by choosing as S the set of all complex polynomials of degree $\leq n$ (for arbitrary fixed $n \in \mathbb{N}$, that is, a finite-dimensional subset in $C(\overline{\mathbb{D}})$), for any $f \in A^1(\mathbb{D})$ satisfying $\text{Re}[f'(z)] > 0$ for all $z \in \mathbb{D}$, there exists a complex polynomial P_n of degree $\leq n$ such that $\text{Re}[P_n'(z)] > 0$ for all $z \in \mathbb{D}$. Taking into account the definition of the norm $\|f\|_{A^1}$ and combining the estimate in Corollary 3.2.9 with Theorem 3.2.1 (iii), we immediately get the estimate

$$\|f - P_n\| \leq c E_{n-1}(f'),$$

i.e., we recover a particular case ($\gamma = 0$) of Theorem 3.2.5, (ii).

Obviously many other choices for X, S, and L can be made.

3.3 Shape-Preserving Approximation by Convolution Polynomials

Let $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disk and let us set $A^*(\mathbb{D}) = \{f : \overline{\mathbb{D}} \to \mathbb{C};\ f$ is analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}, f(0) = 0, f'(0) = 1\}$, i.e., $A^*(\mathbb{D})$ is the class of all normalized functions in $A(\mathbb{D})$.

The aim of this section is to obtain approximation results through convolution polynomials based on various trigonometric kernels (of Fejér, Jackson, Beatson, Cesáro, de la Vallée Poussin mean), producing a Jackson-type approximation rate or best-approximation rate and preserving many properties in geometric function theory such as the coefficients' bounds, positivity of real part, bounded turn, close-to-convexity, starlikeness, convexity, spirallikeness, α-convexity. In addition, some sufficient conditions for starlikeness and univalence of analytic functions are preserved.

3.3.1 Bell-Shaped Kernels and Complex Convolutions

First we consider approximation and geometric properties for complex convolutions, based on the so-called Beatson trigonometric kernels and their generalizations called iterated Beatson kernels.

With the aid of kernels $K_{n,r}(t)$, defined by

$$K_{n,r}(s) = \left(\frac{\sin \frac{ns}{2}}{\sin \frac{s}{2}} \right)^{2r}$$

and with $c_{n,r}$ chosen so that $\frac{1}{\pi} \int_{-\pi}^{\pi} K_{n,r}(s) ds = c_{n,r}$, it is known that the Beatson [32] kernels are defined by

$$B_{n,r}(t) = \frac{n}{2\pi c_{n,r}} \int_{t-\pi/n}^{t+\pi/n} K_{n,r}(s) ds.$$

The following lemma holds.

Lemma 3.3.1. (Gal [129]) *For any* $n, r \in \mathbb{N}$, $n, r \geq 2$ *and* $k \in \{0, 1, \cdots, 2r-2\}$ *it follows that*

$$\int_0^{\pi} t^k B_{n,r+1}(t) dt \leq C n^{-k}.$$

Proof. If $k = 0$ then

$$\int_0^{\pi} t^k B_{n,r+1}(t) dt = \int_0^{\pi} B_{n,r+1}(t) dt \leq \int_0^{2\pi} B_{n,r+1}(t) dt = \pi.$$

Let $k \in \{1, 2, \ldots, 2r-2\}$. Integrating by parts, one obtains

$$\begin{aligned} &\int_0^{\pi} t^k B_{n,r+1}(t) dt \\ &= \frac{n}{2\pi c_{n,r+1}(k+1)} \pi^{k+1} \int_{\pi-\pi/n}^{\pi+\pi/n} K_{n,r+1}(u) du \\ &\quad - \frac{n}{2\pi c_{n,r+1}(k+1)} \int_0^{\pi} t^{k+1} K_{n,r+1}(t+\pi/n) dt \\ &\quad + \frac{n}{2\pi c_{n,r+1}(k+1)} \int_0^{\pi} t^{k+1} K_{n,r+1}(t-\pi/n) dt =: I_1 - I_2 + I_3. \end{aligned}$$

In what follows we will estimate the integrals I_1, I_2, I_3. For that, we will use the following relations (see, e.g., Lorentz [248], p. 57):

$$c_{n,r} \approx n^{2r-1}, \int_0^{\pi} t^k K_{n,r}(t) dt \approx n^{2r-1-k}.$$

First we obtain

$$\begin{aligned}
I_3 &\le Cn^{-2r}\int_0^{\pi} t^{k+1}K_{n,r+1}(t-\pi/n)dt \\
&= Cn^{-2r}\int_{-\pi/n}^{\pi-\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&= Cn^{-2r}\int_{0}^{\pi-\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\quad + Cn^{-2r}\int_{-\pi/n}^{0}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\le 2Cn^{-2r}\int_0^{\pi}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \le Cn^{-2r}n^{2(r+1)-1-(k+1)} = Cn^{-k}.
\end{aligned}$$

Second, we get

$$\begin{aligned}
I_2 &\le Cn^{-2r}\int_0^{\pi} t^{k+1}K_{n,r+1}(t+\pi/n)dt \\
&= Cn^{-2r}\int_{\pi/n}^{\pi+\pi/n}(v-\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\le Cn^{-2r}\int_{\pi/n}^{\pi+\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&= Cn^{-2r}\int_{\pi/n}^{\pi}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\quad + Cn^{-2r}\int_{\pi}^{\pi+\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\le Cn^{-2r}\int_{0}^{\pi}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\quad + Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv \le Cn^{-k} \\
&\quad + Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv.
\end{aligned}$$

If we set

$$J_2 = Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv,$$

then using the substitution $nv/2 = t$, one obtains

$$\begin{aligned}
J_2 &= Cn^{-2r}/n\int_{n\pi/2}^{n\pi/2+\pi/2}(2t/n)^{k+1}(\sin(t)/\sin(t/n))^{2(r+1)}dt \\
&= Cn^{-2r-k-2}\int_{n\pi/2}^{n\pi/2+\pi/2}t^{k+1}(\sin(t)/\sin(t/n))^{2(r+1)}dt.
\end{aligned}$$

But if $t \in [n\pi/2, n\pi/2 + \pi/2]$, then we get $t/n \in [\pi/2, \pi/2 + \pi/(2n)] \subset [0, \pi]$, for all $n \geq 2$, which implies

$$\sin(t/n) \geq \sin(\pi/2 + \pi/(2n)) = \cos(\pi/(2n)) \geq \cos(\pi/4).$$

Since $k \leq 2r - 2 \leq 2r + 1$, it follows that

$$J_2 \leq Cn^{-2r-k-2} \int_{n\pi/2}^{n\pi/2+\pi/2} t^{k+1} dt \leq Cn^{-2r-k-2}(n\pi/2 + \pi/2)^{k+1}$$
$$\leq Cn^{-2r-k-2+k+1} = Cn^{-2r-1} \leq Cn^{-k}.$$

In conclusion, $I_2 \leq Cn^{-k}$.

Now, by the substitution $nu/2 = v$, we get

$$I_1 \leq Cn^{-2r} \int_{\pi-\pi/n}^{\pi+\pi/n} K_{n,r+1}(u) du$$
$$= Cn^{-2r-1} \int_{n\pi/2-\pi/2}^{n\pi/2+\pi/2} (\sin(v)/\sin(v/n))^{2(r+1)} dv$$
$$= Cn^{-2r-1} \int_{n\pi/2-\pi/2}^{n\pi/2} (\sin(v)/\sin(v/n))^{2(r+1)} dv$$
$$+ Cn^{-2r-1} \int_{n\pi/2}^{n\pi/2+\pi/2} (\sin(v)/\sin(v/n))^{2(r+1)} dv =: J_1 + L_1.$$

Since $v \in [n\pi/2 - \pi/2, n\pi/2]$ is equivalent to $v/n \in [\pi/2 - \pi/(2n), \pi/2]$, we have $\sin(v/n) \geq C(v/n)$ and

$$J_1 \leq Cn^{-2r-1} \int_{n\pi/2-\pi/2}^{n\pi/2} (\sin(v)/(v/n))^{2(r+1)} dv$$
$$= Cn \int_{n\pi/2-\pi/2}^{n\pi/2} (\sin(v)/v)^{2(r+1)} dv$$
$$\leq Cn \int_{n\pi/2-\pi/2}^{n\pi/2} (1/v)^{2(r+1)} dv$$
$$\leq Cn(1/((n-1)\pi/2)^{2(r+1)} \leq Cn^{-2r-1} \leq Cn^{-k},$$

for all $n \geq 2$.

Similarly, using the substitution $v/n = t$, since $\cos(\pi/(2n)) \geq \cos(\pi/4)$ for all $n \geq 2$, we obtain

$$L_1 = Cn^{-2r} \int_{\pi/2}^{\pi/2+\pi/(2n)} (\sin(nt)/\sin(t))^{2(r+1)} dt$$
$$\leq Cn^{-2r} \int_{\pi/2}^{\pi/2+\pi/(2n)} (1/\sin(t))^{2(r+1)} dt$$

$$\leq Cn^{-2r}(\pi/(2n))(1/\sin(\pi/2+\pi/(2n)))^{2(r+1)}$$
$$\leq Cn^{-2r-1}(1/\cos(\pi/(2n)))^{2(r+1)} \leq Cn^{-2r-1} \leq Cn^{-k}.$$

From all the estimates for I_1, I_2, and I_3, it follows that $I_1 - I_2 + I_3 \leq Cn^{-k}$, which proves the lemma. □

Consequently, we get the following.

Corollary 3.3.2. (Gal [129]) *If $f \in A^*(\mathbb{D})$ then the convolution polynomials given by*

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{it})B_{m,r}(x-t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{iu})B_{m,r}(u)du,$$

$z = re^{ix} \in \overline{\mathbb{D}}$, $m = [n/r]+1$, satisfy degree $P_n(f)(z) \leq n$, and for all $r \geq 3$, the estimate

$$|f(z) - P_n(f)(z)| \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}}$$

holds for all $z \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$, $n \geq 2$, where

$$\omega_p(f;\delta)_{\partial\mathbb{D}} = \sup\{|\Delta_u^p f(e^{ix})|; |x| \leq \pi, |u| \leq \delta\},$$

and $\Delta_u^p g(x) = \sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}g(x+ku)$.

Proof. Let $r \geq 3$ and $n \geq 2$ be fixed. Because $B_{m,r}(t)$ is even, as in the proof of Theorem 2 in Lorentz [248], p. 56, we get

$$f(z) - P_n(f)(z) = \int_0^{\pi}[2f(z) - f(ze^{it}) - f(ze^{-it})]B_{m,r}(t)dt.$$

Applying the maximum modulus principle, we may take $|z| = 1$ when we have

$$|f(z) - P_n(f)(z)| = |\int_0^{\pi}[2f(z) - f(ze^{it}) - f(ze^{-it})]B_{m,r}(t)dt|$$
$$\leq \int_0^{\pi}|2f(z) - f(ze^{it}) - f(ze^{-it})|B_{m,r}(t)dt.$$

But

$$|2f(z) - f(ze^{it}) - f(ze^{-it})| \leq \omega_2(f;t)_{\partial\mathbb{D}} = \omega_2\left(f; \frac{nt}{n}\right)_{\partial\mathbb{D}}$$
$$\leq C(nt+1)^2\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

which together with Lemma 3.3.1 implies

$$|f(z) - P_n(f)(z)| \leq \omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}}\int_0^{\pi}(nt+1)^2B_{m,r}(t)dt \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}}.$$

□

By Gal [128], we can define the iterated Beatson kernels by recurrence as $B_{n,r,1}(t) := B_{n,r}(t)$,

$$B_{n,r,2}(t) = \frac{n}{2\pi} \int_{t-\pi/n}^{t+\pi/n} B_{n,r,1}(s)ds, \ldots, B_{n,r,p}(t)$$
$$= \frac{n}{2\pi} \int_{t-\pi/n}^{t+\pi/n} B_{n,r,p-1}(s)ds,$$

$p = 2, 3$.

The following generalization of Lemma 3.3.1 holds.

Lemma 3.3.3. (Gal [129]) *For any* $n, r, p \in \mathbb{N}$ *with* $r \geq 2$, $n \geq p+1$, *and* $k \in \{0, 1, \ldots, 2r+2p-4\}$, *we have*

$$\int_0^{\pi} t^k B_{n,r+p,p}(t)dt \leq Cn^{-k}.$$

Proof. The case $p = 1$ is Lemma 3.3.1. First, for simplicity we consider the case $p = 2$. We get

$$B_{n,r,2}(t) = \frac{n}{2\pi} \int_{t-\pi/n}^{t+\pi/n} B_{n,r}(s)ds = \frac{n^2}{(2\pi)^2 c_{n,r}} \int_{t-\pi/n}^{t+\pi/n} \int_{x-\pi/n}^{x+\pi/n} K_{n,r}(s)ds\, dx.$$

For $k = 0$ we have

$$\int_0^{\pi} t^k B_{n,r+2,2}(t)dt \leq \int_0^{2\pi} t^k B_{n,r+2,2}(t)dt \leq \pi.$$

Let $k = 1, 2, \ldots, 2r$. Integrating twice by parts, simple calculations imply

$$\int_0^{\pi} t^k B_{n,r+2,2}(t)dt = B_{n,r+2,2}(\pi)\frac{\pi^{k+1}}{k+1} - B'_{n,r+2,2}(\pi)\frac{\pi^{k+2}}{(k+1)(k+2)}$$
$$+ \frac{1}{(k+1)(k+2)} \int_0^{\pi} t^{k+2} B''_{n,r+2,2}(t)dt$$
$$= \frac{n^2 \pi^{k+1}}{(2\pi)^2 (k+1) c_{n,r+2}} \int_{\pi-\pi/n}^{\pi+\pi/n} \int_{x-\pi/n}^{x+\pi/n} K_{n,r+2}(s)ds\ dx$$
$$- \frac{n^2 \pi^{k+2}}{(2\pi)^2 (k+1)(k+2) c_{n,r+2}}$$
$$\times \left[\int_{\pi}^{\pi+2\pi/n} K_{n,r+2}(s)ds - \int_{\pi-2\pi/n}^{\pi} K_{n,r+2}(s)ds \right]$$
$$+ \frac{n^2}{(2\pi)^2 (k+1))k+2) c_{n,r+2}}$$
$$\times \int_0^{\pi} t^{k+2} [K_{n,r+2}(t+2\pi/n) - 2K_{n,r+2}(t)$$
$$+ K_{n,r+2}(t-2\pi/n)]dt := I_1 - I_2 + I_3.$$

By

$$c_{n,r} \approx n^{2r-1}, \quad \int_0^{\pi} t^k K_{n,r}(t)dt \approx n^{2r-1-k},$$

we get

$$\frac{n^2}{c_{n,r+2}} \approx 1/n^{2r+1}, \quad \int_0^{\pi} t^{k+2} K_{n,r+2}(t)dt \approx n^{2r+1-k}.$$

Reasoning as for I_2 and I_3 in the proof of Lemma 3.3.1, the above relations immediately imply $I_3 \leq Cn^{-k}$.

Also, reasoning as for I_1 in the proof of Lemma 3.3.1, we get $I_2 \leq Cn^{-k}$. It remains to estimate the integral

$$J = \int_{\pi-\pi/n}^{\pi+\pi/n} \int_{x-\pi/n}^{x+\pi/n} K_{n,r+2}(s)ds\ dx.$$

By the mean value theorem, there is $\xi \in [\pi - \pi/n, \pi + \pi/n]$ satisfying

$$\begin{aligned} J &= \frac{2\pi}{n} \int_{\xi-\pi/n}^{\xi+\pi/n} K_{n,r+2}(s)ds \\ &= \frac{2\pi}{n} \int_{\xi-\pi/n}^{\xi} K_{n,r+2}(s)ds + \frac{2\pi}{n} \int_{\xi}^{\xi+\pi/n} K_{n,r+2}(s)ds \\ &= \frac{2\pi}{n^2} \int_{n\xi/2-\pi/2}^{n\xi/2} [\sin(t)/\sin(t/n)]^{2(r+2)} dt \\ &\quad + \frac{2\pi}{n^2} \int_{n\xi/2}^{n\xi/2+\pi/2} [\sin(t)/\sin(t/n)]^{2(r+2)} dt := J_1 + J_2. \end{aligned}$$

We obtain

$$J_1 = \frac{2\pi}{n} \int_{\xi/2-\pi/(2n)}^{\xi/2} [\sin(nv)/\sin(v)]^{2(r+2)} dv$$

and

$$J_2 = \frac{2\pi}{n} \int_{\xi/2}^{\xi/2+\pi/(2n)} [\sin(nv)/\sin(v)]^{2(r+2)} dv.$$

Since $|\sin(nv)| \leq 1$ and $\pi - \pi/n \leq \xi \leq \pi + \pi/n$, it follows that $0 \leq \pi/2 - \pi/n \leq \xi/2 - \pi/(2n) \leq \xi/2 \leq \pi/2 + \pi/(2n) < \pi$, $0 < \pi/2 - \pi/n \leq \xi/2 \leq \xi/2 + \pi/(2n) \leq \pi/2 + \pi/n < \pi$.

So, for $n \geq 3$, $|1/\sin(v)|$ is bounded for both integrals J_1 and J_2, which implies $J_1 \leq Cn^{-2}$, $J_2 \leq Cn^{-2}$.

On the other hand, because $k \leq 2r \leq 2r + 3$, it follows that

$$I_1 = \frac{n^2}{(2\pi)^2} \frac{\pi^{k+1}}{(k+1)c_{n,r+2}} J \leq Cn^{-2} n^{-2r-1} = Cn^{-2r-3} \leq Cn^{-k}.$$

All these prove the statement of Lemma 3.3.3 for the case $p = 2$. For general p, reasoning by recurrence and integrating p times by parts the integral $\int_0^\pi t^k B_{n,r+p,p}(t)dt$, the proof is similar. □

An immediate consequence is the following.

Corollary 3.3.4. (Gal [129]) *If $f \in A^*(\mathbb{D})$ and $p \in \mathbb{N}$ is fixed, then the convolution polynomials given by*

$$P_{n,r,p}(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{iu}) B_{m,r,p}(u) du,$$

$z \in \overline{\mathbb{D}}$, $m = [n/r] + 1$, satisfy degree $P_{n,r,p}(f)(z) \le n$, and moreover, for all $m, r \ge p + 2$, $z \in \overline{\mathbb{D}}$, the estimate

$$|f(z) - P_{n,r,p}(f)(z)| \le C\omega_2 \left(f; \frac{1}{n}\right)_{\partial\mathbb{D}}$$

holds.

Proof. Taking into account Lemma 3.3.3 and the fact that $B_{m,r,p}(u)$ are even, the proof is similar to that of Corollary 3.3.2. □

Remarks. Beatson [32] observed (without proof; see Gonska–Cao [151] for a proof) that $B_{n,r}(t)$ is bell-shaped (recall that a continuous 2π-periodic function on $[-\pi, \pi]$ is called bell-shaped if it decreases on $[0, \pi]$ and is even).

In what follows, first we prove that the above iterated Beatson kernels also are bell-shaped and that the convolution operators based on them transform the convex functions into close-to-convex polynomials. Then, for convolution polynomials based on various trigonometric kernels, we present approximations that preserve geometric properties of analytic functions, such as the coefficients' bounds, real part positivity, bounded turn, close-to-convexity, starlikeness, convexity, α-convexity, spiralikeness, and some sufficient conditions for starlikeness and univalence. The rates of approximation are of Jackson type or of best-approximation kind.

Lemma 3.3.5. (Gal [128]) *The iterated Beatson kernels $B_{n,r,p}$, $n, r, p \in \mathbb{N}$, are bell-shaped.*

Proof. We will reason by mathematical induction. Thus, by Gonska–Cao [151], $B_{n,r,1}(t) := B_{n,r}(t)$ is nonnegative and bell-shaped. Therefore, suppose that $B_{n,r,p-1}(t)$ is nonnegative and bell-shaped. First, by the substitution ($s = -u$) in the integral defining $B_{n,r,p}(t)$, we immediately get that $B_{n,r,p}(t) \ge 0$ and $B_{n,r,p}(-t) = B_{n,r,p}(t)$, for all $t \in \mathbb{R}$.

Then, we obtain

$$B'_{n,r,p}(t) = \frac{n}{2\pi} \left[B_{n,r,p-1}(t + \pi/n) - B_{n,r,p-1}(t - \pi/n)\right].$$

Let $t \in [0, \pi]$. There are three cases: (i) $t \in [\pi/n, \pi - \pi/n]$; (ii) $t \in [0, \pi/n]$; (iii) $t \in [\pi - \pi/n, \pi]$.

Case (i). Since $B_{n,r,p-1}(t)$ decreases on $[0, \pi]$, we obtain $0 \le t - \pi/n < t + \pi/n \le \pi$, and $B_{n,r,p-1}(t + \pi/n) - B_{n,r,p-1}(t - \pi/n) \le 0$.

Case (ii). Since $0 \le \pi/n - t \le \pi/n + t \le 2\pi/n \le \pi$, (for $n \ge 2$) and $B_{n,r,p-1}(t)$ is even and decreases on $[0, \pi]$, it follows that

$$B_{n,r,p-1}(t + \pi/n) - B_{n,r,p-1}(t - \pi/n) = B_{n,r,p-1}(\pi/n + t) \\ -B_{n,r,p-1}(\pi/n - t) \le 0.$$

Case (iii). Since $0 \le \pi - 2\pi/n \le t - \pi/n \le 2\pi - t - \pi/n \le \pi$ (for $n \ge 2$) and $B_{n,r,p-1}(t)$ is even, 2π-periodic and decreases on $[0, \pi]$, it follows that

$$\begin{aligned} &B_{n,r,p-1}(t + \pi/n) - B_{n,r,p-1}(t - \pi/n) \\ &\quad = B_{n,r,p-1}[-(t + \pi/n)] - B_{n,r,p-1}(t - \pi/n) \\ &\quad = B_{n,r,p-1}[2\pi - (t + \pi/n)] - B_{n,r,p-1}(t - \pi/n) \le 0. \end{aligned}$$

□

Corollary 3.3.6. (Gal [128]) *The convolution polynomials defined by*

$$P_{n,r,p}(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{iu}) B_{m,r,p}(u) du,$$

$z \in \overline{\mathbb{D}}$, $m = [n/r] + 1$, *satisfy degree* $P_{n,r,p}(f)(z) \le n$ *and for all* $m, r \ge p + 2$, *the approximation estimate in Corollary 3.3.4 holds. In addition, they transform any convex function* $f \in A^*(\mathbb{D})$ *into close-to-convex polynomials.*

Proof. While the approximation property follows from Corollary 3.3.4, the geometric property is a direct consequence of Suffridge [375], p. 799, Theorem 3. □

3.3.2 Geometric and Approximation Properties of Various Complex Convolutions

Concerning the coefficients of convolution polynomials based on various trigonometric kernels, we present the following theorem.

Theorem 3.3.7. (Gal [129]) *(i) Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic on* $\mathbb{D}$ *and* $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt)$. *Then for* $P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t) dt$, *we have*

$$P_n(f)(z) = a_0 + \sum_{k=1}^{m_n} a_k \rho_{k,n} z^k.$$

If $f(z) = z + \sum_{k=0}^{\infty} \frac{a_k}{z^k},\ 0 < |z| < 1$ *is meromorphic, then*

$$P_n(f)(z) = \rho_{1,n} z + a_0 + \sum_{p=1}^{m_n} \frac{a_p \rho_{p,n}}{z^p}.$$

(ii) If $O_n(f) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt) \geq 0,\ \forall t \in [0, \pi],$ *and* $\frac{1}{\pi} \int_{-\pi}^{\pi} O_n(t) dt = 1,$ *then*

$$|\rho_{k,n}| \leq 1, \text{ for all } k \in \{1, \dots m_n\}.$$

(iii) Let $F_n(t) = \frac{1}{2n} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2$ *denote the Fejér kernel,* $V_n(t) = 2F_{2n}(t) - F_n(t)$ *the generalized de la Valléey Poussin kernel,*

$$J_n(t) = \frac{3}{2n(2n^2+1)} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4$$

the Jackson kernel, and

$$B_{n,2,1}(t) := B_{n,2}(t) = \frac{n}{2\pi} \int_{t-\pi/n}^{t+\pi/n} J_n(t) dt, B_{n,2,p}(t)$$
$$= \frac{n}{2\pi} \int_{t-\pi/n}^{t+\pi/n} B_{n,2,p-1}(t) dt,$$

$p = 2, 3, \dots,$ *the Beatson kernels. We get*

$$F_n(t) = \frac{1}{2} + \sum_{k=1}^{n-1} \frac{n-k}{n} \cos(kt),$$
$$V_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) + \sum_{k=n+1}^{2n} \frac{2n-k}{n} \cos(kt),$$
$$J_n(t) = \frac{1}{2} + \sum_{k=1}^{2n-2} \lambda_{k,n} \cos(kt), \text{ where}$$
$$\lambda_{k,n} = \frac{4n^3 - 6k^2 n + 3k^3 - 3k + 2n}{2n(2n^2+1)}, \text{ if } 1 \leq k \leq n,$$
$$\lambda_{k,n} = \frac{(k-2n) - (k-2n)^3}{2n(2n^2+1)}, \text{ if } n \leq k \leq 2n-2,$$

and for $p = 1, 2, \dots,$

$$B_{n,2,p}(t) = \frac{1}{2} + \sum_{k=1}^{2n-2} \left[\frac{n}{k\pi} sin(k\pi/n) \right]^p \cdot \lambda_{k,n} \cos(kt),$$

where $\lambda_{k,n}$ *are the coefficients in* $J_n(t)$.

Proof. (i) The analytic case is immediate, reasoning similar to the proof of Theorem 1 (ii) in Gal [127].

In the meromorphic case, we have

$$f(ze^{it})O_n(t) = \left[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z_k} \cdot e^{-ikt}\right]\left[\frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n}\cos(pt)\right]$$

$$= \left[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt}\right]\left\{\frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n} \cdot \frac{1}{2}[e^{ipt} + e^{-ipt}]\right\}$$

$$= \left[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt}\right]\left\{\frac{1}{2} + \frac{1}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{ipt} + \frac{1}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{-ipt}\right\}$$

$$= \frac{z}{2}e^{it} + \frac{z}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{it(p+1)} + \frac{z}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{it(1-p)} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt}$$

$$+ \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p=1}^{m_n} \frac{a_k \rho_{p,n}}{z^k} e^{it(p-k)} + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p=1}^{m_n} \frac{a_k \rho_{p,n}}{z^k} e^{-it(k+p)}.$$

Integrating from $-\pi$ to π and reasoning as in Gal [127], Theorem 1 (ii), we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt = \rho_{1,n}z + \left[a_0 + \sum_{p=1}^{m_n} \frac{a_p \rho_{p,n}}{z^p}\right].$$

(ii) By

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(jt)O_n(t)dt = \rho_{j,n}, \text{ for all } j \in \{1, \dots, m_n\},$$

we obtain

$$|\rho_{j,n}| = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(jt)O_n(t)dt \le \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos jt| O_n(t)dt \le \frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt = 1.$$

(iii) The form of $F_n(t)$ is well known (see, e.g., DeVore–Lorentz [91], p. 339).

Also,

$$V_n(t) = 2F_{2n}(t) - F_n(t)$$

$$= 2\left[\frac{1}{2} + \sum_{k=1}^{2n} \frac{2n-k}{2n}\cos(kt)\right] - \left[\frac{1}{2} + \sum_{k=1}^{n} \frac{n-k}{n}\cos(kt)\right]$$

$$= \frac{1}{2} + \sum_{k=1}^{n}\left[\frac{2n-k}{n} - \frac{n-k}{n}\right]\cos(kt) + \sum_{k=n+1}^{2n} \frac{2n-k}{n}\cos(kt)$$

$$= \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) + \sum_{k=n+1}^{2n} \frac{2n-k}{n}\cos(kt).$$

For the Jackson kernel $J_n(t)$, by Matsuoka [269], Lemma 7, pp. 25–26, one obtains

$$(3!)2^{4-4-1}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4=\frac{1}{2}r_{0,n}+\sum_{k=1}^{2n-2}r_{k,n}\cos(kt),$$

where

$$r_{k,n}=\sum_{\nu=1}^{2}(-1)^{\nu+1}\binom{4}{2-\nu}(k-\nu n+1)(k-\nu n)(k-\nu n-1)$$

if $0\le k\le n$, and

$$r_{k,n}=-(k-2n+1)(k-2n)(k-2n-1) \text{ if } n\le k\le 2n-2.$$

Since $(k-\nu n+1)(k-\nu n)(k-\nu n-1)=(k-\nu n)^3-(k-\nu n)$, by simple calculation one obtains

$$\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{n}}\right)^4=\frac{r_{0,n}}{6}+\sum_{k=1}^{2n-2}\frac{r_{k,n}}{3}\cos(kt),$$

where $r_{0,n}=2n(2n^2+1)$, $r_{k,n}=4(k-n)^3-(k-2n)^3-3k+2n=4n^3-6k^2n+3k^3-3k+2n$, if $1\le k\le n$, and $r_{k,n}=(k-2n)-(k-2n)^3$ if $n\le k\le 2n-2$.

At the end, from Gonska–Cao [151], relation (3.2) and Lemma 3.3 (i), we get

$$B_{n,2,1}(t)=\frac{1}{2}+\sum_{k=1}^{2n-2}\left[\frac{n}{2\pi}\sin(\frac{2\pi}{n})\right]\lambda_{k,n}\cos(kt),$$

and by iteration,

$$B_{n,2,p}(t)=\frac{1}{2}+\sum_{k=1}^{2n-2}\left[\frac{n}{2\pi}\sin\frac{2\pi}{n}\right]^p\lambda_{k,n}\cos(kt).$$

The theorem is proved. □

Remark. The kernels $F_n(t), J_n(t)$, and $B_{n,2,p}(t)$ are ≥ 0, while the kernel $V_n(t)$ is not nonnegative on $[0,\pi]$, but satisfies $\frac{1}{\pi}\int_{-\pi}^{\pi}V_n(k)dt=1$. However, by Theorem 3.3.7 (iii), from the expression of $V_n(t)=\frac{1}{2}+\sum_{k=1}^{2n}\mu_{k,n}\cos(kt)$, we easily get that $0\le\mu_{k,n}\le 1$.

Also, taking into account that

$$B_{n,r,1}(t):=B_{n,r}(t)=\frac{n}{2\pi}\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}}c_{n,r}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^{2r}dt,$$

$$B_{n,r,p}(t)=\frac{n}{2\pi}\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}}B_{n,r,p-1}(t)dt,\ \text{where } \frac{1}{\pi}\int_{-\pi}^{\pi}c_n\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^{2r}dt=1,$$

we get $B_{n,r,p}(t) = \frac{1}{2} + \sum_{k=1}^{nr-n} \lambda_{k,n}^{(p)} \cos(kt) \geq 0$ and $\frac{1}{\pi} \int_{-\pi}^{\pi} B_{n,r,p}(t)dt = 1$, that is, by Theorem 3.3.7 (ii), it follows that $|\lambda_{k,n}^{(p)}| \leq 1$ for all $k \in \{0, \ldots, nr - n\}$, $p \geq 1$.

Regarding the preservation of coefficients' bounds, we have the following theorem.

Theorem 3.3.8. (Gal [129]) *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k(f) z^k$ is analytic in $\mathbb{D}$.*

(i) If $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt) \geq 0$ and $\frac{1}{\pi} \int_{-\pi}^{\pi} O_n(t)dt = 1$, then for

$$P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t)dt = \sum_{k=0}^{m_n} a_k(P_n(f)) z^k,$$

we have $|a_k(P_n(f))| \leq |a_k(f)|$, for all $n \in \mathbb{N}$, $k \in \{0, \ldots, m_n\}$.

(ii) If we set

$$V_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) V_n(t)dt = \sum_{k=0}^{2n} a_k(V_n(f)) z^k,$$

where $V_n(t)$ is the kernel in the statement of Theorem 3.3.7 (iii), then we have

$$|a_k(V_n(f))| \leq |a_k(f)| \text{ for all } n \in \mathbb{N},\ k \in \{0, \ldots, 2n\}.$$

Proof. It is straightforward from Theorem 3.3.7 (i), (ii), and the remark after the proof of Theorem 3.3.7. □

Remark. According to Gal [127], for $f \in A(\mathbb{D})$ we have

$$\|f - V_n(f)\|_{\overline{\mathbb{D}}} \leq 4E_n(f),\ n = 1, 2, \ldots,$$

while for $O_n(t) = J_n(t)$, $n \in \mathbb{N}$, or $O_n(t) = B_{n,r,p}(t)$, $n, r \geq p + 2$, $p \in \mathbb{N}$ (see Lemma 3.3.3) setting $P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t)dt$, we obtain

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where $C > 0$ is an absolute constant.

In what follows, let us introduce the following classes of functions: $S_1 = \{f : \mathbb{D} \to \mathbb{C};\ f(z) = z + a_2 z^2 + \ldots$, analytic in $\mathbb{D}$, satisfying $\sum_{k=2}^{\infty} k|a_k| \leq 1\}$,

$S_2 = \{f : \mathbb{D} \to \mathbb{C};\ f(z) = a_1 z + a_2 z^2 + \ldots$, analytic in $\mathbb{D}$, satisfying $|a_1| \geq \sum_{k=2}^{\infty} |a_k|\}$.

According to, e.g., Mocanu–Bulboacă–Sălăgean [273], p. 97, Exercise 4.9.1, if $f \in S_1$ then $|\frac{zf'(z)}{f(z)} - 1| < 1, z \in \mathbb{D}$, and therefore f is starlike (univalent) on $\mathbb{D}$.

Also, according to Alexander [2], p. 22, if $f \in S_2$ then f is starlike (and univalent) in $\mathbb{D}$. As a consequence, both S_1 and S_2 are subsets of the class of univalent starlike functions on $\mathbb{D}$, denoted by $S^*(\mathbb{D})$.

