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Edgar Lee Stout

Polynomial Convexity

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Preface

This book is devoted to an exposition of the theory of polynomially convex sets. A compact subset of $\mathbb{C}^N$ is polynomially convex if it is defined by a family, finite or infinite, of polynomial inequalities. These sets play an important role in the theory of functions of several complex variables, especially in questions concerning approximation. On the one hand, the present volume is a study of polynomial convexity per se, on the other, it studies the application of polynomial convexity to other parts of complex analysis, especially to approximation theory and the theory of varieties.

Not every compact subset of $\mathbb{C}^N$ is polynomially convex, but associated with an arbitrary compact set, say X, is its polynomially convex hull, $\widehat{X}$, which is the intersection of all polynomially convex sets that contain X. Of paramount importance in the study of polynomial convexity is the study of the complementary set $\widehat{X} \setminus X$. The only obvious reason for this set to be nonempty is for it to have some kind of analytic structure, and initially one wonders whether this set always has complex structure in some sense. It is not long before one is disabused of this naive hope; a natural problem then is that of giving conditions under which the complementary set does have complex structure. In a natural class of one-dimensional examples, such analytic structure is found. The study of this class of examples is one of the major directions of the work at hand.

This book is not self-contained. Certainly it is assumed that the reader has some previous exposure to the theory of functions of several complex variables. Here and there we draw on some major results from the theory of Stein manifolds. This seems reasonable in the context: Stein manifolds are the natural habitat of the complex analyst. We draw freely on the elements of real variables, functional analysis, and classical function theory. At certain points in the text, parts of algebraic topology and Morse theory are invoked. For results in algebraic topology that go beyond what one could reasonably expect to meet in an introductory course in the subject, precise references to the textbook literature are given, as are references for Morse theory. In addition, it is necessary to invoke certain results from geometric measure theory, particularly some of the seminal work of Besicovitch on the structure of one-dimensional sets, work that is quite technical. Again, precise references are given as required.

Chapter by chapter, the contents of the book can be summarized as follows. Chapter 1 is introductory and contains the initial definitions of the subject, develops some of the tools that will be used in subsequent chapters, and gives illustrative examples. Chapter 2 is concerned mainly with general properties of polynomially convex sets, for the most part properties that are independent of particular structural requirements. Chapter 3 is a systematic study of the polynomial hull of a one-dimensional set that is connected and has finite length or, more generally, that is contained in a connected set of finite length. For example, in this chapter, it is found that a rectifiable arc is polynomially convex, a result that, despite the simplicity of its formulation, is not at all simple to prove. Also in this chapter the theory of polynomially convex sets is applied to the study of one-dimensional varieties, especially to questions of analytic continuation. Chapter 4 continues the study of the polynomially convex hull of one-dimensional sets, this time admitting sets more general than those considered in Chapter 3, sets that are sometimes termed *geometrically* 1*-rectifiable*. Chapter 5 studies three distinct subjects that do, though, have some connections with one another. The first concerns certain isoperimetric properties of hulls. Next, we present some results on removable singularities. Finally, the hulls of surfaces in strictly pseudoconvex boundaries are considered. Chapter 6 is devoted to approximation questions, mostly on compact sets, but with some consideration of approximation on unbounded sets. Chapter 7 applies ideas of polynomial convexity to the study of one-dimensional subvarieties of strictly pseudoconvex domains, for example the ball. In part, the motivation for this work comes from the well-developed theory of the boundary behavior of holomorphic functions. Chapter 8 is devoted to some additional topics that either further the subject of polynomial convexity itself or are applications of this theory.

As it stands, the book is not short, but it has been necessary to omit certain topics that might naturally have been considered. For example, it is with real reluctance that I omit all discussion of the hulls of two-spheres or of the hulls of sets fibered over the unit circle. The former omission is explained by the highly technical nature of the subject, the latter by a perception that the subject has not yet achieved its definitive form.

Acknowledgments are in order. For many years the mathematics department of the University of Washington has proved to be an excellent place for my work; to it I am truly thankful. Mary Sheetz of that department has been unflaggingly good-humored as she helped with the manuscript of this book, often in the face of very frustrating difficulties. The work on this book was supported in part by the Royalty Research Fund at the University of Washington. Norman Levenberg read much of the text in manuscript and made many helpful suggestions, all of which I appreciated but not all of which I followed. I am indebted to V.M. Gichev, Mark Lawrence, and Jean-Pierre Rosay for permission to include as yet unpublished results of theirs. Other friends and colleagues have made useful comments and suggestions; to all I express my thanks.

The reader will note the great influence of the work of Herbert Alexander on our subject. Over the course of his career, Alexander made many penetrating contributions to the theory of polynomial convexity. His friends and colleagues, who looked forward to his further development of the subject, were appalled to learn of his untimely death at the age of 58, to learn that a distinguished colleague and good friend had been so prematurely taken away. The man is gone but not forgotten; his work will endure.

Edgar Lee Stout
Seattle
May Day, 2006

Contents

Index of Frequently Used Notation

$A^{\perp}$	the space of measures orthogonal to the algebra A.
bE	the boundary of the set E.
$\mathbb{B}_N$	the open unit ball in $\mathbb{C}^N$.
$\mathbb{B}_N(z, r)$	the open ball of radius r centered at the point $z \in \mathbb{C}^N$.
$\mathbb{B}_N(r)$	the open ball of radius r centered at the origin in $\mathbb{C}^N$.
$\mathscr{C}(X)$	the space of continuous $\mathbb{C}$-valued functions on the space X.
$\mathscr{D}(\mathscr{M})$	the space of compactly supported functions of class $\mathscr{C}^\infty$ on $\mathscr{M}$.
$\mathscr{D}^p(\mathscr{M})$	the space of compactly supported forms of degree p and class $\mathscr{C}^\infty$ on $\mathscr{M}$.
$\mathscr{D}^{p,q}(\mathscr{M})$	the space of compactly supported forms of bidegree (p, q) and class $\mathscr{C}^\infty$ on $\mathscr{M}$.
$\mathscr{D}_p(\mathscr{M}) = \mathscr{D}^{p\prime}(\mathscr{M})$	the space of continuous linear functionals on $\mathscr{D}^p(\mathscr{M})$.
$\mathscr{D}_{p,q}(\mathscr{M}) = \mathscr{D}^{(p,q)\prime}(\mathscr{M})$	the space of continuous linear functionals on $\mathscr{D}^{(p,q)}(\mathscr{M})$.
$\mathbb{G}_{N,k}(\mathbb{C})$	the Grassmannian of all k-dimensional complex-linear subspaces of $\mathbb{C}^N$.
$\mathbb{G}_{N,k}(\mathbb{R})$	the Grassmannian of all k-dimensional real-linear subspaces of $\mathbb{R}^N$.
$\check{H}^*$	Čech cohomology.
$H^p_{\text{deR}}(\mathscr{M})$	the pth de Rham cohomology group of the manifold $\mathscr{M}$.
$\Im z$	the imaginary part of the complex number z.
$k_{\text{BM}}(z, w)$	the Bochner–Martinelli kernel.
$\mathscr{L}$	Lebesgue measure.
$\mathscr{O}(\mathscr{M})$	the algebra of functions holomorphic on the complex manifold $\mathscr{M}$.
$\mathbb{P}^N(\mathbb{C})$	N-dimensional complex projective space.

$\mathscr{P}(X)$	the algebra of functions on the set X uniformly approximable by polynomials.
Psh $(\mathscr{M})$	the space of plurisubharmonic functions on the complex manifold $\mathscr{M}$.
$\Re z$	the real part of the complex number z.
$\mathscr{R}$-hull X	the rationally convex hull of X.
$\mathscr{R}(X)$	the algebra of functions on X uniformly approximable by rational functions without poles on X.
$\mathbb{S}^n$	the unit sphere in $\mathbb{R}^{n+1}$.
$\mathbb{T}^N$	the torus $\{(z_1, \ldots, z_N) \in \mathbb{C}^N : \lvert z_1\rvert = \cdots = \lvert z_N\rvert = 1\}$.
$\mathbb{U}^N$	the open unit polydisk in $\mathbb{C}^N$.
$U(N)$	the unitary group.
Λ^p	p-dimensional Hausdorff measure.
$\omega(z)$	the differential form $dz_1 \wedge \cdots \wedge dz_N$.
$\omega_{[k]}(z)$	the differential form $dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_N$.
$\lVert f\rVert_X$	$= \sup\{\lvert f(x)\rvert : x \in X\}$.
$\widehat{E}$	the polynomially convex hull of the compact set E.
$X \Subset Y$	X is a relatively compact subset of Y.
$\emptyset$	the empty set.

Polynomial Convexity

Chapter 1

INTRODUCTION

Introduction. The first chapter is introductory. It presents some of the basic notions of our subject, it assembles some of the tools and techniques that will be used throughout the text, and it presents some examples. Section 1.1 introduces the notion of polynomial convexity and the related notion of rational convexity. Section 1.2 is an introduction to the abstract theory of uniform algebras. Section 1.3 summarizes some parts of the theory of plurisubharmonic functions. Section 1.4 is devoted to the Cauchy–Fantappiè integral, a very general integral formula in the setting of the theory of functions of several complex variables, whether on $\mathbb{C}^N$ or on complex manifolds. Section 1.5 contains a proof of the Oka–Weil approximation theorem based on the Cauchy–Fantappiè formula. Section 1.6 presents several examples, some of which indicate the pathology of the subject, others of which are results that will find application in the sequel. Section 1.7 gives an example of a hull with no analytic structure.

1.1. Polynomial Convexity

The subject of this monograph is the theory of polynomially convex sets, which are defined as follows. We use the usual notation that $\mathbb{C}^N$ is the N-dimensional complex vector space of N-tuples of complex numbers.

Definition 1.1.1. *A compact subset X of $\mathbb{C}^N$ is* polynomially convex *if for each point $z \in \mathbb{C}^N \setminus X$ there is a polynomial P such that*

$$|P(z)| > \sup\{|P(x)| : x \in X\}.$$

Here and throughout this work *polynomials* are understood to be holomorphic polynomials. By considering the power series expansions of entire functions it is evident that in this definition we obtain precisely the same class of sets if we replace polynomials by entire functions on $\mathbb{C}^N$.

An associated notion is that of rational convexity.

Definition 1.1.2. *A compact subset X of $\mathbb{C}^N$ is* rationally convex *if for every point $z \in \mathbb{C}^N \setminus X$ there is a polynomial P such that $P(z) = 0$ and $P^{-1}(0) \cap X = \emptyset$.*

These definitions are meaningful in $\mathbb{C}^N$ for all $N \geq 1$. For $N = 1$, the sets in question are easily understood. Every compact subset of the plane is rationally convex: If $X \subset \mathbb{C}$ is compact and if $x \in \mathbb{C} \setminus X$, then the zero locus of the polynomial P given by $P(z) = z - x$ contains the point x and misses X. Thus, X is rationally convex. Polynomial convexity is only a little more complicated: *The compact subset X of $\mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus X$ is connected.* If $\mathbb{C} \setminus X$ is connected, and if $x \in \mathbb{C} \setminus X$, then Runge's theorem provides a polynomial P such that $P(x) = 1$ and $|P| < \frac{1}{2}$ on X, so X is polynomially convex. On the other hand, if $\mathbb{C} \setminus X$ is not connected, it has a bounded component, say D. The maximum principle implies that for every point $x \in D$ and for every polynomial P, $|P(x)| \leq \sup\{|P(z)| : z \in X\}$, so X is not polynomially convex.

The last argument hints at the utility of introducing certain hulls. In this connection it is helpful to use the notation $\|P\|_X = \sup\{|P(x)| : x \in X\}$.

Definition 1.1.3. *If X is a compact subset of $\mathbb{C}^N$, the* polynomially convex hull *of X is the set $\widehat{X} = \{z \in \mathbb{C}^N : |P(z)| \leq \|P\|_X$ for every polynomial $P\}$. The* rationally convex hull *of X is the set $\mathscr{R}$-hull X of points $z \in \mathbb{C}^N$ such that for every polynomial P, if $P(z) = 0$, then the zero locus $P^{-1}(0)$ meets X.*

It is clear that the polynomially convex hull of a compact set is compact. Only slightly less evident is the compactness of the rationally convex hull of a compact set. That $\mathscr{R}$-hull X is compact follows from the equality

$$\mathscr{R}\text{-hull } X = \cap_P \{z \in \mathbb{C}^N : |P(z)| \geq \inf_{x \in X} |P(x)|\},$$

in which the intersection is taken over all polynomials P. This exhibits $\mathscr{R}$-hull X as a closed set. It is bounded and therefore necessarily compact.

The rationally convex hull $\mathscr{R}$-hull X is a subset of the polynomially convex hull $\widehat{X}$.

We have seen that for a compact subset X in the plane, $\widehat{X}$ is the union of X and all the bounded components of $\mathbb{C} \setminus X$. The description of the polynomially convex and the rationally convex hulls of sets in $\mathbb{C}^N$ is much more complicated than in the planar case. As we shall see, there are characterizations of these hulls, but none of them is very simple.

If X is a compact subset of $\mathbb{C}^N$, then there are certain algebras of functions naturally associated with X. First there is $\mathscr{C}(X)$, the algebra of all continuous $\mathbb{C}$-valued functions on X. Two subalgebras of $\mathscr{C}(X)$ are central to our work.

Definition 1.1.4. *Let X be a compact subset of $\mathbb{C}^N$. The algebra $\mathscr{P}(X)$ is the uniformly closed subalgebra of $\mathscr{C}(X)$ that consists of all the functions that can be approximated uniformly on X by polynomials. The algebra $\mathscr{R}(X)$ is the uniformly closed subalgebra of $\mathscr{C}(X)$ that consists of all the functions that can be approximated uniformly on X by rational functions $r(z)$ of the form $p(z)/q(z)$ with p, q polynomials, q zero-free on X.*

Plainly $\mathscr{P}(X) \subset \mathscr{R}(X)$.

In addition to the algebras $\mathscr{P}(X)$ and $\mathscr{R}(X)$, certain other algebras can be associated naturally to a compact subset X of $\mathbb{C}^N$. The first of these is denoted by $\mathscr{O}(X)$ and consists of all the continuous functions f on X such that for some sequence $\{V_j\}_{j=1,\dots}$ of neighborhoods of X in $\mathbb{C}^N$ and for some sequence $\{f_j\}_{j=1,\dots}$ of functions $f_j \in \mathscr{O}(V_j)$, $j = 1, \dots,$ the sequence $\{f_j\}_{j=1,\dots}$ converges uniformly on X to f. (Here and throughout we use the notation that for an open subset V of $\mathbb{C}^N$ or a complex manifold, $\mathscr{O}(V)$ denotes the algebra of functions holomorphic on V.) The algebra $A(X)$ is the subalgebra of $\mathscr{C}(X)$ that consists of the functions holomorphic on the interior of X. Thus, if the interior of X is empty, then $A(X) = \mathscr{C}(X)$. If D is a bounded open set in $\mathbb{C}^N$, then $A(D)$ is often used to denote the algebra $A(\bar{D})$.

Runge's theorem implies that if X is a compact, polynomially convex set in the plane, then $\mathscr{P}(X) = \mathscr{R}(X) = \mathscr{O}(X)$. The analogous statement is correct in $\mathbb{C}^N$; this is the content of the Oka–Weil approximation theorem proved in Section 1.5 below. In general, for compact sets in the plane the algebras $\mathscr{R}(X)$ and $A(X)$ differ, though Mergelyan's theorem [311] implies that if X is polynomially convex, then the two are equal. This does not extend to $\mathbb{C}^N$.

There are simple examples of polynomially convex sets in $\mathbb{C}^N$. For example, *every compact convex set is polynomially convex.* If $K \subset \mathbb{C}^N$ is a compact, convex set, then for each point $z \in \mathbb{C}^N \setminus K$, there is a real-valued real-linear functional ℓ on $\mathbb{C}^N = \mathbb{R}^{2N}$ with $\ell < 1$ on K and with $\ell(z) = 1$. The functional ℓ is the real part of a complex-linear functional L on $\mathbb{C}^N$. The entire function $F = e^L$ satisfies $|F(z)| > \|F\|_K$. Thus the set K is polynomially convex as claimed.

Important examples among the convex sets are the closed balls and polydisks. The (open) *ball of radius r centered at $z^o \in \mathbb{C}^N$* is the set

$$\mathbb{B}_N(z^o, r) = \left\{ z \in \mathbb{C}^N : |z - z^o| = \Big(\sum_{j=1}^{N} |z_j - z_j^o|^2 \Big)^{1/2} < r \right\}.$$

Similarly, the (open) *polydisk of polyradius $r = (r_1, \dots, r_N)$ centered at z^o* is the set

$$\mathbb{U}^N(z^o, r) = \{ z \in \mathbb{C}^N : |z_1 - z_1^o| < r_1, \dots, |z_N - z_N^o| < r_N \}.$$

The use of $\mathbb{B}_N$ and $\mathbb{U}^N$ to denote, respectively, the open ball of center 0 and radius one and the open polydisk of polyradius $(1, \dots, 1)$ and center 0 will be used consistently below.

Another class of examples is this: *Every compact subset of $\mathbb{R}^N$ is a polynomially convex subset of $\mathbb{C}^N$.* To prove this assertion, let $X \subset \mathbb{R}^N$ be a compact set. The Weierstrass approximation theorem implies that if $x \in \mathbb{R}^N \setminus X$, then there is a polynomial P with $P(x) = 1 > \|P\|_X$. Consequently, $\widehat{X} \cap \mathbb{R}^N = X$. If $w = u + iv \in \mathbb{C}^N$ with $u, v \in \mathbb{R}^N$, $v \neq 0$, then the entire function F defined by $F(z) = \Pi_{j=1}^N e^{-(z_j - u_j)^2}$ satisfies $|F| \leq 1$ on $\mathbb{R}^N$ and $|F(w)| = e^{v_1^2 + \cdots + v_N^2} > 1$, so w is not in $\widehat{X}$. Thus, $X = \widehat{X}$.

Certain graphs are polynomially convex:

Theorem 1.1.5. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, and if $f \in \mathscr{P}(X)$, then the graph of f is a polynomially convex subset of $\mathbb{C}^{N+1}$.*

In particular, if f is a function continuous on the closed unit disk and holomorphic on its interior, then the graph of f is a polynomially convex subset of $\mathbb{C}^2$.

Proof. Denote by Γ the graph $\{(z, f(z)) : z \in X\}$ of f. The set Γ is compact. Let $(z_o, \zeta_o) \in \mathbb{C}^{N+1} \setminus \Gamma$. If $z_o \notin X$, then there is a polynomial P on $\mathbb{C}^N$ such that $|P(z_o)| > \|P\|_X$, so if we consider P as a function on $\mathbb{C}^{N+1}$, it shows that $(z_o, \zeta_o) \notin \widehat{\Gamma}$. If $z_o \in X$, then $\zeta_o \neq f(z_o)$. Let $c = |\zeta_o - f(z_o)|$. There is a polynomial Q such that $\|Q - f\|_X < c/4$. Let

$$\Delta = \{(z, \zeta) \in \mathbb{C}^N \times \mathbb{C} : z \in X \text{ and } |\zeta - Q(z)| \leq c/2\}.$$

This is a compact polynomially convex subset of $\mathbb{C}^N \times \mathbb{C}$, and it contains Γ, for if $(z, \zeta) \in \Gamma$, then $|\zeta - Q(z)| = |f(z) - Q(z)| < c/4$. Also, $(z_o, \zeta_o) \notin \Delta$, for

$$|\zeta_o - Q(z_o)| \geq |\zeta_o - f(z_o)| - |f(z_o) - Q(z_o)| > 3c/4.$$

That is, the compact polynomially convex subset Δ contains Γ and not the point (z_o, ζ_o), so $(z_o, \zeta_o) \notin \widehat{\Gamma}$.

The result is proved.

A polynomially convex subset X of $\mathbb{C}^N$ can be written as the intersection $\cap_P P^{-1}(\bar{\mathbb{U}})$, $\mathbb{U}$ the open unit disk in $\mathbb{C}$, where the intersection extends over all the polynomials P that are bounded by one in modulus on X. A consequence of this simple observation is that if Ω is a neighborhood of X, then there are finitely many polynomials $P_1, \ldots, P_r$ such that the set $\Delta = \cap_{j=1,\ldots,r} P_j^{-1}(\bar{\mathbb{U}})$ satisfies $X \subset \Delta \subset \Omega$. Relatively compact sets of the form Δ are called *polynomial polyhedra*. Thus, an arbitrary polynomially convex set can be approximated by polynomial polyhedra. This is not unlike the process of approximating arbitrary compact convex sets in $\mathbb{R}^N$ by compact convex polyhedra.

The following simple construction yields examples of rationally convex sets. Fix a polynomially convex subset X of $\mathbb{C}^N$, fix a family $\mathscr{F}$ of open subsets of $\mathbb{C}$, and for each $\Omega \in \mathscr{F}$, let P_Ω be a polynomial. The set

$$Z = \cap_{\Omega \in \mathscr{F}} X \setminus P_\Omega^{-1}(\Omega)$$

is a compact, rationally convex set contained in X.

The Oka–Weil approximation theorem implies that this construction can be generalized by replacing each of the polynomials P_Ω by a function f_Ω that is holomorphic on a neighborhood—which may well depend on Ω—of the set X.

There is a natural way to identify $\mathscr{P}(X)$ with $\mathscr{P}(\widehat{X})$, and $\mathscr{R}(X)$ with $\mathscr{R}(\mathscr{R}\text{-hull } X)$.

Consider first the case of $\mathscr{P}(X)$. If X is a compact subset of $\mathbb{C}^N$, there is a natural extension of each function $f \in \mathscr{P}(X)$ to a function $\widehat{f} \in \mathscr{C}(\widehat{X})$. To construct $\widehat{f}$, note that because $f \in \mathscr{P}(X)$, there is a sequence $\{P_j\}_{j=1\ldots}$ of polynomials that converges uniformly on X to f. If y is any point of $\widehat{X}$, then the sequence $\{P_j(y)\}_{j=1\ldots}$ is a Cauchy sequence and so converges. The limit of this sequence is defined to be $\widehat{f}(y)$. The value $\widehat{f}(y)$ is independent of the choice of the sequence of polynomials. This construction gives an extension of $f \in \mathscr{P}(X)$ to a function $\widehat{f}$ defined on all of $\widehat{X}$. By uniform convergence, $\widehat{f}$ is continuous and lies in $\mathscr{P}(\widehat{X})$. By way of the identification of f with $\widehat{f}$, the algebra $\mathscr{P}(X)$ can be identified naturally with the algebra $\mathscr{P}(\widehat{X})$.

A similar construction works for $\mathscr{R}(X)$. If $f \in \mathscr{R}(X)$ then there is a sequence of rational functions $r_j = p_j/q_j$ for certain polynomials p_j and q_j, q_j zero-free on X, that converges uniformly on X to f. If $y \in \mathscr{R}$-hull X, then for each j, $q_j(y) \neq 0$, and the sequence $r_j(y)$ converges. To see convergence, it suffices to note that *for each $y \in \mathscr{R}$-hull X and for each rational function $r = p/q$, q zero-free on X, we have $|r(y)| \leq \|r\|_X$*: If for some choice of polynomials p and q, q zero-free on X, $p(y)/q(y) = 1 > \|p/q\|_X$, then the polynomial $p - q$ vanishes at y but not at any point of X, which contradicts having $y \in \mathscr{R}$-hull X. Thus again, each function $f \in \mathscr{R}(X)$ extends in a natural way to a function $\widehat{f}$ on $\mathscr{R}$-hull X, and the extended function lies in $\mathscr{R}(\mathscr{R}$-hull $X)$. In this way $\mathscr{R}(X)$ is naturally identified with $\mathscr{R}(\mathscr{R}$-hull $X)$.

Certain formal properties of polynomially convex sets are evident. For example, *the intersection of an arbitrary family of polynomially convex sets (or rationally convex sets) is polynomially convex (or rationally convex).*

A union of polynomially convex sets is generally not polynomially convex. There is a curious problem here. It is a result of Kallin [195], which is proved below in Theorem 1.6.20, that *a disjoint union of three closed balls is polynomially convex,* but she also shows that the disjoint union of three disjoint closed polydisks need not be polynomially convex. In the same negative direction, Khudaiberganov and Kytmanov [212] have given an example of three disjoint closed ellipsoids the union of which is not polynomially convex. Mueller [253, 254] also exhibited a nonpolynomially convex disjoint union of three ellipsoids. Whether a disjoint union of four closed balls in $\mathbb{C}^N$ is polynomially convex remains an open question. Rosay [295] has constructed three disjoint convex sets in $\mathbb{C}^2$ whose union is not polynomially convex.

One can consider polynomial convexity in certain natural infinite-dimensional settings. A formulation is as follows. Denote by Λ an index set of arbitrary cardinality. The product space $\mathbb{C}^\Lambda$ has the natural product topology. On this space one can speak of polynomials as the functions that are represented as polynomials in (finitely many) of the coordinate projections. With these polynomials, one can introduce formally the notion of polynomially convex compact subsets of $\mathbb{C}^\Lambda$. An instance of the use of this notion is the paper of Stolzenberg [342].

1.2. Uniform Algebras

The present section is devoted to the theory of uniform algebras, which provides some tools useful in the study of polynomial convexity. For an extensive treatment of the general theory of uniform algebras see [136] or [345].

Although this section is devoted to the theory of uniform algebras, we shall require one fact from the general theory of commutative Banach algebras, the Gel'fand–Mazur theorem. We shall take the shortest possible path to this theorem, for our interests are not in the general theory.

Definition 1.2.1. *A* Banach algebra *is a $\mathbb{C}$-algebra, $\mathscr{A}$, with identity that at the same time is a complex Banach space with respect to a norm $\|\cdot\|$ that satisfies $\|xy\| \leq \|x\|\,\|y\|$ for all $x, y \in \mathscr{A}$ and that satisfies $\|1\| = 1$, where on the left of this equation,* 1 *denotes the* *multiplicative identity of $\mathscr{A}$, while on the right it denotes the multiplicative identity of the field $\mathbb{R}$.*

The multiplicative inequality for the norm implies that multiplication in $\mathscr{A}$ is jointly continuous, i.e., that the map $(x, y) \mapsto xy$ from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$ is continuous.

If λ is a complex number, we shall often identify it with the element $\lambda \cdot 1$ of $\mathscr{A}$, so that we regard $\mathbb{C}$ as a subalgebra of $\mathscr{A}$.

Fix now a Banach algebra $\mathscr{A}$ with identity.

Definition 1.2.2. $\mathscr{A}^{-1}$ *denotes the multiplicative group of invertible elements of the algebra $\mathscr{A}$.*

Lemma 1.2.3. *The group $\mathscr{A}^{-1}$ is open in $\mathscr{A}$.*

Proof. First, $\mathscr{A}^{-1}$ contains the open ball of radius 1 centered at 1: If $\|a-1\| < 1$, then the series $\sum_{n=0}^{\infty}(-1)^n(a-1)^n$ converges in $\mathscr{A}$ by comparison with the geometric series. If we write $a = 1+(a-1)$, we recognize that the sum of the series is a^{-1}, whence $a \in \mathscr{A}^{-1}$. If a is an arbitrary element of $\mathscr{A}^{-1}$, then for each $x \in \mathscr{A}$, $x = a[(a^{-1}x-1)+1]$. The element a is invertible, and, provided x is near a, the element $a^{-1}x-1$ has norm less than one, so $(a^{-1}x-1)+1$ is invertible. Thus, when x is near a, x is also invertible: $\mathscr{A}^{-1}$ is open.

The map $a \mapsto a^{-1}$ is continuous on $\mathscr{A}^{-1}$ as follows from the formula just exhibited for a^{-1}. That formula shows that on the open ball $B = \{a \in \mathscr{A} : \|a-1\| < 1\}$, the map $a \mapsto a^{-1}$ is the composition of the continuous map $x \mapsto \sum_{n=0}^{\infty}(-1)^n x^n$ with the continuous map $a \mapsto a-1$. Thus $a \mapsto a^{-1}$ is continuous on the ball B. This implies that it is continuous everywhere.

Definition 1.2.4. *The* spectrum *of the element a of $\mathscr{A}$ is the set*

$$\sigma_{\mathscr{A}}(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } \mathscr{A}\}.$$

The openness of the set $\mathscr{A}^{-1}$ implies that the spectrum of a is a closed subset of $\mathbb{C}$. It is also bounded: If $|\lambda| > \|a\|$, then $a-\lambda = \lambda(\lambda^{-1}a-1)$. The element $a-\lambda$ is invertible, because $\|\lambda^{-1}a-1\| < 1$. Thus $\sigma_{\mathscr{A}}(a) \subset \{\zeta \in \mathbb{C} : |\zeta| \le \|a\|\}$.

It is more complicated to show that $\sigma_{\mathscr{A}}(a)$ is not empty.

Lemma 1.2.5. *The spectrum of an element $a \in \mathscr{A}$ is a nonempty compact subset of $\mathbb{C}$.*

Proof. Argue by contradiction: Let $a \in \mathscr{A}$, and suppose $\sigma_{\mathscr{A}}(a)$ to be empty, so that for all $\lambda \in \mathbb{C}$, $\lambda - a$ is invertible.

Fix a continuous linear functional ϕ on the Banach space $\mathscr{A}$. The function Φ defined by $\Phi(\lambda) = \phi((\lambda-a)^{-1})$ is a complex-valued function defined on all of $\mathbb{C}$. It is a holomorphic function, as can be seen by computing the derivative directly:

$$\begin{aligned}(\Phi(\lambda+h)-\Phi(\lambda))/h &= \phi((\lambda+h-a)^{-1}-(\lambda-a)^{-1})/h \\ &= -\phi((\lambda+h-a)^{-1}(\lambda-a)^{-1}).\end{aligned}$$

As $h \to 0$, the last quantity tends to $-\phi((\lambda-a)^{-2})$, and Φ is found to be differentiable at λ.

Thus, Φ is an entire function on $\mathbb{C}$ that satisfies $\lim_{\lambda\to\infty}\Phi(\lambda) = 0$. It follows from the maximum principle that Φ vanishes identically. In particular, $\Phi(0) = -\phi(a^{-1})$ vanishes. This is true for all choices of ϕ, so, by the Hahn–Banach theorem, a^{-1} is the zero element of $\mathscr{A}$. This is impossible. We have reached a contradiction, so $\sigma_{\mathscr{A}}(a)$ is necessarily nonempty.

