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S.A. Grigoryan
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Shift-invariant Uniform Algebras on Groups

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Preface

Shift-invariant algebras are uniform algebras of continuous functions defined on compact connected groups, that are invariant under shifts by group elements. They are outgrowths of generalized analytic functions, introduced almost fifty years ago by Arens and Singer, and are the central object of this book. Associated algebras of almost periodic functions of real variables and of bounded analytic functions on the unit disc are also considered and carried along within the shift-invariant framework. The adopted general approach leads to non-standard perspectives, never-asked-before questions, and unexpected properties.

The book is based mainly on our quite recent, some even unpublished, results. Most of its basic notions and ideas originate in [T2]. Their further development, however, can be found in journal or preprint form only.

Basic terminology and standard properties of uniform algebras are presented in Chapter 1. Associated algebras, such as Bourgain algebras, polynomial extensions, and inductive limit algebras are introduced and discussed. At the end of the chapter we present recently found conditions for a mapping between uniform algebras to be an algebraic isomorphism. In Chapter 2 we give fundamentals, various descriptions and standard properties of three classical families of functions – almost periodic functions of real variables, harmonic functions, and H^p-functions on the unit circle. Later on, in Chapter 7, we return to some of these families and extend them to arbitrary compact groups. Chapter 3 is a survey of basic properties of topological groups, their characters, dual groups, functions and measures on them. We introduce also the instrumental for the sequel notion of weak and strong hull of a semigroup.

Chapter 4 is devoted to shift-invariant algebras. We describe the spaces of automorphisms and of peak subgroups of shift-invariant algebras, and show that the algebraic properties of the generating semigroup S have a significant impact on the properties of the associated shift-invariant algebra A_S. For example, whether analogues of the classical Radó's theorem for null-sets of analytic functions, and of Riemann's theorem for removable singularities hold in a shift-invariant algebra A_S depends on specific algebraic properties of the generating semigroup S. Asymptotically almost periodic functions on $\mathbb{R}$, which share many properties with almost periodic functions, are introduced at the end of the chapter. Extendability of linear multiplicative functionals from smaller to larger shift-invariant algebras is the focal point of Chapter 5. The subject is naturally related with the extendability of non-negative semicharacters from smaller to larger semigroups and, equivalently, of their logarithms, called also additive weights. We give necessary and sufficient conditions for extendability of individual weights, as well as of the entire family of weights on a semigroup. These conditions imply various corona-type theorems. For instance, if S is a semigroup of $\mathbb{R}$ containing the origin, then the algebra of almost periodic functions in one real variable with spectrum in S does not have a $\mathbb{C}_+$-corona if and only if all non-negative semicharacters on S are monotone decreasing, or equivalently, if and only if the strong hull of S coincides with the positive half of the group envelope of S. On the other hand, the same conditions imply necessary and sufficient conditions for the related subalgebra of bounded analytic functions on the unit disc $\mathbb{D}$ to possess a $\mathbb{C}_+$-corona and a $\mathbb{D}$-corona. In Chapter 6 we discuss big disc algebras of generalized analytic functions on a compact abelian group G, an important class of shift-invariant algebras, also known as G-disc algebras. We describe their Bourgain algebras, orthogonal measures and primary ideals.

In Chapter 7 we extend the notion of harmonic and H^p-functions to compact abelian groups, and present corresponding Fatou-type theorems. In Chapter 8 we utilize inductive limits of classical algebras to study and generalize shift-invariant algebras on G-discs. In particular, we show that any sequence Φ of inner functions on the unit disc generates an inductive limit algebra, $H^\infty(\mathcal{D}_\Phi)$, of so called Φ-hyper-analytic functions on the associated big disc $\mathcal{D}_\Phi$. They are generalizations of hyper-analytic functions from [T], and similarly to them do not have a G-disc-corona, i.e. there exists a standard dense embedding of the big disc $\mathcal{D}_\Phi$ into the maximal ideal space of $H^\infty(\mathcal{D}_\Phi)$. We introduce also the class of Blaschke algebras, which are inductive limits of sequences of disc algebras connected with finite Blaschke products.

The selection of topics depended entirely on our own research interests. Many other related topics could not be included, or even mentioned. All chapters are provided with historical notes, references, brief remarks, comments, and unsolved problems. We do not necessarily claim credit for any uncited result. It may be an immediate consequence of previous assertions, or, part of the common mathematical knowledge, or, may have a history difficult to be traced.

The book is addressed primarily to those interested in analytic functions and commutative Banach algebras, though it could be useful to a wide range of research mathematicians and graduate students, familiar only with the fundamentals of complex and functional analysis.

Over the years our thinking in the area has been stimulated and encouraged by discussions and communication with several experts, among which we would like to mention Hugo Arizmendi, Richard Aron, Andrew Browder, Joseph Cima, Brian Cole, Joseph Diestel, Evgeniy Gorin, Farhad Jafari, Krzysztof Jarosz, Paul Muhly, Rao Nagisetty, Scott Saccone, Sadahiro Saeki, Anatoly Sherstnev, Andrzej Sołtysiak, Edgar Lee Stout, John Wermer, and Wiesław Żelazko. Special thanks are due to the participants – current and former – of the Analysis seminar at the University of Montana: Gregory St.George, Karel Stroethoff, Elena Toneva, George Votruba, and Keith Yale for their encouragement and support. We also mention with pleasure and gratitude the contribution of our students Tatyana Ponkrateva from Kazan State University, Aaron Luttman and John Case from the University of Montana, and especially Scott Lambert, who read the entire text and suggested many improvements.

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Missoula, Montana

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Chapter 1

Banach algebras and uniform algebras

In this chapter we present a part of the uniform algebra theory we will need, including several important algebraic constructions. Basic notations, terminology, and selected auxiliary results concerning commutative Banach algebras and uniform algebras are presented in the first two sections. The inductive and projective limits of algebras, introduced in more detail, are very convenient tools for describing the structure and revealing the hidden features of specific uniform algebras. Bourgain algebras and polynomial extensions provide powerful methods for constructing new classes of algebras. Further we discuss isomorphisms and homomorphisms between uniform algebras.

1.1 Commutative Banach algebras

A *Banach space* B over the field of complex numbers $\mathbb{C}$ is a linear space over $\mathbb{C}$ (thus, in B there are defined two operations — addition, and multiplication by complex scalars) which is provided with a *norm*, i.e. a non-negative function $\| \,.\, \|\colon\ B \longrightarrow \mathbb{R}_+ = [\,0, \infty)$ with the following properties:

(i) $\|\lambda a\| = |\lambda| \|a\|$ for each $a \in B$ and any complex scalar $\lambda \in \mathbb{C}$.

(ii) $\|a + b\| \leq \|a\| + \|b\|$ for each $a, b \in B$.

(iii) 0 is the only element in B whose norm is zero.

(iv) B is a *complete* space with respect to the topology generated by the norm $\| \,.\, \|$.

By completeness we mean that every Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ of elements in B is convergent.

A Banach space B over $\mathbb{C}$ is called a *Banach algebra*, if B is provided with an associative operation (called *multiplication*) which is distributive with respect to addition, and if the inequality

(v) $\|ab\| \leq \|a\|\|b\|$

holds for every $a, b \in B$. A Banach algebra is *commutative* if its multiplication is commutative, and *with unit* if it possesses a unit element with respect to multiplication (denoted usually by e, or, by 1) such that

(vi) $\|e\| = 1$.

Let B be a commutative Banach algebra with unit. An element f in B is said to be *invertible* if there exists a g in B such that $fg = e$. The element g with this property is uniquely defined. It is denoted by f^{-1} and is called the *inverse element* of f. Hence we have $f^{-1}f = e$ for any invertible element f in B. The set B^{-1} of all invertible elements of B under multiplication is a subgroup of B. A simple example of a commutative Banach algebra with unit is the set of complex numbers $\mathbb{C}$.

Proposition 1.1.1. *Let B be a commutative Banach algebra with unit e. Every element of the open unit ball centered at e is invertible, i.e.*

$$\{h \in B \colon \|h - e\| < 1\} \subset B^{-1}.$$

Proof. Let $\|f\| < 1$, and let $g_n = \sum_{k=0}^{n} f^k$, where $f^0 = e$. If $m < n$, then by (ii) and (v) from the above we have that

$$\begin{aligned}\|g_n - g_m\| &= \Big\| \sum_{k=m+1}^{n} f^k \Big\| \leq \sum_{k=m+1}^{n} \|f^k\| \leq \sum_{k=m+1}^{n} \|f\|^k \\ &= \frac{\|f\|^{m+1} - \|f\|^{n+1}}{1 - \|f\|} \leq \frac{\|f\|^{m+1}}{1 - \|f\|}.\end{aligned}$$

Hence for any $\varepsilon > 0$ and n, m big enough, we have $\|g_n - g_m\| < \varepsilon$, since by $\|f\| < 1$ we have $\lim_{k\to\infty} \|f^{k+1}\| \leq \lim_{k\to\infty} \|f\|^{k+1} = 0$. Thus, $\{g_n\}$ is a Cauchy sequence, and by the completeness of B it converges to an element $g \in B$, i.e. $g = \lim_{n\to\infty} g_n = \sum_{k=0}^{\infty} f^k$. In addition,

$$\begin{aligned}g\,(e - f) &= \Big(\sum_{n=0}^{\infty} f^n \Big)(e - f) = \Big(\lim_{k\to\infty} \sum_{n=0}^{k} f^n \Big)(e - f) \\ &= \lim_{k\to\infty} \sum_{n=0}^{k} (f^n - f^{n+1}) = \lim_{k\to\infty} (e - f^{k+1}) = e - \lim_{k\to\infty} f^{k+1} = e,\end{aligned}$$

since $\lim_{k\to\infty} \|f\|^{k+1} = 0$. Hence $e - f$ is an invertible element of B, as claimed. $\square$

Definition 1.1.2. The *spectrum* of an element f in a commutative Banach algebra B is the set

$$\sigma(f) = \{\lambda \in \mathbb{C}\colon \lambda e - f \notin B^{-1}\}. \tag{1.1}$$

Corollary 1.1.3. *The spectrum $\sigma(f)$ is contained in the disc $\overline{\mathbb{D}}(\|f\|) = \{z \in \mathbb{C}\colon |z| \leq \|f\|\}$ with radius $\|f\|$, centered at* 0.

Proof. Given an $f \in B$, let s be a complex number with $|s| > \|f\|$. Let $g = f/s = (1/s)f$. By the hypothesis $\|g\| = \|f\|/|s| < 1$. Proposition 1.1.1 implies that the element $e - g$ is invertible, and its inverse element is the sum of the convergent series $\sum_{n=0}^{\infty} g^n$. Thus

$$\begin{aligned} e &= (e-g)\sum_{n=0}^{\infty} g^n = (e - f/s)\sum_{n=0}^{\infty} f^n/s^n \\ &= \big((se-f)/s\big)\sum_{n=0}^{\infty} f^n/s^n = (se-f)\sum_{n=0}^{\infty} f^n/s^{n+1}. \end{aligned}$$

Hence $se - f$ is invertible in B. Therefore, $s \notin \sigma(f)$ whenever $|s| > \|f\|$. Consequently, $\sigma(f) \subset \overline{\mathbb{D}}(\|f\|)$, as claimed. □

Corollary 1.1.3 implies that the spectrum of any element f in B is a bounded set in $\mathbb{C}$, and therefore $\mathbb{C} \setminus \sigma(f) \neq \emptyset$. One can see that B^{-1} is an open subset of B, and the correspondence $f \longmapsto f^{-1}$ is a homeomorphism of B^{-1} onto itself. More precisely, B^{-1} is an open group (under multiplication) in B, and the mapping $f \longmapsto f^{-1}\colon B^{-1} \longrightarrow B^{-1}$ is a group automorphism. The spectrum $\sigma(f)$ is a closed and bounded set, thus a compact subset of $\mathbb{C}$. The number

$$r_f = \max\big\{|z|\colon z \in \sigma(f)\big\}$$

is called the *spectral radius* of $f \in B$. Since $r_f \leq \|f\|$, we have $\sigma(f) \subset \overline{\mathbb{D}}(r_f) \subset \overline{\mathbb{D}}(\|f\|)$. The spectral radius r_f can be expressed explicitly in terms of f (e.g.[G1, S4, T2]). Namely,

$$r_f = \lim_{n\to\infty} \sqrt[n]{\|f^n\|} \leq \lim_{n\to\infty} \sqrt[n]{\|f\|^n} = \|f\|. \tag{1.2}$$

Definition 1.1.4. The *peripheral spectrum* of an element f in a commutative Banach algebra B is the set

$$\sigma_\pi(f) = \big\{z \in \sigma(f) : |z| = r_f\big\} = \sigma(f) \cap \mathbb{T}_{r_f}. \tag{1.3}$$

Any commutative Banach algebra B with unit admits a natural representation by continuous functions on a compact topological space. An important role in this representation, as well as in commutative Banach algebra theory in general, is played by complex-valued homomorphisms, i.e. linear multiplicative functionals of the algebra. A *linear multiplicative functional* of B is called any non-zero complex-valued function φ on B with the following properties:

(i) $\varphi(\lambda a + \mu b) = \lambda \varphi(a) + \mu \varphi(b)$,

(ii) $\varphi(ab) = \varphi(a)\varphi(b)$

for every $a,\ b \in B$, and all scalars $\lambda, \mu \in \mathbb{C}$. The set $\mathcal{M}_B$ of all non-zero linear multiplicative functionals of B is called the *maximal ideal space* (or, the *spectrum*) of B.

For a fixed $a \in B$ with $\varphi(a) \neq 0$ we have $\varphi(a) = \varphi(ea) = \varphi(e)\varphi(a)$, thus $\varphi(a)\big(\varphi(e) - 1\big) = 0$. Consequently, $\varphi(e) = 1$ for every linear multiplicative functional φ of B. Since $aa^{-1} = e$ for every $a \in B^{-1}$, we have $1 = \varphi(e) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$, thus $\varphi(a) \neq 0$ for every invertible element $a \in B$.

Lemma 1.1.5. *Every linear multiplicative functional $\varphi \in \mathcal{M}_B$ is continuous on B, and $\|\varphi\| = 1$.*

Proof. Let $f \in B$, and let $|z| > \|f\|$ for some $z \in \mathbb{C}$. Hence, $ze - f \in B^{-1}$ by Corollary 1.1.3. According to the previous remark, $\varphi(ze - f) \neq 0$, and hence $\varphi(f) \neq z\varphi(e) = z$ for every $\varphi \in \mathcal{M}_B$. Consequently, the number $\varphi(f)$ belongs to the disc $\big\{z \in \mathbb{C}\colon |z| \leq \|f\|\big\}$, i.e. $\big|\varphi(f)\big| \leq \|f\|$, and this holds for every $f \in B$. Therefore, the functional φ is bounded, thus continuous, and $\|\varphi\| \leq 1$. By definition, $\|\varphi\|$ is the least number M with $\big|\varphi(f)\big| \leq M\|f\|$ for all $f \in B$. For any such M we have $M \geq 1$, since $1 = \big|\varphi(e)\big| \leq M\|e\| = M$. Hence, $\|\varphi\| \geq 1$, and therefore $\|\varphi\| = 1$. □

Example 1.1.6. (a) Let X be a compact Hausdorff set. The space $C(X)$ of all continuous functions on X under the pointwise operations and the uniform norm $\|f\| = \max\limits_{x \in X} \big|f(x)\big|$ is a commutative Banach algebra. One can easily identify some of the linear multiplicative functionals of $C(X)$. Namely, for a fixed $x \in X$ consider the functional "the point evaluation φ_x at x" in $C(X)$, i.e. $\varphi_x(f) = f(x)$ for every $f \in C(X)$. Clearly, $\varphi_x \in \mathcal{M}_{C(X)}$. Actually, one can show that every element in $\mathcal{M}_{C(X)}$ is of type φ_x for some $x \in X$. Consequently, $\mathcal{M}_{C(X)}$ and X are bijective spaces. We usually identify them as sets without mention, and write them as $\mathcal{M}_{C(X)} \cong X$.

(b) Let $\mathbb{D} = \mathbb{D}(1) = \big\{z\colon |z| < 1\big\}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $A(\mathbb{D})$ denote the space of continuous functions in the closed unit disc $\overline{\mathbb{D}} = \big\{z \in \mathbb{C}\colon |z| \leq 1\big\}$ that are analytic in $\mathbb{D}$. Equipped with pointwise operations and the uniform norm $\|f\| = \max\limits_{x \in \overline{\mathbb{D}}} \big|f(x)\big|$, $A(\mathbb{D})$ is a commutative Banach algebra, called the *disc algebra*. One can easily check that $\overline{\mathbb{D}} \subset \mathcal{M}_{A(\mathbb{D})}$. In fact, $\mathcal{M}_{A(\mathbb{D})} \cong \overline{\mathbb{D}}$.

A net $\{\varphi_\alpha\}$ of functionals in $\mathcal{M}_B$ is said to converge pointwise to an element $\varphi \in \mathcal{M}_B$ if $\varphi_\alpha(f) \longrightarrow \varphi(f)$ for every $f \in B$. The pointwise convergence generates a topology on the maximal ideal space $\mathcal{M}_B$ of B, called the *Gelfand topology.* With respect to it $\mathcal{M}_B$ is a closed subset of the unit sphere S_{B^*} of the space B^* dual to B. By the Banach-Alaoglu theorem, S_{B^*} is a compact space in the weak*-topology, which in this case coincides with the pointwise topology. Under it $\mathcal{M}_B$ is a closed subset of S_{B^*}, and therefore a compact and Hausdorff set.

Let f be an element in a commutative Banach algebra B. The *Gelfand transform* of f is called the function $\widehat{f}$ defined on $\mathcal{M}_B$ by

$$\widehat{f}(\varphi) = \varphi(f), \ \varphi \in \mathcal{M}_B. \tag{1.4}$$

The Gelfand transform $\widehat{f}$ of any $f \in B$ is a continuous function on $\mathcal{M}_B$ with respect to the Gelfand topology. Indeed, if $\varphi_\alpha \longrightarrow \varphi$ then $\varphi_\alpha(f) \longrightarrow \varphi(f)$, and therefore, $\widehat{f}(\varphi_\alpha) \longrightarrow \widehat{f}(\varphi)$. The *Gelfand transformation* $\Lambda\colon B \longrightarrow \widehat{B} \subset C(\mathcal{M}_B)$ is a homomorphism of B onto the Gelfand transform $\widehat{B} = \{\widehat{f}\colon f \in B\}$ of B. If $B = C(X)$, then $\widehat{f}(\varphi_{x_0}) = \varphi_{x_0}(f) = f(x_0)$ for every $x_0 \in X$. Hence, if we identify $\mathcal{M}_{C(X)}$ with X, as in Example 1.1.6(a), then $\widehat{f}$ coincides with f.

Observe that if $\mathcal{M}_B$ possesses a closed and open set K, then the *characteristic function* $\varkappa_K$ of K (i.e. $\varkappa_K(x) = 1$ for $x \in K$ and $\varkappa_K(x) = 0$ otherwise) belongs to $\widehat{B}$ by the famous *Shilov idempotent theorem* (see e.g. [GRS]), which asserts that under the hypotheses there exists a unique element $b \in B$ with $b^2 = b$ (i.e. b is an *idempotent* of the algebra B) whose Gelfand transform is precisely the characteristic function of K, i.e. $\widehat{b} = \varkappa_K$.

There is a good reason to call $\mathcal{M}_B$ the set of maximal ideals of B. A subset J of a commutative Banach algebra B is called an *ideal* of B, if J is a linear subset of B which is closed with respect to multiplication with elements in B, i.e. $ab \in J$ for any $a \in B$ and $b \in J$. Any ideal of an algebra is an algebra on it own. An ideal $J \subset B$ is *proper* if it differs from B, and *maximal*, if it is proper and every proper ideal of B containing J, equals J. By Zorn's Lemma, one can show that any proper ideal of B is contained in some maximal ideal of B (cf. [G1, S4, T2]).

The sets $\{0\}$, B and $aB = \{ab : b \in B\}$ for a fixed $a \in B$, are all ideals. If φ is a linear multiplicative functional of B, then the *null-set* of φ, $\mathrm{Null}\,(\varphi) = \big\{f \in B\colon \varphi(f) = 0\big\}$ is an ideal of B. Indeed, for every $a \in B$ and $b \in \mathrm{Null}\,(\varphi)$, $\varphi(ab) = \varphi(a)\,\varphi(b) = 0$, i.e. $ab \in \mathrm{Null}\,(\varphi)$. Since $\varphi(e) = 1$ we have that $e \notin \mathrm{Null}\,(\varphi)$, and therefore, $\mathrm{Null}\,(\varphi)$ is a proper ideal of B.

The unit e does not belong to any proper ideal J of B, since by assuming the opposite, i.e. $e \in J$, we get $a = ea \in J$ for all $a \in B$, thus $J = B$. The same argument applies to check that proper ideals J do not contain invertible elements of B, i.e. $B^{-1} \cap J = \emptyset$ for any proper ideal J of B. An ideal of B is proper if and only if a is an invertible element of B, since if $a \in B^{-1}$, then $e = a\,a^{-1} \in aB$, a contradiction.

One can easily see that the null-set $\mathrm{Null}\,(\varphi)$ of any linear multiplicative functional φ is a maximal ideal (e.g. [G1, S4, T2]). Actually, every maximal ideal M of B is of type $\mathrm{Null}\,(\varphi_M)$ for some $\varphi_M \in \mathcal{M}_B$, i.e. the set of maximal ideals of B is bijective to the family of null-sets of linear multiplicative functionals on B.

Proposition 1.1.7. *The spectrum of any element f of B coincides with the range of its Gelfand transform $\widehat{f}$, i.e.*

$$\sigma(f) = \widehat{f}(\mathcal{M}_B) = Ran\,(\widehat{f}). \tag{1.5}$$

Proof. Let $z \in \widehat{f}(\mathcal{M}_B)$ and let $\widehat{f}(\varphi) = z$ for some $\varphi \in \mathcal{M}_B$. Hence $z - \varphi(f) = zf(e) - f(\varphi) = 0$, thus $\varphi(ze - f) = 0$, and therefore $ze - f \notin B^{-1}$, as shown prior to Lemma 1.1.5. Consequently $z \in \sigma(x)$. Conversely, if $z \in \sigma(x)$ then $ze - f \notin B^{-1}$ and hence $J = (ze - f)\,B$ is a proper ideal of B by the above remarks. If M is a maximal ideal containing J, then for the corresponding functional $\varphi_{_M}$ we have $\mathrm{Null}\,(\varphi_{_M}) = M \supset J \ni ze - f$, thus $\varphi_{_M}(ze - f) = 0$. Therefore, $z = \varphi_{_M}(ze) = \varphi_{_M}(f) = \widehat{f}(\varphi_{_M})$. □

As a corollary we see that $\sigma(f + g) = (\widehat{f} + \widehat{g})(\mathcal{M}_B) \subset \widehat{f}(\mathcal{M}_B) + \widehat{g}(\mathcal{M}_B) = \sigma(f) + \sigma(g)$, and, similarly, $\sigma(fg) \subset \sigma(g)\,\sigma(g)$ for every $f, g \in B$.

By Proposition 1.1.7 $\max\limits_{z\in\sigma(f)} |z| = \max\limits_{z\in\widehat{f}(\mathcal{M}_B)} |z| = \max\limits_{x\in\mathcal{M}_B} \big|\widehat{f}(x)\big|$, which yields the formula

$$r_{_f} = \max_{x\in\mathcal{M}_B} \big|\widehat{f}(x)\big| = \|\widehat{f}\|_{C(\mathcal{M}_B)}$$

for the spectral radius $r_{_f}$ of any element $f \in B$. Combined with formula (1.2) this identity yields

$$\|\widehat{f}\|_{C(\mathcal{M}_B)} = \max_{x\in\mathcal{M}_B} \big|\widehat{f}(x)\big| = r_{_f} = \lim_{n\to\infty} \sqrt[n]{\|f^n\|}. \tag{1.6}$$

Proposition 1.1.7 implies the following description of the peripheral spectrum (1.3):

$$\sigma_\pi(f) = \big\{\widehat{f}(x) : |\widehat{f}(x)| = r_{_f},\ x \in \mathcal{M}_A\big\}.$$

By the well-known *maximum modulus principle* for analytic functions, the functions in the disc algebra $A(\mathbb{D})$ assume their maximum modulus only at the points in the unit circle $\mathbb{T}$, i.e. the topological boundary $\mathbb{T} = b\mathbb{D}$ of $\overline{\mathbb{D}} \cong \mathcal{M}_{A(\mathbb{D})}$. Sets of this kind are of special interest for commutative Banach algebras. A subset E in the maximal ideal space of a commutative Banach algebra B is called a *boundary* of B if the Gelfand transform $\widehat{f}$ of every element f in B attains the maximum of its modulus $\max\limits_{m\in\mathcal{M}_B} \big|\widehat{f}(m)\big| = \|\widehat{f}\|_{C(\mathcal{M}_B)}$ in E. In other words, E is a boundary for B if for every $f \in B$ there exists a $\varphi_0 \in E$ such that $\big|\widehat{f}(\varphi_0)\big| = \max\limits_{\varphi\in\mathcal{M}_B} \big|\widehat{f}(\varphi)\big|$, i.e. the equality

$$\max_{\varphi\in E} \big|\widehat{f}(\varphi)\big| = \max_{\varphi\in\mathcal{M}_B} \big|\widehat{f}(\varphi)\big|$$

holds for every $f \in B$. Clearly, the maximal ideal space $\mathcal{M}_B$ is a boundary of B. The celebrated Shilov theorem asserts that the intersection ∂B of all closed boundaries of a commutative Banach algebra B is again a boundary, called the

Shilov boundary of B (e.g. [G1, S4, T2]). Clearly, ∂B is the smallest closed boundary of B. This minimal property of the Shilov boundary implies the following characterization of its points.

Corollary 1.1.8. *A point m_0 in $\mathcal{M}_B$ belongs to the Shilov boundary ∂B of a commutative Banach algebra B if and only if for each neighborhood U of m_0 in $\mathcal{M}_B$ there exists a function f in B such that $\max_{m\in\overline{U}} \left|\widehat{f}(m)\right| > \max_{m\in\mathcal{M}_B\setminus U} \left|\widehat{f}(m)\right|$.*

As it is not hard to see, $\partial C(X) = X$. The maximum modulus principle, mentioned above, shows that $\mathbb{T}$ is a boundary for the disc algebra $A(\mathbb{D})$. In fact, $\partial A(\mathbb{D}) \cong \mathbb{T}$.

Let $\mathbb{B}^n = \left\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon \left\|(z_1, z_2, \ldots, z_n)\right\| < 1\right\}$ be the unit ball in $\mathbb{C}^n$ with radius 1 centered at the origin $(0, 0, \ldots, 0) \in \mathbb{C}_n$, let $\mathbb{D}^n$ be the n-polydisc $\left\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon |z_j| \le 1,\ 1 \le j \le n\right\}$, and let $\mathbb{T}^n = \left\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon |z_j| = 1,\ 1 \le j \le n\right\}$ be the n-dimensional torus in $\mathbb{C}^n$, i.e. the *distinguished boundary* of $\mathbb{B}^n$. The Shilov boundary of the *ball algebra* $A(\mathbb{B}^n)$ is homeomorphic to the unit sphere in $\mathbb{C}^n$, which is the topological boundary of $\mathbb{B}^n$, while the Shilov boundary for the *polydisc algebra* $A(\mathbb{D}^n)$ is homeomorphic to $\mathbb{T}^n$, which is a proper subset of the topological boundary $b\mathbb{D}^n$ of $\mathbb{D}^n$.

1.2 Uniform algebras

Algebras of continuous functions have many useful properties. They play a major role in this book. A commutative Banach algebra A over $\mathbb{C}$ is said to be a *uniform algebra* on a compact Hausdorff space X if:

(i) A consists of continuous complex-valued functions defined on X, i.e. $A \subset C(X)$.

(ii) A contains all constant functions on X. In particular the identically equal to 1 function on X belongs to A.

(iii) The operations in A are the pointwise addition and multiplication.

(iv) A is closed with respect to the *uniform norm* in $C(X)$,

$$\|f\| = \max_{x\in X} \left|f(x)\right|,\ f \in A. \tag{1.7}$$

(v) *A separates the points of X*, i.e. for every two points in X there is a function in A with different values at these points.

A uniform algebra A is said to be *antisymmetric* if there are no real-valued functions in A besides the constants. A is a *maximal algebra on X* if there is no proper intermediate uniform algebra on X between A and $C(X)$. A is a *maximal algebra* if the restriction algebra $A|_{\partial A}$ is a maximal algebra on ∂A. According to the celebrated Wermer's maximality theorem, the disc algebra $A(\mathbb{D})$ is a maximal algebra.

A uniform algebra A is called a *Dirichlet algebra* if the space $\operatorname{Re}\left(A\big|_{\partial A}\right)$ of real parts of its elements is uniformly dense in $C_{\mathbb{R}}(\partial A)$, i.e. if every real continuous function on the Shilov boundary ∂A can be approximated on ∂A by real parts of functions in A. An example of a Dirichlet algebra is, for instance, the disc algebra $A(\mathbb{D})$. Indeed, $\operatorname{Re} A(\mathbb{D})$ consists of all real-valued continuous functions on $\overline{\mathbb{D}}$ that are harmonic on $\mathbb{D}$ and the harmonic conjugates of which are extendable continuously on $\mathbb{T}$. Consequently, $\operatorname{Re} A(\mathbb{D})$ contains all continuously differentiable functions on $\mathbb{T}$, and these are dense in $C_{\mathbb{R}}(\mathbb{T})$.

Let $\varphi \in \mathcal{M}_A$. A non-negative Borel measure μ on X for which the equality

$$\varphi(f) = \int\limits_X f(x)\, d\mu(x)$$

holds for every $f \in A$ is called a *representing measure* for φ on X. Note that

$$\int\limits_X f(x)\, g(x)\, d\mu(x) = \varphi\,(fg) = \int\limits_X f(x)\, d\mu(x) \int\limits_X g(x)\, d\mu(x)$$

for any $f, g \in A$, i.e. μ is a multiplicative measure for A. Any representing measure μ of φ on X satisfies the equalities

$$\|\mu\| = \int\limits_X d\mu = \varphi(1) = 1.$$

By the Hahn-Banach theorem the set M_φ of all representing measures for a $\varphi \in \mathcal{M}_A$ is nonempty. Actually, M_φ is isomorphic to the set of all norm-preserving extensions of $\varphi \in \mathcal{M}_A$ from $A \subset C(X)$ onto $C(X)$ (e.g. [G1]).

If A is a Dirichlet algebra, then every $\varphi \in \mathcal{M}_A$ has a unique representing measure on ∂A, i.e. M_φ is a single-point set for every $\varphi \in \mathcal{M}_A$. If not, the difference of every two representing measures of φ will vanish on A, hence on $\operatorname{Re} A$, hence on $C_{\mathbb{R}}(X)$ and therefore it will be the zero measure.

Proposition 1.2.1. *Let A be a uniform algebra on a compact set X. If there is a representing measure μ for some $\varphi \in \mathcal{M}_A$, such that* $\operatorname{supp}(\mu) = X$, *then A is an antisymmetric algebra.*

Proof. Assume that μ is a representing measure for some $\varphi \in A$ with $\operatorname{supp}(\mu) = X$. Let f be a non-constant real-valued function in A, and let $t_1, t_2 \in f(X) \subset \mathbb{R}$, $t_1 \neq t_2$. Without loss of generality we can assume that $t_1 > 0$. Let F be a closed neighborhood of t_1 in $\mathbb{R}_+$, which does not contain t_2. There exists a function $g \in C_{\mathbb{R}}\big(f(X)\big)$ such that $\sup\limits_X |g| = 1$, $g \equiv 1$ on F, and $g < 1$ on $f(X) \setminus F$.

Note that g is a uniform limit of polynomials on $f(X) \subset \mathbb{R}$. Hence, the function $g \circ f$ belongs to A. Since $\operatorname{supp}(\mu) = X$ and $\|\mu\| = \int_X g \circ f \, d\mu = 1$, we have that $0 < \int_X g \circ f \, d\mu = c < 1$. Since μ is a multiplicative measure, then

$$\lim_{n\to\infty} \int_X (g \circ f)^n d\mu = \Big(\int_X \lim_{n\to\infty} (g \circ f)^n d\mu \Big) = \lim_{n\to\infty} c^n = 0.$$

On the other hand, the assumed property $\operatorname{supp}(\mu) = X$ implies that

$$\lim_{n\to\infty} \int_X (g \circ f)^n \, d\mu = \int_{f^{-1}(F)} d\mu > 0,$$

in contradiction with the previous equality. Therefore, every real-valued function in A is constant, and consequently, A is an antisymmetric algebra. $\square$

The space $C(X)$ for a compact Hausdorff set X is a uniform algebra. Let K be a compact subset of the maximal ideal space $\mathcal{M}_A$ of a uniform algebra A on X. Consider the algebra $\widehat{A}\big|_K$ of restrictions of Gelfand transforms $\widehat{f}$, $f \in A$ on K. In general this is not a closed subalgebra of $C(K)$, and therefore $\widehat{A}\big|_K$ is not always a uniform algebra. However, the closure A_K of $\widehat{A}\big|_K$ in $C(K)$ is a uniform algebra with $\mathcal{M}_{A_K} \subset \mathcal{M}_A$. If $\mathcal{M}_{A_K}$ does not meet ∂A, then $\partial A_K = b(\mathcal{M}_{A_K})$, the topological boundary of $\mathcal{M}_{A_K}$ with respect to the Gelfand topology, which is an immediate corollary of the following.

Theorem 1.2.2 (Rossi's Local Maximum Modulus Principle). *If U is an open subset of $\mathcal{M}_A$, then*

$$\sup_{m\in U} \big|\widehat{f}(m)\big| = \max_{m\in bU\cup(\partial A\cap U)} \big|\widehat{f}(m)\big|$$

for every function $f \in A$.

Let A be a uniform algebra on X. As we know from section 1.1, the maximal ideal space $\mathcal{M}_A$ of A is a compact set. Since the point evaluation $\varphi_x : f \longmapsto \varphi(x)$ at any point of X is a linear multiplicative functional, then $\varphi_x \in \mathcal{M}_A$ for every $x \in X$. This allows us to consider X as a subspace of $\mathcal{M}_A$. The Gelfand transform $\widehat{f}$ of an $f \in A$ is continuous on $\mathcal{M}_A$. For any point of $x \in X$ we have $\widehat{f}(\varphi_x) = \varphi_x(f) = f(x)$, and therefore $\widehat{f}$ can be interpreted as a continuous extension of f on $\mathcal{M}_A$. Moreover, in a certain sense M_A is the largest set for natural extension of all functions in A. Recall that according to Lemma 1.1.5 the norm of any $\varphi \in \mathcal{M}_A$ is 1. Therefore, $\big\|\varphi(f)\big\| \leq \|\varphi\|\|f\| = \|f\|$. It follows that the Gelfand transformation $\Lambda : \ A \longrightarrow \widehat{A} \subset C(\mathcal{M}_A) : \ f \longmapsto \widehat{f}$ is an isometric isomorphism. Consequently, the algebra A and its Gelfand transform $\widehat{A}$ are isometrically isomorphic, and hence $\widehat{A}$ is closed in $C(X)$. Since the algebra $A|_{\partial A}$ of restrictions of elements in A on the Shilov boundary ∂A is also isometrically isomorphic to A, we have $A \cong \widehat{A} \cong A\big|_{\partial A}$. For this reason we will not distinguish, for example, the disc algebra $A(\mathbb{D})$ from its restriction algebra $A(\mathbb{T}) = A(\mathbb{D})\big|_{\mathbb{T}}$ on the Shilov boundary $\partial A(\mathbb{D}) = \mathbb{T}$.

Observe, that $\big\|(m_1 - m_2)(f)\big\| = \big\|m_1(f) - m_2(f)\big\| \le \big(\|m_1\| + \|m_2\|\big)\|f\| \le 2\,\|f\|$ for every $m_1, m_2 \in \mathcal{M}_A$, and $f \in A$. Consequently, the norm $\|m_1 - m_2\|$ of the linear functional $m_1 - m_2 \in A^*$ does not exceed 2. Therefore, the diameter of the set $\mathcal{M}_A \subset A^*$ is not greater than 2. The property $\|m_1 - m_2\| < 2$ generates a transitive relation in $\mathcal{M}_A$. It is easy to check that this is an equivalence relation (e.g. [G1],[S4]). The equivalent classes of the set $\mathcal{M}_A$ with respect to this relation are called *Gleason parts* of A. It is clear that points on the extreme ends of a diameter, i.e. for which $\|m_1 - m_2\| = 2$, belong to distinct Gleason parts.

A homomorphism $\Phi\colon A \longrightarrow B$ between two uniform algebras naturally generates an adjoint continuous map $\Phi^*\colon \mathcal{M}_B \longrightarrow \mathcal{M}_A$ between their maximal ideal spaces, defined by

$$\big(\Phi^*(\varphi)\big)(f) = \varphi\left(\Phi(f)\right),\ f \in A,\ \varphi \in \mathcal{M}_B.$$

If $\Phi : A \longrightarrow B$ preserves the norm, i.e. if

$$\big\|\Phi(f)\big\|_B = \|f\|_A$$

for every $f \in A$, then Φ is called an *embedding* of A into B. Clearly, $\Phi^*(\partial B) \subset \mathcal{M}_A$.

Proposition 1.2.3. *Let A and B be uniform algebras, and let $\Phi\colon A \longrightarrow B$ be a homomorphism that does not increase the norm, i.e. for which $\big\|\Phi(f)\big\|_B \le \|f\|_A$, $f \in A$. Then Φ is an embedding of A into B if and only if the range $\Phi^*(\partial B)$ of Φ^* contains the Shilov boundary ∂A.*

Proof. For every $f \in A$ we have

$$\begin{aligned} \max_{m \in \Phi^*(\partial B)} \big|m(f)\big| &= \max_{\varphi \in \partial B} \big|\big(\Phi^*(\varphi)\big)(f)\big| = \max_{\varphi \in \partial B} \big|\varphi\big(\Phi(f)\big)\big| \\ &= \max_{\varphi \in \partial B} \big|\widehat{(\Phi(f))}(\varphi)\big| = \big\|\Phi(f)\big\|_B. \end{aligned} \tag{1.8}$$

If $\partial A \subset \Phi^*(\partial B)$, then

$$\|f\|_A = \max_{\varphi \in \partial A} \big|\widehat{f}(\varphi)\big| = \max_{\varphi \in \partial A} \big|\varphi(f)\big| \le \max_{\varphi \in \Phi^*(\partial B)} \big|\varphi(f)\big| = \big\|\Phi(f)\big\|_B.$$

Therefore, $\|f\|_A = \big\|\Phi(f)\big\|_B$, i.e. Φ preserves the norm.

Conversely, if $\Phi : A \longrightarrow B$ is an isometry, then $\Phi^*(\partial B)$ is a boundary for A, since by (1.8)

$$\max_{\varphi \in \Phi^*(\partial B)} \big|\widehat{f}(\varphi)\big| = \max_{\varphi \in \Phi^*(\partial B)} \big|\varphi(f)\big| = \big\|\Phi(f)\big\|_B = \|f\|_A.$$

Consequently, $\partial A \subset \Phi^*(\partial B)$. □

Corollary 1.2.4. *A homomorphism Φ of A onto B which does not increase the norm is an embedding if and only if $\Phi^*(\partial B) = \partial A$.*

Proof. The arguments from the proof of Proposition 1.2.3 show that it is enough to show that $\Phi^*(\partial B) \subset \partial A$. Suppose that, on the contrary, $\Phi^*(\partial B) \supsetneq \partial A$, and let $\varphi_0 \in \Phi^*(\partial B) \setminus \partial A$. According to Corollary 1.1.8, there is a neighborhood U of φ_0 in $\mathcal{M}_A \setminus \partial A$, such that for every function $f \in A$,

$$\max_{m\in\overline{U}} \left|\widehat{f}(m)\right| \le \max_{m\in\mathcal{M}_A\setminus U} \left|\widehat{f}(m)\right|.$$

In particular,

$$\max_{\Phi^*(\varphi)\in\overline{U}} \left|\widehat{f}\big(\Phi^*(\varphi)\big)\right| \le \max_{\Phi^*(\varphi)\in\mathcal{M}_A\setminus U} \left|\widehat{f}\big(\Phi^*(\varphi)\big)\right|.$$

Since $\widehat{f}\big(\Phi^*(\varphi)\big) = \big(\Phi^*(\varphi)\big)(f) = \varphi\big(\Phi(f)\big) = \widehat{\Phi(f)}(\varphi)$, we have

$$\max_{\varphi\in(\Phi^*)^{-1}(\overline{U})} \left|\widehat{\Phi(f)}(\varphi)\right| \le \max_{\varphi\in(\Phi^*)^{-1}(\mathcal{M}_A\setminus U)} \left|\widehat{\Phi(f)}(\varphi)\right|.$$

By the assumed $\Phi(A) = B$, we see that

$$\max_{\varphi\in(\Phi^*)^{-1}(\overline{U})} \left|\widehat{g}(\varphi)\right| \le \max_{\varphi\in(\Phi^*)^{-1}(\mathcal{M}_A\setminus U)} \left|\widehat{g}(\varphi)\right|$$

for every $g \in B$. Consequently, $(\Phi^*)^{-1}(\mathcal{M}_A \setminus U)$ is a closed boundary of B, and $(\Phi^*)^{-1}(\varphi_0) \subset (\Phi^*)^{-1}(U) \subset \mathcal{M}_B \setminus (\Phi^*)^{-1}(\mathcal{M}_A \setminus U) \subset \mathcal{M}_B \setminus \partial B$, in contradiction with the initially assumed property $\varphi_0 \in \Phi^*(\partial B)$. Hence $\Phi^*(\partial B) \subset \partial A$. $\square$

Every embedding $\Phi\colon\ A(\mathbb{T}) \longrightarrow A(\mathbb{T})$ of the disc algebra onto itself is an isometric isomorphism between $A(\mathbb{T})$ and $\Phi\big(A(\mathbb{T})\big)$. Consequently, the adjoint map $\Phi^*\colon \mathcal{M}_{\Phi(A(\mathbb{T}))} \longrightarrow \mathcal{M}_{A(\mathbb{T})}$ generates a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$, and $\Phi^*\big(\partial(\Phi(A(\mathbb{T}))\big) = \partial A(\mathbb{T}) = \mathbb{T}$, i.e. $\Phi^*(\mathbb{T}) = \mathbb{T}$, $\Phi^*(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$, and hence the function Φ^* is a *finite Blaschke product* (cf. [G2]) on $\mathbb{D}$, i.e.

$$B(z) = e^{i\theta} \prod_{k=1}^{n} \left(\frac{z - z_k}{1 - \overline{z}_k z}\right) \text{ for some } z_k,\ 0 < |z_k| < 1,\ k = 1, 2, \ldots, n.$$

Therefore, for any embedding $\Phi\colon A(\mathbb{T}) \longrightarrow A(\mathbb{T})$ of $A(\mathbb{T})$ onto itself there exists a finite Blaschke product $B(z)$ on $\mathbb{D}$ with $\Phi \circ f = f \circ B$, i.e. such that

$$\Phi\big(f(z)\big) = (f \circ \Phi^*)(z) = f\big(B(z)\big) \text{ for every } \ f \in A(\mathbb{T}). \tag{1.9}$$

Let $A \subset C(X)$ be a uniform algebra on a compact set X. One can easily identify certain points as elements of the Shilov boundary ∂A of a uniform algebra A. A point $x_0 \in X$ is called a *peak point* of a uniform algebra A if there exists a function f in A such that $f(x_0) = 1$ and $\left|\widehat{f}(x)\right| < 1$ for every $x \in \mathcal{M}_A \setminus \{x_0\}$. Clearly, every peak point belongs to the Shilov boundary ∂A. In general the set of peak points is not a boundary for A. However, for algebras with metrizable maximal ideal spaces it is (e.g. [G1, S4]). Moreover, in this case the set of peak points is the *minimal boundary* for A, i.e. it is contained in every boundary of A.

An element $f \in A$ is called a *peaking function of* A if $\|f\| = 1$, and either $\widehat{f}(x) = 1$, or, $|\widehat{f}(x)| < 1$ for any $x \in \mathcal{M}_A$. In this case the set $P(f) = \{x \in \mathcal{M}_A : \widehat{f}(x) = 1\} = \widehat{f}^{-1}(1)$ is called the *peak set* (or, *peaking set*) of A corresponding to $\widehat{f}$. Clearly, every peak point is a peak set of A, and f is a peaking function if and only if $\sigma_\pi(f) = \{1\}$. If $K \subset \mathcal{M}_A$ is such that $K = P(f)$ for some peaking function f, we say that $\widehat{f}$ *peaks on* K. Clearly, K is a peak set if there is a function$f \in A$, such that $\widehat{f}|_K \equiv 1$, and $|\widehat{f}(m)| < 1$ whenever $m \in \mathcal{M}_A \setminus K$.

A point $x \in \mathcal{M}_A$ is called a *generalized peak point of* A (or, a *p-point of* A) if it coincides with the intersection of a family of peak sets of A. Equivalently, x is a p-point of A if for every neighborhood V of x there is a peaking function f with $x \in P(f) \subset V$. The *Choquet boundary* (or, the *strong boundary*) δA of A is the set of all generalized peak points of A. It is a boundary of A, and its closure coincides with the Shilov boundary ∂A of A, i.e. $\overline{\delta A} = \partial A$. Unlike δA, the set of peak points of A in general is not dense in ∂A, unless $\mathcal{M}_A$ is metrizable (cf. [G1, S4]).

Till the end of the section we will assume that $A \subset C(X)$ is a uniform algebra on its maximal ideal space $\mathcal{M}_A = X$. Denote by $\mathcal{F}(A)$ the set of all peaking functions of A. For a fixed point x in X by $\mathcal{F}_x(A)$ denote the set of all peaking functions of A by $P(f) \ni x$, i.e. with$\widehat{f}(x) = 1$.

Lemma 1.2.5. *Let $A \subset C(X)$ be a uniform algebra. If $f, g \in A$ are such that $\|fh\| \le \|gh\|$ for all peaking functions $h \in \mathcal{F}(A)$, then $|f(x)| \le |g(x)|$ on ∂A.*

Proof. Assume that $\|fh\| \le \|gh\|$ for every $h \in \mathcal{F}(A)$, but $|f(x_0)| > |g(x_0)|$ for some $x_0 \in \partial A$. Without loss of generality we may assume that $x_0 \in \delta A$. Choose a $\gamma > 0$ such that $|g(x_0)| < \gamma < |f(x_0)|$, and choose an open neighborhood V of x_0 in X so that $|g(x)| < \gamma$ on V. Let $h \in \mathcal{F}_{x_0}(A)$ be a peaking function of A on X with $P(h) \subset V$. By choosing a sufficiently high power of h we can assume from the beginning that $|g(x)h(x)| < \gamma$ for every $x \in \partial A \setminus V$. Since this inequality obviously holds also on V, we deduce that $\|gh\| < \gamma$. Hence,

$$|f(x_0)| = |f(x_0)h(x_0)| \le \|fh\| \le \|gh\| < \gamma.$$

Therefore, $|f(x_0)| < \gamma$ in contradiction with the choice of γ. Consequently, $|f(x)| \le |g(x)|$ on ∂A. □

Corollary 1.2.6. *If the functions $f, g \in A$ satisfy the equality $\|fh\| = \|gh\|$ for all peaking functions $h \in \mathcal{F}(A)$, then $|f(x)| = |g(x)|$ on ∂A.*

Lemma 1.2.7. *If the functions* $f, g \in A$ *satisfy the inequality*

$$\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) \le \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$$

for all $k \in A$*, then* $|f(x)| \le |g(x)|$ *for every* $x \in \partial A$*.*

Proof. The proof follows the line of proof of Lemma 1.2.5.
Assume that $\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) \le \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$ for every $k \in A$, but $|f(x_0)| > |g(x_0)|$ for some $x_0 \in \partial A$. Without loss of generality we may assume that $x_0 \in \delta A$. Choose a $\gamma > 0$ such that $|g(x_0)| < \gamma < |f(x_0)|$, and choose an open neighborhood V of x_0 in X so that $|g(x)| < \gamma$ on V. Let $R > 1$ be such that $\|f\| \le R$ and $\max_{\xi \in \partial A} |g(\xi)| \le R$. Let $k \in \mathcal{F}_{x_0}(A)$ be a peaking function for A with $P(k) \subset V$. By choosing a sufficiently high power of k we can assume from the beginning that $|g(x)| + |Rk(x)| < R + \gamma$ for every $x \in \partial A \setminus V$. Since this inequality holds also on V, we deduce that $|g(x)| + |Rk(x)| < R + \gamma$ for every $x \in \partial A$. Hence,

$$\begin{aligned} |f(x_0)| + R &= |f(x_0)| + |Rk(x_0)| \\ &\le \max_{\xi \in \partial A} \big(|f(\xi)| + |Rk(\xi)|\big) \le \max_{\xi \in \partial A} \big(|g(\xi)| + R|k(\xi)|\big) < R + \gamma. \end{aligned}$$

Therefore, $|f(x_0)| < \gamma$ in contradiction with the choice of γ. Consequently, $|f(x)| \le |g(x)|$ for every $x \in \partial A$. □

Corollary 1.2.8. *If the functions* $f, g \in A$ *satisfy the equality*

$$\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) = \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$$

for all $k \in A$*, then* $|f(x)| = |g(x)|$ *for every* $x \in \partial A$*.*

The following lemma, due to Bishop, helps to localize elements of uniform algebras.

Lemma 1.2.9 (Bishop's Lemma). *If* $E \subset X$ *is a peak set for* A*, and* $f \not\equiv 0$ *on* E *for some* $f \in A$*, then there exists a peaking function* $h \in \mathcal{F}(A)$ *which peaks on* E *and such that*

$$|f(x)h(x)| < \max_{\xi \in E} |f(\xi)| \tag{1.10}$$

for any $x \in X \setminus E$*.*

Proof. If $f \in A$ and $\max_{\xi \in E} |f(\xi)| = M > 0$. For any natural $n \in \mathbb{N}$ define the set

$$U_n = \big\{x \in X \colon |f(x)| < M\big(1 + 1/2^{n+1}\big)\big\}.$$

Clearly, $E \subset U_n \subset U_{n-1}$ for every $n > 1$. Choose a function $k \in \mathcal{F}(A)$ which peaks on E, and let k_n be a big enough power of k so that $|k_n(x)| < \dfrac{1}{2^n}$ on $X \setminus U_n$.

The function $h = \sum_1^\infty \frac{1}{2^n} k_n$ belongs to $\mathcal{F}(A)$. Moreover, $P(h) = h^{-1}\{R\} = E$, $|h(x)| < 1$ on $X \setminus E$, and $\max_{\xi \in E} \big(|f(\xi)h(\xi)|\big) = M$. We claim that $|f(x)h(x)| < M$ for every $x \notin E$. In what follows, x is a fixed element in $X \setminus E$.

(i) Let $x \notin U_1$. Then $x \notin U_n$ for all $n \in \mathbb{N}$, and hence $|k_n(x)| < \frac{1}{2^n} < 1$ for all $n \in \mathbb{N}$. Hence, $|h(x)| < \sum_1^\infty \frac{1}{2^n} = 1$, thus $|f(x)h(x)| < M$.

(ii) Let $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for every $1 \le i \le n-1$, and $x \notin U_i$ for all $i \ge n$. Hence $|f(x)| < M\big(1 + 1/2^{i+1}\big)$ for every $1 \le i \le n-1$, and $|k_i(x)| < \frac{1}{2^i}$ for all $i \ge n$. Since $x \in U_{n-1}$, we see that $|f(x)| < M\big(1 + 1/2^n\big)$, and

$$|f(x)h(x)| < M\big(1 + 1/2^n\big)\left(\sum_{i=1}^{n-1} \frac{1}{2^i}|k_i(x)| + \sum_{i=n}^{\infty} \frac{1}{2^i}|k_i(x)|\right).$$

Further,

$$\sum_{i=1}^{n-1} \frac{1}{2^i}|k_i(x)| < \sum_{i=1}^{n-1} \frac{1}{2^i} = \big(1 - 1/2^{n-1}\big), \text{ and}$$

$$\sum_{i=n}^{\infty} \frac{1}{2^i}|k_i(x)| \le \sum_{i=n}^{\infty} \frac{1}{2^i}\left(\frac{1}{2^i}\right) = \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{M}{3 \cdot 4^{n-1}} < \frac{1}{2 \cdot 4^{n-1}} = \frac{1}{2^n \cdot 2^{n-1}}.$$

Consequently,

$$\begin{aligned} |f(x)h(x)| &< M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n 2^{n-1}}\right) \\ &\le M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^n}\right)\right) < M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}} \cdot \frac{1}{2}\right) \\ &= M\big(1 + 1/2^n\big)\big(1 - 1/2^n\big) = M\big(1 - 1/2^{2n}\big) < M. \end{aligned}$$

(iii) If $x \in \bigcap_{n=1}^\infty U_n$, then $|f(x)| \le M$, whence $|f(x)h(x)| < M$ since $|h(x)| < 1$ on $X \setminus E$. □

If also $\sigma_\pi(fh) = \sigma_\pi(gh)$ for all $h \in \mathcal{F}(A)$, then we have a much stronger result than in Corollary 1.2.6. Namely,

Lemma 1.2.10. *If $f, g \in A$ satisfy the equality*

$$\sigma_\pi(fh) = \sigma_\pi(gh) \tag{1.11}$$

for every peaking function $h \in A$, then $f(x) = g(x)$ on ∂A.

Proof. Clearly, $\|fh\| = \|gh\|$, since $|z| = \|f\|$ for every $z \in \sigma_\pi(f)$. Corollary 1.2.6 yields $|f(x)| = |g(x)|$ on ∂A. Let $y \in \delta A$. If $f(y) = 0$, then $|g(y)| = |f(y)| = 0$ implies that also $g(y) = 0$. Therefore, we can assume without loss of generality that $f(y) \neq 0$. Choose an open neighborhood V of y in X, and a peaking function $k \in \mathcal{F}_y(A)$ with $P(k) \subset V$. Let $|f(x_V)| = \max_{\xi \in P(k)} |f(\xi)|$ for some $x_V \in P(k)$. By Bishop's Lemma there is a peaking function $h \in \mathcal{F}_y(A)$ with $P(h) = P(k)$, so that the functions fh and gh attain the maxima of their modulus only within $P(h)$. Therefore, by (1.11), $f(x_V) = f(x_V)\, h(x_V) \in \sigma_\pi(fh) = \sigma_\pi(gh)$. Hence, there is a $z_V \in P(h)$ so that

$$f(x_V) = g(z_V)\, h(z_V) = g(z_V). \tag{1.12}$$

Since in every neighborhood $V \ni y$ there are points x_V and z_V in V with $f(x_V) = g(z_V)$, then $f(y) = g(y)$ by the continuity of f and g. Consequently, $f = g$ on $\partial A = \overline{\delta A}$. □

The next lemma is an additive version of Bishop's Lemma (Lemma 1.2.9).

Lemma 1.2.11 (Additive analogue of Bishop's Lemma). *If $E \subset X$ is a peak set for A, and $f \not\equiv 0$ on E for some $f \in A$, then there exists a function $h \in \mathcal{F}(A)$ which peaks on E and such that*

$$|f(x)| + N|h(x)| < \max_{\xi \in E} |f(\xi)| + N \tag{1.13}$$

for any $x \in X \setminus E$ and any $N \geq \|f\|$.

Proof. The proof follows the line of proof of Bishop's Lemma 1.2.9. If $\|f\| = \max_{\xi \in X} |f(\xi)| = R$ and $\max_{\xi \in E} |f(\xi)| = M$, then clearly, $0 < M \leq R$. For any natural $n \in \mathbb{N}$ define the set

$$U_n = \big\{x \in X : |f(x)| < M\big(1 + 1/2^{n+1}\big)\big\}.$$

Clearly, $E \subset U_n \subset U_{n-1}$ for every $n > 1$. Choose a function $k \in \mathcal{F}(A)$ which peaks on E, and let k_n be a big enough power of k so that $R\,|k_n(x)| < \dfrac{M}{2^n}$ on $X \setminus U_n$. The function $h = \displaystyle\sum_1^\infty \frac{1}{2^n} k_n$ belongs to $\mathcal{F}(A)$. Moreover, $P(h) = h^{-1}\{R\} = E$, $|h(x)| < 1$ on $X \setminus E$, and $\max_{\xi \in E} \big(|f(\xi)| + R\,|h(\xi)|\big) = M + R$. We claim that $|f(x)| + R\,|h(x)| < M + R$ for every $x \notin E$. In what follows, x is a fixed element in $X \setminus E$.

(i) Let $x \notin U_1$. Then $x \notin U_n$ for all $n \in \mathbb{N}$, and hence $R\,|k_n(x)| < \dfrac{M}{2^n} < M$ for all $n \in \mathbb{N}$. Hence, $R\,|h(x)| < \displaystyle\sum_1^\infty \frac{M}{2^n} = M$, thus $|f(x)| + R\,|h(x)| < R + M$.

(ii) Let $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for all $1 \le i \le n-1$, and $x \notin U_i$ for each $i \ge n$. Hence $|f(x)| < M\big(1 + 1/2^{i+1}\big)$ for all $1 \le i \le n-1$, and $R|k_i(x)| < \dfrac{M}{2^i}$ for each $i \ge n$. Since $x \in U_{n-1}$, we see that $|f(x)| < M\big(1 + 1/2^n\big)$, and

$$|f(x)| + R\,|h(x)| < M\big(1 + 1/2^n\big) + \sum_{i=1}^{n-1} \frac{R}{2^i}|k_i(x)| + \sum_{i=n}^{\infty} \frac{R}{2^i}|k_i(x)|.$$

Further,

$$\sum_{i=1}^{n-1} \frac{R}{2^i}|k_i(x)| < \sum_{i=1}^{n-1} \frac{R}{2^i} = R\big(1 - 1/2^{n-1}\big), \text{ and}$$

$$\sum_{i=n}^{\infty} \frac{R}{2^i}|k_i(x)| \le \sum_{i=n}^{\infty} \frac{1}{2^i}\left(\frac{M}{2^i}\right) = M\sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{M}{3 \cdot 4^{n-1}} < \frac{M}{2 \cdot 4^{n-1}} = \frac{M}{2^n \cdot 2^{n-1}}.$$

Consequently,

$$\begin{aligned}
|f(x)| + R\,|h(x)| &< M\big(1 + 1/2^n\big) + R\big(1 - 1/2^{n-1}\big) + \frac{M}{2^n 2^{n-1}} \\
&\le M + R\left(\frac{1}{2^n} + 1 - \frac{1}{2^{n-1}} + \frac{1}{2^n \cdot 2^{n-1}}\right) \\
&= M + R\left(1 - \frac{1}{2^n} + \frac{1}{2^n \cdot 2^{n-1}}\right) < M + R.
\end{aligned}$$

(iii) If $x \in \bigcap_{n=1}^{\infty} U_n$, then $|f(x)| \le M$, whence $|f(x)| + R\,|h(x)| < M + R$ since $|h(x)| < 1$ on $X \setminus E$.

Actually, (1.13) holds with any $N > R$ for the function h constructed above. Indeed,

$$\begin{aligned}
|f(x)| + N|h(x)| &= |f(x)| + R\,|h(x)| + (N - R)\,|h(x)| \\
&< \max_{\xi \in E} |f(\xi)| + R + (N - R) = \max_{\xi \in E} |f(\xi)| + N.
\end{aligned}$$

□

Corollary 1.2.12. *Let E be a peak set of A, $x_0 \in E$, $f \in A$, $N \ge \|f\|$, and $\alpha \in \mathbb{T}$ be such that $|f(x_0)| = \max_{\xi \in E} |f(\xi)| > 0$ and $f(x_0) = \alpha\,|f(x_0)|$. If h is the peaking function of A with $P(h) = E$, constructed in Lemma* 1.2.11, *then*

(a) $\big|f(x) + \alpha N h(x)\big| \le |f(x)| + N|h(x)| < \|f + \alpha N h\| = \big|f(x_0) + \alpha N h(x_0)\big| = \big|f(x_0)\big| + N$ *for all $x \in X \setminus E$, and*

(b) $\|f + \gamma N h\| \le \|f + \alpha N h\|$ *for every $\gamma \in \mathbb{T}$.*

Proof. (a) Lemma 1.2.11 implies that $\big|f(x)+\alpha N h(x)\big| \le |f(x)| + N|h(x)| < \max_{\xi\in E}\big(|f(\xi)|+N\big) = |f(x_0)| + N = \big|f(x_0)+\alpha N h(x_0)\big|$ for all $x \in X\setminus E$. Hence, $\|f+\alpha N h\| = \max_{\xi\in E}\big|f(\xi)+\alpha N h(\xi)\big| = \big|f(x_0)+\alpha N h(x_0)\big| = |f(x_0)|+N$, i.e. (a) holds.

(b) By Lemma 1.2.11 and (a), we have

$$\begin{aligned}\|f+\gamma N h\| &= \max_{\xi\in X}\big|f(\xi)+\gamma N h(\xi)\big| \\ &\le \max_{\xi\in X}\big(|f(\xi)|+N|h(\xi)|\big) = |f(x_0)|+N = \|f+\alpha N h\|.\end{aligned}$$

□

If $\sigma_\pi(f+h) = \sigma_\pi(g+h)$ for all $h \in A$, then we have a much stronger result than in Corollary 1.2.8. Namely,

Lemma 1.2.13. *If $f, g \in A$ satisfy the equalities*

(a) $\sigma_\pi(f+h) = \sigma_\pi(g+h)$, *and*

(b) $\max_{\xi\in\partial A}\big(|f(\xi)|+|h(\xi)|\big) = \max_{\xi\in\partial A}\big(|g(\xi)|+|h(\xi)|\big)$

for every $h \in A$, then $f(x) = g(x)$ for every $x \in \partial A$.

Proof. The proof follows the line of proof of Lemma 1.2.10. Let $f, g \in A$ and let $\|f\| = \|g\| = R$. Equality (b) and Corollary 1.2.8 imply that $|f(x)| = |g(x)|$ on ∂A. Let $y \in \delta A$. If $f(y) = 0$, then by $|g(y)| = |f(y)| = 0$ we see that $g(y) = 0$ too. Suppose now that $f(y) \neq 0$. Choose an open neighborhood V of y in X, and a peaking function $k \in \mathcal{F}_y(A)$ with $P(k) \subset V$. There is an $x_V \in P(k)$ so that $|f(x_V)| = \max_{\xi\in P(k)}|f(\xi)| = M \le R$. Let $f(x_V) = \alpha_V M$ for some $\alpha_V \in \mathbb{T}$. By the additive version of Bishop's Lemma we can choose a peaking function $h \in \mathcal{F}_y(A)$ with $P(h) = P(k)$ and such that the function $|f(x)|+|Rh(x)|$ attains its maximum only within $P(h)$. Hence

$$\begin{aligned}&|f(x_V)| + R = M + R = \big|\alpha_V(M+R)\big| = \big|f(x_V)+\alpha_V\, R\big| \\ &= \big|f(x_V)+\alpha_V\, Rh(x_V)\big| \le \|f+\alpha_V Rh\| = \max_{\xi\in\partial A}\big|\big(f+Rh\big)(\xi)|\big) \\ &\le \max_{\xi\in\partial A}\big(|f(\xi)|+|Rh(\xi)|\big) = \max_{\xi\in P(h)}\big(|f(\xi)|+|Rh(\xi)|\big) \\ &= \max_{\xi\in P(h)}\big(|f(\xi)|+R|\big) = |f(x_V)|+R,\end{aligned}$$

and therefore,

$$\big|f(x_V)+\alpha_V\, R\big| = \max_{\xi\in\partial A}\big(|f(\xi)|+|Rh(\xi)|\big) = \|f+\alpha_V Rh\|, \tag{1.14}$$

and, by equality (a), $f(x_V)+\alpha_V R \in \sigma_\pi(f+\alpha_V Rh) = \sigma_\pi(g+\alpha_V Rh)$. Hence, there is a $z_V \in X$ so that

$$f(x_V)+\alpha_V R = g(z_V)+\alpha_V Rh(z_V). \tag{1.15}$$

We may assume that $z_V \in \partial A$, since $\big|g(z_V) + \alpha_V Rh(z_V)\big| = |f(x_V) + \alpha_V R| = \big|f(x_V) + \alpha_V Rh(x_V)\big|$ is the maximum modulus of both functions $g + \alpha_V Rh$, $f + \alpha_V Rh$, and as a peaking set of A (cf. [L1]), the preimage $\big(g + \alpha_V Rh\big)^{-1}\big(g(z_V) + \alpha_V Rh(z_V)\big)$ of the number $g(z_V) + \alpha_V Rh(z_V)$ under the function $g + \alpha_V Rh$ necessarily meets ∂A. By (1.14) and Corollary 1.2.8 we have

$$\begin{aligned}
&\big|g(z_V) + \alpha_V Rh(z_V)\big| \le |g(z_V)| + |Rh(z_V)| \\
&= |f(z_V)| + |Rh(z_V)| \le \max_{\xi\in\partial A} \big(|f(\xi)| + |Rh(\xi)|\big) = \|f + \alpha_V Rh\| \\
&= \max_{\xi\in\delta A} \big(\big|g(\xi) + \alpha_V Rh(\xi)\big|\big) = \big|g(z_V) + \alpha_V Rh(z_V)\big|.
\end{aligned}$$

Hence, $|g(z_V)| + |Rh(z_V)| = \max_{\xi\in\delta A} \big(\big|g(\xi) + \alpha_V Rh(\xi)\big|\big)$. Since the function $\big|g(\xi) + \alpha_V Rh(\xi)\big|$ attains its maximum only within $P(h)$ it follows that $z_V \in P(h)$, thus $h(z_V) = 1$. The equality (1.15) now becomes $f(x_V) + \alpha_V R = g(z_V) + \alpha_V R$, thus $f(x_V) = g(z_V)$. Since in every neighborhood $V \ni y$ there are points x_V and z_V in V with $f(x_V) = g(z_V)$, then $f(y) = g(y)$ by the continuity of f and g. Consequently, $f = g$ on $\partial A = \overline{\delta A}$. □

1.3 Inductive and inverse limits of algebras and sets

In this section we introduce the notion of inductive and inverse systems and their limits, which are used to construct associated algebras. Since we need the technique in some special cases only, we do not present it in its general form, which can be found elsewhere (e.g. [L1], [ES]).

Consider a family $\{A^\alpha\}_{\alpha\in\Sigma}$ of uniform algebras. Suppose that the index set Σ is directed, i.e. Σ is a partially ordered set, and every pair α, β of elements of Σ has a common successor $\gamma \succ \alpha, \beta$ in Σ. Suppose also that for every pair A^α, A^β, $\alpha \prec \beta$, of algebras there is an algebraic homomorphism $\iota_\alpha^\beta\colon A^\alpha \longrightarrow A^\beta$. The family $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is called an *inductive system* (or, inductive spectrum, direct spectrum) of algebras A^α with connecting homomorphisms ι_α^β, if

(i) ι_α^α is the identity mapping on A^α, and

(ii) $\iota_\beta^\gamma \circ \iota_\alpha^\beta = \iota_\alpha^\gamma$ whenever $\alpha \prec \beta \prec \gamma$.

A *chain* of the system $\{A^\alpha, \iota_a^\beta\}_{\alpha\in\Sigma}$ is called any set of type $\nu = \{p^\alpha : p^\alpha \in A^\alpha\}_{\alpha\succ\alpha_\nu}$, such that $\iota_\alpha^\beta(p^\alpha) = p^\beta$ for every $\alpha, \beta \succ \alpha_\nu$. Let $\mathcal{N}$ denote the set of all chains of the system $\{A^\alpha, \iota_a^\beta\}_{\alpha\in\Sigma}$. Consider the following equivalence relation in $\mathcal{N}$: If $\nu_1 = \{p^\alpha\}_{\alpha\succ\alpha_{\nu_1}}$ and $\nu_2 = \{q^\alpha\}_{\alpha\succ\alpha_{\nu_2}} \in \mathcal{N}$, then $\nu_1 \sim \nu_2$ if there exists a $\beta \in \Sigma$, $\beta \succ \alpha_{\nu_1}, \alpha_{\nu_2}$, such that $p^\sigma = q^\sigma$ for every $\sigma \succ \beta$. The set A of equivalence classes of $\mathcal{N}$ with respect to this relation is called the *inductive limit* of the system $\{A^\alpha, \iota_\alpha^\beta\}$, and is denoted by $\varinjlim\, \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$. The equivalent class of a given chain $\nu = \{p^\alpha\}_{\alpha\succ\alpha_\nu} \in \mathcal{N}$ consists of all chains $\eta = \{q^\alpha\}_{\alpha\succ\alpha_\eta} \in \mathcal{N}$ whose coordinates q^α coincide eventually with the coordinates p^α of ν.

Example 1.3.1. (a) Let $\{A^\alpha\}_{\alpha\in\Sigma}$ be a family of uniform algebras, such that $A^\alpha \subset A^\beta$ whenever $\alpha \prec \beta$. Let ι_α^β be the inclusion mapping of A^α into A^β, i.e. $\iota_\alpha^\beta(a) = a \in A^\beta$ for every $a \in A^\alpha$. It is easy to see that in this case the family $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is an inductive system, and $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma} = \bigcup_{\alpha\in\Sigma} A^a$.

(b) Given a uniform algebra A, and an index set Σ, consider the inductive system $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, where $A^\alpha = A$ for every $\alpha \in \Sigma$, and each of ι_α^β, $\alpha, \beta \in \Sigma$, is the identity mapping on A. It is easy to check that the limit $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ of this system is isomorphic to A.

Every coordinate algebra A^β of an inductive system can be mapped naturally into the inductive limit $\varinjlim\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ by a mapping $\iota_\beta\colon A^\beta \longrightarrow \varinjlim\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, defined as follows: if $a^\beta \in A^\beta$, then $\iota_\beta(a^\beta)$ is the equivalent class of the chain $\{\iota_\beta^\gamma(a^\beta)\}_{\gamma\succ\beta} \in \mathcal{N}$.

Let, for instance, $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be the inductive system from Example 1.3.1(a), where $A^\alpha \subset A^\beta$ whenever $\alpha \prec \beta$, and ι_α^β is the inclusion mapping of A^α into A^β. By definition, the inclusion mapping ι_β of a fixed coordinate algebra A^β into $\bigcup_{\alpha\in\Sigma} A^\alpha = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ maps every $a \in A^\beta$ to the equivalence class of the stationary chain $\{a^\gamma\}_{\gamma\succ\beta}$ with $a^\gamma = a$. Since this class is uniquely defined by the element a, it can be identified by a itself, and henceforth $\iota_\beta(a) = a$ for every $a \in A^\beta$.

One can define algebraic operations in an inductive limit of algebras $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ as follows. Let the chains $\nu_1 = \{a^\alpha\}_{\alpha\succ\alpha_1}$ and $\nu_2 = \{b^\alpha\}_{\alpha\succ\alpha_2}$ be representatives of two elements in $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$. Let $\gamma \in \Sigma$, $\gamma \succ \alpha_1, \alpha_2$. The sum $\nu_1 + \nu_2$ is defined as the equivalence class of the chain $\{a^\alpha + b^\alpha\}_{\alpha\succ\gamma} \in \mathcal{N}$. The product in A is defined in a similar way. It is easy to see that the inductive limit $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is an algebra under these operations.

In the case when the index set Σ is the set of natural numbers $\mathbb{N}$ with the natural ordering, $\{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ is called also an *inductive sequence*, and is expressed by the diagram

$$A^1 \xrightarrow{\iota_1^2} A^2 \xrightarrow{\iota_2^3} A^3 \xrightarrow{\iota_3^4} \cdots .$$

The fact that the algebra $A = \varinjlim \{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ is the limit of the inductive sequence $\{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ can be expressed by the diagram

$$A^1 \xrightarrow{\iota_1^2} A^2 \xrightarrow{\iota_2^3} A^3 \xrightarrow{\iota_3^4} \cdots \longrightarrow A.$$

The inverse systems are dual objects to the inductive ones. Consider a family $\{S_\alpha\}_{\alpha\in\Sigma}$ of sets, parametrized by a directed index set Σ. Suppose that for every pair S_α, S_β, $\alpha \prec \beta$, of sets there is a mapping $\tau_\alpha^\beta\colon S_\beta \longrightarrow S_\alpha$. The collection $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is called an *inverse system* (or, inverse spectrum, projective spectrum) of S_α with connecting mappings τ_α^β, if

(i) τ_α^α is the identity on S_α, and

(ii) $\tau_\alpha^\beta \circ \tau_\beta^\gamma = \tau_\alpha^\gamma$ whenever $\alpha \prec \beta \prec \gamma$.

A *chain* of the system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is any element $\{s_\alpha\}_{\alpha\in\Sigma}$ in the Cartesian product $\prod_{\alpha\in\Sigma} S_\alpha$ such that $\tau_\alpha^\beta(s_\beta) = s_\alpha$ whenever $\alpha \prec \beta$. The family of all chains is denoted by $\varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, and is called the *inverse limit* of the system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$. Clearly, S is a subset of the Cartesian product $\prod_{\alpha\in\Sigma} S_\alpha$. The limit $S = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ of an inverse system can be mapped naturally into every coordinate set S_β by the β-coordinate projection $\pi_\beta : S \longrightarrow S_\beta : \pi\big(\{s_\alpha\}_{\alpha\in\Sigma}\big) = s_\beta$.

In the case when $\Sigma = \mathbb{N}$ with the natural ordering, $\{S_n, \iota_n^m\}_{n\in\mathbb{N}}$ is called also *inverse sequence*, and is expressed by the diagram

$$S_1 \xleftarrow{\tau_1^2} S_2 \xleftarrow{\tau_2^3} S_3 \xleftarrow{\tau_3^4} \cdots .$$

The fact that the set S is the limit of the inverse sequence $\{S_n, \iota_n^m\}_{n\in\mathbb{N}}$ can be expressed also by the diagram

$$S_1 \xleftarrow{\tau_1^2} S_2 \xleftarrow{\tau_2^3} S_3 \xleftarrow{\tau_3^4} \cdots \longleftarrow S.$$

Example 1.3.2. (a) Let $\{S_\alpha\}_{\alpha\in\Sigma}$ be a family of sets, such that $S_\alpha \supset S_\beta$ whenever $\alpha \prec \beta$. Let τ_α^β be the inclusion mapping of S_β into S_α, i.e. $\tau_\alpha^\beta(s) = s \in S_\alpha$ for every $s \in S_\beta$. It is easy to see that in this case the family $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is an inverse system, and $\varinjlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma} = \bigcap_{\alpha\in\Sigma} S_a$. By definition, the projection π_α of $\bigcap_{\alpha\in\Sigma} S_a = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ into a fixed coordinate set S_β maps every chain $\{s_\alpha\}_{\alpha\in\Sigma}$ with $s_\alpha = s$ to its β coordinate $s \in S_\beta$, i.e. π_α is the inclusion mapping of $\bigcap_{\alpha\in\Sigma} S_a$ into S_β.

(b) Let S be a set, and let Σ be an index set. Consider the inverse system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, where $S_\alpha = S$ for $\alpha \in \Sigma$, and every τ_α^β, $\alpha, \beta \in \Sigma$, is the identity mapping on S. It is easy to check that the limit $\varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ of this system is bijective to S.

If all coordinate sets S_α are topological spaces, then the inverse limit $S = \varprojlim \{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ can be equipped with the topology inherited on S from the Cartesian product $\prod_{\alpha\in\Sigma} S_\alpha$. If, in addition the mappings $\tau_\alpha^\beta \colon S_\alpha \longrightarrow S_\beta$ are continuous, then so are all projections $\pi_\alpha \colon S \longrightarrow S_\alpha$. One can show that if all S_α are compact sets, and $\tau_\alpha^\beta \colon S_\beta \longrightarrow S_\alpha$ are continuous mappings, then S is also a compact set. If all sets S_α have a particular algebraic structure, and the mappings τ_α^β respect this structure, then, in principle, the inverse limit S inherits this structure. For instance, if all S_α are groups [resp. semigroups], and all τ_α^β are group [resp.

semigroup] homomorphisms, then $S = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is also a group [resp. semigroup], actually a subgroup [resp. subsemigroup] of the Cartesian product $\prod_{\alpha\in\Sigma} S_\alpha$.

Example 1.3.3. Consider the inverse sequence

$$\mathbb{T}_1 \xleftarrow{\tau_1^2} \mathbb{T}_2 \xleftarrow{\tau_2^3} \mathbb{T}_3 \xleftarrow{\tau_3^4} \mathbb{T}_4 \xleftarrow{\tau_4^5} \cdots,$$

where $\mathbb{T}_k = \mathbb{T}$ are unit circles, and $\tau_k^{k+1}(z) = z^2$ on $\mathbb{T}$. Since all $\mathbb{T}_k$ are compact abelian groups, and z^2 is a continuous group homomorphism, the inverse limit $\varprojlim \{\mathbb{T}_{k+1}, z^2\}_{k\in\mathbb{N}}$ is a compact abelian group. Similarly, if $\overline{\mathbb{D}}_k = \overline{\mathbb{D}}$ are closed unit discs, and $\tau_k^{k+1}(z) = z^2$ on $\mathbb{D}$, the limit of the inverse sequence

$$\overline{\mathbb{D}}_1 \xleftarrow{\tau_1^2} \overline{\mathbb{D}}_2 \xleftarrow{\tau_2^3} \overline{\mathbb{D}}_3 \xleftarrow{\tau_3^4} \overline{\mathbb{D}}_4 \xleftarrow{\tau_4^5} \cdots,$$

is a compact abelian semigroup, containing the inverse limit $\varprojlim \{\mathbb{T}_{k+1}, z^2\}_{k\in\mathbb{N}}$.

Let $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be an inductive sequence of uniform algebras $A^\alpha \subset C(X_\alpha)$, and let $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be its limit algebra. The maximal ideal spaces $\mathcal{M}_{A^a}$ can be lined up into an adjoint inverse system, namely $\{\mathcal{M}_{A^\alpha}, (\iota_\alpha^\beta)^*\}_{\alpha\in\Sigma}$, where the mappings $(\iota_\alpha^\beta)^* \colon \mathcal{M}_{A^\beta} \longrightarrow \mathcal{M}_{A^\alpha}$ are the adjoint mappings of $\iota_\alpha^\beta \colon A^\alpha \longrightarrow A^\beta$, defined as $\big((\iota_\alpha^\beta)^*(\varphi)\big)(f) = \varphi\big(\iota_\alpha^\beta(f)\big)$, where $\varphi \in \mathcal{M}_{A^\beta}$, and $f \in A^\alpha$. The inverse limit $\mathcal{M}_A = \varprojlim \big\{\mathcal{M}_{A^\alpha}, (\iota_\alpha^\beta)^*\big\}_{\alpha\in\Sigma}$ of maximal ideal spaces $\mathcal{M}_{A^a}$ is a compact set. Suppose that the adjoint mappings $(\iota_\alpha^\beta)^*$ map the sets $X_\beta \subset \mathcal{M}_{A^\beta}$ onto $X_\alpha \subset \mathcal{M}_{A^\alpha}$ for every $\alpha, \beta \in \Sigma$. There arises an inverse system $\big\{X_\alpha, (\iota_\alpha^\beta)^*|_{X_\alpha}\big\}$, and its limit $X = \varprojlim \big\{X_\alpha, (\iota_\alpha^\beta)^*|_{X_\alpha}\big\}_{\alpha\in\Sigma}$ is a closed subset of $\mathcal{M}_A$.

There is a close relationship between the properties of the limit algebra A and its coordinate algebras A_α (cf. [L1]).

Proposition 1.3.4. *Assume that $\iota_\alpha^\beta(1) = 1$, and that the adjoint mappings $(\iota_\alpha^\beta)^*$ map X_β onto X_α for every $\alpha, \beta \in \Sigma$. Then:*

(i) *$A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ can be viewed as an algebra of continuous functions on X.*

(ii) *The maximal ideal space of the closure $\overline{A}$ of A in $C(X)$ coincides with the set $\mathcal{M}_A$.*

(iii) *If $(\iota_\alpha^\beta)^*$ maps the Shilov boundary ∂A^β onto ∂A^α for every $\alpha \prec \beta$, then the Shilov boundary of $\overline{A}$ is the set $\varprojlim \big\{\partial A^\alpha, (\iota_\alpha^\beta|_{\partial A^a})^*\big\}_{\alpha\in\Sigma}$.*

(iv) *If every A^α is a Dirichlet algebra on X_α, then $\overline{A}$ is a Dirichlet algebra on X.*

Definition 1.3.5. The closure $\overline{A}$ of an inductive limit $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ of algebras is called an *inductive limit algebra*.

Similarly to the case of algebras, we can consider also inductive limits of groups. An *inductive system* of groups is a family $\{G^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, where $\{G^\alpha\}_{\alpha\in\Sigma}$ is a collection of groups parametrized by a directed set Σ, and $\iota_\alpha^\beta\colon G^\alpha \longrightarrow G^\beta$ are group homomorphisms with the properties (i) and (ii). In a similar way one can define the limit of an inductive system of groups $\varinjlim \{G^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, which is again a group.

Example 1.3.6. Let $\Lambda \subset \mathbb{R}_+$ be a basis in $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. Consider the family J of pairs $\{(\gamma, n)\}$, where γ is a finite subset in Λ and n is a natural number. We equip J with the ordering

$$(\gamma, n) \prec (\delta, k) \text{ if and only if } \gamma \subset \delta \text{ and } n < k.$$

For any $(\gamma, n) \in J,\ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ define the set

$$\Gamma_{(\gamma,n)} = \big\{(1/n!)\,(m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k)\colon m_j \in \mathbb{Z},\ j = 1, \ldots k\big\}.$$

Clearly $\Gamma_{(\gamma,n)}$ is a subgroup of $\mathbb{R}$ isomorphic to $\mathbb{Z}^k = \bigoplus_{i=1}^{k} \mathbb{Z}$, and $\Gamma_{(\gamma,n)} \subset \Gamma_{(\delta,k)}$ whenever $(\gamma, n) \prec (\delta, k)$. With the natural inclusions as connecting homomorphisms, the family $\big\{\Gamma_{(\gamma,n)}\big\}_{(\gamma,n)\in J}$ is an inductive system of groups, and

$$\varinjlim \big\{\Gamma_{(\gamma,n)}\big\}_{(\gamma,n)\in J} = \bigcup_J \Gamma_{(\gamma,n)} = \mathbb{R}.$$

Given an inverse system of groups $\{G_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, their dual groups $\widehat{G}_\alpha$ form an adjoint inductive system, $\big\{\widehat{G}_\alpha, (\tau_\alpha^\beta)^*\big\}_{\alpha\in\Sigma}$, where the mappings $(\tau_\alpha^\beta)^*\colon \widehat{G}_\alpha \longrightarrow \widehat{G}_\beta$, $\alpha \prec \beta$, are the adjoint of $\tau_\alpha^\beta\colon G_\beta \longrightarrow G_\alpha$ mappings, defined by $\big((\tau_\alpha^\beta)^*(\chi)\big)(g) = \chi\big(\tau_\alpha^\beta(g)\big)$, where $\chi \in \widehat{G}_\alpha$, and $g \in G_\beta$. Similarly, the adjoint sequence of an inductive sequence of groups is the inverse system of their dual groups. Moreover, the dual group of the inverse limit $\varprojlim \{G_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is the inductive limit of the adjoint system $\varinjlim \big\{\widehat{G}_\alpha, (\tau_\alpha^\beta)^*\big\}_{\alpha\in\Sigma}$, and vice versa, the dual group of an inductive limit $\varinjlim \{G^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is the inverse limit of the adjoint system $\varprojlim \big\{\widehat{G}^\alpha, (\iota_\alpha^\beta)^*\big\}_{\alpha\in\Sigma}$.

Example 1.3.7. Let $\Lambda = \{d_k\}_{k=1}^\infty$ be a sequence of natural numbers. Suppose that $m_k = \prod_{l=1}^k d_l,\ m_0 = 1$, and denote by Γ_Λ the subgroup of $\mathbb{Q}$ that is generated by the numbers $1/m_k,\ k \in \mathbb{N}$. In particular, if $d_k = 2$ for every $k \in \mathbb{Z}$, Γ_Λ is the group of dyadic numbers in $\mathbb{R}$. The group Γ_Λ can be expressed as the inductive limit of groups $\mathbb{Z}$, namely

$$\mathbb{Z}^{(1)} \xrightarrow{\ \zeta_1^2\ } \mathbb{Z}^{(2)} \xrightarrow{\ \zeta_2^3\ } \mathbb{Z}^{(3)} \xrightarrow{\ \zeta_3^4\ } \mathbb{Z}^{(4)} \xrightarrow{\ \zeta_4^5\ } \cdots \longrightarrow \Gamma_\Lambda,$$

where $\zeta_k^{k+1}(m_k) = d_k \cdot m_k,\ m_k \in \mathbb{Z}^{(k)} = \mathbb{Z}$. The corresponding dual groups $\widehat{\mathbb{Z}}^k \cong \mathbb{T}_k = \mathbb{T}$ form an inverse sequence of unit circles

$$\mathbb{T}_1 \xleftarrow{\ \tau_1^2\ } \mathbb{T}_2 \xleftarrow{\ \tau_2^3\ } \mathbb{T}_3 \xleftarrow{\ \tau_3^4\ } \mathbb{T}_4 \xleftarrow{\ \tau_4^5\ } \cdots \longleftarrow G_\Lambda, \tag{1.16}$$

whose limit $G_\Lambda = \widehat{\Gamma}_\Lambda$ is a compact abelian group. Here $\tau_k^{k+1}(z) = (\zeta_k^{k+1})^*(z) = z^{d_k}$. Indeed, $\tau_k^{k+1}(e^{itm}) = e^{it\zeta_k^{k+1}(m)} = e^{itd_k m} = (e^{itm})^{d_k}$ for every point $e^{itm} \in \mathbb{T}_k = \widehat{\mathbb{Z}}^k$. Observe that both G_Λ and the limit $\mathcal{D}_\Lambda = \varprojlim \{\mathbb{D}_{k+1}, z^{d_k}, d_k \in \Lambda\}_{k\in\mathbb{N}}$ of the corresponding inverse sequence of unit discs $\mathbb{D}_k = \mathbb{D}$,

$$\mathbb{D}_1 \xleftarrow{\tau_1^2} \mathbb{D}_2 \xleftarrow{\tau_2^3} \mathbb{D}_3 \xleftarrow{\tau_3^4} \mathbb{D}_4 \xleftarrow{\tau_4^5} \cdots \longleftarrow \mathcal{D}_\Lambda \tag{1.17}$$

are subsets of the limit $\varprojlim \{\overline{\mathbb{D}}_{k+1}, z^{d_k}, d_k \in \Lambda\}_{k\in\mathbb{N}}$ of the inverse sequence

$$\overline{\mathbb{D}}_1 \xleftarrow{\tau_1^2} \overline{\mathbb{D}}_2 \xleftarrow{\tau_2^3} \overline{\mathbb{D}}_3 \xleftarrow{\tau_3^4} \overline{\mathbb{D}}_4 \xleftarrow{\tau_4^5} \cdots \longleftarrow \tag{1.18}$$

Note that $\mathcal{D}_\Lambda$ is an open subset in the compact set $\varprojlim \{\overline{\mathbb{D}}_{k+1}, z^{d_k}, d_k \in \Lambda\}_{k\in\mathbb{N}}$, and its topological boundary is G_Λ.

Given an inverse system of topological spaces $\{X_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, one can consider inductive limits of various function algebras on X_α. For instance the spaces $C(X_\alpha)$ of continuous functions on X_α can be lined up as an adjoint inductive system, namely $\{C(X_\alpha), (\tau_\alpha^\beta)^*\}_{\alpha\in\Sigma}$ where the mappings $(\tau_\alpha^\beta)^* \colon X_\alpha \longrightarrow X_\beta,\ \alpha \prec \beta$ are the adjoint mappings to $\tau_\alpha^\beta \colon X_\beta \longrightarrow X_\alpha$, defined as $\big((\tau_\alpha^\beta)^*(f)\big)(x) = f\big(\tau_\alpha^\beta(x)\big)$, where $f \in C(X_\alpha)$, and $x \in X_\beta$. One can show that

$$\varinjlim \ \{C(X_\alpha), (\tau_\alpha^\beta)^*\}_{\alpha\in\Sigma} = C\big(\varinjlim \{X^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}\big).$$

In particular, the adjoint sequence of an inverse sequence of topological sets

$$X_1 \xleftarrow{\tau_1^2} X_2 \xleftarrow{\tau_2^3} X_3 \xleftarrow{\tau_2^3} X_4 \xleftarrow{\tau_4^5} \cdots \longleftarrow X$$

is the inductive sequence

$$C(X_1) \xrightarrow{(\tau_1^2)^*} C(X_2) \xrightarrow{(\tau_2^3)^*} C(X_3) \xrightarrow{(\tau_3^4)^*} C(X_4) \xrightarrow{(\tau_4^5)^*} \cdots \longrightarrow C(X).$$

We say that two inductive sequences $\{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ and $\{B^n, \jmath_n^m\}_{n\in\mathbb{N}}$ of algebras are *isomorphic* if there exist isomorphisms $\psi_k \colon A_k \longrightarrow B_k$ so that the infinite diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\iota_1^2} & A_2 & \xrightarrow{\iota_2^3} & \mathcal{A}_3 & \xrightarrow{\iota_3^4} & \cdots \\ \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow & & \\ B_1 & \xrightarrow{\jmath_1^2} & B_2 & \xrightarrow{\jmath_2^3} & B_3 & \xrightarrow{\jmath_3^4} & \cdots \end{array} \tag{1.19}$$

is *commutative*, i.e. if all its squares are commutative.

Likewise, two inverse sequences $\{X^n, \sigma_n^m\}_{n\in\mathbb{N}}$ and $\{Y^n, \tau_n^m\}_{n\in\mathbb{N}}$ of sets are isomorphic if there exist bijections $\varphi_k : X_k \longrightarrow Y_k$ so that the infinite diagram

$$\begin{array}{ccccccc} X_1 & \xleftarrow{\sigma_1^2} & X_2 & \xleftarrow{\sigma_2^3} & X_3 & \xleftarrow{\sigma_3^4} & \cdots \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \\ Y_1 & \xleftarrow{\tau_1^2} & Y_2 & \xleftarrow{\tau_2^3} & Y_3 & \xleftarrow{\tau_3^4} & \cdots \end{array} \tag{1.20}$$

is commutative. The proof of the following proposition uses standard algebraic and topological arguments.

Proposition 1.3.8. (i) *The limits A and B of two isomorphic inductive sequences of algebras*

$$A^1 \xrightarrow{\iota_1^2} A^2 \xrightarrow{\iota_2^3} A^3 \xrightarrow{\iota_3^4} \cdots \longrightarrow A, \text{ and}$$

$$B^1 \xrightarrow{\jmath_1^2} B^2 \xrightarrow{\jmath_2^3} B^3 \xrightarrow{\jmath_3^4} \cdots \longrightarrow B$$

are isomorphic algebras. If A^k, B^k possess other algebraic structure, and if ι_k^{k+1}, $\jmath_k^{k+1}$ and ψ_k from (1.19) *respect it, then A and B possess the same structure, and are isomorphic with respect to it.*

(ii) *The limits U and V of two isomorphic inverse sequences of sets*

$$X_1 \xleftarrow{\sigma_1^2} X_2 \xleftarrow{\sigma_2^3} X_3 \xleftarrow{\sigma_3^4} \cdots \longleftarrow X, \text{ and}$$

$$Y_1 \xleftarrow{\tau_1^2} Y_2 \xleftarrow{\tau_2^3} Y_3 \xleftarrow{\tau_3^4} \cdots \longleftarrow Y$$

are bijective. If X_k, Y_k possess any particular algebraic structure, and σ_k^{k+1}, τ_k^{k+1} and φ_k from (1.20) *respect it, then X and Y possess the same structure, and are isomorphic with respect to it; If X_k, Y_k are topological spaces, and σ_k^{k+1}, τ_k^{k+1} are continuous mappings, and φ_k are homeomorphisms, then X and Y are homeomorphic topological spaces.*

1.4 Bourgain algebras of commutative Banach algebras

The norm topology of a commutative Banach algebra is too rough to reveal some of its hidden properties. Therefore, weaker topologies which contain important information about original algebras are also of importance, and they can be used to construct associated algebras.

Let B be a commutative Banach algebra with norm $\|\cdot\|$ and let $A \subset B$ be a linear subspace of B (not necessarily closed), equipped with the restriction $\|\cdot\|_A$ of the norm $\|\cdot\|_B$ on B on A. Let $\pi_A \colon B \longrightarrow B/A$ be the natural projection of B onto the quotient algebra B/A. For every fixed $f \in B$ let $P_f \colon A \longrightarrow fA \subset B$ be the multiplication by $f \in B$ on A. The mapping

$$S_f = \pi_A \circ P_f \colon A \longrightarrow (fA + A)/A \subset B/A \colon g \longmapsto \pi_A(fg)$$

is called a *Hankel type operator* on A corresponding to f. Note that π_A and S_f both are bounded linear operators onto B/A and onto $(fA + A)/A \subset B/A$ correspondingly. Denote by $c_0^w(A)$ the family of weakly null sequences in A. Hence, a sequence $\{\varphi_n\}$ in A belongs to $c_0^w(A)$ if $L(\varphi_n) \longrightarrow 0$ as $n \longrightarrow \infty$ for every bounded linear functional L on A.

Definition 1.4.1. An element f in B is said to be a *Bourgain element* of A with respect to B if its corresponding Hankel type operator $S_f\colon A \longrightarrow B/A$ is completely continuous. The collection of Bourgain elements with respect to B is denoted by A_b^B, and is called the *Bourgain algebra* of A with respect to B.

Clearly, $f \in A_b^B$ if the operator S_f maps every weakly null sequence of A onto a null sequence under the quotient norm on $\pi_A(fA) \subset B/A$. Equivalently, $f \in A_b^B$ if and only if for every weakly null sequence $\{\varphi_n\}_n$ in A there exists a sequence $\{h_n\}_n,\ h_n \in A$ such that $\|f\varphi_n - h_n\|_B \longrightarrow 0$ as $n \longrightarrow \infty$. If $c_0^\tau(A)$ denotes the family of all sequences of elements in A that converge to 0 with respect to a given topology τ on A, then $f \in A_b^B$ if and only if $S_f(c_0^w(A)) \subset c_0^{\|\cdot\|}(B/A)$. A_b^B is a commutative Banach algebra, and if A is an algebra, then $A \subset A_b^B$ (e.g. [CT]). If $A \subset B \subset C$ are commutative Banach algebras then $A_b^B = A_b^C \cap B$.

Proposition 1.4.2. *If the range $S_f(A) = \pi_A(fA)$ of the Hankel type operator S_f corresponding to $f \in B$ is finite-dimensional, then $f \in A_b^B$.*

Proof. If $\{\varphi_n\}_n$ is a weakly null sequence in A, then the sequence $\{f\varphi_n\}_n$ is also weakly null in B, and therefore $\big\{\pi_A(\varphi_n)\big\}_n$ is a weakly null sequence in $\pi_A(fA) \subset B/A$. Hence $\big\{\pi_A(\varphi_n)\big\}_n \in c_0^{\|\cdot\|}\big(\pi_A(fA)\big) \subset c_0^{\|\cdot\|}(B/A)$, since $\pi_A(fA)$ is finite-dimensional. Consequently $f \in A_b^B$. □

As the following example shows, the range of the completely continuous operator S_f need not be finite-dimensional.

Example 1.4.3. Let $A = A(\mathbb{T})$ be the disc algebra on the unit circle $\mathbb{T}$ and let $B = C(\mathbb{T})$. Consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2 z^k}.$$

Clearly, $f \in C(\mathbb{T}) = A_b^B$. We claim that the range of the Hankel type operator S_f corresponding to f is infinite-dimensional. Indeed, let $c_n = \|z^n f + A\|_{B/A}$, and let $g_n(z) = (1/c_n)z^n$. Clearly, $g_n \in A$, and $\|g_n f + A\|_{B/A} = 1$. To see that $\pi_A(fA)$ is not finite-dimensional it is enough to show that the sequence $\{g_n f + A\}_n$ converges weakly to 0 in B/A.

We can evaluate the $(-m)$-th Fourier coefficient $c_{-m}^{g_n f}$ of the function $g_n f$. Namely,

$$c_{-m}^{g_n f} = \int_{\mathbb{T}} g_n(z)\, f(z)\, z^m\, dz = \frac{1}{c_n (n+m)^2},\ \ m, n \geq 1.$$

Hence,

$$c_n \geq \|z^n f + H^2\|_{L^2/H^2} = \sqrt{\sum_{k=1}^{\infty} \frac{1}{(n+k)^4}},$$

and therefore,

$$\frac{1}{n^2 c_n} \le \frac{1}{\sqrt{n^4 \sum_{k=1}^{\infty} \frac{1}{(n+k)^4}}}.$$

Note that

$$\sum_{k=1}^{\infty} \frac{1}{(n+k)^4} \ge \int_{n+1}^{\infty} \frac{1}{x^4}\, dx = \frac{1}{3(n+1)^3},$$

thus

$$\lim_{n\to\infty} n^4 \sum_{k=1}^{\infty} \frac{1}{(n+k)^4} = \infty.$$

Hence, $\lim_{n\to\infty} 1/(n^2 c_n) = 0$, and therefore

$$\lim_{n\to\infty} \int_{\mathbb{T}} g_n(z)\, f(z)\, z^m\, dz = \lim_{n\to\infty} \frac{1}{c_n (n+m)^2} = 0$$

for all $m \in \mathbb{N}$. It now follows that if p is any polynomial with $p(0) = 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{T}} g_n(z)\, f(z)\, p(z)\, dz = 0.$$

Recall that if X is a Banach space and $\{x_n\}_n$ is a bounded sequence in X tending to zero on a norm-dense set of the dual space X^*, then $\{x_n\}_n$ is weakly null. Since the space H_0^1 is isometrically isomorphic to $\big(C(\mathbb{T})/A(\mathbb{T})\big)^*$, and the polynomials p with $p(0) = 0$ are dense in H_0^1, the sequence $\{g_n f + A\}_n$ converges weakly to zero in B/A, as claimed. □

The next proposition asserts that the complete continuity property of Hankel type operators is invariant under algebraic isometries.

Proposition 1.4.4. *Let $A \subset C$ and $B \subset D$ be two pairs of commutative Banach algebras. If $T\colon C \longrightarrow D$ is an isometric algebra isomorphism with $T(A) = B$, then the Hankel type operator $S_{T(f)}\colon B \longrightarrow D/B$ is completely continuous if and only if $S_f\colon A \longrightarrow C/A$ is completely continuous.*

Proof. Note that T maps the set $c_0^w(A)$ onto the set $c_0^w(B)$ because T^* is an isometry. Secondly, for $\psi_n, g_n \in A$ and $f \in C$ we have that

$$\|\psi_n f - g_n\| = \big\|T(\psi_n f) - T(g_n)\big\| = \big\|T(\psi_n)\, T(f) - T(g_n)\big\|.$$

Thus, if $\{\psi_n\} \in c_0^w(A)$ and S_f is completely continuous, then $\big\{T(\psi_n)\big\} \in c_0^w(B)$ and $T(g_n) \in B$. Consequently, $S_{T(f)}$ is completely continuous since all weakly null sequences in B are of type $\big\{T(\psi_n)\big\}$, where $\{\psi_n\} \in c_0^w(A)$. The argument is readily reversible. □

Note that Proposition 1.4.4 holds for the case when T is a Banach algebra isomorphism, i.e. is a continuous isomorphism of the algebraic structure that has a continuous inverse. Moreover, if T is an algebra homomorphism, while $T|_A$ is a topological linear isomorphism, one can show that $S_{T(f)}$ is completely continuous simultaneously with S_f.

Corollary 1.4.5. *In the setting of Proposition* 1.4.4, $T(A_b^C) = \big(T(A)\big)_b^{T(C)} = B_b^D$.

Corollary 1.4.5 is quite natural since Bourgain algebras are defined solely in terms of algebraic and metric conditions. Since the restriction map $T\colon B \longrightarrow B\big|_{\partial B}$ on the Shilov boundary ∂B is an isometric algebra isomorphism. Proposition 1.4.4 and Corollary 1.4.5 yield the following

Corollary 1.4.6. *Let X be a compact Hausdorff space and $A \subset B \subset C(X)$ be uniform algebras. Then:*

(i) *The Hankel type operator $S_{f|_{\partial B}}\colon A\big|_{\partial B} \longrightarrow \big(B\big|_{\partial B}\big)/\big(A\big|_{\partial B}\big)$ is completely continuous if and only if $S_f\colon A \longrightarrow B/A$ is completely continuous.*

(ii) $\big(A\big|_{\partial B}\big)_b^{B|_{\partial B}} = A_b^B\big|_{\partial B}$.

Observe that Proposition 1.4.4 and Corollary 1.4.5 do not hold for an isometry T between A and B that is not extendable as an isometry between the algebras C and D. For example, the Bourgain algebras of H^∞ relative to L^∞ for the unit disc $\mathbb{D}$ and the unit circle $\mathbb{T} = \partial\mathbb{D}$ are given respectively by

$$H^\infty(\mathbb{T})_b^{L^\infty(\mathbb{T})} = H^\infty(\mathbb{T}) + C(\mathbb{T}), \tag{1.21}$$

$$H^\infty(\mathbb{D})_b^{L^\infty(\mathbb{D})} = H^\infty(\mathbb{D}) + C_u(\mathbb{D}) + V(\mathbb{D}), \tag{1.22}$$

where $C_u(\mathbb{D})$ is the space of uniformly continuous functions on $\mathbb{D}$ and $V(\mathbb{D})$ is the ideal of functions in $L^\infty(\mathbb{D})$ that vanish near the boundary, namely, $f \in V(\mathbb{D})$ if for every $\varepsilon > 0$ there is a compact set $K \subset \mathbb{D}$ for which $\operatorname{ess\,sup}_{z\in K}\big|f(z)\big| < \varepsilon$ [CSY, Y]. Here the boundary value mapping $f \longmapsto f^*$ from $H^\infty(\mathbb{D})$ to $H^\infty(\mathbb{T})$ is an isometry which does not extend to the corresponding L^∞ algebras nor does it even extend to the corresponding Bourgain algebras of H^∞. However, the boundary value mapping extends isometrically from $H^\infty(\mathbb{D})$ to the algebra $\mathcal{U}(\mathbb{D}) = [H^\infty(\mathbb{D}), \overline{H}^\infty(\mathbb{D})]$ generated by $H^\infty(\mathbb{D})$ and $\overline{H}^\infty(\mathbb{D})$. Indeed, it extends to the generators of $\mathcal{U}(\mathbb{D})$ and a closure argument provides a further extension to $\mathcal{U}(\mathbb{D})$.

Note that $\mathcal{U}(\mathbb{D}) = \big[H^\infty(\mathbb{D}), \overline{H}^\infty(\mathbb{D})\big] \cong C(\mathcal{M}_{H^\infty(\mathbb{D})})$, and isometries on $H^\infty(\mathbb{D})$ induced by automorphisms of $\mathbb{D}$ extend naturally to isometries of $\mathcal{U}(\mathbb{D})$.

Consider the algebras $H^\infty(\mathbb{T})$ and $H^\infty(\mathbb{D})$. Note that the maximal ideal spaces of the corresponding algebras L^∞ are the corresponding sets ∂H^∞. Since the mapping $\varLambda\colon H^\infty \longrightarrow C(\partial H^\infty)\colon f \longmapsto \widetilde{f}\big|_{\partial H^\infty}$ is an isometry, we have

Corollary 1.4.7. (i) $\big(\widehat{H}^\infty(\mathbb{T})\big|_{\partial H^\infty}\big)_b^{\widehat{L}^\infty(\mathbb{T})|_{\partial H^\infty}} = \widehat{H}^\infty(\mathbb{T})\big|_{\partial H^\infty} + \widehat{C}(\mathbb{T})\big|_{\partial H^\infty}$,

(ii) $\big(\widehat{H}^\infty(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})}\big)_b^{\widehat{L}^\infty(\mathbb{D})|_{\partial H^\infty(\mathbb{D})}} = \widehat{H}^\infty(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})} + \widehat{C_u}(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})}$.

This follows immediately from (1.21), (1.22) and Corollary 1.4.6. Note that Corollary 1.4.7(ii) implies, in particular, that $\widehat{H}^\infty(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})} + \widehat{C_u}(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})}$ is a closed subalgebra of $\widehat{L}^\infty(\mathbb{D})\big|_{\partial H^\infty(\mathbb{D})}$ since Bourgain algebras are automatically closed.

The Bourgain algebra A_b^B contains important information about A. If A is an algebra of continuous functions on a set Ω, then A_b^B contains also information about Ω.

Proposition 1.4.8. *If U_1 and U_2 are biholomorphically equivalent domains in $\mathbb{C}^n$, then the corresponding Bourgain algebras $H^\infty(U_1)_b^{\mathcal{U}(U_1)}$ and $H^\infty(U_2)_b^{\mathcal{U}(U_2)}$ are isometrically isomorphic.*

Proof. Let U_1 and U_2 be biholomorphically equivalent and $\tau\colon U_2 \longrightarrow U_1$ be a biholomorphic mapping. Define the map $T\colon C_b(U_1) \longrightarrow C_b(U_2)$ by

$$\big(T(f)\big)(z_1, z_2, \ldots, z_n) = f\big(\tau(z_1, z_2, \ldots, z_n)\big)$$

for all $f \in C_b(U_1)$ and $(z_1, z_2, \ldots, z_n) \in U_2$. The mapping T is an isometric algebra isomorphism with respect to the sup-norms on U_1 and U_2. Moreover, $T\big(H^\infty(U_1)\big) = H^\infty(U_2)$, $T\big(\mathcal{U}(U_1)\big) = \mathcal{U}(U_2)$. Corollary 1.4.6 now implies that $T\left(H^\infty(U_1)_b^{\mathcal{U}(U_1)}\right) = H^\infty(U_2)_b^{\mathcal{U}(U_2)}$. □

In what follows we apply completely continuous Hankel type operators related with the Bourgain algebras of the corresponding spaces H^∞ relative to the algebras $\mathcal{U} = [H^\infty, \overline{H}^\infty] \subset L^\infty$ on the unit ball $\mathbb{B}^n$ and the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, respectively, to the problem of biholomorphic equivalence of domains in $\mathbb{C}^n$. Recall that every function $f \in H^\infty(\mathbb{D}^n)$ has *radial limits*,

$$\lim_{r\nearrow 1} f\big(r(z_1, z_2, \ldots, z_n)\big) = f^*(z_1, z_2, \ldots, z_n),$$

at almost every point $(z_1, z_2, \ldots, z_n) \in \mathbb{T}^n$, and the *radial boundary value function* $f^*(z_1, z_2, \ldots, z_n)$ of any $f \in H^\infty(\mathbb{D}^n)$ belongs to $H^\infty(\mathbb{T}^n)$ (e.g. [R6], Theorem 2.3.2]).

Lemma 1.4.9. *Let $g \in \mathcal{U}(\mathbb{D}^n)$ and let the corresponding Hankel type operator $S_g : H^\infty(\mathbb{D}^n) \longrightarrow \mathcal{U}(\mathbb{D}^n)/H^\infty(\mathbb{D}^n)$ be completely continuous. If g^* is the boundary value function of g on $\mathbb{T}^n$, then the operator $S_{g^*}\colon H^\infty(\mathbb{T}^n) \longrightarrow \mathcal{U}(\mathbb{T}^n)/H^\infty(\mathbb{T}^n)$ is also completely continuous.*

This follows directly from the fact that every weakly null sequence $\{f_n\}$ in $H^\infty(\mathbb{T}^n)$ is of the form $f_n = \varphi_n^*$ where $\{\varphi_n\}$ is a weakly null sequence in $H^\infty(\mathbb{D}^n)$.

Below we apply the Bourgain algebra technique to provide an alternative proof of the Poincaré Theorem for analytic functions in several complex variables.

Theorem 1.4.10 (Poincaré Theorem). *The sets $\mathbb{B}^n$ and $\mathbb{D}^n$ are not biholomorphically equivalent if $n \geq 2$.*

Proof. Suppose that $\mathbb{D}^n$ and $\mathbb{B}^n$ are biholomorphically equivalent and let $\tau\colon \mathbb{D}^n \longrightarrow \mathbb{B}^n$ be a biholomorphic mapping between them. Define $T\colon C_b(\mathbb{B}^n) \longrightarrow C_b(\mathbb{D}^n)$, as before, by

$$\big(T(f)\big)(z_1, z_2, \ldots, z_n) = f\big(\tau(z_1, z_2, \ldots, z_n)\big)$$

for all $f \in C_b(\mathbb{B}^n)$ and $(z_1, z_2, \ldots, z_n) \in \mathbb{D}^n$. Proposition 1.4.8 implies that

$$T\left(H^\infty(\mathbb{B}^n)_b^{\mathcal{U}(\mathbb{B}^n)}\right) = H^\infty(\mathbb{D}^n)_b^{\mathcal{U}(\mathbb{D}^n)}.$$

Let f be a fixed non-constant function in the algebra $A(\mathbb{B}^n)$. Observe that $\big(T(\overline{f})\big)^*$ exists because $\overline{f} \in \overline{H}^\infty(\mathbb{B}^n)$ and so $T(\overline{f}) \in \overline{H}^\infty(\mathbb{D}^n)$.

We claim that the mapping $S_{T(\overline{f})}\colon H^\infty(\mathbb{D}^n) \longrightarrow \mathcal{U}(\mathbb{D}^n)/H^\infty(\mathbb{D}^n)$ is completely continuous. Note that $\overline{f} \in C(\overline{\mathbb{B}}^n)|_{\mathbb{B}^n}$ and also $\overline{f} \in \mathcal{U}(\mathbb{B}^n)$. An argument from [I] implies that

$$C(\overline{\mathbb{B}}^n)|_{\mathbb{B}^n} \subset H^\infty(\mathbb{B}^n)_b^{L^\infty(\mathbb{B}^n)}.$$

Consequently, by the remark immediately following Definition 1.4.1 we have

$$\overline{f} \in H^\infty(\mathbb{B}^n)_b^{L^\infty(\mathbb{B}^n)} \cap \mathcal{U}(\mathbb{B}^n) = H^\infty(\mathbb{B}^n)_b^{\mathcal{U}(\mathbb{B}^n)},$$

and hence

$$T(\overline{f}) \in T\left(H^\infty(\mathbb{B}^n)_b^{\mathcal{U}(\mathbb{B}^n)}\right) = H^\infty(\mathbb{D}^n)_b^{\mathcal{U}(\mathbb{D}^n)},$$

by Proposition 1.4.4, i.e. $S_{T(\overline{f})}$ is a completely continuous operator, as claimed. From Lemma 1.4.9 it follows that for $n \geq 2$ the radial boundary value function $\big(T(\overline{f})\big)^*$ of the non-constant anti-analytic function $T(\overline{f})$ belongs to $H^\infty(\mathbb{T}^n) = H^\infty(\mathbb{T}^n)_b^{L^\infty(\mathbb{T}^n)} = H^\infty(\mathbb{T}^n)_b^{\mathcal{U}(\mathbb{T}^n)}$ (see [I, Y]), which is impossible. □

Note that the boundary value technique avoids the need of a direct reference to the more complicated Bourgain algebras of $H^\infty(\mathbb{D}^n)$ and $H^\infty(\mathbb{B}^n)$.

The notion of Bourgain algebras can be extended for commutative topological algebras. Recall that a commutative algebra A over $\mathbb{C}$ is called a *topological algebra*, if it is provided by a topology, under which the basic operations in A are continuous. Let B be a commutative topological algebra and A be its subalgebra. Denote by $c_0^{bw}(A)$ the space of bounded weakly null sequences in A.

Definition 1.4.11. The *Bourgain algebra* A_b^B of a commutative topological algebra A relative to B is the set of all elements $f \in B$ for which $S_f\big(c_0^{bw}(A)\big) \subset c_0(B/A)$, i.e. A_b^B consists of all $f \in B$ such that for every $\{\varphi_n\} \in c_0^{bw}(A)$ there exists a sequence $\{g_n\}$ in A for which

$$\lim_{n\to\infty}(\varphi_n f - g_n) = 0. \tag{1.23}$$

It is straightforward to see that if A is an algebra, then $A \subset A_b^B$.

Proposition 1.4.12. *Let $A \subset B$ be commutative topological algebras. Every completely continuous Hankel type operator $S_f\colon A \longrightarrow \pi_A(fA)$ maps bounded weakly Cauchy sequences in A onto Cauchy sequences in B/A.*

Proof. Suppose that $\{g_n\}_n$ is a bounded weakly Cauchy sequence in A for which the sequence $\{\pi_A(fg_n)\}_n$ is not Cauchy in B/A. Then there is a neighborhood U of 0 in B/A such that for every natural $M > 0$ one can find integers $n_M, m_M \geq M$ with $\pi_A(fg_{n_M}) - \pi_A(fg_{m_M}) \notin U$. Hence the sequence $\big\{\pi_A\big(f\,(g_{n_M} - g_{m_M})\big)\big\}_{M=1}^{\infty}$ does not tend to 0 in B/A. By the complete continuity of S_f we have that the bounded sequence $\{g_{n_M} - g_{m_M}\}_{M=1}^{\infty}$ is not weakly null in A. Hence $F(g_{n_M} - g_{m_M})$ does not tend to 0 for some $F \in A^*$. Therefore $\{F(g_n)\}_n$ is not Cauchy, i.e. $\{g_n\}_n$ can not be a weakly Cauchy sequence. □

Note that the dual space B^* does not always separate the points of B for every commutative topological algebra B. Local convexity of B is a sufficient condition for this.

Theorem 1.4.13. *Let B be a commutative topological algebra and A be a subalgebra of B. The Bourgain algebra A_b^B of A relative to B is a closed commutative topological subalgebra of B.*

Proof. Let $f \in A_b^B$. Given a bounded weakly null sequence $\{\varphi_n\} \in c_0^w(A)$, $\varphi_n \in A$, there are elements $h_n \in A$, such that $\varphi_n f - h_n \longrightarrow 0$. Note that $\{h_n\}$ is a bounded weakly null sequence in A, since $\varphi_n f$ is bounded and tends weakly to 0.

Let now $f_1, f_2 \in A_b^B$, and suppose that $\{\varphi_n\}$ is a bounded weakly null sequence in A. According to (1.23) there are $h_n \in A$ such that $\varphi_n f_1 - h_n \longrightarrow 0$. By the above remark $\{h_n\}$ is a bounded weakly null sequence in A. Therefore there are $k_n \in A$ such that $h_n f_2 - k_n \longrightarrow 0$. Now

$$f_1 f_2 \varphi_n - k_n = f_2(f_1\varphi_n - h_n) + (f_2 h_n - k_n) \longrightarrow 0. \tag{1.24}$$

Consequently $f_1 f_2 \in A_b^B$ and hence A_b^B is an algebra.

Let $\{\varphi_n\}$ be a bounded weakly null sequence in A, let $f \in B$ be the limit of elements $f_k \in A_b^B$ and let U be a bounded set in A that contains $\{\varphi_n\}$. For a given neighborhood W of 0 in B let V be a neighborhood of 0 such that $V + V \subset W$. Take a neighborhood V_1 of 0 with $V_1^2 \subset V$. There is a $t > 0$ such that $tU \subset V_1$. Let k_0 be such that $f - f_k \in tV_1$ for all $k \geq k_0$, take such a k and choose $h_n^k \in A$ such that $f_n\varphi_n - h_n^k \longrightarrow 0$ as $n \longrightarrow 0$. Then $f\,\varphi_n - h_n^k = (f - f_k)\,\varphi_n + (f_k\varphi_n - h_n^k) \in t\,V_1 U + V = t\,U\cdot V_1 + V \subset V_1^2 + V \subset V + V \subset W$ for n big enough. Consequently, A_b^B is closed in B. □

Proposition 1.4.14. *Consider the algebra $B = C(\mathbb{D})$ equipped with the compact-open topology on the open unit disc $\mathbb{D}$ and let $A = \mathcal{O}(\mathbb{D})$ be the algebra of analytic functions in $\mathbb{D}$. Then $\mathcal{O}(\mathbb{D})_b^{C(\mathbb{D})} = C(\mathbb{D})$.*

Proof. Note that $C(\mathbb{D})$ and $\mathcal{O}(\mathbb{D})$ both are Frechét algebras. Therefore every weakly null sequence in $C(\mathbb{D})$ is bounded by the Uniform Boundedness Principle (e.g. [R7], Theorem 2.6). First we show that the function $\bar{z}$ belongs to $\mathcal{O}(\mathbb{D})_b^{C(\mathbb{D})}$. The argument is similar to the corresponding one for the case of $A(\mathbb{D})$ (e.g. [CSY]). Given a weakly null sequence $\{\varphi_n\}$ in $\mathcal{O}(\mathbb{D})$, consider the functions $h_n(z) = \dfrac{\varphi_n(z) - \varphi_n(0)}{z} \in \mathcal{O}(\mathbb{D})$. We observe that $h_n(z)$ tends weakly to 0 in $\mathcal{O}(\mathbb{D})$ since the map $\varphi \longmapsto h = \dfrac{\varphi - \varphi(0)}{z}$ from $\mathcal{O}(\mathbb{D})$ into itself, is a continuous linear operator. Since $\varphi_n(z) = z\,h_n(z) + \varphi_n(0)$, then $\overline{z}\,\varphi_n(z) - h_n(z) = (|z|^2 - 1)\ h_n(z) + \overline{z}\varphi_n(0)$. For any fixed $r \in (0,1)$ we have $\max\limits_{z\in\mathbb{D}_r} |\overline{z}\,\varphi_n(z) - h_n(z)| \leq \max\limits_{z\in\mathbb{D}_r} |h_n(z)| + |\varphi_n(0)| \longrightarrow 0$ by Montel's theorem. Therefore $\overline{z}\,\varphi_n(z) - h_n(z)$ tends to 0 in the compact open topology in $\mathbb{D}$. Consequently $\bar{z} \in \mathcal{O}(\mathbb{D})_b^{C(\mathbb{D})}$, as claimed.

Thus $\mathcal{O}(\mathbb{D})_b^{C(\mathbb{D})}$ contains the restrictions of all polynomials in z and $\bar{z}$ on $\mathbb{D}$; therefore it contains the algebra $C(\overline{\mathbb{D}})$ by the Stone-Weierstrass theorem. Since $C(\mathbb{D})$ is the closure of $C(\overline{\mathbb{D}})$ in the compact open topology in $C(\mathbb{D})$ we conclude that $\mathcal{O}(\mathbb{D})_b^{C(\mathbb{D})} = C(\mathbb{D})$. □

The Bourgain algebra construction can be applied to a rather general situation. Note that the class $c_0^w(M)$ of weakly null sequences on a subspace M of a commutative Banach algebra B in the Bourgain algebra construction can be replaced by an arbitrary class of sequences $\mathcal{S}(M)$. To be more precise, let us denote the Bourgain algebra by $A_b^B(c_0^w)$ rather than by A_b^B. We define the space $A_b^B(\mathcal{S})$ in a way analogous to $A_b^B(c_0^w)$ by requiring $\{\varphi_n\} \in \mathcal{S}(A)$ instead of $\{\varphi_n\} \in c_0^w(A)$, i.e.

$$A_b^B(\mathcal{S}) = \big\{f \in B\colon S_f\big(\mathcal{S}(A)\big) \subset c_0^{\|\cdot\|}(B/A)\big\}.$$

If the class $\mathcal{S}$ is contained in c_0^w, i.e. if $\mathcal{S}(M) \subset c_0^w(M)$ for all M, then $A_b^B(\mathcal{S}) \supset A_b^B(c_0^w) = A_b^B$. If $\mathcal{S}$ contains c_0^w, then $A_b^B(\mathcal{S})$ is smaller than A_b^B. For example, if $\mathcal{S}(M)$ is the class B_M^w of weakly bounded sequences in M and A is an algebra, then $A_b^B(B^w)$ is simply the norm closure of A. The basic argument in [CT] can be carried over to this general setting to show that $A_b^B(\mathcal{S})$ is a closed subalgebra of B whenever the class $\mathcal{S}$ satisfies the following properties:

(i) $\mathcal{S}(M) \subset \mathcal{S}(N)$ whenever $M \subset N \subset B$,

(ii) $\mathcal{S}(M) \subset \mathcal{B}^w(M)$ for all M,

(iii) if $\{\varphi_n\} \in \mathcal{S}(A)$ and $f \in B$, then $\{f\varphi_n\} \in \mathcal{S}(B)$,

(iv) if $\{\varphi_n\} \in \mathcal{S}(B)$ and $\psi_n \in A$ with $\|\varphi_n - \psi_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then $\{\psi_n\} \in \mathcal{S}(A)$.

Properties (i) and (ii) are required for all linear subspaces M and N of B, while (iii) and (iv) are needed only for the fixed A and B under consideration. Observe that by (ii) the elements of the sequences in $\mathcal{S}(M)$ are always in M.

1.5 Polynomial extensions of Banach algebras

In this section we derive a method for expanding commutative Banach algebras, in which polynomials play a crucial role.

Let A and B be commutative Banach algebras with units. B is called an *extension* of A, if there is a homomorphism of A into B preserving the unit. Let $A[x]$ be the algebra of polynomials in the variable x over A. As in the scalar case, the *degree* of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with coefficients in A is said to be again the greatest integer n with $a_0 \neq 0$. In this case $a_0 \in A$ is called again the *leading coefficient* of $p(x)$. If $a_0 = 1$ then $p(x)$ is called a *monic polynomial.*

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial in $A[x]$. We construct an associate extension of A as follows. Consider the ideal $I = p(x) \cdot A[x] \subset A[x]$, and provide the quotient algebra $B = A[x]/I$ with the norm

$$\Big\| \sum_{i=0}^{n-1} c_i x^i + p(x) \cdot A[x] \Big\| = \sum_{i=0}^{n-1} \|c_i\| t^i, \tag{1.25}$$

where the number $t > 0$ satisfies the condition

$$t^n \geq \|c_1\| t^{n-1} + \|c_2\| t^{n-2} + \cdots + \|c_n\|.$$

Under the norm (1.25) B is a commutative Banach algebra, and the natural homomorphism of A into B is an embedding. The algebra B is called the *Arens-Hoffman extension* of A associated with $p(x)$. Below we give some of the properties of Arens-Hoffman extensions (see [AH]).

(a) Any element $b \in B = A[x]/I$ can be expressed uniquely in the form $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$, where $a_i \in A$. In other words, the algebra B is a *free module* over A, i.e. the functions $x^1, x^2, \ldots, x^{n-1}$ form an A-basis of B, and the norm of B is equivalent to the component-wise norm $\max \big\{ \|a_i\| \big\}_{i=0}^{n-1}$.

(b) The set of linear multiplicative functionals $\mathcal{M}_B$ is homeomorphic to the set

$$\big\{ (m, z) \in \mathcal{M}_A \times \mathbb{C} \colon z^n + m(a_{n-1}) z^{n-1} + m(a_{n-2}) z^{n-2} + \cdots + m(a_0) = 0 \big\}.$$

(c) Every $b \in B$ is an *integral element* over A, i.e. there exists a polynomial $q(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ with $c_i \in A$, such that $q(b) = 0$.

We observe that according to the property (b) the map $\pi\colon \mathcal{M}_B \longrightarrow \mathcal{M}_A : (m, z) \longrightarrow m$ is surjective. By the Fundamental Theorem of Algebra, $\operatorname{card} \pi^{-1}(m) \leq n$ for every $m \in \mathcal{M}_A$.

As in the scalar case, the *degree* of a polynomial

$$p(x_1, x_2, \ldots, x_n) = \sum a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{1.26}$$

in n variables $x_1, x_2, \ldots, x_n$ with coefficients in A is the greatest among the numbers $i_1 + i_2 + \cdots + i_n$, where $i_1, i_2, \ldots, i_n$ are from (1.26). By $A[x_1, x_2, \ldots, x_n]$ will be denoted the algebra of polynomials in n variables with coefficients in A.

Let A and B be commutative Banach algebras with units. We say that B is a *polynomial extension* of A if there is an isomorphism Φ_A^B of A onto a Banach subalgebra $C \subset B$ with the same unit as B, an $n \in \mathbb{N}$, and a homomorphism Ψ from $A[x_1, x_2, \ldots, x_n]$ onto B, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi_A^B} & B \\ \mathrm{id} \searrow & & \nearrow \Psi \\ & A[x_1, x_2, \ldots, x_n] & \end{array}$$

is commutative, where id: $A \longrightarrow A \cdot 1$ is the natural embedding of A into $A[x_1, x_2, \ldots, x_n]$.

Example 1.5.1. (a) If $A \subset B$ and A is isometrically isomorphic to B, then B is a trivial polynomial extension of A. Indeed, one can take the set of constant polynomials $A \cdot 1$ for $A[x_1, x_2, \ldots, x_n]$ in (1.27).

(b) Let B be the algebra of all complex-valued functions $f(z, t) \in C(X)$, $X = \overline{\mathbb{D}} \times [0, 1]$, such that the functions $z \longmapsto f(z, t)$ belong to the disc algebra for every fixed $t \in [1/4, 1]$, and let $A \subset B$ be such that for any $f \in A$ the function $z \longmapsto f(z, t)$ belongs to the disc algebra for every fixed $t \in [1/2, 1]$. Clearly, A and B are isomorphic, and B is a polynomial extension of A by the property (a).

(c) Let $X = \overline{\mathbb{D}} \times [0, 1]$ be as in part (b) and B be the algebra of functions $f(z, t) \in C(X)$, such that for every fixed t the function $z \longmapsto f(z, t)$ belongs to the disc algebra. Then B is a polynomial extension of the algebra $A = \{f \in B\colon \partial f / \partial z(0, t) = 0\}$, since $B = A + zA$, $\mathcal{M}_A = \mathcal{M}_B$. However, B is not an Arens-Hoffman extension of A.

As part (c) of Example 1.5.1 shows, not all polynomial extensions are necessarily Arens-Hoffman extensions.

Theorem 1.5.2. *If B is a polynomial extension of A, then there is a nested family of Banach algebras,*

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n,$$

such that there is a homomorphism Φ from A_n onto B for which $\Phi\big|_A$ is the isomorphism Φ_A^B from (1.27), and A_i is an Arens-Hoffman extension of A_{i-1}, $1 \leq i \leq n$.