The classes S_1 and S_2 are preserved by some approximation convolution polynomials, as follows.

Theorem 3.3.9. (Gal [129]) *(i) If* $O_n(t)$ *is* $J_n(t)$, $n \in \mathbb{N}$, *in Theorem 3.3.7 (iii), or* $B_{n,r,p}(t)$, $n, r \geq p+2, p \in \mathbb{N}$, *(see the Remark after the proof of Theorem 3.3.7), then setting* $P_n(f)(z) = \int_0^z Q_n(t)dt$, $Q_n(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt$, *we get that* $f \in S_1$ *implies* $P_n(f) \in S_1$. *In addition, if* f' *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f'; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where $C > 0$ *is independent of* f *and* n.

(ii) Define $V_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})V_n(t)dt$, *where* $V_n(t)$ *is the kernel in the statement of Theorem 3.3.7 (iii). Then* $f \in S_1$ *implies* $V_n(f) \in S_1$ *for all* $n \in \mathbb{N}$, *and if, in addition,* f *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - V_n(f)\|_{\overline{\mathbb{D}}} \leq 4E_n(f), n = 1, 2, \ldots.$$

(iii) For the above $V_n(f)(z)$, $f \in S_2$ *implies* $V_n(f) \in S_2$ *for all* $n \in \mathbb{N}$.

(iv) Suppose that the meromorphic function $f(z) = z + \sum_{k=0}^{\infty} \frac{a_k(f)}{z^k}$ *is univalent on* $\{|z| > 1\}$. *Then for*

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt = \rho_{1,n}z + a_0 + \sum_{p=1}^{m_n} \frac{a_p(f)\rho_{p,n}}{z^p},$$

where

$$O_n(t) = \frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n}\cos(pt)$$

can be $J_n(t), V_n(t)$, $n \in \mathbb{N}$ *or* $B_{n,r,p}(t), n, r \geq p+2$, $p \in \mathbb{N}$, *we have*

$$\sum_{k=1}^{\infty} k \cdot |a_k(P_k(f))|^2 \leq 1, \text{ with } a_k(P_n(f)) = a_k(f) \cdot \rho_{k,n}.$$

Proof. (i) Evidently $P_n(f)(0) = P_n'(f)(0) - 1 = 0$. Then by Gal [127], we obtain (see also Lemma 3.3.3, for $B_{n,r,p}(t)$)

$$|f(z) - P_n(f)(z)| = \left|\int_0^z f'(t)dt - \int_0^z Q_n(t)dt\right|$$
$$\leq |z| \cdot \|f' - Q_n\|_{\overline{\mathbb{D}}} \leq \|f' - Q_n\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f'; \frac{1}{n}\right)_{\partial\mathbb{D}}.$$

Take $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n}\cos(kt)$ and $f(z) = z + a_2z^2 + \cdots +$, that is $f'(z) = 1 + 2a_2z + 3a_3z^2 + \ldots$. By Theorem 3.3.7 we obtain $Q_n(f)(z) = 1 + 2a_2\rho_{1,n}z + 3a_3\rho_{2,n}z^2 + \ldots$, which implies $P_n(f)(z) = \int_0^z Q_n(t)dt = z + a_2\rho_{1,n}z^2 + a_3\rho_{2,n}z^3 + \ldots$, and $a_k(P_n(f)) = a_k(f)\rho_{k-1,n}$. Hence,

$$\sum_{k=2}^{m_n} k|a_k(P_n(f))| = \sum_{k=2}^{m_n} k \cdot |a_k(f)| \cdot |\rho_{k-1,n}| \leq \sum_{k=2}^{\infty} k|a_k(f)| \leq 1$$

(since $|\rho_{k-1,n}| \leq 1$), which implies that $P_n(f) \in S_1$.

(ii) Evidently $V_n(f)(0) = 0$. Take $f \in S_1, f(z) = z + a_2z^2 + \ldots$. By Theorem 3.3.7 (i) and (iii), we obtain $V_n(f)(z) = z + A_2z^2 + \ldots$, (here $A_2 = a_2$ if $n \geq 2$), that is, $V_n'(f)(0) = 1$.

By the remark after the proof of Theorem 3.3.7, we obtain

$$\sum_{k=2}^{2n} k|a_k(V_n(f)| \leq \sum_{k=2}^{\infty} k|a_k(f)| \leq 1,$$

that is, $V_n(f) \in S_1$.

Also, by Gal [127], for continuous f on $\overline{\mathbb{D}}$ we obtain

$$\|f - V_n(f)\|_{\overline{\mathbb{D}}} \leq 4E_n(f).$$

(iii) Suppose $f \in S_2,\ f(z) = \sum_{k=1}^{\infty} a_k(f)z^k$, with $|a_1(f)| \geq \sum_{k=2}^{\infty} |a_k(f)|$. From Theorem 3.3.7, it follows that

$$V_n(f)(z) = a_1(f)z + \sum_{k=2}^{2n} \rho_{k,n}a_n(f) \cdot z^k, \quad 0 \leq \rho_{k,n} \leq 1,$$

which implies

$$\sum_{k=2}^{m_n} |a_k(V_n(f))| = \sum_{k=2}^{m_n} |a_k(f)| \cdot |\rho_{k,n}| \leq \sum_{k=2}^{\infty} |a_k(f)| \leq |a_1(f)| = a_1(V_n(f)),$$

and therefore $V_n(f) \in S_2$.

(iv) By the area theorem (see, e.g., Gronwall [165]) we have the formula $\sum_{k=1}^{\infty} k|a_k(f)|^2 \leq 1$.

Also,

$$\sum_{k=1}^{m_n} k \cdot |a_k(P_n(f))|^2 = \sum_{k=1}^{m_n} k \cdot |a_k(f)|^2 \cdot |\rho_{k,n}|^2 \leq \sum_{k=1}^{\infty} k \cdot |a_k(f)|^2 \leq 1,$$

which proves the theorem. □

Now let us introduce other classes of functions by $\mathcal{P} = \{f : \overline{\mathbb{D}} \to \mathbb{C};$ f is analytic in $\mathbb{D}, f(0) = 1, \mathrm{Re}[f(z)] > 0, z \in \mathbb{D}\}$, $S_3 = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f$ is analytic in $\mathbb{D}, f(0) = f'(0) - 1 = 0, |f''(z)| \leq 1, z \in \mathbb{D}\}$, $S_4 = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f$ is analytic in $\mathbb{D}, f(0) = f'(0) - 1 = 0, |f''(z)| \leq \frac{1}{2}, z \in \mathbb{D}\}$ and $\mathcal{R} = \{f : \overline{\mathbb{D}} \to \mathbb{C};$ f is analytic in $\mathbb{D}, f(0) = f'(0) - 1 = 0, \mathrm{Re}f'(z) > 0, z \in \mathbb{D}\}$.

It is well known that $\mathcal{P}$ is the class of analytic functions with positive real part while $\mathcal{R}$ is the class of functions with bounded turn (since $f \in \mathcal{R}$ is equivalent to $|\arg f'(z)| < \frac{\pi}{2}$, $z \in \mathbb{D}$). Also, it is known that $f \in \mathcal{R}$ implies the univalency of f in $\mathbb{D}$, while by Obradović [291], we get that $f \in S_3$ implies f is starlike, univalent in $\mathbb{D}$, and $f \in S_4$ implies f is convex univalent in $\mathbb{D}$.

Concerning the classes $\mathcal{P}, \mathcal{R}$, S_3 and S_4, we present the following.

Theorem 3.3.10. (Gal [129]) *Let* $O_n(t)$ *be* $J_n(t)$, $n \in \mathbb{N}$, *or* $B_{n,r,p}(t)$, $n, r \geq p+2$, $p \in \mathbb{N}$.

(i) Writing

$$P_n(f)(z) = \int_0^z O_n(t)dt, Q_n(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt,$$

we have $P_n(\mathcal{R}) \subset \mathcal{R}$, $P_n(S_3) \subset S_3$, $P_n(S_4) \subset S_4$, *and if, in addition,* f' *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f'; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where $C > 0$ *is independent of* f *and* n.

(ii) Define

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt.$$

We have $P_n(\mathcal{P}) \subset \mathcal{P}$, *and if, in addition,* f *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where $C > 0$ *is independent of* f *and* n.

Proof. (i) The approximation estimate follows as in the proof of Theorem 3.3.9 (i). Also, it is easy to see that $P_n(f)(0) = P_n'(f)(0) - 1 = 0$.

Let $f(z) = F(r\text{cost}, r\text{sint}) + iG(r\text{cost}, r\text{sint})$, $z = re^{it} \in \mathbb{D}$. By $f'(z) = \frac{\partial F}{\partial x}(r\text{cost}, r\text{sint}) + i\frac{\partial G}{\partial x}(r\text{cost}, r\text{sint})$, from the hypothesis we get $\frac{\partial F}{\partial x}(r\text{cost}, r\text{sint}) > 0$, for all $z = re^{it} \in \mathbb{D}$.

Since

$$\begin{aligned} P_n'(f)(z) = Q_n(f)(z) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{iu})O_n(u)du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial F}{\partial x}(r\cos(t+u), r\sin(t+u))O_n(u)du \\ &\quad + i\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial G}{\partial x}(r\cos(t+u), r\sin(t+u))O_n(u)du, \end{aligned}$$

we obtain

$$Re[P_n'(f)(z)] = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial F}{\partial x}(r\cos(t+u), r\sin(t+u)) \cdot O_n(u)du > 0,$$

taking into account that $O_n(u) > 0$ for all $u \in [-\pi, \pi]$ excepting a finite number of points.

Also, $|P_n''(f)(z)| = \frac{1}{\pi}|\int_{-\pi}^{\pi} e^{it}f''(ze^{it})O_n(t)dt| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f''(ze^{it})|O_n(t)dt \leq 1, z \in \mathbb{D}$, for $f \in S_3$.

Analogously we get $P_n(S_4) \subset S_4$.

(ii) If $f = F + iG$, we easily obtain

$$\mathrm{Re}[P_n(f)(z)] = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r\cos(t+u), r\sin(t+u))O_n(u)du,\ z = re^{it} \in \mathbb{D},$$

which by $F(r\cos u, r\sin u) > 0$ for all $z = re^{iu} \in \mathbb{D}$ implies $\mathrm{Re}[P_n(f)(z)] > 0, z \in \mathbb{D}$.

The estimate follows by Gal [127] for $J_n(t)$ and by Lemma 3.3.3 for $B_{n,r,p}(t)$. □

Remarks. (1) The first inclusion in Theorem 3.3.10 (i) can be reformulated in the following way: if $|\arg f'(z)| < \frac{\pi}{2},\ z \in \mathbb{D}$, then

$$|\arg P_n'(f)(z)| < \frac{\pi}{2},\ z \in \mathbb{D}$$

(and $P_n(f)(z)$ is univalent on $\mathbb{D}$).

(2) Since $V_n(t)$ is not nonnegative on $[0, \pi]$, the convolution polynomials $V_n(f)(z)$ based on the kernel $V_n(t)$ do not satisfy Theorem 3.3.10.

Now, for $M > 1$ let us set $S_M = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f$ is analytic on $\mathbb{D}, f(0) = f'(0) - 1 = 0, |f'(z)| < M, z \in \mathbb{D}\}$.

According to, e.g., Mocanu–Bulboacă–Sălăgean [273], p. 111, Exercise 5.4.1, $f \in S_M$ implies that f is univalent on $\mathbb{D}_{\frac{1}{M}} = \{z \in \mathbb{C}; |z| < \frac{1}{M}\}$.

Regarding the preservation of class S_M, we have the following result.

Theorem 3.3.11. (Gal [129]) *Let $O_n(t)$ be $J_n(t)$, $n \in \mathbb{N}$, or $B_{n,r,p}(t)$, $n, r \geq p+2, p \in \mathbb{N}$.*

Writing

$$P_n(f)(z) = \int_0^z Q_n(t)dt, Q_n(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt,$$

we have $P_n(S_M) \subset S_M$, and if, in addition, f' is continuous on $\overline{\mathbb{D}}$, then

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2\left(f'; \frac{1}{n}\right)_{\partial\mathbb{D}}.$$

Proof. We have $P_n(f)(0) = P_n'(f)(0) - 1 = 0$,

$$|P_n'(f)(z)| = \frac{1}{\pi}|\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f'(ze^{it})|O_n(t)dt < M,$$

$z \in \mathbb{D}$, if $|f'(z)| < M,\ z \in \mathbb{D}$. □

The convergence of the derivatives of convolution polynomials is expressed by the following.

Theorem 3.3.12. (Gal [129]) *For*

$$O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt) \geq 0, \ t \in [0, \pi],$$

$\frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt = 1$ *set* $P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt$. *If* f *is analytic on* $\mathbb{D}$ *with* f' *and* f'' *continuous on* $\overline{\mathbb{D}}$, *respectively, then*

$$\|f' - P_n'(f)\|_{\overline{\mathbb{D}}} \leq C\omega_1(f'; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f'\|_{\overline{\mathbb{D}}} \cdot |1 - \rho_{1,n}|, \quad n \in \mathbb{N},$$

and

$$\|f'' - P_n''(f)\|_{\overline{\mathbb{D}}} \leq C\omega_1(f''; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f''\|_{\overline{\mathbb{D}}} \cdot |1 - \rho_{2,n}|, \quad n \in \mathbb{N},$$

where $C > 0$ *is a constant independent of* f *and* n.

Proof. We can write

$$P_n'(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it}) \cdot e^{it} O_n(t)dt,$$

where

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{it} O_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos t O_n(t)dt = \rho_{1,n}$$

(as in the proof of Theorem 3.3.7 (ii)).

From the proof of Theorem 1 (i), in Gal [128], we obtain

$$\begin{aligned}
|P_n'(f)(z) - f'(z)| &= |P_n'(f)(z) - \rho_{1,n}f'(z) + \rho_{1,n}f'(z) - f'(z)| \\
&= |\frac{1}{\pi}\int_{-\pi}^{\pi} e^{it}O_n(t)[f'(ze^{it}) - f'(z)]dt + f'(z)[\rho_{1,n} - 1]| \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)|f'(ze^{it}) - f'(z)|dt + \|f'\|_{\overline{\mathbb{D}}} \cdot |1 - \rho_{1,n}| \\
&\leq C\omega_1(f'; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f'\|_{\overline{\mathbb{D}}} \cdot |1 - \rho_{1,n}|, \ z \in \overline{\mathbb{D}}.
\end{aligned}$$

In addition, by

$$P_n''(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2it}O_n(t) \cdot f''(ze^{it})dt$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2it} O_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t) O_n(t)dt = \rho_{2,n},$$

we get

$$P_n''(f)(z) - f''(z) = P_n''(f)(z) - \rho_{2,n}f''(z) + \rho_{2,n}f''(z) - f''(z),$$

and reasoning as above, it follows that

$$\|P_n''(f) - f''\|_{\overline{\mathbb{D}}} \leq C\omega_1(f''; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f''\|_{\overline{\mathbb{D}}} \cdot |1 - \rho_{2,n}|.$$

□

Remarks. (1) For example, let $O_n(t)$ be the Jackson kernel $J_n(t)$. By Theorem 3.3.7 (iii), we obtain

$$\rho_{1,n} = \frac{4n^3 - 4n}{2n(2n^2+1)} = \frac{2n^2-2}{2n^2+1}, \quad 1 - \rho_{1,n} = \frac{3}{2n^2+1},$$

and

$$\rho_{2,n} = \frac{4n^3 - 22n + 18}{2n(2n^2+1}, \quad 1 - \rho_{2,n} = \frac{24n-18}{4n^3+2n} = \frac{12n-9}{2n^3+n},$$

that is, the order of convergence to zero of $|1-\rho_{1,n}|$ and $|1-\rho_{2,n}|$ is $\frac{1}{n^2}$.

Analogous estimates of $|1-\rho_{1,n}|$ and $|1-\rho_{2,n}|$ hold in the case of Beatson–kernels, $B_{n,r,p}(t), n, r \geq p+2$ (see Gonska–Cao [151], Lemma 3.3).

(2) Evidently $P_n(f)(0) = 0$ and $P_n'(f)(0) = \rho_{1,n}$. Supposing $\rho_{1,n} \neq 0$, the polynomials defined by

$$R_n(f)(z) = \frac{1}{\rho_{1,n}} \cdot P_n(f)(z) = \frac{1}{\rho_{1,n}} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t) dt$$

have the property $R_n(f)(0) = R_n'(f)(0) - 1 = 0$. Also, for $z \in \overline{\mathbb{D}}$ and $f \in A(\mathbb{D})$ we get

$$\begin{aligned}|R_n(f)(z) - f(z)| &= \left| \frac{1}{\rho_{1,n}} P_n(f) - \frac{1}{\rho_{1,n}} f(z) + \frac{1}{\rho_{1,n}} f(z) - f(z) \right| \\ &\leq \frac{1}{|\rho_{1,n}|} \cdot \|P_n(f) - f\|_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \cdot \left| \frac{1}{\rho_{1,n}} - 1 \right|.\end{aligned}$$

If $O_n(t)$ is $J_n(t)$, then we have

$$\begin{aligned}\|R_n(f) - f\|_{\overline{\mathbb{D}}} &\leq \frac{2n^2+1}{2n^2-2} \cdot \|f - P_n(f)\|_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \left| \frac{2n^2+1}{2n^2-2} - 1 \right| \\ &\leq \frac{9}{6} \cdot C\omega_2 \left(f; \frac{1}{n} \right)_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{3}{2n^2-2} \\ &\leq C \left[\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n^2} \right], \quad n \geq 2,\end{aligned}$$

that is, $\|R_n(f) - f\|_{\overline{\mathbb{D}}} \leq C[\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n^2}]$, where $C > 0$ is independent of f and n.

When $O_n(t) = B_{n,r,p}(t)$, for $n, r \geq p+2,\ p \in \mathbb{N}$, a similar estimate holds.

(3) Note that since $V_n(t)$ is not nonnegative, for $O_n(t) = V_n(t)$, Theorem 3.3.12 does not hold.

In what follows, let us consider the classes of univalent functions introduced by Definition 3.1.1, $S^*(\mathbb{D}),\ K(\mathbb{D}),\ \mathcal{C}(\mathbb{D}),\ M_\alpha(\mathbb{D})$, $\alpha \in \mathbb{R}$ and $S_\gamma(\mathbb{D})$, $|\gamma| < \frac{\pi}{2}$, called the classes of normalized starlike, convex, close-to-convex, α-convex, and γ-spirallike functions, respectively.

The next results deal with preservation by convolution polynomials of the corresponding subclasses

$$S^*(\overline{\mathbb{D}}), \quad K(\overline{\mathbb{D}}), \quad \mathcal{C}(\overline{\mathbb{D}}), \quad M_\alpha(\overline{\mathbb{D}}), \text{ and } S_\gamma(\overline{\mathbb{D}}).$$

Theorem 3.3.13. (Gal [129]) *Let* $O_n(t)=\frac{1}{2}+\sum_{k=1}^{m_n}\rho_{k,n}\cos(kt)$ *be* $J_n(t), n\in\mathbb{N}$, *or* $B_{n,r,p}(t)$, *where* $n, r\geq p+2,\ p\in\mathbb{N}$.

(i) If $f\in\mathcal{C}(\overline{\mathbb{D}})$ *and*

$$P_n(f)(z)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(ze^{it})O_n(t)dt, R_n(f)(z)=\frac{1}{\rho_{1,n}}P_n(f)(z),$$

then

$$\|R_n(f)-f\|_{\overline{\mathbb{D}}}\leq C\left[\omega_2\left(f;\frac{1}{n}\right)_{\partial\mathbb{D}}+\|f\|_{\overline{\mathbb{D}}}\cdot\frac{1}{n^2}\right],$$

where $C>0$ *is independent of* f *and* n *and there exists* $n_0=n_0(f)$ *such that* $R_n(f)\in\mathcal{C}(\overline{\mathbb{D}})$ *for all* $n\geq n_0$.

(ii) For $f(z)=zh(z)$, *let us define*

$$P_n(f)(z)=z\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}h(ze^{it})O_n(t)dt.$$

Then,

$$\|P_n(f)-f\|_{\overline{\mathbb{D}}}\leq C\omega_1\left(f;\frac{1}{n}\right)_{\overline{\mathbb{D}}}+\|f\|_{\overline{\mathbb{D}}}\cdot\frac{1}{n},$$

and there is $n_0=n_0(f)$ *such that for all* $n\geq n_0$ *we have*

$$P_n[S^*(\overline{\mathbb{D}})]\subset S^*(\overline{\mathbb{D}}),\ P_n[K(\overline{\mathbb{D}})]\subset K(\overline{\mathbb{D}}),\ P_n[M_\alpha(\overline{\mathbb{D}})]\subset M_\alpha(\overline{\mathbb{D}}),$$
$$\text{and } P_n[S_\gamma(\overline{\mathbb{D}})]\subset S_\gamma(\overline{\mathbb{D}}).$$

Proof. (i) Let $f\in\mathcal{C}(\overline{\mathbb{D}})$. By Remark 2 of Theorem 3.3.12, we obtain $R_n(f)(0)=R_n'(f)(0)-1=0$ and the estimate of $\|f-R_n(f)\|_{\overline{\mathbb{D}}}$.

There is $d\in S^*(\overline{\mathbb{D}})$ that is univalent on $\overline{\mathbb{D}}$ such that

$$Re\left[\frac{zf'(z)}{d(z)}\right]>0,\quad\forall z\in\overline{\mathbb{D}}.$$

Write $h(z)=\frac{z}{d(z)}$. Since $d(0)=0$ and d is univalent, it follows that $d(z)\neq 0$, $\forall z\in\overline{\mathbb{D}},\ z\neq 0$, and $h(z)$ is analytic on $\overline{\mathbb{D}}$ (with $h(z)\neq 0,\forall z\in\overline{\mathbb{D}}$).

Hence, $h(z)$ is continuous on $\overline{\mathbb{D}}$, that is, there is $M>0$ with $|h(z)|\leq M,\forall z\in\overline{\mathbb{D}}$.

By Theorem 3.3.12 and by $\rho_{1,n}\to 1$, we get $R_n'(f)\to f'$, uniformly on $\overline{\mathbb{D}}$, that is, $h(z)\cdot R_n'(f)(z)\to h(z)f'(z)$ uniformly on $\overline{\mathbb{D}}$. Therefore

$$\text{Re}[h(z)R_n'(f)(z)]\to\text{Re}[h(z)f'(z)]>0,$$

uniformly on $\overline{\mathbb{D}}$, that is, there exists $n_0=n_0(f)$ such that

$$\text{Re}[h(z)R_n'(f)(z)]>0$$

for all $n \geq n_0$, that is,

$$R_n(f)(z) \in \mathcal{C}(\overline{\mathbb{D}})$$

for all $n \geq n_0$.

(ii) Let $f \in S^*(\overline{\mathbb{D}})$. Since $f(0) = 0$ and f is univalent on $\overline{\mathbb{D}}$, we get $f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}, z \neq 0$, that is, $f(z) = z \cdot h(z),\ z \in \overline{\mathbb{D}}$, where h is analytic in $\overline{\mathbb{D}}$ and $h(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$.

Write $Q_n(h)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(ze^{it}) O_n(t)dt$ and $P_n(f)(z) = zQ_n(h)(z)$.

By Gal [128], Theorem 1, and by Rubel–Shields–Taylor [329], we obtain

$$\begin{aligned}
|P_n(f)(z) - f(z)| &= |zQ_n(h)(z) - zh(z)| \leq |Q_n(h)(z) - h(z)| \\
&\leq C\omega_1(h; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} \leq C\omega_1\left(h; \frac{1}{n}\right)_{\partial\mathbb{D}} \\
&= C \sup\left\{\left|\frac{f(z_1)}{z_1} - \frac{f(z_2)}{z_2}\right|; |z_1 - z_2| \leq \frac{1}{n}, |z_1| = |z_2| = 1\right\} \\
&= C \sup\left\{|z_2 f(z_1) - z_1 f(z_2)|; |z_1 - z_2| \leq \frac{1}{n}, |z_1| = |z_2| = 1\right\} \\
&\leq C \sup\{|z_2| \cdot |f(z_1) - f(z_2)| + |z_1 - z_2| \cdot |f(z_2)|; \\
&\quad |z_1 - z_2| \leq \frac{1}{n}, |z_1| = |z_2| = 1\} \leq C\omega_1\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}} \\
&\quad + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n} \leq C\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n}.
\end{aligned}$$

In addition, by Theorem 3.3.12, we have $Q_n'(h) \to h'$ uniformly on $\overline{\mathbb{D}}$. Since $|h(z)| > 0,\ \forall z \in \overline{\mathbb{D}}$, and $Q_n(h) \to h$ uniformly on $\overline{\mathbb{D}}$, there are $n_1 = n_1(h)$ and $m > 0$ such that for all $n \geq n_1$ we have

$$|Q_n(h)(z)| > m,\ \forall z \in \overline{\mathbb{D}},\ \text{and therefore } Q_n(h)(z) \neq 0,\ \forall n \geq n_1, \forall z \in \overline{\mathbb{D}}.$$

Obviously $P_n'(f)(z) = zQ_n'(h)(z) + Q_n(h)(z) \to zh'(z) + h(z) = f'(z)$, uniformly on $\overline{\mathbb{D}}$, which implies

$$\begin{aligned}
\frac{zP_n'(f)(z)}{P_n(f)(z)} &= \frac{z[zQ_n'(h)(z) + Q_n(h)(z)]}{zQ_n(h)(z)} \\
&= \frac{zQ_n'(h)(z) + Q_n(h)(z)}{Q_n(h)(z)} \to \frac{zh'(z) + h(z)}{h(z)} \\
&= \frac{f'(z)}{h(z)} = \frac{zf'(z)}{f(z)},\ \text{uniformly on } \overline{\mathbb{D}}.
\end{aligned}$$

As a consequence,

$$\text{Re}\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to \text{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,$$

uniformly on $\overline{\mathbb{D}}$, that is, there exists $n_0 = n_0(f) > n_1$ such that for all $n \geq n_0$, we have

$$\text{Re}\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] > 0, \text{ for all } z \in \overline{\mathbb{D}},$$

that is, since $P_n(f)(0) = P_n'(f)(0) - 1 = 0$, it follows, that $P_n(f) \in S^*(\overline{\mathbb{D}})$.

If $f \in S_\gamma(\overline{\mathbb{D}})$, then by similar reasoning we get $P_n(f) \in S_\gamma(\overline{\mathbb{D}})$.

Now suppose $f \in K(\overline{\mathbb{D}})$ and again set $f(z) = z \cdot h(z)$, where the univalence of f on $\overline{\mathbb{D}}$ implies $h(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$, with h analytic on $\overline{\mathbb{D}}$.

Since $f \in K(\overline{\mathbb{D}})$ if and only if $zf'(z) \in S^*(\overline{\mathbb{D}})$, it follows that $f'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$, that is $|f'(z)| > 0$ for all $z \in \overline{\mathbb{D}}$.

By Theorem 3.3.12, we easily obtain $P_n'(f) \to f'$ and $P_n''(f) \to f''$, uniformly on $\overline{\mathbb{D}}$. Reasoning as above, we have

$$\text{Re}\left[\frac{zP_n''(f)}{P_n'(f)(z)}\right] + 1 \to \text{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1,$$

uniformly on $\overline{\mathbb{D}}$. Therefore, there exists $n_0 = n_0(f)$ such that for all $n \geq n_0$ we have

$$\text{Re}\left[\frac{zP_n''(f)}{P_n'(f)(z)}\right] + 1 > 0, \quad \forall z \in \overline{\mathbb{D}},$$

i.e., $P_n(f) \in K(\overline{\mathbb{D}})$.

The inclusion $P_n[M_\alpha(\overline{\mathbb{D}})] \subset M_\alpha(\overline{\mathbb{D}})$ can be deduced in a similar way, which proves the theorem. □

In what follows we will present some improvements of Theorem 3.3.13.

Theorem 3.3.14. *Suppose $\alpha \in (0,1)$ and $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.*

(i) For any $f \in S_\alpha^*(\mathbb{D}) \cap A^1(\mathbb{D})$ with $f(z) \neq 0$ for all $|z| = 1$, and any $\beta \in (0, \alpha)$, there is $n_0 \geq 1$ (depending on f and β) such that for all $n \geq n_0$, we have

$$\sigma_{2n,n-1}(f) \in S_\beta^*(\mathbb{D}) \text{ and } \|f - \sigma_{2n,n-1}(f)\| \leq C\frac{1}{n}E_n(f'),$$

where $C > 0$ is independent of f and n;

(ii) For any $f \in K_\alpha(\mathbb{D}) \cap A^2(\mathbb{D})$ with $f'(z) \neq 0$ for all $|z| = 1$, and any $\beta \in (0, \alpha)$, there is $n_0 \geq 2$ (depending on f and β) such that for all $n \geq n_0$, we have

$$\sigma_{2n,n-2}(f) \in K_\beta(\mathbb{D}) \text{ and } \|f - \sigma_{2n,n-2}(f)\| \leq C\frac{1}{n^2}E_n(f''),$$

where $C > 0$ is independent of f and n;

(iii) For any $f \in S_\alpha^\gamma(\mathbb{D}) \cap A^1(\mathbb{D})$ with $f(z) \neq 0$ for all $|z| = 1$, and any $\beta \in (0, \alpha)$, there is $n_0 \geq 1$ (depending on f, γ, and β) such that for all $n \geq n_0$, we have

$$\sigma_{2n,n-1}(f) \in S_\beta^\gamma(\mathbb{D}) \text{ and } \|f - \sigma_{2n,n-1}(f)\| \leq C\frac{1}{n}E_n(f'),$$

where $C > 0$ is independent of f and n.

Proof. From Remark 1 after the proof of Theorem 3.2.1, the estimates in Theorem 3.3.14 (i–iii), are immediate.

Also, because $f(0) = f'(0) - 1 = 0$ and for any k we have $T_k(f)(0) = T_k'(f)(0) - 1 = 0$, it follows that $\sigma_{2n,n-p}(f)(0) = f(0) = 0$ and $\sigma'_{2n,n-p}(f)(0) - 1 = f'(0) - 1 = 0$, for all $p \in \mathbb{N}$ and $n \geq p$.

(i) The hypothesis implies $|f(z)| > 0$ for all $z \in \overline{\mathbb{D}}$ with $z \neq 0$, which from the univalence of f in $\mathbb{D}$ means that we can write f in the form $f(z) = zh(z)$, with $h(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$, where h is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$.

Setting $\sigma_{2n,n-1}(f)(z) = zQ_n(f)(z)$, it is clear that $Q_n(f)(z)$ is a polynomial of degree $\leq 2n - 1$.

For $|z| = 1$ we get

$$|f(z) - \sigma_{2n,n-1}(f)(z)| = |z| \cdot |h(z) - Q_n(f)| = |h(z) - Q_n(f)|.$$

Therefore, the uniform convergence on $\overline{\mathbb{D}}$ of $\sigma_{2n,n-1}(f)$ to f, together with the maximum modulus principle, implies the uniform convergence on $\overline{\mathbb{D}}$ of $Q_n(f)(z)$ to $h(z)$.

Because h is continuous in $\overline{\mathbb{D}}$ and $|h(z)| > 0$ for all $z \in \overline{\mathbb{D}}$, there exist an index $n_0 \in \mathbb{N}$ and $a > 0$ depending on h such that $|Q_n(f)(z)| > a > 0$ for all $z \in \overline{\mathbb{D}}$ and all $n \geq n_0$.

Also, for all $|z| = 1$, we get

$$\begin{aligned}|f'(z) - \sigma'_{2n,n-1}(f)(z)| &= |z[h'(z) - Q_n'(f)(z)] + [h(z) - Q_n(f)(z)]| \\ &\geq ||z| \cdot |h'(z) - Q_n'(f)(z)| - |h(z) - Q_n(f)(z)|| \\ &= ||h'(z) - Q_n'(f)(z)| - |h(z) - Q_n(f)(z)||,\end{aligned}$$

which from maximum modulus principle, the uniform convergence of the sequences $\sigma'_{2n,n-1}(f)$ to f' and of $Q_n(f)$ to h, evidently implies the uniform convergence of the sequence $Q_n'(f)$ to h'.

Again, for $|z| = 1$, we obtain

$$\begin{aligned}\frac{z\sigma'_{2n,n-1}(f)(z)}{\sigma_{2n,n-1}(f)} &= \frac{z[zQ_n'(f)(z) + Q_n(f)(z)]}{zQ_n(f)(z)} \\ &= \frac{zQ_n'(f)(z) + Q_n(f)(z)}{Q_n(f)(z)} \to \frac{zh'(z) + h(z)}{h(z)} = \frac{f''(z)}{h(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which by the maximum modulus principle implies

$$\frac{z\sigma'_{2n,n-1}(f)(z)}{\sigma_{2n,n-1}(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly on } \overline{\mathbb{D}}.$$

As a conclusion,

$$\operatorname{Re}\left[\frac{z\sigma'_{2n,n-1}(f)(z)}{\sigma_{2n,n-1}(f)(z)}\right] \to \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha > 0$$

uniformly on $\overline{\mathbb{D}}$, that is, for any $0 < \beta < \alpha$, there exists n_0 such that for all $n \geq n_0$ we have

$$\text{Re}\left[\frac{z\sigma'_{2n,n-1}(f)(z)}{\sigma_{2n,n-1}(f)(z)}\right] > \beta, \text{ for all } z \in \overline{\mathbb{D}},$$

that is, $\sigma_{2n,n-1}(f) \in S^*_\beta(\mathbb{D})$.

In the case (iii), similarly we obtain

$$\text{Re}\left[e^{i\gamma}\frac{z\sigma'_{2n,n-1}(f)(z)}{\sigma_{2n,n-1}(f)(z)}\right] \to \text{Re}\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right] > \alpha > 0$$

uniformly on $\overline{\mathbb{D}}$, which immediately proves the required result.