We can now prove the Gel'fand–Mazur theorem:

Theorem 1.2.6. *A commutative Banach algebra that is a division ring is (isomorphic to) the field of complex numbers.*

Recall that a *division ring* is a ring containing more than one element in which the set of nonzero elements is a multiplicative group.

Proof. If the commutative Banach algebra with identity $\mathscr{A}$ is a division ring, then for each $a \in \mathscr{A}$, the spectrum $\sigma_{\mathscr{A}}(a)$ is necessarily a singleton: If $\lambda \in \sigma_{\mathscr{A}}(a)$, then $\lambda - a$ is not invertible. By hypothesis $\mathscr{A}$ is a division ring, so this implies that $a = \lambda$. The map from $\mathscr{A}$ to $\mathbb{C}$ that takes $a \in \mathscr{A}$ to the unique element of $\sigma_{\mathscr{A}}(a)$ is an isomorphism of $\mathbb{C}$-algebras, so the result is proved.

A special class of Banach algebras is the class of uniform algebras.

Definition 1.2.7. *A* uniform algebra *on the compact Hausdorff space X is a uniformly closed, point-separating subalgebra with identity of the algebra $\mathscr{C}(X)$.*

The condition of point separation is understood in the sense that given distinct points $x, x' \in X$, there is a function, f, in the algebra such that $f(x) \neq f(x')$.

The algebra $\mathscr{C}(X)$ itself is a uniform algebra on the space X.

Other examples of uniform algebras that will play important roles in this work are the algebras $A(D)$ introduced in the preceding section. In particular, there are the *ball algebra* $A(\mathbb{B}_N)$ and the *polydisk algebra* $A(\mathbb{U}^N)$. The special case of the ball algebra (or the polydisk algebra) when $N = 1$ is the *disk algebra* $A(\mathbb{U})$. More generally, for a compact set $X \subset \mathbb{C}^N$, the algebras $\mathscr{P}(X)$, $\mathscr{R}(X)$, $\mathscr{O}(X)$, and $A(X)$ are uniform algebras on X.

If A is a uniform algebra on a compact space X with the uniform norm given by

$$\|f\|_X = \sup\{|f(x)| : x \in X\},$$

then $\|\cdot\|_X$ *is* a norm on A. It satisfies $\|fg\|_X \leq \|f\|_X \|g\|_X$ for all $f, g \in A$, so that with this norm, A is a normed algebra. The algebra A is supposed to be closed, so it is complete in the norm $\|\cdot\|_X$. Thus, equipped with the norm $\|\cdot\|_X$, A is a commutative Banach algebra with identity.

Given a commutative Banach algebra A, a fundamental problem is to determine the *characters* of A, i.e., to determine the nonzero $\mathbb{C}$-linear functionals φ on A that satisfy $\varphi(fg) = \varphi(f)\varphi(g)$. In general, this is a very difficult problem, but in the case of the full algebra $\mathscr{C}(X)$, the answer is easy to obtain.

Theorem 1.2.8. *If X is a compact Hausdorff space and φ is a character on the algebra $\mathscr{C}(X)$, then there is a unique point $x \in X$ for which $\varphi(f) = f(x)$ for each $f \in \mathscr{C}(X)$.*

Proof. Notice that a character φ on any commutative Banach algebra with identity necessarily satisfies $\varphi(1) = 1$.

Assume there to be no $x \in X$ such that $\varphi(f) = f(x)$ for each $f \in \mathscr{C}(X)$. The compactness of X implies the existence of a finite set of functions $g_1, \ldots, g_r \in \mathscr{C}(X)$ such that for each j, $\varphi(g_j) = 0$ and such that for each $x \in X$, $g_j(x) \neq 0$ for some choice of $j = 1, \ldots, r$. If $h_j = g_j / \sum_{j=1}^r g_j \bar{g}_j$, then $1 = \sum_{j=1}^r g_j h_j$. This leads to the contradiction that $1 = \varphi(1) = \varphi(\sum_{j=1}^r g_j h_j) = \sum_{j=1}^r \varphi(g_j)\varphi(h_j) = 0$. Uniqueness is evident, so the result is proved.

The result just proved implies that each nonzero multiplicative linear functional φ on $\mathscr{C}(X)$ satisfies $|\varphi(f)| \leq \|f\|_X$ and so is of norm no more than one. In particular, it is continuous. The norm actually is one, for $\varphi(1) = 1$.

It is a simple matter to see that, more generally, *each character on an arbitrary uniform algebra is continuous and is of norm one.* For this purpose, note that if $f \in A$ satisfies $\|f\|_X < 1$ and if φ is a character of A, then $|\varphi(f)| < 1$. Otherwise, $\varphi(f) = c$ with $|c| \geq 1$. For some $\epsilon > 0$ and for all $x \in X$, we have $|f(x)| < 1 - \epsilon$, for $\|f\|_X < 1$. The series $\frac{1}{c} \sum_{j=0}^{\infty} \left(\frac{f}{c}\right)^j$ therefore converges uniformly on X to an element $g \in A$ that satisfies $1 = (c - f)g$. This yields a contradiction, for it implies that $1 = \varphi(1) = (c - \varphi(f))\varphi(g) = 0$. Thus, each character of a uniform algebra is continuous and of norm one.

Using the boundedness property just established, we obtain the following basic result. The notation $\widehat{f}$ used in it is that established in the last section, so that $\widehat{f}$ is the natural extension of $f \in \mathscr{P}(X)$ to an element of $\mathscr{C}(\widehat{X})$.

Theorem 1.2.9. *If X is a compact subset of $\mathbb{C}^N$, then every character of $\mathscr{P}(X)$ is of the form $f \mapsto \widehat{f}(z)$ for a unique $z \in \widehat{X}$.*

Proof. In light of the natural identification of $\mathscr{P}(X)$ with $\mathscr{P}(\widehat{X})$, it suffices to suppose that X is polynomially convex.

The functions in $\mathscr{P}(X)$ separate points on X, so the point z is uniquely determined, if it exists.

To prove the existence of the desired z, fix a character φ of $\mathscr{P}(X)$. Let $z \in \mathbb{C}^N$ be the point with coordinates $(\varphi(Z_1), \ldots, \varphi(Z_N))$, where Z_j denotes the jth coordinate function on $\mathbb{C}^N$. The point z is in X: Let P be a polynomial. That φ is linear and multiplicative implies that $\varphi(P)$ must be $P(\varphi(Z_1), \ldots, \varphi(Z_N)) = P(z)$. We have $|\varphi(P)| \leq \|P\|_X$, so the point z must lie in X; by assumption X is polynomially convex. The polynomials are dense in $\mathscr{P}(X)$ by definition, so for every $f \in \mathscr{P}(X)$, the value $\varphi(f)$ must be $f(z)$. The proof is complete.

A similarly direct proof shows that *if X is a compact, rationally compact subset of $\mathbb{C}^N$, then each character of the algebra $\mathscr{R}(X)$ is of the form $f \mapsto f(z)$ for a unique point z in X.*

A simple application of the preceding fact is the following.

Theorem 1.2.10. *A compact subset X of $\mathbb{C}^N$ is polynomially convex if $\mathscr{P}(X) = \mathscr{C}(X)$. It is rationally convex if $\mathscr{R}(X) = \mathscr{C}(X)$.*

Proof. Granted that $\mathscr{P}(X) = \mathscr{C}(X)$, it follows that each character of the algebra $\mathscr{P}(X)$ is evaluation at a unique point of X, for these evaluations are the characters of $\mathscr{C}(X)$. However, it was shown above that the characters of $\mathscr{P}(X)$ are precisely the evaluations at points of the polynomially convex hull $\widehat{X}$. Thus, $X = \widehat{X}$; X is polynomially convex as claimed.

The case of $\mathscr{R}(X)$ follows the same lines, for the characters of $\mathscr{R}(X)$ are point evaluations at points of $\mathscr{R}$-hull X.

It is important to recognize that *an element of a uniform algebra A on a compact space is invertible if and only if no character of A annihilates it*. To this end, let $g \in A$, and suppose that no character, χ, of A satisfies $\chi(g) = 0$. Then g is invertible in A. Suppose not. Denote by (g) the principal ideal in A generated by g. It is contained in a maximal ideal, say $\mathfrak{m}$, of A. The ideal $\mathfrak{m}$ is closed: If not, its closure is an ideal, and it is proper, because it is disjoint from the open unit ball in A centered at the identity. Thus $\mathfrak{m}$ is necessarily closed. The field $A/\mathfrak{m}$ has the quotient norm $\|\cdot\|_q$ given by

$$\|h + \mathfrak{m}\|_q = \inf\{\|h + g\|_X : f \in \mathfrak{m}\}.$$

According to the Gel'fand–Mazur theorem, this quotient field is isomorphic to $\mathbb{C}$. Consequently, the quotient map $A \to A/\mathfrak{m}$ is a character of A that annihilates g.

It is occasionally useful to know that a rationally convex set in $\mathbb{C}^N$ can be realized in $\mathbb{C}^{N+1}$ as a polynomially convex set. This is an observation of Rossi [302], which depends on the result obtained in the last paragraph.

Theorem 1.2.11. *If X is a compact rationally convex subset of $\mathbb{C}^N$, then there is a function ψ defined and of class $\mathscr{C}^\infty$ on all of $\mathbb{C}^N$ such that $\psi|X \in \mathscr{R}(X)$ and such that the graph*

$$\Gamma_\psi = \{z \in \mathbb{C}^{N+1} : z_{N+1} = \psi(z_1, \dots, z_N),\ (z_1, \dots, z_N) \in X\}$$

is a polynomially convex subset of $\mathbb{C}^{N+1}$.

Remark. It was Basener [44] who noted that ψ can be chosen to be of class $\mathscr{C}^\infty$.

As shown by the example of a compact set X in $\mathbb{C}$ for which $\mathbb{C} \setminus X$ has infinitely many components, the function ψ cannot generally be chosen to be a rational function.

Proof. The algebra $\mathscr{R}(X)$ is a subset of the algebra $\mathscr{C}(X)$ and so is separable. Let $f_1, \dots$ be a countable dense subset of $\mathscr{R}(X)$. Without loss of generality, each f_n is rational, say $f_n = p_n/q_n$ with p_n and q_n polynomials, q_n zero-free on X. For $n = 1, \dots$ define ε_n by taking $\varepsilon_1 = 1$ and requiring that ε_n decrease rapidly to zero. In particular, we suppose that

$$\varepsilon_n \max\{\|q_n^{-1}\|_X,\ \|q_1 q_n^{-1}\|_X, \dots, \|q_{n-1} q_n^{-1}\|_X, 1\} \ < \ 2^{-(n+1)}\varepsilon_{n-1}.$$

For each n, let φ_n be a function defined and of class $\mathscr{C}^\infty$ on $\mathbb{C}^N$ that agrees on a neighborhood, which depends on n, of X with q_n^{-1}. Define the function ψ by $\psi = \sum_{j=1,\dots} \varepsilon_n \varphi_n$. If ε_n decreases to zero fast enough, ψ will be defined and of class $\mathscr{C}^\infty$ on $\mathbb{C}^N$. The restriction $\psi|X$ does lie in $\mathscr{R}(X)$. We shall show that the closed subalgebra A of $\mathscr{R}(X)$ generated by the polynomials and the function ψ is all of $\mathscr{R}(X)$.

The polynomials p_n all lie in A, so it suffices to show that each of the functions q_n^{-1} lies in A. If q_1^{-1} is not in A, then there is a character χ of A that annihilates q_1. This leads to the equation

$$\chi(q_1\psi - \varepsilon_1) = \chi\left(\sum_{j=2,\dots} \varepsilon_n q_1 q_n^{-1}\right),$$

which implies that

$$-\varepsilon_1 = \chi\Big(\sum_{j=2,\dots} \varepsilon_n q_1 q_n^{-1}\Big).$$

Because χ is of norm one, the choice of the quantities ϵ_n implies that this equality is impossible. Thus, $q_1^{-1} \in A$. Inductively, $q_2^{-1}, q_3^{-1}, \dots$ all lie in A. It follows that, as claimed, $A = \mathscr{R}(X)$.

That Γ_ψ is polynomially convex is seen in the following way. The algebra $A = \mathscr{R}(X)$ is generated by the polynomials and the function ψ, and so is isomorphic to the algebra $\mathscr{P}(\Gamma_\psi)$. However, X is rationally convex, so each character of $\mathscr{R}(X)$ is evaluation at a point of X, whence each character of $\mathscr{P}(\Gamma_\psi)$ is evaluation at some point of Γ_ψ. It follows that Γ_ψ is polynomially convex, as desired.

As a consequence of the boundedness of characters, it follows that characters of uniform algebras admit representing measures.

Definition 1.2.12. *If A is a uniform algebra on the compact Hausdorff space X, and if φ is a character of A, then a* representing measure *for φ is a finite regular Borel measure μ on X of total mass one such that for each $f \in A$, $\varphi(f) = \int_X f(x)\, d\mu(x)$.*

The Hahn–Banach and Riesz representation theorems imply the existence of representing measures: If φ is a character for the algebra A, then because it is of norm one, the Hahn–Banach theorem implies that it extends to a continuous linear functional $\tilde{\varphi}$ on $\mathscr{C}(X)$, $\tilde{\varphi}$ also of norm one. The Riesz representation theorem provides a finite regular Borel measure μ on X, μ of total mass one, such that $\int_X f(x)\, d\mu(x) = \tilde{\varphi}(f)$ for every $f \in \mathscr{C}(X)$. We have $1 \in A$, $\varphi(1) = 1$, and $\|\mu\| = 1$, so μ is necessarily a positive measure.

From time to time in the sequel it will be useful to consider possibly complex-valued measures ν that satisfy $\varphi(f) = \int_X f(x)\, d\nu(x)$ for all $f \in A$. Such a measure will be referred to as a *complex representing measure*.

An argument more complicated than the one just given yields the existence of representing measures with an additional property that is often important.

Definition 1.2.13. *If A is a uniform algebra on the compact Hausdorff space X, and if φ is a character of A, then a* Jensen measure *for φ is a positive finite regular Borel measure μ on X such that for each $f \in A$,* $\log|\varphi(f)| \le \int_X \log|f(x)|\, d\mu(x)$.

The standard example is the measure $\frac{1}{2\pi} d\vartheta$ on the unit circle, which is a Jensen measure for the character on the disk algebra $A(\mathbb{U})$ that evaluates functions at the origin: If $f \in A(\mathbb{U})$, then by the classical inequality of Jensen, $\log|f(0)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\vartheta})|\, d\vartheta$. More generally, if D is a bounded domain[1] in the plane, and if $w \in D$, let μ_w be the harmonic measure (supported by bD) for the point w. Thus, for each function u continuous on $\bar{D}$, harmonic on D, $u(w) = \int u(z)\, d\mu_w(z)$. If $f \in A(D)$, then the subharmonicity of $\log|f|$ on D implies that $\log|f(w)| \le \int \log|f(z)|\, d\mu_w(z)$, so μ_w is a Jensen measure for the point w (with respect to the algebra $A(D)$).

It is easily seen that *a Jensen measure is a representing measure.* Let μ be a Jensen measure for the character φ of A. If $f \in A$ is invertible, so that $1/f \in A$, then $\log|\varphi(f)| \le$

[1] In this work the word *domain* will be reserved for connected open sets.

$\int_X \log|f(x)|\,d\mu(x)$, and $\log|\varphi(1/f)| \le \int_X \log|1/f(x)|\,d\mu(x)$. It follows that for each invertible element of A, $\log|\varphi(f)| = \int_X \log|f(x)|\,d\mu(x)$. Then for $f = u + iv \in A$, it follows that $\Re\varphi(f) = \log|\varphi(e^f)| = \int_X u(x)\,d\mu(x)$. Consequently, $\varphi(f) = \int_X f\,d\mu(x)$, i.e., μ is a representing measure.

That Jensen measures exist was established by Bishop [59].

Theorem 1.2.14. *If A is a uniform algebra on the compact Hausdorff space X, then for each character φ of A there exists a Jensen measure.*

Bishop proved a somewhat stronger statement:

Theorem 1.2.15. *If A is a uniform algebra on the compact Hausdorff space X, if φ is a character of A, if $X_1, \dots, X_n$ are disjoint compact subsets of X, and if $\alpha_1, \dots, \alpha_n$ are positive numbers with $\sum_{j=1}^n \alpha_j = 1$ such that for each $f \in A$, $|\varphi(f)| \le \|f\|_{X_1}^{\alpha_1} \cdots \|f\|_{X_n}^{\alpha_n}$, then there is a Jensen measure μ for φ with the additional property that $\mu(X_j) \ge \alpha_j$ for each $j = 1, \dots, n$.*

Notice that from $\|\mu\| = 1$, $\mu(X_j) \ge \alpha_j$, and $\alpha_1 + \cdots + \alpha_n = 1$ follows $\mu(X_j) = \alpha_j$ for each j.

The proof of these two theorems depends on a geometric form of the Hahn–Banach theorem: *If A and B are disjoint nonempty convex sets in a Banach space $\mathfrak{X}$, and if A is open, there exists a real-valued continuous $\mathbb{R}$-linear functional λ on $\mathfrak{X}$ such that for some $c \in \mathbb{R}$, $\lambda(x) < c \le \lambda(y)$ for every $x \in A$ and every $y \in B$.* This is a standard result, which can be found, e.g., in [309].

Proof. We first prove the existence of Jensen measures. Fix a character φ on the uniform algebra A. Let C_1 be the cone of negative continuous functions on X; this is an open convex set in the space $\mathscr{C}_{\mathbb{R}}(X)$ of real-valued continuous functions on X. Let C_2 be the set $C_2 = \{h \in \mathbb{C}_{\mathbb{R}}(X) :$ for some $f \in A$ with $\varphi(f) = 1$ and for some $r > 0$, $rh(x) \ge \log|f(x)|$ for all $x \in X\}$. The set C_2 is closed under multiplication by positive numbers. It is also closed under addition. Given $h, h' \in C_2$ with associated functions $f, f' \in A$ and associated real numbers r, r' as in the definition of C_2, there is no loss in assuming that both r and r' are rational, say $r = p/q$ and $r' = p'/q'$. Then $\max\{p, p'\}(h+h') \ge \log|f^p f'^{p'}|$, so $h + h' \in C_2$. Moreover, the cones C_1 and C_2 are disjoint: If $h \in C_1 \cap C_2$, then for some positive ϵ, $h < -\epsilon$ on X, and $rh \ge \log|f|$ for some $f \in A$ with $\varphi(f) = 1$. Then $1 = \varphi(f)$, but $|f| \le e^{rh} < e^{-r\epsilon} < 1$. This contradicts the bound on characters: $|\varphi(f)| \le \|f\|_X$. Thus, the cones are disjoint.

The separation theorem provides a continuous linear functional λ on $\mathscr{C}_{\mathbb{R}}(X)$ that separates the convex sets C_1 and C_2, so there is a finite regular Borel measure μ of total mass one such that

$$\int_X h(x)\,d\mu(x) < 0 \quad (h \in C_1) \quad \text{and} \quad \int_X h(x)\,d\mu(x) \ge 0 \quad (h \in C_2).$$

The former inequality implies that μ is a positive measure. From the latter inequality we deduce that μ is a Jensen measure for φ. To do this, let $f \in A$. If $\varphi(f) = 0$, the desired inequality holds. Suppose, therefore, that $\varphi(f) = 1$. (Replace f by $f/\varphi(f)$ if necessary.)

Let $h \in \mathscr{C}_{\mathbb{R}}(X)$ satisfy $h \ge \log|f|$. Then $0 = \log|\varphi(f)| \le \int_X h(x)\,d\mu(x)$. Because

$$\int_X \log|f(x)|\,d\mu(x) = \inf_{h>\log|f|} \int_X h(x)\,d\mu(x),$$

it follows that, as desired,

$$\log|\varphi(f)| \le \int_X \log|f(x)|\,d\mu(x),$$

i.e., that μ is a Jensen measure. Thus, Jensen measures always exist.

The more refined result also follows from the separation theorem. Given finitely many closed subsets $X_1, \dots, X_n$ and positive numbers $\alpha_1, \dots, \alpha_n$ as in the statement of the theorem, so that for each $f \in A$, $|\varphi(f)| \le \|f\|_{X_1}^{\alpha_1} \cdots \|f\|_{X_n}^{\alpha_n}$, introduce a third cone C_3 by

$$C_3 = \left\{ h \in \mathscr{C}_{\mathbb{R}}(X) : \sum_{j=1}^{n} \max_{X_j} \alpha_j h < 0 \right\}.$$

The cone C_3 contains the cone C_1, for all the α's are positive.

The cones C_2 and C_3 are disjoint: If $h \in C_2 \cap C_3$, there is $f \in A$ with $\varphi(f) = 1$ and with $rh \ge \log|f|$ on X for some positive r. This leads to a contradiction, though: From $|\varphi(f)| \le \|f\|_{X_1}^{\alpha_1} \cdots \|f\|_{X_n}^{\alpha_n}$, it follows that

$$0 = \log|\varphi(f)| \le \sum \alpha_j \|f\|_{X_j},$$

which contradicts $h \in C_3$. Thus $C_2 \cap C_3$ is empty.

As before, it follows from the separation theorem and the Riesz representation theorem that there is a positive measure μ of total mass one such that $\int_X h(x)\,d\mu(x) \ge 0$ for $h \in C_2$ and $\int_X h(x)\,d\mu(x) < 0$ for $h \in C_3$. The measure μ is a Jensen measure.

If $\mu(X_j) < \alpha_j$, we reach a contradiction as follows. Denote by g the function that is $\alpha_j - 1$ on the set X_j and α_j on $X \setminus X_j$. For any continuous function h with $h < g$, we have $\sum_{j=1}^n \max_{X_j} \alpha_j h < 0$, whence $\int_X h(x)\,d\mu(x) < 0$. This is true for each choice of h, so $\int_X \; g(x)\,d\mu(s) \le 0$. But then

$$0 \ge \int_X g(x)\,d\mu(x) = \alpha_j - \mu(X_j) > 0.$$

Thus, as claimed, $\mu(X_j) \ge \alpha_j$. The result is proved.

An entirely different proof of the existence of Jensen measures has been given by Duval and Sibony [107].

As a first, very simple, application of Jensen measures in the theory of polynomial convexity we have the following result.

Theorem 1.2.16. *If $X \subset \mathbb{C}^N$ is compact and if $\mathscr{P}(X)$ contains a real-valued function, f, then X is polynomially convex if and only if each fiber $f^{-1}(t)$, $t \in \mathbb{R}$, is polynomially convex. If X is polynomially convex, then $\mathscr{P}(X) = \mathscr{C}(X)$ if and only if for each t, $\mathscr{P}(f^{-1}(t)) = \mathscr{C}(f^{-1}(t))$.*

Proof. Let $z_o \in \widehat{X}$, and let μ be a Jensen measure carried by X for the point z_o so that for all polynomials P, $\log|P(z_o)| \leq \int_X \log|P|\,d\mu$. The same inequality persists when P is replaced by any $g \in \mathscr{P}(X)$, provided we understand by $g(z_o)$ the value at z_o of the natural extension of g to $\widehat{X}$. We will show that the measure μ is concentrated on a fiber $f^{-1}(t)$ for some $t \in \mathbb{R}$. Suppose not. There is then a decomposition

$$f(X) = (f(X) \cap (-\infty, r_o]) \cup (f(X) \cap (r_o, \infty)) = E' \cup E''$$

such that the sets $f^{-1}(E')$ and $f^{-1}(E'')$ both have positive μ measure.

Suppose $f(z_o) \in E'$. (If not, replace f by $2r_o - f$.) There is a sequence $\{p_n\}_{n=1,\dots}$ of polynomials in one variable such that for all n, $|p_n| \leq 1$ on $f(X)$ and such that $p_n(t) \to 1$ for $t \in E'$ and $p_n(t) \to 0$ for $t \in E''$. If $g_n = p_n \circ f$, then $g_n \in \mathscr{P}(X)$, and we have

$$0 = \lim_{n\to\infty} \log|g_n(z_o)| \leq \lim_{n\to\infty} \int_X \log|g_n|\,d\mu = -\infty,$$

a contradiction.

Thus the Jensen measure μ is concentrated on a fiber $f^{-1}(t)$ for some $t \in \mathbb{R}$. Under the assumption that the fibers of f^{-1} are all polynomially convex, it follows that the point x_o lies in X as we wished. Thus, polynomial convexity of the fibers of f^{-1} implies the polynomial convexity of X.

The converse is shorter: If X is polynomially convex, then for every $f \in \mathscr{P}(X)$ and each $t \in \mathbb{R}$, the fiber $f^{-1}(t)$ is polynomially convex, for it is the intersection of the polynomially convex sets $\{x \in X : |f(x) - t| \leq \varepsilon\}$ for $\varepsilon > 0$.

To prove the final statement of the theorem suppose that for each t, $\mathscr{P}(f^{-1}(t)) = \mathscr{C}(f^{-1}(t))$. If $\mathscr{P}(X) \neq \mathscr{C}(X)$, then there are nonzero finite regular Borel measures μ supported by X such that $\int g\,d\mu = 0$ for all $g \in \mathscr{P}(X)$. Let E be the set of all such measures that have norm not more than one. The set E is a convex set that is compact when endowed with the relative weak* topology obtained by regarding the space of measures on X as the dual of the space of continuous functions on X. By the Krein–Milman theorem [309], E is the closed convex hull of its extreme points. In particular, it has extreme points. Let μ be one. The measure μ is concentrated on one of the fibers $f^{-1}(t)$. If it is not, decompose $\mathbb{R}$ into two mutually disjoint measurable sets S and T such that μ has positive mass on each of the sets $f^{-1}(S)$ and $f^{-1}(T)$. There is a sequence $\{p_j(t)\}_{j=1,\dots}$ of polynomials each bounded by one on the compact set $f(X)$ such that $f \circ p_j$ converges pointwise to the characteristic function of $f^{-1}(S)$. For each $g \in \mathscr{P}(X)$,

$$0 = \int g(x) p_j(f(x))\,d\mu(x).$$

The integral tends, as $j \to \infty$, to $\int_{f^{-1}(S)} g(x)\,d\mu(x)$. Thus, the measure $\mu|f^{-1}(S)$ is orthogonal[2] to $\mathscr{P}(X)$. It follows that the measure $\mu|f^{-1}(T)$ is also orthogonal to $\mathscr{P}(X)$. Both of these measures lie in E and, by hypothesis, neither is the zero measure, so we have a contradiction to the assumption that μ is an extreme point of E.

It follows that the measure μ is concentrated on a fiber $f^{-1}(t)$. Every continuous function on this fiber is approximable uniformly by polynomials, so μ is necessarily the zero measure. This also is a contradiction, and the theorem is proved.

For an arc λ in the complex plane, $\mathscr{P}(\lambda) = \mathscr{C}(\lambda)$ by Mergelyan's theorem. Consequently, in the preceding theorem the real-valued function f can be replaced by a function in $\mathscr{P}(X)$ whose values are restricted to lie in an arc in $\mathbb{C}$. Indeed, it suffices for the function f to satisfy $\mathscr{P}(f(X)) = \mathscr{C}(f(X))$, a condition satisfied by every compact set without interior that does not separate the plane.

The book [139] discusses Jensen measures in detail. In the paper [75] a very general approximation theorem for Jensen measures is proved. In a sense that can be made precise, Jensen measures can be approximated by holomorphic images of Lebesgue measure on the unit circle.

An important notion in the general theory of uniform algebras is that of boundary.

Definition 1.2.17. *If A is a uniform algebra on the compact space X, then the subset E of X is a* boundary *for A if $\|f\|_X = \sup\{|f(x)| : x \in E\}$ for every $f \in A$.*

It is plain that the space X is itself a boundary for A; often there are smaller boundaries. Notice that if E is a closed boundary for A, and if φ is a character for A, then there are representing measures and Jensen measures for φ that are concentrated on E.

Remark 1.2.18. Alexander [13] observed that the proof of Theorem 1.2.14 yields a more general statement: *If A is a uniform algebra on the compact Hausdorff space X and $\Gamma \subset X$ is a closed boundary for A, and if σ is a positive measure of total mass one on X, then there exists a positive measure μ of total mass one on Γ such that*

$$\int \log|f(x)|\, d\sigma(x) \le \int \log|f(x)|\, d\mu(x)$$

for all $f \in A$. We shall call such a measure μ a *Jensen measure for the measure σ.*

Rather than appeal to the proof of Theorem 1.2.14, the existence of μ can be deduced from the existence of Jensen measures as follows. The positive measure σ is assumed to have norm not more than one, so it lies in the weak* closed convex hull of the unit point masses concentrated at points of X: There is a net $\{\sigma_\alpha\}_{\alpha\in A}$ of measures of the form $\sigma_\alpha = \sum_{j=1}^{n(\alpha)} \lambda_{\alpha,j}\delta_{x_{\alpha,j}}$ with $x_{\alpha,j}$ a point of X, with $\delta_{x_{\alpha,j}}$ the unit point mass at $x_{\alpha,j}$, and with the λ's nonnegative numbers with sum one that converges in the weak* sense to σ. For each choice of α and j, let $\mu_{\alpha,j}$ be a Jensen measure for the point $x_{\alpha,j}$ supported by the boundary Γ, and let $\mu_\alpha = \sum_{j=1}^{n(\alpha)} \lambda_{\alpha,j}\mu_{\alpha,j}$. By passing to a subnet if necessary, we

[2] A measure μ is said to be *orthogonal to* a uniform algebra A on a compact space X if $\int f\, d\mu = 0$ for all $f \in A$. In this case, we write $\mu \in A^\perp$. A measure orthogonal to A is also said to be an *annihilating measure* for A.

can suppose that the net $\{\mu_\alpha\}_{\alpha\in A}$ converges in the weak* sense to a measure μ. Then for $f \in A$ and for $\delta > 0$,

$$\int \log(\delta + |f(x)|)\, d\sigma(x) = \lim_{\alpha\in A} \int \log(\delta + |f(x)|)\, d\sigma_\alpha(x)$$
$$\leq \lim_{\alpha\in A} \int \log(\delta + |f(x)|)\, d\mu_\alpha(x) = \int \log(\delta + |f(x)|)\, d\mu(x).$$

This inequality holds for all $\delta > 0$, so

$$\int \log|f(x)|\, d\sigma(x) \leq \int \log|f(x)|\, d\mu(x),$$

and μ is seen to be a Jensen measure for σ.