Proof. By (1.27) we have $B = \Psi\bigl(A[x_1, x_2, \dots, x_n]\bigr)$ and $\Psi(A)$ is isomorphic to A. Hence we may assume that $A \subset B$ and that there are $b_1, b_2, \dots, b_n \in B$ such that $B = A[b_1, b_2, \dots, b_n]$. Let B_k be the space of polynomials of $b_1, b_2, \dots, b_n$ in $A[b_1, b_2, \dots, b_n]$, whose degrees do not exceed k. Clearly,

$$B = A[b_1, b_2, \dots, b_n] = \bigcup_{k=0}^{\infty} B_k.$$

By the Baire Category Theorem there exists a $k_0 \geq 0$, such that the space B_{k_0} is a set of the second category in B. Let D_{k_0} be the space of polynomials in variables $x_1, x_2, \dots, x_n$ over A, whose degrees do not exceed k_0. The elements $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, $i_1 + i_2 + \cdots + i_n \leq k$ form a basis of D_{k_0}. Note that D_{k_0} is a free module over A, which is a Banach space under the norm

$$\Bigl\| \sum a_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \Bigr\| = \sum \|a_{i_1,i_2,\dots,i_n}\|. \tag{1.27}$$

The operator $T\colon \sum a_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \longmapsto \sum a_{i_1,i_2,\dots,i_n} b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n}$, is a continuous mapping from D_{k_0} onto the space of second category B_{k_0} of B. Hence $B = B_{k_0}$ by the Open Mapping Theorem. For a fixed $b \in B$ define a linear operator $S\colon D_{k_0} \longrightarrow D_{k_0}$ as follows. For any basis element $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ define $S(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})$ to be an element in $T^{-1}\left(b \cdot T\bigl(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\bigr)\right)$, and

$$S\Bigl(\sum a_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\Bigr) = \sum a_{i_1,i_2,\dots,i_n} S(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}).$$

Hence, $T\bigl(S(d)\bigr) = b\,T(d)$ for all $d \in D_{k_0}$. Consequently,

$$T\bigl(S^2(d)\bigr) = T\bigl(S(S(d))\bigr) = b\,T\bigl(S(d)\bigr) = b^2 T(d),$$

and, by induction, $TS^n = b^n T$ for all $n \in \mathbb{N}$. By the Cayley–Hamilton Theorem (e.g. [L]) there exists a polynomial $q(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ over A with $q(S) = 0$. Hence, $0 = T(q(S(d))) = q(b)\,T$. Since the unit of A is the same as the unit of B, and T preserves the units, we have $q(b) = 0$. Therefore, every $b \in B$ is an integral element over A.

Let the polynomials $q_j(x_j) = x_j^n + a_{1j} x_j^{n-1} + \cdots + a_{nj}$ be such that $q_j(b_j) = 0$ for $1 \leq j \leq n$. Define $A_0 = A$, and let A_j be the Arens-Hoffman extension of A_{j-1} associated with $q_j(x_j)$, $j = 1, 2, \dots, n$. We obtain a nested sequence

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$$

of Banach algebras, where A_j is an Arens-Hoffman extension of A_{j-1}, $1 \leq j \leq n$. Clearly, every $d \in A_n$ has a unique expression in the form

$$d = \sum a_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where $0 \leq i_k \leq n$. Therefore, the norm on A_n is equivalent to the norm (1.27). Hence, the homomorphism $\Phi\colon A_n \longrightarrow B$, defined by

$$\Phi\Big(\sum a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\Big) = \sum \Phi_A^B\big(a_{i_1,i_2,\ldots,i_n}\, b_1^{i_1}\, b_2^{i_2} \cdots b_n^{i_n}\big)$$

is continuous, and $\Phi|_A = \Phi_A^B$, as claimed. □

Corollary 1.5.3. *Any $b \in B$ is an integral element over $\Phi_A^B(A)$.*

Proof. Since, as we have seen above, every $a \in A_n$ is an integral element of A, then a is also an integral element of $\Phi_A^B(A)$. □

Corollary 1.5.4. *If B is a polynomial extension of an algebra A, then B is a finite A-module.*

Proof. Since A_n is a finite A-module and the surjective homomorphism Φ_A^B preserves the algebra A, then B is also a finite A-module. □

Note that the algebra B from Example 1.5.1(b) is a polynomial extension of A, but there are no $b_1, b_2, \ldots, b_n \in B$, $n = 1, 2, \ldots$, such that $B = A[b_1, b_2, \ldots, b_n]$.

If $A \subset B$ are Banach algebras with the same unit, we say that B is a *strong polynomial extension* of B, if there are $b_1, b_2, \ldots, b_n \in B$ such that $B = A[b_1, b_2, \ldots, b_n]$.

Lemma 1.5.5. *Let A be a closed subalgebra of a Banach algebra B, and let $b \in B$ be such that $B = A[b]$. There exist a closed subalgebra D of B which contains A, and such that $\dim B/D = r(b) - 1$, where $r(b)$ is the minimal degree of all monic polynomials $q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over A with $q(b) = 0$.*

Proof. Let $r(b) = k$ and let A_1 be the Arens-Hoffman extension of A associated with a polynomial

$$q(x) = x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0 \tag{1.28}$$

with $q(b) = 0$. If $\Phi\colon A_1 \longrightarrow B = A[b]$ is the homomorphism defined in Theorem 1.5.2 for A_1, then $B = A_1 = A + Ab + Ab^2 + \cdots + Ab^{k-1}$. The algebra A_1 is a free $(k-1)$-dimensional A-module. Therefore, $\mathrm{Null}\,(\Phi) = \big\{p(x) = c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_0 \in A_1 \colon p(b) = 0\big\}$. For a fixed $p(x) \in \mathrm{Null}\,(\Phi)$ let $K_p = \big\{m \in \mathcal{M}_A \colon m\big(c_1^{(p)}\big) = 0\big\}$, where $c_1^{(p)} \in A$ is the leading coefficient of $p(x)$. We claim that the family of sets $\{K_p\}_p$ has a nonempty intersection, i.e.

$$K = \bigcap_{p \,\in\, \mathrm{Null}\,(\Phi)} K_p \neq \emptyset. \tag{1.29}$$

Indeed, if suppose, on the contrary, that $K = \emptyset$, then for every $p \in \mathrm{Null}\,(\Phi)$ one can find an $m \in \mathcal{M}_S$ with $m\big(c_1^{(p)}\big) \neq 0$. Consequently, there are polynomials $p_1(x), p_2(x), \ldots, p_l(x) \in \mathrm{Null}\,(\Phi)$, and elements $d_1, d_2, \ldots, d_l \in A$, such that

$\sum_{i=1}^{l} d_i c_1^{(p_i)} = 1$. Therefore, for the polynomial

$$p(x) = d_1 p_1(x) + d_2 p_2(x) + \cdots + d_l p_l(x)$$

we have that $p(b) = 0$, its degree is $k-1$, and its leading coefficient at x^{k-1} is 1. This contradicts the minimality property of $r(b) = k$. Therefore, $K \neq \emptyset$, as claimed.

Let $p(x) = c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_0$ be a polynomial in $\mathrm{Null}\,(\Phi)$. We claim that $m(c_i) = 0$ for every $m \in K$, and every c_i, $i = 1, 2, \ldots, k$. According to (1.29) we have that $m(c_{k-1}) = 0$. Since $q(b) = 0$, where q is the polynomial (1.28). It follows that

$$b^k = -a_{k-1}b^{k-1} - a_{k-2}b^{k-2} - \cdots - a_0,$$

and therefore,

$$\begin{aligned}0 &= b\,p(b) = b\,(c_{k-1}b^{k-1} + c_{k-2}b^{k-2} + \cdots + c_0) = c_{k-1}b^k + c_{k-2}b^{k-1} + \cdots + c_0 b \\ &= c_{k-1}(-a_{k-1}b^{k-1} - a_{k-2}b^{k-2} - \cdots - a_0) + c_{k-2}b^{k-1} + \cdots + c_0 b \\ &= (c_{k-2} - c_{k-1}a_{k-1})\,b^{k-1} + (c_{k-3} - c_{k-1}a_{k-2})\,b^{k-2} + \cdots + (c_0 - c_{k-1}a_0)\,b \\ &\quad -c_{k-1}a_0.\end{aligned}$$

Consequently, the polynomial

$$\begin{aligned}x\,p(x) = \;& (c_{k-2} - c_{k-1}a_{k-1})\,x^{k-1} + (c_{k-3} - c_{k-1}a_{k-2})\,x^{k-2} + \cdots \\ & +(c_0 - c_{k-1}a_0)\,x - c_{k-1}a_0\end{aligned}$$

vanishes at b, and hence belongs to $\mathrm{Null}\,(\Phi)$. As shown above, then $0 = m\,(c_{k-2} - c_{k-1}a_{k-1}) = m(c_{k-2}) - m(c_{k-1})\,m(a_{k-1}) = m(c_{k-2})$, since $m(c_{k-1}) = 0$. Proceeding inductively, we obtain that $m(c_i) = 0$ for all $m \in K$ and all $i = 1, 2, \ldots, k$, as claimed.

Let $m_0 \in K$, and let $I_0 = \mathrm{Null}\,(m_0) = \big\{a \in A\colon m_0(a) = 0\big\}$ be the corresponding maximal ideal in the algebra A. As we have seen, $m_0(c_i) = 0$ for each coefficient c_i of any polynomial $p \in \mathrm{Null}\,(\Phi)$. Therefore, $\mathrm{Null}\,(\Phi) \subset D_0$, where

$$D_0 = A + I_0 x + I_0 x^2 + \cdots + I_0 x^{k-1}.$$

Hence D_0 is a closed subalgebra of A_1 containing $\mathrm{Null}\,(\Phi)$. Since the mapping $\Phi : A_1 \longrightarrow B$ is surjective, the set

$$D = \Phi(D_0) = A + I_0 b + I_0 b^2 + I_0 b^3 + \cdots + I_0 b^{k-1}$$

is a closed subalgebra in $A[b] = B$. We claim that no linear combination of elements $b, b^2, \ldots, b^{k-1}$ belongs to D. Indeed, if we suppose that $\alpha_1 b + \cdots + \alpha_{k-1}b^{k-1} \in D = A + I_0 b + I_0 b^2 + I_0 b^3 + \cdots + I_0 b^{k-1}$ with $(\alpha_1, \ldots, \alpha_{k-1}) \neq (0, \ldots, 0)$, then

$$\alpha_1 b + \cdots + \alpha_{k-1}b^{k-1} = a + c_1 b + \cdots + c_{k-1}b^{k-1},\ \ c_i \in I_0.$$

Consequently,

$$a + (c_1 - \alpha_1)\, b + \cdots + (c_{k-1} - \alpha_{k-1})\, b^{k-1} = 0.$$

Hence the polynomial $a + (c_1 - \alpha_1)\, x + \cdots + (c_{k-1} - \alpha_{k-1})\, x^{k-1}$ is in $\mathrm{Null}\,(\Phi)$, and therefore, $m_0(c_i - \alpha_i) = 0$, $i = 1, \ldots, k-1$, since $m_0 \in K$. Consequently, $m(\alpha_i) = \alpha_i = 0$ for all $i = 1, \ldots, k-1$, since $m_0(c_i) = 0$. This contradicts the choice of $\alpha_1, \ldots, \alpha_n$, and completes the proof that $\dim B/D = k-1$, as desired. □

Lemma 1.5.6. *There exists a closed subalgebra D of $B = A[b_1, b_2, \ldots, b_n]$, such that $A \subset D \subset B$, and $\dim B/D < \infty$.*

Proof. Let $B_k = A[b_1, b_2, \ldots, b_k]$ and let $r_k(b)$ be the minimal degree of all monic polynomials $q_k(x) = x^l + a_{l-1}x^{l-1} + a_{l-2}x^{l-2} + \cdots + a_0$ over B_{k-1} with $q(b) = 0$. Applying the technique of Lemma 1.5.5, we construct consecutive extensions of A, namely

$$A \subset A_1 \subset A_2 \subset \cdots \subset A_n.$$

Every element $d \in A_n$ admits a unique representation of type

$$d = \sum a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},\ \ i_k \le r_k(b),\ \ k = 1, 2, \ldots, n.$$

Therefore, the norm on A_n is equivalent to the norm (1.27), and the map $\Phi\colon A_n \longrightarrow B$, defined as

$$\Phi\Big(\sum a_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\Big) = \sum a_{i_1,i_2,\ldots,i_n} b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n}$$

is continuous, and there exists a maximal ideal I in A_{n-1} such that

$$\mathrm{Null}\,(\Phi) \subset D_0 = A_{n-1} + I x_n + \cdots + I x_n^m, \tag{1.30}$$

where $m = r_n(b)$ (cf. the proof of Lemma 1.5.5). The algebra D_0 is closed in $A_n \cong B$, and therefore $D = \Phi(D_0)$ is a Banach subalgebra of B with $\dim B/D = m$. □

A bounded linear functional δ on an algebra B is called a *point derivation* of B at $m \in \mathcal{M}_B$, if

$$\delta(ab) = \delta(a)m(b) + m(a)\delta(b) \text{ for every } a, b \in B.$$

One can easily see that $\mathrm{Null}\,(\delta)$ is a subalgebra of B, and $\delta(1) = 0$. The next theorem is well known (cf. [B7], or [D], p.118).

Theorem 1.5.7. *If θ is a linear functional on B such that $\mathrm{Null}\,(\theta)$ is an algebra, then one of the following alternatives holds.*

(i) *If $\theta(1) \neq 0$, then θ is a scalar multiple of a linear multiplicative functional.*

(ii) *If $\theta(1) = 0$, then either there exist $m_1, m_2 \in \mathcal{M}_B$, $m_1 \neq m_2$, such that θ is a scalar multiple of $m_1 - m_2$, or, θ is a point derivation at some $m \in \mathcal{M}_B$.*

Theorem 1.5.8. *Let $B = A[b_1, b_2, \ldots, b_n]$, $b_i \in B$, be a strong polynomial extension of A. If $A \neq B$, then either there exist $m_1, m_2 \in \mathcal{M}_B$, $m_1 \neq m_2$, such that $A \subset \mathrm{Null}\,(m_1 - m_2) = \{b \in B \colon m_1(b) = m_2(b)\}$, or, there exists a non-trivial point derivation $\delta : B \longrightarrow \mathbb{C}$, such that $A \subset \mathrm{Null}\,(\delta) = \{b \in B \colon \delta(b) = 0\}$.*

Proof. Since $J = I + Ix_n + \cdots + Ix_n^m$ is a maximal ideal of the algebra D_0 from (1.30), then J is a closed ideal in D_0. Let A_n be the algebra generated by D_0 and x_n. Clearly, J is an ideal of A_n and $\dim(A_n/J) = m + 1$. Every element $b \in A_n$ generates a linear operator $T_b : A_n/J \longrightarrow A_n/J$ by $T_b\big([a]\big) = [ba]$, where $[a]$ is the coset $[a] = a + J$. The operator T_b is well defined, since J an ideal in A_n. The space of operators $\{T_b \colon b \in A_n\}$ is a commutative subalgebra of the algebra of all linear operators on an $(m + 1)$-dimensional space. As shown in [D] (Theorem 1.8.11), there exists a basis $[a_1], [a_2], \ldots, [a_{m+1}]$, in A_n/J, such that any operator $T_b \in T(A_n)$ has an upper-triangular form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & \cdots & c_{1k} \\ 0 & c_{22} & c_{23} & c_{24} & \cdots & c_{2k} \\ 0 & 0 & c_{33} & c_{34} & \cdots & c_{3k} \\ 0 & 0 & 0 & c_{44} & \cdots & c_{4k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c_{kk} \end{bmatrix}$$

with respect to this basis. The subspace E_i of A_n/J spanned by $[a_1], [a_2], \ldots, [a_i]$, is an invariant subspace of the algebra $T(A_n)$. Therefore, the set $\widetilde{E}_i' = \{a \in A_n \colon [a] \in E_i\}$ is an ideal in A_n for all $i = 1, 2, \ldots, m$. For the algebras $D_i = \widetilde{E}_i' \oplus \mathbb{C} \cdot 1$, $i = 1, 2, \ldots, m$ we have

$$D_0 \subset D_1 \subset \cdots \subset D_m = A_n,$$

and $\dim(D_{i+1}/D_i) = 1$. Since $D = \Phi(D_0)$ is a closed subalgebra in $B = \Phi(A_n)$, and $\Phi : A_n \longrightarrow B$ is a homomorphism, then $\Phi(D_{l-1})$ is a subalgebra of B with codimension 1 that contains A. The result follows from Theorem 1.5.7. □

Let $A \subset B$ be uniform algebras on X. If $x_0 \in X$ is a p-point (or generalized peak point) for B, then there is no non-trivial derivation at x. Indeed, consider an $f \in J_{x_0} = \{f \in B : f(x_0) = 0\}$, and let $F_n = \{x \in X \colon |f(x)| \geq 1/n\}$. Since x_0 is a p-point, then there exist elements $h_n \in B$, such that $\|h_n\| = h_n(x_0) = 1$, and $|h_n(x)| \leq 1/n$ on F_n. The functions $f_n = 1 - h_n$ belong to J_{x_0}, and $\|f_n f - f\| \longrightarrow 0$ as $n \longrightarrow \infty$. For any point derivation δ of B at x_0 we have $\delta(f_n f) = 0$, and hence $\delta(f) = 0$ for every $f \in J_{x_0}$. Since also $\delta(1) = 0$, we see that δ is trivial, as claimed.

A uniform algebra A on X is said to be *polynomially closed*, if A has no strong polynomial extensions that are uniform algebras on X, i.e. if any strong polynomial extension B of A that is a uniform algebra on X coincides with A. The remarks from the above, and Theorem 1.5.8 imply the following

Theorem 1.5.9. *If A is a uniform algebra, such that every point of its maximal ideal space $\mathcal{M}_A$ is a p-point for A, then A is a polynomially closed algebra.*

1.6 Isomorphisms between uniform algebras

In this section we assume that A and B are uniform algebras on their maximal ideal spaces X and Y respectively. We find conditions for peripherally multiplicative and peripherally additive operators to be algebraic isomorphisms.

Definition 1.6.1. An operator $T\colon A \to B$ is said to be

(a) *preserving the peripheral spectra* of algebra elements if

$$\sigma_\pi(Tf) = \sigma_\pi(f) \qquad \text{for every } f \in A. \tag{1.31}$$

(b) σ_π*-additive* if

$$\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g) \qquad \text{for all } f, g \in A. \tag{1.32}$$

(c) σ_π*-multiplicative* if

$$\sigma_\pi\big((Tf)(Tg)\big) = \sigma_\pi(fg) \qquad \text{for all } f, g \in A, \tag{1.33}$$

First we will consider the case of σ_π-additive operators.

Lemma 1.6.2. *If an operator $T\colon A \to B$ is σ_π-additive, then the following equalities hold for all $f, g \in A$.*

(a) $\|Tf + Tg\| = \|f + g\|$,

(b) $T(0) = 0$,

(c) $T(-f) = -Tf$,

(d) $\sigma_\pi(Tf) = \sigma_\pi(f)$,

(e) $\sigma_\pi\big(T(f + g)\big) = \sigma_\pi(Tf + Tg)$,

(f) $\|Tf\| = \|f\|$, *and*

(g) $\|Tf - Tg\| = \|f - g\|$.

If, in addition T is surjective, then it is $\mathbb{R}$-linear.

Proof. The equality (a) is obvious, since $|z| = \|f\|$ for every $z \in \sigma_\pi(f)$. (b) follows from (a) by letting $f = g = 0$. (c) follows from the σ_π-additivity of T. Indeed, $\sigma_\pi\big(Tf + T(-f)\big) = \sigma_\pi(f + (-f)) = \sigma_\pi(f - f) = \sigma_\pi(0) = \{0\}$, hence $Tf + T(-f) = 0$, and therefore, $T(-f) = -Tf$. Equalities (d) and (f) follow from the σ_π-additivity of T and (a) correspondingly, by letting $g = 0$. (e) follows from (d) and the σ_π-additivity of T because of $\sigma_\pi\big(T(f+g)\big) = \sigma_\pi(f+g) = \sigma_\pi(Tf+Tg)$. (g) follows form (a) and (c). The last statement follows from the theorem of Mazur-Ulam [MU] (see also [V]), since, by (b) and (g), $T(0) = 0$, and $\|Tf - Tg\| = \|f - g\|$ for all $f, g \in A$. □

Note that according to equality (f), the operator T preserves the norms of algebra elements, and therefore maps the unit ball of A in the unit ball of B. By (d) T *preserves the peripheral spectra*, and by (g) it *preserves the distances* between algebra elements. It is straightforward to see that the σ_π-additivity of T is equivalent to both (d) and (e).

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In the sequel we will use the following notation:

$s\,\mathcal{F}(A) = \{s\,f \colon f \in \mathcal{F}(A)\}$, where $s \in \mathbb{C}^*$,
$s\,\mathcal{F}_x(A) = \{s\,f \colon f \in \mathcal{F}_x(A)\}$, where $s \in \mathbb{C}^*$,
$\mathbb{C}^* \cdot \mathcal{F}(A) = \{s\,f \colon f \in \mathcal{F}(A),\ s \in \mathbb{C}^*\}$, and
$\mathbb{C}^* \cdot \mathcal{F}_x(A) = \{s\,f \colon f \in \mathcal{F}_x(A),\ s \in \mathbb{C}^*\}$.

The elements of the family $\{s \colon \mathcal{F}(A),\ s \in \mathbb{C}^*\}$ are called $\mathbb{C}^*$*-peaking functions* of A. Clearly, f is a $\mathbb{C}^*$-peaking function if and only if $\sigma_\pi(f)$ is a singleton, and $g \in s\mathcal{F}(A)$ if and only if $\sigma_\pi(g) = \{s\}$. If $f \in s\mathcal{F}(A)$ for some $s \in \mathbb{C}^*$, then the set $P(f) = f^{-1}\{s\}$ is called the *peak set* for f. Obviously, $P(sf) = P(f)$ for any $f \in \mathcal{F}(A)$ and any $s \in \mathbb{C}^*$. Therefore, the collection of peak sets for the classes of peaking functions and of the classes of $\mathbb{C}^*$-peaking functions of A coincide.

Lemma 1.6.3. *If $T : A \to B$ is a surjective operator which preserves the peripheral spectra of algebra elements, then*

$$T\big(s\mathcal{F}(A)\big) = s\mathcal{F}(B) \tag{1.34}$$

for any $s \in \mathbb{C}^*$.

Proof. Indeed, if $s \in \mathbb{C}^*$ then $T\big(s\mathcal{F}(A)\big) \subset s\mathcal{F}(B)$ follows by the preservation of the peripheral spectra by T. Given a $k \in s\mathcal{F}(B),\ s \in \mathbb{C}^*$, let $k = Th$ for some $h \in A$. Then $h \in s\mathcal{F}(A)$ since $\sigma_\pi(h) = \sigma_\pi(Th) = \sigma_\pi(k) = \{s\}$. Hence $k = Th \in T\big(s\mathcal{F}(A)\big)$, and therefore $s\mathcal{F}(B) \subset T\big(s\mathcal{F}(A)\big)$. Consequently, $T\big(s\mathcal{F}(A)\big) = s\mathcal{F}(B)$ for any $s \in \mathbb{C}^*$, as claimed. □

Definition 1.6.4. An operator $T \colon A \to B$ is called *monotone increasing in modulus* if the inequality $|f(x)| \le |g(x)|$ on ∂A implies $|(Tf)(y)| \le |(Tg)(y)|$ on ∂B for every $f, g \in A$.

Lemma 1.6.5. *If a monotone increasing in modulus surjective operator $T \colon A \to B$ preserves the peripheral spectra of algebra elements, then for any generalized peak point $x \in \delta A$ the set*

$$E_x = \bigcap_{h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)} P(Th) \tag{1.35}$$

is non-empty.

Proof. Observe that

$$T\big(s\mathcal{F}(A)\big) = s\mathcal{F}(B) \tag{1.36}$$

for any $s \in \mathbb{C}^*$. Indeed, if $s \in \mathbb{C}^*$ then $T\big(s\mathcal{F}(A)\big) \subset s\mathcal{F}(B)$ follows by the preservation of the peripheral spectra by T. Given a $k \in s\mathcal{F}(B)$, $s \in \mathbb{C}^*$, let $k = Th$ for some $h \in A$. Then $h \in s\mathcal{F}(A)$ since $\sigma_\pi(h) = \sigma_\pi(Th) = \sigma_\pi(k) = \{s\}$. Hence $k = Th \in T\big(s\mathcal{F}(A)\big)$, and therefore $s\mathcal{F}(B) \subset T\big(s\mathcal{F}(A)\big)$. Consequently, $T\big(s\mathcal{F}(A)\big) = s\mathcal{F}(B)$ for any $s \in \mathbb{C}^*$, as claimed.

Let x be a generalized peak point of A. First we show that the family $\big\{P(Th)\colon h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)\big\}$ has the finite intersection property. If $h_1, h_2, \dots, h_n$ belong to $\mathbb{C}^* \cdot \mathcal{F}_x(A)$, and $h_j \in s_j\mathcal{F}_x(A)$, $s_j \in \mathbb{C}^*$, then, clearly, the function $g = h_1 \cdot h_2 \cdots h_n$ belongs to the space $(s_1 \cdot s_2 \cdots s_n) \cdot \mathcal{F}_x(A)$. Since $|h_j(x)| \le |s_j|$, $j = 1, \dots, n$, we have $|g(\xi)| = |h_1(\xi)| \cdot |h_2(\xi)| \cdots |h_n(\xi)| \le \big[\prod_{j\ne k} |s_j|\big] |h_k(\xi)| = \big|\big[\prod_{j\ne k} |s_j|\big] h_k(\xi)\big|$ for every $\xi \in \partial A$ and any fixed $k = 1, \dots, n$. The preservation of peripheral spectra by T implies that $Tg \in (s_1 \cdot s_2 \cdots s_n) \cdot \mathcal{F}(B)$ and $Th_k \in s_k\mathcal{F}(B)$. Hence, $|(Th_k)(y)| \le |s_k|$ for every $y \in Y$. By Lemma 1.6.2(h), T is $\mathbb{R}$-linear, and since it is also monotone increasing in modulus, it follows that $|(Tg)(y)| \le \big|\big(T\big(\big[\prod_{j\ne k} |s_j|\big]\, h_{j_0}\big)(y)\big| = \big[\prod_{j\ne k} |s_j|\big] |(Th_k)(y)| \le |(s_1 \cdot s_2 \cdots s_n)|$ for every $y \in \partial B$. Consequently, for every $y \in Y$ with $|(Tg)(y)| = |s_1 \cdot s_2 \cdots s_n|$ we must have $|(Th_k)(y)| = |s_k|$, which implies $(Th_k)(y) = s_k$, and hence $P(Tg) \subset P(Th_k)$. Since this holds for every $k = 1, \dots, n$, we obtain that $P(Tg) \subset \bigcap_{k=1}^{n} P(Th_j)$. Consequently, the family $\big\{P(Th)\colon h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)\big\}$ has the finite intersection property, as claimed. Hence it has a non-empty intersection, since all of its elements are closed subsets of the compact set Y. ☐

The following lemma provides sufficient conditions for an operator $T\colon A \to B$ to be monotone increasing in modulus.

Lemma 1.6.6. *If a surjective operator $T\colon A \to B$ satisfies the equality*

(ii) $\max_{\eta\in\partial B} \big(|(Tf)(\eta)| + |(Tg)(\eta)|\big) = \max_{\xi\in\partial A} \big(|f(\xi)| + |g(\xi)|\big)$

for every $f, g \in A$, then it is monotone increasing in modulus.

Proof. If $|f(x)| \le |g(x)|$ on ∂A, then, clearly,

$$\max_{\xi\in\partial A} \big(|f(\xi)| + |k(\xi)|\big) \le \max_{\xi\in\partial A} \big(|g(\xi)| + |k(\xi)|\big)$$

for any $k \in A$. By equality (ii) we have

$$\begin{aligned}&\max_{\eta\in\partial B} \big(|(Tf)(\eta)| + |(Tk)(\eta)|\big) = \max_{\xi\in\partial A} \big(|f(\xi)| + |k(\xi)|\big)\\ &\le \max_{\xi\in\partial A} \big(|g(\xi)| + |k(\xi)|\big) = \max_{\eta\in\partial B} \big(|(Tg)(\eta)| + |(Tk)(\eta)|\big).\end{aligned}$$

Now from Lemma 1.2.7 and the surjectivity of T if follows that $|(Tf)(y)| \le |(Tg)(y)|$ on ∂B. ☐

Proposition 1.6.7. *If a surjective operator $T\colon A \to B$ satisfies the equalities*

(i) $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$, *and*

(ii) $\max\limits_{\eta\in\partial B}\big(|(Tf)(\eta)| + |(Tg)(\eta)|\big) = \max\limits_{\xi\in\partial A}\big(|f(\xi)| + |g(\xi)|\big)$

for all $f, g \in A$, then T is bijective.

Proof. We will show that T is injective. If $Tf = Tg$ for some $f, g \in A$, then for any $h \in A$ we have $Tf + Th = Tg + Th$. Consequently, $\sigma_\pi(Tf + Th) = \sigma_\pi(Tg + Th)$. The σ_π-additivity of T implies

$$\sigma_\pi(f + h) = \sigma_\pi(Tf + Th) = \sigma_\pi(Tg + Th) = \sigma_\pi(g + h).$$

By equality (ii) we have

$$\begin{aligned}&\max_{\xi\in\partial A}\big(|f(\xi)| + |h(\xi)|\big) = \max_{\eta\in\partial B}\big(|(Tf)(\eta)| + |(Th)(\eta)|\big)\\ &= \max_{\eta\in\partial B}\big(|(Tg)(\eta)| + |(Th)(\eta)|\big) = \max_{\xi\in\partial A}\big(|g(\xi)| + |h(\xi)|\big)\end{aligned}$$

for every $h \in A$. Lemma 1.2.13 now implies that $f = g$. Hence T is injective, and therefore bijective. □

Lemma 1.6.8. *If the operator $T\colon A \to B$ satisfies the assumptions of Proposition 1.6.7, then for generalized peak point $x \in \delta A$ the set E_x is a singleton and belongs to δB.*

Proof. Let x be a generalized peak point of A. Equality (ii) and Lemma 1.6.6 imply that T is monotone increasing in modulus. It follows from Lemma 1.6.2(d) that the operator T preserves the peripheral spectra of algebra elements. Therefore, T satisfies the hypotheses of Lemma 1.6.5. In the course of its proof we saw that $\{P(Tf)\colon f \in \mathbb{C}^* \cdot \mathcal{F}_x(A)\}$ is a family of peak sets with non-empty intersection, E_x, hence it meets δB (e.g. [L1]).

Since T preserves peripheral spectra of algebra elements, equality (1.34) implies $T^{-1}\big(\mathcal{F}(B)\big) = \mathcal{F}(A)$. We claim that $T^{-1}\big(\mathcal{F}_y(B)\big) \subset \mathcal{F}_x(A)$ for any $y \in E_x \cap \delta B$. Let $y \in E_x \cap \delta B$, $k \in \mathcal{F}_y(B)$, and let $h = T^{-1}(k)$. To show that $h \in \mathcal{F}_x(A)$ it is enough to verify that $h(x) = 1$. Take an open neighborhood V of x and a peaking function $g \in \mathcal{F}_x(A)$ with $P(g) \subset V$. Equality (1.34) yields $Tg \in \mathcal{F}(B)$. Since $y \in E_x \subset P(Tg)$ we have that $(Tg)(y) = 1$, and therefore, $Tg \in \mathcal{F}_y(B)$. Equality (ii) yields

$$\begin{aligned}&k(y) + (Tg)(y) = 2 \geq \max_{\xi\in\partial A}\big(|h(\xi)| + |g(\xi)|\big)\\ &= \max_{\eta\in\partial B}\big(|(Th)(\eta)| + |(Tg)(\eta)|\big) = \max_{\eta\in\partial B}\big(|k(\eta)| + |(Tg)(\eta)|\big) = 2.\end{aligned}$$

Hence $\max\limits_{\xi\in\partial A}\big(|h(\xi)| + |g(\xi)|\big) = 2$, and there must be a $x_V \in \partial A$ with $h(x_V) = 1$ and $g(x_V) = 1$. Therefore, $x_V \in P(g) \subset V$. We deduce that any neighborhood

V of x contains a point x_V with $h(x_V) = 1$. The continuity of h implies that $h(x) = 1$, thus $h \in \mathcal{F}_x(A)$. Consequently, $T^{-1}\big(\mathcal{F}_y(B)\big) \subset \mathcal{F}_x(A)$, as claimed.

Let $y \in E_x \cap \delta B$. If there were a $z \in E_x \setminus \{y\}$, there would be a peaking function $k \in \mathcal{F}_y(B)$ with $|k(z)| < 1$. For any $h \in T^{-1}(k) \cap \mathcal{F}_x(A)$ we have $h \in \mathcal{F}_x(A)$, $k = Th \in \mathcal{F}(B)$, and $P(k) = P(Th) \supset E_x$. Hence the function $k = Th$ is identically equal to 1 on E_x, contradicting $|k(z)| < 1$. This shows that the set E_x does not contain points other than y. □

Let T satisfy the assumptions of Proposition 1.6.7, and let $x \in \delta A$. If $\tau(x)$ denotes the single element of the set E_x, i.e.

$$\{\tau(x)\} = E_x = \bigcap_{h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)} P(Th), \tag{1.37}$$

then there arises a mapping $\tau \colon x \longmapsto \tau(x)$. If $h \in s\mathcal{F}_x(A)$, $s \in \mathbb{C}^*$, then, due to (1.37), $P(Th) \supset E_x = \{\tau(x)\}$, thus $(T(h))(\tau(x)) = s = h(x)$. Therefore, the equality

$$(T(h))(\tau(x)) = h(x) \tag{1.38}$$

holds for every $\mathbb{C}^*$-peaking function $h \in s\mathcal{F}_x(A)$, $s \in \mathbb{C}^*$.

Under the assumptions of Proposition 1.6.7 the operator T is bijective. Let $k \in \mathbb{C}^* \cdot \mathcal{F}_{\tau(x)}(B)$, for some $x \in \delta A$, and let $T^{-1}k = h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)$. By (1.38) we have $k(\tau(x)) = (Th)(\tau(x)) = h(x) = (T^{-1}k)(x)$, and therefore, the equality

$$(T^{-1}k)(x) = k(\tau(x)) \tag{1.39}$$

also holds for every $x \in \delta A$ and any $\mathbb{C}^*$-peaking function $k \in s\mathcal{F}_{\tau(x)}(B)$, $s \in \mathbb{C}^*$.

Lemma 1.6.9. *Let the operator $T \colon A \to B$ satisfy the assumptions of Proposition 1.6.7, and let $f \in A$. If $(Tf)(\tau(x_0)) = 0$ for some $x_0 \in \delta A$, then also $f(x_0) = 0$.*

Proof. Let x_0 be a generalized peak point of A and let f be in A with $\|f\| = \|Tf\| = R$. Choose an open neighborhood U of $\tau(x_0)$ in Y, such that $|(Tf)(y)| < \varepsilon$ on U. Let $k \in \mathcal{F}_{\tau(x_0)}(B)$ be a peaking function of B with $\tau(x_0) \in P(k) \subset U$. By taking a big enough power of k, we may assume from the beginning that $|(Tf)(y)| + R\,|k(y)| < \max_{\eta \in \overline{U}} \big(|(Tf)(\eta)| + R\,|k(\eta)|\big) < \varepsilon + R$ for all $y \in Y \setminus U$. Consequently, $\max_{\eta \in Y} \big(|(Tf)(\eta)| + R\,|k(\eta)|\big) < \varepsilon + R$, and therefore, according to equality (ii),

$$\max_{\xi \in \partial A} \big(|f(\xi)| + |(T^{-1}(Rk))(\xi)|\big) = \max_{\eta \in \partial B} \big(|(Tf)(\eta)| + |Rk(\eta)|\big) < \varepsilon + R.$$

Hence by (1.39) we have

$$|f(x_0)| + R = |f(x_0)| + |Rk(\tau(x_0))| = |f(x_0)| + \big|(T^{-1}(Rk)(x_0)\big| < \varepsilon + R.$$

Thus, $|f(x_0)| < \varepsilon$, and consequently, $f(x_0) = 0$ by the liberty of choice of ε. □

Lemma 1.6.10. *If the operator $T\colon A \to B$ satisfies the assumptions of Proposition 1.6.7, then $|f(x)| \le |Tf(\tau(x))|$ for every $x \in \delta A$ and all $f \in A$.*

Proof. Since T satisfies the assumptions of Proposition 1.6.7, then the equalities

(i) $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$, and

(ii) $\max\limits_{\eta\in\partial B}\big(|(Tf)(\eta)| + |(Tg)(\eta)|\big) = \max\limits_{\xi\in\partial A}\big(|f(\xi)| + |g(\xi)|\big)$

hold for all $f, g \in A$. By Lemma 1.6.2(d) and (e), $\|Tf\| = \|f\|$, and $\|Tf + Tg\| = \big\|T(f + g)\big\| = \|f + g\|$.

Let $x \in \delta A$, $f \in A$, and let $g = Tf \in B$. Without loss of generality we can assume that $g(\tau(x)) \ne 0$, since if $g(\tau(x)) = (Tf)(\tau(x)) = 0$, then also $f(x) = 0$ by Lemma 1.6.9, and the result carries. If U is an open neighborhood of $\tau(x)$ in Y, then by the additive version of Bishop's Lemma (Lemma 1.2.11) we can choose a $k \in B$ with $\tau(x) \in P(k) \subset U$, such that the function $|g(y)| + |k(y)|$ attains its maximum only within $P(k) \subset U$. Let $\eta_U \in P(k)$ and $\beta_U \in \mathbb{T}$ be such that $|g(\eta_U)| = \max\limits_{\eta\in P(k)} |g(\eta)|$, and $g(\eta_U) = \beta_U\big|g(\eta_U)\big|$. According to Corollary 1.2.12,

$$|g(\eta_U)| + 1 = \|g + \beta_U k\|,$$

while $\|g + \gamma k\| \le \|g + \beta_U k\|$ for any $\gamma \in \mathbb{T}$. Let $\alpha \in \mathbb{T}$, be such that $f(x) = \alpha|f(x)|$. Denote $h = T^{-1}(k) \in \mathcal{F}_x(A)$. By equality (1.39) we have $\alpha = \alpha k(\tau(x)) = (T^{-1}(\alpha k))(x)$. Equality (ii) and Corollary 1.2.12 imply

$$\begin{aligned}&|f(x)| + 1 = \overline{\alpha} f(x) + 1 = \big|f(x) + \alpha\big| = \big|f(x) + (T^{-1}(\alpha k))(x)\big| \\ &= \big|\big(f + T^{-1}(\alpha k)\big)(x)\big| \le \|h + T^{-1}(\alpha k)\| = \big\|g + \alpha k\| \le \|g + \beta_U k\| = |g(\eta_U)| + 1,\end{aligned}$$

thus $|f(x)| \le |g(\eta_U)|$. We have obtained that any neighborhood U of $\tau(x)$ contains a point η_U such that $|f(x)| \le |g(\eta_U)|$. The continuity of g implies that $|f(x)| \le |g(\tau(x))| = \big|(Tf)(\tau(x))\big|$. □

Lemma 1.6.11. *If the operator $T\colon A \to B$ satisfies the assumptions of Proposition 1.6.7, then the mapping τ from* (1.37) *is a homeomorphism from δA onto δB.*

Proof. Proposition 1.6.7 implies that T is bijective. Since the equalities (i) and (ii) are symmetric with respect to f and Tf, they hold also for the inverse operator T^{-1}. According to Lemma 1.6.8 there arises a corresponding continuous map $\psi\colon\ deltaB \to \delta A$ such that the equality (1.38), which in this case reduces to $\big(T^{-1}(k)\big)(\psi(\eta)) = k(\eta)$, holds on δB for any $k \in \mathcal{F}_\eta(B)$. Let $x \in \delta A$ and $y = \tau(x) \in \delta B$. If $h \in \mathcal{F}_x(A)$, then, due to (1.32) and (1.38), $k = Th \in \mathcal{F}_y(B)$, therefore $h(\psi(y)) = \big(T^{-1}(k)\big)(\psi(y)) = k(y) = (Th)(y) = (Th)(\tau(x)) = h(x) = 1$, and therefore $\psi(y) \in P(h)$. Since this holds for every $h \in \mathcal{F}_x(A)$ and $\bigcap\limits_{h\in\mathcal{F}_x(A)} P(h) = \{x\}$, we have that $\psi(\tau(x)) = \psi(y) = x$ for every $x \in \delta A$. By similar arguments one can see also that $\tau(\psi(y)) = y$ for any $y \in \delta B$. Consequently, τ and ψ both are injective mappings, and $\psi = \tau^{-1}$.

Let $x \in \delta A$ be a generalized peak point of A, and let $r \in (0,1)$. Choose an open neighborhood V of $\tau(x)$ in δB, and a peaking function $k \in \mathcal{F}_{\tau(x)}(B)$ with $P(k) \subset V$ and $|k(y)| < r$ on $\delta B \setminus V$. If $h = T^{-1}(k)$, then $h \in \mathcal{F}_x(A)$, and according to (1.38), $k(\tau(\xi)) = (Th)(\tau(\xi)) = h(\xi)$ on δA. Note that since $h(x) = 1 > r$, the open set $W = \{\xi \in \delta A\colon |h(\xi)| > r\}$ contains x. According to Lemma 1.6.10, for every $\xi \in W$ we have $|k(\tau(\xi))| = \big|(Th)(\tau(\xi))\big| \geq |h(\xi)| > r$, and therefore, $\tau(\xi) \in V$, since on $\delta B \setminus V$ we have $|k(\eta)| < r$. Consequently, $\tau(W) \subset V$, which proves the continuity of τ. If we consider the operator $T^{-1}\colon B \to A$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$, the same arguments imply that τ^{-1} is also continuous, which completes the proof. □

When applied to the operator $T^{-1}\colon B \to A$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$, Lemma 1.6.10 implies $|g(y)| \leq \big|(T^{-1}g)(\tau^{-1}(y))\big|$ for any $y \in \delta B$ and every $g \in B$, or, if $g = Tf,\ f \in A$, and $y = \tau(x),\ x \in \delta A$, equivalently, $\big|(Tf)(\tau(x))\big| \leq |f(x)|$. Hence we have the following

Corollary 1.6.12. *If the operator $T\colon A \to B$ satisfies the assumptions of Proposition* 1.6.7*, then $|(Tf)(\tau(x))| \leq |f(x)|$ for any $x \in \delta A$ and every $f \in A$.*

Proposition 1.6.13. *If $T\colon A \to B$ satisfies the assumptions of Proposition* 1.6.7*, then the equality*

$$(Tf)(\tau(x)) = f(x) \tag{1.40}$$

holds for every $f \in A$ and every generalized peak point $x \in \delta A$.

Proof. If T satisfies the assumptions of Proposition 1.6.7, then, by Lemma 1.6.8 the mapping τ from (1.37) is well-defined, and, given an $x \in \delta A$, any $\mathbb{C}^*$-peaking function $h \in s\mathcal{F}_x(A),\ s \in \mathbb{C}^*$, satisfies the equality (1.38), i.e. (1.40).

Let x_0 be a generalized peak point of A and let f be in A with $\|f\| = \|Tf\| = R$. Without loss of generality we can assume that $f(x_0) \neq 0$, since in $f(x_0) = 0$, then also $(Tf)(\tau(x_0)) = 0$ by Lemma 1.6.9, applied to the operator $T^{-1}\colon B \to A$, the function $Tf \in B$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$. Let V be an open neighborhood of x_0 in X. By the additive version of Bishop's Lemma we can choose an R-peaking function $h \in R \cdot \mathcal{F}_{x_0}(A)$ so that $x_0 \in P(h) \subset V$, and such that the function $|f(x)| + |h(x)|$ attains its maximum only within $P(h) \subset V$. Let $\xi_V \in P(h)$ and $\alpha_V \in \mathbb{T}$ be such that $f(\xi_V) = \alpha_V |f(\xi_V)|$ and $|f(\xi_V)| + R = \max\limits_{\xi \in P(h)} \big(|f(\xi)| + |h(\xi)|\big) = \max\limits_{\xi \in P(h)} \big|(f + \alpha_V h)(\xi)\big| = \big|f(\xi_V) + \alpha_V R\big|$. Hence, by Corollary 1.2.12,

$$|f(\xi_V)| + R = \big|f(\xi_V) + \alpha_V R\big| = \|f + \alpha_V h\|, \tag{1.41}$$

while $\|f + \gamma h\| \leq \|f + \alpha_V h\|$ for any $\gamma \in \mathbb{T}$. Therefore, $f(\xi_V) + \alpha_V R \in \sigma_\pi(f + \alpha_V h) = \sigma_\pi\big(Tf + T(\alpha_V h)\big)$. Hence there is a point $z_V \in Y$ with

$$f(\xi_V) + \alpha_V R = \big((Tf + T(\alpha_V h)\big)(z_V). \tag{1.42}$$

We may assume that $z_V \in \delta B$. Indeed, $|f(\xi_V)| + R = \|f + \alpha_V h\|$ is the maximum modulus of the function $f+\alpha_V h$, and, according to Lemma 1.6.2(d), of the function $Tf + T(\alpha_V h)$ as well. Therefore, the function $\big|Tf + T(\alpha_V h)\big|$ attains the value $|f(\xi_V)| + R$ at some point of the Choquet boundary δB, and we can choose z_V to be such a point. The surjectivity of τ implies that $z_V = \tau(x_V)$ for some $x_V \in \delta A$. Equality (1.42), Corollary 1.6.12, (ii) and (1.38) imply

$$\begin{aligned}
&\big|f(\xi_V) + \alpha_V R\big| = \big|\big((Tf + T(\alpha_V h)\big)(z_V)\big| = \big|(Tf)(\tau(x_V)) + (T(\alpha_V h))(\tau(x_V))\big| \\
&\le \big|(Tf)(\tau(x_V))\big| + |(T(\alpha_V h))(\tau(x_V))| \le |f(x_V)| + |\alpha h(x_V)| = |f(x_V)| + |h(x_V)| \\
&\le \max_{\xi\in\delta A}\big(|f(\xi)| + |h(\xi)|\big) = |f(\xi_V)| + R = \big|f(\xi_V) + \alpha_V R\big|,
\end{aligned}$$

thus

$$|f(x_V)| + |h(x_V)| = \big|f(\xi_V) + \alpha_V R\big| = \max_{\xi\in\delta A}\big(|f(\xi)| + |h(\xi)|\big).$$

Since this maximum is attained only within $P(h)$, $x_V \in P(h)$, thus $h(x_V) = 1$, and according to (1.41), $(T(\alpha_V h))(z_V) = (T(\alpha_V h))(\tau(x_V)) = \alpha_V h(x_V) = \alpha_V R$. Now equality (1.42) becomes

$$\begin{aligned}
f(\xi_V) + \alpha_V R &= \big((Tf + T(\alpha_V h)\big)(z_V) = (Tf)(z_V) + (T(\alpha_V h))(z_V) \\
&= (Tf)(\tau(x_V)) + \alpha_V h(x_V) = (Tf)(\tau(x_V)) + \alpha_V R,
\end{aligned}$$

thus $f(\xi_V) = (Tf)(\tau(x_V))$. Therefore, any neighborhood V of x_0 contains points ξ_V and x_V such that $f(\xi_V) = (Tf)(\tau(x_V))$. The continuity of f, Tf, and τ implies that $f(x_0) = (Tf)(\tau(x_0))$. □

Theorem 1.6.14. *Let $A \subset C(X)$ and $B \subset C(Y)$ be uniform algebras on their maximal ideal spaces X and Y correspondingly. If a surjective operator $T\colon A \to B$ satisfies the equalities*

(i) $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$, *and*

(ii) $\max\limits_{\eta\in\partial B}\big(|(Tf)(\eta)| + |(Tg)(\eta)|\big) = \max\limits_{\xi\in\partial A}\big(|f(\xi)| + |g(\xi)|\big)$

for every f and g in A, then T is an isometric algebra isomorphism from A onto B.

Proof. Proof. Since the operator T satisfies the hypotheses of Proposition 1.6.7, then Proposition 1.6.13 implies that the equality (1.40), i.e. $(Tf)(\tau(x)) = f(x)$, holds for every $x \in \delta A$ and all $f \in A$. Therefore, the restricted operator $T'\colon A|_{\delta A} \to B|_{\delta B}$ defined by $T'(f|_{\delta A}) = Tf|_{\delta B}$, $f \in A$, is an algebra isomorphism between $A|_{\delta A}$ and δB. Since the Choquet boundary of an algebra is its boundary, $A|_{\delta A} \cong A$ and $B|_{\delta B} \cong B$, and also T is uniquely determined by T'. It follows that T is an algebra isomorphism between A and B. □

Recall that an operator $T\colon A \to B$ is said to be $\mathbb{T}$-*homogeneous*, if the equality $(Tf)(sf) = s(Tf)$ holds for every $f \in A$ and any $s \in \mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$.

Lemma 1.6.15. *Every additive operator T which preserves the peripheral spectra of algebra elements is σ_π-additive, i.e. satisfies the peripheral additivity property* (i) *of Theorem* 1.6.14.

This follows immediately from $\sigma_\pi(Tf+Tg)=\sigma_\pi\big(T(f+g)\big)=\sigma_\pi(f+g)$.

Lemma 1.6.16. *If an operator $T\colon A\to B$ satisfies the equality*

$$\|Tf+\alpha Tg\|=\|f+\alpha g\| \tag{1.43}$$

for every $f,g\in A$ and any $\alpha\in\mathbb{T}$, then T satisfies equality (ii) *of Theorem* 1.6.14.

Proof. Let $f,g\in A$. If $\alpha\in\mathbb{T}$ is such that

$$\max_{\eta\in\partial B}\big(|(Tf)(\eta)|+|(Tg)(\eta)|\big)=\max_{\eta\in\partial B}\big|(Tf)(\eta)+\alpha(Tg)(\eta)\big|,$$

then equality (1.43) implies

$$\begin{aligned}&\max_{\eta\in\partial B}\big(|(Tf)(\eta)|+|(Tg)(\eta)|\big)=\max_{\eta\in\partial B}\big|(Tf)(\eta)+\alpha(Tg)(\eta)\big|=\|Tf+\alpha Tg\|\\&=\|f+\alpha g\|=\max_{\xi\in\partial A}\big|f(\xi)+\alpha g(\xi)\big|\le\max_{\xi\in\partial A}\big(|f(\xi)|+|g(\xi)|\big).\end{aligned}$$

The argument is reversible, and therefore,

$$\max_{\eta\in\partial B}\big(|(Tf)(\eta)|+|(Tg)(\eta)|\big)=\max_{\xi\in\partial A}\big(|f(\xi)|+|g(\xi)|\big)$$

i.e. equality (ii) holds, as claimed. □

Clearly, Lemma 1.6.16 holds for any operator that satisfies the equality $\sigma_\pi(Tf+\alpha Tg)=\sigma_\pi(f+\alpha g)$ for every f,g in A and any $\alpha\in\mathbb{T}$. In particular, it holds for any $\mathbb{T}$-homogeneous operator which is σ_π-additive.

Lemma 1.6.17. *Any $\mathbb{C}$-linear operator $T\colon A\to B$ with $T(1)=1$ which preserves the norms of algebra elements, preserves also their peripheral spectra.*

Proof. Let $f\in A$ and $z_0\in\sigma_\pi(f)$. Then $z_0=f(x_0)$ for some $x_0\in X$, and $|z_0|=|f(x_0)|=\|f\|$. Clearly, $\|f+z_0\|=\max_{\xi\in\partial A}\big|f(x)+z_0\big|=2|z_0|$. The linearity of T yields $T(f+z_0)=Tf+T(z_0)=Tf+z_0T(1)=Tf+z_0$. The norm-preservation of T implies $|(Tf)(y)|\le\|Tf\|=\|f\|=|z_0|$, and hence

$$\|Tf+z_0\|=\|T(f+z_0)\|=\|f+z_0\|=2|z_0|,$$

since $|f(x)|\le|z_0|$ for all $x\in X$. Thus $(Tf)(y_0)=z_0$ for some $y_0\in Y$, and therefore, $z_0\in\sigma_\pi(Tf)$, since $|z_0|=\|f\|=\|Tf\|$.

Conversely, let $u_0 \in \sigma_\pi(Tf)$ for some $f \in A$. Then $u_0 = (Tf)(y_0)$ for some $y_0 \in Y$, and $|u_0| = |(Tf)(y_0)| = \|Tf\|$. The linearity of T implies $T(f + u_0) = Tf + T(u_0) = Tf + u_0T(1) = Tf + u_0$. Therefore, by the norm-preservation of T we have $|f(x)| \leq \|f\| = \|Tf\| = |u_0|$ and

$$\|f + u_0\| = \big\|Tf + T(u_0)\big\| = \big\|T(f + u_0)\big\| = \|Tf + u_0\| = 2|u_0|,$$

since $|(Tf)(y)| \leq |u_0|$ for all $y \in Y$. Hence, $f(x_0) = u_0$ for some $x_0 \in X$, thus $u_0 \in \sigma_\pi(f)$, since $|u_0| = \|Tf\| = \|f\|$. Consequently, $\sigma_\pi(Tf) = \sigma_\pi(f)$. □

The next corollary follows from Theorem 1.6.14 and Lemma 1.6.16.

Corollary 1.6.18. *If a surjective additive operator $T\colon A \to B$ preserves the peripheral spectra of algebra elements and satisfies the equality*

(ii) $\max\limits_{\eta\in\partial B} \big(|(Tf)(\eta)| + |(Tg)(\eta)|\big) = \max\limits_{\xi\in\partial A} \big(|f(\xi)| + |g(\xi)|\big)$

for all $f, g \in A$, then T is an isometric algebra isomorphism.

Theorem 1.6.14 and Lemma 1.6.16 imply the following

Proposition 1.6.19. *If a surjective operator $T\colon A \to B$ satisfies the equality*

(i′) $\sigma_\pi(Tf + \alpha Tg) = \sigma_\pi(f + \alpha g)$

for every f, g in A and any $\alpha \in \mathbb{T}$, then T is an isometric algebra isomorphism.

Proof. Indeed, (i′) implies (1.43) and also (i) of Theorem 1.6.14, and therefore T satisfies both equalities (i) and (ii) of Theorem 1.6.14. □

As mentioned before, the equality (i′) is satisfied automatically by any $\mathbb{T}$-homogeneous operator which is σ_π-additive. The next proposition follows from Proposition 1.6.19.

Proposition 1.6.20. *If $T\colon A \to B$ is a surjective, $\mathbb{T}$-homogeneous, and σ_π-additive operator with $T(1) = 1$, then T is an isometric algebra isomorphism.*

Proposition 1.6.20 and Lemma 1.6.15 imply

Theorem 1.6.21. *Any surjective $\mathbb{C}$-linear operator T between two uniform algebras, which preserves the peripheral spectra of algebra elements, is an isometric algebra isomorphism.*

Corollary 1.6.22. *Any surjective $\mathbb{C}$-linear operator $T\colon A \to B$ with $T(1) = 1$, which preserves the norms of invertible algebra elements, is an isometric algebra isomorphism.*

Proof. Clearly, if T preserves the norms of invertible elements and $T(c) = c$ for any $c \in \mathbb{C}$, then T preserves the norms of all algebra elements. The result follows from Theorem 1.6.21 and Lemma 1.6.17. □

As a consequence from Corollary 1.6.22 we obtain Nagasawa's theorem [N] (also [R3]) for uniform algebras.

Corollary 1.6.23. *Let A and B be uniform algebras, and let $\mathcal{B}_A$ and $\mathcal{B}_B$ be their open unit balls correspondingly. If $F\colon \mathcal{B}_A \to \mathcal{B}_B$ is a biholomorphic mapping such that $F(0) = 0$, and $F(1) = 1$, then F extends on A as an isometric algebra isomorphism.*

Proof. As shown by T. Ransford [R3], F preserves the norms and extends to a $\mathbb{C}$-linear isomorphism from A onto B. Clearly, the extension also preserves the norms. Corollary 1.6.22 implies that this extension is also multiplicative. □

Now we consider the case of σ_π-multiplicative operators.

Lemma 1.6.24. *If an operator $T\colon A \to B$ with $T(1) = 1$ is σ_π-multiplicative, then the equalities*

(a) $\sigma_\pi(Tf) = \sigma_\pi(f)$,

(b) $\|Tf\| = \|f\|$,

(c) $\sigma_\pi\big((Tf)(Tg)\big) = \sigma_\pi\big(T(fg)\big)$, *and*

(d) $\big\|(Tf)(Tg)\big\| = \|fg\|$,

hold for all $f, g \in A$.

Proof. The equality (a) follows from the σ_π-multiplicativity of T by letting $g = 1$. (a) and the σ_π-multiplicativity of T imply (c), because of $\sigma_\pi\big(T(fg)\big) = \sigma_\pi(fg) = \sigma_\pi\big((Tf)(Tg)\big)$. Equalities (b) and (d) follow from (a) and (c) correspondingly, since $|z| = \|f\|$ for every $z \in \sigma_\pi(f)$. □

Note that according to equality (a) the operator T preserves the peripheral ranges of algebra elements. It is straightforward to see that the σ_π-multiplicativity property (1.33) is equivalent to both properties (a) and (c). According to (b), T preserves the norms of algebra elements, and according to (d) it is *norm-multiplicative.*

Lemma 1.6.25. *Any surjective and norm-multiplicative operator $T\colon A \to B$ is monotone increasing in modulus.*

Proof. If $|f(x)| \le |g(x)|$ on ∂A, then clearly $\|fh\| \le \|gh\|$ for any $h \in A$. The surjectivity of T implies that for any $k \in \mathcal{F}(B)$ there is an $h \in A$ such that $k = T(h)$. By the norm-multiplicativity of T we have

$$\big\|(Tf)\cdot k\big\| = \big\|(Tf)(Th)\big\| = \|fh\| \le \|gh\| = \big\|(Tg)(Th)\big\| = \big\|(Tg)\cdot k)\big\|$$

for every $k \in \mathcal{F}(B)$. Now Lemma 1.2.5 implies that $|(Tf)(y)| \le |(Tg)(y)|$ on ∂B. □

Proposition 1.6.26. *Any σ_π-multiplicative surjective operator $T\colon A \to B$ is bijective.*

Proof. If $Tf = Tg$ for some $f, g \in A$, then for any $h \in \mathcal{F}(A)$ we have $(Tf)(Th) = (Tg)(Th)$. Consequently, $\sigma_\pi\big((Tf)(Th)\big) = \sigma_\pi\big((Tg)(Th)\big)$. The σ_π-multiplicativity of T yields

$$\sigma_\pi(fh) = \sigma_\pi\big((Tf)(Th)\big) = \sigma_\pi\big((Tg)(Th)\big) = \sigma_\pi(gh).$$

Lemma 1.2.10 now implies that $f = g$. Therefore, T is injective, and consequently it is bijective. □

Lemma 1.6.27. *Let $T\colon A \to B$ be a surjective σ_π-multiplicative operator with $T(1) = 1$. Then for any generalized peak point $x \in \delta A$ the set E_x defined in (1.35) is a singleton and belongs to δB.*

Proof. According to Lemma 1.6.5, the set E_x is non-empty. Let $x \in \delta A$ be a generalized peak point of A. The σ_π-multiplicativity property of T implies that T is norm-multiplicative. According to Lemma 1.6.25, the operator T is monotone increasing in modulus. Since also $T(1) = 1$, it follows from Lemma 1.6.24(a) that T preserves the peripheral spectra of algebra elements. Hence T satisfies the hypotheses of Lemma 1.6.5. From its proof we know that $\big\{P(Tf)\colon f \in \mathcal{F}_x(A)\big\}$ is a family of peak sets with non-empty intersection, E_x, hence it meets δB (e.g. [L1]). Consequently, $E_x \cap \delta B \neq \emptyset$.

Since T preserves the peripheral ranges of algebra elements, then, by (1.33), we have that $T^{-1}\big(\mathcal{F}(B)\big) = \mathcal{F}(A)$. We claim that $T^{-1}\big(\mathcal{F}_y(B)\big) \subset \mathcal{F}_x(A)$ for any $y \in E_x \cap \delta B$. Let $y \in E_x \cap \delta B$ $k \in \mathcal{F}_y(B)$, and let $h = T^{-1}(k)$. The function h is unique, as it was already shown that T is injective. To show that $h \in \mathcal{F}_x(A)$ it is enough to prove that $h(x) = 1$. Take an open neighborhood V of x and a peaking function $g \in \mathcal{F}_x(A)$ with $P(g) \subset V$. By (1.33) we have $Tg \in \mathcal{F}(B)$. Since $y \in E_x \subset P(Tg)$ we have $(Tg)(y) = 1$, thus $Tg \in \mathcal{F}_y(B)$. Lemma 1.6.24(c) yields

$$k(y)(Tg)(y) = 1 \geq \|hg\| = \big\|(Th)(Tg)\big\| = \big\|k \cdot (Tg)\big\| = 1.$$

Hence $\|hg\| = 1$, and there must be a $x_V \in \partial A$ with $h(x_V) = 1$ and $g(x_V) = 1$. Therefore, $x_V \in P(g) \subset V$. We deduce that any neighborhood V of x contains a point x_V with $h(x_V) = 1$. The continuity of h implies that $h(x) = 1$, so $h \in \mathcal{F}_x(A)$. Consequently, $T^{-1}\big(\mathcal{F}_y(B)\big) \subset \mathcal{F}_x(A)$, as claimed.

Let $y \in E_x \cap \delta B$, and suppose there exists $z \in E_x \setminus \{y\}$. Then there exists a peaking function $k \in \mathcal{F}_y(B)$ such that $|k(z)| < 1$. By what was shown above, if $h = T^{-1}(k)$, then $h(x) = 1$. Thus $E_x \subset P(Th) = P(k)$, which implies $k(z) = 1$ contradicting $|k(z)| < 1$. This shows that the set E_x contains exactly one point. □

We see that under the assumptions of Lemma 1.6.27, for any $x \in \delta A$ the element $\tau(x)$ from (1.37) for which

$$\{\tau(x)\} = E_x = \bigcap_{f \in \mathcal{F}_x(A)} P(Tf)$$

is well-defined, and so is the mapping $\tau\colon x \longmapsto \tau(x)$ from δA into δB. Moreover, the equality (1.38), i.e.

$$(Th)(\tau(x)) = h(x)$$

holds for every $\mathbb{C}^*$-peaking function $h \in \mathbb{C}^* \cdot \mathcal{F}_x(A)$. Under the assumptions of Lemma 1.6.27, the operator T is bijective, by Proposition 1.6.26. Let $k \in s \cdot \mathcal{F}_{\tau(x)}(B)$, for some $x \in \delta A$, and let $T^{-1}k = h \in \mathbb{C} \cdot \mathcal{F}(A)$. By (1.38) we have $k(\tau(x)) = (Th)(\tau(x)) = h(x) = (T^{-1}k)(x)$. Therefore, the equality (1.39), i.e.

$$(T^{-1}k)(x) = k(\tau(x))$$

also holds for every $x \in \delta A$ and any $\mathbb{C}^*$-peaking function $k \in \mathbb{C}^* \cdot \mathcal{F}_{\tau(x)}(B)$ of B.

Lemma 1.6.28. *Let $T\colon A \to B$ be a surjective and σ_π-multiplicative operator with $T(1) = 1$, and let $f \in A$. If $(Tf)(\tau(x_0)) = 0$ for some $x_0 \in \delta A$, then also $f(x_0) = 0$.*

Proof. Let x_0 be a generalized peak point of A and let $f \in A$. Choose an open neighborhood U of $\tau(x_0)$ in Y, such that $|(Tf)(y)| < \varepsilon$ on U. Let $k \in \mathcal{F}_{\tau(x_0)}(B)$ be a peaking function of B with $\tau(x_0) \in P(k) \subset U$. By taking a big enough power of k, we may assume from the beginning that $\big|(Tf)(y)k(y)\big| < \max\limits_{\xi\in\overline{U}} \big(|(Tf)(\eta)k(\eta)|\big) < \varepsilon$ for all $y \in Y \setminus U$. Consequently, $\big\|(Tf)\cdot k\big\| < \varepsilon$, and according to Lemma 1.6.24(d),

$$\big\|f \cdot (T^{-1}k)\big\| = \big\|(Tf)\cdot k\big\| < \varepsilon.$$

Hence by (1.37) we have

$$|f(x_0)| = \big|f(x_0)\, k(\tau(x_0))\big| = \big|f(x_0)\big((T^{-1}k)(x_0)\big)\big| < \varepsilon.$$

Thus, $|f(x_0)| < \varepsilon$, and consequently, $f(x_0) = 0$ by the liberty of choice of ε. □

Lemma 1.6.29. *If $T\colon A \to B$ is a surjective and σ_π-multiplicative operator with $T(1) = 1$, then $|f(x)| \le |(Tf)(\tau(x))|$ for every $x \in \delta A$ and any $f \in A$.*

Proof. The proof follows the line of proof of Lemma 1.43. According to Lemma 1.6.24(b) and (c), $\|Tf\| = \|f\|$, and $\big\|(Tf)(Tg)\big\| = \|fg\|$ for all $f, g \in A$.

Let $x \in \delta A$, $f \in A$, and let $g = Tf \in B$. Without loss of generality we can assume that $g(\tau(x)) \neq 0$, since if $g(\tau(x)) = (Tf)(\tau(x)) = 0$, then, by Lemma 1.6.28, also $f(x) = 0$, and the result carries. If U is an open neighborhood of $\tau(x)$ in Y, then by Bishop's Lemma we can choose a peaking function $k \in \mathcal{F}_{\tau(x)}(B)$ with $P(k) \subset U$, such that the function $\big|(gk)(y)\big|$ attains its maximum only within $P(k) \subset U$. Let $\eta_U \in P(k)$ be such that $|g(\eta_U)| = \max\limits_{\eta\in P(k)} |g(\eta)|$. Denote $h = T^{-1}(k) \in \mathcal{F}_x(A)$. Since T is norm-multiplicative, we have

$$\begin{aligned}|f(x)| &= \big|f(x)\,\big(T^{-1}(k)\big)(x)\big| = \big|\big(f\cdot T^{-1}(k)\big)(x)\big|\\ &\le \|f\cdot T^{-1}(k)\| = \|(Tf)\cdot k\| = \big\|g\,k\big\| = |g(\eta_U)|,\end{aligned}$$

thus $|f(x)| \le |g(\eta_U)|$. We have obtained that any neighborhood U of $\tau(x)$ contains a point η_U such that $|f(x)| \le |g(\eta_U)|$. The continuity of g implies that $|f(x)| \le |g(\tau(x))| = \big|(Tf)(\tau(x))\big|$. □

Lemma 1.6.30. *If $T\colon A \to B$ is a surjective and σ_π-multiplicative operator with $T(1) = 1$, then the mapping τ from (1.37) is a homeomorphism from δA onto δB.*

Proof. According to Proposition 1.6.26, the operator T is bijective. Since the σ_π-multiplicativity condition (1.33) is symmetric with respect to f and Tf, it holds also for the inverse operator T^{-1}. According to Lemma 1.6.27 there arises a corresponding continuous map $\psi\colon \delta B \to \delta A$ such that the equality (1.38), which in this case reduces to $\big(T^{-1}(k)\big)(\psi(\eta)) = k(\eta)$, holds on δB for any $k \in \mathcal{F}_\eta(B)$. Let $x \in \delta A$ and $y = \tau(x) \in \delta B$. If $h \in \mathcal{F}_x(A)$, then, due to Lemma 1.6.24(a) and (1.38), $k = Th \in \mathcal{F}_y(B)$, therefore $h(\psi(y)) = \big(T^{-1}(k)\big)(\psi(y)) = k(y) = (Th)(y) = (Th)(\tau(x)) = h(x) = 1$, and whence $\psi(y) \in P(h)$. Since this holds for every $h \in \mathcal{F}_x(A)$, and $\bigcap\limits_{h \in \mathcal{F}_x(A)} P(h) = \{x\}$, we have that $\psi(\tau(x)) = \psi(y) = x$ for every $x \in \delta A$. By similar arguments one can see that $\tau(\psi(y)) = y$ for any $y \in \delta B$. Consequently, τ and ψ both are injective mappings, and $\psi = \tau^{-1}$.

Let $x \in \delta A$ be a generalized peak point of A, and let $r \in (0, 1)$. Choose an open neighborhood V of $\tau(x)$ in δB, and a peaking function $k \in \mathcal{F}_{\tau(x)}(B)$ with $\tau(x) \in P(k) \subset V$ and $|k(y)| < r$ on $\delta B \setminus V$. If $h = T^{-1}(k)$, then $h \in \mathcal{F}_x(A)$, and according to (1.38), $k(\tau(\xi)) = (Th)(\tau(\xi)) = h(\xi)$ on δA. Note that since $h(x) = 1 > r$, the open set $W = \big\{\xi \in \delta A\colon |h(\xi)| > r\big\}$ contains x. According to Lemma 1.6.29, for any $\xi \in W$ we have $|k(\tau(\xi))| = \big|(Th)(\tau(\xi))\big| \ge |h(\xi)| > r$, and therefore, $\tau(\xi) \in V$, since on $\delta B \setminus V$ we have $|k(\eta)| < r$. Consequently, $\tau(W) \subset V$, which proves the continuity of τ. If we consider the operator $T^{-1}\colon B \to A$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$, the same arguments imply that τ^{-1} is also continuous, which completes the proof. □

When applied to the operator $T^{-1}\colon B \to A$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$, Lemma 1.6.29 implies $|g(y)| \le |(T^{-1}g)(\tau^{-1}(y))|$ for any $y \in \delta B$ and every $g \in B$. By letting $g = Tf,\ f \in A$, and $y = \tau(x),\ x \in \delta A$, we obtain $\big|(Tf)(\tau(x))\big| \le |f(x)|$. Hence we have the following

Corollary 1.6.31. *If $T : A \to B$ is a surjective and σ_π-multiplicative operator with $T(1) = 1$, then $\big|(Tf)(\tau(x))\big| \le |f(x)|$ for any $x \in \delta A$ and every $f \in A$.*

Proposition 1.6.32. *If $T\colon A \to B$ is a surjective and σ_π-multiplicative operator with $T(1) = 1$, then the equality*

$$(Tf)(\tau(x)) = f(x) \tag{1.44}$$

holds for every $f \in A$ and $x \in \delta A$.

Proof. The proof follows the line of proof of Proposition 1.6.13. By Lemma 1.6.27, the mapping τ from (1.37) is well-defined, and every fixed $x \in \delta A$ all peaking functions $h \in \mathcal{F}_x(A)$ satisfy the equality (1.38), i.e. (1.44).

Let x_0 be a generalized peak point of A and let $f \in A$. Without loss of generality we can assume that $f(x_0) \neq 0$, since in $f(x_0) = 0$, then also $(Tf)(\tau(x_0)) = 0$ by Lemma 1.6.28, if applied to the operator $T^{-1}\colon B \to A$, the function $Tf \in B$ and the mapping $\tau^{-1}\colon \delta B \to \delta A$. Let V be an open neighborhood of x_0 in X. By Bishop's Lemma we can choose a peaking function $h \in \mathcal{F}_{x_0}(A)$ so that $x_0 \in P(h) \subset V$, and such that the function $|(fh)(x)|$ attains its maximum only within $P(h) \subset V$. Hence there is a point $\xi_V \in P(h)$ such that

$$|f(\xi_V)| = \big|(fh)(\xi_V)\big| = \|fh\|. \tag{1.45}$$

Therefore, $f(\xi_V) \in \sigma_\pi(fh) = \sigma_\pi\big((Tf)(Th)\big)$. Hence there is a point $z_V \in Y$ with

$$f(\xi_V) = \big((Tf)(Th)\big)(z_V). \tag{1.46}$$

We may assume that $z_V \in \delta B$. Indeed, $|f(\xi_V)| = \big|(fh)(\xi_V)\big|$ is the maximum modulus of the function fh, and, according to Lemma 1.6.24(d), of the function $(Tf)(Th)$ as well. Therefore, the value $f(\xi_V)$ is attained by $(Tf)(Th)$ at some point of the Choquet boundary δB, and we can choose z_V to be such a point. The surjectivity of τ implies that $z_V = \tau(x_V)$ for some $x_V \in \delta A$. Equality (1.46), Corollary 1.6.31 and (1.38) imply

$$\begin{aligned} |f(\xi_V)| &= \big|\big((Tf)(Th)\big)(z_V)\big| = \big|(Tf)(\tau(x_V))\big|\big|(Th)(\tau(x_V))\big| \\ &\leq |f(x_V)||h(x_V)| = \big|(fh)(x_V)\big| \leq \|fh\| = |f(\xi_V)|, \end{aligned}$$

thus $\big|(fh)(x_V)\big| = |f(\xi_V)| = \|fh\| = \max_{\xi \in X}\big|(fh)(\xi)\big|$. Since this maximum is attained only within $P(h)$, $x_V \in P(h)$, and according to (1.38), $(Th)(z_V) = (Th)(\tau(x_V)) = h(x_V) = 1$. Now equality (1.46) becomes $f(\xi_V) = (Tf)(z_V) = (Tf)(\tau(x_V))$. Therefore, any neighborhood V of x_0 contains points ξ_V and x_V such that $f(\xi_V) = (Tf)(\tau(x_V))$. The continuity of f, Tf, and τ implies that $f(x_0) = (Tf)(\tau(x_0))$. □

The next theorem follows from Proposition 1.6.32 in the same way as Theorem 1.6.14 follows from Proposition 1.6.13.

Theorem 1.6.33. *Let $A \subset C(X)$ and $B \subset C(Y)$ be uniform algebras on their maximal ideal spaces X and Y correspondingly. If $T\colon A \to B$ is a surjective and σ_π-multiplicative operator with $T(1) = 1$, then T is an isometric algebra isomorphism.*

Since every multiplicative operator with $T(1) = 1$ which preserves the peripheral spectra of algebra elements is σ_π-multiplicative, we have the following

Corollary 1.6.34. *Any surjective and multiplicative operator T between two uniform algebras with $T(1) = 1$, which preserves the peripheral spectra of algebra elements is an isometric algebra isomorphism.*

Note that if an operator $T\colon A \to B$ preserves the spectra of algebra elements, then it preserves also their peripheral spectra. As a consequence of Theorem 1.6.33 we obtain

Corollary 1.6.35. *If a surjective operator $T\colon A \to B$ with $T(1) = 1$ possesses the σ-multiplicativity property*

$$\sigma\big((Tf)(Tg)\big) = \sigma(fg)$$

for all $f, g \in A$, then T is an isometric algebra isomorphism.

1.7 Notes

The foundations of the theory of commutative Banach algebras were laid on by Gelfand in [G3]. Systematic expositions on uniform algebras are given, for instance, in [B7, G1, L1, P1, S4, P4, T2, Ż], among others. Shilov boundaries were introduced by Shilov in [G3]. A proof of Rossi's Local Maximum Modulus Principle can be found, say, in [S4], or, [AW].

A thorough exposition of inductive limits of uniform algebras is presented in [L1]. The notion of Bourgain algebras was introduced by J. Cima and R. Timoney [CT] in their study of the Dunford-Pettis property of uniform algebras. It is based on a construction of J. Bourgain [B6] involving operators of Hankel type. Most of the results on Bourgain algebras in this chapter, including the alternative proof of the Poincaré theorem are from [TY] and [TY1]. Example 1.4.3 is due to S. Saccone. Bourgain algebras of topological algebras were considered in [AT]. Bourgain algebras of type $A_b^B(\mathcal{S})$, when $\mathcal{S}$ is a class of sequences in A, become less closely associated to A and B as $\mathcal{S}$ becomes smaller. Note that if $A = \{0\}$ and B is with unit, then $A \cup \{1\} \subset A_b^B(\mathcal{S})$ implies $A_b^B(\mathcal{S}) = B$, since $\mathcal{S}_A$ consists only of the zero sequence. The last remark in Section 1.4 was pointed out by K. Yale, who also raised the question whether every closed intermediate algebra between two given algebras $A \subset B$ can be described as an algebra of type $A_b^B(\mathcal{S})$ for certain classes of sequences $\mathcal{S}$. For example, as it is well known, the closed algebras between $A = H^\infty(\mathbb{T})$ and $B = L^\infty(\mathbb{T})$ on the circle $\mathbb{T}$ are characterized in terms of interpolating Blaschke products. It would be very useful to have their alternative descriptions, and also to have a characterization of the closed subalgebras between $H^\infty(\mathbb{D})$ and $L^\infty(\mathbb{D})$ on the unit disc $\mathbb{D}$ in terms of Bourgain type algebras relative to various classes of sequences.

Polynomial and algebraic extensions of Banach algebras were introduced by Arens and Hoffman in [AH]. Most of the results on polynomial extensions of commutative Banach algebras in this chapter are from [G10]. An algebra B is said to be an *integral extension* of its subalgebra A, if every element in B is an integral element over A. Given two uniform algebras A and B on X, such that B is a non-trivial integral extension of A, it is interesting to know if A is necessarily a polynomially closed algebra. Every strong polynomial extension of A is also an integral extension. We are not aware of any examples of integral extensions of a uniform algebra, that are not polynomial extensions. It will be interesting to find descriptions of polynomial and integral extensions of uniform algebras, and, in particular, of shift-invariant algebras, introduced in Chapter 4.

Finding conditions for an operator between Banach algebras which imply its linearity and multiplicativity, is an important question in Banach algebra theory, which still lacks a satisfactory answer. For linear operators to semisimple algebras an answer is suggested by the theorem of Gleason-Kahane-Żelazko (e.g. [Ż]) in terms of spectra of algebra elements. The theorem by Kowalski and Slodkowski [KS] implies also an answer for operators which are not necessarily linear. N. V. Rao and A. K. Roy [RR] have introduced the σ-multiplicativity condition in Corollary 1.6.35 and a mapping similar to $\tau\colon \delta B \to \delta A$ from (1.37). In the case when $A = C(X)$ this condition was considered by Molnár [M2]. Hatori, Miura and Takagi [HMT] have replaced the σ-multiplicativity condition in Corollary 1.6.35 by a similar 'range multiplicativity' condition, $Ran\big((\Phi f)(\Phi g)\big) = Ran(fg)$, where $Ran(f) = f(X)$ is the range of f. The conditions for a σ_π-additive operator between two uniform algebras to be an algebra isomorphism from Section 1.6 are from [RTT]. The arguments apply also for uniform algebras $A \subset C(X)$, where X is not necessarily the maximal ideal space of A, provided the peaking functions of A are replaced by the peaking functions of A on X, and the peripheral spectra of algebra elements are replaced by their *peripheral ranges* $Ran_\pi(f) = \big\{f(x)\colon |f(x)| = \|f\|,\ x \in X\big\}$. The results for σ_π-multiplicative operators from Section 1.6 hold also for Ran_π-multiplicative operators (cf. [LT]).

Chapter 2

Three classical families of functions

This chapter contains the basics of three classical spaces of functions, namely, almost periodic functions in real variables, harmonic functions in the disc, and H^p-functions on the unit circle and disc. Later on, in Chapters 7 and 8, we introduce and study far reaching generalizations of these function spaces on abelian groups.

2.1 Almost periodic functions of one and several variables

Definition 2.1.1. A continuous function f on the real line $\mathbb{R}$ is said to be *almost periodic* on $\mathbb{R}$ if for every $\varepsilon > 0$ there is an $L > 0$ such that within every interval $I \subset \mathbb{R}$ with $|I| \geq L$ there is some $x \in I$ such that $\max_{t\in\mathbb{R}} \left|f(t+x) - f(t)\right| < \varepsilon$.

The number L is called an *ε-period* of f. Every periodic function with period p is clearly almost periodic. In this case we can choose $L = p$. The Bochner theorem (e.g. [B3]) asserts that f is almost periodic on $\mathbb{R}$ if and only if the set of all its $\mathbb{R}$-shifts $f_t(x) = f(x+t),\ t \in \mathbb{R}$, is relatively uniformly compact in the set $C_b(X)$ of bounded continuous functions. Equivalently, f is almost periodic if it can be approximated uniformly on $\mathbb{R}$ by *exponential polynomials*, i.e. functions of type $\sum_{k=1}^{n} a_k e^{is_k x}$, where the a_k are complex, and s_k are real numbers. Note that the functions $\chi^s(x) = e^{isx}$ are continuous characters on $\mathbb{R}$, i.e. $\chi^s(x+y) = \chi^s(x)\,\chi^s(y)$ for any $x, y \in \mathbb{R}$, and $|\chi^s| \equiv 1$ on $\mathbb{R}$. It is easy to see that the set $AP(\mathbb{R})$ of almost periodic functions on $\mathbb{R}$ is an algebra over $\mathbb{C}$. Actually, under the uniform norm $AP(\mathbb{R})$ is a commutative Banach algebra with unit. We provide the next result with a short proof, probably well known.

Lemma 2.1.2. *Let f be an almost periodic function on $\mathbb{R}$. If there are numbers $t_n \in \mathbb{R}$ such that $\lim_n f(x+t_n) = 0$ for all $x \in \mathbb{R}$, then $f \equiv 0$.*

Proof. By Bochner's theorem mentioned above, the space $\{f_t\}$ of t-shifts of f is relatively uniformly compact in $C_b(\mathbb{R})$. Hence, one can find a subsequence of $\mathbb{R}$-shifts $f_{t_n} = f(x+t_n)$ of f, that converge uniformly on $\mathbb{R}$ to an almost periodic function h on $\mathbb{R}$. By $\|f_{t_n}\| = \|f\|$ for all n we have that $\|h\| = \|f\|$. Since $h(x) = \lim_{m\to\infty} f(x+t_{n_m})$, then $h \equiv 0$, and therefore, $f \equiv 0$. □

Let $f(x)$ be an almost periodic function on $\mathbb{R}$, and let $\lambda \in \mathbb{R}$. *Dirichlet coefficients* a_λ^f of f are the numbers

$$a_\lambda^f = \lim_{T\to\infty} \frac{1}{T} \int_y^{y+T} f(x)\, e^{-i\lambda x} dx, \tag{2.1}$$

where the existence of the limit, and also its value, are independent from $y \in \mathbb{R}$. Dirichlet coefficients a_λ^f are non-zero for countably many λ's at most, which are called *Dirichlet exponents* of $f(x)$. The set $\operatorname{sp}(f)$ of Dirichlet's exponents of $f(x)$ is called the *spectrum* of f. Hence, $\operatorname{sp}(f) = \{\lambda \in \mathbb{R}\colon a_\lambda^f \neq 0\}$ is a countable set. It is customary to express the fact that λ_k are the Dirichlet exponents, and the numbers $A_k^f = a_{\lambda_k}^f$, $k = 1, 2, \ldots$ are the Dirichlet coefficients of $f(x)$ by associating a series expansion to f, namely

$$f(x) \sim \sum_{k=1}^{\infty} A_k^f e^{i\lambda_k x}. \tag{2.2}$$

The series in (2.2), not necessarily convergent, is called the *Dirichlet series* of f. It is easy to see that if all Dirichlet coefficients of an $f \in AP(\mathbb{R})$ are zero, then $f \equiv 0$. Consequently, the correspondence (2.2) between almost periodic functions and their Dirichlet series is injective.

Given a subset $\Lambda \subset \mathbb{R}$, by $AP_\Lambda(\mathbb{R})$ we denote the space of all *almost periodic Λ-functions*, namely, almost periodic functions on $\mathbb{R}$ with spectrum contained in the set Λ, i.e.

$$AP_\Lambda(\mathbb{R}) = \{f \in AP(\mathbb{R})\colon\ \operatorname{sp}(f) \subset \Lambda\}.$$

Note that every $f \in AP_\Lambda(\mathbb{R})$ can be approximated uniformly on $\mathbb{R}$ by *exponential Λ-polynomials*, i.e. by functions of type $\sum_{k=1}^{n} a_k e^{is_k x}$, $s_k \in \Lambda$.

Let $\mathbb{C}_+ = \{z\colon\ \operatorname{Im} z > 0\}$ be the upper half-plane. Clearly, every exponential polynomial $\sum_{k=1}^{n} a_k e^{is_k x}$ with $s_k \geq 0$, $k = 1, \ldots, n$, is analytically extendable on $\overline{\mathbb{C}}_+ = \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$ by $\sum_{k=1}^{n} a_k e^{is_k(x+iy)}$, $s_k \in \Lambda$. If $\operatorname{sp}(f) \subset \mathbb{R}_+ = [0,\infty)$ for an

$f \in AP(\mathbb{R})$, then f also can be extended on $\overline{\mathbb{C}}_+$ as a bounded analytic function $\widetilde{f}$. In this case we call f an *analytic almost periodic function* on $\mathbb{R}$. The set of analytic almost periodic functions on $\mathbb{R}$ is denoted by $AP_a(\mathbb{R})$. If $\mathrm{sp}\,(f) \subset \Lambda \subset \mathbb{R}_+ = [0, \infty)$, then f is called an *analytic almost periodic Λ-function* on $\mathbb{R}$, and its extension $\widetilde{f}$ on $\overline{\mathbb{C}}_+$ is called an *analytic almost periodic Λ-function* on $\overline{\mathbb{C}}_+$. In this case the extended function $\widetilde{f}$ on $\overline{\mathbb{C}}_+$ can be approximated uniformly on $\overline{\mathbb{C}}_+$ by exponential polynomials of type $\sum_{k=1}^{n} a_k e^{i s_k z}$, $s_k \in \Lambda$, $z \in \overline{\mathbb{C}}_+$.

If S is an additive subsemigroup of $\mathbb{R}$, containing 0, then the space $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$ is a commutative Banach subalgebra of the algebra $AP(\mathbb{R})$. If, in addition, $S \subset \mathbb{R}_+$, then every function $f \in AP_S(\mathbb{R})$ is analytic almost periodic. In fact, the algebra $\widetilde{AP_S}(\mathbb{R})$ of analytic extensions of functions in $AP_S(\mathbb{R})$ on $\mathbb{C}_+$ is isometrically isomorphic to $AP_S(\mathbb{R})$.

A continuous function f on $\mathbb{R}^n$ is said to be *almost periodic* on $\mathbb{R}^n$, if the set of all its $\mathbb{R}^n$-shifts $f_{(t_1,t_2,\ldots,t_n)}(x_1, x_2, \ldots, x_n) = f(x_1 + t_1, x_2 + t_2, \ldots, x_n + t_n)$, $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, is relatively uniformly compact in $C_b(\mathbb{R}^n)$. Equivalently, f is almost periodic on $\mathbb{R}^n$ if it can be approximated uniformly on $\mathbb{R}^n$ by *exponential polynomials* on $\mathbb{R}^n$, i.e. functions of type

$$\sum a_{k_1,k_2,\ldots,k_n} e^{i\,(s_{k_1}x_1 + s_{k_2}x_2 + \cdots + s_{k_n}x_n)},$$

where $a_{k_1,k_2,\ldots,k_n}$ are complex, and $s_{k_1}, s_{k_2}, \ldots, s_{k_n}$ are real numbers. Note that any function $\chi(x_1, x_2, \ldots, x_n) = e^{i\,(s_{k_1}x_1 + s_{k_2}x_2 + \cdots + s_{k_n}x_n)}$ is a continuous character on $\mathbb{R}^n$, i.e.

$$\chi\,(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \chi\,(x_1, x_2, \ldots, x_n)\,\chi\,(y_1, y_2, \ldots, y_n),$$

for any $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $|\chi| \equiv 1$ on $\mathbb{R}^n$. The set $AP(\mathbb{R}^n)$ of all almost periodic functions on $\mathbb{R}^n$ is a commutative Banach algebra with unit under the uniform norm on $\mathbb{R}^n$. Let S be a semigroup in $\mathbb{R}^n$. A function in $AP(\mathbb{R}^n)$ which can be approximated uniformly on $\mathbb{R}^n$ by exponential polynomials $\sum a_{k_1,k_2,\ldots,k_n} e^{i(s_{k_1}x_1 + s_{k_2}x_2 + \cdots + s_{k_n}x_n)}$ such that $(s_{k_1}, s_{k_2}, \ldots, s_{k_n}) \in S$ is called an *almost periodic S-function on* $\mathbb{R}^n$. The space of almost periodic S-functions on $\mathbb{R}^n$ is denoted by $AP_S(\mathbb{R}^n)$.

The space $C_b(\mathbb{R})$ of bounded continuous functions on the real line $\mathbb{R}$ is a uniform algebra under the sup-norm on $\mathbb{R}$. Clearly, the space $C_0(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ vanishing at $\pm\infty$ is a subalgebra, actually an ideal, of $C_b(\mathbb{R})$ without unit.

Definition 2.1.3. A bounded continuous function f on $\mathbb{R}$ is said to be *asymptotically almost periodic* on $\mathbb{R}$ if there is an almost periodic function f^* on $\mathbb{R}$ such that

$$\lim_{n \longrightarrow \infty} \left| f(x_n) - f^*(x_n) \right| = 0$$

for every sequence $x_n \longrightarrow \pm\infty$. If f^* is analytic almost periodic, then f is said to be an *asymptotically analytic almost periodic* function on $\mathbb{R}$. We denote by $AP^{as}(\mathbb{R})$, resp. $AP_a^{as}(\mathbb{R})$, the spaces of asymptotically almost periodic, resp. asymptotically analytic almost periodic, functions on $\mathbb{R}$.

Observe that the function f^* is uniquely defined, by Lemma 2.1.2. Since, clearly, $h = f - f^* \in C_0(\mathbb{R})$, we have the following

Lemma 2.1.4. *A bounded continuous function f on $\mathbb{R}$ is asymptotically almost periodic, resp. asymptotically analytic almost periodic on $\mathbb{R}$, if there is a unique $f^* \in AP(\mathbb{R})$, resp. $f^* \in AP_a(\mathbb{R})$, and an $h \in C_0(\mathbb{R})$, such that $f = f^* + h$.*

Lemma 2.1.4 implies that $AP^{as}(\mathbb{R}) \cong AP(\mathbb{R}) \oplus C_0(\mathbb{R})$, resp. $AP_a^{as}(\mathbb{R}) \cong AP_a(\mathbb{R}) \oplus C_0(\mathbb{R})$.

Denote by $\varphi\colon \mathbb{D} \longrightarrow \mathbb{C}_+$ the fractional linear transformation $\varphi(z) = i\dfrac{z+1}{1-z}$. Clearly, $e^{ia\varphi(z)} = e^{a\,(z+1)/(z-1)}$ for every $a \in \mathbb{R}$. Let Λ be a subset of $\mathbb{R}$. It is easy to see that the linear span $AP_\Lambda(\mathbb{R})$ of characters e^{iax}, $a \in \Lambda$, on $\mathbb{R}$ is isometrically isomorphic to the space $AP_\Lambda(\mathbb{R}) \circ \varphi \cong H^\infty_\Lambda$, the linear span of the functions $e^{a\,(z+1)/(z-1)} \in H^\infty$, $a \in \Lambda$, on $\mathbb{T} \setminus \{1\}$. The space $C_0(\mathbb{R}) \oplus \mathbb{C} \cdot 1$ is a commutative Banach algebra with unit. Clearly, $C_0(\mathbb{R}) \oplus \mathbb{C} \cdot 1 \cong \big(C_0(\mathbb{R}) \oplus \mathbb{C} \cdot 1\}\big) \circ \varphi = C(\mathbb{T})$, thus its maximal ideal space is homeomorphic to the unit circle $\mathbb{T}$. Observe that $C_0(\mathbb{R}) \cong C_0(\mathbb{R}) \circ \varphi = \big\{f \in C(\mathbb{T})\colon f(1) = 0\big\}$. Therefore, $AP^{as}(\mathbb{R})$ is isometrically isomorphic to the algebra $AP^{as}(\mathbb{R}) \circ \varphi = \big[z,\ 1/z,\ \big\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}\big\}\big]$ on $\mathbb{R}$ generated algebraically by the functions z, $1/z$, and $e^{a\,(z+1)/(z-1)}$, $a \in \mathbb{R}$. Correspondingly, the algebra $AP_a^{as}(\mathbb{R})$ of asymptotically analytic almost periodic functions is isometrically isomorphic to the algebra $AP_a^{as}(\mathbb{R}) \circ \varphi = \big[z,\ 1/z, \big\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\big\}\big]$ on $\mathbb{R}$.

2.2 Harmonic functions in the unit disc

A complex-valued, function u on the unit disc $\mathbb{D}$ is called *complex harmonic*, if it is twice differentiable on $\mathbb{D}$ and satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

If, in addition, u is real-valued, it is called a *harmonic function.* The space of complex-, resp. real-valued harmonic functions in $\mathbb{D}$ we denote by $\mathcal{H}(\mathbb{D})$, resp. $\mathcal{H}_\mathbb{R}(\mathbb{D})$. The space of analytic functions in the open unit disc $\mathbb{D}$ is denoted by $\mathcal{O}(\mathbb{D})$. Every $f \in \mathcal{O}(\mathbb{D})$ is a complex harmonic function in $\mathbb{D}$. Obviously, the real and the imaginary part of any complex harmonic function are real-valued harmonic functions. Therefore, the real (and the imaginary) parts of analytic functions in $\mathbb{D}$ are harmonic functions in $\mathbb{D}$. The converse is also true. Namely, every real-valued harmonic function u in $\mathbb{D}$ is the real part of some function f analytic in $\mathbb{D}$, i.e. $u = \operatorname{Re} f$, or, equivalently, $f = u+iv$, where v is another harmonic function in $\mathbb{D}$, called the *harmonic conjugate of* u. Note that the harmonic conjugate function is defined uniquely up to an additive constant. Clearly, $\mathcal{H}(\mathbb{D}) = \operatorname{Re}\mathcal{H}(\mathbb{D}) + i \operatorname{Im}\mathcal{H}(\mathbb{D}) = \mathcal{H}_{\mathbb{R}}(\mathbb{D}) + i\mathcal{H}_{\mathbb{R}}(\mathbb{D}) = \operatorname{Re}\mathcal{O}(\mathbb{D}) + i \operatorname{Re}\mathcal{O}(\mathbb{D}) = \mathcal{O}(\mathbb{D}) + \overline{\mathcal{O}}(\mathbb{D})$.

For every real r, $0 < r < 1$, the r-*dilation* f_r of f is defined by $f_r(z) = f(rz)$. Given an analytic function f in $\mathbb{D}$, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be its power series expansion, then its r-dilation f_r is analytic on the disc $\mathbb{D}_{1/r} \supset \mathbb{D}$, and admits there the power series expansion

$$f_r(z) = \sum_{n=0}^{\infty} (a_n r^n)\, z^n.$$

Observe that the restriction of the r-dilation f_r on $\mathbb{T}$ coincides with the r-*trace* $f|_{r\mathbb{T}}$ of f, i.e. the restriction of f on the circle $r\mathbb{T} = \{re^{i\theta},\ 0 < \theta \le 2\pi\}$. We will denote the r-trace of f again by f_r, thus $f_r\big|_{\mathbb{T}} = f\big|_{r\mathbb{T}}$, i.e. $f_r(z)\big|_{\mathbb{T}} = f(z)\big|_{r\mathbb{T}} = f(rz)$, where $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. If f possesses a continuous extension on the closed unit disc $\overline{\mathbb{D}}$, then $f\big|_{\mathbb{T}} \in C(\mathbb{T})$ and $\lim_{r\nearrow 1} f_r(e^{i\theta}) = \lim_{r\nearrow 1} f(re^{i\theta}) = f(e^{i\theta})$ for every $\theta \in [0, 2\pi]$.

Lemma 2.2.1. *If $f \in A(\mathbb{D})$, then* $\lim_{r\nearrow 1} \sup_{z\in\mathbb{D}} \big|f_r(z) - f(z)\big| = 0.$

Proof. The maximum modulus principle for analytic functions implies that

$$\sup_{z\in\overline{\mathbb{D}}} \big|f_r(z) - f(z)\big| = \sup_{\zeta\in\mathbb{T}} \big|f_r(\zeta) - f(\zeta)\big|.$$

If we suppose that $\overline{\lim_{r\nearrow 1}} \sup_{\zeta\in\mathbb{T}} \big|f_r(\zeta) - f(\zeta)\big| > \varepsilon$ for some $\varepsilon > 0$, then there are $r_n \nearrow 1$ and $\zeta_n \longrightarrow \zeta_0$ such that $\big|f_{r_n}(\zeta_n) - f(\zeta_0)\big| > \varepsilon$. Therefore,

$$\big|f(r_n\zeta_n) - f(\zeta_0)\big| = \big|f_{r_n}(\zeta_n) - f(\zeta_0)\big| > \varepsilon$$

as $n \longrightarrow \infty$, contradicting the continuity of f on $\overline{\mathbb{D}}$. □

Lemma 2.2.2. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the power series expansion of a function f in the disc algebra $A(\mathbb{D})$, then*

$$a_n = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} f(z)\, \zeta^{-(n+1)}\, d\zeta.$$

Proof. The power series expansion

$$f_r(z) = \sum_{k=0}^{\infty} (a_k r^k)\, z^k$$

of f_r, $0 < r < 1$, converges absolutely to f on $\overline{\mathbb{D}} \subset \mathbb{D}_{1/r}$ as $r \nearrow 1$. Therefore,

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{\mathbb{T}} f_r(\zeta)\,\zeta^{-(n+1)}\,d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \Big(\sum_{k=0}^{\infty}(a_k r^k)\zeta^k\Big)\zeta^{-(n+1)}\,d\zeta \\
&= \sum_{k=0}^{\infty}\frac{a_k r^k}{2\pi i}\int\limits_{\mathbb{T}} \zeta^{k-n-1}\,d\zeta = \sum_{k=0}^{\infty}\frac{a_k r^k}{2\pi}\int\limits_{\mathbb{T}} \zeta^{k-n}\,\frac{d\zeta}{i\zeta} = \sum_{k=0}^{\infty}\frac{a_k r^k}{2\pi}\int\limits_{0}^{2\pi} e^{i(k-n)\theta}\,d\theta \\
&= a_n r^n,
\end{aligned}$$

where $e^{i\theta} = \zeta$. Lemma 2.2.1 implies that

$$\frac{1}{2\pi i}\int\limits_{\mathbb{T}} f(\zeta)\,\zeta^{-(n+1)}\,d\zeta = \lim_{r\nearrow 1}\frac{1}{2\pi i}\int\limits_{\mathbb{T}} f_r(\zeta)\,\zeta^{-(n+1)}\,d\zeta = \lim_{r\nearrow 1} a_n r^n = a_n,$$

as claimed. □

Lemma 2.2.2 implies that for any $f \in A(\mathbb{D})$ and $z \in \mathbb{D}$ we have:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\Big(\int\limits_{\mathbb{T}} f(\zeta)\,\zeta^{-(n+1)}\,d\zeta\Big) z^n.$$

Note that since $|\zeta| = 1$, for every ϱ with $|z| < \varrho < 1$ the series $\sum\limits_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}$ is absolutely convergent in z on the disc $\overline{\mathbb{D}}_{\varrho}$. As a consequence we obtain the well-known Cauchy integral formula

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\Big(\int\limits_{\mathbb{T}} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta\Big) z^n \\
&= \frac{1}{2\pi i}\int\limits_{\mathbb{T}} f(\zeta)\Big(\sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}\Big)\,d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta.
\end{aligned}$$

If $z = re^{i\theta}$ and $\zeta = e^{it}$, then for the r-trace $f_r(\zeta) = f(r\zeta)$ of f we have

$$\begin{aligned}
f_r(e^{i\theta}) &= f(z) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\,\frac{e^{it}}{e^{it} - re^{i\theta}}\,dt \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\,\frac{1}{1 - re^{i(\theta - t)}}\,dt = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\,C_r(e^{i(\theta-t)})\,dt,
\end{aligned} \tag{2.3}$$

where $C_r(\zeta) = \dfrac{1}{1-r\zeta} = \sum\limits_{n=0}^{\infty} r^n\zeta^n$ is the so-called *Cauchy kernel* in $\mathbb{D}$. Consequently,

$$f_r(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, C_r(e^{i(\theta-t)})\, dt, \text{ i.e. } f_r = f \star C_r \tag{2.4}$$

for every $f \in A(\mathbb{D})$. If $u \in \mathcal{H}_{\mathbb{R}}(\mathbb{D})$ is a real-valued harmonic function in $\mathbb{D}$, then there is an analytic function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ such that $u = 2\operatorname{Re} f = f + \overline{f}$. Hence,

$$u(z) = 2\operatorname{Re} a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{a_n z^n},$$

and therefore, for the r-trace u_r of u we have

$$u_r(\zeta) = u(r\zeta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|}\zeta^n, \ |\zeta| = 1,$$

$$\text{where } c_n = \begin{cases} \overline{a}_n & \text{when } n < 0, \\ 2 \operatorname{Re} a_0 & \text{when } n = 0, \\ a_n & \text{when } n > 0. \end{cases} \tag{2.5}$$

The real part u_r of the r-trace of $f(z) = \dfrac{1}{1-z} = \sum\limits_{n=0}^{\infty} z^n \in \mathcal{O}(\mathbb{D})$ admits the expansion $u_r(\zeta) = \sum\limits_{n=-\infty}^{\infty} r^{|n|}\zeta^n + 1$. The function in two variables

$$P_r(\theta) = u_r(e^{i\theta}) - 1 = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = 2\sum_{n=0}^{\infty} r^n\cos(n\theta) - 1$$

is called the *Poisson kernel* in $\mathbb{D}$. It plays a special rôle in harmonic function theory. Observe that

$$\begin{aligned} P_r(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|}\zeta^n = \sum_{n=0}^{\infty} r^n\zeta^n + \sum_{n=1}^{\infty} r^n\overline{\zeta}^n = C_r(\zeta) + \big(C_r(\overline{\zeta}) - 1\big) \\ &= 2\operatorname{Re} C_r(\zeta) - 1 = \operatorname{Re}\big(2C_r(\zeta) - 1\big) = \operatorname{Re}\Big(2\frac{1}{1-r\zeta} - 1\Big) \\ &= \operatorname{Re}\Big(\frac{1+r\zeta}{1-r\zeta}\Big) = \operatorname{Re}\Big(\frac{1+re^{i\theta}}{1-re^{i\theta}}\Big) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \end{aligned}$$

where $\zeta = e^{i\theta}$. Consequently, $P_r(\theta)$ admits the representation

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \operatorname{Re}\left(\frac{1+r\zeta}{1-r\zeta}\right) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \tag{2.6}$$

where $\zeta = e^{i\theta}$ and $0 \le r < 1$.

Lemma 2.2.3. *For any r, $0 \le r < 1$ we have*

$$\inf_{0<\theta\le 2\pi} P_r(\theta) = \frac{1-r}{1+r}, \quad \textit{and} \quad \sup_{0<\theta\le 2\pi} P_r(\theta) = \frac{1+r}{1-r}.$$

Proof. Note that the conformal mapping $\dfrac{1+z}{1-z}$ maps the open unit disc $\mathbb{D}$ onto the open half-plane $\{z\colon\ \operatorname{Re} z > 0\}$. In particular, the circle $\{z = re^{i\theta}\colon 0 < \theta < 2\pi\}$ is mapped onto the circle centered on $\mathbb{R}$ which passes through the points $\dfrac{1-r}{1+r}$ and $\dfrac{1+r}{1-r}$. Since $P_r(\theta) = \operatorname{Re}\left(\dfrac{1-z}{1+z}\right)$, $z = re^{i\theta}$, we see that $\dfrac{1-r}{1+r} \le P_r(\theta) \le \dfrac{1+r}{1-r}$, where both bounds are taken. $\square$

Let u be a real-valued harmonic function on $\overline{\mathbb{D}}$, and $u = (f + \overline{f})/2$ for some $f \in A(\mathbb{D})$. We claim that if u has a continuous extension on $\overline{\mathbb{D}}$, then the r-trace u_r of u has the unique representation

$$u_r(e^{i\theta}) = u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\, P_r(\theta - t)\, dt = u(e^{i\theta}) \star P_r(\theta). \tag{2.7}$$

Indeed, since all negatively indexed Fourier coefficients of the Cauchy kernel $C_r(\zeta) = \sum_{n=0}^{\infty} r^n \zeta^n$ are zero, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} C_r(e^{it})\, e^{-int}\, dt = \begin{cases} r^n & \text{when } n \ge 0, \\ 0 & \text{when } n < 0. \end{cases}$$

Since $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \overline{f}(e^{it})\, C_r(e^{it})\, dt = \dfrac{1}{2\pi}\overline{\displaystyle\int_{-\pi}^{\pi} f(e^{it})\, \overline{C}_r(e^{it})\, dt}$, (2.3) implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\,P_r(\theta-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\,\big(C_r(e^{i(\theta-t)})+\overline{C}_r(e^{i(\theta-t)})-1\big)\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\big(f(e^{it})+\overline{f}(e^{it})\big)\big(C_r(e^{i(\theta-t)})+\overline{C}_r(e^{i(\theta-t)})-1\big)\,dt$$
$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} f(e^{it})\,C_r(e^{i(\theta-t)})\,dt + \frac{1}{4\pi}\int_{-\pi}^{\pi} f(e^{it})\,\overline{C}_r(e^{i(\theta-t)})\,dt$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi} \overline{f}(e^{it})\,C_r(e^{i(\theta-t)})\,dt + \frac{1}{4\pi}\int_{-\pi}^{\pi} \overline{f}(e^{it})\,\overline{C}_r(e^{i(\theta-t)})\,dt - \frac{1}{4\pi}\int_{-\pi}^{\pi} f(e^{it})\,dt$$
$$-\frac{1}{4\pi}\int_{-\pi}^{\pi} \overline{f}(e^{it})\,dt = (1/2)\big(f_r(e^{i\theta})+a_0+\overline{a}_0+\overline{f}_r(e^{i\theta})-a_0-\overline{a}_0\big) = u_r(e^{i\theta}).$$

The expression (2.7) is called the *Poisson integral representation of* u. Denote by H_r the function $H_r(\theta) = \dfrac{1+re^{i\theta}}{1-re^{i\theta}} = 2\,C_r(e^{i\theta})-1$. As we saw in (2.6), $P_r(\theta) = \operatorname{Re}\Big(\dfrac{1+re^{i\theta}}{1-re^{i\theta}}\Big) = \operatorname{Re} H_r(\theta)$. In the same way we can show that

$$u(e^{i\theta})\star H_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\,H_r(\theta-t)\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\big(f(e^{it})+\overline{f}(e^{it})\big)(2\,C_r\big(e^{i\theta})-1\big)\,dt = \frac{1}{2}\,\big(2f_r(e^{i\theta})+2\overline{a}_0-a_0-\overline{a}_0\big)$$
$$= \frac{1}{2}\big(2f_r(\zeta)-2i\ \operatorname{Im} a_0\big) = f_r(\zeta)-i\ \operatorname{Im} f(0),$$

where $\zeta = e^{i\theta}\in\mathbb{T}$. Therefore, if $f(0)$ is real and $f = u+iv$, then for the r-trace of f we have

$$f_r(e^{i\theta}) = f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\,H_r(\theta-t)\,dt = u(e^{i\theta})\star H_r(\theta). \tag{2.8}$$

The function $Q_r = \operatorname{Im} H_r$ is called the *conjugate Poisson kernel* in $\mathbb{D}$. As a corollary from (2.8) we obtain that the r-trace v_r also admits an integral representation involving Q_r. Namely,

$$\begin{aligned} v_r(e^{i\theta}) &= \operatorname{Im} f_r(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\ \operatorname{Im} H_r(\theta - t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\,Q_r(\theta - t)\,dt = u(e^{i\theta}) \star Q_r(\theta). \end{aligned}$$

2.3 The Poisson integral in the unit disc

In this section we present briefly some of the basic properties of the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta},\ 0 \le r < 1,\ -\pi \le \theta \le \pi$$

in the unit disc, which was defined in (2.6).

Let u be a real-valued harmonic function in $\mathbb{D}$, and let $u = \operatorname{Re} f = (f + \overline{f})/2$ for some analytic function f in $\mathbb{D}$. As we saw in (2.7), if u has a continuous extension on $\overline{\mathbb{D}}$, then the r-trace u_r of u admits the Poisson integral representation

$$u_r(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\ P_r(t - \theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{2\pi} u(e^{i(t-\theta)}) P_r(\theta)\,d\theta. \tag{2.9}$$

Similarly,

$$f_r(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\ P_r(t - \theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{2\pi} f(e^{i(t-\theta)}) P_r(\theta)\,d\theta, \tag{2.10}$$

for any $f \in A(\mathbb{D})$.

The Poisson kernel $P_r(\theta)$ has the following properties.

Theorem 2.3.1.

(i) $P_r(\theta) > 0$ *for every* $r \in [0, 1)$.

(ii) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t - \theta)\,d\theta = 1$ *for every* $r \in [0, 1)$.

(iii) *For every* $\delta > 0$, $\lim\limits_{r\nearrow 1}\ \sup\limits_{|\theta|>\delta} P_r(\theta) = 0$.

Given a continuous function f on $\mathbb{D}$, consider the coefficients

$$c_n^f(r) = \frac{c_n^{f_r}}{r^{|n|}} = \frac{1}{2\pi r^{|n|}}\int_{-\pi}^{\pi} f_r(e^{i\theta})\,e^{-in\theta}\,d\theta.$$

Note that $c_n^{f_r} = r^{|n|} c_n^f(r)$. The r-trace f_r has Fourier series expansion on $\mathbb{T}$ of type

$$f_r(\zeta) \sim \sum_{n-\infty}^{\infty} c_n^{f_r} \zeta^n = \sum_{n=-\infty}^{\infty} c_n^f(r)\, r^{|n|} \zeta^n, \ \zeta = e^{i\theta}.$$

If $f \in A(\mathbb{D})$, then $f = \sum_0^\infty a_n z^n$ in $\mathbb{D}$, and

$$c_n^f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=0}^{\infty} a_k e^{ik\theta} \Big)\, e^{-in\theta}\, d\theta = a_n,$$

i.e. the n-th Fourier coefficient c_n^f equals the n-th coefficient in the Taylor series of f. Further,

$$\begin{aligned} c_n^{f_r} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(e^{i\theta})\, e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \Big)\, e^{-in\theta}\, d\theta = a_n r^n, \end{aligned}$$

i.e. $c_n^{f_r} = c_n^f r^n$. Consequently, for any $f \in A(\mathbb{D})$ we have that $c_n^f(r) = c_n^{f_r}/r^n = c_n^f$, i.e. the coefficients $c_n^f(r)$ do not depend on r, $0 \le r < 1$. The following theorem shows that a similar result holds for analytic and harmonic functions in $\mathbb{D}$.

Theorem 2.3.2. *Let $u \in C_{\mathbb{R}}(\mathbb{D})$ be a real-valued function on $\mathbb{D}$. The following conditions are equivalent:*

(i) *u is harmonic.*

(ii) *The coefficients $c_n^u(r) = c_n^{u_r}/r^{|n|}$ do not depend on r.*

Proof. Let $u = f + \overline{f}$, where f is analytic in $\mathbb{D}$. As we saw above, the coefficients $c_n^f(r) = c_n^f = a_n$ do not depend on $r \in (0,1)$. Since, according to (2.5),

$$c_n^u = \begin{cases} \overline{a}_n & \text{when } n < 0, \\ 2\,\mathrm{Re}\, a_0 & \text{when } n = 0, \\ a_n & \text{when } n > 0, \end{cases}$$

the coefficients $c_n^u(r)$ also are constant with respect to $r \in (0,1)$. Hence, (i) implies (ii).

Conversely, assume that the coefficients $c_n^u(r)$ of a function $u \in C(\mathbb{D})$ do not depend on r, i.e. $c_n^u(r) = c_n^u$, $0 \le r < 1$. We claim that the series $\sum_0^\infty c_n^u z^n$

converges in $\mathbb{D}$. Indeed, for every integer n we have

$$\begin{aligned}|c_n^u r^{|n|}| &= |c_n^u(r)\, r^{|n|}| = \Big|\frac{c_n^{u_r}}{r^{|n|}} r^{|n|}\Big| = |c_n^{u_r}| \\ &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} u_r(e^{i\theta})\, e^{-in\theta}\, d\theta\Big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|u_r(e^{i\theta})\big|\, d\theta < \infty.\end{aligned}$$

Fix a $z \in \mathbb{D}$, and let r_0 be a real number with $|z| < r_0 < 1$. We have

$$\sum_0^\infty |c_n^u z^n| = \sum_0^\infty |c_n^u r_0^n| \left|\frac{z}{r_0}\right|^n \le M(r_0) \sum_0^\infty \left|\frac{z}{r_0}\right|^n < \infty,$$

where $M(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|u_r(e^{i\theta})\big|\, d\theta$. Hence the series $\sum_0^\infty c_n^u z^n$ converges absolutely in $\mathbb{D}$. Therefore its sum $f(z) = \sum_0^\infty c_n^u z^n$ is a function analytic in $\mathbb{D}$, and $c_n^f = c_n^u$ for any $n \in \mathbb{Z}_+$. Clearly, the Fourier coefficients of the function $u(z) = 2\operatorname{Re} f(z) = f(z) + \overline{f}(z) = \sum_0^\infty c_n^u z^n + \sum_0^\infty \overline{c}_n^u \overline{z}^n$ coincide with the corresponding Fourier coefficients of u. Consequently, $u = 2\operatorname{Re} f$, hence u is harmonic. □

We recall that the space L^p consist of all Lebesgue measurable functions on $\mathbb{T}$, for which $\int_{\mathbb{T}} |f(\zeta)|^p\, d\zeta < \infty$. L^p is a normed space under the norm

$$\|f\|_p = \Big(\int_X |f(\zeta)|^p\, d\zeta\Big)^{1/p}.$$

Theorem 2.3.3. *If f is a real-valued L^p-function on the unit circle $\mathbb{T}$ for some $p, 1 \le p < \infty$, then the function $\widetilde{f}\colon \mathbb{D} \longmapsto \mathbb{R}$ defined as*

$$\widetilde{f}(z) = \widetilde{f}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, P_r(\theta - t)\, dt = f(e^{it}) \star P_r(\theta) \qquad (2.11)$$

is harmonic on $\mathbb{D}$, and its r-traces $\widetilde{f}_r(\theta) = \widetilde{f}(re^{i\theta})$ converge on $\mathbb{T}$ to f in the L^p-norm as $r \nearrow 1$. If f is also continuous on $\mathbb{T}$, then this convergence is uniform on $\mathbb{T}$.

Proof. Let $\widetilde{f}_r(e^{i\theta}) = \widetilde{f}(re^{i\theta})$ be the r-trace of $\widetilde{f}$. By the Fubini theorem we have

$$\begin{aligned}
c_n^f(r) &= \frac{c_n^{\widetilde{f}_r}}{r^{|n|}} = \frac{1}{2\pi r^{|n|}} \int_{-\pi}^{\pi} f_r(e^{i\theta})\, e^{-in\theta}\, d\theta \\
&= \frac{1}{2\pi r^{|n|}} \int_{-\pi}^{\pi} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})\, P_r(\theta - t)\, dt\Big) e^{-in\theta}\, d\theta \\
&= \frac{1}{2\pi r^{|n|}} \int_{-\pi}^{\pi} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, e^{-in\theta}\, d\theta\Big) f(e^{it})\, dt \\
&= \frac{1}{2\pi r^{|n|}} \int_{-\pi}^{\pi} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_0^{\infty} r^{|k|} e^{ik(\theta - t)}\Big) e^{-in\theta}\, d\theta\Big) f(e^{it})\, dt \\
&= \frac{1}{2\pi r^{|n|}} \int_{-\pi}^{\pi} r^{|n|} e^{-int} f(e^{it})\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})\, e^{-int}\, dt = c_n^f,
\end{aligned}$$

i.e. the coefficients $c_n^f(r)$ do not depend on r. Theorem 2.3.2 implies that the function $\widetilde{f}(re^{i\theta})$ is harmonic in $\mathbb{D}$. First we will consider the second, continuous case. Observe that

$$\widetilde{f}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})\, P_r(\theta - t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i(\theta - t)})\, P_r(t)\, dt.$$

If f is continuous on $\mathbb{T}$, then for every $\varepsilon > 0$ one can find a $\delta > 0$ such that $\big|f(e^{it}) - f(e^{i\theta})\big| < \varepsilon$ whenever $|t - \theta| < \delta$. Consequently,

$$\big|\widetilde{f}(re^{i\theta}) - f(e^{i\theta})\big| = \Big|\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(e^{i(\theta - t)}\big)\, P_r(t)\, dt - f(e^{i\theta}) \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)\, dt\Big|$$

$$\begin{aligned}
&\le \Big|\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(f\big(e^{i(\theta - t)}\big) - f(e^{i\theta})\Big) P_r(t)\, dt\Big| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \big|f\big(e^{i(\theta - t)}\big) - f(e^{i\theta})\big|\, P_r(t)\, dt \\
&= \frac{1}{2\pi} \int_{|t|<\delta} \big|f\big(e^{i(\theta - t)}\big) - f(e^{i\theta})\big|\, P_r(t)\, dt + \frac{1}{2\pi} \int_{|t|>\delta} \big|f\big(e^{i(\theta - t)}\big) - f(e^{i\theta})\big|\, P_r(t)\, dt.
\end{aligned}$$

By Theorem 2.3.1 one can find an r_0 such that

$$\frac{1}{2\pi} \int_{|t|>\delta} P_r(t)\, dt < \frac{\varepsilon}{2\, \|f\|_{\infty}}$$

for any $r > r_0$, where $\|f\|_\infty = \sup_{\mathbb{T}} |f|$. Consequently,

$$\left|\widetilde{f}(re^{i\theta}) - f(e^{i\theta})\right| < \frac{\varepsilon}{2\pi} \int\limits_{|t|<\delta} P_r(t)\,dt + \frac{2\,\|f\|_\infty}{2\pi} \int\limits_{|t|>\delta} P_r(t)\,dt < \frac{\varepsilon}{2\pi} \int\limits_{\pi}^{\pi} P_r(t)\,dt + \varepsilon = 2\varepsilon$$

for every $r > r_0$.

If $f \in L^p,\ 1 \le p < \infty$, then for any $\varepsilon > 0$ one can choose a continuous on $\mathbb{T}$ function g such that

$$\|f - g\|_p = \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|f(e^{i(t)}) - g(e^{it})\right|^p dt\Big)^{1/p} < \varepsilon.$$

Therefore,

$$\|\widetilde{f}_r - f\|_p \le \|\widetilde{f}_r - \widetilde{g}_r\|_p + \|\widetilde{g}_r - g\|_p + \|g - f\|_p$$

for any $r,\ 0 < r < 1$. Since g is continuous on $\mathbb{D}$, then $\|\widetilde{g}_r - g\|_p \le \|\widetilde{g}_r - g\|_\infty < \varepsilon$, as already shown, and therefore, $\|\widetilde{f}_r - f\|_p \le \|\widetilde{f}_r - \widetilde{g}_r\|_p + 2\varepsilon$. By the Hölder inequality, applied to the measure $P_r(\theta - t)\,dt$, if $1/p + 1/q = 1$ then for any $h \in L^q$ with $\|h\| \le 1$ it follows that

$$\begin{aligned}
&\Big|\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(e^{it})\,h(e^{it}) P_r(\theta - t)\,dt\Big| \\
&\le \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|f(e^{it})\right|^p P_r(\theta - t)\,dt\Big)^{1/p} \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|h(e^{it})\right|^q P_r(\theta - t)\,dt\Big)^{1/q} \\
&\le \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|f(e^{i(\theta - t)})\right|^p P_r(t)\,dt\Big)^{1/p}.
\end{aligned}$$

By the Fubini theorem we have

$$\begin{aligned}
\|\widetilde{f}_r - \widetilde{g}_r\|_p^p &= \left\|(f - g)\widetilde{}_r\right\|_p^p = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \Big|\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \big(f(e^{it}) - g(e^{it})\big)\, P_r(\theta - t)\,dt\Big|^p d\theta \\
&\le \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|f(e^{it}) - g(e^{it})\right|^p\, P_r(\theta - t)\,dt\Big)^p d\theta \\
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \Big(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \left|f(e^{it}) - g(e^{it})\right|^p\, d\theta\Big) P_r(\theta - t)\,dt \le \|f - g\|_p^p < \varepsilon.
\end{aligned}$$

Therefore, $\|\widetilde{f}_r - \widetilde{g}_r\|_p < \varepsilon$, and hence $\|\widetilde{f}_r - f\|_p < 3\varepsilon$. Consequently, $\widetilde{f}_r \longrightarrow f$ in the L^p-norm on $\mathbb{T}$, as claimed. □

The function $\widetilde{f}$ from (2.11) is called the *harmonic extension* of f on $\mathbb{D}$.

2.4 Classes of harmonic functions in the unit disc

In this section we introduce several Banach spaces of harmonic functions on the unit disc. We recall that according to Theorem 2.3.2 a function $u \in C_{\mathbb{R}}(\mathbb{D})$ is harmonic in $\mathbb{D}$ if and only if the coefficients $c_n^u(r) = c_n^{u_r}/r^{|n|}$ do not depend on r, where $c_n^{u_r}$ is the n-th Fourier coefficient of u_r, $0 < r < 1$. As the following theorem shows, this result holds also for complex harmonic functions on $\mathbb{D}$.

Theorem 2.4.1. *The coefficients $c_n^f(r) = c_n^{f_r}/r^{|n|}$ of a function $f \in C(\mathbb{D})$ do not depend on $r \in (0,1)$ if and only if f is harmonic in $\mathbb{D}$, or, equivalently, if and only if $f \in \mathcal{O}(\mathbb{D}) + \overline{\mathcal{O}}(\mathbb{D}) = \mathcal{H}(\mathbb{D})$.*

Proof. The remark preceding Theorem 2.3.2 shows that for any $f \in \mathcal{O}(\mathbb{D})$ the coefficients $c_n^f(r)$ do not depend on $r \in (0,1)$. Clearly this is true for any $f \in \overline{\mathcal{O}}(\mathbb{D})$, and also for any $f \in \mathcal{O}(\mathbb{D}) + \overline{\mathcal{O}}(\mathbb{D}) = \mathcal{H}(\mathbb{D})$. Conversely, if the coefficients $c_n^f(r)$ of an $f \in C(\mathbb{D})$ do not depend on $r \in (0,1)$, then the same is true for both $\operatorname{Re} f = (f + \overline{f})/2$ and $\operatorname{Im} f = (f - \overline{f})/(2i)$. Therefore, $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic functions in $\mathbb{D}$ by Theorem 2.3.2. Hence $f \in \mathcal{H}_{\mathbb{R}}(\mathbb{D}) + i\mathcal{H}_{\mathbb{R}}(\mathbb{D}) = \mathcal{H}(\mathbb{D}) = \mathcal{O}(\mathbb{D}) + \overline{\mathcal{O}}(\mathbb{D})$. $\square$

Theorem 2.4.1 implies the following characterization of harmonic functions.

Corollary 2.4.2. *A function $u \in C(\mathbb{D})$ is harmonic on $\mathbb{D}$ if and only if*

$$u_{r_1}(e^{it}) = u(r_1 e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} u_{r_2}(e^{i(t-\theta)}) P_{r_1/r_2}(\theta)\, d\theta, \tag{2.12}$$

for any $0 \le r_1 < r_2 < 1$. Equivalently, if $u\big|_{r_1\mathbb{T}}(e^{it}) = u\big|_{r_2\mathbb{T}}(e^{i\theta} \star P_{r_1/r_2}(\theta))$, i.e. the r_1-trace u_{r_1} of u equals the r_1/r_2-trace of its harmonic extension $(\widetilde{u_{r_2}})_{r_1/r_2}$.

Proof. Without loss of generality we can assume that $u \in \mathcal{H}_{\mathbb{R}}(\mathbb{D})$. Note that the r_2-dilation $u_{r_2}(z) = u(r_2 z)$ of u is harmonic in $\mathbb{D}$. Therefore,

$$\begin{aligned} u_{r_1}(e^{i\theta}) &= u(r_1 e^{i\theta}) = u\big((r_1/r_2)\, r_2 e^{i\theta}\big) \\ &= u_{r_2}\big((r_1/r_2) e^{i\theta}\big) = \frac{1}{2\pi} \int_0^{2\pi} u_{r_2}\big(e^{i(t-\theta)}\big) P_{r_1/r_2}(\theta)\, d\theta, \end{aligned}$$

hence (2.12) holds. Conversely, if (2.12) holds for some $u \in C_{\mathbb{R}}(\mathbb{D})$ and $r = r_1/r_2$, then by the Fubini theorem

$$\begin{aligned}
c_n^u(r_1) &= c_n^{u_{r_1}}/r_1^{|n|} = \frac{1}{2\pi r_1^{|n|}} \int_{-\pi}^{\pi} u_{r_1}(e^{i\theta})\, e^{-in\theta}\, d\theta = \frac{1}{2\pi r_1^{|n|}} \int_{-\pi}^{\pi} u(r_1 e^{i\theta})\, e^{-in\theta}\, d\theta \\
&= \frac{1}{(2\pi)^2} \frac{1}{r_1^{|n|}} \int_{-\pi}^{\pi}\int_0^{2\pi} \big(u_{r_2}(e^{i(\theta-\zeta)})\, P_{r_1/r_2}(\zeta)\, d\zeta\big) e^{-in\theta}\, d\theta \\
&= \frac{1}{2\pi r_1^{|n|}} \int_0^{2\pi} \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} u_{r_2}(e^{i(\theta-\zeta)})\, e^{-in\theta}\, d\theta\Big)\, P_{r_1/r_2}(\zeta)\, d\zeta \\
&= \frac{1}{2\pi r_1^{|n|}} \int_0^{2\pi} \frac{1}{2\pi}\Big(\int_{-\pi}^{\pi} u_{r_2}(e^{is})\, e^{-ins}\, ds\Big)\, P_{r_1/r_2}(\zeta)\, e^{-in\zeta}\, d\zeta \\
&= \frac{1}{2\pi r_1^{|n|}} \int_0^{2\pi} c_n^{u_{r_2}} P_{r_1/r_2}(\zeta)\, e^{-in\zeta}\, d\zeta = \frac{c_n^{u_{r_2}}}{r_1^{|n|}} \Big(\frac{r_1}{r_2}\Big)^{|n|} = \frac{c_n^{u_{r_2}}}{r_2^{|n|}} = c_n^u(r_2).
\end{aligned}$$

Hence, the coefficients $c_n^u(r) = c_n^{u_r}/r^{|n|}$ do not depend on $r \in (0,1)$, as claimed. Therefore, $u \in \mathcal{H}_{\mathbb{R}}(\mathbb{D})$ by Theorem 2.3.2. □

If $u \in C(\overline{\mathbb{D}})$, then (2.12) is equivalent to $u_r = \widetilde{u}_r$. In fact, (2.12) is a characterizing property for the set of harmonic functions in $\mathbb{D}$. As a consequence from Corollary 2.4.2 we obtain that if $f \in \mathcal{O}(\mathbb{D}) \subset \mathcal{H}(\mathbb{D})$, then

$$f_{r_1}(e^{it}) = f(r_1 e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f_{r_2}(e^{i(t-\theta)}) P_{r_1/r_2}(\theta)\, d\theta \tag{2.13}$$

for any $0 \le r_1 < r_2 < 1$.