(ii) Since $f \in K_\alpha(\mathbb{D})$ if and only if $zf'(z) \in S^*_\alpha(\mathbb{D})$, by the reasoning at the point (i), it follows that $f'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$, that is, $|f'(z)| > 0$ for all $z \in \overline{\mathbb{D}}$. Also, by the same type of reasoning as at the above point (i), where f is replaced by f', we get $\sigma'_{2n,n-2}(f) \to f'$ and $\sigma''_{2n,n-2}(f) \to f''$, uniformly on $\overline{\mathbb{D}}$, and

$$\text{Re}\left[\frac{z\sigma''_{2n,n-2}(f)(z)}{\sigma'_{2n,n-2}(f)(z)}\right] + 1 \to \text{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1 > \alpha,$$

uniformly on $\overline{\mathbb{D}}$. Therefore, for any $0 < \beta < \alpha$, there exists $n_0 = n_0(f, \beta)$ such that for all $n \geq n_0$ we have

$$\text{Re}\left[\frac{z\sigma''_{2n,n-2}(f)(z)}{\sigma'_{2n,n-2}(f)(z)}\right] + 1 > \beta, \quad \forall z \in \mathbb{D}.$$

For the case $\alpha = 0$, we have the following corollary.

Corollary 3.3.15. *Suppose* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(i) For any $f \in S^*(\overline{\mathbb{D}})$, *there is* $n_0 \geq 1$ *(depending on* f*) such that for all* $n \geq n_0$, *we have*

$$\sigma_{2n,n-1}(f) \in S^*(\overline{\mathbb{D}}) \text{ and } \|f - \sigma_{2n,n-1}(f)\| \leq C\frac{1}{n}E_n(f'),$$

where $C > 0$ *is independent of* f *and* n.

(ii) For any $f \in K(\overline{\mathbb{D}})$, *there is* $n_0 \geq 2$ *(depending on* f*) such that for all* $n \geq n_0$, *we have*

$$\sigma_{2n,n-2}(f) \in K(\overline{\mathbb{D}}) \text{ and } \|f - \sigma_{2n,n-2}(f)\| \leq C\frac{1}{n^2}E_n(f''),$$

where $C > 0$ *is independent of* f *and* n;

(iii) For any $f \in S_\gamma(\overline{\mathbb{D}})$, *there is* $n_0 \geq 1$ *(depending on* f *and* γ*) such that for all* $n \geq n_0$, *we have*

$$\sigma_{2n,n-1}(f) \in S_\gamma(\overline{\mathbb{D}}) \text{ and } \|f - \sigma_{2n,n-1}(f)\| \leq C\frac{1}{n}E_n(f'),$$

where $C > 0$ *is independent of* f *and* n.

Proof. Recall that f is called analytic on $\overline{\mathbb{D}}$ if in fact f is analytic in an open set (domain) containing $\overline{\mathbb{D}}$. This obviously implies that $f \in A^p(\mathbb{D})$ for all $p \in \mathbb{N}$. The rest of the proof is similar to that of Theorem 3.3.14.

Remarks. (1) The degree of $\sigma_{2n,n-p}(f)$ is $\leq 2n$, while its approximation rate given by Theorem 3.3.14 and Corollary 3.3.15 is $\frac{1}{n^p} E_n(f^{(p)})$, $p = 1, 2$, which could be essentially worse than the best expected, $\frac{1}{(2n)^p} E_{2n-p}(f^{(p)})$, $p = 1, 2$. This is because in general, $E_{2n-p}(F)$ can be essentially smaller than $E_n(F)$. However, applying a well known result of Gaier (see [121] or, e.g., [122], p. 53, that is, $E_n(F) \leq C_k \omega_k(F; 1/n)$ for any arbitrary $k \in \mathbb{N}$), we immediately get that both quantities $\frac{1}{n^p} E_n(f^{(p)})$ and $\frac{1}{(2n)^p} E_{2n-p}(f^{(p)})$ in fact can be estimated by the same expression $C_k \frac{1}{n^p} \omega_k(f^{(p)}; 1/n)$, $p = 1, 2$. In other words, expressed in terms of moduli of smoothness, we can say that the estimates in Theorem 3.3.14 and Corollary 3.3.15 are near to the best approximation.

(2) The estimates in Corollary 3.3.15 essentially improve those in Theorem 3.3.13, where the orders of approximation are $\omega_1(f; 1/n)$ and $\omega_2(f; 1/n)$ only.

(3) The shortcoming of Theorem 3.3.14 and Corollary 3.3.15 is that the index n_0 depends on f. Therefore, it is natural to ask whether Theorem 3.3.14 and Corollary 3.3.15 would be valid for all $n \geq p$ ($p = 1$ or $p = 2$), or at least for all $n \geq n_0$ with n_0 independent of f.

(4) Note that because the differential inequalities that define the starlikeness, convexity, and spirallikeness are nonlinear, Theorem 3.2.2 and its Corollary 3.2.3 cannot be applied to these subclasses of functions.

3.4 Approximation and Geometric Properties of Bernstein Polynomials

In this section, we first estimate the degrees of simultaneous uniform approximation of analytic functions by complex Bernstein polynomials in closed disks. Then, we prove that the complex Bernstein polynomials attached to an analytic function preserve the univalence, starlikeness, convexity, spirallikeness, and other properties in geometric function theory.

Concerning the approximation properties, two approximation results due to Bernstein and Kantorovich concerning the uniform approximation of Bernstein polynomials in the unit disk and in an ellipse, respectively among others, are well known. In addition, Theorem 3.4.1 (iii), (iv), (v) below, give quantitative estimates. Note that an analogue to Theorem 3.4.1 (iii) (and in essence to (iv) too), was obtained by a different method in Ostrovska [292].

All the above-mentioned approximation results can be summarized by the following.

Theorem 3.4.1. *(i) (Bernstein, see, e.g., [Lorentz [247], p. 88]) For the open $G \subset \mathbb{C}$ such that $\overline{\mathbb{D}} \subset G$ and $f : G \to \mathbb{C}$ is analytic in G, the complex Bernstein polynomials $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n)$ uniformly converge to f in $\overline{\mathbb{D}}$.*

(ii) (Kantorovich, see, e.g., [Lorentz [247], p. 90]) If f is analytic in the interior of an ellipse of foci 0 *and* 1*, then $B_n(f)(z)$ converges uniformly to $f(z)$ in any closed set contained in the interior of the ellipse.*

Now let $G \subset \mathbb{C}$ be an open disk of radius $R > 1$ and center 0*, and let us suppose that $f : G \to \mathbb{C}$ is analytic in G, that is, we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in G$.*

(iii) Denoting by $\|\cdot\|$ the uniform norm in $C(\overline{\mathbb{D}})$, for the complex Bernstein polynomials

$$B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n),$$

we have

$$\|B_n(f) - f\| \leq \frac{M_2(f)}{n}, \text{ for all } n \in \mathbb{N},$$

where $0 < M_2(f) = 3\sum_{j=2}^{\infty} j(j-1)|c_j| < \infty$.

(iv) Let $1 \leq r < R$ be arbitrary fixed. For all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$|B_n(f)(z) - f(z)| \leq \frac{M_{2,r}(f)}{n},$$

where $0 < M_{2,r}(f) = \frac{r(1+r)(1+2r)}{2} \sum_{j=2}^{\infty} j(j-1)|c_j| r^{j-2} < \infty$.

(v) For the simultaneous approximation by complex Bernstein polynomials, we have that if $1 \leq r < r_1 < R$ are arbitrary fixed, then for all $|z| \leq r$ and $n, p \in \mathbb{N}$,

$$|B_n^{(p)}(f)(z) - f^{(p)}(z)| \leq \frac{M_{2,r_1}(f) p! r_1}{n(r_1 - r)^{p+1}}$$

where $M_{2,r_1}(f)$ is given in point (iii).

Proof. (iii) Writing $e_k(z) = z^k$, $k = 0, 1, \ldots$, and $\pi_{k,n}(z) = B_n(e_k)(z)$, by the proof of, e.g., Theorem 4.1.1, p. 88 in Lorentz [247], we can write $B_n(f)(z) = \sum_{k=0}^{\infty} c_k \pi_{k,n}(z)$, which immediately implies

$$|B_n(f)(z) - f(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |\pi_{k,n}(z) - e_k(z)|$$

for all $z \in \overline{\mathbb{D}}$.

First we observe that $|\pi_{k,n}(z)| \leq 1$ for all $z \in \overline{\mathbb{D}}$ and all $k = 1, 2, \ldots$. For this purpose, the reasoning is similar to that in the proof of Theorem 1.4.1, pp. 88–89 in Lorentz [247], by taking $R_1 = 1$ there. Let us briefly recall here this reasoning. Indeed, by the relationship (5) in the above-mentioned proof (by taking $a = 0$ there), we get that the generating functions $\Phi(u, z)$ of the polynomials $\pi_{k,n}(z)$ are

$$\begin{aligned}\Phi(u, z) &= \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{k,n}(z) u^k \\ &= \left(1 + \frac{z}{n} u + \cdots + \frac{z}{n^k} \frac{u^k}{k!} + \cdots + \right)^n.\end{aligned}$$

Taking now $|z| = 1$, the coefficients of u^k in the power series

$$1 + \frac{z}{n}u + \cdots + \frac{z}{n^k}\frac{u^k}{k!} + \cdots$$

are majorized by those of

$$\sum_{k=0}^{\infty} \frac{1}{u!}\left(\frac{u}{n}\right)^k = e^{u/n}$$

if $|z| = 1$ and therefore if $|z| \leq 1$. Finally, this proves that the moduli of the coefficients of u^k in $\Phi(u,z) = \sum_{k=0}^{\infty} \frac{1}{k!}\pi_{k,n}(z)u^k$ do not exceed those in $(e^{u/n})^n = e^u$, which proves the required inequality for $|\pi_{k,n}(z)|$.

In what follows, we use the recurrence relationship proved for the real variable case in Andrica [19],

$$\pi_{k+1,n}(z) = \frac{z(1-z)}{n}\pi'_{k,n}(z) + z\pi_{k,n}(z)$$

for all $n \in \mathbb{N}$, $z \in \mathbb{C}$, and $k = 0, 1, \ldots$. Indeed, since it is a simple algebraic manipulation, the relationship in Andrica [19] proved for the real case is valid for complex variables too. Taking into account that the paper Andrica [19] is not very accessible, let us reproduce here the idea of the proof. It consists in the simple algebraic relationship

$$S'_{k,n}(z) = \frac{S_{k+1,n}(z)}{z(1-z)} - n\frac{S_{k,n}(z)}{1-z},$$

which is divided by n^k, where

$$S_{k,n}(z) = \sum_{j=0}^{n} j^k \binom{n}{j} z^j (1-z)^{n-j}.$$

(Note that the cases $z = 0$ and $z = 1$ are trivial in the recurrence for $\pi_{k,n}(z)$.)

From this recurrence, we easily obtain that degree $(\pi_{k,n}(z)) = k$. Also, by replacing k with $k-1$, we get

$$\pi_{k,n}(z) - z^k = \frac{z(1-z)}{n}[\pi_{k-1,n}(z) - z^{k-1}]' + \frac{(k-1)z^{k-1}(1-z)}{n} + z[\pi_{k-1,n}(z) - z^{k-1}],$$

which by Bernstein's inequality for complex polynomials gives

$$\begin{aligned}\|\pi_{k,n} - e_k\| &\leq (k-1)\frac{2}{n}\|\pi_{k-1,n} - e_{k-1}\| + \frac{2(k-1)}{n} + \|\pi_{k-1,n} - e_{k-1}\| \\ &\leq (k-1)\frac{2}{n} \cdot [\|\pi_{k-1,n}\| + \|e_{k-1}\|] + \frac{2(k-1)}{n} + \|\pi_{k-1,n} - e_{k-1}\| \\ &\leq \|\pi_{k-1,n} - e_{k-1}\| + 6\frac{k-1}{n}.\end{aligned}$$

(We used here that for all k and n we have $\|\pi_{k,n}\| \le 1$ and $\|e_k\| \le 1$.)

Now, by taking $k = 1, 2$, in the inequality

$$\|\pi_{k,n} - e_k\| \le \|\pi_{k-1,n} - e_{k-1}\| + 6\frac{k-1}{n},$$

we finally obtain

$$\|\pi_{k,n} - e_k\| \le \frac{6}{n}[(k-1) + (k-2) + \cdots + 1] = \frac{3}{n}(k-1)k.$$

In conclusion, we get

$$|B_n(f)(z) - f(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot \|\pi_{k,n} - e_k\| \le \frac{1}{n} 3 \sum_{k=2}^{\infty} k(k-1)|c_k|.$$

Note that since by hypothesis, $f(z) = \sum_k^{\infty} c_k z^k$ is absolutely convergent in $|z| \le r$ for any $1 \le r < R$, it follows that the power series obtained by differentiating twice, i.e., $f''(z) = \sum_{k=2}^{\infty} k(k-1)c_k z^{k-2}$, also is absolutely convergent for $|z| \le 1$, which implies $\sum_{k=2}^{\infty} k(k-1)|c_k| < +\infty$.

(iv) The relationship (4) in the proof of Theorem 4.1.1, p. 88 in Lorentz [247], shows that $|B_n(e_k)(z)| \le r^k$ if $|z| \le r$, with $r > 1$, for all $k, n \in \mathbb{N}$.

We observe that the above inequality can be proved in a different way, as follows. Indeed, by He [169] (see also Corollary 2.4 in Andrica [19]), we can write

$$B_n(e_k)(z) = \sum_{j=1}^{k} S(k,j) \frac{n(n-1)\cdots[n-(j-1)]}{n^k} e_j(z),$$

where $S(k,j)$ are the Stirling numbers of the second kind and recall that these numbers satisfy $S(k,j) \ge 0$ for all $j, k \in \mathbb{N}$ and

$$\sum_{j=1}^{k} S(k,j) n(n-1)\cdots[n-(j-1)] = n^k \quad \text{for } k, n \in \mathbb{N}.$$

Now, since $S(k,j)n(n-1)\cdots[n-(j-1)] \ge 0$ for all $k, n, j \in \mathbb{N}$ with $1 \le j \le k$, this implies

$$\begin{aligned}
|B_n(e_k)(z)| &\le \sum_{j=1}^{k} S(k,j) \frac{n(n-1)\cdots[n-(j-1)]}{n^k} |e_j(z)| \\
&\le \sum_{j=1}^{k} S(k,j) \frac{n(n-1)\cdots[n-(j-1)]}{n^k} r^j \\
&\le \sum_{j=1}^{k} S(k,j) \frac{n(n-1)\cdots[n-(j-1)]}{n^k} r^k = r^k, \quad \text{for } |z| \le r.
\end{aligned}$$

Denoting by $\|\cdot\|_r$ the norm in $C(\overline{\mathbb{D}}_r)$, where $\overline{\mathbb{D}}_r = \{z \in \mathbb{C}; |z| \le r\}$, one observes that by a linear transformation, Bernstein's inequality in the closed unit disk becomes $|P_k'(z)| \le \frac{k}{r}\|P_k\|_r \le k\|P_k\|_r$ for all $|z| \le r$, $r \ge 1$ (at this point I would like to thank professors J. Szabados and T. Erdelyi, of the Alfred Renyi Institute of Mathematics of the Hungarian Academy of Science and the Department of Mathematics, Texas A & M University, College Station, USA, respectively, for bibliographical reference).

Therefore, repeating the reasoning from the above point (iii), we get

$$\begin{aligned}|\pi_{k,n}(z) - e_k(z)| &\le (k-1)\frac{r(1+r)}{n}\|\pi_{k-1,n} - e_{k-1}(z)\|_r \\ &\quad + \frac{r^{k-1}(1+r)(k-1)}{n} + r|\pi_{k-1,n}(z) - e_{k-1}(z)| \\ &\le (k-1)\frac{r(1+r)}{n} \cdot [\|\pi_{k-1,n}\|_r + \|e_{k-1}\|_r] \\ &\quad + \frac{r^{k-1}(1+r)(k-1)}{n} + r|\pi_{k-1,n}(z) - e_{k-1}(z)| \\ &\le r|\pi_{k-1,n}(z) - e_{k-1}(z)| + \left[2r(1+r)r^{k-1} + (1+r)r^{k-1}\right]\frac{k-1}{n}.\end{aligned}$$

(We used here that for all $k, n \in \mathbb{N}$ and $|z| \le r$ we have $|\pi_{k,n}(z)| \le r^k$ and $|e_k(z)| \le r^k$.)

Now, by taking $k = 1, 2, \ldots,$ in the inequality

$$|\pi_{k,n}(z) - e_k(z)| \le r|\pi_{k-1,n}(z) - e_{k-1}(z)| + (1+r)(1+2r)r^{k-1}\frac{k-1}{n},$$

by recurrence we easily obtain

$$\begin{aligned}|\pi_{k,n}(z) - e_k(z)| &\le \frac{(1+r)(1+2r)}{n}\left[r^{k-1} + 2r^{k-1} + \cdots + (k-1)r^{k-1}\right] \\ &\le \frac{(1+r)(1+2r)}{n} \cdot \frac{k(k-1)}{2}r^{k-1} \\ &\le \frac{r(1+r)(1+2r)}{2n} \cdot k(k-1)r^{k-2}.\end{aligned}$$

As a conclusion, for all $|z| \le r$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned}|B_n(f)(z) - f(z)| &\le \sum_{k=0}^{\infty} |c_k| \cdot |\pi_{k,n}(z) - e_k(z)| \\ &\le \frac{r(1+r)(1+2r)}{2n}\sum_{k=2}^{\infty} k(k-1)|c_k|r^{k-2}.\end{aligned}$$

Note that since by hypothesis, $f(z) = \sum_k^{\infty} c_k z^k$ is absolutely and uniformly convergent in $|z| \le r$ for any $1 \le r < R$, it follows that the power series obtained by differentiating twice, i.e., $f''(z) = \sum_{k=2}^{\infty} k(k-1)c_k z^{k-2}$, also is absolutely convergent for $|z| \le r$, which implies $\sum_{k=2}^{\infty} k(k-1)|c_k|r^{k-2} < +\infty$.

(v) Denoting by γ the circle of radius $r_1 > 1$ and center 0, since for any $|z| \le r$ and $v \in \gamma$, we have $|v - z| \ge r_1 - r$, by Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|B_n^{(p)}(f)(z) - f^{(p)}(z)| = \frac{p!}{2\pi}\left|\int_\gamma \frac{B_n(f)(v) - f(v)}{(v-z)^{p+1}}dv\right|$$
$$\le \frac{M_{2,r_1}(f)}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1 - r)^{p+1}} = \frac{M_{2,r_1}(f)}{n}\frac{p! r_1}{(r_1 - r)^{p+1}},$$

which proves the theorem. □

Remarks. (1) For $r = 1$, the estimate in Theorem 3.4.1 (iv) becomes the estimate in Theorem 3.4.1 (iii) but obtained by a different method of proof.

2) By Theorem 4.1.1, p. 88 in Lorentz [247], in fact it follows that for this $R > 1$, $B_n(f) \to f$ uniformly in any closed disk included in $\{|z| < R\}$, which by the well known Weierstrass theorem (see, e.g., Kohr–Mocanu [198], p. 18., Theorem 1.1.6) implies that the sequences of the derivatives of any order of complex Bernstein polynomials converge uniformly on compacts in G (including the closed unit disk) to the corresponding derivatives of f. Theorem 3.4.1 (v) expresses in addition quantitative estimates for these convergence processes.

The first geometric properties of the Bernstein polynomials are consequences of Theorem 3.4.1 and can be expressed by the following.

Theorem 3.4.2. *Let us suppose that $G \subset \mathbb{C}$ is open such that $\overline{\mathbb{D}} \subset G$ and $f : G \to \mathbb{C}$ is analytic in G.*

(i) If f is univalent in $\overline{\mathbb{D}}$, then there exists an index n_0 depending on f such that for all $n \ge n_0$, the complex Bernstein polynomials $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n)$ are univalent in $\overline{\mathbb{D}}$.

(ii) If $f(0) = f'(0) - 1 = 0$ and f is starlike in $\overline{\mathbb{D}}$, that is,

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \textit{ for all } z \in \overline{\mathbb{D}},$$

then there exists an index n_0 depending on f such that for all $n \ge n_0$, the complex Bernstein polynomials are starlike in $\overline{\mathbb{D}}$.

If $f(0) = f'(0) - 1 = 0$ and f is starlike only in $\mathbb{D}$, then for any disk of radius $0 < r < 1$ and center 0, denoted by $\mathbb{D}_r$, there exists an index $n_0 = n_0(f, \mathbb{D}_r)$ such that for all $n \ge n_0$, the complex Bernstein polynomials $B_n(f)(z)$ are starlike in $\overline{\mathbb{D}}_r$, that is,

$$\text{Re}\left(\frac{zB_n'(f)(z)}{B_n(f)(z)}\right) > 0 \textit{ for all } z \in \overline{\mathbb{D}}_r.$$

(iii) If $f(0) = f'(0) - 1 = 0$ and f is convex in $\overline{\mathbb{D}}$, that is,

$$\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) + 1 > 0 \textit{ for all } z \in \overline{\mathbb{D}},$$

then there exists an index n_0 *depending on* f *such that for all* $n \geq n_0$*, the complex Bernstein polynomials are convex in* $\overline{\mathbb{D}}$*.*

If $f(0) = f'(0) - 1 = 0$ *and* f *is convex only in* $\mathbb{D}$*, then for any disk of radius* $0 < r < 1$ *and center* 0*, denoted by* $\mathbb{D}_r$*, there exists an index* $n_0 = n_0(f, \mathbb{D}_r)$ *such that for all* $n \geq n_0$*, the complex Bernstein polynomials* $B_n(f)(z)$ *are convex in* $\overline{\mathbb{D}}_r$*, that is,*

$$\text{Re}\left(\frac{zB_n''(f)(z)}{B_n'(f)(z)}\right) + 1 > 0 \text{ for all } z \in \overline{\mathbb{D}}_r.$$

(iv) If $f(0) = f'(0) - 1 = 0$*,* $f(z) \neq 0$*, for all* $z \in \overline{\mathbb{D}} \setminus \{0\}$ *and* f *is spirallike of type* $\gamma \in (-\pi/2, \pi/2)$ *in* $\overline{\mathbb{D}}$*, that is,*

$$\text{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right) > 0 \text{ for all } z \in \overline{\mathbb{D}},$$

then there exists an index n_0 *depending on* f *and* γ *such that for all* $n \geq n_0$ *we have* $B_n(f)(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}} \setminus \{0\}$*, and* $B_n(f)(z)$ *are spirallike of type* γ *in* $\overline{\mathbb{D}}$*.*

If $f(0) = f'(0) - 1 = 0$*,* $f(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$ *and* f *is spirallike of type* γ *only in* $\mathbb{D}$*, then for any disk of radius* $0 < r < 1$ *and center* 0*, denoted by* $\mathbb{D}_r$*, there exists an index* $n_0 = n_0(f, \mathbb{D}_r, \gamma)$ *such that for all* $n \geq n_0$*, the Bernstein polynomials* $B_n(f)(z)$ *are non-zero for all* $z \in \overline{\mathbb{D}}_r \setminus \{0\}$ *and they are spirallike of type* γ *in* $\overline{\mathbb{D}}_r$*, that is,*

$$\text{Re}\left(e^{i\gamma}\frac{zB_n'(f)(z)}{B_n(f)(z)}\right) > 0 \text{ for all } z \in \overline{\mathbb{D}}_r.$$

Proof. (i) It is immediate from the uniform convergence in Theorem 3.4.1 and a well known result concerning sequences of analytic functions converging locally uniformly to a univalent function (see, e.g., Kohr–Mocanu [198], p. 130, Theorem 4.1.17 or Graham-Kohr [162], Theorem 6.1.18).

For the proof of the next points (ii), (iii) and (iv), let us make some general useful remarks. By Remark 2 after the proof of Theorem 3.4.1, we get that for $n \to \infty$, we have $B_n(f)(z) \to f(z)$, $B_n'(f)(z) \to f'(z)$ and $B_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}$. In all that follows, set $P_n(f)(z) = \frac{B_n(f)(z)}{nf(1/n)}$.

By $f(0) = f'(0) - 1 = 0$ and the univalence of f, we get $nf(1/n) \neq 0$, $P_n(f)(0) = \frac{f(0)}{nf(1/n)} = 0$, $P'(f)(0) = \frac{B_n'(f)(0)}{nf(1/n)} = 1$, $n \geq 2$, $nf(1/n) = \frac{f(1/n)-f(0)}{1/n}$ converges to $f'(0) = 1$ as $n \to \infty$, which means that for $n \to \infty$, we have $P_n(f)(z) \to f(z)$, $P_n'(f)(z) \to f'(z)$ and $P_n''(f)(z) \to f''(z)$, uniformly in $\overline{\mathbb{D}}$.

(ii) By hypothesis we get $|f(z)| > 0$ for all $z \in \overline{\mathbb{D}}$ with $z \neq 0$, which from the univalence of f in $\mathbb{D}$ implies that we can write $f(z) = zg(z)$, with $g(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$, where g is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$.

Writing $P_n(f)(z)$ in the form $P_n(f)(z) = zQ_n(f)(z)$, obviously $Q_n(f)(z)$ is a polynomial of degree $\leq n - 1$.

Let $|z| = 1$. We have

$$|f(z) - P_n(f)(z)| = |z| \cdot |g(z) - Q_n(f)(z)| = |g(z) - Q_n(f)(z)|,$$

which by the uniform convergence in $\overline{\mathbb{D}}$ of $P_n(f)$ to f and by the maximum modulus principle implies the uniform convergence in $\overline{\mathbb{D}}$ of $Q_n(f)(z)$ to $g(z)$.

Since g is continuous in $\overline{\mathbb{D}}$ and $|g(z)| > 0$ for all $z \in \overline{\mathbb{D}}$, there exist an index $n_1 \in \mathbb{N}$ and $a > 0$ depending on g such that $|Q_n(f)(z)| > a > 0$ for all $z \in \overline{\mathbb{D}}$ and all $n \geq n_0$.

Also, for all $|z| = 1$, we have

$$\begin{aligned}|f'(z) - P_n'(f)(z)| &= |z[g'(z) - Q_n'(f)(z)] + [g(z) - Q_n(f)(z)]| \\ &\geq ||z| \cdot |g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)|| \\ &= ||g'(z) - Q_n'(f)(z)| - |g(z) - Q_n(f)(z)||,\end{aligned}$$

which from the maximum modulus principle and the uniform convergence of $P_n'(f)$ to f' and of $Q_n(f)$ to g evidently implies the uniform convergence of $Q_n'(f)$ to g'.

Then, for $|z| = 1$, we get

$$\begin{aligned}\frac{zP_n'(f)(z)}{P_n(f)} &= \frac{z[zQ_n'(f)(z) + Q_n(f)(z)]}{zQ_n(f)(z)} \\ &= \frac{zQ_n'(f)(z) + Q_n(f)(z)}{Q_n(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} = \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)},\end{aligned}$$

which again from the maximum modulus principle implies

$$\frac{zP_n'(f)(z)}{P_n(f)} \to \frac{zf'(z)}{f(z)}, \text{ uniformly in } \overline{\mathbb{D}}.$$

Since $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)$ is continuous in $\overline{\mathbb{D}}$, there exists $\alpha \in (0, 1)$ such that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \text{ for all } z \in \overline{\mathbb{D}}.$$

Therefore

$$\mathrm{Re}\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] \geq \alpha > 0$$

uniformly on $\overline{\mathbb{D}}$, i.e., for any $0 < \beta < \alpha$, there is n_0 such that for all $n \geq n_0$ we have

$$\mathrm{Re}\left[\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] > \beta > 0, \text{ for all } z \in \overline{\mathbb{D}}.$$

Since $P_n(f)(z)$ differs from $B_n(f)(z)$ only by a constant, this proves the first part in (ii).

For the second part, the proof is identical to that of the first part, with the only difference being that instead of $\overline{\mathbb{D}}$, we reason for $\overline{\mathbb{D}}_r$.

(iv) Obviously we have

$$\mathrm{Re}\left[e^{i\gamma}\frac{zP_n'(f)(z)}{P_n(f)(z)}\right] \to \mathrm{Re}\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right],$$

uniformly in $\overline{\mathbb{D}}$. We also note that since f is univalent in $\overline{\mathbb{D}}$, by the above point (i), there exists n_1 such that $B_n(f)(z)$ is univalent in $\overline{\mathbb{D}}$ for all $n \geq n_1$ which by $B_n(f)(0) = 0$ implies $B_n(f)(z) \neq 0$ for all $z \in \overline{\mathbb{D}} \setminus \{0\}$, $n \geq n_1$. For the rest, the proof is identical to that from the above point (ii).

(iii) For the first part, by hypothesis there is $\alpha \in (0, 1)$ such that

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}$. It is not difficult to show that this fact is equivalent to the fact that for any $\beta \in (0, \alpha)$, the function $zf'(z)$ is starlike of order β in $\overline{\mathbb{D}}$ (see, e.g., Mocanu–Bulboaca–Salagean [271], p. 77), which implies $f'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$, i.e., $|f'(z)| > 0$ for all $z \in \overline{\mathbb{D}}$. Also, by the same type of reasoning as that from the above point (ii), we get

$$\mathrm{Re}\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 \to \mathrm{Re}\left[\frac{zf''(z)}{f'(z)}\right] + 1 \geq \alpha > 0,$$

uniformly in $\overline{\mathbb{D}}$. As a conclusion, for any $0 < \beta < \alpha$, there is n_0 depending on f such that for all $n \geq n_0$ we have

$$\mathrm{Re}\left[\frac{zP_n''(f)(z)}{P_n'(f)(z)}\right] + 1 > \beta > 0 \text{ for all } z \in \overline{\mathbb{D}}.$$

The proof of the second part in (iii) is similar, which proves the theorem. □

Remarks. (1) Let us recall that geometrically, the starlikeness/convexity of f on $\mathbb{D}_r \subset \mathbb{D}$ means that the image through f of any closed disk of center 0 included in $\mathbb{D}_r$ is a starlike/convex set in $\mathbb{C}$ (here starlikeness of a set means with respect to the origin).

(2) The results in Theorem 3.4.2 state that the complex Bernstein polynomials $B_n(f)(z)$ preserve the univalence, starlikeness, convexity, and spirallikeness only for sufficiently large values of n, that is, only for $n \geq n_0$, where the value $n_0 = n_0(f)$ cannot be in general specified. Their shortcoming is that they do not say anything about the cases of small values of n (that is for $n < n_0$) or for specified values of n. In what follows, we will present such kinds of results. Also, the next results represent interesting and simple ways of construction for particular polynomials that are univalent starlike or convex in the unit disk.

For this purpose, we introduce four subclasses of functions defined similarly to S_1 and S_3, S_4 and S_M, considered in Section 3.3.2, just before Theorems 3.3.9, 3.3.10, and 3.3.11, respectively.

Thus, for $M > 1$, let us define

$$\overline{S}_M = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f \in A^1(\mathbb{D}), f(0) = f'(0) - 1 = 0, |f'(z)| < M, \text{ for all } z \in \overline{\mathbb{D}}\},$$

then

$$\overline{S}_3 = \{f : \overline{\mathbb{D}} \to \mathbb{C}; f \in A^2(\mathbb{D}), f(0) = f'(0) - 1 = 0, |f''(z)| \le 1, \text{ for all } z \in \overline{\mathbb{D}}\},$$

$$\overline{S}_4 = \left\{f : \overline{\mathbb{D}} \to \mathbb{C}; f \in A^2(\mathbb{D}), f(0) = f'(0) - 1 = 0, |f''(z)| \le \frac{1}{2}, \text{ for all } z \in \overline{\mathbb{D}}\right\},$$

and

$$\overline{S}_1 = \left\{f : \overline{\mathbb{D}} \to \mathbb{C}; f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \text{ analytic in } \mathbb{D}, \text{ satisfying } \sum_{k=2}^{\infty} k|a_k| \le 1\right\}.$$

It is evident that $\overline{S}_M \subset S_M$, $\overline{S}_3 \subset S_3$, $\overline{S}_4 \subset S_4$, and $\overline{S}_1 \subset S_1$, which shows that $f \in \overline{S}_M$ implies that f is univalent in $\{z \in \mathbb{C}; |z| < \frac{1}{M}\}$, $f \in \overline{S}_3$ implies that f is starlike (and univalent) in $\mathbb{D}$, $f \in \overline{S}_4$ implies that f is convex (and univalent) in $\mathbb{D}$, and $f \in \overline{S}_1$ implies that f is starlike (and univalent) in $\mathbb{D}$.

Also, a key tool in our proofs will be the following.

Theorem 3.4.3. (see, e.g., Stancu [361], p. 258, Exercise 4.20) *If $f : E \to \mathbb{C}$, $E \subset \mathbb{C}$, is of C^m-class on the compact convex set E, then for all distinct points $z_0, \dots, z_m \in E$, there exists a point $\xi \in \overline{\mathrm{conv}\{z_0, \dots, z_m\}}$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ such that*

$$[z_0, \dots, z_m; f] = \lambda \frac{f^{(m)}(\xi)}{m!},$$

where $[z_0, \dots, z_m; f]$ denotes the divided difference, and it is defined as in the real case.

Theorem 3.4.4. *Let $f \in \overline{S}_M$, $M > 1$. If $n \in \mathbb{N}$ satisfies*

$$\frac{n|f(1/n)|}{3^{n-1}} \le 1,$$

then $B_n(f)$ is univalent in $\{z \in \mathbb{C}; |z| < \frac{n|f(1/n)|}{M3^{n-1}}\}$.