The simplest example of nontrivial boundary arises in the context of the algebra $A(D)$, D a bounded open subset of $\mathbb{C}^N$. According to the maximum modulus theorem, the topological boundary of D is a boundary for $A(D)$. A thinner example is the *distinguished boundary* for the polydisk algebra. If by $\mathbb{T}$ we denote the unit circle in the complex plane, then for each function f in the polydisk algebra $A(\mathbb{U}^N)$, $\|f\|_{\overline{\mathbb{U}}^N} = \|f\|_{\mathbb{T}^N}$, so that the torus $\mathbb{T}^N$ is a boundary for the polydisk algebra. It is easy to see that no closed boundary for $A(\mathbb{U}^N)$ is smaller than $\mathbb{T}^N$. The dimension of $\mathbb{T}^N$ is N, while that of the topological boundary $b\mathbb{U}^N$ of $\mathbb{U}^N$ is $2N - 1$.

Uniform algebras have unique minimal closed boundaries:

Theorem 1.2.19. *If A is a uniform algebra on the compact space X, then there is a unique minimal closed boundary for A.*

The minimal closed boundary for A is called the *Shilov boundary* for A, for it was G.E. Shilov who introduced it. The following proof of its existence is that given in [180].

Proof. Denote by Γ the intersection of all closed boundaries for A. This is a certain closed subset of X and so is compact. It is plainly contained in every closed boundary for A, so if Γ is a boundary, then it is the Shilov boundary for A.

That it is a boundary depends on the following observation: If $f_1, \dots, f_q$ are elements of A, and if $U = \{x \in X : |f_1(x)|, \dots, |f_q(x)| < 1\}$, then either U meets every closed boundary for A or else for every closed boundary Γ, $\Gamma \setminus U$ is a boundary. To see this, assume that Γ_o is a closed boundary for A but that the closed set $\Gamma_o \setminus U$ is not one. Thus there is a function $f \in A$ with $\|f\|_X = 1$ but $|f| < 1$ on the compact set $\Gamma_o \setminus U$. We can replace f by a high power, f^k, of it and achieve that $|f| < \eta$ on $\Gamma_o \setminus U$, where $\eta > 0$ is so small that $\eta|f_j| < 1$ on $\Gamma_o \setminus U$ for each $j = 1, \dots, q$. Then $|ff_j| < 1$ on $\Gamma_o \setminus U$, and the same inequality holds on U. The set Γ_o is a boundary for A, so it follows that $|ff_j| < 1$ on all of X. This implies that the points at which $|f|$ takes the value one must lie in U, so U meets each boundary, as desired.

To see that the set Γ is a boundary for the algebra A, suppose it is not, so that there is an element $f \in A$ such that $\|f\|_X = 1$ but $\|f\|_\Gamma = r < 1$. Let $E = \{x \in X : |f(x)| = 1\}$. The set E is compact and nonempty. If $x_o \in E$, there is a boundary, say Γ_o, that misses the point x_o. Then for finitely many functions $f_1, \dots, f_q$ in A, the set $U_{x_o} = \{x \in X :$

$|f_1(x)|, \ldots, |f_q(x)| < 1\}$ is a neighborhood of x_o that is disjoint from Γ. Accordingly, by the last paragraph, $\Gamma \setminus U_{x_o}$ is a boundary for A. Applying this process a finite number of times yields finitely many open sets $U_1, \ldots, U_s$ of the form just considered the union of which covers the set E and such that $\Gamma \setminus \cup_{j=1}^s U_j$ is a boundary for A. This is impossible, because $\|f\|_X = 1$ and $\|f\|_{\Gamma \setminus \cup_{j=1}^s U_j} < 1$. The theorem is proved.

Definition 1.2.20. *If A is a uniform algebra on a compact space X, then the point $x \in X$ is a* peak point *for A if $f(x) = 1$ and $|f| < 1$ on $X \setminus \{x\}$. The subset E of X is a* peak set *for A if there is $f \in A$ with $f = 1$ on E and $|f| < 1$ on $X \setminus E$.*

Each peak point is contained in the Shilov boundary, and each peak set meets the Shilov boundary.

In general, there are no peak points for A: A peak point is a closed subset of X of type G_δ, i.e., it is the intersection of a *countable* family of open sets. The general compact space contains no points that are sets of type G_δ. This problem is essentially irrelevant for our purposes, for we shall work almost exclusively on compact subsets of $\mathbb{C}^N$, which are metrizable. Points in a metrizable space are G_δ's.

Remark. If $X \subset \mathbb{C}^N$ is a compact, polynomially convex set, and if $z_o \in X$ is a peak point for $\mathscr{P}(X)$, one cannot conclude that there is a polynomial P such that $P(z_o) = 1$ and $|P(z)| < 1$ for $z \in X \setminus \{z_o\}$. A simple example is this. Let Δ^+ and Δ^- be, respectively, the closed disks $\{\zeta \in \mathbb{C} : |\zeta - i| \leq 1\}$ and $\{\zeta \in \mathbb{C} : |\zeta + i| \leq 1\}$. These two disks are externally tangent at the origin, and if X is their union, then the origin is a peak point for $\mathscr{P}(X)$. To see this, define f^+ on Δ^+ by $f^+(\zeta) = e^{i\zeta}$, and define f^- on Δ^- by $f^-(\zeta) = e^{-i\zeta}$. These functions agree at the origin, and on $\Delta^+ \setminus \{0\}$, $|f^+| < 1$, and on $\Delta^- \setminus \{0\}$, $|f^-| < 1$. By Mergelyan's theorem, the function F on X that agrees with f^+ on Δ^+ and with f^- on Δ^- lies in the algebra $\mathscr{P}(X)$.

However, there is no polynomial P with $P(0) = 1$ and $|P| < 1$ on $X \setminus \{0\}$. Suppose there is. Let $u = \Re P$, $v = \Im P$. The peaking hypothesis implies that the partial derivative $u_y(0) = \frac{\partial u}{\partial y}(0)$ vanishes and that the directional derivative $D_\alpha u = \operatorname{grad} u \cdot \alpha$ of u in the direction of the vector $(1, 1)$ also vanishes at the origin. From this we find that the partial derivative $u_x(0)$ also vanishes. The Cauchy–Riemann equations yield that the partial derivatives v_x and v_y vanish at the origin, and finally, we conclude that the derivative $P'(0)$ vanishes. Consequently, $P(\zeta) = 1 + \zeta^q Q(\zeta)$ for some integer $q > 1$ and some polynomial Q with $Q(0) \neq 0$. That P is of this form implies that P maps X onto a neighborhood of the point $1 \in \mathbb{C}$, so $|P|$ does not attain its maximum over X at the origin. Contradiction.

There are local notions of peak point and peak set that occasionally are useful.

Definition 1.2.21. *The set $E \subset X$ is a* local peak set *for the uniform algebra A on the compact set X if there is a neighborhood U of E in X such that for some $f \in A$, $f = 1$ on E and $|f| < 1$ on $U \setminus E$.*

The notion of local peak point is defined in a similar way.

An important result is that restrictions of uniform algebras to peak sets are closed.

Definition 1.2.22. *If A is an algebra of $\mathbb{C}$-valued functions on a set S, and if S_o is a subset of S, the restriction algebra $A|S_o$ is the algebra of all restrictions $f|S_o$ for $f \in A$.*

Theorem 1.2.23. [55] *Let A be a uniform algebra on the compact space X. If $E \subset X$ is a peak set for A, then the algebra of restrictions $A|E$ is closed in $\mathscr{C}(E)$, and for each $f \in A$, there is $\tilde{f} \in A$ such that $\tilde{f}|E = f|E$ and $\|\tilde{f}\|_X = \|f\|_E$.*

Granted that $A|E$ is closed in $\mathscr{C}(E)$, the open mapping theorem of Banach space theory implies the existence of a constant C such that for each $f \in A$, there is $g \in A$ with $g|E = f|A$ and with $\|g\|_X \leq C\|f\|_E$. Part of the conclusion of the theorem is that C can be taken to be one.

Proof. By hypothesis there is a function $\varphi \in A$ such that $\varphi = 1$ on E and $|\varphi(x)| < 1$ for all $x \in X \setminus E$. Consider an $f \in A$, which we assume to satisfy $\|f\|_E = 1$. By replacing f by $\varphi^q f$ for a large positive integer q, we can suppose that $\|f\|_X < \frac{3}{2}$. For all $n = 1, \ldots,$ let $E_n = \{x \in X : |f(x)| < 1 + 2^{-n}\}$. There is then $f_n \in A$ with $f_n|A = 1 = \|f_n\|_X$ and $|f_n| < 1$ on $X \setminus E_n$. For a large positive integer $\nu(n)$, the function $f_n^{\nu(n)}$ is identically one on E and satisfies $|f_n^{\nu(n)} f| < 2^{-n}$ on $X \setminus E_n$. For $\tilde{f}$ we can take the sum $\tilde{f} = \sum_{n=1,\ldots} 2^{-n} f_n^{\nu(n)} f$: If $|f(x)| \leq 1$, then $|\tilde{f}(x)| \leq 1$. If $|f(x)| > 1$, choose n_o such that $x \in E_{n_o}$ but $x \notin E_k$ for any $k > n_o$. Then $|\tilde{f}(x)| < (1 + 2^{-n_o}) \sum_{n=1,\ldots,n_o} 2^{-n} + 2^{-n_o} \sum_{n=n_o+1,\ldots} 2^{-n} = 1$. Thus, $\tilde{f}$ is a norm-preserving extension of $f|E$.

That $A|E$ is closed in $\mathscr{C}(E)$ is now easily proved. Let $\{f_n\}_{n=1,\ldots}$ be a sequence in A such that the sequence of restrictions $\{f_n|E\}_{n=1,\ldots}$ converges in $\mathscr{C}(E)$ to g. Suppose $\|f_n - g\|_E \leq 2^{-(n+1)}$. Then $\|f_{n+1} - f_n\|_E < 2^{-n}$, and by the preceding paragraph, there is a function $h_n \in A$ with $h_n|E = (f_{n+1} - f_n)|_E$ and $\|h_n\| < 2^{-n}$. Accordingly, the series $f_1 + \sum_{n=1,\ldots} h_n$ converges in A to an element $\tilde{f}$ with $\tilde{f}|E = g$. This completes the proof.

The spectrum plays an important role in the general theory of uniform algebras.

Definition 1.2.24. *If A is a uniform algebra on the compact Hausdorff space X, then the* spectrum, *$\Sigma(A)$, of A is the set of all characters of A endowed with the weak* topology, so that a net $\{\varphi_\iota\}_{\iota \in I}$ in $\Sigma(A)$ converges to $\varphi \in \Sigma(A)$ if for each $f \in A$, the net $\{\varphi_\iota(f)\}_{\iota \in I}$ converges to $\varphi(f)$.*

Alternatively, a neighborhood basis of $\varphi \in \Sigma(A)$ is provided by the sets of the form

$$W(f_1, \ldots, f_r; \epsilon) = \{\psi \in \Sigma(A) : |\psi(f_1) - \varphi(f_1)|, \ldots, |\psi(f_r) - \varphi(f_r)| < \epsilon\}$$

in which r ranges through the positive integers, $f_1, \ldots, f_r$ range through A, and ϵ ranges through $(0, \infty)$.

The space $\Sigma(A)$ is evidently a Hausdorff space. It is also compact, as can be seen in the following way. Denote by $\mathbb{C}^A$ the Cartesian product of A copies of the complex plane endowed with the product topology. (As a set, $\mathbb{C}^A$ is the set of all functions from A into $\mathbb{C}$.) Define a map $\eta : \Sigma(A) \to \mathbb{C}^A$ by $\eta(\psi) = \{\psi(f) : f \in A\}$. By the definition of the topologies, the map η is continuous. It is plainly one-to-one. The set $\eta(\Sigma(A))$ is a subset of the product $\prod_{f \in A} \bar{\mathbb{U}}_f$ if $\bar{\mathbb{U}}_f = \{\zeta \in \mathbb{C} : |\zeta| \leq \|f\|_X\}$, which, by the Tychonov theorem, is a compact space. The range of η is closed: Let $\{z_f : f \in A\}$ be a limit point of $\eta(\Sigma(A))$. Thus for some net $\{\psi_\iota\}_{\iota \in I}$ in $\Sigma(A)$, $\psi_\iota(f) \to z_f$ for each $f \in A$. Define a functional $\varphi : A \to \mathbb{C}$ by $\varphi(f) = z_f$. This functional is linear and multiplicative, and it lies in

$\Sigma(A)$, for $\varphi(1) = 1$. As a closed subset of a compact space, $\eta(\Sigma(A))$ is itself compact. To conclude that $\Sigma(A)$ is compact, it is enough to notice that η is a homeomorphism, as follows from the continuity of the inverse, η^{-1}. This map *is* continuous, for if $\{\eta(\psi_\iota)\}$ converges to $\eta(\varphi)$, then $\psi_\iota(f) \to \varphi(f)$, so that $\psi_\iota \to \varphi$ in $\Sigma(A)$.

The spectra of the algebras $\mathscr{P}(X)$ and $\mathscr{R}(X)$ have been identified above with the spaces $\widehat{X}$ and $\mathscr{R}$-hull X, respectively. It is usually difficult to determine a concrete representation for the spectrum of a uniform algebra.

If A is a uniform algebra on the compact space X, and if $\Sigma(A)$ is its spectrum, there is a representation of A in $\mathscr{C}(\Sigma(A))$, the *Gel'fand transform*, which is constructed in this way. The map, which is the Gel'fand transform, $f \mapsto \widehat{f}$ is defined by the condition that for $f \in A$, $\widehat{f}(\varphi) = \varphi(f)$. The definition of the topology on $\Sigma(A)$ implies that $\widehat{f}$ is a continuous function. The algebra $\widehat{A}$ of Gel'fand transforms is a uniform algebra on $\Sigma(A)$. To see this, let $x \mapsto \varepsilon_x$ be the map that takes $x \in X$ to the character ε_x on A given by $\varepsilon_x(f) = f(x)$. This is a continuous map. If $f \in A$, then $\|f\|_X = \sup_{x\in X}|\varepsilon_x(f)| \leq \sup_{\varphi\in\Sigma(A)}|\widehat{f}(\varphi)|$, so that $\|f\|_X \leq \|\widehat{f}\|_{\Sigma(A)}$. But also, if $\alpha \in \mathbb{C}$ satisfies $|\alpha| > \|f\|_X$, then $f - \alpha$ is invertible in A, so $(f-\alpha)g = 1$ for a certain $g \in A$. This implies $1 = (\widehat{f} - \alpha)\widehat{g}$, so $\widehat{f}$ omits the value α. Consequently, $\|f\|_{\Sigma(A)} \leq \|f\|_X$. The two norms are seen to be the same. It follows that the Gel'fand transform is an isometry, and that the algebra $\widehat{A}$ of transforms is a uniform algebra on $\Sigma(A)$.[3]

In the case of the algebra $\mathscr{P}(X)$, the Gel'fand transform is the natural extension of $f \in \mathscr{P}(X)$ to a function on the hull $\widehat{X}$, which was given above.

If A is a uniform algebra on the compact space X with spectrum Σ, then for each $f \in A$, the spectrum of f is the set $\widehat{f}(\Sigma)$.

A simple fact about uniform algebras is that holomorphic functions operate:

Theorem 1.2.25. *If A is a uniform algebra, if $f \in A$, and if φ is holomorphic on a neighborhood of $\sigma_A(f)$, then $\varphi \circ f \in A$.*

Proof. Let D be a bounded open set in $\mathbb{C}$ that contains $\sigma_A(f)$ and on the closure of which φ is holomorphic. Assume bD to be smooth, so that for a suitable orientation of bD,

$$\varphi(z) = \frac{1}{2\pi i}\int_{bD}\frac{\varphi(\zeta)}{\zeta - z}\,d\zeta$$

for $z \in D$. For each fixed $\zeta \in bD$, the function $(\zeta - f)^{-1}$ belongs to A, for $\zeta \notin \sigma_A(f)$. The map $\zeta \mapsto (\zeta - f)^{-1}$ from $\mathbb{C}\setminus\sigma_A(f)$ to A is continuous, so $\varphi \circ f$ can be approximated uniformly by Riemann sums for the A-valued integral

$$\frac{1}{2\pi i}\int_{bD}\frac{\varphi(\zeta)}{\zeta - f}\,d\zeta.$$

These Riemann sums are in A, so $\varphi \circ f \in A$ as claimed.

A multivariate extension of this result will be obtained below as a consequence of the Oka–Weil theorem. The formulation of this extension will be in terms of functions

[3] Care is required here. The Gel'fand transform is naturally defined for all commutative Banach algebras. In general, it is *not* an isometry, though, as we have just seen, it is in the case of uniform algebras.

holomorphic on a neighborhood of the joint spectrum of several elements of the algebra A, which is defined as follows.

Definition 1.2.26. *If A is a uniform algebra and $f_1, \dots, f_r \in A$, the* joint spectrum *of the set $\{f_1, \dots, f_r\}$ is the subset*

$$\sigma_A(f_1, \dots, f_r) = \{(\chi(f_1), \dots, \chi(f_r)) : \chi \text{ a character of } A\}$$

of $\mathbb{C}^r$.

Alternatively phrased, $\sigma_A(f_1, \dots, f_r)$ is the image of the spectrum $\Sigma(A)$ of A under the map $\tilde{f} : \Sigma(A) \to \mathbb{C}^r$ defined by $\tilde{f}(\chi) = (\hat{f}_1(\chi), \dots, \hat{f}_r(\chi))$. This map is continuous, so the joint spectrum is compact.

In general, the joint spectrum is not polynomially convex; it is when $f_1, \dots, f_r$ generate A.

We conclude this section on general uniform algebras with a remark about logarithms, which is essentially a theorem of Bruschlinsky [74]. If X is a compact space that satisfies the condition that the Čech cohomology group $\check{H}^1(X, \mathbb{Z})$ vanishes, then every continuous zero-free function f on X has a logarithm.

The shortest route to this conclusion is via sheaf theory. Let $\mathscr{C}_X$ and $\mathscr{C}_X^*$ denote, respectively, the sheaf of germs of continuous $\mathbb{C}$-valued functions and the sheaf of germs of continuous zero-free $\mathbb{C}$-valued functions on X. There is then an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathscr{C}_X \xrightarrow{E} \mathscr{C}_X^* \to 0$$

in which the map E is the exponential map given by $E(f) = e^{2\pi i f}$. The associated cohomology sequence contains the segment

$$\cdots \to \mathscr{C}(X) \to \mathscr{C}^*(X) \xrightarrow{E} \check{H}^1(X, \mathbb{Z}) \to \check{H}^1(X, \mathscr{C}_X) \to \cdots .$$

The group $\check{H}^1(X, \mathscr{C}_X)$ vanishes, for the sheaf $\mathscr{C}_X$ is fine. Consequently, there is an isomorphism

$$\check{H}^1(X, \mathbb{Z}) = \mathscr{C}^*(X)/E(\mathscr{C}(X)). \tag{1.1}$$

In particular, when the cohomology group $\check{H}^1(X, \mathbb{Z})$ vanishes, each zero-free continuous function is an exponential.

This result can be established by a longer argument that appears to avoid the sheaf-theoretic formalism. Assume $\check{H}^1(X, \mathbb{Z})$ to vanish, and consider a zero-free $f \in \mathscr{C}(X)$. There is a finite open cover $\mathscr{U} = \{U_1, \dots, U_q\}$ of X such that for each j, there is $g_j \in \mathscr{C}(U_j)$ such that on U_j, $f = e^{2\pi i g_j}$. For each choice of j, k, the difference $g_j - g_k = \gamma_{jk}$ is a continuous $\mathbb{Z}$-valued function on $U_{jk} = U_j \cap U_k$. The γ_{jk} define a $\mathbb{Z}$-valued 1-cocycle associated with the covering $\mathscr{U}$ of X. The vanishing of $\check{H}^1(X, \mathbb{Z}) = 0$ implies that this cocycle is a coboundary: There are continuous $\mathbb{Z}$-valued functions γ_j on U_j for each j such that $\gamma_{jk} = \gamma_j - \gamma_k$. Define $\tilde{g}_j$ by $\tilde{g}_j = g_j - \gamma_j$. The functions $\tilde{g}_j$ and $\tilde{g}_k$ agree on U_{jk} so that there is a continuous function g on X that satisfies $g = \tilde{g}_j$ on U_j. The function g satisfies $f = e^{2\pi i g}$; f is an exponential.

An immediate consequence of the equality (1.1) is that if X is a compact space and if Y is a closed subset of X such that the map $\check{H}^1(Y,\mathbb{Z}) \to H^1(X,\mathbb{Z})$ induced by the inclusion $Y \hookrightarrow X$ is an isomorphism, then a zero-free function f on X has a continuous logarithm exactly when the restriction $f|Y$ of f to Y has a logarithm.

1.3. Plurisubharmonic Functions

Plurisubharmonic functions play an important role in the theory of polynomial convexity. Although polynomially convex sets are defined in terms of polynomial inequalities, it turns out that they can as well be defined in terms of plurisubharmonic functions. In this section we recall the notion of plurisubharmonic function and establish some of the more immediate connections between these functions and polynomial convexity. We shall see further relations of this kind as the theory develops.

It will be convenient to preface our discussion of plurisubharmonic functions with some remarks about integration on balls and spheres.

For $p = 1, \ldots$ and $r > 0$, denote by $V_p(r)$ the volume of the r-ball in $\mathbb{R}^p$, i.e., the Lebesgue measure of the set $\{x \in \mathbb{R}^p : |x| < r\}$. (Here $|x|$ denotes the Euclidean norm, so that $|x| = (x_1^2 + \cdots + x_p^2)^{1/2}$.) We shall show that

$$V_p(r) = \frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} r^p, \tag{1.2}$$

where $\Gamma(t)$ denotes the Eulerian integral $\int_0^\infty e^{-s} s^{t-1}\, ds$ ($\Re t > 0$), which satisfies the functional equation $\Gamma(t+1) = t\Gamma(t)$ and takes the values $n!$ at $t = n+1$ and $\pi^{1/2}$ at $t = \frac{1}{2}$.

The formula (1.2) is correct for $p = 1, 2$, as we know from elementary geometry or from calculus. Assuming it true for a given value of p, we prove it for $p+2$:

$$\begin{aligned} V_{p+2}(r) &= \int_{|x|<r} dx \\ &= \int_{x_{p+1}^2 + x_{p+2}^2 < r^2} \left\{ \int_{x_1^2 + \cdots + x_p^2 < r^2 - x_{p+1}^2 - x_{p+2}^2} dx_1 \cdots dx_p \right\} dx_{p+1} dx_{p+2}. \end{aligned}$$

Passing to polar coordinates in the (x_{p+1}, x_{p+2})-space and using the induction hypothesis and the functional equation for the Γ-function leads to

$$\begin{aligned} V_{p+2}(r) &= 2\pi \int_0^r V_p(\sqrt{r^2 - s^2}) s\, ds \\ &= 2\pi \frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} \int_0^r (r^2 - s^2)^{p/2} s\, ds \\ &= \frac{\pi^{p/2+1}}{\Gamma(\frac{p+2}{2}+1)} r^{p+2}. \end{aligned}$$

This proves the formula (1.2).

It is curious that the induction from p to $p+2$ is so easy, while that from p to $p+1$, which can be carried out, involves much more complicated integrals.

The calculation just given shows that the volume of the ball $\mathbb{B}_N(0,r)$ in $\mathbb{C}^N$ is $\frac{\pi^N}{N!}r^{2N}$, that of $\mathbb{B}_N$, $\frac{\pi^N}{N!}$.

The boundary, $b\mathbb{B}_N$, of the ball $\mathbb{B}_N$ is the unit sphere $\mathbb{S}^{2N-1}$. On $\mathbb{S}^{2N-1}$ there is a unique positive measure σ with $\sigma(\mathbb{S}^{2N-1}) = 1$ that is invariant under rotations. It is determined by the condition that for every continuous $\mathbb{C}$-valued function g on $\mathbb{S}^{2N-1}$,

$$\int_{\mathbb{S}^{N-1}} g(z)\,d\sigma(z) = \frac{N!}{\pi^N}\int_{\mathbb{B}_n} g\left(\frac{z_1}{|z|},\dots,\frac{z_N}{|z|}\right) d\mathscr{L}(z). \tag{1.3}$$

(Here, and often in the sequel, $\mathscr{L}$ denotes the Lebesgue measure on the underlying Euclidean space.) This is clear: The right side of (1.3) defines a rotation-invariant functional ψ on the space $\mathscr{C}(\mathbb{S}^{2N-1})$ of continuous functions on $\mathbb{S}^{2N-1}$ that satisfies $\psi(1) = 1$. There is only one such functional.

Denote by dS the surface area measure on $\mathbb{S}^{2N-1}$. The measure dS is rotation invariant and so differs from $d\sigma$ by a constant. The polar-coordinates formula for integration on $\mathbb{C}^N$ gives

$$\frac{\pi^N}{N!} = \int_0^1\int_{\mathbb{S}^{2N-1}} r^{2N-1}\,dS\,dr = \frac{1}{2N}\int_{\mathbb{S}^{2N-1}} dS,$$

whence $\int_{\mathbb{S}^{2N-1}} dS = \frac{2\pi^N}{(N-1)!}$, i.e., the area of $\mathbb{S}^{2N-1}$ is $\frac{2\pi^N}{(N-1)!}$. It follows that $d\sigma = \frac{(N-1)!}{2\pi^N}dS$.

There is another integration formula for odd-dimensional spheres, which involves the unitary group, $U(N)$. Let dg denote the Haar measure[4] on $U(N)$ normalized to have total mass one, so that dg is the unique positive measure on $U(N)$ that is $U(N)$-invariant and of total mass one. Let T denote the circle $\{(e^{i\vartheta},0,\dots,0)\}$, which is contained in $b\mathbb{B}_N$. Then for an integrable function F on $b\mathbb{B}_N$,

$$\int_{b\mathbb{B}_N} F\,dS = \frac{\pi^{N-1}}{(N-1)!}\int_{U(N)}\int_{g(T)} F\,ds\,dg, \tag{1.4}$$

in which ds denotes arc length along the various circles $g(T)$. The case of integrable F follows from the case of continuous F. For the continuous case, it is sufficient to note that the right-hand side of (1.4) is a $U(N)$-invariant functional of F and so, to within a constant, is integration against dS. The constant is evaluated by taking for F the function identically one.

We now turn to the theory of plurisubharmonic functions itself. We begin with the definition.

Definition 1.3.1. *A function u defined on a domain Ω in $\mathbb{C}^N$ with values in $[-\infty,\infty)$ is* plurisubharmonic *if it is upper semicontinuous and if for each complex line λ in $\mathbb{C}^N$, the restriction $u|(\Omega\cap\lambda)$ is subharmonic on the open subset $\Omega\cap\lambda$ of the line λ.*

[4] A good discussion of Haar measure is given by Nachbin [257].

Thus an upper semicontinuous function u is plurisubharmonic on Ω if for every pair of vectors a and b in $\mathbb{C}^N$ with $a \in \Omega$,

$$u(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\vartheta} b)\, d\vartheta \tag{1.5}$$

for all sufficiently small $r > 0$. For a function of class $\mathscr{C}^2$, this is equivalent to the condition that the Levi form

$$\mathscr{L}_u(z; w) = \sum_{j,k=1\dots,N} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k$$

be nonnegative for each point $z \in \Omega$. The $\mathscr{C}^2$ function u is said to be *strictly plurisubharmonic* if its Levi form is positive definite at each point of its domain.

From the inequality (1.5) it follows that plurisubharmonic functions are *subharmonic* in the usual sense of potential theory: By that inequality and the integration formula (1.4), we get that if u is plurisubharmonic and r is a small positive number, then

$$u(z_o) \leq \frac{(N-1)!}{2\pi^N} \int_{b\mathbb{B}_N} u(z_o + rz)\, dS(z). \tag{1.6}$$

The function u is upper semicontinuous, so u is seen to be a subharmonic function on the space $\mathbb{R}^{2N} = \mathbb{C}^N$.

Notice that if we integrate the inequality (1.6) against $r^{2N-1}dr$, $0 < r < R$, we obtain

$$u(z_o) \leq \frac{N!}{\pi^N R^{2N}} \int_{\mathbb{B}_N(z_o,R)} u(z)\, d\mathscr{L}(z). \tag{1.7}$$

The notion of plurisubharmonic function extends immediately to complex manifolds. If $\mathscr{M}$ is a complex manifold, then the real-valued function u on $\mathscr{M}$ is plurisubharmonic if it is plurisubharmonic with respect to every set of local holomorphic coordinates in $\mathscr{M}$.

For detailed treatments of the theory of plurisubharmonic functions see [157], [180], or [287].

Among the plurisubharmonic functions are the functions $\log|f|$ and $|f|$ for holomorphic f. The set of all plurisubharmonic functions on a complex manifold is a convex cone: *If u and v are plurisubharmonic functions with common domain, then for nonnegative constants α and β, the function $\alpha u + \beta v$ is also plurisubharmonic.*

The elementary theory of plurisubharmonic functions parallels that of subharmonic functions rather closely. In particular, plurisubharmonic functions enjoy the following properties:

A. If $\{u_j\}_{j=1,\dots}$ is a monotonically decreasing sequence of plurisubharmonic functions defined on a domain Ω, then the function u defined by $u(z) = \lim_{j\to\infty} u_j(z)$ is also plurisubharmonic.

B. If $\{u_\alpha\}_{\alpha\in A}$ is an arbitrary collection of plurisubharmonic functions on a domain Ω, and if $u(z) = \sup_{\alpha\in A} u_\alpha(z)$, then the upper regularization of u defined by

$$u^*(w) = \lim_{\varepsilon\to 0^+} \left(\sup_{|w-z|<\varepsilon} u(z)\right)$$

is plurisubharmonic or else identically $+\infty$.