Denote by $\mathcal{H}^p_\varrho(\mathbb{D})$ the space of harmonic functions $u \in \mathcal{H}(\mathbb{D})$ for which

$$\begin{aligned}
\|u\|_{p,\varrho} &= \sup_{z\in\mathbb{D}} \big(\widetilde{|u|^p}\big|_{\varrho\mathbb{T}}\big)^{1/p} = \sup_{\substack{r\,\in\,(0,1)\\ t\,\in\,[0,2\pi]}} \big(\widetilde{|u_r(e^{i\theta})|^p}\big|_{\varrho\mathbb{T}}\big)^{1/p} \\
&= \sup_{\substack{r\,\in\,(0,1)\\ t\,\in\,[0,2\pi]}} \Big(\frac{1}{2\pi}\int_0^{2\pi} |u_r(e^{i(t-\theta)})|^p P_\varrho(\theta)\, d\theta\Big)^{1/p} < \infty,
\end{aligned}$$

where $0 \le \varrho < 1$ and $1 \le p < \infty$, and $\widetilde{u}$ is the harmonic extension (2.11) of u. If $\varrho = 0$, then

$$\|u\|_{p,0} = \sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |u_r(e^{i\theta})|^p\, d\theta\Big)^{1/p} = \Big(\frac{1}{2\pi}\int_0^{2\pi} |u_r(e^{i\theta})|^p\, d\theta\Big)^{1/p} = \|u\|_p,$$

since $p_0(\theta) \equiv 1$. For this reason we will write $\mathcal{H}^p(\mathbb{D})$ instead of $\mathcal{H}^p_0(\mathbb{D})$. We claim that the spaces $\mathcal{H}^p_\varrho(\mathbb{D})$, ϱ, $0 \le \varrho < 1$ are mutually isometrically isomorphic. Indeed,

$$\sup_{0\le\theta\le 2\pi} P_\varrho(\theta) = \frac{1+\varrho}{1-\varrho}$$

according to Lemma 2.2.3. Hence,

$$\|u\|_{p,\varrho} \le \left(\frac{1+\varrho}{1-\varrho}\right)^{\frac{1}{p}} \sup_{\substack{r\,\in\,(0,1)\\ t\,\in\,[0,2\pi]}} \left(\frac{1}{2\pi}\int_0^{2\pi} |u_r(e^{i(t-\theta)})|^p\, d\theta\right)^{\frac{1}{p}} = \left(\frac{1+\varrho}{1-\varrho}\right)^{\frac{1}{p}} \|u\|_p.$$

Similarly, the inequality

$$\inf_{0\le\theta\le 2\pi} P_\varrho(\theta) = \frac{1-\varrho}{1+\varrho}$$

in Lemma 2.2.3 implies that $\|u\|_{p,\varrho} \ge \left(\frac{1-\varrho}{1+\varrho}\right)^{\frac{1}{p}} \|u\|_p$. Hence the norms $\|u\|_p$ and $\|u\|_{p,\varrho}$ on the space $\mathcal{H}^p_\varrho(\mathbb{D})$ are equivalent. Therefore, $\mathcal{H}^p_\varrho(\mathbb{D}) \cong \mathcal{H}^p_0(\mathbb{D}) = \mathcal{H}^p(\mathbb{D})$. This proves the following lemma.

Lemma 2.4.3. *For every $0 \le \varrho < 1$ and $1 \le p < \infty$ there are constants $0 < c_1 < c_2 < \infty$, depending on ϱ, such that*

$$c_1\|u\|_{p,\varrho} \le \|u\|_p \le c_2\|u\|_{p,\varrho}.$$

Consequently, all the spaces $\mathcal{H}^p_\varrho(\mathbb{D})$, $0 \le \varrho < 1$, are mutually isometrically isomorphic.

Recall that the Hardy space H^p on the unit circle is the set of analytic functions in the unit disc, whose restrictions on concentric circles centered at the origin are uniformly bounded in the L^p-norm. Consequently, the spaces $\mathcal{H}^p_{\mathbb{R}}(\mathbb{D}) = \operatorname{Re} H^p = \operatorname{Re}(H^p + \overline{H}^p)$, and $\mathcal{H}^p(\mathbb{D}) = \mathcal{H}^p_{\mathbb{R}}(\mathbb{D}) + i\mathcal{H}^p_{\mathbb{R}}(\mathbb{D})$, can be defined in a similar way. On the other hand, $\operatorname{Re} H^p = \operatorname{Re}(H^p + \overline{H}^p) = \operatorname{Re}\mathcal{H}^p(\mathbb{D}) = \mathcal{H}^p_{\mathbb{R}}(\mathbb{D})$. Clearly, $\mathcal{H}^p(\mathbb{D}) \cap \mathcal{O}(\mathbb{D}) = H^p$, $\mathcal{H}^p_{\mathbb{R}}(\mathbb{D}) = \operatorname{Re} H^p$, and $\mathcal{H}^p(\mathbb{D}) = \mathcal{H}^p_{\mathbb{R}}(\mathbb{D}) + i\mathcal{H}^p_{\mathbb{R}}(\mathbb{D}) = H^p + \overline{H}^p$.

Let $p \ge 1$ and let $L^p_\varrho(\mathbb{T})$ be the completion of the space $C_{\mathbb{R}}(\mathbb{T})$ of real-valued continuous functions on $\mathbb{T}$ under the norm

$$\|u\|_{p,\varrho} = \sup_{t\in[0,2\pi]} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|u(e^{i(t-\theta)})\right|^p P_\varrho(\theta)\, d\theta\right)^{\frac{1}{p}}.$$

Applying the same arguments as in Lemma 2.4.3, we see that the spaces $L^p_\varrho(\mathbb{T})$, $0 \le \varrho < 1$, are mutually isometrically isomorphic. Therefore, $L^p_\varrho(\mathbb{T}) \cong L^p_0(\mathbb{T}) \cong L^p$ for any $0 \le \varrho < 1$. Given an $u \in L^p_\varrho(\mathbb{T})$, the integral

$$\frac{1}{2\pi}\int_0^{2\pi} u(e^{i(t-\theta)})\, P_\varrho(\theta)\, d\theta$$

is called also the *Poisson integral* of u. The next two theorems are direct consequences of Lemma 2.4.3 (cf. [H3], Ch. 3).

Theorem 2.4.4. *Let u be a harmonic function in $\mathbb{D}$, and $\varrho \in [0,1)$. Then*

(a) *u equals the Poisson integral of a function in $L^p_\varrho(\mathbb{T}) \cong L^p$ for some p, $1 < p \leq \infty$, if and only if $u \in \mathcal{H}^p_\varrho(\mathbb{D}) \cong \mathcal{H}^p(\mathbb{D})$.*

(b) *u equals the Poisson integral of a function in $L^1_\varrho(\mathbb{T}) \cong L^1$ if and only if $u \in \mathcal{H}^1_\varrho(\mathbb{D}) \cong \mathcal{H}^1(\mathbb{D})$, while the r-traces u_r of u converge to u in the $L^1_\varrho(\mathbb{T})$-norm, as $r \nearrow 1$.*

(c) *If $u \in C_{\mathbb{R}}(\mathbb{T})$, then the r-traces u_r of u converge uniformly to u as $r \nearrow 1$.*

(d) *u equals the Poisson integral of a regular real Borel measure on $\mathbb{T}$ if and only if $u \in \mathcal{H}^1(\mathbb{D})$.*

Theorem 2.4.5. *Let u be a harmonic function in $\mathcal{H}^p_\varrho(\mathbb{D}) \cong \mathcal{H}^p(\mathbb{D})$, $1 \leq p \leq \infty$. The limit $u^*(\zeta) = \lim\limits_{r\nearrow 1} u(r\zeta)$ exists for almost every θ with respect to the Lebesgue measure on $\mathbb{T}$, and is a function in $L^p_\varrho(\mathbb{T}) \cong L^p$. Moreover:*

(a) *If $p > 1$, then u equals the Poisson integral of u^*, i.e. u coincides with the harmonic extension $\widetilde{u^*}$ of u^*.*

(b) *If $p = 1$, then u equals the Poisson integral of a regular Borel measure on $\mathbb{T}$, whose absolutely continuous component with respect to the Lebesque measure on $\mathbb{T}$ is $u^*\, d\theta$.*

(c) *If u is also bounded on $\mathbb{D}$, then u^* is bounded on $\mathbb{T}$, and u coincides with the Poisson integral of u^*, i.e. u equals the harmonic extension $\widetilde{u^*}$ of u^*.*

2.5 Notes

Almost periodic functions were introduced by H. Bohr [B5], who, together with Besicovitch [B] and Jessen [J1], established their basic properties. Bohr discovered the almost periodicity property in the course of his study of Dirichlet series of analytic functions. For more detailed exposition on almost periodic functions we refer to the books of Corduneanu [C2] and Loomis [L4].

Closely related to asymptotically almost periodic functions on $\mathbb{R}$ is the class of weakly almost periodic functions on $\mathbb{R}$. A function $f \in C_b(\mathbb{R})$ is *weakly almost periodic*, if the set of all $\mathbb{R}$-shifts, $f_t(x) = f(x+t)$, $t \in \mathbb{R}$, is relatively weakly compact in $C_b(\mathbb{R})$ (e.g. [B8, E]). Let $AP_w(\mathbb{R})$ denote the set of weakly almost periodic functions on $\mathbb{R}$. One can show that $AP(\mathbb{R}) \subset AP^{as}(\mathbb{R}) \subset AP_w(\mathbb{R})$. Moreover, $AP_w(\mathbb{R}) = AP(\mathbb{R}) + C(\overline{\mathbb{R}})|_{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. Therefore, $AP_w(\mathbb{R}) \circ \varphi$ is isometrically isomorphic to the subalgebra of H^∞ on $\mathbb{T}$, generated by $e^{a\,(z+1)/(z-1)}$, $a \in \mathbb{R}$, and the set of functions in $C(\mathbb{T} \setminus \{1\})$ that possess one-sided limits at 1 along $\mathbb{T}$. The algebra $AP_{w,a}(\mathbb{R}) \cong AP_{w,a}(\mathbb{R}) \circ \varphi$

of analytically extendable on $\mathbb{C}_+$ weakly almost periodic functions on $\mathbb{R}$ is isometrically isomorphic to the subalgebra of $H^\infty \cap C(\overline{\mathbb{D}} \setminus \{1\})$ on $\mathbb{R}$ generated algebraically by the functions $e^{a\,(z+1)/(z-1)}$, $a \in \mathbb{R}_+$, and the set of functions in $H^\infty \cap C(\overline{\mathbb{D}} \setminus \{1\})$ that possess one-sided limits at 1 along $\mathbb{T}$.

The classical Hardy space H^p on the unit circle $\mathbb{T}$ can be defined in three equivalent ways. Firstly, H^p is the set of functions in L^p on the unit circle, whose negative Fourier coefficients are zero. Secondly, H^p is the completion of the space of polynomials on the unit circle under the L^p-norm. And thirdly, H^p is the set of analytic functions in the unit disc, whose restrictions on concentric circles centered at the origin are uniformly bounded in the L^p-norm. As we saw in Section 2.4, the Hardy space H^p is closely related to the space $\mathcal{H}^p(\mathbb{D})$.

There are many classical books on harmonic functions, Poisson integrals, and H^p-spaces. For more detailed exposition see [G2, H3].

Chapter 3

Groups and semigroups

The main properties of topological groups and semigroups, their characters and semicharacters are outlined in this chapter. Some of the basic features of various spaces of functions, measures and operators on groups are presented too. Semigroup algebras $\ell^1(S)$, introduced at the end of the chapter, are closely related with the underlying semigroups S.

3.1 Topological groups and their duals

Let G be a *group*, i.e. a set provided by an associative operation $(g,h) \longmapsto g\,h$, an identity (or neutral) element $\imath \in G$, and an inverse operation $g \longmapsto g^{-1}$. The defining property of the identity element is that $\imath\, g = g\, \imath = g$ for all $g \in G$, while the defining property of the inverse element $g^{-1} \in G$ is that $g\, g^{-1} = g^{-1}g = \imath$. The identity element $\imath$ is uniquely defined and is called the *unit element of* G. For any $g \in G$ the element g^{-1} is also uniquely defined and is called the *inverse element of* g in G. If the group operation is commutative, i.e. if $g\,h = h\,g$ for all $g, h \in G$, then G is called a *commutative*, or *abelian* group. We use mainly the additive notation for the group operation, $(g,h) \longmapsto g + h$, if the group is commutative, instead of the multiplicative one, $(g,h) \longmapsto g\,h$. The identity element in this case is denoted by 0, rather than by $\imath$, and is called the *zero element*, rather than the *unit element* of G. In the sequel we will consider only groups that are abelian.

Let G_1 and G_2 be two groups. A *homomorphism* from G_1 to G_2 is called any multiplicative mapping $\varphi\colon G_1 \longrightarrow G_2$, i.e. for which $\varphi(g\,h) = \varphi(g)\,\varphi(h)$ for any $g, h \in G$. If a homomorphism φ is one-to-one, i.e. $\varphi(g) = \varphi(h)$ if and only if $g = h$, it is called an *isomorphism* of G_1 into G_2. If, in addition, φ is surjective, i.e. if $f(G_1) = G_2$, then φ is called an *isomorphism* between G_1 and G_2. In this case we write $G_1 \cong G_2$.

If a group G is a topological space, then it is called a *topological group* if the mapping $(g,h) \longmapsto g\,h^{-1}$ from $G \times G$ to G is continuous. This happens if and only if the functions $(g,h) \longmapsto g\,h$ from $G \times G$ to G, and $g \longmapsto g^{-1}$ from G to G are continuous. Let G be an abelian topological group G. A continuous complex-valued function χ on G is called a *character* of G, if $|\chi| \equiv 1$, and $\chi(x+y) = \chi(x)\,\chi(y)$ (under the additive notation), or, $\chi(x\,y) = \chi(x)\,\chi(y)$ (under the multiplicative notation) for any $x, y \in G$. In other words, the characters of G are complex homomorphisms of G into $\mathbb{T}$. The *dual group*, $\widehat{G}$, of G is the set of all characters of G. It is easy to see that, if G is abelian, then $\widehat{G}$ is also an abelian group under the pointwise multiplication $\big(\chi_1\,\chi_2\big)(g) = \chi_1(g)\chi_2(g)$ for every $g \in G$, where $\chi_1, \chi_2 \in \widehat{G}$. The inverse element to a $\chi \in \widehat{G}$ is the character $\chi^{-1}(g) = \overline{\chi(g)}$.

Example 3.1.1. (a) The dual group of $\mathbb{Z}$ is bijective to the unit circle $\mathbb{T}$. Indeed, any character of $\mathbb{Z}$ has the form $\chi^z : n \longmapsto z^n \in \mathbb{T}$ for some $z \in \mathbb{T}$, i.e. $\chi^z(n) = z^n$. Hence $\widehat{\mathbb{Z}} \cong \mathbb{T}$.

(b) The characters on $\mathbb{T}$ are of the form $\chi^n : z \longmapsto z^n \in \mathbb{T}$ for some integer $n \in \mathbb{Z}$, i.e. $\chi^n(z) = z^n$. Therefore, $\widehat{\mathbb{T}} \cong \mathbb{Z}$. The characters of any subgroup G of $\mathbb{T}$ are also of type $z \longmapsto z^n,\ z \in G$.

(c) The dual group of $\mathbb{Z}^n$ is bijective to the n-torus $\mathbb{T}^n$, i.e. $\widehat{\mathbb{Z}^n} \cong \mathbb{T}^n$.

(d) The dual group of $\mathbb{T}^n$ is bijective to $\mathbb{Z}^n$, i.e. $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$.

Let G_0 be a closed subgroup of a given abelian topological group G. *Cosets* of G_0 are called the sets of type $g\,G_0 = \{g\,h \colon h \in G_0\}$, $g \in G$. It is clear that the family of all cosets of G_0 is a group under the operation $(g\,G_0)(h\,G_0) = (g\,h)\,G_0$. This group is denoted by G/G_0, and is called the *quotient group* (or, *factor group*) of G modulo G_0. Under the topology generated by the sets of type $UG_0 = \{hg \colon h \in U,\ g \in G_0\}$, where U are open sets in G, G/G_0 is a topological group. The map $\pi_{G_0} \colon g \longmapsto g\,G_0$, is a continuous group homomorphism from G onto G/G_0.

Definition 3.1.2. Given a character $\chi \in \widehat{G}$, the set $\mathrm{Ker}\,(\chi) = \{x \in G \colon \chi(x) = 1\}$ is called the *kernel* of χ.

Observe that, since χ is a continuous function on G, its kernel is a closed subgroup of G, containing 0. Clearly, $0 \in \mathrm{Ker}\,(\chi)$ for any $\chi \in \widehat{G}$. The following lemma shows that the kernels determine to a great extent their characters.

Lemma 3.1.3. *If χ_1 and χ_2 are two characters on G with* $\mathrm{Ker}\,(\chi_1) = \mathrm{Ker}\,(\chi_2)$*, then either $\chi_2 = \chi_1$, or $\chi_2 = \overline{\chi}_1$.*

Proof. Let $G_0 = \mathrm{Ker}\,(\chi_1) = \mathrm{Ker}\,(\chi_2)$. The mapping $\widetilde{\chi}_1 \colon g\,G_0 \longmapsto \chi_1(g)$ is an isomorphism from the quotient group G/G_0 into $\mathbb{T}$, and in fact, $(\widetilde{\chi}_1 \circ \pi_{G_0})(g) = \chi_1(g)$ for all $g \in G$. Likewise, $\widetilde{\chi}_2 \circ \pi_{G_0} = \chi_2$, where $\widetilde{\chi}_2 \colon G/G_0 \longrightarrow \mathbb{T} \colon g\,G_0 \longmapsto \chi_1(g)$. Therefore, $\widetilde{\chi}_1(G) \cong \widetilde{\chi}_2(G) \cong G/G_0$. Since both $\widetilde{\chi}_1(G)$ and $\widetilde{\chi}_2(G)$ are subgroups of $\mathbb{T}$, without loss of generality we can assume that actually $\widetilde{\chi}_1(G) = \widetilde{\chi}_2(G)$. The mapping $\widetilde{\chi} = \widetilde{\chi}_2 \circ \widetilde{\chi}_1^{-1}$ is an isomorphism from $\widetilde{\chi}_1(G)$ onto $\widetilde{\chi}_1(G)$, i.e. $\widetilde{\chi}$ is a character on $\widetilde{\chi}_1(G) \subset \mathbb{T}$. Hence $\widetilde{\chi}$ is a function of type $z \longmapsto z^n$ for some $n \in \mathbb{Z}$. Only two of the functions z^n map $\widetilde{\chi}_1(\Gamma)$ isomorphically onto itself, namely the identity $z \longmapsto z$, and its conjugate $z \longmapsto \overline{z}$. Therefore, $\widetilde{\chi}(z) \equiv z$, or $\widetilde{\chi}(z) \equiv \overline{z}$. Hence, $\chi_2 = \widetilde{\chi}_2 \circ \pi_{G_0}(\chi_2) = \widetilde{\chi}_2 \circ \widetilde{\chi}_1^{-1} \circ \widetilde{\chi}_1 \circ \pi_{G_0} = \widetilde{\chi} \circ \widetilde{\chi}_1 \circ \pi_{G_0} = \widetilde{\chi} \circ \chi_1$. Consequently, either $\chi_2 = \chi_1$, or $\chi_2 = \overline{\chi}_1$, as claimed. □

The real line $\mathbb{R}$ is a group under addition. For every $a \in \mathbb{R}$ the function $\chi^a \colon x \longmapsto e^{iax} \in \mathbb{T}$ is a character of $\mathbb{R}$. As we show below, every character on $\mathbb{R}$ has this form.

Lemma 3.1.4. *The kernel of every non-trivial character χ on $\mathbb{R}$ is of the form $\mathbb{Z}x_0$ for some $x_0 > 0$.*

Proof. If χ is a non-trivial character on $\mathbb{R}$, then $\mathrm{Ker}\,(\chi)$ is a closed subgroup of $\mathbb{R}$, containing 0. We claim that 0 is an isolated point in $\mathrm{Ker}\,(\chi)$. Indeed, suppose that there exists a sequence $\{x_n\}$ in $\mathrm{Ker}\,(\chi)$ that converges to 0. Consider the sets $\mathbb{Z}\,x_n = \{kx_n \colon k \in \mathbb{Z}\} \subset \mathrm{Ker}\,(\chi)$. One can easily see that the set $\bigcup_{n=1}^{\infty} \mathbb{Z}\,x_n$ is dense in $\mathbb{R}$, and so is $\mathrm{Ker}\,(\chi)$. Therefore $\mathrm{Ker}\,(\chi) = \mathbb{R}$, since $\mathrm{Ker}\,(\chi)$ is closed in $\mathbb{R}$. Hence χ is the trivial character on $\mathbb{R}$, contrary to its choice. Hence 0 is an isolated point in $\mathrm{Ker}\,(\chi)$. Hence, there is a minimal positive number in $\mathrm{Ker}\,(\chi)$, say, x_0, since $\mathrm{Ker}\,(\chi)$ is a closed subset in $\mathbb{R}$. We claim that $\mathrm{Ker}\,(\chi) = \mathbb{Z}\,x_0$. Indeed, suppose that there is an $a \in \mathrm{Ker}\,(\chi) \setminus \mathbb{Z}\,x_0$. Without loss of generality we can assume that $a > 0$. Note that $a + kx_0 \in \mathrm{Ker}\,(\chi)$ for every $k \in \mathbb{Z}$. If $n \in \mathbb{Z}$ is such that $a \in \big[nx_0, (n+1)x_0\big)$, then $a - nx_0 \in \mathrm{Ker}\,(\chi)$, and $0 \le a - nx_0 < x_0$, in contradiction with the minimality property of x_0. Therefore, $\mathrm{Ker}\,(\chi) = \mathbb{Z}\,x_0$. □

Lemma 3.1.5. *Let χ_1 and χ_2 be two characters on $\mathbb{R}$. If there is a sequence $x_n \longrightarrow 0$ such that $\chi_1(x_n) = \chi_2(x_n)$ for every n, then $\chi_1 \equiv \chi_2$.*

Proof. Let $\chi_1, \chi_2 \in \widehat{\mathbb{R}}$, $\chi_1 \neq \chi_2$. If $\chi = \chi_1 \overline{\chi}_2 \in \widehat{G}$, then $\chi(x_n) = 1$ for all n, i.e. $\{x_n\}_n \subset \mathrm{Ker}\,(\chi)$. Lemma 3.1.4 implies that $\chi \equiv 1$, i.e. $\chi_1 = \chi_2$, in contradiction with the choice of χ_1 and χ_2. □

Theorem 3.1.6. *For every character χ on $\mathbb{R}$ there is a real number $a \in \mathbb{R}$ such that χ has the form $\chi = \chi^a \colon x \longmapsto e^{iax}$. Thus, $\widehat{\mathbb{R}} \cong \mathbb{R}$.*

Proof. If $\chi \equiv 1$, the statement holds with $a = 0$. Let χ be a non-trivial character of $\mathbb{R}$, and let x_0 be the minimal positive number in $\mathrm{Ker}\,(\chi)$, which exists by Lemma 3.1.4. Consider the character $\chi_1(x) = e^{i(2\pi/x_0)x}$ of $\mathbb{R}$. The minimal positive number in $\mathrm{Ker}\,(\chi_1)$ is also x_0, and therefore, $\mathrm{Ker}\,(\chi_1) = \mathrm{Ker}\,(\chi) = \mathbb{Z}\,x_0$. Lemma 3.1.3 implies that either $\chi = \chi_1$, or, $\chi_1 = \overline{\chi}$. Hence χ is of type $\chi(x) = e^{iax}$ with $a = 2\pi/x_0$ in the first case, and $a = -(2\pi/x_0)$ in the second. □

Lemma 3.1.7. *If $\mathrm{Ker}\,(\chi^a) \subset \mathrm{Ker}\,(\chi^b)$ for some $a, b \in \Gamma = \widehat{G}$, then b is an integer multiple of a.*

Proof. Let $G_0 = \mathrm{Ker}\,(\chi^a)$. If $\chi^a = \widetilde{\chi}^a \circ \pi_{G_0}$, then $\widetilde{\chi}^a$ maps continuously and isomorphically G/G_0 in $\mathbb{T}$. Let $\chi^b = \widetilde{\chi}^b \circ \pi_{G_0}$ where $\widetilde{\chi}^b$ maps continuously and homomorphically G/G_0 in $\mathbb{T}$. Consider the mapping $\gamma = \widetilde{\chi}^b \circ (\widetilde{\chi}^a)^{-1} \colon (\widetilde{\chi}^a \circ \pi_{G_0})(g) \longmapsto (\widetilde{\chi}^b \circ \pi_{G_0})(g)$ from $G/G_0 \subset \mathbb{T}$ onto $G/\mathrm{Ker}\,(\chi^b) \subset \mathbb{T}$, such that $\widetilde{\chi}^b = \gamma \circ \widetilde{\chi}^a$ on $\pi_{G_0}(G) = G/G_0$. Therefore, $\mathrm{Ker}\,(\gamma) = \gamma^{-1}\{1\} = \big(\widetilde{\chi}^b \circ (\widetilde{\chi}^a)^{-1}\big)^{-1}\big(\{1\}\big) = \big(\widetilde{\chi}^a \circ (\widetilde{\chi}^b)^{-1}\big)\big(\{1\}\big) = \widetilde{\chi}^a\big(\mathrm{Ker}\,(\widetilde{\chi}^b)\big)$. As a finite subgroup of $\mathbb{T}$, the set $\widetilde{\chi}^a\big(\mathrm{Ker}\,(\widetilde{\chi}^b)\big)$ is of type $\{e^{ik\,(2\pi/n)}\}_{k=1}^n$ for some $n \in \mathbb{N}$. Hence, by Lemma 3.1.3, $\gamma(z) = z^n$, or $\gamma(z) = \overline{z}^n = z^{-n}$. Consequently, $\chi^b = \widetilde{\chi}^b \circ \pi_{G_0} = \gamma \circ \widetilde{\chi}^a \circ \pi_{G_0} = (\widetilde{\chi}^a)^m \circ \pi_{G_0} = \widetilde{\chi}^{ma} \circ \pi_{G_0} = \chi^{ma}$, where $m = \pm n$. □

It is easy to see that if a belongs to the group envelope Γ_K of K, then $\mathrm{Ker}\,(\chi^a) \supset \bigcap\limits_{b\in K} \mathrm{Ker}\,(\chi^b)$.

Proposition 3.1.8. *If $a \in \Gamma$ and* $\mathrm{Ker}\,(\chi^a) \subset \bigcap\limits_{b\in K} \mathrm{Ker}\,(\chi^b)$ *for some subset K of Γ, then a belongs to the group envelope Γ_K of K.*

Proof. Let Γ_K be the group generated by K. Since $\mathrm{Ker}\,(\chi^a) \subset \mathrm{Ker}\,(\chi^b)$, any $b \in K$ is of type $b = n_b a$ for some $n_b \in \mathbb{Z}$ by Lemma 3.1.7. Hence $K \subset \mathbb{Z}a$, thus $\Gamma_K \subset \mathbb{Z}a$, so that $\Gamma_K = \mathbb{Z}\,(ma)$ for some $m \in \mathbb{N}$. We claim that, in fact, $m = 1$. If we suppose that $m > 1$, then $\Gamma_K = \mathbb{Z}\,(ma)$ is a proper subgroup of $\mathbb{Z}a$. The function $\gamma \colon \mathbb{Z}a \longrightarrow \mathbb{T} : na \longmapsto e^{i\,(2\pi n)/m}$ is a character on $\mathbb{Z}a$, and $\mathrm{Ker}\,(\gamma) = \mathbb{Z}(ma) \supset K$. Further, γ can be extended to a character $\widetilde{\gamma}$ on Γ. Since $a \notin \mathrm{Ker}\,(\gamma)$, and $\widetilde{\gamma} \in \widehat{\Gamma} \cong \widehat{\widehat{G}} \cong G$, there is a $g \in G$, so that $\chi^a(g) = \widetilde{\gamma}(a) = \gamma(a) = e^{i\,(2\pi/m)} \neq 1$, while $\chi^b(g) = \widetilde{\gamma}(b) = 1$ for all $b \in K$. Therefore, $g \in \Big(\bigcap\limits_{b\in K} \mathrm{Ker}\,(\chi^b)\Big) \setminus \mathrm{Ker}\,(\chi^a)$, contrary to the hypothesis on a. Hence, $\Gamma_K = \mathbb{Z}a$, and consequently, $a \in \Gamma_K$. □

Any element $g \in G$ gives rise to a continuous character g^* on $\widehat{G}$ by the rule $g^*(\chi) = \chi(g)$, $\chi \in \widehat{G}$. The celebrated *Pontryagin's duality theorem* asserts that if G is a locally compact group, then $\widehat{G}$ is also a locally compact group under the compact-open topology in $C(G)$, and every continuous character of $\widehat{G}$ has the form g^* for a suitable element $g \in G$. Moreover, if all three groups $G, \widehat{G}$, and $\widehat{\widehat{G}}$ are equipped by their compact-open topologies, then G is homeomorphically isomorphic to $\widehat{\widehat{G}}$ via the mapping $g \longmapsto g^*$. Consequently, we can identify every locally compact topological group G with its double dual group $\widehat{\widehat{G}} \cong G$.

Example 3.1.9. (a) If G is a compact group, then $\widehat{G}$ is a discrete group under the compact-open topology, which in this case is the uniform topology on G. If G is a discrete group, then the compact-open topology on $\widehat{G}$ is generated by the pointwise convergence. Clearly, with respect to it, $\widehat{G}$ is a compact group.

(b) Let G be an abelian topological group. Denote by G_d the same group equipped by the discrete topology. Its dual group $\widehat{G_d}$ is a compact group, containing the dual group $\widehat{G}$ of G with the original topology as a dense subgroup.

The group $\widehat{G_d}$ is called the *Bohr compactification* of the group $\widehat{G}$. We will denote it by $b(G)$. In particular, the group of real numbers $\mathbb{R}$ can be embedded isomorphically and densely into its Bohr compactification $b(\mathbb{R}) = \widehat{\mathbb{R}_d}$, since $\widehat{\mathbb{R}} = \mathbb{R}$ by Theorem 3.1.6. Almost periodic functions f on $\mathbb{R}$ can be extended naturally as continuous functions $\widetilde{f}$ on the Bohr compactification $b(\mathbb{R}) = \widehat{\mathbb{R}_d}$ of $\mathbb{R}$. The Fourier coefficients $c_k^{\tilde{f}}$ of the function extended in this way, $\widetilde{f}$, on $b(\mathbb{R})$ equal the Dirichlet coefficients A_k^f of the original function f. Moreover, the maximal ideal space $\mathcal{M}_{AP(\mathbb{R})}$ of the algebra of almost periodic functions on $\mathbb{R}$ is homeomorphic to the Bohr compactification $b(\mathbb{R})$ of $\mathbb{R}$.

In the sequel we will consider often the following situation. Let Γ be a discrete abelian group. Its dual group $G = \widehat{\Gamma}$ is a compact abelian group, and the dual group $\widehat{G}$ of G is isomorphic to Γ, by Pontryagin's duality theorem. The character $a^* \in \widehat{G}$ corresponding to $a \in \Gamma$ we will denote by χ^a. Thus, for the elements of the dual group $\Gamma \cong \widehat{G}$ we may use interchangeably two notations, namely a and χ^a. If we use the additive notation for the group operation of G, then the elements of the dual group $\Gamma \cong \widehat{G}$ will be denoted by a, b, etc. If, alternatively, we use the multiplicative notation for the group operation of Γ, then its elements will be denoted by χ^a, χ^b, etc. In this respect the expressions "$\chi^a\chi^b$ for $\chi^a, \chi^b \in \widehat{G}$", and "$a + b$ for $a, b \in \Gamma$", are equivalent. Now the equality $\chi^a\chi^b = \chi^{a+b}$, for $a, b \in \Gamma$ makes perfect sense.

Any compact abelian group G possesses a unique probability measure σ that is invariant under G-shifts. It is called the *Haar measure* of G. More precisely, σ is a positive regular Borel measure on G, such that

$$\sigma(G) = 1, \text{ and } \sigma(hF) = \sigma(F) \tag{3.1}$$

for every $h \in G$ and every Borel set $F \subset G$. Equivalently, σ is defined by the properties

$$\int_G d\sigma = 1, \text{ and } \int_G f(g\,h)\,d\sigma(g) = \int_G f(g)\,d\sigma(g)$$

for every $f \in C(G)$ and $h \in G$. Note that (3.1) implies that the Haar measure σ on G is also inverse invariant, i.e. $\sigma(F^{-1}) = \sigma(F)$ for every Borel set $F \subset G$, where $F^{-1} = \{g^{-1} \in G \colon g \in F\}$. To see this, consider the measure σ' on G defined by $\sigma'(F) = \sigma(F^{-1})$. One can easily show that σ' is also a normalized and translation-invariant positive Borel measure on G. The uniqueness property of σ implies that $\sigma' = \sigma$. Consequently,

$$\int_G f(g^{-1})\,d\sigma(g) = \int_G f(g)\,d\sigma(g)$$

for all continuous functions f on G.

Given an $h \in G$, the *h-shift* of f is called the function $f_h(g) = f(h\,g)$. The map $f \longmapsto f_h$ is an isometric isomorphism of $C(G)$ onto itself. The invariant property of the Haar measure σ implies $\int\limits_G f_h\, d\sigma = \int\limits_G f\, d\sigma$. Henceforth, for any character on G we have

$$\int\limits_G \chi\, d\sigma = \int\limits_G \chi_h d\sigma = \int\limits_G \chi(h\,g)\, d\sigma(g) = \chi(h) \int\limits_G \chi(g)\, d\sigma(g) = \chi(h) \int\limits_G \chi\, d\sigma.$$

Consequently, $\int\limits_G \chi\, d\sigma = 0$ if $\chi \not\equiv 1$. Clearly, $\int\limits_G 1\big|_G\, d\sigma = \int\limits_G d\sigma = 1$.

Example 3.1.10. The Haar measure σ on the unit circle $\mathbb{T}$ is defined by the equality

$$\int\limits_{\mathbb{T}} f\, d\sigma = \frac{1}{2\pi} \int\limits_0^{2\pi} f(e^{i\theta})\, d\theta, \;\; f \in C(\mathbb{T}).$$

Theorem 3.1.11. *Let G_0 be a proper closed subgroup of a compact abelian group G. Then:*

(a) *There is a non-trivial character χ on G such that $\chi \equiv 1$ on G_0.*

(b) *Every character of G_0 possesses a continuous extension on G as a character.*

(c) *Given a character γ on G_0, there is a one-to-one correspondence between the set of all character extensions of γ on G, and the dual group $(G/G_0)\widehat{\ }$ of the quotient group G/G_0.*

Proof. (a) One of the corollaries from Pontryagin's duality theorem is that every group with more than one element possesses non-trivial characters. Let χ denote a non-trivial character of the quotient group G/G_0. Therefore, the composition $\chi \circ \pi_{G_0} \colon G \longrightarrow \mathbb{T}$ is a non-trivial character of G, and, clearly, it is identically equal to 1 on G_0.

(b) The restriction mapping on G_0 generates a homomorphism ψ from $\widehat{G}$ into $\widehat{G}_0$. We claim that ψ is surjective. If $\psi(\widehat{G})$ is a proper subset of $\widehat{G}_0$, then there is a $\chi \in \widehat{G}_0 \setminus \psi(\widehat{G})$. Note that $\chi\,\gamma \not\equiv 1$ for any $\gamma \in \psi(\widehat{G})$, since $\chi \neq \overline{\gamma} \in \psi(\widehat{G})$. Therefore, $\int\limits_{G_0} \chi\,\gamma\, d\sigma_0 = 0$ for every $\gamma \in \psi(\widehat{G})$, where σ_0 is the Haar measure on G_0. The Stone-Weierstrass approximation theorem, applied to the algebra generated by the elements in $\psi(\widehat{G}) \subset \widehat{G}_0$, yields $\int\limits_{G_0} \chi\, f\, d\sigma_0 = 0$ for every $f \in C(G_0)$. In particular, $0 = \int\limits_{G_0} \chi\,\overline{\chi}\, d\sigma_0 = \int\limits_G 1\, d\sigma_0$, which is impossible. Hence $\psi(\widehat{G}) = \widehat{G}_0$, and consequently every $\gamma \in \widehat{G}_0$ is the restriction of some character $\chi \in \widehat{G}$.

(c) Note that if χ_1 and χ_2 are two extensions of a $\gamma \in \widehat{G}_0$ on G, then $\chi_1\big|_{G_0} \equiv \chi_2\big|_{G_0}$, thus $\overline{\chi}_2\,\chi_1\big|_{G_0} \equiv 1$, and therefore $\operatorname{Ker}(\overline{\chi}_2\,\chi_1) \supset G_0$. Consequently, $\chi_2 = \chi\,\chi_1$, where $\chi = \overline{\chi}_2\,\chi_1 \in \widehat{G}$, and $\operatorname{Ker}(\chi) \supset G_0$. Conversely, if χ_1 is a character extension of γ on G, and if $\operatorname{Ker}(\chi) \supset G_0$ for some $\chi \in \widehat{G}$, then $\chi\,\chi_1$ also extends γ on G. Therefore, the set of all possible character extensions of γ on G is bijective to the orthogonal group $G_0^{\perp} = \big\{\chi \in \widehat{G}\colon \chi\big|_{G_0} \equiv 1\big\}$, which is isomorphic to $(G/G_0)\widehat{\ }$. □

Definition 3.1.12. A compact group G is said to be *solenoidal*, if there is an isomorphism of the group $\mathbb{R}$ of real numbers in G, whose range is a proper dense subgroup of G.

Clearly, $\mathbb{T}$ is not a solenoidal group. The next theorem gives a criterion for a group to be solenoidal.

Theorem 3.1.13. *A compact group $G \not\cong \mathbb{T}$ is solenoidal if and only if there is an isomorphism from $\widehat{G}$ into $\mathbb{R}$ with a dense range.*

Proof. Let G be a solenoidal group, and let j be an isomorphism from $\mathbb{R}$ into G such that its closure $\big[j(\mathbb{R})\big]$ coincides with G. Clearly, j generates naturally an adjoint homomorphism $j^*\colon \widehat{G} \longrightarrow \widehat{\mathbb{R}_d} = \mathbb{R}$ by $\big(j^*(\chi)\big)(x) = \chi\big(j(x)\big)$. If $j^*(\chi) = j^*(\gamma)$, $\chi, \gamma \in \widehat{G}$, then $\chi\big(j(x)\big) = \big(j^*(\chi)\big)(x) = \big(j^*(\gamma)\big)(x) = \gamma\big(j(x)\big)$ for every $x \in \mathbb{R}$. Consequently $\chi = \gamma$, since the range $j(G)$ of j is dense in G. Hence, j^* is an isomorphism from $\widehat{G}$ onto a subgroup of $\mathbb{R}$.

Conversely, suppose that there exists an isomorphism φ from $\widehat{G}$ onto a proper dense subgroup of $\mathbb{R}$. For every $x \in \mathbb{R}$ the mapping $\chi \longmapsto e^{i\,\varphi(\chi)\,x}$ is a character on $\widehat{G}$, i.e. belongs to $\widehat{\widehat{G}} \cong G$. Denote by $j_\imath(x)$ the corresponding element in G. We have that $\chi\big(j_\imath(x)\big) = e^{i\,\varphi(\chi)\,x}$ for any $\chi \in \widehat{G}$. There arises a mapping $j_\imath\mathbb{R} \longrightarrow \widehat{\widehat{G}} : x \longmapsto j_\imath(x)$. Clearly, $j_\imath(0) = \imath \in G$. It is easy to check that $j_\imath$ maps $\mathbb{R}$ homomorphically into G. We claim that the range $j_\imath(\mathbb{R})$ is dense in G. Indeed, if χ, γ are two different characters on G with equal values on the range $j_\imath(\mathbb{R})$, then $\chi\big(j_\imath(x)\big) = \gamma\big(j_\imath(x)\big)$ for every $x \in \mathbb{R}$. Since $\chi\big(j_\imath(x)\big) = e^{i\,\varphi(\chi)\,x}$, and, also, $\gamma\big(j_\imath(x)\big) = e^{i\,\varphi(\gamma)\,x}$, we have $e^{i\,\varphi(\chi)\,x} = e^{i\,\varphi(\gamma)\,x}$ for every $x \in \mathbb{R}$. Hence, $\varphi(\chi) = \varphi(\gamma)$, and therefore $\chi = \gamma$, since φ is an isomorphism. According to Theorem 3.1.11(c) this is is possible only if $j_\imath(\mathbb{R})$ is dense in G. □

The mapping $j_\imath\colon \mathbb{R} \longmapsto G$ with $j_\imath(0) = \imath \in G$, defined in the proof of Theorem 3.1.13 is called the *standard embedding of* $\mathbb{R}$ *into* G via $\imath$.

Let G be a solenoidal group, and let S be an additive semigroup of $\widehat{G}_+ = \widehat{G} \cap [0, \infty)$. Denote by Γ the subgroup $S - S$ of $\widehat{G} \subset \mathbb{R}$ generated by S. For simplicity we assume that $2\pi \in S$. The kernel $\operatorname{Ker}(\chi^{2\pi})$ of $\chi^{2\pi}$ is a compact subgroup of G, which we will denote by K, i.e. $K = \{g \in G\colon \chi^{2\pi}(g) = 1\}$. Denote by g_t the element $j_\imath(t) \in G$, where $j_\imath\colon \mathbb{R} \longrightarrow G$ is the standard embedding of $\mathbb{R}$ into G via $\imath$. It is clear that $g_n \in K$ for every $n \in \mathbb{Z}$. The Cartesian product $\widetilde{G} = K \times \mathbb{R}$ is a locally compact abelian group, and the map $\pi\colon \widetilde{G} \to G\colon (g,t) \longmapsto g_t\, g$ is a group homomorphism. The kernel of π is the subgroup $\operatorname{Ker}(\pi) = \{(g_n, -n) \in \widetilde{G} : n \in \mathbb{Z}\}$ of $\widetilde{G}$. For each $n \in \mathbb{Z}$ the set $K_n = K \times [n, n+1)$ is a fundamental domain for π. Therefore, $\pi\colon \widetilde{G} \longrightarrow G$ generates a countably-sheeted covering over G without singularities. The group G can be recovered from any set of type $K_n = K \times [n, n+1]$ by identifying the points $(g, n+1)$ and $(g_1 g, n)$, $g \in G$.

Let $\Gamma \subset \mathbb{R}$, $\Gamma \ncong \mathbb{Z}$. Since $\widehat{b(\widehat{\Gamma})} = \widehat{\widehat{\Gamma_d}} \cong \Gamma_d \subset \mathbb{R}$, Theorem 3.1.13 implies the following

Corollary 3.1.14. *If $\Gamma \ncong \mathbb{Z}$ is an additive subgroup of $\mathbb{R}$, then its Bohr compactification $b(\widehat{\Gamma}) = \widehat{\Gamma_d}$ is a solenoidal group.*

In particular, $b(\widehat{\mathbb{Q}}) = \widehat{\mathbb{Q}_d}$ and $b(\mathbb{R}) = b(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}_d}$ are solenoidal groups.

3.2 Functions and measures on groups

Let G be a compact Hausdorff group. The set $P(G)$ of finite linear combinations $\sum_{j=1}^{m} d_j \chi^{a_j}$, $a_j \in \widehat{G} = \Gamma$, $d_j \in \mathbb{C}$ of characters of G is a separating self-conjugate subalgebra of $C(G)$. By the Stone-Weierstrass approximation theorem $P(G)$ is dense in $C(G)$ in the uniform topology. If Λ is a fixed subset of $\widehat{G} = \Gamma$, a *Λ-polynomial* is any finite linear combination $\sum_{j=1}^{m} d_j \chi^{a_j}$ with $a_j \in \Lambda$. The set of all Λ-polynomials is denoted by $P_\Lambda(G)$.

In the case when the group envelope $\Gamma = S - S$ is a discrete subgroup of $\mathbb{R}$ the dual group G of Γ is compact, by Pontryagin's duality theorem. If, in addition, Γ is dense in $\mathbb{R}$, then G is a solenoidal group, thus it contains a dense homomorphic image of the real line $\mathbb{R}$. Every continuous character on $\mathbb{R}$ can be extended to a continuous character on G, and therefore, every almost periodic S-function f on $\mathbb{R}$ can be extended to a continuous function $\widetilde{f}$ on G. The Fourier coefficients $c_k^{\widetilde{f}}$ of the function $\widetilde{f}$ on G, extended in this way, coincide with the Dirichlet coefficients A_k^f of f.

A continuous function f on a topological group Γ is said to be *almost periodic on Γ* if the set of all its G-shifts $f_h(g) = f(h\, g)$, $h \in \Gamma$, is relatively uniformly compact in $C_b(\Gamma)$. Equivalently, f is almost periodic on Γ if it can be approximated uniformly on Γ by linear combinations $\sum_{k=1}^{n} a_k \chi_k \in P(\Gamma),\ a_k \in \mathbb{C}$ of characters χ_k on Γ. Under the uniform norm on Γ the space $AP(\Gamma)$ of almost periodic functions on Γ is a commutative Banach algebra with unit. Let S be a subsemigroup of $\widehat{\Gamma}$. Uniform limits on Γ of linear combinations of characters χ^a in S are called *almost periodic S-functions on Γ*. The set of almost periodic S-functions on Γ will be denoted by $AP_S(\Gamma)$.

Given a p: $1 \leq p \leq \infty$, the expression $\|f\|_p = \Big(\int_G |f|^p\Big)^{\frac{1}{p}} d\sigma$ is a norm on $C(G)$, called the *$L^p(G,\sigma)$-norm.* Indeed, if $\|f\|_p = 0$ for an $f \in C(G)$, then $\int_G |f|^p \, d\sigma = 0$, and therefore, $f \equiv 0$ on the support of σ, i.e. on G. The completion $L^p(G,\sigma)$ of $C(G)$ under the $L^p(G,\sigma)$-norm is a Banach space. Actually, $L^p(G,\sigma)$ coincides with the set of all Borel functions f on G with finite $L^p(G,\sigma)$-norms, under the understanding that we identify functions that coincide σ-almost everywhere on G. If $p > 1$, then the dual space, i.e. the set of all bounded linear functionals of $L^p(G,\sigma)$, is isomorphic to the space $L^q(G,\sigma)$, where q is such that $1 < q < \infty$, and $1/p + 1/q = 1$. Given a bounded Borel function f on G, the $L^\infty(G,\sigma)$-norm is defined by

$$\|f\|_\infty = \operatorname{ess\,sup}_G \big|f(g)\big|,$$

which is the smallest number $\lambda \geq 0$ such that $\sigma\big(\big\{g \in G\colon \big|f(g)\big| \geq \lambda\big\}\big) = 0$. Under the agreement that we identify bounded Borel functions that coincide σ-almost everywhere on G, i.e. for which $\|f - g\|_\infty = 0$, then the space $L^\infty(G,\sigma)$ of all bounded Borel functions on G is a Banach space under the L^∞-norm. It is the dual space of $L^1(G,\sigma)$.

The space $C(G)$ can be provided with an inner product, namely,

$$\langle f, g\rangle = \int_G f\overline{g}\, d\sigma.$$

Under the $L^2(G,\sigma)$-norm, $\|f\|_2 = \Big(\int_G |f|^2 d\sigma\Big)^{\frac{1}{2}}$, $C(G)$ is a pre-Hilbert space. Hence the completion $L^2(G,\sigma)$ of $C(G)$ in this norm is a Hilbert space. For every $\chi^a, \chi^b \in \widehat{G}$ we have $\langle \chi^a, \chi^b\rangle = \int_G \chi^{a-b}\, d\sigma = 0$ if $a \neq b$, and $\langle \chi, \chi\rangle = \int_G d\sigma = 1$. Therefore, the characters on G are mutually orthogonal functions of unit $L^2(G,\sigma)$-norm. Being dense in $C(G)$ under the uniform norm, the linear combinations of characters on G are also dense in $L^2(G,\sigma)$ under the $L^2(G,\sigma)$-norm. Moreover, the set of characters $\widehat{G}$ on G is an orthonormal basis for the space $L^2(G,\sigma)$. The Hilbert space theory implies that every $f \in L^2(G,\sigma)$ has a unique representation as a series of type

$$f = \sum_{a\in\Gamma} c_a^f \chi^a, \text{ where } c_a^f = \int_G f\chi^{-a}\, d\sigma = \int_G f\overline{\chi}^a\, d\sigma, \tag{3.2}$$

called its *Fourier series of* f, and $\|f\|_2^2 = \sum_{a\in\Gamma} |c_a^f|^2$. The numbers c_a^f are called the *Fourier coefficients* of f. The definition of Fourier coefficients c_a^f makes sense for any $L^1(G,\sigma)$-function f. The fact that the numbers c_a^f are the Fourier coefficients of f we indicate by

$$f \sim \sum_{a\in\widehat{G}} c_a^f \chi_a.$$

Note that the above series may not be convergent, and therefore, in general it can not be interpreted as a function.

Theorem 3.2.1. *If all Fourier coefficients of a function* $f \in L^1(G,\sigma)$ *are* 0, *then* f *is* σ*-almost everywhere* 0 *on* G.

Proof. Consider the measure μ with $d\mu = f\, d\sigma$ on G, i.e. whose values on Borel sets $F \subset G$ are defined by $\mu(F) = \int_F f\, d\sigma$, or equivalently, for which $\int_G g\, d\mu = \int_G gf\, d\sigma$ for every $f \in C(G)$. The function $\Phi(g) = \int_G g\, d\mu$ is a continuous linear functional on $C(G)$. Since $\Phi(\chi^a) = \int_G f\chi^a d\sigma = 0$ for every $a \in \Gamma$, the Stone-Weierstrass approximation theorem implies that $\Phi(g) = 0$ for every function $g \in C(G)$. According to the Riesz representation theorem for bounded linear functionals on $C(G)$, the dual space of $C(G)$ is isometrically isomorphic to the set $M(G)$ of regular Borel measures on G. Therefore, $\Phi \equiv 0$, thus $\mu = f\, d\sigma$ is the zero measure on G, and hence $f \equiv 0$ σ-almost-everywhere on G. □

As an immediate corollary to Theorem 3.2.1 we obtain that the Fourier series of functions in $L^1(G,\sigma)$ are uniquely defined. If a runs in Γ, then the Fourier coefficient c_a^f generates a function $\widehat{f}$ on Γ, namely $\widehat{f}(a) = c_a^f$, called the *Fourier transform* of f.

Similarly, with any regular Borel measure $\mu \in M(G)$ on G we associate a unique, not necessarily convergent, series, namely,

$$\mu \sim \sum_{a\in\widehat{G}} c_a^\mu \chi^a, \text{ where } c_a^\mu = \int_G \chi^{-a}\, d\mu,$$

called the *Fourier-Stieltjes series* of μ. The numbers c_a^μ are called the *Fourier-Stieltjes coefficients* of μ. By arguments similar to the ones used in the proof of Theorem 3.2.1, one can see that μ is the zero measure on G if and only if all its Fourier-Stieltjes coefficients c_a^μ are 0. If a runs in Γ, the Fourier-Stieltjes coefficient c_a^μ gives rise to a function $\widehat{\mu}$ on Γ, namely, $\widehat{\mu}(a) = c_a^\mu$, called the *Fourier-Stieltjes transform* of μ.

Theorem 3.2.2. *If the Fourier transform $\widehat{f}$ of a function $f \in L^1(G,\sigma)$ coincides with the Fourier-Stieltjes transform $\widehat{\mu}$ of a Borel measure $\mu \in M(G)$, then $d\mu = f\,d\sigma$, i.e. $\int\limits_G \varphi\,d\mu = \int\limits_G \varphi f\,d\sigma$ for every $\varphi \in C(G)$.*

Proof. Consider the measure $\mu_1 \in M(G)$ with $d\mu_1 = f\,d\sigma$. For every $a \in \Gamma$ we have $\widehat{\mu}_1(a) = c_a^{\mu_1} = \int\limits_G \chi^{-a}\,d\mu_1 = \int\limits_G \chi^{-a} f\,d\sigma = c_a^f = \widehat{f}(a) = \widehat{\mu}(a)$. For the Fourier-Stieltjes coefficients of the measure $\nu = \mu_1 - \mu \in M(G)$ we have $c_a^\nu = \widehat{\nu}(a) = \widehat{\mu}_1 - \widehat{\mu} = 0$ for every $a \in \Gamma$. Consequently, $\nu = 0$, and therefore $\mu_1 = \mu$. □

The *spectrum* of a function $f \in L^1(G,\sigma)$ is called the set $\mathrm{sp}\,(f)$ of all $a \in \Gamma$, such that $c_a^f \neq 0$. Similarly, for a given measure $\mu \in M(G)$ the set $\mathrm{sp}\,(\mu) = \{a \in \Gamma\colon c_a^\mu \neq 0\}$ is called the *spectrum* of μ. It is easy to see that any function $f \in L^1(G,\sigma)$ whose spectrum $\mathrm{sp}\,(f)$ consists of finitely many elements $a_1, a_2, \ldots, a_m$ of Γ, is a finite linear combination of the corresponding characters $\chi^{a_j} \in \Gamma = \widehat{G}$, i.e. is of type $\sum\limits_{j=1}^m d_j \chi^{a_j}$ with some $d_j \in \mathbb{C}$.

The space $C(G)$ can be equipped with a multiplicative operation. Namely, given two functions $f, h \in C(G)$ their *convolution* $f_1 \star f_2 \in C(G)$ is defined by

$$(f_1 \star f_2)(g) = \int\limits_G f_1(g\,h^{-1})\,f_2(h)\,d\sigma(h). \tag{3.3}$$

Since for any $f \in C(G)$,

$$\int\limits_G f(h^{-1})\,d\sigma(h) = \int\limits_G f(h)\,d\sigma(h),$$

one can easily check that $(f_1 \star f_2)(g) = (f_2 \star f_1)(g)$, i.e. the convolution is a commutative operation in $C(G)$. It is easy to see that

$$c_a^{f_1 \star f_2} = c_a^{f_1} c_a^{f_2} \tag{3.4}$$

for every $f, g \in L^1(G,\sigma)$. Indeed,

$$\begin{aligned}
c_a^{f_1\star f_2} &= \int_G (f_1 \star f_2)(g)\,\chi^{-a}(g)\,d\sigma(g) \\
&= \int_G \Big(\int_G f_1(h^{-1}g)\,f_2(h)\,d\sigma(h)\Big)\chi^{-a}(g)\,d\sigma(g) \\
&= \int_G \Big(\int_G f_1(h^{-1}g)\,\chi^{-a}(g)\,d\sigma(g)\Big) f_2(h)\,d\sigma(h) \\
&= \int_G \Big(\int_G f_1(h^{-1}g)\,\chi^{-a}(h^{-1}g)\,d\sigma(g)\Big) f_2(h)\,\chi^{-a}(h)\,d\sigma(h) \\
&= c_a^{f_1} \int_G f_2(h)\,\chi^{-a}(h)\,d\sigma(h) = c_a^{f_1} c_a^{f_2}.
\end{aligned}$$

In particular, as an immediate consequence we obtain that

$$\operatorname{sp}(f_1 \star f_2) = \operatorname{sp}(f_1) \cap \operatorname{sp}(f_2). \tag{3.5}$$

Let $f \in C(G)$ and $\mu \in M(G)$. The *convolution* $f \star \mu$ of f and μ is the function $f \star \mu \in C(G)$ defined by

$$(f \star \mu)(g) = \int_G f(g\,h^{-1})\,d\mu(h).$$

The *convolution* $\mu \star \nu$ of two measures $\mu, \nu \in M(G)$ is the measure $\mu \star \nu$ on G defined by

$$(\mu \star \nu)(F) = \int_G \nu(g^{-1}F)\,d\mu(g)$$

for every Borel set $F \subset G$. Equivalently, $\mu \star \nu$ is defined by the property

$$\int_G f\,d\,(\mu \star \nu) = \int_G \int_G f(g\,h)\,d\mu(g)\,d\nu(h),$$

where $f \in C(G)$. Consequently, $\|\mu \star \nu\| \le \|\mu\|\|\nu\|$.

Similarly, for every $\mu, \nu \in M(G)$ we have

$$c_a^{\mu\star\nu} = c_a^{\mu} c_a^{\nu}. \tag{3.6}$$

Indeed,

$$\begin{aligned}
c_a^{\mu\star\nu} &= \int_G \chi^{-a} d(\mu \star \nu) = \int_G \chi^{-a}(gh)\,d(\mu(g)\,d\nu(h) \\
&= \int_G\int_G \chi^{-a}(gh)\,d(\mu(g)\,d\nu(h) = \int_G \chi^{-a}(g)\,d(\mu(g) \int_G \chi^{-a}(h)\,d\nu(h) = c_a^{\mu} c_a^{\nu}.
\end{aligned}$$

In particular we see that

$$\mathrm{sp}\,(\mu \star \nu) = \mathrm{sp}\,(\mu) \cap \mathrm{sp}\,(\nu). \tag{3.7}$$

The following lemma shows that the mapping $f \longmapsto f_h$, where $h \in G$, is a continuous operator in $L^p(G, \sigma),\ 1 \le p \le \infty$.

Lemma 3.2.3. *Let $f \in L^p(G, \sigma),\ 1 \le p \le \infty$. If $h_n \longrightarrow h$, where $h_n, h \in G$, then $\|f_{h_n} - f_h\|_p \longrightarrow 0$.*

Proof. Since the space $L^p(G, \sigma)$ is the closure of $C(G)$ in the $L^p(G, \sigma)$-norm, then for every $\varepsilon > 0$ there is a $g \in C(G)$ such that $\|f - g\|_p < \varepsilon/3$. We have that $\|f_h - g_h\|_p = \|f - g\|_p < \varepsilon/3$ for every $h \in G$. Clearly, $\|g_{h_n} - g_h\|_\infty \longrightarrow 0$ as $h_n \longrightarrow h$. Hence $\|g_{h_n} - g_h\|_p \longrightarrow 0$ too, and therefore there is an $N > 0$ such that $\|g_{h_n} - g_h\|_p < \varepsilon/3$ when $n \ge N$. Therefore, for every $n \ge N$ we have

$$\|f_{h_n} - f_h\|_p \le \|f_{h_n} - g_{h_n}\|_p + \|g_{h_n} - g_h\|_p + \|f_h - g_h\|_p \le 3\,(\varepsilon/3) = \varepsilon. \qquad \square$$

Note that the mapping $f \longmapsto f_h$, where $h \in G$, is an isometric isomorphism on $L^p(G, \sigma),\ 1 \le p \le \infty$.

The convolution in $C(G)$ can be extended on the space $L^1(G, \sigma)$, under which it becomes a commutative Banach algebra. namely, the convolution of two functions f_1, f_2 in $L^1(G, \sigma)$ is defined in exactly the same way, as in the continuous case, namely,

$$(f_1 \star f_2)(g) = \int\limits_G f_1(g\,h^{-1})\, f_2(h)\, d\sigma(h).$$

Theorem 3.2.4. *If $f_1, f_2 \in L^1(G, \sigma)$, then $f_1 \star f_2 \in L^1(G, d\sigma)$, and*

$$\|f_1 \star f_2\|_1 \le \|f_1\|_1 \|f_2\|_1.$$

Therefore, with the convolution as a multiplication, the space $L^1(G, \sigma)$ is a commutative Banach algebra.

Proof. Let $f_1 \in L^1(G, \sigma)$ and $\{\varphi_n\}_n$ be a sequence of continuous functions on G, such that $\|\varphi_n - f_1\|_1 \longrightarrow 0$ as $n \longrightarrow 0$. Then $\varphi_n \star f_2 \in L^1(G, \sigma)$, and by the Fubini theorem

$$\begin{aligned}
&\|\varphi_n \star f_2 - \varphi_m \star f_2\|_1 = \int\limits_G \big|(\varphi_n - \varphi_m) \star f_2\big|\, d\sigma \\
&\le \int\limits_G \Big(\int\limits_G \big|\varphi_n(g\,h^{-1}) - \varphi_m(g\,h^{-1})\big|\big|f_2(h)\big|\, d\sigma(h)\Big) d\sigma(g) \\
&= \int\limits_G \Big(\int\limits_G \big|\varphi_n(g\,h^{-1}) - \varphi_m(g\,h^{-1})\big|\, d\sigma(g)\Big)\big|f_2(h)\big|\, d\sigma(h) = \|\varphi_n - \varphi_m\|_1 \|f_2\|_1.
\end{aligned}$$

Therefore, $\{\varphi_n \star f_2\}_n$ is a Cauchy sequence in $L^1(G,\sigma)$, and consequently its limit $f_1 \star f_2$ is also in $L^1(G,\sigma)$. In addition,

$$\begin{aligned}\|f_1 \star f_2\|_1 &\le \int_G \int_G \big|f_1(g\,h^{-1})\, f_2(h)\big|\, d\sigma(h) d\sigma(g)\\ &= \int_G \Big(\int_G \big|f_1(g\,h^{-1})\big|\, d\sigma(g)\Big) \big|f_2(h)\big|\, d\sigma(h) = \|f_1\|_1 \|f_2\|_1.\end{aligned}$$

□

Theorem 3.2.5. *The convolution $f_1 \star f_2$ is a continuous function on G in the following cases.*

(a) *$f_1 \in L^1(G,\sigma)$ and $f_2 \in L^\infty(G,\sigma)$.*

(b) *$f_1 \in L^p(G,\sigma)$ and $f_2 \in L^q(G,\sigma)$, where $1 < p, q < \infty$, $1/p + 1/q = 1$.*

Proof. (a) Let $f_1 \in L^1(G,\sigma)$ and $f_2 \in L^\infty(G,\sigma)$. Hölder's inequality implies

$$\big|(f_1 \star f_2)(g)\big| = \Big| \int_G f_1(g\,h^{-1})\, f_2(h)\, d\sigma(h)\Big| \le \|f_1\|_1 \|f_2\|_\infty.$$

By the previous remark, for any $g_n, g \in G$ we have

$$\big|(f_1 \star f_2)(g_n) - (f_1 \star f_2)(g)\big| = \Big| \int_G \big(f_1(g_n h^{-1})\, f_2(h) - f_1(gh^{-1})\, f_2(h)\big)\, d\sigma(h)\Big|$$

$$\le \Big| \int_G \big((f_1)_{g_n}(h^{-1}) - (f_1)_g(h^{-1})\big)\, f_2(h)\, d\sigma(h)\Big| \le \big\|(f_1)_{g_n} - (f_1)_g\big\|_1 \|f_2\|_\infty.$$

If $g_n \longrightarrow g$, then $\big\|(f_1)_{g_n} - (f_1)_g\big\|_1 \longrightarrow 0$ by Lemma 3.2.3, thus $(f_1 \star f_2)(g_n) \longrightarrow (f_1 \star f_2)(g)$ as $g_n \longrightarrow g$. Consequently, $f \star g \in C(G)$.

(b) Let $f_1 \in L^p(G,\sigma)$ and $f_2 \in L^q(G,\sigma)$, where $1 < p, q < \infty$, $1/p + 1/q = 1$. In the same way as (a) we obtain by Hölder's inequality that

$$\big|(f_1 \star f_2)(g)\big| = \Big| \int_G f_1(g\,h^{-1})\, f_2(h)\, d\sigma(h)\Big| \le \|f_1\|_p \|f_2\|_q.$$

The argument from (a) applies to obtain also that

$$\big|(f_1 \star f_2)(g_n) - (f_1 \star f_2)(g)\big| \le \big\|(f_1)_{g_n} - (f_1)_g\big\|_p \|f_2\|_q.$$

Consequently, $(f_1 \star f_2)(g_n) \longrightarrow (f_1 \star f_2)(g)$ as $g_n \longrightarrow g$, by Lemma 3.2.3. This proves that $f_1 \star f_2 \in C(G)$. □

3.3 Bochner-Fejér operators on groups

Bochner-Fejér operators on compact groups, introduced in this section, play a role in the approximation of functions on groups, similar to the role of Fejér kernels for functions on the unit circle. As mentioned in the previous section, given a compact abelian group G, the set $P(G)$ of linear combinations of characters on G is dense in the space $L^p(G,\sigma)$, $1 \le p \le \infty$, i.e. every function $f \in L^p(G,\sigma)$ can be approximated by linear combinations of characters on G in the $L^p(G,\sigma)$-norm. We show in this section that f can also be approximated in the $L^p(G,\sigma)$-norm by $\operatorname{sp}(f)$-polynomials, i.e. by finite linear combinations of characters $\sum_{j=1}^{m} d_j \chi^{a_j}$ on G, with $a_j \in \operatorname{sp}(f)$.

Lemma 3.3.1. *Let $\{U_i\}_{i\in I}$ be a neighborhood basis in G of the unit element $\imath$. For every $\varepsilon \in (0,1)$ and every $i \in I$ there exists a linear combination of characters, ψ_i^ε, on G with the following properties:*

(i) $\psi_i^\varepsilon(g) \ge 0$ *on G.*

(ii) $\psi_i^\varepsilon(g) < \varepsilon$ *for every $g \in G \setminus U_i$.*

(iii) $\displaystyle\int_G \psi_i^\varepsilon(g)\, d\sigma(g) = 1.$

Proof. For every neighborhood $U_i \ni \imath$ there is a nonnegative continuous function $\psi_i \in C(G)$ such that $\psi_i(\imath) = 1$ and $\psi_i\big|_{G\setminus U_i} \equiv 0$, by Urysohn's Lemma. Clearly, $\int_G \psi_i\, d\sigma > 0$. According to the Stone-Weierstrass approximation theorem, for every $\varepsilon \in (0,1)$ one can find a linear combination $P_i = \sum_j c_j \chi^{a_j}$ in $P(G)$ such that $\max_G \big|P_i(g) - \psi_i(g)\big| < \varepsilon$. Since $\psi_i \ge 0$, we have

$$\max_G \big|\operatorname{Re} P_i(g) - \psi_i(g)\big| \le \max_G \big|P_i(g) - \psi_i(g)\big| < \varepsilon.$$

Therefore, $Q_i(g) = \operatorname{Re} P_i(g) + \varepsilon \ge 0$ on G, while $Q_i(g)|_{G\setminus U_i} \le \varepsilon$, and

$$\int_G Q_i\, d\sigma \ge \int_G \psi_i\, d\sigma > 0.$$

Note that since $\operatorname{Re} P_i = (1/2)(P_i + \overline{P_i})$ is a linear combination of characters on G, so is Q_i. Now the function

$$\psi_i^\varepsilon = Q_i\Big(\int_G Q_i\, d\sigma\Big)^{-1}$$

satisfies the properties (i), (ii), (iii). □

If ψ_i^ε is one of the functions from Lemma 3.3.1, then for every $p\colon 1 \le p \le \infty$ we define the operator $K_i^\varepsilon\colon L^p(G,\sigma) \longrightarrow L^p(G,\sigma)$ by

$$K_i^\varepsilon(f)(g) = \big(f \star \psi_i^\varepsilon\big)(g) = \int\limits_G f(g\,h^{-1})\,\psi_i^\varepsilon(h)\,d\sigma(h). \tag{3.8}$$

Such a set is called a family of *Bochner-Fejér operators* on G. We define a partial order in the family $\{K_i^\varepsilon\}$, by setting $K_i^\varepsilon \prec K_j^\eta$ if $U_j \subset U_i$, and $\eta \le \varepsilon$.

Theorem 3.3.2. *Let $f \in L^p(G,\sigma)$, $1 \le p \le \infty$. For every $\varepsilon > 0$ and $i \in I$ the functions $K_i^\varepsilon(f)$ possess the following properties.*

(a) *If $p < \infty$, then $K_i^\varepsilon(f) \in L^p(G,\sigma)$ and $\big\|K_i^\varepsilon(f)\big\|_p \le \|f\|_p$.*

(b) *$K_i^\varepsilon(f)$ are linear combinations of characters on G.*

(c) *$\operatorname{sp}\big(K_i^\varepsilon(f)\big) \subset \operatorname{sp}(f)$, thus $K_i^\varepsilon(f)$ are $\operatorname{sp}(f)$-polynomials on G.*

(d) *If $p < \infty$, then $\lim\limits_{\prec} \big\|K_i^\varepsilon(f) - f\big\|_p = 0$.*

(e) *If $f \in C(G)$, then $\big\|K_i^\varepsilon(f)\big\|_\infty \le \|f\|_\infty$ and $\lim\limits_{\prec} \big\|K_i^\varepsilon(f) - f\big\|_\infty = 0$.*

Proof. (a) If $p = 1$, Theorem 3.2.4 implies that $K_i^\varepsilon(f) = f \star \psi_i^\varepsilon \in L^1(G,\sigma)$, and $\|f \star \psi_i^\varepsilon\|^1 \le \|f\|_1\|\psi_i^\varepsilon\|_1 = \|f\|_1$. Let $1 < p < \infty$, and let $\varphi \in L^q(G,\sigma)$, where $1/p + 1/q = 1$. We have

$$\begin{aligned}\Big|\int\limits_G K_i^\varepsilon(f)\,\varphi\,d\sigma\Big| &= \Big|\int\limits_G \Big(\int\limits_G f(g\,h^{-1})\,\psi_i^\varepsilon(h)\,d\sigma(h)\Big)\varphi(g)\,d\sigma(g)\Big| \\ &= \Big|\int\limits_G \Big(\int\limits_G f(g\,h^{-1})\,\varphi(g)\,d\sigma(g)\Big)\psi_i^\varepsilon(h)\,d\sigma(h)\Big| \\ &\le \int\limits_G \Big|\int\limits_G f(g\,h^{-1})\,\varphi(g)\,d\sigma(g)\Big|\psi_i^\varepsilon(h)\,d\sigma(h).\end{aligned}$$

Hölder's inequality implies that the inner integral does not exceed $\|f\|_p\|\varphi\|_q$. Hence,

$$\Big|\int\limits_G K_i^\varepsilon(f)\,\varphi\,d\sigma\Big| \le \|f\|_p\|\varphi\|_q \int\limits_G \psi_i^\varepsilon(f)\,d\sigma = \|f\|_p\|\varphi\|_q.$$

Since this holds for any $\varphi \in L^q(G,\sigma)$, the mapping $\varphi \longmapsto \int\limits_G K_i^\varepsilon(f)\,\varphi\,d\sigma$ is a bounded linear functional on $L^q(G,\sigma)$. Hence, $K_i^\varepsilon(f) \in L^p(G,\sigma)$ and $\|K_i^\varepsilon(f)\|_p \le \|f\|_p$. This proves (a).

From $K_i^\varepsilon(f) = f \star \psi_i^\varepsilon$, we see that $\operatorname{sp}\big(K_i^\varepsilon(f)\big) = \operatorname{sp}(f) \cap \operatorname{sp}(\psi_i^\varepsilon)$ by (3.5). Consequently, $\operatorname{sp}\big(K_i^\varepsilon(f)\big)$ is a finite subset of $\operatorname{sp}(f)$, i.e. $K_i^\varepsilon(f)$ is a $\big(\operatorname{sp}(f)\big)$-polynomial, which proves (b) and (c).

(d) For any $\varphi \in L^q(G,\sigma)$ we have

$$\Big|\int_G \big(K_i^\varepsilon(f) - f\big)\,\varphi\,d\sigma\Big| = \Big|\int_G \Big(\int_G (f(g\,h^{-1}) - f(hg))\,\psi_i^\varepsilon(h)\,d\sigma(h)\Big)\,\varphi(g)\,d\sigma(g)\Big|$$
$$\le \int_G \Big|\int_G (f(g\,h^{-1}) - f(g))\,\varphi(g)\,d\sigma(g)\Big|\psi_i^\varepsilon(h)\,d\sigma(h).$$

Hölder's inequality implies that the inner integral does not exceed $\|f_h - f\|_p\|\varphi\|_q$. For every $f \in L^p(G,\sigma)$, $1 \le p < \infty$, and any $\varepsilon > 0$ there exists a basis neighborhood U_i of $\imath$, such that $\|f - f_h\|_p < \varepsilon$ for all $h \in U_i$. Indeed, by $\big|\varphi(h) - \varphi(g)\big| \le \|f_h - f_g\|_p$ we see that the function $\varphi(h) = \|f - f_h\|_p$ is continuous on G. Since $\varphi(\imath) = 0$, there certainly exists a neighborhood U_i of $\imath$ such that $\varphi(h) = \|f - f_h\|_p < \varepsilon$, as claimed. The properties of ψ_i^ε established in Lemma 3.3.1 yield

$$\Big|\int_G \Big(\int_G (f(g\,h^{-1}) - f(g))\,\varphi(g)\,d\sigma(g)\Big)\,\psi_i^\varepsilon(h)\,d\sigma(h)\Big|$$
$$\le \Big|\int_{U_i} \Big(\int_G \Big(f(g\,h^{-1}) - f(g)\Big)\,\varphi(g)\,d\sigma(g)\Big)\,\psi_i^\varepsilon(h)\,d\sigma(h)\Big|$$
$$+\Big|\int_{G\setminus U_i} \Big(\int_G (f(g\,h^{-1}) - f(g))\,\varphi(g)\,d\sigma(g)\Big)\,\psi_i^\varepsilon(h)\,d\sigma(h)\Big|.$$

The former expression does not exceed $\varepsilon\,\|\varphi\|_q$, while the latter one is not greater than $2\varepsilon\,\|f\|_p\|\varphi\|_q$. Since ε can be chosen arbitrarily small, this shows that the mapping $\varphi \longrightarrow \int_G \big(K_i^\varepsilon(f) - f\big)\,\varphi\,d\sigma$ is a bounded linear functional on $L^q(G,\sigma)$, and its norm is $\big\|K_i^\varepsilon(f) - f\big\|_p \le 2\varepsilon\,\|f\|_p$. This proves (d).

(e) If $f \in C(G)$, the argument from part (d) implies

$$\big\|K_i^\varepsilon(f) - f\big\|_\infty \le \int_G \big|\big(f(g\,h^{-1}) - f(g)\big)\big|\,\psi_i^\varepsilon(h)\,d\sigma(h)$$
$$\le \int_{U_i} \big|\big(f(g\,h^{-1}) - f(\gamma)\big)\big|\,\psi_i^\varepsilon(h)\,d\sigma(h) + \int_{G\setminus U_i} \big|f(g\,h^{-1}) - f(g)\big|\,\psi_i^\varepsilon(h)\,d\sigma(h).$$

The former integral does not exceed ε, while the previous one does not exceed $2\varepsilon\,\|f\|_\infty$. This completes the proof of (e) since ε can be chosen small enough. □

As an immediate consequence from Theorem 3.3.2 we obtain the following

Corollary 3.3.3. *Any* $f \in L^p(G,\sigma)$, $1 \le p < \infty$ *can be approximated by* $\mathrm{sp}\,(f)$*-polynomials* K_i^ε *with respect to the* $L^p(G,\sigma)$*-norm.*

Theorem 3.3.4. *For every regular Borel measure $\mu \in M(G)$ on G there exists a net of linear combinations of characters, $\{p_\alpha\}_\alpha$, $p_\alpha \in P(G)$ with $\|p_\alpha d\sigma\| \le \|\mu\|$ and* $\mathrm{sp}\,(p_\alpha) \subset \mathrm{sp}(\mu)$, *such that the measures $p_\alpha d\sigma$ converge to μ in the weak*-topology on $M(G)$.*

Proof. Let $\{\psi_i^\varepsilon\}_{i,\varepsilon}$ be the family of functions defined in Lemma 3.3.1. Consider the functions

$$P_i^\varepsilon(g) = \int_G \psi_i^\varepsilon(g^{-1}h)\, d\mu(h).$$

Note that since ψ_i^ε are linear combinations of characters on G, so are P_i^ε, and $\mathrm{sp}\,(P_i^\varepsilon) \subset \mathrm{sp}\,(\mu)$. Indeed, if $\psi_i^\varepsilon = \sum_j c_j \chi^{a_j}$, then

$$\begin{aligned} P_i^\varepsilon(g) &= \int_G \psi_i^\varepsilon(g^{-1}h)\, d\mu(h) = \int_G \sum_j c_j \chi^{a_j}(g^{-1}h)\, d\mu(h) \\ &= \sum_j c_j \Big(\int_G \chi^{a_j}(h)\, d\mu(h) \Big) \chi^{a_j}(g^{-1}) = \sum_j c_j \Big(\int_G \chi^{a_j}(h)\, d\mu(h) \Big) \chi^{-a_j}(g). \end{aligned}$$

Thus, if $a \in \mathrm{sp}\,(P_i^\varepsilon)$, we have

$$0 \ne \int_G P_i^\varepsilon(g)\, \chi^{-a}(g)\, d\sigma(g) = \sum_j c_j \int_G \chi^{a_j}(h)\, d\mu(h) \int_G \chi^{-(a_j+a)}(g)\, d\sigma(g).$$

Therefore, $\int_G \chi^{a_j}(h)\, d\mu(h) \int_G \chi^{-(a_j+a)}(g)\, d\sigma(g) \ne 0$ for some $a_j \in \Gamma$. Consequently, $a = -a_j$, and $\int_G \chi^{a_j}(h)\, d\mu(h) = \int_G \chi^{-a}(h)\, d\mu(h) \ne 0$. Hence $a \in \mathrm{sp}\,(\mu)$.

Consider the measures μ_i^ε defined by $d\mu_i^\varepsilon = P_i^\varepsilon d\sigma \in M(G)$. Fubini's theorem implies

$$\|\mu_i^\varepsilon\| = \int_G |P_i^\varepsilon|\, d\sigma \le \int_G \int_G |\psi_i^\varepsilon|\, d\,|\mu| d\sigma = \int_G \Big(\int_G \psi_i^\varepsilon\, d\sigma \Big) d\,|\mu| = \|\mu\|,$$

where $d\,|\mu|$ is the total variation of μ. For any $f \in C(G)$ we have

$$\begin{aligned} &\int_G f(g)\, d\mu_i^\varepsilon(g) = \int_G f(g)\, P_i^\varepsilon(g)\, d\sigma(g) = \int_G f(g) \Big(\int_G \psi_i^\varepsilon(g^{-1}h)\, d\mu(h) \Big)\, d\sigma(g) \\ &= \int_G \Big(\int_G f(g)\, \psi_i^\varepsilon(g^{-1}h)\, d\sigma(g) \Big) d\mu(h) = \int_G \Big(\int_G f(h\, g^{-1})\, \psi_i^\varepsilon(g)\, d\sigma(g^{-1}) \Big) d\mu(h) \\ &= \int_G \Big(\int_G f(h\, g^{-1})\, \psi_i^\varepsilon(g)\, d\sigma(g) \Big) d\mu(h) = \int_G K_i^\varepsilon(f)(h) d\mu(h), \end{aligned}$$

since the Haar measure σ is inverse invariant. Theorem 3.3.2(e) implies that

$$\int_G f\, d\mu_i^\varepsilon = \int_G K_i^\varepsilon(f)\, d\mu \longrightarrow \int_G f\, d\mu$$

with respect to the ordering '$\prec$' in the family $\{K_i^\varepsilon\}$. Since this is true for every $f \in C(G)$, and $M(G) \cong C(G)^*$, it follows that the measures $\mu_i^\varepsilon \in M(G)$ converge to μ in the weak*-topology on $M(G)$. □

3.4 Semigroups and semicharacters

This section provides a survey of the basic properties of semigroups and their semicharacters.

A *semigroup* is a set S provided with an associative operation $(a, b) \longmapsto ab$, $a, b \in S$. If it is commutative, i.e. if $ab = ba$ for all $a, b \in S$, then S is said to be *commutative*. In this case we use the additive notation, $(a, b) \longmapsto a + b$ for the semigroup operation, rather than the multiplicative one. In the sequel we will consider only commutative semigroups.

An element $\iota \in S$ called *identity element* for S, if $\iota + a = a$ for every $a \in S$. If it exists, the identity element of S is uniquely defined. It is denoted by 0, and is called the *zero* of S. Under the multiplicative notation, ι is the identity element of S if $\iota\, a = a$ for every $a \in S$. In this case it is denoted by 1 or $\imath$, and is called the *unit* of S. An element $a \in S$ is said to be *invertible* in S if $a + b = 0$ for some $b \in S$ (resp. $ab = \iota$ under the multiplicative notation). In this case the element b is said to be *inverse* to a. Every $a \in S$ has at most one inverse, and if it exists it is denoted by $-a$ (resp. a^{-1} under the multiplicative notation). If S has an identity element, and if every element of S is invertible, then, clearly, S is a group.

An element $\jmath \in S$ is called an *idempotent element* (or, just *idempotent*) of S, if $\jmath + \jmath = \jmath$ (resp. $\jmath\jmath = \jmath$ under the multiplicative notation). Clearly, the identity element is idempotent in any semigroup S. Note that if G is a group, and S is a subsemigroup of G containing the identity element, then it is the only idempotent of S.

Example 3.4.1. (a) The set of positive numbers $(0, \infty)$ is a semigroup under addition. It contains neither a zero element, nor idempotents. However, when endowed with the multiplication, $(0, \infty)$ is a semigroup with unit 1.

(b) The sets $[0, \infty) = \mathbb{R}_+$, and $[0, \infty] = \overline{\mathbb{R}}_+$ are semigroups under addition, with zero element 0. $\mathbb{R}_+$ has only one idempotent, 0, while $[0, \infty]$ has two idempotents, 0 and ∞.

(c) The semigroup $\mathbb{R}_+ = [0, \infty)$ under multiplication has two idempotents, 0 and 1, while the semigroup $(0, \infty]$ under multiplication has two idempotents, 1 and ∞.

(d) The sets $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+ = \mathbb{N} \cap \{0\}$, $2\mathbb{Z} = \{2n \colon n \in \mathbb{Z}\}$, $\mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbb{R}_+ \times \mathbb{R}_+$, $\{(n,m) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon m \geq \sqrt{n}\}$, $\{(n,m) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon m \geq 1\}$, $\{(n,m) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon n/m \leq \sqrt{2}\}$ are semigroups under addition. So is the set $\{\infty\} \cup \mathbb{Z}_+$ endowed with the following operation: the usual addition on $\mathbb{Z}_+$, $\infty x = x$ for every $x \in \{\infty\} \cup \mathbb{Z}_+$.

(e) Let β be an irrational number. The set $\mathbb{Z}_+ + \mathbb{Z}_+\beta = \{n + m\beta \colon n, m \in \mathbb{Z}_+\}$ is a semigroup in $\mathbb{R}_+$, while $\mathbb{Z}_+ + \mathbb{Z}\beta = \{n + m\beta \colon n \in \mathbb{Z}_+,\ m \in \mathbb{Z}\}$ is a semigroup in $\mathbb{R}$.

(f) The set $S = \mathbb{R}_+$ endowed with the operation $x \cdot y = \max\{x, y\}$ is a commutative semigroup. Its idempotents are the characteristic functions $\varkappa_{[0,x)}$ and $\varkappa_{[0,x]}$, $x \in \mathbb{R}_+$.

The set of all idempotents of a semigroup S is denoted by I_S. It is a semigroup with respect to the operation inherited from S. One can introduce a natural order in I_S, namely, $\iota_1 \prec \iota_2$ if and only if $\iota_1 + \iota_2 = \iota_1$. Clearly, the zero element succeeds every other idempotent of S. In the semigroup $[0, \infty]$ from Example 3.4.1(b) we have $0 \succ \infty$, while in the semigroups from Example 3.4.1(c) we have $1 \succ 0$ in $[0, \infty)$, and $1 \succ \infty$ in $(0, \infty]$.

A subset $H \subset S$ is called a *subsemigroup* of S, if it is closed under the semigroup operation, i.e. if $H + H \subset H$. Equivalently, H is a subsemigroup of S if the sum $a+b$ of every pair of elements $a, b \in H$ belongs to H. Given an idempotent $\iota \in I_S$, the set $G_S^\iota = \{a \in S \colon a + \iota = a \text{ and } a + b = \iota \text{ for some } b \in S\}$ is a group with identity element ι. Actually G_S^ι is the maximal subgroup of S that contains ι. If S possesses 0, then the set $S \cap (-S)$ of invertible elements in S coincides with G_S^0, which is the maximal subgroup of S containing 0. It is called the *group kernel* of S, and is denoted by G_S. Clearly, if S does not possess idempotents other than 0, then G_S is the only maximal subgroup of S.

Let S_1 and S_2 be two semigroups. A map $\varphi \colon S_1 \longrightarrow S_2$ is called a *semigroup homomorphism*, if it respects the semigroup operation, i.e. if $\varphi(a+b) = \varphi(a) + \varphi(b)$ for any pair of elements $a, b \in S$. The set $\mathrm{Hom}\,(S_1, S_2)$ of semigroup homomorphisms from S_1 to S_2 is denoted also by $H(S_1, S_2)$. A homomorphism that is injective, i.e. for which $\varphi(a) \neq \varphi(b)$ whenever $a \neq b$, is called an *embedding* of S_1 in S_2. If, in addition, φ is surjective, i.e. if $f(G_1) = G_2$, then φ is called a (*semigroup*) *isomorphism* between S_1 and S_2. In this case we write $S_1 \cong S_2$.

We say that S is a semigroup with *cancellation law*, if $a + c = b + c$ implies $a = b$ for any $a, b, c \in S$. The cancellation law holds on the set $\mathbb{R}_+$ with addition, while it fails on $\mathbb{R}$ with the operation $x \cdot y = \max\{x, y\}$, considered in Example 3.4.1(f). It is easy to see that the cancellation law holds on every group. Clearly, the cancellation law holds on any subsemigroup of a group. As the next proposition shows, the converse is also true.

Proposition 3.4.2. *A semigroup S with 0 can be embedded into a group if and only if the cancellation law holds on S.*

Proof. If φ is an embedding of S into a group, then obviously $\varphi(S)$ is a semigroup with cancellation law, and so is S.

Conversely, assume that S is a semigroup with cancellation law. A group into which S can be embedded can be obtained by applying the classical construction for creating the set of integers by natural numbers. Indeed, the Cartesian product $S \times S$ is a semigroup under the standard operation $(a, b) + (c, d) = (a + c, b + d)$. Consider on $S \times S$ the equivalence relation $(a, b) \sim (c, d)$ if and only if $a+d = b+c$. It is clear that $(a, b) \sim (a, b)$ and that $(a, b) \sim (c, d)$ implies $(c, d) \sim (a, b)$ for every (a, b), $(c, d) \in S \times S$. The relation '$\sim$' is also transitive, i.e. if $(a, b) \sim (c, d)$, and $(c, d) \sim (k, l)$, then $(a, b) \sim (k, l)$. Indeed, $(a + l) + d = (a + d) + l = (b + c) + l = (c + l) + b = (d + k) + b = (b + k) + d$. The cancellation law on S implies that $a + l = b + k$, i.e. $(a, b) \sim (k, l)$. Denote by $\mathit{\Gamma}_S$ the set of equivalence classes $[(a, b)]$ of elements $(a, b) \in S \times S$, i.e. $[(a, b)] = \{(c, d) \in S \times S\colon (c, d) \sim (a, b)\}$. Define an operation on $\mathit{\Gamma}_S$ by $[(a, b)] + [(c, d)] = [(a + c, b + d)]$. It is easy to check that this operation is associative and commutative. We claim that under this operation $\mathit{\Gamma}_S$ is a group. Indeed, it is easy to see that for any $c, d \in S$ the elements (c, c) and (d, d) are equivalent, i.e. belong to one and the same equivalence class, say $[(c, c)] \in \mathit{\Gamma}_S$. Note that $(0, 0) \in [(c, c)]$. The cancellation law on S implies that $[(a, b)] + [(c, c)] = [(a + c, b + c)] = [(a, b)]$ for any a, $b \in S$. Hence, $[(c, c)] = [(0, 0)]$ is the identity element of $\mathit{\Gamma}_S$, and the class $[(b, a)] \in \mathit{\Gamma}_S$ is inverse to $[(a, b)] \in \mathit{\Gamma}_S$. Therefore, $\mathit{\Gamma}_S$ is a group, and the map $a \longmapsto [(a, 0)]$ embeds the semigroup S into $\mathit{\Gamma}_S$. □

The group $\mathit{\Gamma}_S$ constructed in Proposition 3.4.2 is called the *group envelope* of S. Formally, one can think of its elements $[(a, b)]$ as 'differences' $a - b$, so that $\mathit{\Gamma}_S = \{a - b\colon a, b \in S\} = S - S$. Therefore, any semigroup S with cancellation law and 0 can be assumed to be a subsemigroup of its group envelope $\mathit{\Gamma}_S = S - S$.

Example 3.4.3. Let β be an irrational number. Observe that $\mathbb{Z} + \mathbb{Z}\beta \in \mathbb{R}$ is the group envelope of the semigroups $\mathbb{Z}_+ + \mathbb{Z}_+\beta$ and $\mathbb{Z}_+ + \mathbb{Z}\beta$. The group kernel of $\mathbb{Z}_+ + \mathbb{Z}_+\beta$ is $\{0\}$, while $\{0\} + \mathbb{Z}\beta$ is the group kernel of $\mathbb{Z}_+ + \mathbb{Z}\beta$.

A subsemigroup J of S is said to be a (*semigroup*) *ideal* of S , if $J + S \subset J$, i.e. if, given an element $a \in J$, its sum $a + b$ with any element $b \in S$ belongs to J. If J_1 and J_2 are two ideals of S, then the sets $J_1 \cap J_2$ and $J_1 + J_2$ are also ideals of S. It is easy to see that groups do not contain proper ideals.

Example 3.4.4. (a) Intervals of type (c, ∞) and $[c, \infty)$, $c \in \mathbb{R}_+$, are ideals of the semigroup $\mathbb{R}_+$ endowed with addition. In fact, every ideal of $\mathbb{R}_+$ is of this type, which follows immediately from the fact that if J is an ideal of $\mathbb{R}_+$, and if $r \in J$, then J contains every $s > r$, since $s = r + (s - r)$ and $s - r \in \mathbb{R}_+$. By the same token, the ideals of the semigroup $\mathbb{Z}_+$ are sets of type $\{k \in \mathbb{N}\colon k \geq n\}$, where $n \geq 0$.

(b) Any set of type $(x, y) + \mathbb{R}_+ \times \mathbb{R}_+$ is an ideal of the semigroup $\mathbb{R}_+ \times \mathbb{R}_+$ endowed with addition. In fact, any ideal of $\mathbb{R}_+ \times \mathbb{R}_+$ is a union of such ideals. If

J is an ideal in $\mathbb{R}_+ \times \mathbb{R}_+$, and $(x, y) \in J$, then $(x, y) + \mathbb{R} \times \mathbb{R} \subset J$. In particular, the sets $(0, \infty) \times \mathbb{R}_+$, $\mathbb{R}_+ \times (0, \infty)$, $\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(0, 0)\}$ are ideals in $\mathbb{R}_+ \times \mathbb{R}_+$. The ideals of the semigroup $\mathbb{Z}_+ \times \mathbb{Z}_+$ have a similar description.

(c) The sets of type $(0, y) + \mathbb{R} \times \mathbb{R}_+$ are ideals of the semigroup $\mathbb{R} \times \mathbb{R}_+$ under addition. It is easy to see that every ideal of $\mathbb{R} \times \mathbb{R}_+$ is of type $\mathbb{R} \times (c, \infty)$, or, $\mathbb{R} \times [c, \infty)$, $c \in \mathbb{R}_+$. The ideals of the semigroup $\mathbb{Z} \times \mathbb{Z}_+$ are of similar type.

(d) The sets $\big(\mathbb{Z}_+ + \mathbb{Z}_+\beta\big) \setminus \mathbb{Z}_+$ and $\big(\mathbb{Z}_+ + \mathbb{Z}_+\beta\big) \setminus \mathbb{Z}_+ b$ are ideals in the semigroup $\big(\mathbb{Z}_+ + \mathbb{Z}_+\beta\big)$ under addition, where β is an irrational number.

Proposition 3.4.5. *Any ideal J of a semigroup S that meets the group kernel G_S coincides with S. Therefore, the set $J_S = S \setminus G_S$ contains any proper ideal of S.*

Proof. Assume that $J \cap G_S \neq \emptyset$, and let $a \in J \cap G_S$. Then a possesses an inverse element $-a \in S$, and therefore $0 = a - a = a + (-a) \in J$. Consequently, $S = S + 0 \subset J \subset S$, thus $J = S$ as claimed. □

Example 3.4.6. (a) Let $J \subset S$ be an ideal of a semigroup S. For any element $a \in S$, the set $a + J = \{a + b,\ b \in J\} \subset J$ is also an ideal of S.

(b) The complement $J_S = S \setminus G_S$ of the group kernel of S is an ideal of S. Indeed, if $a \in J_S$ and $b \in S$ is such that $a + b \notin J_S$, then $a + b \in G_S$, and hence it possesses an inverse element $-(a + b) \in G_S$. Hence $0 = (a + b) + \big(-(a + b)\big) = a + \big(b - (a + b)\big)$, i.e. a is an invertible element of S, which is impossible since $a \in J_S = S \setminus G_S$.

Definition 3.4.7. A proper ideal of a semigroup S is called

(a) a *maximal ideal* of S if it is not contained in any other proper ideal of S.

(b) a *minimal ideal* of S if it does not contain any other ideal of S.

Theorem 3.4.8. *Let S be a semigroup with* 0*. Then:*

(i) *$J_S = S \setminus G_S$ is the only maximal ideal of S.*

(ii) *If the cancellation law holds on S, then it does not have proper minimal ideals.*

Proof. (i) As we saw in Example 3.4.6(b), J_S is an ideal in S. It is a maximal ideal, in fact the only maximal ideal in S, since, by Proposition 3.4.5, J_S contains every proper ideal of S.

(ii) Suppose that the cancellation law holds on S, and let J be a proper minimal ideal of S. Fix an $a \in J$ and consider the ideal $a + J \subset J$. By the minimality of J we have that $J = a + J$. Therefore, $a = a + b$ for some $b \in J$. By the cancellation property, $b = 0$, thus $0 \in J$ and therefore, $J = S$, contrary to hypothesis on J. □

Definition 3.4.9. Let S be a semigroup with the cancellation property.

(i) The *strong hull* $[S]_s$ of S is the set of elements $a \in \Gamma_S$ for which there is an $m_a \in \mathbb{N}$ such that $m_a a \in S$. If $[S]_s = S$, then S is called a *strongly enhanced semigroup.*

(ii) The *weak hull* $[S]_w$ of S is the set of elements $a \in \Gamma_S$ for which there is an $n \in \mathbb{N}$ such that $ma \in S$ for every $m \geq n$. If $[S]_w = S$, then S is called a *weakly enhanced semigroup.*

Similarly, by replacing Γ_S by any semigroup P containing S, one can define the *strong hull,* $[S]_s^P$*, of* S *in* P, the *weak hull,* $[S]_w^P$*, of* S *in* P, and the notion of strongly and weakly enhanced semigroups in P. One can readily see that $S \subset [S]_w \subset [S]_s \subset \Gamma_S$, $[S]_s = [S]_s^{\Gamma_S}$, and $[S]_w = [S]_w^{\Gamma_S}$.

Example 3.4.10. Consider the semigroup $S = \{(m,n)\colon m,n > 0\} \cup \{(0,n)\colon n \geq 2\} \cup \{(2k,0)\colon k \geq 0\} \subset \mathbb{Z}^2$. It is easy to see that $\Gamma_S = \mathbb{Z}^2$, $[S]_w = \{(m,n)\colon m \geq 0,\ n > 0\} \cup \{(2k,0)\colon k \geq 0\}$, and $[S]_s = \{(m,n)\colon m \geq 0,\ n \geq 0\}$. Hence, $S \neq [S]_w \neq [S]_s$, and therefore the semigroup S is neither weakly, nor strongly enhanced.

Proposition 3.4.11. *Let* S *be a semigroup, and* $a \in \Gamma_S \setminus S$*. If* $m,n \in \mathbb{N}$ *are two relatively prime integers, such that the elements* na *and* ma *belong to* S*, then* $a \in [S]_w$*.*

Proof. Without loss of generality we can assume that $m > n > 1$. We claim that under the hypotheses there is an $N \in \mathbb{N}$ such that every natural number $M \geq N$ can be expressed in the form $M = rn + sm$ with some $r,s \in \mathbb{N}$. If $m = n+1$, then we can choose $N = n\,(n+1)+1 = n^2+(n+1)$. If $M \geq N$ and $M = rn+s\,(n+1)$ for some $r,s \in \mathbb{N}$, then $M+1 = (r-1)\,n+(s+1)(n+1)$, which proves the claim in the case when $r > 1$. If $r = 1$, then $M = n+s\,(n+1)$, and hence $M+1 = (s+1)(n+1)$. Thus $M+1 = (s+1)(n+1) = (s+1-n)(n+1)+n\,(n+1)$. Note that $s+1-n > 0$ since $(s+1)(n+1) = M+1 \geq N > n\,(n+1)$. In general, $kn + lm = 1$ for some $k,l \in \mathbb{Z}\setminus\{0\}$, since m and n are relatively prime integers. Without loss of generality we may assume that $k > 0$. If so, then $l < 0$ and $kn = (-l)\,m+1$. Now the previous argument applies to the integers $(-l)\,m$ and $(-l)\,m+1 = kn$, and we obtain that every natural number $M > \big((-l)\,m\big)\big((-l)\,m+1\big) = \big((-l)\,m\big)\,kn$ can be expressed in the form $M = r\big((-l)\,m\big) + s\,(kn) = \big(r\,(-l)\big)\,m + (sk)\,n$ with some $r,s \in \mathbb{N}$.

Assume now that m,n are relatively prime natural numbers, such that na and ma belong to S. As we saw above, there is an $N \in \mathbb{N}$ such that every natural number $M > N$ has the form $M = rn + sm$ with some $r,s \in \mathbb{N}$. Therefore, $Ma = (rn + sm)\,a = r(na) + s(ma) \in S$ for every $M > N$, and consequently $a \in [S]_w$. □

Definition 3.4.12. A complex-valued function φ on S is called a *semicharacter* of S, if $\varphi(0) = 1$, $|\varphi| \leq 1$ and $\varphi(a+b) = \varphi(a)\,\varphi(b)$ for any $a,b \in S$.

Semicharacters on S are complex homomorphisms of G into $\overline{\mathbb{D}}$, the closed unit disc in $\mathbb{C}$, considered as a semigroup under multiplication. If φ is a semicharacter of S and $\iota \in I_S$ is an idempotent, then $\varphi(\iota) = \varphi(\iota + \iota) = \varphi(\iota)\,\varphi(\iota)$, wherefrom $\varphi(\iota)$ is either 0 or 1. If $\varphi(\iota) = 0$, then necessarily, $\varphi \equiv 0$ on the subgroup G_S^ι, since $\varphi(a) = \varphi(a + \iota) = \varphi(a)\,\varphi(\iota) = 0$ for every $a \in G_S^\iota$. If $\varphi(\iota) = 1$, then $|\varphi| \equiv 1$ on G_S^ι. Indeed, if $a \in G_S^\iota$, then $a + b = \iota$ for some $b \in G_S^\iota$. Hence $1 = \varphi(\iota) = \varphi(a + b) = \varphi(a)\,\varphi(b)$, and therefore, $|\varphi(a)| = |\varphi(b)| = 1$, since $|\varphi(a)|, |\varphi(b)| \leq 1$. In particular, $|\varphi| \equiv 1$ on the group kernel G_S of S for any semicharacter φ of S, since $\varphi(0) = 1$.

Example 3.4.13. (a) The function $1\big|_S = \varkappa_S \equiv 1$ is a semicharacter of S, called the *trivial semicharacter* of S.

(b) For any $x \in \mathbb{R}$ the function e^{-x} is a semicharacter of the semigroup $\mathbb{R}_+ = [0, \infty)$ endowed with addition. Likewise, the function $n \longmapsto e^{-n}$ is a semicharacter of the semigroup $\mathbb{Z}_+$ under addition.

(c) The characteristic function $\varkappa_{G_S}$ of the group kernel G_S of S, defined by

$$\varkappa_{G_S}(a) = \begin{cases} 1 & \text{when } a \in G_S, \\ 0 & \text{when } a \in J_S = S \setminus G_S \end{cases}$$

is a semicharacter of S.

A semicharacter ω of S with $\omega^2 = \omega$ is called an *idempotent semicharacter* of S. We denote by $\mathcal{I}_S$ the set of all idempotent semicharacters of S that are not identically equal to 0 on S. It is easy to see that 0 or 1 are the only possible values of any idempotent semicharacter.

Let $\mathrm{Null}\,(\omega)$ be the *null-set* of a semicharacter φ of S, i.e. $\mathrm{Null}\,(\varphi) = \{a \in S \colon \varphi(a) = 0\}$, and let $\mathrm{supp}\,(\varphi)$ be the support set of φ, i.e. $\mathrm{supp}\,(\varphi) = \{a \in S \colon \varphi(a) \neq 0\}$. The proof of the following lemma is straightforward.

Lemma 3.4.14. *Let ω be an idempotent semicharacter of S. Then*

(a) $\mathrm{supp}\,(\omega)$ *is a subsemigroup of S,*

(b) $\mathrm{Null}\,(\omega)$ *is an ideal of S,*

(c) $\mathrm{supp}\,(\omega) \cup \mathrm{Null}\,(\omega) = S$,

(d) $\mathrm{supp}\,(\omega) \cap \mathrm{Null}\,(\omega) = \emptyset$.

We define a natural partial order in the set $\mathcal{I}_S$, by letting $\omega_1 \prec \omega_2$ if $\omega_1\omega_2 = \omega_1$. Clearly, $\omega_1 \prec \omega_2$ if and only if $\mathrm{supp}\,(\omega)_1 \subset \mathrm{supp}\,(\omega)_2$, or, equivalently, if $\mathrm{Null}\,(\omega_1) \supset \mathrm{Null}\,(\omega_2)$. The unit semicharacter $\varkappa_S \equiv 1$ is a maximal element of $\mathcal{I}_S$, while the semicharacter $\varkappa_{G_S}$ from Example 3.4.13(c) is the minimal element of $\mathcal{I}_S$.

Example 3.4.15. (a) Consider the subsemigroup $\mathbb{R}_+$ endowed with addition. The origin 0 is the only subsemigroup of $\mathbb{R}_+$, whose complement is an ideal of $\mathbb{R}_+$.

Lemma 3.4.14 shows that the only non-trivial idempotent semicharacter of $\mathbb{R}_+$ is the characteristic function, $\delta_0 = \varkappa_{\{0\}}$, of the origin 0. The same result holds for the semigroup $\mathbb{Z}_+$.

(b) The sets $\mathbb{R}_+ \times \{0\}$, $\{0\} \times \mathbb{R}_+$ and the origin $(0,0)$ are the only semigroups of $\mathbb{R}_+ \times \mathbb{R}_+$ under addition, whose complement in $\mathbb{R}_+ \times \mathbb{R}_+$ is an ideal. By Lemma 3.4.14 the characteristic functions, $\varkappa_{\mathbb{R}_+ \times \{0\}}$, $\varkappa_{\{0\} \times \mathbb{R}_+}$, and $\delta_{(0,0)} = \varkappa_{\{(0,0)\}}$ of these sets are the only non-trivial idempotent semicharacters of $\mathbb{R}_+ \times \mathbb{R}_+$ endowed with addition. A similar result holds for the semigroup $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(c) The set $\mathbb{R} \times \{0\}$ is the only semigroup of $\mathbb{R} \times \mathbb{R}_+$ under addition whose complement in $\mathbb{R} \times \mathbb{R}_+$ is an ideal. Lemma 3.4.14 implies again that the characteristic function $\varkappa_{\mathbb{R} \times \{0\}}$ is the only non-trivial idempotent semicharacter of $\mathbb{R} \times \mathbb{R}_+$. A similar result holds for the semigroup $\mathbb{Z} \times \mathbb{Z}_+$.

(d) Let β be an irrational number. The sets $\mathbb{Z}_+$, $\mathbb{Z}_+\beta$ and $\{0\}$ are the only semigroups whose complement in the semigroup $\mathbb{Z}_+ + \mathbb{Z}_+\beta$, endowed with addition, is an ideal. Hence, their characteristic functions $\chi_{\mathbb{Z}_+}$, $\chi_{\mathbb{Z}_+\beta}$ and $\delta_0 = \chi_{\{0\}}$ are the only idempotent semicharacters of $\mathbb{Z}_+ + \mathbb{Z}_+\beta$.

3.5 The set of semicharacters

The set of semicharacters of a semigroup is a semigroup on its own. Here we present some of its basic properties. All semigroups in this section are considered to be with cancellation property and 0.

Let S be a semigroup with zero. The set $H(S, \overline{\mathbb{D}})$ of all semicharacters of S is denoted also by $H(S)$. It is easy to see that $H(S)$ is a commutative semigroup under the pointwise multiplication on S. Namely, $\big(\varphi_1\varphi_2\big)(a) = \varphi_1(a)\,\varphi_2(a)$ for every $a \in S$ and $\varphi_1, \varphi_2 \in H(S)$. The unit of $H(S)$ is the semicharacter $\chi_S \equiv 1$. Clearly, the set $\mathcal{I}_S$ of idempotent semicharacters of S is a subsemigroup of $H(S)$.

Example 3.5.1. All semigroups below are considered to be endowed with addition.

(a) The set of semicharacters of $S = \mathbb{Z}_+$ is isomorphic to the closed unit disc $\overline{\mathbb{D}}$, i.e. $H(\mathbb{Z}_+) \cong \overline{\mathbb{D}}$. Indeed, any $z \in \overline{\mathbb{D}}$ generates a semicharacter $\varphi_z \in H(\mathbb{Z}_+)$, defined by

$$\varphi_z(n) = z^n, \; n \in \mathbb{Z}_+, \; 0^0 = 1.$$

Conversely, if $\varphi \in H(\mathbb{Z}_+)$, then $\varphi(1) \in \overline{\mathbb{D}}$, and $\varphi(n) = \varphi(1)^n$. Therefore, the correspondence $z \longrightarrow \varphi_z$ is an isomorphism between $H(\mathbb{Z}_+)$ and $\overline{\mathbb{D}}$.

(b) $H\big(\mathbb{Z}_+ \times \mathbb{Z}_+\big) \cong H(\mathbb{Z}_+) \times H(\mathbb{Z}_+) \cong \overline{\mathbb{D}} \times \overline{\mathbb{D}} = \overline{\mathbb{D}}^2$, the closed bi-disc in $\mathbb{C}^2$. Functions of type

$$(n, m) \longmapsto z_1^n z_2^m,$$

where $z_1, z_2 \in \overline{\mathbb{D}}$, and $n, m \in \mathbb{Z}_+$, are typical semicharacters of $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(c) $H\big(\mathbb{Z}\times\mathbb{Z}_+\big)\cong H(\mathbb{Z})\times H(\mathbb{Z}_+)\cong\mathbb{T}\times\overline{\mathbb{D}}$, the solid torus in $\mathbb{R}^3$. Typical semicharacters of $\mathbb{Z}\times\mathbb{Z}_+$ are the functions

$$(n,m)\longmapsto z_1^n z_2^m,$$

where $|z_1|=1,\ z_2\in\overline{\mathbb{D}}$, and $n\in\mathbb{Z},\ m\in\mathbb{Z}_+$.

(d) Let β be an irrational number, and let $S=\mathbb{Z}_++\mathbb{Z}_+\beta$. It is clear that $\mathbb{Z}_++\mathbb{Z}_+\beta\cong\mathbb{Z}_+\times\mathbb{Z}_+$. Therefore, $H(\mathbb{Z}_++\mathbb{Z}_+\beta)\cong H(\mathbb{Z}_+\times\mathbb{Z}_+)\cong\overline{\mathbb{D}}^2$, the closed bi-disc in $\mathbb{C}^2$. Functions of type

$$n+m\beta\longmapsto z_1^n z_2^m,$$

where $z_1,z_2\in\overline{\mathbb{D}}$, and $n,m\in\mathbb{Z}_+$, are typical semicharacters on $\mathbb{Z}_++\mathbb{Z}_+\beta$.

(e) Likewise, $\mathbb{Z}+\mathbb{Z}_+\beta\cong\mathbb{Z}\times\mathbb{Z}_+$, and therefore, $H(\mathbb{Z}+\mathbb{Z}_+\beta)\cong H(\mathbb{Z}\times\mathbb{Z}_+)\cong\mathbb{T}\times\overline{\mathbb{D}}$, the solid torus in $\mathbb{R}^3$, where β is an irrational number. Typical semicharacters on $\mathbb{Z}+\mathbb{Z}_+\beta$ are given by

$$n+m\beta\longmapsto z_1^n z_2^m,$$

where $|z_1|=1,\ z_2\in\overline{\mathbb{D}}$, and $n\in\mathbb{Z},\ m\in\mathbb{Z}_+$.

A semicharacter φ of S with $\big|\varphi(a)\big|=1$ for every $a\in S$ is called a *character* of S. Clearly, any character is a homomorphism of S into the unit circle $\mathbb{T}$. The set $H(S,\mathbb{T})$ of characters of S is a subgroup of $H(S)$ with unit element $\varkappa_S\equiv 1$. Every $\varphi\in H(S,\mathbb{T})$ has an inverse element, namely the character defined by $\varphi^{-1}(a)=1/\varphi(a)=\overline{\varphi(a)},\ a\in S$.

Proposition 3.5.2. (a) *The group $H(S,\mathbb{T})$ of characters of S coincides with the group kernel of the semigroup $H(S)$.*

(b) *$H(S)\setminus H(S,\mathbb{T})$ is the only maximal ideal of the semigroup $H(S)$.*

(c) *If S is not a group, then $H(S)\cdot\varkappa_{G_S}$ is the only minimal ideal of $H(S)$.*

Proof. (a) By definition, a semicharacter φ of S belongs to the group kernel of the semigroup $H(S)$ if and only if there is a $\psi\in H(S)$ with $\varphi\psi=1$. This happens if and only if $|\varphi|\equiv 1$, i.e. if φ is a character of S.

(b) That $H(S)\setminus H(S,\mathbb{T})$ is the only maximal ideal of the semigroup $H(S)$ follows directly from part (a) and Theorem 3.4.8(iii).

(c) Clearly, $H(S)\cdot\varkappa_{G_S}$ is an ideal of $H(S)$. Let $J\subset H(S)\cdot\varkappa_{G_S}$ be an ideal of $H(S)$. The conjugate function $\overline{\varphi}:\overline{\varphi}(a)=\overline{\varphi(a)}$ of any $\varphi\in J$ is also a semicharacter of S. Therefore $|\varphi|^2=\varphi\overline{\varphi}\in J$. From $|\varphi|^2\equiv 1$ on G_S, we have that $|\varphi|^2\varkappa_{G_S}=\varkappa_{G_S}$, since $\varkappa_{G_S}$ is the characteristic function of G_S. Hence, $\varkappa_{G_S}=|\varphi|^2\chi_{G_S}\in J$, therefore $H(S)\cdot\varkappa_{G_S}\subset J\subset H(S)\cdot\varkappa_{G_S}$, and thus, $H(S)\cdot\varkappa_{G_S}=J$. Consequently, $H(S)\cdot\varkappa_{G_S}$ is the minimal ideal of $H(S)$. ☐

Proposition 3.5.2 and Theorem 3.4.8(ii) below imply that, in general, the cancellation law does not hold on the semigroup $H(S)$.

Proposition 3.5.3. *The group of characters, $H(S,\mathbb{T})$, of S is isomorphic to the dual group $\widehat{\Gamma}_S$ of the group envelope Γ_S of S.*

Proof. Every character $\chi \in H(S,\mathbb{T})$ can be extended uniquely to a character on Γ_S by

$$\widetilde{\chi}(b) = \frac{\chi(a)}{\chi(c)} \text{ whenever } b = a - c \in \Gamma_S,\ a, c \in S.$$

The extended function, $\widetilde{\chi}$, is of modulus 1, and is multiplicative on Γ_S. Indeed, if $b, b_1 \in \Gamma$ and $b = a - c,\ b_1 = a_1 - c_1$, where $a, a_1, c, c_1 \in S$, then

$$\begin{aligned}\widetilde{\chi}(b+b_1) &= \widetilde{\chi}((a+a_1)-(c+c_1)) = \frac{\chi(a+a_1)}{\chi(c+c_1)}\\ &= \frac{\chi(a)\,\chi(a_1)}{\chi(c)\,\chi(c_1)} = \Big(\frac{\chi(a)}{\chi(c)}\Big)\Big(\frac{\chi(a_1)}{\chi(c_1)}\Big) = \widetilde{\chi}(b)\,\widetilde{\chi}(b_1).\end{aligned}$$

Thus, the extended function $\widetilde{\chi}$ is a character of the group $\Gamma_S \supset S$, as claimed. Since the restriction of every character of Γ_S on S is a character, we conclude that $H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$, as claimed. □

For every semicharacter $\varphi \in H(S)$ the modulus $|\varphi|\colon |\varphi|(a) = |\varphi(a)|$ is a non-negative semicharacter of S. Indeed, it is easy to see that $0 \le |\varphi|(a) \le 1$, and $|\varphi|(a+b) = |\varphi|(a)\,|\varphi|(b)$ for any $a, b \in S$.

Theorem 3.5.4. *For every φ in $H(S)$ there is a character $\chi \in H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$, such that φ can be expressed in the form $\varphi = |\varphi|\,\chi$ on S.*

Proof. If $S_1 = \operatorname{supp}(\varphi) = S \setminus \operatorname{Null}(\varphi)$, let $\Gamma_S = S - S$ and $\Gamma_{S_1} = S_1 - S_1$ be the group envelopes of S and S_1 respectively, equipped with the discrete topologies. As a character on S_1, the function $\varphi'\colon S_1 \to \mathbb{T}$, defined by $\varphi'(a) = \varphi(a)/|\varphi(a)|$ can be extended, by Proposition 3.5.3, to a character $\widetilde{\varphi}'$ on the group Γ_{S_1}. Further, as a character of the group Γ_{S_1}, $\widetilde{\varphi}'$ can be extended on the larger group $\Gamma_S \supset \Gamma_{S_1}$, by Theorem 3.1.11. The function $\chi = \widetilde{\varphi}'|_S$ is a character of S, and, clearly $\varphi = |\varphi|\,\chi$ on S. □

The expression $\varphi = |\varphi|\,\chi$ from Theorem 3.5.4 is called a *polar decomposition* of $\varphi \in H(S)$ (cf. [AS1]). In general it is not unique. However, if $\varphi(a) \neq 0$ for every $a \in S$, it is, as the following proposition asserts.

Proposition 3.5.5. *The polar decomposition $\varphi = |\varphi|\,\chi$ of a semicharacter $\varphi \in H(S)$ is uniquely determined if and only if $\varphi(a) \neq 0$ for any $a \in S$.*

Proof. Let $\varphi = |\varphi|\,\chi_1 = |\varphi|\,\chi_2$ for some $\chi_1, \chi_2 \in \widehat{\Gamma}_S$. If $\varphi(a) \neq 0$ on S, then $\chi_1(a) = \chi_2(a)$ for every $a \in S$. Consequently, $\chi_1|_S = \chi_2|_S$, thus $\chi_1 = \chi_2$, since S generates Γ_S.

Conversely, assume on the contrary, that $\varphi(a) = 0$ for some $a \in S$, i.e. that $\mathrm{Null}\,(\varphi) \neq \emptyset$. Denote by Γ_φ the group envelope of the semigroup $\mathrm{supp}\,(\varphi) = S \backslash \mathrm{Null}\,(\varphi)$ with the discrete topology, i.e. $\Gamma_\varphi = \mathrm{supp}\,(\varphi) - \mathrm{supp}\,(\varphi)$. We can extend $\varphi/|\varphi|$ to a character χ on Γ_φ. Note that $\Gamma_\varphi \neq \Gamma_S$ since $a \notin \Gamma_\varphi$. As a character on the semigroup $\Gamma_\varphi \subset \Gamma_S$, χ possesses an extension on Γ_S by Theorem 3.1.11. The set of all possible character extensions of χ on Γ_S is isomorphic to the group $\Gamma_\varphi^\perp \cong \Gamma_S/\Gamma_\varphi$, which certainly contains more than one element. Therefore, the polar decomposition of φ is not unique. □

Proposition 3.5.6. *If $S \subset P$ are semigroups with cancellation law and 0, then every semicharacter φ of S can be extended uniquely to a semicharacter on the strong hull $[S]_s^P$ of S in P.*

Proof. By Theorem 3.5.4 every $\varphi \in H(S)$ has a polar decomposition $\varphi = |\varphi|\,\chi$ for some character χ of Γ_S. Hence φ is extendable on $[S]_s^P$ as an element of $H\big([S]_s^P\big)$ if and only if $|\varphi|$ is extendable there. For any $a \in [S]_s^P \setminus S$ there is an $m_a \in \mathbb{N}$ such that $m_a a \in S$. The function $\big|\widetilde{\varphi}(a)\big| = \big|\varphi(m_a a)\big|^{1/m_a}$ is a semicharacter extension of $|\varphi|$ on the strong hull $[S]_s^P$ of S in P. □

As an immediate corollary from Proposition 3.5.6 and Example 3.4.15(a) we see that semigroups S whose strong hulls $[S]_s^P$ in P contain $\mathbb{R}_+$ do not have non-trivial idempotent semicharacters besides $\varkappa_{\{0\}}$.

It is easy to see that $\mathcal{I}_S = \mathcal{I}_{[S]_s^P}$ for every subsemigroup $S \subset \Gamma_S$. Indeed, any idempotent semicharacter ω of S can be extended uniquely to an idempotent semicharacter on $[S]_s^P$. Namely, for any $a \in [S]_s^P$ define $\widetilde{\omega}(a) = \omega(n_a a)$, where $n_a a \in S$. Equivalently, the extension $\widetilde{\omega}$ on $[S]_s^P$ is given by

$$\widetilde{\omega}(a) = \begin{cases} 1 & \text{when } a \in [\mathrm{supp}\ \ (\omega)]_s^P, \\ 0 & \text{when } a \in [\mathrm{Null}\,(\omega)]_s^P. \end{cases}$$

Proposition 3.5.7. *The restriction mapping on the group kernel G_S is a surjective homomorphism from the semicharacter semigroup $H(S)$ onto the dual group $\widehat{G}_S$.*

Proof. We only need to prove that $H(S)|_{G_S} = \widehat{G}_S$. Given a $\chi \in \widehat{G}_S$, consider the function φ_χ on S, defined by

$$\varphi_\chi(a) = \begin{cases} \chi(a) & \text{when } a \in G_S, \\ 0 & \text{when } a \in J_S = S \setminus G_S. \end{cases}$$

Clearly, $\varphi_\chi\big|_{G_S} = \chi$, and one can easily check that φ_χ is a semicharacter of S. □

Let $H\big(S, [0,1]\big)$ denote the semigroup of all *non-negative semicharacters* of S. Clearly $\mathcal{I}_S \subset H\big(S, [0,1]\big) \subset H(S)$. The modulus $|\varphi|$ of any semicharacter φ of S also belongs to $H\big(S, [0,1]\big)$. Given an idempotent semicharacter $\omega \in \mathcal{I}_S$, let $H_\omega(S)$ be the set of all $\varphi \in H\big(S, [0,1]\big)$ with the same support as the support of ω. Let $\Gamma_\omega \subset \Gamma_S$ be the group envelope of $\mathrm{supp}\,(\omega)$ with the discrete topology. Note that

$H_\omega\big(\operatorname{supp}(\omega)\big)$ is isomorphic to the set $H\big(\operatorname{supp}(\omega),(0,1]\big)$ of positive semicharacters of the semigroup $\operatorname{supp}(\omega)$. The uniqueness of the polar decomposition (Proposition 3.5.5) implies that $H\big(S,\overline{\mathbb{D}}^*\big)\cong H\big(S,(0,1]\big)\times\widehat{\varGamma}_S$, where $\overline{\mathbb{D}}^*=\overline{\mathbb{D}}\setminus\{0\}$. Likewise,

$$H\big(\operatorname{supp}(\omega),\overline{\mathbb{D}}^*\big)\cong H\big(\operatorname{supp}(\omega),(0,1]\big)\times\widehat{\varGamma}_\omega=H_\omega(S)\times\widehat{\varGamma}_\omega$$

for every idempotent semicharacter $\omega\in\mathcal{I}_S$.

The symbol $\sqcup$ is used below to indicate disjoint unions of sets.

Proposition 3.5.8. *The semigroup $H(S)$ of semicharacters of S can be expressed as the disjoint union*

$$H(S)\cong\bigsqcup_{\omega\in\mathcal{I}_S}H\big(\operatorname{supp}(\omega),\overline{\mathbb{D}}^*\big)\cong\bigsqcup_{\omega\in\mathcal{I}_S}H_\omega(S)\times\widehat{\varGamma}_\omega\cong\bigsqcup_{\omega\in\mathcal{I}_S}H\big(\operatorname{supp}(\omega),(0,1]\big)\times\widehat{\varGamma}_\omega. \tag{3.9}$$

Proof. Note that the polar decomposition, $\varrho\,\chi$, $\varrho\in H\big(\operatorname{supp}(\omega),(0,1]\big)$, $\chi\in\widehat{\varGamma}_\omega$, of non-vanishing on $\operatorname{supp}(\omega)$ semicharacters is uniquely defined, and hence $\big\{\varrho\,\chi\colon\ \varrho\in H\big(\operatorname{supp}(\omega),(0,1]\big),\ \chi\in\widehat{\varGamma}_\omega\big\}\cong H_\omega(S)\times\widehat{\varGamma}_\omega$. Therefore,

$$\begin{aligned}H(S)&=\bigcup\{\varrho\,\chi\colon\ \varrho\in H\big(S,[0,1]\big),\ \chi\in\widehat{\varGamma}_S\big\}=\bigsqcup_{\omega\in\mathcal{I}_S}\{\varrho\,\chi\colon\ \varrho\in H_\omega(S),\ \chi\in\widehat{\varGamma}_S\big\}\\&\cong\bigsqcup_{\omega\in\mathcal{I}_S}\{\varrho\,\chi\colon\ \varrho\in H_\omega(S),\ \chi\in\widehat{\varGamma}_\omega\big\}\cong\bigsqcup_{\omega\in\mathcal{I}_S}H_\omega(S)\times\widehat{\varGamma}_\omega.\end{aligned}$$

□

Let $\varkappa_{G_S}$ and $\varkappa_S\equiv 1$ be the only idempotent semicharacters of S. Since their supports are G_S and S correspondingly, we have $\varGamma_{\varkappa_{G_S}}=G_S$ and $\varGamma_{\varkappa_S}=\varGamma_S$. Hence, $H_{\varkappa_{G_S}}(S)\cong H\big(G_S,(0,1]\big)\cong\{\varkappa_{G_S}\}$, and $H_{\varkappa_S}(S)\cong H\big(S,(0,1]\big)$. Therefore, $H\big(\operatorname{supp}(\varkappa_{G_S}),\overline{\mathbb{D}}^*\big)\cong H\big(\operatorname{supp}(\varkappa_{G_S}),(0,1]\big)\times\widehat{\varGamma}_{\varkappa_{G_S}}\cong H_{\varkappa_{G_S}}(S)\times\widehat{G}_S\cong\{\varkappa_{G_S}\}\times\widehat{G}_S$, while $H(\operatorname{supp}\big(\varkappa_S),\overline{\mathbb{D}}^*\big)=H(S,\overline{\mathbb{D}}^*)\cong\big(H(S),(0,1]\big)\times\widehat{\varGamma}_S$ respectively. In this case the identity (3.9) becomes

$$\begin{aligned}H(S)&\cong\big(H\big(\operatorname{supp}(\varkappa_{G_S}),(0,1]\big)\times\widehat{\varGamma}_{\varkappa_{G_S}}\big)\sqcup\big(H\big(\operatorname{supp}(\varkappa_S),(0,1]\big)\times\widehat{\varGamma}_{\varkappa_S}\big)\\&=\big(\{\varkappa_{G_S}\}\times\widehat{G}_S\big)\sqcup\big(H(S,(0,1])\times\widehat{\varGamma}_S\big).\end{aligned} \tag{3.10}$$

If, in addition, S has a trivial group kernel, so that $G_S=\{0\}$, then $\{\varkappa_{G_S}\}\times\widehat{G}_S=(\varkappa_{\{0\}},1)$, thus $H\big(S,[0,1]\big)\setminus H\big(S,(0,1]\big)=\{\varkappa_{\{0\}}\}$, and (3.10) becomes

$$H(S)\cong(\varkappa_{\{0\}},1)\sqcup\big(H(S,(0,1])\times\widehat{\varGamma}_S\big)\cong\big(H(S,[0,1])\times\widehat{\varGamma}_S\big)/\big(\{0\}\times\widehat{\varGamma}_S\big), \tag{3.11}$$

where we regard the points in the set $\big(\{0\}\times\widehat{\varGamma}_S\big)/\big(\{0\}\times\widehat{\varGamma}_S\big)$ to be identified with the semicharacter $(\varkappa_{\{0\}},1)$. Clearly, $\varkappa_{\{0\}}$ is the delta function δ_0.

Definition 3.5.9. The set $\overline{\mathbb{D}}_G=([0,1]\times G)/(\{0\}\times G)$ is called the *G-disc*, or, the (*big*) disc over G. The set $\overline{\mathbb{D}}_G^*=(0,1]\times G$ is the *punctured G-disc.*

The points of the G-disc $\overline{\mathbb{D}}_G$ will be denoted by $r \diamond g$, where $r \in [0,1]$ and $g \in G$, under the agreement that we regard the points $0 \diamond g$, $g \in G$ as identified. The point obtained in this way is called the *origin* of the G-disc $\overline{\mathbb{D}}_G$ and is denoted by ω, i.e. $0 \diamond g = \omega$ for every $g \in G$. Therefore, $\overline{\mathbb{D}}_G = [0,1] \diamond G / 0 \diamond G$. If we denote $\omega = 0 \diamond G / 0 \diamond G \in \overline{\mathbb{D}}_G$, then $\overline{\mathbb{D}}_G^* = \overline{\mathbb{D}_G} \setminus \{\omega\}$. The points of type $1 \diamond g$, $g \in G$, are denoted simply by g. The non-negative number r is called the *modulus* of the point $r \diamond g$, and we write $|r \diamond g| = r$. Clearly, the topological boundary of the G-disc $\overline{\mathbb{D}}_G$ is the set $1 \diamond G \cong G$.

Let the group kernel G_S of S be $\{0\}$, and $H\big(\operatorname{supp}(\omega), (0,1]\big) \cong (0,1]$ for every idempotent semicharacter $\omega \in \mathcal{I}_S$, $\omega \neq \delta_0$. In this case the identity (3.9) becomes

$$H(S) = \big\{(\varkappa_{\{0\}}, \varkappa_S)\big\} \sqcup \bigsqcup_{\omega \in \mathcal{I}_S \setminus \varkappa_{\{0\}}} (0,1] \times \widehat{\varGamma}_\omega = \big\{(\varkappa_{\{0\}}, \varkappa_S)\big\} \sqcup \bigsqcup_{\omega \in \mathcal{I}_S \setminus \varkappa_{\{0\}}} \overline{\mathbb{D}}^*_{\widehat{\varGamma}_\omega}, \quad (3.12)$$

i.e. $H(S)$ is the union of a single point and a family of punctured $\widehat{\varGamma}_\omega$-discs $\overline{\mathbb{D}}^*_{\widehat{\varGamma}_\omega}$, $\omega \in \mathcal{I}_S \setminus \varkappa_{\{0\}}$.

Example 3.5.10. All semigroups below are considered to be endowed with addition.