Proof. As in the case of the real Bernstein polynomials, we can write (see, e.g., Lorentz [247], p. 12)

$$B'_n(f)(z) = \sum_{k=0}^{n-1} [k/n, (k+1)/n; f] \binom{n-1}{k} z^k (1-z)^{n-1-k}.$$

Passing to the modulus and taking into account Theorem 3.4.3 too, for all $z \in \overline{\mathbb{D}}$ we get

$$|B'_n(f)(z)| \leq M \sum_{k=0}^{n-1} \binom{n-1}{k} |z|^k (1+|z|)^{n-1-k} \leq M 2^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2^k} \leq 3^{n-1} M,$$

since it is easy to show that

$$2^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{2^k} = 2^{n-1} \left(\frac{3}{2}\right)^{n-1} = 3^{n-1}.$$

Set $P_n(f)(z) = \frac{B_n(f)(z)}{nf(1/n)}$. Since $f \in \overline{S}_M$, we have $f(0) = f'(0) - 1 = 0$, which implies $P_n(f)(0) = \frac{f(0)}{nf(1/n)} = 0$ and $P'(f)(0) = \frac{B'_n(f)(0)}{nf(1/n)} = 1$.

We get

$$|P'_n(f)(z)| \leq \frac{M3^{n-1}}{n|f(1/n)|},$$

so the hypothesis implies $\frac{M3^{n-1}}{n|f(1/n)|} > 1$, which proves that $P_n(f)$ is univalent in $\{z \in \mathbb{C}; |z| < \frac{n|f(1/n)|}{M3^{n-1}}\}$. Since $P_n(f)$ differs from $B_n(f)$ by a nonzero constant, this proves the theorem. □

Remark. The case $n = 1$ in Theorem 3.4.4 is trivial, since $f(0) = 0$ implies $B_1(f)(z) = zf(1)$.

The second result is the following.

Theorem 3.4.5. *Let $f \in \overline{S}_3$. If $n \in \mathbb{N}$ satisfies*

$$1 \leq \frac{2n|f(1/n)|}{(1-1/n)3^{n-2}},$$

then $B_n(f)$ is starlike (and univalent) in $\mathbb{D}$.

Proof. Suppose $f \in \overline{S}_3$. As in the real case, we can write (see, e.g., Lorentz [247], p. 12)

$$B''_n(f)(z) = \left(1 - \frac{1}{n}\right) \sum_{k=0}^{n-2} [k/n, (k+1)/n, (k+2)/n; f] \binom{n-2}{k} z^k (1-z)^{n-2-k}$$

for all $n \geq 2$. Applying Theorem 3.4.3 and the same reasoning as in the proof of Theorem 3.4.4, we get

$$|B''_n(f)(z)| \leq (1 - \frac{1}{n})\frac{3^{n-2}}{2},$$

which implies (by keeping the notation in the proof of Theorem 3.4.4) that

$$|P''_n(f)(z)| \leq \left(1 - \frac{1}{n}\right) \frac{3^{n-2}}{2n|f(1/n)|},$$

for all $z \in \overline{\mathbb{D}}$.

If n is as in the hypothesis, it follows that $P_n(f) \in \overline{S}_3$, which implies that $P_n(f)$ is starlike (and univalent) in $\mathbb{D}$, and since it differs from $B_n(f)$ by the nonzero constant $\frac{1}{nf(1/n)}$, we get that $B_n(f)$ is starlike (and univalent) in $\mathbb{D}$. □

An immediate consequence of Theorem 3.4.5 is the following.

Corollary 3.4.6. *For any $m \geq 2$, the Bernstein polynomials of degree 3, $B_3(f_m)(z)$, attached to $f_m \in \overline{S}_3$ and given by $f_m(z) = z + \frac{z^m}{m(m-1)}$, are starlike (and univalent) in $\mathbb{D}$.*

Proof. It is easy to see that $f_m \in \overline{S}_3$ for all $m \geq 2$. Also, for $n = 3$ in Theorem 3.4.5, we easily get

$$\frac{2n|f_m(1/n)|}{(1 - 1/n)3^{n-2}} = 1 + \frac{1}{m(m-1)3^{m-1}} > 1,$$

which proves the corollary. □

Concerning the class $\overline{S}_4$, we present the following.

Corollary 3.4.7. *Let $f \in \overline{S}_4$.*

(i) If $n \in \mathbb{N}$ satisfies

$$1 \leq \frac{2n|f(1/n)|}{(1 - 1/n)3^{n-2}},$$

then $B_n(f)$ is convex (and univalent) in $\mathbb{D}$;

(ii) For any $m \geq 2$, the Bernstein polynomials of degree 3, $B_3(g_m)(z)$, attached to $g_m \in \overline{S}_4$ and given by $g_m(z) = z + \frac{z^m}{2m(m-1)}$, are convex (and univalent) in $\mathbb{D}$.

Proof. (i) Reasoning as in the proof of Theorem 3.4.5, we get

$$|P''_n(f)(z)| \leq \left(1 - \frac{1}{n}\right) \frac{3^{n-2}}{4n|f(1/n)|}$$

for all $z \in \overline{\mathbb{D}}$.

Now, if $(1-\frac{1}{n})\frac{3^{n-2}}{4n|f(1/n)|} \le \frac{1}{2}$, then according to Theorem 3 in Obradović [291], it follows that $P_n(f)$ is convex (and univalent) in $\mathbb{D}$, which proves the convexity of $B_n(f)$ too.

(ii) It is easy to see that $g_m \in \overline{S}_4$ for all $m \ge 2$. Also, for $n = 3$ in the above point (i), we easily get

$$\frac{2n|g_m(1/n)|}{(1-1/n)3^{n-2}} = 1 + \frac{1}{2m(m-1)3^{m-1}} > 1,$$

which proves the corollary. □

The next result shows that for $n \ge 4$, Theorem 3.4.5 and Corollaries 3.4.6–3.4.7 in fact cannot hold.

Corollary 3.4.8. *If $f \in \overline{S}_3$ or $f \in \overline{S}_4$, then the inequality (appearing in Corollaries 3.4.6 and 3.4.7)*

$$1 \le \frac{2n|f(1/n)|}{(1-1/n)3^{n-2}}$$

does not hold for $n \ge 4$.

Proof. Set $g(z) = f(z) - z$. Since $f \in \overline{S}_3$ or $f \in \overline{S}_4$ implies $f(0) = f'(0) - 1 = 0$, we get $g(0) = g'(0) = 0$. Also, $g''(z) = f''(z)$ for all $|z| \le 1$.

By Theorem 3.4.3, we have

$$|g'(z)| = |g'(z) - g'(0)| \le |z| \cdot |\lambda| \cdot |g''(\xi)|$$

and

$$|g(z)| = |g(z) - g(0)| \le |z| \cdot |\mu| \cdot |g'(\eta)|,$$

where $|\lambda| \le 1$ and $|\mu| \le 1$.

Suppose first $f \in \overline{S}_3$. It follows that $|g''(z)| \le 1$, which combined with the above two relationships immediately implies $|g'(z)| \le 1$ and $|g(z)| \le |z|$, for all $|z| \le 1$. This implies $|f(z)| = |z + g(z)| \le 2|z|$ for all $|z| \le 1$ and $n|f(1/n)| \le 2$ for all $n \in \mathbb{N}$. Then the inequality in the statement becomes $(1-\frac{1}{n})3^{n-2} \le 4$, which is true for $n \le 3$ but is false for $n \ge 4$.

Similarly, if $f \in \overline{S}_4$, then by $|f''(z)| \le \frac{1}{2}$, for all $|z| \le 1$, we obtain $|g'(z)| \le \frac{1}{2}$ and $|g(z)| \le |z|/2$, for all $|z| \le 1$. This implies $|f(z)| = |z + g(z)| \le \frac{3}{2}|z|$, which in the inequality in the statement leads to $(1-\frac{1}{n})3^{n-2} \le 3$, which again it is true for $n \le 3$ but is false for $n \ge 4$ and proves the corollary. □

Remark. From Corollary 3.4.8 it is natural to ask what happens with the geometric properties of the Bernstein polynomials of degrees ≥ 4 with respect to the classes $\overline{S}_3$, $\overline{S}_4$, and $\overline{S}_1$.

Positive results will be given using suitable subclasses of $\overline{S}_3$, $\overline{S}_4$, and $\overline{S}_1$.

Theorem 3.4.9. *Let $n \geq 2$ and suppose that $f \in A^n(\mathbb{D})$.*

(i) If $f(0) = f'(0)-1 = 0$, $\|f^{(k)}\| \leq \frac{1}{e}$, for all $k = 2, \ldots, n$ and $n|f(1/n)| \geq 1$, then $B_n(f)$ is starlike (univalent) in $\mathbb{D}$. Here recall that $\|\cdot\|$ denotes the uniform norm on $C(\overline{\mathbb{D}})$.

(ii) If $f(0) = f'(0) - 1 = 0$, $\|f^{(k)}\| \leq \frac{1}{2e}$, for all $k = 2, \ldots, n$ and $n|f(1/n)| \geq 1$, then $B_n(f)$ is convex (univalent) in $\mathbb{D}$.

Proof. We will use the following well known relationship (see, e.g., Lorentz [247], p. 13):

$$B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \Delta_{1/n}^k f(0) z^k = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{n^k} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] z^k.$$

Set $P_n(f)(z) = \frac{B_n(f)(z)}{nf(1/n)}$.

(i) It is obvious that an f satisfying the conditions in the statement belongs to $\overline{S}_3$.

Differentiating $B_n(f)(z)$ twice, passing to the modulus, and taking into account Theorem 3.4.3 and the hypothesis, we get

$$\begin{aligned}
|P_n''(f)(z)| &\leq \frac{1}{n|f(1/n)|} \sum_{k=2}^{n} \binom{n}{k} \frac{k(k-1)}{n^k} \|f^{(k)}\| \\
&\leq \frac{1}{e} \sum_{k=2}^{n} \binom{n}{k} \frac{k(k-1)}{n^k} = \frac{n-1}{n \cdot e} \sum_{k=2}^{n} \binom{n-2}{k-2} \frac{1}{n^{k-2}} \\
&= \frac{n-1}{n \cdot e} \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{1}{n^j} = \frac{(n-1)(1+1/n)^n}{n \cdot e (1+1/n)^2} \leq \frac{1}{e}(1+1/n)^n \leq 1,
\end{aligned}$$

for all $z \in \overline{\mathbb{D}}$.

This implies that $P_n(f) \in \overline{S}_3$ and that $B_n(f)$ is starlike (univalent) in $\mathbb{D}$.

(ii) Obviously, an f satisfying the conditions in the statement belongs to $\overline{S}_4$.

By similar reasoning to that in the above point (i), we obtain $|P_n''(f)(z)| \leq \frac{1}{2}$, for all $|z| \leq 1$, that is $P_n(f) \in \overline{S}_4$, which immediately implies that $B_n(f)$ is convex (univalent) in $\mathbb{D}$ and proves the theorem. □

Remark. Simple examples of f satisfying Theorem 3.4.9 (i) and (ii) are given by $f_{n,m}$ and $g_{n,m}$ below, defined respectively by

$$f_{n,m}(z) = z + \frac{n}{e(n+1)} \left(z - \frac{1}{n}\right) \frac{z^m}{m(m-1)\cdots(m-n+1)C_n}$$

and

$$g_{n,m}(z) = z + \frac{n}{2e(n+1)} \left(z - \frac{1}{n}\right) \frac{z^m}{m(m-1)\cdots(m-n+1)C_n},$$

for any $m > n$, where $C_n \geq n + 1 + 1/n$.

Indeed, it is easy to show that $f_{n,m}(0) = g_{n,m}(0) = f'_{n,m}(0) - 1 = g'_{n,m}(0) - 1 = 0$, $nf_{n,m}(1/n) = ng_{n,m}(1/n) = 1$. Then for all $k = 2, \ldots, n$, we have

$$f^{(k)}_{n,m}(z) = \frac{n}{e(n+1)}\left[\left(z - \frac{1}{n}\right)\frac{z^m}{m(m-1)\cdots(m-n+1)C_n}\right]^{(k)}$$

$$= \frac{n}{e(n+1)}\left[\left(z - \frac{1}{n}\right)\frac{z^{m-k}m(m-1)\cdots(m-k+1)}{m(m-1)\cdots(m-n+1)C_n} + k\frac{z^{m-k+1}m(m-1)\cdots(m-k+2)}{m(m-1)\cdots(m-n+1)C_n}\right],$$

which immediately implies

$$|f^{(k)}_{n,m}(z)| \leq \frac{1}{e}\left[\frac{1}{C_n} + \frac{n}{C_n}\right] \leq \frac{1}{e}$$

for all $|z| \leq 1$.

Similarly, we get $|g^{(k)}_{n,m}(z)| \leq \frac{1}{2e}$, for all $k = 2, \ldots, n$ and $|z| \leq 1$.

The class of functions for which the attached Bernstein polynomials of degree $n \geq 3$ are starlike in $\mathbb{D}$ can be enlarged, as can be seen by the following result.

Theorem 3.4.10. *Let $n \geq 2$. If $f \in A^n(\mathbb{D})$ satisfies $f(0) = f'(0) - 1 = 0$, $\|f^{(k)}\| \leq \frac{n-1}{n(e-1)}$ for all $k = 2, \ldots, n$ and $n|f(1/n)| \geq 1$, then $B_n(f)$ is starlike (univalent) in $\mathbb{D}$.*

Proof. Integrating from 0 to x with respect to t the identity

$$\sum_{j=0}^{n-2}\binom{n-2}{j}t^j = (1+t)^{n-2},$$

it follows that

$$x\sum_{j=0}^{n-2}\binom{n-2}{j}x^j\frac{1}{j+1} = \frac{(1+x)^{n-1}-1}{n-1},$$

and taking $x = \frac{1}{n}$, we get

$$\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{1}{n^j}\cdot\frac{1}{j+1} = \frac{n}{n-1}\left[\left(1+\frac{1}{n}\right)^{n-1} - 1\right]$$

$$= \frac{n}{n-1}\left[\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n - 1\right] \leq \frac{n(e-1)}{n-1}.$$

Writing $B_n(f)(z) = \sum_{k=0}^{n} a_k z^k$ with $a_k = \binom{n}{k} \Delta_{1/n}^k f(0)$ and reasoning as in the proof of Theorem 3.4.9 (i), we have

$$\sum_{k=2}^{n} k|a_k| = \sum_{k=2}^{n} k \binom{n}{k} |\Delta_{1/n}^k f(0)|$$
$$= \sum_{k=2}^{n} k \binom{n}{k} \frac{k!}{n^k} |[0, 1/n, \ldots, k/n; f]| \leq \sum_{k=2}^{n} k \binom{n}{k} \frac{\|f^{(k)}\|}{n^k}$$
$$\leq \frac{n-1}{n(e-1)} \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{1}{n^j} \cdot \frac{1}{j+1} \leq 1$$

for all $|z| \leq 1$. This implies that $P_n(f)(z) = \frac{B_n(f)(z)}{n|f(1/n)|} \in \overline{S}_1$, that is, $B_n(f)$ is starlike in $\mathbb{D}$. □

Remarks. (1) Since for any $n \geq 3$ we obviously have $\frac{1}{e} < \frac{n-1}{n(e-1)}$, it follows that Theorem 3.4.10 is more general than Theorem 3.4.9 (i).

(2) The influence of critical points on the univalence of polynomials in the unit disk (see, e.g., Robertson [323]), could be applied to Bernstein polynomials too, at least for the simplest cases, e.g., $n = 2, 3$.

(3) If we consider $f(z) = \sum_{k=1}^{\infty} c_k z^k$, with $c_1 = 1$, $z \in \overline{\mathbb{D}}$, then by the proof of Theorem 3.4.1 (iii), we have $\|\pi_{n,k}\| \leq 1$ and by the Bernstein's inequality we easily get for all $|z| \leq 1$,

$$|B_n'(f)(z)| \leq \sum_{k=1}^{\infty} |c_k| \cdot |\pi_{k,n}'(z)| \leq \sum_{k=1}^{\infty} |c_k| k, \; |B_n''(f)(z)| \leq \sum_{k=1}^{\infty} |c_k|(k-1)k.$$

Then, denoting by R_M (with $M > 1$), R_3 and R_4, the sets of all functions f as above, satisfying in addition $\sum_{k=1}^{\infty} |c_k| k < M$, $\sum_{k=2}^{\infty} |c_k|(k-1)k \leq 1$ and $\sum_{k=2}^{\infty} |c_k|(k-1)k \leq 1/2$, respectively, we easily get $R_M \subset \overline{S}_M$, $R_3 \subset \overline{S}_3$, $R_4 \subset \overline{S}_4$ and $B_n(U) \subset U$, for any $U = R_M$, $U = R_3$, $U = R_4$ and all $n \in \mathbb{N}$.

(4) Another type of property preserved by the Bernstein polynomials (related somehow to the variation-diminishng property) was pointed out in Schmeisser [341] and can be briefly stated as follows: the sets $(-\infty, 0)$, $(-\infty, 0]$, $(1, +\infty)$, $[1, +\infty)$, $\mathbb{R} \setminus (0, 1)$, $\mathbb{R} \setminus (0, 1]$, $\mathbb{R} \setminus [0, 1]$ are Bernstein invariant where by definition, a set $U \subset \mathbb{C}$ is called Bernstein invariant if for any polynomial P with complex coefficients having all its zeros in U, all the Bernstein polynomials $B_n(P)$ of positive degree have their zeros in U.

At the end of this section, it is of interest to point out (without proof) the following approximation properties of the complex Bernstein polynomials.

In this sense, for $R \geq 1$, let us define by $\mathbb{A}_R$ the space of all functions defined and analytic in the open disk of center 0 and radius R denoted by $\mathbb{D}_R$ (obviously $\mathbb{D}_1 = \mathbb{D}$). Setting $r_j = R - \frac{R-1}{j}$, $j \in \mathbb{N}$, and for $f \in A_R$, $\|f\|_j = \max\{|f(z)|; |z| \leq r_j\}$, since $r_1 = 1$ and $r_j \nearrow R$, it is well known that $\{\| \cdot \|_j, j \in \mathbb{N}\}$ is a countable family of increasing seminorms on $\mathbb{A}_R$ and that

$\mathbb{A}_R$ becomes a metrizable complete locally convex space (Fréchet space) with respect to the metric

$$d(f,g)=\sum_{j=1}^{\infty}\frac{1}{2^j}\cdot\frac{\|f-g\|_j}{1+\|f-g\|_j},\quad f,g\in\mathbb{A}_R.$$

It is well known that $\lim_{n\to\infty}d(f_n,f)=0$ is equivalent to the fact that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on compacts in $\mathbb{D}_R$.

Theorem 3.4.11. (Gal [135]) *Consider $f\in\mathbb{A}_R$, $R>1$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$.*

Let the complex Bernstein polynomials be given by

$$B_n(f)(z)=\sum_{k=0}^{n}\binom{n}{k}z^k(1-z)^{n-k}f(k/n),$$

and let us define their iterates by $B_n^{(1)}(f)(z)=B_n(f)(z)$ and $B_n^{(m)}(f)(z)=B_n[B_n^{(m-1)}(f)](z)$, for any $m\in\mathbb{N}$, $m\geq 2$.

(i) The following Voronovskaya-type result in the compact unit disk holds:

$$\left|B_n(f)(z)-f(z)-\frac{z(1-z)}{2n}f''(z)\right|\leq\frac{|z(1-z)|}{2n}\cdot\frac{10M(f)}{n}$$

for all $n\in\mathbb{N},|z|\leq 1$, where $0<M(f)=\sum_{k=3}^{\infty}k(k-1)(k-2)^2|c_k|<\infty$.

(ii) For any $r\in[1,R)$, the following Voronovskaya-type result for compact disks holds

$$\left|B_n(f)(z)-f(z)-\frac{z(1-z)}{2n}f''(z)\right|\leq\frac{5(1+r)^2}{2n}\cdot\frac{M_r(f)}{n}$$

for all $n\in\mathbb{N},|z|\leq r$, where $M_r(f)=\sum_{k=3}^{\infty}|c_k|k(k-1)(k-2)^2r^{k-2}<\infty$.

(iii) For any fixed $n\in\mathbb{N}$, we have

$$\lim_{m\to\infty}d[B_n^{(m)}(f),B_1(f)]=0;$$

(iv) If $lim_{n\to\infty}\frac{m_n}{n}=0$, then

$$\lim_{n\to\infty}d[B_n^{(m_n)}(f),f]=0.$$

Moreover, for any fixed $q\in\mathbb{N}$, the following estimates hold:

$$\|B_n^{(m)}(f)-f\|_q\leq\frac{m}{n}\sum_{k=2}^{\infty}|c_k|k(k-1)r_q^k$$

and

$$d[B_n^{(m)}(f), f] \leq \frac{m}{n} \sum_{k=2}^{\infty} |c_k| k(k-1) r_q^k + \frac{1}{2^q},$$

where $\sum_{k=2}^{\infty} |c_k| k(k-1) r_q^k < \infty$.

(v) If $\lim_{n\to\infty} \frac{m_n}{n} = \infty$, *then*

$$\lim_{n\to\infty} d[B_n^{(m_n)}(f), B_1(f)] = 0.$$

3.5 Bibliographical Notes and Open Problems

Theorem 3.2.6, Corollary 3.2.7, Theorem 3.2.8, Corollaries 3.2.9–3.2.12, Theorem 3.3.14, Corollary 3.3.15, Theorems 3.4.1 (iv), (v), 3.4.2, 3.4.4, 3.4.5, Corollaries 3.4.6–3.4.8, and Theorems 3.4.9, 3.4.10 appear for the first time here.

Open Problem 3.5.1. Suggested by the real case (see Chapter 1), we can formally define the concepts of *costarlike* and *coconvex* approximation as follows. Given $f \in A^*(\mathbb{D})$, find polynomials P_n and Q_n of degree $\leq n$ ($n \in \mathbb{N}$) of good approximation for f such that

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] \text{Re}\left[\frac{zP_n'(z)}{P_n(z)}\right] > 0, \ \forall z \in \mathbb{D},$$

and

$$\text{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] \text{Re}\left[\frac{zP_n''(z)}{P_n'(z)} + 1\right] > 0, \ \forall z \in \mathbb{D},$$

respectively.

Similarly, we can define the concept of cobound turn approximation as the problem of finding polynomials R_n of degree $\leq n$ ($n \in \mathbb{N}$) of good approximation to f such that $\text{Re}[f'(z)]\text{Re}[R_n'(z)] > 0$ for all $z \in \mathbb{D}$.

Open Problem 3.5.2. An open question is whether the inclusions in Theorem 3.3.13 remain true by considering the open unit disk $\mathbb{D}$ instead of $\overline{\mathbb{D}}$.

Also, since a shortcoming of Theorem 3.3.13 is that the preservations of the classes hold beginning with an index $n_0 = n_0(f)$ (depending on f), it would be interesting to find subclasses of functions f, such that to get preservation results for all $n \geq n_0$, with a specified n_0 independent of f.

Open Problem 3.5.3. It is an open question whether the approximation polynomials considered by the theorems in Sections 3.2 and 3.3 preserve the subordination and also, the distorsions of $f(z)$.

Open Problem 3.5.4. It is also natural to consider the following problem concerning improvements in starlike and convex approximation:

For $f \in A(\mathbb{D})$ (or $f \in A^*(\mathbb{D})$), construct a sequence of complex polynomials $P_n(f)(z)$, $n = 1, 2, \ldots$, with the degree of $P_n(f)(z) \leq n$, of the convolution form

$$P_n(f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})Q_n(x-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{iu})Q_n(u)du,$$

$z = re^{ix} \in \overline{\mathbb{D}}$ (or of nonconvolution form) such that for some $p \geq 1$,

$$|f(z) - P_n(f)(z)| \leq C\omega_p\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}},$$

and moreover, if f is starlike (convex) on $\mathbb{D}$, then all $P_n(f)(z)$ are starlike (convex, respectively) on $\mathbb{D}$.

Open Problem 3.5.5. For functions $f : \overline{\mathbb{D}} \to \mathbb{C}$, analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, it is natural to attach other Bernstein-type polynomials too, as follows:

$$Q_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k(1-z)^{n-k} f(e^{\frac{2k\pi i}{n}}), \quad z \in \overline{\mathbb{D}},$$

or of the form

$$P_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k(1-z)^{n-k} f(e^{\frac{k\pi i}{n}}), \quad z \in \overline{\mathbb{D}}.$$

It would be interesting to study the approximation and geometric properties of these complex polynomials too.

Open Problem 3.5.6. In the hypothesis of Theorem 3.4.1 (iii), it is natural to ask the open question whether for $f \notin \mathcal{P}_1$, actually we have $\|B_n(f) - f\| \sim \frac{1}{n}$ in the closed unit disk, with constants depending on f but independent of n. This is suggested by the phenomenon in the real case, when it is known that (see Knoop–Zhou [194] and Totik [387]) $\|B_n(f)-f\|_{\infty,C[0,1]} \sim \omega_2^{\varphi}(f;1/\sqrt{n})_{\infty}$, with absolute constants.

Open Problem 3.5.7. Find geometric (and approximation) properties for the complex versions (that is, simply replacing the real variable x by the complex variable z) of the Bernstein-type polynomials introduced just before Theorem 1.3.3, that is, for Stancu, Soardi, q-Bernstein, Bernstein–Chlodowsky–Stancu, Lupas^2, Durrmeyer, Lupas–Durrmeyer, Lazarevic–Lupas, Mache, and Paltanea–Berens–Xu polynomials.

The methods in the proofs of Theorem 3.4.2 and Theorems 3.4.4–3.4.10 (in the case of shape preservation) and the considerations in Theorem 3.4.1 (in the case of approximation properties) could be useful for most of them.

4

Shape-Preserving Approximation by Complex Multivariate Polynomials

In this shorter chapter we extend a few results in Chapter 3 to the case of functions of several complex variables.

For simplicity and without loss of generality, sometimes we consider the case of two complex variables.

4.1 Introduction

In this section we recall some well-known concepts and results in geometric function theory in several complex variables.

Definition 4.1.1. (i) (see, e.g., Graham-Kohr [162], Chapter 6) Let $\mathbb{C}^n$ denote the space of n-complex variables $z = (z_1, \ldots, z_n),\ z_j \in \mathbb{C},\ j = 1, \ldots, n.$

The open unit polydisk (of center 0 and radius 1) is defined by

$$P(0;1) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}; |z_j| < 1, \forall j = 1, \ldots, n\} = \{z \in \mathbb{C}; \|z\|_\infty < 1\},$$

where $\|z\|_\infty$ is the norm on $\mathbb{C}^n$ given by $\|z\|_\infty = \max\{|z_j|; j = 1, \ldots, n\}$.

The open unit Euclidean ball is defined by $B(0;1) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; \|z\|_E < 1\}$, with $\|z\|_E = \langle z, z\rangle^{1/2} = \sqrt{\sum_{k=1}^n |z_k|^2}$, where $\|\cdot\|_E$ is the Euclidean norm generated by the scalar product on $\mathbb{C}^n$, $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j},\ \forall z = (z_1, \ldots, z_n),\ w = (w_1, \ldots, w_n)$. Here $\overline{w_j}$ denotes the conjugate of w_j in $\mathbb{C}$.

For $1 \le p \le +\infty$, the unit ball $B_p(0;1)$ is defined with respect to the special norm $\|z\|_p = \left[\sum_{j=1}^n |z_j|^p\right]^{1/p}, |z = (z_1, \ldots, z_n)$, i.e., $B_p(0;1) = \{z = (z_1, ..., z_n) \in \mathbb{C}^n; \|z\|_p < 1\}$, where for $p = +\infty$ we recapture the above norm $\|\cdot\|_\infty$ that defines the polydisk $P(0;1) = B_\infty(0;1)$. Also, obviously we have $\|\cdot\|_E = \|\cdot\|_2$.

(ii) (see, e.g., Andreian Cazacu [18]) Let Ω be a domain in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$. We say that f is holomorphic on Ω if f is continuous on Ω and holomorphic

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in each variable separately (when the others are fixed). Equivalently, f is holomorphic on Ω if for each point $a = (a_1, \ldots, a_n) \in \Omega$, there exists a neighborhood of a such that we have

$$f(z) = \sum_{j_1,\ldots,j_n=0}^{\infty} c_{j_1,\ldots,j_n}(z-a_1)^{j_1}\cdots(z-a_n)^{j_n}, \quad \forall z \in \Omega,$$

where the series converges absolutely and uniformly on each compact subset of Ω.

Using the multi-index notation $|j| = j_1 + j_2 + \cdots + j_n$, $z^j = z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$, $j! = j_1!j_2!\ldots j_n!$, for any $j = (j_1, \ldots, j_n), j_k \in \{0, 1, \ldots, \}$ it is well known that for, e.g., $a = 0$, we can write $c_{j_1,\ldots,j_n} = \frac{\partial^{|j|} f(0)}{\partial z_1^{j_1} \ldots \partial z_n^{j_n}} \frac{1}{j!} := D^j f(0)/j!$, and therefore we get the Taylor form $f(z) = \sum_{j_1,\cdots,j_n=0}^{\infty} \frac{D^j f(0)}{j!} z^j$.

(iii) (see, e.g., Graham–Kohr [162], Chapter 6) Let $f : \Omega \to \mathbb{C}^n, f = (f_1, \ldots, f_n), f_j : \Omega \to \mathbb{C}, |j = 1, \ldots, n$, where Ω is a domain in $\mathbb{C}^n$. The mapping f is called holomorphic on Ω if all its components $f_j, |j = 1, \ldots, n$ are holomorphic on Ω (according the above point (ii)). In this case, the differential $Df(z)$ at $z \in \Omega$ is a complex linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$ that can be identified with the complex matrix $Df(z) = (A_{i,j})_{i,j=1,\ldots,n}$, with $A_{i,j} = \frac{\partial f_j}{\partial z_i}(z)$ for all $z = (z_1, \ldots, z_n) \in \Omega$. In this case, the relation $f(z+h) = f(z) + Df(z) + o(\|h\|_E)$ holds in a sufficiently small neighborhood of the origin in $\mathbb{C}^n$.

Denote by $L(\mathbb{C}^n, \mathbb{C}^n)$ the class of all bounded complex-linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ and the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$ by I. Also, we denote by $\mathrm{Hol}(\Omega)$ the class of all holomorphic mappings from Ω to $\mathbb{C}^n$. If $0 \in \Omega$ then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I denotes the unitary $n \times n$ matrix. If $f \in \mathrm{Hol}(\Omega)$ and $z \in \Omega$, then we say that f is nonsingular at z if $Df(z)$ is invertible. The mapping f is called nonsingular on Ω if $Df(z)$ is invertible at any $z \in \Omega$.

(iv) (see, e.g., Graham–Kohr [162], Chapter 6) A function $f \in \mathrm{Hol}(\Omega)$ is called locally biholomorphic on Ω if $J_f(z) := \det[Df(z)] \neq 0$ for all $z \in \Omega$. We say that $f \in \mathrm{Hol}(\Omega)$ is biholomorphic on Ω if is a holomorphic mapping from the domain Ω onto a domain Ω' and has a holomorphic inverse defined on Ω'. (In this case, Ω and Ω' are called biholomorphically equivalent.)

Note that in fact, the concept of biholomorphic mapping on $\Omega \subset \mathbb{C}^n$ is equivalent to the concept of injective (univalent) holomorphic mapping on $\Omega \subset \mathbb{C}^n$.

(v) (see, e.g., Graham–Kohr [162], Chapter 6) If $f \in \mathrm{Hol}(\Omega)$, then for each $k = 1, 2, \ldots$, there exists a bounded symmetric k-linear mapping $D^k f(z_0) : \Pi_{i=1}^k \mathbb{C}^n \to \mathbb{C}$ (i.e., of k variables, linear with respect to each one) called the kth order Fréchet derivative of f at z_0, such that we can write

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(z_0)[(z-z_0)^k]$$

for all z in a neighborhood of z_0. Here we have the notation $D^0 f(z_0)(h^0) = f(z_0)$, and for $k \geq 1$, $D^k f(z_0)(h^k) = D^k f(z_0)(h, h, ..., h)$, where $h \in \mathbb{C}^n$ is taken $k$times. Also, D^1 is denoted by D.

(vi) (see, e.g., Graham–Kohr [162], Chapter 6) We say that $f \in \mathrm{Hol}(\Omega)$, $\Omega \subset \mathbb{C}^n$, is starlike with respect to $z_0 \in \Omega$ if f is biholomorphic on Ω and $f(\Omega)$ is a starlike domain with respect to $f(z_0)$, i.e., for all $t \in [0, 1], z \in \Omega$, we have $(1 - t)f(z_0) + tf(z) \in f(\Omega)$. If $0 \in \Omega$ and $f(0) = 0$, then we simply say that f is a starlike function on Ω.

(vii) (see, e.g., Graham–Kohr [162], Chapter 6) We say that $f \in \mathrm{Hol}(\Omega)$, $\Omega \subset \mathbb{C}^n$, is convex on Ω, if f is biholomorphic on Ω and $f(\Omega)$ is a convex domain, i.e., for all $t \in [0, 1], z, u \in \Omega$, we have $(1 - t)f(z) + tf(u) \in f(\Omega)$.

It is a well-known fact that there are results in the theory of functions of one complex variable that can be extended to several complex variables, while others cannot be so extended.

Thus, recall that in the theory of several complex variables, for example the Cauchy formula, the maximum modulus theorem, and Schwarz's lemma hold, while the Riemann mapping theorem fails, due to Poincaré's result [307], which states that for $n \geq 2$, the unit ball $B(0; 1)$ and the unit polydisk $P(0; 1)$ are not biholomorphic equivalent. Also, Bloch's theorem for normalized holomorphic mappings fails to exist in $\mathbb{C}^n$, for $n \geq 2$; see, e.g., Harris [166].