C. A plurisubharmonic function on a connected open set in $\mathbb{C}^N$ is either identically $-\infty$ or else is locally integrable with respect to Lebesgue measure on $\mathbb{C}^N$.

D. If u is a plurisubharmonic function on the domain Ω, there is a decreasing sequence $\{u_j\}_{j=1,\dots}$ of functions of class $\mathscr{C}^\infty$ on Ω with $u(z) = \lim_{j\to\infty} u_j(z)$ for all z and with the property that if K is a compact subset of Ω, then all but finitely many of the functions u_j are plurisubharmonic on a neighborhood of K. (In general, it is not possible to have the functions u_j plurisubharmonic on all of Ω. In this connection, see [119] and the references cited there.)

E. If u is a plurisubharmonic function on a domain Ω and if $\chi : \mathbb{R} \to \mathbb{R}$ satisfies $\chi', \chi'' \geq 0$, then $\chi \circ u$ is plurisubharmonic on Ω.

A slightly less well known result concerning plurisubharmonic functions is the following result, which is in essence a result of Hartogs [161].

Theorem 1.3.2. *Let Ω be a domain in $\mathbb{C}^N$, let K be a compact subset of Ω, and let g be a continuous function on Ω. If $\{u_k\}_{k=1,\dots}$ is a sequence of plurisubharmonic functions that is locally uniformly bounded on Ω and that satisfies*

$$\limsup_{k\to\infty} u_k(z) \leq g(z) \textit{ for all } z \in \Omega,$$

then for each $\varepsilon > 0$ there is a k_ε such that for $k > k_\varepsilon$ and $z \in K$, $u_k(z) < g(z) + \varepsilon$.

Proof. (See Hörmander [180].) We first suppose that the function g is constant. Without loss of generality, we can suppose that the sequence is uniformly bounded on Ω. It can then be supposed that $g = C$ with $C < 0$ and that each u_k is negative on Ω.

Choose a $\delta < \frac{1}{3}\mathrm{dist}(K, \mathbb{C}^N \setminus \Omega)$. If $z_o \in K$, then by (1.7),

$$u_k(z_o) \leq \frac{N!}{\pi^N \delta^{2N}} \int_{\mathbb{B}_N(z_o,\delta)} u_k(z)\, d\mathscr{L}(z).$$

Fatou's lemma implies that

$$\limsup_{k\to\infty} \int_{B_N(z_o,\delta)} u_k(z)\, d\mathscr{L}(z) \leq \int_{\mathbb{B}_N(z_o,\delta)} C\, d\mathscr{L}(z).$$

Thus there is $k(z_o)$ large enough that for $k > k(z_o)$,

$$\int_{\mathbb{B}_N(z_o,\delta)} u_k(z)\, d\mathscr{L}(z) < \frac{\pi^N \delta^{2N}}{N!}(C + \varepsilon/2).$$

If $|w - z_o| < r$ for an $r < \delta$, then, because the u's are negative, we have, for large k,

$$\begin{aligned} u_k(w) &\le \frac{N!}{\pi^N(\delta+r)^{2N}} \int_{\mathbb{B}_N(w,\delta+r)} u_k(z)\, d\mathscr{L}(z) \\ &\le \frac{N!}{\pi^N(\delta+r)^{2N}} \int_{\mathbb{B}_N(z_o,\delta)} u_k(z)\, d\mathscr{L}(z) \\ &\le \frac{\delta^{2N}}{(\delta+r)^{2N}}(C+\varepsilon/2). \end{aligned}$$

This is less than $C + \varepsilon$. Thus, for each $z_o \in K$, we have found a neighborhood of z_o on which $u_k \le C + \varepsilon$ provided k is big enough. Compactness now implies the result.

Having established the result when the function g is constant, we derive the general case. Let K, ε, and g be as given in the theorem. Because the function g is continuous, compactness yields finitely many compact sets $E_1, \dots, E_q$ with union K and corresponding constants $c_1, \dots, c_q$ such that for each j and all $x \in E_j$, $g(x) < c_j < g(x) + \varepsilon/2$. By the special case of the result that we have proved, there is an integer k_j such that $u_k(x) < c_j + \varepsilon/2$ if $j > k_j$, $x \in \bar{E}_j$. With $k > \max\{k_1, \dots, k_q\}$, we have that for all $x \in K$, $u_k(x) < g(x) + \varepsilon$. The theorem is proved.

Remark 1.3.3. There is an immediate extension of the preceding result in which the open subset Ω of $\mathbb{C}^N$ is replaced by a connected complex manifold. This extension follows from the theorem, because each compact subset of a complex manifold $\mathscr{M}$ is contained in a finite union of coordinate patches in $\mathscr{M}$.

A different treatment of this result was given by Lelong [219].

Associated with a compact subset X of a complex manifold $\mathscr{M}$ is its *hull with respect to the family of plurisubharmonic functions on* $\mathscr{M}$, which is denoted by $\text{Psh-hull}_{\mathscr{M}} X$ and is defined by

$$\text{Psh-hull}_{\mathscr{M}} X = \cap_u \{z \in \mathscr{M} : u(z) \le \sup_X u\},$$

in which the intersection is taken over the family of all functions u plurisubharmonic on $\mathscr{M}$. It is not evident from the definition that the set $\text{Psh-hull}_{\mathscr{M}} X$ is compact. In this connection, see Corollary 1.3.12 below.

Theorem 1.3.4. *For a domain Ω in $\mathbb{C}^N$, the following conditions are equivalent:*

(a) $-\log \operatorname{dist}(z, \mathbb{C}^N \setminus \Omega)$ *is a plurisubharmonic function of z, $z \in \Omega$.*

(b) *There is a continuous plurisubharmonic function $\varrho : \Omega \to \mathbb{R}$ such that for each $c \in \mathbb{R}$, the sublevel set $\Omega_c = \{z \in \Omega : \varrho(z) < c\}$ is a relatively compact subset of Ω.*

(c) *For each compact subset K of Ω, the hull* $\text{Psh-hull}_\Omega K$ *is a relatively compact subset of Ω.*

This is a standard result; see [157], [180], or [287].

The function ϱ of part (b) is called a *plurisubharmonic exhaustion function for* Ω.

Definition 1.3.5. *A domain that possesses the equivalent properties (a), (b), and (c) is said to be* pseudoconvex.

One of the central problems of complex analysis for the first half of the twentieth century was the *Levi problem*, the problem of showing that the class of pseudoconvex domains in $\mathbb{C}^N$ coincides with the class of domains of holomorphy. This identity was finally established in the early 1950s by Oka, by Bremermann, and by Norguet. Recent treatments of this result are considerably simpler than the original ones. See in particular the development given by Range [287]. Although we do not include the details of the solution of the Levi problem, the result will be invoked at several points in the development below.

The notion of Runge domain will be used frequently below. Recall the definition.

Definition 1.3.6. *If $\mathscr{M}$ is a complex manifold, the domain Ω in $\mathscr{M}$ is a* Runge domain *if $\mathscr{O}(\mathscr{M})|\Omega$ is dense in $\mathscr{O}(\Omega)$.*

A basic property of pseudoconvex domains is this:

Theorem 1.3.7. *If $\varrho : \Omega \to \mathbb{R}$ is a continuous plurisubharmonic exhaustion function for Ω, so that each sublevel set $\Omega_c = \{z \in \Omega : \varrho(z) < c\}$ is relatively compact in Ω, then each of the regions Ω_c is a Runge domain in Ω.*

For this result, one can consult [180].

A fundamental connection between polynomial convexity and plurisubharmonic functions is established by the following result. The original source of the result is not clear.

Theorem 1.3.8. *If X is a compact, polynomially convex subset of $\mathbb{C}^N$, then there is a nonnegative plurisubharmonic function, v, on $\mathbb{C}^N$ with $\lim_{z\to\infty} v(z) = \infty$, with $X = v^{-1}(0)$, and with the additional properties that v is of class $\mathscr{C}^\infty$ on $\mathbb{C}^N$ and strictly plurisubharmonic on $\mathbb{C}^N \setminus X$. The function v can be chosen to satisfy $v(z) = |z|^2$ for z near infinity. Conversely, if v is a nonnegative plurisubharmonic function on $\mathbb{C}^N$ such that $\lim_{z\to\infty} v(z) = \infty$, then the set $v^{-1}(0)$ is polynomially convex.*

Proof. (See [127].) Fix a nonnegative function χ of class $\mathscr{C}^\infty$ on $\mathbb{R}$ with the properties that $\chi(t) = 0$ if $t < \frac{1}{2}$ and $\chi(1) = 1$. Require also that χ' and χ'' be nonnegative and strictly positive on $t > \frac{1}{2}$. Given a point $z \in \mathbb{C}^N \setminus X$, there is a polynomial P_z such that $P_z(z) = 1$ and $|P_z| < \frac{1}{4}$ on X. The function $|P_z|^2$ is of class $\mathscr{C}^\infty$ and is plurisubharmonic. If $\varepsilon_z > 0$ is sufficiently small, then the function η_z defined by $\eta_z(w) = \chi(|P_z(w)^2| + \epsilon_z|w|^2)$ is plurisubharmonic and of class $\mathscr{C}^\infty$ on $\mathbb{C}^N$. It vanishes on a neighborhood of X, and is *strictly* plurisubharmonic on a neighborhood W_z of the point z. A countable number of the neighborhoods W_z, say $W_1, \ldots$, cover $\mathbb{C}^N \setminus X$. Let $\eta_1, \ldots$ be the associated functions. If $\{\delta_j\}_{j=1,\ldots}$ is a sequence of positive numbers that decrease sufficiently rapidly to zero, then the function u defined by $u = \sum_{j=1,\ldots} \delta_j \eta_j$ is a nonnegative plurisubharmonic function of class $\mathscr{C}^\infty$ with X as its zero set that is strictly plurisubharmonic on $\mathbb{C}^N \setminus X$. It satisfies $\lim_{w\to\infty} u(w) = \infty$.

To obtain the function v of the statement of the theorem, fix an $R > 0$ so large that the set X is contained in the ball $\mathbb{B}_N(R)$. Let $\eta : \mathbb{R} \to [0, \infty)$ be a smooth function with $\eta(t) = 0$ on $[0, R)$ and with $\eta(t) = t^2$ when $t > 3R$. Require also that η' and η''

be nonnegative. Let $\rho : \mathbb{R} \to [0, 1]$ satisfy $\rho(t) = 0$ if $t > 3R$ and $\rho(t) = t$ when $t \in [0, 2R)$. The function v we desire can be defined by $v(w) = \eta(|w|) + \varepsilon\rho(|w|)u(w)$ for a sufficiently small positive ε.

This completes the proof of one implication of the theorem.

We defer the proof of the final statement of the theorem for the moment; it will be contained in a more general result, Theorem 1.3.11, below.

The preceding argument provides a result somewhat more general than the one stated in that $\mathbb{C}^N$ can be replaced by a general complex manifold: *If $\mathscr{M}$ is a complex manifold with strictly plurisubharmonic exhaustion function ϱ, and if X is a compact subset of $\mathscr{M}$ that is $\mathscr{O}(\mathscr{M})$-convex, then there is a nonnegative plurisubharmonic function v on $\mathscr{M}$ that is of class $\mathscr{C}^\infty$ on $\mathscr{M}$ and that satisfies $v^{-1}(0) = X$ and the further condition that $v = \varrho$ off a compact subset of $\mathscr{M}$.*

It is important to notice that Theorem 1.3.8 does not exhibit every polynomially convex set as the zero locus of a nonnegative smooth function that is strictly plurisubharmonic on all of $\mathbb{C}^N$ or even on a neighborhood of the set. Sets for which such functions exist are quite special. See the discussion of totally real sets in Chapter 6.

On domains of holomorphy, plurisubharmonic functions can be approximated by plurisubharmonic functions of particularly simple form. The following result was stated by Bremermann [71].

Theorem 1.3.9. *If Ω is a domain of holomorphy in $\mathbb{C}^N$, and if u is a continuous plurisubharmonic function on Ω, then for each compact subset K of Ω and for every $\varepsilon > 0$, there are finitely many holomorphic functions $f_1, \dots, f_r$ on Ω such that for suitable positive constants c_j,*

$$u(z) \le \max_{j=1,\dots,r} c_j \log|f_j(z)| \le u(z) + \varepsilon.$$

In the event that Ω is a Runge domain in $\mathbb{C}^N$, the holomorphic functions f_j can be taken to be polynomials.

Proof. (Sibony [330].) Introduce the domain Ω^* in $\mathbb{C}^{N+1}$ defined by

$$\Omega^* = \{(z, w) \in \mathbb{C}^N \times \mathbb{C} : |w| < e^{-u(z)}\}.$$

This domain is pseudoconvex and so a domain of holomorphy.

For $z_o \in \Omega$, define the function f_{z_o} by $f_{z_o}(w) = \sum_{k=0}^\infty e^{ku(z_o)} w^k$, which is defined and holomorphic in the planar domain $\{w \in \mathbb{C} : |w| < e^{-u(z_o)}\}$. The domain Ω^* is a domain of holomorphy, so there is a function $F \in \mathscr{O}(\Omega^*)$ with $F(z_o, w) = f_{z_o}(w)$ for all $w \in \mathbb{C}$ with $|w| < e^{-u(z_o)}$. The function F admits an expansion $F(z, w) = \sum_{k=0}^\infty a_k(z) w^k$ with coefficients $a_k \in \mathscr{O}(\Omega)$ that satisfy

$$\limsup_{k\to\infty} \frac{\log|a_k(z)|}{k} \le u(z)$$

for all $z \in \Omega$ by Hadamard's formula for the radius of convergence of a power series.

The theorem of Hartogs established above, Theorem 1.3.2, implies that for $\varepsilon > 0$, there is k_o large enough that for $k > k_o$, $\frac{\log|a_k(z)|}{k} \le u(z) + \varepsilon$ for all $z \in K$. By the choice of

F, $\limsup_{k\to\infty} \frac{\log|a_k(z_o)|}{k} = u(z_o)$, whence by continuity, $\limsup_{k\to\infty} \frac{\log|a_k(z)|}{k} > u(z_o) - \varepsilon$ for all z in a neighborhood of z_o. By compactness, a finite number of choices of the point z_o will yield a cover of K by the corresponding neighborhoods. The theorem follows.

Corollary 1.3.10. *If X is a compact subset of $\mathbb{C}^N$ and $x_o \in \widehat{X}$, then for each Jensen measure μ for x_o carried by X and for each plurisubharmonic function u defined on a neighborhood of $\widehat{X}$,*

$$u(x_o) \leq \int u(z)\, d\mu(z).$$

Proof. By the monotone convergence theorem, it suffices to prove that the desired inequality holds when u is a continuous plurisubharmonic function. Accordingly, let u be such a function, and let $\varepsilon > 0$ be given. By the preceding theorem, there are polynomials $P_1, \ldots, P_r$ and positive constants $c_1, \ldots, c_r$ such that on a neighborhood of $\widehat{X}$ the inequalities

$$u - \varepsilon < \max_{j=1,\ldots,r} c_j P_j < u$$

are satisfied. Then for each k,

$$\int u(z)\, d\mu(z) \geq \int \max_{j=1,\ldots,r} c_j \log|P_j|\, d\mu \geq c_k \log|P_k(x_o)|.$$

It follows that, as desired,

$$u(x_o) \leq \int u(z)\, d\mu(z).$$

We can now complete the proof of Theorem 1.3.8. What remains to be proved is the final assertion. It is a consequence of a more general fact:

Theorem 1.3.11. *If X is a compact subset of $\mathbb{C}^N$, then $\widehat{X}$ coincides with* Psh-hull$_{\mathbb{C}^N}$ X.

Proof. For every polynomial P, the function $|P|$ is plurisubharmonic on $\mathbb{C}^N$, whence the inclusion $\widehat{X} \supset$ Psh-hull$_{\mathbb{C}^N}$ X.

For the reverse inclusion, let p be a point of $\widehat{X}$. There is a Jensen measure μ for p supported by X. The corollary just proved shows that for every plurisubharmonic function u on $\mathbb{C}^N$, $u(p) \leq \int_X u(x)\, d\mu(x)$, which implies the inequality $u(p) \leq \sup_X u(x)$, whence $p \in$ Psh-hull$_{\mathbb{C}^N}$ X. The theorem is proved.

More generally, if $\Omega \subset \mathbb{C}^N$ is a pseudoconvex domain and X is a compact subset of Ω, then the $\mathscr{O}(\Omega)$-convex hull of X coincides with Psh-hull X.

Corollary 1.3.12. *If $X \subset \mathbb{C}^N$ is a compact set, then* Psh-hull$_{\mathbb{C}^N}$ *X is compact.*

We finish this section with a further simple remark about the relation of polynomially convex sets and pseudoconvexity: *The interior of a polynomially convex set is pseudoconvex.* In fact, the result is more general: In [180, Corollary 2.5.7], it is observed that if $\{\Omega_\alpha\}_{\alpha\in A}$ *is a family of domains of holomorphy, then the interior of* $\bigcap_{\alpha\in\mathscr{A}_1}\Omega_\alpha$ *is a domain of holomorphy.*

1.4. The Cauchy–Fantappiè Integral

In this section we derive a very general integral formula, the *Cauchy–Fantappiè* formula, which yields explicit multidimensional integral formulas of Cauchy type in a wide variety of settings. This integral will be applied in the next section to derive the Oka–Weil approximation theorem and at other points in the sequel.

The general integral formula can be formulated on an arbitrary complex manifold. Thus, let $\mathscr{M}$ be an N-dimensional complex manifold, which might be a domain in $\mathbb{C}^N$. Let $f = (f_1, \dots, f_N) : \mathscr{M} \to \mathbb{C}^N$ be a holomorphic map and let $\varphi = (\varphi_1, \dots, \varphi_N) : \mathscr{M} \to \mathbb{C}^N$ be a smooth map. The precise degree of smoothness is not important here; it suffices for φ to be of class $\mathscr{C}^2$. Define forms $\omega'(\varphi)$ and $\omega(f)$ by

$$\omega'(\varphi) = \sum_{j=1}^{N} (-1)^{j-1} \varphi_j \bar\partial\varphi_1 \wedge \cdots \wedge \widehat{\bar\partial\varphi_j} \wedge \cdots \wedge \bar\partial\varphi_N$$

and

$$\omega(f) = df_1 \wedge \cdots \wedge df_N.$$

(The hat indicates the omission of the term under it.) Thus, $\omega(f)$ is a holomorphic form of bidegree $(N, 0)$, and $\omega'(\varphi)$ is a form of bidegree $(0, N-1)$. Set

$$\varphi \cdot f = \varphi_1 f_1 + \cdots + \varphi_N f_N,$$

so that $\varphi \cdot f$ is a smooth complex-valued function on $\mathscr{M}$. On the manifold

$$\mathscr{M}_{\varphi\cdot f} = \{z \in \mathscr{M} : \varphi(z) \cdot f(z) \neq 0\},$$

define the *Cauchy–Fantappiè form* $\Omega_{\varphi;f}$ by

$$\Omega_{\varphi;f} = \frac{\omega'(\varphi) \wedge \omega(f)}{(\varphi \cdot f)^N}.$$

The main result about these forms is given by the following statement.

Theorem 1.4.1. *Let $\mathscr{M}$ be a complex manifold of dimension N, let $\varphi, \psi : \mathscr{M} \to \mathbb{C}^N$ be $\mathscr{C}^2$ maps, and let $f : \mathscr{M} \to \mathbb{C}^N$ be a holomorphic map.*

(a) *The forms $(\varphi \cdot f)^{-N}\omega'(\varphi)$ and $(\psi \cdot f)^{-N}\omega'(\psi)$ are $\bar\partial$-closed on $\mathscr{M}_{\varphi\cdot f}$ and $\mathscr{M}_{\psi\cdot f}$, respectively.*

(b) *There is a form Θ of bidegree $(0, N-2)$ on $\mathscr{M}_{\varphi\cdot f} \cap \mathscr{M}_{\psi\cdot f}$ such that*

$$\bar\partial\Theta = (\varphi \cdot f)^{-N}\omega'(\varphi) - (\psi \cdot f)^{-N}\omega'(\psi).$$

The application of this result is based on the following reformulation.

Corollary 1.4.2.

(a) *The forms* $\Omega_{\varphi \cdot f}$ *and* $\Omega_{\psi \cdot f}$ *are* $\bar{\partial}$*-closed and* d*-closed.*

(b) *On* $\mathscr{M}_{\varphi \cdot f} \cap \mathscr{M}_{\psi \cdot f}$,

$$\Omega_{\varphi \cdot f} - \Omega_{\psi \cdot f} = \bar{\partial}\left(\Theta \wedge \omega(f)\right) = d\left(\Theta \wedge \omega(f)\right),$$

i.e., the forms $\Omega_{\varphi \cdot f}$ *and* $\Omega_{\psi \cdot f}$ *are* $\bar{\partial}$*- and* d*-cohomologous on their common domain of definition.*

Before we proceed to the proof of the theorem, a few remarks are in order about the forms involved.

If H is the form given by $H = \sum_{j=1}^{N} \varphi_j df_j$, then $\bar{\partial} H = \sum_{j=1}^{N} \bar{\partial}\varphi_j \wedge df_j$, so that

$$(\bar{\partial} H)^{N-1} = (N-1)! \sum_{j=1}^{N} \bar{\partial}\varphi_1 \wedge df_1 \wedge \cdots \wedge \widehat{\bar{\partial}\varphi_j \wedge df_j} \wedge \cdots \wedge \bar{\partial}\varphi_N \wedge df_N,$$

whence

$$H \wedge (\bar{\partial} H)^{N-1} = (-1)^{\frac{N(N-1)}{2}} (N-1)! \omega'(\varphi) \wedge \omega(f).$$

The forms $\omega'(\varphi)$ and $\omega(f)$ admit natural expressions as determinants. Given differential forms α_{jk}, $1 \leq j, k \leq N$, define the determinant of forms

$$\det(\alpha_{jk}) = \det \begin{vmatrix} \alpha_{11} & \dots & \alpha_{1N} \\ \vdots & & \vdots \\ \alpha_{N1} & & \alpha_{NN} \end{vmatrix}$$

to be $\sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \alpha_{\sigma(1)1} \wedge \cdots \wedge \alpha_{\sigma(N)N}$, where the summation is over $\mathfrak{S}_N$, the symmetric group of degree N, and where $\varepsilon(\sigma)$ denotes the sign of the permutation σ. We understand that all the members of a given column are forms of the same degree. Thus, if for each k, α_{jk} is a form of degree d_j, then $\det(\alpha_{jk})$ is a form of degree $d_1 + \cdots + d_N$. In particular, some of the columns may consist of functions, which are forms of degree zero. It must be borne constantly in mind that the formal properties of these determinants differ from those of determinants with entries from commutative rings.

Short calculations show that

$$\omega(f) = \frac{1}{N!} \det \begin{vmatrix} df_1 & \dots & df_1 \\ \vdots & & \vdots \\ df_N & & df_N \end{vmatrix}$$

and

$$\omega'(\varphi) = \frac{1}{(N-1)!} \det \begin{vmatrix} \varphi_1 & \bar{\partial}\varphi_1 \dots & \bar{\partial}\varphi_1 \\ \vdots & \vdots & \vdots \\ \varphi_N & \bar{\partial}\varphi_N & \bar{\partial}\varphi_N \end{vmatrix}.$$

Lemma 1.4.3. *If* $\widetilde{\varphi}_j = \frac{\varphi_j}{\varphi \cdot f}$, *then* $\omega'(\widetilde{\varphi}) = (\varphi \cdot f)^{-N}\omega'(\varphi)$.

Proof. By definition,

$$\begin{aligned}\omega'(\widetilde{\varphi}) &= \sum_{j=1}^{N}(-1)^{j-1}\frac{\varphi_j}{\varphi \cdot f}\bar{\partial}\left(\frac{\varphi_1}{\varphi \cdot f}\right) \wedge \cdots \wedge \bar{\partial}\widehat{\left(\frac{\varphi_j}{\varphi \cdot f}\right)} \wedge \cdots \wedge \bar{\partial}\left(\frac{\varphi_N}{\varphi \cdot f}\right) \\ &= (\varphi \cdot f)^{-(2N-1)}\sum_{j=1}^{N}(-1)^{j-1}\varphi_j[\varphi \cdot f\bar{\partial}\varphi_1 - \varphi_1\bar{\partial}(\varphi \cdot f)] \wedge \cdots \wedge \widehat{[j]} \\ &\qquad \wedge \cdots \wedge [\varphi \cdot f\bar{\partial}\varphi_N - \varphi_N\bar{\partial}(\varphi \cdot f)].\end{aligned}$$

We have $\bar{\partial}(\varphi \cdot f) \wedge \bar{\partial}(\varphi \cdot f) = 0$, which implies that this sum contains two kinds of nonzero terms: those that contain exactly one factor $\bar{\partial}(\varphi \cdot f)$ and those that contain no such factors. The latter terms have sum

$$(\varphi \cdot f)^{-N}\sum_{j=1}^{N}(-1)^{j-1}\varphi_j\bar{\partial}\varphi_1 \wedge \cdots \wedge \widehat{\bar{\partial}\varphi_j} \wedge \cdots \wedge \bar{\partial}\varphi_N,$$

so the lemma will be proved if we can show that the sum of the terms of the former kind is zero. This can be done in the following way.

Except for a factor of $-(\varphi \cdot f)^{N-2}$, the sum of the terms of the first kind is

$$\begin{aligned}&\sum_{j=1}^{N}(-1)^{j-1}\varphi_j\sum_{\substack{k=1,\\ k\neq j}}^{N}\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{k-1} \wedge \varphi_k\bar{\partial}(\varphi \cdot f) \wedge \cdots \wedge \bar{\partial}\varphi_{j-1} \wedge \widehat{\bar{\partial}\varphi_j} \\ &\qquad \wedge \bar{\partial}\varphi_{j+1} \wedge \cdots \wedge \bar{\partial}\varphi_N.\end{aligned}$$

In this we can replace $\bar{\partial}(\varphi \cdot f)$ by $f_k\bar{\partial}\varphi_k + f_j\bar{\partial}\varphi_j$ to find that the sum is

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{\substack{k=1,\\ k\neq j}}^{N}(-1)^{j-1}\varphi_j\varphi_k\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{k-1} \wedge [f_k\bar{\partial}\varphi_k + f_j\bar{\partial}\varphi_j] \wedge \cdots \\ &\qquad \wedge \bar{\partial}\varphi_{j-1} \wedge \widehat{\bar{\partial}\varphi_j} \wedge \bar{\partial}\varphi_{j+1} \wedge \cdots \wedge \bar{\partial}\varphi_N.\end{aligned}$$

Fix a pair of integers p and q with $1 \leq p < q \leq N$, and consider the coefficient of $\varphi_p\varphi_q$ in the last sum. It is

$$\begin{aligned}&\{(-1)^{p-1}\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{p-1} \wedge \widehat{\bar{\partial}\varphi_p} \wedge \bar{\partial}\varphi_{p+1} \wedge \cdots \wedge \bar{\partial}\varphi_{q-1} \\ &\qquad \wedge [f_p\bar{\partial}\varphi_p + f_q\bar{\partial}\varphi_q] \wedge \bar{\partial}\varphi_{q+1} \wedge \cdots \wedge \bar{\partial}\varphi_N\} \\ &+ \{(-1)^{q-1}\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{p-1} \wedge [f_p\bar{\partial}\varphi_p + f_q\bar{\partial}\varphi_q] \wedge \bar{\partial}\varphi_{p+1} \wedge \cdots \wedge \bar{\partial}\varphi_{q-1} \\ &\qquad \wedge \widehat{\bar{\partial}\varphi_q} \wedge \bar{\partial}\varphi_{q+1} \wedge \cdots \wedge \bar{\partial}\varphi_N\}.\end{aligned}$$

The terms in braces differ by sign, so their sum is zero. The lemma is proved.

The proof we give for Theorem 1.4.1 is based on geometric notions. Given φ and ψ as in the theorem, introduce $\widetilde{\varphi}$ and $\widetilde{\psi}$ by

$$\widetilde{\varphi} = (\varphi \cdot f)^{-1}\varphi \quad \text{and} \quad \widetilde{\psi} = (\psi \cdot f)^{-1}\psi,$$

so that $\widetilde{\varphi} \cdot f = 1 = \widetilde{\psi} \cdot f$.

Consider the map $H : [0, 1] \times \mathscr{M} \to \mathbb{C}^N \times \mathbb{C}^N$ given by

$$H(t, z) = \left(t\widetilde{\varphi}(z) + (1 - t)\widetilde{\psi}(z), f(z)\right).$$

With coordinates $\xi = (\xi_1, \ldots, \xi_N)$ on the first factor of $\mathbb{C}^N \times \mathbb{C}^N$ and coordinates $\zeta = (\zeta_1, \ldots, \zeta_N)$ on the second, the range of H is contained in the complex submanifold[5] $\mathscr{V} = \{(\xi, \zeta) \in \mathbb{C}^N \times \mathbb{C}^N : \xi \cdot \zeta = \xi_1\zeta_1 + \cdots + \xi_N\zeta_N = 1\}$ of $\mathbb{C}^N \times \mathbb{C}^N$, and H is a smooth homotopy through $\mathscr{V}$ connecting the maps $\widetilde{\Phi} = (\widetilde{\varphi}, f)$ and $\widetilde{\Psi} = (\widetilde{\psi}, f)$ of $\mathscr{M}$ into $\mathscr{V}$.