(a) If S is the semigroup $S = \mathbb{N} \times \mathbb{N} \cup (0,0) \subset \mathbb{Z}^2$, then $G_S = \{(0,0)\}$, and $\varGamma_S = \mathbb{Z}^2$. The only non-trivial idempotent semicharacter of S is $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$. According to (3.11),

$$\begin{aligned} H(S) &\cong \big\{(\varkappa_{\{(0,0)\}}, 1)\big\} \sqcup \big(H(S,(0,1]) \times \widehat{\varGamma}_S\big) \\ &\cong \big\{(\varkappa_{\{(0,0)\}}, 1)\big\} \sqcup \big(H(\mathbb{N} \times \mathbb{N} \cup (0,0), (0,1]) \times \widehat{\mathbb{Z}^2}\big) \\ &\cong \big\{((0,0),(1,1))\big\} \sqcup \big((0,1] \times \mathbb{T}\big)^2 \\ &\cong \{(0,0)\} \sqcup (\overline{\mathbb{D}}^*)^2 \cong \overline{\mathbb{D}}^2 / \big((\{0\} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \{0\})\big). \end{aligned}$$

A typical non-trivial semicharacter of $\mathbb{N} \times \mathbb{N} \cup (0,0) \subset \mathbb{Z}^2$ is given by

$$(n,m) \longmapsto z_1^n z_2^m,$$

where $z_1,\ z_2 \in \overline{\mathbb{D}}$, and $n, m \in \mathbb{N}$. Observe that the points $(z_1, 0)$ and $(0, z_2)$ generate the semicharacter $\varkappa_{(0,0)} = \delta_{(0,0)}$, then the points of the set $\overline{\mathbb{D}}^2$ belonging to $\{0\} \times \overline{\mathbb{D}}$ and $\overline{\mathbb{D}} \times \{0\}$ are identified.

(b) The semigroup $S = \mathbb{Z} \times \mathbb{N} \cup (0,0)$ has also a single non-trivial idempotent semicharacter, namely, $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$, while $G_S = \{(0,0)\}$, and $\varGamma_S = \mathbb{Z}^2$. By

(3.10) we have

$$\begin{aligned}H(S) &\cong \{(\varkappa_{\{(0,0)\}}, \varkappa_S)\} \sqcup H\big(S,(0,1]\big) \times \widehat{\varGamma}_S\\ &\cong \{(\varkappa_{\{(0,0)\}}, 1)\} \sqcup H\big(\mathbb{Z}\times\mathbb{N}\cup(0,0),(0,1]\big) \times \widehat{\mathbb{Z}^2}\\ &\cong \big\{(\varkappa_{\{(0,0)\}}, (1,1))\big\} \sqcup H\big(\mathbb{Z},(0,1]\big) \times H\big(\mathbb{N},(0,1]\big) \times \mathbb{T}^2\\ &\cong \big\{\{1\}\times\{0\}\times\{(1,1)\}\big\} \sqcup \{1\}\times(0,1]\times\mathbb{T}^2\\ &\cong \big\{\{0\}\times\{(1,1)\}\big\} \sqcup (0,1]\times\mathbb{T}^2\\ &\cong \big([0,1]\times\mathbb{T}^2\big)/\big(\{0\}\times\mathbb{T}^2\big) \cong \overline{\mathbb{D}}_{\mathbb{T}^2},\end{aligned}$$

the closed $\mathbb{T}^2$-disc. Consequently, $H\big(\mathbb{Z}\times\mathbb{N}\cup(0,0)\big)$ is bijective to the closed $\mathbb{T}^2$-disc in $\mathbb{R}^3$. Typical non-trivial semicharacters of S are of type

$$(n,m) \longmapsto z_1^n z_2^m,$$

where $|z_1| = 1$, $z_2 \in \overline{\mathbb{D}}$, and $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Note that every point of type $(z_1, 0) \in \mathbb{T}\times\{0\}$ generates the semicharacter $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$, which implies that the points of the set $\mathbb{T}\times\overline{\mathbb{D}}$ belonging to $\mathbb{T}\times\{0\}$ are identified.

(c) Non-trivial idempotent semicharacters of the semigroup $S = \mathbb{Z}\times\mathbb{N}\cup\mathbb{Z}_+\times\{0\}$ are $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$ and $\varkappa_{\mathbb{Z}_+\times\{0\}}$, while $G_S = \{(0,0)\}$, and $\varGamma_S = \mathbb{Z}^2$. By (3.12), we obtain

$$\begin{aligned}H(S) &\cong \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup H\big(\mathbb{Z}_+,(0,1]\big) \times \widehat{\mathbb{Z}} \sqcup H\big(S,(0,1]\big) \times \widehat{\varGamma}_S\\ &= \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup (0,1]\times\mathbb{T} \sqcup H\big(S,(0,1]\big) \times \widehat{\mathbb{Z}^2}\\ &\cong \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup \overline{\mathbb{D}}^* \sqcup H\big(S,(0,1]\big) \times \mathbb{T}^2 \cong \overline{\mathbb{D}} \sqcup (0,1]\times\mathbb{T}^2.\end{aligned}$$

Typical non-trivial examples of semicharacters on S are

$$(n,m) \longmapsto z_1^n z_2^m, \text{ where } |z_1| = 1,\ z_2 \in \overline{\mathbb{D}},\ n \in \mathbb{Z},\ m \in \mathbb{N}$$

$$\text{and } n \longmapsto z_1^n, \text{ where } z_1 \in \overline{\mathbb{D}},\ n \in \mathbb{Z}_+.$$

Proposition 3.5.11. *The set of semicharacters on $\mathbb{R}_+ = \mathbb{R}\cap[0,\infty)$ under addition is bijective to the G-disc $\overline{\mathbb{D}}_G$, where G is the Bohr compactification, $G = b(\mathbb{R})$, of $\mathbb{R}$, i.e.*

$$H(\mathbb{R}_+) \cong \overline{\mathbb{D}}_{b(\mathbb{R})}. \tag{3.13}$$

Proof. Clearly, $\varGamma_{\mathbb{R}_+} = \mathbb{R}$. The only non-trivial idempotent semicharacters of $S = \mathbb{R}_+$ is $\varkappa_{\{0\}}$. In addition, $H\big(\mathbb{R}_+,[0,1]\big) \cong [0,1]$. Indeed, for every $\rho \in [0,1]$ the function $\varphi(x) = \rho^x$ belongs to $H\big(\mathbb{R}_+,[0,1]\big)$. Conversely, we claim that every $\varphi \in H\big(\mathbb{R}_+,[0,1]\big)$ is of this type. Clearly, $\rho = \varphi(1) \in [0,1]$. For any $m, n \in \mathbb{N}$ we have $\varphi(m/n)^n = \varphi\big((m/n)\,n\big) = \varphi(m) = \varphi(1)^m$, and therefore, $\varphi(m/n) = \varphi(1)^{m/n}$. If

λ is an irrational number, and $p/q < \lambda < m/n$, then $\varphi(p/q) > \varphi(\lambda) > \varphi(m/n)$, and hence

$$\begin{aligned}\varphi(1)^{\lambda} &= \varphi(1)^{\sup_{p/q}\{p/q<\lambda\}} = \inf_{p/q<\lambda} \varphi(1)^{p/q} = \inf_{p/q<\lambda} \varphi(p/q) \geq \varphi(\lambda)\\ &\geq \sup_{m/n>\lambda} \varphi(m/n) = \sup_{m/n>\lambda} \varphi(1)^{m/n} = \varphi(1)^{\inf_{m/n}\{m/n<\lambda\}} = \varphi(1)^{\lambda}.\end{aligned}$$

Therefore, $\varphi(\lambda) = \rho^{\lambda}$ with $\rho = \varphi(1) \in [0,1]$. Hence, $H\big(\mathbb{R}_+, [0,1]\big) \cong [0,1]$, as claimed. Together with (3.11) this implies that

$$H(\mathbb{R}_+) \cong \overline{\mathbb{D}}_{\widehat{\mathbb{R}_d}} = \overline{\mathbb{D}}_{b(\mathbb{R})},$$

as claimed. □

It is easy to see that if the point $r \diamond g \in \overline{\mathbb{D}}_G$ corresponds to a semicharacter $\varphi \in H(\mathbb{R}_+)$ with a polar decomposition $|\varphi|\,\chi$, where $|\varphi| \in H\big(\mathbb{R}_+, [0,1]\big)$ and $\chi \in \widehat{\mathbb{R}}_d$, then to the semicharacter $|\varphi| \in H\big(\mathbb{R}_+, [0,1]\big) \cong [0,1]$ corresponds the point $r \diamond 1 \in \overline{\mathbb{D}}_G$, $r \in [0,1]$, while to the character χ corresponds the point $1 \diamond g \in \overline{\mathbb{D}}_G$.

A similar result holds for any semigroup of type Γ_+ with $\Gamma \subset \mathbb{R}$.

Proposition 3.5.12. *Let Γ be a subgroup of $\mathbb{R}_d$, and let $\Gamma_+ = \Gamma \cap [0,\infty)$. The set of semicharacters $H(\Gamma_+)$ is bijective to the G-disc $\overline{\mathbb{D}}_G = [0,1] \diamond G/\{0\} \diamond G$, where $G = \widehat{\Gamma}$, i.e.*

$$H(\Gamma_+) \cong \overline{\mathbb{D}}_G = \overline{\mathbb{D}}_{\widehat{\Gamma}}. \tag{3.14}$$

Proof. Let Γ be dense in $\mathbb{R}$. Every semicharacter $\varphi \in H(\Gamma_+, [0,1])$ can be extended to a semicharacter $\widetilde{\varphi}$ in $H(\mathbb{R}_+, [0,1])$ by setting, say, $\widetilde{\varphi}(x) = \inf\big\{\varphi(a)\colon a \leq x,\ a \in \Gamma_+\big\}$. This extension is unique, since if two semicharacters in $H(\mathbb{R}_+, [0,1])$ coincide on a dense subset of $\mathbb{R}_+$, they are equal. Hence $H(\Gamma_+, [0,1]) \cong H(\mathbb{R}_+, [0,1]) \cong [0,1]$. Since the group kernel of Γ_+ is $\{0\}$, and Γ_+ does not have non-trivial idempotent semicharacters, (3.11) implies that

$$H(\Gamma_+) \cong \big(H\big(\Gamma_+, [0,1]\big) \times \widehat{\Gamma}\big)/\big(\{0\} \times \widehat{\Gamma}\big) \cong \big([0,1] \times \widehat{\Gamma}\big)/\big(\{0\} \times \widehat{\Gamma}\big) = \overline{\mathbb{D}}_{\widehat{\Gamma}}.$$

If Γ is not dense in $\mathbb{R}$, then $\Gamma \cong \mathbb{Z}$, and $H(\mathbb{Z}_+) \cong \overline{\mathbb{D}}$, as we saw in Example 3.5.1(a). □

Corollary 3.5.13. *If S is a semigroup in $\mathbb{R}_+$ such that $[S]_s = \Gamma_+$ for some subgroup Γ of $\mathbb{R}$, then $H(S)$ is bijective to the G-disc $\overline{\mathbb{D}}_G$, where $G = \widehat{\Gamma}$.*

Proof. Indeed, Proposition 3.5.12 implies, $H(S) = H\big([S]_s\big) = H(\Gamma_+) \cong \overline{\mathbb{D}}_G$. □

Example 3.5.14. All semigroups below are considered to be endowed with addition.

(a) $H\big(\mathbb{R}_+ \times \mathbb{R}_+\big) \cong H(\mathbb{R}_+) \times H(\mathbb{R}_+) \cong \overline{\mathbb{D}}_{b(\mathbb{R})} \times \overline{\mathbb{D}}_{b(\mathbb{R})} = \overline{\mathbb{D}}^2_{b(\mathbb{R})}$, the closed $b(\mathbb{R})$-bi-disc.

(b) Let $\Gamma_i \subset \mathbb{R}$, $i = 1, \dots, n$ be additive subgroups of $\mathbb{R}$, and consider the semigroup $S = (\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+ \subset \mathbb{R}^n_+$. Proposition 3.5.12 yields

$$\begin{aligned} H(S) &= H\big((\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+\big) \\ &\cong H\big((\Gamma_1)_+\big) \times H\big((\Gamma_2)_+\big) \times \cdots \times H\big((\Gamma_n)_+\big) \cong \overline{\mathbb{D}}_{\widehat{\Gamma}_1} \times \overline{\mathbb{D}}_{\widehat{\Gamma}_2} \times \cdots \times \overline{\mathbb{D}}_{\widehat{\Gamma}_n}. \end{aligned}$$

(c) $H\big(\mathbb{R} \times \mathbb{R}_+\big) \cong H(\mathbb{R}) \times H(\mathbb{R}_+) \cong b(\mathbb{R}) \times \overline{\mathbb{D}}_{b(\mathbb{R})}$, the solid torus over $b(\mathbb{R})$.

(d) Let $S = (0, \infty) \times (0, \infty) \cup (0, 0) \subset \mathbb{R}^2$. The only non-trivial idempotent semicharacter of S is $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$, while $G_S = \{(0, 0)\}$, and $\Gamma_S = \mathbb{R}^2$. As in Example 3.5.10(a) we obtain that

$$\begin{aligned} H(S) &\cong \{(\varkappa_{\{(0,0)\}}, \varkappa_S)\} \sqcup \big((0, 1]) \times b(\mathbb{R})\big)^2 \cong \{(\omega, \omega)\} \sqcup \big(\overline{\mathbb{D}}^*_{b(\mathbb{R})}\big)^2 \\ &\cong \big(\overline{\mathbb{D}}_{b(\mathbb{R})}\big)^2 / \big((\{\omega\} \times \overline{\mathbb{D}}_{b(\mathbb{R})}) \sqcup (\overline{\mathbb{D}}_{b(\mathbb{R})} \times \{\omega\})\big). \end{aligned}$$

(e) The semigroup $S = \mathbb{R} \times (0, \infty) \cup (0, 0)$ also has a single non-trivial idempotent semicharacter, namely, $\varkappa_{\{(0,0)\}}$, $G_S = \{(0, 0)\}$, while $\Gamma_S = \mathbb{R}^2$. Similarly to Example 3.5.10(b) we obtain that

$$H(S) \cong \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup (0, 1] \times b(\mathbb{R})^2 \cong \big([0, 1] \times b(\mathbb{R})^2\big) / \big(\{0\} \times b(\mathbb{R})^2\big) \cong \overline{\mathbb{D}}_{b(\mathbb{R})^2}.$$

Consequently, $H\big(\mathbb{R} \times (0, \infty) \cup (0, 0)\big)$ is bijective to the $b(\mathbb{R})^2$-disc $\overline{\mathbb{D}}_{b(\mathbb{R})^2}$, the closed $b(\mathbb{R})^2$-disc.

(f) Non-trivial idempotent semicharacters of the semigroup

$$S = \mathbb{R} \times (0, \infty) \cup [0, \infty) \times \{0\}$$

are $\varkappa_{\{(0,0)\}} = \delta_{(0,0)}$ and $\varkappa_{\mathbb{R}_+} \times \{0\}$, while $G_S = \{(0, 0)\}$, and $\Gamma_S = \mathbb{R}^2$. Similarly to Example 3.5.10(c) we obtain

$$\begin{aligned} H(S) &\cong \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup H\big(\mathbb{R}_+, (0, 1]\big) \times \widehat{\mathbb{R}_d} \sqcup H\big(S, (0, 1]\big) \times \widehat{\Gamma}_S \\ &= \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup (0, 1] \times b(\mathbb{R}) \sqcup H\big(S, (0, 1]\big) \times \widehat{\mathbb{R}^2_d} \\ &\cong \big\{(\varkappa_{\{(0,0)\}}, \varkappa_S)\big\} \sqcup \overline{\mathbb{D}}^*_{b(\mathbb{R})} \sqcup H\big(S, (0, 1]\big) \times (b(\mathbb{R}))^2 \\ &\cong \overline{\mathbb{D}}_{b(\mathbb{R})} \sqcup (0, 1] \times \big(b(\mathbb{R})\big)^2. \end{aligned}$$

3.6 The semigroup algebra $\ell^1(S)$ of a semigroup

Any semigroup with cancellation law and 0 generates in a standard way its semigroup algebra, described in this section. Its properties reflect the properties of the original semigroup.

Let S be a commutative semigroup with cancellation law and 0, and let $\Gamma_S = S - S$ be its group envelope. Consider the linear space $\ell^1(S)$ of complex valued functions f on S, for which

$$\|f\|_1 = \sum_{a \in S} \big|f(a)\big| < \infty.$$

Equivalently, $\ell^1(S)$ is the space of functions f on S with countable support, and such that $\sum_{a\in\ \mathrm{supp}(f)} \big|f(a)\big| < \infty$. It is well known, that under the pointwise operations and with the norm

$$\|f\|_1 = \sum_{a\in S} \big|f(a)\big|,$$

$\ell^1(S)$ is a Banach space. The characteristic function, δ_a, of any element $a \in S$ obviously is in $\ell^1(S)$, since $\delta_a(a) = 1$ and $\delta_a(b) = 0$ for any $b \neq a$. Clearly, $\|\delta_a\|_1 = 1$.

Given $f,\ g \in \ell^1(S)$ the set $\mathrm{supp}\,(f) + \mathrm{supp}\,(g) = \big\{c \in S \colon c = a+b,\ a \in \mathrm{supp}\,(f),\ b \in \mathrm{supp}\,(g)\big\}$ is at most countable, and the following inequalities hold:

$$\begin{aligned}\sum_{c\in S}\Big|\sum_{a,c-a\in S} f(a)\,g(c-a)\Big| &\le \sum_{c\in S}\Big(\sum_{a,c-a\in S}\big|f(a)\big|\big|g(c-a)\big|\Big)\\ &\le \sum_{a\in S}\sum_{b\in S}\big|f(a)\big|\big|g(b)\big| = \sum_{a\in S}\big|f(a)\big|\sum_{b\in S}\big|g(b)\big| = \|f\|_1\|g\|_1.\end{aligned} \tag{3.15}$$

As consequences from them we obtain that

(i) for every $c \in S$ the number $\sum_{a,b\in S} f(a)\,g(b)$ is finite, and therefore the convolution $f \star g$ defined by $(f\star g)(c) = \sum_{a,c-a\in S} f(a)\,g(b)$ of f and g, is a well-defined function on S, and

(ii) $\|f\star g\|_1 = \sum_{c\in S}\big|(f\star g)(c)\big| = \sum_{c\in S}\Big|\sum_{a,c-a\in S} f(a)\,g(c-a)\Big| \le \|f\|_1\|g\|_1.$

In particular, (ii) implies that for any $f, g \in \ell^1(S)$ their convolution $f\star g$ also belongs to $\ell^1(S)$. Note that $\delta_a \star \delta_b = \delta_{a+b}$ for any $a,\ b \in S$. Indeed,

$$(\delta_a\star\delta_b)(p) = \sum_{c,p-c\in S} \delta_a(c)\,\delta_b(p-c) \neq 0$$

only in the case when $c = a$ and $b = d$, i.e. only if $p = a+b$, and $(\delta_a\star\delta_b)(a+b) = \sum_{c+d=a+b} \delta_a(c)\,\delta_b(d) = \delta_a(a)\,\delta_b(b) = 1$. Hence, $\delta_a\star\delta_b$ is the characteristic function of the point $a+b$, i.e. $\delta_a\star\delta_b = \delta_{a+b}$.

Proposition 3.6.1. *The space $\ell^1(S)$ with multiplication $f\star g$ and the unit element δ_0 is a commutative Banach algebra over $\mathbb{C}$.*

Proof. The inequality $\|f\star g\|_1 \le \|f\|_1\|g\|_1$ is established already in (ii). Given an $f \in \ell^1(S)$, for any $c \in S$ we have

$$(f\star\delta_0)(c) = \sum_{a,c-a\in S} f(a)\,\delta_0(c-a) = \sum_{a+0=c} f(a)\,\delta_0(0) = f(c),$$

thus $f \star \delta_0 = f$ for every $f \in \ell^1(S)$. Therefore, the function δ_0 is indeed the unit element of $\ell^1(S)$. □

Since $\Big(\sum\limits_{a\in S} f(a)\,\delta_a\Big)(c) = \sum\limits_{a\in S} f(a)\,\delta_a(c) = f(c)\,\delta_c(c) = f(c)$ for every $c \in S$, we see that any $f \in \ell^1(S)$ can be expanded as a pointwise convergent series on S, namely $f = \sum\limits_{a\in S} f(a)\,\delta_a = \sum\limits_{a\in \operatorname{supp}(f)} f(a)\,\delta_a$. Actually, this series also converges to f in the $\ell^1(S)$-norm. Indeed, since $\|f\|_1 = \sum\limits_{a\in S} \big|f(a)\big| < \infty$, for any $\varepsilon > 0$ we can choose an $n \in \mathbb{N}$ for which

$$\sum_{i>n} \big|f(a_i)\big| = \|f\|_1 - \sum_{i=1}^{n} \big|f(a_i)\big| < \varepsilon,$$

where $\{a_i\}_{i=1}^{\infty}$ is any enumeration of $\operatorname{supp}(f)$. Now,

$$\begin{aligned} \Big\|f - \sum_{i=1}^{n} f(a_i)\,\delta_{a_i}\Big\|_1 &= \Big\|\sum_{i>n} f(a_i)\,\delta_{a_i}\Big\|_1 = \sum_{c\in S}\ \Big|\sum_{i>n} f(a_i)\,\delta_{a_i}(c)\Big| \\ &\le \sum_{c\in S}\sum_{i>n} \big|f(a_i)\,\delta_{a_i}(c)\big| = \sum_{i>n} \big|f(a_i)\delta_{a_i}(a_i)\big| = \sum_{i>n} \big|f(a_i)\big| < \varepsilon. \end{aligned}$$

Theorem 3.6.2. *Let S be a semigroup with cancellation law and 0. The maximal ideal space $\mathcal{M}_{\ell^1(S)}$ of the algebra $\ell^1(S)$ is homeomorphic to the set $H(S)$ of semicharacters of S with pointwise convergence.*

Proof. Let $m : \ell^1(S) \longrightarrow \mathbb{C}$ be a linear multiplicative functional of $\ell^1(S)$. We associate with m the function $\varphi_m\colon\ S \longrightarrow \mathbb{C}$, by $\varphi_m(a) = m(\delta_a)$. Clearly, $\varphi_m(a + b) = m(\delta_{a+b}) = m(\delta_a \star \delta_b) = m(\delta_a)\,m(\delta_b) = \varphi_m(a)\,\varphi_m(b)$ for every $a, b \in S$. Also, $\varphi_m(0) = m(\delta_0) = 1$, since δ_0 is the unit of $\ell^1(S)$. Moreover, φ_m maps S into the closed unit disc $\overline{\mathbb{D}}$ of $\mathbb{C}$. Indeed, $\big|\varphi_m(a)\big| = |m(\delta_a)| \le \|\delta_a\|_1 = 1$, since the functional m is of unit norm. Hence, φ_m is a semicharacter of S, i.e. $\varphi_m \in H(S)$.

Conversely, with any semicharacter φ of S we associate the linear functional $m_\varphi\colon \ell^1(S) \longrightarrow \mathbb{C}$ defined by $m_\varphi(f) = \sum\limits_{a\in S} f(a)\,\varphi(a)$ for any $f = \sum\limits_{a\in S} f(a)\,\delta_a \in \ell^1(S)$. This is a well-defined function in $\ell^1(S)$, since

$$\Big|\sum_{a\in S} f(a)\,\varphi(a)\Big| \le \sum_{a\in S} \Big|f(a)\,\varphi(a)\Big| \le \sum_{a\in S} \big|f(a)\big| = \|f\|_1 < \infty.$$

Note that $m_\varphi(\delta_a) = \sum\limits_{c\in S} \delta_a(c)\,\varphi(c) = \varphi(a)$. Clearly, $m_\varphi(\delta_0) = \varphi(0) = 1$, and

$$m_\varphi(\delta_a \star \delta_b) = m_\varphi(\delta_{a+b}) = \varphi(a + b) = \varphi(a)\,\varphi(b) = m_\varphi(\delta_a)\,m_\varphi(\delta_b).$$

Since the elements in $\ell^1(S)$ are $\ell^1(S)$-limits of finite linear combinations of functions $\delta_a,\ a \in S$, it follows that m_φ is also multiplicative on $\ell^1(S)$. Hence, m_φ is a linear multiplicative functional of $\ell^1(S)$.

It is easy to check that $\varphi_{m_\varphi} = \varphi$, and $m_{\varphi_m} = m$. Indeed, $\varphi_{m_\varphi}(a) = m_\varphi(\delta_a) = \varphi(a)$ for any $a \in S$, i.e. $\varphi_{m_\varphi} = \varphi$. Since $m_{\varphi_m}(\delta_a) = \varphi_m(a) = m(\delta_a)$ for every $a \in S$, it follows that $m_{\varphi_m}(p) = m(p)$ for every finite linear combination of functions δ_a. By continuity, $m_{\varphi_m}(f) = m(f)$ for any $f \in \ell^1(S)$. Consequently, the correspondence $\mathcal{M}_{\ell^1(S)} \longrightarrow H(S)\colon\ m \longmapsto \varphi_m$ is one-to-one. If $m_\alpha \longrightarrow m$ in $\mathcal{M}_{\ell^1(S)}$, then $\varphi_{m_\alpha}(a) = m_\alpha(\delta_a) \longrightarrow m(\delta_a) = \varphi_m(a)$ for every $a \in S$, i.e. $\varphi_{m_\alpha} \longrightarrow \varphi_m$ pointwise in $H(S)$. Conversely, if $\varphi_\alpha \longrightarrow \varphi$ pointwise in $H(S)$, then $m_{\varphi_\alpha}(\delta_a) = \varphi_\alpha(a) \longrightarrow \varphi(a) = m_\varphi(\delta_a)$. Hence, $m_{\varphi_\alpha}(\delta_a) \longrightarrow m_\varphi(\delta_a)$ for every $a \in S$, and therefore, $m_{\varphi_\alpha}(p) \longrightarrow m_\varphi(p)$ for every finite linear combination p of functions δ_a. Let $f = \sum\limits_{a\in S} f(a)\,\delta_a$ be an arbitrary function in $\ell^1(S)$. Given an $\varepsilon > 0$, let $\Big\| f - \sum\limits_{i=1}^n f(a_i)\,\delta_{a_i} \Big\|_1 < \varepsilon$ for some n, where $a_i \in \operatorname{supp}(f)$. Now

$$
\begin{aligned}
&\big|m_{\varphi_\alpha}(f) - m_\varphi(f)\big| \\
&\le \Big|m_{\varphi_\alpha}(f) - m_{\varphi_\alpha}\Big(\sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big)\Big| + \Big|m_{\varphi_\alpha}\Big(\sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big) - m_\varphi\Big(\sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big)\Big| \\
&\quad + \Big|m_\varphi\Big(\sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big) - m_\varphi(f)\Big| \\
&\le \Big\|f - \sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big\|_1 + \Big|\sum_{i=1}^n f(a_i)\, m_{\varphi_\alpha}(\delta_{a_i}) - \sum_{i=1}^n f(a_i)\, m_\varphi(\delta_{a_i})\Big| \\
&\quad + \Big\|f - \sum_{i=1}^n f(a_i)\,\delta_{a_i}\Big\|_1 \\
&\le \sum_{i=1}^n \big|f(a_i)\big|\big|\varphi_\alpha(\delta_{a_i}) - \varphi(\delta_{a_i})\big| + 2\varepsilon,
\end{aligned}
$$

thus $m_{\varphi_\alpha}(f) \longrightarrow m_\varphi(f)$ for every $f \in \ell^1(S)$, and hence $m_{\varphi_\alpha} \longrightarrow m_\varphi$ in $\mathcal{M}_{\ell^1(S)}$. Consequently $\mathcal{M}_{\ell^1(S)}$ and $H(S)$ are homeomorphic spaces under the corresponding topologies. □

Since every character on S belongs to $H(S)$, the group of characters $H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$ is a subset of $\mathcal{M}_{\ell^1(S)}$. Note that if $S = \Gamma_S$, then $\mathcal{M}_{\ell^1(\Gamma_S)} \cong H(\Gamma_S) = H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$.

Let $\varphi \in H(S)$ be a semicharacter of S, and let $\varphi = \varrho\chi$ be its polar decomposition, where $\varrho = |\varphi| \in H(S,[0,1])$ is the modulus of φ, and $\chi \in \widehat{\Gamma}_S$ (cf.

Theorem 3.5.4). Every z in the closed upper half-plane $\overline{\mathbb{C}}_+$ generates a semicharacter $\varphi^{(z)} \in H(S)$ by

$$\varphi^{(z)}(a) = \chi(a)\,\varrho(a)^{-iz}.$$

Note that if $x \in \mathbb{R}$, then $\varphi^{(x)}(a) = \chi(a)\,\varrho(a)^{-ix}$, and therefore, $\left|\varphi^{(x)}(a)\right| = \left|\chi(a)\right|\left|\varrho(a)^{-ix}\right| = 1$, i.e. $\varphi^{(x)} \in \widehat{\Gamma}_S$. If $\varphi \in H(S,\mathbb{T})$ is a character of S, then $|\varrho| \equiv 1$, thus $\varphi^{(z)} = \chi = \varphi$ for every $z \in \overline{\mathbb{C}}_+$. If φ is an idempotent semicharacter of S, then $\varrho^{-iz} = \varrho$, and therefore, $\varphi^{(z)} = \chi\,\varrho = \varphi$ for every $z \in \overline{\mathbb{C}}_+$. If $\varphi = \omega\chi$, where ω is an idempotent semicharacter, and χ is a character of S, then $\operatorname{supp}(\varphi^{(z)}) = \operatorname{supp}(\omega)$. If φ is not an idempotent semicharacter, then the mapping $z \longmapsto \varphi^{(z)}$ is a continuous embedding of $\overline{\mathbb{C}}_+$ into $H(S) \cong \mathcal{M}_{\ell^1(S)}$, such that $\varphi^{(i)} = \varphi$.

Lemma 3.6.3. *Let φ be a non-idempotent semicharacter of S. For any $f \in \ell^1(S)$ the mapping $z \longmapsto \widehat{f}(\varphi^{(z)})$ is a bounded analytic function in $\mathbb{C}_+$, continuous up to the boundary $\mathbb{R}$.*

Proof. If $f = \sum_{a\in S} f(a)\,\delta_a \in \ell^1(S)$, and φ is a non-idempotent semicharacter of S, then

$$\widehat{f}(m_\varphi) = m_\varphi(f) = m_\varphi\Big(\sum_{a\in S} f(a)\,\delta_a\Big) = \sum_{a\in S} f(a)\,m_\varphi(\delta_a) = \sum_{a\in S} f(a)\,\varphi(a),$$

where $m_\varphi \in H(S)$ is the linear multiplicative functional of $\ell^1(S)$ associated with φ. Hence,

$$\widehat{f}\big(m_{\varphi^{(z)}}\big) = \sum_{a\in S} f(a)\,\varphi^{(z)}(a) = \sum_{a\in S} f(a)\,\chi(a)\,\varrho(a)^{-iz}. \tag{3.16}$$

This is an absolutely convergent Dirichlet series in $z \in \overline{\mathbb{C}}_+$. Indeed, since

$$\left|\varrho(a)^{-iz}\right| = \left|\varrho(a)^{y\,-\,ix}\right| = \left|\varrho(a)^{y}\right|\left|\varrho(a)^{-ix}\right| = \left|\varrho(a)^{y}\right|\left|e^{-ix\ln\varrho(a)}\right| = \left|\varrho(a)^{y}\right| \le 1$$

for any $z \in \overline{\mathbb{C}}_+$, we see that

$$\Big|\sum_{a\in S} f(a)\,\chi(a)\,\varrho(a)^{-iz}\Big| \le \sum_{a\in S}\left|f(a)\right| = \|f\|_1 < \infty.$$

Consequently, $\widehat{f}\big(m_{\varphi^{(z)}}\big)$ is a bounded analytic function in $\mathbb{C}_+$, which is continuous on $\overline{\mathbb{C}}_+$, as claimed. $\square$

Proposition 3.6.4. *Let φ be a semicharacter of S. Denote by $H_\varphi(S,\mathbb{T})$ the set of semicharacters ψ of S with the same support as φ, and for which $|\psi|\big|_{\operatorname{supp}\varphi} \equiv 1$. Then*

$$\big|\widehat{f}(m)\big| \le \sup_{\chi\in H_{\varphi_m}(S,\mathbb{T})} \big|\widehat{f}(m_\chi)\big|$$

for every $f \in \ell^1(S)$ and $m \in \mathcal{M}_{\ell^1(S)}$.

Proof. Let $f \in \ell^1(S)$, and let $\varphi = \varrho\chi \in H(S)$. By the previous lemma $F(z) = \widehat{f}\big(m_{\varphi^{(z)}}\big)$ is a bounded and analytic function in $\overline{\mathbb{C}}_+$. By the Phragmen-Lindelöf theorem, $F(z)$ attains the maximum of its modulus on the boundary $\mathbb{R}$. Therefore, for any $z \in \mathbb{C}$ we have

$$\begin{aligned}\big|F(z)\big| &\le \sup_{z\in\mathbb{R}} \big|F(z)\big| = \sup_{y=0} \big|\widehat{f}\big(m_{\varphi^{(x+iy)}}\big)\big| \\ &= \sup_{x\in\mathbb{R}} \Big|\sum_{a\in S} f(a)\,\varphi^{(x+iy)}(a)\Big| = \sup_{x\in\mathbb{R}} \Big|\sum_{a\in S} f(a)\,\chi(a)\,\varrho(a)^{-ix}\Big|.\end{aligned}$$

Note that for every $x \in \mathbb{R}$ the function $\chi\varrho^{-ix}$ is a character of the semigroup $\operatorname{supp}(\varphi) \subset S$, and $\sum\limits_{a\in S} f(a)\,\chi(a)\,\varrho(a)^{-ix} = \sum\limits_{a\in S} f(a)\big(\chi\varrho^{-ix}\big)(a) = m_{\chi\varrho^{-ix}}(f) = \widehat{f}\big(m_{\chi\varrho^{-ix}}\big)$. Consequently,

$$\big|F(z)\big| \le \sup_{x\in\mathbb{R}} \big|\widehat{f}(m_{\chi\varrho^{-ix}})\big| \le \sup_{\psi\in H_\varphi(S,\mathbb{T})} \big|\widehat{f}(m_\psi)\big|$$

for any $z \in \mathbb{C}_+$. Therefore, for any $m \in \mathcal{M}_{\ell^1(S)}$ we have

$$\big|\widehat{f}(m)\big| = \big|\varphi_m(f)\big| = \big|\varphi_m^{(i)}(f)\big| = \big|\widehat{f}(m_{\varphi_m^{(i)}})\big| \le \sup_{\psi\in H_{\varphi_m}(S,\mathbb{T})} \big|\widehat{f}(m_\psi)\big|. \quad \square$$

Theorem 3.6.5. *The group $H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$ of characters of S is a boundary for the algebra $\ell^1(S)$.*

Proof. Clearly, $\ell^1(S)$ is a Banach subalgebra of $\ell^1(\Gamma_S)$. Let $f \in \ell^1(S)$ and $p = \sum\limits_{i=1}^{n} f(a)\,\delta_{a_i} \in \ell^1(S)$ be a finite linear combination of functions δ_{a_i}, $a_i \in S$. The spectral radius formula (1.6) yields

$$\begin{aligned}\sup_{m\in\mathcal{M}_{\ell^1(S)}} \big|\widehat{p}(m)\big| &= \lim_{n\to\infty} \sqrt[n]{\|p^{\star n}\|_{\ell^1(S)}} = \lim_{n\to\infty} \sqrt[n]{\|p^{\star n}\|_{\ell^1(\Gamma_S)}} \\ &= \sup_{m\in\mathcal{M}_{\ell^1(\Gamma_S)}} \big|\widehat{p}(m)\big| = \sup_{\varphi\in\widehat{\Gamma}_S} \big|\widehat{p}(m_\varphi)\big| = \sup_{\varphi_m\in H(S,\mathbb{T})} \big|\widehat{p}(m)\big|,\end{aligned}$$

where $p^{\star n} = \underbrace{p \star p \star \cdots \star p}_{n}$. Since the finite linear combinations of functions δ_a, $a \in S$ are $\ell^1(S)$-dense in $\ell^1(S)$, their Gelfand transforms are uniformly dense on $\mathcal{M}_{\ell^1(S)}$ in $\widehat{\ell^1}(S)$, and therefore, $\sup\limits_{m\in\mathcal{M}_{\ell^1(S)}} \big|\widehat{f}(m)\big| \le \sup\limits_{\varphi_m\in H(S,\mathbb{T})} \big|\widehat{f}(m)\big|$ for any $f \in \ell^1(S)$. Hence, the group $\widehat{\Gamma}_S \cong H(S,\mathbb{T})$ of characters of S is a closed boundary for the algebra $\ell^1(S)$, as claimed. $\square$

In fact, as the next corollary shows, $H(S,\mathbb{T})$ is the smallest closed boundary for $\ell^1(S)$, i.e. it is its Shilov boundary.

Corollary 3.6.6. *The group of characters* $H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$ *of* S *is homeomorphic to the Shilov boundary* $\partial\ell^1(S)$ *of* $\ell^1(S)$*.*

Proof. Since, by Theorem 3.6.5 the group $H(S,\mathbb{T}) \cong \widehat{\Gamma}_S$ is a boundary for $\ell^1(S)$, then $\widehat{\ell^1}(S) \cong \widehat{\ell^1}(S)|_{\widehat{\Gamma}_S} \subset C(\widehat{\Gamma}_S)$. Note that for every $g \in \widehat{\Gamma}_S$ the g-shift $T_g\colon \widehat{\Gamma}_S \longrightarrow \widehat{\Gamma}_S\colon \chi \longmapsto \chi_g = \chi(g)\,\chi$ generates an automorphism T_g^* of $\widehat{\ell^1}(S)$. Therefore, $T_g^*\big(\widehat{\ell^1}(S)\big) = \widehat{\ell^1}(S)$, and $T_g\big(\partial(\ell^1(S)\big) = \partial\ell^1(S)$. This is impossible if $\partial\ell^1(S) \neq \widehat{\Gamma}_S$. □

If ω is an idempotent semicharacter of S, then Proposition 3.6.4 and Theorem 3.6.5 imply that $\sup\limits_{\varphi_m \in H_\omega(S,\mathbb{T})} \big|\widehat{f}(m)\big| \leq \sup\limits_{\varphi_m \in \widehat{\Gamma}_S} \big|\widehat{f}(m)\big|$ for every $f \in \ell^1(S)$.

3.7 Notes

A standard reference on topological groups is the classical book by Pontryagin [P4]. Results on functions and measures on topological groups can be found in many books (e.g. [G1], [L4], [R5]). Bochner-Fejér's operators were considered initially in the almost periodic setting (e.g. [P2]). The semigroup algebras $\ell^1(S)$ of topological semigroups are studied in [AS1, HZ, H2], among others.

Chapter 4

Shift-invariant algebras on compact groups

In this chapter we introduce shift-invariant algebras, the main objects of this book. These are uniform algebras on a compact connected group G, consisting of continuous functions on G, whose spectrum is contained in a semigroup S of the dual group $\widehat{G}$. If $\widehat{G}$ is a subgroup of $\mathbb{R}$, and $S \subset \mathbb{R}_+$, then the maximal ideal space of the corresponding shift-invariant algebra is the G-disc, or, big disc over G. In this chapter we describe two important models of shift-invariant algebras, namely, by the means of almost periodic functions on $\mathbb{R}$, and by the means of H^∞-functions on the unit circle. The set of automorphisms, and the peak groups of shift-invariant algebras are also characterized. Extensions on G-discs and groups of several classical theorems of Complex Analysis, such as Radó's theorem for null-sets and the Riemann theorem for removable singularities of analytic functions, are stated and proved. It is shown that these extensions hold for some semigroups, while in general they fail. In principle we state all results for general shift-invariant algebras A_S, though they apply automatically to the particular cases of algebras AP_S of almost periodic functions, and of H_S^∞-algebras. Asymptotically almost periodic functions combine the properties of classical almost periodic functions on $\mathbb{R}$, and of continuous functions on $\mathbb{R}$ that vanish at infinity.

4.1 Algebras of S-functions on groups

Let G be a compact connected abelian group and let $S^* = \{\chi^a\}_{a\in S}$ be a subsemigroup of the dual group $\widehat{G}$, containing the unit element $\chi^0 \equiv 1$. We will assume that the index set S is provided with the additive operation, induced from the multiplication in S^*. In particular, we will assume that $\chi^a\chi^b = \chi^{a+b}$ for every $a, b \in S$. Therefore, S becomes a semigroup isomorphic to S^*. We suppose that $S^* - S^* = \widehat{G}$, i.e. that S^* generates the dual group $\widehat{G}$. The group envelope

$\Gamma_S = S - S \cong S^* - S^*$, which we will denote in this section by Γ instead of by Γ_S, in this case coincides with $\widehat{G}$. In the sequel we will not distinguish between S and S^* [resp. between Γ and $\widehat{G}$], and will denote S^* rather by S [resp. $\widehat{G}$ by Γ]. Since $\widehat{G}$ separates the points of G, so does S^*. If the group envelope $\Gamma = S - S$ of S differs from $\widehat{G}$, then we could replace, right from the beginning, $\widehat{G}$ by the group Γ, and the dual group $\widehat{\widehat{G}} \cong G$ – by the dual $\widehat{\Gamma}$ of the group Γ.

Definition 4.1.1. Finite linear combinations over $\mathbb{C}$ of characters $\chi^a \in \widehat{G}$ with $a \in S$ are called *S-polynomials* on G. Continuous functions on G with $\mathrm{sp}\,(f) \subset S$ are called *S-functions* on G.

Equivalently, $f \in C(G)$ is an S-function on G if and only if its Fourier coefficients

$$c_a^f = \int_G f(g)\,\overline{\chi}^a(g)\,d\sigma \tag{4.1}$$

are zero whenever $a \in \Gamma_S \setminus S$. The set of all S-functions on G we denote by A_S. Obviously, any S-polynomial is an S-function on G. Theorem 3.3.2 yields the following

Proposition 4.1.2. *The algebra A_S of continuous S-functions on G coincides with the algebra of uniform limits of S-polynomials on G.*

As a consequence we see that A_S is a uniform algebra on the group G.

Example 4.1.3. (a) Algebras A_S of S-functions are natural generalizations of polydisc algebras $A(\mathbb{T}^n)$, $n \in \mathbb{N}$. Actually, if $G = \mathbb{T}^n$, $\Gamma = \widehat{G} = \mathbb{Z}^n$, and $S = \mathbb{Z}_+^n$, the algebra A_S on G is exactly the polydisc algebra $A_{\mathbb{Z}_+^n} = A(\mathbb{T}^n)$ on $\mathbb{T}^n$. $\mathbb{Z}_+$-functions on $\mathbb{T}^n$ are restrictions on $\mathbb{T}^n$ of usual analytic functions in n variables in the polydisc $\overline{\mathbb{D}}^n$, continuous up to the boundary $\mathbb{T}^n$.

(b) For any $a \in S \subset \mathbb{R}$ denote by $\psi_a \in H^\infty$ the singular function $\varphi_a(z) = e^{ia\,(1+z)/(1-z)}$ on the unit disc $\mathbb{D}$. Denote by H_S^∞ the Banach algebra on $\mathbb{D}$ generated by the functions $\psi_a(z)$, $a \in S$, equipped by the sup-norm on $\mathbb{D}$. Clearly, $H_S^\infty \subset H^\infty \cap C(\overline{\mathbb{D}}_G \setminus \{1\})$. Note that the fractional linear transformation $\varphi(z) = i\,(1+z)/(1-z)$ maps the unit disc $\overline{\mathbb{D}}$ conformally onto the extended upper half plane $\overline{\mathbb{C}}_+$. Moreover, $e^{\varphi(z)} = e^{i\,(1+z)/(1-z)}$, and $\left(e^{\varphi(z)}\right)^a = e^{ia\,(1+z)/(1-z)} = \psi_a(z)$ for every $a \in S$. Hence, H_S^∞ is isometrically isomorphic to the algebra A_S of S-functions on $G = \widehat{\Gamma}$.

(c) The Riemann surface $\mathcal{R}_{Log}$ of the function $\mathrm{Log}(z)$, $z \in \mathbb{C}$, admits the following parametrization: $\mathcal{R}_{Log} = \{(r,\theta)\colon 0 < r < \infty,\ \theta \in \mathbb{R}\}$, i.e. $\mathcal{R}_{Log} \approx (0,\infty) \times \mathbb{R}$. The space $\overline{\mathcal{R}}_{Log} \cong \big([0,\infty) \times \mathbb{R}\big)/\big(\{0\} \times \mathbb{R}\big)$, which contains $\mathcal{R}_{Log}$ as a dense subset, admits the parametrization $\overline{\mathcal{R}}_{Log} = \{(r,\theta)\colon 0 \le r < \infty,\ \theta \in \mathbb{R}\}$, with the agreement that $(0,\theta_1) = (0,\theta_2)$ for every $\theta_1, \theta_2 \in \mathbb{R}$. The mapping $(r,\theta) \longmapsto \mathrm{Log}(re^{i\theta}) = \ln r + i\theta$ is a continuous lifting of $\mathrm{Log}(z)$ on $\mathcal{R}_{Log}$, which maps $\mathcal{R}_{Log}$ homeomorphically onto $\mathbb{C}$. Every natural power z^n of z can be lifted on $\mathcal{R}_{Log}$ as $\widetilde{z}^n(r,\theta) = r^n e^{in\theta}$. The functions $\widetilde{z}^n$ can be extended on $\overline{\mathcal{R}}_{Log}$ by letting $\widetilde{z}^n\big(\{0\}\times\mathbb{R}\big)/\big(\{0\}\times\mathbb{R}\big) = 0$. Consider the portion $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ of $\overline{\mathcal{R}}_{Log}$ above the closed unit disc $\overline{\mathbb{D}}$, i.e. $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}}) = \big\{(r,\theta)\in\overline{\mathcal{R}}_{Log}\colon r\le 1\big\}$. Any function $\widetilde{z}^n$ maps $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ continuously onto $\overline{\mathbb{D}}$. We can define also functions $\widetilde{z}^a$ on $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ for every real number $a\ge 0$, namely, $\widetilde{z}^a(r,\theta) = r^a e^{ia\theta}$, which map $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ continuously onto $\overline{\mathbb{D}}$. Let S be an additive subsemigroup of $\mathbb{R}_+$, containing 0, and let $G=\widehat{\Gamma}_S$. Clearly, the algebra generated by the functions $\widetilde{z}^a$, $a\in S$, on $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ is isometrically isomorphic to the shift-invariant algebra A_S of S-functions on G.

(d) Let $\Gamma=\mathbb{Z}^2=\mathbb{Z}\times\mathbb{Z}$ be the two-dimensional integer lattice in $\mathbb{R}^2$. The group of characters of Γ is the two-dimensional torus $G=\mathbb{T}^2$. For a fixed irrational number $\beta>0$ we set $\Gamma_+^\beta = \{(n,m)\in\mathbb{Z}^2\colon \beta n+m\ge 0\}\subset\mathbb{R}_+$. The uniform algebra A_β on $\mathbb{T}^2$ generated by the characters χ^a, $a\in\Gamma_+^\beta$ is also a shift-invariant algebra, which contains the bi-disc algebra (e.g. [T2]).

We recall that for any semigroup S, $0\in S\subset\mathbb{R}_+$ the algebra $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$ is generated by the characters e^{-iat}, $a\in S$. Since the correspondence $e^{-iat}\longmapsto\chi^a$ extends to an isometric isomorphism between the algebras $AP_S(\mathbb{R})$ and $A_S\subset C(b(\mathbb{R}))$, we obtain

Proposition 4.1.4. *If G is a solenoidal group, and S is an additive subsemigroup of $\Gamma=\widehat{G}\subset\mathbb{R}$, containing 0, then the algebra A_S of S-functions on G is isometrically isomorphic to the algebra $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$.*

Similarly, if S is a semigroup of $\mathbb{R}^n$ containing the origin, then the algebra $AP_S(\mathbb{R}^n)$ of almost periodic S-functions on $\mathbb{R}^n$ is generated by the characters $e^{i(s_{k_1}x_1+s_{k_2}x_2+\cdots+s_{k_n}x_n)}$, $(s_{k_1},s_{k_2},\dots,s_{k_n})\in S$. The correspondence $e^{i(s_{k_1}x_1+s_{k_2}x_2+\cdots+s_{k_n}x_n)}\longmapsto\chi^{(s_{k_1},s_{k_2},\dots,s_{k_n})}$ extends to an isometric isomorphism between the algebras $AP_S(\mathbb{R}^n)$ and $A_S\subset C\big(b(\mathbb{R})^n\big)$. Therefore we have the following

Proposition 4.1.5. *Let G be a compact abelian group, whose dual group $\widehat{G}$ is a subgroup of $\mathbb{R}^n$, and let S be an additive subsemigroup of $\mathbb{R}^n$, containing the origin $(0,0,\dots,0)$, then the algebra A_S of S-functions on G is isometrically isomorphic to the algebra $AP_S(\mathbb{R}^n)$ of almost periodic S-functions on $\mathbb{R}^n$.*

Example 4.1.6. (a) Let $G=\mathbb{R}^2$ and $S=\mathbb{R}_+\times\mathbb{R}_+$. The algebra $AP_{\mathbb{R}^2_+}(\mathbb{R}^2)$ of almost periodic $\mathbb{R}^2_+$-functions on $\mathbb{R}^2$ consists of continuous functions on $\mathbb{R}^2$ that can be approximated uniformly on $\mathbb{R}^2$ by exponential polynomials of type

$$\sum a_{k_1,k_2} e^{i(s_{k_1}x_1+s_{k_2}x_2)},\ s_{k_1},s_{k_2}\in\mathbb{R}_+.$$

These are precisely the analytic almost periodic functions in two real variables.

(b) If $G=\mathbb{R}^2$ and $S=\mathbb{R}\times\mathbb{R}_+$, then the functions in the algebra $AP_{\mathbb{R}\times\mathbb{R}_+}(\mathbb{R}^2)$ of almost periodic $\mathbb{R}\times\mathbb{R}_+$-functions on $\mathbb{R}^2$ are approximable uniformly on $\mathbb{R}^2$ by exponential polynomials of type $\sum a_{k_1,k_2} e^{i(s_{k_1}x_1 + s_{k_2}x_2)}$ with $s_{k_1} \in \mathbb{R}$, $s_{k_2} \in \mathbb{R}_+$. These are precisely the almost periodic functions on $\mathbb{R}^2$, analytic in the second variable x_2.

(c) Let G be a compact abelian group. Denote by Γ_τ the dual group $\Gamma = \widehat{G}$ equipped by some topology τ, so that Γ_τ is a locally compact topological group, and for which $\widehat{\Gamma_\tau}$ is bijective to $\widehat{\Gamma}$. Then $C(G) \cong AP(\Gamma_\tau)$. If S is a semigroup of Γ, then the shift-invariant algebra A_S is isometrically isomorphic to the algebra $AP_S(\Gamma_\tau)$ of almost periodic S-functions on Γ_τ.

We can begin the construction of the algebra A_S with the semigroup S, instead of with G. Let S be a semigroup with cancellation law and 0, and let $\Gamma = S - S$ be the group envelope of S. Unless otherwise indicated, in the sequel we will assume that besides the identity Γ has no elements of finite orders and is equipped with the discrete topology. The dual group $G = \widehat{\Gamma}$ of Γ, is a compact connected abelian group. By Pontryagin's duality theorem, $\widehat{G} \cong \Gamma$. Therefore, every $a \in S$ generates a unique character $\chi^a = a^*$ on G, and the semigroup S is isomorphic to the semigroup $S^* = \{a^* = \chi^a : a \in S\} \subset \widehat{G}$. Note that if $S = \mathbb{R}_+$, then $\Gamma = \mathbb{R}_d$, and the group $G = \widehat{\mathbb{R}_d}$ coincides with the Bohr compactification $b(\mathbb{R})$ of $\mathbb{R}$.

Definition 4.1.7. A uniform algebra A on G is said to be *shift-invariant* if the g-shifts f_g, defined as $f_g(h) = f(gh)$, $g \in G$, of every element $f \in A$ belong to A.

Any algebra of type A_S is shift-invariant. Indeed, since $\chi^a_g(h) = \chi^a(h\,g) = \chi^a(h)\,\chi^a(g)$, any g-shift χ^a_g of a character χ^a, $a \in S$, belongs to A_S, and so does the g-shift P_g of any S-polynomial $P(h) = \sum_{k=1}^{n} c_k \chi^{a_k}(h)$. Since any S-function $f \in A_S$ is uniformly approximable on G by S-polynomials, the g-shifts f_g of S-functions f also belong to A_S. Therefore, A_S is shift-invariant, as claimed. The converse is also true. Namely

Proposition 4.1.8. *A uniform algebra A on a compact group G is shift-invariant on G if and only if A is an algebra of type A_S for some subsemigroup $S \subset \widehat{G}$.*

Proof. By the preceding remark, we only need to prove the necessity. Let A be a shift-invariant algebra on G, and let S be the semigroup in Γ generated by the set $\bigcup_{f \in A} \operatorname{sp}(f)$. Definition 4.1.1 implies that $A \subset A_S$.

We claim, conversely, that if f is a fixed function in A, then any character $\chi^a \in \widehat{G}$ with $a \in \operatorname{sp}(f)$ belongs to A. Indeed, for any fixed $a \in \operatorname{sp}(f)$ the function

$$\widetilde{f^a}(g) = c_a^{f_g} = \int_G f_g(h)\,\overline{\chi}^a(h)\,d\sigma(h) \tag{4.2}$$

is uniformly approximable by its Riemann sums

$$\sum_j f_g(h_j^*)\,\overline{\chi}^a(h_j^*)\Delta h_j.$$

Each one of these Riemann sums is a function in g that belongs to A, since $f_g(h_j^*) = f(g\,h_j^*) = f_{h_j^*}(g) \in A$. Therefore, $\widetilde{f^a} \in A$. By

$$\begin{aligned}\widetilde{f^a}(g) &= \int_G f_g(h)\,\overline{\chi}^a(h)\,d\sigma(h) = \int_G f(g\,h)\,\overline{\chi}^a(h)\,d\sigma(h)\\ &== \int_G f(h)\,\overline{\chi}^a(h\,g^{-1})\,d\sigma(h) = \chi^a(g)\int_G f(h)\,\overline{\chi}^a(h)\,d\sigma(h) = c_a^f\,\chi^a(g),\end{aligned}$$

we have that $\chi^a \in A$, as claimed. Hence for every $a \in S$ the corresponding character χ^a belongs to A. Proposition 4.1.2 implies that $A_S \subset A$. Consequently, $A = A_S$. ☐

Any algebra $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$, where S is a semigroup of $\mathbb{R}$ is invariant under $\mathbb{R}$-shifts, i.e. the function $f_t(x) = f(x+t)$ belongs to A for any $f \in A$ and $t \in \mathbb{R}$. Restricted to $\jmath_\imath(\mathbb{R}) \subset G$, the arguments from the proof of Proposition 4.1.8 imply the following

Proposition 4.1.9. *A uniform algebra $A \subset AP(\mathbb{R})$ is invariant under $\mathbb{R}$-shifts if and only if A is an algebra of type $AP_S(\mathbb{R})$ for some semigroup $S \subset \mathbb{R}$.*

4.2 The maximal ideal space of a shift invariant algebra

In this section we describe the maximal ideal spaces of shift-invariant algebras on compact groups. Let S be a semigroup with cancellation law and 0, and let $\Gamma = S - S$ be its group envelope. Denote by G the dual group of Γ.

Theorem 4.2.1. *The maximal ideal space $\mathcal{M}_{A_S}$ of the algebra A_S is homeomorphic to the semigroup $H(S)$ of semicharacters of S.*

Proof. If $m \in \mathcal{M}_{A_S}$ is a linear multiplicative functional on A_S, then the function $\varphi_m(a) = m(\chi^a)$, $a \in S$, is a semicharacter on S. Indeed, $\varphi_m(a+b) = m(\chi^{a+b}) = m(\chi^a\chi^b) = \varphi_m(a)\,\varphi_m(b)$, and also $|\varphi_m(a)| = |m(\chi^a)| \le \|\chi^a\| = 1$.

Conversely, if $\varphi \in H(S)$ is a semicharacter of S, then the function m_φ: $m_\varphi\Big(\sum_{k=1}^n c_k\chi^{a_k}\Big) = \sum_{k=1}^n c_k\varphi(a_k)$ is a linear multiplicative functional on the algebra $P_S(G)$ of S-polynomials on G. Note that φ also generates a linear multiplicative functional m_φ^1 on the algebra $\ell^1(S) \supset P_S(G)$. Let $p(g) = \sum_{k=1}^n c_k\chi^{a_k}(g)$. For any

$\gamma \in \widehat{\Gamma}$ and $a \in S \subset \Gamma$ we have $\widehat{\chi^a}(m_\gamma) = m_\gamma(\chi^a) = \chi^a(\gamma)$, and therefore, $\widehat{p}(\gamma) = \sum_{k=1}^{n} c_k \chi^{a_k}(\gamma)$. Since $G = \widehat{\Gamma}$ is a boundary of $\ell^1(S)$ by Theorem 3.6.5, for every $p \in P_S(G)$ we have

$$\begin{aligned} |m_\varphi(p)| &= |m_\varphi^1(p)| \leq \max_{\psi \in \mathcal{M}_{\ell^1(S)}} |\widehat{p}(\psi)| = \max_{\gamma \in \widehat{\Gamma}} |\widehat{p}(m_\gamma)| = \max_{\gamma \in G\widehat{\Gamma}} |m_\gamma(p)| \\ &= \max_{\gamma \in G\widehat{\Gamma}} \Big| \sum_{k=1}^{n} c_k \gamma(a_k) \Big| = \max_{\gamma \in G\widehat{\Gamma} \cong G} \Big| \sum_{k=1}^{n} c_k \chi^{a_k}(\gamma) \Big| = \|\widehat{p}\|_{C(G)} = \|p\|_{A_S}. \end{aligned}$$

Consequently, $\|m_\varphi\| \leq 1$ on the dense subset $P_S(G)$ of A_S. Hence, m_φ can be extended by continuity from $P_S(G)$ on its uniform closure A_S as a linear multiplicative functional. Therefore, the mapping $m \longmapsto \varphi_m$ is a bijection between $\mathcal{M}_{A_S}$ and $H(S)$, and $\varphi \longmapsto m_\varphi$ is its inverse mapping.

Let $m_\alpha \longrightarrow m$ in the Gelfand topology of $\mathcal{M}_{A_S}$, i.e. $m_\alpha(f) \longrightarrow m(f)$ for every $f \in A_S$. Hence $\varphi_{m_\alpha}(a) = m_\alpha(\chi^a) \longrightarrow m(\chi^a) = \varphi_m(a)$ for every $a \in S$, i.e. φ_{m_α} converges to φ_m pointwise on S. Conversely, if $\varphi_{m_\alpha}(a) \longrightarrow \varphi_m(a)$ for every $a \in S$, then $m_\alpha(\chi^a) \longrightarrow m(\chi^a)$, and hence $m_\alpha(p) \longrightarrow m(p)$ for every S-polynomial p. Since the set $P_S(G)$ of S-polynomials is uniformly dense in A_S, one can see that $m_\alpha(f) \longrightarrow m(f)$ for every $f \in A_S$. Consequently, $\mathcal{M}_{A_S}$ is homeomorphic to the semigroup $H(S)$ equipped with pointwise convergence. □

As a consequence from Theorem 4.2.1 and (3.10), we obtain

$$\mathcal{M}_{A_S} \cong H(S) \cong \bigsqcup_{\omega \in \mathcal{I}_S} H_\omega(S) \times \widehat{\Gamma}_\omega \cong \bigsqcup_{\omega \in \mathcal{I}_S} H\big(\operatorname{supp}(\omega), (0,1]\big) \times \widehat{\Gamma}_\omega. \tag{4.3}$$

Theorem 4.2.2. *The Shilov boundary ∂A_S of the algebra A_S is homeomorphic to the group G.*

Proof. Since $A_S \subset C(G)$, it is clear that G is a boundary for A_S. Suppose that ∂A_S is a proper subset of G. Let $h \in G \setminus \partial A_S$, and let $g \in \partial A_S$. Then $h\,g^{-1}\partial A_S$ is also a boundary for A_S, since A_S is shift-invariant on G. Consequently, $\partial A_S \cap (h\,\partial A_S)$ is a boundary of A_S. Note that $h \in h\,g^{-1}\partial A_S$, while $h \notin \partial A_S$. Hence the boundary $\partial A_S \cap (h\,\partial A_S)$ is a proper subset of ∂A_S, in contradiction with its minimality property. □

The Gelfand transform, $\widehat{f}$, of any element $f \in A_S$ is a continuous function on $\mathcal{M}_{A_S}$, and the Gelfand transform $\widehat{A}_S = \{\widehat{f} \colon f \in A_S\}$ of A_S is a uniform algebra on $\mathcal{M}_{A_S}$. It is easy to show that a shift-invariant algebra A_S is antisymmetric if and only if the group kernel G_S is trivial, i.e. if $S \cap (-S) = \{0\}$ (cf. [AS1, T2]). As Arens and Singer [AS1] have shown , A_S is a maximal algebra if and only if $G_S = \{0\}$, $S \cup (-S) = \widehat{G}$, and the partial order generated by the semigroup S in $\Gamma = \widehat{G}$ is Archimedean (cf. [AS1, G1]). In this case, $\Gamma \subset \mathbb{R}$ and $S = \Gamma_+$. If Γ is dense in $\mathbb{R}$, then $G = \widehat{\Gamma}$ is a solenoidal group. If Γ is not dense in $\mathbb{R}$, then it is isomorphic to $\mathbb{Z}$, and G is isomorphic to the unit circle $\mathbb{T}$. In this case we also have $\widehat{G} \cong \mathbb{Z}$, $S \subset \mathbb{Z}_+$, and therefore the elements of the algebra A_S can be approximated uniformly on $\mathbb{T}$ by the usual polynomials. Hence they can be extended on the open unit disc $\mathbb{D}$ as analytic functions, continuous up to the boundary. Therefore A_S is isometrically isomorphic to the disc algebra $A(\mathbb{D})$ and $\mathcal{M}_{A_S} = \overline{\mathbb{D}}$.

In the sequel we will identify $\mathcal{M}_{A_S}$ with $H(S)$ without mention. Since $H(S)$ is a semigroup under pointwise multiplication $(\varphi\,\psi)(a) = \varphi(a)\,\psi(a),\ a \in S$, so is the maximal ideal space $\mathcal{M}_{A_S}$ of A_S, and $\chi^0 \equiv 1$ is its unit element.

Let P be a semigroup with $S \subset P \subset \Gamma_S$. Since $A_S \subset A_P$, we may consider that $\widehat{A}_S \subset \widehat{A}_P \subset C(\mathcal{M}_{A_P})$. Since $\Gamma_S = \widehat{G}$, the Gelfand transforms of the characters $\chi^a,\ a \in S$, separate the points of $\mathcal{M}_{A_P}$, and therefore, $\mathcal{M}_{A_P} \subset \mathcal{M}_{A_S}$, while $\partial A_P = G$. However, in general $\mathcal{M}_{A_P} \neq \mathcal{M}_{A_S}$.

If G is a compact group with $\widehat{G} \subset \mathbb{R}$ and $\Gamma_+ = \Gamma \cap \mathbb{R}_+$, then A_{Γ_+} is called the *G-disc algebra*, or, the *big disc algebra on G*. The Γ_+-functions are called also *analytic functions on G*, *analytic Γ_+-functions on G*, or, *generalized analytic functions* in the sense of Arens-Singer on G. From (3.13) it follows that $S = \mathbb{R}_+ = \mathbb{R} \cap [0, \infty)$, then

$$\mathcal{M}_{\mathbb{R}_+} \cong H(\mathbb{R}_+) \cong \big([0,1] \times G\big)/\big(\{0\} \times G\big) = [0,1] \diamond G/\{0\} \diamond G = \overline{\mathbb{D}}_G, \qquad (4.4)$$

where $G = b(\mathbb{R})$ is the Bohr compactification of $\mathbb{R}$, i.e. $\mathcal{M}_{A_{\mathbb{R}_+}}$ is bijective with the closed G-disc $\overline{\mathbb{D}}_G$. This bijection is materialized by the polar decomposition mapping $m \longmapsto \varphi_m = |\varphi_m|\,\chi_m \longmapsto r \diamond g$, where r is the corresponding real number to $|\varphi_m| \in H\big(\mathbb{R}_+, [0,1]\big) \cong [0,1]$, while $g \in G \cong \widehat{\Gamma}_d$ is the element in G that corresponds to $\chi_m \in \widehat{\Gamma}_d$. The topology near any point φ in $H(\mathbb{R}_+)$ agrees with the topology near the corresponding point $r \diamond g \in \overline{\mathbb{D}}_G$. Indeed, if $\varphi \in H(\mathbb{R}_+)$ is such that $\varphi \neq \chi_{\{0\}}$, then $\varphi \neq 0$ on $\mathbb{R}_+$. Let $r \diamond g,\ r > 0$, and $r_\alpha \diamond g_\alpha$ be the points corresponding to φ and φ_α in $\overline{\mathbb{D}}_G$. Clearly, $\varphi_\alpha \longrightarrow \varphi$ if and only if $|\varphi_\alpha| \longrightarrow |\varphi|$, and $\varphi_\alpha/|\varphi_\alpha| \longrightarrow \varphi/|\varphi|$ pointwise on $\mathbb{R}_+$, i.e. if and only if $r_\alpha \longrightarrow r$ and $g_\alpha \longrightarrow g$. Likewise, $\varphi_a \longrightarrow \varkappa_{\{0\}} = \chi^0 \in H(\mathbb{R}_+)$ if and only if $|\varphi_a| \longrightarrow 0$ pointwise on $\mathbb{R}_+ \setminus \{0\}$, i.e. if and only if $r_\alpha \longrightarrow 0$, wherefrom $r_\alpha \diamond g_\alpha \longrightarrow \omega$, the origin of $\mathbb{D}_G$. We have obtained the following

Proposition 4.2.3. *Let $G = b(\mathbb{R})$ be the Bohr compactification of $\mathbb{R}$. The maximal ideal space $\mathcal{M}_{A_{\mathbb{R}_+}}$ of the G-disc algebra $A_{\mathbb{R}_+}$ is homeomorphic to the G-disc $\overline{\mathbb{D}}_G = [0,1] \diamond G/\{0\} \diamond G$.*

A similar argument applies to any G-disc algebra A_{Γ_+} with $\Gamma \subset \mathbb{R}$.

Theorem 4.2.4. *Given a subgroup Γ of $\mathbb{R}_d$, the maximal ideal space $\mathcal{M}_{A_{\Gamma_+}}$ of the G-disc algebra A_{Γ_+} is homeomorphic to the G-disc $\overline{\mathbb{D}}_G = [0,1] \diamond G/\{0\} \diamond G$, where $G = \widehat{\Gamma}$.*

Proof. Indeed, Proposition 3.5.12 yields $\mathcal{M}_{A_{\Gamma_+}} \cong H(\Gamma_+) \cong \overline{\mathbb{D}}_G$. $\square$

If S is a semigroup of $\mathbb{R}_+$, $\Gamma = S - S$, and $G = \widehat{\Gamma}$, then $\overline{\mathbb{D}_G} = \mathcal{M}_{A_{\Gamma_+}} \subset \mathcal{M}_{A_S}$ and $\partial A = G = \widehat{\Gamma}_S$, by one of the remarks following Theorem 4.2.2. However, in general, $\mathcal{M}_{A_S}$ does not necessarily coincide with $\overline{\mathbb{D}_G}$.

Example 4.2.5. Theorem 4.2.1 and Examples 3.5.10 and 3.5.14 yield:

$\mathcal{M}_{(A_{\mathbb{Z}_+\times\mathbb{Z}_+})} \cong H(\mathbb{Z}_+ \times \mathbb{Z}_+) \cong \overline{\mathbb{D}}^2$, the closed bidisc in $\mathbb{C}^2$.

$\mathcal{M}_{(A_{\mathbb{R}_+\times\mathbb{R}_+})} \cong H(\mathbb{R}_+ \times \mathbb{R}_+) \cong \overline{\mathbb{D}}^2_{b(\mathbb{R})}$, the closed $b(\mathbb{R})$-bi-disc.

$\mathcal{M}_{(A_{(\Gamma_1)_+\times(\Gamma_2)_+\times\cdots\times(\Gamma_n)_+})} \cong H\big((\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+\big) \cong \overline{\mathbb{D}}_{\widehat{\Gamma}_1} \times \overline{\mathbb{D}}_{\widehat{\Gamma}_2} \times \cdots \times \overline{\mathbb{D}}_{\widehat{\Gamma}_n}$, where $\Gamma_i \subset \mathbb{R}$, $i = 1, \dots, n$ are additive subgroups of $\mathbb{R}$.

$\mathcal{M}_{(A_{\mathbb{Z}\times\mathbb{Z}_+})} \cong H(\mathbb{Z} \times \mathbb{Z}_+) \cong \mathbb{T} \times \overline{\mathbb{D}}$, the solid torus in $\mathbb{R}^3$.

$\mathcal{M}_{(A_{\mathbb{R}\times\mathbb{R}_+})} \cong H(\mathbb{R} \times \mathbb{R}_+) \cong b(\mathbb{R}) \times \overline{\mathbb{D}}_{b(\mathbb{R})}$, the solid $b(\mathbb{R})$-torus.

$\mathcal{M}_{(A_{\mathbb{N}\times\mathbb{N}\cup(0,0)})} \cong H(\mathbb{N} \times \mathbb{N} \cup (0,0)) \cong (0,0) \cup (\overline{\mathbb{D}}^* \times \overline{\mathbb{D}}^*) \cong \overline{\mathbb{D}}^2/\big((\{0\} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \{0\})\big)$.

$\mathcal{M}_{(A_{(0,\infty)\times(0,\infty)\cup(0,0)})} \cong H\big((0,\infty) \times (0,\infty) \cup (0,0)\big) \cong (\omega,\omega) \cup \big(\overline{\mathbb{D}}^*_{b(\mathbb{R})}\big)^2 \cong (\overline{\mathbb{D}}_{b(\mathbb{R})})^2/\big((\{\omega\} \times \overline{\mathbb{D}}_{b(\mathbb{R})}) \cup (\overline{\mathbb{D}}_{b(\mathbb{R})} \times \{\omega\})\big)$,

$\mathcal{M}_{(A_{\mathbb{Z}\times\mathbb{N}\cup(0,0)})} \cong H\big(\mathbb{Z} \times \mathbb{N} \cup (0,0)\big) \cong \overline{\mathbb{D}}_{\mathbb{T}^2}$, the closed $\mathbb{T}^2$-disc.

$\mathcal{M}_{(A_{\mathbb{R}\times(0,\infty)\cup(0,0)})} \cong H(\mathbb{R} \times (0,\infty) \cup (0,0)) \cong \overline{\mathbb{D}}_{b(\mathbb{R})^2}$, the closed $b(\mathbb{R})^2$-disc.

If β is an irrational number, then:

$\mathcal{M}_{(A_{(\mathbb{Z}_+ + \mathbb{Z}_+\beta)})} \cong H(\mathbb{Z}_+ + \mathbb{Z}_+\beta) \cong \overline{\mathbb{D}}^2$, the closed bi-disc.

$\mathcal{M}_{(A_{(\mathbb{Z}_+ + \mathbb{Z}_\beta)})} \cong H(\mathbb{Z}_+ + \mathbb{Z}\beta) \cong \mathbb{T} \times \overline{\mathbb{D}}$, the solid torus in $\mathbb{R}^3$.

The Gelfand transforms $\widehat{f}$ of an element $f \in A_{\Gamma_+}$ is continuous in the G-disc $\overline{\mathbb{D}}_G = \mathcal{M}_{A_{\Gamma_+}}$, and $A(\mathbb{D}_G) = \widehat{A}_{\Gamma_+} = \{\widehat{f}\colon f \in A_{\Gamma_+}\}$ is a uniform algebra on $\overline{\mathbb{D}}_G$. Clearly, $\widehat{f}$ is a continuous extension of f on $\overline{\mathbb{D}}_G$. In particular, the Gelfand transform of any $\chi^a \in \Gamma_+$ is the function $\widehat{\chi^a}(r \diamond g) = r^a g(a)$, which maps $\overline{\mathbb{D}}_G$ into $\overline{\mathbb{D}}$. Every Γ_+-polynomial $P(g) = \sum_{k=1}^n c_k \chi^{a_k}(g)$, can be extended on $\overline{\mathbb{D}}_G$ by $\widehat{P}(r \diamond g) = \sum_{k=1}^n c_k \widehat{\chi^{a_k}}(r \diamond g)$, where $a_k \in \Gamma_+$ and $r \diamond g \in \overline{\mathbb{D}}_G$. Since Γ_+-functions f on G are uniform limits of Γ_+-polynomials on the Shilov boundary $\partial A_{\Gamma_+} = G$, their Gelfand transforms $\widehat{f}$ are approximable by Gelfand transforms of Γ_+-polynomials $\widehat{P}$ on $\overline{\mathbb{D}}_G$ with respect to the uniform norm $\|f\| = \max_{r\diamond g \in \overline{\mathbb{D}}_G} \big|f(r \diamond g)\big|$ in $C(\overline{\mathbb{D}}_G)$.

Example 4.2.6. (a) The Gelfand transforms of the G-disc algebra $A(\mathbb{D}_g) = \widehat{A}_{\Gamma_+}(\mathbb{D}_G)$ of analytic Γ_+-functions are natural generalizations of the disc algebra $A(\mathbb{D})$. Actually, with $G = \mathbb{T}$ and $\Gamma = \widehat{G} = \mathbb{Z}$, the algebra $A(\mathbb{D}_G)$ coincides with the $\mathbb{Z}_+$-disc algebra $A(\mathbb{D}_{\mathbb{T}}) = \widehat{A_{\mathbb{Z}_+}} = A(\mathbb{D})$. Indeed, in this case $\overline{\mathbb{D}}_G = \overline{\mathbb{D}}$, $r \diamond g = re^{i\theta}$ is the standard trigonometric form of $z \in \mathbb{C}$, and $\chi^n(r \diamond g) = \chi^n(r\, e^{i\theta}) = r^n e^{in\theta} = z^n$ for every $n \in \mathbb{Z}_+$. Therefore, analytic $\mathbb{Z}_+$-functions are standard analytic functions in $\mathbb{D}$ continuous up to the boundary $\mathbb{T}$. Similarly, $A(\mathbb{D}_{\mathbb{T}^2}) = \widehat{A}_{\mathbb{Z}_+ \times \mathbb{Z}_+}$ is the bi-disc algebra of analytic functions in two complex variables on the closed unit bi-disc in $\mathbb{C}^2$.

(b) Let S be a semigroup in $\mathbb{R}_+$. Since, as we saw in Example 4.1.3(b), $H^\infty_S \cong A_S$, we have that $\mathcal{M}_{H^\infty_S} \cong \overline{\mathbb{D}_G}$.

(c) The maximal ideal space of the algebra from Example 4.1.3(c), generated by the functions $\widetilde{z}^a$, $a \in S$, on $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ is the G-disc $\overline{\mathbb{D}}_G$. Consequently, the portion of $\overline{\mathcal{R}}_{Log}(\overline{\mathbb{D}})$ above the closed unit disc $\overline{\mathbb{D}}$ of $\mathcal{R}_{Log}$ can be embedded densely and homeomorphically into the G-disc $\overline{\mathbb{D}}_G$.

(d) The algebra A_β from Example 4.1.3(d) can be obtained in a different way. Consider the compact set $\overline{\mathbb{D}}' = \left\{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| \le 1,\ |z_1| = |z_2|^\beta\right\}$ which contains $\mathbb{T}^2$. The set $\mathbb{D}' = \overline{\mathbb{D}}' \setminus \left(\mathbb{T}^2 \cup \{(0,0)\}\right)$ is a three-dimensional manifold with one-dimensional complex structure (i.e. a CR-manifold). Let $\widetilde{A}_\beta$ be the set of continuous functions on $\overline{\mathbb{D}}'$ which are CR-functions on $\mathbb{D}'$ (see [G13]). It can be shown that the restriction of $\widetilde{A}_\beta$ on $\mathbb{T}^2$ coincides with the algebra A_β. Since $\overline{\mathbb{D}}'$ is the space of maximal ideals of A_β, we have that $\overline{\mathbb{D}}' = \overline{\mathbb{D}}_{\mathbb{T}^2}$, and the Gelfand transform of A_β coincides with $\widetilde{A}_\beta$.

(e) Let Γ be a subgroup of $\mathbb{R}_d$. The algebra $AP_{\Gamma_+}(\mathbb{R})$ of almost periodic Γ_+-functions is isometrically isomorphic to the algebra A_{Γ_+} of analytic functions on $b(\mathbb{R})$. Therefore, the maximal ideal space of $AP_{\Gamma_+}(\mathbb{R})$ is homeomorphic to the closed $b(\mathbb{R})$-disc $\overline{\mathbb{D}}_{b(\mathbb{R})}$.

(f) Let $S = \mathbb{R}^n_+ \subset \mathbb{R}^n$. The algebra $AP_a(\mathbb{R}^n)$ of analytic almost periodic functions on $\mathbb{R}^n$ is isometrically isomorphic to the algebra $A_{\mathbb{R}^n_+}$ of analytic $\mathbb{R}^n_+$-functions on the group $b(\mathbb{R})^n$. Therefore, the maximal ideal space of the algebra of analytic almost periodic functions in n variables is homeomorphic to the closed $b(\mathbb{R})$-polydisc $\overline{\mathbb{D}}^n_{b(\mathbb{R})}$.

(g) Let $\Gamma_i \subset \mathbb{R}$, $i = 1, \ldots, n$ be additive subgroups of $\mathbb{R}$, and $S = (\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+ \subset \mathbb{R}^n_+$. The algebra $AP_{(\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+}(\mathbb{R}^n)$ of $(\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+$-almost periodic functions on $\mathbb{R}^n$ is isometrically isomorphic to the algebra $A_{(\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+}$ of analytic functions on the group $\widehat{\Gamma}_1 \times \widehat{\Gamma}_2 \times \cdots \times \widehat{\Gamma}_n$. Therefore, the maximal ideal space of the algebra of $(\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+$-almost periodic functions in n variables is homeomorphic to the closed polydisc $\overline{\mathbb{D}}_{\widehat{\Gamma}_1} \times \overline{\mathbb{D}}_{\widehat{\Gamma}_2} \times \cdots \times \overline{\mathbb{D}}_{\widehat{\Gamma}_n}$.

We recall that the strong hull $[S]_s$ of a semigroup S is the set of elements $a \in \widehat{G}$ for which there is an $m_a \in \mathbb{N}$ such that $m_a a \in S$. The weak hull $[S]_w$ of S is the set of elements $a \in \Gamma_S$ for which there is an $m_a \in \mathbb{N}$ such that $na \in S$ for every $n \geq m_a$ (cf. Definition 3.4.9). Clearly, $S \subset [S]_w \subset [S]_s \subset \Gamma$. Let S be a semigroup of Γ, and suppose that P is an additive subsemigroup of Γ, that contains 0, and is a subset of $[S]_s$. Proposition 3.5.6 implies that every semicharacter $\varphi \in H(S)$ can be extended uniquely on P as a semicharacter in $H(P)$. Therefore, we have

Proposition 4.2.7. *If S is a semigroup with cancellation law and 0, and P is a subsemigroup of Γ with $S \subset P \subset [S]_s \subset \Gamma$, then $\mathcal{M}_P = \mathcal{M}_S$.*

Proof. Since $A_S \subset A_P$, the restriction mapping $r\colon H(P) \longrightarrow H(S)\colon \varphi \longmapsto \varphi\big|_S$ maps $\mathcal{M}_{A_P} \cong H(P)$ continuously into $\mathcal{M}_{A_S} \cong H(S)$. Since by Proposition 3.5.6 every semicharacter on S admits a unique semicharacter extension on P, r is a bijection of $H(P)$ onto $H(S)$. □

Since $S \subset [S]_w \subset [S]_s$, Proposition 4.2.7 implies

Corollary 4.2.8. $\mathcal{M}_{A_{[S]_w}} = \mathcal{M}_{A_S}$ *for every semigroup $S \subset \widehat{G}$.*

Since, according to Corollary 4.2.8, $\mathcal{M}_{A_S} = \mathcal{M}_{A_{[S]_w}} = \mathcal{M}_{A_{[P]_w}} = \mathcal{M}_{A_P}$, we have the following

Corollary 4.2.9. *Let $S, P \subset \widehat{G}$ be two subsemigroups of $\widehat{G}$ such that $S - S = P - P = \widehat{G}$. If $[S]_w = [P]_w$, then $\mathcal{M}_{A_S} = \mathcal{M}_{A_P}$.*

Proposition 4.2.10. *Let $a \in \widehat{G} \setminus S$, and $S_a = S + \mathbb{Z}_+ a \subset \widehat{G}$. Then $\mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_S}$ if and only if $a \in [S]_w$.*

Proof. We need to prove the necessity part only, since the sufficiency follows from Corollary 4.2.8. Assume that $\mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_S}$ for some $a \in \widehat{G}$. Note that in this case the element $(-a)$ does not belong to S. Otherwise, the non-invertible function χ^{-a} in A_S will be invertible in A_{S_a}, in contradiction with the assumption $\mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_S}$. The same is true for all elements of type $(-n)\,a,\ n \in \mathbb{N}$. First we will show that $a \in [S]_s$. Suppose, on the contrary, that $\mathbb{N}a \cap S = \emptyset$. We have $S_a = S + \mathbb{Z}_+ a = J_a \cup \mathbb{Z}_+ a$ with $J_a = S \setminus \{0\} + \mathbb{N}a$, which is a semigroup ideal in S_a. The assumption $\mathbb{N}a \cap S = \emptyset$ implies that $J_a \cap (\mathbb{Z}_+ a) = \emptyset$. The functions

$$\gamma_1(c) = \begin{cases} 1 & \text{when } c \in \mathbb{Z}_+ a, \\ 0 & \text{when } c \in J_a, \end{cases}$$

and

$$\gamma_2(c) = \begin{cases} 1 & \text{when } c = 0, \\ 0 & \text{when } c \in S_a \setminus \{0\}, \end{cases}$$

are two different elements in $H(S_a) = H(S_a, \overline{\mathbb{D}})$ with one and the same restriction on S, namely,

$$\gamma(c) = \begin{cases} 1 & \text{when } c = 0, \\ 0 & \text{when } c \in S \setminus \{0\}. \end{cases}$$

Because of

$$\widehat{\chi}^c(m_1) = \gamma_1(c) = \gamma_2(c) = \widehat{\chi}^c(m_2) \text{ for all } c \in S,$$

for the linear multiplicative functionals $m_i \in \mathcal{M}_{A_S}$ corresponding to γ_i, $i = 1, 2$, we have $m_1 = m_2$. Since $\gamma_1 \neq \gamma_2$ on S_a, we must have $\widehat{\chi}^a(m_1) = \gamma_1(a) \neq \gamma_2(a) = \widehat{\chi}^a(m_2)$. Consequently, $m_1 \neq m_2$ as elements of $\mathcal{M}_{A_{S_a}}$, contradicting the assumption that $\mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_S}$. This completes the proof that $\mathbb{N}a \cap S \neq \emptyset$, i.e. that $a \in [S]_s$, as claimed.

Suppose that $a \notin [S]_w$ for some $a \in \widehat{G}$. Fix a prime number n. Proposition 3.4.11 implies that n divides every $m \in \mathbb{N}$ with $m \geq n$ and $ma \in S$. Define the set

$$S_1 = \{b \in \widehat{G} \colon b + c = k\, a \text{ for some } c \in S \text{ and } k \in \mathbb{N}\}.$$

Note that $S_1 = \big(\mathbb{N}\,(na) - S\big) \cap S$ is a semigroup of S containing 0. Its complement $S \setminus S_1$ is a semigroup ideal of S. We claim that $S \setminus S_1 \neq \emptyset$. Let $u, v \in S$ be such that $a = u - v$. If we assume that $v \in S_1$, then by definition there is a $w \in S$ with $v + w = p\, a$, $p \in \mathbb{N}$. Then $u + w = a + v + w = (p+1)\, a$, i.e. $u \in S_1$. According to Proposition 3.4.11 there is an $N \in \mathbb{N}$ such that $Ma = \big(r\, p + s\,(p+1)\big)\, a = r\,(v + w) + s\,(u + w) \in S$ for every $M > N$, in contradiction with the supposed $a \notin [S]_w$. Hence $S \setminus S_1 \neq \emptyset$, as claimed.

One can see that $S_a \setminus (\mathbb{N}a + S_1)$ is a semigroup ideal in S_a and if n does not divide $q \geq n$, then $(qa + S_1) \cap S = \emptyset$. Note that $\mathbb{N}\,(na) \cap S \subset S_1$. The function

$$\gamma_0(c) = \begin{cases} 1 & \text{when } c \in S_1, \\ 0 & \text{when } c \in S \setminus S_1 \end{cases}$$

belongs to $H(S)$ and can be extended to an element of $H(S_a)$ by

$$\gamma_3(c) = \begin{cases} \gamma_0(c) & \text{when } c \in S, \\ e^{2k\pi i/n} & \text{when } c = k\, a,\ k \in \mathbb{N}. \end{cases}$$

Also the function

$$\gamma_4(c) = \begin{cases} \gamma_0(c) & \text{when } c \in S, \\ 1 & \text{when } c = ka,\ k \in \mathbb{N} \end{cases}$$

can be extended as a semicharacter on S_a. Now γ_3 and γ_4 are two different elements in $H(S_a)$, whose restrictions on S are equal to γ_0. This is impossible since $H(S_a) = \mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_S} = H(S)$. Consequently, $a \in [S]_w$. ☐

Corollary 4.2.11. *Under the assumptions of Proposition* 4.2.10, $\mathcal{M}_{A_S} = \mathcal{M}_{A_P}$ *if and only if* $[S]_w = [P]_w$.

Proof. If $\mathcal{M}_{A_S} = \mathcal{M}_{A_P}$, then $a \in [S]_w$ for every $a \in P \setminus S$ by Proposition 4.2.10. Consequently, $P \subset [S]_w$ and hence $[P]_w \subset [S]_w$. The opposite inclusion follows immediately from Corollary 4.2.9. ☐

A uniform algebra A is said to be *analytic* on its maximal ideal space $\mathcal{M}_A$ if any function $f \in \widehat{A}$ that vanishes on an open subset of $\mathcal{M}_A \setminus \partial A$ vanishes identically on $\mathcal{M}_A$. In this section we will assume that all algebras are analytic. Let G be a compact abelian group and let S be a generating subsemigroup of the dual group $\Gamma = \widehat{G}$. It is easy to see that if a shift-invariant algebra A_S on G is analytic, then S does not contain subgroups other than $\{0\}$, i.e. $S \cap (-S) = \{0\}$.

Proposition 4.2.12. *Let $S \subset P \subset \widehat{G}$ be two subsemigroups of $\widehat{G}$ such that $S - S = \widehat{G}$ and $\mathcal{M}_{A_P} = \mathcal{M}_{A_S}$. If the algebra A_{S_a} is analytic for some $a \in P \setminus S$, then $\mathcal{M}_{A_S} = \mathcal{M}_{A_{S_a}}$, and therefore, $a \in [S]_w$ by Proposition* 4.2.10.

Proof. Clearly $\mathcal{M}_{A_S} \subset \mathcal{M}_{A_{S_a}}$ for any $a \in P \setminus S$, since $\widehat{\chi}^a$ is continuous on $\mathcal{M}_{A_P} = \mathcal{M}_{A_S}$. Suppose that $\mathcal{M}_{A_S} \neq \mathcal{M}_{A_{S_a}}$ for some $a \in P \setminus S$ and let $a = c - d$ for some $c, d \in S$. The natural inclusions $A_S \xrightarrow{i} A_{S_a} \hookrightarrow A_P$ generate adjoint mappings $\mathcal{M}_{A_P} \longrightarrow \mathcal{M}_{A_{S_a}} \xrightarrow{i^*} \mathcal{M}_{A_S}$ on the respective maximal ideal spaces. Note that by $\mathcal{M}_{A_S} = \mathcal{M}_{A_P}$, we have that $i^*(\mathcal{M}_{A_{S_a}}) = \mathcal{M}_{A_S}$. We claim that the mapping $i^* : \mathcal{M}_{A_{S_a}} \longrightarrow \mathcal{M}_{A_S}$ is injective. Observe that the restriction of i^* on $\mathcal{M}_{A_S} \subset \mathcal{M}_{A_{S_a}}$ is the identity mapping on $\mathcal{M}_{A_S}$. Assume that for some $m_0 \in \mathcal{M}_{A_S}$ the set $(i^*)^{-1}(m_0)$ contains a point, say m_1, different from m_0. Since $m_0(\widehat{\chi}^b) = m_1(\widehat{\chi}^b)$ for every $b \in S$, then necessarily $m_0(\widehat{\chi}^a) \neq m_1(\widehat{\chi}^a)$. Thus

$$m_0(\widehat{\chi}^{a+d}) = m_0(\widehat{\chi}^c) = m_1(\widehat{\chi}^c) = m_1(\widehat{\chi}^{a+d}),$$

and therefore,

$$m_0(\widehat{\chi}^a)\, m_0(\widehat{\chi}^d) = m_0(\widehat{\chi}^c) = m_1(\widehat{\chi}^a) m_1(\widehat{\chi}^d) = m_1(\widehat{\chi}^a)\, m_0(\widehat{\chi}^d).$$

Hence $m_0(\widehat{\chi}^d) = m_0(\widehat{\chi}^c) = 0$, since $m_0(\widehat{\chi}^a) \neq m_1(\widehat{\chi}^a)$, i.e. $m_0 \in \mathrm{Null}\,(\widehat{\chi}^c)$. Because of $(i^*)^{-1}\big(\mathrm{Null}\,(\widehat{\chi}^c)\big) = \mathrm{Null}_{S_a}(\widehat{\chi}^c) = \big\{m \in \mathcal{M}_{A_{S_a}} : m(\widehat{\chi}^c) = 0\big\}$, i^* maps the set $\mathcal{M}_{A_{S_a}} \setminus \mathrm{Null}_{S_a}(\widehat{\chi}^c)$ homeomorphically onto the set $\mathcal{M}_{A_S} \setminus \mathrm{Null}\,(\widehat{\chi}^c)$, i.e. the restriction of i^* on the set $\mathcal{M}_{A_{S_a}} \setminus \mathrm{Null}_{S_a}(\widehat{\chi}^c)$ is the identity mapping. By the analyticity of S_a, $\mathcal{M}_{A_{S_a}} \setminus \mathrm{Null}_{S_a}(\widehat{\chi}^c)$ is dense in $\mathcal{M}_{A_{S_a}}$. Let $m_\alpha \longrightarrow m_1$, $m_\alpha \in \mathcal{M}_{A_{S_a}} \setminus \mathrm{Null}_{S_a}(\widehat{\chi}^c) = \mathcal{M}_{A_S} \setminus \mathrm{Null}\,(\widehat{\chi}^c)$. If we choose a convergent subsequence $m_{\alpha_\beta} \in \mathcal{M}_{A_S} \setminus \mathrm{Null}\,(\widehat{\chi}^c)$, then $m_{\alpha_\beta} \longrightarrow m_1$, contrary to $m_{\alpha_\beta} = (i^*)(m_{\alpha_\beta}) \longrightarrow (i^*)(m_1) = m_0 \neq m_1$. ☐

If $\Gamma \subset \mathbb{R}$, and $[S]_s = \Gamma_+$, then Proposition 4.2.7 implies that $\mathcal{M}_{A_S} \cong \mathcal{M}_{A_{\Gamma_+}} \cong \overline{\mathbb{D}_G}$, the G-disc over $G = \widehat{\Gamma}$. We have obtained the following corona type results.

Corollary 4.2.13. *Let $\Gamma \subset \mathbb{R}$, $G = \widehat{\Gamma}$, and let S be a semigroup of $\mathbb{R}$ with $[S]_s = \Gamma_+$. Then:*

(a) *$\mathcal{M}_{A_S} \cong \overline{\mathbb{D}}_G$, thus the shift-invariant algebra A_S does not have a $\mathbb{C}_+$-corona. Namely, the mapping $\widetilde{\jmath}_\imath$ densely embeds the upper half-plane $\overline{\mathbb{C}}_+$ into its maximal ideal space $\mathcal{M}_{A_S}$.*

(b) *The algebra $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$ does not have a $\mathbb{C}_+$-corona. Namely, the mapping $\widetilde{\jmath}_\imath$ densely embeds the upper half-plane $\overline{\mathbb{C}}_+$ into its maximal ideal space $\mathcal{M}_{AP_S(\mathbb{R})}$.*

(c) *The algebra H^∞_S does not have a $\overline{\mathbb{C}}_+$-corona. Namely, $\widetilde{\jmath}_\imath(\overline{\mathbb{C}}_+)$ is dense in the maximal ideal space $\mathcal{M}_{H^\infty_S}$.*

Corollary 4.2.14. *Let $\Gamma_i \subset \mathbb{R}$, $i = 1, \dots, n$ be dense additive subgroups of $\mathbb{R}$, $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and $G = \widehat{\Gamma}_1 \times \widehat{\Gamma}_2 \times \cdots \times \widehat{\Gamma}_n$. If S is a semigroup of $\mathbb{R}^n$ with $[S]_s = (\Gamma_1)_+ \times (\Gamma_2)_+ \times \cdots \times (\Gamma_n)_+$, then:*

(a) *The shift-invariant algebra A_S does not have a $\mathbb{C}^n_+$-corona$\mathcal{M}_{A_S} \cong \overline{\mathbb{D}}_{\widehat{\Gamma}_1} \times \overline{\mathbb{D}}_{\widehat{\Gamma}_2} \times \cdots \times \overline{\mathbb{D}}_{\widehat{\Gamma}_n}$. Namely, the mapping $\left(\widetilde{\jmath}_\imath^{(1)}, \widetilde{\jmath}_\imath^{(2)}, \dots, \widetilde{\jmath}_\imath^{(n)}\right)$ densely embeds the Cartesian product $\mathbb{C}^n_+$ into its maximal ideal space $\mathcal{M}_{A_S}$.*

(b) *The algebra $AP_S(\mathbb{R})$ of almost periodic S-functions, on $\mathbb{R}^n$ does not have a $\mathbb{C}^n_+$-corona. Namely, the mapping $\left(\widetilde{\jmath}_\imath^{(1)}, \widetilde{\jmath}_\imath^{(2)}, \dots, \widetilde{\jmath}_\imath^{(n)}\right)$ densely embeds $\mathbb{C}^n_+$ into its maximal ideal space $\mathcal{M}_{AP_S(\mathbb{R}^n)}$.*

Corollary 4.2.15. *If S is a semigroup in $\mathbb{R}^k_+$ with $[S]_s^{\mathbb{R}^k} = \mathbb{R}^k_+$, then the maximal ideal space $\mathcal{M}_{AP_S(\mathbb{R}^k)}$ of the algebra $AP_S(\mathbb{R}^k)$ of almost periodic S-functions on $\mathbb{R}^k$ is homeomorphic to the $b(\mathbb{R})$-polydisc $\overline{\mathbb{D}}^n_{b(\mathbb{R})} \cong \mathcal{M}_{AP_{\mathbb{R}^k_+}(\mathbb{R}^k)} = \mathcal{M}_{AP_a(\mathbb{R}^k)}$.*

Corollary 4.2.14 implies that in the setting of Corollary 4.2.15 the Cartesian product $\overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+ \times \cdots \times \overline{\mathbb{C}}_+ \subset \mathbb{C}^k$ can be embedded densely into the closed $b(\mathbb{R})$-polydisc $\overline{\mathbb{D}}^n_{b(\mathbb{R})} \sim \mathcal{M}_{AP_S(\mathbb{R}^k)} = \mathcal{M}_{AP_a(\mathbb{R}^k)}$. Therefore, the algebra $AP_S(\mathbb{R}^k)$ does not have a $(\mathbb{C}_+)^k$-corona.

The third part of Corollary 4.2.13 implies that the algebra H^∞_S does not have a $\mathbb{D}$-corona either. Consider the mapping $\pi\colon \mathcal{M}_{H^\infty} \longrightarrow \mathbb{D}$ defined by $\pi(\varphi) = \varphi(\mathrm{id}) \in \overline{\mathbb{D}}$. It is easy to see that π is a bijection on the set $\widetilde{\mathbb{D}} = \pi^{-1}(\mathbb{D}) \subset \mathcal{M}_{H^\infty}$. Therefore, $\widetilde{\mathbb{D}}$ and $\mathbb{D}$ are homeomorphic sets. Since $H^\infty_S \subset H^\infty$, there is a mapping $\pi_S\colon \mathcal{M}_{H^\infty} \longrightarrow \mathcal{M}_{H^\infty_S}$ so that $\pi_S \circ \pi^{-1}$ is the standard embedding of $\mathbb{D}$ into $\mathcal{M}_{H^\infty_S}$. If $\varphi(x) = i\,(1+z)/(1-z)$, then $\pi_S \circ \pi^{-1} = \widetilde{\jmath}_\imath \circ \varphi\colon \mathbb{D} \longrightarrow \mathcal{M}_{H^\infty_S}$, and therefore, $\pi_S \circ \pi^{-1}$ maps $\mathbb{D}$ densely into $\mathcal{M}_{H^\infty_S}$. By Corollary 4.2.13(c) it follows that H^∞_S does not have a $\mathbb{D}$-corona, as claimed.

Theorem 4.2.16. *If G is a solenoidal group, and S is an additive subsemigroup of $\Gamma_+ \subset \mathbb{R}_+$ that contains 0 and generates $\Gamma = \widehat{G}$, then there is a continuous mapping from $\mathcal{M}_{H^\infty}$ onto the closed G-disc $\overline{\mathbb{D}}_G$.*

Proof. The maximal ideal space of the algebra H^∞_S is the G-disc $\overline{\mathbb{D}}_G$. Since $H^\infty_S \cong A_S \subset A_{\Gamma_+} \cong H^\infty_{\Gamma_+} \subset H^\infty$, the restriction mapping $r\colon m \longmapsto m|_{H^\infty_S}$ maps $\mathcal{M}_{H^\infty}$ into $\mathcal{M}_{H^\infty_S} \cong \mathcal{M}_{A_S} = \overline{\mathbb{D}}_G$. If φ_z is the evaluation functional on H^∞ at some point $z \in \mathbb{D}$, then $r(\varphi_z) \in \mathbb{D}_G$. Hence, $r(\mathcal{M}_{H^\infty}) \supset \mathbb{D}_G$. Since $\mathbb{D}^*_G \cong \mathbb{C}_+$ is dense in the G-disc $\overline{\mathbb{D}}_G$, and $r(\mathcal{M}_{H^\infty})$ is closed in $\overline{\mathbb{D}}_G$, then $r(\mathcal{M}_{H^\infty}) \subset \overline{\mathbb{D}}_G$ too, i.e. $r(\mathcal{M}_{H^\infty}) = \overline{\mathbb{D}}_G$. ☐

4.3 Automorphisms of shift-invariant algebras

The study of automorphisms of a uniform algebra is of considerable importance for the algebra. In this section we describe automorphisms of shift-invariant algebras on compact groups.

Let G be a compact abelian group and let S be a subsemigroup generating its dual group $\widehat{G}$, containing the unit element $\chi^0 \equiv 1$, and such that $S \cap (-S) = \{0\}$. Hence, $\Gamma_S = S - S = \widehat{G}$, thus, S separates the points of G, and S contains no non-trivial subgroups. Therefore, the shift-invariant algebra A_S is antisymmetric.

An *automorphism* of an algebra A is any isometric isomorphism Φ of A onto itself. The adjoint mapping Φ^* of Φ defined by $\big(\Phi^*(m)\big)(f) = m\big(\Phi(f)\big)$, is a homeomorphism of the maximal ideal space $\mathcal{M}_A$ onto itself.

For example, adjoint mappings Φ^* of automorphisms Φ of the disc algebra $A(\mathbb{D}) = A_{\mathbb{Z}_+}$ are Möbius transformations, i.e.

$$\Phi^*(z) = C\frac{z - z_0}{1 - \overline{z}_0 z}, \ |C| = 1, \ |z_0| < 1.$$

Note that if the origin 0 is a fixed point of a Möbius transformation Φ^*, then $\Phi^*(z) = Cz$ for some constant C with $|C| = 1$. As the following lemma shows, the same fact holds for automorphisms of the subalgebra $A_0(\mathbb{D}) = \big\{f \in A(\mathbb{D})\colon f'(0) = 0\big\}$ of the disc algebra $A(\mathbb{D})$. It is easy to check that if a Möbius transformation $\psi(z) = C\dfrac{z - z_0}{1 - \overline{z}_0 z}$ belongs to $A_0(\mathbb{D})$, i.e. if $\psi'(0) = 0$, then $z_0 = 0$, thus $\psi(z) = Cz$. Observe, that $A_0(\mathbb{D}) = A_S$, where S is the subsemigroup $\mathbb{Z}_+ \setminus \{1\} = \{0, 2, 3, 4, \dots\}$ of $\mathbb{Z}$.

Lemma 4.3.1. *The adjoint mapping of an automorphism of the algebra* $A_0(\mathbb{D}) = \big\{f \in A(\mathbb{D})\colon f'(0) = 0\big\}$ *fixes the origin.*

Proof. If $\Phi : A_0(\mathbb{D}) \longrightarrow A_0(\mathbb{D})$ is an automorphism, then its adjoint Φ^* is a homeomorphism of the unit disc $\overline{\mathbb{D}}$ onto itself, and

$$f\big(\Phi^*(z)\big) = \big(\Phi(f)\big)(z) \text{ for any } \varphi \in A_0(\mathbb{D}). \tag{4.5}$$

Moreover, Φ^* is an analytic function of the unit disc $\overline{\mathbb{D}}$ onto itself, i.e. Φ^* is a Möbius transformation. Applied to the function $f(z) = z^2$, (4.5) implies $\big(\Phi(f)\big)(z) = f\big(\Phi^*(z)\big) = \big(\Phi^*(z)\big)^2 = (\Phi^*)^2(z)$, i.e. $\Phi(f) = (\Phi^*)^2$. Therefore, $(\Phi^*)^2 \in A_0(\mathbb{D})$, and hence $\big((\Phi^*)^2\big)'(0) = 0$. Consequently, $2\Phi^*(0)(\Phi^*)'(0) = 0$, thus $\Phi^*(0) = 0$, or $(\Phi^*)'(0) = 0$. In the second case $\Phi^*(z) = Cz$, and therefore, $\Phi^*(0) = 0$ too. $\square$

Observe that the adjoint mapping of any automorphism Φ of a shift-invariant algebra A_S on a group G maps the maximal ideal space $\mathcal{M}_{A_S}$ homeomorphically onto itself, and maps idempotent semicharacters of S to idempotent semicharacters of S. Indeed, if $\psi \in \mathcal{I}_S \subset H(S) \cong \mathcal{M}_{A_S}$ and $m_\psi \in \mathcal{M}_{A_S}$ is the corresponding linear multiplicative functional on A_S, then for any $a \in S$ we have $\big((\Phi^*(m_\psi))(\chi^a)\big)^2 = \big(M_\psi(\Phi(\chi^a))\big)^2 = (m_\psi)^2\big(\Phi(\chi^a)\big) = m_\psi\big(\Phi(\chi^a)\big) = \big(\Phi^*(m_\psi)\big)(\chi^a)$, i.e. $\big(\Phi^*(m_\psi)\big)^2 = \Phi^*(m_\psi)$, thus the semicharacter of S corresponding to $\Phi^*(m_\psi)$ is idempotent. Hence, Φ^* maps $\mathcal{I}_S$ onto itself.

An automorphism Φ of A_S is said to be *inner*, if there is a $\tau \in \mathrm{Hom}\,(S,S)$ and an element $g_0 \in G$ such that $\Phi(\chi^a) = \chi^a(g_0)\,\chi^{\tau(a)}$ for every $\chi^a \in S$. Arens and Singer [AS1] have shown that in the case when G is a solenoidal group and S is a semigroup in $\mathbb{R}$ with $S \cup (-S) = \widehat{G}$, then every automorphism φ of the algebra A_S is inner.

Example 4.3.2. Not every automorphism Φ of the disc algebra $A(\mathbb{D})$ is inner. However, any automorphism Φ whose adjoint fixes the origin, i.e. for which $\Phi^*(z) = Cz$, $|C| = 1$ is inner. Indeed, for every $z \in \overline{\mathbb{D}}$ we have $\big(\Phi(f)\big)(z) = f\big(\Phi^*(z)\big) = f(Cz)$. If $\chi^n \in \mathbb{Z}_+$ is the function $\chi^n(z) = z^n$, $n \geq 0$, then $\big(\Phi(\chi^n)\big)(z) = \big(\Phi^*(z)\big)^n = (Cz)^n = C^n\chi^n(z)$, hence $\Phi(\chi^n) = C^n\chi^n = \chi^n(C)\,\chi^n$, i.e. Φ is an inner automorphism, since $C \in \mathbb{T}$.

If the group envelope Γ_S of S is dense in $\mathbb{R}$, then G is a solenoidal group, and the algebra A_S is a subalgebra of the $\widehat{\Gamma}_S$-disc algebra $A_{(\Gamma_S)_+}$ on the group $G = \widehat{\Gamma}_S$. Since $\Gamma_S = \widehat{G}$, the characters χ^a, $a \in S$, separate the points of $\mathbb{D}_G$, and therefore, $\mathbb{D}_G \subset \mathcal{M}_{A_S}$, and $\partial A = G$. Let $z = x + iy \in \overline{\mathbb{C}}_+$. For any $a \in S$ the mapping $a \longmapsto e^{-ay}\chi^a\big(j_\imath(x)\big)$ is a semicharacter of S, where $j_\imath$ is the standard embedding of $\mathbb{R}$ into G. Denote by $m_{(x,y)}$ its corresponding linear multiplicative functional on A_S. There arises a mapping $\widetilde{j}_\imath \colon \overline{\mathbb{C}}_+ \longrightarrow \mathcal{M}_{A_S} : x + iy \longmapsto m_{(x,y)}$. One can show that $\widetilde{j}_\imath$ is an embedding of $\overline{\mathbb{C}}_+$ into $\mathcal{M}_{A_S}$. For any $a \in S$ we have $\widehat{\chi}^a\big(\widetilde{j}_\imath(z)\big) = \widehat{\chi}^a\big(\widetilde{j}_\imath(x+iy)\big) = \widehat{\chi}^a\big(m_{(x,y)}\big) = m_{(x,y)}(\chi^a) = e^{-ay}\chi^a\big(j_\imath(x)\big) = e^{-ay}e^{iax} = e^{ia(x+iy)} = e^{iaz}$, i.e. any $\chi^a \in S$ can be extended analytically on the range $\widetilde{j}_\imath(\overline{\mathbb{C}}_+)$. Consequently, for any $f \in A_S$ the analytic almost periodic function $\widehat{f}|_{j_\imath(\overline{\mathbb{C}}_+)}$ is an analytic extension of f on $\mathbb{C}_+$. Note that $\widehat{\chi^a} \neq 0$ on $\widetilde{j}_\imath(\overline{\mathbb{C}}_+)$, i.e. the maximal ideals of A_S on which $\widehat{\chi^a}$ vanishes are outside $\widetilde{j}_\imath(\overline{\mathbb{C}}_+)$.

Theorem 4.3.3. *If G is a group with $\widehat{G} \subset \mathbb{R}$, and A_S is a shift-invariant algebra on G with $\Gamma_S = \widehat{G}$, then either $A_S \cong A(\mathbb{D})$, or every automorphism of the algebra A_S is inner.*

Proof. If the group Γ_S generated by S is not dense in $\mathbb{R}$, then A_S is a subalgebra of the disc algebra $A(\mathbb{D})$. If we assume that A_S differs from $A(\mathbb{D})$, then, as it is easy to see, $1 \notin S$, thus $A_S \subset A_0(\mathbb{D})$. In the same way as for $A_0(\mathbb{D})$ one can see that in this case any automorphism is a composition operator generated by a Möbius transformation that fixes the origin, thus it is an inner automorphism.

If the group envelope Γ_S of S is dense in $\mathbb{R}$, then the algebra A_S is a subalgebra of the $\widehat{\Gamma}_S$-disc algebra $A_{\widehat{\Gamma}_S}$. Let Φ be an automorphism of A_S onto itself. By the remark preceding Theorem 4.3.3, the function $\widehat{\chi}^a$ does not vanish on $\widetilde{\jmath}_\imath(\mathbb{C}_+) \subset \mathcal{M}_{A_S}$ for any $a \in S$. We claim that $\widehat{\Phi(\chi^a)}$ does not vanish on $\widetilde{\jmath}_\imath(\mathbb{C}_+)$ either. Indeed, if $m \in \text{Null}\,\big(\widehat{\Phi(\chi^a)}\big) \subset \mathcal{M}_{A_S}$, then $0 = m\big(\Phi(\chi^a)\big) = \big(\Phi^*(m)\big)(\chi^a) = \widehat{\chi}^a\big(\Phi^*(m)\big)$, i.e. $\Phi^*(m) \notin \widetilde{\jmath}_\imath(\mathbb{C}_+)$. On the other hand, $\text{Null}\,\big(\Phi^*(m)\big) = \big\{a \in S\colon \big(\Phi^*(m)\big)(\chi^a) = 0\big\}$ is a semigroup ideal in S. Consider the function

$$\gamma(b) = \begin{cases} 0 & \text{when } \chi^b \in \text{ Null}\,\big(\Phi^*(m)\big), \\ 1 & \text{when } \chi^b \in S \setminus \text{Null}\,\big(\Phi^*(m)\big). \end{cases}$$

Clearly, $\gamma \in \mathcal{I}_S$, and $\gamma\,\Phi^*(m) = \Phi^*(m)$. Thus $(\Phi^*)^{-1}(\gamma)$ is a non-trivial idempotent semicharacter of S, and $\big((\Phi^*)^{-1}(\gamma)\big)\,m = m$. Hence, $\varphi_m(a) = m(\chi^a) = \big((\Phi^*)^{-1}(\gamma)\big)(\chi^a)\,m(\chi^a)$ is a non-invertible semicharacter on S, thus $\text{Null}\,(\varphi_m) \neq \emptyset$, i.e. $\widehat{\chi}^a(\varphi_m) = \varphi_m(a) = 0$ for some $a \in S$. Therefore, $m \notin \widetilde{\jmath}_\imath(\mathbb{C}_+)$ by the remark before Theorem 4.3.3. We have obtained that $\widehat{\Phi(\chi^a)} \neq 0$ on $\widetilde{\jmath}_\imath(\mathbb{C}_+)$, as claimed. Consequently, the bounded analytic function $\widetilde{\Phi(\chi^a)}(z)$ does not have zeros in $\overline{\mathbb{C}}_+$. Moreover, $\big|\Phi(\chi^a)\big| \equiv 1$ on $j_\imath(\mathbb{R})$. Indeed, $\big|\big(\Phi(\chi^a)\big)(g)\big| = \big|\chi^a(\Phi^*(g))\big| = 1$ for every $g \in G$, since $|\chi^a| \equiv 1$, and $\Phi^*(G) = G$, because on $G = \partial A_S$. By Besicovitch's theorem [B], $\widehat{\Phi(\chi^a)}\big(\widetilde{\jmath}_\imath(z)\big) = \widetilde{\Phi(\chi^a)}(z) = Ce^{isz} = C\widehat{\chi^s}\big(\widetilde{\jmath}_\imath(z)\big)$, where $s \geq 0$, $C \in \mathbb{C}$, $|C| = 1$. Hence, $C = \chi^a(g_0)$ for some $g_0 \in G$. As it it is easy to see, $s \in S$, and the mapping $\tau\colon S \longrightarrow S\colon \chi^a \longmapsto \chi^s$ is a homomorphism. □

4.4 p-groups and peak groups of shift-invariant algebras

In this section we give conditions for a subgroup of the carrier group of a shift-invariant algebra to be a peak set or a p-set for the algebra.

Let $A \subset C(X)$ be a uniform algebra on a compact set X. Recall that a closed set K in X is a peak set (for A), if there is a function $f \in A$, such that $f|_K \equiv 1$, and $|\widehat{f}(m)| < 1$ whenever $m \in \mathcal{M}_A \setminus K$. Intersections of peak sets are called *p-sets*.

Let G be a connected compact abelian group and let S be a semigroup that contains 0 and generates the dual group $\widehat{G}$. Here we give necessary and sufficient conditions for a closed subgroup H of G to be a peak set for the algebra A_S, i.e. to be a p-group for A_S.

Example 4.4.1. The subgroup $H = \{(z_1, z_2) \in \mathbb{T}^2\colon\ z_1 = z_2\}$ of the torus $\mathbb{T}^2$ is not a peak set for the bi-disc algebra $A(\mathbb{T}^2) \cong A_{\mathbb{Z}_+^2}$. Indeed, assume, on the contrary, that H is a peak set, and let $f \in A(\mathbb{T}^2)$ peak on H, i.e. $f|_{\{z_1=z_2\}} = f(z,z) \equiv 1$, while $\big|\widehat{f}(z_1, z_2)\big| < 1$ whenever $z_1 \neq z_2$, where $\widehat{f}$ is the analytic extension of f on the bi-disc $\overline{\mathbb{D}}^2$. Clearly, the function $v(z) = \widehat{f}(z,z) = \widehat{f}|_H$ is analytic in $z \in \mathbb{D}$. By $v(z) \equiv 1$ on $\mathbb{T}$ we see that $v(z) \equiv 1$ on $\overline{\mathbb{D}}$, and in particular, $\widehat{f}(0,0) = v(0) = 1$.

Since $A(\mathbb{T}^2)$ is shift-invariant on $\mathbb{T}^2$, the function

$$F(u_1,u_2) = \int\limits_{|z|=1} \frac{f(zu_1, zu_2)}{z}\, dz = \int\limits_H f(z_1u_1, z_2u_2)\, d\sigma(z_1, z_2)$$

belongs to $A(\mathbb{T}^2)$. The function $w(z) = \widehat{f}(u_1z, u_2z)$ is analytic in $z \in \mathbb{D}$, and therefore

$$F(u_1,u_2) = \int\limits_{|z|=1} \frac{w(z)}{z}\, dz = w(0) = \widehat{f}(0,0) = v(0) = 1$$

for every $(u_1, u_2) \in \mathbb{T}^2$. This is impossible, since if $(u_1, u_2) \in \mathbb{T}^2 \setminus H$, then

$$\big|F(u_1,u_2)\big| \leq \int\limits_H \big|f(zu_1, zu_2)\big|\, d\sigma(z_1, z_2) < 1.$$

Consequently, there are no functions in $A(\mathbb{T}^2)$ that peak on H, and hence H is not a peak group for $A(\mathbb{T}^2)$.

If A_S is a shift-invariant algebra on G, then every $g \in G$ is a p-point for A_S. Indeed, $\{g\} = \bigcap \big\{\mathrm{Ker}\,(\widehat{\chi}^a)\colon \chi^a(g) = 1,\ a \in S\big\}$, since the characters χ^a separate the points of $\mathcal{M}_{A_S}$. For every $a \in S$ the kernel $\mathrm{Ker}\,(\widehat{\chi}^a) = \big\{g \in \mathcal{M}_{A_S} \colon \widehat{\chi}^a(g) = 1\big\}$ is a peak set for A_S, since the function $h = (1 + \widehat{\chi}^a)/2 \in \widehat{A}_S$ is identically equal to 1 exactly on $\mathrm{Ker}\,(\widehat{\chi}^a) \subset \mathcal{M}_{A_S}$.

If $g \in G$ is a peak point for the shift-invariant algebra A_S on G, so is any point $g_0 \in G$. Indeed, if $f \in A_S$, $S \subset \mathbb{R}_+$, is a peaking function at g for A_S, then $f_{g_0^{-1}g}(h) = f(g_0^{-1}gh)$ peaks at g_0. Therefore, peak points for A_S are either all points in G, or none of them. For instance, if the group Γ is countable, then $\imath$ is a peak point for A_S. Indeed, let $\{\chi^{a_i}\}_{i\in\mathbb{N}}$ be an enumeration of S. Then the function

$$f(g) = \sum_{i=1}^{\infty} \frac{1}{2^n} \chi^{a_i}(g)$$

belongs to A_S, and its Gelfand transform peaks at $\imath$, i.e. $\widehat{f}(\imath) = 1$, while $\big|\widehat{f}(r \diamond g)\big| < 1$ at any other point $r \diamond g \in \overline{\mathbb{D}_G}$. Hence, if S is countable, or, more generally, if G is a metrizable group, every point of G is a peak point for A_S.

If $K \subset G$, we denote by $K^\perp = \{\chi \in \widehat{G} \colon \chi|_K \equiv 1\} = \big\{\chi \in \widehat{G} \colon\ \mathrm{Ker}\,(\chi) \supset K\big\}$ the *orthogonal set* of K. Let H be a closed subgroup of G, and let $\pi_H \colon G \longrightarrow G/H$ be the natural projection from G onto the quotient group G/H. We recall that a natural isomorphism from the dual group $(G/H)\widehat{\ }$ to the set $H^\perp = \big\{\chi \in \widehat{G}\colon \chi|_H \equiv 1\big\} = \big\{\chi \in \widehat{G}\colon\ \mathrm{Ker}\,(\chi) \supset H\big\}$ can be obtained as follows. Observe that every $\chi \in H^\perp$ generates a character $\widetilde{\chi} \in (G/H)\widehat{\ }$, defined as $\widetilde{\chi}\big(\pi_H(g)\big) = \chi(g)$.

Clearly, $\chi \longmapsto \widetilde{\chi}$ is an isomorphism of $H^\perp$ onto $(G/H)\widehat{\ }$, and its inverse is the mapping $\gamma \longmapsto \gamma \circ \pi_H \colon (G/H)\widehat{\ } \longrightarrow H^\perp$.

Consider the semigroup $H_S^\perp = H^\perp \cap S = \{\chi \in S \colon \chi|_H \equiv 1\}$, which is isomorphic to the subsemigroup $\widetilde{H}_S^\perp = \{\widetilde{\chi} \colon \chi \in H_S^\perp\}$ of the group $(G/H)\widehat{\ }$. The subalgebra $A_{H_S^\perp}$ of A_S generated by $H_S^\perp$ is isometrically isomorphic to the shift-invariant algebra $A_{\widetilde{H}_S^\perp}$ on the quotient group $G/H \cong \pi_H(G)$. Note that, in general, the set $H_S^\perp$ does not separate the H-cosets of G. For instance, in the case of the group $H = \big\{(z_1, z_2) \in \mathbb{D}^2 \colon z_1 = z_2\big\} \subset \mathbb{T}^2$, considered in Example 4.4.1, and $S = \mathbb{Z}_+^2$, the set $H_{\mathbb{Z}_+^2}^\perp = \{1\}$ clearly does not separate the H-cosets of $\mathbb{T}^2$.

Lemma 4.4.2. *A closed group $H \subset G$ is a peak set [resp. a p-set] for the algebra A_S if and only if the unit $\pi_H(H) = \pi_H(\imath)$ of the group $G/H = \pi_H(G)$ is a peak point [resp. a p-point] for $A_{\widetilde{H}_S^\perp}$.*

Proof. Let $\pi_H(\imath)$ be a peak point for the algebra $A_{\widetilde{H}_S^\perp}$, and let $f \in A_{\widetilde{H}_S^\perp}$ be such that $\widehat{f}$ peaks on $\pi_H(\imath)$. Then $\widehat{f} \circ \pi_H$ is a function in $\widehat{A}_S$ that peaks on $H = \pi_H^{-1}\big(\pi_H(\imath)\big) = \pi_H^{-1}\big(\pi_H(H)\big)$.

Conversely, let H be a peak set for A_S, and let $f \in A_S$ be such that $\widehat{f}$ peaks on H, namely, $\widehat{f}|_H \equiv 1$, while $\big|\widehat{f}(m)\big| < 1$ for $m \in \mathcal{M}_{A_S} \setminus H$. Since A_S is shift-invariant on G, the function $F(g) = \displaystyle\int_H f(hg)\, d\sigma(h)$ also belongs to A_S, where σ is the Haar measure on H. The Gelfand transform

$$\widehat{F}(m) = \int_H \widehat{f}_h(m)\, d\sigma(h)$$

of F is constant on every H-coset of $\mathcal{M}_{A_S}$, and $\big|\widehat{F}(m)\big| \le \displaystyle\int_H \big|\widehat{f}_h(m)\big|\, d\sigma(h) < 1$ for any $m \in \mathcal{M}_{A_S} \setminus H$. Therefore, $\widetilde{F} \colon \pi_H(g) \longmapsto \widehat{F}(g)$ is a well-defined continuous function on $\mathcal{M}_{A_{\widetilde{H}_S^\perp}}$, which belongs to $\widehat{A}_{\widetilde{H}_S^\perp}$, and peaks on $\pi_H(\imath) \in G/H$. Similar arguments apply to p-sets. □

Proposition 4.4.3. *A closed group $H \subset G$ is a p-set for A_S if and only if the set $\widetilde{H}_S^\perp$ separates the points of G/H. If the quotient group G/H is metrizable, the same condition is necessary and sufficient for H to be a peak set for A_S.*

Proof. If the set $\widetilde{H}_S^\perp$ separates the points of G/H, so does the algebra $A_{\widetilde{H}_S^\perp}$, associated with $\widetilde{H}_S^\perp$. Since the Shilov boundary of $A_{\widetilde{H}_S^\perp}$ is homeomorphic to the group $G/H \cong \pi_H(G)$, then $\pi_H(\imath) \in \pi_H(G)$ is a p-point for $A_{\widetilde{H}_S^\perp}$, by the remark following Example 4.4.1. Hence, by Lemma 4.4.2, H is a p-set for A_S. Conversely, if the set $\widetilde{H}_S^\perp$ does not separate the points of G/H, neither does $A_{\widetilde{H}_S^\perp}$. Therefore,

$\pi_H(\imath)$ is not a p-point for $A_{\widetilde{H}_S^\perp}$. Consequently, H is not a p-set for A_S, by Lemma 4.4.2. In the case when G/H is metrizable, similar arguments apply for peak sets. □

Observe that the algebra $A(\mathbb{T}^2)$ in Example 4.4.1 does not contradict Proposition 4.4.3, since the set $H^\perp_{\mathbb{Z}^2_+} \cong \{1\}$ clearly does not separate the points of the group $\mathbb{T}^2/H \cong \mathbb{T}$.

Theorem 4.4.4. *If G is a compact abelian group, H is a closed subgroup of G and S is a generating subsemigroup of the dual group $\widehat{G}$, then the following conditions are equivalent.*

(i) *H is a p-set for A_S.*

(ii) *The semigroup $\widetilde{H}_S^\perp$ separates the points of $G/H = \pi_H(G)$, i.e. $H_S^\perp$ separates the H-cosets of G.*

(iii) *The dual group $(G/H)\widehat{}$ coincides with the group envelope $\Gamma_{\widetilde{H}_S^\perp} \subset (G/H)\widehat{}$ of $\widetilde{H}_S^\perp$.*

(iv) *H coincides with the group $(H_S^\perp)^\perp \cong \bigcap_{\chi\in H_S^\perp} \operatorname{Ker}(\chi) = \big\{g \in G\colon \chi(g) = 1$ for all $\chi \in H_S^\perp\big\}$.*

If, in addition, G/H is a metrizable group, then the above conditions are equivalent to

(v) *H is a peak set for A_S.*

Proof. The equivalence of (i), (ii), and (iii) is already shown in Proposition 4.4.3. If the semigroup $\widetilde{H}_S^\perp$ separates the points of G/H, so does its group envelope $\Gamma_{\widetilde{H}_S^\perp} \subset (G/H)\widehat{}$. By Pontryagin's duality theorem both groups coincide. Conversely, if $\widetilde{H}_S^\perp = (G/H)\widehat{}$, then $\Gamma_{\widetilde{H}_S^\perp}$ separates the points of G/H. Assume that $\widetilde{H}_S^\perp$ does not separate the points of G/H. Then there is a $\pi_H(g) \in \pi_H(G) \setminus \pi_H(\imath)$ such that $\widetilde{\chi}^a\big(\pi_H(g)\big) = 1$ for all $a \in S$. Let $\widetilde{\chi} \in \Gamma_{\widetilde{H}_S^\perp}$ be such that $\widetilde{\chi}\big(\pi_H(g)\big) \neq 1$. Since, clearly, $\chi \notin S$, then $\chi = \chi^a/\chi^b$ for some $a, b \in S$. Therefore, $\chi^a\big(\pi_H(g)\big) \neq \chi^b\big(\pi_H(g)\big)$, in contradiction with the property $\widetilde{\chi}^a\big(\pi_H(g)\big) = 1$ for all $a \in S$. This proves that (iii) and (iv) are equivalent.

If $\widetilde{H}_S^\perp$ generates the dual group $(G/H)\widehat{}$, then $(H_S^\perp)^\perp = \big\{g \in G\colon \gamma\circ\pi_H(g) = 1$ for all $\gamma \in (G/H)\widehat{}\big\} = \pi_H^{-1}(\imath) = H$. If, on the other hand, $\Gamma_{\widetilde{H}_S^\perp}$ is a proper subgroup of $(G/H)\widehat{}$, then the group $(H_S^\perp)^\perp = \big\{g \in G\colon \gamma \circ \pi_H(g) = 1$ for all $\gamma \in \Gamma_{\widetilde{H}_S^\perp}\big\}$ contains properly the group $\big\{g \in G : \gamma \circ \pi_H(g) = 1$ for all $\gamma \in (G/H)\widehat{}\big\}$ which is isomorphic to H. This proves that (iv) and (v) are equivalent. □

Analogous results hold for peak subsets of G relative to A_S. Let X be a compact set, and $A \subset C(X)$ be a uniform algebra on X, which does not necessarily coincide with $\mathcal{M}_A$. A closed set K in X is called a peak subset of X, relative to A, if there is a function $f \in A$, such that $f\big|_K \equiv 1$, and $\big|f(x)\big| < 1$ whenever $x \in X \setminus K$. Intersections of peak subsets of X for A are called *p-subsets of X relative to A*. Clearly, peak sets and p-sets for A are peak subsets and p-subsets of $\mathcal{M}_A$ relative to $\widehat{A}$ correspondingly.

Example 4.4.5. If $a \in S$, then the kernel $\operatorname{Ker}(\chi^a) = \big\{g \in G \colon \chi(g) = 1\big\}$ is a peak subset of G relative to A_S. Indeed, the function $h = (1+\chi^a)/2 \in A_S$ is identically equal to 1 exactly on $\operatorname{Ker}(\chi^a) \subset G$. However, kernels of characters $\chi^a,\ a \in S$, are not always peak sets relative to A_S.

By using similar arguments, one can obtain results similar to the above for peak subsets of G, relative to A_S.