Concerning geometric function theory in several complex variables, there are results in univalent function theory in one complex variable that cannot be extended to higher dimensions (e.g., the univalent Bloch theorem, see, e.g., Harris [166]). However, as was conjectured by Cartan [62], many properties of starlike and convex functions in one complex variable can be extended to several complex variables. For example, Matsuno [268] proved that if $f : B(0; 1) \to \mathbb{C}^n$ is a locally biholomorphic mapping such that $f(0) = 0$, then f is starlike on $B(0; 1)$ if and only if

$$Re[\langle [Df(z)]^{-1} f(z), z \rangle] > 0, \forall z \in B(0; 1), z \neq 0.$$

However, the theory presents some special characteristics in a sense that depends much on the norm considered on $\mathbb{C}^n$. As for examples, Alexander's result that connects the starlike functions with the convex functions is not true in several complex variables if in $\mathbb{C}^n$ we consider the unit ball with respect to the Euclidean norm $\| \cdot \|_E$, but it is true with respect to the norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$.

Especially the theory of convex mappings is very dependent on the norm considered in $\mathbb{C}^n$. For example, Suffridge [377], [378], showed that a convex function on the unit ball $B_1(0; 1)$ necessarily is linear.

A very detailed description of geometric function theory in several complex variables can be found, for example, in Graham–Kohr [162].

In what follows, we present only a few facts that are used to extend some results in Chapter 3 to higher dimensions.

Theorem 4.1.2. *(i) (see, e.g., Graham–Kohr [162], Chapter 6, Problem 6.2.5) If* $f_1, \ldots, f_n : \mathbb{D} \to \mathbb{C}$ *are normalized starlike functions on the unit* *disk* $\mathbb{D}$*, then for any* $1 \le p \le +\infty$*, the function* $f : B_p(0;1) \to \mathbb{C}^n$ *defined by* $f(z) = (f_1(z_1), \ldots, f_n(z_n))$ *for all* $z = (z_1, \ldots, z_n) \in B_p(0;1)$ *is starlike on* $B_p(0;1)$*.*

But if the above functions $f_1, \ldots, f_n$ *are convex on* $\mathbb{D}$*, then the function* f *is not necessarily convex on* $B(0;1)$*.*

(ii) (Suffridge [376]) Let $f : P(0;1) \to \mathbb{C}^n$ *be a locally biholomorphic mapping such that* $f(0) = 0$*. Then* f *is convex on* $P(0;1)$ *if and only if there exist convex functions* $g_1, \ldots, g_n$ *on the unit disk* $\mathbb{D}$ *such that* f *is of the form*

$$f(z) = M(g_1(z_1), \ldots, g_n(z_n)), z = (z_1, \ldots, z_n) \in P(0;1),$$

where $M \in L(\mathbb{C}^n, \mathbb{C}^n)$ *is a nonsingular transformation.*

(iii) (Roper–Suffridge [324]) Let $f : B(0;1) \to \mathbb{C}^n$ *be a normalized holomorphic mapping on* $B(0;1)$*. If* f *satisfies the condition* $\sum_{k=2}^{+\infty} k^2 \frac{\|D^k f(0)\|_E}{k!} \le 1$*, then* f *is convex.*

Set $K_1^n = \{f : \overline{B(0;1)} \to \mathbb{C}^n; f$ is holomorphic normalized on $B(0;1)$, continuous on $\overline{B(0;1)}$, and $\sum_{k=2}^{+\infty} k^2 \frac{\|D^k f(0)\|_E}{k!} \le 1\}$.

4.2 Bernstein-Type Polynomials Preserving Univalence

By analogy with the real case, we can consider two kinds of Bernstein-type polynomials of several complex variables $z_1, \ldots, z_p$, as follows.

For $R > 0$, let us consider the open polydisk $P(0;R) = \{(z_1, \ldots, z_p) \in \mathbb{C}^p; |z_k| < R, k = 1, \ldots, p\}$ and the closed polydisk $\overline{P(0;R)} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p; |z_k| \le R, k = 1, \ldots, p\}$. For simplicity, everywhere in this section we denote $P(0;R)$ by P_R and $\overline{P(0;R)}$ by $\overline{P}_R$.

If f is an analytic complex-valued function in P_1 and continuous in $\overline{P}_1$, we first consider the Bernstein polynomials attached to f by

$$\begin{aligned} &B_{n_1,\ldots,n_p}(f)(z_1, \ldots, z_p) \\ &\quad = \sum_{k_1=0}^{n_1} \cdots \sum_{k_p=0}^{n_p} \binom{n_1}{k_1} \cdots \binom{n_p}{k_p} z_1^{k_1}(1-z_1)^{n_1-k_1} \cdots z_p^{k_p}(1-z_p)^{n_p-k_p} \\ &\quad\quad \times f(k_1/n_1, \ldots, k_p/n_p) \end{aligned}$$

for all $|z_j| \le 1$, $j = 1, ..., p$.

Second, let us consider

$$B_n(f)(z_1, \ldots, z_p) = \sum_{k_i \ge 0, k_1+\cdots+k_p \le n} f(k_1/n, \ldots, k_p/n) p_{k_1,\ldots,k_p}(z_1, \ldots, z_p),$$

where

$$p_{k_1,\ldots,k_p}(z_1, \ldots, z_p) = \binom{n}{k_1, \ldots, k_p} z_1^{k_1} \cdots z_p^{k_p}(1 - z_1 - \cdots - z_p)^{n-k_1-\cdots-k_p},$$

and

$$\binom{n}{k_1, \ldots, k_p} = \frac{n!}{k_1! \cdots k_p!(n - k_1 - \cdots - k_p)!}.$$

The main result is the following.

Theorem 4.2.1. *Let us suppose that $F : P_R \to \mathbb{C}^p$, with $R > 1$, is analytic in P_R and univalent in $\overline{P}_1$, i.e., $F = (f_1, \ldots, f_p)$, with $f_k : P_R \to \mathbb{C}$ analytic in P_R for all $k = 1, \ldots, p$.*

(i) There exists an index $n_0 \in \mathbb{N}$ such that for all $n_1 > n_0, \ldots, n_p > n_0$, the Bernstein polynomials

$$\begin{aligned} &B_{n_1,\ldots,n_p}(F)(z_1, \ldots, z_p) \\ &\quad = (B_{n_1,\ldots,n_p}(f_1)(z_1, \ldots, z_p), \ldots, B_{n_1,\ldots,n_p}(f_p)(z_1, \ldots, z_p)) \end{aligned}$$

are univalent in $\overline{P}_1$.

(ii) There exists an index $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the Bernstein polynomials

$$B_n(F)(z_1, \ldots, z_p) = (B_n(f_1)(z_1, \ldots, z_p), \ldots, B_n(f_p)(z_1, \ldots, z_p))$$

are univalent in $\overline{P}_1$.

Proof. For the simplicity of notation, we consider $p = 2$, but the proof for $p > 2$ is absolutely similar. Let $F = (f, g)$, where $f, g : P_R \to \mathbb{C}$ are analytic in P_R.

(i) In this case, the Bernstein polynomials can be written by the formula $B_{m,n}(F)(z_1, z_2) = (B_{m,n}(f)(z_1, z_2), B_{m,n}(g)(z_1, z_2))$, where

$$B_{m,n}(f)(z_1, z_2) = \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{m}{k} \binom{n}{j} z_1^k (1 - z_1)^{m-k} z_2^j (1 - z_2)^{n-j} f(k/m, j/n),$$

($B_{m,n}(g)(z_1, z_2)$ can be defined similarly).

We will prove that

$$B_{m,n}(f)(z_1, z_2) \to f(z_1, z_2) \quad \text{and} \quad B_{m,n}(g)(z_1, z_2) \to g(z_1, z_2)$$

as $m, n \to \infty$, uniformly in $\overline{P}_1 = \{(z_1, z_2); |z_1| \leq 1, |z_2| \leq 1\}$. This will imply that $B_{m,n}(F) \to F$, uniformly in $\overline{P}_1$, which by the univalence of uniformly convergent sequences to a univalent function, see, e.g., Theorem 6.1.18 in Graham–Kohr [162], will imply that there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, the Bernstein polynomials $B_{m,n}(F)$ are univalent in $\overline{P}_1$.

We use the ideas of the proof in the case of the Bernstein polynomials of one complex variable in Lorentz [247], pp. 88–89. Thus, since f is analytic in P_R, we can write $f(z_1, z_2) = \sum_{k,j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1, z_2) \in P_R$, and therefore there exists $R_1 > 1$ such that $\sum_{k,j=0}^{\infty} |c_{k,j}| R_1^{k+j}$ is finite.

Since it is well known that $B_{m,n}(f)(z_1, z_2)$ converges uniformly to $f(z_1, z_2)$ in the square $S = \{(z_1, z_2); z_1, z_2 \in \mathbb{R}, 0 \leq z_1 \leq 1, 0 \leq z_2 \leq 1\} \subset \overline{P}_1$, by Vitali's convergence theorem (see, e.g., Graham–Kohr [162], Theorem 6.1.16), it will be sufficient to prove that the polynomials $B_{m,n}(f)(z_1, z_2)$ are bounded in $\overline{P}_{R_1}$.

Set $\Phi_{r,s}(z_1, z_2) = \Phi_r(z_1) \cdot \Phi_s(z_2)$, where $\Phi_r(z_1) = z_1^r$ and $\Phi_s(z_2) = z_2^s$ and

$$\pi_{r,s}(z_1, z_2) = B_{m,n}(\Phi_{r,s})(z_1, z_2).$$

It is immediate that

$$B_{m,n}(\Phi_{r,s})(z_1, z_2) = B_m(\Phi_r)(z_1) \cdot B_n(\Phi_s)(z_2),$$

where B_m and B_n are the Bernstein polynomials of one complex variable that appear in the proof of Theorem 4.1.1, page 88 in Lorentz [247]. Since we also can write

$$B_{m,n}(f)(z_1, z_2) = \sum_{k,j=0}^{\infty} c_{k,j} \pi_{k,j}(z_1, z_2),$$

and since by the relation (4) in the proof of Theorem 4.1.1, page 88 in Lorentz [247], we have $|\pi_r(z_1)| \leq R_1^r$ and $|\pi_s(z_2)| \leq R_1^s$ for all $|z_1| \leq R_1$, $|z_2| \leq R_1$, by the analyticity of f in $\overline{P}_{R_1}$, it immediately follows that there exists $M > 0$ such that $|B_{m,n}(f)(z_1, z_2)| \leq M$ for all $(z_1, z_2) \in \overline{P}_{R_1}$ and all $m, n \in \mathbb{N}$.

The reasoning for $B_{m,n}(g)(z_1, z_2)$ is similar.

(ii) In this case, the Bernstein polynomials can be written by the formula $B_n(F)(z_1, z_2) = (B_n(f)(z_1, z_2), B_n(g)(z_1, z_2))$, where

$$B_n(f)(z_1, z_2) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} z_1^k z_2^j (1 - z_1 - z_2)^{n-k-j} f(k/n, j/n)$$

($B_{m,n}(g)(z_1, z_2)$ can be defined similarly).

We will prove that $B_n(f)(z_1, z_2) \to f(z_1, z_2)$ and $B_n(g)(z_1, z_2) \to g(z_1, z_2)$ as $n \to \infty$, uniformly in $\overline{P}_1 = \{(z_1, z_2); |z_1| \leq 1, |z_2| \leq 1\}$. This will imply that $B_n(F) \to F$, uniformly in $\overline{P}_1$, which by the univalence of uniformly convergent sequences to univalent function, see, e.g., Theorem 6.1.18 in Graham–Kohr [162], will imply that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the Bernstein polynomials $B_n(F)$ are univalent in $\overline{P}_1$.

We use the ideas of proof in the case of the Bernstein polynomials of one complex variable in Lorentz [247], pp. 88–89. Thus, since f is analytic in P_R, we can write $f(z_1, z_2) = \sum_{k,j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1, z_2) \in P_R$, and therefore there exists $R_1 > 1$ such that $\sum_{k,j=0}^{\infty} |c_{k,j}| R_1^{k+j}$ is finite.

Since it is well known that $B_n(f)(z_1, z_2)$ converges uniformly to $f(z_1, z_2)$ in the triangle $T = \{(z_1, z_2); z_1, z_2 \in \mathbb{R}, 0 \leq z_1, 0 \leq z_2, z_1 + z_2 \leq 1\} \subset \overline{P}_1$ by Vitali's convergence theorem (see, e.g., Graham–Kohr [162], Theorem 6.1.16), it will be sufficient to prove that the polynomials $B_n(f)(z_1, z_2)$ are bounded in $\overline{P}_{R_1}$.

Set $\Phi_{r,s}(z_1, z_2) = z_1^r \cdot z_2^s$ and

$$\pi_{r,s}(z_1, z_2) = B_n(\Phi_{r,s})(z_1, z_2).$$

Since we can write

$$B_n(f)(z_1, z_2) = \sum_{r,s=0}^{\infty} c_{r,s}\pi_{r,s}(z_1, z_2),$$

it will be sufficient to show that

$$|\pi_{r,s}(z_1, z_2)| \le R_1^{r+s} \text{ for all } (z_1, z_2) \in \overline{P}_{R_1}.$$

Writing $p_{n,k,j}(z_1, z_2) = \binom{n}{k}\binom{n-k}{j} z_1^k z_2^j (1 - z_1 - z_2)^{n-k-j}$, let us consider the generating function

$$\begin{aligned}
&\Phi(u, v, z_1, z_2)\\
&\quad= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \pi_{r,s}(z_1, z_2)\frac{u^r}{r!}\frac{v^s}{s!}\\
&\quad= \sum_{k=0}^{n}\sum_{j=0}^{n-k} p_{n,k,j}(z_1, z_2)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{k}{n}\right)^r\left(\frac{j}{n}\right)^s\frac{u^r}{r!}\frac{v^s}{s!}\\
&\quad= \sum_{k=0}^{n}\sum_{j=0}^{n-k} p_{n,k,j}(z_1, z_2)e^{ku/n}e^{jv/n}\\
&\quad= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}\left(z_1e^{u/n}\right)^k\left(z_2e^{v/n}\right)^j(1 - z_1 - z_2)^{n-k-j} := S.
\end{aligned}$$

Writing $z_1e^{u/n} = A$, $z_2e^{v/n} = B$ and $1 - z_1 - z_2 = C$, we get

$$\begin{aligned}
S &= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}A^kB^jC^{n-k-j}\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}A^kC^{n-k}\binom{n-k}{j}\left(\frac{B}{C}\right)^j\\
&= \sum_{k=0}^{n}\binom{n}{k}A^kC^{n-k}\sum_{j=0}^{n-k}\binom{n-k}{j}\left(\frac{B}{C}\right)^j = \sum_{k=0}^{n}\binom{n}{k}A^kC^{n-k}\left(1 + \frac{B}{C}\right)^{n-k}\\
&= (A + B + C)^n = [1 - z_1 - z_2 + z_1e^{u/n} + z_2e^{v/n}]^n,
\end{aligned}$$

that is,

$$\begin{aligned}
\Phi(u, v, z_1, z_2) &= [1 - z_1 - z_2 + z_1e^{u/n} + z_2e^{v/n}]^n\\
&= \left(1 + \sum_{r=1}^{\infty} z_1\frac{u^r}{n^r r!} + \sum_{s=1}^{\infty} z_2\frac{v^s}{n^s s!}\right)^n.
\end{aligned}$$

Since for $|z_1| \geq 1$, $|z_2| \geq 1$ we have $|z_1| \leq |z_1|^k$ and $|z_2| \leq |z_2|^k$ for all $k = 1, 2, \ldots$, it follows that the coefficients of u^r and v^s in the power series $1 + \sum_{r=1}^{\infty} z_1 \frac{u^r}{n^r r!} + \sum_{s=1}^{\infty} z_2 \frac{v^s}{n^s s!}$ are majorized by those of

$$1 + \sum_{r=1}^{\infty} \frac{(R_1 u)^r}{n^r r!} + \sum_{s=1}^{\infty} \frac{(R_1 v)^s}{n^s s!}$$

if $|z_1| = |z_2| = R_1$, and therefore if $|z_1| \leq R_1$, $|z_2| \leq R_1$.

But we observe that the series

$$1 + \sum_{r=1}^{\infty} \frac{(R_1 u)^r}{n^r r!} + \sum_{s=1}^{\infty} \frac{(R_1 v)^s}{n^s s!},$$

is a subpart of the series

$$\left(1 + \sum_{r=1}^{\infty} \left(\frac{R_1 u}{n}\right)^r \frac{1}{r!}\right) \left(1 + \sum_{s=1}^{\infty} \left(\frac{R_1 v}{n}\right)^s \frac{1}{s!}\right) = e^{R_1 u/n} \cdot e^{R_1 v/n}.$$

This implies that the moduli of the coefficients of $u^r v^s$ in $\Phi(u, v, z_1, z_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \pi_{r,s}(z_1, z_2) \frac{u^r}{r!} \frac{v^s}{s!}$ do not exceed those in

$$[e^{R_1 u/n} \cdot e^{R_1 v/n}]^n = e^{R_1 u} e^{R_1 v},$$

which proves the theorem. □

Remarks. By the proofs of Theorem 4.2.1 and Theorem 3.4.1 (iii) for an analytic function $f : P_R \to \mathbb{C}$ with $R > 1$, we easily can deduce the estimate

$$\|B_{m,n}(f) - f\| \leq M_2(f) \left[\frac{1}{m} + \frac{1}{n}\right] \text{ for all } m, n \in \mathbb{N},$$

where the constant $M_2(f) > 0$ is independent of m, n and $\|\cdot\|$ denotes the uniform norm in $C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$.

4.3 Shape-Preserving Approximation by Other Types of Polynomials

First we introduce the moduli of smoothness we need in the approximation process.

For simplicity and without loss of generality, we consider the case of two complex variables.

Definition 4.3.1. *Let $B_p(0; 1)$, $P(0; 1) \subset \mathbb{C}^2$, $1 \leq p \leq \infty$, where $P(0; 1) = B_\infty(0; 1)$.*

(i) If $f : \overline{P(0;1)} \to \mathbb{C}$, then the second-order partial moduli of smoothness of f on the distinguished boundary $\partial_0 P(0;1) = \{(z_1, z_2); |z_1| = |z_2| = 1\}$ (which is obviously different from the whole boundary $\partial P(0;1)$) can be defined by

$$\omega_2^{(z_1)}(f;\delta)_{\partial_0 P(0;1)} = \sup\{|f(z_1 e^{iu}, z_2) + f(z_1 e^{-iu}, z_2) - 2f(z_1, z_2)|; \\ |z_1| = |z_2| = 1, |u| \le \delta\},$$
$$\omega_2^{(z_2)}(f;\delta)_{\partial_0 P(0;1)} = \sup\{|f(z_1, z_2 e^{iv}) + f(z_1, z_2 e^{-iv}) - 2f(z_1, z_2)|; \\ |z_1| = |z_2| = 1, |v| \le \delta\},$$

for any $\delta \ge 0$.

(ii) If $f : \overline{P(0;1)} \to \mathbb{C}$, then a modulus of continuity of f on $\overline{P(0;1)}$ can be defined by

$$\omega_1(f;\delta,\eta)_{\overline{P(0;1)}} = \sup\{|f(z,w) - f(u,v)|; (z,w),(u,v) \in \overline{P(0;1)}, \\ |z-u| \le \delta, |w-v| \le \eta\},$$

for any $\delta, \eta \ge 0$.

(iii) If $f : \overline{B_p(0;1)} \to \mathbb{C}$, then we can define another modulus of continuity by

$$\omega_1(f;\delta)_{\overline{B_p(0;1)}} = \sup\{|f(A) - f(B)|; A = (z,w), B = (u,v) \in \overline{B_p(0;1)}, \\ \|A - B\|_p \le \delta\}.$$

If $p = +\infty$, since $B_\infty(0;1) = P(0;1)$, then it is easy to see that $\omega_1(f;\delta)_{\overline{B_\infty(0;1)}} = \omega_1(f;\delta,\delta)_{\overline{P(0;1)}}$.

(iv) If $f : \overline{B_p(0;1)} \to \mathbb{C}$, then

$$E_{n,m}(f) = \inf\{\|f - P\|_{\overline{B_p(0;1)}}; P \text{ is polynomial of degree } \le n \text{ in } z_1 \\ \text{and of degree} \le m \text{ in } z_2\}$$

is called the best approximation of f by complex polynomials $P(z_1, z_2)$ of degree $\le n$ with respect to z_1 and of degree $\le m$ with respect to z_2. Here $\|f - P\|_{\overline{B_p(0;1)}} = \sup\{|f(A) - P(A)|; A \in \overline{B_p(0;1)}\}$.

A polynomial $P^*_{n,m}$ satisfying $E_{n,m}(f) = \|f - P^*_{n,m}\|_{\overline{B_p(0;1)}}$ will be called a polynomial of best approximation.

Note that since the class of complex polynomials of degree $\le n$ in z_1 and of degree $\le m$ in z_2 is finite-dimensional, by, e.g., Singer [357], p. 91, Corollary 2.2, if $f : \overline{B_p(0;1)} \to \mathbb{C}$ is continuous on $\overline{B_p(0;1)}$, then for any $n, m \in \{0, 1, \ldots,\}$ there exists a polynomial of best approximation $P^*_{n,m}$.

A mapping $f : \Omega \to \mathbb{C}^n$, where we suppose, that $\Omega \subset \mathbb{C}^n$ and $f(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$, will be called a generalized polynomial of degree d_k in the variable z_k, $k = 1, \ldots, n$, if each $f_j, j = 1, \ldots, n$ is a polynomial with respect to the variables $z_1, \ldots, z_n$. If we denote by

$m_{j,k}$ the degree of the polynomial f_j with respect to the variable z_k, then $d_k = \max_{j=1,\ldots,n} m_{j,k}$, for all $k = 1, \ldots, n$.

Based on the results in Section 3.3 and on the above Theorem 4.1.2, we present the following result.

Theorem 4.3.2. *(i) Let* $f : \overline{B_p(0;1)} \to \mathbb{C}^n$ *with* $1 \le p \le +\infty$ *be fixed, of the form* $f(z) = (f_1(z_1), \ldots, f_n(z_n))$, *where each* $f_k : \overline{\mathbb{D}} \to \mathbb{C}$, $k = 1, \ldots, n$, *is analytic, normalized in* $\mathbb{D}$, *and continuous in* $\overline{\mathbb{D}}$.

Set $\Omega_m(t) = \frac{(m!)^2}{2(2m)!}[2\cos(t/2)]^{2m}$ *the de la Vallée Poussin kernel and define the polynomial of degree* $\le m_k$ *in* z_k *given by* $P_{m_k}(z_k) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_k(z_k e^{it})\Omega_{m_k}(t)dt$ *for all* $k = 1, \ldots, n$.

If all f_k, $k = 1, \ldots, n$ *are starlike on* $\mathbb{D}$, *then for any* $m_1, \ldots, m_n \in \mathbb{N}$, *the mappings* $P_{m_1,\ldots,m_n}(f) : \overline{B_p(0;1)} \to \mathbb{C}^n$ *defined by* $P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n) = (P_{m_1}(z_1), \ldots, P_{m_n}(z_n))$ *are starlike (generalized polynomials) on* $B_p(0;1)$ *(of degree* $\le m_k$, *with respect to* z_k, $k = 1, \ldots, n$*), and the following estimate holds:*

$$\|f(z_1, \ldots, z_n) - P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n)\|_p \le 3\left[\sum_{j=1}^{n} \omega_1(f_j; 1/\sqrt{m_j})_{\overline{\mathbb{D}}}\right],$$

for all $z = (z_1, \ldots, z_n) \in \overline{B_p(0;1)}$.

Let us consider the subclasses of starlike functions S_1, S_2, *and* S_3 *as defined in Section* 3.3 *and suppose that* f_k, $k = 1, \ldots, n$ *belong all to some subclass* S_{j_0}, $j_0 \in \{1, 3\}$ *(not necessarily the same* j_0 *for all* k*). Then the generalized polynomials defined by* $P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n) = (P_{m_1}(z_1), \ldots, P_{m_n}(z_n))$, *where each* $P_{m_k}(z_k) = \int_0^{z_k} Q_{m_k}(t)dt$, *with* $Q_{m_k}(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_k'(z_k e^{it})O_{m_k}(t)dt$ *and* $O_{m_k}(t)$ *the normalized Jackson or a generalized Beatson kernel, are starlike on* $B_p(0;1)$. *In addition, if* f_k', $k = 1, \ldots, n$, *are all continuous on* $\overline{\mathbb{D}}$, *then the following estimate holds:*

$$\|f(z_1, \ldots, z_n) - P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n)\|_p \le C\left[\sum_{j=1}^{n} \omega_2(f_j'; 1/m_j)_{\partial\mathbb{D}}\right],$$

for all $z = (z_1, \ldots, z_n) \in \overline{B_p(0;1)}$, *where* $C > 0$ *is independent of* f *and* n.

If in the above construction we replace $O_{m_k}(t)$ *by the generalized de la Valleé Poussin kernel* $V_{m_k}(t) = 2F_{2m_k}(t) - F_{m_k}(t)$, *where* $F_{m_k}(t)$ *is the normalized Fejér kernel, and* f_k, $k = 1, \ldots, n$ *all belong to some subclass* S_{j_0}, $j_0 \in \{1, 2\}$ *(not necessarily the same* j_0 *for all* k*), then all* $P_{m_1,\ldots,m_n}(f)$ *are starlike on* $B_p(0;1)$, *and the following better estimate holds*

$$\|f(z_1, \ldots, z_n) - P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n)\|_p \le 4\left[\sum_{j=1}^{n} E_{m_k}(f_k)\right],$$

for all $z = (z_1, \ldots, z_n) \in \overline{B_p(0;1)}$*. Here* $E_{m_k}(f_k)$ *denotes the best approximation of* f_k *by polynomials of degree* $\leq m_k$*.*

(ii) Suppose $f : \overline{P(0;1)} \to \mathbb{C}^n$ *is a locally biholomorphic mapping, continuous on* $\overline{P(0;1)}$*,* $f(0) = 0$*, and* f *is convex on* $P(0;1)$*, i.e., necessarily of the form*

$$f(z_1, \ldots, z_n) = M(g_1(z_1), \ldots, g_n(z_n)), (z_1, \ldots, z_n) \in \overline{P(0;1)},$$

where $M \in L(\mathbb{C}^n, \mathbb{C}^n)$*,* $M = (a_{i,j})_{i,j=1,\ldots,n}$ *is nonsingular, and all* $g_k : \overline{\mathbb{D}}$ *are convex on* $\mathbb{D}$ *and continuous on* $\overline{\mathbb{D}}$*.*

As at the above point (i), define $P_{m_k}(z_k) = \frac{1}{\pi} \int_{-\pi}^{\pi} g_k(z_k e^{it}) \Omega_{m_k}(t) dt$ *for all* $k = 1, \ldots, n$*. Then the mappings* $P_{m_1,\ldots,m_n}(f) : \overline{P(0;1)} \to \mathbb{C}^n$ *defined by* $P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n) = M(P_{m_1}(z_1), \ldots, P_{m_n}(z_n))$ *are convex (generalized polynomials) on* $P(0;1)$ *(of degree* $\leq m_k$*, with respect to* z_k*,* $k = 1, \ldots, n$*) and the following estimate holds:*

$$\|f(z_1, \ldots, z_n) - P_{m_1,\ldots,m_n}(f)(z_1, \ldots, z_n)\|_\infty \leq 3C \left[\sum_{j=1}^{n} \omega_1(f_j; 1/\sqrt{m_j})_{\overline{\mathbb{D}}} \right]$$

for all $z = (z_1, \ldots, z_n) \in \overline{P(0;1)}$*, where* $C = \max_{i,j=1,\ldots,n} |a_{i,j}| > 0$*.*

(iii) Let f *be holomorphic normalized on* $B(0;1)$*,* $f = (G, H)$ *with* $G, H : \overline{B(0;1)} \to \mathbb{C}$*,* $f(z_1, z_2) = (G(z_1, z_2), H(z_1, z_2))$*.*

Suppose that $O_n(t)$ *is the normalized Jackson kernel or a normalized generalized Beatson kernel. Define*

$$P_{n,m}(f)(z_1, z_2) = (P_1(z_1, z_2), P_2(z_1, z_2)),$$

where

$$P_1(z_1, z_2) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} O_n(u) O_m(v) G(z_1 e^{iu}, z_2 e^{iv}) du\, dv$$

and

$$P_2(z_1, z_2) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} O_n(u) O_m(v) H(z_1 e^{iu}, z_2 e^{iv}) du\, dv.$$

Also, for the generalized de la Vallée Poussin kernel $V_n(t)$ *defined at the above point (i), let us define*

$$R_{n,m}(f)(z_1, z_2) = (R_1(G)(z_1, z_2), R_2(H)(z_1, z_2)),$$

where

$$R_1(G)(z_1, z_2) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [2F_{2n}(u) - F_n(u)][2F_{2m}(v) - F_m(v)] G(z_1 e^{iu}, z_2 e^{iv}) du\, dv,$$

and

$$R_2(H)(z_1, z_2) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[2F_{2n}(u) - F_n(u)][2F_{2m}(v) - F_m(v)]H(z_1e^{iu}, z_2e^{iv})du\,dv.$$

Then, the generalized polynomials $P_{n,m}(f)(z_1, z_2), R_{n,m}(f)(z_1, z_2) \in K_1^2$ *satisfy the estimates*

$$\begin{aligned}\|f(z_1, z_2) - P_{n,m}(f)(z_1, z_2)\|_2 &\leq C[\omega_2^{(z_1)}(G; 1/n)_{\partial_0 P(0;1)} + \omega_2^{(z_1)}(H; 1/n)_{\partial_0 P(0;1)}] \\ &\quad + C[\omega_2^{(z_2)}(G; 1/n)_{\partial_0 P(0;1)}] + \omega_2^{(z_2)}(H; 1/m)_{\partial_0 P(0;1)}]\end{aligned}$$

and

$$\|f(z_1, z_2) - R_{n,m}(f)(z_1, z_2)\|_2 \leq 10[E_{n,m}(G) + E_{n,m}(H)]$$

for all $(z_1, z_2) \in \overline{B(0;1)}$. *Here* $\partial_0 P(0;1) = \{(z_1, z_2); |z_1| = |z_2| = 1\}$ *is the distinguished boundary of* $P(0;1)$ *and* $E_{n,m}$ *is given by Definition 4.3.1 (iv).*

Proof. (i) By the inequality $\|(z_1, ..., z_n)\|_p \leq \sum_{k=1}^m |z_k|$, it is an immediate consequence of Theorem 4.1.2 (i) combined with Theorem 3.1.3 (viii) for the de la Valleé Poussin kernel, with Theorem 3.3.9 (i),(ii), (iii), for the subclasses S_1 and S_2, and with Theorem 3.3.10, (i), for the subclass S_3. Also, we note that $B_p(0;1) \subset P(0;1)$, and in fact the estimates are valid for all $z \in \overline{P(0;1)} = \overline{\mathbb{D}}^n$.

(ii) It is an immediate consequence of Theorem 4.1.2 (ii) combined with Theorem 3.1.3 (viii).

(iii) Concerning the approximation error by $P_{n,m}(f)$, we get

$$\begin{aligned}\|f(z_1, z_2) - P_{n,m}(f)(z_1, z_2)\|_2 &\leq |G(z_1, z_2) - P_{n,m}(G)(z_1, z_2)| \\ &\quad + |H(z_1, z_2) - P_{n,m}(H)(z_1, z_2)|.\end{aligned}$$

But

$$\begin{aligned}&G(z_1, z_2) - P_{n,m}(G)(z_1, z_2) \\ &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} O_n(u)O_m(v)\left[G(z_1, z_2) - G(z_1e^{iu}, z_2e^{iv})\right]du\,dv \\ &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} O_n(u)O_m(v) \\ &\quad \times [G(z_1, z_2) - G(z_1, z_2e^{iv}) + G(z_1, z_2e^{iv}) - G(z_1e^{iu}, z_2e^{iv})]du\,dv \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} O_n(u)\left[\frac{1}{\pi}\int_0^{\pi}(2G(z_1, z_2) - G(z_1, z_2e^{iv})\right. \\ &\qquad \left. - G(z_1, z_2e^{-iv}))O_m(v)dv\right]du + \frac{1}{\pi}\int_{-\pi}^{\pi} O_m(v) \\ &\quad \left[\frac{1}{\pi}\int_0^{\pi}(2G(z_1, z_2e^{iv}) - G(z_1e^{iu}, z_2e^{iv}) - G(z_1e^{-iu}, z_2e^{iv}))O_n(u)du\right]dv.\end{aligned}$$

From the maximum modulus principle (see, e.g., Andreian Cazacu [18], p. 69), the maximum is attained on the distinguished boundary of $P(0;1)$, denoted by $\partial_0 P(0;1)$. Therefore, it suffices to prove the estimates for $|z_1| = |z_2| = 1$ in order for them to be valid for all $z_1, z_2 \in \overline{P(0;1)}$, and since obviously $B(0;1) \subset P(0;1)$, the estimate will be valid for all $z_1, z_2 \in \overline{B(0;1)}$.

Therefore, from the above relationship, for $|z_1| = |z_2| = 1$ we immediately get (reasoning as in the case if one complex variable)

$$|G(z_1, z_2) - P_{n,m}(G)(z_1, z_2)| \leq C[\omega_2(G; 1/m)_{\partial_0 P(0;1)} + \omega_2(G; 1/n)_{\partial_0 P(0;1)}],$$

with $C > 0$ independent of f, n, and m.