Introduce the holomorphic $(2N - 1)$-form η on $\mathbb{C}^N \times \mathbb{C}^N$ by

$$\eta = \sum_{j=1}^{N} (-1)^{j-1} \xi_j d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_N \wedge d\zeta_1 \wedge \cdots \wedge \zeta_N.$$

The form η is not closed, for $d\eta = N d\xi_1 \wedge \cdots \wedge d\xi_N \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_N$. However, if $\iota : \mathscr{V} \to \mathbb{C}^N \times \mathbb{C}^N$ denotes the inclusion, then $\iota^*\eta$, as a holomorphic form of maximal degree on $\mathscr{V}$, is closed. The function f is holomorphic, so Lemma 1.4.3 shows that

$$\Omega_{\varphi; f} = \frac{\omega'(\varphi) \wedge \omega(f)}{(\varphi \cdot f)^N} = \widetilde{\Phi}^*\eta = \widetilde{\Phi}^*\iota^*\eta$$

and

$$\Omega_{\psi; f} = \frac{\omega'(\psi) \wedge \omega(f)}{(\psi \cdot f)^N} = \widetilde{\Psi}^*\eta = \Psi^*\iota^*\eta.$$

The form $\iota^*\eta$ is a closed form on $\mathscr{V}$, and the maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are homotopic through $\mathscr{V}$, so it follows that the forms $\Omega_{\varphi; f}$ and $\Omega_{\psi; f}$ are cohomologous:

$$\Omega_{\varphi; f} = \Omega_{\psi; f} + d\alpha \tag{1.8}$$

for a suitable form α. (This conclusion is based on the *homotopy formula* for forms, which we explain below.) The result (1.8) is sufficient to yield the general integral formula below, but it is not quite the result (b) of the theorem. To obtain the result (b), it seems to be necessary to examine the mechanism of the homotopy formula, rather than merely invoke the final result.

[5]Denote by π_ξ the projection of $\mathscr{V}$ into the first factor of $\mathbb{C}^N$. The range of π_ξ is $\mathbb{C}^N \backslash \{0\}$, and π_ξ has maximal rank. For $\xi_0 \in \mathbb{C}^N \backslash \{0\}$, the fiber $\pi_\xi^{-1}(\xi_0)$ is the affine subspace $\{\zeta \in \mathbb{C}^N : \xi_0 \cdot \zeta = 1\}$ of $\mathbb{C}^N$. We see that $\mathscr{V}$ is a holomorphic fiber bundle over $\mathbb{C}^N \backslash \{0\}$. It is called the *Leray fibration*. A projective version of this bundle was introduced by Leray [221].

The setting for the homotopy formula for forms is this. Fix a smooth n-dimensional manifold $\mathscr{N}$ and a form μ on $\mathscr{N} \times \mathbb{R}$, μ of degree $p \geq 1$. Write

$$\mu = \mu_1 + dt \wedge \nu, \tag{1.9}$$

where t is the coordinate on $\mathbb{R}$. The form μ_1 is of degree p, ν of degree $p-1$, and we assume that μ_1 does not involve dt. Denote by $\mathscr{E}^r$ the space of forms of degree r, and define an operator

$$I : \mathscr{E}^p(\mathscr{N} \times \mathbb{R}) \to \mathscr{E}^{p-1}(\mathscr{N})$$

as follows. For each $t \in \mathbb{R}$, let $\iota_t : \mathscr{N} \to \mathscr{N} \times \mathbb{R}$ be the embedding given by $\iota_t(x) = (x, t)$. Then ι_{t*} denotes the induced map of tangent spaces. Given a p-form μ on $\mathscr{N} \times \mathbb{R}$, decompose it according to (1.9) and then for $p-1$ vector fields $X_1 \ldots, X_{p-1}$ on $\mathscr{N}$, set $I\mu(X_1, \ldots, X_{p-1}) = \int_0^1 \nu(\iota_{t*}X_1, \ldots, \iota_{t*}X_{p-1})\, dt.$

Theorem 1.4.4. *(*Homotopy formula for forms.*) For a p-form μ on the product $\mathscr{N} \times \mathbb{R}$,*

$$\iota_1^*\mu - \iota_0^*\mu = d(I\mu) + I(d\mu).$$

Proof. We can use a partition of unity to reduce to the situation in which the support of μ is contained in a coordinate patch.

Suppose that in terms of local coordinates $x_1, \ldots, x_n$ on $\mathscr{N}$,

$$\mu = f\,dx^I = f\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Then because $d\mu = \theta + \frac{\partial f}{\partial t} dt \wedge dx^I$ with θ independent of dt,

$$I(d\mu) = \left(\int_0^1 \frac{\partial f}{\partial t}(\cdot, t)\, dt \right) dx^I = f(\cdot, 1)dx^I - f(\cdot, 0)dx^I = \iota_1^*\mu - \iota_0^*\mu.$$

In this case, $I\mu = 0$, and the result is seen to be correct.

If, on the other hand, for some $J = (j_1, \ldots, j_{p-1})$,

$$\mu = f\,dt \wedge dx^J,$$

then $\iota_0^*\mu = 0 = \iota_1^*\mu$. Also,

$$I(d\mu) = I\left(-\sum_{j=1}^n \frac{\partial f}{\partial x_j} dt \wedge dx_j \wedge dx^J \right) = -\sum_{j=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_j}(\cdot, t)\, dt \right) dx_j \wedge dx^J,$$

and

$$d(I\mu) = d\left(\int_0^1 f(\cdot, t) dt \right) dx^J = \sum_{j=1}^n \int_0^1 \frac{\partial f}{\partial x_j}(\cdot, t)\, dt)\, dx_j \wedge dx^J.$$

These equations give $I(d\mu) + d(I\mu) = 0$, so the theorem is proved.

A corollary of this is the fact used above that *if μ is closed, then $\iota_1^*\mu - \iota_2^*\mu$ is exact.* Another corollary, which is the form in which we shall use the result, is this:

Corollary 1.4.5. *If $\mathcal{N}$ and $\mathcal{N}'$ are smooth manifolds, and if $\varphi_0, \varphi_1 : \mathcal{N} \to \mathcal{N}'$ are smoothly homotopic smooth maps, then for any closed form μ on $\mathcal{N}'$, $\varphi_0^*\mu - \varphi_1^*\mu$ is exact.*

The proof will yield a bit more than we have stated.

Proof. The hypothesis that φ_0 and φ_1 are smoothly homotopic means that there is a smooth map $H : \mathcal{N} \times \mathbb{R} \to \mathcal{N}'$ with $H(x, 0) = \varphi_0(x)$, $H(x, 1) = \varphi_1(x)$. If $\iota_0, \iota_1 : \mathcal{N} \to \mathcal{N} \times \mathbb{R}$ are the inclusions as above, then $\varphi_0 = H \circ \iota_0$ and $\varphi_1 = H \circ \iota_1$, whence

$$\varphi_0^*\mu - \varphi_1^*\mu = \iota_0^* H^*\mu - \iota_1^* H^*\mu.$$

The homotopy formula for forms shows that this is $dI(H^*\mu) + I(dH^*\mu)$. Granted that μ is closed, $H^*\mu$ is also closed. This finishes the proof of the lemma.

Notice that if the form μ is closed, the proof gives an explicit formula for a form with differential $\iota_1^*\mu - \iota_0^*\mu$.

We can now obtain the $\bar\partial$-exactness statement (b) of Theorem 1.4.1. To do this, we apply the construction in the proof of the homotopy formula to find an explicit expression from which (b) will follow. The pullback of the form η on $\mathcal{V}$ (or $\mathbb{C}^N \times \mathbb{C}^N$) by the map $H : \mathcal{M} \times \mathbb{R} \to \mathbb{C}^N \times \mathbb{C}^N$ is

$$H^*\eta = \sum_{j=1}^{N} (-1)^{j-1} \left[t\frac{\varphi_j}{\varphi \cdot f} + (1-t)\frac{\psi_j}{\psi \cdot f} \right] d\left[t\frac{\varphi_1}{\varphi \cdot f} + (1-t)\frac{\psi_1}{\psi \cdot f} \right] \wedge \cdots$$
$$\cdots \wedge [j] \wedge \cdots \wedge d\left[t\frac{\varphi_N}{\varphi \cdot f} + (1-t)\frac{\psi_N}{\psi \cdot f} \right] \wedge \omega(f).$$

The form $\omega(f)$ has bidegree $(N, 0)$, so all the terms $d\varphi_k$ and $d\psi_k$ in this expression can be replaced by $\bar\partial\varphi_k$ and $\bar\partial\psi_k$ respectively. We can then write

$$H^*\eta = F(z,t)\mu \wedge \omega(f) + dt \wedge \nu \wedge \omega(f),$$

where F is a function, μ a form that does not involve dt, and ν a form of bidegree $(0, N-2)$ that depends on t. In terms of the operator I considered above, we have

$$\Omega_{\varphi;f} - \Omega_{\psi;f} = d(IH^*\eta) = \bar\partial \left\{ \int_0^1 \nu \, dt \right\} \wedge \omega(f),$$

where by $\int_0^1 \nu\, dt$ we mean the $(0, N-2)$-form obtained by integrating the coefficients of ν with respect to t. Call this form Θ. It is a $(0, N-2)$-form with

$$\bar\partial\Theta = (\varphi \cdot f)^{-N}\omega'(\varphi) - (\psi \cdot f)^{-N}\omega'(\psi), \tag{1.10}$$

and our proof is complete.

We are now able to prove a very general integral representation formula. Denote by c_N the constant $c_N = \frac{(-1)^{\frac{1}{2}N(N-1)}(N-1)!}{(2\pi i)^N}$.

Definition 1.4.6. *If $\mathscr{M}$ is a complex manifold, if $z \in \mathscr{M}$, and if $f_1, \dots, f_r \in \mathscr{O}(\mathscr{M})$, then $f_1, \dots, f_r$ are said to* generate the ideal $\mathscr{I}_z$ *if z is the only common zero of the f's and if, in addition, in terms of local coordinates $\zeta_1, \dots, \zeta_N$ defined near z, the matrix of partial derivatives $\left(\frac{\partial f_j}{\partial \zeta_k}\right)_{\substack{1\le j\le n\\ 1\le k\le N}}$ has rank N at z.*

When $r = N$, the rank condition is the condition that $df_1 \wedge \cdots \wedge df_N$ not vanish at z.

Theorem 1.4.7. *Let $\mathscr{M}$ be an N-dimensional complex manifold, let $D \subset \mathscr{M}$ be a relatively compact smoothly bounded domain, and let $f_1, \dots, f_N \in \mathscr{O}(\mathscr{M})$ generate the ideal $\mathscr{I}_z$, $z \in D$. If $\varphi_1, \dots, \varphi_N$ are smooth functions on $\mathscr{M}$ such that $\varphi \cdot f$ does not vanish on bD, then for every $F \in \mathscr{O}(\mathscr{M})$,*

$$F(z) = c_N \int_{bD} F\Omega_{\varphi;f}.$$

Proof. By taking $\psi = \bar{f} = (\bar{f}_1, \dots, \bar{f}_N)$ in Corollary 1.4.2, we find that the forms $\Omega_{\varphi;f}$ and $\Omega_{\bar{f};f}$ are $\bar\partial$ cohomologous on a neighborhood of bD: For some smooth $(N, N-2)$-form χ defined near bD, $\Omega_{\varphi;f} - \Omega_{\bar{f};f} = \bar\partial\chi$, whence, by the holomorphicity of F and type considerations, $F\Omega_{\varphi;f} = F\Omega_{\bar{f};f} + d(F\chi)$. Stokes's theorem implies that $c_N \int_{bD} F\Omega_{\varphi;f} = c_N \int_{bD} F\Omega_{\bar{f};f}$. By hypothesis, $df_1 \wedge \cdots \wedge df_N = \omega(f)$ is not zero at z, so a neighborhood of z is mapped biholomorphically onto a neighborhood of 0 in $\mathbb{C}^N$ by f. The form $F\Omega_{\bar{f};f}$ is closed on $\mathscr{M} \setminus \{z\}$, so Stokes's theorem yields $c_N \int_{bD} F\Omega_{\varphi;f} = c_N \int_{\{\zeta\in\mathscr{M}:|f(\zeta)|=\varepsilon\}} F\Omega_{\bar{f};f}$. In the integral on the right make the change of variable $w = f(\zeta)$ to obtain

$$c_N \int_{bD} F\Omega_{\varphi;f} = c_N \int_{|w|=\varepsilon} F\left(f^{-1}(w)\right)\Omega_{\bar{w};w}.$$

We have $\Omega_{\bar{w};w} = \frac{\omega'(\bar{w})\wedge\omega(w)}{|w|^{2N}} = \epsilon^{-2N}\omega'(\bar{w}) \wedge \omega(w)$ on $|w| = \epsilon$, so that the last integral is

$$\epsilon^{-2N}\left\{\int_{|w|=\epsilon} \{F(f^{-1}(w)) - F(z)\}\omega'(w) \wedge \omega(w)\right\} + \epsilon^{-2N}F(z)\int_{|w|<\epsilon} N\omega(\bar{w}) \wedge \omega(w),$$

as follows from Stokes's theorem and the equality $d\omega'(\bar{w}) = N\omega(\bar{w})$. The first of these terms tends to zero as $\epsilon \to 0$, for the area of $|w| = \epsilon$ is const ϵ^{2N-1}, the coefficients of $\omega'(\bar{w}) \wedge \omega(w)$ are $O(w)$, and $|F(f^{-1}(w)) - F(z)| \to 0$ as $\epsilon \to 0$. From $\omega(\bar{w}) \wedge \omega(w) = (-1)^{\frac{1}{2}N(N-1)} d\bar{w}_1 \wedge dw_1 \wedge \cdots \wedge d\bar{w}_N \wedge dw_N$, we obtain the value of $\epsilon^{-2N}F(z)\int_{|w|<\epsilon} N\omega(\bar{w}) \wedge \omega(w)$ to be $(-1)^{\frac{1}{2}N(N-1)}(2i)^N N\frac{\pi^N}{N!}F(z)$, and the theorem is proved.

Given the vast literature of the subject, one must be hesitant to assign authorship to this theorem. Theorem 1.4.1 seems first to have been formulated by Koppelman [210]. Gleason [144] has given a more general integral formula, which he obtained by quite different methods, and Aĭzenberg [2] has given a formula for domains in $\mathbb{C}^N$ that is essentially equivalent to the formula above. There is also a paper of Harvey [164] that takes a very general view of integral formulas. Conceptually, it is rather near Gleason's work, though in detail it is entirely distinct. Finally, mention must be made of the earlier seminal work of Leray [221] in which the notion of Cauchy–Fantappiè form seems first to have been introduced. Leray's work was developed further by Norguet [266].

The literature of integral formulas has grown enormously in recent years. We refer the interested reader to the book of Aĭzenberg and Juzhakov [3], which has an extensive list of references, to that of Griffiths and Harris [155], and to the encyclopedia paper by Henkin [169]. The reports in the Norguet seminars, e.g., [267], contain much interesting material on integral formulas. The subject of integral formulas in several complex variables has now developed to the point that it is entirely reasonable to speak of "the method of integral formulas" as Henkin does [169].

A special case of the Cauchy–Fantappiè integral is important to notice: the Bochner–Martinelli formula.

This integral formula is valid on a bounded domain $D \subset \mathbb{C}^N$ on which Stokes's theorem is valid. It is obtained by fixing a point $w \in D$ and letting the holomorphic map f in the Cauchy–Fantappiè formula be the translation given by $f(z) = z - w$ and the smooth map φ be the conjugate of f, so that $\varphi(z) = \overline{w - z}$. Then $\varphi(z) \cdot f(z) = |z - w|^2$, and the Cauchy–Fantappiè formula yields the integral formula that for g holomorphic on $\bar{D}$,

$$g(w) = c_N \int_{bD} g(z) \frac{\omega'(\overline{z - w}) \wedge \omega(z)}{|z - w|^{2N}}. \tag{1.11}$$

This is the *Bochner–Martinelli* integral formula for the domain D. In the one-dimensional case, $N = 1$, this is just the usual Cauchy integral formula for the domain D.

It is worth observing that to obtain the Bochner–Martinelli formula, not all of the details of the proof of the full Cauchy–Fantappiè formula are necessary. In essence, we need only the remark that the form $k_{\mathrm{BM}}(z, w) = \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}}$ is closed where it is defined. Then Stokes's theorem can be applied exactly as in the proof of the Cauchy–Fantappiè formula to yield the formula. As for the degree of smoothness on bD, all that is required is that Stokes's theorem be valid on D.

The Bochner–Martinelli formula, like the Cauchy integral formula, reproduces holomorphic functions from their boundary values.

It is well to point out explicitly that the hypothesis that $g \in \mathscr{C}^1(\bar{D})$ is still in force in the formula (1.11). However, for some domains, this formula can be established under milder regularity hypotheses on g. Let D be a domain with $\mathscr{C}^1$ boundary, so that for some $\mathscr{C}^1$ function, Q, defined on a neighborhood V of bD, $D \cap V = \{w \in V : Q(w) < 0\}$ and $dQ \neq 0$ on bD. If $g \in A(D)$ and if $z \in D$, then for sufficiently small $\varepsilon > 0$, the corollary yields

$$g(z) = c_N \int_{\{Q=-\varepsilon\}} g(w) k_{\mathrm{BM}}(z, w).$$

We can take the limit as $\varepsilon \to 0^+$ to obtain the desired formula (1.11). The formula (1.11) will hold for $g \in A(D)$ under conditions on bD less stringent than that it be of class $\mathscr{C}^1$, but of course, the weaker the hypothesis, the more involved the proof becomes. We shall not pursue the question of establishing the Bochner–Martinelli formula under the weakest possible hypotheses.

The formula (1.11) was given in [237] and in [67]. There seems to have been a minor controversy about the priority of the discovery of the formula. See [67, p. 652, note added in proof] and [240, p. 117].

There is the following important difference between the Cauchy integral formula in one variable and the Bochner–Martinelli formula. If f is an arbitrary integrable function on the curve γ in $\mathbb{C}$, then the function F given by $F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$ is holomorphic off γ, as is clear from the holomorphy in z of the Cauchy kernel $\frac{1}{\zeta - z}$ for fixed ζ. The corresponding integrals of Bochner–Martinelli type do not have the analogous property. If D is a bounded smoothly bounded domain in $\mathbb{C}^N$, then given an integrable function g on bD, the function G given by

$$G(z) = c_N \int_{bD} g(w) k_{\mathrm{BM}}(z, w)$$

is, generally speaking, not holomorphic anywhere in $\mathbb{C}^N \setminus bD$.

It is, however, harmonic there. This is most easily seen from the expression (1.11). Write $G(z) = G_1(z) + \cdots + G_N(z)$ with

$$G_j(z) = (-1)^{j-1} c_N \int_{bD} g(w) \frac{\bar{w}_j - \bar{z}_j}{|w - z|^{2N}}\, d\bar{w}_1 \wedge \cdots \wedge \widehat{d\bar{w}_j} \wedge \cdots \wedge d\bar{w}_N \wedge \omega(w).$$

For fixed w, $\frac{1}{|w-z|^{2N-2}}$ is harmonic in $\mathbb{C}^N \setminus \{w\}$. The same is therefore true of its derivative $\frac{\partial}{\partial z_j} \frac{1}{|w-z|^{2N-2}} = (N-1)\frac{\bar{w}_j - \bar{z}_j}{|w-z|^{2N}}$. Consequently, each of the functions G_j is harmonic off bD.

It is sometimes useful to express the Bochner–Martinelli integral as an integral against the surface area measure on bD. The next lemma allows this. It is useful to use the notation that $\omega_{[k]}(z)$ denotes the form $dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_N$. The form $\omega_{[k]}(\bar{z})$ is defined in a similar way.

Lemma 1.4.8. *Let D be a bounded domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^1$ and with $\mathscr{C}^1$ defining function Q. As functionals on $\mathscr{C}(bD)$,*

$$\omega_{[k]}(z) \wedge \omega(\bar{z}) = (-1)^{k-1+\frac{1}{2}N(N-1)} (2i)^N \frac{\partial Q}{\partial z_k} |\mathrm{grad}\ Q|^{-1} dS$$

and

$$\omega_{[k]}(\bar{z}) \wedge \omega(z) = (-1)^{k-1+\frac{1}{2}N(N-1)} (2i)^N \frac{\partial Q}{\partial \bar{z}_k} |\mathrm{grad}\ Q|^{-1} dS.$$

The statement means that for every continuous function f on bD, the integral of f against the *differential form* on the left is the same as the integral of f against the *measure* on the right.

The two formulas are equivalent by complex conjugation.

Proof. Denote by ι the inclusion $bD \hookrightarrow \mathbb{C}^n$. At a point of bD where $\iota^* dz_k \neq 0$, we have $\frac{\partial Q}{\partial z_k} \neq 0$, and conversely. Near such points, bD is given by

$$y_k = H(z_1, \dots, z_{k-1}, x_k, z_{k+1}, \dots, z_N)$$

or else by

$$x_k = H(z_1, \dots, z_{k-1}, y_k, z_{k+1}, \dots, z_N).$$

Consider the former case, and let

$$\Phi(z_1, \dots, z_{k-1}, x_k, z_{k+1}, \dots, x_N) = (z_1, \dots, z_{k-1}, x_k + iH, z_{k+1}, \dots, z_N).$$

Then

$$\begin{aligned}
&\Phi^*\iota^*(\omega_{[k]}(z) \wedge \omega(\bar{z})) \\
&\quad = \Phi^*\iota^*\omega_{[k]}(z) \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{k-1} \wedge (dx_k - i\,dH) \wedge d\bar{z}_{k+1} \wedge \cdots \wedge d\bar{z}_N, \\
&\quad = \Phi^*\iota^*\omega_{[k]}(z) \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{k-1} \wedge \left(1 - i\frac{\partial H}{\partial x_k}\right) dx_k \wedge d\bar{z}_{k+1} \wedge \cdots \wedge d\bar{z}_N.
\end{aligned}$$

However, Q is a defining function for bD, so $\frac{\partial H}{\partial x_k} = -\frac{\partial Q}{\partial x_k}\left[\frac{\partial Q}{\partial y_k}\right]^{-1}$, whence

$$\begin{aligned}
\Phi^*\iota^*(\omega_{[k]}(z) \wedge \omega(\bar{z})) = 2i\frac{\partial Q}{\partial z_k}\left[\frac{\partial Q}{\partial y_k}\right]^{-1} \Phi^*\iota^*\omega_{[k]}(z) \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{k-1} \\
\wedge dx_k \wedge d\bar{z}_{k+1} \wedge \cdots \wedge d\bar{z}_N.
\end{aligned} \tag{1.12}$$

To complete the proof, compute the pullback $\Phi^* dS$ of the volume form. It is given by

$$\begin{aligned}
\Phi^* dS = \pm\left(\frac{i}{2}\right)^{N-1} J_\Phi dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1} \wedge dx_k \\
\wedge dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_N \wedge d\bar{z}_N,
\end{aligned} \tag{1.13}$$

where the Jacobian J_Φ is given by

$$J_\Phi = \sqrt{1 + |\text{grad } H|^2} = \left|\frac{\partial Q}{\partial y_k}\right|^{-1} |\text{grad } Q| \tag{1.14}$$

and where the sign is + if the parameterization Φ is orientation-preserving, − if orientation-reversing. If we compare (1.12) and (1.13) using (1.14), we find that

$$\begin{aligned}
\Phi^*\iota^*\omega_{[k]}(z) \wedge \omega(\bar{z})] = \pm(-1)^{k-1+\frac{1}{2}N(N+1)}(2i)^N \frac{\partial Q}{\partial z_k}\left|\frac{\partial Q}{\partial y_k}\right| \\
\times\left[\frac{\partial Q}{\partial y_k}\right]^{-1} |\text{grad } Q|^{-1}\Phi^* dS.
\end{aligned} \tag{1.15}$$

The final observation is that along the part of bD where $\frac{\partial Q}{\partial y_k} > 0$, Φ is orientation-reversing, along the part where $\frac{\partial Q}{\partial y_k} < 0$, Φ is orientation-preserving. To verify this, it suffices to verify that for fixed $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_N$, the partial map

$$x_k \mapsto (z_1, \ldots, z_{k-1}, x_k + iH(z_1, \ldots, z_{k-1}, x_k, z_{k+1}, \ldots, z_N), z_{k+1}, \ldots, z_N)$$

is orientation-preserving when $\frac{\partial Q}{\partial y_k} < 0$, orientation-reversing in the contrary case. This means that the matter reduces to a question in the plane, and then the result is easily verified. Thus, in (1.15) on the right we have $\pm\left|\frac{\partial Q}{\partial y_k}\right|\left[\frac{\partial Q}{\partial y_k}\right]^{-1} = 1$, so the lemma is proved.

The lemma implies that if f is a function on bD and if $F : \mathbb{C}^N \setminus bD \to \mathbb{C}$ is the Bochner–Martinelli integral of f, then

$$F(w) = \frac{(N-1)!}{\pi^N} \int_{bD} \frac{f(z)}{|z-w|^{2N}} \frac{1}{|\operatorname{grad} Q|} \sum_{k=1}^{N} \frac{\partial Q}{\partial \bar{z}_k}(z)(\bar{z}_k - \bar{w}_k)\, dS(z). \tag{1.16}$$

In terms of the vector field Ξ defined on a neighborhood of bD by

$$\Xi = \frac{1}{|\operatorname{grad} Q|} \sum_{k=1}^{N} \frac{\partial Q}{\partial \bar{z}_k} \frac{\partial}{\partial z_k},$$

the integral (1.16) is

$$F(w) = \frac{-(N-2)!}{\pi^N} \int_{bD} f(z)\, \Xi \frac{1}{|z-w|^{2(N-1)}}\, dS(z). \tag{1.17}$$

This formula goes back to [238]. We are implicitly supposing that $N \geq 2$. The correct version in the plane is the classical formula

$$\frac{1}{2\pi i} \int_{bD} f(\zeta) \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi} \int_{bD} f(\zeta) \left(\frac{\partial}{\partial \nu} + i \frac{\partial}{\partial \tau}\right) \log|\zeta - z|\, ds(\zeta).$$

See [277, p. 110].

The vector field Ξ has a clear geometric meaning. If $z_k = x_{2k-1} + ix_{2k}$, then a calculation shows that

$$\Xi = \frac{1}{4|\operatorname{grad} Q|}(\operatorname{grad} Q + iJ\operatorname{grad} Q)$$

if J denotes the complex structure operator on $\mathbb{C}^N$, i.e., the operator on vector fields determined by $J\frac{\partial}{\partial x_{2k-1}} = \frac{\partial}{\partial x_{2k}}$ and $J\frac{\partial}{\partial x_{2k}} = -\frac{\partial}{\partial x_{2k-1}}$. The vector $\frac{\operatorname{grad} Q}{|\operatorname{grad} Q|}$ is the outer unit normal to bD, and $\frac{J\operatorname{grad} Q}{|\operatorname{grad} Q|}$ is a unit vector that is tangent to bD. The former vector field we denote by $\frac{\partial}{\partial \nu}$, the latter by $\frac{\partial}{\partial \tau}$. Expressed in these terms, the formulas (1.16) and (1.17) are equivalent to

$$F(w) = \frac{-(N-1)!}{4\pi^N} \int_{bD} f(z) \left(\frac{\partial}{\partial \nu} + i\frac{\partial}{\partial \tau}\right) \frac{1}{|z-w|^{2(N-1)}}\, dS(z). \tag{1.18}$$

The integral $\int_{bD} f(z) \frac{\partial}{\partial\nu} \frac{1}{|z-w|^{2(N-1)}}\, dS(z)$ is the *double layer potential* with moment f; such integrals play a role in potential theory. See, e.g., [246].

These formulas suggest a natural way of defining the Bochner–Martinelli integral of a measure: If μ is a finite regular Borel measure on bD, the *Bochner–Martinelli integral of* μ is the harmonic function on $\mathbb{C}^N \setminus bD$ defined by

$$F(w) = \frac{-(N-2)!}{4\pi^N} \int_{bD} \left(\frac{\partial}{\partial\nu} + i \frac{\partial}{\partial\tau} \right) \frac{1}{|w-z|^{2(N-1)}}\, d\mu(z). \tag{1.19}$$

There is the obvious possibility of replacing the measure μ by more general functionals, e.g., by distributions or by analytic functionals, granted suitable smoothness properties for bD.

To conclude this section on integral formulas, we notice that implicit in what we have done above is a proof of an integral representation formula for *smooth* but not necessarily holomorphic functions. For this, fix a bounded domain D in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$, and fix a function f that is defined and of class $\mathscr{C}^2$ on a neighborhood of $\bar{D}$. Let w be a point of D. We have noted above that the form $\frac{\omega'(\overline{w-z})\wedge\omega(z)}{|z-w|^{2N}}$ is d- and $\bar\partial$-closed on $\mathbb{C}^N \setminus \{w\}$. If D_ε denotes the domain obtained from D by excising from D the closed ball of radius ε centered at the point w, then by Stokes's theorem,

$$\begin{aligned} &\int_{bD} f(z) \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}} - \int_{\{z:|z-w|=\varepsilon\}} f(z) \frac{\omega'(\overline{w-z}) \wedge \omega(z)}{|z-w|^{2N}} \\ &\qquad = \int_{D_\varepsilon} \bar\partial f(z) \wedge \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}}. \end{aligned} \tag{1.20}$$

Passing to the limit as $\varepsilon \to 0^+$ yields the formula

$$f(w) = c_N \left(\int_{bD} f(z) \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}} - \int_D \bar\partial f(z) \wedge \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}} \right). \tag{1.21}$$

In the planar case, $N = 1$, this formula is habitually referred to as the *generalized Cauchy integral formula*.

1.5. The Oka–Weil Theorem

The approximation theorem of Runge states that if K is a compact subset of the complex plane, then every function holomorphic on a neighborhood of K can be approximated uniformly on K by rational functions with poles off K. When K is polynomially convex, the rational approximating functions can be chosen to be polynomials.

A natural N-dimensional analogue of Runge's theorem is the *Oka–Weil theorem*, which is formulated as follows.

Theorem 1.5.1. *If the compact set K in $\mathbb{C}^N$ is polynomially convex and if f is a function holomorphic on a neighborhood of K, then given $\varepsilon > 0$ there is a polynomial P with $\|f - P\|_K < \varepsilon$.*

In classical function theory, Runge's approximation theorem is often derived by examining the Riemann sums for certain Cauchy integrals. An analogous process based on the Cauchy–Fantappiè integral can be used to obtain the Oka–Weil approximation theorem in $\mathbb{C}^N$.

The integral formula we need for this purpose is one valid on convex domains in $\mathbb{C}^N$.