Proposition 4.4.6. *A closed group $H \subset G$ is a p-subset of G relative to A_S if and only if the unit $\pi_H(\imath)$ of the group $G/H = \pi_H(G)$ is a p-point of G/H relative to $A_{\widetilde{H}_S^\perp}$. If G/H is a metrizable group, then $G/H = \pi_H(G)$ is a peak point of G/H relative to $A_{\widetilde{H}_S^\perp}$*

Theorem 4.4.7. *If G is a compact abelian group, H is a closed subgroup of G and S is a generating subsemigroup of the dual group $\widehat{G}$, then the following statements are equivalent.*

(i) *H is a p-subset of G relative to A_S.*

(ii) *The semigroup $\widetilde{H}_S^\perp$ separates the points of $G/H = \pi_H(G)$, i.e. $H_S^\perp$ separates the H-cosets of G.*

(iii) *The dual group $(G/H)\widehat{}$ coincides with the group envelope $\Gamma_{\widetilde{H}_S^\perp} \subset (G/H)\widehat{}$ of $\widetilde{H}_S^\perp$.*

(iv) *H coincides with the group $(H_S^\perp)^\perp \cong \bigcap\limits_{\chi \in H_S^\perp} \operatorname{Ker}(\chi) = \big\{g \in G : \chi(g) = 1 \text{ for all } \chi \in H_S^\perp\big\}$.*

If, in addition, G/H is a metrizable group, then the above conditions are equivalent to

(v) *H is a peak subset of G relative to A_S.*

Denote by $\wp(S)$ the class of all *p-groups for A_S*, i.e. closed subgroups of G that are p-subsets of G relative to A_S.

Proposition 4.4.8. *If $H \in \wp(S)$, then* $H = \bigcap\limits_{\chi^a \in H_S^\perp} \operatorname{Ker}(\chi^a)$.

Proof. Note that the point $\pi_H(\imath)$ belongs to $\pi_H(G) = G/H$, while the set $\widetilde{H}_S^\perp$ separates the points of $\pi_H(G)$ by Theorem 4.4.4. Consequently, $\pi_H(H) = \pi_H(\imath) = \bigcap_{\widetilde{\chi}^a \in \widetilde{H}_S^\perp} \operatorname{Ker}(\widetilde{\chi}^a)$. Therefore, $H = \pi_H^{-1}\big(\pi_H(H)\big) = \bigcap_{\chi^a \in H_S^\perp} \operatorname{Ker}(\chi^a)$, as claimed. □

Observe that if $S = \widehat{G}$, then $A_S = A_{\widehat{G}} = C(G)$, and $H_{\widehat{G}}^\perp$ separates the points of G/H, since $H_{\widehat{G}}^\perp = H^\perp \cong (G/H)\widehat{}$ for any subgroup $H \subset G$. Theorem 4.4.4 yields that H is a p-set for $A_{\widehat{G}}$. Hence, $\wp(\widehat{G})$ is the class of all closed subgroups of G.

We recall that the weak hull $[S]_w$ of S is the set of elements $a \in \Gamma_S$ for which there is an $m_a \in \mathbb{N}$ such that $na \in S$ for all $n \geq m_a$. S is weakly enhanced if $[S]_w = S$, i.e. if S coincides with its weak hull. (cf. Definition 3.4.9).

Proposition 4.4.9. $\wp(S) = \wp\big([S]_w\big)$ *for any semigroup* $S \subset G$.

Proof. Clearly, $A_S \subset A_{[S]_w}$ since $S \subset [S]_w$, and therefore, $\wp(S) \subset \wp\big([S]_w\big)$. Conversely, if $H \notin \wp(S)$, then, by Theorem 4.4.4, the set $\widetilde{H}_S^\perp$ does not separate the points of $\pi_H(G)$. Hence, there is a $g \in G$ such that $\pi_H(g) \neq \pi_H(\imath)$, and $\widetilde{\chi}^a\big(\pi_H(g)\big) = 1$ for all $\chi^a \in H_S^\perp$. If $a \in H_{[S]_w}^\perp$, then there is an $m_a \in \mathbb{N}$ such that $na \in S$ for all $n \geq m_a$. Therefore $\chi^{na}(g) = \widetilde{\chi}^{na}\big(\pi_H(g)\big) = 1$ for any $n \geq m_a$. This can happen only if $\widetilde{\chi}^a\big(\pi_H(g)\big) = 1 = \widetilde{\chi}^a\big(\pi_H(\imath)\big)$. Consequently, $\widetilde{H}_{[S]_w}^\perp$ does not separate the points of $\pi_H(G)$, i.e. $H \notin \wp\big([S]_w\big)$. Hence, $\wp(S) \supset \wp\big([S]_w\big)$. □

Theorem 4.4.10. *Let S be a weakly enhanced semigroup of Γ. All closed subgroups of G are p-subsets of G relative to A_S if and only if $\widehat{G} = S \cup (-S)$. If G is metrizable, the same result holds for peak subsets of G relative to A_S.*

Proof. Suppose that $\widehat{G} = S \cup (-S)$ and let H be a closed subgroup of G. The group $H_S^\perp$ separates the points of G/H, since $H_S^\perp = H_{(-S)}^\perp = H_{S\cup(-S)}^\perp = H_{\widehat{G}}^\perp = (G/H)\widehat{}$. Therefore, $H \in \wp(S)$ by Proposition 4.4.3.

Conversely, let every closed subgroup H of G be a p-subset of G relative to A_S. Hence, $\operatorname{Ker}(\chi^a) \in \wp(S)$ for any $a \in \Gamma \setminus S$, and $\operatorname{Ker}(\chi^a) = \bigcap_{\chi^b \in \left(\operatorname{Ker}(\chi^a)\right)_S^\perp} \operatorname{Ker}(\chi^b)$ by Proposition 4.4.8. Denote $K = \big(\operatorname{Ker}(\chi^a)\big)_S^\perp \subset S$. Lemmas 3.1.8 and 3.1.7 imply that $\Gamma_K = \mathbb{Z}a$, while $\chi^a \notin K$. Therefore there are $m, n \in \mathbb{Z}$ with $m > n$ and n not a divisor of m, so that na and ma belong to the set $\{a\colon \chi^a \in K\} \subset S$. Hence $rn + sm = 1$ for some $r, s \in \mathbb{Z}$. If $m > 0$ and $n < 0$ (or vice versa), then $r, s \in \mathbb{N}$ and hence $a = (rn + sm)\,a = r\,(na) + s\,(ma) \in S$, contrary to the choice of a. If $n, m \in \mathbb{Z}_+$, then $a \in [S]_w = S$ by Proposition 3.4.11, while if $n, m \in \mathbb{Z}_-$, then $(-n)(-a)$ and $(-m)(-a)$ both belong to $-S$, where $-n, -m \in \mathbb{Z}_+$. Proposition 3.4.11 implies again that $-a \in [S]_w = S$, i.e. $a \in (-S)$. □

Example 4.4.11. Let $G = \mathbb{T}^2$. Consider the following semigroups of $\Gamma = \widehat{G} \cong \mathbb{Z}^2$:

(i) $S_1 = \{(m,n) \in \mathbb{Z}^2 \colon m \in \mathbb{Z},\ n > 0\} \cup \{(m,n) \in \mathbb{Z}^2 \colon m \geq 0,\ n \geq 0\} = (\mathbb{Z} \times \mathbb{N}) \cup (\mathbb{Z}_+ \times \mathbb{Z}_+)$.

(ii) $S_2 = \{(m,n) \in \mathbb{Z}^2 \colon m \in \mathbb{Z},\ n > 0\} \cup \{(m,n) \in \mathbb{Z}^2 \colon m \leq 0,\ n \geq 0\} = (\mathbb{Z} \times \mathbb{N}) \cup \big((-\mathbb{Z}_+) \times \mathbb{Z}_+\big)$.

(iii) $S_3 = \{(m,n) \in \mathbb{Z}^2 \colon m \in \mathbb{Z},\ n < 0\} \cup \{(m,n) \in \mathbb{Z}^2 \colon m \geq 0,\ n \leq 0\} = \big(\mathbb{Z} \times (-\mathbb{N})\big) \cup \big(\mathbb{Z}_+ \times (-\mathbb{Z}_+)\big)$.

(iv) $S_4 = \{(m,n) \in \mathbb{Z}^2 \colon m \in \mathbb{Z},\ n < 0\} \cup \{(m,n) \in \mathbb{Z}^2 \colon m \leq 0,\ n \leq 0\} = \big(\mathbb{Z} \times (-\mathbb{N})\big) \cup \big((-\mathbb{Z}_+) \times (-\mathbb{Z}_+)\big)$.

It is easy to see that $S_i \cup (-S_i) = \mathbb{Z}^2 \cong \widehat{G}$ for any $i = 1,2,3,4$. By Theorem 4.4.14 all closed subgroups of $\mathbb{T}^2$ are peak groups relative to the algebras A_{S_i}, $i = 1,2,3,4$.

Example 4.4.12. Let $G = \mathbb{T}^2$, so that $\Gamma = \mathbb{Z}^2$. Given a fixed irrational number $\beta > 0$, consider the semigroup $\Gamma_+^\beta = \{(n,m) \in \mathbb{Z}^2 \colon \beta n + m \geq 0\}$ from Example 4.1.3, and the associated algebra A_β on $\mathbb{T}^2$ generated by Γ_+^β. Here again $\Gamma_+^\beta \cup (-\Gamma_+^\beta) \cong \mathbb{Z}^2$, and by Theorem 4.4.14 all closed subgroups of $\mathbb{T}^2$ are peak groups relative to the algebra A_β.

Example 4.4.13. Let $G = \mathbb{T}^2$, $\Gamma = \mathbb{Z}^2$, and $S = \mathbb{Z}_+^2 = \{(m,n) \colon n \geq 0,\ m \geq 0\}$. For a fixed $k \in \mathbb{Z}$ consider the semigroup $G_k = \{(z_1, z_1^k) \colon |z| = 1\} \subset A(\mathbb{T}^2)$ and its corresponding algebra A_{G_k}. If $k < 0$, the function $(1 + z_1^{-k} z_2)/2$ belongs to the algebra $A_{\mathbb{Z}_+^2}$, and peaks on G_k. Hence, G_k is a peak subset of $\mathbb{T}^2$ relative to the algebra $A_{\mathbb{Z}_+^2}$. If $k > 0$, as in Example 4.4.1 one can see that every character $z_1^n z_2^m$ that is identically equal to 1 on G_k, is identically equal to 1 on $\mathbb{T}^2$. Therefore, the set $(G_k)_{\mathbb{Z}_+^2}^\perp \cong \{1\}$ does not separate the points of $\mathbb{T}^2/G_k$, and consequently, G_k is not a p-group, nor a peak group for $A_{\mathbb{Z}_+^2}$.

Theorem 4.4.14. *Let S and S_1 be two weakly enhanced semigroups of $\widehat{G}$ containing 0. Then $\wp(S) = \wp(S_1)$ if and only if $S \cup (-S) = S_1 \cup (-S_1)$.*

Proof. If $S \cup (-S) = S_1 \cup (-S_1)$, then $H_S^\perp = H_{S\cup(-S)}^\perp = H_{S_1\cup(-S_1)}^\perp = H_{S_1}^\perp$ for every subgroup $H \subset G$. Therefore, the points of G/H are separated by $H_S^\perp$ if and only if they are separated by $H_{S_1}^\perp$, thus $H \in \wp(S)$ if and only if $H \in \wp(S_1)$, by Proposition 4.4.3.

Conversely, assume that $\wp(S) = \wp(S_1)$. Since $\operatorname{Ker}(\chi^a) \in \wp(S) = \wp(S_1)$ for any $a \in S$, we have that $\operatorname{Ker}(\chi^a) = \bigcap_{\chi^b \in (\operatorname{Ker}(\chi^a))_{S_1}^\perp} \operatorname{Ker}(\chi^b)$ by Proposition 4.4.8. As in the proof of Theorem 4.4.14 we see that $a \in S_1$, or $a \in (-S_1)$. Therefore, $S \subset S_1 \cup (-S_1)$. Similarly, $(-S) \subset S_1 \cup (-S_1)$. Therefore, $S \cup (-S) \subset S_1 \cup (-S_1)$. The opposite inclusion holds by a symmetry argument. □

Theorem 4.4.14 shows that for weakly enhanced semigroups S and S_1 the equalities $\wp(S) = \wp(S_1) = \wp(\widehat{G})$ hold if and only if $S \cup (-S) = S_1 \cup (-S_1) = \Gamma = \widehat{G}$.

Corollary 4.4.15. *Let S and S_1 be two weakly enhanced semigroups of $\widehat{G}$ containing 0, for which $S \cup (-S) = S_1 \cup (-S_1) \neq \Gamma$, and $G_S = G_{S_1} = \{0\}$. Suppose that the only subgroups in the set $S \cup (-S)$ are of type $\mathbb{Z}c,\ c \in S$. Then $\wp(S) = \wp(S_1)$ if and only if either $S = S_1$, or $S = (-S_1)$, or, equivalently, if and only if $A_S = A_{S_1}$, or $A_S = \overline{A}_{S_1}$.*

Proof. If $S = \pm S_1$, then clearly $S \cup (-S) = S_1 \cup (-S_1)$, and therefore $\wp(S) = \wp(S_1)$ by Theorem 4.4.14.

If $S \cup (-S) = S_1 \cup (-S_1)$, then $S = (S \cap S_1) \cup \big(S \cap (-S_1)\big)$. Assume that $S \cap S_1$ and $S \cap (-S_1)$ both are nonempty. Suppose first S is not contained in any subgroup of type $\mathbb{Z}c,\ c \in S$. Then there are $a \in S \cap S_1$ and $b \in S \cap (-S_1)$ that do not belong to any subgroup of type $\mathbb{Z}c,\ c \in S$. By the hypotheses, the group $\Gamma_{\{a,b\}}$ generated by a and b is not a subset of $S \cup (-S)$. Therefore, there are $m, n \in \mathbb{Z} \setminus \{0\}$, such that $c = ma + nb \notin S \cup (-S)$. Note that either $m, n \in \mathbb{N}$, or $m, n \in (-\mathbb{N})$. Indeed, if $m > 0$ and $n < 0$, then ma and nb both are in S_1 contrary to $G_S = \{0\}$. If $m < 0,\ n > 0$ we have $ma,\ nb \in (-S_1)$. In both cases $c = ma + nb \in S_1 \cup (-S_1)$, contrary to the choice of c. If m, n both are in $\mathbb{N}$, or in $(-\mathbb{N})$, we obtain that ma, nb both are in S, or in $(-S)$ correspondingly. In either case $c = ma + nb \in S \cup (-S)$, which is impossible. Therefore, either $S \cap S_1 = \emptyset$, or $S \cap (-S_1) = \emptyset$. Consequently, either $S = S_1$, or $S = (-S_1)$. If $S \subset \mathbb{Z}c$ for some $c \in S$, then $S \cup (-S) \subset \mathbb{Z}c$, and therefore $S \cup (-S)$ is a group, thus $\Gamma \cong \widehat{G}$, which contradicts the assumption on S. ☐

For non-weakly enhanced semigroups we have the following

Corollary 4.4.16. *Let S and S_1 be two semigroups of $\widehat{G}$ containing 0, such that $[S]_w \cup [-S]_w = [S_1]_w \cup [-S_1]_w \neq \Gamma$, and $[S]_w \cap [-S]_w = [S_1]_w \cap [-S_1]_w = \{0\}$. Suppose that the only subgroups in $[S]_w \cup [-S]_w$ are of type $\mathbb{Z}c$ for some $c \in S$. Then $\wp(S) = \wp(S_1)$ if and only if either $[S]_w = [S_1]_w$, or $[S]_w = [-S_1]_w$, i.e. if either $A_{[S]_w} = A_{[S_1]_w}$, or $A_{[S]_w} = \overline{A}_{[S_1]_w}$.*

Observe that the set $[S]_w \cup [-S]_w$ does not contain subgroups other than $\mathbb{Z}c$, if and only if it does not contain subgroups of type $\Gamma_{\{a,b\}}$, $a, b \in S$, other than $\mathbb{Z}c$, i.e. if for every $a, b \in S$ there are $m, n \in \mathbb{Z} \setminus \{0\}$, such that $\mathbb{Z}(na+mb) \cap \big(S \cap (-S)\big) = \{0\}$, provided $\mathbb{Z}(na + mb) \neq \mathbb{Z}c$ for any $c \in S$.

If the group $\Gamma_{\{a,b\}} \subset \Gamma$ generated by two elements $a, b \in S$ is not of type $\mathbb{Z}c$, $c \in S$, then we can define a homomorphism $\zeta \colon \Gamma_{\{a,b\}} \longrightarrow \mathbb{Z}^2$, by $na + mb \longmapsto (n, m)$. One can easily see that $\mathbb{Z}\,(na + mb) \cap (S \cap (-S)) = \{0\}$ for some $m, n \in \mathbb{Z} \setminus \{0\}$ if and only if the set $\zeta(\Gamma_{\{a,b\}}) \cap S$ is situated between two non-collinear rays initiating at 0, the angle between which is less than π.

If the dual group $\Gamma = \widehat{G}$ is countable, or, more generally, if the group G is metrizable, then all results in this section hold also for peak groups, instead of for p-groups relative to A_S.

4.5 Radó's and Riemann's theorems on G-discs

Many theorems of classical complex analysis have natural extensions for functions in general uniform algebras. Let U be an open set in the maximal ideal space $\mathcal{M}_A$ of a uniform algebra A. Consider the uniform closure A_U of the restriction of Gelfand transform $\widehat{A}$ of A on U. It is clear that A_U is a uniform algebra on $\overline{U} \subset \mathcal{M}_A$. A continuous function f on U is said to be A-*holomorphic* on U if for every $x \in U$ there is a neighborhood V of x so that the restriction $f\big|_V$ belongs to A_V, i.e. if $f\big|_V$ can be approximated uniformly on V by Gelfand transforms of functions in A. The set of all A-holomorphic functions on U is denoted by $\mathcal{O}_A(U)$.

The classical Radó's theorem for the disc algebra asserts that if a function f is continuous on the closed unit disc $\overline{\mathbb{D}}$ and analytic in the complement of its null-set $\mathrm{Null}\,(f)$ in $\mathbb{D}$, then f is analytic on the whole unit disc $\mathbb{D}$.

Definition 4.5.1. A uniform algebra A has *Radó's property*, if every function f, which is continuous on $\mathcal{M}_A$ and A-holomorphic on $\mathcal{M}_A \backslash \mathrm{Null}\,(f)$ belongs to $\widehat{A}$.

Radó's theorem implies that the disc algebra $A(\mathbb{D})$ has Radó's property. However, the Radó's property fails for the algebra $A_0(\mathbb{D})$ of functions f continuous on $\overline{\mathbb{D}}$, analytic in $\mathbb{D}$, and whose derivative vanishes at 0. Observe that this algebra is of type A_S with $S = \mathbb{Z}_+ \setminus \{1\} = \{0, 2, 3, 4 \dots\} \subset \mathbb{Z}$, whose weak hull $[S]_w$ is $\mathbb{Z}_+ \neq S$.

Theorem 4.5.2. *Let G be a compact connected abelian group and S be a subsemigroup of $\widehat{G}$ such that $S - S = \widehat{G}$, and $S \cap (-S) = \{0\}$. Then the algebra A_S has the Radó property if and only if the semigroup S is weakly enhanced.*

Proof. Suppose that the algebra A_S has the Radó property. We claim that S is weakly enhanced. Let $a \in [S]_w \subset \widehat{G}$. Then there is an $n \in \mathbb{N}$ such that $ak \in S$ for all $k \in \mathbb{N},\ k \geq n$. Extend χ^a on $\mathcal{M}_{A_S} \supset G$ as follows:

$$\widetilde{\chi}^a(m) = m(a) = \begin{cases} \dfrac{\widehat{\chi}^{a\,(n+1)}(m)}{\widehat{\chi}^{an}(m)} & \text{when } \widehat{\chi}^{an}(m) \neq 0, \\ 0 & \text{when } \widehat{\chi}^{an}(m) = 0. \end{cases}$$

If $\widehat{\chi}^{an}(m) \neq 0$, then $\widehat{\chi}^a(m) \neq 0$ for every $m \in U$ in some neighborhood $U \subset \mathcal{M}_S$ of m. Since $\widehat{\chi}^{a\,(n+1)}$ and $\widehat{\chi}^{an}$ are in A_S, $\widetilde{\chi}^a$ belongs to $(A_S)_U$. Hence the function $\widetilde{\chi}^a$ is continuous on $\mathcal{M}_{A_S}$ and A_S-holomorphic outside its null-set $\mathrm{Null}\,(\widetilde{\chi}^a)$. The Radó property for A_S implies that $\widetilde{\chi}^a\big|_G \in \widehat{A}_S$, thus $\chi^a \in A_S$, i.e. $a \in S$. Consequently, S is weakly enhanced.

Conversely, suppose that S is a weakly enhanced semigroup of $\widehat{G}$. We will show that A_S has Radó's property. For simplicity we will assume that G is a separable group. Let $f \in C(\mathcal{M}_S) \cap \mathcal{O}_{A_S}\big(\mathcal{M}_{A_S} \setminus \mathrm{Null}\,(f)\big)$, and let $\{h_j\}_{j=1}^{\infty}$ be a countable dense subset of G, so that $h_1 = \imath$. Denote by $f_j = f_{h_j}$ the h_j-shifts of f, i.e. $f_{h_j}(g) = f(gh_j)$, $g \in G$. For every $j \in \mathbb{N}$ consider the algebras $A_j = [A_S, f_1, \dots, f_j]$ with maximal ideal space $\mathcal{M}_j = \mathcal{M}_{A_S}$. We have $A_j = [A_{j-1}, f_j]$, $A_0 = A_S$ and f_j is A_{j-1}-holomorphic on $\mathcal{M}_{j-1} \setminus \mathrm{Null}\,(f_j)$. By the arguments from [G1] (II, Theorem 6.3) we conclude that $\partial A_j = \partial A_{j-1}$. If $(i_n^{n+1})^* \colon \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ is the adjoint map to the inclusion $i_n^{n+1} \colon A_n \hookrightarrow A_{n+1}$, then $(i_n^{n+1})^*$ is surjective and $(i_n^{n+1})^*(\partial A_{n+1}) = \partial A_n$.

We obtain two sequences

$$A_S \subset A_0 \subset A_1 \subset \cdots \subset A_k \subset \cdots$$

and

$$\mathcal{M}_{A_S} = \mathcal{M}_0 \xleftarrow{(i_0^1)^*} \mathcal{M}_1 \xleftarrow{(i_1^2)^*} \mathcal{M}_2 \xleftarrow{(i_2^3)^*} \cdots .$$

The closure $A_f = \big[\bigcup_{j=1}^{\infty} A_j\big]$ in $C(G)$ is an inductive limit algebra with $\mathcal{M}_{A_f} = \mathcal{M}_{A_S}$, and $\partial A_f = \partial A_S = G$ (cf. Section 1.3). Moreover, A_f is a shift-invariant uniform algebra on G, and hence, is generated algebraically by a semigroup of $\widehat{G}$, actually, by the semigroup S_f generated by the elements of S and $\mathrm{sp}\,(f|_G)$, i.e. $A_f = A_{S_f}$. Hence, $\mathcal{M}_{A_{S_a}} = \mathcal{M}_{A_f} = \mathcal{M}_{A_S}$ for every $a \in \mathrm{sp}\,(f) \subset S_f$, and hence, $a \in [S]_w$ according to Proposition 4.2.10. Hence, $S_f = [S]_w = S$ because S is weakly enhanced. Therefore, $\mathrm{sp}\,(f|_G) \subset S$, thus $f|_G \in A_S$, and consequently, $\widehat{f} \in \widehat{A}_S$. □

Theorem 4.5.2 can be used to explore integral closedness phenomena in uniform algebras. Recall that every continuous solution of a polynomial equation with coefficients that are analytic functions in $\mathbb{C}$ is an analytic function.

Definition 4.5.3. A uniform algebra A is *integrally closed* if every continuous function on $\mathcal{M}_A$ satisfying a polynomial equation of type

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,\ a_j \in A$$

belongs to $\widehat{A}$.

For example, the disc algebra, the polydisc algebra, and the algebra of analytic $\widehat{G}_+$-functions on a G-disc over a group G with ordered dual, are integrally closed algebras.

Theorem 4.5.4. *Under the assumptions of Theorem 4.5.2 the algebra A_S is integrally closed if and only if the semigroup S is weakly enhanced.*

Proof. Suppose that A_S is integrally closed and let $a \in [S]_w$. Then there is an $n_a \in \mathbb{N}$ such that $ma \in S$ for every $m \geq n_a$. Consider the equation

$$x^{n_a} - \widehat{\chi}^{a^{n_a}} = 0. \tag{4.6}$$

Being a continuous solution of (4.6) on $\mathcal{M}_S$, the function χ^a belongs to A_S, thus $a \in S$. Hence, $[S]_w = S$, i.e. S is weakly enhanced.

Conversely, if S is weakly enhanced, then Radó's theorem holds on A_S according to Theorem 4.5.2. This implies that the algebra A_S is integrally closed (cf. [G5]). □

Example 4.5.5. Let $S = \{(m,n)\colon m,n > 0\} \cup \{(0,n)\colon n \geq 2\} \cup \{(2k,0)\colon k \geq 0\} \subset \mathbb{Z}^2$ be the non-weakly enhanced semigroup from Example 3.4.10. Theorems 4.5.2 and 4.5.4 imply that the algebra A on the bi-disc $\overline{\mathbb{D}}^2$ that is generated by the functions $z^m w^n$, where $(m,n) \in S$ neither has Radó's property, nor is integrally closed. For instance, the function $f(z,w) = w$ is continuous on $\mathcal{M}_A = \overline{\mathbb{D}}^2$ and A-holomorphic outside $\text{Null}\,(f) = \{(z,w)\colon w \neq 0\}$ and also satisfies the polynomial equation $x^2 - w^2 = 0$, where $w^2 \in A$. However $f \notin A$.

The classical theorem of Riemann for removable singularities of analytic functions on $\mathbb{D}$ asserts that if a function $f \in A(\mathbb{D})$ is analytic in the complement of one point $a \in \mathbb{D}$, and bounded in a neighborhood of a, then f is analytic on the whole unit disc $\mathbb{D}$. Note that single points in the complex plane $\mathbb{C}$ are zeros of particular analytic functions.

Definition 4.5.6. A uniform algebra A has the *Riemann property* if, given an $f \in \widehat{A}$ with $\text{Null}\,(f) \cap \partial A = \emptyset$, every bounded A-holomorphic function on $\mathcal{M}_A \backslash \text{Null}\,(f)$ can be extended on $\mathcal{M}_A$ as an element in $\widehat{A}$.

The classical theorem of Riemann implies that the disc algebra $A(\mathbb{D})$ possesses the Riemann property.

Definition 4.5.7. Let S be a semigroup of $\widehat{G}$. The *bounded hull* $[S]_b$ of S is the set of elements $a \in \widehat{G}$ for which there are $b, c \in S$ with $a = b - c$, such that $\widehat{\chi}^b/\widehat{\chi}^c$ is bounded on the set $\mathcal{M}_{A_S} \backslash \text{Null}\,(\widehat{\chi}^c)$. S is said to be *boundedly enhanced* if $[S]_b = S$.

A straightforward check shows that $S \subset [S]_w \subset [S]_s \subset [S]_b \subset \widehat{G}$. Note that any semigroup $S \subset \widehat{G}$ with $S \cup (-S) = \widehat{G},\ S \cap (-S) = \{0\}$ is simultaneously weakly, strongly, and boundedly enhanced.

Theorem 4.5.8. *Let G be a compact connected abelian group and let S be a subsemigroup of $\widehat{G}$ such that $S - S = \widehat{G}$ and $S \cap (-S) = \{0\}$. The algebra A_S has the Riemann property if and only if the semigroup S is boundedly enhanced.*

Proof. Let the algebra A_S have the Riemann property. Fix an element $a \in [S]_b \subset \widehat{G}$ and let $a = b - c$, where $b, c \in S$ be such that the function $\gamma = \widehat{\chi}^b/\widehat{\chi}^c$ is bounded on $\mathcal{M}_S \backslash\ \text{Null}\,(\widehat{\chi}^c)$. Since γ is bounded and A_S-holomorphic on $\mathcal{M}_{A_S} \backslash \text{Null}\,(\widehat{\chi}^c)$, it belongs to $\widehat{A}_S$ by the Riemann property. Hence, $\gamma|_G = \chi^b/\chi^c = \chi^a \in A_S$, i.e. $a \in S$. Consequently, the semigroup S is boundedly enhanced.

Conversely, assume that the semigroup S is boundedly enhanced, and let f be a bounded function in $\mathcal{O}_{A_S}\big(\mathcal{M}_{A_S}\backslash \mathrm{Null}\,(g)\big)$ for some $g \in \widehat{A}_S$. Denote $A_0 = [A_S, f]$ and $\mathcal{M}_0 = \mathcal{M}_{A_0}$. The adjoint mapping $(i^0_{-1})^*$ of the inclusion $i^0_{-1}\colon A_S \hookrightarrow A_0$ maps $\mathcal{M}_0$ into $\mathcal{M}_{A_S}$. Since f is A_S-holomorphic on $\mathcal{M}_{A_S}\backslash \mathrm{Null}\,(g)$ we have that $(i^0_{-1})^*(\mathcal{M}_0) \supset \mathcal{M}_{A_S}\backslash \mathrm{Null}\,(g)$, and hence, $(i^0_{-1})^*(\mathcal{M}_0) = \mathcal{M}_{A_S}$, because $\big[\mathcal{M}_{A_S} \setminus \mathrm{Null}\,(g)\big] = \mathcal{M}_{A_S}$ by the analyticity of A_S. Observe that for every $f_1 \in \widehat{A}_0$ we have $f_1 g \in \mathcal{O}_{A_S}\big(\mathcal{M}_{A_S}\backslash \mathrm{Null}\,(f_1 g)\big)$, and consequently, $f_1 g \in \widehat{A}_S$ by Theorem 4.5.2. Hence, the function $(f_1 g)/g = f_1$ is bounded on $\mathcal{M}_{A_S}\backslash \mathrm{Null}\,(g)$. By the Glicksberg general version of Schwarz's lemma [G5] (Th. 4.1) then

$$\begin{aligned}\sup\big|f_1\big(\mathcal{M}_{A_S} \setminus \mathrm{Null}\,(g)\big)\big| &= \sup\Big|\frac{f_1 g}{g}\,\big(\mathcal{M}_{A_S} \setminus \mathrm{Null}\,(g)\big)\Big| \\ &= \sup\Big|\frac{f_1 g}{g}\,(\partial A_S \setminus \mathrm{Null}\,(g))\Big| = \sup\Big|\frac{f_1 g}{g}(\partial A_S)\Big|.\end{aligned}$$

We conclude that $\partial A_0 = \partial A_S$, because $\big[\mathcal{M}_{A_S}\backslash \mathrm{Null}\,(g)\big] = \mathcal{M}_{A_S}$ by the analyticity of A_S.

As in the proof of Theorem 4.5.2 we will assume that G is a separable group. Let $\{h_j\}_{j=1}^{\infty}$, $h_1 = \imath$ be a countable dense subset of G and let $f_j = f_{h_j}$ be the h_j-shift of f. Consider the algebra $A_1 = [A_S, f, f_1]$ with maximal ideal space $\mathcal{M}_1$. The functions $f = f_0$ and f_1 are A_S-holomorphic on the set $\mathcal{M}_{A_S} \setminus \big(\mathrm{Null}\,(g) \cup h_1^{-1}\cdot \mathrm{Null}(g)\big) = \mathcal{M}_{A_S} \setminus \big(\mathrm{Null}\,(g) \setminus h_1^{-1}\cdot \mathrm{Null}\,(g)\big)$. By the analyticity of A_S the adjoint map $(i^1_0)^*\colon \mathcal{M}_1 \longrightarrow \mathcal{M}_{A_S}$ to the inclusion $i^1_0\colon A_S \hookrightarrow A_1 = [A_S, f_0, f_1]$ maps $\mathcal{M}_1$ onto $\mathcal{M}_0 = \mathcal{M}_{A_S}$. In a similar way as in Theorem 4.5.2, we see that $\partial A_1 = \partial A_0 = \partial A_S = G$. By the same arguments we obtain two sequences

$$A_S \subset A_0 \subset A_1 \subset \cdots \subset A_k \subset \cdots,$$

and

$$\mathcal{M}_{A_S} \xleftarrow{(i^0_{-1})^*} \mathcal{M}_0 \xleftarrow{(i^1_0)^*} \mathcal{M}_1 \xleftarrow{(i^2_1)^*} \mathcal{M}_2 \xleftarrow{(i^3_2)^*} \cdots,$$

where $A_k = [A_S, f, f_1, \ldots, f_j]$, and $\mathcal{M}_k = \mathcal{M}_{A_k}$. Proceeding inductively as before, we obtain

$$\mathcal{M}_{A_S} = (i^0_{-1})^* \circ (i^1_0)^*(\mathcal{M}_1) = (i^0_{-1})^* \circ (i^1_0)^* \circ (i^2_1)^*(\mathcal{M}_2) = \cdots,$$

and $\partial A_k = \partial A_S$. As in the proof of Theorem 4.5.2, $A_f = \big[\bigcup_{j=1}^{\infty} A_j\big]$ is an inductive limit algebra generated algebraically by the semigroup $S_f = \big[S,\, \mathrm{sp}(f|_G)\big]$, i.e. $A_f = A_{S_f}$. Since every $(i^k_{k-1})^*$ maps $\mathcal{M}_k$ onto $\mathcal{M}_{k-1}$, the adjoint projection $\psi\colon \mathcal{M}_{A_{S_f}} \longrightarrow \mathcal{M}_{A_S}$ to the natural inclusion $i\colon A_S \hookrightarrow A_f$ maps $\mathcal{M}_{A_{S_f}}$ onto $\mathcal{M}_{A_S}$, and $\partial A_f = \partial A_S = G$ (cf. Section 1.3). Consequently, A_f is a shift-invariant uniform algebra on G.

If now $a \in \operatorname{sp}(f)$, then $a = b - c$ for some $b, c \in S$, thus $a + c = b$. The function $\widehat{\chi}^{a+c}/\widehat{\chi}^c \in A_S$ is continuous and bounded on $\mathcal{M}_{A_S} \backslash \operatorname{Null}(\widehat{\chi}^c) = \psi\big(\mathcal{M}_{A_{S_f}} \backslash \operatorname{Null}(\widehat{\chi}^c)\big)$, since the function $i(\widehat{\chi}^{a+c})/(\widehat{\chi}^c) = i(\widehat{\chi}^{a+c})/i(\widehat{\chi}^c) \in A_f$ is continuous and bounded on $\mathcal{M}_{A_{S_f}} \backslash \operatorname{Null}(\widehat{\chi}^c)$. Hence, $a \in [S]_b$, i.e. $S \subset S_f \subset [S]_b = S$ since S is boundedly enhanced. Consequently, $S_f = S$, thus $\operatorname{sp}(f|_G) \subset S$, and therefore, $f \in A_S$. $\square$

4.6 Asymptotically almost periodic functions in one variable

We recall that a function $f \in C_b(\mathbb{R})$ is *asymptotically almost periodic*, if there is a unique $f^* \in AP(\mathbb{R})$ and $h \in C_0(\mathbb{R})$ such that $f = f^* + h$ (cf. Lemma 2.1.4). Hence, for the uniform algebra $AP^{as}(\mathbb{R})$ of asymptotically almost periodic functions on $\mathbb{R}$ we have $AP^{as}(\mathbb{R}) \cong AP(\mathbb{R}) \oplus C_0(\mathbb{R})$. It is easy to see that the set $C_0(\mathbb{R})$ is an ideal in $AP^{as}(\mathbb{R})$. In this section we study in more detail the space of asymptotically almost periodic functions on $\mathbb{R}$.

The maximal ideal space $\mathcal{M}_{AP^{as}(\mathbb{R})}$ of $AP^{as}(\mathbb{R})$ has a sophisticated structure. Let $G = b(\mathbb{R})$ be the Bohr compactification of $\mathbb{R}$. The restrictions of a linear multiplicative functional $m \in \mathcal{M}_{AP^{as}(\mathbb{R})}$ on the algebras $AP(\mathbb{R})$ and $\mathbb{C} \cdot 1 \oplus C_0(\mathbb{R})$ generate linear multiplicative functionals $m' = m\big|_{AP(\mathbb{R})} \in \mathcal{M}_{AP(\mathbb{R})} \cong G$ and $m'' = m\big|_{\mathbb{C}\cdot 1 \oplus C_0(\mathbb{R})} \in \mathcal{M}_{\mathbb{C}\cdot 1\oplus C_0(\mathbb{R})} \cong \mathbb{T}$. For any $f = f^* + h \in AP^{as}(\mathbb{R})$ we have $m(f) = m(f^* + h) = m(f^*) + m(h) = m'(f^*) + m''(h)$. Therefore, there arises an injective mapping $\varPhi\colon m \longmapsto (m', m'')$ of $\mathcal{M}_{AP^{as}(\mathbb{R})}$ into the set $\mathcal{M}_{AP(\mathbb{R})} \times \mathcal{M}_{\mathbb{C}\cdot 1\oplus C_0(\mathbb{R})} \cong G \times \mathbb{T}$. Hence, without loss of generality, we can consider that $\varPhi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$ is embedded into $G \times \mathbb{T}$. Below we describe in more detail the set $\varPhi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$.

Let $m = (m', m'') \in \mathcal{M}_{AP(\mathbb{R})} \times \mathcal{M}_{\mathbb{C}\cdot 1\oplus C_0(\mathbb{R})}$. If $m''|_{C_0(\mathbb{R})} \equiv 0$, then $m(f) = m'(f^*) + m''(h) = m'(f^*)$, and hence $m(f) = m'(f^*)$ is a linear multiplicative functional on $AP^{as}(\mathbb{R})$. Now m' coincides with the point evaluation m_g in $\widehat{AP}(\mathbb{R})$ at some point $g \in G$, while $m''(g) = g(\infty) = (g \circ \varphi)(1) = 0$ for any $g \in \mathbb{C}\cdot 1 \oplus C_0(\mathbb{R})$. We assume that m'' acts as 'the evaluation' in $\mathbb{C} \cdot 1 \oplus C_0(\mathbb{R})$ at $\infty \in \overline{\mathbb{R}}$, i.e. as the point evaluation in $C(\mathbb{T}) \cong (\mathbb{C} \cdot 1 \oplus C_0(\mathbb{R})) \circ \varphi$ at $1 \in \mathbb{T}$, where φ is the fractional linear transformation $\varphi : \mathbb{D} \longrightarrow \mathbb{C}_+ : \varphi(z) = i\,\dfrac{z+1}{1-z}$. Consequently, $m(f) = m'(f^*) + m''(h) = \widehat{f^*}(g) + \widehat{h}(1)$, and hence, without loss of generality, we can assume that $G \times \{1\}$ is injectively embedded into $\varPhi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$.

If $m''\big|_{C_0(\mathbb{R})} \not\equiv 0$, then m'' acts as the evaluation in $C_0(\mathbb{R})$ at some point x of $\mathbb{R}$, i.e. $m''(h) = h(x)$ for any $h \in C_0(\mathbb{R})$. If $f \in AP(\mathbb{R})$, then $fh \in C_0(\mathbb{R})$, and hence $f(x)\,h(x) = (fh)(x) = m(fh) = m(f)\,m(h) = m(f)\,h(x)$. Consequently, $m(f) = f(x)$, i.e. $m = m_x$ is the point evaluation in $AP^{as}(\mathbb{R})$ at $x \in \mathbb{R}$. Now we have $m(f) = m_x(f) = m_x(f^* + h) = m'_x(f^*) + m''_x(h) = \widehat{f}^*(m'_x) + h(x) = \widehat{f}^*\big(j_\imath(x)\big) + h(x)$, where $j_\imath$ is the standard dense embedding of $\mathbb{R}$ into G through $\imath$. Therefore, $\Phi(m_x) = (m'_x, m''_x)$, where m''_x is the point evaluation in $C_0(\mathbb{R})$ at $x \in \mathbb{R}$, while m'_x is the evaluation in $\widehat{AP}(\mathbb{R})$ at $j_\imath(x)$. Thus the set $\widetilde{\mathbb{R}} = \big\{\big(j_\imath(x), x\big) \colon x \in \mathbb{R}\big\}$ is contained in $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$. Since, clearly, $\widetilde{\mathbb{R}} \cong \mathbb{R}$, the set $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$ and the disjoint union $\big(G \times \{1\}\big) \sqcup \widetilde{\mathbb{R}} \cong G \sqcup \mathbb{R}$ are bijective. Note that G and $\mathbb{R}$ keep their own topologies in $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$. Since G is compact, all its accumulation points in $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$ belong to G. However, the set $\widetilde{\mathbb{R}}$ might have accumulation points in $G \times \{1\} \cong \Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big) \setminus \widetilde{\mathbb{R}}$. As we show below, this has a significant impact on the topology of $\mathcal{M}_{AP^{as}(\mathbb{R})}$. Let $m_\alpha \in \widetilde{\mathbb{R}}$ and $m_\alpha \longrightarrow m_0 \in G \times \{1\}$. If $\Phi(m_\alpha) = \big(j_\imath(x_\alpha), x_\alpha\big)$, $x_\alpha \in \mathbb{R}$, then for any $h \in C_0(\mathbb{R})$ we have $h(x_\alpha) = m''(h) = m_\alpha(h) = \widehat{h}(m_\alpha) \longrightarrow \widehat{h}(m_0) = m_0(h) = m''_0(h) = 0$, wherefrom $x_\alpha \longrightarrow \pm\infty$. Clearly, for any $\varepsilon > 0$ and any $h \in C_0(\mathbb{R})$ the set $\big\{x \in \mathbb{R} \colon |\widehat{h}(x)| < \varepsilon\big\}$ is open in $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$, and necessarily contains the set $G \times \{1\} \cong G$. This can happen if and only if the sets $G \sqcup \big\{x \in \mathbb{R} \colon |x| > N,\ N \in \mathbb{N}\big\} \subset G \sqcup \widetilde{\mathbb{R}}$ are open in $\Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$. Since $\mathcal{M}_{AP^{as}(\mathbb{R})} \cong \Phi\big(\mathcal{M}_{AP^{as}(\mathbb{R})}\big)$, we obtain the following

Theorem 4.6.1. *The maximal ideal space $\mathcal{M}_{AP^{as}(\mathbb{R})}$ of the algebra of asymptotically almost periodic functions $AP^{as}(\mathbb{R})$ on $\mathbb{R}$ is homeomorphic to the disjoint union $G \sqcup \mathbb{R}$ provided with the topology generated by the standard open sets on G and $\mathbb{R}$ correspondingly, and the sets of type $G \sqcup \big\{x \in \mathbb{R} \colon |x| > N,\ N \in \mathbb{N}\big\}$.*

In other words, $\mathcal{M}_{AP^{as}(\mathbb{R})}$ is homeomorphic to the disjoint union of the group $G = b(\mathbb{R})$ and a copy of the line $\mathbb{R}$ that winds around G above the set $j_\imath(\mathbb{R}) \subset G$ and approaches it as $x \longrightarrow \pm\infty$. Equivalently, $\mathcal{M}_{AP^{as}(\mathbb{R})}$ is homeomorphic to the set $G \sqcup \mathbb{T}$ provided with the topology generated by the standard open sets on G, $\mathbb{T} \setminus \{1\}$, and the sets of type $G \sqcup \big(U \setminus \{1\}\big)$, where U is an open subset of $\mathbb{T}$ containing 1. Observe that the closure of $b(\mathbb{R})$ in $\mathcal{M}_{AP^{as}(\mathbb{R})}$ is $b(\mathbb{R})$ itself, while the closure of $\mathbb{R}$ in $\mathcal{M}_{AP^{as}(\mathbb{R})}$ is the entire space $\mathcal{M}_{AP^{as}(\mathbb{R})}$. Therefore, $\mathbb{R}$ is dense in $\mathcal{M}_{AP^{as}(\mathbb{R})}$, while $b(\mathbb{R})$ is not.

If we assume that the set $G \sqcup \mathbb{R}$ is equipped by the topology from Theorem 4.6.1, then the Gelfand transform of any function $f \in AP(\mathbb{R}) \subset AP^{as}(\mathbb{R})$ is the function $\widehat{f} \in C(G \sqcup \mathbb{R})$ such that $\widehat{f}|_G$ coincides with the natural continuous extension $\widetilde{f}$ of f on $G = b(\mathbb{R})$, and $\widehat{f}|_{\mathbb{R}} \equiv f$. If $f \in C_0(\mathbb{R}) \subset AP^{as}(\mathbb{R})$, its Gelfand transform is the function $\widehat{f} \in C(G \sqcup \mathbb{R})$ such that $\widehat{f}|_G \equiv 0$, and $\widehat{f}|_{\mathbb{R}} \equiv f$. The algebra $\widehat{AP^{as}}(\mathbb{R})$ consists of all continuous functions in $C(G \sqcup \mathbb{R})$ of type $\widetilde{f} + h$, where $f \in AP(\mathbb{R})$ and $h \in C_0(\mathbb{R})$. Equivalently, $\widehat{AP^{as}}(\mathbb{R}) = \big\{\widetilde{f} + h \colon f \in AP(\mathbb{R}),\ h \in C(\mathbb{T}),\ h(1) = 0\big\} \subset C(G \sqcup \mathbb{T})$.

We recall that the algebra $AP_a(\mathbb{R})$ of analytic almost periodic functions on $\mathbb{R}$ is isomorphic to the algebra $A_{\mathbb{R}_+} \cong AP_{\mathbb{R}_+}(\mathbb{R})$ of analytic $\mathbb{R}_+$-functions on $\mathbb{R}$.

Consequently, $\mathcal{M}_{AP_a(\mathbb{R})} \cong \mathcal{M}_{A_{\mathbb{R}_+}} \cong \overline{\mathbb{D}}_G$, the G-disc over $G = b(\mathbb{R})$. Theorem 4.6.1 now yields the following

Theorem 4.6.2. *The maximal ideal space of the algebra*

$$AP_a^{as}(\mathbb{R}) \cong AP_a(\mathbb{R}) \oplus C_0(\mathbb{R}) \cong \left[z,\ 1/z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\right\}\right]$$

of asymptotically analytic almost periodic functions on $\mathbb{R}$ *is homeomorphic to the set* $\overline{\mathbb{D}}_{\beta\mathbb{R}} \sqcup \mathbb{T}$ *provided with the topology generated by the standard open sets on* $\overline{\mathbb{D}}_{\beta\mathbb{R}}$, $\mathbb{T}\setminus\{1\}$, *and sets of type* $\overline{\mathbb{D}}_{\beta\mathbb{R}} \sqcup \big(U\setminus\{1\}\big)$, *where* U *is an open subset of* $\mathbb{T}$ *containing* $1 \in \mathbb{T}$.

Let $A_0(\mathbb{C}_+)$ and $A_0(\mathbb{C}_-)$ be the algebras of functions in $C_0(\mathbb{R})$, that possess analytic extensions in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_- = -\,\mathbb{C}_+$ correspondingly. If a function in $AP^{as}(\mathbb{R})$ is analytically extendable in the upper half-plane $\mathbb{C}_+$, it is called *analytic asymptotically almost periodic*. One can see that any analytic asymptotically almost periodic function f admits a unique decomposition of the form $f = f^* + h$, where $f^* \in AP_a(\mathbb{R})$ and $h \in A_0(\mathbb{C}_+)$. The algebra $AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$ of analytic asymptotically almost periodic functions is isometrically isomorphic to the subalgebra $\left[z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\right\}\right]$ of $H^\infty \cap C\big(\overline{\mathbb{D}} \setminus \{1\}\big)$ generated by the functions z and $e^{a\,(z+1)/(z-1)}$, $a \in \mathbb{R}_+$, on $\mathbb{R}$. Likewise, the algebra $AP(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$ of all $f \in C_b(\mathbb{R})$ with a unique decomposition of type $f = f^* + h$, where $f^* \in AP(\mathbb{R})$ and $h \in A_0(\mathbb{C}_+)$, is isometrically isomorphic to the algebra $\left[z, \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}\right\}\right]$ on $\mathbb{R}$. Since $\mathcal{M}_{A_0(\mathbb{C}_+)} \cong \overline{\mathbb{D}}$, we have also the following

Theorem 4.6.3. *The maximal ideal space of the algebra*

$$AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+) \cong \left[z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\right\}\right] \subset H^\infty \cap C(\overline{\mathbb{D}} \setminus 1)$$

of analytic asymptotically almost periodic functions on $\mathbb{R}$ *(or on* $\overline{\mathbb{C}}_+$*) is homeomorphic to the set* $\overline{\mathbb{D}}_{b(\mathbb{R})} \sqcup \overline{\mathbb{D}}$ *provided with the topology generated by the standard open sets on* $\overline{\mathbb{D}}_{b(\mathbb{R})}$, $\overline{\mathbb{D}}\setminus\{1\}$, *and sets of type* $\overline{\mathbb{D}}_{b(\mathbb{R})} \sqcup \big(U \setminus \{1\}\big)$, *where* U *is an open subset of* $\mathbb{D}$ *containing* $1 \in \overline{\mathbb{D}}$.

Let $\pi : \mathcal{M}_{H^\infty} \longrightarrow \mathbb{D}$ be the mapping defined by $\pi(\varphi) = \varphi(\mathrm{id}) \in \overline{\mathbb{D}}$, which provides a homeomorphism between $\widetilde{\mathbb{D}} = \pi^{-1}(\mathbb{D}) \subset \mathcal{M}_{H^\infty}$ and $\mathbb{D}$.

Corollary 4.6.4. (a) *The mapping* π *maps the unit disc* $\mathbb{D} = \pi(\widetilde{\mathbb{D}})$ *densely in the maximal ideal space of the algebra* $AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$ *of analytic asymptotically almost periodic functions on* $\mathbb{R}$. *Hence the algebra*

$$\left[z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\right\}\right] \subset H^\infty$$

does not have a $\mathbb{D}$*-corona.*

(b) *The image* $\widetilde{j}_\imath(\overline{\mathbb{C}}_+)$ *of the upper half-plane* $\overline{\mathbb{C}}_+$ *under the embedding* $\widetilde{j}_\imath$ *is not dense in the maximal ideal space of the algebra* $AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$. *Hence the algebra*

$$\left[z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\right\}\right] \subset H^\infty$$

has a $\mathbb{C}_+$*-corona.*

In a similar way we obtain

Theorem 4.6.5. *The maximal ideal space of the algebra*

$$AP(\mathbb{R}) \oplus A_0(\mathbb{C}_+) \cong \left[z,\ \left\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}\right\}\right]$$

on $\mathbb{T}$ *is homeomorphic to the set* $G \sqcup \overline{\mathbb{D}}$ *provided with the topology generated by the standard open sets on* G, $\overline{\mathbb{D}} \setminus \{1\}$, *and sets of type* $G \sqcup (U \setminus \{1\})$, *where* U *is an open subset of* $\overline{\mathbb{D}}$ *containing* $1 \in \overline{\mathbb{D}}$.

The arguments used in the proof of Theorem 4.2.16 yield the following result.

Proposition 4.6.6. *There is a continuous mapping* r *from* $\mathcal{M}_{H^\infty}$ *onto the set*

$$\overline{\mathbb{D}}_{b(\mathbb{R})} \sqcup \overline{\mathbb{D}} \cong \mathcal{M}_{AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+)}$$

equipped with the topology described in Theorem 4.6.3.

Observe that the space $AP^{as}(\mathbb{R})$ of asymptotically almost periodic functions is a uniform algebra, invariant under $\mathbb{R}$-shifts, i.e. the function $f_t(x) = f(x+t)$ belongs to $AP^{as}(\mathbb{R})$ for any $f \in AP^{as}(\mathbb{R})$ and $t \in \mathbb{R}$. The uniform algebras $AP^{as}_a(\mathbb{R})$, $C_0(\mathbb{R})$, $A_0(\mathbb{C}_+)$ and $A_0(\mathbb{C}_-)$ also are $\mathbb{R}$-invariant. If S is an additive subgroup of $\mathbb{R}$, then the set $AP_S(\mathbb{R}) \oplus C_0(\mathbb{R})$ is an $\mathbb{R}$-invariant uniform subalgebra of $AP^{as}(\mathbb{R})$ which contains $C_0(\mathbb{R})$. As the following theorem implies, every $\mathbb{R}$-invariant subalgebra of $AP^{as}(\mathbb{R})$ containing $C_0(\mathbb{R})$ is of this type.

Theorem 4.6.7. *For any* $\mathbb{R}$*-invariant subalgebra* A *of* $AP^{as}(\mathbb{R})$ *there is a unique semigroup* $S \subset \mathbb{R}$, *and a closed* $\mathbb{R}$*-invariant subalgebra* B *of* $C_0(\mathbb{R})$, *so that* A *admits a decomposition of type* $A = AP_S(\mathbb{R}) \oplus B$.

Proof. According to Lemma 2.1.4, for every $f \in A$ there is a unique almost periodic function f^* on $\mathbb{R}$, and an $h \in C_0(\mathbb{R})$, such that $f = f^* + h$. Fix an $\varepsilon > 0$, and choose a positive number t such that $\left|f^*(x) - f^*(x+t)\right| < \varepsilon$ for all $x \in \mathbb{R}$, and $|h(x)| < \varepsilon$ for all $|x| > t$. This is possible, since $g \in AP(\mathbb{R})$, and $h \in C_0(R)$. Consider the function

$$f^N = \frac{1}{N}\sum_{n=1}^{N} f_{nt},$$

where $f_{nt}(x) = f(x+nt)$. Clearly, $f^N \in A$, since A is $\mathbb{R}$-invariant. One can easily check that $|f^N - f^*| < 2\varepsilon$ for N big enough. Hence the function f^* can be approximated uniformly by functions of type $f^N \in A$, and therefore, $f^* \in A$.

Thus, $h = f - f^* \in A$, and therefore, $A = AP^{as}(\mathbb{R}) \cap A = \big(AP(\mathbb{R}) \cap A\big) \oplus B \subset AP^{as}(\mathbb{R})$, where $B = C_0(\mathbb{R}) \cap A$ is an $\mathbb{R}$-invariant ideal in A. Since the algebra $AP(\mathbb{R}) \cap A$ is $\mathbb{R}$-invariant, by Proposition 4.1.9 there is a semigroup $S \subset \mathbb{R}$ such that $AP(\mathbb{R}) \cap A = AP_S(\mathbb{R})$. Consequently, $A = AP_S(\mathbb{R}) \oplus B$. $\square$

Note that in this case B is an ideal in A, while $AP_S(\mathbb{R})$ and B both are closed subalgebras of A.

If $S \subset \mathbb{R}_+$, then the algebra $AP_S(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$ is an antisymmetric uniform subalgebra of the algebra $AP_a(\mathbb{R}) \oplus A_0(\mathbb{C}_+)$ of analytic asymptotically almost periodic functions on $\mathbb{R}$, containing $A_0(\mathbb{C}_+)$. The following theorem implies that every antisymmetric subalgebra of $AP^{as}(\mathbb{R})$ containing $A_0(\mathbb{C}_+)$ is of this type.

Theorem 4.6.8. *For any antisymmetric $\mathbb{R}$-invariant subalgebra A of $AP^{as}(\mathbb{R})$ there exist a unique semigroup $S \subset \mathbb{R}_+$, and a closed antisymmetric $\mathbb{R}$-invariant subalgebra B of $C_0(\mathbb{R})$, so that A admits a decomposition of type $A = AP_S(\mathbb{R}) \oplus B$.*

The proof follows the same lines as the proof of Theorem 4.6.7, by taking into account the antisymmetry of A. Similarly as before, B is an ideal in A, while $AP_S(\mathbb{R})$ and B are closed subalgebras of A.

Theorem 4.6.8 implies that any antisymmetric subalgebra of the algebra

$$\big[z, \big\{e^{a\,(z+1)/(z-1)},\ a \in \mathbb{R}_+\big\}\big] \subset H^\infty \cap A(\overline{\mathbb{D}} \setminus 1),$$

is of type $\big[\big\{e^{a\,(z+1)/(z-1)}, a \in S\big\}\big] \oplus B$, where S is an additive semigroup in $\mathbb{R}$, and B is an antisymmetric closed subalgebra of the space $\big\{f \in A(\mathbb{D}) : f(1) = 0\big\}$.

Using similar arguments as in Theorem 4.6.8, one can show the following

Theorem 4.6.9. *Let A be an antisymmetric $\mathbb{R}$-invariant uniform subalgebra of the algebra $AP_w(\mathbb{R})$ of weakly almost periodic functions on $\mathbb{R}$ which is invariant under $\mathbb{R}$-shifts. Then there is a unique semigroup $S \subset \mathbb{R}$, and a closed $\mathbb{R}$-invariant subalgebra B of $C(\overline{\mathbb{R}})|_{\mathbb{R}}$, such that $A = AP_S(\mathbb{R}) \oplus B$.*

Note that in this case both algebras $AP_S(\mathbb{R})$, of almost periodic S-functions on $\mathbb{R}$, and $B \subset C(\overline{\mathbb{R}})\big|_{\mathbb{R}}$ are closed subalgebras of A.

4.7 Notes

The class of shift-invariant algebras is more general than the class of G-disc algebras of generalized analytic functions, introduced by Arens and Singer [AS1]. Algebras of S-functions with $S \neq \Gamma_+$ were considered in [T2]. The description of automorphisms for shift-invariant algebras is from [GPT]. For G-disc algebras generated by weakly archimedean ordered semigroups the result is due to Arens [A]. The results on peak groups for shift-invariant algebras are from [GT4]. Radó's and Riemann's theorem were proven originally by classical arguments. Their versions in a uniform algebra setting were considered by Glicksberg [G5]. In [GPT] (see also [T2]) they have been extended to shift-invariant algebras. Integrally closed uniform algebras were studied by Glicksberg [G5]. A uniform algebra A possesses the *weak Riemann property* if, given a function $g \in A$ with $\mathrm{Null}\,(g) \cap \partial A = \emptyset$, every bounded function, A-holomorphic on $\mathcal{M}_A \backslash \mathrm{Null}\,(g)$, can be extended continuously on $\mathcal{M}_A$. Using similar arguments as in Theorem 4.5.8, one can show that a shift-invariant algebra A_S on G possesses the weak Riemann property if and only if the weak and the strong hulls of S coincide. The results on asymptotically almost periodic functions are from [GT4].

It would also be interesting to have a characterization of semigroups S and S_1 of $\Gamma = \widehat{G}$ containing 0, with the same families of p-sets, or, peak sets, of G relative to A_S and A_{S_1}. This problem seems to be related with the existence of an automorphism Ψ of Γ, with $\Psi(S) = S_1$.

Chapter 5

Extension of semicharacters and additive weights

The central theme of this chapter is the extendability of linear multiplicative functionals from smaller to larger shift-invariant algebras. It is closely related to extendability of non-negative semicharacters and their logarithms (additive weights) from smaller to larger semigroups. We give necessary and sufficient conditions for extendability of additive weights in terms of properties such as monotonicity, and in terms of purely algebraic properties of their semigroup domains. As immediate corollaries we obtain necessary and sufficient conditions for the corresponding algebras of almost periodic functions and of H^∞-functions to possess coronae.

5.1 Extension of non-vanishing semicharacters

Let G be a compact abelian group, and let $S \subset P$ be two semisubgroups of the dual group $\widehat{G}$. For the corresponding shift-invariant algebras we have $A_S \subset A_P \subset C(G)$. Clearly, a linear multiplicative functional of A_S can be extended to a linear multiplicative functional on A_P if and only if the corresponding semicharacter φ on S can be extended to a semicharacter on P.

Let S be an additive subsemigroup with cancellation law and 0, and let $\Gamma_S = S - S$ be the group envelope of S. The S-*order*, on Γ_S is defined by $b \succ a$ if and only if $b - a \in S$. Any non-negative semicharacter $\varphi \in H(S)$ is monotone decreasing with respect to the S-order on S. Indeed, if $b \succ a$ for some $a, b \in S$, then $b = a + c$ for some $c \in S$. Therefore, $\varphi(b) = \varphi(a)\,\varphi(c) \leq \varphi(a)$, since $0 \leq \varphi \leq 1$.

If P is a semigroup in Γ_S containing S, we equip Γ_S with the P-order. By the above remark every non-negative semicharacter on P is monotone decreasing with respect to the P-order. Consequently, for a non-negative semicharacter φ on

S to possess a semicharacter extension on P, φ needs to be monotone decreasing on S with respect to the P-order. In fact, this condition is also sufficient.

Proposition 5.1.1. *A positive semicharacter $\varphi \in H\big(S,(0,1]\big)$ on S has a unique semicharacter extension on a semigroup $P \supset S$ if and only if φ is monotone decreasing with respect to the P-order on S.*

Proof. By the previous remarks, we only need to prove the sufficiency. Assume that φ is a monotone decreasing positive semicharacter on S. If $b \in P \subset \Gamma_S = S - S$, then $b = a - c$ for some $a, c \in S$. Clearly, $a \succ c$, and $\widetilde{\varphi}(b) = \varphi(a)/\varphi(c)$ is a positive homomorphic extension of φ on P. Since $\varphi(a) \le \varphi(c)$, we have that $0 \le \widetilde{\varphi}(b) \le 1$, i.e. $\widetilde{\varphi}$ is a positive semicharacter on P. □

Proposition 5.1.2. *A non-vanishing semicharacter $\varphi \in H(S, \overline{\mathbb{D}}^*)$ on S has a unique semicharacter extension on a semigroup $P \supset S$ if and only if its modulus $|\varphi| \in H(S)$ is monotone decreasing with respect to the P-order on S.*

Proof. Suppose $\varphi \in H(S)$ does not vanish on S. By Theorem 3.5.4 and Proposition 3.5.5, φ has a unique polar decomposition $\varphi = |\varphi|\,\chi$ with some character χ of Γ_S. Hence, φ is extendable on P as an element of $H(P)$ if and only if $|\varphi|$ is extendable. By Proposition 5.1.1 this happens if and only if $|\varphi|$ is monotone decreasing on S with respect to the P-order on S. □

Consider the particular case when $S \subset \mathbb{R}$, and $P = \Gamma_+ = \Gamma \cap [0, \infty)$, where $\Gamma \subset \mathbb{R}$ is the group envelope Γ_S of S, equipped with the discrete topology.

Proposition 5.1.3. *Let S be a semigroup of $\mathbb{R}_+$ containing 0 and let $\Gamma = \Gamma_S = S - S$ be its group envelope. A non-negative semicharacter $\varphi \in H\big(S,[0,1]\big)$ on S has a unique semicharacter extension on Γ_+ if and only if φ is monotone decreasing on S.*

Proof. Note that the Γ_+-order coincides with the standard order on Γ_+ inherited from $\mathbb{R}$. Because of Proposition 5.1.2 and the remarks preceding Proposition 5.1.1, we only need to prove the sufficiency part of the statement for vanishing on S semicharacters. Assume that φ is a monotone decreasing semicharacter of S with $\varphi(a) = 0$ for some $a \in S$. Assume $\varphi(b) \ne 0$ for some $b \in S \setminus \{0\}$. If $n \in \mathbb{N}$ is such that $nb > a$, then $\varphi(nb) = \varphi(b)^n \ne 0$ in contradiction with the monotonicity of φ on $S \subset \mathbb{R}_+$. Therefore, the only monotone decreasing semicharacter of S that vanishes on S is the characteristic function $\varkappa_{\{0\}} \in H\big(S,[0,1]\big)$ of 0 in S. Applied to the semigroup Γ_+, the same argument shows that the only monotone decreasing semicharacter of Γ_+ that vanishes on Γ_+ is the characteristic function $\varkappa_{\{0\}}$ of 0 in Γ_+. Clearly, $\varkappa_{\{0\}} \in H\big(S,[0,1]\big)$ is the only possible semigroup extension of $\varkappa_{\{0\}}$ from S on Γ_+. □

Corollary 5.1.4. *Under the assumptions of Proposition* 5.1.3*, a semicharacter $\varphi \in H(S)$ possesses a unique semicharacter extension on Γ_+ if and only if its modulus $|\varphi|$ is monotone decreasing on S.*

Proof. If φ possesses a semigroup extension on Γ_+, then so does $|\varphi|$, and therefore it is monotone decreasing by Proposition 5.1.3.

Let, conversely, $|\varphi| \in H(S,[0,1])$ be monotone decreasing on S. If φ does not vanish on S, then it possesses a semigroup extension on Γ_+ by Proposition 5.1.2. If φ vanishes in S, then by the arguments from the proof of Proposition 5.1.3 it coincides with $\varkappa_{\{0\}}$, and possesses a unique semigroup extension on Γ_+. □

Example 5.1.5. Let S be the semigroup $\Gamma_t = \{0\} \cup [t,\infty) \subset \mathbb{R}_+$, where $t > 0$. Clearly, the group envelope $\Gamma_{\Gamma_t} = \Gamma_t - \Gamma_t$ of Γ_t is $\mathbb{R}$. We claim that every semicharacter $\varphi \in H\big(S,[0,1]\big)$ is monotone decreasing on Γ_t. Indeed, let $a > b$, $a, b \in \Gamma_t$, and let $n \in \mathbb{N}$ be such that $n(a-b) \geq \nu$, thus $n(a-b) \in \Gamma_t$. Hence, $\varphi(a)^n = \varphi(na) = \varphi\big(nb + n(a-b)\big) = \varphi(nb)\,\varphi\big(n(a-b)\big) \geq \varphi(nb) = \varphi(b)^n$, and therefore, $\varphi(a) \geq \varphi(b)$. Consequently, every non-negative semicharacter φ of Γ_t is monotone decreasing. By Proposition 5.1.3 φ can be extended on $\mathbb{R}_+$ as a semicharacter in $H(\mathbb{R}_+)$.

Example 5.1.6. Let $\beta > 0$ be an irrational number. Consider the two-dimensional semigroup $S^\beta = \{n + m\beta \colon n, m \in \mathbb{Z}_+\} \subset \mathbb{R}$. The group envelope of S^β is $\Gamma^\beta = S^\beta - S^\beta = \{n+m\beta \colon n, m \in \mathbb{Z}\}$. Let $P = \Gamma^\beta_+ = \Gamma^\beta \cap \mathbb{R}_+ = \{n+m\beta \geq 0 : n, m \in \mathbb{Z}\}$. Clearly, $S^\beta \neq \Gamma^\beta_+$. For instance, if $\beta > 1$, then the positive number $\beta - \lfloor\beta\rfloor \in \Gamma^\beta_+ \setminus S^\beta$, where $\lfloor\beta\rfloor$ is the greatest integer preceding β. For a fixed $b \in (0,1)$ the function $\varphi(n+m\beta) = b^n$, $n+m\beta \in S^\beta$, is a homomorphism from S^β to $(0,1] \subset \overline{\mathbb{D}}$. Thus, φ is a non-negative semicharacter on S^β, i.e. $\varphi \in H\big(S,[0,1]\big) \subset H(S)$. We claim that φ is not monotone decreasing on S^β. Indeed, $\varphi(m\beta) = 0$, while $\varphi(n) = b^n > 0$ for every $n > m\beta$. The natural (and only) homomorphic extension $\widetilde{\varphi}$ of φ on Γ^β_+ is given by $\widetilde{\varphi}(n + m\beta) = b^n$, $n, m \in \mathbb{Z}$, $n + m\beta \geq 0$. However, $\widetilde{\varphi} \notin H(\Gamma^\beta_+)$ since, for instance, $\widetilde{\varphi}(\beta - \lfloor\beta\rfloor) = b^{-\lfloor\beta\rfloor} > 1$. Consequently, φ can not be extended as a semicharacter on $\mathbb{R}$.

Proposition 5.1.7. *Let S be a semigroup of $\mathbb{R}_+$ containing 0 and let $\Gamma = S - S$ be its group envelope. The maximal ideal space $\mathcal{M}_{A_S}$ of the algebra A_S of S-functions on $G = \widehat{\Gamma}$ is homeomorphic to the G-disc $\overline{\mathbb{D}}_G$ if and only if all non-negative semicharacters on S are monotone decreasing.*

Proof. By Corollary 5.1.4, the above conditions are necessary and sufficient for every semicharacter $\varphi \in H(S)$ to be extended uniquely as a semicharacter in $H(\Gamma_+) \cong \overline{\mathbb{D}}_G$. □

According to Proposition 4.1.4 the space $AP_S(\mathbb{R})$ of almost periodic S-functions on $\mathbb{R}$ is a uniform algebra isometrically isomorphic to the algebra A_S of S-functions on $G = \widehat{\Gamma}$. Propositions 5.1.7 and 4.2.4 yield the following

Proposition 5.1.8. *Let S be a semigroup of $\mathbb{R}_+$ containing 0, whose group envelope $\Gamma = S - S$ is dense in $\mathbb{R}$. The maximal ideal space $\mathcal{M}_{AP_S(\mathbb{R})}$ of the algebra $AP_S(\mathbb{R})$ of almost periodic S-functions is homeomorphic to the G-disc $\overline{\mathbb{D}}_G$ if and only if all non-negative semicharacters on S are monotone decreasing.*

Note that any dense in $\mathbb{R}_+$ semigroup of type $\Gamma_+ = \Gamma \cap \mathbb{R}_+$ satisfies the assumptions of Proposition 5.1.8.

We recall that if G is a solenoidal group, then there is a natural embedding $j_\imath : \mathbb{R} \longrightarrow G$ of the real line $\mathbb{R}$ into G with a dense range, so that $j_\imath(0) = \imath$. If $S \subset \mathbb{R}_+$, the restrictions of S-functions on $j_\imath(\mathbb{R})$ are almost periodic S-functions, that admit analytic extension on the upper half-plane $\mathbb{C}_+$. Clearly, the set $(0,1] \times \mathbb{R}$ can be embedded densely in the G-disc $\overline{\mathbb{D}}_G = [0,1] \diamond G/\{0\} \diamond G$. Since $(0,1] \times \mathbb{R} \cong [0,\infty) \times \mathbb{R} \cong \overline{\mathbb{C}}_+$, the upper half-plane $\overline{\mathbb{C}}_+$ also can be embedded densely in the G-disc $\overline{\mathbb{D}}_G$. Note that $\widehat{\Gamma_S} \subset \mathcal{M}_{A_S}$ for every semigroup $S \subset \mathbb{R}_+$. The embedding $j_\imath \colon \mathbb{R} \longrightarrow \widehat{\Gamma_S}$ extends naturally to an embedding $\widetilde{j_\imath}$ of the upper half-plane $\mathbb{C}_+$ into the maximal ideal space $\mathcal{M}_{A_S}$ of the algebra A_S (and, together, of $AP_S(\mathbb{R})$). Since the closure of the range $\widetilde{j_\imath}(\mathbb{C}_+)$ in $\mathcal{M}_{A_S}$ is homeomorphic to the G-disc $\overline{\mathbb{D}}_G$, we see that $\widetilde{j_\imath}(\mathbb{C}_+)$ is dense in $\mathcal{M}_{A_S}$ if and only if $\mathcal{M}_{A_S} = \overline{\mathbb{D}}_G$. Therefore, we have the following

Corollary 5.1.9. *Under the assumptions of Proposition* 5.1.8 *the upper half-plane $\mathbb{C}_+$ can be embedded densely via $\widetilde{j_\imath}$ in the maximal ideal space $\mathcal{M}_{AP_S(\mathbb{R})}$ of the algebra $AP_S(\mathbb{R})$ of almost periodic S-functions if and only if all non-negative semicharacters on S are monotone decreasing. Consequently, if all non-negative semicharacters on S are monotone decreasing, then the algebra $AP_S(\mathbb{R})$ does not have a $\mathbb{C}_+$-corona.*

Note that $\theta = \ln \varrho$ is an additive function from S to $(-\infty, 0]$ for any semicharacter $\varrho \in H_+(S)$. It is straightforward to see that the necessary and sufficient condition in Corollary 5.1.9 is equivalent to the following: Every additive positive function θ on S is monotone increasing, i.e. of type $\theta(a) = y_\theta a$ for some $y_\theta \in [0,\infty)$, or $\theta(a) = \infty$, for every $a \neq 0$. In the case when $S = \mathbb{R}_+$ this form of the condition for the dense embedding of $\mathbb{C}_+$ into $\mathcal{M}_{AP_S(\mathbb{R}_+)}$ has been given by Boettcher [B4].

Proposition 5.1.7 and the remarks preceding Corollary 5.1.9 imply the following corona type theorem for the algebra H^∞_S.

Proposition 5.1.10. *Let S be a semigroup of $\mathbb{R}_+$ containing* 0. *The unit disc $\mathbb{D}$ is dense in the maximal ideal space of the algebra H^∞_S via the fractional linear transformation $\widetilde{j_\imath} \circ \varphi$, where $\varphi(z) = i\,\dfrac{1+z}{1-z}$ if and only if all non-negative semicharacters on S are monotone decreasing. Consequently, the algebra H^∞_S does not have a $\mathbb{D}$-corona if and only if all non-negative semicharacters on S are monotone decreasing.*

In particular, since every non-negative semicharacter on $\Gamma_+ \subset \mathbb{R}_+$, where Γ is a dense subgroup of $\mathbb{R}$, is monotone decreasing, the algebra $H^\infty_{\Gamma_+}$ does not have a $\mathbb{D}$-corona, i.e. the open unit disc $\mathbb{D}$ can be embedded densely in the maximal ideal space $\mathcal{M}_{H^\infty_{\Gamma_+}}$ via the mapping $\widetilde{j_\imath} \circ \varphi$ from Proposition 5.1.10.

5.2 Extension of additive weights and semicharacters on semigroups

In this section we prove that the monotonicity condition of a semicharacter, considered in the previous section, is also sufficient for the extendability of semicharacters of a semigroup.

Theorem 5.2.1. *Let $S \subset P$ be two semigroups with cancellation law and 0. A semicharacter $\varphi \in H(S)$ possesses a semigroup extension on P if and only if $|\varphi|$ is monotone decreasing with respect to the P-order on S.*

We present a proof of this theorem based on so-called additive weights.

Definition 5.2.2. A function θ on a semigroup S with 0 is called an *additive weight* on S, if it is additive, takes values in the extended half-line $\overline{\mathbb{R}}_+ = [0, \infty]$, and $\theta(0) = 0$.

Hence, the values of any additive weight θ on S may be either non-negative, or ∞, while $\theta(0) = 0$, and $\theta(a + b) = \theta(a) + \theta(b)$ for every $a, b \in S$. Denote by $\Theta(S)$ the set of additive weights on S. It is clear that with the pointwise addition $\Theta(S)$ is a semigroup.

Proposition 5.2.3. *The semigroup of additive weights $\Theta(S)$ of a semigroup S is isomorphic to the semigroup $H\big(S, [0, 1]\big)$ of non-negative semicharacters of S.*

Proof. Indeed, given an additive weight $\theta \in \Theta(S)$, the function $r_\theta \colon a \longmapsto e^{-\theta(a)}$, with $e^{-\infty} = 0$, is multiplicative on S, $e^{-\theta(0)} = e^0 = 1$, and $\big|e^{-\theta(a)}\big| \leq 1$ on S. Therefore, $\varrho_\theta = e^{-\theta} \in H\big(S, [0, 1]\big)$. As we saw above, for any $\varrho \in H\big(S, [0, 1]\big)$ the function $\theta_\varrho \colon a \longmapsto -\ln \varrho(a)$ is an additive weight on S. It is clear that $\theta \longmapsto \varrho_\theta$ is a bijection from $\Theta(S)$ onto $H\big(S, [0, 1]\big)$ that preserves the operations, i.e. is an isomorphism. $\square$

Example 5.2.4. (a) Let $S = \mathbb{Z}_+$, and let $x_0 \in [0, \infty)$. Clearly, the function $\theta_{x_0}(n) = x_0\, n$, $n \in \mathbb{Z}_+$ is an additive weight on $\mathbb{Z}_+$. The function

$$\theta_\infty(n) = \begin{cases} 0 & \text{when } n = 0, \\ \infty & \text{when } n \in \mathbb{N} \end{cases}$$

is also an additive weight on $\mathbb{Z}_+$. Actually, any additive weight on $\mathbb{Z}_+$ is of type θ_{x_0} for some $x_0 \in [0, \infty]$. Therefore, in this case $\Theta(\mathbb{Z}_+)$ is isomorphic to the semigroup $[0, \infty]$.

(b) For $S = \mathbb{R}_+ = [0, \infty)$ the set $\Theta(\mathbb{R}_+)$ is also isomorphic to $[0, \infty]$, since all additive weights on $\mathbb{R}_+$ are of type $\theta_{x_0}(x) = x_0 x$ for some $x \in [0, \infty)$, or

$$\theta_\infty(x) = \begin{cases} 0 & \text{when } x = 0, \\ \infty & \text{when } n \in (0, \infty). \end{cases}$$

Let P be an additive semigroup with cancellation law and 0, and let $S \subset P$ be a semigroup of P containing 0. According to Proposition 5.1.1, a necessary condition for a semicharacter $\varphi \in H\big(S, [0,1]\big)$ to possess a semigroup extension on P is that φ be monotone decreasing with respect to the P-order on S. Consequently, for a weight $\theta \in \varTheta(S)$ to possess a weight extension on P it is necessary for θ to be monotone increasing with respect to the same order on S. The next theorem shows that this condition is also sufficient.

Theorem 5.2.5. *Let $S \subset P$ be two additive subsemigroups with cancellation law and 0. An additive weight $\theta \in \varTheta(S)$ possesses a weight extension if and only if θ is monotone increasing with respect to the P-order on S.*

Proof. The necessity is clear from the previous remarks. To prove the sufficiency, suppose that θ is an increasing additive weight on S. Fix an $a \in P \setminus S$ and let $S_a = \mathbb{Z}_+ a + S = \big\{na + b \colon b \in S,\ n \in \mathbb{Z}_+\big\}$. To extend θ on S_a as, say, $\widetilde{\theta}_a \in \varTheta(S_a)$, we first define $\widetilde{\theta}_a$ at a.

If $\mathbb{N}a + P \cap S = \emptyset$, we define $\widetilde{\theta}_a(a) = \infty$. Assume that $\mathbb{N}a + P \cap S \neq \emptyset$. Note that $c \in \mathbb{N}a + P \cap S$ if and only if $c \in S$ and $na + p = c$ for some $n \in \mathbb{N}$, $p \in P$ i.e. $c \in S$ and $na \prec c$ for some $n \in \mathbb{N}$. Observe that if $c \in \mathbb{N}a + P \cap S$, then the set $\varSigma = \big\{(n, b, c) \colon n \in \mathbb{N},\ b, c \in S,\ \theta(b) < \infty,\ c \in \mathbb{N}a + P \cap S,\ na + b \prec c\big\}$ is nonempty. Indeed, any triple $(n, 0, c)$ with $c \in \mathbb{N}a + P \cap S$ belongs to $\varSigma$. Set

$$\widetilde{\theta}_a(a) = \inf_{(n,b,c) \in \varSigma} \frac{\theta(c) - \theta(b)}{n}. \tag{5.1}$$

The condition $\theta(b) \neq \infty$ is needed here to avoid the undeterminate expression of type $\infty - \infty$ in the numerator. Observe that $0 \le \widetilde{\theta}_a(a) \le 1$. Indeed, if $(n, b, c) \in \varSigma$, then $c \succ na + b$, and hence $c = (na + b) + p = b + (na + p)$ for some $p \in P$. Consequently, $c \succ b$, since $na + p \in P$. Therefore, $\theta(c) \ge \theta(b)$, and hence $\widetilde{\theta}_a(a) \ge 0$. Also, $\theta(c) - \theta(b) \le 1$ since both $\theta(b)$, $\theta(c) \le 1$. Note that the number $\widetilde{\theta}_a(a)$ is well-defined, since if $na + b = c \in S$ for some $b \in S$ with $\theta(b) < \infty$ and $c \in \mathbb{N}a + P \cap S$, then $\widetilde{\theta}_a(a) = \dfrac{\theta(c) - \theta(b)}{n}$. Indeed, if $(m, b', c') \in \varSigma$, then $ma + b' \prec c'$ and therefore $mc + nb' = m(na + b) + nb' = n(ma + b') + mb \prec nc' + mb$. The additivity and monotonicity of θ imply

$$m\theta(c) + n\theta(b') \le n\theta(c') + m\theta(b),$$

whence

$$\frac{\theta(c) - \theta(b)}{n} \le \frac{\theta(c') - \theta(b')}{m}.$$

Consequently, $\dfrac{\theta(c) - \theta(b)}{n} = \displaystyle\inf_{(n,b',c') \in \varSigma} \frac{\theta(c') - \theta(b')}{n} = \widetilde{\theta}_a(a)$. This shows that the number $\widetilde{\theta}_a(a)$ is well-defined, as claimed.

Now define $\widetilde{\theta}_a$ by the equation $\widetilde{\theta}_a(na+b) = n\widetilde{\theta}_a(a) + \theta(b)$ for any $na+b \in S_a$. This definition is unambiguous. Indeed, if $na+b = ma+d \in S_a$ with, say $m > n$, then $(m-n)a + d = b$. As noticed above, then $\widetilde{\theta}_a(a) = \dfrac{\theta(b)-\theta(d)}{m-n}$, thus $m\widetilde{\theta}_a(a) + \theta(d) = n\widetilde{\theta}_a(a) + \theta(b)$. One can readily check that $\widetilde{\theta}_a$ is an additive weight on S_a. We claim that $\widetilde{\theta}_a$ is increasing on S_a. Indeed, let

$$na + b \succ ma + b', \tag{5.2}$$

be two elements of S_a, where b, $b' \in S$, n, $m \in \mathbb{N}$. There are three possible cases for m and n.

(a) If $m = n$, then $b \succ b'$ by the cancellation law, and therefore $\theta(b) \geq \theta(b')$ by the assumed monotonicity of θ.

(b) If $n > m$, then (5.2) becomes $ka + b \succ b'$, with $k = n - m$, and what we need to show is that $\widetilde{\theta}_a(ka+b) \geq \theta(b')$. It is clear that if $\widetilde{\theta}_a(a) = \infty$, we have $\widetilde{\theta}_a(ka+b) = k\widetilde{\theta}_a(a) + \theta(b) = \infty \geq \theta(b')$, as needed. If $\widetilde{\theta}_a(a) < \infty$, then according to (5.1), for every $\varepsilon > 0$ one can find a $c \in S$ with $\theta(c) < \infty$ and $d \in \mathbb{N}a + P \cap S$, so that $(n, c, d) \in \Sigma$ and

$$\frac{\theta(d) - \theta(c)}{n} < \widetilde{\theta}_a(a) + \frac{\varepsilon}{k}.$$

Because $(n, c, d) \in \Sigma$, we have $d \succ na + c$, and therefore

$$kd + nb \succ k(na + c) + nb = n(ka + b) + kc \succ nb' + kc,$$

since $ka + b \succ b'$. Additivity and monotonicity of θ imply that $k\theta(d) + n\theta(b) \geq n\theta(b') + k\theta(c)$, thus

$$\frac{k\big(\theta(d) - \theta(c)\big)}{n} + \theta(b) \geq \theta(b').$$

Consequently,

$$\widetilde{\theta}_a(ka+b) = k\widetilde{\theta}_a(a) + \theta(b) > \frac{k\big(\theta(d) - \theta(c)\big)}{n} - \varepsilon + \theta(b) \geq \theta(b') - \varepsilon.$$

Since ε was arbitrarily chosen, we conclude that $\widetilde{\theta}_a(ka+b) \geq \theta(b')$, as desired.

(c) If $n < m$, then (5.2) becomes $b \succ la + b'$ with $l = m - n$, and what we are to show is that $\theta(b) \geq \widetilde{\theta}_a(la + b')$. Note that in this case $\theta(b) \geq \theta(b')$, since $b \succ b'$. If $\theta(b') = \infty$, then also $\theta(b) = \infty$, thus $\theta(b) \geq \theta(la + b')$. Assume that $\theta(b') < \infty$. If also $\widetilde{\theta}_a(a) = \infty$, then

$$\frac{\theta(b) - \theta(b')}{l} \geq \widetilde{\theta}_a(a),$$

since $(l, b', b) \in \Sigma$, and hence $\theta(b) = \infty$. Therefore, $\theta(b) = \infty \geq \theta(la + b')$, as needed.

If $\widetilde{\theta}_a(a)$ and $\theta(b')$ are finite numbers, then $l\,\widetilde{\theta}_a(a) \leq \theta(b)-\theta(b')$, and therefore, $\widetilde{\theta}_a(la+b') = l\widetilde{\theta}_a(a) + \theta(b') \leq \theta(b) - \theta(b') + \theta(b') = \theta(b)$. This completes the proof of the monotonicity of $\widetilde{\theta}_a$. We have shown that $\widetilde{\theta}_a \in \Theta(S_a)$.

Consider the family of increasing weight extensions of $\theta \in \Theta(S)$ on semigroups H, $S \subset H \subset P$, ordered by inclusion. It is easy to see that every chain of its elements has a largest element. Zorn's Lemma implies that this family has a maximal element, say, $\widetilde{\theta}$. Denote by $S_{\widetilde{\theta}}$ the domain of $\widetilde{\theta}$, which is a subsemigroup of P. If $S_{\widetilde{\theta}}$ is a proper subsemigroup of P, then, as shown above, for any $a \in P \setminus S_F$ there is an increasing additive extension of $\widetilde{\theta}$ on the semigroup $(S_{\widetilde{\theta}})_a = \mathbb{Z}_+ a + S_{\widetilde{\theta}}$, properly containing $S_{\widetilde{\theta}}$. This contradicts the maximality property of $\widetilde{\theta}$. Therefore, $S_{\widetilde{\theta}} = P$, i.e. $\widetilde{\theta} \in \Theta(P)$. □

The proof of Theorem 5.2.1 now follows directly from the relationship between the additive weights and semicharacters on S, described in Proposition 5.2.3.

Corollary 5.2.6. *Let S and P be as in Theorem 5.2.1. If every $\varphi \in H(S)$ has monotone decreasing modulus $|\varphi|$ with respect to the P-order on S, then the restriction $r\colon \mathcal{M}_{A_P} \longrightarrow \mathcal{M}_{A_S} : m \longmapsto m\big|_{A_S}$ is a continuous mapping from $\mathcal{M}_{A_P}$ onto $\mathcal{M}_{A_S}$.*

It is clear that $r\colon m \longmapsto m\big|_{A_S}$ is a continuous mapping from $\mathcal{M}_{A_P}$ into $\mathcal{M}_{A_S}$. The surjectivity of r follows from Theorem 5.2.1.

Corollary 5.2.7. *Let S be a dense semigroup in $\mathbb{R}_+^k$. If every semicharacter $\varphi \in H(S)$ has monotone decreasing modulus $|\varphi|$ with respect to the $\mathbb{R}_+^k$-order on S, then the restriction $r\colon \overline{\mathbb{D}}^n_{b(\mathbb{R})} \longrightarrow \mathcal{M}_{A_S}$ is a continuous mapping from $\overline{\mathbb{D}}^n_{b(\mathbb{R})}$ onto $\mathcal{M}_{A_S}$.*

As a consequence we see that under the assumptions of Corollary 5.2.7 there is an embedding of $\mathbb{C}_+^n$ into the space $\mathcal{M}_{A_S} = \mathcal{M}_{AP(\mathbb{R}^n)}$ with a dense image.

Corollary 5.2.8. *If $0 \in S \subset P \subset \Gamma_S$ are two semigroups, then $H(S) = H(P)$ if and only if*

(a) *Every positive semicharacter on S is monotone decreasing with respect to the P-order on S, and*

(b) *Every idempotent semicharacter on S possesses a unique extension on P, which is an idempotent semicharacter of P.*

Proof. If (a) holds, then every non-vanishing semicharacter on S has a unique extension on P by Proposition 5.1.2. Let (b) hold, i.e. suppose that every idempotent semicharacter on S can be extended uniquely on P, and let $\psi \in H(S)$. By (a), ψ possesses a semicharacter extension on P. We claim that this extension is unique. Assume that $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are two semicharacter extensions of ψ

on P. Clearly, $\operatorname{supp}(\widetilde{\psi}_1) = \operatorname{supp}(\widetilde{\psi}_1)$, since otherwise the characteristic functions $\varkappa_{\operatorname{supp}(\widetilde{\psi}_i)}$, $i = 1, 2$ will be two different extensions of the idempotent semicharacter $\varkappa_{\operatorname{supp}(\psi)}$. Note that the set $P\setminus \operatorname{supp}(\widetilde{\psi}_1)$ is an ideal in P. Consider the set $K = \{a \in \operatorname{supp}(\widetilde{\psi}_1)\colon \widetilde{\psi}_1(a) = \widetilde{\psi}_2(a)\}$. Clearly, $\operatorname{supp}(\psi_1) \subset K \subset \operatorname{supp}(\widetilde{\psi}_1)$, where the inclusions are proper. One can easily check that $\operatorname{supp}(\widetilde{\psi}_1) \setminus K$ is an ideal in $\operatorname{supp}(\widetilde{\psi})$, and consequently, it is an ideal in P. Therefore $\varkappa_{(\operatorname{supp}(\widetilde{\psi}_1)\setminus K)}$ is an idempotent semicharacter of P, which extends $\varkappa_{\operatorname{supp}(\widetilde{\psi})}$ on P, and is different from the extension $\varkappa_{\operatorname{supp}(\widetilde{\psi})}$. □

If S is a semigroup of $\mathbb{R}^2_+$ that meets the sets $\mathbb{R}_+ \times \{0\}$ and $\{0\} \times \mathbb{R}_+$, then every idempotent semicharacter on S which is monotone decreasing with respect to the $\mathbb{R}^2$-order has unique semicharacter extensions on $\mathbb{R}^2_+$, namely the characteristic functions $\varkappa_{\{0\}}$, $\varkappa_{\mathbb{R}\times\{0\}}$ and $\varkappa_{\{0\}\times\mathbb{R}}$, i.e. the idempotent semicharacters of $\mathbb{R}^2_+$. Therefore, Corollary 5.2.8 implies the following.

Corollary 5.2.9. *Let S be a semigroup of $\mathbb{R}^2_+$ that meets the sets $\mathbb{R}_+ \times \{0\}$ and $\{0\} \times \mathbb{R}_+$, and for which $\Gamma_S = \mathbb{R}^2$. Then $H(S) \cong H(\mathbb{R}^2_+) \cong \overline{\mathbb{D}}_{b(\mathbb{R})^2}$, the $b(\mathbb{R})$-bi-disc, where $b(\mathbb{R})$ is the Bohr compactification of $\mathbb{R}$, if and only if every semicharacter $\varphi \in H(S)$ has monotone decreasing modulus $|\varphi|$ with respect to the $\mathbb{R}^2_+$-order on S.*

It is straightforward to see that the necessary and sufficient condition in Corollary 5.2.9 is equivalent to the following one: Every additive positive function θ on $S \subset \mathbb{R}^2$ to be of type $\theta(x_1, x_2) = y_1x_1 + y_2x_2$ for some $y_1, y_2 \in [0, \infty]$, considered by Boettcher [B4].

5.3 Semigroups with extendable additive weights

In this section we provide characterizations of semigroups with the property that any of its additive weights possesses a weight extension on larger semigroups. Let $S \subset P$ be two semigroups with cancellation law and 0, and let $\Gamma_S = S - S \subset P - P = \Gamma_P$ be their group envelopes correspondingly. Note that in general P may not be a subset of Γ_S.

Proposition 5.3.1. *Any weight $\theta \in \Theta(S)$ has a unique weight extension on the strong hull $[S]^P_s$ of S in P.*

Proof. If θ is an additive weight on S, then the associated semicharacter on S, $\chi(a) = e^{-\theta(a)}$, has a unique semicharacter extension $\widetilde{\chi}$ on $[S]^P_s$ by Proposition 3.5.6. The additive weight $\widetilde{\theta} = -\log \widetilde{\chi}$ corresponding to $\widetilde{\chi}$ is an additive weight on $[S]^P_s$ extending θ. □

Definition 5.3.2. Let $S \subset P$ be two semigroups with cancellation law and 0. We say that S is *complete* in P if $\Gamma_S \cap P = S$, i.e. if $P \setminus \Gamma_S = P \setminus S$.

If S is a complete subsemigroup in P, then $P = (P \cap \Gamma_S) \sqcup (P \setminus \Gamma_S) = S \sqcup (P \setminus \Gamma_S)$. Note that the notion of completeness of semigroups makes sense only in the case when $P \not\subset \Gamma_S$, since if $S \subset P \subset \Gamma_S$, then S is complete in P if and only if $S = P$.

Example 5.3.3. Let S be the two-dimensional semigroup $S^\beta = \{n + m\beta \colon n, m \in \mathbb{Z}_+\} \subset \mathbb{R}$ considered in Example 5.1.6, where β is an irrational number. The group envelope of S_β is given by $\Gamma^\beta = S^\beta - S^\beta = \big\{n + m\beta \colon n, m \in \mathbb{Z}\big\}$. Let $P = \Gamma^\beta_+ = \Gamma^\beta \cap \mathbb{R}_+ = \big\{n + m\beta \geq 0 \colon n, m \in \mathbb{Z}\big\}$. Clearly, $S^\beta \neq \Gamma^\beta_+ = \Gamma^\beta \cap P$, since, for instance, the positive number $\beta - \lfloor \beta \rfloor \in \Gamma^\beta_+ \setminus S^\beta$. Therefore, S^β is not complete in P.

Example 5.3.4. Let $S = \mathbb{R}_+ \times \{0\}$, and $P = \mathbb{R} \times \mathbb{R}_+$, endowed with addition. Then $\Gamma_S = \mathbb{R} \times \{0\}$, and $\Gamma_S \cap P = \mathbb{R} \times \{0\} \neq S$. Therefore, $\mathbb{R}_+$ is not complete in $\mathbb{R} \times \mathbb{R}_+$.

Given a weight $\theta \in \Theta(S)$, consider the semigroups

$$\theta^{-1}(\mathbb{R}) = \big\{a \in S \colon \theta(a) < \infty\big\}, \text{ and } \theta^{-1}\{\infty\} = \big\{a \in S \colon \theta(a) = \infty\big\}.$$

The set $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P = \big\{b \in P \colon b + d \in \theta^{-1}(\mathbb{R}) \text{ for some } d \in P\big\}$ is a semigroup of P. Moreover, $P \setminus \big(\theta^{-1}(\mathbb{R}) - P\big)$ is an ideal in P.

Lemma 5.3.5. *If S is a complete semigroup in P, then $\theta^{-1}(\mathbb{R})$ is a semigroup in $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$, while $\theta^{-1}\{\infty\}$ is a semigroup in $P \setminus \big(\theta^{-1}(\mathbb{R}) - P\big)$.*

Proof. The first part is clear, since $0 \in P$ and $a + 0 \in \theta^{-1}(\mathbb{R})$ for every $a \in \theta^{-1}(\mathbb{R})$. To prove the second part, assume, on the contrary, that there is an $a \in \theta^{-1}\{\infty\}$ and $d \in P$ such that $c = a + d \in \theta^{-1}(\mathbb{R})$. Hence, $d = c - a \in \Gamma_S \cap P = S$, since S is a complete semigroup in P. So, $d \in S$, thus $a + d \in \theta^{-1}\{\infty\}$, since $\theta^{-1}\{\infty\}$ is an ideal of S. Consequently, $a + d = c \in \theta^{-1}(\mathbb{R}) \cap \theta^{-1}\{\infty\} = \emptyset$, which is impossible. □

Lemma 5.3.6. *If S is a complete semigroup in P, then:*

(i) $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ *is a strongly enhanced complete semigroup in P.*

(ii) $\theta^{-1}(\mathbb{R})$ *is a complete semigroup in $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ and in P.*

(iii) *The strong hulls $\big[\theta^{-1}(\mathbb{R})\big]^P_s$ and $[S]^P_s$ of $\theta^{-1}(\mathbb{R})$ and S in P correspondingly, are complete semigroups in P.*

(iv) *If S is strongly enhanced in P, then so is $\theta^{-1}(\mathbb{R})$.*

Proof. (i) If $b \in P$ and $b = a - c$, where $a, c \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$, then $b + c = a \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$. Hence $(b + c + P) \cap \theta^{-1}(\mathbb{R}) \neq \emptyset$, thus $b \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ by definition. Hence, $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ is complete in P. The equality $\big[\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P\big]_s^P = \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ follows directly from the definition of $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$.

(ii) Let $b = a - c \in P$, where $a, c \in \theta^{-1}(\mathbb{R}) \subset S$. Since S is complete in P, then $b \in S = \theta^{-1}(\mathbb{R}) \cup \theta^{-1}\{\infty\}$. Note that b can not belong to $\theta^{-1}\{\infty\}$, since if it did, then $\theta(a) = \theta(b + c) = \infty + \theta(c) = \infty$, contradicting the assumption $a \in \theta^{-1}(\mathbb{R})$. Therefore, $b \in \theta^{-1}(\mathbb{R})$.

(iii) Let $b = a - c \in P$, where $a, c \in [S]_s^P$. There is an $n \in \mathbb{N}$ such that $na, nc \in S$, and hence $nb = na - nc \in P \cap \Gamma_S = S$, since S is complete. Therefore, $nb \in S$, thus $b \in [S]_s^P$, and consequently, $[S]_s^P$ is complete. The proof for the hull $\big[\theta^{-1}(\mathbb{R})\big]_s^P$ follows the same lines.

(iv) Let $b \in P$ be such that $nb \in \theta^{-1}(\mathbb{R}) \subset S$ for some $n \in \mathbb{N}$. Hence $b \in [S]_s^P = S$ by the hypothesis on S. Note that $b \notin \theta^{-1}\{\infty\}$, since $n\theta(b) = \theta(nb) < \infty$. Therefore, $b \in \theta^{-1}(\mathbb{R})$. □

Given a $\theta \in \Theta(S)$, let $\Gamma_{\theta^{-1}(\mathbb{R})}$ be the group envelope of $\theta^{-1}(\mathbb{R})$, i.e. $\Gamma_{\theta^{-1}(\mathbb{R})} = \theta^{-1}(\mathbb{R}) - \theta^{-1}(\mathbb{R})$.

Lemma 5.3.7. *If S is a strongly enhanced complete semigroup in P, then the group $\Gamma_{\theta^{-1}(\mathbb{R})} + \mathbb{Z}\, b$ is isomorphic to $\Gamma_{\theta^{-1}(\mathbb{R})} \times \mathbb{Z}$ for any $b \in \big(\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P\big) \setminus \theta^{-1}(\mathbb{R})$.*

Proof. Assume that $c + nb = d + kb$, for some $n < k$ and $b, c \in \Gamma_{\theta^{-1}(\mathbb{R})} = \theta^{-1}(\mathbb{R}) - \theta^{-1}(\mathbb{R})$. Then $(k - n)\, b = c - d \in \theta^{-1}(\mathbb{R}) \cap P = \theta^{-1}(\mathbb{R})$, since, by Lemma 5.3.6(ii), $\theta^{-1}(\mathbb{R})$ is complete in P. Therefore, $b \in \big[\theta^{-1}(\mathbb{R})\big]_s^P = \theta^{-1}(\mathbb{R})$ by Lemma 5.3.6(iv). This contradicts the hypothesis on b. We conclude that the group $\Gamma_{\theta^{-1}(\mathbb{R})} + \mathbb{Z}\, b$ is isomorphic to $\theta^{-1}(\mathbb{R}) \times \mathbb{Z}$, as claimed. □

Below we consider a particular case of weight extensions. Let Γ be an additive subgroup of $\mathbb{R}$. Denote by S a subsemigroup in $\Gamma_+ = \{x \in \Gamma \colon x \geq 0\}$ that contains 0 and generates Γ. The set

$$S \times \{0\} = \big\{(x, 0) \in \Gamma \times \mathbb{Z} \colon x \in S\big\} \subset \mathbb{R} \times \mathbb{Z}$$

is a semigroup in the group $\Gamma \times \mathbb{Z}$. Let P be a semigroup in $\Gamma \times \mathbb{Z}$ which contains the set $S \times \{0\}$ and the point $(0, 1)$, and for which $P \cap \big(\Gamma \times \{0\}\big) = S \times \{0\}$. We will describe the weight extensions of the x-projection

$$\pi_1 : S \times \{0\} \longrightarrow S \colon \pi_1(x, 0) = x$$

on P. We claim that for any $(x, n) \in P \setminus (0, 0)$ either x or n are non-negative.

Indeed, if both x and n are negative, then $(0,-n) = (-n)(0,1) \in P$, since $(0,1) \in P$ and $(-n) > 0$. Hence $(x,-n)+(0,n) = (x,0) \in P$, thus $(x,0) \in P \cap \big(\Gamma \times \{0\}\big) = S \times \{0\}$, and therefore, $x \in S \subset \Gamma_+$, i.e. $x \geq 0$, in contradiction with the choice of x. Consider the two numbers

$$\lambda_P = \begin{cases} -\infty & \text{if } n \geq 0 \text{ for all } (x,n) \in P \\ \sup\limits_{\substack{(x,n)\,\in\,P,\\ n\,<\,0}} \big\{x/n\big\} & \text{if there are } (x,n) \in P \text{ with } n < 0 \end{cases}$$

$$\text{and} \tag{5.3}$$

$$\Lambda_P = \inf\big\{x/n : (x,n) \in P,\ n > 0\big\}.$$

Lemma 5.3.8. *If P is a semigroup with $S \times \{0\} \subset P \subset \Gamma \times \mathbb{Z}$ and $P \cap \big(\Gamma \times \{0\}\big) = S \times \{0\}$, then $\lambda_P \leq \Lambda_P \leq 0$.*

Proof. This is clear in the case when $x \geq 0$ for all $(x,n) \in P$, or if $n \geq 0$ for all $(x,n) \in P$. Let $b_1 = (x_1,n_1)$ with $n_1 < 0$ and $b_2 = (x_2,n_2)$ with $n_2 > 0$, be two elements of P. As shown above, then $x_1 > 0$. We will show that $\dfrac{x_1}{n_1} \leq \dfrac{x_2}{n_2}$. If we suppose, on the contrary, that $\dfrac{x_1}{n_1} > \dfrac{x_2}{n_2}$, then $x_1 n_2 < n_1 x_2$, since $n_1 < 0$ and $n_2 > 0$. The elements $-n_1 b_2 = (-n_1 x_2, -n_1 n_2)$ and $n_2 b_1 = (n_2 x_1, n_2 n_1)$ belong to P since both $-n_1$ and n_2 are positive. Therefore, $n_2 b_1 - n_1 b_2 = (n_2 x_1 - n_1 x_2, 0) \in P \cap (\Gamma \times \{0\})$, and hence $(n_2 x_1 - n_1 x_2, 0) \in S \times \{0\} \subset \Gamma_+ \times \{0\}$, in contradiction with the already obtained $x_1 n_2 < n_1 x_2$. Consequently, $\dfrac{x_1}{n_1} \leq \dfrac{x_2}{n_2}$, whenever $n_1 < 0$ and $n_2 > 0$. It is easy to see now that $\lambda_P \leq \Lambda_P \leq 0$. ☐

Example 5.3.9. If $P = \Gamma \times \mathbb{Z}_+$, then $\lambda_P = -\infty$, while $\Lambda_P = 0$. If $P = \Gamma \times \mathbb{Z}$, then $\lambda_P = \Lambda_P = 0$.

Proposition 5.3.10. *Let $S \subset \mathbb{R}_+$ be a semigroup with 0, and $\Gamma_S = S - S \subset \mathbb{R}$ is the group envelope of S. Suppose that P is a semigroup of $\Gamma_S \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{Z}$ with $P \cap \big(\Gamma \times \{0\}\big) = S \times \{0\} \subset \mathbb{R}_+ \times \{0\}$. Then:*

(i) *For every real number α, $\lambda_P \leq \alpha \leq \Lambda_P$, the map $\phi\colon P \longrightarrow \mathbb{R}$, $\phi(x,n) = x - \alpha n$ is an additive weight on P.*

(ii) *For any weight $\phi \in \Theta(P)$ the number $\alpha = -\phi(0,1)$ satisfies the inequalities $\lambda_P \leq \alpha \leq \Lambda_P$. If, in addition, $\phi(x,0) = x$ for all $x \in S$, then ϕ can be expressed as $\phi(x,n) = x - \alpha n$, $(x,n) \in P$.*

Proof. (i) Let α be a real number with $\lambda_P \leq \alpha \leq \Lambda_P$. Clearly, the function ϕ from (i) is additive. We will show that $\phi \geq 0$ on P. Let $b = (x,n)$ be a fixed element in P. If $n = 0$, we have $\phi(x,0) = x \in S \subset \mathbb{R}_+$, and hence $\phi(b) \geq 0$. Let $n \neq 0$. If $n > 0$, then $\dfrac{x}{n} \geq \Lambda_P \geq \alpha$, and hence $x - \alpha n \geq 0$. If $n < 0$, then $x > 0$, thus $\dfrac{x}{n} \leq \lambda_P \leq \alpha$, and hence $x - \alpha n \geq 0$. Therefore, ϕ is an additive weight on P.

(ii) For any $b=(x,n)\in P$ we have

$$\phi(b)=\phi\big((x,0)+(0,n)\big)=\phi(x,0)+\phi(0,n)=x+n\,\phi(0,1)=x-n\,\alpha,$$

where $\alpha=-\phi(0,1)$. Moreover, $0\le\phi(b)=\phi(x,n)=x-\alpha n$. If $n>0$ we have $\alpha\le x/n$, thus $\alpha\le\Lambda_P$. If $n<0$, then $x/n\le\alpha$, and hence $\lambda_P\le\alpha$. Therefore $\lambda_P\le\alpha\le\Lambda_P$, as claimed. □

As we saw in Example 5.1.6, not every character in the semigroup S_α can be extended on P. Therefore, non-completeness of S might be an obstacle to the extendability on larger semigroups of semicharacters of S, and together, of additive weights of S.

Theorem 5.3.11. *If S is a complete semigroup in P, then every weight $\theta\in\Theta(S)$ possesses a weight extension on P.*

Proof. Let $\theta\in\Theta(S)$. First we extend θ on $\big(\theta^{-1}(\mathbb{R})–P\big)\cap P$. Because of Proposition 5.3.1, without loss of generality we can assume that S is strongly enhanced in P. We can extend θ additively on $\Gamma_{\theta^{-1}(\mathbb{R})}=\theta^{-1}(\mathbb{R})–\theta^{-1}(\mathbb{R})$ by

$$\widetilde{\theta}(a-b)=\theta(a)-\theta(b),\ a,b\in\theta^{-1}(\mathbb{R}).$$

Note that $\widetilde{\theta}\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)\subset\mathbb{R}$. For any $b\in\big((\theta^{-1}(\mathbb{R})–P)\cap P\big)\setminus\theta^{-1}(\mathbb{R})$ the mapping

$$\tau(a+nb)=\big(\widetilde{\theta}(a),n\big)$$

is a surjective homomorphism from $\Gamma_{\theta^{-1}(\mathbb{R})}+\mathbb{Z}\,b$ onto $\widetilde{\theta}\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)\times\mathbb{Z}\subset\mathbb{R}\times\mathbb{Z}$. Consider the semigroup

$$\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P=\big(\Gamma_{\theta^{-1}(\mathbb{R})}+\mathbb{Z}\,b\big)\cap P.$$

It is easy to see that $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ is strongly enhanced, and $(0,1)\in\tau\big((\Gamma_{\theta^{-1}(\mathbb{R})})_b^P\big)$. Since $\Gamma_{\theta^{-1}(\mathbb{R})}+\mathbb{Z}\,b$ is a group, $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ is a complete semigroup in P. Moreover, $\tau\big((\Gamma_{\theta^{-1}(\mathbb{R})})_b^P\big)\cap\big(\widetilde{\theta}(\Gamma_{\theta^{-1}(\mathbb{R})})\times\{0\}\big)=\widetilde{\theta}\big(\Gamma_{\theta^{-1}(\mathbb{R})}\cap P\big)\times\{0\}\subset\widetilde{\theta}(S)\times\{0\}$, since $\Gamma_{\theta^{-1}(\mathbb{R})}\cap P\subset S$ by the completeness of S. In fact, $\Gamma_{\theta^{-1}(\mathbb{R})}\cap P=\theta^{-1}(\mathbb{R})$, since the values of $\widetilde{\theta}$ on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ are in $\mathbb{R}_+$. Therefore, $\tau\big((\Gamma_{\theta^{-1}(\mathbb{R})})_b^P\big)\cap\big(\widetilde{\theta}((\Gamma_{\theta^{-1}(\mathbb{R})})_b^P)\times\{0\}\big)=\widetilde{\theta}\big(\theta^{-1}(\mathbb{R})\big)\times\{0\}$. By Proposition 5.3.10 there exist an additive weight ϕ on the semigroup $\tau\big((\Gamma_{\theta^{-1}(\mathbb{R})})_b^P\big)\subset\mathbb{R}\times\mathbb{Z}$, whose restriction on $\widetilde{\theta}\big(\theta^{-1}(\mathbb{R})\big)\times\{0\}$ is the projection

$$\pi_1\colon\widetilde{\theta}\big(\theta^{-1}(\mathbb{R})\big)\times\{0\}\longrightarrow[0,\infty):\pi_1(x,0)=x.$$

The restriction of the function $\psi=\phi\circ\tau\colon a+nb\longmapsto\phi\big(\widetilde{\theta}(a),n\big)$ on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ is an additive weight on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$, and its restriction on $\theta^{-1}(\mathbb{R})$ coincides with θ. Note that ψ takes only finite values on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$. Indeed, since $b \in \big(\theta^{-1}(\mathbb{R})–P\big) \cap P$, there exists a $d \in P$ such that $b + d = a \in \theta^{-1}(\mathbb{R})$. Therefore, $d = a - b \in \big(\theta^{-1}(\mathbb{R}) + \mathbb{Z}\big) \cap P \subset \big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$. Since both b and d belong to $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ and ψ is an additive weight on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$, we have

$$\psi(b) \leq \psi(b + d) = \theta(b + d) = \theta(a) < \infty.$$

Consequently, ψ takes only finite values on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$.

Let $\big\{(S_i, \theta_i)\big\}_{i\in I}$ be the family of pairs of complete semigroups S_i strongly enhanced in P, with $\theta^{-1}(\mathbb{R}) \subset \big(\theta^{-1}(\mathbb{R})–P\big) \cap P$, and weights $\theta_i \in \Theta(S_i)$, such that $\theta_i|_{\theta^{-1}(\mathbb{R})} \equiv \theta$. One can define a partial order in this family, by $(S_i, \theta_i) \prec (S_j, \theta_j)$ when $S_i \subset S_j$ and $\theta_i \equiv \theta_j$ on S_i. For any chain $\big\{(S_j, \theta_j)\big\}_{j\in J}$ the set $S' = \bigcup_j S_j$ is a complete semigroup in $\big(\theta^{-1}(\mathbb{R})–P\big) \cap P$, strongly enhanced in P, and the function θ' defined on S' by $\theta'|_{S_j} \equiv \theta_j$ for any $j \in J$, is an additive weight on S' such that $\theta' = \theta$ on $\theta^{-1}(\mathbb{R})$. The pair $(S', \theta') \in \big\{(S_i, \theta_i)\big\}_{i\in I}$ is the largest element of the chain. By Zorn's Lemma there exists a maximal element (S^*, θ^*) in the family $\big\{(S_i, \theta_i)\big\}_{i\in I}$. It is clear that $S^* = \theta^{-1}(\mathbb{R})–P$, since otherwise one can obtain a proper extension of θ^* on a semigroup of type $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)_b^P$ for some $b \in \big((\theta^{-1}(\mathbb{R})–P)\cap P\big)\setminus S^*$. Consequently, θ possesses an extension on $\big(\theta^{-1}(\mathbb{R})–P\big)\cap P$ as an additive weight on $\theta^{-1}(\mathbb{R})–P$. On $P \setminus \big(\theta^{-1}(\mathbb{R})–P\big)$ we extend θ' as ∞. □

Corollary 5.3.12. *If S is a complete semigroup in $\mathbb{R}_+^k$, then the restriction $r\colon \overline{\mathbb{D}}_{b(\mathbb{R})}^n \longrightarrow \mathcal{M}_{A_S}$ is a continuous mapping from $\overline{\mathbb{D}}_{b(\mathbb{R})}^n$ onto $\mathcal{M}_{A_S}$.*

As a consequence we see that under the assumptions of Corollary 5.3.12 there is an embedding of $\mathbb{C}_+^n$ into the space $\mathcal{M}_{A_S} = \mathcal{M}_{AP(\mathbb{R}^n)}$ with a dense image, thus the algebra $AP(\mathbb{R}^n)$ does not have a $\mathbb{C}_+^n$-corona.

If all θ-values are finite, θ can be extended as a weight on $\big(\theta^{-1}(\mathbb{R})–P\big) \cap P$ with finite values. To show this, in the family $\big\{(S_i, \theta_i)\big\}_{i\in I}$ from the above we should consider weights $\theta_i \in \Theta(S_i)$ which take finite values only.

Corollary 5.3.13. *For any non-invertible element $a \in S$ there exists a weight $\theta \in \Theta(S)$, such that $\theta(na) = n$.*

Proof. Indeed, $\mathbb{Z}_+a$ is a complete semigroup in S, and by Theorem 5.3.11 the weight $\theta(na) = n$ of $\mathbb{Z}_+a$ can be extended as a weight on S. □

Given a weight $\theta \in \Theta(S)$ consider the following two semigroups of $\Gamma_{\theta^{-1}(\mathbb{R})}$:

$$\Gamma_{\theta^{-1}(\mathbb{R})}^{+} = \big\{a - b\colon a, b \in \theta^{-1}(\mathbb{R}),\, \theta(a) - \theta(b) \geq 0\big\},$$

$$\Gamma^-_{\theta^{-1}(\mathbb{R})} = \{a - b\colon a, b \in \theta^{-1}(\mathbb{R}),\ \theta(a) - \theta(b) < 0\}.$$

We recall that if $a, b \in S$, then $a \prec b$ with respect to the P-order on S if and only if there is a $c \in P$ such that $a + c = b$.

Proposition 5.3.14. *A weight θ on S is monotone increasing with respect to the P-order on S if and only if*

(i) $\Gamma^-_{\theta^{-1}(\mathbb{R})} \cap P = \varnothing$, *and*

(ii) $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap \theta^{-1}\{\infty\} = \varnothing$.

Proof. If $\theta \in \Theta(S)$ is an increasing weight on S, then we will show that (i) and (ii) are satisfied. Assume, on the contrary, that $\Gamma^-_{\theta^{-1}(\mathbb{R})} \cap P \neq \varnothing$, and let $p \in \Gamma^-_{\theta^{-1}(\mathbb{R})} \cap P$. Then $p = a - b$, where $a, b \in \theta^{-1}(\mathbb{R})$, and $\theta(a) - \theta(b) < 0$. Since $a = b + p$, $p \in P$, we have that $a \succ b$, and consequently, $\theta(a) \geq \theta(b)$, a contradiction. If $a \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$, then $a + p \in \theta^{-1}(\mathbb{R})$ for some $p \in P$. Therefore, $a \prec a + p = b \in \theta^{-1}(\mathbb{R})$, and hence $\theta(a) \leq \theta(b) < \infty$ by the monotonicity of θ. Consequently, $a \notin \theta^{-1}\{\infty\}$.

Let now (i) and (ii) both hold. Assume that $\theta \in \Theta(S)$ be such that $\theta(a) < \theta(b)$ for some $a, b \in S$, $a \succ b$. Consider first the case when both $a, b \in \theta^{-1}(\mathbb{R})$. Since $a \succ b$, there is a $p \in P$ for which $b + p = a$. Therefore, $p = a - b \in P$, and $\theta(a) - \theta(b) < 0$, i.e. $p \in \Gamma^-_{\theta^{-1}(\mathbb{R})}$, in contradiction with (i). Let now $\theta(b) = \infty$, i.e. $b \in \theta^{-1}\{\infty\}$, while $a \in \theta^{-1}(\mathbb{R})$. From $b + p = a$, $p \in P$ we have $b \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$. Therefore, if $b = a + p$, then $b \in \big(\theta^{-1}(\mathbb{R}) - P\big) \cap \theta^{-1}\{\infty\} = \varnothing$. □

Theorem 5.3.15. *A weight $\theta \in \Theta(S)$ can be extended as an additive weight on P if and only if*

(i) $\Gamma^-_{\theta^{-1}(\mathbb{R})} \cap P = \varnothing$ *and*

(ii) $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap \theta^{-1}\{\infty\} = \varnothing$.

While this result follows from Proposition 5.3.14 and Theorem 5.2.5, here we give a more direct proof.

Proof. Assume that (i) and (ii) both hold. According to (i) the semigroup $\Gamma_{\theta^{-1}(\mathbb{R})} \cap P = \Gamma^+_{\theta^{-1}(\mathbb{R})} \cap P$ is a semigroup in $\Gamma^+_{\theta^{-1}(\mathbb{R})}$. Let $\widetilde{\theta}\colon \Gamma_{\theta^{-1}(\mathbb{R})} \to \mathbb{R}\colon \widetilde{\theta}(a - b) = \theta(a) - \theta(b)$, $a, b \in \theta^{-1}(\mathbb{R})$ be the natural extension of θ on $\Gamma_{\theta^{-1}(\mathbb{R})}$. Since $\widetilde{\theta}(a - b) \geq \theta(a) - \theta(b) \geq 0$ on $\Gamma^+_{\theta^{-1}(\mathbb{R})}$, the restriction of $\widetilde{\theta}$ on $\Gamma_{\theta^{-1}(\mathbb{R})} \cap P$ is an additive weight with $\widetilde{\theta}(a) = \theta(a)$ for $a \in \theta^{-1}(\mathbb{R})$. One can see that $\Gamma_{\theta^{-1}(\mathbb{R})} \cap P$ is a complete semigroup in both $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ and P. By Theorem 5.3.11, $\widetilde{\theta}$ can be extended on $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ as an additive weight. Since $\big(\theta^{-1}(\mathbb{R}) - P\big) \cap P$ is an ideal in P, we can further extend $\widetilde{\theta}$ as an additive weight on P, by setting $\widetilde{\theta}|_{P\setminus(\theta^{-1}(\mathbb{R})-P)} \equiv \infty$. (ii) implies that this is an extension of θ on P.

Let now $\widetilde{\theta} \in \Theta(P)$ be an extension of a $\theta \in \Theta(S)$. As before, for any $a-b \in \Gamma_{\theta^{-1}(\mathbb{R})} \cap P$, $a, b \in \theta^{-1}(\mathbb{R})$ we have

$$0 \le \widetilde{\theta}(a-b) = \theta(a) - \theta(b).$$

Therefore, $a-b \in \big(\theta^{-1}(\mathbb{R})-P\big) \cap P$, and hence $\Gamma_{\theta^{-1}(\mathbb{R})} \cap P \subset \big(\theta^{-1}(\mathbb{R})-P\big) \cap P$. Consequently, $\Gamma^{-}_{\theta^{-1}(\mathbb{R})} \cap P = \emptyset$, i.e. (i) holds. By definition, for any $a \in \big(\theta^{-1}(\mathbb{R})-P\big) \cap P$ there exists some $b \in P$ such that $b + a = c \in \theta^{-1}(\mathbb{R})$. Since $a, b \in P$, and $\widetilde{\theta} \in \Theta(P)$, we have $\widetilde{\theta}(a) \le \widetilde{\theta}(a+b) = \theta(c) < \infty$, i.e. $\big(\theta^{-1}(\mathbb{R})-P\big) \cap P \subset \theta^{-1}(\mathbb{R})$, and therefore, $\big(\theta^{-1}(\mathbb{R})-P\big) \cap \theta^{-1}\{\infty\} = \emptyset$, i.e. (ii) holds, too. □

Let θ be an additive weight on S, and let $\Sigma_P(\theta)$ be the set of all weight extensions of θ on P, i.e.

$$\Sigma_P(\theta) = \big\{\psi \in \Theta(P)\colon \psi\big|_S \equiv \theta\big\}.$$

Observe that $\Sigma_P(\theta)$ is a convex set in $\Theta(P)$, i.e. if $\theta_1, \theta_2 \in \Sigma_P(\theta)$ and $0 < t < 1$, then $t\theta_1 + (1-t)\theta_2 \in \Sigma_P(\theta)$.

Proposition 5.3.16. *If S is a strongly enhanced complete semigroup in P, $b \in P\setminus S$, and $\theta \in \Theta(S)$, then*

$$\big\{\psi(b)\colon \psi \in \Sigma_P(\theta)\big\} = [\lambda_b, \Lambda_b],$$

where λ_b and Λ_b are the corresponding numbers (5.3) *for θ with respect to the semigroup $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)^P_b \cap \big(\theta^{-1}(\mathbb{R})-P\big) = \big(\Gamma_{\theta^{-1}(\mathbb{R})} + \mathbb{Z}b\big) \cap \big(\theta^{-1}(\mathbb{R})-P\big) \cap P$.*

Proof. The restriction of any $\psi \in \Sigma_P(\theta)$ on the semigroup $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)^P_b \cap \big(\theta^{-1}(\mathbb{R})-P\big)$ is an additive weight, and as we saw in the proof of Theorem 5.3.11, it is of type $\psi = \phi \circ \tau$, where $\tau\colon \Gamma_{\theta^{-1}(\mathbb{R})} + \mathbb{Z}b \longrightarrow \mathbb{R}\times\mathbb{Z}$ is the mapping $\tau(a+nb) = (\psi(a), n)$, and ϕ is an additive weight on $\tau\big(\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)^P_b\big) \subset \mathbb{R}\times\mathbb{Z}$. By Proposition 5.3.10, ϕ can be expressed as

$$\phi(x,n) = x - \alpha n,\ (x,n) \in P,$$

where $\alpha \in [\lambda_b, \Lambda_b]$. Therefore, $\psi(a+nb) = \psi(a) - \alpha n$, and hence, $\psi(b) = -\alpha$. Clearly, the correspondence $\psi \longmapsto -\psi(b) = \alpha$ maps Σ_P into the interval $[\lambda_b, \Lambda_b]$.

Conversely, for every $\alpha \in [\lambda_b, \Lambda_b]$ the mapping $\psi(a+nb) = \psi(a) - \alpha n$ is an additive weight on $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)^P_b \cap \big(\theta^{-1}(\mathbb{R})-P\big)$, by Proposition 5.3.10. Since $\big(\Gamma_{\theta^{-1}(\mathbb{R})}\big)^P_b \cap \big(\theta^{-1}(\mathbb{R})-P\big)$ is a complete semigroup in $\big(\theta^{-1}(\mathbb{R})-P\big) \cap P$, ψ can be extended as an additive weight on $\big(\theta^{-1}(\mathbb{R})-P\big) \cap P$, by Theorem 5.3.11. On $P \setminus \big(\theta^{-1}(\mathbb{R})-P\big)$ we can extend ψ as ∞. Clearly, different $\alpha_1, \alpha_2 \in [\lambda_b, \Lambda_b]$ generate different extensions ψ_1, ψ_2 of ψ on P, since $\psi_i(b) = -\alpha_i$, $i = 1, 2$. □

Observe that the complement $J_S = S \setminus G_S$ of the group kernel $G_S = \{a \in S : a+b = 0 \text{ for some } b \in S\}$ of S is an ideal, namely the maximal ideal of S. The following proposition gives an alternative sufficient condition for weight extensions on $P \supset S$ of additive weights on S.

Proposition 5.3.17. *Let P be a semigroup that contains S. If J_S is an ideal in P, then any weight $\theta \in \Theta(S)$ can be extended to an additive weight on P.*

Proof. Suppose that J_S is an ideal of P. Since $\theta^{-1}\{\infty\} = \{a \in S\colon \theta(a) = \infty\}$ is an ideal in S, then $\theta^{-1}\{\infty\} \subset J_S$. If $\theta^{-1}\{\infty\}= J_S$, then $\theta^{-1}(\mathbb{R}) = G_S$, and therefore the weight θ is identically equal to 0 on G_S, and identically equal to ∞ on $\theta^{-1}\{\infty\}= J_S$. As an ideal in P, J_S is contained in the maximal ideal $J_P = P \setminus G_P$ of P, and therefore the function

$$\widetilde{\theta}(a) = \begin{cases} 0 & \text{when } a \in G_P, \\ \infty & \text{when } a \in J_P \end{cases}$$

is an additive weight on P. Clearly, $\widetilde{\theta}$ is a weight extension of θ on P.

If $\theta^{-1}\{\infty\}$ properly contains J_S, then there is a $c \in J_S$ with $\theta(c) < \infty$. Since J_S is an ideal in P, we have that $a + c \in J_S$ for any $a \in P$, and we can extend θ on P as

$$\widetilde{\theta}(a) = \theta(a + c) - \theta(c),\ a \in P.$$

It is easy to see that $\widetilde{\theta}$ is a well-defined function on P. Indeed, if d is another element in J_S with $\theta(d) < \infty$, then $\theta(a+c)+\theta(d) = \theta\big((a+c)+d\big) = \theta\big((a+d)+c\big) = \theta(a+d)+\theta(c)$, since $c,\ d,\ a+c$, and $a+d$ belong to $J_S \subset S$. Therefore, $\theta(a+c) - \theta(c) = \theta(b+c) - \theta(b)$. We claim that $\widetilde{\theta}$ is an additive weight on P. Indeed, if $\theta(c) < \infty$ for $c \in J_S$, then $2c \in J_S$, and $\theta(2s) = 2\theta(s) < \infty$. Hence, for any $a, b \in P$, we have

$$\begin{aligned} \widetilde{\theta}(a+b) &= \theta(a+b+c) - \theta(c) = \theta(a+b+2c) - \theta(2c) \\ &= \theta\big((a+c)+(b+c)\big) - 2\theta(c) \\ &= \theta(a+c) - \theta(c) + \theta(b+c) - \theta(c) = \widetilde{\theta}(a) + \widetilde{\theta}(b). \end{aligned}$$

Therefore, $\widetilde{\theta}$ is additive on P. $\widetilde{\theta}$ is also non-negative on P, since, given an $a \in P$, for any $n \in \mathbb{N}$ we have

$$n\widetilde{\theta}(a) = \widetilde{\theta}(na) = \theta(na+c) - \theta(c) \geq -\theta(c).$$

Hence, $\widetilde{\theta}(a) \geq -\dfrac{\widetilde{\theta}(c)}{n}$ for any $n \in \mathbb{N}$. By letting $n \longrightarrow \infty$ we see that $\widetilde{\theta}(a) \geq 0$. Consequently, $\widetilde{\theta}$ is an additive weight on P. □

Corollary 5.3.18. *If J_S is an ideal in P, and θ is an additive weight on S, such that $\theta^{-1}\{\infty\}$ is a proper subset of J_S, then the weight extension of θ on P constructed in Proposition* 5.3.17 *is uniquely defined.*

Proof. Let $\theta \in \Theta(S)$ and let $\widetilde{\theta}$ be the extension of θ defined in Proposition 5.3.17 by some element $c \in J_S \setminus \overset{-1}{\theta}\{\infty\}$. If c is as in the proof of Proposition 5.3.17, then $a + c \in J_S$ whenever $a \in P$, and for any extension $\theta_1 \in \Theta(P)$ of θ on P we have $\theta_1(a) + \theta(c) = \theta_1(a) + \theta_1(c) = \theta_1(a + c) = \theta(a + c)$, and hence $\theta_1(a) = \theta(a + c) - \theta(c) = \widetilde{\theta}(a)$. □

Example 5.3.19. If S is the semigroup $S = (\mathbb{Z}_+ \times \mathbb{N}) \cup \{(0,0)\}$, endowed with addition, then $G_S = \{(0,0)\}$ and $J_S = \mathbb{Z}_+ \times \mathbb{N}$. Since J_S is an ideal in $P = \mathbb{Z}_+ \times \mathbb{Z}_+$, then every additive weight on S can be extended to an additive weight on P. Let $\theta_\infty^S \in \Theta(S)$ be the maximal weight on S, for which $\theta_\infty^S \equiv \infty$ on $\mathbb{Z}_+ \times \mathbb{N}$ and $\theta_\infty^S(0,0) = 0$. Now $\mathbb{Z}_+ \times \mathbb{N} = \theta^{-1}(\infty)$, and therefore θ_∞^S can be extended on P as

$$\widetilde{\theta}(n,m) = \begin{cases} 0 & \text{when } m = 0, \\ \infty & \text{when } m \neq 0. \end{cases}$$

Note that the maximal weight θ_∞^P on P is an alternative weight extension of θ on P.