By similar reasoning we have

$$|H(z_1, z_2) - P_{n,m}(H)(z_1, z_2)| \leq C[\omega_2(H; 1/m)_{\partial_0 P(0;1)} + \omega_2(H; 1/n)_{\partial_0 P(0;1)}].$$

Concerning the approximation error by $R_{n,m}(f)(z_1, z_2)$, first we prove that if $f = (p_{n,m}, q_{n,m})$, where $p_{n,m}(z_1, z_2), q_{n,m}(z_1, z_2)$ are polynomials of degree $\leq n$ in z_1 and of degree $\leq m$ in z_2, we have $R_{n,m}(f)(z_1, z_2) = f(z_1, z_2)$. It suffices to prove this for any f of the form $f(z_1, z_2) = (z_1^j z_2^k, z_1^s z_2^t)$, where $j, s \in \{0, 1, \ldots, n\}$ and $k, t \in \{0, 1, \ldots, m\}$.

This is immediate from the case of one complex variable (see, e.g., Gal [127], p. 425), which proves the first estimate in (iii).

Then, we have

$$\begin{aligned}[2F_{2n}(u) - F_n(u)][2F_{2m}(v) - F_m(v)] &= 4F_{2n}(u)F_{2m}(u) - 2F_{2n}(u)F_m(v) \\ &\quad - 2F_{2m}(v)F_n(u) + F_n(u)F_m(u).\end{aligned}$$

By the case of one complex variable (see Gal [127], p. 424), for all $r, s \in \mathbb{N}$ we immediately get

$$\left| \frac{1}{\pi^2} \int_{-\pi}^{\pi} F_r(u)F_s(v)[A(z_1 e^{iu}, z_2 e^{iv}) - B(z_1 e^{iu}, z_2 e^{iv})] du\, dv \right| \leq \|A - B\|_{\overline{P(0;1)}}$$

for all $z_1, z_2 \in \overline{P(0;1)}$, where $A, B : \overline{P(0;1)} \to \mathbb{C}$ and $\|A - B\|_{\overline{P(0;1)}} = \sup\{|A(z_1, z_2) - B(z_1, z_2)|; (z_1, z_2) \in \overline{P(0;1)}\}$.

This immediately implies $|R_1(A)(z_1, z_2) - R_1(B)(z_1, z_2)| \leq 9\|A - B\|_{\overline{P(0;1)}}$, i.e.,

$$\|R_1(A) - R_1(B)\|_{\overline{P(0;1)}} \leq 9\|A - B\|_{\overline{P(0;1)}}.$$

Therefore,

$$\|f(z_1, z_2) - R_{n,m}(f)(z_1, z_2)\|_2 \leq \|G - R_1(G)\|_{\overline{P(0;1)}} + \|H - R_1(H)\|_{\overline{P(0;1)}}.$$

Denote by $G^*_{n,m}$ the polynomial of best approximation of G of degree $\leq n$ in z_1 and by $H^*_{n,m}$ the polynomial of best approximation of H.

By the above conclusions, we get

$$\begin{aligned}\|G - R_1(G)\|_{\overline{P(0;1)}} &\leq \|G - G^*_{n,m}\|_{\overline{P(0;1)}} + \|R_1(G^*_{n,m}) - R_1(G)\|_{\overline{P(0;1)}} \\ &\leq E_{n,m}(G) + 9\|G - G^*_{n,m}\|_{\overline{P(0;1)}} = 10E_{n,m}(G).\end{aligned}$$

Similarly,

$$\|H - R_1(H)\|_{\overline{P(0;1)}} \leq 10E_{n,m}(H),$$

which proves the estimate in (iii). □

Remark. It is easy to see that the results in Chapter 3, Theorem 3.2.6, Corollary 3.2.7, Theorem 3.2.8, and Corollaries 3.2.9–3.2.12 remain valid without any change if we consider there that $\Omega, \Omega_j \subset \mathbb{C}^m$. For example, corresponding to Corollary 3.2.12, we get (by taking $m = 2$ for simplicity) the following.

Theorem 4.3.3. *For $(X, \|\cdot\|_X)$ a complex normed space and $S \subset X$ a linear subspace, let $L : X \to \mathcal{F}(\mathbb{D} \times \mathbb{D})$, where and $(\mathcal{F}(\mathbb{D} \times \mathbb{D}), \|\cdot\|_{\mathcal{F}})$ is a certain normed space of complex-valued functions defined on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. If L is a linear bounded operator such that there exists $e \in S$ with $\text{Re}[L(e)(z)] \geq 1$ for all $z \in \mathbb{D} \times \mathbb{D}$ and if we set $M_0(\text{Re}[L]) = \{f \in X : \text{Re}[L(f)(z)] > 0,$ for all $z \in \mathbb{D} \times \mathbb{D}\}$, then for any $f \in M_0(\text{Re}[L])$ we have*

$$E_{S,M_0(\text{Re}[L])}(f; X) \leq (1 + \|e\|_X + |||L||| \cdot \|e\|_X)E_S(f; X).$$

Remark. In order to get an application of Theorem 4.3.3, let us particularize the spaces X, S and the operator L.

For $d \in \{0, 1, \ldots\}$, let us consider $X = A^d(\mathbb{D} \times \mathbb{D})$, the space of d-continuously differentiable functions $f : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$ on the closed unit bidisk $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, endowed with the norm $\|f\|_{A^d(\mathbb{D}\times\mathbb{D})} = \max\{\|f^{(k,s)}\|; 0 \leq k, s, k+s \leq d\}$, where $f^{(k,s)}(z) = \frac{\partial^{k+s} f}{\partial z_1^k \partial z_2^s}(z), z = (z_1, z_2)$ and $\|\cdot\|$ denotes the uniform norm on $C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$.

Take $L : A^d(\mathbb{D} \times \mathbb{D}) \to C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$, defined by $L(f)(z) = \frac{\partial^{k_0+s_0} f}{\partial z_1^{k_0} \partial z_2^{s_0}}(z),\ z = (z_1, z_2)$, where k_0, s_0 are fixed with $k_0 + s_0 \leq d$. Then obviously L is linear and bounded, and $e(z) = z_1^{k_0} z_2^{s_0},\ \forall z = (z_1, z_2) \in \mathbb{D} \times \mathbb{D}$ satisfies $\text{Re}[L(e)(z)] = 1$.

Choose as S the set of all complex polynomials of degree $\leq n$ in z_1 and of degree $\leq m$ in z_2 (for arbitrary fixed $n, m \in \mathbb{N}$, which is a finite-dimensional set in $C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$).

As a consequence of Theorem 4.3.3, we get that for any $f \in A^d(\mathbb{D} \times \mathbb{D})$ satisfying $\text{Re}[\frac{\partial^{k_0+s_0} f}{\partial z_1^{k_0} \partial z_2^{s_0}}(z)] > 0$, for all $z = (z_1, z_2) \in \mathbb{D} \times \mathbb{D}$, there exists a complex polynomial $P_{n,m}$ of degree $\leq n$ in z_1 and of degree $\leq m$ in z_2 such that $\text{Re}[\frac{\partial^{k_0+s_0} P_{n,m}}{\partial z_1^{k_0} \partial z_2^{s_0}}(z)] > 0$ for all $z \in \mathbb{D} \times \mathbb{D}$, and the estimate

$$\|f - P_{n,m}\|_{A^d(\mathbb{D}\times\mathbb{D})} \leq 3E_{n,m}(f)$$

holds, where $E_{n,m}(f)$ denotes the best (unrestricted) approximation of f by complex polynomials of degree $\leq n$ in z_1 and of degree $\leq m$ in z_2 with respect to the norm $\|\cdot\|_{A^d(\mathbb{D}\times\mathbb{D})}$.

For our purpose, instead of $A^d(\mathbb{D}\times\mathbb{D})$, in what follows it will also be useful to introduce the following notations.

We let $A^{p,q}(\mathbb{D}\times\mathbb{D})$, $p, q \in \{0, 1, \ldots\}$, denote the set of all functions $f(z_1, z_2)$, $f : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$ such that all the partial derivatives exist: $f^{(k,l)}(z_1, z_2) = D^{(k,l)}(f)(z_1, z_2) = \frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l}(z_1, z_2)$, continuous on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, for all $0 \leq k \leq p$, $0 \leq l \leq q$ (it is easy to see that $A^{p,q}(\mathbb{D} \times \mathbb{D})$ coincides with $A^d(\mathbb{D} \times \mathbb{D})$, where $d = p + q$). The norm on $A^{p,q}(\mathbb{D} \times \mathbb{D})$ is given by $\|f\|_{A^{p,q}(\mathbb{D}\times\mathbb{D})} = \max\{\|D^{(k,l)}(f)\|; 0 \leq k \leq p, 0 \leq l \leq q\}$, where $\|\cdot\|$ denotes the uniform norm on $C(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$.

On the other hand, for $h \in A(\mathbb{D})$ and $r \in \mathbb{N}$, we set

$$\omega_r(h;\delta) = \sup\{|\Delta_u^r h(e^{ix})|; |x| \leq \pi, |u| \leq \delta\},$$

with $\Delta_u^r h(x) = \sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k} h(x + ku)$, while for $g : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$, $g := g(z_1, z_2)$, we set

$$\omega_r(g;\delta,0) = \sup_{z_2\in\overline{\mathbb{D}}} \omega_r(g_{z_2};\delta)$$

and

$$\omega_r(g;0,\delta) = \sup_{z_1\in\overline{\mathbb{D}}} \omega_r(g_{z_1};\delta),$$

where for fixed z_2, the function g_{z_2} is defined by $g_{z_2}(z_1) = g(z_1, z_2)$ for all $z_1 \in \overline{\mathbb{D}}$, and similarly, for fixed z_1, the function g_{z_1} is defined by $g_{z_1}(z_2) = g(z_1, z_2)$ for all $z_2 \in \overline{\mathbb{D}}$.

Looking more closely at the proof of Theorem 1 in Beutel–Gonska [41] (see Theorem 2.3.1 in Chapter 2), below we prove that it remains valid for tensor product of complex linear continuous operators defined on $A^d(\mathbb{D})$:

Theorem 4.3.4. *Let us consider that $A^p(\mathbb{D})$ is endowed with the norm*

$$\|f\|_{A^p(\mathbb{D})} := \max\{\|f^{(j)}\|; j = 0, \ldots, p\},$$

where $\|g\|$ denotes the uniform norm of g in $C(\overline{\mathbb{D}})$.

(i) If $L : A^p(\mathbb{D}) \to A(\mathbb{D})$ is a linear and continuous operator, then for $f \in A^{p,q}(\mathbb{D} \times \mathbb{D})$, $L[f_{z_2}]$ is q-times continuously differentiable with respect to $z_2 \in \overline{\mathbb{D}}$ and it commutes as a tensor product with the partial differentiation of $f(z_1, z_2) \in A^{p,q}(\mathbb{D} \times \mathbb{D})$ with respect to z_2, i.e.,

$$D^{(0,l)} \circ_{z_1} L =_{z_1} L \circ D^{(0,l)}, \quad l = 0, \ldots, q, \quad z_1 \in \overline{\mathbb{D}}.$$

Here $[D^{(0,l)} \circ_{z_1} L](f) := \frac{\partial^l L[f_{z_2}]}{\partial z_2^l}$, $[_{z_1} L \circ D^{(0,l)}](f) = L\left[\left(\frac{\partial^l f_{z_2}}{\partial z_2^l}\right)_{z_2}\right]$, $f_{z_2}(z_1) = f(z_1, z_2)$.

(ii) If $M : A^q(\mathbb{D}) \to A(\mathbb{D})$ is a linear and continuous operator, then for $f \in A^{p,q}(\mathbb{D} \times \mathbb{D})$, $L[f_{z_1}]$ is p-times continuously differentiable with respect to $z_1 \in \overline{\mathbb{D}}$ and it commutes as tensor product with the partial differentiation of $f(z_1, z_2) \in A^{p,q}(\mathbb{D} \times \mathbb{D})$ with respect to z_1, i.e.,

$$D^{(k,0)} \circ_{z_2} M =_{z_2} M \circ D^{(k,0)}, k = 0, ..., p, \quad z_2 \in \overline{\mathbb{D}}.$$

Here $[D^{(k,0)} \circ_{z_2} M](f) := \frac{\partial^k M[f_{z_1}]}{\partial z_1^k}$, $[_{z_2} M \circ D^{(k,0)}](f) = M\left[\left(\frac{\partial^k f_{z_1}}{\partial z_1^k}\right)_{z_1}\right]$, $f_{z_1}(z_2) = f(z_1, z_2)$.

(iii) Let $L : A^p(\mathbb{D}) \to A^q(\mathbb{D})$ and $M : A^p(\mathbb{D}) \to A^q(\mathbb{D})$ be linear operators satisfying the estimates

$$|[g - L(g)]^{(k)}(z_1)| \le \sum_{r=1}^{m} \alpha_{r,k}(z_1)\omega_r(g^{(p)}; \rho_1(z_1))$$

for all $z_1 \in \overline{\mathbb{D}}$, $0 \le k \le p$, and $g \in A^p(\mathbb{D})$, where $C_1 \ge \alpha_{r,k}(z_1) \ge 0$, $c_1 \ge \rho_1(z_1) \ge 0$ for all $1 \le r \le m$, $0 \le k \le p$, $z_1 \in \overline{\mathbb{D}}$,

and

$$|[h - M(h)]^{(l)}(z_2)| \le \sum_{s=1}^{t} \beta_{s,l}(z)\omega_s(h^{(q)}; \rho_2(z_2))$$

for all $z_2 \in \overline{\mathbb{D}}$, $0 \le l \le q$, and $h \in A^q(\mathbb{D})$, where $C_2 \ge \beta_{s,l}(z_2) \ge 0$, $c_2 \ge \rho_2(z_2)) \ge 0$ for all $1 \le s \le t$, $0 \le l \le q$, $z_2 \in \overline{\mathbb{D}}$.

Then, for all $z_1, z_2 \in \overline{\mathbb{D}}$, $f \in A^{p,q}(\mathbb{D} \times \mathbb{D})$, $0 \le k \le p$, $0 \le l \le q$, we have

$$\begin{aligned}
&|[f - ({}_{z_1}L \circ_{z_2} M)(f)]^{(k,l)}(z_1, z_2)| \\
&\le \sum_{r=1}^{m} \alpha_{r,k}(z_1)\omega_r(f^{(p,l)}; \rho_1(z_1), 0) \\
&\quad + \|D^{(k)} \circ L\| \sup_{0 \le i \le p} \sum_{s=1}^{t} \beta_{s,l}(z_2)\omega_s(f^{(i,q)}; 0, \rho_2(z_2))
\end{aligned}$$

and

$$\begin{aligned}
&|[f - ({}_{z_1}L \circ_{z_2} M)(f)]^{(k,l)}(z_1, z_2)| \\
&\le \sum_{s=1}^{t} \beta_{r,k}(z_2)\omega_s^*(f^{(k,q)}; 0, \rho_1(z_2)) \\
&\quad + \|D^{(l)} \circ M\| \sup_{0 \le j \le q} \sum_{r=1}^{m} \alpha_{r,k}(z_1)\omega_r(f^{(p,j)}; \rho_1(z_1), 0).
\end{aligned}$$

Proof. (i) and (ii) are immediate from the linearity and continuity of L and M.

(iii) First we prove that the estimates satisfied by L and M imply their continuity. We give the proof only for L, since the case of M is similar. The proof here follows mainly the proof in the real case of the Theorem 1 in Beutel–Gonska [41].

Since, in general, we have

$$\omega_r(g^{(p)};\delta) \leq 2^r \|g^{(p)}\|,$$

we get for all $0 \leq k \leq p$,

$$\begin{aligned}\|[L(g)]^{(k)}\| &\leq \|[L(g)-g]^{(k)}\| + \|g^{(k)}\| \\ &\leq \sum_{r=1}^{m} \sup_{z_1\in\overline{\mathbb{D}}} \omega_r(g^{(p)}; \sup_{z_1\in\overline{\mathbb{D}}} \rho_1(z_1)) + \|g^{(k)}\| \\ &\leq C\|g^{(p)}\| + \|g^{(k)}\| \leq C\|g\|_{A^p(\mathbb{D})},\end{aligned}$$

which immediately implies $\|L(g)\|_{A^p(\mathbb{D})} \leq C\|g\|_{A^p(\mathbb{D})}$, i.e., L is a continuous operator.

As a consequence, by the above point (i) we obtain $D^{(0,l)} \circ_{z_1} L =_{z_1} L \circ D^{(0,l)}$, $l = 0, \ldots, q$, which for $f \in A^{p,q}(\mathbb{D}\times\mathbb{D})$ implies

$$\begin{aligned}&|[f - ({}_{z_1}L \circ_{z_2} M)(f)]^{(k,l)}(z_1,z_2)| \\ &\quad\leq |[D^{(k,l)} \circ (Id -_{z_1} L) \circ D^{(0,l)}](f)(z_1,z_2)| \\ &\qquad + |[D^{(k,0)} \circ_{z_1} L \circ D^{(0,l)} \circ (Id -_{z_2} M)](f)(z_1,z_2)| := U + V.\end{aligned}$$

By the estimate satisfied by L, we immediately obtain

$$U \leq \sum_{r=1}^{m} \alpha_{r,k}(z_1)\omega_r(f^{(p,l)};\rho_1(z_1),0)$$

for all $z_1, z_2 \in \overline{\mathbb{D}}$, $0 \leq k \leq p$, $0 \leq l \leq q$, and $f \in A^{p,q}(\mathbb{D}\times\mathbb{D})$.

Now let us set $G = [D^{(0,l)} \circ (\mathrm{Id} -_{z_2} M)](f)$. By $f \in A^{p,q}(\mathbb{D}\times\mathbb{D})$, we get $f -_{z_2} M(f) \in A^{p,q}(\mathbb{D}\times\mathbb{D})$, $D^{(0,l)}[f -_{z_2} M(f)] \in A^{p,q-l}(\mathbb{D}\times\mathbb{D})$, and $G_{z_2} \in A^p(\mathbb{D})$ for all $z_2 \in \overline{\mathbb{D}}$, $0 \leq l \leq q$. From this and from the properties of M, denoting by $\|\!|\cdot|\!\|$ the operator norm, $\|\cdot\|$ the uniform norm and $\|\cdot\|_{z_1}$ the uniform norm with respect to z_1, we get

$$\begin{aligned}V &\leq \| \left(\frac{d^k}{dz_1^k} \circ L\right)(G_{z_2})\| \leq \|\!|\frac{d^k}{dz_1^k} \circ L|\!\| \cdot \|G_{z_2}\|_{A^p(\mathbb{D})} \\ &= \|\!|D^k \circ L|\!\| \cdot \sup_{0\leq i\leq p} \|\frac{d^i}{dz_1^i}[D^{(0,l)} \circ (Id -_{z_2} M)(f)]_{z_2}\|_{z_1} \\ &= \|\!|D^k \circ L|\!\| \cdot \sup_{0\leq i\leq p} \|D^{(0,l)} \circ (Id -_{z_2} M)(f^{(i,0)}(\cdot, z_2))\|_{z_1} \\ &= \|\!|D^k \circ L|\!\| \cdot \sup_{0\leq i\leq p} \sup_{z_1\in\overline{\mathbb{D}}} |D^{(0,l)} \circ (Id -_{z_2} M)(f^{(i,0)})(z_1,z_2)| \\ &\leq \|\!|D^k \circ L|\!\| \cdot \sup_{0\leq i\leq p} \sup_{z_1\in\overline{\mathbb{D}}} \sum_{s=1}^{t} \beta_{s,l}(z_2)\omega_s((f^{(i,q)})_{z_1};\rho_2(z_2)) \\ &= \|\!|D^k \circ L|\!\| \cdot \sup_{0\leq i\leq p} \sum_{s=1}^{t} \beta_{s,l}(z_2)\omega_s(f^{(i,q)};0,\rho_2(z_2)),\end{aligned}$$

which proves the first estimate.

The proof of second estimate is similar, which proves the theorem. □

Remark. Recall that by Remark 1 after the proof of Theorem 3.2.1 (iii) in Chapter 3, it easily follows that in fact, for univariate functions f, we have

$$\begin{aligned}&\|f^{(k)} - \sigma^{(k)}_{2n,n-p}(f)\|\\&\quad = \|f^{(k)} - \sigma_{2n-k,n-p}(f^{(k)})\|\\&\quad \le C_p E_{n+p-k}(f^{(k)}) \le C_p(n+p-k)^{-p+k} E_n(f^{(p)}) \le C_p n^{-p+k} E_n(f^{(p)})\end{aligned}$$

for all $k = 0, \ldots, p$ and $n > p$.

But obviously, the polynomial $\sigma_{2n,n-p}(f)$ is linear and continuous as a function of $f \in A^d(\mathbb{D})$. Combining this with the above Theorem 4.3.4 (since writing $L = \sigma_{2n,n-p}$ we have $|||D^k \circ L||| \le C$) and with the well-known relationships $E_{2n}(f) \le C_r \omega_r\left(f; \frac{1}{2n}\right)$ (see Gaier [121] or Gaier [122], p. 53) and

$$\omega_r\left(f; \frac{1}{2n}\right) \sim \omega_r\left(f; \frac{1}{2n+1}\right) \sim \omega_r\left(f; \frac{1}{n}\right)$$

(equivalences with respect to constants depending only on r), we immediately get the following result:

Theorem 4.3.5. *For any $f \in A^{p,q}(\mathbb{D} \times \mathbb{D})$ and for $0 \le k \le p$, $0 \le l \le q$, there exists a sequence of bivariate polynomials $Q_{n,m}(f)(z_1, z_2)$ of degree $\le n$ in z_1 and of degree $\le m$ in z_2 such that for all $n > p$, $m > q$, we have*

$$\begin{aligned}\|[f - Q_{n,m}]^{(k,l)}\| \le\ & c_{p,r}\left(\frac{1}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p,l)}; \frac{1}{n}, 0\right)\\&+ c_{p,q,r,s}\left(\frac{1}{m}\right)^{q-l} \cdot \sup_{0 \le i \le p} \omega_s\left(f^{(i,q)}; 0, \frac{1}{m}\right).\end{aligned}$$

Here $c_{p,r} > 0$ and $c_{p,q,r,s} > 0$ are constants independent of n, m, and f.

Theorem 4.3.5 allows us to prove a Shisha-type result for complex functions of two complex variables, as follows.

Theorem 4.3.6. *Let h_1, h_2, p, q, r, s be positive integers, $0 \le h_1 \le p$, $0 \le h_2 \le q$, and let $f \in A^{p,q}(\mathbb{D} \times \mathbb{D})$. Consider the continuous functions $\alpha_{i,j} : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{R}$, $i = h_1, \ldots, p$, $j = h_2, \ldots, q$, assume $a_{h_1,h_2}(z_1, z_2) = 1$ for all $z_1, z_2 \in \overline{\mathbb{D}}$, and define the differential operator*

$$L = \sum_{i=h_1}^{p} \sum_{j=h_2}^{q} \alpha_{i,j}(z_1, z_2) \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j}.$$

Also, set

$$\begin{aligned}M^{k,l}_{n,m}(f) =\ & c_{p,r}\left(\frac{1}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p,l)}; \frac{1}{n}, 0\right)\\&+ c_{p,q,r,s}\left(\frac{1}{m}\right)^{q-l} \cdot \sup_{0 \le i \le p} \omega_s\left(f^{(i,q)}; 0, \frac{1}{m}\right),\end{aligned}$$

$$P_{n,m}(L;f)=\sum_{i=h_1}^{p}\sum_{j=h_2}^{q} l_{i,j}M_{n,m}^{i,j},$$

where $c_{p,r}, c_{p,q,r,s}$ *are the constants in Theorem 4.3.5 and we have the formula* $l_{i,j}=\sup_{z_1,z_2\in\overline{\mathbb{D}}}\{|\alpha_{i,j}(z_1,z_2)|\}<\infty$.

If $\mathrm{Re}[L(f)(z_1,z_2)]\geq 0, \forall z_1,z_2\in\mathbb{D}$, *then for all* n,m *integers with* $n>p$, $m>q$, *there exists a bivariate polynomial* $Q_{n,m}(f)(z_1,z_2)$, *of degrees* $\leq n$ *in* z_1 *and* $\leq m$ *in* z_2, *satisfying* $\mathrm{Re}[L(Q_{n,m})(f)(z_1,z_2)]\geq 0, \forall z_1,z_2\in\mathbb{D}$, *and*

$$\|f-Q_{n,m}(f)\|\leq\frac{P_{n,m}(L;f)}{(h_1)!(h_2)!}+c_{p,r}\frac{1}{n^p}\omega_r\left(f^{(p,0)};\frac{1}{n},0\right)$$
$$+c_{p,q,r,s}\frac{1}{m^q}\sup_{0\leq i\leq p}\omega_s\left(f^{(i,q)};\frac{1}{m}\right). \tag{4.1}$$

Proof. From Theorem 4.3.5, there exists the polynomial $Q_{n,m}(f)(z_1,z_2)$ such that we have

$$\left\|\left[f+P_{n,m}(L;f)\frac{z_1^{h_1}}{h_1!}\frac{z_2^{h_2}}{h_2!}\right]^{(k,l)}-Q_{n,m}^{(k,l)}(f)\right\|\leq M_{n,m}^{k,l},$$

which implies

$$\|f^{(k,l)}-Q_{n,m}^{(k,l)}(f)\|\leq\frac{P_{n,m}(L;f)}{(h_1-k)!(h_2-l)!}+M_{n,m}^{k,l}(f),$$

for all $0\leq k\leq h_1, 0\leq h_2$.

Taking now $k=l=0$, it follows that

$$\|f-Q_{n,m}(f)\|\leq\frac{P_{n,m}(L;f)}{(h_1)!(h_2)!}+M_{n,m}^{0,0}(f)$$
$$=\frac{P_{n,m}(L;f)}{(h_1)!(h_2)!}+c_{p,r}\frac{1}{n^p}\omega_r\left(f^{(p,0)};\frac{1}{n},0\right)$$
$$+c_{p,q,r,s}\frac{1}{m^q}\sup_{0\leq i\leq p}\omega_s\left(f^{(i,q)};0,\frac{1}{m}\right), \tag{4.2}$$

which proves the estimate in the statement.

On the other hand, we have

$$L[Q_{n,m}(f)](z_1,z_2)=L(f)(z_1,z_2)+P_{n,m}(L;f)+\sum_{i=h_1}^{p}\sum_{j=h_2}^{q}\alpha_{i,j}(z_1,z_2)$$
$$\times\left[Q_{n,m}(z_1,z_2)-f(z_1,z_2)-P_{n,m}(L;f)\frac{z_1^{h_1}}{h_1!}\frac{z_2^{h_2}}{h_2!}\right]^{(i,j)},$$

which implies the equality of the real parts of the left side and right side.

Reasoning in what follows exactly as in the proof of one complex variable of Theorem 3.2.2, we arrive at the desired conclusion. □

Remark. If $\mathrm{Re}[L(f)(z_1, z_2)] > 0, \forall z_1, z_2 \in \mathbb{D}$, then for all n, m integers with $n > p$, $m > q$, following the proof of Theorem 4.3.6, it is easy to see that the approximation bivariate polynomial $Q_{n,m}(f)(z_1, z_2)$ also satisfies $\mathrm{Re}[L(Q_{n,m})(f)(z_1, z_2)] > 0,\ \forall z_1, z_2 \in \mathbb{D}$.

4.4 Bibliographical Notes and Open Problems

Theorems 4.2.1 and 4.3.2–4.3.6 are new.

Open Problem 4.4.1. By analogy with the other types of Bernstein polynomials of one complex variable in Open Problem 3.5.6, we can consider the Bernstein type polynomials of two complex variables

$$B^*_{m,n}(f)(z_1, z_2) = \sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j} z_1^k(1-z_1)^{m-k} z_2^j(1-z_2)^{n-j} f(e^{\frac{2k\pi i}{m}}, e^{\frac{2j\pi i}{n}}) \tag{4.3}$$

and

$$B^*_{n}(f)(z_1, z_2) = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j} z_1^k z_2^j(1-z_1-z_2)^{n-k-j} f(e^{\frac{2k\pi i}{n}}, e^{\frac{2j\pi i}{n}}) \tag{4.4}$$

(here $i^2 = -1$).

The approximation and geometric properties of $B^*_{m,n}(f)(z_1, z_2)$ and $B^*_n(f)(z_1, z_2)$ are open questions.

Open Problem 4.4.2. In the paper of Roper–Suffridge [324], it was proved that if $f : B(0; 1) \to \mathbb{C}^2$ is a normalized holomorphic mapping on $B(0; 1)$ satisfying the condition $\sum_{p=2}^{+\infty} p^2 \frac{\|D^p f(0)\|}{p!} \le 1$, then f is convex. Here $D^p f(0)$ is the Fréchet derivative, that is, a p-variable mapping $D^p f(0) : \Pi_{j=1}^{p}\mathbb{C}^2 \to \mathbb{C}^2$, linear with respect to each variable and symmetric.

Taking into account the notation after Theorem 4.1.2, denote by K_1^2 the class of all normalized holomorphic mappings on $B(0; 1)$ satisfying the above condition.

Let $K_n(t)$ be the de la Vallée Poussin trigonometric kernel given by $K_n(t) = \Omega_n(t) = \frac{(n!)^2}{(2n)!}(2\cos[t/2])^{2n}$.

Writing $f(z_1, z_2) = (F(z_1, z_2), G(z_1, z_2))$, define the convolution generalized polynomials

$$P_{n,m}(f)(z_1, z_2) = (R_{n,m}(F)(z_1, z_2), S_{n,m}(G)(z_1, z_2)),$$

where

$$R_{n,m}(F)(z_1, z_2) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_n(u) K_m(v) F(z_1 e^{iu}, z_2 e^{iv}) du\, dv$$
$$= \sum_{i,j=0}^{\infty} A_{i,j}^{n,m} z_1^i z_2^j$$

and

$$S_{n,m}(G)(z_1, z_2) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K_n(u) K_m(v) G(z_1 e^{iu}, z_2 e^{iv}) du\, dv$$
$$= \sum_{k,l=0}^{\infty} B_{k,l}^{n,m} z_1^k z_2^l.$$

An open question is whether the relationship

$$\|D^p P_{n,m}(f)(0)\| \le \|D^p f(0)\|, \quad \forall p,$$

holds.

If the answer is positive, then $f \in K_1^2$ would imply $P_{n,m}(f) \in K_1^2$, i.e. this would allow that starting from a convex function one could construct another convex function (generalized polynomial) in a simpler way than that given by the so-called Roper–Suffridge operator in Roper–Suffridge [324]. For example, first one may start with the particular case $F(z_1, z_2) = g(z_1)h(z_2)$, $G(z_1, z_2) = p(z_1)q(z_2)$, when the Taylor expansion of $F(z_1, z_2)$ becomes in fact the product of the Taylor expansions of $g(z_1)$ and $h(z_2)$, the Taylor expansion of $G(z_1, z_2)$ becomes in fact the product of the Taylor expansions of $p(z_1)$ and $q(z_2)$, and $R_{n,m}(F)(z_1, z_2)$, $S_{n,m}(G)(z_1, z_2)$ becomes the product of the corresponding convolution polynomials attached to the univariate functions $g(z_1)$, $h(z_2)$ and $p(z_1)$, $q(z_2)$, respectively.

5

Appendix: Some Related Topics

In this chapter we present some topics concerning shape-preserving approximations not necessarily of polynomial or spline type, but of great importance and very related to the previous chapters.

An important topic that is not included in this section is the shape-preserving approximation by spline functions (for reasons explained in the preface), but many details in that subject can be found, for example, in the books of de Boor [49], Schumaker [344], Chui [69], DeVore–Lorentz [91], Kvasov [218], and in the surveys of Leviatan [229] and Kocić–Milovanović [197]. In Sections 5.1 and 5.2 we present some shape-preserving results for general (positive) linear operators defined on $C[a, b]$ and for some concrete (real and complex) nonpolynomial operators, respectively. We can mention here the contributions of (in alphabetical order) Andrica, Badea, Cottin, Gal, Gavrea, Gonska, Kacsó, Karlin, Kocić, Lacković, Lupaş, Popoviciu, Tzimbalario, Zhou, and others.

Because of its close connection with the shape-preserving properties, in Sections 5.1 and 5.2 we also consider the *variation-diminishing property.*

In Section 5.3 Leviatan's result in [228] and some shape-preserving properties of Bernstein polynomials are extended to monotone and convex functions defined on $[-1, 1]$, with values in an ordered vector space. The first contribution belongs to Anasstasiou–Gal [9], while the second one appears for the first time here.

Section 5.4 contains some shape-preserving properties of the complex singular integrals of Poisson–Cauchy and Gauss–Weierstrass integrals, with applications to complex PDE.

5.1 Shape-Preserving Approximation by General Linear Operators on $C[a, b]$

In this section we present some interesting results concerning the preservation of monotonicity and convexity by general linear operators on the space $C[a, b]$.

S.G. Gal, *Shape-Preserving Approximation by Real and Complex Polynomials*,
DOI: 10.1007/978-0-8176-4703-2_5,

Let $a \leq x_0 < x_1 < \cdots < x_m \leq b$ and $P_j : [a,b] \to \mathbb{R}_+,\ j = 0,1,\ldots,m$, be continuous functions. The linear operator $F_m(f)(x) = \sum_{j=0}^m P_j(x)f(x_j)$ is said to be of interpolation type.

The first result concerns interpolation-type operators.

Theorem 5.1.1. (see Popoviciu [313]) *Considering the sequence of functions defined by* $\Phi_0 = \sum_{k=0}^m P_k$ *and* $\Phi_1 = \sum_{j=1}^m P_j$, $\Phi_i = \sum_{i=j+2}^m (x_k - x_{i+1})P_k,\ i = 0,\ldots,m-2$, *the operators (defined above)* F_m *preserve the (usual) convexity of* f *if and only if the functions* Φ_0, Φ_1 *are polynomials of degree* ≤ 1 *on* $[a,b]$ *and* $\Phi_i, i = 0,\ldots,m-2$, *are (usual) convex on* $[a,b]$*. Here usual convexity means* 1*-convexity according to Definition 1.1.1 (i).*

More generally, we have the following result.