Fix a convex function Q of class $\mathscr{C}^2$ on $\mathbb{C}^N$. Thus, given holomorphic coordinates $z_1, \dots, z_N$ on $\mathbb{C}^N$ with $z_j = x_{2j-1} + ix_{2j}$, the real Hessian matrix

$$H_Q = \left(\frac{\partial^2 Q}{\partial x_j \partial x_k}\right)_{1\le j,k\le 2N}$$

is nonnegative: $\sum_{j,k=1}^N \frac{\partial^2 Q}{\partial x_j \partial x_k}(z) y_j y_k \ge 0$ for all $z \in \mathbb{C}^N$ and all $y \in \mathbb{R}^{2N}$. (For the theory of convex functions, see [182, 293].)

Let $D = \{z \in \mathbb{C}^N : Q(z) < 0\}$, so that D is an open convex subset of $\mathbb{C}^N$, which we assume to be nonempty. Then dQ is zero-free on bD. To see this, assume Q to take its minimum at the point $z_0 \in D$, and consider a point $w_0 \in bD$. The function $h : (0, \infty) \to \mathbb{R}$ given by $h(t) = Q(z_0 + t(w_0 - z_0))$ is convex and satisfies $h(0) = Q(z_0) < 0 = Q(w_0) = h(1)$. Thus, h' is not identically zero in $(0, 1)$. Because h is convex and of class $\mathscr{C}^2$, $h'' \ge 0$, so h' is nondecreasing on $(0, 1)$, whence $h'(1) \ne 0$. This implies that $dQ(w_0) \ne 0$.

Introduce the complex gradient $\nabla_{\mathbb{C}} Q : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\nabla_{\mathbb{C}} Q = \left(\frac{\partial Q}{\partial z_1}, \dots, \frac{\partial Q}{\partial z_N}\right).$$

This is a map of class $\mathscr{C}^1$ with the property that if $w \in D$ and $z \in bD$, then

$$\nabla_{\mathbb{C}} Q(z) \cdot (z - w) = \sum_{j=1}^N \frac{\partial Q}{\partial z_j}(z)(z_j - w_j) \ne 0,$$

for if $z_j = x_{2j-1} + ix_{2j}$ as above, and $w_j = y_{2j-1} + iy_{2j}$, then

$$\Re\big(\nabla_{\mathbb{C}} Q(z) \cdot (z - w)\big) = \frac{1}{2} \sum_{j=1}^{2N} \frac{\partial Q}{\partial x_j}(z)(x_j - y_j).$$

Thus the equation of the real tangent plane to bD at z is $\Re\big(\nabla_{\mathbb{C}} Q(z) \cdot (z - w)\big) = 0$; by convexity, this plane does not pass through D.

If Q were of class $\mathscr{C}^3$ rather than $\mathscr{C}^2$, the general Cauchy–Fantappiè integral formula would yield that for $F \in A(D)$, $w \in D$,

$$F(w) = c_N \int_{bD} F(z) \frac{\omega'(\nabla_{\mathbb{C}} Q(z)) \wedge \omega(z)}{[\nabla_{\mathbb{C}} Q(z) \cdot (z - w)]^N}. \tag{1.22}$$

This formula *is* correct with Q only of class $\mathscr{C}^2$. Given that Q is of class $\mathscr{C}^2$, then on a large ball in $\mathbb{C}^N$, Q can be approximated in the $\mathscr{C}^2$ sense by functions $\tilde{Q}$ of class

$\mathscr{C}^\infty$. The formula is correct if $\tilde{Q}$ replaces Q in it. But because the formula involves only second-order derivatives of Q, and $\tilde{Q}$ approximates Q in the $\mathscr{C}^2$ sense, the formula is correct with Q.

The formula (1.22) is implicit in the paper of Leray [221]; see also [266]. The formula was derived ab initio by Aĭzenberg [2] using Stokes's theorem, though Aĭzenberg gave the details only in $\mathbb{C}^2$. An explicit derivation of it from Leray's work is given in [346].

It is of interest to note, following [2], that the formula (1.22) does not require the full hypothesis of convexity; it is sufficient for D to be *lineally convex* in the sense that at each point $z \in bD$, the maximal complex subspace of the tangent space $T_z(bD)$ is disjoint from D. Analytically, this is simply the condition that if G is a defining function for bD, then for no $z \in bD$ does $\nabla_{\mathbb{C}} G(z) \cdot (z - w) = 0$ for some $w \in D$. This geometric property is weaker than convexity.

The form $\omega'(\nabla_{\mathbb{C}} Q) \wedge \omega(z)$ that appears in formula (1.22) admits an expression in terms of Q and its derivatives as follows.

Lemma 1.5.2. *The forms* $\partial Q \wedge (\bar{\partial}\partial Q)^{N-1}$ *and*

$$(-1)^{\frac{1}{2}N(N-1)}(N-1)!\omega'(\nabla_{\mathbb{C}} Q) \wedge \omega(z)$$

coincide.

Proof. The proof is a direct calculation: If $Q_j = \frac{\partial Q}{\partial z_j}$, then

$$\begin{aligned}\partial Q \wedge (\bar{\partial}\partial Q)^{N-1} &= \left(\sum_{j=1}^{N} Q_j dz_j\right) \wedge (N-1)! \sum_{j-1}^{N} \bar{\partial} Q_1 \wedge dz_1 \\ &\qquad \wedge \cdots \wedge [j] \cdots \wedge \bar{\partial} Q_N \wedge dz_N \\ &= (N-1)!(-1)^{\frac{1}{2}(N-1)(N-2)} \sum_{j=1}^{N} Q_j dz_j \wedge \bar{\partial} Q_1 \\ &\qquad \wedge \cdots \wedge [j] \wedge \cdots \wedge \bar{\partial} Q_N \wedge dz_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge dz_N \\ &= (N-1)!(-1)^{\frac{1}{2}N(N-1)} \omega'(\nabla_{\mathbb{C}} Q) \wedge \omega(z).\end{aligned}$$

The lemma is proved.

The constant c_N is $\frac{(-1)^{\frac{1}{2}N(N-1)}(N-1)!}{(2\pi i)^N}$, so the Cauchy–Fantappiè formula (1.22) can be written as

$$F(w) = \frac{(-1)^{N-1}}{(2\pi i)^N} \int_{bD} F(z) \frac{\partial Q \wedge (\partial\bar{\partial} Q)^{N-1}}{[\nabla_{\mathbb{C}} Q(z) \cdot (z - w)]^N}. \tag{1.23}$$

The integral formula we have for convex domains assumes a particularly simple form in the special case of the ball $\mathbb{B}_N$. The ball admits the strictly convex defining function Q given by $Q(z) = |z|^2 - 1$.

With this choice of Q and with $\langle w, z\rangle$ again the Hermitian inner product on $\mathbb{C}^N$, equation (1.23) and Lemma 1.4.8 yield the formula

$$F(w) = \frac{(N-1)!}{2\pi^N} \int_{\mathbb{S}^{2N-1}} \frac{F(z)dS(z)}{(1 - \langle w, z\rangle)^N} \tag{1.24}$$

for $F \in A(\mathbb{B}_N)$ and $w \in \mathbb{B}_N$. This formula is often referred to as the *Cauchy integral formula for the ball* or, alternatively, as the *Szegő integral representation* for the ball.

The formula (1.24) seems to have appeared in function theory surprisingly late. It does not appear in the classical book of Osgood or that of Bochner and Martin on the theory of functions of several complex variables. It is contained in some work of Hua from the 1940s and 1950s. In this connection, see Hua's book [184]. A direct derivation of the formula (1.24), which depends on the theory of Hilbert spaces with kernel functions, was given by Bungart [76]. Neither Bungart's derivation of the formula nor that of Hua exhibits it as an instance of the Cauchy–Fantappiè formula.

We now take up the proof of the Oka–Weil theorem. The proof depends on a division lemma.

Lemma 1.5.3. *If P is a polynomial in N complex variables, there exist polynomials $p_1, \dots, p_N$ in $2N$ complex variables such that for all $z, w \in \mathbb{C}^N$,*

$$P(z) - P(w) = \sum_{r=1}^{N} (z_r - w_r) p_r(z, w).$$

Proof. One can give purely algebraic proofs for this. Alternatively, it is an immediate consequence of the fundamental theorem of calculus:

$$P(z) - P(w) = \int_0^1 \frac{\partial}{\partial t} P\,(w + t(z - w))\,dt = \sum_{r=1}^{N} (z_r - w_r) \int_0^1 P_r\,(w + t(z - w))\,dt,$$

wherein we use the notation P_r to denote the derivative of P with respect to the rth variable. The integrals here are polynomials in z and w.

Notice that this simple analytic process yields a corresponding decomposition for arbitrary functions holomorphic on a convex domain in $\mathbb{C}^N$.

Proof of the Oka–Weil theorem. Let the function f is holomorphic on the bounded neighborhood W of K. Given ε, we are to find a polynomial P with $\|f - P\|_K < \varepsilon$. The assumption that K is polynomially convex implies the existence of a finite number of polynomials, say $f_1, \dots, f_q$, such that for each j, $|f_j| < \frac{1}{2}$ on K but $\max_{1 \le j \le q} |f_j(z)| > 1$ for each $z \in bW$. (The boundary bW is compact, for D is bounded.) Assume that among the f_j there are enough functions to guarantee that if $F(z) = \big(f_1(z), \dots, f_q(z)\big)$, then F is one-to-one and regular on $\mathbb{C}^N$. The unit polydisk $\mathbb{U}^q$ is convex, so there is a smoothly bounded convex domain D with $F(K) \subset D \subset \mathbb{U}^q$. (For D we can take the domain $\{\zeta \in \mathbb{C}^q : |\zeta_1|^{2l} + \cdots + |\zeta_q|^{2l} < c\}$ for suitably large positive integral l and for c suitably near, but less than, one.) We can assume in addition that bD meets the manifold $F(W)$ transversally. For some smooth strictly convex function Q defined on $\mathbb{C}^q$, $D = \{\zeta \in \mathbb{C}^q : Q(\zeta) < 0\}$ and $dQ \neq 0$ on bD. If $\zeta \in bD$, the quantity $\sum_{j=1}^q Q_j(\zeta)(\zeta_j - \eta_j)$ vanishes for no point η in D. (We use Q_j to denote $\frac{\partial Q}{\partial \zeta_j}$.)

The function f_j is a polynomial, so there are polynomials f_{jk} such that for $z, w \in \mathbb{C}^N$,

$$f_i(z) - f_j(w) = \sum_{k=1}^N (z_k - w_k) f_{jk}(z, w).$$

Then

$$\begin{aligned}\sum_{j=1}^q Q_j\,(F(z))\big(f_j(z) - f_j(w)\big) &= \sum_{k=1}^N (z_k - w_k) \sum_{j=1}^q f_{jk}(z, w) Q_j\,(F(z)) \\ &= \sum_{k=1}^N (z_k - w_k) \varphi_k(z, w)\end{aligned}$$

if $\varphi_k(z, w) = \sum_{j=1}^q f_{jk}(z, w) Q_j\,(F(z))$.

For a fixed $w \in \{w \in W : Q\,(F(w)) < 0\}$, the Cauchy–Fantappiè integral formula gives

$$f(w) = c_N \int_{b\{z \in W : F(z) \in D\}} f(z) \frac{\omega'(\varphi) \wedge \omega(z)}{\left(\sum\limits_{k=1}^N (z_k - w_k)\varphi_k(z, w)\right)^N} \tag{1.25}$$

if $\omega'(\varphi)$ is the form

$$\sum_{k=1}^N (-1)^{k-1} \varphi_k(z, w) \bar{\partial}_z \varphi_1(z, w) \wedge \cdots \wedge [k] \wedge \cdots \wedge \bar{\partial}_z \varphi_N(z, w),$$

which has coefficients that depend polynomially on w. Taking Riemann sums in the integral, (1.25) yields that on K, f admits uniform approximation by functions of the form

$$\sum_{r=1}^{r_0} f(z^{(r)}) h_r(w) \left[\sum_{k=1}^N (z_k^{(r)} - w_k) \varphi_k(z^{(r)}, w)\right]^{-N},$$

where h_r is a polynomial and $z^{(1)}, \ldots, z^{(r_0)}$ are some points in $b\{z \in W : F(z) \in D\}$.

What has to be seen then is that a function g of the form

$$g(w) = \left(\sum_{k=1}^N (z_k^{(r)} - w_k) \varphi_k(z^{(r)}, w)\right)^{-1}$$

admits polynomial approximation on K. The map F is polynomial, so it is sufficient to see that the function g_ζ given for fixed ζ with $Q(\zeta) = 0$ by

$$g_\zeta(\eta) = \left(\sum_{j=1}^{q} Q_{\bar{j}}(\zeta)(\zeta_j - \eta_j)\right)^{-1}$$

can be approximated uniformly on compacta in D by polynomials. However, this is clear: If E is such a compact set, there is a ball B that contains E and on which g_ζ is holomorphic because g_ζ is holomorphic off a certain affine hyperplane in $\mathbb{C}^q$ that is disjoint from D. We are done, for g_ζ admits a power series expansion in B.

This theorem was proved by Weil [361] and Oka [271]. The present approach using Cauchy–Fantappiè integrals has been noticed by several mathematicians. See [287].

It is worth dwelling on two of the ingredients of the proof of the Oka–Weil theorem. One is the possibility of division provided by Lemma 1.5.3.

This kind of division is possible much more generally: If $D \subset \mathbb{C}^N$ is a domain of holomorphy, then for each function f holomorphic on D, there exist functions g_j holomorphic on $D \times D$ such that

$$f(z) - f(w) = \sum_{j=1}^{N} g_j(z, w)(z_j - w_j). \tag{1.26}$$

For this kind of division to hold, it is not necessary that D be a domain of holomorphy. A theorem of Ortega [274] shows that *for the decomposition (1.26) to be possible for all functions f, it is necessary and sufficient that the envelope of holomorphy of D be one-sheeted.*

The second main ingredient of the proof of the Oka–Weil theorem is the hypothesis of convexity: The compact set on which approximation is to occur needs to be convex with respect to the proposed algebra of approximating functions. This is a situation that often arises.

The general principle is that when the conditions of convexity and division are satisfied, the mechanism of the proof given above for the Oka–Weil theorem can be invoked to yield an approximation theorem.

A particular case of this situation arises in connection with rational convexity. What emerges is that *if K is a compact rationally convex subset of $\mathbb{C}^N$, then every function f that is holomorphic on a neighborhood of K can be approximated uniformly on K by rational functions with no poles on K.*

The proof of this assertion runs as follows. Let K be a rationally convex compact subset of $\mathbb{C}^N$, which we suppose to be a subset of the polydisk $\mathbb{U}^N$. Let F be holomorphic on the bounded neighborhood W of K. By compactness and rational convexity, there exist finitely many polynomials $p_1, \dots, p_r$ such that for each $x \in bW$, $\min_j |p_j(x)| < \frac{1}{2}$ and for each $x \in K$, $\min_j |p(x)| > 2$. Let $q = N + r$, and let F be the rational map from $\mathbb{C}^N$ to $\mathbb{C}^q$ given by $F(z) = (z_1, \dots, z_N, \frac{1}{p_1(z)}, \dots, \frac{1}{p_r(z)})$. The map F is holomorphic on the open set $W_o = W \setminus \{z : p_1(z) \cdots p_r(z) = 0\}$. If $x \in bW_o$, then $\liminf_{z \to x, z \in W_o} \|f(z)\| \geq 2$. The map F carries a neighborhood of K biholomorphically onto a closed complex submanifold $\mathscr{M}$ of the polydisk $\mathbb{U}^q$. Choose a strictly convex domain D with bD smooth and with $F(K) \subset D \subset \mathbb{U}^q$ that has the property that $\mathscr{M}$ is transversal to bD. We can write

$$\frac{1}{p_j(z)} - \frac{1}{p_j(w)} = \frac{p_j(w) - p_j(z)}{p_j(z)p_j(w)}$$

and then decompose the numerator as

$$p_j(w) - p_j(z) = \sum_{k=1}^{N}(z_k - w_k)p_{j,k}(z, w)$$

with polynomials $p_{j,k}$ to obtain a decomposition

$$\frac{1}{p_j(z)} - \frac{1}{p_j(w)} = \sum_{k=1}^{N}(z_k - w_k)f_{jk}(z, w)$$

with f_{jk} the rational function $p_{jk}(z, w)/p_j(z)p_j(w)$. If we denote by Q a strictly convex defining function for the domain D, then our situation is precisely parallel to that of the proof of the Oka–Weil theorem given above. We can execute a parallel argument using Riemann sums to approximate the appropriate Cauchy–Fantappiè integral and discover that the function f can be approximated uniformly on K by rational functions.

The Oka–Weil theorem will be used repeatedly in the rest of this work. As a first application, we deduce a geometric fact about polynomially convex sets.

Corollary 1.5.4. *If the compact polynomially convex set X in $\mathbb{C}^N$ is of the form $X = X' \cup X''$ with X' and X'' disjoint and compact, then each of X' and X'' is polynomially convex.*

Proof. The sets X' and X'' are both compact, so there are disjoint open sets V' and V'' with $V' \supset X'$ and $V'' \supset X''$. If $V = V' \cup V''$, and if $f = 1$ on V', $f = 0$ on V'', then f is holomorphic on V. From the Oka–Weil theorem, it follows that there is a polynomial P such that $\|P - f\|_X < \frac{1}{4}$. Consequently, the polynomially convex hull $\widehat{X''}$ is disjoint from X'. From the inclusion $\widehat{X''} \subset X$, the set X'' is seen to be polynomially convex. In a similar way, X' is polynomially convex.

Another result in the same spirit is this:

Corollary 1.5.5. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, then every component of X is also polynomially convex.*

By a *component* of X we understand a maximal connected subset of X.

Proof. This observation is a consequence of the preceding corollary, of the result that an arbitrary intersection of polynomially convex subsets is polynomially convex, and the elementary topological fact [252, p. 17] or [261, p. 107], that in a locally compact Hausdorff space, X, each component X_o of X is the intersection of all the open and closed subsets of X that contain it.

A related result is the following.

Corollary 1.5.6. [24] *If X is a compact subset of $\mathbb{C}^N$ and E is a component of $\widehat{X}$, then $E = \widehat{E \cap X}$.*

Proof. We can write $E = \cap_{j=1,\ldots} K_j$, where each K_j is an open and closed subset of $\widehat{X}$, for E is a component of X. As noted above, each of the sets K_j is polynomially convex. By the Oka–Weil theorem, for every $\varepsilon > 0$, there is a polynomial P such that $|P - 1| < \varepsilon$ on K_j and $|P| < \varepsilon$ on $\widehat{X} \setminus K_j$. If μ is a Jensen measure supported in X for a point $s \in K_j$, then μ must be concentrated on $K_j \cap X$. It follows that $K_j \subset \widehat{K_j \cap X}$, so the two sets must be equal, for K_j is polynomially convex. Consequently, the set E is polynomially convex.

It is shown in Section 1.2 that if f is an element of the uniform algebra A and if φ is a function holomorphic on a neighborhood of the spectrum $\sigma_A(f)$, then $\varphi \circ f \in A$. The Oka–Weil theorem implies an extension of this result:

Theorem 1.5.7. *If A is a uniform algebra, if $f_1, \ldots, f_r \in A$, and if φ is holomorphic on a neighborhood of the joint spectrum $\sigma_A(f_1, \ldots, f_r)$, then the function g defined by $g = \varphi(f_1, \ldots, f_r)$ is in A.*

Proof. If the functions $f_1, \ldots f_r$ generate A, then $\sigma_A(f_1, \ldots, f_r)$ is polynomially convex. The Oka–Weil theorem provides a sequence $\{P_j\}_{j=1,\ldots}$ of polynomials with $P_j \to \varphi$ uniformly on $\sigma_A(f_1, \ldots, f_r)$. The functions $P_j(f_1, \ldots, f_r)$ lie in A, so the same is true of $\varphi(f_1, \ldots, f_r)$.

In case the functions $f_1, \ldots, f_r$ do not generate A but A is finitely generated, let $f_1, \ldots, f_r, f_{r+1}, \ldots, f_s$ generate A, and let $\pi : \mathbb{C}^s \to \mathbb{C}^r$ be the projection. The function $\varphi \circ \pi$ is holomorphic on a neighborhood of $\sigma_A(f_1, \ldots, f_s)$, so by the last paragraph, the function $\varphi \circ \pi(f_1, \ldots, f_s) = \varphi(f_1, \ldots, f_r)$ is in A.

The case that the algebra A is not finitely generated requires a more involved argument based on a lemma of Arens and Calderón [39]:

Lemma 1.5.8. *Let A be a uniform algebra, let $f_1, \ldots, f_r \in A$, and let Ω be an open set in $\mathbb{C}^r$ that contains $\sigma_A(f_1, \ldots, f_r)$. There exist $f_{r+1}, \ldots, f_s \in A$ such that if $\pi : \mathbb{C}^s \to \mathbb{C}^r$ is the projection, then $\pi(\sigma_A(f_1, \ldots, f_s))\widehat{\ } \subset \Omega$.*

Proof. The set $\widehat{\sigma}_A(f_1, \ldots, f_r) \setminus \Omega$ is compact. Let $z^o \in \widehat{\sigma}_A(f_1, \ldots, f_r) \setminus \Omega$. The functions $\widehat{f_1} - z_1^o, \ldots, \widehat{f_r} - z_r^o$ have no common zero on the spectrum of A, so the ideal generated by them is all of A. In particular, there are functions $g_1, \ldots, g_r$ in A such that

$$1 = (f_1 - z_1^o)g_1 + \cdots + (f_r - z_r^0)g_r.$$

This equation continues to hold in the subalgebra B of A generated by $f_1, \ldots, f_r, g_1, \ldots, g_r$. The joint spectrum $\sigma_B(f_1, \ldots, f_r, g_1, \ldots, g_r)$ is polynomially convex. If $\eta : \mathbb{C}^{2r} \to \mathbb{C}^r$ is the projection onto the first r coordinates, then $\eta(\sigma_B(f_1, \ldots, f_r, g_1, \ldots, g_r))$ does not contain the point z^o and so omits a neighborhood of it. Thus, by compactness, we can find a finite number of functions $f_{r+1} \ldots, f_s$ in A such that if C is the subalgebra of A generated by $f_1 \ldots, f_s$, then the projection $\pi : \mathbb{C}^s \to \mathbb{C}^r$ onto the first r coordinates takes the polynomially convex set $\widehat{\sigma}_C(f_1, \ldots, f_s)$ into Ω. Because $\widehat{\sigma}_C(f_1, \ldots, f_s) \supset \sigma_A(f_1, \ldots, f_s)$, the lemma is proved.

Proof of Theorem 1.5.7 concluded. Let Ω be an open set in $\mathbb{C}^r$ that contains the joint spectrum $\sigma_A(f_1, \ldots, f_r)$, and let $\varphi \in \mathscr{O}(\Omega)$. Choose $f_{r+1}, \ldots, f_s$ as in the lemma, so that $\widehat{\sigma}_A(f_1, \ldots, f_s)$ projects under π into Ω. The function $\varphi \circ \pi$ is holomorphic on $\pi^{-1}(\Omega)$, and so can be approximated uniformly on $\widehat{\sigma}_A(f_1, \ldots, f_s)$ by polynomials. Then as above, it follows that the function $\varphi(f_1, \ldots, f_r)$ is in A. The theorem is proved.

It should be noted that the general result proved by Arens and Calderón contains Theorem 1.5.7 as a special case but is more general in that it applies to arbitrary commutative Banach algebras, not only uniform algebras. There is a yet more general result, the so-called *holomorphic functional calculus*, which provides homomorphisms from algebras of holomorphic functions into commutative Banach algebras. This is developed in detail in the books [69] and [345].

One consequence of Theorem 1.5.7 that we shall need in the sequel is a version of the Shilov idempotent theorem:

Theorem 1.5.9. *If A is a uniform algebra with spectrum X, and if there is a decomposition $X = X' \cup X''$ of X into the union of two mutually disjoint closed (and therefore open) subsets, then there is an element $f \in A$ with $f = 1$ on X' and $f = 0$ on X''.*

This depends on a simple lemma:

Lemma 1.5.10. *Let Y be a compact space and A an algebra of continuous functions on Y that contains 1 and that separates points on Y. If Y' and Y'' are disjoint closed subsets of Y, then there is a* finite *subset $\mathscr{F} = \{h_1, \ldots, h_r\}$ of A such that the map $F : Y \to \mathbb{C}^r$ given by $F(y) = (h_1(y), \ldots, h_r(y))$ satisfies $F(Y') \cap F(Y'') = \emptyset$.*

Proof. If $y' \in Y'$ and $y'' \in Y''$, there is $h_{y',y''} \in A$ such that $h_{y',y''}(y') = 0$ and $h_{y',y''}(y'') = 2$. By compactness, there is a finite collection $\mathscr{F}_{y'}$ in A such that each $h \in \mathscr{F}_{y'}$ satisfies $|h| < \frac{1}{2}$ on a neighborhood of y' and such that for each $y'' \in Y''$, $\max\{|h(y'')| : h \in \mathscr{F}_{y'}\} > \frac{3}{2}$. For the family $\mathscr{F}$ of the theorem we can take the union of a finite number of the families $\mathscr{F}_{y'}$ for $y' \in Y'$.

Proof of Theorem 1.5.9. Apply the lemma to the algebra A: There are finitely many elements, $f_1, \ldots, f_r$ of A such that if $F : X \to \mathbb{C}^r$ is the map with coordinates $f_1, \ldots, f_r$, then $F(X') \cap F(X'') = \emptyset$. The set $F(X)$ is the joint spectrum $\sigma_A(f_1, \ldots, f_r)$, and if we define g to be 1 on a neighborhood of $F(X')$ and 0 on a neighborhood of $F(X'')$, then g is holomorphic on a neighborhood of $F(X)$, so by Theorem 1.5.7, the composition $g \circ F$ lies in A.

The theorem is proved.

1.6. Some Examples

We turn now to a few examples that will serve to exhibit some of the complexities involved in the theory of polynomially convex sets.

1.6.1. Metric Conditions

A natural first question is whether there are purely *metric* conditions that are sufficient to guarantee polynomial convexity. The question is natural, but there is no major result in this direction.

The obvious metric conditions to impose are in terms of the Hausdorff measures induced on $\mathbb{C}^N$ by the Euclidean metric. For each real number $p \geq 0$, introduce the constant $\gamma_p = \frac{2^p\Gamma(p/2+1)}{\pi^{p/2}}$.

Definition 1.6.1. *For a positive real number p, the p-*dimensional Hausdorff measure *of the subset E of* $\mathbb{R}^N$ *is the number* $\Lambda^p(E)$ *defined by* $\Lambda^p(E) = \lim_{\varepsilon\to 0^+} \Lambda^p_\varepsilon(E)$, *where* $\Lambda^p_\varepsilon(E) = \inf\left\{\gamma_p \sum_{j=1,\dots} d_j^p\right\}$, *in which the infimum is extended over all countable collections* $\{B_1, B_2, \dots\}$ *of sets* B_j *of diameter* d_j *that cover E and for which* d_j *is no more than* ε. *The* 0*-dimensional Hausdorff measure of a set is understood to be its cardinality if this cardinality is finite,* ∞ *otherwise.*

The set function Λ^p is an outer measure on $\mathbb{R}^N$ with respect to which all Borel sets are measurable.

The constant γ_p is introduced so that on sets E in $\mathbb{R}^N$, the N-dimensional Hausdorff measure agrees with Lebesgue measure. More generally, for a smooth p-dimensional submanifold $\mathscr{M}$ of $\mathbb{R}^N$, the p-dimensional measure is the usual volume computed with respect to the Riemannian metric on $\mathscr{M}$ induced from the Euclidean metric on $\mathbb{R}^N$. For a rectifiable curve in $\mathbb{R}^N$, the length coincides with the one-dimensional measure.

If in the definition of the Hausdorff measures Λ^p the sets B_j are required to be *balls* instead of being allowed to be arbitrary sets, the resulting construction yields the *spherical measure* $\mathscr{S}^p$. For each p there is the inequality $\Lambda^p \leq \mathscr{S}^p \leq 2^p\Lambda^p$.

We shall need few of the properties of the Hausdorff measures. It will be useful to notice, though, that if $h : \mathbb{R}^N \to \mathbb{R}^M$ satisfies the Lipschitz condition $|h(x) - h(x')| \leq C|x - x'|$, then $\Lambda^p(h(S)) \leq C^p\Lambda^p(S)$ for each $S \subset \mathbb{R}^N$. This is immediate from the definition. In particular, Lipschitz maps carry sets of zero p-dimensional measure to sets of zero p-dimensional measure.

A set E with $\Lambda^1(E) = 0$ is said to be a set of *zero length*.

An introduction to the theory of Hausdorff measures is in [337], a more extensive introduction is given in [114], and a comprehensive treatment can be found in the treatise [115] of Federer.

The first fact we establish is very simple:

Theorem 1.6.2. *A compact subset X of* $\mathbb{C}^N$ *with* $\Lambda^1(X) = 0$ *is polynomially convex and satisfies* $\mathscr{P}(X) = \mathscr{C}(X)$.

Proof. To prove this, it suffices, because of the Stone–Weierstrass theorem, to show that $\mathscr{P}(X)$ contains enough real-valued functions to separate the points of X.

Let x, y be distinct points of X. There is a linear functional ψ on $\mathbb{C}^N$ that satisfies $\psi(x) \neq \psi(y)$. The set X has zero length, so the image $\psi(X)$ also has zero length. It is compact and totally disconnected, i.e., its only connected subsets are points, so there exist disjoint open subsets of the plane, V_x and V_y, such that $\psi(x) \in V_x$ and $\psi(y) \in V_y$ and such that $V_x \cup V_y \supset \psi(X)$. Because $\psi(X)$ does not separate the plane, Runge's theorem provides a polynomial P such that $|P|$ is small on $V_x \cap \psi(X)$ and $|P-1|$ is small on $V_y \cap \psi(X)$. Consequently, $\mathscr{P}(X)$ contains a real-valued function that is zero at x and one at y. The result follows.

Plainly, a set E with $\Lambda^1(E)$ nonzero but finite need not be polynomially convex, as is shown by every smooth simple closed curve in the complex plane.