We will show that the condition in Proposition 5.3.17 is also necessary.

Theorem 5.3.20. *Let S be a strongly enhanced semigroup in P. All weights in $\Theta(S)$ have extensions as additive weights on P if and only if $J_S = S \setminus G_S$ is an ideal in the semigroup $\Gamma_S \cap P$.*

Proof. If J_S is an ideal of $\Gamma_S \cap P$, then by Proposition 5.3.17 any $\theta \in \Theta(S)$ can be extended as an additive weight $\widetilde{\theta}$ on $\Gamma_S \cap P$. Since, clearly, $\Gamma_S \cap P$ is a complete semigroup in P, Theorem 5.3.11 implies that $\widetilde{\theta}$ can be extended further on P as an additive weight.

Conversely, suppose that every $\theta \in \Theta(S)$ has an extension on P as an additive weight. We claim that no $b \in J_S$ is invertible in P. Observe that every additive weight vanishes at invertible elements of S. By Corollary 5.3.13 there is a weight in $\Theta(S)$ extending the weight $\theta\colon \mathbb{Z}_+ b \longrightarrow \mathbb{Z}_+$ by $\theta(nb) = n$. According to our assumption, θ can be extended further as an additive weight $\widetilde{\theta}$ on P. Hence $\widetilde{\theta}(b) = \theta(b) = 1 \neq 0$, and therefore, b can not be invertible in P. Consequently, $J_S \cap G_P = \emptyset$, thus $J_S \subset J_P$.

Suppose that J_S is not an ideal in $\Gamma_S \cap P$. Then there are elements $a \in (\Gamma_S \cap P) \setminus S$ and $b \in J_S$, such that $a + b \notin J_S$. We claim that $a + b \notin G_S$. If we assume, on the contrary, that $a + b \in G_S$, then $c = a + b$ is invertible in S, and $a + (b - c) = 0$, i.e. $(b - c)$ is invertible in P, which contradicts $b - c = b + (-c) \in J_S + G_S \subset J_S$. Consequently, $a + b \notin G_S$, and therefore, $a + b \notin S$. Hence, $n\,(a + b) \notin S$ for any $n \in \mathbb{N}$, since S is strongly enhanced in P. We claim that the elements a and b are linearly independent over $\mathbb{Z}$. First we will show that if a is invertible in P, then $-a \notin S$. If we assume that $-a \in S$, then $-a$ is not invertible, since $a \notin S$, and by Corollary 5.3.13 there is a weight $\psi \in \Theta(S)$ extending the weight $\theta\bigl(n(-a)\bigr) = n$ defined on $\mathbb{Z}_+(-a)$, and according to our supposition, ψ can be extended further as an additive weight $\widetilde{\psi}$ on P, and further on P. This is impossible, since for any extension $\widetilde{\theta}_1$ of θ_1 we have

$$\psi_1(a) = -\widetilde{\psi}_1(-a) = -1 < 0,$$

in contradiction with $\widetilde{\psi} \geq 0$. Therefore, for any $a \in (\varGamma_S \cap P) \setminus S$ we have $-a \notin S$. If we suppose that $a + b \in G_S$, then $c = a + b$ is invertible in S, and hence $-c = -a - b \in S$. Then $-a = b + (-c) \in J_S + G_S = S$, a contradiction. Consequently, $n\,(a+b) \notin S$ for any $n \in \mathbb{Z} \setminus \{0\}$, since S is strongly enhanced in P. Assume that $na + lb = 0$ and let $l > 0$. Since $b \in J_S$ we have $-na \in S$, and therefore, either $-a$, or a belongs to S, since the semigroup S is strongly enhanced in P. As shown above, neither one of these cases is possible. We conclude, that the elements a and b are linearly independent over $\mathbb{Z}$. Consider the group $\mathbb{Z}a + \mathbb{Z}b$ spanned by a and b. Note that if $na + kb \in (\mathbb{Z}a + \mathbb{Z}b) \cap S$, then $n \leq k$. Indeed, if we assume that $k < n$, then $n\,(a + b) = na + kb + (n - k)\,b \in S$, which is impossible. Consider the additive weight ψ on $(\mathbb{Z}a + \mathbb{Z}b) \cap S$, defined by

$$\psi\colon (\mathbb{Z}a + \mathbb{Z}b) \cap S \longrightarrow [0, \infty)\colon \psi\,(na + kb) = k - n.$$

Clearly $nb \in (\mathbb{Z}a + \mathbb{Z}b) \cap S$ for any $n \in \mathbb{Z}_+$, and therefore, $\psi(kb) = k$. Since the semigroup $(\mathbb{Z}a + \mathbb{Z}b) \cap S$ is complete in S, ψ has an extension $\widetilde{\psi}$ as an additive weight on S, and further on P. For $0 < k < n$ we have

$$0 = \widetilde{\psi}\,(na + nb) = \widetilde{\psi}\,(na + kb) + \widetilde{\psi}\,\big((n - k)b\big) = \widetilde{\psi}\,(na + kb) + (n - k).$$

Thus $\widetilde{\psi}\,(na + kb) = -(n - k) = k - n < 0$, which is impossible. Hence, J is an ideal in $\varGamma_S \cap P$. □

Corollary 5.3.21. *Let S be a strongly enhanced semigroup in $\mathbb{R}_+$. All weights in $\varTheta(S)$ have extensions as additive weights on $\mathbb{R}_+$ if and only if $S \setminus \{0\}$ is an ideal in the semigroup $(\varGamma_S)_+ = \varGamma_S \cap \mathbb{R}_+$.*

Note that the sufficient condition in Proposition 5.3.17 is also necessary if $\varGamma_S \cap P = P$, since then $P \subset \varGamma_S$.

Corollary 5.3.22. *Let $S \subset P \subset \varGamma_S$ and let S be strongly enhanced in P. The following conditions are equivalent.*

(i) *$J_S = S \setminus G_S$ is an ideal in P.*

(ii) *All weights in $\varTheta(S)$ have weight extensions on P.*

(iii) *All semicharacters in $H(S)$ possess semicharacter extension on P.*

(iv) *All semicharacters in $H(S)$ are monotone decreasing.*

(v) *Any linear multiplicative functional of the algebra A_S can be extended to a linear multiplicative functional on A_P.*

Corollary 5.3.23. *Let S and P be as above. If $J_S = S \setminus G_S$ is an ideal in P, then the restriction $r\colon m \longmapsto m\big|_{A_S}$ is a continuous mapping from $\mathcal{M}_{A_P}$ onto $\mathcal{M}_{A_S}$.*

Clearly, $r\colon m \longmapsto m\big|_{A_S}$ is a continuous mapping from $\mathcal{M}_{A_P}$ into $\mathcal{M}_{A_S}$. The surjectivity of r follows from Corollary 5.3.22.

Corollary 5.3.24. *Let S be a strongly enhanced semigroup in $\mathbb{R}^k_+$. The above restriction r maps the $b(\mathbb{R})$-polydisc onto $\mathcal{M}_{A_S}$ if and only if $J_S = J \setminus S$ is an ideal in $\mathbb{R}^k_+$.*

As a consequence we see that in the setting of Corollary 5.3.24 the standard embedding of $\mathbb{C}^n_+$ into the space $\mathcal{M}_{A_S} = \mathcal{M}_{AP(\mathbb{R}^n)}$ has a dense image (equivalently, the algebra A_S does not have a $\mathbb{C}^n_+$-corona) if and only if $J_S = J \setminus S$ is an ideal in $\mathbb{R}^k_+$.

5.4 Weights on algebras generated by Archimedean ordered semigroups

The natural partial order generated by a semigroup S on itself reflects the properties of the semigroup. Recall that if S is a semigroup with 0 and cancellation law, the standard S-order is a partial pseudo-order on S, such that

$$a \prec b \ \text{ if } \ a + c = b \ \text{ for some } \ c \in S.$$

This order possesses the following properties:

(i) If $a \prec b$ and $b \prec c$, then $a \prec c$.

(ii) $a \prec a$ for every $a \in S$.

(iii) If $a \prec b$ and $c \in S$, then $a + c \prec b + c$.

(iv) $0 \prec a$ for every $a \in S$.

The S-order is an *order* on S, namely,

(v) $a \prec b$ and $b \prec a$ imply $a = b$,

if and only if the group kernel G_S of S is trivial, i.e. if $G_S = S \cap (-S) = \{0\}$.

The S-order $\prec$ is *total*, i.e.

(vi) For every $a, b \in S$ either $a \prec b$, or $b \prec a$ holds,

if and only if the group envelope $\Gamma_S = S - S$ of S equals $S \cup (-S)$.

A total order $\prec$ on S is *Archimedean*, i.e.

(vii) For every $a, b \in S$, $a \neq 0$ there exist an $m \in \mathbb{N}$ such that $b \prec ma$, if and only if $S \cup (-S) = \Gamma_S$ and S can be embedded in $\mathbb{R}$.

The S-order can be extended naturally in the group envelope Γ_S of S by letting

$$a \prec b \text{ if } b - a \in S.$$

If the S-order is total on S, then its natural extension on Γ_S is also total. Indeed, if $a, b \in \Gamma_S$, and $a = a_1 - a_2$, $b = b_1 - b_2$, where $a_i, b_i \in S, 1 = 1, 2$, then $a_1 + b_2$ and $b_1 + a_2$ belong to S. If, say, $a_1 + b_2 \prec b_1 + a_2$, then $a_1 + b_2 + c = b_1 + a_2$ for some $c \in S$. Therefore, $b - a = (b_1 - b_2) - (a_1 - a_2) = c \in S$, i.e. $a \prec b$. The S-order on Γ_S can be expanded in Γ_S by letting

$$a \prec^* b \text{ if and only if } n(b-a) \in S \text{ for some } n \in \mathbb{N}$$

for any $a, b \in \Gamma_S$. Restricted on S, the order $\prec^*$ has the following property

$$a \prec^* b \text{ if and only if } na + c = nb \text{ for some } n \in \mathbb{N} \text{ and } c \in S.$$

Clearly, $a \prec b$ implies $a \prec^* b$. The opposite, however, is not always true. One can check that the properties (i)–(v) from the above hold for the order $\prec'$ too.

Example 5.4.1. It is easy to see that the S-order on the semigroup $S = \mathbb{Z}_+ \times \mathbb{Z}_+$ under addition is not total. Given an irrational number $\beta > 0$, we can define an order $\prec_\beta$ on S by

$$(n, k) \prec_\beta (m, l) \text{ if } n + \alpha k < m + \alpha l.$$

Note that β defines an embedding ϕ_β of S into $\mathbb{R}_+$ by $\phi_\beta(n, k) = n + \beta k$. One can check that the order $\prec_\beta$ is total, i.e. for any $a, b \in S$ either $a \prec_\beta b$, or $b \prec_\beta a$ holds. This order is different from the natural order $\prec$ on S, and also from its extension $\prec^*$. However, $\prec_\beta$ coincides with the natural order on the semigroup

$$\Gamma_+^\beta = \big\{(n, m) \in \mathbb{Z} \times \mathbb{Z} \colon n + \beta m \geq 0\big\},$$

on which it is both total and Archimedean.

Clearly, if $0 \prec^* a$, then $na \in S$, i.e. $a \in [S]_s$, thus $-a \in [-S]_s$. Hence, $(\Gamma_S)_+ = \{a \in \Gamma_S : 0 \prec^* a\} \subset [S]_s \setminus [-S]_s$, and $(\Gamma_S)_- = \{a \in \Gamma_S : a \prec^* 0\} \subset [-S]_s \setminus [S]_s$. Therefore, $(\Gamma_S)_+ \cup (\Gamma_S)_- \subset [S]_s \cup [-S]_s$. The order $\prec^*$ generated by S on Γ_S is total if $(\Gamma_S)_+ \cap \big(-(\Gamma_S)_+\big) = \{0\}$ and $(\Gamma_S)_+ \cup \big(-(\Gamma_S)_+\big) = \Gamma_S$. This happens if and only if $(\Gamma_S)_+ = [S]_s$. We have obtained the following

Proposition 5.4.2. *The $\prec^*$-order in Γ_S is Archimedean if and only if the following conditions hold.*

(i) $(\Gamma_S)_+ = [S]_s$.

(ii) $[S]_s \cap [-S]_s = \{0\}$.

(iii) *For every $a, b \in (\Gamma_S)_+$, $a \neq 0$ there exists an $m \in \mathbb{N}$ such that $b \prec^* ma$.*

There is a close connection between the orders in the group envelope Γ_S, and the additive weights on S. Since every weight $\theta \in \Theta(S, \mathbb{R})$ can be extended on the group envelope Γ_S of S, say as $\widetilde{\theta}\colon \Gamma_S \longmapsto \mathbb{R}$, we can define an order on Γ_S by $a \prec_\theta b$ if and only if $\widetilde{\theta}(a) \le \widetilde{\theta}(b)$. The order $\prec_\theta$ is total if and only if the weight θ is injective on S. If this is the case, the mapping $\widetilde{\theta}\colon \Gamma_S \longmapsto \mathbb{R}$ is an embedding, and therefore the order generated by θ in Γ_S is total and Archimedean.

Note that if $\psi \in \Theta(S)$, then $\psi(a) \ge 0$ if and only if $0 \prec^* a$, i.e. if $a \in (\Gamma_S)_+$. If $G_S = \{0\}$, then for every $a \in S \setminus \{0\}$ there is a weight $\theta \in \Theta(S, \mathbb{R})$ so that $0 < \theta(a) < \infty$. Indeed, the semigroup $\mathbb{Z}_+ a$ generated by a is complete in S. By Theorem 5.3.11 and Proposition 5.3.1 there is an additive weight θ on $[S]_s$ with values in $\mathbb{R}_+$, such that $\theta(na) = n$. In particular, $\theta(a) = 1$.

Proposition 5.4.3. *If the order $\prec^*$ of the group Γ_S is total and Archimedean, then there is a homomorphic embedding of Γ_S into $\mathbb{R}$.*

Proof. Let θ be a weight in $\Theta\big((\Gamma_S)_+\big)$ with $0 < \theta(a) < \infty$ for some $a \in S \setminus \{0\}$. By the Archimedean property of $\prec^*$, given a $b \in (\Gamma_S)_+$ there is an $m \in \mathbb{Z}_+$, so that $b \prec^* ma$. Therefore, $\theta(b) \le \theta(ma) = m\theta(a) < \infty$, i.e. $\theta \in \Theta(S, \mathbb{R})$. On the other hand we have also $a \prec' kb$ for some $k \in \mathbb{Z}_+$, thus $\theta(b) > 0$. Consequently, θ is a homomorphic embedding of $(\Gamma_S)_+$ into $\mathbb{R}_+$. Its natural extension, $\widetilde{\theta} : \widetilde{\theta}(-a) = -\theta(a)$, on Γ_S embeds Γ_S homomorphically into $\mathbb{R}$. $\square$

Theorem 5.4.4. *The following conditions are equivalent.*

(i) *The order $\prec^*$ generated by S on Γ_S is Archimedean.*

(ii) *The weight semigroup $\Theta(S)$ is isomorphic to $[0, \infty]$.*

(iii) *The semigroup $\Theta(S, \mathbb{R})$ of weights with values in $\mathbb{R}$ is isomorphic to $\mathbb{R}_+ = [0, \infty)$.*

Proof. First we show that (i) implies (ii). Assume that $\prec^*$ is an Archimedean order on Γ_S. By Proposition 5.4.2(i) we have that $(\Gamma_S)_+ = [S]_s$. If $\theta \in \Theta(S)$ is such that $\theta \not\equiv 0$, and $\theta \not\equiv \infty$ on $S \setminus \{0\}$, then there is an $a \in S \setminus \{0\}$ with $0 < \theta(a) < \infty$. Since the order $\prec^*$ on Γ_S is Archimedean, for every $b \in (\Gamma_S)_+$ there exists an $m \in \mathbb{N}$ such that $b \prec^* ma$. Therefore, $\theta(b) < \infty$ on $(\Gamma_S)_+$, i.e. $\theta \in \Theta(S, \mathbb{R})$. We claim that if θ_1 is also an additive weight on S with $\theta(a) \equiv \theta_1(a)$, then $\theta \equiv \theta_1$. Assume that θ and θ_1 are extended naturally on Γ_S as additive functions, namely, by $\theta(b - c) = \theta(b) - \theta(c)$, and $\theta_1(b - c) = \theta_1(b) - \theta_1(c)$ respectively. Let, say, $a \prec^* c$ for some $c \in (\Gamma_S)_+$, and assume that $\theta(c) < \theta_1(c)$. There are numbers $m, n \in \mathbb{Z}$ so that

$$\theta(c) - (m/n)\,\theta(a) = \theta(c) - (m/n)\,\theta_1(a) < 0 < \theta_1(c) - (m/n)\,\theta(a).$$

Hence, $\theta(c) < 0 < \theta_1(c)$, contrary to the fact that the extension on Γ_S of the weights in $\Theta(S)$ are non-negative on $[S]_s = (\Gamma_S)_+$. Consequently, $\theta(c) = \theta_1(c)$ for any $c \in \Gamma_S$ with $a \prec^* c$. The same argument applies for any $c \prec^* a$. Hence, $\theta = \theta_1$

on Γ_S, and consequently, every weight $\psi \in \Theta(S)$ can be expressed uniquely in the form $\psi = \psi(a)\,\theta$. Therefore, the mapping $\psi \longmapsto \psi(a)$ is a bijection between the set $\{\psi \in \Theta(S) \colon \psi(b) \neq \infty,\ b \in S\}$ and $\mathbb{R}_+$. If there is no $a \in S$ with $\theta(a) < \infty$, then $\theta \equiv \infty$ on $S \setminus \{0\}$, which corresponds to the case when $\theta(a) = \infty$. Consequently, $\Theta(S) \cong [0, \infty]$, which proves (ii).

Since, obviously, (ii) implies (iii), it remains to show that (iii) implies (i). Observe that the semigroup $\Theta(S)$ is conic over $\mathbb{R}_+$. If (i) holds, then the set $\Theta(S, \mathbb{R})$ can be parametrized by a real parameter λ, namely, $\Theta(S, \mathbb{R}) = \{\theta_\lambda,\ \lambda \in \mathbb{R}_+\}$. Fix a $\lambda \in \mathbb{R}_+$ and let $\theta_\lambda(a) = 1$ for some $a \in S \setminus \{0\}$. We claim that the elements of $\Theta(S, \mathbb{R})$ are of type $r\,\theta_\lambda$ for some real r, i.e. that $\Theta(S, \mathbb{R}) \cong \mathbb{R}_+\,\theta_\lambda$. Because of $\Theta(S, \mathbb{R}) \cong \mathbb{R}_+$ we have that $\theta_{n\lambda} = n\theta_\lambda$. Since $n\theta_{(m/n)\lambda} = \theta_{n(m/n)\lambda} = \theta_{m\lambda} = m\theta_\lambda$, we obtain that $\theta_{(m/n)\lambda} = (m/n)\,\theta_\lambda$ for every rational number m/n. If α is an irrational number, and if $p/q < \alpha < m/n$, then

$$\theta_{(p/q)\lambda} = (p/q)\,\theta_\lambda \leq \alpha\theta_\lambda \leq (m/n)\,\theta_\lambda = \theta_{(m/n)\lambda}.$$

Therefore,

$$\sup_{p/q<\alpha} (p/q)\theta_\lambda \leq \theta_{\alpha\lambda} \leq \inf_{m/n>\alpha} (p/q)\theta_\lambda,$$

and hence $\theta_{\alpha\lambda} = \alpha\theta_\lambda$. Consequently, $\Theta(S, \mathbb{R}) = \mathbb{R}_+\,\theta_\lambda$, as desired. Since $\Theta(S, \mathbb{R})$ separates the points of the semigroup S, so does θ_λ. Therefore, if $a, b \in S$, $a \neq b$, then either $\theta_\lambda(a) < \theta_\lambda(b)$, or $\theta_\lambda(b) < \theta_\lambda(a)$. We assume that θ_λ is extended naturally as an additive function on Γ_S by $\theta_\lambda(a - b) = \theta_\lambda(a) - \theta_\lambda(b)$. Since θ_λ separates the points in S, so does its extension $\widetilde{\theta}_\lambda$ on Γ_S. Hence $\widetilde{\theta}_\lambda$ is an embedding of Γ_S into $\mathbb{R}$. If $\widetilde{\Gamma}_+ = \{c \in \Gamma_S \colon \widetilde{\theta}_\lambda(c) \geq 0\}$, then $\Gamma_S = \widetilde{\Gamma}_+ \cup (-\widetilde{\Gamma}_+)$, and $\widetilde{\Gamma}_+ \cap (-\widetilde{\Gamma}_+) = \{0\}$. Therefore, Γ_S is isomorphic to a subgroup of $\mathbb{R}$ and the usual order $\leq$ on $\mathbb{R}$ corresponds to the order $\prec^*$. Consequently, $\prec^*$ is an Archimedean order, and $(\Gamma_S)_+ = \widetilde{\Gamma}_+ = [S]_s$ by Proposition 5.4.2. This proves that (iii) implies (i). □

Since the property (iii) in Proposition 5.4.2 holds for the order $\prec^*$ generated by a semigroup $S \subset \mathbb{R}_+$, we have the following

Corollary 5.4.5. *If $S \subset \mathbb{R}_+$, then $\Theta(S, \mathbb{R}) \cong \mathbb{R}_+$ if and only if $[S]_s = (\Gamma_S)_+$.*

As a corollary $\Theta(S, \overline{\mathbb{R}}) \cong H\big(S, [0, 1]\big)$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Since $[0, \infty] \cong [0, 1]$, Theorem 5.4.4 yields

Corollary 5.4.6. *If S is a semigroup in $\mathbb{R}_+$, then the semigroup $H\big(S, [0, 1]\big)$ of non-negative semicharacters on S is isomorphic to the semigroup $[0, 1]$ if and only if $[S]_s = (\Gamma_S)_+$.*

Note that $H(S) \cong \overline{\mathbb{D}}_G$, where $G = \widehat{\Gamma}_S$, if and only if $H\big(S, [0, 1]\big)$ is isomorphic to $[0, 1]$. As shown in Proposition 4.2.7, $[S]_s \supset (\Gamma_S)_+$ is a sufficient condition for the set $H(S) \cong \mathcal{M}_{A_S}$ to be homeomorphic to $\overline{\mathbb{D}}_G$. The following corollary shows that this condition is also necessary.

Corollary 5.4.7. *Let S be a semigroup in $\mathbb{R}_+$. The following conditions are equivalent.*

(i) $[S]_s = (\Gamma_S)_+$.

(ii) *The semigroup $H(S)$ is isomorphic to the G-disc $\overline{\mathbb{D}}_G$, $G = \widehat{\Gamma}_S$.*

(iii) *Any linear multiplicative functional of the algebra A_S can be extended uniquely to a linear multiplicative functional on $A_{(\Gamma_S)_+}$.*

(iv) *All weights in $\Theta(S)$ have unique weight extensions on $(\Gamma_S)_+$.*

(v) *All semicharacters in $H(S)$ have unique semicharacter extension on $(\Gamma_S)_+$.*

(vi) *All non-negative semicharacters on S are monotone decreasing.*

This is a direct consequence from Proposition 4.2.7 and Corollary 5.4.6.

Example 5.4.8. Let $\beta > 0$ be an irrational number. Consider the two-dimensional semigroup $S^\beta = \{n + m\beta \colon n, m \in \mathbb{Z}_+\} \subset \mathbb{R}$ endowed with addition. Clearly, $[S^\beta]_s = S^\beta$, and the group envelope of S^β is $\Gamma_{S^\beta} = \Gamma^\beta = S^\beta - S^\beta = \{n + m\beta \colon n, m \in \mathbb{Z}\}$. It is easy to see that in this case $[S^\beta]_s \neq (\Gamma_{S^\beta})_+ = \{n + \beta m \geq 0\}$. For instance, if $\beta > 1$, then the positive number $\beta - \lfloor \beta \rfloor \in (\Gamma_{S^\beta}) + \backslash [S^\beta]_s$, where $\lfloor \beta \rfloor$ is the greatest integer preceding β. As we saw in Example 5.1.6, not every semicharacter on S^β is monotone decreasing, and not all semicharacters on S^β are extendable on $(\Gamma_{S^\beta})_+$.

Consider the mapping $\pi \colon \mathcal{M}_{H^\infty} \longrightarrow \overline{\mathbb{D}} \colon \varphi \longmapsto \varphi(\mathrm{id}) \in \overline{\mathbb{D}}$, which is bijective on the set $\widetilde{\mathbb{D}} = \pi^{-1}(\mathbb{D}) \subset \mathcal{M}_{H^\infty}$. Therefore, $\widetilde{\mathbb{D}}$ and $\mathbb{D}$ are homeomorphic sets. The following theorem, which is a direct consequence of Corollary 5.4.7 shows that the condition in Corollary 4.2.13 is also necessary.

Theorem 5.4.9. *If S is a semigroup of $\mathbb{R}_+$ with Γ_S dense in $\mathbb{R}$, then the following conditions are equivalent.*

(i) $[S]_s = (\Gamma_S)_+$.

(ii) *The maximal ideal spaces $\mathcal{M}_{A_S}$ and $\mathcal{M}_{AP_S(\mathbb{R})}$ corresponding to the shift-invariant algebra A_S, and the algebra $AP_S(\mathbb{R})$ of almost periodic functions with spectrum in S, are homeomorphic to the G-disc $\overline{\mathbb{D}}_G$ with $G = \widehat{\Gamma}_S$.*

(iii) *The algebras A_S and $AP_S(\mathbb{R})$ do not have a $\mathbb{C}_+$-corona, i.e. the mapping $\widetilde{j}_\imath$ embeds densely the upper half-plane $\mathbb{C}_+$ in $\mathcal{M}_{A_S} = \mathcal{M}_{AP_S(\mathbb{R})}$.*

(iv) *The algebra $H^\infty_S \subset H^\infty$ on the unit disc $\mathbb{D}$, generated by the singular functions $\left\{e^{ia\,(1+z)/(1-z)},\ a \in S\right\}$ and endowed with the sup-norm on $\mathbb{D}$, does not have a $\mathbb{D}$-corona, i.e. the unit disc $\mathbb{D} = \pi(\widetilde{\mathbb{D}})$ is dense in the maximal ideal space of H^∞_S.*

(v) *The algebra H^∞_S does not have a $\mathbb{C}_+$-corona, i.e. the image $\widetilde{j}_\imath(\mathbb{C}_+)$ of the upper half-plane $\mathbb{C}_+$ under the embedding $\widetilde{j}_\imath$ is dense in its maximal ideal space $\mathcal{M}_{H^\infty_S}$.*

Let $S \subset \mathbb{R}_+$, $\theta \in \Theta(S)$ and $g \in G$. Consider the embedding $\mathbb{C}_+ \longrightarrow \mathcal{M}_{A_S} \colon z \longmapsto \varphi_\theta^g(z)$, where

$$\big(\varphi_\theta^g(z)\big)(a) = e^{-iz\,\theta(a)}\chi^a(g), \; a \in S.$$

Theorem 5.4.10. *If S is a semigroup of $\mathbb{R}_+$ with Γ_S dense in $\mathbb{R}$, then the following conditions are equivalent.*

(i) $[S]_s = (\Gamma_S)_+$.

(ii) *There is a $\theta_0 \in \Theta(S, \mathbb{R})$ and a $g \in G$ such that the set $\varphi_{\theta_0}^g(\mathbb{C}_+)$ is dense in $\mathcal{M}_{A_S}$.*

(iii) *The set $\varphi_\theta^g(\mathbb{C}_+)$ is dense in $\mathcal{M}_{A_S}$ for all $g \in G$ and all $\theta \in \Theta(S, \mathbb{R})$.*

Proof. If $[S]_s = (\Gamma_S)_+$, then $\Theta(S, \overline{\mathbb{R}}) \cong \mathbb{R}_+\,\theta_0 \cup \{\infty\}$, and $\mathcal{M}_{A_S} \cong \overline{\mathbb{D}}_G$. In this case one can choose $\varphi_\theta^i = \widetilde{\jmath}_\imath$ for θ_0 in (ii). This proves that (i) implies (ii).

Assume that (ii) holds, and let $\theta \colon S \longrightarrow \mathbb{R}_+$ be the identity weight, i.e. $\theta(a) = a$, $a \in S$. As a semicharacter on S, the function $a \longmapsto e^{i(1+i)\,a}$ belongs to $\mathcal{M}_{A_S}$. Since $\varphi_{\theta_0}^g(\mathbb{C}_+)$ is dense in $\mathcal{M}_{A_S}$, there exists a sequence $\{z_n\}$ in $\mathbb{C}_+$ and $g_n \in G$ such that

$$e^{iz_n\theta_0(a)}\chi^a(g_n) \longrightarrow e^{i(1+i)\,a}$$

for all $a \in S$. Hence, if $z_n = x_n + iy_n$, then

$$\big|e^{iz_n\theta_0(a)}\chi^a(g)\big| = \big|e^{-y_n\theta_0(a)}\big| \longrightarrow e^{-a}, \text{ as } n \to \infty$$

for all $a \in S$. Thus $\lambda = \lim_{n\to\infty} y_n = a/\theta_0(a)$. It is clear that λ is independent of $a \neq 0$. Therefore, $\theta_0(a) = \lambda\,a = \lambda\theta(a)$. By the same arguments as in Theorem 5.4.4 one can show that the set $\Theta(S, \overline{\mathbb{R}})$ of non-zero weights $\theta \in \Theta(S)$ is isomorphic to $\mathbb{R}_+\,\theta_0 \cup \{\infty\}$, which implies (i), according to Theorem 5.4.4. Consequently, (ii) implies (i).

Clearly, (iii) implies (ii). If (i) holds, then any $\theta_0 \in \Theta(S, \overline{\mathbb{R}})$ is of type $\lambda\theta$, and $\varphi_{\theta_0}^\imath(z)(a) = \big(\varphi_\theta^\imath(z)(a)\big)^\lambda$, thus $\varphi_{\theta_0}^\imath(z) = \big(\varphi_\theta^\imath(z)\big)^\lambda$, and hence, $\varphi_{\theta_0}^\imath(\mathbb{C}) = \varphi_\theta^\imath(\mathbb{C})$. Similarly, $\varphi_{\theta_0}^g(z)(a) = \varphi_{\theta_0}^\imath(z)\cdot\chi^a(g)$, i.e. $\varphi_{\theta_0}^g(z) = \varphi_{\theta_0}^\imath(z)\cdot g$ thus $\varphi_{\theta_0}^g(\mathbb{C}) = \varphi_{\theta_0}^\imath(\mathbb{C})\cdot g$. This proves (iii), since the multiplication by $g \in G$ is a homeomorphism of $\mathcal{M}_{A_S}$ onto itself. □

Let $g = \imath$, and let $\theta_{id} \in \Theta(S)$ be the identity mapping on S, i.e. $\theta_{id}(a) = a$, $a \in S \subset \mathbb{R}$. For any $a \in S$ we have that

$$\widehat{\chi}^a\big(\varphi_{\theta_{id}}^\imath(z)\big) = \big(\varphi_{\theta_{id}}^\imath(z)\big)(a) = e^{-iaz}$$

is an analytic periodic S-function on every line $\mathbb{R}_{y_0} = \{x + iy_0 \in \mathbb{C} \colon x \in \mathbb{R}\}$. Therefore, if $\theta_0 = \lambda_0\theta_{id}$, we have $\big(\varphi_{\theta_0}^g(z)\big)(a) = \big(\varphi_{\theta_{id}}^\imath(z)\big)(\lambda_0 a)\,\chi^a(g)$, and the algebra $A_S \circ \varphi_{\theta_{id}}^g = \{f \circ \varphi_{\theta_{id}}^g \colon f \in A_S\}$ coincides with the algebra $AP_S(\mathbb{R})$ of bounded almost periodic S-functions on $\mathbb{R}$, which are analytic on $\mathbb{C}_+$.

5.5 Notes

The study of extendability properties of linear multiplicative functionals from smaller to larger shift-invariant algebras was initiated in [GT1]. The idea of involving monotonicity in this subject originated in [T2]. The solution of the related problem for semicharacter extension of semicharacters on arbitrary semigroups in terms of monotonicity (Theorem 5.2.1), is due to K. Ross [R3] (see also [CP], [K]). In the case of semigroups of $\mathbb{R}$, alternative sufficient conditions for semigroup extension of semicharacters were considered in [B4], and for arbitrary semigroups — in [GT2] and [P3]. The proof of Theorem 5.2.5 presented here is due to Sherstnev [S3] (see also [TG]). Most of the results on characterizations of semigroups with the property that any of its additive weights possesses an extension on larger semigroups are from [**?**] and [TG]. The necessary and sufficient conditions in Corollary 5.1.9, Proposition 5.1.10 and Theorem 5.4.9 for existence of $\mathbb{C}_+$- and $\mathbb{D}$-coronae of the considered classes of algebras of almost periodic functions and of H^∞-functions are from [GT4].

Chapter 6

G-disc algebras

G-disc algebras are also called big disc algebras, or algebras of generalized analytic functions. They are special classes of shift-invariant algebras on compact groups, generated by 'one half' of their dual groups. Their properties, including the description of their Bourgain algebras and primary ideals, are presented in this chapter. While all the results are given for general shift-invariant algebras A_{Γ_+}, they apply automatically to the particular cases of algebras AP_{Γ_+} of almost periodic functions, and of $H^\infty_{\Gamma_+}$-algebras.

6.1 Analytic functions on groups and G-discs

Let G be a compact group such that its dual group $\Gamma = \widehat{G}$ is isomorphic to a subgroup of $\mathbb{R}$. Without loss of generality we will assume that $\Gamma \subset \mathbb{R}$.

Definition 6.1.1. The G-disc algebra is said to be the shift-invariant algebra A_{Γ_+} on G, where $\Gamma_+ = \Gamma \cap \mathbb{R}_+$ is the non-negative part of Γ. The elements in A_{Γ_+}, the Γ_+*-functions*, are also called *generalized analytic functions* in the sense of Arens-Singer on G, or, *analytic functions on* G, for short. The Gelfand transform $\widehat{A}_{\Gamma_+}$ of A_{Γ_+} is denoted also by $A(\mathbb{D}_G)$. Its elements are called *analytic functions on the* G*-disc* $\overline{\mathbb{D}}_G$.

We recall that a function $f \in C(G)$ belongs to A_{Γ_+} if and only if its Fourier coefficients

$$c_a^f = \int\limits_G f(g)\, \overline{\chi}^a(g)\, d\sigma$$

are zero for all $a \in \Gamma \setminus \Gamma_+$. As a shift-invariant algebra on G, any G-disc algebra A_{Γ_+} possesses the following properties (cf. Section 4.1).

(i) The maximal ideal space $\mathcal{M}_{A_{\Gamma_+}}$ is homeomorphic to the closed G-disc $\overline{\mathbb{D}}_G = \big([0,1] \diamond G\big)/\{0\} \diamond G = \big([0,1] \times G\big)/\big(\{0\} \times G\big) \cong H(\Gamma_+)$.

(ii) The Shilov boundary ∂A_{Γ_+} is the group G.

(iii) $A(\mathbb{D}_G) = \widehat{A}_{\Gamma_+} \cong A_{\Gamma_+}$.

(iv) The algebra $A(\mathbb{D}_G)$ satisfies the *local maximum modulus principle*, i.e. given an analytic function f on $\overline{\mathbb{D}}_G$, considered as a subset of the G-plane $\mathbb{C}_G = \big([0,\infty) \times G\big)/\big(\{0\} \times G\big)$ and a compact set $K \subset \overline{\mathbb{D}}_G$, then

$$\big|f(r_0 \diamond g_0)\big| \leq \max_{r\diamond g \in bK} \big|f(r \diamond g)\big|$$

for every point $r_0 \diamond g_0 \in K$.

(v) If an analytic function on $\mathbb{D}_G$ vanishes on a non-void open subset of $\overline{\mathbb{D}}_G$ then it vanishes identically on $\overline{\mathbb{D}}_G$.

(vi) A_{Γ_+} is an antisymmetric algebra, i.e. any real-valued analytic function on G is constant.

(vii) The upper half-plane $\mathbb{C}_+$ can be embedded densely in the G-disc $\overline{\mathbb{D}}_G$ via the natural mapping $\widetilde{\jmath}_\imath \colon \mathbb{C}_+ \longrightarrow \mathcal{M}_{A_{\Gamma_+}}$, and the function $\widehat{f}|_{\widetilde{\jmath}_\imath(\mathbb{C}_+)}$ is analytic on $\mathbb{C}_+$ for any $f \in A$.

An essential part of the classical theory of analytic functions in $\mathbb{D}$ is based on the maximality property of the disc algebra $A(\mathbb{D})$. The next theorem says that, likewise, the algebra of analytic functions on G is a maximal algebra on G.

Proposition 6.1.2. *A shift-invariant algebra A_S on G is maximal if and only if $S \cong \Gamma_+$, i.e. if A_S coincides with the G-disc algebra.*

Proof. It is obvious that if S is a proper subsemigroup of Γ_+, then A_{Γ_+} properly contains A_S. Hence, A_S can not be maximal. Conversely, we claim that any G-disc algebra is maximal. Indeed, assume that $B \neq C(G)$ is a closed subalgebra of $C(G)$ with $A_{\Gamma_+} \subset B \subset C(G)$. Since $B \supset A_{\Gamma_+}$, we see that B contains every χ^a, $a \in \Gamma_+$. Because $B \neq C(G)$ there exists a $c \in \Gamma_+ \setminus \{0\}$ such that $\chi^{-c} \notin B$. Hence, χ^c has no inverse element in B, and therefore there is a linear multiplicative functional $\varphi \in \mathcal{M}_B$, such that $\varphi(\chi^c) = 0$. By the Archimedean property of $\mathbb{N}$, for any $a \in \Gamma_+$ there is an $m \in \mathbb{N}$ such that $ma \geq c$. Hence, if $b = ma - c \in \mathbb{N}$ we have $\chi^b \in B$, and therefore $\varphi(\chi^a)^m = \varphi(\chi^{ma}) = \varphi(\chi^c\chi^b) = \varphi(\chi^c)\,\varphi(\chi^b) = 0$. Consequently, $\varphi(\chi^a) = 0$ for every $a \in \Gamma_+ \setminus \{0\}$, and therefore, any representing measure μ of φ vanishes identically on both sets $\big\{\chi^a \colon a \in \Gamma_+ \setminus \{0\}\big\}$, and $\big\{\overline{\chi}^a \colon a \in \Gamma_+ \setminus \{0\}\big\} = \big\{\chi^a \colon a \in \Gamma \setminus \Gamma_+\big\}$. Hence, $\int\limits_G \chi^a(g)\, d\mu = 0$ for all $a \in \Gamma \setminus \{0\}$. Since, on the other hand, $\int\limits_G \chi^0(g)\, d\mu = \int\limits_G 1\, d\mu = \varphi(1) = 1$, we see that $\int\limits_G \chi^a\, d\mu = \int\limits_G \chi^a\, d\sigma$ for any $\chi^a \in \widehat{G}$. By the Stone-Weierstrass theorem $\int\limits_G f\, d\mu = \int\limits_G f\, d\sigma$ for every

$f \in C(G)$. The Riesz representation theorem implies that $\mu = \sigma$. Consequently, $\varphi(f) = \int_G f\, d\sigma$ for every $f \in B$. If $f \in B$ and $a \in \Gamma_+ \setminus \{0\}$ then also $f\chi^a \in B$ and

$$\int_G f\,\chi^a\, d\sigma = \varphi(f\chi^a) = \varphi(f)\,\varphi(\chi^a) = 0.$$

Thus $c^f_{-a} = \int_G f(g)\,\overline{\chi}^{-a}(g)\, d\sigma = \int_G f(g)\,\chi^a(g)\, d\sigma = 0$ for all $a \in \Gamma_+ \setminus \{0\}$, i.e. $f \in A_{\Gamma_+}$. Hence, $B \subset A_{\Gamma_+}$, and therefore, $B = A_{\Gamma_+}$. Consequently, A_{Γ_+} is a maximal algebra. $\square$

The maximality of A_{Γ_+} implies that if h is a continuous function on G, which is not an analytic function on G, then every continuous function on G can be approximated uniformly on G by functions of type $\sum_{k=1}^{n} f_k(g)\, h^k(g)$, where $f_k \in A_{\Gamma_+}$. Since $AP_{\Gamma_+}(\mathbb{R}) \cong A_{\Gamma_+}$, as a corollary from Proposition 6.1.2 we obtain that the algebra $AP_{\Gamma_+}(\mathbb{R}) \subset AP_a(\mathbb{R})$ of analytic almost periodic Γ-functions on $\mathbb{R}$ also is a maximal algebra.

Theorem 6.1.3. *Let $S \subset \Gamma_+$. A shift-invariant algebra A_S on G is a Dirichlet algebra, i.e. every real-valued continuous function f on G can be approximated by real parts of analytic functions on G, if and only if S generates Γ, i.e. if $S - S = \Gamma$.*

Proof. If $\Gamma_S = S - S \neq \Gamma$, then A_S does not separate the points of G, and neither does $\operatorname{Re} A_S$. Therefore, its closure $\left[\operatorname{Re} A_S\right]$ does not coincide with $C(G)$, thus A_S is not a Dirichlet algebra. Conversely, let $S - S = \Gamma$, and assume, on the contrary, that A_S is not a Dirichlet algebra. Then $\operatorname{Re} A_S$ is not dense in the space $C_{\mathbb{R}}(G)$ of real-valued continuous functions on G, hence $\left[\operatorname{Re} A_S\right]$ is a proper subspace of $C_{\mathbb{R}}(G)$. By the Hahn-Banach theorem there exists a non-zero linear positive functional φ on $C_{\mathbb{R}}(G)$ that vanishes identically on $\left[\operatorname{Re} A_S\right]$. By the Riesz representation theorem, φ can be expressed as the integration with respect to a Borel measure, say μ, on G. We have that $\int_G g\, d\mu = 0$ for all $g \in \operatorname{Re} A_S$, thus $\int_G f\, d\mu = \int_G \operatorname{Re} f\, d\mu + i \int_G \operatorname{Im} f\, d\mu = 0$ for every $f \in A_S$. Since the measure μ is real-valued, then also $\int_G \overline{f}\, d\mu = \overline{\int_G f\, d\mu} = 0$ for every $f \in A_S$. In particular, for any character $\chi \in S$ we have $\int_G \chi\, d\mu = 0$, and, $\int_G \overline{\chi}\, d\mu = \overline{\int_G \chi\, d\mu} = 0$. Since $\widehat{G} = \Gamma = S - S$, $\int_G \chi\, d\mu = 0$ for any $\chi \in \Gamma$. Therefore, $\int_G h\, d\mu = 0$ for any linear combination h of characters of G. Since by the Stone-Weierstrass theorem these combinations are dense in $C(G)$, we deduce that $\int_G f\,d\mu = 0$ for every continuous function $f \in C(G)$. Hence $\varphi(g) = \int_G g\,d\mu = 0$ for any $g \in C_{\mathbb{R}}(G)$. Therefore, φ is the zero functional on $C_{\mathbb{R}}(G)$, contradicting its choice. Consequently, A_S is a Dirichlet algebra if $S - S = \Gamma$. □

In particular, any G-disc algebra A_{Γ_+} is a Dirichlet algbera. Assume that $S - S = \Gamma$. Since A_{Γ_+} is a Dirichlet algebra, and G is its Shilov boundary, every linear multiplicative functional of A_S has a unique representing measure on G. Therefore, for any fixed point $r \diamond g \in \mathbb{D}_G \subset \mathcal{M}_S$ there is a unique positive measure $m_{r\diamond g}$ on G with $\operatorname{supp}(m_{r\diamond g}) = G$, and such that

$$\widehat{f}(r \diamond g) = \int_G f\,dm_{r\diamond g}$$

for every $f \in A_S$. In particular, $\widehat{\chi}^a(r \diamond g) = \int_G \chi^a\,dm_{r\diamond g} = r^a \chi^a(g)$ for all $a \in S$. Note that $\int_G \overline{\chi}^a\,dm_{r\diamond g} = r^{|a|}\overline{\chi}^a(g)$ for any $a \in S$.

Proposition 6.1.4. *The convolution of the representing measures of two points in $\overline{\mathbb{D}}_G$ for the G-disc algebra A_{Γ_+} is the representing measure of their product.*

Proof. Let m_i be the representing measures of the points $r_i \diamond g_i \in \overline{\mathbb{D}}_G,\ i = 1, 2$. By definition,

$$\int_G f\,d(m_1 \star m_2) = \int_G\int_G f(gh)\,dm_1(g)\,dm_2(h) \tag{6.1}$$

for all $f \in C(G)$. Therefore,

$$\begin{aligned}\int_G \chi^a\,d(m_1 \star m_2) &= \widehat{\chi}^a(r_1 \diamond g_1)\,\widehat{\chi}^a(r_2 \diamond g_2) = r_1^a\chi^a(g_1)\,r_2^a\chi^a(g_2)\\ &= (r_1r_2)^a\chi^a(g_1g_2) = \chi^a\big((r_1 \diamond g_1)(r_2 \diamond g_2)\big),\end{aligned}$$

for every $a \in \Gamma_+$. Consequently, $m_1 \star m_2$ is the representing measure of the point $(r_1 \diamond g_1)(r_2 \diamond g_2)$. □

Denote by m_r the representing measure of the point $r \diamond \imath$, i.e. $m_r = m_{r\diamond\imath}$, where $\imath$ is the unit element of G. Since $m_{r\diamond g} = m_{1\diamond g} \star m_{r\diamond\imath}$ for any $g \in G$, then

(6.1) implies that

$$\begin{aligned}\widehat{f}(r \diamond g) &= \int_G f\, d\, m_{r\diamond g} = \int_G f\, d\,(m_{1\diamond g} \star m_{r\diamond \imath}) \\ &= \int_G \int_G f(hk)\, dm_{1\diamond g}(h)\, dm_{r\diamond\imath}(k) = \int_G f(gk)\, d\, m_r(k) = \widetilde{f}_r(g)\end{aligned} \tag{6.2}$$

for any $f \in C(G)$. Here $\widetilde{f}_r$ is denotes the convolution $\widetilde{f}_r = f \star m_r$ of f and m_r. It is easy to see that

$$\widetilde{f}_{r_1} = f \star m_{r_1} = (f \star m_{r_2}) \star m_{r_1/r_2} = \widetilde{f}_{r_2} \star m_{r_1/r_2} = (\widetilde{f}_{r_2})\widetilde{\ }_{r_1/r_2},$$

whenever $r_1 < r_2$.

6.2 Bourgain algebras of G-disc algebras

Bourgain elements and Bourgain algebras were introduced in Section 1.4. Let G be a compact abelian group with unit element $\imath$, and let S be a subsemigroup of $\widehat{G}$ containing the unit character $\chi^0 \equiv 1$. For a fixed character $\chi \in \widehat{G}$ denote by $\mathcal{P}_\chi$ the set $\chi S \setminus S$.

Proposition 6.2.1. *Any character $\chi \in \widehat{G}$, for which the set $\mathcal{P}_\chi$ is finite, is a Bourgain element of A_S with respect to $C(G)$.*

Proof. Note that the characters on G are linearly independent in $C(G)$. Since the algebra A_S is generated linearly by $S \subset C(G)$, the sets $\mathcal{P}_\chi$ and $\pi_{A_S}(\mathcal{P}_\chi)$ have the same cardinality. Therefore,

$$\dim\big(S_\chi(A_S)\big) = \dim\big(\pi_{A_S}(\chi A_S)\big) = \mathrm{card}\,\big(\pi_{A_S}(\mathcal{P}_\chi)\big) = \mathrm{card}\,(\mathcal{P}_\chi) < \infty.$$

By Proposition 1.4.2 the Hankel type operator S_χ is completely continuous. Hence χ belongs to $(A_S)_b^{C(G)}$ as claimed. □

Note that for any $\chi \in S$ the set $\mathcal{P}_{\overline{\chi}}$ has the same cardinality as $\chi\mathcal{P}_{\overline{\chi}} = S \setminus \chi S = \{\gamma \in S \colon \gamma \notin \chi S\}$, which is the set of all predecessors of χ in S, i.e. of all elements γ in S which precede χ with respect to the S-order on $\widehat{G}$. If, in addition, $S - S = \widehat{G}$ and every $\chi \in S$ has finitely many predecessors in S, then every character $\chi \in \widehat{G}$ has finitely many predecessors in S. Then Proposition 6.2.1 yields $\big(A_S^{C(G)}\big)_b = C(G)$, and therefore the corresponding algebra A_S possesses the Dunford-Pettis property (cf. [CT]).

Corollary 6.2.2. *If $\chi \in S$ is such that $S \setminus \{1\} \subset \chi S$, then $\overline{\chi}$ is a Bourgain element of A_S with respect to $C(G)$.*

Proof. Since $\chi\mathcal{P}_{\overline{\chi}} = S \setminus \chi S = \big(\{1\} \cup (S \setminus \{1\})\big) \setminus \chi S \subset \big(\{1\} \cup \chi S\big) \setminus \chi S = \{1\}$, we have that $\mathcal{P}_{\overline{\chi}} = \{\overline{\chi}\}$. Hence $\overline{\chi} \in (A_S)_b^{C(G)}$ by Proposition 6.2.1. □

Corollary 6.2.3. *If A_S is a maximal algebra, and the set $\mathcal{P}_\chi$ is finite for some character $\chi \in \widehat{G} \setminus S$, then $(A_S)_b^{C(G)} = C(G)$.*

Proof. Indeed, $\chi \in (A_S)_b^{C(G)}$ by Proposition 6.2.1. However, $\chi \notin A_S$, since $\chi \notin S$. Consequently $(A_S)_b^{C(G)} = C(G)$ by the maximality of A_S. $\square$

Example 6.2.4. If H is a finite group, $G = (H \oplus \mathbb{Z})\widehat{}$ and $S \cong H \oplus \mathbb{Z}_+$, then $(A_S)_b^{C(G)} = C(G)$. Indeed, for every character $\chi_{(h,n)} \in \widehat{G}$, where $h \in H$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}&\operatorname{card}\big(\mathcal{P}_{\chi_{(h,n)}}\big) = \operatorname{card}\big((h,n)(H \oplus \mathbb{Z}_+) \setminus H \oplus \mathbb{Z}_+\big)\\&= \operatorname{card}\big((hH \oplus (n+\mathbb{Z}_+)) \setminus H \oplus \mathbb{Z}_+\big) = \operatorname{card}\big((H \oplus (n+\mathbb{Z}_+)) \setminus H \oplus \mathbb{Z}_+\big)\\&= \operatorname{card}\big(H \oplus ((n+\mathbb{Z}_+) \ \setminus \mathbb{Z}_+)\big) = \operatorname{card} H + n < \infty.\end{aligned}$$

Proposition 6.2.1 implies that $\chi_{(h,n)} \in (A_S)_b^{C(G)}$ for every $h \in H$ and $n \in \mathbb{Z}$. Consequently $\widehat{G} = H \oplus \mathbb{Z} \subset (A_S)_b^{C(G)}$, wherefrom $(A_S)_b^{C(G)} = C(G)$.

The following theorem gives a description of Bourgain algebras for some G-disc algebras.

Theorem 6.2.5. *If G is a solenoidal group whose dual group $\widehat{G} \cong \Gamma$ is divisible by an integer $n \neq 1$ belonging to Γ, then the Bourgain algebra $(A_{\Gamma_+})_b^{C(G)}$ of the G-disc algebra A_{Γ_+} coincides with A_{Γ_+}.*

The spaces $\mathbb{R}$, $\mathbb{Q}$, and the group of dyadic numbers $\{m/2^n : m \in \mathbb{Z},\ n \in \mathbb{N}\}$ are examples of groups Γ satisfying the hypotheses of Theorem 6.2.5. Without loss of generality we can assume that $1 \in \Gamma_+$, thus $1/n \in \Gamma_+$, i.e. $\chi^{1/n} \in \widehat{G}_+$. Clearly, Γ_+ is a subset of $(A_{\Gamma_+})_b^{C(G)}$. First we prove two auxiliary results.

Lemma 6.2.6. *The sequence of real-valued functions $\varphi_n(x) = \Big|\dfrac{1+e^{i\,(x/n)}}{2}\Big|^{2n}$ converges pointwise to 1 as $n \longrightarrow \infty$ for every $x \in \mathbb{R}$.*

Proof. Fix an $x \in \mathbb{R}$. Since $e^{i\,(x/n)} \neq -1$ for n big enough, we have

$$\varphi_n(x) = \Big(\Big|\frac{1+e^{i\,(x/n)}}{2}\Big|^2\Big)^n = \Big(\frac{2+2\cos(x/n)}{4}\Big)^n = \cos^{2n}(x/(2n) \longrightarrow 1$$

as $n \longrightarrow \infty$. $\square$

Note that the convergence in Lemma 6.2.6 is not necessarily uniform on $\mathbb{R}$, since, for instance, $\varphi_n(x) = 0$ if $x = \pi n$ for any integer n.

Lemma 6.2.7. *In the setting of Theorem 6.2.5, $\psi_n(g) = \Big|\dfrac{1+\chi^{1/n}(g)}{2}\Big|^{2n}$ converges pointwise to 1 as $n \longrightarrow \infty$ for every $g \in G$.*

Proof. Let $j_\imath\colon \mathbb{R} \longrightarrow G$ be the standard embedding of the real line onto a dense subgroup of G such that $j_\imath(0) = \imath$. Then $\chi^{1/n}\big(j_\imath(x)\big) = e^{i\,(x/n)}$ and $\psi_n(j_\imath(x)) = \varphi_n(x)$ for every real x. Hence $\varphi_n(x) \longrightarrow 1$ as $n \longrightarrow \infty$ by Lemma 6.2.6.

Consider the neighborhood U of $\imath$, defined as $U = (\chi^1)^{-1}\{e^{it}\colon -\pi/4 < t < \pi/4\} \subset G$. Note that if $\sqrt[n]{[\,\cdot\,]}$ is the principal value of the n-th root considered on the set $\{e^{it}\colon -\pi/4 < t < \pi/4\}$, then $\chi^{1/n}(h) = \sqrt[n]{\chi^1(h)}$ on U. For a given $g \in G$ there is an $h_g \in U$ such that $g = j_{h_g}(x)$ for some $x \in \mathbb{R}$, where $j_h = hj_\imath$ is the standard dense embedding of $\mathbb{R}$ into G with $j_h(0) = h$. Hence, if $\chi^1(h_g) = e^{is}$ for some s, $-\pi/4 < s < \pi/4$, then $\chi^{1/n}(h_g) = e^{i(s/n)}$, and therefore,

$$\begin{aligned}\psi_n(g) &= \psi_n(j_{h_g}(x)) = \Big|\frac{1+\chi^{1/n}(j_{h_g}(x))}{2}\Big|^{2n} \\ &= \Big|\frac{1+\chi^{1/n}(h_g)\,\chi^{1/n}(j_\imath(x))}{2}\Big|^{2n} = \Big|\frac{1+e^{i\,(s+x)/n}}{2}\Big|^{2n}.\end{aligned}$$

Consequently, $\psi_n(g) = \varphi_n(s+x) \longrightarrow 1$ as $n \longrightarrow \infty$, by Lemma 6.2.6. □

The remark following Lemma 6.2.6 shows that the convergence in Lemma 6.2.7 may not be uniform.

Proof of Theorem 6.2.5. Suppose that $\overline{\chi}^3 \in (A_{\Gamma_+})_b^{C(G)}$ and consider the sequence $\xi_n(g) = \psi_n(g) - 1$, where ψ_n is the function from Lemma 6.2.7. Since $\{\chi^1\xi_n\}_n$ converges pointwise to 0 on the compact group G, it is weakly null in A_{Γ_+}. Since $\overline{\chi}^3 \in (A_{\Gamma_+})_b^{C(G)}$, there are functions $h_n \in A_{\Gamma_+}$ such that $\|\overline{\chi}^3\chi^1\xi_n - h_n\| < 1/n$ for every n, where $\|\cdot\|$ is the sup-norm on G. By integrating, if necessary, over $\operatorname{Ker}\big(\chi^{1/n}\big)$, we can assume that h_n is constant on $\operatorname{Ker}\big(\chi^{1/n}\big)$-cosets in G, thus $h_n = q_n\big(\chi^{1/n}\big)$ for some polynomial q_n. Since

$$(\chi^1\psi_n)(g) = \big(\chi^{1/n}(g)\big)^n\Big(\frac{1+\chi^{1/n}(g)}{2}\Big)^n\Big(\frac{1+\overline{\chi}^{1/n}(g)}{2}\Big)^n = p_n\big(\chi^{1/n}(g)\big),$$

where p_n is the polynomial $p_n(z) = \Big(\dfrac{1+z}{2}\Big)^{2n}$, we have that $\chi^1\psi_n \in A_{\Gamma_+}$, and therefore, $\xi_n \in A_{\Gamma_+}$ too. If S_k is the k-th partial sum of p_n, then the j-th Cesáro mean

$$\sigma_j^{p_n} = \frac{S_0 + S_1 + \cdots + S_j}{j+1}$$

of p_n for $j = 2n$ equals

$$\sigma_{2n}^{p_n}(z) = \frac{1}{4^n(2n+1)}\sum_{k=0}^{2n}(2n-k+1)\binom{2n}{k} z^k.$$

Hence

$$\begin{aligned} 4^n(2n+1)\,\sigma_{2n}^{p_n}(z) &= \sum_{k=0}^{2n}\binom{2n}{k}z^k + \sum_{k=0}^{2n-1}(2n-k)\binom{2n}{k}z^k \\ &= (1+z)^{2n} + 2n\,(1+z)^{2n-1} = (2n+1+z)\,(1+z)^{2n-1}. \end{aligned}$$

Now

$$\begin{aligned} &\|\overline{\chi}^3\chi^1\xi_n - h_n\| = \max\nolimits_{g\in G}\left|(\overline{\chi}^3\chi^1\xi_n)(g) - h_n(g)\right| \\ &= \max\nolimits_{g\in G}\left|(\chi^1\xi_n)(g) - (\chi^3 h_n)(g)\right| = \max\nolimits_{g\in G}\left|(\chi^1\psi_n)(g) - \chi^1(g) - \chi^3(g)\,h_n(g)\right| \\ &= \max\nolimits_{g\in G}\left|p_n\big(\chi^{1/n}(g)\big) - \chi^1(g) - \big(\chi^{1/n}(g)\big)^{3n} q_n\big(\chi^{1/n}(g)\big)\right| \\ &= \max\nolimits_{z\in\mathbb{T}}\left|p_n(z) - z^n - z^{3n}q_n(z)\right|. \end{aligned}$$

Note that $\sigma_{2n}^{p_n(z)-z^n}(z) = \sigma_{2n}^{p_n(z)-z^n-z^{3n}q_n(z)}(z)$, because the Cesáro mean σ_{2n} depends only on the first $2n$ terms of the Taylor series. Since the inequality $\max\limits_{z\in\mathbb{T}}\left|\sigma_n^f(z)\right| \le \max\limits_{z\in\mathbb{T}}\left|f(z)\right|$ holds for every $f \in A(\mathbb{T})$, we obtain

$$\begin{aligned} &\max\nolimits_{z\in\mathbb{T}}\left|\sigma_{2n}^{p_n(z)-z^n}(z)\right| = \max\nolimits_{z\in\mathbb{T}}\left|\sigma_{2n}^{p_n(z)-z^n-z^{3n}q_n(z)}(z)\right| \\ &\le \max\nolimits_{z\in\mathbb{T}}\left|p_n(z) - z^n - z^{3n}q_n(z)\right| = \|\overline{\chi}^3\chi^1\xi_n - h_n\| < 1/n, \end{aligned}$$

i.e. $\left\|\sigma_{2n}^{p_n(z)-z^n}\right\| \longrightarrow 0$ as $n \longrightarrow \infty$. However, $\sigma_{2n}^{p_n(z)-z^n}(z) = \sigma_{2n}^{p_n(z)}(z) - z^n(n+1)/(2n+1)$, and thus $\sigma_{2n}^{p_n(z)-z^n}(-1) \longrightarrow 1/2$ as $n \longrightarrow \infty$ for odd n, contrary to the already obtained $\left\|\sigma_{2n}^{p_n(z)-z^n}\right\| \longrightarrow 0$. Hence $\|\overline{\chi}^3\chi^1\xi_n - h_n\| \not\longrightarrow 0$ for any $h_n \in A_{\Gamma_+}$, and therefore $\overline{\chi}^3 \notin (A_{\Gamma_+})_b^{C(G)}$. The maximality of A_{Γ_+} implies that $(A_{\Gamma_+})_b^{C(G)} = A_{\Gamma_+}$. □

6.3 Orthogonal measures to G-disc algebras

Orthogonal measures are important tools for studying subspaces of uniform algebras. In this section we describe the set of orthogonal measures to a G-disc algebra. We use these results to characterize all primary ideals in shift-invariant algebras.

Let G be a compact group, whose dual $\Gamma = \widehat{G}$ is a subgroup of $\mathbb{R}$. A measure $\mu \in M(G)$ is orthogonal to the G-disc algebra A_{Γ_+} if and only if $\mathrm{sp}\,(\mu) \subset \Gamma_+ \setminus \{0\}$. The space of measures orthogonal to A_{Γ_+} is denoted by $A_{\Gamma_+}^{\perp}$. The measures in $A_{\Gamma_+}^{\perp}$ are called also *analytic measures* with respect to the algebra A_{Γ_+} (e.g. [G1]). The celebrated F. and M. Riesz theorem asserts that the space of orthogonal measures to the disc algebra $A(\mathbb{D})$ is isometrically isomorphic to the space $H_0^1 = zH^1$, where H^1 is the classical Hardy space on $\mathbb{T}$, i.e. $A(\mathbb{D})^{\perp} \cong H_0^1$. More precisely, $A(\mathbb{D})^{\perp} = H_0^1\,d\sigma$.

Let $A_b(\mathbb{C}_+)$ be the algebra of bounded continuous functions on $\overline{\mathbb{C}}_+$ that are analytic in the upper half-plane $\mathbb{C}_+$. By the classical Phragmen-Lindelöf principle (e.g.[G2]), for any $f \in A_b(\mathbb{C}_+)$,

$$\sup_{z\in\overline{\mathbb{C}}_+} \big|f(z)\big| = \sup_{t\in\mathbb{R}} \big|f(t)\big|. \tag{6.3}$$

Hence the restriction $A_b(\mathbb{R}) = \big\{f|_{\mathbb{R}}\colon f \in A_b(\mathbb{C}_+)\big\}$ of $A_b(\mathbb{C}_+)$ on $\mathbb{R}$ is a closed algebra in the sup-norm. Let w be the fractional linear transformation $w(t) = \dfrac{i-t}{i+t}$ that maps $\mathbb{R}$ into the unit circle $\mathbb{T}$. Denote by $H^1(\mathbb{R}) = H^1 \circ w$ the space of functions in $L^1\Big(\mathbb{R}, \dfrac{dt}{1+t^2}\Big)$ of type $h \circ w$ with $h \in H^1$, and let $H^1_0(\mathbb{R}) = H^1_0 \circ w = (zH^1) \circ w = w \cdot (H^1 \circ w) = w \cdot H^1(\mathbb{R})$. Since $A(\mathbb{D})^\perp = H^1_0\, d\sigma$, then the space $H^1_0 \circ w\, d\,(w^*\sigma) = H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ equals the space $A_b(\mathbb{R})^\perp$ of measures on $\mathbb{R}$ orthogonal to $A_b(\mathbb{R})$.

Let $AP_{\Gamma_+}(\mathbb{R})^\perp$ be the set of measures $\mu \in M(\mathbb{R})$ on $\mathbb{R}$ that are orthogonal to the algebra $AP_{\Gamma_+}(\mathbb{R})$ of almost periodic functions on $\mathbb{R}$ with spectrum in $\Gamma_+ = \Gamma \cap \mathbb{R}_+$, Since $AP_{\Gamma_+}(\mathbb{R})$ is generated by the functions e^{iat}, $a \in \Gamma_+$, we see that $\mu \in AP_{\Gamma_+}(\mathbb{R})^\perp$ if and only if $\displaystyle\int_{\mathbb{R}} e^{iat}\, d\mu = 0$ for every $a \in \Gamma_+$.

Proposition 6.3.1. *For any measure $\mu \in M(\mathbb{R})$, the following are equivalent:*

(i) $\displaystyle\int_{\mathbb{R}} e^{iat}\, d\mu = 0$ *for every* $a \in \Gamma_+$,

(ii) $\mu \in H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$,

thus $AP_{\Gamma_+}(\mathbb{R})^\perp = H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$.

Proof. If $\mu \in H^1_0 \cdot \dfrac{dt}{1+t^2}$, then clearly $\mu \in A_b(\mathbb{R})^\perp$, which implies (ii). If $\mu \in AP_{\Gamma_+}(\mathbb{R})^\perp$, then

$$\int_{\mathbb{R}} e^{iat}\, d\mu(t) = 0, \text{ for any } a \in \Gamma_+, \tag{6.4}$$

and hence $\displaystyle\int_{\mathbb{R}} e^{iat}\, d\mu(t) = 0$ for all $a \in \mathbb{R}_+$ by the continuity argument. Consequently, the Fourier transform of μ vanishes on $\mathbb{R}_+$, and therefore μ belongs to $H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ (e.g. [G1]). □

Let $j_\imath\colon \mathbb{R} \longrightarrow G$ be the standard embedding of $\mathbb{R}$ into a dense subgroup of G with $j_\imath(0) = \imath$. Since $A_{\Gamma_+}\big|_{j(\mathbb{R})} \cong A_{\Gamma_+} \circ j_\imath = AP_{\Gamma_+}(\mathbb{R})$, we obtain the following

Corollary 6.3.2. *A measure μ on $\mathbb{R}$ is orthogonal to algebra $A_{\Gamma_+} \circ j_\imath$ if and only if $\mu \in H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$. Equivalently, $\big(A_{\Gamma_+}\big|_{j(\mathbb{R})}\big)^\perp = H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$.*

Without loss of generality we can assume that $2\pi \in \Gamma$. Denote $K = \mathrm{Ker}\,(\chi^{2\pi}) = \big\{g \in G\colon \chi^{2\pi}(g) = 1\big\}$. The Cartesian product $\widetilde{G} = K \times \mathbb{R}$ is a locally compact abelian group, contained in the set $K \times \overline{\mathbb{C}}_+$. Consider the algebra $A_b(K \times \overline{\mathbb{C}}_+)$ of functions $f(g,z)$ that are continuous on $K \times \overline{\mathbb{C}}_+$, and analytic in z, i.e. for every fixed $g \in K$ the function $z \longmapsto f(g,z)$ belongs to $A_b(\mathbb{C}_+)$. Let $A_b(\widetilde{G}) = A_b(K \times \overline{\mathbb{C}}_+)\big|_{\widetilde{G}}$ be the restriction of the space $A_b(K \times \overline{\mathbb{C}}_+)$ on $\widetilde{G} = K \times \mathbb{R}$. According to (6.3) for every $f \in A_b(K \times \overline{\mathbb{C}}_+)$ we have

$$\sup_{(g,z)\in K\times\overline{\mathbb{C}}_+} \big|f(g,z)\big| = \sup_{(g,x)\in\widetilde{G}} \big|f(g,x)\big|.$$

Therefore, $A_b(K \times \overline{\mathbb{C}}_+)$ and $A_b(\widetilde{G})$ are isometric and isomorphic uniform algebras.

Denote by $M(G)$ and $M(\widetilde{G})$ the spaces of finite Borel measures on G and $\widetilde{G}$ correspondingly. Clearly, the set $A_b(\widetilde{G})^\perp$ of measures on $\widetilde{G}$ orthogonal to the algebra $A_b(\widetilde{G})$ is contained in $M(\widetilde{G})$. Let $M_0^1(\widetilde{G})$ be the set of measures on $\widetilde{G}$, for which there exist probability measures ν on K, such that

$$h(g,t) \cdot \Big(d\nu(g) \times \frac{dt}{1+t^2}\Big), \tag{6.5}$$

where $h(g,t)$ are functions in $L^1\Big(d\nu(g) \times \dfrac{dt}{1+t^2}\Big)$ such that the functions $t \longmapsto h(g,t)$ are in $H_0^1(\mathbb{R})$ for ν-almost every $g \in K$.

Proposition 6.3.3. *A measure $\mu \in M(\widetilde{G})$ is orthogonal to the algebra $A_b(\widetilde{G})$ if and only if $\mu \in M_0^1(\widetilde{G})$, i.e. $A_b(\widetilde{G})^\perp = M_0^1(\widetilde{G})$.*

Proof. We embed naturally $C(K)$ into $A_b(\widetilde{G})$ by $f(g,t) = f(g)$, $f \in C(K)$. According to a version of the Krein-Milman theorem (e.g. [R7]), every measure $\mu \in A_b(\widetilde{G})^\perp$ has the form $d\mu(g,t) = d\nu(g) \times d\nu_g(t)$, where ν is a probability measure on K, and $\{\nu_g\}_{g\in K}$ is a family of measures on $\mathbb{R}_g = \{g\} \times \mathbb{R} \subset \widetilde{G}$, such that $\nu_g \in A_b(\widetilde{G})^\perp$ for ν-almost every $g \in K$. Since the restriction of $A_b(\widetilde{G})$ on $\mathbb{R}_g$ is $A_b(\mathbb{R})$, then by Proposition 6.3.1 we have that $\nu_g(t) \in H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$. The measures μ and $d\nu(g) \times \dfrac{dt}{1+t^2}$ are mutually absolutely continuous. Consequently, $d\mu(g,t) = h(g,t) \cdot \Big(d\nu(g) \times \dfrac{dt}{1+t^2}\Big)$ with a $h(g,t) \in L^1\Big(d\nu(g) \times \dfrac{dt}{1+t^2}\Big)$

such that $h(g,t)\cdot\dfrac{dt}{1+t^2} = d\nu_g(t) \in H_0^1(\mathbb{R})\cdot\dfrac{dt}{1+t^2}$ for ν-almost all $g\in K$. Therefore, $\mu\in M_0^1(\widetilde{G})$.

Conversely, if $\mu\in M_0^1(\widetilde{G})$ then $d\mu(g,t) = h(g,t)\cdot\Big(d\nu(g)\times\dfrac{dt}{1+t^2}\Big)$, where $h(g,\,\cdot\,)\in H_0^1(\mathbb{R}) = A_b(\mathbb{R})^\perp$ for ν-almost all $g\in K$. For any $f\in A_b(\widetilde{G})$ we have

$$\int\limits_{\widetilde{G}} f\,d\mu = \int\limits_K \Big(\int\limits_{\mathbb{R}} f(g,t)\,h(g,t)\,\frac{dt}{1+t^2}\Big)\,d\nu(g),$$

and the inner integral vanishes for ν-almost all $g\in K$. Therefore, $\mu\in A_b(\widetilde{G})^\perp$. Consequently, $M_0^1(\widetilde{G}) = A_b(\widetilde{G})^\perp$, as claimed. □

For a given measure $\mu\in A_b(\widetilde{G})^\perp$ the probability measure ν on K in (6.5) can be determined as follows. The measure $\mu' = |\mu|/\|\mu\|$ on $\widetilde{G}$ generates a positive linear functional F on $C(K)\subset A_b(\widetilde{G})$ with unit norm, namely

$$F(f) = \int\limits_{\widetilde{G}} f\,d\mu',\ \ f\in C(K).$$

We now take ν to be the probability measure on K with $F(f) = \int\limits_K f\,d\nu$ for every $f\in C(K)$, existing by the Riesz representation theorem.

Consider the homomorphism $\pi_G\colon \widetilde{G}\longrightarrow G$ defined by $\pi_G(g,t) = g_t\,g$, where $g_t = j_\imath(t)\in G,\ t\in\mathbb{R}$. Clearly, $g_n\in K$ for every $n\in\mathbb{Z}$. The kernel Ker$\,(\pi_G)$ of π is the subgroup $\mathcal{P} = \big\{(g_n,-n)\in\widetilde{G}\colon n\in\mathbb{Z}\big\}$ (cf. Section 3.1), and the group G can be obtained from the set $K\times[n,n{+}1]$ by identifying the points $(g,n{+}1)$ and $(g_1\,g,n)$. Actually, π_G generates a countably-sheeted covering without singularities of $\widetilde{G}$ onto G. Observe that for any $\chi^a\in\Gamma_+$ we have $(\chi^a\circ\pi_G)(g,t) = e^{iat}\chi^a(g)\in A_b(\widetilde{G})$. Consequently, the adjoint mapping to $\pi_G\colon\widetilde{G}\longrightarrow G$, π_G^*, maps the G-disc algebra A_{Γ_+} into $A_b(\widetilde{G})$.

Proposition 6.3.4. *The algebra* $\widetilde{A}_{\Gamma_+} = \{f\circ\pi_G\colon f\in A_{\Gamma_+}\} = A_{\Gamma_+}\circ\pi_G$ *coincides with the subalgebra of functions in* $A_b(\widetilde{G})$ *which are invariant under shifts by elements of the group* Ker$\,(\pi_G)$.

Proof. Since Ker(π_G) is the kernel of the homomorphism $\pi_G\colon\widetilde{G}\longrightarrow G$, the algebra $\widetilde{A}_{\Gamma_+} = A_{\Gamma_+}\circ\pi_G$ is Ker(π_G)-invariant. Let $\widetilde{f}\in A_b(\widetilde{G})$ be a Ker(π_G)-invariant function on $\widetilde{G}$. Since the quotient group $\widetilde{G}/\text{Ker}(\pi_G)$ is isomorphic to G, there is a continuous function f on G such that $f\circ\pi_G = \widetilde{f}$. We claim that $f\in A_{\Gamma_+}$. Observe that the homomorphism π_G generates an isometric isomorphism between the uniform algebras $[\widetilde{A}_{\Gamma_+}, \widetilde{f}] \subset A_b(\widetilde{G})$ on $\widetilde{G}$ and $[A_{\Gamma_+}, f]$ on G. The maximality of A_{Γ_+} implies that $[A_{\Gamma_+}, f]$ coincides either with A_{Γ_+}, or with $C(G)$. In the second case we get $\chi^a \circ \pi_G \in A_b(\widetilde{G})$ for all $a \in \Gamma$, which is impossible, since $(\chi^a \circ \pi_G)(g,t) = e^{iat}\chi^a(g) \notin A_b(\widetilde{G})$ for any $a < 0$. Therefore, $[A_{\Gamma_+}, f] = A_{\Gamma_+}$, and hence $\widetilde{f} \in \widetilde{A}_{\Gamma_+}$. □

The projection $\pi_G\colon \widetilde{G} \longrightarrow G$ can be extended on $K \times \overline{\mathbb{C}}_+$ as a mapping $\widetilde{\pi}_G\colon K \times \overline{\mathbb{C}}_+ \longrightarrow \overline{\mathbb{D}}_G^* = \overline{\mathbb{D}}_G \setminus \{\omega\}$ by $\widetilde{\pi}_G(g, t+iy) = e^{-y} g\, g_t$. Note that $\widetilde{\pi}_G$ generates a countably-sheeted covering without singularities and without a boundary from $K \times \mathbb{C}_+ \subset K \times \overline{\mathbb{C}}_+$ onto $\mathbb{D}_G^*$. The mapping $f \longmapsto f \circ \widetilde{\pi}_G$ is an embedding of $\widehat{A}_{\Gamma_+}$ into $A_b(K \times \overline{\mathbb{C}}_+)$.

For any $\mu \in M(G)$ one can define a net of measures $\{\mu_\alpha\} \subset M(G)$ with the following properties.

(i) $\mathrm{sp}\,(\mu_\alpha)$ is finite, and is contained in $\mathrm{sp}\,(\mu)$.

(ii) $\|\mu_\alpha\| \leq \|\mu\|$.

(iii) The net $\{\mu_\alpha\}$ converges in the weak*-topology to μ.

If μ is an analytic measure for the algebra A_{Γ_+} on G, then $\mathrm{sp}\,(\mu) \subset \Gamma_+ \setminus \{0\}$, and so are all μ_α according to (i). Moreover, $d\mu_\alpha = f_\alpha\, d\sigma$, where σ is the normalized Haar measure of G, and $f_\alpha = \sum_{a\in S} c_a \chi^a$ is a Γ_+-polynomial on G with $a \in \mathrm{sp}\,(\mu_\alpha) \subset \mathrm{sp}\,(\mu)$.

Any Borel set $E \subset \widetilde{G} = K \times \mathbb{R}$ can be expressed in the form $E = \bigcup_{n=-\infty}^{\infty} E \cap \widetilde{G}_n$, where $\widetilde{G}_n = K \times [n, n+1)$. A measure $\mu \in M(G)$ can be lifted naturally to $\widetilde{G}$ as a locally finite measure $\widetilde{\mu} \in M(\widetilde{G})$, defined by

$$\widetilde{\mu}(E) = \sum_{n=-\infty}^{\infty} \mu\big(\pi_G(E \cap \widetilde{G}_n)\big)$$

for every Borel set $E \subset \widetilde{G}$. The arising mapping $\mu \longmapsto \widetilde{\mu}$ maps $M(G)$ onto the space $M_{\mathrm{Ker}(\pi_G)}(\widetilde{G}) \subset M(\widetilde{G})$ of locally finite measures on $\widetilde{G}$ that are invariant under shifts by elements of $\mathrm{Ker}\,(\pi_G)$. Note that if σ and τ are the normalized Haar measures on the groups G and K respectively, then $d\widetilde{\sigma} = d\tau \times dt$. In general, $\widetilde{\mu}$ has the form $d\widetilde{\mu}(g,t) = \widetilde{f}(g,t)(d\nu \times dt)$, where ν is a probability measure on K and the function $f \in L^1\big(d\nu(g) \times dt\big)$ is invariant under $\mathrm{Ker}\,(\pi_G)$-shifts. For every Borel set $E \subset K \times [0,1]$ we have $\widetilde{\mu}(E) = \mu\big(\pi_G(E)\big)$. Therefore, $d\widetilde{\mu}\big|_{K\times[0,1]} = \widetilde{f}(g,t)(d\nu \times dt)$, where $f \in L^1\big(d\nu(g) \times dt\big)\big|_{[0,1]}$, and hence $\int_{K\times[0,1]} f \circ \pi_G\, d\widetilde{\mu} = \int_G f\, d\mu$ for every $f \in C(G)$.

Denote by $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$ the set of measures $\widetilde{\mu} \in M_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$ for which $\dfrac{w(t)}{1+t^2} \cdot d\widetilde{\mu}(g,t) \in M^1_0(\widetilde{G})$, where $w\colon \mathbb{R} \longrightarrow \mathbb{T}$ is the fractional linear transformation $w(t) = \dfrac{i-t}{i+t}$. Clearly, $\widetilde{\sigma} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$, since $w\,d\nu \times \dfrac{w(t)}{1+t^2} \in M^1_0(\widetilde{G})$.

Proposition 6.3.5. *The mapping $\mu \longmapsto \widetilde{\mu}$ maps $A^{\perp}_{\Gamma_+} \subset M(G)$ into $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G}) \subset M(\widetilde{G})$.*

Proof. For any $\nu \in M(G)$ the measure $\dfrac{w(t)}{1+t^2} \cdot d\widetilde{\nu}(g,t)$ belongs to $M(\widetilde{G})$, and satisfies the inequalities

$$\frac{1}{2}\,\|\nu\|_G \leq \left\|\frac{w(t)}{1+t^2}\,d\widetilde{\nu}\right\|_{\widetilde{G}} \leq 4\,\|\nu\|_G, \tag{6.6}$$

where $\|\cdot\|_G$ and $\|\cdot\|_{\widetilde{G}}$ are the standard norms in $M(G)$ and $M(\widetilde{G})$ respectively. Hence, if a net of measures $\{\mu_\alpha\} \subset M(G)$ converges in the weak*-topology to a measure $\mu \in M(G)$, then the net $\left\{\dfrac{w(t)}{1+t^2} \cdot d\widetilde{\mu}_\alpha\right\} \subset M(\widetilde{G})$ also converges in the weak*-topology to the measure $\dfrac{w(t)}{1+t^2} \cdot d\widetilde{\mu} \in M(\widetilde{G})$. Therefore, the properties (i), (ii), (iii) from the above imply that it is enough to prove that $\widetilde{\mu} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$ for measures $\mu \in A^{\perp}_{\Gamma_+}$ with finite spectrum only. If $\mu \in A^{\perp}_{\Gamma_+}$ has a finite spectrum $\mathrm{sp}(\mu)$, then $d\mu = f\,d\sigma$, where $f = \sum_{k=1}^{m} c^f_{a_k}\chi^{a_k}$, $a_k \in \Gamma_+$. For every fixed $g \in K$ the function $\widetilde{f}(g,t) = (f \circ \pi_G)(g,t)$ belongs to $H^1(\mathbb{R})$. Hence, $d\widetilde{\mu}(g,t) = \widetilde{f}(g,t) \cdot d\widetilde{\sigma} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. □

We will consider the functions in H^1 on $\mathbb{T}$ as H^1-functions on $\mathbb{R}$ that are periodic with period 1, i.e. that H^1 is a subset of $H^1(\mathbb{R})$.

Lemma 6.3.6. *If $f \in H^1$ and the function $e^{-iat}f(t)$ belongs to $H^1 \hookrightarrow H^1(\mathbb{R})$ for some $a > 0$, then $\displaystyle\int_0^1 f(t)\,dt = 0$.*

Proof. Any $f \in H^1 \hookrightarrow H^1(\mathbb{R})$ can be expressed as a formal Fourier series $f(t) \sim c^f_0 + \sum_{n=1}^{\infty} c^f_n e^{i2\pi nt}$, where $c^f_n = \displaystyle\int_0^1 f(t)\,e^{-i2\pi nt}\,dt$. Clearly, the Fourier series of the function $e^{-iat}f(t)$ is

$$e^{-iat}f(t) \sim c^f_0 e^{-iat} + \sum_{n=1}^{\infty} c^f_n e^{i(2\pi n - a)t}.$$

If $e^{-iat}f(t) \in H^1 \hookrightarrow H^1(\mathbb{R})$, for some $a > 0$, then also $c_0^f e^{-iat} \in H^1 \hookrightarrow H^1(\mathbb{R})$, which is possible if and only if $c_0^f = 0$, i.e. if $\int\limits_0^1 f(t)\,dt = 0$. □

Recall that the point evaluation at the origin ω of a G-disc $\overline{\mathbb{D}}_G$ is a linear multiplicative functional of the G-disc algebra A_{Γ_+}. Denote by $\mathcal{J}_\omega$ the corresponding maximal ideal, i.e. $\mathcal{J}_\omega = \{f \in A_S : \widehat{f}(\omega) = 0\}$ of A_{Γ_+}. The following theorem describes the space $\mathcal{J}_\omega^\perp$ of measures on G that are orthogonal to $\mathcal{J}_\omega$.

Theorem 6.3.7. *For a measure μ on G the following statements are equivalent.*

(i) $\int \chi^a\,d\mu = 0$ *for all* $a \in \Gamma_+ \setminus \{0\}$, *i.e.* $\mu \in \mathcal{J}_\omega^\perp$.

(ii) $\widetilde{\mu} \in M^1_{Ker(\pi_G)}(\widetilde{G})$.

Equivalently, $\mathcal{J}_\omega^\perp \cong M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$.

Proof. We will show that the lifting $\mu \longmapsto \widetilde{\mu}$ maps $\mathcal{J}_\omega^\perp$ onto $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. Recall that the representing measure of $\mathcal{J}_\omega$ is the normalized Haar measure σ of G (e.g. [G1]). Since $\dim(A_{\Gamma_+}/\mathcal{J}_\omega) = 1$, we have $\mathcal{J}_\omega^\perp = A_{\Gamma_+}^\perp + \mathbb{C}\sigma$. The mapping $\mu \longmapsto \widetilde{\mu}$ is a linear map of $A_{\Gamma_+}^\perp$ into $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$, and σ to $d\widetilde{\sigma} = d\tau \times dt \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. Therefore, it maps $\mathcal{J}_\omega^\perp = A_{\Gamma_+}^\perp + \mathbb{C}\sigma$ into $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$.

Let $\mu \in M(G)$ be such that $\widetilde{\mu} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. We claim that $\mu \in \mathcal{J}_\omega^\perp$. Indeed, let

$$d\widetilde{\mu}(g,t) = \widetilde{f}(g,t) \cdot \big(d\nu(g) \times dt\big), \tag{6.7}$$

where $\widetilde{f}$ is $\mathrm{Ker}(\pi_G)$-invariant on $\mathbb{R}$ and the functions $t \longmapsto \widetilde{f}(g,t)$ belong to $H^1 \hookrightarrow H^1(\mathbb{R})$ for ν-almost all $g \in K$ (cf. the proof of Proposition 6.3.5). The function

$$f_{(a)}(t) = \int\limits_K \chi^a(g)\,\widetilde{f}(g,t)\,d\nu(g) \tag{6.8}$$

belongs to $H^1 \hookrightarrow H^1(\mathbb{R})$ for all $a \in \Gamma_+$. Since $\widetilde{\mu}$ is $\mathrm{Ker}(\pi_G)$-invariant, we have

$$d\widetilde{\mu}(g,t) = d\widetilde{\mu}(g_n\,g, t-n) = \widetilde{f}(g_n\,g, t-n) \cdot \big(d\nu(g_n\,g) \times dt\big), \tag{6.9}$$

where $(g_n, -n) \in \operatorname{Ker}(\pi_G)$. Comparing (6.7) and (6.9) we see that $\widetilde{f}(g_n\, g, t-n)\, d\nu(g_n\, g) = \widetilde{f}(g,t)\, d\nu(g)$ for almost every $t \in \mathbb{R}$. Hence, $\widetilde{f}(g, t+n)\, d\nu(g) = \widetilde{f}(g_n\, g, t)\, d\nu(g_n\, g)$, and therefore

$$\begin{aligned} f_{(a)}(t+n) &= \int_K \chi^a(g)\, \widetilde{f}(g, t+n)\, d\nu(g) \\ &= \int_K \chi^a(g)\, \widetilde{f}(g_n\, g, t)\, d\nu(g_n\, g) = \int_K \chi^a(g_{-n}\, g)\, \widetilde{f}(g,t)\, d\nu(g) \\ &= \int_K e^{-ian} \chi^a(g)\, \widetilde{f}(g,t)\, d\nu(g) = e^{-ian} f_{(a)}(t), \end{aligned} \tag{6.10}$$

since $\chi^a(g_{-n}\, g) = e^{-ian}\chi^a(g)$. Therefore, $f_{(a)}(t+n) = e^{-ian} f_{(a)}(t)$. Let $\varphi_a(t) = e^{iat} f_{(a)}(t)$. For $a > 0$ the functions e^{iat} and $f_{(a)}(t)$ belong to $H^1(\mathbb{R})$. Therefore, $\varphi_a(t)$ is a periodic function with period 1 in $H^1 \hookrightarrow H^1(\mathbb{R})$. Applied to $\varphi_a(t)$, Lemma 6.3.6 yields $\int_0^1 \varphi_a(t)\, dt = 0$. Hence, for any $a \in \Gamma_+ \setminus \{0\}$ we have

$$\begin{aligned} \int_G \chi^a\, d\mu &= \int_{K\times[0,1]} \chi^a \circ \pi_G\; d\widetilde{\mu} = \int_{K\times[0,1]} (\chi^a \circ \pi_G)(g,t)\, \widetilde{f}(g,t)\big(d\nu(g) \times dt\big) \\ &= \int_{K\times[0,1]} e^{iat}\chi^a(g)\, \widetilde{f}(g,t)\big(d\nu(g)\times dt\big) = \int_0^1 e^{iat}\Big(\int_K \chi^a(g)\, \widetilde{f}(g,t)\, d\nu(g)\Big)\, dt \\ &= \int_0^1 e^{iat} f_{(a)}(t)\, dt = \int_0^1 \varphi_a(t)\, dt = 0. \end{aligned}$$

Consequently $\mu \perp \chi^a$ for any $a \in \Gamma_+ \setminus \{0\}$, thus $\mu \in \mathcal{J}_\omega^\perp$. □

6.4 Primary ideals of G-disc algebras

Finding descriptions of various types of ideals is an important and interesting issue in uniform algebra theory. A proper ideal of an algebra is called a *primary ideal* if it is contained in exactly one maximal ideal of the algebra. The primary ideals of the disc algebra $A(\mathbb{D})$ have simple descriptions. Namely, these are the ideals of type $\left(\dfrac{z-z_0}{1-\overline{z}_0 z}\right)^n \cdot A(\mathbb{D})$, where $z_0 \in \overline{\mathbb{D}}$ and $n \in \mathbb{N}$. Each ideal of this type is contained in the maximal ideal $J_{z_0} = \big\{f \in A(\mathbb{D})\colon f(z_0) = 0\big\}$. In this section we describe all primary ideals of G-disc algebras A_{Γ_+} of analytic functions on a solenoidal group G.

Let G be a solenoidal group, i.e. a compact group, whose dual group $\widehat{G}$ is dense in $\mathbb{R}$. As in Section 6.3 we assume that $2\pi \in \Gamma = \widehat{G}$. Recall that every maximal ideal of a G-disc algebra A_{Γ_+} is a point evaluation, i.e. of type $\mathcal{J}_{r\diamond g} = \{f \in A_{\Gamma_+} : \widehat{f}(r \diamond g) = 0\}$ for some $r \diamond g \in \overline{\mathbb{D}}_G$. Therefore, an ideal $I \subset A_{\Gamma_+}$ is primary if and only if $I \subset \mathcal{J}_{r\diamond g}$ for some $r \diamond g \in \overline{\mathbb{D}}_G$. In the sequel we will consider closed ideals only.

We recall that the *hull* of an ideal I of the algebra A_{Γ_+} is the set

$$\text{hull}\,(I) = \{r \diamond g \in \overline{\mathbb{D}}_G \colon \widehat{f}(r \diamond g) = m_{(r\diamond g)}(f) = 0 \text{ for all } f \in I\}.$$

Clearly, $I \subset A_{\Gamma_+}$ is a primary ideal if and only if its hull is a singleton. First we describe the primary ideals of A_{Γ_+}, whose hulls coincide with the origin $\omega = 0 \diamond g \in \mathbb{D}_G$ of the G-disc, i.e. which are subideals of the maximal ideal $\mathcal{J}_\omega = \{f \in A_{\Gamma_+} : \widehat{f}(\omega) = 0\}$.

With any non-negative number $b \in \mathbb{R}_+$ we associate two ideals,

$$\mathcal{J}_\omega(b^+) = \Big[\bigcup_{\substack{a \in \Gamma_+ \\ a > b}} \chi^a \mathcal{J}_\omega\Big], \text{ and } \mathcal{J}_\omega(b^-) = \bigcap_{\substack{a \in \Gamma_+ \\ a < b}} \chi^a \mathcal{J}_\omega,$$

where [.] is the closure of the enclosed set. Note that $\mathcal{J}_\omega(0^+) = \mathcal{J}_\omega$. Indeed, if $f \in \mathcal{J}_\omega$, then for any $\varepsilon > 0$ there is a Γ_+-polynomial $g = \sum_{a\in\Gamma_+} c_a^f \chi^a$ such that $\widehat{g}(\omega) = 0$, and $\|f - g\| < \varepsilon$. Clearly, $g \in \mathcal{J}_\omega(0^+)$. Since $\mathcal{J}_\omega(0^+)$ is closed and ε is arbitrary, it follows that $f \in \mathcal{J}_\omega(0^+)$. Therefore, $\mathcal{J}_\omega(0^+) = \mathcal{J}_\omega$. Similarly, one can see that $\mathcal{J}_\omega(0^-) = A_{\Gamma_+}$

Definition 6.4.1. An ideal I of the algebra A_{Γ_+} is called

(i) *right-continuous*, if the set $\bigcup_{a\in\Gamma_+\setminus\{0\}} \chi^a I$ is dense in I,

(ii) *left-continuous*, if the set $\bigcap_{a\in\Gamma_+\setminus\{0\}} \overline{\chi}^a I$ coincides with I.

Lemma 6.4.2. *Let $b \in \mathbb{R}_+$. Then:*

(a) *$\mathcal{J}_\omega$ and $\mathcal{J}_\omega(b^+)$ are right-continuous ideals.*

(b) *$\mathcal{J}_\omega(b^-)$ is a left-continuous ideal.*

(c) *If $b \in \Gamma_+$, then $\mathcal{J}_\omega(b^+) = \chi^b \mathcal{J}_\omega$, while $\mathcal{J}_\omega(b^-) = \chi^b A_{\Gamma_+}$.*

Proof. The right-continuity of the ideal $\mathcal{J}_\omega = \mathcal{J}_\omega(0^+)$ follows from Definition 6.4.1. If $b > 0$ and $a > b$, $a \in \Gamma_+$, then $\mathcal{J}_\omega(b^+) = \chi^a \mathcal{J}_\omega$ is right-continuous, and so is $\mathcal{J}_\omega(b^+)$. This proves (a), while (b) follows directly from Definition 6.4.1. If

$b \in \Gamma_+$, then $\chi^b \mathcal{J}_\omega = \big[\bigcup_{\substack{a \in \Gamma_+ \\ a>0}} \chi^{b+a}\mathcal{J}_\omega\big] = \big[\bigcup_{\substack{a \in \Gamma_+ \\ a>b}} \chi^{a}\mathcal{J}_\omega\big] = \mathcal{J}_\omega(b^+)$. One can show that $\mathcal{J}_\omega(b^-) = \chi^b A_{\Gamma_+}$ by similar arguments. This proves (c). □

Theorem 6.4.3. *Any primary ideal J of the algebra A_{Γ_+} which is contained in $\mathcal{J}_\omega$ is either of type $\mathcal{J}_\omega(b^-)$, or of type $\mathcal{J}_\omega(b^+)$ for some non-negative real number b.*

For the proof we need several results from the theory of analytic functions. We recall that any positive measure μ on $\mathbb{R}$, which is singular with respect to the Lebesgue measure, generates a singular inner function on $\mathbb{C}_+$ by

$$\exp\Big(-i\int_{\mathbb{R}} \frac{tz+1}{z-t}\,d\mu(t)\Big).$$

It is well known (e.g. [G2]), that any non-vanishing function $f \in H^1$ admits a unique inner-outer factorization of type

$$f(z) = Ce^{i\,s(f)\,z}B_f(z)S_f(z)F_f(z), \tag{6.11}$$

on $\mathbb{D}$, where $|C| = 1$, $s(f) \geq 0$, S_f is a singular inner function, F_f is an outer function, and B_f is a Blaschke product in $\mathbb{D}$ whose zeros coincide with the zeros of f. If f is continuous on $\overline{\mathbb{D}}$, then the support of the singular measure μ_f which generates S_f is contained in the set of zeros Null (f) of f on $\mathbb{T}$.

Note that $H^1 \cong H^1(\mathbb{D}) \cong H^1(\mathbb{C}_+) \subset H^1(\mathbb{R})$. As in Section 6.3 we will assume that H^1 is embedded into $H^1(\mathbb{R})$, i.e. we will regard the functions in H^1 as functions in $H^1(\mathbb{R})$ that are periodic with period 1. We will indicate by $H^1_{\mathbb{R}}$ the image of H^1 in $H^1(\mathbb{R})$.

Lemma 6.4.4. *The natural analytic extension $\widetilde{f}$ of any function $f \in AP_{\Gamma_+}(\mathbb{R})$ on $\overline{\mathbb{C}}_+$ admits a unique inner-outer factorization of type*

$$\widetilde{f}(z) = Ce^{i\,s(f)\,z}B_f(z)\,S_f(z)\,F_f(z), \tag{6.12}$$

where $|C| = 1$, $s(f) = \inf\big(\mathrm{sp}\,(f)\big)$, and the functions B_f, S_f, F_f are analogous to those in (6.11)*.*

Proof. Since (6.11) holds for f, and $AP_{\Gamma_+}(\mathbb{R}) \subset H^1_{\mathbb{R}}$, we need to show only that $s(f) = \inf\big(\mathrm{sp}\,(f)\big)$. It is well known that for every analytic almost periodic function h on $\overline{\mathbb{C}}_+$ the function $h(z)/e^{ibz}$ is bounded on the upper half-plane $\mathbb{C}_+$ if and only if $b \leq a_0 = s(h) = \inf\big(\mathrm{sp}\,(h)\big)$. Therefore, $a_0 \geq s(f)$ since all functions in (6.12) are bounded on $\overline{\mathbb{C}}_+$ (cf. [B]). Hence, $g(t) = f(t)/e^{ia_0t}$ belongs to $AP_{\Gamma_+}(\mathbb{R})$, and $0 \leq s(g) = s(f) - a_0$. Thus, $s(f) \geq a_0$, and consequently, $s(f) = a_0$. □

Corollary 6.4.5. *Let π_G be the natural projection $\pi_G\colon \widetilde{G} = K \times \mathbb{R} \longrightarrow G$, where $K = \operatorname{Ker}(\chi^{2\pi})$. For any $f \in A_{\Gamma_+}$ the function $\widetilde{f}(g,t) = (f \circ \pi_G)(g,t)$ can be expressed on $\widetilde{G}$ as*

$$\widetilde{f}(g,t) = e^{iat}\chi^a(g)\, h(g,t), \tag{6.13}$$

where $s(h_g) = 0$, $a = s(f) = \inf\big(\operatorname{sp}(f)\big)$, and $h_g(t) = h(g,t)$ belongs to the space $H^1_{\mathbb{R}}$ for every $g \in K$.

Proof. Clearly, the function $\widetilde{f}_g(t) = \widetilde{f}(g,t)$ belongs to $AP_{\Gamma_+}(\mathbb{R})$ for every $g \in K$, and $\operatorname{sp}(\widetilde{f}_g) = \operatorname{sp}(f)$. Hence, $\operatorname{sp}(\widetilde{f}_g)$ does not depend on $g \in K$. Now (6.13) follows from Lemma 6.4.4. ☐

If B_1 and B_2 are two Blaschke products on $\overline{\mathbb{C}}_+$, then the quotient B_1/B_2 is also a Blaschke product if and only if the set $\operatorname{Null}(B_2)$ of zeros of B_2, counting the multiplicities, is contained in $\operatorname{Null}(B_1)$. Let S_1 and S_2 be two singular inner functions on $\mathbb{C}_+$, and let μ_1 and μ_2 be positive singular measures on $\mathbb{R}$ generating S_1 and S_2, respectively. The function S_1/S_2 is also an inner function on $\mathbb{C}_+$ if and only if $\mu_1 - \mu_2$ is a positive measure on $\mathbb{R}$ (cf. [G2]).

Lemma 6.4.6. *Let F be a family of bounded analytic functions in $\mathbb{C}_+$ which do not vanish simultaneously at any $z \in \overline{\mathbb{C}}_+$. If $u(t) \in L^\infty(\mathbb{R}, dt)$ is a unimodular function, i.e. $\big|u(t)\big| = 1$ almost everywhere on $\mathbb{R}$, such that $fu \in H^1_{\mathbb{R}}$ for all $f \in F$, then there is a real number $a \in \mathbb{R}$ and a function $k(t) \in H^1_{\mathbb{R}}$ with $s(k) = \inf\big(\operatorname{sp}(k)\big) = 0$, such that $u(t) = k(t)\, e^{iat} \in H^1_{\mathbb{R}}$.*

Proof. Let $f \in F$, $f \neq 0$. Since f and fu are in $H^1_{\mathbb{R}}$, they admit factorizations of type (6.11). The function u can be expressed as a meromorphic function on $\mathbb{C}_+$. Namely,

$$u(z) = \frac{(fu)(z)}{f(z)} = e^{iaz}\,\frac{B_{fu}(z)\, S_{fu}(z)}{B_f(z)\, S_f(z)}, \tag{6.14}$$

for every $z \in \mathbb{C}_+$, where $a = s\,(fu) - s\,(f)$. Here we assume that B_{fu}/B_f and S_{fu}/S_f are irreducible fractions. Therefore, B_{fu} and B_f have no common zeros, and also the measures μ_{fu} and μ_f, that generate the singular factors S_{fu} and S_f correspondingly, are mutually singular. We claim that, $B_f \equiv S_f \equiv 1$. Assume, on the contrary, that $B_f(z) \not\equiv 1$, i.e. $B_f(z_0) = 0$ for some $z_0 \in \overline{\mathbb{C}}_+$. Let $g \in F$ be such that $g(z_0) \neq 0$. Since the function B_g/B_f is not a Blaschke product, then $g\,u \notin H^1$, contradicting the properties of F.

Assume now that $S_f \not\equiv 1$ and let $t_0 \in \operatorname{supp}(\mu_f)$. If $g \in F$ is such that $g(t_0) \neq 0$, then t_0 does not belong to the support of the measure μ_g generating the singular factor S_g of g. Therefore, S_g/S_f is not a singular inner function. Consequently, $g\,u \notin H^1_{\mathbb{R}}$, contradicting the hypotheses on F. Hence, $u(z) = e^{iaz}k(z)$, where $k(z) = B_{fu}(z)\, S_{fu}(z)$. This yields $k(t) = e^{-iat}u(t) \in H^1_{\mathbb{R}}$, and $s\big(k(t)\big) = s\big(e^{-iat}u(t)\big) = s\,(B_{fu}S_{fu}) = 0$, as claimed. ☐

Corollary 6.4.7. *In the setting of Lemma* 6.4.6*, for any non-vanishing function* $h \in L^1\Big(\mathbb{R}, \dfrac{dt}{1+t^2}\Big)$ *with* $f\,h \in H^1_{\mathbb{R}}$ *for all* $f \in F$*, there exists a function* $k \in H^1_{\mathbb{R}}$ *with* $s(k) = 0$*, such that* $h(t) = k(t)\,e^{iat}$ *for some* $a \in \mathbb{R}$*.*

Proof. It is known that a positive function $g \in L^1\Big(\mathbb{R}, \dfrac{dt}{1+t^2}\Big)$ coincides with the modulus of a function in $H^1_{\mathbb{R}}$ if and only if

$$\int\limits_{\mathbb{R}} \log g(t)\,\frac{dt}{1+t^2} > -\infty.$$

For every non-zero $f \in F$ we have

$$-\infty < \int\limits_{\mathbb{R}} \log\big|f(t)\,h(t)\big|\,\frac{dt}{1+t^2} = \int\limits_{\mathbb{R}} \log\big|f(t)\big|\,\frac{dt}{1+t^2} + \int\limits_{\mathbb{R}} \log\big|h(t)\big|\,\frac{dt}{1+t^2}.$$

Since the first summand is finite, so is the second one. Hence there exists an outer function $\Phi \in H^1_{\mathbb{R}}$, such that $\big|\Phi(t)\big| = \big|h(t)\big|$ for almost all $t \in \mathbb{R}$. Therefore, $h = \Phi u$ for some unimodular function u. It is easy to see that u satisfies the hypotheses of Lemma 6.4.6 with the family $\Phi F \subset A_b(\mathbb{C}_+)$. □

As a direct consequence from Corollary 6.4.7 we see that $e^{-iat}h(t) \in H^1_{\mathbb{R}}$, and $s\left(e^{-iat}h(t)\right) = 0$. For any subset $F \subset L^1\Big(\mathbb{R}, \dfrac{dt}{1+t^2}\Big)$ define the number

$$s(F) = \inf\big\{s(f)\colon f \in F,\ f \neq 0\big\}.$$

Corollary 6.4.8. *The number a from Corollary* 6.4.7 *is not smaller than* $s(F)$*, i.e.* $a \geq -s(F)$*.*

Proof. Since $s(k) = 0$ and $f(t)\,k(t)\,e^{iat} \in H^1_{\mathbb{R}}$ for any $f \in B$, we have that $s(f) + a \geq 0$. Hence $a \geq -s(f)$ for any $f \in F$, $f \neq 0$. □

Corollary 6.4.9. *In the setting of Corollary* 6.4.8 *the function h belongs to* $H^1_{\mathbb{R}}$ *if* $s(F) = 0$*.*

Proof of Theorem 6.4.3. Let J be a primary ideal of A_{Γ_+} that is contained in $\mathcal{J}_\omega$. Let $f \in J$, $f \neq 0$, and $\mu \in J^\perp$. Since the measure $f\,d\mu$ belongs to $\mathcal{J}_\omega^\perp$, Theorem 6.3.7 implies that $(f\,d\mu)^\sim \subset M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. Therefore, $(f\,d\mu)^\sim$ is of type

$$(f\,d\mu)^\sim(g,t) = f^1(g,t)\big(d\nu(g) \times dt\big), \tag{6.15}$$

where f^1 is $\mathrm{Ker}(\pi_G)$-invariant, and the functions $f^1_{(g)} = f^1(g,t)$ belong to $H^1_{\mathbb{R}}$ for ν-almost all $g \in K$. Since $(f\,d\mu)^{\sim} = \widetilde{f}\,d\widetilde{\mu}$, where $\widetilde{f} = f \circ \pi_G$, we have $d\widetilde{\mu}(g,t) = h(g,t)\big(d\nu(g) \times dt\big)$, where $h(g,t) = f^1(g,t)/\widetilde{f}(g,t)$. Let $h_{(g)}(t) = h(g,t)$, and $\widetilde{f}_{(g)}(t) = \widetilde{f}(g,t)$. By (6.15), for all $f \in J$ and for ν-almost every g we have

$$\widetilde{f}_{(g)} h_{(g)} = f^1_{(g)} \in H^1_{\mathbb{R}}. \tag{6.16}$$

Since $\widetilde{\pi}_G(K \times \overline{\mathbb{C}}_+) = \overline{\mathbb{D}}^*_G$, and $J \subset \mathcal{J}_\omega$, then for every $g \in K$ the restriction of $\widetilde{J} = \{\widehat{f} \circ \pi_G : f \in J\}$ on the set $\{g\} \times \overline{\mathbb{C}}_+$ satisfies the hypotheses of Lemma 6.4.6. By applying Corollary 6.4.7 to (6.16) we obtain $h_{(g)}(t) = k_{(g)}(t)\, e^{ia_g t}$, with $k_{(g)}(t) \in H^1_{\mathbb{R}}$, and $s\big(k_{(g)}(t)\big) = 0$ for almost every $g \in K$. Corollary 6.4.8 implies that $-s\big(\widetilde{f}_{(g)}\big) \le a_g$, for almost every g. Since $s\big(\widetilde{f}_{(g)}\big)$ equals $\inf\big(\mathrm{sp}\,(f)\big)$, which is independent of $g \in K$, $f \in J$, we have that $s(J) = \inf\big\{\mathrm{sp}\,(f)\colon f \in J,\ f \neq 0\big\} \ge -a_g$ for ν-almost every g. Therefore, $d\widetilde{\mu}(g,t) = k(g,t)\, e^{ia_g t}\big(d\nu(g) \times dt\big)$, where $k(g,t) = k_g(t) \in H^1_{\mathbb{R}}$, and $s(J) \ge -a_g$ for ν-almost all $g \in K$. Hence if $a > s(J) \ge -a_g$, then $a + a_g > 0$, which implies that the measure $d(\widetilde{\mu}^a)(g,t) = \chi^a(g)\, e^{iat}\, d\widetilde{\mu}(g,t)$ is in $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$, i.e. the measure $\chi^a\, d\mu$ belongs to $\mathcal{J}^\perp_\omega$. Let $b > s(J)$. Since $\varGamma = \widehat{G}$ is dense in $\mathbb{R}$, there exist numbers $a, c \in \varGamma_+ \setminus \{0\}$ with $b = a + c$, and $a > s(J)$. Because of $\chi^a dm \in \mathcal{J}^\perp_\omega$, we have $0 = \displaystyle\int_G \chi^c \chi^a\, d\mu = \int_G \chi^b\, d\mu$. Since μ is an arbitrary measure in $J^\perp$, we deduce that $\chi^b \in J$, and therefore, $\mathcal{J}_\omega\big(s(J)^+\big) \subset J$. There are two possible cases.

Case 1. $s(J) \notin \mathrm{sp}\,(f)$ for any $f \in J$. Let $f \in J$ and $\varepsilon > 0$. By Theorem 3.3.2 there is a $\varGamma_+$-polynomial $g = \displaystyle\sum_{i=1}^n c_i \chi^{a_i} \in A_{\varGamma_+}$ with $\mathrm{sp}\,(g) \subset \mathrm{sp}\,(f)$, so that $\|f - g\|_G < \varepsilon$. Since $a_i \in \mathrm{sp}\,(f)$, $1 \le i \le n$, we see that $a_i > s(J)$. Therefore, χ^{a_i} belongs to $\mathcal{J}_\omega\big(s(J)^+\big)$, and so does g. Since $\mathcal{J}_\omega\big(s(J)^+\big)$ is closed and $\varepsilon > 0$ is arbitrary, we see that $f \in \mathcal{J}_\omega\big(s(J)^+\big)$. Therefore, $J = \mathcal{J}_\omega\big(s(J)^+\big)$.

Case 2. $s(J) \in \mathrm{sp}\,(f)$ for some $f \in J$. We claim that in this case $J = \chi^{s(J)} A_{\varGamma_+}$. Since $s(J) \in \mathrm{sp}\,(f) = \inf\big(\mathrm{sp}\,(f)\big)$, by the Besicovitch theorem (e.g. [B]) we have that $f = \chi^{s(J)} h$, where $h \in A_{\varGamma_+}$, $\widehat{h}(\omega) \neq 0$. Consider the ideal $J_1 = hA_{\varGamma_+} + J$ of $A_{\varGamma_+}$. Since $J \subset \mathcal{J}_\omega$, and $\widehat{h}(\omega) \neq 0$, J_1 coincides with the algebra $A_{\varGamma_+}$. Hence there is a $k \in A_{\varGamma_+}$ and $\varphi \in J$ so that $hk + \varphi \equiv 1$. By multiplying with $\chi^{s(J)}$, we get $fk + \chi^{s(J)}\varphi = \chi^{s(J)}$. Since f and φ belong to J, it follows that $\chi^{s(J)} \in J$, and therefore, $\chi^{s(J)} A_{\varGamma_+} \subset J$. For every $g \in J$ we have $\inf\big(\mathrm{sp}\,(g)\big) \ge s(J)$, and therefore, $g/\chi^{s(J)} \in A_{\varGamma_+}$. Hence $g \in \chi^{s(J)} A_{\varGamma_+}$, i.e. $J \subset \chi^{s(J)} A_{\varGamma_+}$. Thus, $J = \chi^{s(J)} A_{\varGamma_+} = \mathcal{J}_\omega\big(s(J)^+\big)$. □

As a consequence we obtain the following complete description of primary ideals inside $\mathcal{J}_\omega$.

Corollary 6.4.10. *If J is a primary ideal of the algebra A_{Γ_+} that is contained in $\mathcal{J}_\omega$, then either $J = \chi^{s(J)}\mathcal{J}_\omega$, or, $J = \chi^{s(J)}A_{\Gamma_+}$.*

The proof follows immediately from Lemma 6.4.2(c) and Theorem 6.4.3.

Next we describe primary ideals of a G-disc algebra A_{Γ_+} which are contained in maximal ideals other than $\mathcal{J}_\omega$, where ω is the origin of the G-disc $\mathbb{D}_G$.

Let J be a primary ideal of A_{Γ_+}, which is contained in a maximal ideal $\mathcal{J}_{r\diamond g}$, $r>0$. Denote by $\widehat{J} = \{\widehat{f}\colon f \in J\}$ the Gelfand transform of J on $\overline{\mathbb{D}}_G$. Without loss of generality we can assume that $r \diamond g = j_\imath(z_0)$ for some $z_0 \in \mathbb{C}_+ = \overline{\mathbb{C}}_+ \setminus \mathbb{R}$. Denote $J(\overline{\mathbb{C}}_+) = \widehat{J}|_{\widetilde{j}_\imath(\overline{\mathbb{C}}_+)}$ and $J(\mathbb{R}) = \widehat{J}|_{j_\imath(\mathbb{R})} = J(\overline{\mathbb{C}}_+)|_{j_\imath(\mathbb{R})}$. The set $J(\overline{\mathbb{C}}_+)$ is a subalgebra of $A_b(\mathbb{C}_+)$, the algebra of bounded analytic functions in $\mathbb{C}_+$. Let $\mathrm{ord}_{z_0} f$ be the multiplicity of the zero of $\widehat{f}\circ j_\imath$, $f \in J$ at z_0. Set $\mathrm{ord}_{z_0} J = \inf_{f\in J}\big(\mathrm{ord}_{z_0} f\big)$. Denote by $J(\mathbb{R})^\perp = J^\perp \cap M\big(j_\imath(\mathbb{R})\big)$ the space of measures on $j_\imath(\mathbb{R}) \subset G$ which are orthogonal to $J(\mathbb{R})$. Recall that $H_0^1(\mathbb{R}) = w \cdot H^1(\mathbb{R}) = w\cdot (H^1 \circ w) = H_0^1 \circ w$, where $w(t) = \dfrac{i-t}{i+t} : \mathbb{R} \to \mathbb{T}$. Denote by u the unimodular function $u(z) = \dfrac{z-z_0}{z-\overline{z}_0}$.

Lemma 6.4.11. *If J is a primary ideal of A_{Γ_+} contained in some maximal ideal $\mathcal{J}_{r\diamond g}$, $r>0$, then*

$$J(\mathbb{R})^\perp = \overline{u}^n \cdot H_0^1(\mathbb{R}) \cdot \frac{dt}{1+t^2},$$

where $u(t) = \dfrac{t-z_0}{t-\overline{z}_0}$, and $n = \mathrm{ord}_{z_0} J$.

Proof. For any $f \in J$ we have $\widehat{f}\circ j_\imath(z) = u^n(z) f_0(z)$, where $z \in \overline{\mathbb{C}}_+$ and $f_0 \in A_b(\mathbb{C}_+)$. Hence the set $J_n(\overline{\mathbb{C}}_+) = \overline{u}^n \cdot J(\overline{\mathbb{C}}_+)$ is a subspace of $A_b(\mathbb{C}_+)$. The definition of n implies that for every $z \in \overline{\mathbb{C}}_+$ there is function $f \in J_n(\overline{\mathbb{C}}_+)$ which does not vanish at $j_\imath(z)$. Therefore, we can apply Lemma 6.4.6 to $J_n(\overline{\mathbb{C}}_+)$. Since $J \subset \mathcal{J}_{r\diamond g}$, $r>0$, we have $s(J) = 0$, thus $s\left(J_n(\overline{\mathbb{C}}_+)\right) = s(J) = 0$. Applying Lemma 6.4.6, Corollary 6.4.7 and Corollary 6.4.8 to $J_n(\mathbb{R}) = J_n(\overline{\mathbb{C}}_+)\big|_{j_\imath(\mathbb{R})}$, we obtain $J_n(\mathbb{R})^\perp = H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$. Since $J(\mathbb{R})^\perp = \overline{u}^n \cdot J_n(\mathbb{R})^\perp$, we conclude that $J(\mathbb{R})^\perp = \overline{u}^n \cdot H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$, as claimed. □

Note that the restriction of the covering $\widetilde{\pi}_G\colon K \times \overline{\mathbb{C}}_+ \longrightarrow \overline{\mathbb{D}}_G^*$ on $\{g\}\times\overline{\mathbb{C}}_+$, $g \in K$, generates an embedding of the upper half-plane $\overline{\mathbb{C}}_+$ into $\overline{\mathbb{D}}_G$.

Lemma 6.4.12. *Let J be a primary ideal of A_{Γ_+} contained in some $\mathcal{J}_{r\diamond g}$, $r>0$. If μ is a measure on G which is orthogonal to J and for which $\mu|_{j(\mathbb{R})} = 0$, then $\mu \in A_{\Gamma_+}^\perp$.*

Proof. The set $\widetilde{\mathbb{R}} = \pi_G^{-1}\big(j_\imath(\mathbb{R})\big) = \big\{(g,t) \in \widetilde{G} \colon \pi_G(g,t) \in j_\imath(\mathbb{R})\big\}$ can be expressed as $\widetilde{\mathbb{R}} = \bigcup_{n=-\infty}^{\infty} \{g_n\} \times \mathbb{R}$. Since $\mu\big|_{j_\imath(\mathbb{R})} = 0$, for the lifting $\widetilde{\mu}$ of μ on $K \times \overline{\mathbb{C}}_+$ (cf. Section 6.3), we have $\widetilde{\mu}\big|_{\widetilde{\mathbb{R}}} = 0$. Since for any $f \in J$ the measure $f\,d\mu$ belongs to $A_{\Gamma_+}^{\perp}$, it follows that $\widetilde{f}\,d\widetilde{\mu} = f^1(g,t)\,\big(d\nu(g) \times dt\big)$ belongs to the space $M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$ introduced in Section 6.3, and hence $\widetilde{\mu} = h(g,t)\big(d\nu(g) \times dt\big)$ with $h = f^1/\widetilde{f}$, where $\widetilde{f} = \widehat{f} \circ \widetilde{\pi}_G$. We claim that $\widetilde{\mu} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. By the assumptions we have that $\nu\big(\{g_n\}_{n=-\infty}^{\infty}\big) = 0$. Hence, for ν-almost all $g \in K$ the space $\widehat{J} \circ \widetilde{\pi}_G\big|_{\{g\}\times\overline{\mathbb{C}}_+}$ does not vanish at any point of $\{g\} \times \overline{\mathbb{C}}_+$, and $s\left(\widehat{J} \circ \widetilde{\pi}_G\big|_{\{g\}\times\overline{\mathbb{C}}_+}\right) = s(J) = 0$. As in the proof of Theorem 6.4.3 we see that $h_{(g)}(t) = h(g,t) \in H^1_{\mathbb{R}}$ for ν-almost all $g \in K$, i.e. $\widetilde{\mu} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$, as desired. Therefore, $\mu \in \mathcal{J}_\omega^{\perp}$ by Theorem 6.3.7. In fact, $\mu \in A_{\Gamma_+}^{\perp} \subset \mathcal{J}_\omega^{\perp}$. Indeed, since $\mathcal{J}_\omega^{\perp} = A_{\Gamma_+}^{\perp} + \mathbb{C}\sigma$, we see that $\mu = \mu_1 + c\sigma$ for some measure $\mu_1 \in A_{\Gamma_+}^{\perp}$ and $c \in \mathbb{C}$. Let $\widehat{f}(\omega) \neq 0$ for some $f \in J$. Then $0 = \int_G f\,d\mu = \int_G f\,d\mu_1 + c\widehat{f}(\omega) = c\widehat{f}(\omega)$, since $\int_G f\,d\mu_1 = 0$. Hence $c = 0$, and consequently, $\mu = \mu_1 \in A_{\Gamma_+}^{\perp}$. ☐

Lemma 6.4.13. *If J is a primary ideal of A_{Γ_+}, such that $J \subset \mathcal{J}_{r\diamond g}$ for some $r > 0$, then $J^{\perp} = A_{\Gamma_+}^{\perp} + J(\mathbb{R})^{\perp}$.*

Proof. Note that since $J(\mathbb{R}) = \widehat{J}\big|_{j_\imath(\mathbb{R})}$, we have $J(\mathbb{R})^{\perp} \subset J^{\perp} \cap M\big(j_\imath(\mathbb{R})\big)$. Clearly, $A_{\Gamma_+}^{\perp} \subset J^{\perp}$, wherefrom $A_{\Gamma_+}^{\perp} + J(\mathbb{R})^{\perp} \subset J^{\perp}$.

Conversely, let $\mu \in J^{\perp}$, and let $\mu' = \mu\big|_{j_\imath(\mathbb{R})}$. For any $f \in J$ we have $f\,d\mu \in A_{\Gamma_+}^{\perp}$, and therefore, $\widetilde{f}\,d\widetilde{\mu} \in M^1_{\mathrm{Ker}(\pi_G)}(\widetilde{G})$. Hence $d\widetilde{\mu} = h(g,t)\big(d\nu(g) \times dt\big)$, and $\widetilde{f}_{(g)}(t)\,f^1_{(g)}(t) = \widetilde{f}(g,t)\,h(g,t) \in H^1_{\mathbb{R}}$ for ν-almost all $g \in K$ (cf. (6.16)). The support of $\widetilde{\mu}'$ is inside $\widetilde{\mathbb{R}} = \bigcup_{n=-\infty}^{\infty} \{g_n\} \times \mathbb{R}$, and

$$d\widetilde{\mu}' = \sum_{n=-\infty}^{\infty} h(g_n,t)\big(d\nu(g_n) \times dt\big).$$

For the measure $\gamma = \mu - \mu'$, we have $\gamma \in J^{\perp}$ and $\gamma\big|_{j_\imath(\mathbb{R})} = 0$. Therefore, $\gamma \in A_{\Gamma_+}^{\perp}$ by Lemma 6.4.12. Hence, $\mu = \gamma + \mu'$, where $\gamma \in A_{\Gamma_+}^{\perp}$. Note that if $f\,d\mu \in A_{\Gamma_+}^{\perp}$, then $f\,d\mu' \in \big(A_{\Gamma_+}|_{j_\imath(\mathbb{R})}\big)^{\perp} = H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ by Corollary 6.3.2. It follows that $\int_{j_\imath(\mathbb{R})} \widehat{f} \circ j\,d\mu' = 0$ for any $f \in J$, and hence $\mu' \in J(\mathbb{R})^{\perp}$. Consequently, $\mu = \gamma + \mu' = \gamma + \mu|_{j(\mathbb{R})} \in A_{\Gamma_+}^{\perp} + J(\mathbb{R})^{\perp}$, as claimed. ☐

While the sum in Lemma 6.4.13 might not be direct, the space $J(\mathbb{R})^\perp$ has a direct complement in $A_{\Gamma_+}^\perp$. Indeed, if $\mu \in A_{\Gamma_+}^\perp$, then $\mu' = \mu\big|_{j_\imath(\mathbb{R})} \in H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2} \subset A_{\Gamma_+}^\perp$. Hence $\mu = \mu_* + \mu|_{j_\imath(\mathbb{R})}$, where $\mu_*|_{j_\imath(\mathbb{R})} = 0$. Therefore,

$$A_{\Gamma_+}^\perp = (A_{\Gamma_+}^\perp)_* \oplus H_0^1(\mathbb{R}) \cdot \frac{dt}{1+t^2}$$

where $(A_{\Gamma_+}^\perp)_* = \{\mu_* : \mu \in A_{\Gamma_+}^\perp\}$. Since $J(\mathbb{R})^\perp \supset H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$, we see that $J^\perp = A_{\Gamma_+}^\perp + J(\mathbb{R})^\perp = (A_{\Gamma_+}^\perp)_* \oplus J(\mathbb{R})^\perp$.

Theorem 6.4.14. *If J is a primary ideal of A_{Γ_+}, which is contained in some maximal ideal $\mathcal{J}_{r\diamond g}$, $r > 0$, then $J^\perp = \overline{u}^n \cdot A_{\Gamma_+}^\perp$, where $u(g)$ is the unimodular Borel function on G defined by*

$$u(g) = \begin{cases} u(t) = \dfrac{t - z_0}{t - \overline{z}_0} & \text{when } g = g_t \in j_\imath(\mathbb{R}), \\ 1 & \text{when } g \notin j_\imath(\mathbb{R}). \end{cases}$$

Proof. Indeed, Lemma 6.4.11 implies that $J^\perp = (A_{\Gamma_+}^\perp)_* + \overline{u}^n \cdot H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ $= \overline{u}^n\Big(u^n \left(A_{\Gamma_+}^\perp\right)_* \oplus H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}\Big) = \overline{u}^n A_{\Gamma_+}^\perp$, since $u^n \cdot (A_{\Gamma_+}^\perp)_* = (A_{\Gamma_+}^\perp)_*$. □

The next theorem shows that every primary ideal J in A_{Γ_+} which is contained in some $\mathcal{J}_{r\diamond g}$, $r > 0$, is uniquely determined by a natural parameter $n \in \mathbb{N}$.

Theorem 6.4.15. *If J is a primary ideal of A_{Γ_+} contained in some maximal ideal $\mathcal{J}_{r\diamond g}$, $r > 0$, then $J = I_n = \{f \in A_{\Gamma_+} :\ \mathrm{ord}_{z_0} f \geq n\}$, where $n = \mathrm{ord}_{z_0} J$.*

Proof. The inclusion $J \subset I_n$ is clear. We claim that $I_n \subset J$, or, equivalently, that $I_n^\perp \supset J^\perp$. If $f \in I_n$, then $f\big(j_\imath(t)\big) = u^n(t) f_0(t)$, where $f_0(t) \in A_b(\mathbb{R})$. Hence $\overline{u}^n \cdot H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2} \subset I_n^\perp$, and thus $J_{j_\imath(\mathbb{R})}^\perp \subset I_n^\perp$. Clearly, $A_{\Gamma_+}^\perp \subset I_n^\perp$. Therefore, $I_n^\perp \supset J_{j_\imath(\mathbb{R})}^\perp + A_{\Gamma_+}^\perp = J^\perp$. Consequently, $I_n \subset J$. □

Corollary 6.4.16. *Every primary ideal J in $\mathcal{J}_{r\diamond g}$, $r > 0$, has a finite codimension in A_{Γ_+}.*

Proof. By Theorem 6.4.15, $J = I_n$ with $n = \mathrm{ord}_{z_0} J$. Let $f \in A_{\Gamma_+}$ be a function with $\mathrm{ord}_{z_0} f = n - 1$. The ideal $I_{n-1} = \{f \in A_{\Gamma_+} :\ \mathrm{ord}_{z_0} f \geq n-1\}$ is isomorphic to $\mathbb{C}f + J$. Similarly, $I_{n-2} = \mathbb{C}g + \mathbb{C}f + J$, where $\mathrm{ord}_{z_0} g = n - 2$. Proceeding inductively, we obtain that $A_{\Gamma_+} = \mathbb{C}f_0 + \mathbb{C}f_1 + \cdots + \mathbb{C}f_{n-1} + J$, where $\mathrm{ord}_{z_0} f_k = k$. □

It remains to describe primary ideals $J \subset A_{\Gamma_+}$ that are contained in maximal ideals of type $\mathcal{J}_g = \mathcal{J}_{1\diamond g}$, where g is in the Shilov boundary $\partial A_{\Gamma_+} = G$. Without loss of generality we can assume that $g = \imath = j_\imath(0) \in j_\imath(\mathbb{R})$. Let $S_f(t)$ be the singular component of a function $f \in H^1_{\mathbb{R}}$, and let μ_f be a positive singular measure on $\mathbb{R}$ that generates $S_f(t)$. The measure μ_f can be expressed uniquely as the sum of two singular measures, namely, $\mu_f = \mu'_f + c_f\delta_0$, where δ_0 is the Dirac measure at $j_\imath(0) = \imath \in j_\imath(\mathbb{R})$. Hence, $S_f(t) = S_f^+(t)\, S_f^-(t)$, where $S_f^+(t)$ and $S_f^-(t)$ are singular inner functions generated by the measures μ'_f and $c_f\delta_0$ correspondingly. Observe that $S_f^-(t) = e^{-ic_f/t}$.