Theorem 5.1.2. (see Tzimbalario [390]) *Let* $L : C[a,b] \to C[a,b]$ *be a continuous linear operator. Denoting by* $K^r[a,b]$ *the class of all convex functions of order* r *on* $[a,b]$*, necessary and sufficient conditions for the implication*

$$f \in K^r[a,b] \Longrightarrow L(f) \in K^r[a,b]$$

are as follows:

(i) If p *is polynomial of degree* $\leq r-1$*, then so is* $L(f)$*.*
(ii) $L(\phi_c^{r-1}) \in K^r[a,b]$ *for every* $c \in [a,b]$ *where* $\phi_c^r(x) = 0$ *if* $x \in [a,c)$ *and* $\phi_c^r(x) = (x-c)^r$ *if* $x \in [c,b]$*.*

Remark. Theorem 5.1.2 was proved in [390] in the more general setting of convex functions with respect to a Chebyshev system.

In order to state another result related to Theorem 5.1.2, we need the concept of one-sided strong local maximum (OSLM) for real functions.

Definition 5.1.3. (see Kocić–Lacković [196]) The function $\phi \in C[a,b]$ has the *OSLM* property at the point $x_0 \in (a,b)$ if there exists $h > 0$ such that for any $x \in (x_0 - h, x_0 + h) \subset [a,b]$, we have $\phi(x) \leq \phi(x_0)$ and $\phi(x) < \phi(x_0)$ at least in one of the intervals $(x_0 - h, x_0)$ and $(x_0, x_0 + h)$. We denote by $C_0[a,b]$ the class of all functions $f \in C[a,b]$ having the (OSLM) property at least at a point.

Theorem 5.1.4. (see Kocić–Lacković [196]) *If* $(A_\lambda)_\lambda$ *is a family of continuous linear operators* $A_\lambda : C[a,b] \to S[c,d]$ *(where* $S[c,d]$ *is a normed subspace of real-valued functions defined on* $[c,d]$*) satisfying the conditions*

(i) every A_λ *preserves the affine functions;*
(ii) for every $\phi \in C_0[a,b]$*, there exist at least one* λ_0 *and* $y_0 \in [c,d]$ *such that* $A_{\lambda_0}(\phi)(y_0) < 0$*;*
(iii) $A_\lambda(\sigma_c) \geq 0$ *for all* $c \in [a,b]$ *and all* λ*, where* $\sigma_c(x) = |x-c|$*, for all* $x \in [a,b]$*,*

then $A_\lambda(f) \geq 0$ *for all* $\lambda \Leftrightarrow f$ *is (usual) convex on* $[a,b]$*.*

It is also worth mentioning the following interesting equivalence result that connects the convex approximation by positive linear operators with other classical results.

Theorem 5.1.5. (see Andrica–Badea [20]) *The following statements are equivalent:*

(i) There is a sequence of positive linear polynomial operators reproducing affine functions that preserve (usual) convexity;
(ii) Jensen's inequality for (usual) convex functions;
(iii) Korovkin's theorem on the space $C[a, b]$;
(iv) Jensen's inequality for positive linear functionals on $C[a, b]$.

Other kinds of (positive) linear operators were considered by Karlin [192], of the form $T : C[a, b] \to C[c, d]$, given by $T(f)(x) = \int_a^b K(x, y)f(y)dy, x \in [c, d]$.

Theorem 5.1.6. (see Karlin [192]) *If the kernel $K : [c, d] \times [a, b] \to \mathbb{R}$ is a totally positive function of order 3 and satisfies the conditions*

$$\int_a^b K(x, y)dy = 1, \int_a^b yK(x, y)dy = Ax + B, \ A > 0,$$

then T preserves the (usual) convexity of $f \in C[a, b]$.

Recall here that $K(x, y)$ is called totally positive of order 3 if for all $1 \leq m \leq 3$ and for all $c \leq x_1 < \cdots < x_m \leq d$, $a \leq y_1 < \cdots < y_m \leq b$, we have

$$\Delta_K(x_1, ..., x_m; y_1, ..., y_m) \geq 0,$$

where $\Delta_K(x_1, \ldots, x_m; y_1, \ldots, y_m)$ denotes the determinant of the matrix given by $(K(x_i, y_j))_{i,j=1,\ldots,m}$.

In what follows, we present some interesting relationships of the shape-preserving property of positive linear operators with other properties, such as the global smoothness preservation property and the variation-diminishing property.

For the simplicity of presentation, we consider operators of the form $T : C[0, 1] \to C[0, 1]$. First, recall that T is said to have the global smoothness preservation property (with respect to the first-order modulus of smoothness) if

$$\omega_1(T(f); \delta)_\infty \leq C\omega_1(f; \delta)_\infty, \ \forall f \in C[-1, 1], \delta \geq 0,$$

where $\omega_1(f; \delta)_\infty = \sup\{|f(x) - f(y)|; |x - y| \leq \delta\}$ and $C > 0$ is independent of f and δ.

Also, recall that the least concave majorant of $\omega_1(f; \delta)_\infty$, for $f \in C[0, 1]$ can be defined by (see, e.g., Anastassiou–Gal [7], p. 233)

$$\overline{\omega}_1(f;t)_\infty = \sup\left\{\frac{(t-x)\omega_1(f;y)_\infty + (y-t)\omega_1(f;x)_\infty}{y-x}; 0 \le x \le t \le y \le 1\right\},$$

$\overline{\omega}_1(f;t)_\infty = \omega_1(f,1)_\infty$, for $t > 1$.

It is evident by definition that $\omega_1(f;t)_\infty \le \overline{\omega_1}(f;t)_\infty$. By, e.g., Anastassiou-Cottin–Gonska [6] (see also Anastassiou–Gal [7], pp. 237–238, Theorem 7.2.3) we have $\overline{\omega}_1(f;t)_\infty \le 2\omega_1(f;t)_\infty$, since f is defined on a compact subinterval of $\mathbb{R}$.

As a consequence, without loss of generality, the global smoothness preservation property can be expressed in terms of $\overline{\omega}_1(f;t)_\infty$ too. Let us also mention that for the Bernstein polynomials we have $\omega_1(B_n(f);\delta)_\infty \le \overline{\omega}_1(f;\delta)_\infty \le 2\omega_1(f;\delta)_\infty$ for all $f \in C[0,1]$ and $\delta \ge 0$ (see Anastassiou–Cottin–Gonska [6]).

A first connection between the variation-diminishing property and the shape-preserving property is given by the following result.

Theorem 5.1.7. *(i)* (Cottin–Gavrea–Gonska–Kacsó–Zhou [72]) *If the linear operator $T : C[0,1] \to C[0,1]$ is strongly variation-diminishing, then it preserves the positivity and monotonicity of f;*

(ii) (Gavrea–Gonska–Kacsó [146]) *Let $T_n : C[a,b] \to \Pi_n$ be a positive linear operator satisfying the following conditions: degree$(T_n(e_i)) = i$, $i = 0,1,\ldots,n$. If T_n has the strong variation-diminishing property, then T_n preserves all the convexities of order $i = 0,1,\ldots,n$.*

(iii) (Gavrea–Gonska–Kacsó [146]) *Let us consider $I = (a,b)$ or $I = (a,+\infty)$ with $a \ge 0$, $[\alpha,\beta] \subset [0,+\infty)$ and the continuous weight function $w : I \to \mathbb{R}_+$. If $A : C(I) \to \mathbb{R}$ is a linear and positive definite functional (i.e., if $f \ge 0$ and $A(f) = 0$, then $f = 0$) with the property that there exists a subspace $C_w^{[\alpha,\beta]}(I) \subset C(I)$ such that for any $f \in C_w^{[\alpha,\beta]}(I) \subset C(I)$, the value $L(f)(x) = A_t[t^x w(t) f(t)]$ is well-defined (here A_t means that the functional A is evaluated with respect to t), then*

$$Z_{(\alpha,\beta)}[L(f)] \le S_I[f]$$

for all $f \in C_w^{[\alpha,\beta]}(I)$. We recall that $Z_{(\alpha,\beta)}[g]$ denotes the number of zeros of g in (α,β), while $S_I[f]$ is defined by Definition 1.1.1 (vi).

Let us establish the following notation:

$M[0,1]$ is the set of all monotone functions on $[0,1]$, $M^+[0,1]$ is the set of all increasing functions on $[0,1]$, $M^-[0,1]$ is the set of all decreasing functions on $[0,1]$, $\mathrm{Lip}_M[0,1] = \{f \in C[-1,1]; |f(x)-f(y)| \le M|x-y|, x,y \in [0,1]\}$, $\mathrm{Lip}[0,1] = \bigcup_{M>0} \mathrm{Lip}_M[0,1]$, $\|f\|_{\mathrm{Lip}} = \sup_{|x-y|>0} \frac{|f(x)-f(y)|}{|x-y|}$.

The next three results present the relationships between the shape-preserving property and global smoothness preservation.

Theorem 5.1.8. (Cottin–Gavrea–Gonska–Kacsó–Zhou [72]) *Let us consider $T : C[0,1] \to C[0,1]$, a positive linear operator that maps $C^1[0,1]$ into $\mathrm{Lip}[0,1]$ and reproduces constant functions. If $T(M^+ \bigcap C^1[0,1]) \subset M^+$ or*

$T(M^+ \bigcap C^1[0,1]) \subset M^-$, then $\omega_1(T(f);\delta)_\infty \le \overline{\omega}_1(f;c\delta)_\infty$, $\forall \delta \in [0,1]$, $f \in C[0,1]$, and the best constant c is $|T(e_1)|$, $e_1(x) = x$.

Theorem 5.1.9. (Cottin–Gavrea–Gonska–Kacsó–Zhou [72]) *Let us consider $T : C[0,1] \to C[0,1]$, a positive linear operator that maps $C^1[0,1]$ into* Lip$[0,1]$ *and satisfies $T(e_i) = e_i$, $i = 0, 1$, where $e_i(x) = x^i$, $x \in [0,1]$. Then*

$$\omega_1(T(f);\delta)_\infty \le \overline{\omega}_1(f;\delta)_\infty \le 2\omega_1(f;\delta)_\infty, \text{ for all } \delta \in [0,1],\ f \in C[0,1],$$

if and only if

$$T(M^+[0,1] \cap C^1[0,1]) \subset M^+[0,1].$$

Theorem 5.1.10. (Cottin–Gavrea–Gonska–Kacsó–Zhou [72]) *Let us consider $T : C[0,1] \to C[0,1]$, a positive linear operator that maps $C^1[0,1]$ into* Lip$[0,1]$ *and satisfies $T(e_i) = e_i$, $i = 0, 1$. If $T(M[0,1] \cap C[0,1]) \subset M[0,1]$, then*

$$\omega_1(T(f);\delta)_\infty \le \overline{\omega}_1(f;\delta)_\infty \le 2\omega_1(f;\delta)_\infty, \text{ for all } \delta \in [0,1],\ f \in C[0,1].$$

Remark. The proofs of Theorems 5.1.7–5.1.10 can be found in the original mentioned papers or in the book Anastassiou–Gal [7], Section 20.1, pp. 485–491.

5.2 Some Real and Complex Nonpolynomial Operators Preserving Shape

In this section, first we present a few of the best-known real nonpolynomial approximation operators that preserve the convexity properties of the approximated function and some of which are strongly variation-diminishing too, as follows.

(1) The Gauss–Weierstrass operators (introduced and used by Weierstrass [398] in order to prove his famous result on the approximation of continuous functions by algebraic polynomials), attached to bounded $f : \mathbb{R} \to \mathbb{R}$, given by

$$W_n(f)(x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{+\infty} e^{-n(u-x)^2/2} f(u)du,$$

preserve the convexity of any order of f (see Butzer–Nessel [52]).

(2) The Favard–Szász–Mirakjan operators introduced by Mirakjan [271] and studied by Favard [114] and Szász [382], attached to bounded $f : [0,\infty) \to \mathbb{R}$ and defined by

$$F_n(f)(x) = e^{-nx} \sum_{k=0}^{+\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

preserve the convexity of any order of f (see Lupaş [256]) and are strongly variation-diminishing (see Lupaş [258]).

(3) The Baskakov [30] operators, attached to bounded $f : [0,\infty) \to \mathbb{R}$ and defined by

$$V_n(f)(x) = (1+x)^{-n} \sum_{k=0}^{+\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right),$$

preserve the convexity of any order of f (see Lupaş [257] and Ibragimov–Gadzijev [180]) and are strongly variation-diminishing (see Lupaş [258]).

(4) The Meyer–König–Zeller [270] operators, attached to bounded functions $f : [0,1) \to \mathbb{R}$ and defined by

$$Z_n(f)(x) = \sum_{k=0}^{+\infty} \binom{n+k}{k} (1-x)^{n+1} x^k f\left(\frac{k}{n+k}\right),\ x \in [0,1), \quad Z_n(f)(1) = f(1),$$

preserve the sign, monotonicity, and usual convexity of the approximated function (see Lupaş [256]) and are strongly variation-diminishing operators (see Cimoca–Lupaş [70].

(5) Jakimovski–Leviatan [186] operators defined by

$$P_n(f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{+\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0,$$

where $p_k(x)$ are the so-called Appel polynomials given by the relationship $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$, with g analytic in a disk, $g(1) \neq 1$, preserve the usual convexity of f (see Wood [401]).

(6) The gamma operators introduced by Lupaş–Müller [262] are defined by

$$G_n(f)(x) = \frac{x^{n+1}}{n!} \int_0^{+\infty} e^{-xt} t^n f\left(\frac{n}{t}\right) dt,$$

preserve the convexity of f and are strongly variation-diminishing (see Lupaş–Müller [262]).

(7) The Bleimann–Butzer–Hahn operators introduced by Bleimann, Butzer, and Hahn [46], attached to bounded $f : [0,\infty) \to \mathbb{R}$, are defined by

$$BBH_n(f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n+1-k}\right),$$

satisfy the rate of convergence

$$|BBH_n(f)(x) - f(x)| \leq (1+\sqrt{3})\omega_1\left(f; \sqrt{\frac{x(1+x)^2}{n+2}}\right)_{\infty},\ x \in (0,+\infty), n \in \mathbb{N},$$

and preserve the convexities of higher order of f (see Abel–Ivan [1]).

The methods of proof for the shape-preserving properties and for the strongly variation-diminishing property of these operators follow in general, those in the case of classical Bernstein polynomials, i.e., by representing the derivative of a given order as a sum of products between positive quantities and finite (or divided) differences of the same order of f and Descarte's rule of signs, respectively.

A large number of other real nonpolynomial approximation operators preserve the convexity properties of the approximated function, but because they do not represent the main topic of this monograph, we close the list here.

At the end of this section, for the complex Favard–Szász–Mirakjan operator (obtained from the well-known real version of it, simply replacing the real variable x by the complex one z), given by

$$S_n(f)(z) = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} f(j/n),$$

we present some very recent results.

Note that the first result concerning the convergence of complex $S_n(f)(z)$ to the function f belonging to a class of analytic functions in a parabolic domain of the complex plane satisfying a suitable exponential-type growth condition was proved in Dressel–Gergen–Purcell [99], but without any estimate of the approximation error and any shape-preserving properties.

The main results can be summarized by the following.

Theorem 5.2.1. (Gal [134]) *Let $G \subset \mathbb{C}$ be the open disk of radius $R > 1$ and center 0. Let us suppose that $f : \overline{G} \cup [R, +\infty) \to \mathbb{C}$ is continuous on $\overline{G} \cup [R, +\infty)$, analytic in G, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in G$, and that there exist $M, C, B > 0$ and $A \in (\frac{1}{R}, 1)$ with the property $|c_k| \le M \frac{A^k}{k!}$ for all $k = 1, 2, \ldots$, (which implies $|f(z)| \le M e^{A|z|}$ for all $z \in G$) and $|f(x)| \le C e^{Bx}$ for all $x \in [R, +\infty)$.*

(i) Let $1 \le r < \frac{1}{A}$ be arbitrary fixed. For all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$|S_n(f)(z) - f(z)| \le \frac{C_{r,A}}{n},$$

where $C_{r,A} = \frac{M}{2r} \sum_{k=2}^{\infty} (k+1)(rA)^k < \infty$.

(ii) For the simultaneous approximation we have that if $1 \le r < r_1 < \frac{1}{A}$ are arbitrary fixed, then for all $|z| \le r$ and $n, p \in \mathbb{N}$,

$$|S_n^{(p)}(f)(z) - f^{(p)}(z)| \le \frac{p! r_1 C_{r_1,A}}{n(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ is given at the above point (i).

(iii) Let $1 \le r < \frac{1}{A}$ be arbitrary fixed. The following Voronovskaya-type result holds:

$$\left| S_n(f)(z) - f(z) - \frac{z}{2n} f''(z) \right| \le \frac{3MA|z|}{2r^2 n^2} \sum_{k=2}^{\infty} k(rA)^{k-1}, \text{ for all } n \in \mathbb{N},\ |z| \le r.$$

(iv) If $f(0) = f'(0) - 1 = 0$ *and* f *is starlike (convex, spirallike of type* $\gamma \in (-\pi/2, \pi/2)$*, respectively) in* $\overline{\mathbb{D}}$*, then there exists an index* n_0 *depending on* f *(and on* γ *for spirallikeness) such that for all* $n \geq n_0$*, the complex Favard–Szász–Mirakjan operators* $S_n(f)(z)$*, are starlike (convex, spirallike of type* γ*, respectively) in* $\overline{\mathbb{D}}$*.*

If $f(0) = f'(0) - 1 = 0$ *and* f *is starlike (convex, spirallike of type* $\gamma \in (-\pi/2, \pi/2)$*, respectively) only in* $\mathbb{D}$*, then for any disk of radius* $0 < r < 1$ *and center* 0*, denoted by* $\mathbb{D}_r$*, there exists an index* $n_0 = n_0(f, \mathbb{D}_r)$ *(*n_0 *depends on* γ *too in the case of spirallikeness) such that for all* $n \geq n_0$*, the complex Favard–Szász–Mirakjan operators* $S_n(f)(z)$ *are starlike (convex, spirallike of type* γ*, respectively) in* $\overline{\mathbb{D}}_r$*.*

Remark. For other complex operators, see Open Question 5.5.5 in Section 5.5.

5.3 Shape-Preserving Polynomial Approximation in Ordered Vector Spaces

In this section, some shape-preserving properties due to Leviatan in Sections 1.6, 1.7 and of Bernstein polynomials in Section 1.3, are extended to the abstract setting, i.e., to functions with values in ordered vector spaces.

Similar to the case of real-valued functions, first we introduce the following concepts.

Definition 5.3.1. (Gal [124]) (i) Let $(X, \|\cdot\|)$ be a real normed space.

A generalized algebraic polynomial of degree $\leq n$ with coefficients in X is an expression of the form $P_n(x) = \sum_{k=0}^{n} c_k x^k$, where $c_k \in X,\ k = 0, \ldots, n$, and $x \in [a, b]$.

(ii) The uniform kth Ditzian–Totik modulus of smoothness of $f : [-1, 1] \to X$ is given by

$$\omega_k^{\phi}(f; \delta)_{+\infty} = \sup_{0 \leq h \leq \delta} \|\overline{\Delta}_{h\phi(x)}^{k} f(x)\|_{+\infty},$$

where $\phi^2(x) = 1 - x^2$ and $\overline{\Delta}_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + kh/2 - jh)$ if $x,\ x \pm kh/2 \in [-1, 1]$, $\overline{\Delta}_h^k f(x) = 0$, otherwise. Here $\|f\|_\infty = \sup\{\|f(x)\|; x \in [-1, 1]\}$.

(iii) The uniform kth modulus of smoothness of $f : [-1, 1] \to X$ is given by

$$\omega_k(f; \delta)_\infty = \sup_{0 \leq h \leq \delta} \{\sup\{\|\Delta_h^k f(x)\|; x, x + kh \in [-1, 1]\}\}.$$

Here $\Delta_h^k f(x) = \sum_{j=0}^{n} (-1)^{k-j} \binom{k}{j} f(x + jh)$.

The main results contain some shape-preserving approximation properties by generalized polynomials for $f : [-1, 1] \to X$, where $(X, \|\cdot\|, \leq)$ is a vector space endowed with the structure of an ordered linear space. We thus extend two classical results in the case of real functions of real variables in Sections 1.7 and 1.3.

The main tool used in our proofs is the following well-known result in functional analysis.

Theorem 5.3.2. (see, e.g., Muntean [279], p. 183) *Let* $(X, \|\cdot\|)$ *be a normed space over the real or complex numbers and denote by* X^* *the conjugate space of* X*. Then* $\|x\| = \sup\{|x^*(x)| : x^* \in X^*, \||x^*\|| \leq 1\}$*, for all* $x \in X$*.*

Let us suppose that $(X, \|\cdot\|, \leq_X)$ is a normed space such that $\leq_X$ is an order relation on X that satisfies the conditions

$$x \leq_X y,\ 0 \leq \alpha,\ \text{imply } \alpha x \leq_X \alpha y;$$

$$x \leq_X y \text{ and } u \leq_X v \text{ imply } x + u \leq_X y + v.$$

The following concepts are well known.

Definition 5.3.3. Let $f : [a, b] \to X$.

(i) f is said to be increasing on $[a, b]$ if $x \leq y$ implies $f(x) \leq_X f(y)$.

(ii) f is called convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \leq_X \lambda f(x) + (1-\lambda)f(y), \forall x, y \in [a, b], \lambda \in [0, 1].$$

We present the first main result.

Theorem 5.3.4. (Anastassiou–Gal [9]) *For any convex function* $f : [-1, 1] \to X$ *and every* $n \in \mathbb{N}$*, there is a convex generalized algebraic polynomial* $A_n(x)$*, of degree* $\leq n$*, such that*

$$\|f - A_n\|_\infty \leq C\omega_2^\phi(f; 1/n)_\infty$$

and

$$|f(x) - A_n(x)| \leq C\omega_2(f; \sqrt{1-x^2}/n)_\infty,\ x \in [-1, 1].$$

If, in addition, f *is increasing, then so is* A_n*. Here* $C > 0$ *is an absolute constant.*

Proof. As in the case of real-valued functions, we define the generalized algebraic polynomial of degree $\leq n$ by

$$A_n(f)(x) = f(-1) + \sum_{j=0}^{n-1} s_j[R_j(x) - R_{j+1}(x)],$$

where $s_j = \frac{f(\xi_{j+1})-f(\xi_j)}{\xi_{j+1}-\xi_j}$, $j = 0, \ldots, n-1$, and $-1 = \xi_0 < \cdots < \xi_j < \cdots < \xi_n = 1$, $R_j(x)$ are defined as in Leviatan [228], pp. 472–473.

Simple calculation gives us

$$A_n(x) = f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j+1}) R_j(x).$$

Writing $F_j(x) = R_j(x) - R_{j+1}(x)$, in Leviatan [228] it is proved that $F_j(x)$ are increasing, while $R_j(x)$ are convex, real-valued functions on $[-1, 1]$.

Suppose $x, y \in [-1, 1]$, $x \le y$, and f is increasing on $[-1, 1]$. We get $0_X \le s_j$, which implies

$$\begin{aligned} A_n(f)(x) &= f(-1) + \sum_{j=0}^{n-1} s_j F_j(x) \le_X f(-1) + \sum_{j=0}^{n-1} s_j F_j(y) \\ &= A_n(f)(y), \end{aligned}$$

that is, $A_n(f)$ is increasing.

Suppose now that f is convex on $[-1, 1]$. By $\xi_{j-1} < \xi_j < \xi_{j+1}$, we get $\xi_j = \lambda \xi_{j-1} + (1 - \lambda)\xi_{j+1}$, with suitable $\lambda \in (0, 1)$. Then, by the relationship

$$\begin{aligned} s_j - s_{j-1} = &\frac{1}{\lambda(1-\lambda)(\xi_{j+1} - \xi_j)} \\ &\times [(1-\lambda) f(\xi_{j+1}) + \lambda f(\xi_{j-1}) - f((1-\lambda)\xi_{j+1} + \lambda \xi_{j-1})], \end{aligned}$$

we obtain $0_X \le_X s_j - s_{j-1}$ and therefore

$$\begin{aligned} A_n(f)[\lambda x + (1-\lambda) y] &= f(-1) + s_0 R_0(\lambda x + (1-\lambda) y) \\ &\quad + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j[\lambda x + (1-\lambda) y] \\ &\le_X f(-1) + s_0[\lambda R_0(x) + (1-\lambda) R_0(y)] \\ &\quad + \sum_{j=1}^{n-1} (s_j - s_{j-1})[\lambda R_j(x) + (1-\lambda) R_j(y)] \\ &= \lambda A_n(f)(x) + (1-\lambda) A_n(f)(y), \end{aligned}$$

that is, $A_n(f)$ is convex on $[-1, 1]$.

In order to prove the estimates in the statement, let $x^* \in B_1$ and set $g(x) = x^*[f(x)], x \in [-1, 1]$. By Leviatan [228], we have

$$|g(x) - A_n(g)(x)| \le C\omega_2(g; \sqrt{1 - x^2}/n)_\infty, \quad x \in [-1, 1],$$

and

$$\|g - A_n(g)\|_\infty \le C\omega_2^\phi(g; 1/n)_\infty.$$

By $|\Delta_h^2 g(x)| = |x^*[\Delta_h^2 f(x)]|$ and by

$$\begin{aligned}
&|g(x+h\phi(x)) - 2g(x) + g(x-h\phi(x))| \\
&\quad = |x^*[f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))]| \\
&\quad \leq |||x^*||| \cdot \|f(x+f\phi(x)) - 2f(x) + f(x+h\phi(x))\| \\
&\quad \leq \|f(x+h\phi(x)) - 2f(x) + f(x+h\phi(x))\|,
\end{aligned}$$

we obtain $\omega_2(g;\sqrt{1-x^2}/n)_\infty \leq \omega_2(f;\sqrt{1-x^2}/n)_\infty$ and $\omega_2^\phi(g;1/n)_\infty \leq \omega_2^\phi(f;1/n)_\infty$.

Also, we get

$$\begin{aligned}
&|x^*[f(x) - A_n(f)(x)]| \leq C\omega_2(f;\sqrt{1-x^2}/n)_\infty, x \in [-1,1] \\
&|x^*[f(x) - A_n(f)(x)]| \leq C\omega_2^\phi(f;1/n)_\infty, \forall x \in [-1,1],
\end{aligned}$$

since $A_n(g)(x) = x^*[A_n(f)(x)]$ from the linearity of x^*.

Passing to the supremum with $x^* \in B_1$, from Theorem 5.3.2 we obtain the desired estimates. □

Remarks. 1) The x^*-method in this section was used to obtain error estimates in approximation of vector-valued functions for the first time for Bernstein, Kantorovich, and Lagrange generalized polynomials in the paper Gal [124] and in the book Gal [123], pp. 24–26. Also, very recently, this method was used by Anastassiou–Gal [10] to obtain results on best-approximation generalized polynomials.

2) If, on $\mathbb{R}$, we define the new order $x \leq_\varphi y$ iff $\varphi(x) \leq \varphi(y)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is any fixed discontinuous solution of the Cauchy functional equation $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{R}$, then by Theorem 5.3.4 we get new results in shape-preserving approximation by polynomials, for real-valued functions.

Finally, for the Bernstein operators, let us recall the estimates obtained in the recent paper Gal [124] (see also the book Gal [123], pp. 24–25), for $f : [0,1] \to X$, continuous on $[0,1]$:

(i)

$$C_1\omega_2^\phi\left(f;\frac{1}{\sqrt{n}}\right)_\infty \leq \|B_n(f) - f\|_\infty \leq C_2\omega_2^\phi\left(f;\frac{1}{\sqrt{n}}\right)_\infty,$$

where $\|f\|_\infty = \sup\{\|f(x)\|; x \in [0,1]\}$ *and* $C_1, C_2 > 0$ *are absolute constants.*

(ii)

$$\|B_n(f)(x) - f(x)\| \leq M\left[\frac{x(1-x)}{n}\right]^{\alpha/2} \forall x \in [0,1]$$

if and only if $\omega_2(f;\delta)_\infty = O(\delta^\alpha)$, where $\alpha \leq 2$. Here, the moduli of smoothness are defined in Definition 5.3.1 (ii), (iii), and the Bernstein type operators attached to f are given by $B_n(f)(x) = \sum_{k=0}^n p_{n,k}(x)f(\frac{k}{n})$.

Similar to Theorem 1.3.1 (i) in Section 1.3, we have

Theorem 5.3.5. *First, $f : [0,1] \to X$ will be called k-convex if all the kth forward differences $\Delta_h^k f(t) \geq 0_X$, $0 \leq h \leq (b-a)/k$, $t \in [a, b-kh]$. (Here $\Delta_h^k f(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(t+jh)$, for all $k = 0, 1, \ldots$).*

If f is k convex on $[0,1]$, then so is $B_n(f)$.

Proof. We have

$$\Delta_h^k B_n(f)(t) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) \Delta_h^k p_{n,j}(x),$$

where $\Delta_h^k p_{n,j}(x) = \binom{n}{j} \Delta_h^k [x^j(1-x)^{n-j}]$. Applying the mean value theorem in real analysis, there exists $\xi \in [0,1]$, such that

$$\Delta_h^k [x^j(1-x)^{n-j}] = h^k [\xi^j (1-\xi)^{n-j}]^{(k)},$$

and replacing back in the expression of $B_n(f)$, we obtain (by similar reasoning to that in the case of real-valued functions)

$$\Delta_h^k B_n(f)(x) = n(n-1)(n-k+1) \frac{1}{n^k} \sum_{j=0}^{n-k} \Delta_{1/n}^k f(j/n) p_{n-k,j}(\xi) \geq 0_X,$$

since all $\Delta_{1/n}^k f(j/n) \geq 0_X$. $\square$

5.4 Complex Nonpolynomial Convolutions Preserving Shape

The main idea of this section belongs to a series of recent papers of Anastassiou–Gal ([11]–[15]) and consists in the "complexification" of several real convolution operators, a procedure that keeps the rate of approximation in the real case but induces the preservation property of some conditions in geometric function theory. More exactly, if $K : \mathbb{R} \to \mathbb{R}$ is a real integrable "even kernel," it is known that its convolution with an integrable function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$J(x) = \int_{-\infty}^{+\infty} K(t) f(x-t) dt = \int_{-\infty}^{+\infty} K(t) f(x+t) dt$$

(if the kernel K and f are 2π-periodic, then the above integral is replaced by $\int_{-\pi}^{+\pi}$).

The "complexification" means to replace $x+t$ by $ze^{it} = re^{i(x+t)}$ under f (where $z = re^{ix}$), which generates the complex convolution

$$P(z) = \int_{-\infty}^{+\infty} K(t) f(ze^{it}) dt = \int_{-\infty}^{+\infty} K(t) f(ze^{-it}) dt$$

(or $P(z) = \int_{-\pi}^{+\pi} K(t) f(ze^{it}) dt$ in the case of 2π-periodicity).

The complex convolution integral $P(z)$ has in general the same approximation properties as the real one $J(x)$ has, but in addition, for many choices of K, it preserves the analyticity of f and some geometric properties of f in the unit disk.

The sections in Chapter 3 treated the case of convolution complex polynomials. In this section, we will present some classical nonpolynomial complex convolution operators. Interesting applications to complex PDE also will be presented.

Recall that $A^*(\mathbb{D}) = \{f \in A(\mathbb{D}) : f(0) = f'(0) - 1 = 0\}$.

Definition 5.4.1. (Anastassiou–Gal [13]) Let $f \in A^*(\mathbb{D})$. For $\xi > 0$, the complex singular integrals defined by

$$P_\xi(f)(z) = \frac{1}{2\xi}\int_{-\infty}^{+\infty} f(ze^{iu})e^{-|u|/\xi}\,du, \quad z \in \overline{\mathbb{D}},$$
$$Q_\xi(f)(z) = \frac{\xi}{\pi}\int_{-\pi}^{\pi} \frac{f(ze^{iu})}{u^2+\xi^2}\,du, \quad z \in \overline{\mathbb{D}},$$
$$Q^*_\xi(f)(z) = \frac{\xi}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{u^2+\xi^2}\,du, \quad z \in \overline{\mathbb{D}},$$
$$R_\xi(f)(z) = \frac{2\xi^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{iu})}{(u^2+\xi^2)^2}\,du, \quad z \in \overline{\mathbb{D}},$$
$$W_\xi(f)(z) = \frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^{\pi} f(ze^{iu})e^{-u^2/\xi}\,du, \quad z \in \overline{\mathbb{D}},$$
$$W^*_\xi(f)(z) = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/\xi}\,du, \quad z \in \overline{\mathbb{D}},$$

are described as follows : $P_\xi(f)$ is of Picard type, $Q_\xi(f)$, $Q^*_\xi(f)$, and $R_\xi(f)$ are of Poisson–Cauchy type and $W_\xi(f)$, $W^*_\xi(f)$ are of Gauss–Weierstrass type.