Corollary 1.6.3. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, and if E is a subset of $\mathbb{C}^N$ with zero length such that the set $Y = X \cup E$ is compact, then Y is polynomially convex, and $\mathscr{P}(Y) = \{f \in \mathscr{C}(Y) : f|X \in \mathscr{P}(X)\}$.*

It is not required that the set $\bar{E}$ have zero length; it could have positive $2N$-dimensional measure.

Proof. Suppose Y not to be polynomially convex and let $z_o \in \widehat{Y} \setminus Y$. Thus $z_o \notin X$. Therefore there is a polynomial P such that $\Re P < -1$ on X and $\Re P(z_o) > 1$. We have $P(\widehat{Y}) \subset P(Y)\widehat{\ }$. The set E has length zero and the polynomial P is locally a Lipschitz map from $\mathbb{C}^N$ to $\mathbb{C}$, so the set $P(E)$ also has zero length. Consequently, there is an $x_o \in (0, 1)$ such that the line $L_{x_o} = \{x_o + iy : y \in \mathbb{R}\}$ is disjoint from the set $P(Y)$ and so disjoint from $P(Y)\widehat{\ }$. Let S^+ and S^- be the parts of $P(Y)\widehat{\ }$ to the right and to the left, respectively, of the line L_{x_o}. Then $\widehat{Y} = [\widehat{Y} \cap P^{-1}(S^+)] \cup [\widehat{Y} \cap P^{-1}(S^-)] = T^+ \cup T^-$. The sets T^+ and T^- are both open and closed subsets of $\widehat{Y}$, and so each is polynomially convex. We have $z_o \in T^+$. The polynomial convexity of $E \cap T^+$ implies that $z_o \in E$, which contradicts the hypothesis that $z_o \notin Y$. The set Y is thus polynomially convex, as claimed.

It remains to show that $\mathscr{P}(X \cup E) = \{f \in \mathscr{C}(X \cup E) : f|X \in \mathscr{P}(X)\}$. Suppose, to this end, that μ is a finite regular Borel measure on Y that is orthogonal to $\mathscr{P}(Y)$. If $z_o \in Y \setminus X$, then there is a polynomial P such that $P(z_o) = 1$ and $\Re P < -1$ on X. It follows that if U is an open and closed subset of $P(X \cap E) \cap \{\Re P > -1\}$ that contains the point 1, then $\mathscr{P}(P(Y))$ contains a function g such that $g = 1$ on U, $g = 0$ on $P(Y) \setminus U$. Then for all positive integers m we have $0 = \int [g \circ P]^m \, d\mu$. If we let $m \to \infty$, then, by Lebesgue's dominated convergence theorem, $\mu(P^{-1}(U)) = 0$. This is true for all neighborhoods U of $P(z_o) = 1$ and for all polynomials of the kind considered, so the measure μ is carried by the set X. Consequently, for every measure $\mu \in \mathscr{P}(Y)^\perp$, $\int g \, d\mu = 0$ if $g \in \mathscr{C}(Y)$ satisfies $g|X \in \mathscr{P}(X)$.

The corollary is proved.

There are results for rational convexity analogous to the preceding ones about polynomial convexity. Before giving them, it is convenient to recall certain results about the algebra $\mathscr{R}(X)$ for compact subsets X of the plane. First we need an approximation theorem in the plane, the *Hartogs–Rosenthal theorem* [163].

Theorem 1.6.4. *If X is a compact subset of the plane with measure zero, then $\mathscr{R}(X) = \mathscr{C}(X)$.*

Proof. If for the compact planar set X of Lebesgue measure zero, $\mathscr{R}(X)$ differs from $\mathscr{C}(X)$, then the Hahn–Banach and Riesz representation theorems yield a nonzero finite regular Borel measure μ on X such that $\int_X r(\zeta) \, d\mu(\zeta) = 0$ for every rational function r with no poles on X. In particular, the Cauchy transform, $\tilde{\mu}$, of μ defined by

$$\tilde{\mu}(z) = \int_X \frac{d\mu(\zeta)}{\zeta - z}$$

vanishes when $z \in \mathbb{C} \setminus X$.

The theorem is thus a consequence of the following lemma.

Lemma 1.6.5. *If ν is a finite regular Borel measure on $\mathbb{C}$, then the Cauchy transform of ν is locally integrable on $\mathbb{C}$. If it vanishes almost everywhere, then ν is the zero measure.*

Proof. That $\tilde{\nu}$ is locally integrable is a consequence of Fubini's theorem: If Δ is a disk of finite radius in the plane, then

$$\int_\Delta |\tilde{\nu}(z)|\, d\mathscr{L}(z) \leq \int_X \int_\Delta \frac{d\mathscr{L}(z)}{|\zeta - z|}\, d|\nu|(\zeta),$$

so that the local integrability of $\tilde{\nu}$ follows from the local integrability of $1/z$.

That ν is the zero measure when $\tilde{\nu} = 0$ almost everywhere can be shown by proving that for every compactly supported smooth function g on the plane, $\int g\, d\nu = 0$. This is so, for the function g can be expressed by the generalized Cauchy integral formula as

$$g(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial g}{\partial \bar{\zeta}}(\zeta) \frac{d\mathscr{L}(\zeta)}{\zeta - z}, \tag{1.27}$$

whence

$$\int g(z)\, d\nu(z) = -\frac{1}{\pi} \int \frac{\partial g}{\partial \bar{\zeta}}(\zeta) \tilde{\nu}(\zeta)\, d\mathscr{L}(\zeta). \tag{1.28}$$

The function $\tilde{\nu}$ vanishes almost everywhere, so this quantity is zero. The proof is complete.

Another proof of this theorem is contained in Lemma 5.1.3.

Two important facts emerge from the ideas of the proof just given.

Corollary 1.6.6. *If ν is a finite measure on the plane and E is a compact set in the plane, then $\tilde{\nu} = 0$ a.e. $[d\mathscr{L}]$ off E implies that* $\operatorname{supp} \nu \subset E$*, and moreover, $\tilde{\nu} = 0$ a.e. $[d\mathscr{L}]$ off E if and only if ν annihilates the algebra $\mathscr{R}(E)$.*

Proof. Suppose that $\tilde{\nu}$ vanishes off the set E. Apply the formula (1.28) to a smooth function on $\mathbb{C}$ that has compact support and that vanishes on a neighborhood of E. The conclusion is that $\int g\, d\nu = 0$ for all such functions g, whence the support of ν is contained in E.

If ν is orthogonal to $\mathscr{R}(E)$, then $\tilde{\nu}$ vanishes at every point of $\mathbb{C} \setminus E$. Conversely, if $\tilde{\nu}$ vanishes a.e. $[d\mathscr{L}]$ off E, then ν is supported by E, as follows from the preceding paragraph. If f is a rational function with no poles on X, write, for $z \in E$,

$$f(z) = \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for a suitable domain D that contains E and on whose closure f is holomorphic. Then

$$\int f\,d\nu = \frac{-1}{2\pi i}\int_{bD} f(\zeta)\tilde{\nu}(\zeta)\,d\zeta = 0,$$

so ν is orthogonal to $\mathscr{R}(X)$.

We will need a few additional facts about the algebra $\mathscr{R}(X)$ for a compact subset X of the plane. We shall not go deeply into this subject; much more information about these algebras can be found in the books [378], [136], and [345].

We need to determine the Cauchy transform of a product $\varphi\mu$ for a compactly supported smooth function φ on $\mathbb{C}$. The answer is given by

$$\widetilde{\varphi\mu} = \varphi\tilde{\mu} - \tilde{\sigma},$$

where σ is the measure determined by $d\sigma = -\frac{1}{\pi}\tilde{\mu}\frac{\partial\varphi}{\partial\bar{z}}d\mathscr{L}$. This follows from a direct calculation of $\tilde{\sigma}(w)$:

$$\begin{aligned}\tilde{\sigma}(w) &= -\frac{1}{\pi}\int_{\mathbb{C}}\tilde{\mu}(z)\frac{\partial\varphi}{\partial\bar{z}}(z)\frac{1}{z-w}\,d\mathscr{L}(z)\\ &= -\frac{1}{\pi}\int_{\mathbb{C}}\int\frac{\partial\varphi}{\partial\bar{z}}(z)\frac{1}{z-w}\frac{1}{\zeta-z}\,d\mu(\zeta)\,d\mathscr{L}(z)\\ &= -\frac{1}{\pi}\int\int_{\mathbb{C}}\Big\{\frac{1}{z-w}+\frac{1}{\zeta-z}\Big\}\frac{1}{\zeta-w}\frac{\partial\varphi}{\partial\bar{z}}(z)\,d\mathscr{L}(z)\,d\mu(\zeta)\\ &= \int\frac{1}{\zeta-w}\{\varphi(w)-\varphi(\zeta)\}\,d\mu(\zeta).\end{aligned}$$

(The last step is an application of the generalized Cauchy integral formula.) This gives the result.

It now follows that measures in $\mathscr{R}(X)^\perp$ have the *decomposition property*: Such a measure can be decomposed into a sum of measures with small support each of which is also in $\mathscr{R}(X)^\perp$. *Let $U_1,\dots,U_r$ be a collection of open sets in $\mathbb{C}$ with $\cup_{j=1,\dots,r}U_j\supset X$. If $\mu\in\mathscr{R}(X)^\perp$, then $\mu=\sum_{j=1,\dots,r}\mu_j$, where each $\mu_j\in\mathscr{R}(X)^\perp$, and supp $\mu_j\subset \bar{U}_j\cap X$.* To see this, let φ_j be a smooth function on $\mathbb{C}$ with support in U_j, chosen so that $\sum_{j=1,\dots,r}\varphi_j=1$ on X. Define a measure μ_j by

$$d\mu_j = \varphi_j d\mu - \frac{1}{\pi}\frac{\partial\varphi_j}{\partial\bar{z}}d\mathscr{L}.$$

Then $\mu=\mu_1+\cdots+\mu_r$. If σ_j is defined by

$$d\sigma_j = -\frac{1}{\pi}\tilde{\mu}\frac{\partial\varphi_j}{\partial\bar{z}}d\mathscr{L},$$

then $\tilde{\mu}_j=\varphi_j\tilde{\mu}-\tilde{\sigma}_j$, which vanishes off $\bar{U}_j\cap X$ and so $\tilde{\mu}_j\in\mathscr{R}(\bar{U}\cap X)^\perp\subset\mathscr{R}(X)^\perp$.

Dually, for compact subsets X of the plane, the algebra $\mathscr{R}(X)$ has the *localization property*: If $X=\cup_{j=1,\dots,r}V_j$, where each V_j is an open subset of X, then the function

$f \in \mathscr{C}(X)$ belongs to $\mathscr{R}(X)$ if for each j, $f|\bar{V}_j \in \mathscr{R}(\bar{V}_j)$. This is so, for if $f \notin \mathscr{R}(X)$, then there is a measure $\mu \in \mathscr{R}(X)^\perp$ with $\int f\,d\mu \neq 0$. According to the previous paragraph, μ can be decomposed as a sum of measures $\mu_j \in \mathscr{R}(X)^\perp$ with $\operatorname{supp}\mu_j \subset \bar{V}_j \cap X$. Then $\int f\,d\mu = \sum \int f\,d\mu_j = 0$. Contradiction.

Note: The localization property just established is a purely one-dimensional phenomenon; there is no analogue in C^N, for Kallin [194] has constructed compact polynomially convex sets X for which the polynomial algebra $\mathscr{P}(X)$ is not a local algebra. See also the discussion in [345].

Some additional properties of the algebras $\mathscr{R}(X)$ are given in Chapter 6.

Theorem 1.6.7. *If X is a compact subset of $\mathbb{C}^N$ with $\Lambda^2(X) = 0$, then X is rationally convex and satisfies $\mathscr{R}(X) = \mathscr{C}(X)$.*

Proof. Let x and y be distinct points of the set X. There is a complex linear functional φ on $\mathbb{C}^N$ such that $\varphi(x) \neq \varphi(y)$. The set $\varphi(X)$ is a compact subset of the plane that has measure zero, so by the Hartogs–Rosenthal theorem, $\mathscr{R}(\varphi(X)) = \mathscr{C}(\varphi(X))$. Consequently, $\mathscr{R}(\varphi(X))$ contains a real-valued function g with $g(\varphi(x)) \neq g(\varphi(y))$. The composition $g \circ \varphi$ lies in $\mathscr{R}(X)$, so that $\mathscr{R}(X)$ contains a real-valued function that separates x and y. The Stone–Weierstrass theorem implies that $\mathscr{R}(X) = \mathscr{C}(X)$, whence X is rationally convex.

Corollary 1.6.8. *If X is a rationally convex subset of $\mathbb{C}^N$ and E is a subset of $\mathbb{C}^N$ of zero two-dimensional Hausdorff measure such that the set $Y = X \cup E$ is compact, then the set Y is rationally convex, and $\mathscr{R}(Y) = \{f \in \mathscr{C}(Y) : f|X \in \mathscr{R}(X)\}$.*

The corollary implies in particular that the union of a polynomially convex set and a set of zero two-dimensional Hausdorff measure is rationally convex if it is compact.

Proof. Let $z_o \in \mathscr{R}$-hull Y. We shall show that if $z_o \notin X$, then $z_o \in E$. This is so, for if $z_o \notin X$, then because X is rationally convex, there is a polynomial P that is zero-free on X but that vanishes at z_o. The set $P(E)$ has area zero, so it follows from the localization property above and the theorem of Hartogs and Rosenthal that there is a function $g \in \mathscr{R}(P(Y))$ with $g(0) = 1$ and $|g| < 1$ on $P(E) \setminus \{0\}$. Consequently, if μ is a representing measure for the point z_o with respect to the algebra $\mathscr{R}(Y)$, μ supported by Y, then necessarily μ is supported by $P^{-1}(0)$. If T is an affine automorphism of $\mathbb{C}^N$ that leaves the point z_o fixed but is sufficiently near the identity, then the polynomial $Q_T = P \circ T$ has the properties we have required of P, so it follows that each representing measure μ as above is also supported on $Q_T^{-1}(0)$. But the intersection of all these zero loci is z_o. Consequently, $z_o \in E$, and Y is seen to be rationally convex.

We can now see that $\mathscr{R}(Y)$ consists precisely of those continuous functions on Y that restrict to X as elements of $\mathscr{R}(X)$. To this end, consider a measure $\mu \in \mathscr{R}(Y)^\perp$. Fix a point $z_o \in Y \setminus X$. There is a polynomial P with $P(z_o) = 0$ and $|P| > 1$ on X. The set $P(E)$ has zero planar measure, for the set E has two-dimensional measure zero. It follows that the algebra $\mathscr{R}(P(Y))$ contains a function g with $g = 1$ on a neighborhood U of z_o in $P(Y)$, and with $|g| < 1$ on $P(Y) \setminus U$. (Again, use the localization property and the Hartogs–Rosenthal theorem.) It follows as in the case of polynomial convexity that the measure μ must be supported by X. Thus if $g \in \mathscr{C}(Y)$ satisfies $g|X \in \mathscr{R}(X)$, and if $\mu \in \mathscr{R}(Y)^\perp$, then $\int g\,d\mu = 0$. This is correct for every choice of μ, so $g \in \mathscr{R}(Y)$.

As concerns Theorems 1.6.4 and 1.6.7, there is a measure-theoretic nicety to be remarked. For Theorem 1.6.4, what is required is not really that $\Lambda^1(X) = 0$ but only that for each linear functional ψ on $\mathbb{C}^N$, the set $\psi(X)$ have zero length. It would suffice for $\psi(X)$ to have zero length for almost all choices of ψ or, indeed, even for N linearly independent choices of ψ. A parallel remark holds for Theorem 1.6.7. Sets E in $\mathbb{R}^N$ with the property that for almost all projections $\pi : \mathbb{R}^N \to \mathbb{R}^N$ of rank p the p-dimensional measure of $\pi(E)$ is zero are said to be sets of *p-dimensional integral-geometric measure* zero. This is not precisely the class of subsets E of vanishing p-dimensional Hausdorff measure. For the comparison of the Hausdorff measures and the integral-geometric measures, see [114] and [115].

1.6.2. Arcs and Cantor Sets

It is possible to construct some examples of complicated polynomially convex hulls following a procedure used by Wermer [367] and Rudin [307]. The method is based on the observation that if h is a bounded measurable function with bounded support in the plane, then the convolution of h with the Cauchy kernel is continuous on the Riemann sphere and holomorphic off the support of h. That is, for a bounded measurable function h on $\mathbb{C}$ with bounded support E, the function H defined on the Riemann sphere $\mathbb{C}^*$ by $H(z) = \int_E \frac{h(\zeta)}{\zeta - z}\, d\mathscr{L}(\zeta)$ is continuous. We have already observed that the function H is locally integrable. Morera's theorem implies that it is holomorphic on the set $\mathbb{C} \setminus E$. It is also holomorphic at infinity. That it is continuous on $\mathbb{C}^*$ is a standard fact about convolutions, which can be proved as follows. Suppose $h(z) = 0$ when $|z| > R$. Let $\varepsilon > 0$. Choose a continuous function g on the plane with compact support such that $\int_{\{|\zeta|<2R\}} \left|\frac{1}{\zeta} - g(\zeta)\right| d\mathscr{L}(\zeta) < \varepsilon$. If $z, z' \in \mathbb{C}$ satisfy $|z|, |z'| < R$, then

$$\begin{aligned} |H(z) - H(z')| &\leq \|h\|_\infty \int_{\{|\zeta|<R\}} \left| \frac{1}{\zeta - z} - \frac{1}{\zeta - z'} \right| d\mathscr{L}(\zeta) \\ &\leq \|h\|_\infty \int_{\{|\zeta|<R\}} \left\{ \left| \frac{1}{\zeta - z} - g(\zeta - z) \right| + |g(\zeta - z) - g(\zeta - z')| \right. \\ &\qquad\qquad \left. + \left| \frac{1}{\zeta - z'} - g(\zeta - z') \right| \right\} d\mathscr{L}(\zeta). \end{aligned}$$

The first and third terms contribute at most $\pi R^2 \varepsilon \|h\|_\infty$, and the contribution of the middle term can be made arbitrarily small by making $|z - z'|$ small. Consequently, the function H is continuous.

It is clear that $\lim_{z\to\infty} H(z) = 0$ and that $\lim_{z\to\infty} zH(z) = \int h(\zeta)\, d\mathscr{L}(\zeta)$. Thus, if the function h is nonnegative and positive on a set of positive measure, then H is not constant on the sphere.

Now fix a bounded measurable function h with bounded support such that the associated function H is not constant. Define G on $\mathbb{C}^*$ by $G(z) = zH(z)$. Also, let $\alpha \in \mathbb{C} \setminus E$ be a point where H does not vanish, and define the function F by $F(z) = \frac{H(z) - H(\alpha)}{z - \alpha}$. Let $\Phi : \mathbb{C}^* \to \mathbb{C}^3$ be the map $\Phi(z) = (F(z), G(z), H(z))$. This map is continuous and is holomorphic on $\mathbb{C}^* \setminus E$. Moreover, it is one-to-one, and so a homeomorphism of the two-sphere $\mathbb{C}^*$ onto the set $\Sigma = \Phi(\mathbb{C}^*)$. To see that it is one-to-one, let z, z' be distinct points of $\mathbb{C}^*$. If $H(z) = H(z')$ and $G(z) = G(z')$, then either $z = z'$ or else $H(z) = H(z') = 0$. In the latter case, $F(z) \neq F(z')$, whence the assertion.

The set Σ, which is topologically a two-sphere, has the property that if P is a function holomorphic on a neighborhood in $\mathbb{C}^3$ of Σ, then every value that P assumes on Σ is assumed also at some point of $\Gamma = \Phi(E)$. For this, suppose that $P \circ \Phi$ does not vanish on the set E but that it does have zeros on $\mathbb{C}^*$. There are only finitely many of these zeros, and we can write $P \circ \Phi = pq$ where p is the polynomial that vanishes at the zeros of $P \circ \Phi$ that lie in the finite plane, each counted with multiplicity, and q is a function continuous and zero-free on $\mathbb{C}$, holomorphic on $\mathbb{C}^* \setminus E$. The function q is continuous and zero-free on $\mathbb{C}$ and so has a logarithm there. But this is impossible, for q necessarily vanishes at the point at infinity. Contradiction.

The property of Σ established in the preceding paragraph is stronger than the statement that Σ is contained in the rationally convex hull of the set Γ.

Appropriate choices of the initial function h lead to some surprising examples. For example, if h is the characteristic function of a compact, perfect,[6] totally disconnected subset of $\mathbb{C}$ that has positive Lebesgue measure, then the set Γ is a set homeomorphic to the Cantor set, and its polynomially convex hull and its rationally convex hull both contain Σ.

Similarly, if h is the characteristic function of an arc in the plane that has positive Lebesgue measure, then we obtain an arc, Γ, in $\mathbb{C}^3$ with $\widehat{\Gamma}$ and $\mathscr{R}$-hull Γ both containing Σ.

It is obvious, but perhaps worth noting, that we have identified neither the polynomially convex hull nor the rationally convex hull of Γ.

If we start with a totally disconnected, perfect subset E of $\mathbb{C}^*$, and if h is its characteristic function, then, with H and G as above, the map $\Psi = (G, H) : \mathbb{C}^* \to \mathbb{C}^2$ is no longer one-to-one, but the set $\Psi(E)$ is seen without difficulty to be a perfect, totally disconnected set the rational and polynomially convex hulls of which contain the set $\Psi(\mathbb{C}^*)$. (Note that, topologically, $\Psi(E)$ is the set obtained from E by collapsing the subset $E \cap H^{-1}(0)$ to a point.)

We finish this discussion by mentioning some other examples of the kind of phenomenon we have been discussing. First, Bagby and Gauthier [42] have found an arc in $\mathbb{C}^N$, $N \geq 2$, that has finite 2-dimensional measure but that is not rationally convex. Vitushkin [359] has exhibited a Cantor set in $\mathbb{C}^2$ the polynomially convex hull of which contains an open set. Building on Vitushkin's construction, Byčkov [78] produced a Cantor set the polynomially convex hull of which is the closure of a domain. Sibony [333] found a set of Hausdorff dimension one with hull the closure of an open set. (A set E has *Hausdorff dimension* p if $p = \inf\{s : \Lambda^s(E) = 0\}$.) Globevnik in [146] constructs an arc in the boundary of the unit ball in $\mathbb{C}^6$ that is not polynomially convex; it is the boundary of an analytic disk contained in the ball. Finally, Jöricke [192] has constructed a Cantor set in $b\mathbb{B}_2$ that is neither polynomially convex nor rationally convex.

[6] A *perfect* set is one each point of which is a limit point. All compact perfect totally disconnected subsets of $\mathbb{R}^N$ are homeomorphic to the usual middle-third set of Cantor.

1.6.3. Lipschitz Graphs

In the preceding section we applied the Cauchy–Fantappiè integral to obtain the Oka–Weil theorem. In this section we apply it to obtain two other approximation theorems, which yield additional classes of polynomially convex sets.

Theorem 1.6.9. *Let X be a compact subset of $\mathbb{R}^N$, and let $\psi : X \to \mathbb{R}^N$ be a map that satisfies the Lipschitz condition*

$$|\psi(x) - \psi(x')| \leq c|x - x'|$$

for all $x, x' \in X$ and for some fixed $c \in [0, 1)$. If Γ_ψ denotes the graph $\{x + i\psi(x) : x \in X\}$ in $\mathbb{C}^N$, then every continuous function on Γ_ψ can be approximated uniformly by polynomials.

Theorem 1.6.10. *Let X be a compact set in $\mathbb{C}^N$, and let $R = (R_1, \ldots, R_N) : X \to \mathbb{C}^N$ be a map that satisfies the Lipschitz condition $|R(z) - R(z')| < c|z - z'|$ for all $z, z' \in X$ and some fixed $c \in [0, 1)$. Every continuous function on X can be approximated uniformly by polynomials in $z_1, \ldots, z_N$ and $\bar{z}_1 + R_1, \ldots, \bar{z}_N + R_N$.*

Theorem 1.6.10 was established by Weinstock [364] under the hypothesis that the map R is of class $\mathscr{C}^1$. His paper contains a note added in proof that shows, following a remark of Browder, Cole, and Wermer, that the result can be obtained under the Lipschitz hypothesis of the stated theorem. The result, under more stringent regularity hypotheses, is given in [183]. Theorem 1.6.9 was proved by Frih and Gauthier [134] with methods derived from those of the proof of the earlier Theorem 1.6.10 by Weinstock, Wermer, Browder, and Cole.

Throughout the discussion of Theorems 1.6.9 and 1.6.10, we assume that the maps ψ and R are defined on all of $\mathbb{R}^N$ and $\mathbb{C}^N$, respectively, and satisfy a Lipschitz condition with constant c on the entire space. That this is permissible is a consequence of a theorem of Kirszbraun: *A Lipschitz map from a set $E \subset \mathbb{R}^n$ to $\mathbb{R}^m$ has an extension to a map from R^n to $\mathbb{R}^m$ with the same Lipschitz constant.* For this fact, we refer to [115, p. 201]. That a Lipschitz map from E into $\mathbb{R}^m$ admits a Lipschitz extension to all of $\mathbb{R}^n$ *with a somewhat larger Lipschitz constant* is a rather elementary point, which can be found in [115, p. 202].

Theorem 1.6.10 implies the approximation result of Theorem 1.6.9 *for maps ψ that satisfy the stronger condition $|\psi(x) - \psi(x')| \leq k|x - x'|$ for some $k \in [0, \frac{1}{2})$*. To see this, let $X \subset \mathbb{R}^N$ be compact, and let $\psi : X \to \mathbb{R}^N$ satisfy a Lipschitz condition with Lipschitz constant $k \in [0, \frac{1}{2})$. Apply Theorem 1.6.10 to the set Γ_ψ and the map $R : \Gamma_\psi \to \mathbb{C}^N$ given by $R(z) = 2i\psi(x)$ if $z = x + iy$ with $x, y \in \mathbb{R}^N$. The map R satisfies a Lipschitz condition with constant less than one on Γ_ψ, so it follows that polynomials in $z_1, \ldots, z_N$ and $\bar{z}_1 + R_1, \ldots, \bar{z}_N + R_N$ are dense in $\mathscr{C}(\Gamma_\psi)$. We have $\bar{z}_j + R_j(z) = x_j - iy_j + 2i\psi_j(x)$, whence *on the set* Γ_ψ, $\bar{z}_j + R_j(z) = z_j$. Thus, the polynomials are dense in $\mathscr{C}(\Gamma_\psi)$ as claimed.

The condition that the Lipschitz constants be strictly less than one is necessary in both theorems: If $R(z) = -\bar{z}$, then R satisfies a Lipschitz condition of constant one on all of $\mathbb{C}^N$, but polynomials in $z_1, \ldots, z_N$ are generally not dense in $\mathscr{C}(X)$, $X \subset \mathbb{C}^N$ compact. A corresponding example for Theorem 1.6.9 is provided by the map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$\psi(x_1, x_2) = (x_2, -x_1)$. Then $\Gamma_\psi = \{(\zeta, -i\zeta) : \zeta \in \mathbb{C}\}$, and on compact sets of Γ_ψ with nonvoid interior (in Γ_ψ) we cannot approximate all continuous functions by polynomials.

Corollary 1.6.11. *With X and ψ as in Theorem 1.6.9, the graph Γ_ψ is polynomially convex.*

Corollary 1.6.12. *With X and R, as in Theorem 1.6.10, the set*

$$X_R = \{(z_1, \ldots, z_N, \bar{z}_1 + R_1(z), \ldots, \bar{z}_N + R_N(z)) : z \in X\}$$

in $\mathbb{C}^{2N}$ is polynomially convex and satisfies $\mathscr{P}(X_R) = \mathscr{C}(X_R)$.

For the approximation assertion, note that Theorem 1.6.10 implies that every continuous function on X_R can be approximated by polynomials.

Proof of Theorem 1.6.9. The map ψ of the theorem is defined on all of $\mathbb{R}^N$ with Lipschitz constant $c \in [0, 1)$. Impose for the moment the further condition that ψ be a map of class $\mathscr{C}^2$. Given ψ, construct the map $\varphi : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\varphi_j(z, w) = \bar{z}_j + 2i\psi_j(\Re z) - \bar{w}_j - 2i\psi_j(\Re w). \tag{1.29}$$

Set $\Phi(z, w) = (z - w) \cdot \varphi(z, w) = \sum_{j=1}^N (z_j - w_j)\varphi_j(z, w)$. Let $\Phi_j = \varphi_j/\Phi^N$, and introduce the Cauchy–Fantappiè form $\Omega_{\varphi; z-w}$.

The function Φ satisfies

$$\Re\Phi(z, w) \geq \frac{1 - c^2}{2}|z - w|^2, \tag{1.30}$$

as can be seen by using the two inequalities

$$2|a|\,|b| \leq |a|^2 + |b|^2 \text{ and } 2|a|\,|b| \leq c^2|a|^2 + \frac{1}{c^2}|b|^2$$

together with the Lipschitz condition satisfied by ψ to write

$$\Re\,\Phi(z, w) \geq |z - w|^2 - |\Im\, z - \Im\, w|^2 - c^2|\Re\, z - \Re\, w|^2 = (1 - c^2)|\Re\, z - \Re\, w|^2$$

and

$$\Re\Phi(z, w) \geq |z - w|^2 - c^2|\Im\, z - \Im\, w| - |\Re\, z - \Re\, w|^2 = (1 - c^2)|\Im\, z - \Im\, w|^2,$$

whence the desired inequality. We also have the estimate

$$|\Phi_j(z, w)| \leq (1 + 2c)\left(1 - \frac{c^2}{2}\right)^{-N} |z - w|^{1-2N}, \tag{1.31}$$

which follows from (1.30) and the definition of Φ_j, using the Lipschitz condition on the map ψ. The estimate (1.31) implies that for fixed w, $\Phi_j(\cdot, w)$ is locally integrable and that for a compact set E,

$$\sup_{w \in C^N} \int_E |\Phi_j(z, w)|\, d\mathscr{L}(z) < \infty. \tag{1.32}$$

With these facts at our disposal, arguing as at the end of Section 1.4 yields that if g is a function on $\mathbb{C}^N$ of class $\mathscr{C}^1$ with compact support, then

$$g(w) = -c_N \int_{\mathbb{C}^N} \bar{\partial} g(z) \wedge \Omega_{\varphi; z-w}. \tag{1.33}$$

Next, there is an analogue for our kernel of Lemma 1.6.5, which concerns measures with identically vanishing Cauchy transforms. Specifically, the following result holds:

Lemma 1.6.13. *If μ is a finite regular Borel measure on $\mathbb{C}^N$ with compact support, then for almost every $z \in \mathbb{C}^N$ and for each $j = 1, \ldots, N$, the integral $\int |\Phi_j(z, w)|\, d|\mu|(w)$ is finite. If $\int \Phi_j(z, w)\, d\mu(w) = 0$ for almost all points z and each $j = 1, \ldots, N$, then μ is the zero measure.*

Proof. The support, E, of μ is a compact set, so by (1.32),

$$\int \int_E |\Phi_j(z, w)|\, d\mathscr{L}(z)\, d|\mu|(w) < \infty,$$

whence by Fubini's theorem, for almost every point $z \in E$, $\int |\Phi_j(z, w)|\, d|\mu|(w) < \infty$. It is evident that for every $w \in \mathbb{C}^N \setminus E$, this integral is finite, so we have the first assertion of the lemma.