Let J be a primary ideal of the algebra A_{Γ_+}, such that $J \subset \mathcal{J}_\imath = \mathcal{J}_{j_\imath(0)}$, and let h be a function in $h \in L^1\left(\mathbb{R}, \dfrac{dt}{1+t^2}\right) \setminus H^1(\mathbb{R})$ such that $fh \in H^1_{\mathbb{R}}$ for all $f \in J(\mathbb{R})$. Using the same arguments as in Lemma 6.4.6, Corollary 6.4.7 and 6.4.8, one can show that

$$h(t) = e^{ic_h/t} k(t), \tag{6.17}$$

where $c_h > 0$, $k(t) \in H^1_{\mathbb{R}}$, $S_k^-(t) \equiv 1$, and $c_h \leq \inf_{f \in J} c_f$.

For any $c \geq 0$ let $I_c = e^{ic/t} \cdot H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$. Since $e^{-ic'/t} \cdot H^1_0(\mathbb{R}) \subset H^1_0(\mathbb{R})$ for all $c' \geq 0$, we have $I_c \supset e^{-ic'/t} \cdot I_c = I_{c-c'}$ for any $c' : 0 \leq c' \leq c$. Therefore, $I'_c \subset I_c$ when $0 \leq c' \leq c$.

Theorem 6.4.17. *For any primary ideal J of A_{Γ_+} contained in $\mathcal{J}_\imath = \mathcal{J}_{j_\imath(0)}$ there exists a non-negative number $c \geq 0$ such that $J^\perp = A^\perp_{\Gamma_+} + \mathbb{C}\delta_0 + I_c$.*

Proof. We perform the proof in four steps.

Step 1. Let $\mu \in J(\mathbb{R})^\perp$ be a measure on $\mathbb{R}$, which is singular with respect to the Lebesgue measure. We claim that $\mu = c\delta_0$ for a complex number $c \in \mathbb{C}$. Indeed, suppose, on the contrary, that $\mathrm{sp}\,(\mu)$ contains a point $t_0 \in \mathbb{R},\ t_0 \neq 0$, and consider a function $f \in J$ with $f(t_0) \neq 0$. The non-zero measure $f\,d\mu$ belongs to $(A_{\Gamma_+})^\perp_{j(\mathbb{R})}$, and is singular with respect to Lebesgue measure. However, this is impossible, since $f\,d\mu \in H^1_0(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ by Proposition 6.3.1. Consequently, $\mathrm{sp}(\mu) = \{0\}$, and therefore $\mu \in \mathbb{C}\delta_0$.

Step 2. Let $\mu \in J(\mathbb{R})^\perp$ be a measure that is absolutely continuous with respect to $\dfrac{dt}{1+t^2}$, i.e. $d\mu = h(t) \cdot \dfrac{dt}{1+t^2}$, where $h(t) \in L^1\left(\mathbb{R}, \dfrac{dt}{1+t^2}\right)$. Since

$$f\,d\mu \in H^1_0(\mathbb{R}) \cdot \frac{dt}{1+t^2}$$

for every $f \in J$, it follows that $fh \in H^1_0(\mathbb{R})$. By (6.17) we have

$$h(t) = e^{ic/t} k(t), \tag{6.18}$$

where $c \le s(J) = \inf\{c_f : f \in J(\mathbb{R})\}$, and $k \in H^1(\mathbb{R})$. We claim that $k \in H_0^1(\mathbb{R})$. Indeed, for every $f \in J_{j_\imath(\mathbb{R})}$ we have

$$0 = \int_{\mathbb{R}} f(t)\, h(t)\, \frac{dt}{1+t^2} = \widehat{f}(i)\, e^{c}\, k(i).$$

Since we can choose an $f \in J$ with $\widehat{f}(i) \neq 0$, we deduce that $k(i) = 0$, i.e. $k \in H_0^1(\mathbb{R})$. Thus, $\mu \in I_c$, and consequently, $J(\mathbb{R})^\perp \subset I_c$.

Step 3. Let μ be an arbitrary measure in $J^\perp$, and let $\mu' = \mu\big|_{j_\imath(\mathbb{R})}$. Since $f\,d\mu \in A_{\Gamma_+}^\perp$ for any $f \in J$, then $f\,d\mu' \in A_{\Gamma_+}^\perp\big|_{j_\imath(\mathbb{R})} = H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$ by Corollary 6.3.2. Therefore, $\displaystyle\int_{j_\imath(\mathbb{R})} f\,d\mu' = 0$ for any $f \in J$, and hence $\mu' \in J(\mathbb{R})^\perp$. Steps 1 and 2 imply that $d\widetilde{\mu}' = c\delta_0 + e^{ic/t}k(t) \cdot \dfrac{dt}{1+t^2}$, where $k \in H_0^1(\mathbb{R})$, and $c \le s(J) = c_0$. The argument from the proof of Lemma 6.4.12 shows that the measure $\gamma = \mu - \mu'$ belongs to $A_{\Gamma_+}^\perp$. Therefore, $\mu = \gamma + \mu' \in A_{\Gamma_+}^\perp + \mathbb{C}\delta_0 + I_c$. Since $I_c \subset I_{c_0}$, we see that $J^\perp \subset A_{\Gamma_+}^\perp + \mathbb{C}\delta_0 + I_{c_0}$.

Step 4. We claim that $I_{c_0} \subset J^\perp$. Let $\mu \in I_{c_0}$. Any function $f \in J$ can be expressed on $\mathbb{R}$ as $f(t) = e^{-ic_0/t}k(t)$, where $k \in H_0^1(\mathbb{R})$. Similarly,

$$d\mu = e^{ic_0/t}h(t) \cdot \frac{dt}{1+t^2},$$

where $h \in H_0^1(\mathbb{R})$, and hence $f\,d\mu \in H_0^1(\mathbb{R}) \cdot \dfrac{dt}{1+t^2}$. This implies that $I_{c_0} \subset J^\perp$. Therefore, $J^\perp = A_{\Gamma_+}^\perp + \mathbb{C}\delta_0 + I_{c_0}$, as claimed. □

6.5 Notes

Generalized analytic functions, as well as G-disc algebras, were introduced by Arens and Singer in [AS1]. They have been intensively studied afterwards by Hoffman, Helson, Lowdenslager, Kaufman, de Leeuw, Glicksberg, Gamelin, Muhly, Curto, Xia, Asmar, Montgomery-Smith, Yale, Grigoryan, Tonev and others. Systematic expositions on G-disc algebras are given, for instance, in [G1, T2], where one can find also a complete bibliography on the matter.

The results on Bourgain algebras of G-disc algebras are from [TY, TY1]. Most of the results on orthogonal measures to G-disc algebras and primary ideals of G-disc algebras are from [G15].

Chapter 7

Harmonicity on groups and G-discs

In this chapter we extend the notions of harmonic and H^p-functions for compact groups G and on corresponding G-discs. We explore also their boundary behavior, and prove corresponding Fatou type theorems. The results hold for general shift-invariant algebras A_S, and consequently to the particular cases of of H_S^∞-algebras and algebras AP_S of almost periodic functions, considered earlier.

7.1 Harmonic functions on groups and G-discs

The basic idea in extending the notion of harmonicity on groups and G discs is to preserve the main properties of harmonic functions on the unit disc $\mathbb{D}$, presented in Chapter 2.

Let G be a compact abelian group whose dual group $\Gamma = \widehat{G}$ is a subgroup of $\mathbb{R}$. We call the set $\overline{\mathbb{D}}_G(r) = \mathbb{D}_g^{[0,r]} = \left\{\varrho \diamond g \in \mathbb{D}_G \colon \varrho \le r\right\}$ a *G-disc with radius* $r < 1$.

We will denote by $m_{r\diamond g}$ the representing measure on G of the point $r \diamond g \in \mathbb{D}_G$ for the G-disc algebra $A(\mathbb{D}_G)$. The measure $m_{r\diamond \imath}$ we will denote also by m_r. Note that in the classical setting $dm_r(\theta) = P_r(\theta)\, d\theta$, where

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

is the Poisson kernel on the unit disc $\mathbb{D}$. A real-valued harmonic function u on $\mathbb{D}$ can be defined in three standard ways. Firstly, u is harmonic if it is a solution of the Laplace equation, i.e. u is the real part of a holomorphic function on $\mathbb{D}$.

Secondly, u is harmonic, if the Fourier coefficients $c_n^{u_r}$ of its r-traces u_r do not depend on r (cf. Theorem 2.4.1). And thirdly, u is harmonic if the equations

$$u_{r_1}(e^{it}) = u(r_1 e^{it}) = \frac{1}{2\pi}\int_0^{2\pi} u_{r_2}\big(e^{i(t-\theta)}\big)\, P_r(\theta)\, d\theta = u_{r_2}\big(e^{i(t-\theta)}\big) \star m_r(e^{i\theta}), \quad (7.1)$$

hold for every $r_1, r_2,\ 0 \le r_1 < r_2 < 1$, where $r = r_1/r_2$ (cf. Corollary 2.4.2). We will follow the second and third way to generalize the notion of harmonicity on G-discs $\mathbb{D}_G$ and groups. For approaches based on the first definition of harmonicity see, e.g. ([ABG]).

For a $f \in C(\mathbb{D}_G)$ denote by f_r the r-trace of f on $r \diamond G$, namely $f_r(g) = f(r \diamond g)$.

Definition 7.1.1. A set $\big\{f^{(r)}\big\}_{r\in[0,1)}$ of functions on G is said to be a *harmonic family of functions* if

$$f^{(r_1)}(g) = \big(f^{(r_2)} \star m_r\big)(g) = \int_G f^{(r_2)}(gh^{-1})\, dm_r(h)$$

for any $0 \le r_1 < r_2 < 1$, where $r = r_1/r_2$.

Therefore, if $\big\{f^{(r)}\big\}_{r\in[0,1)}$ is a holomorphic family of functions, then

$$f^{(r_1)}(g) = \int_G f^{(r_2)}(gh)\, dm_r(h) = \int_G f^{(r_2)}(gh^{-1})\, dm_r(h) = \big(f^{(r_2)} \star m_r\big)(g)$$

for any $0 \le r_1 < r_2 < 1$, where $r = r_1/r_2$.

Definition 7.1.2. A function $f \in C(\mathbb{D}_G)$ is said to be harmonic on $\mathbb{D}_G$, if the family $\{f_r\}_{r\in[0,1)}$ of its r-traces is a harmonic family of functions, i.e. if

$$f_{r_1}(g) = (f_{r_2} \star m_r)(g) = \int_G f_{r_2}(gh)\, dm_r(h) \quad (7.2)$$

for any $0 \le r_1 < r_2 < 1$, where $r = r_1/r_2$.

In other words, f is harmonic on $\mathbb{D}_G$ if and only if $f\big|_{r_1\diamond G} = (f\big|_{r_2\diamond G}) \star m_{r_1/r_2}$, whenever $r_1, r_2,\ 0 \le r_1 < r_2 < 1$. We denote by $\mathcal{H}(\mathbb{D}_G)$ the space of harmonic functions on $\mathbb{D}_G$. It is called a *Stepanov space*. Clearly, $\mathcal{H}(\mathbb{D}_G) = \mathcal{H}_{\mathbb{R}}(\mathbb{D}_G) + i\mathcal{H}_{\mathbb{R}}(\mathbb{D}_G)$, where $\mathcal{H}_{\mathbb{R}}(\mathbb{D}_G)$ is the space of real-valued harmonic functions on $\mathbb{D}_G$. Observe that every function $f \in \mathcal{O}(\mathbb{D}_G)$ satisfies the condition $f_{r_1} = f_{r_2} \star m_{r_1/r_2}(g), 0 < r_1 < r_2 < 1$. Therefore, $\mathcal{O}(\mathbb{D}_G) \subset \mathcal{H}(\mathbb{D}_G) = \mathcal{H}_{\mathbb{R}}(\mathbb{D}_G) + i\mathcal{H}_{\mathbb{R}}(\mathbb{D}_G)$, and also $\mathcal{O}(\mathbb{D}_G) + \overline{\mathcal{O}}(\mathbb{D}_G) \subset \mathcal{H}(\mathbb{D}_G)$. Consequently, $\operatorname{Re} f \in \mathcal{H}_{\mathbb{R}}(\mathbb{D}_G)$ for every $f \in \mathcal{O}(\mathbb{D}_G)$. If $u \in \mathcal{H}(\mathbb{D})$, then the function $u \circ \chi^a$ is harmonic on $\mathbb{D}_G$ for every $a \in \Gamma$. Indeed, if

$u = \operatorname{Re} f$ for some $f \in \mathcal{O}(\mathbb{D})$, then $f \circ \chi^a \in \mathcal{O}(\mathbb{D}_G)$. Hence, $u \circ \chi^a = (\operatorname{Re} f) \circ \chi^a = \operatorname{Re}(f \circ \chi^a) \in \mathcal{H}_{\mathbb{R}}(\mathbb{D}_G)$. The above remark shows that the space $\mathcal{H}_{\mathbb{R}}(\mathbb{D}_{\mathbb{T}})$ of real-valued harmonic functions on $G = \mathbb{T}$ coincides with the space of classical harmonic functions $\mathcal{H}_{\mathbb{R}}(\mathbb{D})$ on the unit disc $\mathbb{D}$. However, contrary to the classical situation, not every real-valued harmonic function on $\mathbb{D}_G$ is the real part of a function in $\mathcal{O}(\mathbb{D}_G)$. To see this, consider the operator $T\colon \mathcal{O}(\mathbb{D}_G) \longrightarrow \mathcal{H}(\mathbb{D}_G): \ f \longrightarrow \operatorname{Re} f$. Note that T is linear over $\mathbb{R}$. If $f \in \mathcal{O}(\mathbb{D}_G)$ and $f \sim \sum_{a\in S} c_a^f \chi^a$ is its Fourier expansion, the Fourier series of $T(f)$ is given by $\sum_{a\in\Gamma} c_a^{T(f)} \chi^a$, where

$$c_a^{T(f)} = \begin{cases} c_a^f/2 & \text{when } a > 0, \\ \overline{c}_a^f/2 & \text{when } a < 0, \\ \operatorname{Re} c_0 & \text{when } a = 0. \end{cases}$$

Theorem 7.1.3. *The operator $T\colon \mathcal{O}(\mathbb{D}_G) \longrightarrow \mathcal{H}(\mathbb{D}_G)\colon f \longrightarrow \operatorname{Re} f$ is surjective if and only if $\Gamma = \widehat{G}$ is isomorphic to $\mathbb{Z}$.*

Proof. If $\Gamma \cong \mathbb{Z}$, then $\mathbb{D}_G$ is the unit disc $\mathbb{D}$ in $\mathbb{C}$, and $\mathcal{O}(\mathbb{D}_G)$ is the space of analytic functions on $\mathbb{D}$, i.e. $\mathcal{O}(\mathbb{D}_G) = \mathcal{O}(\mathbb{D})$, thus $\mathcal{H}(\mathbb{D}_G) = \mathcal{H}(\mathbb{D})$. According to Theorem 2.3.2, in this case $T\big(\mathcal{O}(\mathbb{D})\big) = \mathcal{H}(\mathbb{D})$, as claimed.

If Γ is not isomorphic to $\mathbb{Z}$, then Γ is dense in $\mathbb{R}$. Assume that $p_1(z), p_2(z)$ are two polynomials on $\mathbb{C}$ with $p_1(0) = p_2(0) = 0$. We claim that for any r_1, r_2, $0 < r_1, r_2 < 1$, and each $a \in \Gamma \setminus \{0\}$ there is a $b \in \Gamma$ such that:

(i) If $\left|\chi^b(r \diamond g)\right| \le r_2$, then $\left|\chi^a(r \diamond g)\right| \le r_1$ for some point $r \diamond g \in \overline{\mathbb{D}}_G$.

(ii) There is a point $t \diamond h \in \overline{\mathbb{D}}_G$ such that $\left|\chi^a(t \diamond h)\right| \le r_1$ and $\left|p_2(\chi^b(t \diamond h))\right| > (1/2) \max_{\mathbb{D}_G} \left|p_2 \circ \chi^b\right|$.

Indeed, if $s = \sqrt[a]{r}$, then $\left|\chi^a(r \diamond g)\right| \le 1$ on $\mathbb{D}_G(s)$. Note that if $b \longrightarrow 0$, then $r^b \longrightarrow 1$, thus the set $\chi^b\big(\mathbb{D}_G(s)\big)$ expands to the open unit disc $\mathbb{D}$ as $b \longrightarrow 0$. Hence, we can choose a $c \in \mathbb{R}$ such that $\max |p_2|$ on $\chi^c\big(\mathbb{D}_G(r)\big)$ is greater than $(1/2) \max_{\mathbb{D}_G} \left|p_2 \circ \chi^c\right|$. Consider a sequence of polynomials $\big\{p_n(z)\big\}_{n=1}^{\infty}$ on $\mathbb{D}$ such that

(i) $p_n(0) = 0$,

(ii) $\max_{\mathbb{D}} \left|p_n(z)\right| > n$, and

(iii) $\max_{\mathbb{D}} \left|\operatorname{Re}(p_n(z))\right| \le 1/4^n$.

Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\left|p_n(z)\right| \le 1/4^n$ whenever $|z| \le t_n$. By the previous remarks we can choose a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that:

(a) $k_n a_n < a_{n-1}$, $n \geq 2$, where k_n is the order of $p_n(z)$.

(b) If $r \diamond g \in \overline{\mathbb{D}}_G$ and $\left|\chi^{a_n}(r \diamond g)\right| \leq t_n$, then $\left|\chi^{a_{n-1}}(r \diamond g)\right| \leq t_{n-1}$, and there is a point $r_n \diamond g_n \in \overline{\mathbb{D}}_G$, such that $\left|\chi^{a_{n-1}}(r_n \diamond g_n)\right| \leq t_{n-1}$, and $\left|p_n\big(\chi^{a_n}(r_n \diamond g_n)\big)\right| > (1/2) \max\limits_{\mathbb{D}} |p_n|$.

By definition, $\left|\chi^{a_1}(r_n \diamond g_n)\right| \leq t_1$ for all n. Consider the function

$$h = \sum_{n=1}^{\infty} T(p_n \circ \chi^{a_n}).$$

Since $T(p_n \circ \chi^{a_n}) = \operatorname{Re}(p_n \circ \chi^{a_n}) \in \mathcal{H}(\mathbb{D}_G)$, property (iii) from the above implies that $\sup\limits_{\mathbb{D}_G} |h| \leq \sum\limits_{n=1}^{\infty}(1/4^n) < 1$, thus h is a well-defined harmonic function on $\mathbb{D}_G$. Suppose, on the contrary, that there is an analytic function $f \in \mathcal{O}(\mathbb{D}_G)$ with $T(f) = h$. If $G_k = \operatorname{Ker}(\chi^{a_k}) = \big\{g \in G \colon \chi^{a_k}(g) = 1\big\}$, and σ_k is the Haar measure of G_k, consider the function

$$f_m(r \diamond g) = \int\limits_{G_m} f(r \diamond gk)\, d\sigma_m(k),$$

on $\mathbb{D}_G$. Note that $\operatorname{sp}(f_m) = \big\{a \in \operatorname{sp}(f) \colon a/a_m \in \mathbb{N}\big\}$. Therefore,

$$\begin{aligned}
&\operatorname{Re} f_m(r \diamond g) = \int\limits_{G_m} \operatorname{Re} f(r \diamond gk)\, d\sigma_m(k) = \int\limits_{G_m} h(r \diamond gk)\, d\sigma_m(k) \\
&= \sum_{n=1}^{\infty} \int\limits_{G_m} \big(T(p_n \circ \chi^{a_n})\big)(r \diamond gk)\, d\sigma_m(k) = \sum_{n=1}^{\infty} T\Big(\int\limits_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k)\Big) \\
&= T\Big(\sum_{n=1}^{\infty} \int\limits_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k)\Big) = \operatorname{Re}\Big(\sum_{n=1}^{\infty} \int\limits_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k)\Big).
\end{aligned}$$

Property (a) from the above implies

$$\sum_{n=1}^{\infty} \int\limits_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k) = \sum_{n=1}^{m} \int\limits_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k).$$

Consequently, $\operatorname{Re} f_m(r \diamond g) = \sum\limits_{n=1}^{m} \int\limits_{G_m} \operatorname{Re}(p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k)$. Therefore,

$f_m(r \diamond g) = \sum_{n=1}^{m} \int_{G_m} (p_n \circ \chi^{a_n})(r \diamond gk)\, d\sigma_m(k) + f(\omega)$, and hence

$$\begin{aligned}\sup_{\mathbb{D}_{G_m}} \left|f - f(\omega)\right| &\geq \sup_{\mathbb{D}_{G_m}} \left|f_m - f(\omega)\right| \geq \left|f_m(r_n \diamond g_n) - f(\omega)\right| \\ &\geq \left|p_m(\chi^{a_m}(r_n \diamond g_n))\right| - \Big|\sum_{n=1}^{m-1} p_n\big(\chi^{a_n}(r \diamond gh)\big)\Big| \\ &\geq (1/2)\max_{\mathbb{D}} |p_m| - 1 > (m/2) - 1.\end{aligned}$$

Since $\overline{\mathbb{D}}_{G_k} \subset \overline{\mathbb{D}}_{G_1}$ we obtain that f is unbounded at $\omega \in \overline{\mathbb{D}}_{G_1}$, in contradiction with $f \in \mathcal{O}(\mathbb{D}_G)$. Consequently, $h \notin T\big(\mathcal{O}(\mathbb{D}_G)\big)$, and hence $T\big(A(\mathbb{D}_G)\big)$ is a proper subalgebra of $\mathcal{H}(\mathbb{D}_G)$. ☐

Let $A\big(\mathbb{D}_G(r)\big)$ be the uniform closure $\big[A(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)}\big]$ of the restriction of the algebra $A(\mathbb{D}_G)$ on the G-disc $\overline{\mathbb{D}}_G(r)$ with radius $r < 1$. Since $A\big(\mathbb{D}_G(r)\big)$ is generated by the semigroup $\big\{\widehat{\chi}^a|_{\overline{\mathbb{D}}_G(r)} : a \in \Gamma_+\big\} \cong \Gamma_+$, it is isometrically isomorphic to the G-disc algebra $A_{\Gamma_+} = A(\mathbb{D}_G)$. Let $\mathcal{O}(\mathbb{D}_G)$ denote the set of continuous functions on the open G-disc, that are locally approximable on $\mathbb{D}_G$ by analytic functions in $\mathbb{D}_G$. Clearly, the restriction algebra $\mathcal{O}(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)}$ contains $A(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)}$. It is easy to see that the set $r \diamond G = \{r \diamond g \colon g \in G\}$ is the Shilov boundary of the uniform closures of both these algebras. Consequently, $A(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)} \subset \mathcal{O}(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)} \subset \big[A(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)}\big] = A\big(\mathbb{D}_G(r)\big)$, by the maximality of the algebra $A\big(\mathbb{D}_G(r)\big) \cong A_{\Gamma_+}$.

Proposition 7.1.4. *A function $f \in C(\mathbb{D}_G)$ belongs to the class $\mathcal{O}(\mathbb{D}_G)$, if and only if the restriction of f on every closed G-disc $\overline{\mathbb{D}}_G(r)$ belongs to the algebra $A\big(\mathbb{D}_G(r)\big)$.*

Proof. The remark from the above implies that $\mathcal{O}(\mathbb{D}_G)|_{\overline{\mathbb{D}}_G(r)} \subset A\big(\mathbb{D}_G(r)\big)$ for any $r < 1$. Conversely, given a $\varrho \diamond g \in \mathbb{D}_G$, let r_1 be a positive number with $\varrho < r_1 < 1$. If the restriction of $f \in C(\mathbb{D}_G)$ on the G-disc $\overline{\mathbb{D}}_G(r_1)$ belongs to the algebra $A_{\Gamma_+}\big(\mathbb{D}_G(r_1)\big)$, then f is an analytic function on the open set $\mathbb{D}_G(r_1)$, and therefore, f belongs to $\mathcal{O}(\mathbb{D}_G)$. ☐

For any $f \in C(\mathbb{D}_G)$ and $r < 1$ define the dilation f_r of f by $f_r(\varrho \diamond g) = f_{r\diamond\imath}(\varrho \diamond g) = f(r\varrho \diamond g)$, $0 \leq \varrho < 1$. Note that $f \longmapsto f_r$ maps the algebra $C\big(\mathbb{D}_G(r)\big)$ isometrically and isomorphically onto $C(\mathbb{D}_G)$. The inverse mapping is given by $f \longmapsto f_{1/r} \colon C(\mathbb{D}_G) \longrightarrow C\big(\mathbb{D}_G(r)\big) \colon f_{1/r}(\varrho \diamond g) = f\big((\varrho/r) \diamond g\big)$, $0 \leq \varrho < r$.

Proposition 7.1.5. *A function $f \in C(\mathbb{D}_G)$ belongs to $\mathcal{O}(\mathbb{D}_G)$, if and only if it is harmonic on $\mathbb{D}_G$ and at least one of its r-traces $f_r = f\big|_{r\diamond G}$, $0 < r < 1$ belongs to A_{Γ_+}.*

Proof. Proof. By the remarks before Theorem 7.1.3, it is enough to prove only the sufficiency part. Assume that $f_r \in A_{\Gamma_+}$ for some $r \in (0,1)$. If $r_1 < r$, then

$f_{r_1} = f_r \star m_{r_1/r}$, according to Definition 7.1.2. By (3.7) we have that $\operatorname{sp}(f_{r_1}) \subset \operatorname{sp}(f_r) \subset \Gamma_+$. Hence, $f_{r_1} \in A_{\Gamma_+}$. Note that since m_r is real-valued, then

$$\int_G \chi^a dm_r = \int_G \overline{\chi}^a dm_r = r^a.$$

Therefore, $\operatorname{sp}(m_r) = \Gamma$, and by (3.7) we have $\operatorname{sp}(f_{r_1}) = \operatorname{sp}(f_r)$ for every $r_1 > r$, since $f_r = f_{r_1} \star m_{r/r_1}$. We obtain that all r-traces f_r of f belong to A_{Γ_+}. Note that the restriction of f on every closed G-disc $\overline{\mathbb{D}}_G(r)$ coincides with the function $\widehat{f_r} \in A(\mathbb{D}_G)$, i.e. f belongs to the algebra $A\big(\mathbb{D}_G(r)\big)$. Proposition 7.1.4 now implies that $f \in \mathcal{O}(\mathbb{D}_G)$. $\square$

Since the Gelfand transform $\widehat{\chi}^a(r \diamond g) = r^{|a|}\chi^a(g)$ of any χ^a, $a \in \Gamma_+$, belongs to $\mathcal{O}(\mathbb{D}_G)$, it is a harmonic function on $\mathbb{D}_G$. Therefore, every Γ_+-polynomial $\sum_{k=1}^{n} d_k \chi^{a_k}$ on G admits a harmonic extension on $\mathbb{D}_G$.

Lemma 7.1.6. *If u is harmonic on $\mathbb{D}_G$, then*

$$\|u_{r_1}\|_\infty = \sup_{g\in G} |u_{r_1}| \le \sup_{g\in G} |u_{r_2}| = \|u_{r_2}\|_\infty$$

for every r_1, r_2, $0 \le r_1 \le r_2 < 1$.

Proof. According to (7.2), $u_{r_1}(g) = u_{r_2} \star m_r(g)$, where $r = r_1/r_2$, then

$$\|u_{r_1}\|_\infty = \|u_{r_2} \star m_r\| \le \|u_{r_2}\|_\infty \|m_r\| = \|u_{r_2}\|_\infty,$$

since m_r is a probability measure. Consequently, $\|m_r\| = 1$. $\square$

Let $\mathcal{H}_c(\overline{\mathbb{D}}_G) = \mathcal{H}(\mathbb{D}_G) \cap C(\overline{\mathbb{D}}_G)$ be the space of continuous functions on $\overline{\mathbb{D}}_G$ that are harmonic on $\mathbb{D}_G$.

Lemma 7.1.7. *Every continuous function on G can be extended uniquely on $\overline{\mathbb{D}}_G$ as a harmonic function on $\overline{\mathbb{D}}_G$, i.e. $C_{\mathbb{R}}(G) \cong \mathcal{H}_c(\overline{\mathbb{D}}_G)$.*

Proof. If f is a continuous function on G, then by the Stone-Weierstrass theorem there is a sequence of linear combinations of characters $p_n \in P(G)$ on $\overline{\mathbb{D}}_G$ uniformly converging on G to f. Lemma 7.1.6 implies that $\sup_G |p_n - p_m| = \sup_{\overline{\mathbb{D}}_G} |p_n - p_m|$, and therefore, $\{p_n\}_n$ is a Cauchy sequence in the sup-norm on $\overline{\mathbb{D}}_G$. Let $\widetilde{f} \in C_{\mathbb{R}}(\overline{\mathbb{D}}_G)$ be the uniform limit of this sequence. Since $(p_n)_{r_1} \in \mathcal{O}(\overline{\mathbb{D}}_G) \subset \mathcal{H}_c(\overline{\mathbb{D}}_G)$, we deduce that $\widetilde{f} \in \mathcal{H}_c(\overline{\mathbb{D}}_G)$. Hence $\widetilde{f}$ is a harmonic function on $\overline{\mathbb{D}}_G$, and its restriction on G is f. $\square$

For any $f \in C(\overline{\mathbb{D}}_G)$ and $r < 1$, the trace $f_r(g) = f(r \diamond g)$ of f on $r \diamond G$ admits the series expansion

$$f_r(g) \sim \sum_{a\in\Gamma} c_a^f(r)\, \chi^a r^{|a|}(g),$$

where $c_a^f(r) = \frac{1}{r^{|a|}} \int\limits_G f_r(g)\,\chi^{-a}(g)\,d\sigma(g) = \frac{1}{r^{|a|}}\,c_a^{f_r}$. As in the classical setting, the numbers $c_a^f(r)$ in general depend continuously on $r \in (0,1)$.

Theorem 7.1.8. *A function $u \in C(\mathbb{D}_G)$ is harmonic on $\mathbb{D}_G$ if and only if the coefficients $c_a^u(r)$ are independent of r for any $a \in \mathbb{R}$.*

Proof. Let $u \in \mathcal{H}(\mathbb{D}_G)$, $0 \le r_1 < r_2 < 1$, and $r = \frac{r_1}{r_2}$. The Fubini theorem implies that

$$\begin{aligned}
c_a^u(r_1) &= \frac{1}{r_1^{|a|}} \int\limits_G u_{r_1}(g)\,\chi^{-a}(g)\,d\sigma(g) = \frac{1}{r_1^{|a|}} \int\limits_G \Big(\int\limits_G u_{r_2}(gh)\,dm_r(h)\Big)\,\chi^{-a}(g)\,d\sigma(g)\\
&= \frac{1}{r_1^{|a|}} \int\limits_G \Big(\int\limits_G u_{r_2}(gh)\,\chi^{-a}(g)\,d\sigma(g)\Big)\,dm_r(h) = \Big(\frac{r_2}{r_1}\Big)^{|a|} c_a^u(r_2) \int\limits_G \chi^a(h)\,dm_r(h)\\
&= \Big(\frac{r_2}{r_1}\Big)^{|a|} c_a^u(r_2)\chi^a(r \diamond \imath) = \Big(\frac{r_2}{r_1}\Big)^{|a|} \Big(\frac{r_1}{r_2}\Big)^{|a|} c_a^u(r_2) = c_a^u(r_2).
\end{aligned}$$

Conversely, let $f \in C(\mathbb{D}_G)$ be a function with coefficients $c_a^f(r) = c(a,f)$, $a \in \Gamma$, independent of r. If $v = f_\varrho$ is the ϱ-dilation of f, i.e. $v(s \diamond g) = f(\varrho s \diamond g)$ for some ρ, $0 < \rho < 1$, then $c_a^f(r\varrho) = c_a^f(r) = c(a,f)$, $v \in C(\overline{\mathbb{D}}_G)$, and for all $a \in \mathbb{R}$ and $0 < \varrho < 1$ we have

$$\begin{aligned}
c_a^{f_\varrho}(r) &= c_a^v(r) = \frac{c_a^{v_r}}{r^{|a|}}\\
&= \frac{1}{r^{|a|}} \int\limits_G v_r(g)\,\chi^{-a}(g)\,d\sigma(g) = \frac{1}{r^{|a|}} \int\limits_G f(r\varrho \diamond g)\,\chi^{-a}(g)\,d\sigma(g)\\
&= \frac{1}{r^{|a|}} \int\limits_G f_{r\varrho}(g)\,\chi^{-a}(g)\,d\sigma(g) = \frac{1}{r^{|a|}} c_a^{f_{r\varrho}} = \frac{1}{r^{|a|}} (\varrho r)^{|a|} c_a^f(r\varrho) = \varrho^{|a|} c(a,f).
\end{aligned}$$

By Lemma 7.1.7 there exists a $u \in \mathcal{H}_c(\overline{\mathbb{D}}_G)$, such that $u \equiv v$ on G. Thus $c_a^u(r) = c(a,u) = c_a^u(1) = c_a^v(1) = c_a^v(r)$ for all $a \in \Gamma \subset \Gamma_+$ and $r \in (0,1)$. The uniqueness of Fourier series implies that $u \equiv v$ on $\overline{\mathbb{D}}_G$, thus $v \in \mathcal{H}_c(\overline{\mathbb{D}}_G)$. Hence, $v_{r_1}(g) = \int\limits_G v_{r_2}(gh)\,dm_r(h)$, where $r = r_1/r_2$ if $0 \le r_1 < r_2 < 1$. Consequently,

$$f_{r_1}(\varrho \diamond g) = v_{r_1}(1 \diamond g) = v_{r_1}(g) = \int\limits_G v_{r_2}(gh)\,dm_r(h) = \int\limits_G f_{r_2}\big(\varrho \diamond (gh)\big)\,dm_r(h).$$

By letting $\varrho \nearrow 1$ we obtain $f_{r_1}(g) = \int\limits_G f_{r_2}(gh)\,dm_r(h) = (f_{r_2} \star m_r)(g)$, and therefore, $f \in \mathcal{H}_c(\overline{\mathbb{D}}_G)$. $\square$

7.2 L^p-harmonicity on groups and G-discs

There are close connections between harmonic functions on the unit disc $\mathbb{T}$ and the Hardy space H^p on the unit circle. Hardy space H^p can be defined in two standard ways. Firstly, H^p-functions are defined as functions in $L^p(\mathbb{T})$ whose negative Fourier coefficients are zero. Secondly, H^p-functions are defined as analytic functions in $\mathbb{D}$, whose restrictions on circles centered at the origin are uniformly bounded in the L^p-norm. Following these definitions leads to essentially different classes of functions on groups. For approaches based on the first definition of H_p-functions i.e. based on their Fourier coefficients see e.g. [HL], also [G10]). We will follow the second definition to generalize the notion of Hardy spaces on groups. The resulting spaces are called Hardy-Bohr spaces. They are closely related with analytic almost periodic functions on $\mathbb{R}$.

Definition 7.2.1. Let G be a compact group whose dual group Γ is a subgroup of $\mathbb{R}$, and let $m_{(1/e)}$ be the representing measure of the point $(1/e)\diamond \imath \in \mathbb{D}$, i.e. $m_{(1/e)} = m_{(1/e)\diamond\imath}$. Consider the following spaces.

(a) The space $\mathcal{H}^p(\mathbb{D}_G)$ of L^p-harmonic functions on $\mathbb{D}_G$, $1 \le p < \infty$ is the set of harmonic functions $u \in \mathcal{H}(\mathbb{D}_G)$ for which

$$\begin{aligned} \|u\|_p &= \Big(\sup_{\substack{0\le r<1\\ g\in G}} \int_G \big|u_r(gh)\big|^p\, dm_{(1/e)}(h)\Big)^{1/p} \\ &= \Big(\sup_{\substack{0\le r<1\\ g\in G}} \int_G \big|u_r(gh^{-1})\big|^p\, dm_{(1/e)}(h)\Big)^{1/p} < \infty. \end{aligned} \tag{7.3}$$

(b) The space $\mathcal{H}^\infty(\mathbb{D}_G)$ of L^∞-harmonic functions on $\mathbb{D}_G$ is the set of harmonic functions $u \in \mathcal{H}(\mathbb{D}_G)$ with

$$\|u\|_\infty = \sup_{\substack{0\le r<1\\ g\in G}} \big|u_r(g)\big| < \infty. \tag{7.4}$$

The space $\mathcal{H}^p(\mathbb{D}_G)$ is called also *Stepanov's p-space*. Note that $\mathcal{H}^p(\mathbb{D}_G) = \mathcal{H}^p_{\mathbb{R}}(\mathbb{D}_G) + i\mathcal{H}^p_{\mathbb{R}}(\mathbb{D}_G)$, where $\mathcal{H}^p_{\mathbb{R}}(\mathbb{D}_G)$ is the set of the real-valued functions in $\mathcal{H}^p(\mathbb{D}_G)$.

Lemma 7.2.2. *Let $f \in \mathcal{H}^p(\mathbb{D}_G)$, and $r, r_1, r_2 \in [0,1)$, $r_1 < r_2$, $1 \le p < \infty$. Then:*

(a) $\displaystyle\sup_{g\in G}\int_G \big|f_{r_1}(gh)\big|^p\, dm_r(h) \le \sup_{g\in G}\int_G \big|f_{r_2}(gh)\big|^p\, dm_r(h)$, *i.e.*

$$\sup_{g\in G}\big(|f_{r_1}|^p \star m_r\big)(g) \le \sup_{g\in G}\big(|f_{r_2}|^p \star m_r\big)(g).$$

(b) $\int\limits_G |f_{r_1}|^p \, d\sigma \le \int\limits_G |f_{r_2}|^p \, d\sigma$, *i.e.* $\|f_{r_1}\|_p \le \|f_{r_2}\|_p$.

(c) *For every* r, $0 < r < 1$, *there exist real constants* $0 < c_1 < c_2 < \infty$, *such that*

$$c_1 \sup_{g\in G} \big(|f_{r_1}|^p \star m_r\big)(g) < \sup_{g\in G} \big(|f_{r_1}|^p \star m_{1/e}\big)(g) < c_2 \sup_{g\in G} \big(|f_{r_1}|^p \star m_r\big)(g).$$

Proof. (a) We recall that $m_{(r_1\diamond g_1)} \star m_{(r_2\diamond g_2)} = \mu_{(r_1\diamond g_1)\cdot(r_2\diamond g_2)} = m_{(r_1 r_2 \diamond g_1 g_2)}$ for every $r_1 \diamond g_1$, $r_2 \diamond g_2 \in \mathbb{D}_G$. In particular, $m_{r_1} \star m_{r_2} = m_{(r_1 r_2)}$, i.e. for any function $f \in C(G)$ we have

$$\int\limits_G \int\limits_G f(gh) \, dm_{r_1}(g) \, dm_{r_2}(h) = \int\limits_G f(g) \, dm_{(r_1 r_2)}.$$

Since the function f is harmonic on $\mathbb{D}_G$, Hölder's inequality and the Fubini theorem imply

$$\begin{aligned}
&\sup_{g\in G} \int\limits_G \big|f_{r_1}(gh)\big|^p \, dm_r(h) = \sup_{g\in G} \int\limits_G \Big| \int\limits_G f_{r_2}(ghk) \, dm_{(r_1/r_2)}(k) \Big|^p \, dm_r(h) \\
&\le \sup_{g\in G} \int\limits_G \int\limits_G \big|f_{r_2}(ghk)\big|^p \, dm_{(r_1/r_2)}(k) \, dm_r(h) \\
&= \sup_{g\in G} \int\limits_G \int\limits_G \big|f_{r_2}(ghk)\big|^p \, dm_r(h) \, dm_{(r_1/r_2)}(k) \le \sup_{g\in G} \int\limits_G |f_{r_2}(gh)|^p \, dm_r(h),
\end{aligned}$$

as claimed.

(b) By the same argument we have

$$\begin{aligned}
\int\limits_G \big|f_{r_1}(g)\big|^p \, d\sigma(g) &\le \int\limits_G \Big(\int\limits_G \big|f_{r_2}(gh)\big|^p \, dm_{(r_1/r_2)}(h) \Big) \, d\sigma(g) \\
&= \int\limits_G \Big(\int\limits_G \big|f_{r_2}(gh)\big|^p \, d\sigma(g) \Big) \, dm_{(r_1/r_2)}(h) \le \int\limits_G |f_{r_2}(g)|^p \, d\sigma(g).
\end{aligned}$$

(c) Since $m_{(1/e)}$ and m_r are mutually absolutely continuous measures for any r, $0 < r < 1$, there is a Borel function K_r such that $dm_r = K_r \cdot dm_{(1/e)}$. In fact, $K_r\big|_{j_\imath(\mathbb{R})}$ is a quotient of two Poisson kernels on $\mathbb{R}$, where $j_\imath \colon \mathbb{R} \longrightarrow G$ is the standard embedding of $\mathbb{R}$ into G with $j_\imath(0) = \imath$. Hence, $\sup_{g\in G} \big|K_r(g)\big| < \infty$, and also

$\sup_{g\in G}\left|\frac{1}{K_r(g)}\right| < \infty$. Therefore,

$$\begin{aligned}\sup_{g\in G}\int_G \left|f_{r_1}(gh)\right|^p dm_r(h) &= \sup_{g\in G}\int_G \left|f_{r_1}(gh)\right|^p K_r(g)\, dm_{(1/e)}(h)\\ &\le \sup_{g\in G}\left|K_r(g)\right| \sup_{g\in G}\int_G \left|f_{r_1}(gh)\right|^p dm_{(1/e)}(h)\\ &\le \sup_{g\in G}\left|K_r(g)\right| \sup_{g\in G}\left|\frac{1}{K_r(g)}\right| \sup_{g\in G}\int_G \left|f_{r_1}(gh)\right|^p dm_{(1/e)}(h).\end{aligned}$$

Consequently, (b) holds with $c_1 = \dfrac{1}{\sup_{g\in G}\left|K_r(g)\right|}$, and $c_2 = \sup_{g\in G}\left|\dfrac{1}{K_r(g)}\right|$. □

Theorem 7.2.3. *Under the $\mathcal{H}^p(\mathbb{D}_G)$-norm from Definition* 7.2.1, *$\mathcal{H}^p(\mathbb{D}_G)$, $1 \le p \le \infty$ is a Banach space.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the $\mathcal{H}^p(\mathbb{D}_G)$-norm. If $1 \le p < \infty$, Lemma 7.2.2 and Hölder's inequality imply that for every r, $0 < r < 1$, we have

$$\begin{aligned}\left|(u_n)_{r_1}(g) - (u_m)_{r_1}(g)\right| &= \left|\int_G \big((u_n)_{r_2}(gh) - (u_m)_{r_2}(gh)\big) K_r(g)\, dm_{(1/e)}(h)\right|\\ &\le \sup_{g\in G}\left|K_r(g)\right|\|u_n - u_m\|_p.\end{aligned}$$

Therefore, for every $0 < r < 1$ the sequence $\{(u_n)_r\}_{n=1}^{\infty}$ converges uniformly on G to a continuous function $u_r \in C(G)$. Let $u(r \diamond g) = u_r(g)$. We claim that $u \in \mathcal{H}(\mathbb{D}_G)$. Indeed, if $0 < r_1 < r_2 < 1$, and $r = r_1/r_2$, then

$$\begin{aligned}u_{r_1}(g) &= \lim_{n\to\infty}(u_n)_{r_1} = \lim_{n}\int_G (u_n)_{r_2}(gh)\, dm_r(h)\\ &== \int_G \lim_{n\to\infty}(u_n)_{r_2}(gh)\, dm_r(h) = \int_G u_{r_2}(gh)\, dm_r(h),\end{aligned}$$

and consequently, $u \in \mathcal{H}(\mathbb{D}_G)$. If $p = \infty$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{D}_G$ to a continuous function u. The proof of harmonicity of u follows the same arguments as in the case $p < \infty$. □

Let $\mathcal{H}_c^p(\overline{\mathbb{D}}_G)$ be the closure of the space $\mathcal{H}_c(\overline{\mathbb{D}_G}) = \mathcal{H}(\mathbb{D}_G) \cap C(\overline{\mathbb{D}}_G)$ in the $\mathcal{H}^p(\mathbb{D}_G)$-norm. The next theorem is obvious.

Theorem 7.2.4. *$\mathcal{H}_c^p(\overline{\mathbb{D}}_G)$ is a closed proper subspace of $\mathcal{H}^p(\mathbb{D}_G)$ for every $p \in [1, \infty]$.*

For every $1 \le p \le \infty$ denote $H^p(\mathbb{D}_G) = \mathcal{H}^p(\mathbb{D}_G) \cap \mathcal{O}(\mathbb{D}_G)$. The following theorem and its proof is due to Hoffman ([H2]).

Theorem 7.2.5. *If f is a function in $\mathcal{H}^p(\mathbb{D}_G)$, or in $H^p(\mathbb{D}_G)$, $1 \le p \le \infty$, then:*

(a) *The limit $f^*(g) = \lim\limits_{r \nearrow 1} f_r(g)$ exists for $m_{r\diamond g}$-almost every $g \in G$, $r \diamond g \in \mathbb{D}_G$.*

(b) $f^* \in \bigcap\limits_{r\diamond g \in \mathbb{D}_G} L^p(G, m_{r\diamond g})$.

(c) $\lim\limits_{r \nearrow 1} \|f_r\|_{L^p(G,\, m_{(r\diamond g)})} = \|f^*\|_{L^p(G,\, m_{(r\diamond g)})}$ *for every $r \diamond g \in \mathbb{D}_G$ and $1 \le p < \infty$.*

(d) $\lim\limits_{r \nearrow 1} \|f_r\|_{L^\infty(G,\, m_{(r\diamond g)})} = \|f^*\|_{L^\infty(G,\, m_{(r\diamond g)})}$ *for every $r \diamond g \in \mathbb{D}_G$.*

(e) *If $p = 1$, then the sequence of measures $f_r\, dm_{(1/e)}$ converges in the weak*-topology to a measure $\mu \in M(G)$ as $r \nearrow 1$, and $\|\mu\| = \lim\limits_{r \nearrow 1} \big\|f_r\, dm_{(1/e)}\big\|$, where $\|\cdot\|$ is the total variation norm on the space of Borel measures $M(G)$ on G.*

Note that if $f \in \mathcal{H}^p(\mathbb{D}_G)$, then (7.2) holds for all $0 \le r_1 < r_2 < 1$.

Definition 7.2.6. The space of boundary values $f^* = \lim\limits_{r \nearrow 1} f_r$ of functions f in $H^p(\mathbb{D}_G)$, which exists by Theorem 7.2.5(a) is called the *Hardy-Bohr space $H^p(G)$* on G.

Denote by $\mathcal{H}^p(G)$ the space of limits $f^*(g) = \lim\limits_{r \nearrow 1} f_r(g)$ of functions in $\mathcal{H}^p(\mathbb{D}_G)$, existing by Theorem 7.2.5(a). Let $\mathcal{H}^p_c(G)$ be the space of restrictions on G, i.e. of limits $f^* = \lim\limits_{r \nearrow 1} f_r$ of functions f in $\mathcal{H}^p_c(\overline{\mathbb{D}}_G)$. The next theorem is a direct consequence from Theorem 7.2.5 and Theorem 7.2.4.

Theorem 7.2.7. (a) *The Hardy-Bohr space $H^p(G)$ is a proper subspace of the space $\mathcal{H}^p(G)$ for every $p \in [1, \infty]$.*

(b) *For every p, $1 \le p < \infty$, $\mathcal{H}^p_c(G)$ and $H^p(G)$ are Banach spaces with respect to the norm $\|f^*\|_p = \sup\limits_{g \in G} \Big(\int\limits_G \big|f^*(gh)\big|^p\, dm_{(1/e)}(h)\Big)^{1/p} = \|f\|_p$.*

Denote by $L^p\big(G, m_{(1/e)}\big)$ the space of all $m_{(1/e)}$-integrable functions on G with finite $L^p\big(G, m_{(1/e)}\big)$-norm.

Definition 7.2.8. For any $1 < p < \infty$ and $q = \dfrac{p}{p-1}$ let $\mathcal{L}^{p,q}(G)$ be the set of integrable functions on G, with the property that the function

$$(f \star u)(g) = \int\limits_G f(gh)\, u(h)\, dm_{(1/e)}(h) \tag{7.5}$$

is continuous on G for any $u \in L^q\big(G, m_{(1/e)}\big)$.

Theorem 7.2.9. *For any* $1 < p < \infty$, $\mathcal{L}^{p,q}(G)$ *is a Banach subspace of* $L^p(G, m_{(1/e)})$ *with the norm*

$$\|f\|_p = \sup_{g\in G}\Big(\int_G \big|f(gh)\big|^p \, dm_{(1/e)}(h)\Big)^{1/p}. \tag{7.6}$$

Proof. Let $f \in \mathcal{L}^{p,q}(G)$. First we will show that $\|f\|_p < \infty$. For any $g \in G$ consider the bounded linear functional F_g, $g \in G$, on $L^q\big(G, m_{(1/e)}\big)$, defined by

$$F_g(u) = (f \star u)(g) = \int_G f(gh)\, u(h)\, dm_{(1/e)}(h).$$

According to (7.3), $\big|F_g(u)\big| \le C = \sup_{g\in G}\big|(f \star u)(g)\big|$ for every F_g, $g \in G$. The Banach-Steinhaus theorem, applied to the family $\{F_g\}_{g\in G}$, implies

$$\sup_{g\in G} \|F_g\|'_q < \infty, \tag{7.7}$$

where $\|F_g\|'_q$ is the norm of F_g in the dual space of $L^q(G, m_{(1/e)})$. Consequently,

$$\|f\|_p = \sup_{g\in G}\Big(\int_G \big|f(gh)\big|^p \; dm_{(1/e)}(h)\Big)^{1/p} = \sup_{g\in G} \|F_g\|'_q < \infty,$$

as claimed. It remains to show that the space $\mathcal{L}^{p,q}(G)$ is complete with respect to the norm (7.6). Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^{p,q}(G)$. For any fixed $g \in G$ the family of functions $\big\{f_n(gh)\big\}$ is a Cauchy sequence in the space $L^p\big(G, m_{(1/e)}\big)$. Let $f_0(gh)$ be the limit of this sequence. By (7.3) we have that the functions

$$(f_n \star u)(g) = \int_G f_n(gh)\, u(h)\, dm_{(1/e)}(h)$$

form a Cauchy sequence in $C(G)$ for any fixed $u \in L^q(G, m_{(1/e)})$. Therefore the function

$$(f_0 \star u)(g) = \int_G f_0(gh)\, u(h)\, dm_{(1/e)}(h)$$

is continuous on G. Consequently, $f_0 \in \mathcal{L}^{p,q}(G)$. □

Let $\widetilde{\jmath}_\imath \colon \overline{\mathbb{C}}_+ \longrightarrow \mathbb{D}_G$ be the natural extension on $\overline{\mathbb{C}}_+$of the standard embedding $\jmath_\imath \colon \mathbb{R} \longrightarrow G$ of $\mathbb{R}$ onto a dense subgroup of G, with $\jmath_\imath(0) = \imath$. Namely, $\jmath_\imath(t) = g_t \in G = \widehat{\varGamma}$ for any $t \in \mathbb{R}$, where $g_t(a) = e^{iat}$, $a \in \varGamma = \widehat{G}$. Denote by

$$P_{(t_0,y)}(t)\, dt = \frac{1}{\pi}\frac{y}{y^2 + (t_0 - t)^2}\, dt,\ y > 0 \tag{7.8}$$

the Poisson measure on $\mathbb{C}$, and let $m_{(e^{-y}\diamond g_{t_0})}$ be the representing measure of the point $e^{-y}\diamond g_{t_0} \in \mathbb{D}_G$. One can easily see that $dm_{(e^{-y}\diamond g_{t_0})} \circ \widetilde{\jmath}_\imath(t+iy) = P_{(t_0,y)}\, dt$, i.e.

the measure $m_{(e^{-y}\diamond g_{t_0})}$ is the image of the Poisson measure via the embedding $\widetilde{j}_\imath$. In particular, the measure $m_{(1/e)}$ is the image of the measure $\dfrac{1}{\pi}\dfrac{dt}{1+t^2}$. Therefore the corresponding L^q-spaces can be identified, i.e.

$$L^q\big(G, m_{(1/e)}\big) \cong L^q\Big(\mathbb{R}, \frac{dt}{1+t^2}\Big). \tag{7.9}$$

Note that the measures $m_{(e^{-y}\diamond g_t)}$ and $m_{(1/e)}$ are mutually absolutely continuous. If $K_{(e^{-y}\diamond g_t)} = dm_{(e^{-y}\diamond g_t)}/dm_{(1/e)}$ is the Radon-Nikodym derivative of $m_{(e^{-y}\diamond g_t)}$ with respect to $m_{(1/e)}$, then $dm_{(e^{-y}\diamond g_t)} = K_{(e^{-y}\diamond g_t)}\, dm_{(1/e)}$. Denote by $\mathcal{K}$ the linear space generated by the functions $K_{(e^{-y}\diamond g_t)}$, $t \in \mathbb{R}$, $y > 0$.

Lemma 7.2.10. *For any* $1<q<\infty$ *the space* $\mathcal{K}$ *is a dense subspace in* $L^q(G, m_{(1/e)})$.

Proof. Let $f \in L^p\big(G, m_{(1/e)}\big)$ with $\int_G f K\, dm_{(1/e)} = 0$ for all $K \in \mathcal{K}$. Identities (7.8) and (7.9) imply that

$$\frac{1}{\pi}\int_{\mathbb{R}} f(g_t)\,\frac{y\, dt}{y^2 + (t_0 - t)^2} = 0$$

for every $y > 0$ and $t_0 \in \mathbb{R}$. Therefore, $f \equiv 0$, because the Poisson integral of a non-zero function can not be 0. Since the dual space of $L^q\big(G, m_{(1/e)}\big)$ is $L^p\big(G, m_{(1/e)}\big)$, the bipolar theorem implies that $\mathcal{K}$ is dense in $L^q\big(G, m_{(1/e)}\big)$, as claimed. □

Lemma 7.2.11. *Every function* $f \in \mathcal{L}^{p,q}(G)$, $1 < p < \infty$, *can be extended uniquely to a function* $\widetilde{f} \in \mathcal{H}^p(\mathbb{D}_G)$ *such that* $f(g) = \lim_{r\nearrow 1} \widetilde{f}_r(g)$ *for almost all* $g \in G$ *with respect to any measure* $m_{r\diamond g}$, $r \diamond g \in \mathbb{D}_G$.

Proof. Let $K_r = K_{(r\diamond\imath)}$. Since $dm_r = K_r dm_{(1/e)}$, and $K_r \in L^p(G, m_{(1/e)})$, the definition of the space $\mathcal{L}^{p,q}(G)$ implies that the function

$$\widetilde{f}_r(g) = (f \star m_r)(g) = \int_G f(gh)\, dm_r(h)$$

is continuous on G. We have that $K_{(r\diamond g_t)} \longrightarrow K_{(r_0\diamond g_t)}$ in the $L^q(G, m_{(1/e)})$-norm, as $r \longrightarrow r_0$. Therefore, the function $\widetilde{f}(r\diamond g) = \widetilde{f}_r(g)$ is continuous on $\mathbb{D}_G$. For the Fourier coefficient $c_a^{\widetilde{f}}(r)$, $a \in \Gamma$ we have

$$c_a^{\widetilde{f}}(r) = \frac{1}{r^{|a|}}\int_G\int_G f(gh)\,\chi^{-a}(h)\, d\sigma(h)\, dm_r(g) = \frac{c_a^f}{r^{|a|}}\int_G \chi^a(g)\, dm_r(g) = c_a^f.$$

Theorem 7.1.8 implies that f is a harmonic function in $\mathbb{D}_G$. Moreover, $\widetilde{f} \in \mathcal{H}^p(\mathbb{D}_G)$ since

$$\sup_{\substack{g\in G\\ r<1}}\Big(\int\limits_G |\widetilde{f}_r(gh)|^p\, dm_{(1/e)}(h)\Big)^{\frac{1}{p}} \le \sup_{\substack{g\in G\\ r<1}}\Big(\int\limits_G |\int\limits_G f(ghk)\, dm_r(k)|^p\, d\mu_{(1/e)}(h)\Big)^{\frac{1}{p}}.$$

The Fubini theorem and Hölder's inequality, applied to the inner integral, yield $\|\widetilde{f}\|_p \le \|f\|_p$. □

Lemma 7.2.11 implies that $\mathcal{L}^{p,q}(G) \subset \mathcal{H}^p(\mathbb{D}_G)$. The opposite inclusion also is true.

Theorem 7.2.12. *All spaces of type $\mathcal{L}^{p,q}(G)$, $q > 1$ are isometrically isomorphic to the space $\mathcal{H}^p(\mathbb{D}_G)$.*

Proof. Let $f \in \mathcal{H}^p(G)$ and $\widetilde{f} \in \mathcal{H}^p(\mathbb{D}_G)$ be such that $f = \lim\limits_{r\nearrow 1} \widetilde{f}_r$, where $\widetilde{f}_r = \widetilde{f}|_{r\diamond G}$ is the r-trace of $\widetilde{f}$ on $r \diamond G$. For any $r \diamond g \in \mathbb{D}_G$ we have

$$\widetilde{f}_{r\diamond g}(h) = \int\limits_G f(hk)\, dm_{(1/e)\diamond g}(k) = \int\limits_G f(hk)\, K_{(1/e)\diamond g}(k)\, dm_{(1/e)}(k) \tag{7.10}$$

is continuous on G. For a fixed $u \in L^q\big(G, m_{(1/e)}\big)$ consider the function

$$(f \star u)(g) = \int\limits_G f(hg)\, u(h)\, dm_{(1/e)}(h). \tag{7.11}$$

Obviously,

$$\sup_{g\in G}\big|(f \star u)(g)\big| = \|f \star u\|_\infty \le \|f\|_p \|u\|_q \tag{7.12}$$

for any $u \in L^q\big(G, m_{(1/e)}\big)$. Since $\mathcal{K}$ is dense in $L^q\big(G, m_{(1/e)}\big)$, then by (7.10) and (7.12) we see that $f \star u$ can be approximated uniformly on G by linear combinations of functions $\widetilde{f}_{r\diamond g}$, $r \diamond g \in \mathbb{D}_G$. Hence $f \star u$ is continuous on G for any $u \in L^q\big(G, m_{(1/e)}\big)$, and therefore, $f \in \mathcal{L}^{p,q}(G)$. Consequently, $\mathcal{H}^p(\mathbb{D}_G) \cong \mathcal{L}^{p,q}(G)$, by Lemma 7.2.11. □

Define a topology τ on the space $\mathcal{L}^{p,q}(G) \cong \mathcal{H}^p(\mathbb{D}_G)$ as follows. Choose the neighborhood basis of a function $f \in \mathcal{L}^{p,q}(G)$ to be the family of sets

$$U(f; u_1, \ldots, u_n, \varepsilon) = \big\{g \in \mathcal{L}^{p,q}(G)\colon \|f \star u_i - g \star u_i\|_\infty < \varepsilon,\ 1 \le i \le n\big\},$$

where $\varepsilon > 0$ and $u_1, \ldots, u_n \in L^q\big(G, m_{(1/e)}\big)$. We say that a net $\{f_\alpha\}_{\alpha\in\Sigma} \subset \mathcal{L}^{p,q}(G)$ τ-converges to $f \in \mathcal{L}^{p,q}(G)$ if

$$\lim_{\alpha\in\Sigma} \|f_\alpha \star u - f \star u\|_\infty = 0$$

for every $u \in L^q\big(G, m_{(1/e)}\big)$. Denote by τ-$\lim\limits_{\alpha\in\Sigma} f_\alpha$ and $\mathcal{H}^p$-$\lim\limits_{\alpha\in\Sigma} f_\alpha$ the limits of $\{f_\alpha\}_\Sigma$ with respect to the corresponding topologies.

Theorem 7.2.13. *The space $\mathcal{H}^p(\mathbb{D}_G) \cong \mathcal{L}^{p,q}(G)$, $1 < p < \infty$, is τ-complete, i.e. if for every $u \in L^q(G, m_{(1/e)})$ the family $\{f_\alpha \star u\}_{\alpha\in\Sigma}$ is a Cauchy net in $C(G)$, then there is a function $f_0 \in \mathcal{L}^{p,q}(G) \cong \mathcal{H}^p_S(\mathbb{D}_G)$ such that $f_0 = \tau\text{-}\lim_{\alpha\in\Sigma} f_\alpha$, i.e. $f_0 \star u = \lim_{\alpha\in\Sigma}(f_\alpha \star u)$ for every $u \in \mathcal{L}^q(G, m_{(1/e)})$.*

Proof. Let, as before, K_r be the Radon-Nikodym derivative of the measure m_r with respect to $m_{(1/e)}$. Clearly, the function

$$\widehat{f_\alpha}(r \diamond g) = f_\alpha \star K_r(g) = \int_G f_\alpha(hg)\, K_r(h)\, dm_{(1/e)}(h)$$

belongs to $\mathcal{H}^p(\mathbb{D}_G)$. Since $K_r \in L^q\big(G, m_{(1/e)}\big)$, the limit

$$\widehat{f}_r(g) = \widehat{f}(r \diamond g) = \mathcal{H}^p\text{-}\lim_{\alpha\in\Sigma} \widehat{f}_\alpha(r \diamond g) \tag{7.13}$$

exists for every fixed $r < 1$. Note that $\widehat{f_\alpha} \in \mathcal{H}^p(\mathbb{D}_G)$ for every $\alpha \in \Sigma$. Lemma 7.2.2 and (7.13) imply that $\widehat{f} \in \mathcal{H}(\mathbb{D}_G)$. It remains to show that $f \in \mathcal{H}^p(G)$, i.e. that $\widehat{f} \in \mathcal{H}^p(\mathbb{D}_G)$.

Let $F\colon L^q\big(G, m_{(1/e)}\big) \longrightarrow C(G)$ be the linear operator defined by $F(u) = \mathcal{H}^p\text{-}\lim_{\alpha\in\Sigma} f_\alpha \star u$. Consider an extension $\widetilde{F(u)} \in \mathcal{H}_c(\overline{\mathbb{D}}_G)$ of the function $F(u)$ on $\overline{\mathbb{D}}_G$. According to Lemma 7.1.6

$$\lim_{\alpha\in\Sigma} \sup_{\overline{\mathbb{D}}_G} \big|\widetilde{F(u)} - \widehat{f}_\alpha \star u\big| = \lim_{\alpha\in\Sigma} \big\|F(u) - f_\alpha \star u\big\|_\infty = 0.$$

Consequently,

$$\mathcal{H}^p\text{-}\lim_{r\nearrow 1}(\widehat{f} \star u)_r = \lim_{r\nearrow 1} \mathcal{H}^p\text{-}\lim_{\alpha\in\Sigma}(f_\alpha \star u)_r = F(u).$$

The Banach-Steinhaus theorem, applied to the family of bounded linear operators

$$F_r\colon L^q\big(G, m_{(1/e)}\big) \longmapsto C(G)\colon F_r(u) = (\widehat{f} \star u)_r,$$

with $\|F_r\| = \|f_r\|_p$, implies that the family $\{F_r\}_{r\in(0,1)}$ is uniformly bounded. Let B be the unit ball in $L^q(G, m_{(1/e)})$. We have

$$\begin{aligned}\|F_r\| &= \sup_{g\in G}\sup_{u\in B}\Big|\int_G \widehat{f}_r(gh)\, u(h)\, dm_{(1/e)}(h)\Big| \\ &= \sup_{g\in G}\Big(\int_G \big|\widehat{f}(gh)\big|^p\, dm_{(1/e)}(h)\Big)^{1/p} < c,\end{aligned}$$

where the positive constant c does not depend on r. Hence, $\widehat{f} \in \mathcal{H}^p(\mathbb{D}_G)$. Consequently, $f \in \mathcal{H}^p(G) \cong \mathcal{L}^{p,q}(G)$. ☐

Theorem 7.2.14. *For any $1 < p < \infty$ the space $C(G)$ is τ-dense in $\mathcal{H}^p(\mathbb{D}_G) \cong \mathcal{L}^{p,q}(G)$.*

Proof. If $f \in \mathcal{H}^p(\mathbb{D}_G)$, then $\widehat{f_r} \in C(G)$ and $f = \tau\text{-}\lim_{r \nearrow 1} \widehat{f_r}$. □

All results in this section admit analogues for harmonic S-functions, where S is a semigroup of Γ.

Definition 7.2.15. *Hardy S-space* on G is called the space $\mathcal{H}^p_S(G)$, $1 < p < \infty$ of L^p-harmonic functions in $\mathbb{D}_G$ with spectrum in S, i.e.

$$\mathcal{H}^p_S(G) = \big\{f \in \mathcal{H}^p(\mathbb{D}_G) \cong \mathcal{L}^{p,q}(G):\ \operatorname{sp}(f) \subset S\big\}.$$

If $S \subset \Gamma_+$, then the set $\mathcal{H}^p(\mathbb{D}_G)$ is called *Hardy S-space on G*, or, *Hardy-Helson-Lowdenslager space*, and is denoted by $H^p_S(G)$. By the technique from the above used to study the space $\mathcal{H}^p(G)$, one can obtain the following theorems.

Theorem 7.2.16. (a) *For any $1 < p < \infty$ the space $\mathcal{H}^p_S(G)$, resp. $H^p_S(G)$, coincide with the space of $\mathcal{H}^p$-limits $f^* = \mathcal{H}^p\text{-}\lim_{r \nearrow 1} f_r$ with $f_r \in \mathcal{H}^p_S(\mathbb{D}_G)$, resp $f_r \in H^p_S(\mathbb{D}_G)$.*

(b) *For any $1 < p < \infty$ the spaces $\mathcal{H}^p_S(G)$ and $H^p_S(G)$ are τ-closed subspaces of $\mathcal{L}^{p,q}(G) \cong \mathcal{H}^p(G)$ for every p, $1 < p < \infty$.*

(c) *For any $1 < p < \infty$ the spaces $\mathcal{H}^p_S(G)$ and $H^p_S(G)$ are Banach spaces under the norm*

$$\|f\|_p = \sup_{g \in G} \Big(\int_G \big|f(gh)\big|^p \, dm_{(1/e)}(h) \Big)^{1/p}.$$

Theorem 7.2.17. (a) *The algebra A_S is τ-dense in the space $\mathcal{H}^p_S(G)$.*

(b) *The $\mathcal{H}^p(G)$-closure of A_S is a proper subspace of $\mathcal{H}^p_S(G)$.*

7.3 L^1-harmonic functions on groups and G-discs

Let $M(G)$ be the space of regular Borel measures on G. If $m_r = m_{r \diamond \imath}$ is the representing measure of the point $r \diamond \imath \in \overline{\mathbb{D}}_G$ for $A(\mathbb{D}_G)$, then, as it is easy to check,

$$\Big| \int_G \chi^a dm_r \Big| = r^{|a|} \tag{7.14}$$

for every $a \in \Gamma \subset \mathbb{R}$ and every r, $0 \le r < 1$. Therefore, $\operatorname{sp}(m_r) = \Gamma$ for every measure of type m_r.

Given a $\mu \in M(G)$, consider the measures μ_r and m^f, defined by

(a) $\mu_r = \mu \star m_r$, and

(b) $d\mu^f = f \, d\sigma$, $f \in L^1(G, \sigma)$.

Clearly μ_r and μ^f belong to $M(G)$. Since $\operatorname{sp}(\mu \star \nu) = \operatorname{sp}(\mu) \cap \operatorname{sp}(\nu)$ for $\mu, \nu \in M(G)$ (cf. (3.7)), and $\operatorname{sp}(m_r) = \varGamma$, we see that $\operatorname{sp}(\mu_r) = \operatorname{sp}(\mu)$.

For any $f \in C(G)$ let $\widetilde{f}_r$ be the function

$$\widetilde{f}_r(g) = (f \star m_r)(g) = \int_G f(gh)\, dm_r(h).$$

Clearly, $\widetilde{f}_r \in C(G)$.

Lemma 7.3.1. *If $f \in \mathcal{H}^1(\mathbb{D}_G)$, and $\mu^{f_r}_{(1/e)} = \mu^{f_r} \star m_{(1/e)}$, then*

$$d\mu^{f_r}_{(1/e)} = f_{(r/e)}\, d\sigma = d\mu^{f_{(r/e)}}.$$

Proof. Recall that the a-th Fourier-Stieltjes coefficient of the convolution of two measures equals the product of the a-th coefficients of both measures (cf. (3.7)). By (7.14) we see that the a-th Fourier-Stieltjes coefficient of the measure $\mu^{f_r}_{(1/e)} = \mu^{f_r} \star m_{(1/e)}$ equals $(c^f_a)\, r^{|a|}(1/e)^{|a|} = (r/e)^{|a|} c^f_a$. Theorem 7.1.8 implies that $(r/e)^{|a|} c^f_a$ is also the a-th coefficient of the measure $d\mu^{f_{(r/e)}} = f_{(r/e)}\, d\sigma$. Consequently both measures coincide. □

Let $M^1(G)$ be the set of all $\mu \in M(G)$, such that for every $r \in (0,1)$ the measure $\mu_r = \mu \star m_r$ can be expressed in the form $d\mu_r = f^{(r)}\, d\sigma = d\mu^{f^{(r)}}$ for some function $f^{(r)} \in C(G)$ with

$$\sup_{\substack{h \in G \\ 0<r<1}} \int_G \left| f^{(r)}(g) \right| dm_{(1/e)}(gh) < \infty. \tag{7.15}$$

To every $\mu \in M^1(G)$ we assign the function f_μ on $\mathbb{D}_G$, defined by $f_\mu(r \diamond g) = \widetilde{f}^{(r)}(g)$, where for every fixed $r \in (0,1)$ the function $f^{(r)} \in C(G)$ is such that $d\mu^{f^{(r)}} = f^{(r)}\, d\sigma$.

Theorem 7.3.2. *Let $\mu \in M^1(G)$ be a measure, such that $d\mu_r = d\mu^{f^{(r)}} = f^{(r)} d\sigma$ with $f^{(r)} \in C(G)$ for each $r \in (0,1)$. The map $\mu \longmapsto f_\mu$ is a bijective mapping between $M^1(G)$ and $\mathcal{H}^1(\overline{\mathbb{D}}_G)$.*

Proof. Let $\mu \in M^1(G)$. We claim that the function f_μ belongs to $\mathcal{H}^1(\mathbb{D}_G)$. Because of $(f_\mu)_r\, d\sigma = f^{(r)}\, d\sigma = d\mu^{f^{(r)}} = d\mu_r$, we have that $c^\mu_a r^{|a|} = c^{f_\mu}_a r^{|a|}$ for all $a \in \varGamma$. Together with (7.15) and Theorem 7.1.8, this implies that $f_\mu \in \mathcal{H}^1(\overline{\mathbb{D}}_G)$.

Conversely, given an $f \in \mathcal{H}^1(\overline{\mathbb{D}}_G)$, consider the family of measures $\{\mu_r\}_{r \in (0,1)}$ with $d\mu_r = f_r\, d\sigma$. Clearly, $f_r \in C(G)$ for any r, $0 < r < 1$. Since

$$\begin{aligned} \|\mu_r\| &= \int_G \left| f_r(g) \right| d\sigma(g) = \int_G \int_G \left| f_r(gh) \right| d\sigma(g)\, dm_{(1/e)}(h) \\ &\le \int_G \Big(\sup_{0<r<1} \int_G \left| f_r(gh) \right| dm_{(1/e)}(h) \Big)\, d\sigma(g) \le \|f\|_1, \end{aligned}$$

we can choose $r_n \nearrow 1$ so that the sequence $\{\mu_{r_n}\}_n$ is weak*-convergent to a measure $\nu \in M(G)$. Since $\nu_r = \nu \star m_r = w^*\text{-}\lim_n \mu^{f_{r_n}} \star m_r$, and the weak*-limit of $\{\mu^{f_{r_n}} \star m_r\}$ as $r_n \nearrow 1$ equals $\big(L^\infty\text{-}\lim_n f_r(r_n \diamond g)\big)\, d\sigma(g) = f_r(g)\, d\sigma(g) = \mu^{f_r}(g)$, we obtain that $d\nu_r = d\mu^{f_r} = f_r\, d\sigma$ for all $r \in (0,1)$. Therefore, $\nu \in M^1(G)$, and $f = f_\nu$. Since $\operatorname{sp}(m_r) = \Gamma$, the uniqueness of the Fourier-Stieltjes expansion implies that ν does not depend on the choice of the sequence $r_n \nearrow 1$. □

Theorem 7.3.2 shows that $M^1(G) = \big\{\mu \in M(G) \mu_r = f_r d\sigma,\ r \in (0,1),\ f \in \mathcal{H}^1(\overline{\mathbb{D}}_G)\big\}$. Clearly, for any $f \in C(G)$ the measure $d\mu^f = f\, d\sigma \in M^1(G)$. Hence the closure of $M^1(G)$ in $M(G)$ contains measures of type $g\, d\sigma,\ g \in L^1(G,\sigma)$. However, there are functions integrable with respect to the Haar measure σ on G, that are not extendable as harmonic functions on $\mathbb{D}_G$. This implies that $M^1(G)$ is not closed in $M(G)$.

For any $f \in \mathcal{H}^1(\mathbb{D}_G)$ denote by $M_{(1/e)}(f)$ the family of measures $\mu_h^f,\ h \in G$ of type

$$d\mu_h^f(g) = w^*\text{-}\lim_{r\nearrow 1} f_r(g)\, dm_{(1/e)}(h^{-1}g),\ h \in G,$$

which exist according to Theorem 7.2.5(e). It is obvious that $M_{(1/e)}(f+g) = M_{(1/e)}(f) + M_{(1/e)}(g)$. Let $M^1_{(1/e)}(G)$ be the space of all families of type $M_{(1/e)}(f)$, $f \in \mathcal{H}^1(\mathbb{D}_G)$.

Theorem 7.3.3. (a) *Endowed with the norm* $\big\|M_{(1/e)}(f)\big\| = \sup_{h\in G} \|\mu_h^f\|$, *the set* $M^1_{(1/e)}(G)$ *is a Banach space.*

(b) $\big\|M_{(1/e)}(f)\big\| = \|f\|_1$ *for every* $f \in \mathcal{H}^1(\mathbb{D}_G)$.

Proof. Theorem 7.2.5(e) implies that $\|\mu_h^f\| = \lim_{r\nearrow 1} \int_G |f_r(gh)|\, dm_{(1/e)}(g)$. Thus, $\big\|M_{(1/e)}(f)\big\| = \|f\|_1$, which proves (b). On the other hand, the mapping $M_{(1/e)}(f) \longmapsto f$ is an isometric isomorphism between $M_{(1/e)}(G)$ and the space $\mathcal{H}^1(\mathbb{D}_G)$, which proves (a). □

The following theorem and its proof are similar to the corresponding results in the classical setting (cf. [H3]).

Theorem 7.3.4. *For a* $f \in \mathcal{H}^1(\mathbb{D}_G)$ *let* $f^* = \lim_{r\nearrow 1} f_r$ *be the boundary value function, existing by Theorem* 7.2.5. *Then:*

(a) $d\mu_h^f(g) = f^*(g)\, dm_{(1/e)}(h^{-1}g) + d\nu^f(g)$, *where* ν^f *is a singular measure with respect to the measure* $m_{(1/e)}(h^{-1}g)$.

(b) $f(r \diamond h) = (f^* \star K_r)(h) + (K_r \star \nu^f)(h)$, *where* K_r *is the Radon-Nikodym derivative of* μ_r *with respect to* $m_{(1/e)}$.

Example 7.3.5. Without loss of generality we can assume that $2\pi \in \Gamma$. Let $K = \{g \in G \colon \chi^{2\pi}(g) = 1\} = \operatorname{Ker}(\chi^{2\pi})$. Recall that the group G can be obtained from the set $K_0 = K \times [0,1]$ by identifying the points $(g,1)$ and $(g_1\, g, 0)$, $g \in G$. Here g_t is the element $j_\imath(t) \in G$, where $j_\imath \colon \mathbb{R} \longrightarrow G$ is the homomorphic embedding of $\mathbb{R}$ into G with $j_\imath(0) = \imath$. The mapping $\psi \colon K \times [0,1] \longrightarrow G$ is defined by $\psi(g,t) = g\, g_t$. Fix a $g_0 \in K$, and let $f(g,t)$ be a function on $K_0 = K \times [0,1]$, such that:

(a) $f(g,0) = f(g,1) = 0$ for all $h \in K$, and $f(g_0,t) = 0$ for all $t \in [0,1]$.

(b) $f(g,t)$ is a continuous function on $K_0 \setminus (g_0, 1/2)$.

(c) $f \geq 0$ and $\displaystyle\int_0^1 f(g,t)\, dt \longrightarrow 1$ whenever $g \longrightarrow g_0$.

Consider the atomic measure $\delta_{(g_0,1/2)}$ concentrated at the point $(g_0, 1/2)$. From (c) it follows that $\delta_{(g_0,1/2)} = w^*\text{-}\lim\limits_{g \to g_0} \mu(g)$, where $d\mu(g) = f(g,t)\, dt$. If $\widetilde{f} = f \circ \psi$, then the function $g(r \diamond h) = \big(\widetilde{f}^* \star K_r\big)(h) + \big(K_r \star \delta_{(g_0,1/2)}\big)(h)$ belongs to $\mathcal{H}^1(\mathbb{D}_G)$.

7.4 The space $\mathcal{H}^p(\mathbb{D}_G)$ as Banach algebra

For any $p \in [1,\infty)$ the space $\mathcal{H}^p(\mathbb{D}_G)$ possesses a natural multiplication. Namely, given $f, h \in \mathcal{H}^p(\mathbb{D}_G)$, their *Hadamard product* $f \times h$ is the function defined by

$$\big(f \times h\big)\big((r_1 r_2) \diamond g\big) = (f_{r_1} \star h_{r_2})(g) = \int_G f_{r_1}(gk^{-1})\; h_{r_2}(k)\, d\sigma(k).$$

Note that if $s_1 s_2 = r_1 r_1$, then

$$\begin{aligned} f_{r_1} \star h_{r_2} &= \big(f_{s_1} \star m_{(r_1/s_1)}\big) \star \big(h_{s_2} \star m_{(r_2/s_2)}\big) \\ &= \big(f_{s_1} \star\; h_{s_2}\big) \star \big(m_{(r_1/(s_1)} \star m_{(r_2/(s_2)}\big) \\ &= f_{s_1} \star h_{s_2} \star m_{(r_1 r_2)/(s_1 s_2)} = f_{s_1} \star h_{s_2} \star m_1 = f_{s_1} \star h_{s_2}. \end{aligned}$$

Consequently, Hadamard's product is a well-defined operation in $\mathcal{H}^p(\mathbb{D}_G)$.

Theorem 7.4.1. *The Hadamard product $f \times h$ of any $f, h \in \mathcal{H}^p(\mathbb{D}_G)$, $1 \leq p < \infty$, belongs to the space $\mathcal{H}^p(\mathbb{D}_G)$, and*

$$\|f \times h\|_p \leq \|f\|_p\, \|h\|_p,$$

where $\|\cdot\|_p$ is the L^p-norm defined in Definition 7.2.1.

Proof. For any $p \in [1,\infty)$ we have

$$\int_G \big|\big(f \times h\big)\big((r_1 r_2) \diamond g\big)\big|^p dm_{(1/e)}(g) \leq \int_G \int_G \big|f_{r_1}(gk^{-1})\, h_{r_2}(k)\big|^p d\sigma(k)\, dm_{(1/e)}(g)$$

$$\leq \int_G \big|h_{r_2}(k)\big|^p \Big(\int_G \big|f_{r_1}(gk^{-1})\big|^p dm_{(1/e)}(g)\Big) d\sigma(k) \leq \|f\|_p^p \int_G \big|h_{r_2}(k)\big|^p d\sigma(k).$$

Since the Haar measure σ is invariant, and $m_{(1/e)}$ is a probability measure on G, we have

$$\begin{aligned}\int_G \left|h_{r_2}(k)\right|^p d\sigma(k) &= \int_G\int_G \left|h_{r_2}(k)\right|^p d\sigma(k)\, dm_{(1/e)}(g)\\ &= \int_G\int_G \left|h_{r_2}(kg^{-1})\right|^p d\sigma(k)\, dm_{(1/e)}(g)\\ &= \int_G \Big(\int_G \left|h_{r_2}(kg^{-1})\right|^p dm_{(1/e)}(g)\Big) d\sigma(k) \le \|h\|_p^p.\end{aligned}$$

Therefore, $\|f \times h\|_p^p \le \|f\|_p^p\ \|h\|_p^p$, as claimed. □

As an immediate consequence from Theorem 7.4.1 we obtain that $\mathcal{H}^p(\mathbb{D}_G)$ is a Banach algebra with multiplication given by the Hadamard product. We will characterize the ideals of this algebra.

Lemma 7.4.2. *Any one-dimensional ideal J of the algebra $\mathcal{H}^p(\mathbb{D}_G)$ is of type $J = \mathbb{C}\chi^a$ for some $a \in \Gamma$.*

Proof. It is easy to see that $c_a^f\, \chi^a = f \times \chi^a$ for any $f \in J$ and $a \in \Gamma$. Indeed,

$$\begin{aligned}&\big(f \times \chi^a\big)\big((r_1 r_2) \diamond g\big) = \int_G f_{r_1}(gk^{-1})\, \chi^a_{r_2}(k)\, d\sigma(k) = \int_G f_{r_1}(gk)\, \chi^a_{r_2}(k^{-1})\, d\sigma(k)\\ &= \int_G f_{r_1}(gk)\, \overline{\chi}^a_{r_2}(k)\, d\sigma(k) = r_2^{|a|} \int_G f_{r_1}(k)\, \overline{\chi}^a(kg^{-1})\, d\sigma(k)\\ &= r_2^{|a|}\overline{\chi}^a(g^{-1}) \int_G f_{r_1}(k)\, \overline{\chi}^a(k)\, d\sigma(k) = r_2^{|a|}\chi^a(g) \int_G f_{r_1}(k)\, \overline{\chi}^a(k)\, d\sigma(k)\\ &= r_2^{|a|}\chi^a(g)\, c_a^{f_{r_1}} = r_1^{|a|} r_2^{|a|}\chi^a(g)\, c_a^f(r_1) = (r_1 r_2)^{|a|}\chi^a(g)\, c_a^f = \chi^a\big((r_1 r_2)\diamond g\big)\, c_a^f.\end{aligned}$$

Hence, $f \times \chi^a = c_a^f\, \chi^a$, as claimed. It follows that $c_a^f\, \chi^a$ for any $f \in J$ and $a \in \Gamma$, and therefore, any finite linear combination of type $\sum\limits_{\substack{f \in J\\ a \in \Gamma}} c_a^f\, \chi^a$ belongs to J. Since $\dim(J) = 1$, there is an $a \in \Gamma$ such that $c_b^f = 0$ for all $b \in \Gamma \setminus \{a\}$ and all $f \in J$. Consequently, $J = \mathbb{C}\chi^a$. □

Let J be a closed ideal of the algebra $\mathcal{H}^p(\mathbb{D}_G)$. Recall that the hull of J is the set $\operatorname{hull}(J)$ of all $a \in \Gamma$, such that $c_a^f = 0$ for every $f \in J$.

Theorem 7.4.3. *The mapping $J \longmapsto \operatorname{hull}(J)$ is a bijection between the family of weakly closed ideals of the algebra $\mathcal{H}^p(\mathbb{D}_G)$ and the family of subsets of the dual group Γ.*

Proof. Suppose that J is a weakly closed ideal of $\mathcal{H}^p(\mathbb{D}_G)$. As we saw in Lemma 7.4.2, if $a \notin \operatorname{hull}(J)$, then $\mathbb{C}\chi^a \subset J$. Hence, the set $I = \{f \in \mathcal{H}^p(\mathbb{D}_G)\colon\ \operatorname{sp}(f) \cap \operatorname{hull}(J) = \emptyset\}$ is contained in J. Hence the weak closure $[I]_w$ of I in $\mathcal{H}^p(\mathbb{D}_G)$ also is contained in J. Since any function $f \in J \subset \mathcal{H}^p(\mathbb{D}_G)$ can be approximated weakly by linear combinations p of characters of G with $\operatorname{sp}(p) \subset \operatorname{sp}(f)$, we obtain that $[I]_w = J$. Hence, the mapping $J \longmapsto \operatorname{hull}(J)$ is bijective. Conversely, let $K \subset \Gamma$, and $I_K = \{f \in \mathcal{H}^p(\mathbb{D}_G)\colon\ \operatorname{sp}(f) \cap K = \emptyset\}$. One can easily check that in this case the ideal I_K is weakly closed, and $\operatorname{hull}(I_K) = K$. □

Note that since the convolution of two functions in $L^p(G,\sigma)$ belongs to $C(G)$, Hadamard's product of functions in $\mathcal{H}^p(\mathbb{D}_G)$ is in $\mathcal{H}_c(\overline{\mathbb{D}}_G)$.

Theorem 7.4.4. *The space $\mathcal{H}^p_c(\overline{\mathbb{D}}_G)$ is the minimal closed ideal in the algebra $\mathcal{H}^p(\mathbb{D}_G)$ with empty hull, which contains $\mathcal{H}_c(\overline{\mathbb{D}}_G)$.*

Proof. From $\mathcal{H}_c(\overline{\mathbb{D}}_G) \subset \mathcal{H}^p(\overline{\mathbb{D}}_G)$ we see that $\mathcal{H}^p_c(\overline{\mathbb{D}}_G)$ is a closed ideal in $\mathcal{H}^p(\mathbb{D}_G)$. It is obvious that $\mathcal{H}^p_c(\overline{\mathbb{D}}_G)$ has an empty hull, and that $\mathcal{H}^p_c(\overline{\mathbb{D}}_G)$ is the minimal closed ideal of $\mathcal{H}^p(\mathbb{D}_G)$ that contains $\mathcal{H}_c(\overline{\mathbb{D}}_G)$. □

As an immediate consequence from Theorems 7.4.3 and 7.4.4 we obtain the following.

Corollary 7.4.5. *Let M be a closed maximal ideal of the algebra $\mathcal{H}^p(\mathbb{D}_G)$. Then either there is an element $a \in \Gamma$ for which $M = \{f \in \mathcal{H}^p(\mathbb{D}_G)\colon c^f_a = 0\}$, or, M is a closed hyperspace in $\mathcal{H}^p(\mathbb{D}_G)$ that contains $\mathcal{H}(\overline{\mathbb{D}}_G)$.*

We recall that, given a semigroup $S \subset \Gamma$, $\mathcal{H}^p_S(\mathbb{D}_G) \cong \mathcal{L}^{p,q}_S(G) = \{f \in \mathcal{L}^{p,q}(G)\colon\ \operatorname{sp}(f) \in S\}$. Since the space $\mathcal{H}^p_S(\mathbb{D}_G)$ is an ideal in the algebra $\mathcal{H}^p(\mathbb{D}_G)$, then any ideal of $\mathcal{H}^p_S(\mathbb{D}_G)$ is also an ideal in $\mathcal{H}^p(\mathbb{D}_G)$. Therefore, the problem for characterizing ideals in $\mathcal{H}^p_S(\mathbb{D}_G)$ can be reduced to characterizing these ideals in $\mathcal{H}^p(\mathbb{D}_G)$, whose hull contains the set $\Gamma \setminus S$.

7.5 Fatou type theorems for families of harmonic measures on groups

Fatou's theorem on radial limits of harmonic functions in $\mathbb{D}$ is part of measure theory, rather than of function theory. This is remarkably evident in the case $p = 1$, when the theorem of Fatou is expressed entirely in terms of measures.

Definition 7.5.1. A set of Borel measures $\widetilde{\mu} = \{\mu^{(r)}\}_{r\in(0,1)}$ is said to be a *harmonic family of measures* on G, if $\mu^{(r_1)} = \mu^{(r_2)} \star m_{(r_1/r_2)}$ for any $0 < r_1 < r_2 < 1$. The space of all harmonic families of measures on G will be denoted by $\mathcal{M}_{\mathcal{H}}(G)$.

Clearly, $\mathcal{M}_{\mathcal{H}}(G)$ is a linear space. Any harmonic family $\widetilde{\mu}$ can be interpreted as a measure-valued map $\widetilde{\mu}\colon (0,1) \longrightarrow M(G) : r \longmapsto \mu^{(r)}$.

Let $\mu \in M(G)$ be a Borel measure on G. The family $\widetilde{\mu} = \{\mu \star m_r\}_{r\in(0,1)}$ is a standard example of a harmonic family of measures on G. Indeed, since $m_{r_1 r_2} = m_{r_1} \star m_{r_2}$, we have $\mu \star m_{r_1} = \mu \star m_{r_2} \star m_{(r_1/r_2)} = \mu_{r_2} \star m_{(r_1/r_2)}$.

Lemma 7.5.2. *If $\widetilde{\mu} \in \mathcal{M}_{\mathcal{H}}(G)$ is a harmonic family of measures on the group G, then*

(a) $\widetilde{\mu} : r \longmapsto \mu^{(r)}$ *is a continuous mapping from* $(0,1)$ *to* $M(G)$*, and*

(b) $\|\mu^{(r_1)}\| \leq \|\mu^{(r_2)}\|$ *whenever* $r_1 < r_2$.

Proof. Without loss of generality we can assume that $\mu^{(r)} = \mu_0 \star m_r$ for some $\mu_0 \in M(G)$. If this is not the case, we can replace, right from the beginning, the given harmonic family by $\widetilde{\nu} = \{\nu^{(r)}\}_{r\in(0,1)}$, where r_0 is a fixed number in $(0,1)$, and $\nu^{(r)} = \mu^{(r_0)} \star m_r = \mu^{(r_0 r)}$. If $r' \longrightarrow r$, then

$$\lim_{r'\to r} \left\|\mu^{(r')} - \mu^{(r)}\right\| \leq \|\mu_0\| \lim_{r'\to r} \|m_r' - m_r\|.$$

Since $L^1\text{-}\lim\limits_{r'\to r} (dm_{r'}/dm_r) = 1$ (cf. Lemma 7.2.11), we have that $m_r = \lim\limits_{r'\to r} m_{r'}$, and therefore, $\lim\limits_{r'\to r} \|m_r' - m_r\| = 0$. This proves the first part of the statement. The second part is obvious. □

We recall that given an $f \in C(G)$, the function $\widetilde{f}(r \diamond g) = (f \star m_r)(g) = f_r(g)$ is continuous on $\overline{\mathbb{D}}_G$ and harmonic on $\mathbb{D}_G$ (e.g. Lemma 7.2.11). Consider the family $\{\mu_r^f\}_{r\in(0,1)} \in \mathcal{M}_{\mathcal{H}}(G)$, where $d\mu^f = f \star d\sigma$, thus $d\mu_r^f = d\mu^f \star dm_r = (f \star d\sigma) \star dm_r$. Hence, $d\mu_r^f(g) = (f \star dm_r) \star d\sigma(g) = \widetilde{f}(r \diamond g)\, d\sigma(g)$. Consequently, the family $\{\mu_r^f\}_{r\in(0,1)}$ is uniquely defined by the harmonic extension $\widetilde{f}$ of f. Note that the mapping $r \longmapsto \mu_r^f$, defined on $(0,1)$, can be extended naturally as a continuous mapping from $[0,1]$ to $M(G)$.

Lemma 7.5.3. *Let $f \sim \sum\limits_{a\in\Gamma} c_a^f \chi^a$ be the Fourier series expansion of a function $f \in L^1(G,\sigma)$. For any $r \in [0,1]$ there exists a function $f_r \in L^1(G,\sigma)$ such that*

(a) $f_r \sim \sum\limits_{a\in\Gamma} c_a^f \chi^a r^{|a|}$*, and*

(b) $r \longmapsto f_r$ *is a continuous map from* $[0,1]$ *to* $L^1(G,\sigma)$.

Proof. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on G which converges to f in the $L^1(G,\sigma)$-norm, then $\mu^f = \lim\limits_{n\to\infty} \mu^{f_n}$, where $d\mu^{f_n} = f_n\, d\sigma$, and $d\mu^f = f\, d\sigma$. Therefore, the harmonic family of measures $\widetilde{\mu}_n = \left\{(f_n)_r\, d\sigma\right\}_{r\in(0,1)}$ can be extended to a continuous map from $[0,1]$ into $M(G)$. By Theorem 7.3.3(b) the sequence of measures $\left\{\mu_n^{(r)}\right\}_n = \left\{\mu^{(f_n)_r}\right\}_n$ converges to a measure $\mu^{(r)}$ as $n \longrightarrow \infty$. Hence, the sequence $(f_n)_r$ converges to a function $f_r \in L^1(G,\sigma)$ uniformly on $[0,1]$. Therefore, the sequence $\left\{(f_n)_r \colon n \in \mathbb{N},\ r \in [0,1]\right\}_{n=1}^{\infty} \subset C\big([0,1], C(G)\big)$ converges to the continuous function $r \longmapsto f_r$ in $C\big([0,1], L^1(G,\sigma)\big)$. Since $(f_n)_1 = f_n \longrightarrow f$, the Fourier series of $(f_n)_1$ converges coefficient-wise to the Fourier series of f. Note that $f_0 = f(0) = c_0^f$. Consequently, the Fourier series of f_r is $\sum c_a^f \chi_a r^{|a|}$. □

Corollary 7.5.4. *Let $\widetilde{\mu} = \{\mu^{(r)}\}_{r\in(0,1)}$ be a harmonic family of measures on G, such that $d\mu^{(r_0)} = d\mu^f = f\,d\sigma$ for some $f \in L^1(G,\sigma)$ and $r_0 \in (0,1)$. Then the map $\widetilde{\mu} : (0,1) \longmapsto M(G)$ can be extended to a continuous map $\widetilde{\mu}'\colon [0,1] \longmapsto M(G)$.*

Definition 7.5.5. Let $p \geq 1$. A harmonic family of measures $\widetilde{\mu} = \{\mu^{(r)}\}_{r\in(0,1)}$ on G is called L^p-harmonic, if:

(a) $d\mu^{(r)} = d\mu^{f_r} = f_r\,d\sigma$ for some $f_r \in L^p(G,\sigma)$ for every $r \in (0,1)$.

(b) $\sup\limits_{0<r<1} \|f_r\|_p < \infty$.

We denote by $\mathcal{M}^p_{\mathcal{H}}(G)$ the space of L^p-harmonic families of measures on G.

The properties of the spaces $\mathcal{M}^p_{\mathcal{H}}(G)$ resemble very much the properties of the spaces $L^p(G,\sigma)$.

Theorem 7.5.6 (Fatou's theorem for the space $\mathcal{M}^p_{\mathcal{H}}(G)$)**.** *Let $\widetilde{\mu} = \big\{\mu^{(r)}\big\}_{r\in(0,1)}$ be an L^p-harmonic family of measures on G. Then*

(a) *If $1 < p \leq \infty$, there exists a function $f \in L^p(G,\sigma)$ such that $\mu^{(r)} = \mu^f \star m_r$, where $d\mu^f = f\,d\sigma$, and the map $\widetilde{\mu}\colon (0,1) \longmapsto M(G)$ can be extended to a continuous map $\widetilde{\mu}' : [0,1] \longmapsto M(G)$.*

(b) *If $p = 1$, there exists a $\mu_0 \in M(G)$ such that $\mu^{(r)} = \mu_0 \star m_r$.*

The proof makes use of Lemma 7.5.3, and in general follows the lines of proof of the classical Fatou theorem on radial limits of harmonic functions on $\mathbb{D}$ (cf. [Hof], p. 55).

A measure μ on G is called an *analytic measure* (or, *Γ_+-measure*), if $\mathrm{sp}\,(\mu) \subset \Gamma_+$. The class of all analytic measures on G will be denoted by $M_{\Gamma_+}(G)$. If Γ is isomorphic to $\mathbb{Z}$, then $G \cong \mathbb{T}$, and any measure in $M_{\mathbb{Z}_+}(\mathbb{T})$ is absolutely continuous with respect to the Haar measure σ on G. According to Riesz's representation theorem, its Radon-Nikodym derivative belongs to the Hardy space H^1. If Γ is not isomorphic to $\mathbb{Z}$ however, there are analytic measures on G that are singular with respect to σ.

Theorem 7.5.7. *Let $\mu \in M_{\Gamma_+}(G)$ be a regular Borel measure on G, such that $\mu \star m_r$ is absolutely continuous with respect to the Haar measure σ on G for some $r \in (0,1)$. Then there exists a function $f \in L^1(G,\sigma)$ such that $\mathrm{sp}\,(f) \subset \Gamma_+$ and $d\mu = d\mu^f = f\,d\sigma$.*

For the proof we need several properties of Γ_+-analytic measures on G. Without loss of generality we can assume that $2\pi \in \Gamma$. Let $K = \mathrm{Ker}\,(\chi^{2\pi}) = \big\{g \in G\colon \chi^{2\pi}(g) = 1\big\}$. Given a $t \in \mathbb{R}$, choose $g_t \in G$ so that $\chi^a(g_t) = e^{iat}$ for each

$a \in \Gamma$. The map $t \longmapsto g_t$ is an imbedding of the group $\mathbb{R}$ into G. This map generates a homomorphism $\pi\colon K \times \mathbb{R} \longrightarrow G$, defined as $\pi(g,t) = g\, g_t$. We recall that the kernel of π is the set $\operatorname{Ker}(\pi) = \big\{(g_n, -n)\big\}_{n=1}^{\infty}$ (cf. Section 3.1). Clearly, the set $K \times [0,1)$ is a fundamental domain of π, and therefore we can identify the group G with the set $K \times [0,1)$.

Denote by $\widetilde{M}(K \times \mathbb{R})$ the set of locally finite measures on $K \times \mathbb{R}$ which are invariant with respect to shifts by elements in $\operatorname{Ker}(\pi)$. Note that the restriction of the space $\widetilde{M}(K \times \mathbb{R})$ on the set $K \times [0,1)$ coincides with the space $M(G)$. In particular, if σ_K is the Haar measure on the group K, then the restriction of the measure $d\sigma_K \times dt$ on $K \times [0,1)$ is the Haar measure σ.

Let H^1 be the Hardy space on the unit circle $\mathbb{T}$. The fractional linear transformation $w(z) = \dfrac{i-z}{i+z}$ maps $\mathbb{R}$ onto $\mathbb{T}\setminus\{1\}$. Every function in the space $H^1 \circ w = \{f \circ w\colon f \in H^1\}$ can be extended as an analytic function in the upper half-plane $\mathbb{C}_+$.

Let $\widetilde{M}^1(K \times \mathbb{R})$ be the space of measures on $K \times \mathbb{R}$ of type $f(g,t)\,\big(d\nu(g) \times dt\big)$, where:

(a) ν is a probability measure on K.

(b) $f(g,t) \in L^1\Big(d\nu(g) \times \dfrac{dt}{1+t^2}\Big)$.

(c) For ν-almost all $g \in K$ the function $t \longmapsto f(g,t)$ belongs to $H^1 \circ w$.

(d) The measure $f(g,t)\, d\nu(g)$ belongs to $\widetilde{M}(K \times \mathbb{R})$.

(e) $\big|f(g,t)\big|\,\big(d\nu(g) \times dt\big)$ and $d\nu(g) \times dt$ are mutually absolutely continuous measures.

The proof of the next lemma is straightforward (cf. [G10], Theorem 3.2).

Lemma 7.5.8. *The restriction of the space $\widetilde{M}^1(K \times \mathbb{R})$ on $K \times [0,1]$ is isometrically isomorphic to $M_{\Gamma_+}(G)$.*

Lemma 7.5.9. *For any $\mu \in M_{\Gamma_+}(G)$, let $d\mu' = f(g,t)\big(d\nu(g) \times dt\big)$ be the measure in $\widetilde{M}^1(K \times \mathbb{R})$, for which $\mu'\big|_{K \times [0,1]} = \mu$, where f is as in (b) from the above. If $\mu^{(r)} = \mu \star m_r$, then $\big(d(\mu')^{(r)}\big)(g,t) = f_r(g,t)\,\big(d\nu(g) \times dt\big)$, where*

$$f_r(g,t) = \frac{1}{\pi} \int_{\mathbb{R}} f(g, x+t) \frac{y_r\, dx}{y_r^2 + x^2}, \quad \textit{with } y_r = -\ln r.$$

Proof. Observe that linear combinations of the measures $d\mu^{\chi^a} = \chi^a d\sigma$, $a \in \Gamma_+$, are weak*-dense in $M_{\Gamma_+}(G)$. Since the convolution operator preserves weak*-convergence, it suffices to prove the lemma only for measures of type $d\mu^{\chi^a} = \chi^a d\sigma,\ a \in \Gamma_+$. Note that $d(\mu^{\chi^a})'(g,t) = \chi^a(g)\, e^{iat} r^{|a|}\big(d\sigma_K(g) \times dt\big) \in \widetilde{M}^1(K \times \mathbb{R})$ is the corresponding measure to $d\mu^{\chi^a}$ on $K \times \mathbb{R}$. For every $r \in (0,1)$ the corresponding measure to $d(\mu^{\chi^a})^{(r)} = d\mu^{\chi^a} \times dm_r = \chi^a r^{|a|} d\sigma$ is , $\big(d(\mu^{\chi^a})'\big)(g) = \chi^a(g)\, e^{iat} r^{|a|}\big(d\sigma_K(g) \times dt\big)$. Hence,

$$e^{iat} r^{|a|} = \frac{1}{\pi} \int\limits_{\mathbb{R}} e^{ia(t+x)} \frac{y_r\, dx}{y_r^2 + x^2},$$

which completes the proof. □

Note that for any $\mu \in M_{\Gamma_+}(G)$ we have $\int\limits_G \chi^a d\mu = 0,\ a \in \Gamma_+ \setminus \{0\}$, i.e. the measures $\mu \in M_{\Gamma_+}(G)$ are orthogonal to the ideal $\mathcal{J}_\omega$ of the G-disc algebra A_{Γ_+}, generated by characters $\chi^a,\ a \in \Gamma_+ \setminus \{0\}$. By the Lebesgue decomposition theorem, the measure μ can be expressed as $\mu = \mu^f + \mu_s$ with $\mu^f, \mu_s \in M_{\Gamma_+}(G)$, where $d\mu^f = f\, d\sigma$ is the absolutely continuous component, and μ_s is the singular component of μ with respect to σ (e.g. [G10]).

Proof of Theorem 7.5.7. It is enough to show that if a measure $\mu \in M_{\Gamma_+}(G)$ is singular with respect to σ, and $\mu^{(r)}$ is absolutely continuous with respect to σ for some $r \in (0,1)$, then $\mu = 0$. If it were, the corresponding measure $\mu' \in \widetilde{M}^1(K \times \mathbb{R})$ would have the form $d\mu'(g,t) = f(g,t)\big(d\nu(g) \times dt\big)$, where ν is a singular measure with respect to the Haar measure σ_K of K. For any $r \in (0,1)$ we have also $d(\mu')^{(r)}(g,t) = f_r(g,t)\big(d\nu(g) \times dt\big)$. Hence the measure $(\mu')^{(r)}\big|_{K\times[0,1]}$ is singular with respect to σ, in contradiction with the definition of the space $M^1(K \times \mathbb{R})$. □

Definition 7.5.10. A harmonic family of measures on G is said to be a harmonic family of analytic measures on G, if $\widetilde{\mu} = \{\mu^{(r)}\}_{r\in(0,1)}$, where $\mu^{(r)}$ are analytic measures on G for any $r \in (0,1)$. We denote by $\mathcal{M}^p_{\mathcal{H},\Gamma_+}(G)$ the space L^p-harmonic Γ_+-measure families on G.

Theorems 7.5.6 and 7.5.7 imply the following

Corollary 7.5.11.. *For any $\mu \in \mathcal{M}^p_{\mathcal{H},\Gamma_+}(G),\ p \geq 1$, there exists a measure $\mu_0 \in M_{\Gamma_+}(G)$ such that $\mu^{(r)} = \mu_0 \star m_r$.*

7.6 Notes

Most of the results in this chapter are from [G14]. Theorem 7.1.3 [resp. 7.3.2, 7.2.3 and 7.4.1] is due to Milaszewicz [M] [resp. [M1]]. Fatou type theorems for spaces $\mathcal{H}^p(\mathbb{D}_G)$ and $H^p(\mathbb{D}_G),\ 1 < p \leq \infty$ were considered first by Hoffman [H1, H2], who has developed the appropriate technique utilized also in this chapter. The boundary behavior of bounded generalized analytic functions on a G-disc were studied also in [KT] and [T2].

Chapter 8

Shift-invariant algebras and inductive limit algebras on groups

Uniform algebras that can be expressed as inductive limits of standard simpler algebras are of particular interest. For instance, some G-disc algebras are inductive limits of sequences of disc algebras, connected with finite Blaschke products, called also Blaschke inductive limit algebras. Here we show, among others, that only G-disc algebras, and the spaces $H^\infty(\mathbb{D}_G)$ with $G \subset \mathbb{Q}$, can be expressed as limits of countable inductive sequences of algebras of type $A(\mathbb{D})$ and H^∞ correspondingly. We study also inductive limits of sequences of spaces of type H^∞ and prove corresponding corona theorems. Further, we establish relationships between Bourgain algebras of coordinate algebras, and the Bourgain algebra of their inductive limit, and also between H^∞-spaces on $\mathbb{D}$, as coordinate algebras in an inductive sequence, and their inductive limit. While we state all results for general shift-invariant algebras A_S, they apply automatically to the particular cases of algebras AP_S of almost periodic functions, and of H^∞_S-algebras.

8.1 Inductive limits of H^∞-algebras

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions in the unit disc $\mathbb{D} \subset \mathbb{C}$. Every $f \in H^\infty(\mathbb{D})$ possesses a boundary value function f^* on $\mathbb{T}$, and the correspondence $f \longmapsto f^*$ is an isometric isomorphism between the spaces $H^\infty(\mathbb{D})$ and H^∞ on $\mathbb{T}$ (cf. Theorem 2.4.4). Suppose that $I = \{i_k^{k+1}\}_{k=1}^\infty$ is a sequence of homomorphisms $i_k^{k+1} : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D})$. Consider the inductive sequence

$$H^\infty(\mathbb{D}_1) \xrightarrow{i_1^2} H^\infty(\mathbb{D}_2) \xrightarrow{i_2^3} H^\infty(\mathbb{D}_3) \xrightarrow{i_3^4} \cdots \tag{8.1}$$

of algebras $H^\infty(\mathbb{D}_k) \cong H^\infty(\mathbb{D})$. Any adjoint mapping $(i_k^{k+1})^*: \mathcal{M}_k \longleftarrow \mathcal{M}_{k+1}$ maps the maximal ideal space of $H^\infty(\mathbb{D}_{k+1})$ into the maximal ideal space of $H^\infty(\mathbb{D}_k)$. The maximal ideal space of the limit algebra $[\varinjlim \{H^\infty(\mathbb{D}_k), i_k^{k+1}\}_{k\in\mathbb{N}}]$ is the limit $\mathcal{M}_I$ of the inverse sequence

$$\mathcal{M}_1 \xleftarrow{(i_1^2)^*} \mathcal{M}_2 \xleftarrow{(i_2^3)^*} \mathcal{M}_3 \xleftarrow{(i_3^4)^*} \mathcal{M}_4 \xleftarrow{(i_4^5)^*} \cdots \longleftarrow \mathcal{M}_I.$$

We recall that, according to Carleson's $\mathbb{D}$-corona theorem ([C]), the open unit disc $\mathbb{D}$ is a dense subset of every $\mathcal{M}_k$. In general, the mappings $(i_k^{k+1})^*$ do not necessarily map the open disc $\mathbb{D}$ onto itself. However, the most interesting situations arise when they do. In this section we will assume that the mappings $(i_k^{k+1})^*$ are inner non-constant functions on $\mathbb{D}$, and hence they map $\mathbb{D}$ onto itself.

Example 8.1.1. Consider the inverse sequence $\{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$, where $\overline{\mathbb{D}}_k = \overline{\mathbb{D}}$, and $\tau_k^{k+1}(z) = z^{d_k}$ on $\overline{\mathbb{D}}_k$. By Example 1.3.7, the inverse limit $\varprojlim \{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$ is a compact set, containing the compact abelian group $G_\Lambda = \varprojlim \{\mathbb{T}_{k+1}, z^{d_k}\}_{k\in\mathbb{N}}$ and the open set $\mathcal{D}_\Lambda = \varprojlim \{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$ from (1.16) and (1.17). In fact, the set $\varprojlim \{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$ coincides with the G_Λ-disc $\overline{\mathbb{D}}_{G_\Lambda} = \big([0,1]\times G_\Lambda\big)/\big(\{0\}\times G_\Lambda\big)$ over the group $G_\Lambda = \widehat{\Gamma}_\Lambda$. The connecting homomorphisms of the adjoint inductive sequence $\big\{H^\infty(\mathbb{D}_k), i_k^{k+1}\big\}_1^\infty$ of algebras $H^\infty(\mathbb{D}_k) \cong H^\infty(\mathbb{D})$ are the compositions $i_k^{k+1} = (\tau_k^{k+1})^*: H^\infty(\mathbb{D}_k) \longrightarrow H^\infty(\mathbb{D}_{k+1})$: $i_k^{k+1}(f) = f \circ \tau_k^{k+1}$, i.e. $\big(i_k^{k+1}(f)\big)(z) = f(z^{d_k})$ for $z \in \mathbb{D}_{k+1}$. Note that the elements of the component algebras $H^\infty(\mathbb{D}_k)$ can be interpreted as continuous functions on the G_Λ-disc $\mathbb{D}_{G_\Lambda}$. The uniform closure

$$H^\infty(\mathcal{D}_\Lambda) = \big[\varinjlim \big\{H^\infty(\mathbb{D}_k), (z^{d_k})^*;\ d_k \in \Lambda\big\}_{k\in\mathbb{N}}\big]$$

of the limit of the inductive sequence $\big\{H^\infty(\mathbb{D}_k), (z^{d_k})^*; d_k \in \Lambda\big\}_{k\in\mathbb{N}}$ in the space $C_b(\mathbb{D}_{G_\Lambda})$ of bounded continuous functions on $\mathbb{D}_{G_\Lambda}$, is a commutative Banach algebra of functions on $\mathbb{D}_{G_\Lambda}$, which is an inductive limit algebra (cf. Definition 1.3.5).

Example 8.1.2. Let $B = \{B_k\}_{k=1}^\infty$ be a sequence of finite Blaschke products $B_k: \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$, i.e.

$$B_k(z) = e^{i\theta_k} \prod_{l=1}^{n_k} \Big(\frac{z - z_l^{(k)}}{1 - \overline{z}_l^{(k)} z}\Big) \text{ for some } \ z_l^{(k)} \in \mathbb{D}.$$

Consider the inductive sequence

$$H^\infty(\mathbb{D}_1) \xrightarrow{i_1^2} H^\infty(\mathbb{D}_2) \xrightarrow{i_2^3} H^\infty(\mathbb{D}_3) \xrightarrow{i_3^4} \cdots, \tag{8.2}$$

of algebras $H^\infty(\mathbb{D})$, where the connecting homomorphisms $i_k^{k+1}: H^\infty(\mathbb{D}_k) \longrightarrow H^\infty(\mathbb{D}_{k+1})$ are compositions by the Blaschke products B_k, namely, $i_k^{k+1} = B_k^*$, so that $i_k^{k+1}(f) = f \circ B_k$ for every $k \in \mathbb{N}$. Denote by $\mathcal{D}_B = \varprojlim \{\mathbb{D}_{k+1}, B_k\}_{k\in\mathbb{N}}$ the limit of the inverse sequence $\{\mathbb{D}_{k+1}, B_k\}_{k=1}^{\infty}$. The uniform closure

$$H^\infty(\mathcal{D}_B) = \left[\varinjlim \left\{H^\infty(\mathbb{D}_k), B_k^*\right\}_{k\in\mathbb{N}}\right]$$

in $C_b(\mathcal{D}_B)$ of the limit of the inductive system $\left\{H^\infty(\mathbb{D}_k), B_k^*\right\}_{k=1}^{\infty}$, is a commutative Banach algebra of functions on $\mathcal{D}_B$, which is an inductive limit algebra.

Let $\Lambda = \{d_k\}_{k=1}^{\infty}$ be the sequence of orders of Blaschke products $\{B_k\}_{k=1}^{\infty}$ in Example 8.1.2, and let $\Gamma_\Lambda \subset \mathbb{Q}$ be the group generated by the numbers $1/m_k,\ k = 0, 1, 2, \dots$, where $m_k = \prod_{l=1}^{k} d_l,\ m_0 = 1$. Denote by $\mathcal{T}_k$ the standard d_k-sheeted lifting of the unit circle $\mathbb{T}$ to the Riemann surface $\mathcal{R}_{d_k}$ of the function z^{1/d_k}. Clearly $\mathbb{T}_k \cong \mathbb{T}$, and the diagram

$$\begin{array}{ccc} \mathcal{T}_k & \xleftarrow{\widetilde{B}_k} & \mathcal{T}_{k+1} \\ \psi_k \downarrow & & \psi_{k+1} \downarrow \\ \mathbb{T} & \xleftarrow{B_k} & \mathbb{T} \end{array}$$

is commutative for every $k = 0, 1, 2, \dots$, where ψ_k is the natural covering mapping $\psi_k : \mathcal{T}_k \longrightarrow \mathbb{T}$. Therefore, the infinite diagram

$$\begin{array}{ccccccc} \mathcal{T}_1 & \xleftarrow{\widetilde{B}_1} & \mathcal{T}_2 & \xleftarrow{\widetilde{B}_2} & \mathcal{T}_3 & \xleftarrow{\widetilde{B}_3} & \cdots \\ \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow & & \\ \mathbb{T}_1 & \xleftarrow{B_1} & \mathbb{T}_2 & \xleftarrow{B_2} & \mathbb{T}_3 & \xleftarrow{B_3} & \cdots \end{array}$$

is commutative, and hence the inverse sequence of circles $\mathbb{T}$

$$\mathbb{T}_1 \xleftarrow{B_1} \mathbb{T}_2 \xleftarrow{B_2} \mathbb{T}_3 \xleftarrow{B_3} \mathbb{T}_4 \xleftarrow{B_4} \cdots, \tag{8.3}$$

where $\widetilde{B}_k$ is the natural lifting of B_k on $\mathcal{T}_{k+1}$ (cf. Section 1.3), is isomorphic to the inverse sequence of sets $\mathcal{T}_k \subset \mathcal{R}_{d_k}$

$$\mathcal{T}_1 \xleftarrow{\widetilde{B}_1} \mathcal{T}_2 \xleftarrow{\widetilde{B}_2} \mathcal{T}_3 \xleftarrow{\widetilde{B}_3} \mathcal{T}_4 \xleftarrow{\widetilde{B}_4} \cdots. \tag{8.4}$$

In particular, if in Example 8.1.2 we take $B_k(z) = \tau_k^{k+1}(z) = z^{d_k}$, we obtain that the sequences

$$\mathcal{T}_1 \xleftarrow{\widetilde{\tau}_1^2} \mathcal{T}_2 \xleftarrow{\widetilde{\tau}_2^3} \mathcal{T}_3 \xleftarrow{\widetilde{\tau}_3^4} \mathcal{T}_4 \xleftarrow{\widetilde{\tau}_4^5} \cdots, \tag{8.5}$$

and

$$\mathbb{T}_1 \xleftarrow{\tau_1^2} \mathbb{T}_2 \xleftarrow{\tau_2^3} \mathbb{T}_3 \xleftarrow{\tau_3^4} \mathbb{T}_4 \xleftarrow{\tau_4^5} \cdots \longleftarrow G_\Lambda \tag{8.6}$$

are (topologically) isomorphic. Since on the other hand the sequences (8.4) and (8.5) are also isomorphic, so are the sequences

$$\mathbb{T}_1 \xleftarrow{B_1} \mathbb{T}_2 \xleftarrow{B_2} \mathbb{T}_3 \xleftarrow{B_3} \mathbb{T}_4 \xleftarrow{B_4} \cdots \longleftarrow \mathcal{T}_B \tag{8.7}$$

and

$$\mathbb{T}_1 \xleftarrow{\tau_1^2} \mathbb{T}_2 \xleftarrow{\tau_2^3} \mathbb{T}_3 \xleftarrow{\tau_3^4} \mathbb{T}_4 \xleftarrow{\tau_4^5} \cdots \longleftarrow G_\Lambda. \tag{8.8}$$

Proposition 1.3.8 implies that the set $\mathcal{T}_B$ is homeomorphic to the group G_Λ. Clearly, for the adjoint sequence we have

$$\mathbb{Z}^1 \xrightarrow{\widehat{B_1}} \mathbb{Z}^2 \xrightarrow{\widehat{B_2}} \mathbb{Z}^3 \xrightarrow{\widehat{B_3}} \cdots \longrightarrow \widehat{G}_\Lambda = \Gamma_\Lambda \subset \mathbb{Q}. \tag{8.9}$$

We have obtained the following

Lemma 8.1.3. *The inverse limit $\mathcal{T}_B = \varprojlim \{\mathbb{T}_{k+1}, B_k\}_{k\in\mathbb{N}}$ in (8.7) can be equipped with the structure of a compact abelian group isomorphic to G_Λ, whose dual group is $\Gamma_\Lambda = \widehat{G}_\Lambda \subset \mathbb{Q}$.*

Let $B = \{B_k\}_{k=1}^\infty$ be a sequence of finite Blaschke products on $\overline{\mathbb{D}}$. The limit $\mathcal{D}_B$ of the inverse sequence

$$\mathbb{D}_1 \xleftarrow{B_1} \mathbb{D}_2 \xleftarrow{B_2} \mathbb{D}_3 \xleftarrow{B_3} \mathbb{D}_4 \xleftarrow{B_4} \cdots \longleftarrow \mathcal{D}_B$$

is a Hausdorff space. The limit of the inductive sequence $\left\{H^\infty(\mathbb{D}_k), \beta_k^{k+1}\right\}_1^\infty$ of algebras $H^\infty(\mathbb{D}_k)$, the connecting homomorphisms of which are the composition operators $\beta_k^{k+1} = B_k^* \colon H^\infty(\mathbb{D}_k) \longrightarrow H^\infty(\mathbb{D}_{k+1}) : \big(\beta_k^{k+1}(f)\big)(z_{k+1}) = f\big(B_k(z_{k+1})\big)$, is an algebra of functions on $\mathcal{D}_B$ whose closure

$$H^\infty(\mathcal{D}_B) = \left[\varinjlim \left\{H^\infty(\mathbb{D}_k), \beta_k^{k+1}\right\}_{k\in\mathbb{N}}\right]$$

in $C_b(\mathcal{D}_B)$ is an inductive limit algebra.

We recall that a point $z_0 \in \mathbb{D}$ is said to be *singular* for a finite Blaschke product B, if $\operatorname{card} B^{-1}(z_0) < \operatorname{ord}(B)$.

Theorem 8.1.4. *Let $B = \{B_k\}_{k=1}^\infty$ be a sequence of finite Blaschke products on the unit disc $\overline{\mathbb{D}}$, each one with at most one singular point $z_0^{(k)}$, and such that $B_k(z_0^{(k+1)}) = z_0^{(k)}$. The corresponding inductive limit algebra*

$$H^\infty(\mathcal{D}_B) = \left[\varinjlim \left\{H^\infty(\mathbb{D}_k), \beta_k^{k+1}\right\}_{k\in\mathbb{N}}\right]$$

generated by B, is isometrically isomorphic to the algebra $H^\infty(\mathcal{D}_\Lambda)$, where $\Lambda = \{d_k\}_{k=1}^\infty$ with $d_k = \operatorname{ord} B_k$.

For the proof we need the following

Lemma 8.1.5. *If B is a finite Blaschke product with the only singular point $z_0 \in \mathbb{D}$, then*

$$B(z) = \frac{\tau_\theta(z)^m + B(z_0)}{1 + \overline{B(z_0)}\tau_\theta(z)^m},$$

where $m = \operatorname{ord} B$ and $\tau_\theta = e^{i\theta}\dfrac{z - z_0}{1 - \overline{z}_0 z}$ for some θ, $0 \le \theta < 2\pi$.

Proof. The restriction of B on $\mathbb{D}\setminus\{z_0\}$ generates a holomorphic covering from $\mathbb{D}\setminus\{z_0\}$ onto $\mathbb{D}\setminus\{B(z_0)\}$. The composition $\varphi\circ B$, where $\varphi(z) = \dfrac{z - B(z_0)}{1 - \overline{B(z_0)}z}$, generates a non-ramified m-sheeted holomorphic covering from $\mathbb{D}\setminus\{z_0\}$ onto $\mathbb{D}\setminus\{0\}$. There exists a biholomorphic map $\sigma\colon \mathbb{D}\setminus\{z_0\} \longrightarrow \mathbb{D}\setminus\{0\}$, such that $(\varphi\circ B)(z) = \sigma(z)^m$ (cf. [G13]). Clearly, $\sigma = \tau_\theta$ for some θ, $0 \le \theta < 2\pi$, i.e. $\varphi\big(B(z)\big) = \big(\tau_\theta(z)\big)^m$. Hence

$$B(z) = \varphi^{-1}\big(\tau_\theta(z)\big)^m = \frac{\tau_\theta(z)^m + B(z_0)}{1 + \overline{B(z_0)}\tau_\theta(z)^m}. \qquad \square$$

Proof of Theorem 8.1.4. By Lemma 8.1.5 we see that for every Möbius transformation φ_k on $\mathbb{D}$ with $\varphi_k(z_0^{(k)}) = 0$ there exists another Möbius transformation φ_{k+1} on $\mathbb{D}$ for which the diagram

$$\begin{array}{ccc} \mathbb{D} & \xleftarrow{B_k} & \mathbb{D} \\ \varphi_k \downarrow & & \varphi_{k+1} \downarrow \\ \mathbb{D} & \xleftarrow{\tau_k^{k+1}(z)=z^{d_k}} & \mathbb{D} \end{array}$$

is commutative. Hence, $\varphi_k \circ B_k = (\varphi_{k+1})^{d_k}$ and $\varphi_k(z_0^{(k)}) = 0$. Choose φ_0 to be the identity map on $\mathbb{D}$. By Lemma 8.1.5 we can define inductively a sequence $\{\varphi_k\}_{k=1}^\infty$ of Möbius transformations on $\mathbb{D}$, with adjoint isometric automorphisms φ_k^* on $H^\infty(\mathbb{D})$ such that the diagram

$$\begin{array}{ccc} H^\infty(\mathbb{D}) & \xrightarrow{\beta_k^{k+1}=B_k^*} & H^\infty(\mathbb{D}) \\ \varphi_k^* \uparrow & & \varphi_{k+1}^* \uparrow \\ H^\infty(\mathbb{D}) & \xrightarrow{(\tau_k^{k+1})^*} & H^\infty(\mathbb{D}) \end{array}$$

is commutative. Here $B_k^* \circ \varphi_k^* = \varphi_{k+1}^* \circ \big((\,\cdot\,)^{d_k}\big)^*$. Therefore, the infinite diagram

$$\begin{array}{ccccccc} H^\infty(\mathbb{D}) & \xrightarrow{\beta_1^2} & H^\infty(\mathbb{D}) & \xrightarrow{\beta_2^3} & H^\infty(\mathbb{D}) & \xrightarrow{\beta_3^4} & \cdots \\ \varphi_1^* \uparrow & & \varphi_2^* \uparrow & & \varphi_3^* \uparrow & & \\ H^\infty(\mathbb{D}) & \xrightarrow{(\tau_1^2)^*} & H^\infty(\mathbb{D}) & \xrightarrow{(\tau_2^3)^*} & H^\infty(\mathbb{D}) & \xrightarrow{(\tau_3^4)^*} & \cdots, \end{array}$$

where $\beta_k^{k+1} = B_k^*$ are the composition operators on $H^\infty(\mathbb{D})$ defined by $B_k^*(f) = f\circ B_k$, is also commutative. Therefore, the inductive sequences

$$H^\infty(\mathbb{D}) \xrightarrow{\beta_1^2} H^\infty(\mathbb{D}) \xrightarrow{\beta_2^3} H^\infty(\mathbb{D}) \xrightarrow{\beta_3^4} \cdots \longrightarrow H^\infty(\mathcal{D}_B), \text{ and}$$

$$H^\infty(\mathbb{D}) \xrightarrow{(\tau_1^2)^*} H^\infty(\mathbb{D}) \xrightarrow{(\tau_2^3)^*} H^\infty(\mathbb{D}) \xrightarrow{(\tau_3^4)^*} \cdots \longrightarrow H^\infty(\mathcal{D}_\Lambda),$$

with $\tau_k^{k+1}(z) = z^{d_k}$, are isomorphic. Consequently, $H^\infty(\mathcal{D}_B) \cong H^\infty(\mathcal{D}_\Lambda)$, as claimed. $\square$

Corollary 8.1.6. *If the Blaschke products* B_k *in Theorem* 8.1.4 *are of type* $B_k(z) = z^{d_k}\,\varphi_k(z)$, *where* φ_k *are Möbius transformations and* $d_k > 1$, *then the algebra* $H^\infty(\mathcal{D}_B)$ *is isometrically isomorphic to the algebra* $H^\infty(\mathcal{D}_\Lambda)$, *where* $\Lambda = \{\frac{1}{d_k}\}_{k=1}^{\infty}$.

Corollary 8.1.7. *If every Blaschke product* B_k *in Theorem* 8.1.4 *is a Möbius transformation, then the algebra* $H^\infty(\mathcal{D}_B)$ *is isometrically isomorphic to the algebra* $H^\infty(\mathbb{D}) \cong H^\infty$.

Indeed, Theorem 8.1.4 implies that $H^\infty(\mathcal{D}_B) \cong H^\infty(\mathcal{D}_\Lambda)$ with $\Lambda=\{1,1,\dots\}$. Therefore, $\Gamma_\Lambda = \mathbb{Z}$ and $G_\Lambda = \mathbb{T}$.