The approximation and some geometric properties of these operators can be summarized by the next result. We keep the notation of Section 3.3 for the sets $S_1, S_2, S_3, S_4, S_M, \mathcal{P}, \mathcal{R}, \mathcal{S}^*(\mathbb{D}), \mathcal{K}(\mathbb{D})$. Also, for $f \in A(\mathbb{D})$ we set $\omega_1(f;\delta)_{\infty,\overline{D}} = \sup\{|f(u) - f(v)|; u, v \in \overline{D}, |u - v| \le \delta\}$ and

$$\omega_2(f;\delta)_{\infty,\partial D} = \sup\{|f(ze^{-iu}) - 2f(z) + f(ze^{iu})|; |u| \le \delta, |z| = 1\}.$$

Theorem 5.4.2. (Anastassiou–Gal [13]) *(1) For all $z \in \overline{\mathbb{D}}$ and $\xi > 0$ we have*

$$|P_\xi(f)(z) - f(z)| \le C\omega_2(f;\xi)_{\infty,\partial\mathbb{D}},$$

$$P_\xi(S_2) \subset S_2 \quad \text{and} \quad P_\xi(\mathcal{P}) \subset \mathcal{P},$$

$(1+\xi^2)P_\xi(S_1) \subset S_1$, $(1+\xi^2)P_\xi(S_M) \subset S_{M(1+\xi^2)}$ *and* $(1+\xi^2)P_\xi(S^*_{3,\xi}) \subset S_3$, *where*

$$S^*_{3,\xi} = \left\{f \in S_3;\ |f''(z)| \le \frac{1}{1+\xi^2},\ \forall z \in \mathbb{D}\right\} \subset S_3.$$

Also, $P_\xi\big(S^*_{3,1}\big) \subset S_3$ *and* $f \in S_M$ *implies that* $P_\xi(f)$ *is univalent in* $\big\{z \in \mathbb{C};\ |z| < \frac{1}{2M}\big\}$ *for all* $\xi \in (0,1]$.

(2)

$$|Q_\xi(f)(z) - f(z)| \le C\frac{\omega_2(f;\xi)_{\infty,\partial\mathbb{D}}}{\xi}, \quad \forall x \in \overline{\mathbb{D}},\ \xi \in (0,1],$$
$$|Q^*_\xi(f)(z) - f(z)| \le C\frac{\omega_2(f;\xi)_{\infty,\partial\mathbb{D}}}{\xi}, \quad \forall z \in \overline{\mathbb{D}},\ \xi \in (0,1],$$
$$|R_\xi(f)(z) - f(z)| \le C\omega_1(f;\xi)_{\infty,\overline{\mathbb{D}}}, \quad \forall z \in \overline{\mathbb{D}},\ \xi \in (0,1];$$

$Q^*_\xi(\mathcal{P}) \subset \mathcal{P}$, $R_\xi(\mathcal{P}) \subset \mathcal{P}$;

$$\frac{1}{b_1(\xi)} \cdot Q_\xi(S_{3,b_1(\xi)}) \subset S_3, \quad \frac{1}{b^*_1(\xi)} \cdot Q^*_\xi(S_{3,b^*_1(\xi)}) \subset S_3,$$
$$\frac{1}{c_1(\xi)} \cdot R_\xi(S_{3,c_1(\xi)}) \subset S_3, \quad \frac{1}{b_1(\xi)}Q_\xi(S_M) \subset S_{M/|b_1(\xi)|},$$
$$\frac{1}{b^*_1(\xi)}Q^*_\xi(S_M) \subset S_{M/|b^*_1(\xi)|}, \quad \frac{1}{c_1(\xi)}R_\xi(S_M) \subset S_{M/|c_1(\xi)|},$$

where $S_{3,a} = \{f \in S_3; |f''(z)| \le a\}$, $b_1(\xi) = \frac{2\xi}{\pi}\int_0^\pi \frac{\cos u}{u^2+\xi^2}\,du$, $b^*_1(\xi) = \frac{2\xi}{\pi}\int_0^{+\infty} \frac{\cos u}{u^2+\xi^2}\,du$, *and* $c_1(\xi) = \frac{4\xi^3}{\pi}\int_0^\infty \frac{\cos u}{(u^2+\xi^2)^2}\,du$.

Also, if $f \in S_{3,\frac{1}{e}}$, *then* $Q^*_\xi(f) \in S_3$ *for all* $\xi \in (0,1]$, *and if* $f \in S_M$ $(M > 1)$, *then* $Q^*_\xi(f)$ *is univalent in* $\big\{z \in \mathbb{C};\ |z| < \frac{1}{eM}\big\}$, *for all* $\xi \in (0,1]$.

If $f \in S_{3,\frac{2}{e}}$, *then* $R_\xi(f) \in S_3$ *for all* $\xi \in (0,1]$, *and if* $f \in S_M$, *then* $R_\xi(f)$ *is univalent in* $\big\{|z| < \frac{2}{eM}\big\}$, *for all* $\xi \in (0,1]$.

(3)

$$|W_\xi(f)(z) - f(z)| \le C\frac{\omega_2(f;\xi)_{\infty,\partial\mathbb{D}}}{\xi}, \quad z \in \overline{\mathbb{D}},\ \xi \in (0,1],$$
$$|W^*_\xi(f)(z) - f(z)| \le C\omega_1(f;\sqrt{\xi})_{\infty,\overline{\mathbb{D}}}, \quad z \in \overline{\mathbb{D}},\ \xi \in (0,1],$$
$$W_\xi(K(\mathbb{D})) \subset K(\mathbb{D}), W_\xi(S^*(\mathbb{D})) \subset S^*(\mathbb{D}),$$
$$W^*_\xi(\mathcal{P}) \subset \mathcal{P}, \quad \frac{1}{d_1(\xi)}W_\xi(S_{3,d_1(\xi)}) \subset S_3,$$
$$\frac{1}{d^*_1(\xi)}W^*_\xi(S_{3,d^*_1(\xi)}) \subset S_3, \quad \frac{1}{d_1(\xi)}W_\xi(S_M) \subset S_{M/|d_1(\xi)|},$$
$$\frac{1}{d^*_1(\xi)}W^*_\xi(S_M) \subset S_{M/|d^*_1(\xi)|},$$

for any $\xi > 0$, *where* $d_1(\xi) = \frac{1}{\sqrt{\pi\xi}} \cdot \int_{-\pi}^{\pi} e^{-u^2/\xi}\cos u\,du$ *and* $d^*_1(\xi) = e^{-\xi/4}$.

If $f \in S_{3,\frac{1}{e^{1/4}}}$ *then* $W^*_\xi(f) \in S_3$ *for all* $\xi \in (0,1]$, *and if* $f \in S_M$ $(M > 1)$, *then* $W^*_\xi(f)$ *is univalent in* $\big\{z \in \mathbb{C};\ |z| < \frac{1}{Me^{1/4}}\big\}$, *for all* $\xi \in (0,1]$.

(4) Let us define

$$S^*_{4,\xi} = \left\{ f \in S_3;\ |f''(z)| \leq \frac{1}{2(1+\xi^2)},\ \forall z \in \mathbb{D} \right\}$$

and

$$S_{4,a} = \left\{ f \in S_4; |f''(z)| \leq \frac{|a|}{2}, \forall z \in \mathbb{D} \right\}.$$

We have

$(1+\xi^2)P_\xi(S^*_{4,\xi}) \subset S_4$ *for all* $\xi > 0$ *and* $P_\xi\big(S^*_{4,1}\big) \subset S_4$ *for all* $\xi \in (0,1]$*;*

$\frac{1}{b_1(\xi)} \cdot Q_\xi(S_{4,b_1(\xi)}) \subset S_4$, $\frac{1}{b_1^*(\xi)} \cdot Q^*_\xi(S_{4,b_1^*(\xi)}) \subset S_4$ *and* $\frac{1}{c_1(\xi)} \cdot R_\xi(S_{4,c_1(\xi)}) \subset S_4$ *for all* $\xi > 0$*, and if* $f \in S_{4,\frac{1}{e}}$*, then* $Q^*_\xi(f) \in S_4$*; if* $f \in S_{4,\frac{2}{e}}$*, then* $R_\xi(f) \in S_4$ *for all* $\xi \in (0,1]$*;*

$W_\xi(S_{4,d_1(\xi)}) \subset S_4$, $\frac{1}{d_1^*(\xi)} W^*_\xi(S_{4,d_1^*(\xi)}) \subset S_4$ *for all* $\xi > 0$*, and if* $f \in S_{4,\frac{1}{e^{1/4}}}$*, then* $W^*_\xi(f) \in S_4$ *for all* $\xi \in (0,1]$.

Remarks. (1) The nice properties

$$W_\xi(K(\mathbb{D})) \subset K(\mathbb{D}),\ W_\xi(S^*(\mathbb{D})) \subset S^*(\mathbb{D}),$$

are direct consequences of Theorem 3.1.5, while the others are proved by some calculations.

(2) Let us mention here, without details, that approximation and geometric preservation properties of other nonpolynomial complex convolution operators were studied recently, as follows: the Jackson-type generalized singular complex integrals of those in Definition 5.4.1 in Anastassiou–Gal [11], the complex Post–Widder operator in Anastassiou–Gal [12], the complex rotation-invariant integral operators in Anastassiou–Gal [14] and a complex spline operator in Anastassiou–Gal [15].

In what follows, we present two simple applications of the complex Poisson–Cauchy and Gauss–Weierstrass convolution integrals to complex partial differential equations. Let us recall the Cauchy problem for the following two classical evolution equations in real variables:

(i) the one dimensional heat equation, given by

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), t \geq 0,\ u(0,x) = f(x),\ x \in \mathbb{R},$$

where $f \in \mathrm{BUC}(\mathbb{R})$ the space of all bounded uniformly continuous functions on $\mathbb{R}$. It is well known that its unique bounded solution is given by the semigroup of linear operators given by

$$T(t)(f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(x-y) e^{-y^2/(2t)} dy$$

(see, e.g., Goldstein [150], p. 23).

(ii) the case of the Laplace equation, given by

$$\frac{\partial^2 u}{\partial t^2}(t,x)+\frac{\partial^2 u}{\partial x^2}(t,x)=0, t\geq 0,\ u(0,x)=f(x),\ x\in\mathbb{R},$$

where $f\in \mathrm{BUC}(\mathbb{R})$ has its unique bounded solution given by the semigroup of linear operators given by

$$S(t)(f)(x)=\frac{t}{\pi}\int_{-\infty}^{+\infty}\frac{f(x-y)}{t^2+y^2}dy$$

(see, e.g., Goldstein [150], p. 23).

Recall that if $(X, \|\cdot\|_X)$ is a Banach space, then $T(t): X\to X,\ t\geq 0$, is said to be a strongly continuous semigroup of linear operators on X if it satisfies the following conditions (see, e.g., Goldstein [150], p. 5):

(1) $T(t+s)(f)=T(t)[T(s)(f)]$ for all $t,s\geq 0$, $f\in X$.

(2) $T(0)(f)=f$ (or more general, $\lim_{t\searrow 0}T(t)(f)=f$) for all $f\in X$.

(3) For any fixed $f\in X$, as a function of t, $T(t)(f)$ is continuous on $\mathbb{R}_+$.

(4) $|||T(t)|||_X=\sup_{\|f\|_X\leq 1}\{\|T(t)(f)\|_X\}<+\infty$ for all $t\geq 0$.

If $|||T(t)|||_X\leq 1$, then we call it a strong contraction semigroup of linear operators.

(5) The (infinitesimal) generator of T is the operator A defined by the formula

$$A(f)=\lim_{t\searrow 0}\frac{T(t)(f)-f}{t}$$

for all $f\in X$ where this limit exists.

Now, it is natural to ask what happens if we "complexify" the corresponding semigroups of operators for the heat and Laplace equations, i.e., what partial differential equations correspond to the complexified semigroups of operators?

First we complexify the semigroup $(T(t), t\geq 0)$ attached to the above real heat equation, as follows. Let $f\in A(\mathbb{D})$. We have $f(z)=\sum_{k=0}^{\infty}a_k z^k$ for all $z\in\mathbb{D}$ and also $\|f\|=\sup\{|f(z)|; z\in\overline{\mathbb{D}}\}$ is a norm. It is well known that $(A(\mathbb{D}), \|\cdot\|)$ is a Banach space.

For $t\in\mathbb{R}$, $t>0$, let us consider the complex singular integral

$$W_t^*(f)(z)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}f(ze^{-iu})e^{-u^2/(2t)}\,du,\quad z\in\overline{\mathbb{D}}.$$

Evidently $W_t^*(f)(z)$ is the singular integral of Gauss–Weierstrass type in Definition 5.4.1 (i).

The first aim is to show that the above-defined complex singular integral defines a strong contraction semigroup on $A(\mathbb{D})$, its infinitesimal generator is calculated, and applications to some Cauchy problems are obtained.

We present below without proofs the main results.

Theorem 5.4.3. (Gal–Gal–Goldstein [138]) *(i)*

$$|W_t^*(f)(z) - W_s^*(f)(z)| \le C_s|t-s|, \ \forall z \in \overline{D}, \ t \in V_s \subset (0, +\infty),$$

where $C_s > 0$ is a constant independent of z, t, and f, and V_s is a neighborhood of s.

(ii)

$$\|W_t^*(f)\| \le \|f\|, \forall t > 0, \ f \in A(\mathbb{D}).$$

(iii) $W_{t+s}^*(f)(z) = W_t^*[W_s^*(f)(z)]$ for all $t, s > 0$, $f \in A(\mathbb{D})$, $z \in \overline{\mathbb{D}}$.

Remark. An immediate consequence is that $(W_t^*, t \ge 0)$ is a contraction strongly continuous semigroup of linear operators on $A(\mathbb{D})$.

Concerning the generator, we have the following.

Theorem 5.4.4. (Gal–Gal–Goldstein [138]) *The generator A of the semigroup $(W_t^*, t \ge 0)$ is given by*

$$A(f)(z) = \frac{1}{2}\frac{\partial^2 f}{\partial \varphi^2}(z) = \frac{1}{2}\left[\frac{\partial^2}{\partial \varphi^2}[U(r\cos\varphi, r\sin\varphi)] + i\frac{\partial^2}{\partial \varphi^2}[V(r\cos\varphi, r\sin\varphi)]\right]$$

for any $f \in A(\mathbb{D})$, $f(z) = U(r\cos\varphi, r\sin\varphi) + iV(r\cos\varphi, r\sin\varphi)$, $z = re^{i\varphi} \in \mathbb{D}$, $z \ne 0$, where φ is the principal value of the argument of z and $A(f)(0) = 0$.

An immediate consequence of Theorem 5.4.4 and of Theorem 5.4.2, point (3), is the following.

Theorem 5.4.5. (Gal–Gal–Goldstein [138]) *(i) The unique solution $u(t,z)$ $(t \ge 0, z \in \overline{\mathbb{D}})$ in $A(\mathbb{D})$ as a function of z of the Cauchy problem with complex spatial variable*

$$\frac{\partial u}{\partial t}(t,z) = \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}[u(t,z)], \ z = re^{i\varphi} \in \mathbb{D}, \ z \ne 0, \ \frac{\partial u}{\partial t}(t,0) = 0,$$
$$u(0,z) = f(z), \ z \in \overline{\mathbb{D}}, \ f \in A(\mathbb{D}),$$

is given by

$$u(t,z) = W_t^*(f)(z) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/(2t)}\,du.$$

(ii) As a function of z, the above solution $u(t,z)$ satisfies if $f \in S_{3,\frac{1}{e^{1/4}}}$, then $u(t,z) \in S_3$ for all $t \in (0,1]$, and if $f \in S_M$ $(M > 1)$, then $u(t,z)$ is univalent in $\left\{z \in \mathbb{C}; |z| < \frac{1}{Me^{1/4}}\right\}$ for all $t \in (0,1]$.

(iii) As a function of z, the above solution $u(t,z)$ satisfies if $f \in S_{4,\frac{1}{e^{1/4}}}$, then $u(t,z) \in S_4$ for all $t \in (0,1]$.

Remarks. (1) Theorem 5.4.5 (ii) and (iii) says that for all $t \in (0,1]$, the solution $u(t,z)$ preserves as a function of z the properties of the boundary function $f(z)$ in some subclasses of starlike, univalent, or convex functions.

(2) In fact, supposing that f is analytic in an open set G including $\overline{\mathbb{D}}$, for all $t \in (0, t_f]$ with sufficiently small $t_f > 0$ (depending on f), the solution $u(t,z) = W_t^*(f)(z)$ in Theorem 5.4.5 preserves as a function of z the starlikeness, convexity, and spirallikeness of the boundary function f (exactly as do the Bernstein polynomials in Theorem 3.4.2). This is immediate if we reason exactly as in the proof of Theorem 3.4.2, by replacing there the Bernstein polynomials $B_n(f)(z)$ with $W_t^*(f)(z)$, taking into account that by Theorem 4.1 in Anastassiou–Gal [13] we have that $S(f)(t,z) = e^{t/4} \cdot W_t^*(f)(z)$ satisfies $S(f)(t,0) = f(0)$, $S'(f)(t,0) = f'(0)$ and that by Theorem 5.4.2 (3), we obviously have $\lim_{t \to 0} W_t^*(f)(z) = f(z)$, uniformly with respect to $z \in \overline{\mathbb{D}}$ (note that in fact one can easily prove that $\lim_{t \to 0} W_t^*(f)(z) = f(z)$, uniformly with respect to $z \in K$, for any compact disk $K \subset G$).

In what follows we complexify the real semigroup of operators $(S(t), t \geq 0)$ attached to the real Laplace equation. We obtain the complex operator

$$Q_t^*(f)(z) = \frac{t}{\pi} \int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{u^2 + t^2}\, du, \quad z \in \overline{\mathbb{D}},$$

called a singular integral complex operator of Poisson–Cauchy type.

Theorem 5.4.6. (Gal–Gal–Goldstein [138]) *We have (i)* $Q_{t+s}^*(f)(z) = Q_t^*[Q_s^*(f)(z)],\ \forall t, s > 0,\ f \in A(\mathbb{D}),\ z \in \overline{\mathbb{D}}$.

(ii) $|Q_t^*(f)(z) - Q_s^*(f)(z)| \leq C_s|t - s|,\ \forall z \in \overline{\mathbb{D}},\ t \in V_s \subset (0, +\infty)$, *where* $C_s > 0$ *is a constant independent of* z, t, *and* f, *and* V_s *is a neighborhood of (fixed)* s.

(iii)

$$\|Q_t^*(f)\| \leq \|f\|,\ \forall\ t > 0,\ f \in A(\mathbb{D}).$$

(iv) $\lim_{t \searrow 0} Q_t^*(f) = f$, *for any* $f \in A(\mathbb{D})$.

Remark. From this theorem it immediately follows that $(Q_t^*, t \geq 0)$ is a contraction strongly continuous semigroup of linear operators on $A(\mathbb{D})$.

We have the following application.

Theorem 5.4.7. (Gal–Gal–Goldstein [138]) *(i) The unique solution, denoted by* $v(t,z)$ $(t \geq 0, z \in \overline{\mathbb{D}})$, *in* $A(\mathbb{D})$ *as a function of* z *of the Cauchy problem with complex spatial variable*

$$\frac{\partial^2}{\partial t^2}[v(t,z)] + \frac{\partial^2}{\partial \varphi^2}[v(t,z)] = 0,\ z = re^{i\varphi} \in \mathbb{D},\ z \neq 0,\ \frac{\partial^2 u}{\partial t^2}(t,0) = 0,$$

$$v(0,z) = f(z),\ z \in \overline{D},\ f \in A(\mathbb{D}),$$

is given by

$$v(t,z)=Q_t^*(f)(z)=\frac{t}{\pi}\int_{-\infty}^{+\infty}f(ze^{-iu})\frac{du}{t^2+u^2}.$$

(ii) As a function of z, $v(t,z)$ *satisfies if* $f\in S_{3,\frac{1}{e}}$, *then* $v(t,z)\in S_3$ *for all* $t\in(0,1]$, *and if* $f\in S_M(M>1)$, *then* $v(t,z)$ *is univalent in* $\{z\in\mathbb{C};\ |z|<\frac{1}{eM}\}$ *for all* $t\in(0,1]$.

(iii) As a function of z, *the above solution* $v(t,z)$ *satisfies if* $f\in S_{4,\frac{1}{e}}$, *then* $v(t,z)\in S_4$ *for all* $t\in(0,1]$.

Remarks. (1) The proofs of Theorems 5.4.3–5.4.7 and other details can be found in Gal–Gal–Goldstein [138].

(2) Theorem 5.4.7 (ii) and (iii) says that for all $t\in(0,1)$, the solution $v(t,z)$ preserves as a function of z the properties of the boundary function $f(z)$ in some subclasses of starlike, univalent, or convex functions.

(3) In fact, supposing that f is analytic in an open set G including $\overline{\mathbb{D}}$ for all $t\in(0,t_f]$ with sufficiently small $t_f>0$ (depending on f), the solution $v(t,z)=Q_t^*(f)(z)$ in Theorem 5.4.7 preserves as a function of z the starlikeness, convexity, and spirallikeness of the boundary function f (exactly as do the Bernstein polynomials in Theorem 3.4.2). This is immediate if we reason exactly as in the proof of Theorem 3.4.2, by replacing there the Bernstein polynomials $B_n(f)(z)$ with $Q_t^*(f)(z)$, taking into account that by Theorem 3.1 in Anastassiou–Gal [13] we have that $S(f)(t,z)=\frac{1}{b_1^*(t)}Q_t^*(f)(z)$ satisfies $S(f)(t,0)=f(0)$, $S'(f)(t,0)=f'(0)$ and that by Theorem 5.4.2 (2), it follows that $\lim_{t\to 0}Q_t^*(f)(z)=f(z)$, uniformly with respect to $z\in\overline{\mathbb{D}}$, since there we have $\omega_2(f;t)/t\le Ct\|f''\|_{\overline{\mathbb{D}}}$ (note that in fact one can easily prove that $\lim_{t\to 0}Q_t^*(f)(z)=f(z)$, uniformly with respect to $z\in K$, for any compact disk $K\subset G$).

5.5 Bibliographical Notes and Open Problems

Note 5.5.1. Theorem 5.3.5, Remark 2 after Theorem 5.4.5, and Remark 3 after Theorem 5.4.7 are new.

Note 5.5.2. An important concept in approximation theory is that of width of a subset in a Banach space, introduced by Kolmogorov [199] (see, e.g., for details, Lorentz [247], Chapter 9, pp. 132–149). In a series of papers, Konovalov–Leviatan [200, 201] and Gilewicz–Konovalov–Leviatan [148] introduce the concept of shape-preserving width of a weighted Sobolev space by

$$d_n(\Delta_+^s W_{p,\alpha}^r,\Delta_+^s L_q)_{L_q}:=\inf_{M^n\in\mathcal{M}^n}\sup_{x\in\Delta_+^s W_{p,\alpha}}\inf_{y\in M^n\cap\Delta_+^s L_q}\|x-y\|_{L_q},$$

where $\mathcal{M}^n$ is the set of all linear manifolds M^n in L_q with $\dim(M^n)\le n$ satisfying $M^n\cap\Delta_+^s L_q\ne\emptyset$, $W_{p,\alpha}^r$ denotes a class of Sobolev type on $[-1,1]$,

$\Delta^s_+ W^r_{p,\alpha} := \{x \in W^r_{p,\alpha}; \Delta^s_+ x(t) \geq 0\}$, and $\Delta^s_+ L_q := \{x \in L_q[-1,1]; \Delta^s_+ x(t) \geq 0\}$, $1 \leq p, q \leq \infty$, $0 \leq \alpha < \infty$, $r, n \in \mathbb{N}$, $s \in \mathbb{N}_0$.

Two-sided estimates (exact orders) of $d_n(\Delta^s_+ W^r_{p,\alpha}, \Delta^s_+ L_q)_{L_q}$ were obtained in Konovalov–Leviatan [200] for $s = 0, 1, 2$, in Konovalov–Leviatan [201] for $3 \leq s \leq r+1$, and in Gilewicz–Konovalov–Leviatan [148] for $s > r+1$.

Note 5.5.3. It is worth mentioning other interesting topics that refer to (local) saturation of k-convex linear operators (that is, linear operators preserving k-convexity); for details see, e.g., Cárdenas-Morales and Garrancho [57–59], and to the shape-preserving properties of some linear Bernstein-type operators that fix polynomials; see, e.g., Cárdenas-Morales, Muñoz-Delgado, and Garrancho [60].

Open Problem 5.5.4. In what follows, we would like to bring attention a possible interesting new direction in constructive approximation of functions by operators. Thus, in two very recent papers (Bede–Nobuhara–Fodor–Hirota [35] and Bede–Nobuhara–Dankova–Di Nola [36]), the authors consider a new idea for construction of nonlinear nonpolynomial operators by replacing the usual pair of operations (sum, product) by other pairs suggested in fuzzy set theory and image processing. Then, they apply the idea to the Shepard-type interpolation operator and give interesting concrete applications to image-processing experiments. Let us briefly describe their theoretical results.

First, starting form the classical linear and positive Shepard operator attached to a function $f : [0,1] \to \mathbb{R}$ and to equidistant nodes, given by

$$S_{n,\lambda}(f)(x) = \sum_{k=0}^{n} f(k/n) \cdot \frac{|x - k/n|^{-\lambda}}{\sum_{k=0}^{n} |x - k/n|^{-\lambda}},$$

where $\lambda \geq 1$, $n \in \mathbb{N}$, they replace “sum” by “max,” so that they obtain for the pair (max, product), the following Shepard-type nonlinear operator (see Bede–Nobuhara–Fodor–Hirota [35]):

$$S^{(M)}_{n,\lambda}(f)(x) = \frac{\max_{0 \leq k \leq n}\{f(k/n) \cdot |x - k/n|^{-\lambda}\}}{\max_{0 \leq k \leq n}\{|x - k/n|^{-\lambda}\}}.$$

In the same paper, for positive f, the authors prove the following Jackson-type estimate:

$$|S^{(M)}_{n,\lambda}(f)(x) - f(x)| \leq \frac{3}{2}\omega_1(f; 1/n)_\infty,$$

valid for all $x \in [0,1]$, $n \in \mathbb{N}$.

Comparing with the estimates given by the classical Shepard operator in Szabados [380], we note that for $1 \leq \lambda \leq 2$, the operator $S^{(M)}_{n,\lambda}(f)$ gives essentially better estimates.

Second, in Bede–Nobuhara–Dañkova–Di Nola [36], replacing the usual pair (sum, product) by the pair (max, min), the authors define the following Shepard-type nonlinear operator:

$$S^{(M-m)}_{n,\lambda}(f)(x) = \max_{0 \leq k \leq n} \frac{\min\{f(k/n), |x - k/n|^{-\lambda}\}}{\max_{0 \leq k \leq n}\{|x - k/n|^{-\lambda}\}},$$

for which they prove the estimate (much worse than that in the case of $S_{n,\lambda}^{(M)}(f)(x)$)

$$|S_{n,\lambda}^{(M-m)}(f)(x) - f(x)| \leq \max\{\omega_1(f;1/n)_\infty, 1/3^\lambda\},$$

valid for all $f : [0,1] \to [0,1],\ x \in [0,1],\ n \in \mathbb{N}$.

It is then natural to ask how the above idea could be applied to other linear and positive operators. In what follows, we sketch this for the first time for some Bernstein-type operators.

Thus, if in the classical Bernstein polynomials $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot f(k/n)$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, we use the pair (max, product), then we get

$$P_n(f)(x) = \max_{0 \leq k \leq n} \{p_{n,k}(x) f(k/n)\},$$

while for the pair (max, min), we get

$$Q_n(f)(x) = \max_{0 \leq k \leq n} \{\min\{p_{n,k}(x), f(k/n)\}\}.$$

Unfortunately, the approximation properties of these two nonlinear operators seem to be very bad, because it is easy to see that convergence does not hold even for constant functions.

In order to get operators with better approximation potential, we write the classical Bernstein polynomials in the form

$$B_n(f)(x) = \sum_{k=0}^{n} \frac{f(k/n) \cdot p_{n,k}(x)}{\sum_{k=0}^{n} p_{n,k}(x)}.$$

Then, applying the pair (max, product), we get

$$B_n^{(M)}(f)(x) = \frac{\max_{0 \leq k \leq n}\{f(k/n) \cdot p_{n,k}(x)\}}{\max_{0 \leq k \leq n}\{p_{n,k}(x)\}},$$

while for the pair (max, min), we obtain

$$B_n^{(M-m)}(f)(x) = \max_{0 \leq k \leq n} \frac{\min\{f(k/n), p_{n,k}(x)\}}{\max_{0 \leq k \leq n}\{p_{n,k}(x)\}}.$$

Note that each $B_n^{(M)}(f)(x)$ is a piecewise rational function of degree $\leq n$.

In the case of $B_n^{(M)}(f)(x)$, with f positive, by taking into account the inequality (valid for the positive numbers $A_i, B_i, i = 0, \ldots, n$)

$$|\max_{0 \leq i \leq n}\{A_i\} - \max_{0 \leq i \leq n}\{B_i\}| \leq \max_{0 \leq i \leq n}\{|A_i - B_i|\}$$

and

$$\begin{aligned} &|f(k/n) - f(x)| p_{n,k}(x) \\ &\quad \leq p_{n,k}(x)\omega_1(f; |k/n - x|)_\infty \leq p_{n,k}(x)[1 + |k/n - x|/\delta_n]\omega_1(f;\delta_n)_\infty \end{aligned}$$

(valid for any $\delta_n > 0$), we easily can deduce

$$
\begin{aligned}
&|B_n^{(M)}(f)(x) - f(x)| \\
&\quad \le \frac{|\max_{0\le k\le n}\{f(k/n)\cdot p_{n,k}(x)\} - \max_{0\le k\le n}\{f(x)\cdot p_{n,k}(x)\}|}{\max_{0\le k\le n}\{p_{n,k}(x)\}} \\
&\quad \le \frac{\max_{0\le k\le n}\{|f(k/n) - f(x)|\cdot p_{n,k}(x)\}}{\max_{0\le k\le n}\{p_{n,k}(x)\}} \\
&\quad \le \left(1 + \frac{1}{\delta_n}\cdot\frac{\max_{0\le k\le n}\{p_{n,k}(x)|k/n - x|\}}{\max_{0\le k\le n}\{p_{n,k}(x)\}}\right)\omega_1(f;\delta_n)_\infty.
\end{aligned}
$$

Therefore, the convergence property of $B_n^{(M)}(f)(x)$ is controlled by the ratio $r_n(x) = \max_{0\le k\le n}\{p_{n,k}(x)|k/n - x|\}/\max_{0\le k\le n}\{p_{n,k}(x)\}$.

It is left to the reader to see whether this ratio $r_n(x)$ is at least of order $O(1/\sqrt{n})$ (as we would expect). Of course, a better order would imply an essentially better approximation order of $B_n^{(M)}(f)(x)$ than that given by the classical Bernstein operator $B_n(f)(x)$.

Also, we could consider shape-preserving properties for $B_n^{(M)}(f)(x)$. In this sense, first we would need some suitable concepts of shapes in accordance with the operator's form. For example, we could consider the max-convexity property of f defined by the inequality (valid for all $x, y \in [0,1]$, $\alpha \in [0,1]$)

$$
f[\max\{\alpha x, (1-\alpha)y\}] \le \max\{\alpha f(x), (1-\alpha)f(y)\},
$$

and as an open question we could ask whether the Bernstein-type operator $B_n^{(M)}(f)(x)$ keeps this property.

Replacing the pair of operations (sum, product) with (max, product) in the definitions of classical operators in Section 5.2, Examples 2, 3, 4, and 7, we can obtain the nonlinear variants of Favard–Szasz–Mirakjan, Baskakov, Meyer–König–Zeller, and Bleimann–Butzer–Hahn operators, given by

$$
\begin{aligned}
F_n^{(M)}(f)(x) &= \frac{\sup_{k\in\mathbb{N}_0}\{f(k/n)\cdot(nx)^k/k!\}}{\sup_{k\in\mathbb{N}_0}\{(nx)^k/k!\}}, \\
V_n^{(M)}(f)(x) &= \frac{\sup_{k\in\mathbb{N}_0}\{f(k/n)\cdot\binom{n+k-1}{k}x^k/(1+x)^{n+k}\}}{\sup_{k\in\mathbb{N}_0}\{\binom{n+k-1}{k}x^k/(1+x)^{n+k}\}}, \\
Z_n^{(M)}(f)(x) &= \frac{\sup_{k\in\mathbb{N}_0}\{\binom{n+k}{k}x^k f[k/(n+k)]\}}{\sup_{k\in\mathbb{N}_0}\{\binom{n+k}{k}x^k\}}, \\
BBH_n^{(M)}(f)(x) &= \frac{\sup_{0\le k\le n}\{f[k/(n+1-k)]\cdot q_{n,k}(x)\}}{\sup_{0\le k\le n}\{q_{n,k}(x)\}},
\end{aligned}
$$

respectively, where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $q_{n,k}(x) = \binom{n}{k}x^k$.

Remaining as open questions are the approximation and shape-preserving properties for these nonlinear operators too.

Open Problem 5.5.5. Find geometric (and approximation) properties for the complex versions (that is, simply replacing the real variable x by the complex variable z) of the following classical nonpolynomial Bernstein-type operators: Baskakov, Meyer–König–Zeller, Jakimovski–Leviatan, gamma, Bleimann–Butzer–Hahn, Bernstein–Kantorovich, and so on.

The methods in the proofs of Theorem 3.4.2 and Theorems 3.4.4–3.4.10 (in the case of shape preservation) and the considerations in Theorem 3.4.1 (in the case of approximation properties) could be useful for most of them.

Note, for example, that in Wood [400] is proved the uniform convergence of the complex form for the generalized Bernstein operator of Jakimovski–Leviatan. However, the estimate of the approximation errors and the shape-preserving properties for this complex operator still remain to be studied.

Open Problem 5.5.6. In the very recent book Ban–Gal [29], pp. 193–200, for functions that are not monotone (or convex, respectively), the degree (and its complementary concept, the defect) of monotonicity (or of convexity, respectively) is introduced. Reasoning exactly as in Ban–Gal [29], Open Question 5.2, pp. 208–209, it is natural to ask how the degree of monotonicity (or convexity, respectively) of a function is preserved by the most important Bernstein-type operators.

Open Problem 5.5.7. If f is univalent (starlike, convex) in $\mathbb{D}$, then find the radius of univalence (starlikeness, convexity, respectively) independent of $\xi > 0$ and f for all $Q_{\xi}^{*}(f)z)$ and $W_{\xi}^{*}(f)(z)$ in Definition 5.4.1 and their partial sums.

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