Let $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_k, \dots\}$ be a sequence of non-constant inner functions on $\mathbb{D}$. Consider the inverse sequence

$$\mathbb{D}_1 \xleftarrow{\varphi_1} \mathbb{D}_2 \xleftarrow{\varphi_2} \mathbb{D}_3 \xleftarrow{\varphi_3} \mathbb{D}_4 \xleftarrow{\varphi_4} \cdots \longleftarrow \mathcal{D}_\Phi, \tag{8.10}$$

where $\mathbb{D}_k = \mathbb{D}$. The limit $\varinjlim \{H_k^\infty, \varphi_k^*\}_{k\in\mathbb{N}}$ of the adjoint inductive sequence

$$H_1^\infty \xrightarrow{\varphi_1^*} H_2^\infty \xrightarrow{\varphi_2^*} H_3^\infty \xrightarrow{\varphi_3^*} \cdots$$

of algebras $H_k^\infty = H^\infty(\mathbb{D}_k) \cong H^\infty(\mathbb{D})$, where φ_k^*, $k \in \mathbb{N}$, are the composition operators $\varphi_k^*(f) = f \circ \varphi_k$ on H^∞, is a subalgebra of $C_b(\mathcal{D}_\Phi)$.

Definition 8.1.8. The inductive limit algebra

$$H^\infty(\mathcal{D}_\Phi) = \left[\varinjlim \{H^\infty, \varphi_k^*\}_{k\in\mathbb{N}}\right] \subset C_b(\mathcal{D}_\Phi)$$

is called the *algebra of Φ-hyper-analytic functions* on $\mathcal{D}_\Phi$.

Carleson's $\mathbb{D}$-corona theorem for the space H^∞ on the unit circle [C] asserts that, given $f_1, \dots, f_k$ in H^∞ with $\sum_{j=1}^{k} |f_j| \geq \sigma > 0$ on $\mathbb{D}$, there exist functions $g_1, \dots, g_k$ in H^∞ such that $\sum_{j=1}^{k} f_j g_j = 1$ on $\mathbb{D}$. If $\|f_j\|_\infty \leq 1$, then g_j can be chosen to satisfy the estimates $\|g_j\| \leq C(k,\sigma)$ for some constant $C(k,\sigma) > 0$. Below we state and solve the $\mathcal{D}_\Phi$-corona problem for the algebra $H^\infty(\mathcal{D}_\Phi)$, where $\mathcal{D}_\Phi = \varprojlim \{\overline{\mathbb{D}}_{k+1}, \varphi_k\}_{k\in\mathbb{N}}$.

Theorem 8.1.9. *If* $f_1, f_2, \dots, f_n$ *are Φ-hyper-analytic functions on* $\mathcal{D}_\Phi$, *for which* $\|f_j\| \leq 1$, *and*

$$\left|f_1(x)\right| + \cdots + \left|f_n(x)\right| \geq \delta > 0 \text{ for every } x \in \mathcal{D}_\Phi, \tag{8.11}$$

then there is a constant $K(n,\delta)$ *and Φ-hyper-analytic functions* $g_1, \dots, g_n$ *on* $\mathcal{D}_\Phi$ *with* $\|g_j\| \leq K(n,\delta)$, *such that the equality*

$$f_1(x)\, g_1(x) + \cdots + f_n(x)\, g_n(x) = 1$$

holds for every point x *in the set* $\mathcal{D}_\Phi$.

We observe that the adjoint mappings $\varphi_j^*\colon H_j^\infty \longrightarrow H_{j+1}^\infty$ are isometric isomorphisms. So are the mappings $\iota_j^k\colon H_j^\infty \longrightarrow H_k^\infty$ defined by $\iota_j^k = \varphi_j^* \circ \varphi_{j+1}^* \circ \cdots \circ \varphi_k^*$. Because $\big(\varphi_j^*(f)\big)(z) = f\big(\varphi_j(z)\big)$ whenever $z \in \mathbb{D}_{j+1}$, $j \in \mathbb{N}$ and $f \in H_j^\infty$ we have $\big(\iota_j^k(f)\big)(z) = f\big(\varphi_j \circ \varphi_{j+1} \circ \cdots \circ \varphi_k)(z)\big)$, whenever $z \in \mathbb{D}_{k+1}$. Consequently, every coordinate algebra H_j^∞ can be embedded isometrically and isomorphically into $\varinjlim \big\{H^\infty, \varphi_k^*\big\}_{k\in\mathbb{N}} \subset H^\infty(\mathcal{D}_\Phi)$ via the natural inclusions $\iota_j\colon H_j^\infty \longrightarrow H^\infty(\mathcal{D}_\Phi)$. Moreover, if $z^* \in \mathbb{D}_j$, then $f(z^*) = \big(\iota_j(f)\big)(x^*)$, where $x^* \in \mathcal{D}_\Phi$ is the chain $x^* = (z_1, z_2, \ldots, z_j, \ldots)$ of the spectrum (8.10) with $z_j = z^*$ and $\varphi_n(z_{n+1}) = z_n$ for $n \geq j$.

Proof. Without loss of generality we can assume that $\|f_j\| \leq 1/2$ for all $f_j \in H^\infty(\mathcal{D}_\Phi)$ in (8.11) and that $\delta \leq 1/2$. Let $C(n, \delta/2)$ be the corresponding Carleson's constant and let $c = \max\big\{1, C\big(n, \delta/2\big)\big\}$. By the definition of the space $H^\infty(\mathcal{D}_\Phi)$ there are integers $n_j \in \mathbb{N}$ and functions $\widetilde{f}_j \in H_{n_j}^\infty$, such that

$$\big\|f_j - \iota_{n_j}(\widetilde{f}_j)\big\|_\infty = \sup_{x\in\mathcal{D}_I} \big|f_j(x) - \big(\iota_{n_j}(\widetilde{f}_j)\big)(x)\big| < \frac{\delta}{2cn},\ j = 1, \ldots, n.$$

Replacing $\widetilde{f}_j$ by $\iota_{n_j}^m(\widetilde{f}_j)$, we can assume from the beginning that all $\widetilde{f}_j \in H_m^\infty$ for some $m \geq n_j$, $j = 1, 2, \ldots, n$. The inequality (8.11) implies that for every $z^* \in \mathbb{D}$ we have

$$\begin{aligned}&\big|\widetilde{f}_1(z^*)\big| + \cdots + \big|\widetilde{f}_n(z^*)\big| = \big|\big(\iota_m(\widetilde{f}_1)\big)(x^*)\big| + \cdots + \big|\big(\iota_m(\widetilde{f}_n)\big)(x^*)\big| \\ &\geq\geq \sum_{j=1}^n \big|f_j(x^*)\big| - \sum_{j=1}^n \big|f_j(x^*) - \big(\iota_m(\widetilde{f}_j)\big)(x^*)\big| \geq \delta - \frac{\delta}{2c} \geq \frac{\delta}{2} > 0,\end{aligned}$$

where, as before, $x^* = (z_1, z_2, \ldots, z_m, \ldots)$ is a chain of the spectrum (8.10) with $z_m = z^*$ and $\varphi_n(z_{n+1}) = z_n$ for $n \geq m$. Therefore, $|\widetilde{f}_1| + \cdots + |\widetilde{f}_n| \geq \delta/2 > 0$ on $\mathbb{D}$ for the bounded analytic functions $\widetilde{f}_1, \ldots, \widetilde{f}_n$ on $\mathbb{D}$. In addition,

$$\|\widetilde{f}_j\|_\infty = \big\|\iota_m(\widetilde{f}_j)\big\|_\infty \leq \|f_j\|_\infty + \big\|f_j - \iota_m(\widetilde{f}_j)\big\|_\infty \leq \|f_j\|_\infty + \frac{\delta}{2cn} \leq 1.$$

According to Carleson's $\mathbb{D}$-corona theorem for the space H^∞ there exist functions $h_1, \ldots, h_n \in H^\infty$ with $\|h_j\|_\infty \leq C\big(n, \delta/2\big) \leq c$ such that $\widetilde{f}_1 h_1 + \cdots + \widetilde{f}_n h_n = 1$ on $\mathbb{D}$. Hence,

$$\begin{aligned}1 &= (\widetilde{f}_1 h_1 + \cdots + \widetilde{f}_n h_n)(z^*) = \iota_m(\widetilde{f}_1 h_1 + \cdots + \widetilde{f}_n h_n)(x^*) \\ &= \big(\iota_m(\widetilde{f}_1)\,\iota_m(h_1) + \cdots + \iota_m(\widetilde{f}_n)\,\iota_m(h_n)\big)(x^*)\end{aligned}$$

on $\mathcal{D}_\Phi$, and $\big\|\iota_m(h_j)\big\|_\infty = \|h_j\|_\infty \leq c$. Though the function

$$F = f_1 \iota_m(h_1) + \cdots + f_n \iota_m(h_n) \in H^\infty(\mathcal{D}_\Phi)$$

is not necessarily equal to 1 on $\mathcal{D}_\Phi$, it is invertible in $H^\infty(\mathcal{D}_\Phi)$. Indeed,

$$\begin{aligned} \|1-F\|_\infty &= \Big\| \sum_j \iota_m(\widetilde{f}_j)\,\iota_m(h_j) - \sum_j f_j \iota_m(h_j)\Big\|_\infty \\ &\le \sum_j \big\|\iota_m(\widetilde{f}_j) - f_j\big\|_\infty \big\|\iota_m(h_j)\big\|_\infty \le \frac{\delta}{2cn}\, c\,n = \frac{\delta}{2} < 1. \end{aligned} \tag{8.12}$$

Now the identity $f_1g_1+\cdots+f_ng_n = 1$ holds on $\mathcal{D}_\Phi$ with $g_j = \iota_m(h_j)/F \in H^\infty(\mathcal{D}_\Phi)$, $j = 1,\ldots n$. Note that $\|F^{-1}\|_\infty \le \dfrac{1}{1-\delta/2} = \dfrac{2}{2-\delta}$, since $|F(x)| \ge 1 - \dfrac{\delta}{2}$ on $\mathcal{D}_\Phi$ according to (8.12). If $K(n,\delta) = \dfrac{2}{2-\delta}\max\big\{1, C(n,\delta/2)\big\}$, then $\|g_j\|_\infty \le \big\|\iota_m(h_j)\big\|_\infty \|F^{-1}\|_\infty \le \dfrac{2c}{2-\delta} = \dfrac{2}{2-\delta}\max\big\{1, C(n,\delta/2)\big\} = K(n,\delta)$. This completes the proof. □

8.2 Blaschke inductive limits of disc algebras

Let $\Lambda \subset \mathbb{R}_+$ be a basis in $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. As we saw in Example 1.3.6, $\mathbb{R}$ can be expressed as the limit

$$\mathbb{R} = \varinjlim \{\Gamma_{(\gamma,n)}\}_{(\gamma,n)\in J}$$

of the inductive system $\{\Gamma_{(\gamma,n)}\}_{(\gamma,n)\in J}$ of subgroups $\Gamma_{(\gamma,n)}$ of $\mathbb{R}$ isomorphic to $\mathbb{Z}^k = \bigoplus_{i=1}^{k} \mathbb{Z}$, defined by

$$\Gamma_{(\gamma,n)} = \big\{(1/n!)\,(m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k) \colon m_j \in \mathbb{Z},\ j = 1,\ldots,k\big\}.$$

Let $P_{(\gamma,n)} = (\Gamma_{(\gamma,n)})_+$ be the non-negative half-group

$$P_{(\gamma,n)} = \big\{(1/n!)\,(m_1\gamma_1 + \cdots + m_k\gamma_k) \in \Gamma_{(\gamma,n)} \colon m_1\gamma_1 + \cdots + m_k\gamma_k \ge 0)\big\}$$

of $\Gamma_{(\gamma,n)}$. The $b(\mathbb{R})$-disc algebra $A_{b(\mathbb{R})}$ can be expressed as an inductive limit algebra

$$A_{b(\mathbb{R})} = \big[\varinjlim \big\{A_{P_{(\gamma,n)}}(\mathbb{D}_G)\big\}_{(\gamma,n)\in J}\big],$$

under inclusions, where $A_{P_{(\gamma,n)}}$ is the algebra of $P_{(\gamma,n)}$-functions on $G = b(\mathbb{R})$, the Bohr compactification of $\mathbb{R}$.

Similarly, every shift-invariant algebra A_S with $S \subset \mathbb{R}_+$ can be expressed as an inductive limit of algebras $A_{P_{(\gamma,n)}}$ of analytic $P_{(\gamma,n)}$-functions on $b(\mathbb{R})$. Note that $\Gamma_{(\gamma,n)} \cong \mathbb{Z}^k$, where $k = \operatorname{card}(\gamma)$.

Let B be the algebra of linear combinations of functions $\widetilde{z}^{1/n},\ n \in \mathbb{N}$ on the set $\mathcal{R}^*_{Log}(\overline{\mathbb{D}})$, considered in Example 4.2.6(c). For a fixed $n \in \mathbb{N}$ denote by $A^{(n)}$ the algebra of polynomials of $\widetilde{z}^{1/n}$. Clearly, $B = \bigcup\limits_{n\in\mathbb{N}} A^{(n)}$, and $A^{(n)} \subset A^{(m)}$ whenever $n \leq m$. Consequently, B can be expressed as the limit of the inductive system $\{A^{(n)}, \iota_n^m\}_{n\in\mathbb{N}}$, where $\mathbb{N}$ is considered with the usual order, namely,

$$B = \bigcup_{n\in\mathbb{N}} A^{(n)} = \varinjlim \, \{A^{(n)}, \iota_n^m\}_{n\in\mathbb{N}}.$$

Uniform algebras that can be represented as inductive limits of disc algebras $A(\mathbb{D})$ are of particular interest. Let $I = \{i_k^{k+1}\}_{k=1}^{\infty}$ be a sequence of homomorphisms $i_k^{k+1} \colon A(\mathbb{D}) \longrightarrow A(\mathbb{D})$. Consider the inductive sequence

$$A(\mathbb{T}_1) \xrightarrow{\;i_1^2\;} A(\mathbb{T}_2) \xrightarrow{\;i_2^3\;} A(\mathbb{T}_3) \xrightarrow{\;i_3^4\;} \cdots \tag{8.13}$$

of disc algebras $A(\mathbb{T}_k) = A(\mathbb{T})$ with connecting homomorphisms $i_k^{k+1} \colon A(\mathbb{T}_k) \longrightarrow A(\mathbb{T}_{k+1})$. Every adjoint mapping $(i_k^{k+1})^* \colon \mathcal{M}_k \longleftarrow \mathcal{M}_{k+1}$ maps the maximal ideal space $\mathcal{M}_{k+1} \cong \overline{\mathbb{D}}$ of $A(\mathbb{T}_{k+1})$ into the maximal ideal space $\mathcal{M}_k \cong \overline{\mathbb{D}}$ of $A(\mathbb{T}_k)$. Since $i_k^{k+1}(f) = f \circ (i_k^{k+1})^* \in A(\mathbb{T}_{k+1})$ for every $f \in A(\mathbb{T}_k)$, the mapping $(i_k^{k+1})^*$ is an analytic function preserving the unit disc. The inverse limit

$$\overline{\mathbb{D}}_1 \xleftarrow{\;(i_1^2)^*\;} \overline{\mathbb{D}}_2 \xleftarrow{\;(i_2^3)^*\;} \overline{\mathbb{D}}_3 \xleftarrow{\;(i_3^4)^*\;} \overline{\mathbb{D}}_4 \xleftarrow{\;(i_4^5)^*\;} \cdots \longleftarrow \mathcal{D}_I$$

is the maximal ideal space of the inductive limit algebra

$$\mathcal{A}(\mathcal{D}_I) = \big[\varinjlim \, \{A(\mathbb{T}_k), i_k^{k+1}\}_{k\in\mathbb{N}}\big].$$

In general, the mappings $(i_k^{k+1})^*$ do not necessarily map the unit circle $\mathbb{T}_{k+1}$ onto itself. The most interesting situations, though, occur when they do, and this is what we will assume in this section. In this case the mapping $(i_k^{k+1})^*$ becomes a finite Blaschke product B_k on $\overline{\mathbb{D}}$.

Let $\{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$ be the inverse sequence with $\overline{\mathbb{D}}_k = \overline{\mathbb{D}}$ and $\tau_k^{k+1}(z) = z^{d_k}$ on $\overline{\mathbb{D}}_k$. As we saw in Example 8.1.1, the limit $\varprojlim \, \{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$ of the inverse sequence $\{\overline{\mathbb{D}}_{k+1}, \tau_k^{k+1}\}_{k\in\mathbb{N}}$, is the G_Λ-disc $\overline{\mathbb{D}}_{G_\Lambda} = ([0,1] \times G_\Lambda)/(\{0\} \times G_\Lambda)$ over the group $G_\Lambda = \widehat{\Gamma}_\Lambda$. There arises an adjoint inductive sequence $\{A(\mathbb{D}_k), i_k^{k+1}\}_{k\in\mathbb{N}}$ of algebras $A(\mathbb{D}) \cong A(\mathbb{T})$ with connecting homomorphisms $i_k^{k+1} = (\tau_k^{k+1})^* \colon A(\mathbb{D}_k) \longrightarrow A(\mathbb{D}_{k+1})$, defined by $\big(i_k^{k+1}(f)\big)(z) = \big(f(z)\big)^{d_k}$. The elements of the coordinate algebras $A(\mathbb{D}_k)$ can be interpreted as continuous functions on G_Λ. The uniform closure

$$\mathcal{A}(\mathbb{D}_{G_\Lambda}) = \big[\varinjlim \, \{A(\mathbb{D}_k), i_k^{k+1}\}_{k\in\mathbb{N}}\big]$$

in $C(\overline{\mathbb{D}}_{G_\Lambda})$ of the inductive limit of the sequence $\{A(\mathbb{D}_k), i_k^{k+1}\}_{k\in\mathbb{N}}$ and the corresponding restriction algebra $\big[\varinjlim \, \{A(\mathbb{T}_k), i_k^{k+1}\}_{k\in\mathbb{N}}\big]$ are isometrically isomorphic to the G_Λ-disc algebra $A_{\Gamma_{\Lambda+}}$, i.e., to the algebra of analytic functions on the G_Λ-disc (cf. [T1]).

In a similar way, if $\{K_l\}_{l=1}^\infty$ is a sequence of connected compact sets in the complex plane $\mathbb{C}$ with $\tau_l^{l+1}(K_{l+1}) = K_l$ for every $l \in \mathbb{N}$, then the closure of the inductive limit $\varinjlim \left\{A(K_l), i_l^{l+1}\right\}_{l\in\mathbb{N}}$ in $C(\mathcal{K}_\Lambda)$ is the algebra $\mathcal{A}(\mathcal{K}_\Lambda)$ of analytic $\Gamma_{\Lambda+}$-functions on the compact set $\mathcal{K}_\Lambda = \varprojlim \{K_{l+1}, \tau_l^{l+1}\}_{l\in\mathbb{N}}$ in the G_Λ-plane $\mathbb{C}_{G_\Lambda}$ over the group G_Λ (e.g. [L1]).

Consider an inductive sequence of disc algebras

$$A(\mathbb{T}_1) \xrightarrow{i_1^2} A(\mathbb{T}_2) \xrightarrow{i_2^3} A(\mathbb{T}_3) \xrightarrow{i_3^4} \cdots, \tag{8.14}$$

with the the natural embeddings $i_k^{k+1} \colon A(\mathbb{T}_k) \longrightarrow A(\mathbb{T}_{k+1})$ as connecting homomorphisms. Since $\mathcal{M}_{i_k^{k+1}(A(\mathbb{T}_k))} = \overline{\mathbb{D}}$ and $\partial(i_k^{k+1}(A(\mathbb{T}_k))) = \mathbb{T}$, then according to the remarks following Lemma 8.1.5, there are finite Blaschke products $B_k : \mathbb{D} \longrightarrow \mathbb{D}$ such that $i_k^{k+1} = B_k^*$ for every $k \in \mathbb{N}$, i.e.

$$i_k^{k+1}(f) = f \circ B_k,$$

where

$$B_k(z) = e^{i\theta_k} \prod_{l=1}^{n_k} \Big(\frac{z - z_l^{(k)}}{1 - \overline{z}_l^{(k)} z}\Big), \ |z_l^{(k)}| < 1.$$

Let $B = \{B_k\}_{k=1}^\infty$ be the sequence of finite Blaschke products corresponding to the mappings i_k^{k+1}, i.e. $(B_k)^*(z) = i_k^{k+1}(z)$. Let $\Lambda = \{d_k\}_{k=1}^\infty$ be the sequence of orders of Blaschke products $\{B_k\}_{k=1}^\infty$ and let $\Gamma_\Lambda \subset \mathbb{Q}$ be the group generated by $1/m_k, \ k = 0, 1, 2, \ldots,$ where $m_k = \prod_{l=1}^{k} d_l, \ m_0 = 1$.

Consider the inverse sequence

$$\overline{\mathbb{D}}_1 \xleftarrow{B_1} \overline{\mathbb{D}}_2 \xleftarrow{B_2} \overline{\mathbb{D}}_3 \xleftarrow{B_3} \overline{\mathbb{D}}_4 \xleftarrow{B_4} \cdots \longleftarrow \mathcal{D}_B.$$

The inverse limit $\mathcal{D}_B = \varprojlim \ \{\overline{\mathbb{D}}_{k+1}, B_k\}_{k\in\mathbb{N}}$ is a Hausdorff compact space. The limit of the adjoint inductive sequence $\left\{A(\mathbb{D}_k), \beta_k^{k+1}\right\}_1^\infty$ of disc algebras $A(\mathbb{D}_k)$, whose connecting homomorphisms are the composition operators $\beta_k^{k+1} = B_k^* \colon A(\mathbb{D}_k) \longrightarrow A(\mathbb{D}_{k+1})\colon (\beta_k^{k+1}(f))(z_{k+1}) = f(B_k(z_{k+1}))$, is an algebra of functions on $\mathcal{D}_B$.

Definition 8.2.1. The *Blaschke inductive limit of disc algebras* is the closure

$$\mathcal{A}(\mathcal{D}_B) = \left[\varinjlim \left\{A(\mathbb{D}_k), \beta_k^{k+1}\right\}_{k\in\mathbb{N}}\right]$$

of the inductive limit algebra $\left[\varinjlim \left\{A(\mathbb{D}_k), b_k^{k+1}\right\}_{k\in\mathbb{N}}\right]$ in $C(\mathcal{D}_B)$.

Clearly, the algebra $\mathcal{A}(\mathcal{D}_B)$ is isometrically isomorphic to the algebra

$$\left[\varinjlim \left\{A(\mathbb{T}_k), \beta_k^{k+1}\right\}_{k\in\mathbb{N}}\right].$$

Proposition 8.2.2. *Let $B = \{B_k\}_{k=1}^{\infty}$ be a sequence of finite Blaschke products and let*

$$\mathcal{A}(\mathcal{D}_B) = \left[\varinjlim \left\{A(\mathbb{D}_k), B_k^*\right\}_{k\in\mathbb{N}}\right]$$

be the corresponding Blaschke inductive limit algebra. Then:

(i) *$\mathcal{A}(\mathcal{D}_B)$ is a uniform algebra on the compact set*

$$\mathcal{D}_B = \varprojlim \left\{\overline{\mathbb{D}}_{k+1}, B_k\right\}_{k\in\mathbb{N}}.$$

(ii) *The maximal ideal space of $\mathcal{A}(\mathcal{D}_B)$ is $\mathcal{D}_B$.*

(iii) *$\mathcal{A}(\mathcal{D}_B)$ is a Dirichlet algebra.*

(iv) *$\mathcal{A}(\mathcal{D}_B)$ is a maximal algebra.*

(v) *The Shilov boundary of $\mathcal{A}(\mathcal{D}_B)$ is homeomorphically isomorphic to G_Λ, and its dual group is isomorphic to the group*

$$\Gamma_\Lambda \cong \bigcup_{k=0}^{\infty} \frac{1}{m_k}\mathbb{Z} \subset \mathbb{Q},$$

where $m_k = \prod_{l=1}^{k} d_l$, $m_0 = 1$, and $d_k = \operatorname{ord} B_k$.

Indeed, under our assumptions B_k maps $\mathbb{T}_{k+1}$ onto $\mathbb{T}_k$ and $\mathbb{D}_{k+1}$ onto $\mathbb{D}_k$. Since the Shilov boundary of every component algebra $A(\mathbb{D}_k)$ is the unit circle $\mathbb{T}_k$, and the maximal ideal space is the disc $\overline{\mathbb{D}}_k$, the properties (i), (ii), (iii) follow from Proposition 1.3.4, while (iv) follows from the next proposition.

Proposition 8.2.3. *The inductive limit of maximal algebras is a maximal algebra.*

Proof. Let $\mathcal{A} = \left[\varinjlim \{A^\sigma, i_\sigma^\tau\}_{\sigma\in\Sigma}\right]$, where A^σ are maximal algebras. The maximal ideal space of $\mathcal{A}$ is the inverse limit $\mathcal{M}_\mathcal{A} = \varprojlim \left\{\mathcal{M}_\sigma, (i_\tau^\sigma)^*\right\}_{\sigma\in\Sigma}$, where $\mathcal{M}_\sigma$ are the maximal ideal spaces of the algebras A^σ. If $h \in C(\mathcal{M}_A) \setminus A$, then the algebra $A[h]$ generated by A and h coincides with $\left[\varinjlim \left\{A^\sigma[h_\sigma], (i_\sigma^\tau)^{**}\right\}_{\sigma\in\Sigma}\right]$. Since $i_\tau^\sigma(A^\sigma) \subset A^\tau$ and $h \notin \mathcal{A}$, it follows that $h_\sigma \notin A^\sigma$ for every $\sigma \in \Sigma$. By the maximality of A^σ we have that $A^\sigma[h_\sigma] = C(\mathcal{M}_\sigma)$, $\sigma \in \Sigma$. Hence,

$$\mathcal{A}[h] = \left[\varinjlim \left\{A^\sigma[g], (i_\sigma^\tau)^{**}\right\}_{\sigma\in\Sigma}\right] = \left[\varinjlim \left\{C(\mathcal{M}_\sigma), (i_\sigma^\tau)^{**}\right\}_{\sigma\in\Sigma}\right] = C(\mathcal{M}_\mathcal{A}).$$

Consequently, $\mathcal{A}$ is a maximal algebra, as claimed. $\square$

Theorem 8.2.4. *Let G be a compact abelian group with dual group $\widehat{G}$ isomorphic to a subgroup Γ of $\mathbb{R}$. The G-disc algebra A_{Γ_+} is a Blaschke inductive limit of disc algebras if and only if Γ is isomorphic to some subgroup of $\mathbb{Q}$.*

Proof. The first part of the theorem is proven already in Proposition 8.2.2. Let $\widehat{G} \cong \Gamma \subset \mathbb{Q}$ and let $\{a_i\}_{i=1}^{\infty}$ be an enumeration of Γ. Without loss of generality we can assume that $a_1 = 1$. Let $\Gamma^{(1)} = \mathbb{Z},\ \Gamma^{(2)} = \mathbb{Z} + a_2\mathbb{Z},\ \Gamma^{(3)} = \mathbb{Z} + a_2\mathbb{Z} + a_3\mathbb{Z}$, etc. Since $\mathbb{Z} \subset \Gamma^{(k)}$ and $\Gamma^{(k)}$ is isomorphic to $\mathbb{Z}$, there is an $m_k \in \mathbb{N}$, such that $\Gamma^{(k)} = (1/m_k)\,\mathbb{Z}$. By $\Gamma^k \subset \Gamma^{k+1}$ we have that $d_{k+1} = (m_{k+1}/m_k) \in \mathbb{Z}$. The inclusion $i_k^{k+1} \colon \Gamma^{(k)} \hookrightarrow \Gamma^{(k+1)}$ generates a mapping $\widetilde{i_k^{k+1}} \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $\widetilde{i_k^{k+1}}(1) = d_{k+1}$, and therefore $\widetilde{i_k^{k+1}}(n) = d_{k+1}n,\ n \in \mathbb{Z}^{(k)}$. Clearly, the group

$$\Gamma \cong \bigcup_{k=1}^{\infty} (1/m_k)\,\mathbb{Z} = \varinjlim\, \{\Gamma^{(k)}, \widetilde{i_k^{k+1}}\}_{k\in\mathbb{N}} \subset \mathbb{Q}$$

is generated by the numbers $1/m_k,\ k \in \mathbb{N}$. As we saw at the beginning of this section, the Blaschke inductive limit algebra $\mathcal{A}(\mathcal{D}_\Lambda)$ corresponding to the sequence $\Lambda = \{d_k\}_1^{\infty}$ coincides with the Γ_Λ-disc algebra $A_{\Gamma_{\Lambda+}}$. □

Similarly to the case of inductive limits of algebras $H^\infty(\mathbb{D}) \cong H^\infty$, we obtain the following analogues to Theorem 8.1.4 and its corollaries.

Theorem 8.2.5. *If $B = \{B_k\}_{k=1}^{\infty}$ is a sequence of finite Blaschke products on $\overline{\mathbb{D}}$, each one with at most one singular point $z_0^{(k)}$ and such that $B_k\big(z_0^{(k+1)}\big) = z_0^{(k)}$, then the algebra $\mathcal{A}(\mathcal{D}_B)$ is isometrically isomorphic to the algebra $A_{(\Gamma_\Lambda)_+}$ with $\Lambda = \{d_k\}_{k=1}^{\infty}$, where $d_k = \operatorname{ord} B_k$.*

Corollary 8.2.6. *If every Blaschke product B_k in Theorem 8.2.5 is a Möbius transformation, then the algebra $A(\mathcal{D}_B)$ is isometrically isomorphic to the disc algebra $A_{\mathbb{Z}_+} = A(\mathbb{T})$.*

8.3 Blaschke inductive limit algebras of annulus type

Let $\mathbb{D}^{[r,1]}$ be the annulus region $\mathbb{D}^{[r,1]} = \big\{z \in \mathbb{C} \colon r \le |z| \le 1\big\}$, with topological boundary $b\mathbb{D}^{[r,1]} = \mathbb{T}_r \cup \mathbb{T} = \big\{z \in \mathbb{C} \colon |z| = r\big\} \cup \big\{|z| = 1\big\}$. Denote by $A\big(\mathbb{D}^{[r,1]}\big)$ the uniform algebra of continuous functions on $\mathbb{D}^{[r,1]}$, analytic in its interior. Note that $A\big(\mathbb{D}^{[r,1]}\big)$ coincides with $R\big(\mathbb{D}^{[r,1]}\big)$, the algebra of uniform limits of rational functions on $\mathbb{D}^{[r,1]}$. By a well-known result of Bishop (e.g. [S4]), the Shilov boundary of $A\big(\mathbb{D}^{[r,1]}\big)$ is the topological boundary $b\mathbb{D}^{[r,1]} = \mathbb{T}_r \cup \mathbb{T}$. The restriction of $A\big(\mathbb{D}^{[r,1]}\big)$ on $b\mathbb{D}^{[r,1]}$ is a maximal algebra, such that $\operatorname{codim}\big(\operatorname{Re}\,(A(\mathbb{D}^{[r,1]})\big|_{b\mathbb{D}^{[r,1]}})\big) = 1$. These results can be extended to the case of analytic Γ_+-functions on compact groups (cf. [G9]).

Let G be a solenoidal group, i.e. a compact abelian group such that its dual group Γ is isomorphic to a dense subgroup of $\mathbb{R}$. Let $\mathbb{D}_G^{[r,1]} = [r,1] \diamond G,\ 0 < r < 1$, be the G-annulus, and let $(\mathbb{D}_G^{[r,1]})$ be the uniform algebra on $\mathbb{D}_G^{[r,1]}$, generated by the functions $\widehat{\chi^a}$, $a \in \Gamma$. Then:

(a) $\mathbb{D}_G^{[r,1]}$ is the maximal ideal space of $R(\mathbb{D}_G^{[r,1]})$.

(b) $b\mathbb{D}_G^{[r,1]} = (\{r\} \times G) \cup (\{1\} \times G)$ is the Shilov boundary of $R(\mathbb{D}_G^{[r,1]})$.

(c) $R(\mathbb{D}_G^{[r,1]})$ is a maximal algebra with $\operatorname{codim}\big(\operatorname{Re}(R(\mathbb{D}_G^{[r,1]})|_{b\mathbb{D}_G^{[r,1]}})\big) = 1$.

Consequently, $R(\mathbb{D}_G^{[r,1]})$ coincides with $A(\mathbb{D}_G^{[r,1]})$, the Blaschke inductive limit algebra of annulus type, consisting of continuous functions on $\mathbb{D}_G^{[r,1]}$ that are locally approximable by analytic functions in $\mathbb{D}_G^{(r,1)}$.

Let $\Lambda = \{d_k\}_{k=1}^{\infty}$ be a sequence of natural numbers, $\tau_k^{k+1}(z) = z^{d_k}$, and $r \in (0,1]$ be a fixed number. Consider the sets

$$E_k = \overline{\mathbb{D}}^{[r^{1/m_k},1]} = \big\{z \in \overline{\mathbb{D}} \colon r^{1/m_k} \le |z| \le 1\big\} = (\tau_1^2 \circ \tau_2^3 \circ \cdots \circ \tau_{k-1}^k)^{-1}\big(\overline{\mathbb{D}}^{[r,1]}\big),$$

where $m_k = \prod_{l=1}^{k} d_l$, $m_0 = 1$, and $E_1 = \mathbb{D}^{[r,1]}$. There arises an inverse sequence

$$\mathbb{D}^{[r,1]} \xleftarrow{\tau_1^2} E_2 \xleftarrow{\tau_2^3} E_3 \xleftarrow{\tau_3^4} E_4 \xleftarrow{\tau_4^5} \cdots$$

of compact subsets of $\overline{\mathbb{D}}$. Consider the adjoint inductive sequence

$$A\big(\mathbb{D}^{[r,1]}\big) \xrightarrow{i_1^2} A(E_2) \xrightarrow{i_2^3} A(E_3) \xrightarrow{i_3^4} \cdots, \tag{8.15}$$

where the embeddings $i_k^{k+1} \colon A(E_k) \longrightarrow A(E_{k+1})$ are the adjoints to z^{d_k} composition operators, namely, $(i_k^{k+1} \circ f)(z) = f(z^{d_k})$. Let G_Λ denote the compact abelian group whose dual group $\Gamma_\Lambda = \widehat{G}_\Lambda$ is the subgroup of $\mathbb{Q}$ generated by the set Λ.

Lemma 8.3.1. *The uniform algebra* $\big[\varinjlim\, \{A(E_k), i_k^{k+1}\}_{k\in\mathbb{N}}\big]$ *is isometrically isomorphic to* $A\big(\mathbb{D}_G^{[r,1]}\big)$, *the Blaschke inductive limit algebra of annulus type.*

Proof. Let $a_k = 1/m_k$, where, as before, $m_k = \prod_{l=1}^{k} d_l$, $m_0 = 1$. Consider the algebras $A^k\big(\mathbb{D}_G^{[r,1]}\big) = \big\{g \circ \widehat{\chi}^{a_k} \colon g \in A(E_k)\big\} \subset A\big(\mathbb{D}_G^{[r,1]}\big)$, $k = 1, 2, \ldots$, where $\widehat{\chi}^{a_k} \in \Gamma_+$. Clearly, $A^k\big(\mathbb{D}_G^{[r,1]}\big) \subset A^{k+1}\big(\mathbb{D}_G^{[r,1]}\big)$ and $A\big(\mathbb{D}_G^{[r,1]}\big) = \big[\bigcup_{k=0}^{\infty} A^k\big(\mathbb{D}_G^{[r,1]}\big)\big]$. We thus have an inductive sequence

$$A^1\big(\mathbb{D}_G^{[r,1]}\big) \xhookrightarrow{j_1^2} A^2\big(\mathbb{D}_G^{[r,1]}\big) \xhookrightarrow{j_2^3} \cdots \hookrightarrow A\big(\mathbb{D}_G^{[r,1]}\big), \tag{8.16}$$

where j_k^{k+1} is the natural inclusion of $A^k\big(\mathbb{D}_G^{[r,1]}\big)$ into $A^{k+1}\big(\mathbb{D}_G^{[r,1]}\big)$. We claim that the inductive sequences (8.15) and (8.16) are isomorphic. Indeed, $\widehat{\chi}^{a_k}$ maps $\mathbb{D}_G^{[r,1]}$

onto E_k, and the mapping φ_k defined by $\varphi_k(f\circ\widehat{\chi}^{a_k}) = f$ maps $A^k\big(\mathbb{D}_G^{[r,1]}\big)$ isometrically and isomorphically onto $A(E_k)$. In addition, $i_k^{k+1}\circ\varphi_k = \varphi_{k+1}\big|_{A_k\left(\mathbb{D}_G^{[r,1]}\right)} = \varphi_{k+1}\circ j_k^{k+1}$, i.e. the diagram

$$\begin{array}{ccc} A^k\big(\mathbb{D}_G^{[r,1]}\big) & \xrightarrow{j_k^{k+1}} & A^{k+1}\big(\mathbb{D}_G^{[r,1]}\big) \\ {\scriptstyle\varphi_k}\downarrow & & {\scriptstyle\varphi_{k+1}}\downarrow \\ A(E_k) & \xrightarrow{i_k^{k+1}} & A(E_{k+1}) \end{array}$$

is commutative. Therefore the infinite diagram

$$\begin{array}{ccccccc} A^1\big(\mathbb{D}_G^{[r,1]}\big) & \xrightarrow{j_1^2} & A^2\big(\mathbb{D}_G^{[r,1]}\big) & \xrightarrow{j_2^2} & A^3\big(\mathbb{D}_G^{[r,1]}\big) & \xrightarrow{j_2^3} & \cdots \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & {\scriptstyle\varphi_3}\downarrow & & \\ A(E_1) & \xrightarrow{i_1^2} & A(E_2) & \xrightarrow{i_2^3} & A(E_3) & \xrightarrow{i_3^4} & \cdots \end{array}$$

is commutative too, and hence the inductive sequences (8.15) and (8.16) are isomorphic, as claimed. Consequently, $A\big(\mathbb{D}_G^{[r,1]}\big) \cong \big[\varinjlim \{A(E_k), i_k^{k+1}\}_{k\in\mathbb{N}}\big]$, as claimed. $\square$

Let $B = \{B_k\}_{k=1}^{\infty}$ be a sequence of finite Blaschke products on $\overline{\mathbb{D}}$ and let $d_k = \operatorname{ord} B_k$. Define inductively a sequence of sets F_k as follows: $F_1 = \mathbb{D}^{[r,1]}$,

$$F_{n+1} = B_n^{-1}(F_n) = \{z\in\overline{\mathbb{D}} : B_n(z)\in F_n\} = (B_1\circ B_2\circ\cdots\circ B_n)^{-1}(\mathbb{D}^{[r,1]})$$

for $n\geq 0$. Consider the adjoint sequences

$$\mathbb{D}^{[r,1]} \xleftarrow{B_1} F_2 \xleftarrow{B_2} F_3 \xleftarrow{B_3} \cdots \longleftarrow \mathcal{D}_B^{[r,1]} \subset \mathcal{D}_B, \tag{8.17}$$

where $\mathcal{D}_B = \varprojlim \{\overline{\mathbb{D}}_{k+1}, B_k\}_{k\in\mathbb{N}}$, and

$$A\big(\mathbb{D}^{[r,1]}\big) \xrightarrow{\beta_1^2} A(F_2) \xrightarrow{\beta_2^3} A(F_3) \xrightarrow{\beta_3^4} \cdots, \tag{8.18}$$

where $\beta_k^{k+1} = B_k^*$, i.e. $(\beta_k^{k+1}\circ f)(z) = f\big(B_k(z)\big)$.

Theorem 8.3.2. *If the Blaschke products B_n do not have singular points on the sets F_n for any $n\in\mathbb{N}$, then $\mathcal{D}_B^{[r,1]}\cong\mathbb{D}_G^{[r,1]}$, and the algebra*

$$\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big) = \big[\varinjlim \{A(F_n), B_n^*\}_{n\in\mathbb{N}}\big]$$

is isometrically isomorphic to $A\big(\mathbb{D}_G^{[r,1]}\big)$, the Blaschke inductive limit algebra of annulus type.

For the proof we need a version of a well-known result on Riemann surfaces, which we provide here with a short proof.

Lemma 8.3.3. *Suppose that the d_k-sheeted holomorphic covering $B_k : F_{k+1} \longrightarrow F_k$ does not have singular points, and there exists a biholomorphic mapping ψ_k from F_k onto E_k. Then there exists a biholomorphic mapping $\psi_{k+1} : F_{k+1} \longrightarrow E_{k+1}$ such that the diagram*

$$\begin{array}{ccc} F_k & \overset{B_k}{\longleftarrow} & F_{k+1} \\ \psi_k \downarrow & & \psi_{k+1} \downarrow \\ E_k & \overset{z^{d_k}}{\longleftarrow} & E_{k+1} \end{array}$$

is commutative, i.e. $\psi_k \circ B_k = (\psi_{k+1})^{d_k}$, where $d_k = \operatorname{ord} B_k$.

Proof. The function z^{d_k} generates a bijection $\widetilde{z^{d_k}}$ from E_{k+1} onto the d_k-sheeted covering $\widetilde{E}_k$ over E_k. Likewise, the map $\psi_k \circ B_k \colon F_{k+1} \longrightarrow E_k$ generates a bijection $(\psi_k \circ B_k)^{\sim}$ from F_{k+1} to $\widetilde{E}_k$. Therefore the map $\psi_{k+1} = (\widetilde{z^{d_k}})^{-1} \circ (\psi_k \circ B_k)^{\sim}$ is a well-defined bijection from F_{k+1} onto E_{k+1}, so that the diagram

$$\begin{array}{ccc} F_{k+1} & & \\ \psi_{k+1} \downarrow & \searrow^{(\psi_k \circ B_k)^{\sim}} & \\ E_{k+1} & \overset{\widetilde{z}^{d_k}}{\longrightarrow} & \widetilde{E}_k \end{array}$$

is commutative. Since both component mappings of ψ_{k+1} are locally holomorphic, so is ψ_{k+1}. □

Proof of Theorem 8.3.2. Let ψ_1 be the identity map on $\mathbb{D}^{[r,1]} = E_1 = F_1$. By Lemma 8.3.3 we can define inductively biholomorphic mappings $\psi_k \colon F_k \longrightarrow E_k$ for every $k \in \mathbb{N}$, such that $\psi_k \circ B_k = (\psi_{k+1})^{d_k}$. Consequently,

$$\mathbb{D}_G^{[r,1]} = \underset{\longleftarrow}{\lim} \{E_{n+1}, z^{d_n}\}_{n\in\mathbb{N}} \cong \underset{\longleftarrow}{\lim} \{F_{n+1}, B_n\}_{n\in\mathbb{N}} = \mathcal{D}_B^{[r,1]} \subset \mathcal{D}_B.$$

The adjoint mapping ψ_k^* maps the algebra $A(E_k)$ isometrically and isomorphically onto $A(F_k)$. Hence the infinite diagram

$$\begin{array}{ccccccc} A(E_1) & \overset{(z^{d_1})^*}{\longrightarrow} & A(E_2) & \overset{(z^{d_2})^*}{\longrightarrow} & A(E_3) & \overset{(z^{d_3})^*}{\longrightarrow} & \cdots \\ \psi_1^* \downarrow & & \psi_2^* \downarrow & & \psi_3^* \downarrow & & \\ A(F_1) & \overset{\beta_1^2}{\longrightarrow} & A(F_2) & \overset{\beta_2^3}{\longrightarrow} & A(F_3) & \overset{\beta_3^4}{\longrightarrow} & \cdots \end{array}$$

is commutative, and therefore, the inductive sequences (8.15) and (8.18) are isomorphic. Consequently,

$$\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big) = \big[\underset{\longrightarrow}{\lim} \{A(F_n), \beta_n^{n+1}\}_{n\in\mathbb{N}}\big] = \big[\underset{\longrightarrow}{\lim} \{A(E_n), i_n^{n+1}\}_{n\in\mathbb{N}}\big] \cong A\big(\mathbb{D}_G^{[r,1]}\big).$$

□

The following properties of the algebra $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$ follow directly from Theorem 8.3.2, Proposition 8.2.3, and the results in [G12].

(a) The maximal ideal space of $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$ is homeomorphic to the G-annulus $\mathbb{D}_G^{[r,1]}$.

(b) The Shilov boundary of $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$ is the set $b\mathbb{D}_G^{[r,1]} = r \diamond G \cup 1 \diamond G$.

(c) $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$ is a maximal algebra on its Shilov boundary.

(d) $\operatorname{codim}\big(\operatorname{Re}(\mathcal{A}(\mathcal{D}_B^{[r,1]})\big|_{b\mathbb{D}_G^{[r,1]}})\big) = 1$.

Let F be a closed subset of the unit disc $\overline{\mathbb{D}}$. Denote by $A(F)$ the algebra of all continuous functions on F that are analytic in the interior of F. Recall that $A(F)$ coincides with the uniform closure on F of the restrictions of Gelfand transforms of the elements in $A(\mathbb{T})$ on F. i.e. $A(F) = \widehat{A}(\mathbb{D})|_F$.

Let $B = \{B_1, B_2, \ldots, B_n, \ldots\}$ be a sequence of finite Blaschke products on $\overline{\mathbb{D}}$, and let $0 < r < 1$. Define inductively a sequence of compact sets D_n in $\overline{\mathbb{D}}$, as follows. $D_{n+1} = B_n^{-1}(D_n)$, for $n \geq 1$, $D_1 = \mathbb{D}^{[0,r]} = \overline{\mathbb{D}}(r) = \big\{z \in \mathbb{D}\colon |z| \leq r\big\}$. There arises an inverse sequence

$$\overline{\mathbb{D}}(r) \xleftarrow{B_1} D_2 \xleftarrow{B_2} D_3 \xleftarrow{B_3} D_4 \xleftarrow{B_4} \cdots \longleftarrow \overline{\mathcal{D}}_B(r) \tag{8.19}$$

of subsets of $\mathbb{D}$. The inductive limit

$$\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big) = \big[\varinjlim \{A(D_n), B_n^*\}_{n\in\mathbb{N}}\big] \tag{8.20}$$

is a uniform algebra on its maximal ideal space

$$\varprojlim \{D_n, B_{n-1}\}_{k\in\mathbb{N}} = \overline{\mathcal{D}}_B(r) \subset \mathcal{D}_B.$$

Any Blaschke product $B(z) = e^{i\theta}\prod\limits_{k=1}^{n}\Big(\dfrac{z - z_k}{1 - \overline{z}_k z}\Big)$, $|z_k| < 1$, of order n generates an n-sheeted covering over each simply connected domain $V \subset \mathbb{D}$ which is free of singular points of B. Thus the set $F = B^{-1}(V) \subset \mathbb{D}$ is biholomorphic with a disjoint collection of n copies of V, i.e. $F \cong V \times F_n$, where $F_n = \{1, 2, \ldots, n\}$, and the algebra $A(F)$ is isomorphic to a subalgebra of the algebra

$$A^{(n)}(V) = A(V) \oplus A(V) \oplus \cdots \oplus A(V) \cong A(V \times F_n),$$

where $A(V \times F_n)$ is the algebra of all continuous functions $f(z,k)$ on $\overline{V} \times F_n$ such that $f(\,\cdot\,,k) \in A(V)$, $k = 1, 2, \ldots, n$. Clearly, $\overline{V} \times F_n$ is the set of maximal ideals of algebra $A(F)$, and $A(F)|_{\overline{V}\times\{k\}} \cong A(V)$ for every $k = 1, 2, \ldots, n$. Hence $A(F) \subset A^n(V) = A(V \times F_n) \subset C(\overline{V} \times F_n)$. The space $C(F_n)$ also can be considered as a subalgebra of $A^n(V)$ consisting of all functions $f \in A^n(V)$ that are constant on the sets $\overline{V} \times \{k\}$, $k \in F_n$.

Proposition 8.3.4. *Let $B = \{B_1, B_2, \dots, B_n, \dots\}$ be a sequence of finite Blaschke products on $\overline{\mathbb{D}}$ and let $0 < r < 1$. Suppose that the set D_n does not contain singular points of B_{n-1} for every $n \in \mathbb{N}$. Then:*

(i) *There is a compact set Y such that the maximal ideal space*

$$\mathcal{M}_{\mathcal{A}(\overline{\mathcal{D}}_B(r))} = \overline{\mathcal{D}}_B(r) = \varprojlim \{D_{n+1}, B_n\}_{n\in\mathbb{N}}$$

is homeomorphic to the Cartesian product $\overline{\mathbb{D}}(r) \times Y$.

(ii) *$\mathcal{A}(\overline{\mathcal{D}}_B(r))$ is isometrically isomorphic to an algebra of functions $f(x,y) \in C(\overline{\mathbb{D}}(r) \times Y)$, such that $f(\cdot\,, y) \in A(\overline{\mathbb{D}}(r))$ for every $y \in Y$.*

(iii) *$\mathcal{A}(\overline{\mathcal{D}}_B(r))\big|_{\overline{\mathbb{D}}(r)\times\{y\}} \cong A(\mathbb{D}(r))$ for every $y \in Y$.*

Proof. Consider the adjoint inductive sequence

$$A(\overline{\mathbb{D}}(r)) \xrightarrow{B_1^*} A(D_2) \xrightarrow{B_2^*} A(D_3) \xrightarrow{B_3^*} \cdots \longrightarrow \mathcal{A}(\overline{\mathcal{D}}_B(r)). \tag{8.21}$$

Observe that the set $D_{m+1} = B_m^{-1}(D_m)$ is biholomorphic to $\overline{\mathbb{D}}(r) \times F_{d_m}$ for $m \geq 1$, where $d_m = \operatorname{ord} B_m$. There arises a mapping $j_k : \overline{\mathbb{D}}(r) \times F_{d_{k+1}} \longrightarrow \overline{\mathbb{D}}(r) \times F_{d_k}$ such that the diagram

$$\begin{array}{ccc} D_k & \xleftarrow{B_k} & D_{k+1} \\ {\scriptstyle I_k}\downarrow & & {\scriptstyle I_{k+1}}\downarrow \\ \overline{\mathbb{D}}(r) \times F_{d_k} & \xleftarrow{j_k} & \overline{\mathbb{D}}(r) \times F_{d_{k+1}} \end{array}$$

commutes. Here I_k is the natural biholomorphic mapping $I_k : D_k \longrightarrow \overline{\mathbb{D}}(r) \times F_{d_k}$. Note that j_k maps $\overline{\mathbb{D}}(r)$ onto $\overline{\mathbb{D}}(r)$, and $F_{d_{k+1}}$ onto F_{d_k}. Hence, the adjoint diagram

$$\begin{array}{ccc} A(D_k) & \xrightarrow{B_k^*} & A(D_{k+1}) \\ {\scriptstyle I_k^*}\uparrow & & {\scriptstyle I_{k+1^*}}\uparrow \\ A(\overline{\mathbb{D}}(r) \times F_{d_k}) & \xrightarrow{j_k^*} & A(\overline{\mathbb{D}}(r) \times F_{d_{k+1}}) \end{array}$$

is commutative for every $k \in \mathbb{N}$, and therefore the infinite diagram

$$\begin{array}{ccccccc} A(D_1) & \xrightarrow{B_1^*} & A(D_2) & \xrightarrow{B_2^*} & A(D_3) & \xrightarrow{B_3^*} & \cdots \\ {\scriptstyle I_1^*}\uparrow & & {\scriptstyle I_2^*}\uparrow & & {\scriptstyle I_3^*}\uparrow & & \\ A(\overline{\mathbb{D}}(r) \times F_{d_1}) & \xrightarrow{j_1^*} & A(\overline{\mathbb{D}}(r) \times F_{d_2}) & \xrightarrow{j_2^*} & A(\overline{\mathbb{D}}(r) \times F_{d_3}) & \xrightarrow{j_3^*} & \cdots \end{array}$$

is commutative. Hence, the inductive sequence (8.21) is isomorphic to the sequence

$$A(\mathbb{D}(r)) \xrightarrow{j_1^*} A(\overline{\mathbb{D}}(r) \times F_{d_2}) \xrightarrow{j_2^*} A(\overline{\mathbb{D}}(r) \times F_{d_3}) \xrightarrow{j_3^*} \cdots. \tag{8.22}$$

Consider the inductive sequence

$$\mathbb{C} \xrightarrow{j_1^*} C(F_{d_2}) \xrightarrow{j_2^*} C(F_{d_3}) \xrightarrow{j_3^*} \cdots \tag{8.23}$$

of algebras $A\big(\overline{\mathbb{D}}(r)\times F_{d_k}\big)$ restricted on F_{d_k}. A straightforward check shows that $\mathcal{F}=\big[\varinjlim\,\{C(F_{d_n}),j_n^*\}_{n\in\mathbb{N}}\big]$ is a commutative C^*-algebra. Therefore $\mathcal{F}=C(Y)$, where $Y=\varprojlim\,\{F_{d_{n+1}},j_n|_{F_{d_{n+1}}}\}_{n\in\mathbb{N}}$. Note that the inductive sequence (8.19) is isomorphic to the sequence

$$\overline{\mathbb{D}}(r)\xleftarrow{j_1}\overline{\mathbb{D}}(r)\times F_{d_2}\xleftarrow{j_2}\overline{\mathbb{D}}(r)\times F_{d_3}\xleftarrow{j_3}\overline{\mathbb{D}}(r)\times F_{d_4}\xleftarrow{j_4}\cdots\longleftarrow\overline{\mathbb{D}}(r)\times Y.$$

It is clear that the algebra

$$\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)=\big[\varinjlim\,\{A(D_n),B_n^*\}_{n\in\mathbb{N}}\big]\cong\big[\varinjlim\,\{A\big(\overline{\mathbb{D}}(r)\times F_{d_n}\big),j_n^*\}_{n\in\mathbb{N}}\big]$$

is a subalgebra of $A\big(\overline{\mathbb{D}}(r)\times Y\big)$ such that $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)|_{\overline{\mathbb{D}}(r)\times\{y\}}=A\big(\mathbb{D}(r)\big)$ for every $y\in Y$. $\square$

Note that the set Y from Proposition 8.3.4 is homeomorphic to the set $\big\{\{y_n\}_{n=1}^\infty,\ y_n\in(B_1\circ B_2\circ\cdots\circ B_{n-1})^{-1}(0)\big\}$. Let $\mathbb{T}_r=r\mathbb{T}=\big\{|z|=r\big\}$. Since

$$b\overline{\mathcal{D}}_B(r)=\varprojlim\,\big\{bD_{n+1},B_n|_{bD_{n+1}}\big\}_{n\in\mathbb{N}}\cong\mathbb{T}_r\times Y,\tag{8.24}$$

Proposition 8.3.4 implies the following.

Corollary 8.3.5. *In the setting of Proposition* 8.3.4, *the only singleton Gleason parts of the algebra* $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)$ *are the points on the Shilov boundary* $b\,\overline{\mathcal{D}}_B(r)\cong\mathbb{T}_r\times Y$.

Proposition 8.3.6. *Let* $B=\{B_1,B_2,\ldots,B_n,\ldots\}$ *be a sequence of finite Blaschke products on* $\overline{\mathbb{D}}$, *and let* $0<r<1$. *Suppose that for every* $n\in\mathbb{N}$ *the set of singular points for* B_{n-1} *in* D_n *is the set* $(B_1\circ B_2\circ\cdots\circ B_{n-1})^{-1}(0)$, *and assume that all its points have one and the same order* $d_{n-1}>1$. *Then:*

(i) *There is a compact set* Y *such that the maximal ideal space*

$$\mathcal{M}_{\mathcal{A}\left(\overline{\mathcal{D}}_B(r)\right)}=\overline{\mathcal{D}}_B(r)=\varprojlim\ \{D_{n+1},B_n|_{D_{n+1}}\}_{k\in\mathbb{N}}$$

is homeomorphic to the Cartesian product $\overline{\mathbb{D}}_{G_\Lambda}(r)\times Y$, *where* $\Lambda=\{d_k\}_{k=1}^\infty$ *is the sequence of the orders of* B_k.

(ii) *The algebra* $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)$ *on* $\overline{\mathcal{D}}_B(r)$ *is isometrically isomorphic to an algebra of functions* $f(x,y)\in C\big(\overline{\mathbb{D}}_{G_\Lambda}(r)\times Y\big)$, *such that* $f(\,\cdot\,,y)\in A\big(\overline{\mathbb{D}}_{G_\Lambda}(r)\big)$ *for every* $y\in Y$.

(iii) $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)|_{\overline{\mathbb{D}}(r)\times\{y\}}=A\big(\mathbb{D}_{G_\Lambda}(r)\big)$ *for every* $y\in Y$.

Proof. The set $F=(B_1\circ B_2\circ\cdots\circ B_{n-1})^{-1}(D_n)\subset\overline{\mathbb{D}}$ is biholomorphic to a disjoint collection of d_n copies of D_n, i.e. $F\approx D_n\times F_{d_n}$, $F_{d_n}=\{1,2,\ldots,d_n\}$. In addition, the algebra $A(F)$ is isomorphic to a subalgebra of the algebra

$$A^{d_n}(D_n)=A(D_n)\oplus A(D_n)\oplus\cdots\oplus A(D_n)\cong A(D_n)\times F_{d_n}.$$

Moreover, $\mathcal{A}(F)|_{D_n\times\{k\}} \cong A(D_n)$ for every $k = 1, 2, \ldots, d_n$. Hence $\mathcal{A}(F) \subset A^{d_n}(D_n) = A(D_n \times F_{d_n}) \subset C(D_n \times F_{d_n})$, and $D_n \times F_{d_n}$ is the set of maximal ideals of $\mathcal{A}(F)$. Consider the space $C(F_{d_n})$ as a subalgebra of $A^{d_n}(D_n)$ consisting of all functions $f \in A^{d_n}(D_n)$ that are constant on the sets $D_n^{(r)} \times \{k\}$, $k \in F_{d_n}$. As in the proof of Proposition 8.3.4, we see that $\mathcal{F} = \big[\varinjlim \{C(F_{d_n}), B_n^*\}_{n\in\mathbb{N}}\big] = C(Y)$, where $Y = \varprojlim \{F_{d_{n+1}}, B_n\}_{n\in\mathbb{N}}$, and (8.19) is isomorphic to the sequence

$$\overline{\mathbb{D}}(r) \xleftarrow{j_1} \overline{\mathbb{D}}(r) \times F_{d_2} \xleftarrow{j_2} \overline{\mathbb{D}}(r) \times F_{d_3} \xleftarrow{j_3} \cdots \longleftarrow \overline{\mathbb{D}}_{G_\Lambda}(r) \times Y.$$

Consequently, the limit $\overline{\mathcal{D}}_B(r)$ of the inverse sequence (8.19) is isomorphic to $\overline{\mathbb{D}}_{G_\Lambda}(r) \times Y$. Moreover, the algebra $\mathcal{A}\big(\mathcal{D}_B(r)\big) = \big[\varinjlim \{A(D_n), B_n^*\}_{n\in\mathbb{N}}\big]$ is a subalgebra of $C\big(\overline{\mathbb{D}}_{G_\Lambda}(r) \times Y\big)$ such that $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)\big|_{\overline{\mathbb{D}}_{G_\Lambda}(r)\times\{y\}} = A\big(\mathbb{D}_{G_\Lambda}(r)\big)$ for every $y \in Y$. $\square$

Note that, as before, the set Y from the above is homeomorphic to the set $\big\{\{y_n\}_{n=1}^{\infty},\ y_n \in (B_1 \circ B_2 \circ \cdots \circ B_{n-1})^{-1}(0)\big\}$.

8.4 Parts of Blaschke inductive limit algebras

We have seen already various links between Blaschke inductive limit algebras and G-disc algebras. In this section we describe the Gleason parts of Blaschke inductive algebras, and use them to find necessary and sufficient conditions for a Blaschke inductive limit algebra to be isometrically isomorphic to a G-disc algebra.

Let A be a uniform algebra on the compact set X. While every point in the Shilov boundary ∂A is itself a Gleason part (e.g. [S4]), the opposite is not always true, i.e. there are singleton Gleason parts outside the Shilov boundary of A. For instance, if G is a solenoidal group, then the origin $\omega = \big(\{0\} \times G\big)/\big(\{0\} \times G\big) \in \mathbb{D}_G$ of the G-disc $\mathbb{D}_G$ is a singleton Gleason part for the G-disc algebra A_{Γ_+}. Of course $\omega \notin \partial\, A_{\Gamma_+} = G$.

The celebrated theorem of Wermer (e.g. [S4]) states that an analytic disc can be embedded in every non-singleton Gleason part of the maximal ideal space of a Dirichlet algebra. Therefore it is of particular interest to locate singleton Gleason parts of an algebra, and especially those of them that do not belong to the Shilov boundary.

Observe that, as it follows from the results in Section 8.3, all singleton Gleason parts of the algebra $A\big(\mathcal{D}_B^{[r,1]}\big)$ belong to the Shilov boundary $b\mathbb{D}_G^{[r,1]}$. On the other hand, by (8.24), Propositions 8.3.4 and 8.3.6 we have the following

Corollary 8.4.1. *Let* $B = \{B_1, B_2, \dots, B_n, \dots\}$ *be a sequence of finite Blaschke products on* $\overline{\mathbb{D}}$, D_n *are as in* (8.19), *and let* $0 < r < 1$. *In the setting of Proposition* 8.3.6 *there are no singleton Gleason parts of the algebra* $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)$ *in the set* $\mathcal{M}_{\mathcal{A}\left(\overline{\mathcal{D}}_B(r)\right)} \setminus \big(b\overline{\mathcal{D}}_B(r) \cup (\{\omega\} \times Y)\big)$, *where* ω *is the origin of the* G_Λ*-disc* $\overline{\mathbb{D}}_{G_\Lambda}$, *and* Y *is the set* $Y = \big\{\{y_n\}_{n=1}^\infty,\ y_n \in (B_1 \circ B_2 \circ \cdots \circ B_{n-1})^{-1}(0)\big\}$ *from Proposition* 8.3.6.

As an immediate consequence we obtain that the algebra $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)$ is isomorphic to a G-disc algebra if and only if Y is a singleton set.

Corollary 8.4.2. *In the setting of Corollary* 8.4.1 *the algebra* $\mathcal{A}\big(\overline{\mathcal{D}}_B(r)\big)$ *is isomorphic to a G-disc algebra if and only if every Blaschke product* B_n *has a single singular point* $z_0^{(n)}$ *in* $D_n^{(r)}$ *such that* $B_n\big(z_0^{(n)}\big) = z_0^{(n+1)}$ *for all* n *big enough.*

Given a sequence of Blaschke products $B = \{B_n\}_{n=1}^\infty$ on $\overline{\mathbb{D}}$, consider the Blaschke inductive limit algebra $\mathcal{A}(\mathcal{D}_B) = \big[\varinjlim \{A(\mathbb{D}_k), \beta_k^{k+1}\}_{k\in\mathbb{N}}\big]$ on the compact set $\mathcal{D}_B = \varprojlim \{\overline{\mathbb{D}}_k, B_{k-1}\}_{k\in\mathbb{N}}$, where $\beta_k^{k+1} = B_k^*$. Recall that the Shilov boundary of $\mathcal{A}(\mathcal{D}_B)$ is the group $\mathcal{T}_B = \varprojlim \{\mathbb{T}_k, B_{k-1}\}_{k\in\mathbb{N}}$.

Definition 8.4.3. We denote by $\mathcal{B}_r$ the family of Blaschke products on $\overline{\mathbb{D}}$ whose zeros are inside the disc $\overline{\mathbb{D}}(r) = r\overline{\mathbb{D}} = \big\{|z| \le r\big\}$. The set of elements in $\mathcal{B}_r$ that vanish at 0 will be denoted by $\mathcal{B}_r^0 \subset \mathcal{B}_r$.

Theorem 8.4.4. *Suppose that* $B_n \in \mathcal{B}_r^0$ *and* $\operatorname{ord} B_n > 1$ *for every* $n \in \mathbb{N}$. *Then there is only one singleton Gleason part in the set* $\mathcal{D}_B \setminus \mathcal{T}_B$.

We need several preliminary results for the proof. Given two points m_1 and m_2 in $\mathcal{D}_B = \mathcal{M}_{A(\mathcal{D}_B)}$, consider the Gleason metric

$$d(m_1, m_2) = \sup_{\substack{\|f\| < 1 \\ f \in A(\mathcal{D}_B)}} \big|m_1(f) - m_2(f)\big|.$$

Lemma 8.4.5. *Let* $m_1 = (z_1, z_2, \dots)$, *where* $z_k = B_k(z_{k+1})$, *and* $m_2 = (w_1, w_2, \dots)$, *where* $w_k = B_k(w_{k+1})$, *be the chains in the sequence* $\{\overline{\mathbb{D}}_k, B_{k-1}\}_{k\in\mathbb{N}}$ *of the points* m_1, $m_2 \in \mathcal{D}_B$ *correspondingly. Then*

$$\frac{4\,d\,(m_1, m_2)}{4 + d^2(m_1, m_2)} = \lim_{k\to\infty} \left| \frac{z_k - w_k}{1 - \overline{w}_k z_k} \right|. \tag{8.25}$$

Proof. Let $z_k, w_k \in \mathbb{D}$ denote the restrictions of m_1 and m_2 on $A(\mathbb{D}_k)$ respectively. Define

$$d_k(m_1, m_2) = \sup_{\substack{\|f\| < 1 \\ f \in A(\mathbb{D}_k)}} \big|m_1(f) - m_2(f)\big| = d\,(z_k, w_k).$$

Since $A(\mathbb{D}_k) \subset A(\mathbb{D}_{k+1})$ and $\mathcal{A}(\mathcal{D}_B) = \big[\bigcup_{k=1}^{\infty} A(\mathbb{D}_k)\big]$, we have that

$$d_k(m_1, m_2) \le d_{k+1}(m_1, m_2) \le d\,(m_1, m_2), \text{ and } d\,(m_1, m_2) = \lim_{k\to\infty} d_k(m_1, m_2).$$

Note that (cf. [G2]),

$$\frac{4\,d_k(m_1, m_2)}{4 + d_k^2(m_1, m_2)} = \left|\frac{z_k - w_k}{1 - \overline{w}_k z_k}\right|.$$

Consequently,

$$\frac{4\,d\,(m_1, m_2)}{4 + d^2(m_1, m_2)} = \lim_{k\to\infty} \frac{4\,d_k(m_1, m_2)}{4 + d_k^2(m_1, m_2)} = \lim_{k\to\infty} \left|\frac{z_k - w_k}{1 - \overline{w}_k z_k}\right|. \qquad \square$$

Lemma 8.4.6. *For every* $\rho \in [0, 1]$ *let* $\alpha(\rho) = \sup_{\substack{|z| \le \rho \\ |z_0| \le r}} \left|\frac{z - z_0}{1 - \overline{z}_0 z}\right|$. *Then for every* $B \in \mathbb{B}_r$,

$$\max_{|z|<\rho} \big|B(z)\big| < \big(\alpha(\rho)\big)^{\operatorname{ord} B}.$$

Proof. Standard properties of Möbius transformations imply that $\alpha(\rho) \le 1$ and $\alpha(\rho) = 1$ only if $\rho = 1$. Consequently, if $|z| \le \rho$ for any $B \in \mathcal{B}_r$ we have

$$\big|B(z)\big| = \Big|\prod_{k=1}^{n} \Big(\frac{z - z_0}{1 - \overline{z}_0 z}\Big)\Big| \le \big(\alpha(\rho)\big)^n. \qquad \square$$

We observe that since $B_n(0) = 0$ for every $n \in \mathbb{N}$, the chain $\omega = (0, 0, \dots)$ of elements in the system $\{\overline{\mathbb{D}}_k, B_{k-1}\}_{k\in\mathbb{N}}$ belongs to $\mathcal{D}_B = \varprojlim \{\overline{\mathbb{D}}_k, B_{k-1}\}_{k\in\mathbb{N}}$, i.e. the maximal ideal space of $\mathcal{A}(\mathcal{D}_B)$. The point ω is called also the *origin* of $\mathcal{D}_B$.

Proposition 8.4.7. *Suppose that* $B_n \in \mathcal{B}_r^0$ *and* $\operatorname{ord} B_n > 1$ *for every* $n \in \mathbb{N}$. *Then* ω *is a singleton Gleason part of* $\mathcal{A}(\mathcal{D}_B)$ *in* $\mathcal{D}_B \setminus \mathcal{T}_B$.

Proof. Let $m = (z_1, z_2, \dots)$ be a point in $\mathcal{D}_B$ and let $d(\omega, m) = d$. By (8.25)

$$\frac{4\,d\,(\omega, m)}{4 + d^2(\omega, m)} = \lim_{n\to\infty} |z_n| = \frac{4\,d}{4 + d^2} = c \le 1.$$

By Schwarz's lemma, $|z_n| = \big|B_n(z_{n+1})\big| < |z_{n+1}|$, and hence $|z_n| \le c$ for every $n \in \mathbb{N}$. Hence,

$$|z_n| = \big|B_n(z_{n+1})\big| = |z_{n+1}|\Big|\Big(\frac{B_n}{z}\Big)(z_{n+1})\Big| < |z_{n+1}|\big(\alpha(c)\big)^{\operatorname{ord} B_n - 1} < c\,\alpha(c),$$

and consequently,

$$c = \lim_{n\to\infty} |z_n| \le c\,\alpha(c) \le c.$$

Therefore, $\alpha(c) = 1$ and thus $1 = c = 4/(4 + d^2)$, i.e. $d = d(\omega, m) = 2$, i.e. m and ω belong to different Gleason parts. $\square$

It remains to show that ω is the only singleton Gleason part of $\mathcal{A}(\mathcal{D}_B)$. For the following lemma, probably well known, we provide a short proof.

Lemma 8.4.8. *Let $B \subset \mathbb{B}_r$ and let W be a simply connected domain in $\mathbb{D}$, such that $\mathbb{D}^{[0,r]} \subset W \subset \mathbb{D}$. Let $K = \mathbb{D} \setminus W$ and $K_B = \mathbb{D} \setminus B^{-1}(W)$. If the boundary bW of W is a piecewise smooth curve, then the covering mapping $K_B \longrightarrow K$ generated by B does not have singular points.*

Proof. Let $z_0 \in K$. Consider a simply connected domain $\widetilde{W}$, $W \subset \widetilde{W} \subset \mathbb{D}$ with a piecewise smooth boundary $b\widetilde{W}$ that contains z_0. $B^{-1}(\widetilde{W})$ also has a piecewise smooth boundary $bB^{-1}(\widetilde{W}) = B^{-1}(b\widetilde{W})$. Since all zeros of B belong to $\mathbb{D}^{[0,r]} \subset W \subset \widetilde{W}$, the Argument Principle for analytic functions implies that every turn around the curve $bB^{-1}(\widetilde{W})$ generates N rotations around the curve $b\widetilde{W}$, where $N = \operatorname{ord} B$. Therefore, $\operatorname{card} B^{-1}(z_0) = \operatorname{ord} B$, i.e. z_0 is not a singular point for B. □

Proof of Theorem 8.4.4. By Proposition 8.4.7, it is enough to show that the point ω is the only singleton Gleason part for $A(\mathcal{D}_B)$. Let $m \in \mathcal{D}_B,\ m = (z_1, z_2, \dots, z_n, \dots) \neq \omega$. As we saw in the proof of Proposition 8.4.7, $|z_n| < |z_{k+1}|$ for every $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} |z_n| = 1$. Therefore, without loss of generality we can assume that

$$|z_1| > r + \varepsilon, \text{ where } \varepsilon = (1 - r)/2. \tag{8.26}$$

Consider the simply connected domains

$$W_{n+1} = B_n^{-1}(W_n),\ W_0 = \mathbb{D}^{[0,r+\varepsilon/2]}, \tag{8.27}$$

and let

$$K_0 = \mathbb{D}^{[0,r+\varepsilon/2]},\ K_{n+1} = B_n^{-1}(K_n) = \mathcal{D}_B \setminus W_{n+1}.$$

Lemma 8.4.8 implies that B_n has no singularities on K_{n+1}. According to Theorem 8.3.2, $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$ is isomorphic to $A\big(\mathbb{D}_{G_\Lambda}^{[r,1]}\big)$. Clearly $\mathcal{M}_{\mathcal{A}\left(\mathcal{D}_B^{[r,1]}\right)} \subset \mathcal{D}_B$, and $A(\mathcal{D}_B)|_{\mathcal{M}_{\mathcal{A}\left(\mathcal{D}_B^{[r,1]}\right)}}$ is a uniform subalgebra of $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$. The point m belongs to the interior of $\mathcal{M}_{\mathcal{A}\left(\mathcal{D}_B^{[r,1]}\right)}$ since $z_n \in \operatorname{int} K_n$ for every $n \in \mathbb{N}$. If we assume that m is the only point in its Gleason part relative to $\mathcal{A}(\mathcal{D}_B)$, then

$$\sup_{\substack{\|f\| = 1 \\ f \in \mathcal{A}(\mathcal{D}_B^{[r,1]})}} \big|\widetilde{f}(m_1) - \widetilde{f}(m)\big| \geq \sup_{\substack{\|f\| = 1 \\ f \in \mathcal{A}(\mathcal{D}_B)}} \big|\widetilde{f}(m_1) - \widetilde{f}(m)\big| = 2$$

for every $m_1 \in \mathcal{M}_{\mathcal{A}\left(\mathcal{D}_B^{[r,1]}\right)}$, i.e. m is the only point in its Gleason part relative to $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$, which is impossible. Hence, m does not belong to any singleton Gleason part of $\mathcal{A}\big(\mathcal{D}_B^{[r,1]}\big)$. □

Corollary 8.4.9. *Let $B \in \mathbb{B}_r$, $B(0) \neq 0$, and $B_k(z) = z^{d_k} B^{c_k}$, $d_k > 1$. Then there is only one singleton Gleason part in the set $\mathcal{D}_B \setminus \mathcal{T}_B$.*

Proposition 8.4.10. *Let B be a finite Blaschke product with $B(0) = 0$. Consider the stationary sequence $B = \{B, B, \dots\}$. If the Blaschke inductive limit algebra $\mathcal{A}(\mathcal{D}_B) = \left[\varinjlim \left\{A(\mathbb{D}_k), B_k\right\}_{k\in\mathbb{N}}\right]$, $\mathbb{D}_k = \mathbb{D}$, $B_k = B$ is isometrically isomorphic to a G-disc algebra, then necessarily $B(z) = c\, z^n$, where $c \in \mathbb{D}$, $|c| = 1$, and $n \in \mathbb{N}$.*

We precede the proof by several preliminary results.

Lemma 8.4.11. *Consider the inductive sequence of algebras*

$$A(\mathbb{D}_1) \xrightarrow{B^*} A(\mathbb{D}_2) \xrightarrow{B^*} A(\mathbb{D}_3) \xrightarrow{B^*} \cdots \longrightarrow \mathcal{A}(\mathcal{D}_B),$$

where B is a Blaschke product B with $B(0) = 0$. For every $n \in \mathbb{N}$ there exists an automorphism $I_n : \mathcal{A}(\mathcal{D}_B) \longrightarrow \mathcal{A}(\mathcal{D}_B)$ such that

$$I_n\big(i_1(A(\mathbb{D}_1))\big) = i_n\big(A(\mathbb{D}_n)\big), \tag{8.28}$$

where i_n is the natural embedding $i_n \colon A(\mathbb{D}_n) \longrightarrow \mathcal{A}(\mathcal{D}_B)$.

Proof. We prove the statement in the case $n = 2$. For $n > 2$ it follows the same lines. For every $n \in \mathbb{N}$ denote by I_2^n the identity mapping of $A(\mathbb{D}_n)$ onto $A(\mathbb{D}_{n+1})$. For any $n \in \mathbb{N}$ we have that $I_2^n(z|_{\mathbb{D}_n}) = z|_{\mathbb{D}_{n+1}}$, $\big\|I_2^n(f)\big\| = \|f\|$ for each $f \in A(\mathbb{D}_n)$, and hence the diagram

$$\begin{array}{ccc} A(\mathbb{D}_n) & \xrightarrow{B^*} & A(\mathbb{D}_{n+1}) \\ {\scriptstyle I_2^n}\downarrow & & {\scriptstyle I_2^{n+1}}\downarrow \\ A(\mathbb{D}_{n+1}) & \xrightarrow{B^*} & A(\mathbb{D}_{n+2}) \end{array}$$

commutes. Consequently, the infinite diagram

$$\begin{array}{ccccccc} A(\mathbb{D}_1) & \xrightarrow{B^*} & A(\mathbb{D}_2) & \xrightarrow{B^*} & A(\mathbb{D}_3) & \xrightarrow{B^*} & \cdots \\ {\scriptstyle I_2^1}\downarrow & & {\scriptstyle I_2^2}\downarrow & & {\scriptstyle I_2^3}\downarrow & & \\ A(\mathbb{D}_2) & \xrightarrow{B^*} & A(\mathbb{D}_3) & \xrightarrow{B^*} & A(\mathbb{D}_4) & \xrightarrow{B^*} & \cdots \end{array}$$

is commutative, and hence the given inductive sequence is isomorphic to

$$A(\mathbb{D}_2) \xrightarrow{B^*} A(\mathbb{D}_3) \xrightarrow{B^*} A(\mathbb{D}_4) \xrightarrow{B^*} \cdots .$$

There arises an isometric isomorphism from the algebra $\varinjlim \left\{A(\mathbb{D}_k), B_k^*\right\}_{k\in\mathbb{N}}$ onto itself, that can be extended as an isometric isomorphism I_2 from the algebra $\mathcal{A}(\mathcal{D}_B) = \left[\varinjlim \left\{A(\mathbb{D}_k), B_k^*\right\}_{k\in\mathbb{N}}\right]$ onto itself. It is straightforward to check that I_2 satisfies the identity (8.28). □

Corollary 8.4.12. *If $B(0) = 0$, then the origin $\omega = (0, 0, \dots) \in \mathcal{M}_{\mathcal{A}(\mathcal{D}_B)}$ is a fixed point of the mapping $I_n^* : \mathcal{M}_{\mathcal{A}(\mathcal{D}_B)} \longrightarrow \mathcal{M}_{\mathcal{A}(\mathcal{D}_B)}$ adjoint to the automorphism I_n from Lemma* 8.4.11.

Proof. We observe that according to Proposition 8.4.7 and Corollary 8.4.9, the point ω is the only singleton Gleason part of the algebra $\mathcal{A}(\mathcal{D}_B)$ outside its Shilov boundary. Since I_n is an automorphism, it preserves all properties of the algebra $\mathcal{A}(\mathcal{D}_B)$. Therefore, the point $I_n^*(\omega)$ is also a singleton Gleason part of $\mathcal{A}(\mathcal{D}_B)$ that is outside the Shilov boundary. Hence, $I_n^*(\omega) = \omega$, as claimed. □

For the following lemma, probably well known, we provide a short proof.

Lemma 8.4.13. *Let X be a connected compact Hausdorff set. If $\lim_{n\to\infty} \|e^{\psi_n} - 1\| = 0$ for some $\psi_n \in C(X)$, then there are $k_n \in \mathbb{Z}$ such that the functions $\varphi_n = \psi_n - 2\pi i k_n$ converge uniformly to 0 on X.*

Proof. If $\psi_n = u_n + iv_n$, then $e^{\psi_n} = e^{u_n}(\cos v_n + i \sin v_n)$. Since $\lim_{n\to\infty} \|e^{\psi_n} - 1\| = 0$, we have that $e^{u_n} \sin v_n \longrightarrow 0$, while $e^{u_n} \cos v_n \longrightarrow 1$ uniformly on X. Therefore, e^{u_n} is a bounded sequence on X and, consequently, $\cos v_n \longrightarrow 1$, $\sin v_n \longrightarrow 0$ uniformly on X. The connectedness of X implies that for every $n \in \mathbb{N}$ there is a $k_n \in \mathbb{Z}$ such that $\|v_n - 2\pi k_n\| < 1$. Thus $v_n - 2\pi k_n \longrightarrow 0$ because of $\sin v_n \longrightarrow 0$, and hence $\cos(v_n - 2\pi k_n) \longrightarrow 1$, which yields $e^{u_n} \longrightarrow 1$, i.e. $u_n \longrightarrow 0$. Consequently, $\varphi_n = \psi_n - 2\pi i k_n \longrightarrow 0$ uniformly on X, as claimed. □

We observe that the mapping $i_1^* \colon \mathcal{M}_{\mathcal{A}(\mathcal{D}_B)} \longrightarrow \overline{\mathbb{D}}$, adjoint to the inclusion $i_1 \colon A(\mathbb{D}) \longrightarrow \mathcal{A}(\mathcal{D}_B)$, maps the Shilov boundary $\partial A(\mathcal{D}_B) = \mathcal{T}_B$ onto $\mathbb{T} = \partial A(\mathbb{D})$.

Lemma 8.4.14. *Let B be a finite Blaschke product with $B(0) = 0$. If S is an isomorphism between the Blaschke inductive limit algebra $\mathcal{A}(\mathcal{D}_B)$ and a G-disc algebra A_{Γ_+}, then the set $(S \circ i_1)\big(A(\mathbb{T})\big)$ contains a character χ_0 of the group $G = \partial A_{\Gamma_+}$.*

Proof. Observe first that $\left|(S \circ i_1)(z|_{\mathbb{T}})\right| \neq 0$ on G. Indeed, since the identity on $\mathbb{T}$ does not vanish on the Shilov boundary of $A(\mathbb{D})$, its image $i_1\big(\mathrm{id}(\mathbb{T})\big) \subset \mathcal{A}(\mathcal{D}_B) \neq 0$ on the Shilov boundary $\mathcal{T}_B$ of $\mathcal{A}(\mathcal{D}_B)$, and therefore $\left|(S \circ i_1)\big(\mathrm{id}(\mathbb{T})\big)\right| \neq 0$ on $\partial A_{\Gamma_+} = G$. By van Kampen's theorem [vK] there is a $\chi_0 \in \widehat{G}$ and $\varphi \in C(G)$, such that $(S \circ i_1)(z|_{\mathbb{T}}) = \chi_0 \, e^{\varphi}$. The Arens-Royden theorem (e.g. [S4]) implies that $\chi_0 \in A_{\Gamma_+}$. We claim that $\chi_0 \in (S \circ i_1)\big(A(\mathbb{T})\big)$.

Let χ be a fixed element in $\Gamma_+ = \widehat{G} \cap A_{\Gamma_+}$. Given an $\varepsilon > 0$ one can find an $n \in \mathbb{N}$ so that $d\big((S \circ i_n)\big(A(\mathbb{T})\big), \chi\big) < \varepsilon$, where $d(\,\cdot\,,\,\cdot\,)$ is the uniform distance in $A_{\Gamma_+} \subset C(G)$. If I_n is the automorphism defined in Lemma 8.4.11, we have

$$d\left((S \circ i_1)\big(A(\mathbb{T})\big),\ S I_n^{-1} S^{-1} \chi\right) = d\left(i_1\big(A(\mathbb{T})\big),\ I_n^{-1} S^{-1} \chi\right) =$$

$$d\left((I_n \circ i_1)\big(A(\mathbb{T})\big),\ S^{-1} \chi\right) = d\left(i_n\big(A(\mathbb{T})\big),\ S^{-1} \chi\right) = d\left((S \circ i_n)\big(A(\mathbb{T})\big),\ \chi\right) < \varepsilon,$$

Being an automorphism of the G-disc algebra A_{Γ_+} onto itself, the operator $S I_n^{-1} S^{-1}$ maps χ_0 onto a function of type $c\,\chi_1$, where $\chi_1 \in A_{\Gamma_+}$ is again a character on G, and $c \in \mathbb{C}$, $|c| = 1$ (see Section 4.3). Thus $d\left((S \circ i_1)\big(A(\mathbb{T})\big), c\,\chi_1\right) < \varepsilon$, and hence $d\big((S\circ i_1)\big(A(\mathbb{T})\big),\chi_1\big) < \varepsilon$. Therefore, for every $\varepsilon > 0$ one can find a character $\chi_\varepsilon \in \widehat{G}\cap S\big(\mathcal{A}(\mathcal{D}_B)\big)\big|_G$ such that

$$d\left(i_1\big(A(\mathbb{T})\big), S^{-1}\chi_\varepsilon\right) = d\left((S\circ i_1)\big(A(\mathbb{T})\big),\chi_\varepsilon\right) < \varepsilon. \tag{8.29}$$

Let $\chi_n \in A_{\Gamma_+}$ be such that $d\big((S\circ i_1)\big(A(\mathbb{T})\big),\chi_n\big) < 1/n$, and suppose that $\big\|(S\circ i_1)(f_n)-\chi_n\big\| < 1/n$ for some $f_n \in A(\mathbb{T})$. Hence $f_n \neq 0$ on G, and by van Kampen's theorem [vK] there is an $m_n \in \mathbb{Z}$, and a $\psi_n \in C(\mathbb{T})$, such that $f_n = z^{m_n}e^{\psi_n}$. We have

$$\big\|(S\circ i_1)(f_n)-\chi_n\big\| = \big\|(S\circ i_1)(z^{m_n})\exp\left((S\circ i_1)(\psi_n)-\chi_n\right)\big\| < 1/n, \tag{8.30}$$

where $i_1(f_n) = i_1\big(z^{m_n}e^{\psi_n}\big) \in i_1\big(A(\mathbb{T})\big)$. Consequently,

$$\big\|(\chi_0)^{m_n}\exp\left(m_n\varphi + (S\circ i_1)(\psi_n)\right) - \chi_n\big\| < 1/n.$$

This can happen only if $(\chi_0)^{m_n} = \chi_n$. We have obtained that

$$\big\|\exp\left(m_n\varphi + (S\circ i_1)(\psi_n)\right) - 1\big\| < 1/n.$$

By Lemma 8.4.13 we have that, as $n \longrightarrow \infty$, the functions $m_n\varphi + (S\circ i_1)(\psi_n) - 2\pi i k_n$ converge uniformly to 0 for some $k_n \in \mathbb{Z}$. Note that $(S\circ i_1)(\psi_n) - 2\pi i k_n \in (S\circ i_1)(C(\mathbb{T})) \subset C(\mathcal{T}_B) = \big[\varinjlim\,\{C(\mathbb{T}_k), B_k^*\}_{k\in\mathbb{N}}\big]$. Consequently, $\big\|\varphi + \big((S\circ i_1)(\psi_n) - 2\pi i k_n\big)/m_n\big\| \longrightarrow 0$, and hence $\varphi \in (S\circ i_1)\big(C(\mathbb{T})\big)$. From $(S\circ i_1)\big(\mathrm{id}(\mathbb{T})\big) = \chi_0\, e^{\varphi}$ we see that $\chi_0 \in (S\circ i_1)\big(C(\mathbb{T})\big)$. It remains to show that $\chi_0 \in (S\circ i_1)\big(A(\mathbb{T})\big)$. Suppose that $S^{-1}\chi_0 \notin i_1\big(A(\mathbb{T})\big) \subset i_1\big(C(\mathbb{T})\big)$ and take a $g \in C(\mathbb{T})$ such that $i_1(g) = S^{-1}\chi_0$. Then $g \notin A(\mathbb{T})$, and therefore, by the maximality of the disc algebra $A(\mathbb{T})$, the algebra $\big[A(\mathbb{T}), g\big]$ on $\mathbb{T}$ generated by $A(\mathbb{T})$ and g coincides with $C(\mathbb{T})$. Observe that $i_1\big(C(\mathbb{T})\big) = i_1\big([A(\mathbb{T}), g]\big) = \big[i_1\big(A(\mathbb{T})\big), i_1(g)\big] = \big[i_1\big(A(\mathbb{T})\big), S^{-1}\chi_0\big] \subset i_1\big(C(\mathbb{T})\cap\mathcal{A}(\mathcal{D}_B)\big)\big|_{\mathcal{T}_B}$. However, this contradicts the antisymmetry property of the G-disc algebra $A_{\Gamma_+} \cong \mathcal{A}(\mathcal{D}_B)$. We conclude that $S^{-1}\chi_0 \in i_1\big(A(\mathbb{T})\big)$, i.e. $\chi_0 \in (S\circ i_1)\big(A(\mathbb{T})\big)$, as claimed. □

Proof of Proposition 8.4.10. Let $i_1^*\colon \mathcal{M}_{\mathcal{A}(\mathcal{D}_B)} \longrightarrow \overline{\mathbb{D}}$ denote the adjoint mapping to i_1, $i_1^*(z_1, z_2, \dots) = z_1$, where $(z_1, z_2, \dots)$ is a chain in $\{\overline{\mathbb{D}}_k, B_{k-1}\}_{k\in\mathbb{N}}$. Observe that $i_1(\omega) = 0$. According to Lemma 8.4.14 the set $\big(S\circ i_1)(A(\mathbb{D})\cap\widehat{G}\big)$ contains a character $\chi_0 \in \widehat{G}$. Let $S^{-1}\chi_0 = \big[\big(h, h\circ B, h\circ B^2, h\circ B^3, \dots\big)\big] \in \mathcal{A}(\mathcal{D}_B)$, where $h \in A(\mathbb{T})$. Note that for the Gelfand transform $(S^{-1}\chi_0)^\frown$ we have $0 = \big((S^{-1}\chi_0)^\frown\big)(\omega) = \big(i_1(h)\big)(\omega) = h\big(i_1^*(\omega)\big) = h(0)$. Suppose that $B(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Then $\big((S^{-1}\chi_0)^\frown\big)(0, z_0, \dots) = h(0) = 0$, and therefore $(0, z_0, \dots) = \omega$, since ω is the only zero of the function $(S^{-1}\chi_0)^\frown$ in $\mathcal{M}_{\mathcal{A}(\mathcal{D}_B)}$. Hence, $z_0 = 0$, i.e. 0 is the only zero of the Blaschke product B. Consequently, $B(z) = c\,z^m$ for some $m \in \mathbb{N}$, $c \in \mathbb{C}$, $|c| = 1$. □

Theorem 8.4.4 and Proposition 8.4.10 imply the following:

Theorem 8.4.15. *Let B be a finite Blaschke product on $\mathbb{D}$. The Blaschke inductive limit algebra $A(\mathcal{D}_B)$ is isometrically isomorphic to a G-disc algebra if and only if $B(z)$ is adjoint to a power z^m of z, i.e. if and only if there is an $m \in \mathbb{N}$ and a Möbius transformation $\tau \colon \mathbb{D} \longrightarrow \mathbb{D}$ such that $(\tau^{-1} \circ B \circ \tau)(z) = z^m$.*

8.5 H^∞-type spaces on compact groups

In this section we study algebras of bounded analytic functions on G-discs, that are limits of inductive sequences of H^∞ spaces on groups. What differentiates these algebras from the algebras H^∞_Φ of Φ-hyper-analytic functions, considered in Section 8.1, is that the connecting homomorphisms are not necessarily adjoint homomorphisms to continuous mappings of $\mathbb{D}$ into itself.

Let G be a compact abelian group, whose dual group $\Gamma = \widehat{G}$ is a subgroup of $\mathbb{R}$. We recall that the Hardy space $H^\infty(G)$ on G is the space of boundary value functions on G of elements in $H^\infty(\mathbb{D}_G)$, i.e.

$$H^\infty(G) = \big\{f^* \colon f \in H^\infty(\mathbb{D}_G)\big\},$$

and $\|f^*\|_\infty = \lim_{r\nearrow 1} \sup_{g\in G} \big|f_r(g)\big|$, for every $f \in H^\infty(G)$, where f_r is the r-trace of f. $H^\infty(G)$ is a closed subalgebra of $L^\infty(G, \sigma)$ (e.g. [H2]),

Let I be a directed set, i.e. I is a partially ordered set such that every pair i_1 and i_2 in I has a common successor $i_3 \in I$, such that $i_1 \prec i_3$ and $i_2 \prec i_3$. Let $\{\Gamma_i\}_{i\in I}$ be a family of subgroups of Γ parametrized by I, such that $\Gamma_{i_1} \subset \Gamma_{i_2}$ whenever $i_1 \prec i_2$. Under the natural inclusions $\{\Gamma_i\}_{i\in I}$ becomes an inductive system of groups. Suppose that Γ coincides with the inductive limit of the system $\{\Gamma_i\}_{i\in I}$, i.e. $\Gamma = \varinjlim \{\Gamma_i\}_{i\in I}$. Let $H^\infty_{\Gamma_i}(G)$ denote the space of functions $f \in H^\infty(G)$ with $\operatorname{sp}(f) \subset \Gamma_i$. Clearly, $H^\infty_{\Gamma_i}(G)$ is a closed subalgebra of $H^\infty(G)$, and $H^\infty_{\Gamma_i}(G) \subset H^\infty_{\Gamma_j}(G)$ if and only if $\Gamma_i \subset \Gamma_j$. Therefore the family $\big\{H^\infty_{\Gamma_i}(G)\big\}_{i\in I}$ of subalgebras in $H^\infty(G)$ is ordered by inclusion. Let $\Phi = \{\varphi_i^j\}_{i,j\in I}$ be the family of inclusions $\varphi_i^j : \Gamma_i \hookrightarrow \Gamma_j$, $i \prec j$. It is easy to see that the $\|\cdot\|_\infty$-closure $H^\infty_I(G)$ of the set $\bigcup_{i\in I} H^\infty_{\Gamma_i}(G) = \varinjlim \big\{H^\infty_{\Gamma_i}(G)\big\}_{i\in I}$ coincides with the Hardy space $H^\infty(\mathcal{D}_\Phi)$ of *Φ-hyper-analytic functions* on G, where $\mathcal{D}_F = \mathbb{D}_G$. Clearly, $H^\infty_I(G)$ is a commutative Banach algebra. The next theorem provides a criteria for the algebra $H^\infty(G)$ to be of type $H^\infty_I(G)$.

Theorem 8.5.1. *Let G be a solenoidal group such that its dual group $\Gamma = \widehat{G}$ is the inductive limit of a family $\{\Gamma_i\}_{i\in I}$ of subgroups of Γ, i.e. $\Gamma = \varinjlim \{\Gamma_i\}_{i\in I}$. Let $H^\infty_{\Gamma_i}(G)$ be the space of functions in $H^\infty(G)$ with spectra in Γ_i, $i \in I$. Then the following statements are equivalent.*

(a) $H^\infty(G) = H^\infty_I(G)$.

(b) $H^\infty(G) = \bigcup_{i\in I} H^\infty_{\Gamma_i}(G)$.

(c) *Every countable subgroup Γ_0 in Γ is contained in some group from the family $\{\Gamma_i\}_{i\in I}$.*

For the proof we need the following lemma.

Lemma 8.5.2. *Let $r \in (0,1)$ and let m_r be the representing measure on G of the point $r \diamond \imath \in \mathbb{D}_G$. Then*

$$\lim_{j\to\infty} \sup_{g\in G} m_r(V_j g) = 0 \tag{8.31}$$

for every nested family $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ of neighborhoods of the unit element $\imath$ with $\bigcap_{j=1}^{\infty} V_j = \{\imath\}$.

Proof. Assume, on the contrary, that $\lim_{j\to\infty} \sup_{g\in G} m_r(V_j g) > 0$, and let $\{g_j\}_1^\infty$ be a sequence in G with $m_r(V_j g_j) \longrightarrow c > 0$. By the compactness of G there is a subsequence of $\{g_j\}_1^\infty$, say $\{h_n\}_1^\infty$, that converges to a point $h \in G$. Note that $m_r(V_n h) \geq \lim_{k\to\infty} m_r(V_k h_k) = c$ for every integer n, since $V_n h \supset V_k h_k$ for k big enough. Consequently, $m_r(h) = \lim_{k\to\infty} m_r(V_k h_k) = c$. Consider the Lebesgue decomposition of the measure $m_r - \delta_{r\diamond\imath}$ with respect to the point evaluation δ_h at h, namely $m_r - \delta_{r\diamond\imath} = c\delta_h + \nu$, where the measure ν is singular with respect to δ_h. Since $m_r - \delta_{r\diamond\imath} \perp A_{\Gamma_+}$, the Dirac measure δ_h (as well as ν) is orthogonal to A_{Γ_+}, according to Ahern's theorem (cf. [G1], Ch. II, Cor. 7.8]), which is impossible. $\square$

Proof of Theorem 8.5.1. Clearly, (b) implies (a). Let $f \in H^\infty(G)$. Parseval's identity implies that the spectrum $\mathrm{sp}\,(f)$ of f is countable. Therefore the group Γ_f generated by $\mathrm{sp}\,(f)$ is also countable. By the supposition there is a group $\Gamma_j \in \{\Gamma_i\}_{i\in I}$ that contains Γ_f. Hence $f \in H^\infty_{\Gamma_j}(G) \subset \bigcup_I H^\infty_{\Gamma_i}(G)$. This shows that (c) implies (b).

To prove that (a) implies (c), assume first that Γ is a countable group. Then $G = \widehat{\Gamma}$ is a separable metric group. Let $\{h_n\}_1^\infty$ be a sequence of different points in G with $h_n \longrightarrow \imath$, and let $\{B_n\}_1^\infty$ be a family of disjoint (metric) balls centered at h_n, not containing $\imath$, and such that for every neighborhood V of $\imath$ there is a natural number N such that $B_n \subset V$ for all $n \geq N$. Consider a function $f_n \in A_{\Gamma_+}$ such that $\|f_n\|_\infty = f_n(h_n) = 1$, $f_n(\imath) = 0$ and $|f_n| < 1/2^n$ on $G \setminus B_n$. Such a function exists since the points of G are peak points for A_{Γ_+}. By (6.2) we have

$$\begin{aligned}\big|(f_n)_r(g)\big| &= \big|f_n(r\diamond g)\big| = \Big|\int_G f_n(g\,h)\,dm_r(h)\Big| \leq \int_G \big|f_n(g\,h)\big|\,dm_r(h)\\ &= \int_{g^{-1}B_n} \big|f_n(g\,h)\big|\,dm_r(h) + \int_{G\setminus g^{-1}B_n} \big|f_n(g\,h)\big|\,dm_r(h) \leq m_r(g^{-1}B_n) + 1/2^n.\end{aligned}$$

The hypotheses on the balls B_n imply that

$$\sum_{n=k}^{\infty} \big|(f_n)_r(g)\big| \le m_r(g^{-1}V_k) + 1/2^{k-1} < 2,$$

where $V_k = \bigcup_{n=k}^{\infty} B_n$. It follows from (8.31) that for every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $m_r(g^{-1}V_k) < \varepsilon$ for all $g \in G$. Consequently, the series $\sum_{n=1}^{\infty}(f_n)_r$ converges uniformly on G to a function $\widetilde{f}_r \in A_{\Gamma_+}$. Clearly $\|\widetilde{f}_r\|_\infty < 2$. Therefore the function $\widetilde{f} = \sum_{n=1}^{\infty} f_n$ belongs to the algebra $H^\infty(G)$. Since, by the hypothesis, $H^\infty(G) = H_I^\infty(G)$, the function $\widetilde{f}$ is in $H_I^\infty(G)$. Then there is a $j \in I$ and $f \in H_{\Gamma_j}^\infty(G)$, such that

$$\|\widetilde{f} - f\|_\infty < 1/16.$$

Recall that the group Γ_j is the dual group of the quotient group G/G_j, where $G_j = \Gamma_j^\perp = \big\{g \in G \ : \ \chi^a(g) = 1 \text{ for all } a \in \Gamma_j\big\}$. Therefore, the space $H_{\Gamma_j}^\infty(G)$ coincides with the space of G_j-periodic functions in $H^\infty(G)$, i.e. $u \in H_{\Gamma_j}^\infty(G)$ if and only if $u \in H^\infty(G)$ and $u(h) = u(g\,h) = u_g(h)$ for all $g \in G_j$, $h \in G$. Consequently, $f = f_g$ for some $g \in G_j$, and hence

$$\|\widetilde{f} - \widetilde{f}_g\|_\infty \le \|\widetilde{f} - f\|_\infty + \|\widetilde{f}_g - f_g\|_\infty < 1/8 \tag{8.32}$$

for every $g \in G_j$. Suppose that $\Gamma_j \ne \Gamma$, i.e. $G_j \ne \{\imath\}$. Fix a $g_0 \in G_j \setminus \{\imath\}$. By the continuity of $\widetilde{f}$ on $G \setminus \{\imath\}$ the set

$$V = \big\{h \in G \setminus \{\imath\} \ \big| \ \big|\widetilde{f}(h) - \widetilde{f}(g_0)\big| < 1/16\big\}$$

is an open neighborhood of $g_0 \ne \imath$. By the construction of $\widetilde{f}$, there are g_1 and g_2 in $g_0^{-1}V \setminus \{\imath\}$ such that $|\widetilde{f}(g_1)| > 15/16$ and $|\widetilde{f}(g_2)| < 1/16$. Now

$$\big|\widetilde{f}(g_i) - \widetilde{f}_{g_0}(g_i)\big| \le \big\|\widetilde{f} - \widetilde{f}_{g_0}\big\|_\infty \le 1/8 \ \text{ for } i = 1, 2$$

implies

$$\big|\widetilde{f}_{g_0}(g_1)\big| > 13/16 \ \text{ and } \ \big|f_{g_0}(g_2)\big| < 3/16.$$

Consequently,

$$\big|\widetilde{f}_{g_0}(g_1) - \widetilde{f}_{g_0}(g_2)\big| > 5/8,$$

which is impossible since g_0g_1 and g_0g_2 belong to V. Thus, $G_j = \{\imath\}$, i.e. $\Gamma = \Gamma_j$ is a group in the family $\{\Gamma_i\}_{i\in I}$.

Suppose now that Γ is uncountable and that Γ_0 is a countable subgroup of Γ. Now the argument from the above applies to the countable group Γ_0 and the family $\{\Gamma_0 \cap \Gamma_i\}_{i\in I}$ instead of to the group Γ and the family $\{\Gamma_i\}_{i\in I}$, to obtain that $\Gamma_0 = \Gamma_j \cap \Gamma_0 \subset \Gamma_i$ for some $i \in I$. This completes the proof that (a) implies (c). □

Example 8.5.3. Let $\Gamma = \mathbb{Q}$ be the group of discrete rational numbers. Assume that $\{\Gamma_i\}_{i\in I}$ is an inductive system of subgroups of $\mathbb{Q}$ such that $\mathbb{Q} = \varinjlim \{\Gamma_i\}_{i\in I}$. Theorem 8.5.1 implies that if $\mathbb{Q}$ is not one of the groups in the family $\{\Gamma_i\}_{i\in I}$, then $H_I^\infty(G) \neq H^\infty(G)$.

Example 8.5.4. Let $\Gamma = \mathbb{R}$ and let $\Lambda \subset \mathbb{R}_+$ be a basis in $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. As we saw in Example 1.3.6, $\mathbb{R}$ can be expressed as the limit

$$\mathbb{R} = \varinjlim \left\{\Gamma_{(\gamma,n)}\right\}_{(\gamma,n)\in J}$$

of the inductive system $\left\{\Gamma_{(\gamma,n)}\right\}_{(\gamma,n)\in J}$ of subgroups

$$\Gamma_{(\gamma,n)} = \left\{(1/n!)\,(m_1\gamma_1 + m_2\gamma_2 + \cdots + m_k\gamma_k)\colon m_j \in \mathbb{Z},\ j = 1,\ldots k\right\}$$

of $\mathbb{R}$ that are isomorphic to $\mathbb{Z}^k = \bigoplus_{i=1}^{k} \mathbb{Z}$. Given an $(\gamma, n) \in J$, consider the set

$$H^\infty_{(\gamma,n)}(G) = \left\{f \in H^\infty(G)\colon\ \mathrm{sp}\,(f) \subset \Gamma_{(\gamma,n)}\right\}.$$

The closure $H_J^\infty(G)$ of the set $\bigcup_{(\gamma,n)\in J} H^\infty_{(\gamma,n)}(G) = \varinjlim \left\{H^\infty_{(\gamma,n)}(G)\right\}_{(\gamma,n)\in J}$ under the $\|\cdot\|_\infty$-norm is a subalgebra of $H^\infty(G)$.

There arises the question of whether $H_J^\infty(G)$ coincides with $H^\infty(G)$. As the next theorem shows, in general this is not true.

Theorem 8.5.5. *The space* $H_J^\infty(G) = \varinjlim \left\{H^\infty_{(\gamma,n)}(G)\right\}_{(\gamma,n)\in J}$ *is a proper closed subalgebra of* $H^\infty(G)$.

Proof. The inclusion $H_J^\infty(G) \subset H^\infty(G)$ is proven essentially in [CMX]. Assume, on the contrary, that

$$H^\infty(G) = H_J(G) = \varinjlim \left\{H^\infty_{(\gamma,n)}(G)\right\}_{(\gamma,n)\in J}.$$

According to Theorem 8.5.1 the countable subgroup $\mathbb{Q} \subset \mathbb{R}$ is among the members of the family $\left\{\Gamma_{(\gamma,n)}\right\}_{(\gamma,n)\in J}$, which is impossible since $\Gamma_{(\gamma,n)}$ is isomorphic to $\mathbb{Z}^k$ for some $k \in \mathbb{N}$. □

8.6 Bourgain algebras of inductive limit algebras on groups

In this section we apply the technique of inductive limit algebras to Bourgain algebras. Let $\mathbb{B}$ be a commutative Banach algebra and let $\{A_\sigma\}_{\sigma\in\Sigma}$ be a family of closed subalgebras of $\mathbb{B}$, whose union $\bigcup_{\sigma\in\Sigma} A_\sigma$ is an algebra. Denote by A the closure of $\bigcup_{\sigma\in\Sigma} A_\sigma$ in $\mathbb{B}$. Clearly, A is a commutative Banach algebra in $\mathbb{B}$. The following theorem establishes a relationship between the Bourgain algebra of A and the Bourgain algebras of A_σ.

Theorem 8.6.1. *Let $\{A_\sigma\}_{\sigma\in\Sigma}$, $\{B_\sigma\}_{\sigma\in\Sigma}$ be two families of closed subspaces of a commutative Banach algebra $\mathbb{B}$ such that the B_σ are algebras, and $A_\sigma \subset B_\sigma$ for every $\sigma \in \Sigma$. Let $A = \big[\bigcup_{\sigma\in\Sigma} A_\sigma\big]$ be a (in general linear) subspace, and let $B = \big[\bigcup_{\sigma\in\Sigma} B_\sigma\big]$ be a subalgebra of $\mathbb{B}$. If for every $\sigma \in \Sigma$ there is a bounded linear operator $r_\sigma \colon B \longrightarrow B_\sigma$, such that*

(i) $r_\sigma\big|_{B_\sigma} = \mathrm{id}_{B_\sigma}$,

(ii) $r_\sigma(fg) = f\cdot r_\sigma(g)$ *for every* $f \in B_\sigma$, $g \in B$,

(iii) $\sup_{\sigma\in\Sigma} \|r_\sigma\| < \infty$,

then $A_b^B \subset \big[\bigcup_{\sigma\in\Sigma} (A_\sigma)_b^{B_\sigma}\big]$.

Proof. Let $f \in B$ be a Bourgain element for A. Fix a $\sigma \in \Sigma$, and consider a weakly null sequence $\{\varphi_n\}$ in $A_\sigma \subset A$. Then $\{\varphi_n\}$ is also a weakly null sequence in A since $F|_{A_\sigma} \in A_\sigma^*$ for any $F \in A^*$. Therefore, one can find $h_n \in A$ such that $\|f\varphi_n - h_n\| \longrightarrow 0$. Hence,

$$\big\|r_\sigma(f)\varphi_n - r_\sigma(h_n)\big\| = \big\|r_\sigma(f\varphi_n) - r_\sigma(h_n)\big\| \le \|r_\sigma\|\|f\varphi_n - h_n\| \to 0.$$

Consequently, $r_\sigma(f)$ is a Bourgain element for A_σ, i.e. $r_\sigma(f) \in (A_\sigma)_b^{B_\sigma}$ for every $\sigma \in \Sigma$. Note that under the hypotheses every $f \in B$ is approximable by the elements $r_\sigma(f)$ in the norm of $\mathbb{B}$. Indeed, let $f_{\sigma_n} \in B_{\sigma_n}$ be such that $f_{\sigma_n} \longrightarrow f$. Then $\big\|f - r_{\sigma_n}(f)\big\| \le \|f - f_{\sigma_n}\| + \big\|r_{\sigma_n}(f_{\sigma_n}) - r_{\sigma_n}(f)\big\| \le \|f - f_{\sigma_n}\| + \sup\|r_{\sigma_n}\|\|f_{\sigma_n} - f\|$. Hence $r_{\sigma_n}(f) \longrightarrow f$, and consequently, $f \in \big[\bigcup_{\sigma\in\Sigma} (A_\sigma)_b^{B_\sigma}\big]$. □

Let G be a compact abelian group, and let P be a subsemigroup of $\Gamma = \widehat{G}$, such that $P \cup (-P) = \Gamma$ and $P \cap (-P) = \{0\}$. We equip Γ with the P-order, i.e. $\chi_1 \succ \chi_2$ if and only if $\chi_1 - \chi_2 \in P$. Let Σ be a directed set, and let $\{\Gamma_\sigma\}_{\sigma\in\Sigma}$ be a family of subgroups of $\Gamma = \widehat{G}$ indexed by Σ and directed by inclusions. We will interpret the subgroups $\Gamma_\sigma \subset \Gamma$ as subspaces of $C(G)$. If Γ coincides with the inductive limit of the system $\{\Gamma_\sigma\}_{\sigma\in\Sigma}$, i.e. $\Gamma = \varinjlim \{\Gamma_\sigma\}_{\sigma\in\Sigma} = \bigcup_{\sigma\in\Sigma} \Gamma_\sigma$, then $\Gamma_+ = \varinjlim \{(\Gamma_\sigma)_+\}_{\sigma\in\Sigma} = \bigcup_{\sigma\in\Sigma} (\Gamma_\sigma)_+$. Denote by $A_{(\Gamma_\sigma)_+}$ the space of functions

$f \in A_{\Gamma_+}$ with $\operatorname{sp}(f) \subset \Gamma_\sigma$. Clearly, $A_{(\Gamma_\sigma)_+}$ is a closed subalgebra of A_{Γ_+}, and $A_{(\Gamma_\sigma)_+} \subset A_{(\Gamma_\tau)_+}$ whenever $\sigma \prec \tau$. Therefore, $\{A_{(\Gamma_\sigma)_+}\}_{\sigma\in\Sigma}$ is an inductive system of algebras and $A_{\Gamma_+} = [\varinjlim \{A_{(\Gamma_\sigma)_+}\}_{\sigma\in\Sigma}] = [\bigcup_{\sigma\in\Sigma} A_{(\Gamma_\sigma)_+}]$.

Corollary 8.6.2. *Let $\Gamma = \varinjlim \{\Gamma_\sigma\}_{\sigma\in\Sigma}$ and $G = \widehat{\Gamma}$. Bourgain elements for the G-disc algebra A_{Γ_+} can be approximated uniformly on G by Bourgain elements for the algebras $A_{(\Gamma_\sigma)_+}$, $\sigma \in \Sigma$.*

Proof. Fix a $\sigma \in \Sigma$ and let $K_\sigma = \Gamma_\sigma^\perp = \bigcap_{\chi\in\Gamma\sigma} \{g \in G\colon \chi(g) = 1\}$. The mappings $r_\sigma\colon C(G) \longrightarrow C(G_\sigma)$ defined by

$$\big(r_\sigma(f)\big)(g) = \int_{K_\sigma} f(gh)\, d\sigma(h),$$

where $d\sigma$ is the Haar measure on K_σ satisfies the conditions (i), (ii), (iii) of Theorem 8.6.1. In particular, $\|r_\sigma\| \le 1$ for every $\sigma \in \Sigma$. □

Suppose $\Gamma = \varinjlim \{\Gamma_\sigma\}_{\sigma\in\Sigma}$ as before. Let $H^\infty_{\Gamma_\sigma}(G) = \{f \in H^\infty(G)\colon\ \operatorname{sp}(f) \subset \Gamma_\sigma\}$. Note that $H^\infty_{\Gamma_\sigma}(G)$ is a closed subalgebra of $H^\infty(G)$, and $H^\infty_{\Gamma_\sigma}(G) \subset H^\infty_{\Gamma_\tau}(G)$ if and only if $\Gamma_\sigma \subset \Gamma_\tau$. Therefore, the family $\{H^\infty_{\Gamma_\sigma}(G)\}_{\sigma\in\Sigma}$ of subalgebras of $H^\infty(G)$ is ordered by inclusions. Denote by H^∞_G the $L^\infty(G,\sigma)$-closure $H^\infty_\Gamma(G)$ of the set $\bigcup_{\sigma\in\Sigma} H^\infty_{\Gamma_\sigma}(G) = \varinjlim \{H^\infty_{\Gamma_\sigma}(G)\}_{\sigma\in\Sigma}$. Note that H^∞_G and $H^\infty(G) \cong H^\infty(\mathbb{D}_G)$ are commutative Banach subalgebras of $L^\infty(G,\sigma)$ that are different from each other, unless $G = \mathbb{T}$ (e.g. [T2, GT]). The mappings $r_\sigma : H^\infty_G \longrightarrow H^\infty_{\Gamma_\sigma}(G)$ defined by $\big(r_\sigma(f)\big)(g) = \int_{K_\sigma} f(gh)\, d\sigma(h)$ are bounded linear operators from H^∞_G onto $H^\infty_{\Gamma_\sigma}(G)$ satisfying the hypotheses of Theorem 8.6.1.

Corollary 8.6.3. *Let $\Gamma = \varinjlim \{\Gamma_\sigma\}_{\sigma\in\Sigma}$ and $G = \widehat{\Gamma}$. Any Bourgain element for H^∞_G can be approximated in the L^∞-norm on G by Bourgain elements for $H^\infty_{\Gamma_\sigma}(G)$, $\sigma \in \Sigma$.*

Let $\Gamma = \mathbb{Q}$ be the group of rational numbers with the discrete topology and let $G = b(\mathbb{Q}) = \widehat{\mathbb{Q}_d}$ be its Bohr compactification. Note that $\mathbb{Q}$ can be presented as the limit of the inverse sequence $\{\mathbb{Q}_n, z^{n/k}\}_{n\in\mathbb{N}}$, where $\mathbb{Q}_n = \{m/n\colon m \in \mathbb{Z}\} \cong \mathbb{Z}$ and $n \succ k$ if n is a multiple of k (e.g. [T2]). For any $n \in \mathbb{N}$ consider the algebra $H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q})\big) = H^\infty \circ \chi^{1/n} = \{f \circ \chi^{1/n}\colon f \in H^\infty\} = \{f \in H^\infty(G) : \operatorname{sp}(f) \subset \mathbb{Q}_n\}$. Clearly, $H^\infty_{b(\mathbb{Q}_n)} = H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big)$ is a closed subalgebra of $H^\infty(G)$.

Definition 8.6.4. The closure $H^\infty_{b(\mathbb{Q})}$ of the inductive limit of the system

$$\{H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q})\big), (z^{n/k})^*\}_{n\in\mathbb{N}}$$

with respect to the $\|\cdot\|_\infty$-norm is called the algebra of *hyper-analytic functions* on $G = b(\mathbb{Q}_n)$.

Clearly, $H^\infty_{b(\mathbb{Q})}$ is an inductive limit algebra. As shown in [T], its maximal ideal space resembles the maximal ideal space of H^∞ on the unit circle. In particular, it has no $\mathbb{D}_G$-corona. Let $L^\infty_{\mathbb{Q}}\big(b(\mathbb{Q})\big)$ denote the algebra $\big[\bigcup_{n\in\mathbb{N}} L^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big)\big]_\infty$, where $L^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q})\big) = \big\{f \in L^\infty(G,\sigma)\colon\ \mathrm{sp}\,(f) \subset \mathbb{Q}_n\big\}$. Corollary 8.6.3 implies that the Bourgain algebra of $H^\infty_{b(\mathbb{Q})}$ relative to $L^\infty_{\mathbb{Q}}\big(b(\mathbb{Q})\big)$ is a proper subalgebra of $L^\infty_{\mathbb{Q}}\big(b(\mathbb{Q})\big)$. More precisely,

Corollary 8.6.5. *The Bourgain algebra of $H^\infty_{b(\mathbb{Q})}$ is contained in the algebra $H^\infty_{b(\mathbb{Q})} + C(b(\mathbb{Q}))$.*

Proof. For the Bourgain algebra of $H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big)$ relative to $L^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big)$ we have $\big(H^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))\big)_b^{L^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))} \cong (H^\infty)_b^{L^\infty} \cong H^\infty + C(\mathbb{T})$ (e.g. [CJY]. Hence,

$$\big(H^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))\big)_b^{L^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))} \cong H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big) + C(\mathbb{T})\circ\chi^{1/n} \subset H^\infty_{b(\mathbb{Q})} + C(G).$$

Corollary 8.6.3 implies

$$\begin{aligned}\big(H^\infty_{b(\mathbb{Q})}\big)_b^{L^\infty_{\mathbb{Q}}(b(\mathbb{Q}))} &\subset \Big[\bigcup_{n\in\mathbb{N}} H^\infty_{\mathbb{Q}_n}\big(b(\mathbb{Q}_n)\big)\Big]_b^{L^\infty_{\mathbb{Q}}(b(\mathbb{Q}))} \\ &= \Big[\bigcup_{n\in\mathbb{N}} \big(H^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))\big)_b^{L^\infty_{\mathbb{Q}_n}(b(\mathbb{Q}_n))}\Big] \subset H^\infty_{b(\mathbb{Q})} + C\big(b(\mathbb{Q})\big).\end{aligned}$$

□

Denote by $V(\overline{\mathbb{D}}_G, G)$ the ideal of $L^\infty(G,\sigma)$-functions on $\overline{\mathbb{D}}_G$ converging uniformly to 0 as $m = r \diamond g \longrightarrow G$.

Corollary 8.6.6. *The Bourgain algebra of the algebra $H^\infty_{\mathbb{Q}}(\mathbb{D}_G) = (H^\infty_{\mathbb{Q}})\hat{}$ of hyper-analytic functions on $\mathbb{D}_G$ is a subset of the algebra $H^\infty_{\mathbb{Q}}(\mathbb{D}_G) + C(\overline{\mathbb{D}}_G) + V(\overline{\mathbb{D}}_G, G)$.*

Let $\varGamma = \mathbb{R}$ be the group of real numbers with the discrete topology and let $G = b(\mathbb{R}) = \widehat{\mathbb{R}_d}$ be its Bohr compactification. As we saw in Example 1.3.6, the dual group $\widehat{G} = \mathbb{R}$ can be expressed as the inductive limit of an increasing system of groups $\mathbb{R}_\sigma,\ \sigma \in \varSigma$, isomorphic to $\mathbb{Z}^{k_\sigma},\ k_\sigma \in \mathbb{N}$, i.e. $\mathbb{R} = \bigcup_{\sigma\in\varSigma} \mathbb{R}_\sigma$, where $\mathbb{R}_\sigma \cong \mathbb{Z}^{k_\sigma}$ for some $k_\sigma \in \mathbb{N},\ k_\sigma \geq 2$ (cf. Section 8.5). Theorem 8.6.1 implies that

$$\big(H^\infty_{b(\mathbb{R})}\big)_b^{L^\infty(b(\mathbb{R}))} \subset \Big[\bigcup_{\sigma\in\varSigma} \big(H^\infty_{\mathbb{Z}^{k_\sigma}}(b(\mathbb{Z}^{k_\sigma}))\big)_b^{L^\infty_{\mathbb{Z}^{k_\sigma}}(b(\mathbb{Z}^{k_\sigma}))}\Big].$$

A function $f \in B$ is said to be a *wc*-element [resp. a *c*-element] for A, if for every sequence $\{\varphi_n\}_n \in A,\ \|\varphi_n\| \leq 1$ the sequence $\pi_A(f\varphi_n)$ contains a weakly convergent [resp. norm convergent] subsequence in B/A, i.e. if for every sequence $\{\varphi_n\}_n \in A$, $\|\varphi_n\| \leq 1$ there are elements $h_n \in A$ such that the sequence $f\varphi_n - h_n \in B$ contains a weakly convergent [resp. norm convergent] subsequence. We denote by A^B_{wc} and A^B_c the sets of all wc-elements and c-elements for A in B correspondingly. Clearly, $A^B_c \subset A^B_{wc}$, also $A^B_c \subset A^B_b$.

Theorem 8.6.7. *Under the assumptions of Theorem* 8.6.1,

$$A^B_{wc} \subset \big[\bigcup_{\sigma\in\Sigma} (A_\sigma)^{B_\sigma}_{wc}\big] \text{ and } A^B_c \subset \big[\bigcup_{\sigma\in\Sigma} (A_\sigma)^{B_\sigma}_{c}\big].$$

Proof. Let $f \in B$ be a wc-element for A. Fix a $\sigma \in \Sigma$, and consider a sequence $\{\varphi_n\}$ in the unit ball of $A_\sigma \subset A$. Clearly, every φ_n belongs to the unit ball of A. Therefore one can find $h_n \in A$ such that the sequence $f\varphi_n - h_n$ contains a weakly convergent subsequence in B, which we will denote again by $f\varphi_n - h_n$. Hence the sequence

$$r_\sigma(f)\,\varphi_n - r_\sigma(h_n) = r_\sigma(f\varphi_n) - r_\sigma(h_n) = r_\sigma(f\varphi_n - h_n)$$

converges weakly in B_σ. Consequently, $r_\sigma(f)$ is a wc-element for A_σ. Similar arguments apply for c-elements of A. □

Corollary 8.6.8. *If the hypotheses of Corollary* 8.6.3 *are met, then*

(a) $\big(H^\infty_G\big)^{L^\infty_\Gamma(G)}_{wc} \subset \Big[\bigcup_{\sigma\in\Sigma} \big(H^\infty_{\Gamma_\sigma}(G)\big)^{L^\infty_{\Gamma_\sigma}(G)}_{wc}\Big]$, *and*

(b) $\big(H^\infty_G\big)^{L^\infty_\Gamma(G)}_{c} \subset \Big[\bigcup_{\sigma\in\Sigma} \big(H^\infty_{\Gamma_\sigma}(G)\big)^{L^\infty_{\Gamma_\sigma}(G)}_{c}\Big]$.

The next result follows from Corollary 8.6.8 in the same way as Corollary 8.6.5 follows from Theorem 8.6.1.

Corollary 8.6.9. *The algebra* $H^\infty_{b(\mathbb{Q})} + C\big(b(\mathbb{Q})\big)$ *contains the spaces* $\big(H^\infty_{b(\mathbb{Q})}\big)^{L^\infty_{\mathbb{Q}}(b(\mathbb{Q}))}_{wc}$ *and* $\big(H^\infty_{b(\mathbb{Q})}\big)^{L^\infty_{\mathbb{Q}}(b(\mathbb{Q}))}_{c}$.

A uniform algebra $A \subset C(X)$ is said to be *tight* [resp. *strongly tight*] if every $f \in C(X)$ is a wc-element [resp. c-element] for A, i.e. if $(A)^{C(X)}_{wc} = C(X)$ [resp. $(A)^{C(X)}_{c} = C(X)$] (cf. [CG, S]). Corollary 8.6.9 implies that the algebra $H^\infty_{b(\mathbb{Q})}$ is neither tight, nor strongly tight.

8.7 Notes

Most of the results in this section are from [GT, GT3, T3]. The idea for involving inductive sequences of disc algebras and of H^∞-spaces in exploring the structure of algebras of generalized analytic functions originates in from [T] and

[T2]. In particular, as shown in [T], the inductive limit algebra H^∞_G of the sequence $\big\{H^\infty_{(1/n)\mathbb{Z}}(b(\mathbb{Q})), (z^{n/k})^*\big\}_{n\in\mathbb{N}}$ coincides with the algebra of hyper-analytic functions on G introduced there. In this setting the set $\mathcal{D}_\Phi$ defined in (8.10) coincides with the open big disc $\mathbb{D}_G$, the algebra $H^\infty(\mathcal{D}_\Phi)$ of Φ-hyper-analytic functions on $\mathcal{D}_\Phi$, introduced in Definition 8.1.8, coincides with the algebra H^∞_G of hyper-analytic functions, and Theorem 8.1.9 becomes the $\mathbb{D}_G$-corona theorem for the algebra H^∞_G of hyper-analytic functions on G, considered and proved in [T].

The idea for Blaschke algebras appeared in [T1] in connection with the study of the big disc algebra A_G with $\widehat{G} \cong \mathbb{Q}$. G-disc algebras which are inductive limits of disc algebras were considered also in [T2].

Most of the results in Section 8.5 are from [GT]. The difference between the algebra $H^\infty_\Phi(\mathcal{D}_\Phi)$ and the space $H^\infty(G) = \big\{f^*\colon f \in H^\infty(\mathbb{D}_G)\big\}$ of boundary values of bounded analytic functions in $\mathbb{D}_G$, and also from the weak*-closure of A_{Γ_+} in $L^\infty(G,\sigma)$ is shown in [T2]. Algebras that are similar to $H^\infty_\Phi(G)$ were introduced by Curto, Muhly and Xia in [CMX] in connection with their study of Wiener-Hopf operators with almost periodic symbols. The algebra $H^\infty(G)$ is isometrically isomorphic to the algebra $H^\infty_{AP_{\Gamma_+}(\mathbb{R})}(\mathbb{R}) \subset H^\infty(\mathbb{R})$ of boundary values of almost periodic Γ-functions on $\mathbb{R}$, extendable analytically on the upper half plane. Similarly, the algebra $H^\infty_J(\mathbb{D}_G) = \varinjlim \big\{H^\infty_{(\gamma,n)}(G)\big\}_{(\gamma,n)\in J}$ from Section 8.5 is isomorphic to the algebra $H^\infty_J(\mathbb{R}) = \varinjlim \big\{H^\infty_{(\gamma,n)}(\mathbb{R})\big\}_{(\gamma,n)\in J} \subset H^\infty_{AP_{\Gamma_+}(\mathbb{R})}(\mathbb{R})$. By Theorem 8.5.1 these two algebras are different. This answers one of the questions raised by Curto, Muhly and Xia in [CMX]. Bourgain algebras of inductive limit algebras were studied in [T3] The concept of tightness of an algebra was introduced by Cole and Gamelin [CG] (see also [S]).

In principle, any G-disc algebra $A(\mathbb{D}_G)$ can be expressed as the limit of an, in general uncountable, inductive system of disc algebras $\big\{A(\mathbb{D}_r),\ r \in \widehat{G}_+ \subset \mathbb{R}_+\big\}$. Indeed, assume that $A(\mathbb{D}_r)$ is realized as an algebra of continuous functions in the closed upper half-plane $\mathbb{C}_+$, generated by the single function e^{-ir}, i.e. $A(\mathbb{D}_r) \cong [e^{-ir}]_{r\in\Gamma_+}$. For any $r < s$, set $b = (s-r)/s+r)$. The function $i^r_s(z) = \dfrac{b-z}{1-bz}$ maps $\mathbb{D}_r$ onto $\mathbb{D}_s$, and $i^s_r(e^{-ir}) = e^{-is}$. Therefore, for the G-disc algebra $A(\mathbb{D}_G)$ we have $A(\mathbb{D}_G) \cong \big[\varinjlim \big\{A(\mathbb{D}_r), i^s_r\big\}_{r\in\Gamma_+}\big]$. However, the connecting homomorphisms i^s_r are not necessarily adjoint to mappings of the unit disc $\mathbb{D}_G$ into itself.

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