Assume now that $\int \Phi_j(z, w)\, d\mu(w) = 0$ for almost every $z \in \mathbb{C}^N$. We prove $\mu = 0$. To do this, it suffices, by duality, to prove that for every compactly supported $\mathscr{C}^1$ function g on $\mathbb{C}^N$, $\int g\, d\mu = 0$. By equation (1.33) applied to g,

$$\begin{aligned}
\int g(w)\, d\mu(w) &= -c_N \int \left\{ \int_{C^N} \bar{\partial} g(z) \wedge \Omega_{\varphi; z-w} \right\} d\mu(w) \\
&= -c_N \sum_{j=1}^{N} (-1)^{j-1} \int_{\mathbb{C}^N} \left\{ \int \Phi_j(z, w)\, d\mu(w) \right\} \bar{\partial} g(z) \wedge \bar{\partial}_z \varphi_1 \\
&\qquad \wedge \cdots \wedge \widehat{\bar{\partial}_z \varphi_j} \wedge \cdots \wedge \bar{\partial}_z \varphi_N \wedge \omega(z) \\
&= 0.
\end{aligned}$$

The lemma is proved.

Proof of Theorem 1.6.9 concluded. If the theorem is false, there is a finite regular Borel measure μ on Γ_ψ with $\int f\, d\mu = 0$ for every f in the algebra $\mathscr{P}(\Gamma_\psi)$, μ not the zero measure.

On the set Γ_ψ, $\bar{z}_j + 2i\psi_j(\Re z) = z_j$, so that for fixed w, $\varphi_j(\cdot, w) \in \mathscr{P}(\Gamma_\psi)$. Consequently, for fixed w, $\Phi(\cdot, w) \in \mathscr{P}(\Gamma_\psi)$. For $\nu = 1, \ldots,$ let

$$Q_\nu(z, w) = \left(\Phi(z, w) + \tfrac{1}{\nu}\right)^{-1}.$$

Because $\Re\Phi(z, w) \geq 0$, Runge's theorem implies that $Q_\nu(\cdot, w)$ lies in $\mathscr{P}(\Gamma_\psi)$, for $\frac{1}{\zeta}$ can be approximated uniformly by polynomials on compacta in the half-plane $\Re\zeta > 0$. We have $\lim_{\nu\to\infty} Q_\nu(z, w) = \Phi^{-1}(z, w)$ for each $z \neq w$.

If $w \in \mathbb{C}^N$ is a point for which $\int |\Phi_j(z,w)|\, d|\mu|(z)$ exists, then

$$\int \Phi_j(z,w)\, d\mu(z) = \lim_{\nu\to\infty} \int \varphi_j(z,w) Q_\nu^N(z,w)\, d\mu(z) = 0,$$

for by hypothesis, μ annihilates the algebra $\mathscr{P}(\Gamma_\psi)$. The symmetry $\Phi_j(z,w) = -\Phi_j(w,z)$ implies that for almost every w, $\int \Phi_j(w,z)\, d\mu(z) = 0$, whence, by Lemma 1.6.13, $\mu = 0$, as we wished to show.

To complete the proof, we have to pass from the case of ψ's of class $\mathscr{C}^2$ to ψ's that satisfy a Lipschitz condition.

In this case, the argument proceeds essentially as above. The issue is to see that if μ is a measure on Γ_ψ that annihilates $\mathscr{P}(\Gamma_\psi)$, then it annihilates each of the functions $\Phi_j(z,w)$ for fixed w, and then to deduce that μ is the zero measure.

This depends on a smoothing process. Let χ be a nonnegative even function on $\mathbb{R}^N$ with compact support and with $\int_{\mathbb{R}^N} \chi = 1$, and for $\varepsilon > 0$, let $\chi_\varepsilon(x) = \varepsilon^{-N}\chi(x/\varepsilon)$. Given the Lipschitz map ψ, put

$$\psi_\epsilon(x) = \chi_\epsilon * \psi(x) = \int_{\mathbb{R}^N} \chi_\epsilon(y)\psi(x-y)\, d\mathscr{L}(y).$$

The map ψ_ϵ is smooth, and it satisfies the same Lipschitz condition as ψ:

$$|\psi_\epsilon(x) - \psi_\epsilon(x')| = \left| \int_{\mathbb{R}^N} \chi_\epsilon(y)\big(\psi(x-y) - \psi(x'-y)\big)\, d\mathscr{L}(y) \right| \le c|x-x'|.$$

As $\epsilon \to 0$, ψ_ϵ converges to ψ uniformly on compacta in $\mathbb{R}^N$. Construct the functions Φ^ϵ and Φ_j^ϵ and the form $\Omega_{\psi_\epsilon;z-w}$ associated with the map ψ_ϵ as Φ, Φ_j, $\Omega_{\varphi;z-w}$ were associated to ψ. There is a bound $|\Phi_j^\epsilon(z,w)| \le C|z-w|^{1-2N}$ for an absolute constant C. The first-order derivatives of the maps ψ_ϵ are bounded uniformly in ϵ, and given a $\mathscr{C}^1$ function g with compact support, we can write, for a measure μ,

$$\int_X g\, d\mu = \int_X \left\{ \sum_{j=1}^N (-1)^{j-1} \int_{C^N} \Phi_j^\epsilon(z,w) S_\epsilon(z)\, d\mathscr{L}(z) \right\} d\mu(w),$$

where S_ϵ is a certain function bounded independently of the index ϵ. This follows from the representation (1.33) for g in terms of the kernel $\Omega_{\psi_\epsilon;z-w}$ together with the uniform boundedness of the first derivatives of the functions ψ_ε. Apply Fubini's theorem to get

$$\int_X g\, d\mu = \sum_{j=1}^N (-1)^{j-1} \int_{C^N} \left\{ \int_X \Phi_j^\epsilon(z,w)\, d\mu(w) \right\} S_\epsilon(z)\, d\mathscr{L}(z).$$

Pass to the limit as $\epsilon \to 0^+$ to find that for some bounded function S,

$$\int g\, d\mu = \sum_{j=1}^N (-1)^{j-1} \int_{C^N} \left\{ \int_X \Phi_j(z,w)\, d\mu(w) \right\} S(z)\, d\mathscr{L}(z).$$

The analysis used in the $\mathscr{C}^2$ case applies equally well in the present case to show that if μ is orthogonal to the algebra $\mathscr{P}(\Gamma_\psi)$, then $\int_{\Gamma_\psi} \Phi_j(z, w)\, d\mu(z) = 0$ for almost all w, and hence, by the symmetry $\Phi_j(z, w) = -\Phi_j(w, z)$, it follows that μ is the zero measure.

This completes the proof of Theorem 1.6.9.

The proof of Theorem 1.6.10 can be executed in precisely parallel steps, working this time with the functions

$$\varphi_j(z, w) = \bar{z}_j + R_j(z) - \bar{w}_j - R_j(w)$$

and constructing the associated Φ, Φ_j, and the kernel $\Omega_{\varphi;z-w}$.

In connection especially with Theorem 1.6.9, it is worth remarking that if $\psi : \mathbb{R}^N \to \mathbb{R}^N$ is of class $\mathscr{C}^1$ and satisfies $|\psi(x) - \psi(x')| \le c|x - x'|$ for all $x, x' \in \mathbb{R}$, some $c \in [0, 1)$, then Γ_ψ is a $\mathscr{C}^1$ manifold and is totally real in the sense of the following definition:

Definition 1.6.14. *The k-dimensional $\mathscr{C}^1$ submanifold $\mathscr{M}$ of the open set Ω in $\mathbb{C}^N$ is* totally real *at the point $p \in \mathscr{M}$ if the $\mathbb{R}$-affine subspace of $\mathbb{C}^N$ through p and tangent at p to $\mathscr{M}$ contains no complex line.*

Equivalently, in terms of the J operator that defines the complex structure on $\mathbb{C}^N$, $\mathscr{M}$ is totally real at p if for no nonzero tangent vector $v \in T_p\mathscr{M}$ is $Jv \in T_p\mathscr{M}$, so that $T_p\mathscr{M} \cap JT_p\mathscr{M} = \{0\}$.

Observe that total reality is an *open* condition in the sense that if $\mathscr{M}$ is totally real at a point, it is totally real in a neighborhood of the point.

The total reality of Γ_ψ is seen as follows. Set $\Psi(x) = x + i\psi(x)$, so that $\Gamma_\psi = \Psi(\mathbb{R}^N)$. Suppose that $\Psi(0) = 0$ and that the tangent space $T_0\Gamma_\psi$ is not totally real. Then there is $v \in T_0\Gamma_\psi$ such that $Jv \in T_0\Gamma_\psi,\ v \ne 0$. The tangent space $T_0\Gamma_\psi$ is $d\Psi_0(\mathbb{R}^N)$ if $d\Psi_0$ denotes the differential at 0 of Ψ. If M denotes the matrix $\left[\frac{\partial\psi_j}{\partial x_k}(0)\right]_{j,k=1}^N$, then $d\Psi_0(x) = x + iMx$. The tangent vector v is $d\Psi_0(x_0)$ for some $x_0 \in \mathbb{R}^N$, and Jv is $d\Psi_0(y_0)$. Thus

$$x_0 + iMx_0 = i(y_0 + iMy_0),$$

which yields $x_0 = -M^2x_0$. The hypothesis that $|\psi(x) - \psi(x')| \le c|x - x'|$ implies that the matrix M is of norm no more than c: $|Mx| \le c|x|$, which leads to $|x_0| \le c^2|x_0|$. This is not possible unless $x_0 = 0$, which yields $v = 0$. Thus, as claimed, Γ_ψ is totally real.

Corollary 1.6.15. *If Σ is a totally real submanifold of $\mathbb{C}^N$ and if $p_0 \in \Sigma$, then there is a compact neighborhood K of p_0 in Σ that is polynomially convex and satisfies $\mathscr{P}(K) = \mathscr{C}(K)$.*

Proof. Assume first that $\dim \Sigma = N$. Choose holomorphic coordinates on $\mathbb{C}^N$ such that $p_0 = 0$ and $T_0\Sigma = \mathbb{R}^N$. Thus, near 0, Σ is a graph $\Sigma = \{x + i\varphi(x) : x \in \mathbb{R}^N\}$ with $\varphi(0) = 0$ and $d\varphi(0) = 0$. From $d\varphi(0) = 0$ follows $|\varphi(x) - \varphi(x')| < \frac{1}{2}|x - x'|$ when $x, x' \in \mathbb{R}^N$, $|x - x'| < \delta$. The result is a consequence of Theorem 1.6.9.

If $\dim \Sigma < N$, fix $p_0 \in \Sigma$. In a neighborhood U of p_0 in $\mathbb{C}^N$, there is a $\mathscr{C}^1$ totally real submanifold of dimension N that contains $\Sigma \cap$ U, so the result in this case follows from the N-dimensional case.

It is to be emphasized that Corollary 1.6.15 is entirely local. For example, the torus $\mathbb{T}^N$ in $\mathbb{C}^N$ is totally real and so locally polynomially convex. It is not polynomially convex. Indeed, there is no compact N-dimensional topological manifold in $\mathbb{C}^N$ that is polynomially convex. See Corollary 2.3.5 below.

Totally real manifolds are locally polynomially convex—this was derived above as a consequence of the local approximation theorem and a general fact from functional analysis. It is well to notice that there is a direct route to the result, one that does not require the analysis that went into the proof of the approximation theorem.

Consider initially the case of N-dimensional totally real submanifolds of $\mathbb{C}^N$. Thus, let $\Sigma \subset \mathbb{C}^N$ be an N-dimensional totally real submanifold of class $\mathscr{C}^1$. Suppose $0 \in \Sigma$ and $T_0\Sigma = \mathbb{R}^N$. Near 0, Σ is the graph of a function $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ with $\varphi(0) = 0$, $d\varphi(0) = 0$. Consequently, there is a constant $r_0 > 0$ so small that if $x', x'' \in \mathbb{R}^N$ satisfy $|x'|, |x''| \leq r_0$, then $|\varphi(x') - \varphi(x'')| < \frac{1}{2}|x' - x''|$. Let $z^0 = x^0 + iy^0 \in \mathbb{C}^N \setminus \Sigma$ with $|x^0| \leq r_0$. Set

$$\begin{aligned}P(z) &= \sum_{j=1}^N \bigl(z_j - (x_j^0 + i\varphi_j(x^0))\bigr)^2 \\ &= \sum_{j=1}^N \bigl(((x_j - x_j^0)^2 - (y_j - \varphi_j(x_0))^2) + 2i(x_j - x_j^0)(y_j - \varphi_j(x^0))\bigr).\end{aligned}$$

Then

$$P(z^0) = -\sum_{j=1}^N \bigl(y_j^0 - \varphi_j(x_0)\bigr)^2,$$

which is strictly negative, because $z^0 \notin \Sigma$. If $z = x + iy \in \Sigma$ with $|x| \leq r_0$, then, because $y = \varphi(x)$, we have

$$\Re P(z) = \sum_{j=1}^N (x_j - x_j^0)^2 - (\varphi_j(x) - \varphi_j(x_0))^2,$$

and this quantity is strictly positive, for

$$\sum_{j=1}^N \bigl(\varphi_j(x) - \varphi_j(x_0)\bigr)^2 \leq \tfrac{1}{4}|x - x_0|^2.$$

Thus, the point z^0 is not in the polynomially convex hull of $\Sigma_{r_0} = \{z \in \Sigma : |x| \leq r_0\}$. If $z^0 \in \mathbb{C}^N$ satisfies $|x^0| > r_0$, then z^0 is not in $\widehat{\Sigma}_{r_0}$ either, and we conclude that Σ_{r_0} is polynomially convex.

If Σ has dimension less than N, again let $0 \in \Sigma$. There is an N-dimensional totally real manifold $\widetilde{\Sigma}$ with $0 \in \widetilde{\Sigma}$ and with $\widetilde{\Sigma}$ containing a neighborhood of 0 in Σ. Let $\widetilde{K} \subset \widetilde{\Sigma}$ be a compact neighborhood of $0 \in \widetilde{\Sigma}$ that is polynomially convex. There is considerable latitude in the choice of $\widetilde{\Sigma}$, so finitely many $\widetilde{\Sigma}$'s exist, say $\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_q$, with associated $\widetilde{K}_1, \ldots, \widetilde{K}_q$, such that the polynomially convex set $\widetilde{K}_1 \cap \cdots \cap \widetilde{K}_q$ is a neighborhood of 0 in Σ.

Corollary 1.6.16. *Let f be a function of class $\mathscr{C}^1$ on the complex plane. If $\bar\partial f$ does not vanish at the point $(0, f(0))$, then there is a compact neighborhood K of $0 \in \mathbb{C}$ such that polynomials in z and f are dense in $\mathscr{C}(K)$.*

This result was first proved by Wermer [370].

Proof. The hypothesis implies that the graph $\Gamma_f = \{(z, f(z)) : z \in \mathbb{C}\}$ is totally real near 0, whence the result.

There is an N-dimensional version of this due to Weinstock [364].

Corollary 1.6.17. *If the Jacobian determinant $\det\big[\frac{\partial f_j}{\partial \bar z_k}(0)\big]_{j,k=1}^N$ of the $\mathscr{C}^1$ map $f : \mathbb{C}^N \to \mathbb{C}^N$ does not vanish, then there is a compact neighborhood K of $0 \in \mathbb{C}^N$ such that the polynomials in $z_1, \dots, z_N, f_1, \dots, f_N$, are dense in $\mathscr{C}(K)$.*

Proof. With the given hypotheses, the graph $\Gamma_f = \{(z, f(z)) \in \mathbb{C}^{2N} : z \in \mathbb{C}^N\}$ is totally real near the point $(0, f(0))$. Assume without loss of generality that $f(0) = 0$. It is sufficient to show that the tangent plane $T_0\Gamma_f$ is totally real. Denote by D' the matrix $\big[\frac{\partial f_j}{\partial z_k}(0)\big]_{j,k=1}^N$ and by D'' the matrix $\big[\frac{\partial f_j}{\partial \bar z_k}(0)\big]_{j,k=1}^N$. The tangent plane $T_0\Gamma_f$ is the plane

$$\{(z, D'z + D''\bar z) : z \in \mathbb{C}^N\},$$

as follows from the decomposition $d = \partial + \bar\partial$. We will show that this plane is totally real. In the contrary case, there exists $w \in T_0\Gamma_f$, $w \neq 0$, such that $Jw \in T_0\Gamma_f$. This implies the existence of $\alpha, \beta \in \mathbb{C}^N$ with $\alpha = i\beta$ and $D'\alpha + D''\bar\alpha = iD'\beta + iD''\bar\beta$. Substituting the first of these into the second yields

$$iD'\beta - iD''\bar\beta = iD'\beta + iD''\bar\beta.$$

The matrix D'' is nonsingular, so β must be zero, and this means that α is 0. Thus, $T_0\Gamma_f$ is totally real, and the corollary is proved.

Other approximation theorems in the spirit of Theorem 1.6.9 have been given by Sakai [315] and Weinstock [365].

It was shown above that totally real manifolds are locally polynomially convex and that each continuous function on such a manifold can be approximated locally uniformly by polynomials. In special cases, uniform approximation can occur on manifolds that are not totally real. A simple example is the following.

Let $\Sigma = \{(z, w) \in \mathbb{C}^2 : w = z^2 + \bar z^2 + 2z\bar z\}$. This is a smooth quadratic surface in $\mathbb{C}^2$ that is totally real except at the origin, where the tangent plane is the z-axis. Let Σ_1 be the compact disk $\{(z, w) \in \Sigma : |z| \leq 1\}$, a neighborhood in Σ of the origin. The disk Σ_1 is polynomially convex and satisfies $\mathscr{P}(\Sigma_1) = \mathscr{C}(\Sigma_1)$. The polynomial convexity is a consequence of the equality of these two algebras. That each continuous function on Σ_1 can be approximated by polynomials is equivalent to the observation that if $\mathscr{A}$ is the subalgebra of $\mathscr{C}(\bar{\mathbb{U}})$ generated by z and x^2, $z = x + iy$, then $\mathscr{A} = \mathscr{C}(\bar{\mathbb{U}})$, as follows from Theorem 1.2.16: The element g of $\mathscr{A}$ given by $g(z) = x^2$ is real-valued, so $\mathscr{A} = \mathscr{C}(\bar{\mathbb{U}})$ if and only if for each $t \in \mathbb{R}$, $\mathscr{P}(g^{-1}(t)) = \mathscr{C}(g^{-1}(t))$. For each t, the fiber $E_t = g^{-1}(t)$, if not empty, is either a compact vertical interval or the disjoint union of two (possibly degenerate) such intervals. In either case, $\mathscr{P}(E_t) = \mathscr{C}(E_t)$, as follows from Mergelyan's theorem, whence the equality of $\mathscr{P}(\Sigma)$ and $\mathscr{C}(\Sigma)$.

1.6.4. Certain Unions

A union of polynomially convex sets is generally not polynomially convex. There is, however, an important result that affirms the polynomial convexity of a union of two polynomially convex sets under suitable hypotheses. It goes back to work of E. Kallin [195, 196] and is often referred to as *Kallin's lemma*. See also [345], [128], and [276].

An ingredient of the proof of Kallin's lemma is a fact about peak points for polynomial algebras on the plane.

Lemma 1.6.18. *If X is a compact, polynomially convex subset of the complex plane, then every point of bX is a peak point for the algebra $\mathscr{P}(X)$.*

Proof. Without loss of generality, assume that the origin is a point of bX and that X is a subset of the open unit disk.

Let $\{z_k\}_{k=1,\dots}$ be a sequence in $\mathbb{C}\setminus X$ that converges to the origin, and for each k, let λ_k be an arc in the Riemann sphere from z_k to infinity that misses X. Fix a point $z_o \in X \setminus \{0\}$, and for each k, let ℓ_k be a branch of $\log(z - z_k)$ defined on $\mathbb{C} \setminus \lambda_k$, the ℓ_k chosen so that the sequence $\ell_k(z_o)$ converges. The sequence $\{\ell_k\}_{k=1,\dots}$ converges pointwise on $X \setminus \{0\}$ to a continuous branch of $\log z$. We shall denote the limit function by $\log z$.

The function ψ defined by $\psi(z) = \frac{\log z}{\log z - 1}$, $z \in X \setminus \{0\}$, $\psi(0) = 1$, is continuous on X, and is holomorphic on the interior of X. Moreover, $\psi(0) = 1 > |\psi(z)|$ for all $z \in X \setminus \{0\}$. Mergelyan's theorem shows that $\psi \in \mathscr{P}(X)$, so 0 is a peak point for the algebra $\mathscr{P}(X)$. The lemma is proved.

A disjoint union of two compact, convex subsets of $\mathbb{C}^N$ is polynomially convex, because the sets can be separated by a linear functional. Kallin's lemma is an analogous result for polynomially convex sets that can be separated in a suitable sense by polynomials:

Theorem 1.6.19. *Let X_1 and X_2 be compact, polynomially convex subsets of $\mathbb{C}^N$. Let p be a polynomial such that the polynomially convex subsets $Y_j = (p(X_j))\widehat{\ }$, $j = 1, 2$, of $\mathbb{C}$ meet at most at the origin, which is a boundary point for each of them. If the set $p^{-1}(0)\cap(X_1\cup X_2)$ is polynomially convex, then the set $X = X_1 \cup X_2$ is polynomially convex. If, in addition, $\mathscr{P}(X_1) = \mathscr{C}(X_1)$ and $\mathscr{P}(X_2) = \mathscr{C}(X_2)$, then $\mathscr{P}(X_1 \cup X_2) = \mathscr{C}(X_1 \cup X_2)$.*

The hypothesis that $P^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex is satisfied in particular when this set is empty.

Proof. Let $x \in (X_1 \cup X_2)\widehat{\ }$, and let μ be a representing measure for x supported by the set $X_1 \cup X_2$. We have $p((X_1 \cup X_2)\widehat{\ }) \subset (p(X_1 \cup X_2))\widehat{\ } \subset Y_1 \cup Y_2$, so $p(x) \in Y_1 \cup Y_2$. (Note that the set $Y_1 \cup Y_2$ is polynomially convex.)

If $p(x) \neq 0$, suppose $p(x) \in Y_1$. Mergelyan's theorem provides a function $g \in \mathscr{P}(Y_1 \cup Y_2)$ with $g(p(x)) = 1$ and $g|Y_2 = 0$. For every polynomial q and for each positive integer k,

$$|q^k(x)| = |q^k(x)g(p(x))| \leq \|q\|_{X_1}^k \int |g \circ p|\, d\mu.$$

Take the kth roots of both sides and let $k \to \infty$ to reach $|q(x)| \leq \|q\|_{X_1}$, so that $x \in \widehat{X}_1 = X_1$.

If $p(x) = 0$, let $g \in \mathscr{P}(Y_1 \cup Y_2)$ satisfy $g(0) = 1$, $|g| < 1$ on $(Y_1 \cup Y_2) \setminus \{0\}$. For each polynomial q, $q(x) = q(x)[g(p(x))]^k = \int q[g \circ p]^k \, d\mu$, which tends to $\int_{p^{-1}(0)\cap(X_1\cap X_2)} q\, d\mu$ as $k \to \infty$. Consequently, $x \in (p^{-1}(0) \cap (X_1 \cap X_2))\widehat{\ } \subset X_1 \cup X_2$. It follows that $X_1 \cup X_2$ is polynomially convex.

For the second part of the theorem, let μ be a measure on $X_1 \cup X_2$ that is orthogonal to the algebra $\mathscr{P}(X_1 \cup X_2)$, so that for every polynomial q we have $\int q\, d\mu = 0$.

We first show that $\mu|(X_1 \setminus p^{-1}(0)) \in \mathscr{P}(X_1)^\perp$, so that because $\mathscr{P}(X_1) = \mathscr{C}(X_1)$, $\mu|(X_1 \setminus p^{-1}(0))$ is the zero measure. For this, let $h \in \mathscr{P}(Y_1 \cup Y_2)$ be the function with $h|Y_2 = 0$ and with $h(z) = z$, $z \in Y_1$. If $h_n = h^{1/n}$, $n = 1, \ldots$, then $h_n \in \mathscr{P}(Y_1 \cup Y_2)$, as follows from Mergelyan's theorem, and for every polynomial q, $0 = \int (h_n \circ p)q\, d\mu \to \int_{X_1\cap p^{-1}(0)} pq\, d\mu$. Thus, as desired, $\mu|(X_1 \setminus p^{-1}(0))$ is the zero measure. In the same way, $\mu|(X_2 \setminus p^{-1}(0))$ is the zero measure.

We now have that the measure μ is concentrated on the set

$$(X_1 \cup X_2) \setminus ((X_1 \setminus p^{-1}(0)) \cup (X_2 \setminus p^{-1}(0)) = (X_1 \cup X_2) \cap p^{-1}(0).$$

Because $\mathscr{P}(X_1) = \mathscr{C}(X_1)$, it follows that $\mu = 0$, so, as claimed, $\mathscr{P}(X_1 \cup X_2) = \mathscr{C}(X_1 \cup X_2)$.

The paper of dePaepe [276] gives a useful survey of the applications of this result.

As an atypically simple application of Theorem 1.6.19, notice that the union of two closed balls with disjoint interiors is polynomially convex, even if they have a point in common.

More generally, if X_1 and X_2 are convex sets that intersect only at one point, say the origin, and if X_1 and X_2 have the property that for some linear functional φ on $\mathbb{C}^N$, $\Re\varphi \leq 0$ on X_1, $\Re\varphi \geq 0$ on X_2, and $\Re\varphi^{-1}(0)$ meets X_1 only in a point, then the union $X_1 \cup X_2$ is polynomially convex.

It is not true in general that the union of two compact, convex sets that share only one point is polynomially convex. Various examples are known, and Section 8.1 below gives a systematic discussion of a class of examples. A simple example is this: Let $X_1 = \{z \in \mathbb{C}^2 : z_1 = \bar{z}_2, |z_2| \leq 2\}$ and $X_2 = \{z \in \mathbb{C}^2 : z_1 = 2\bar{z}_2, |z_2| \leq 2\}$. Each of these sets is compact and convex, and they intersect only at the origin. Their union, $X = X_1 \cup X_2$, is not polynomially convex. To see this, let $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{C}^2$ be the map given by $\psi(\zeta) = (\zeta, 1/\zeta)$. If A is the annulus $\{\zeta \in \mathbb{C} : 1 \leq |\zeta| \leq \sqrt{2}\}$, then the inner boundary of A goes under ψ into X_1, and the outer boundary of A goes into X_2. Consequently, $\widehat{X} \supset \psi(A)$, and X is seen not to be polynomially convex.

The original application of the theorem is a result of Kallin [195]:

Theorem 1.6.20. *A union of three mutually disjoint closed balls is polynomially convex.*

Proof. We consider first the case that $N = 2$. Let the balls be B_1, B_2, and B_3 with respective radii $r_1 = 1 \geq r_2 \geq r_3$. By proper choice of coordinates, we can suppose that B_1 is the closed unit ball centered at the origin. With a further unitary change of coordinates, we can suppose the center of B_2 to be $(\alpha, 0)$ for some $\alpha \in \mathbb{C}$. Then by rotations in the z_1 and z_2 axes, we can suppose that the center of B_3 is the point $\beta = (\beta_1, \beta_2)$ with β_1 and β_2 real. This transformation brings the center of B_2 to the point $(\gamma, 0)$ for a $\gamma \in \mathbb{C}$.

Introduce the polynomial φ given by $\varphi(z) = z_1^2 + z_2^2$. Let $z = x + iy$ with $x, y \in \mathbb{R}^2$. The polynomial φ carries B_1 onto the closed unit disk in $\mathbb{C}$, and we shall show that $\Re\varphi > 1$ on B_3. For this, notice that because $\Re\varphi$ is pluriharmonic, its minimum over the closed ball B_3 occurs on bB_3. We therefore consider the constrained extremal problem $\min \Re\varphi(z)$ subject to $z \in bB_3$. That is, we seek the minimum of $\Re\varphi(z)$ subject to $|z - \beta|^2 = r_3^2$. Treat this problem by the method of Lagrange multipliers: The minimum of $\Re\varphi(z)$ subject to $|z - \beta|^2 = r_3^2$ is achieved when $\operatorname{grad} \Re\varphi(z) = \lambda \operatorname{grad} |z - \beta|^2$ for some choice of the multiplier λ, which is equivalent to the system (in which $z_j = x_j + iy_j$)

$$\begin{aligned} x_1 &= \lambda(x_1 - \beta_1), \\ -y_1 &= \lambda y_1, \\ x_2 &= \lambda(x_2 - \beta_2), \\ -y_2 &= \lambda y_2. \end{aligned}$$

This system admits solutions only when $\lambda = -1$ or when $y_1 = y_2 = 0$. The case $\lambda = -1$ leads to $x_j = \beta_j/2$, which is impossible, for we are considering only points on bB_3, which is given by $|x - \beta|^2 + |y|^2 = r_3^2$. This equation cannot be satisfied when $x = \beta/2$, because $|\beta| > 1 + r_3$, for B_1 and B_3 are disjoint. Thus, the solution to our extremal problem occurs when $y = 0$. We are then looking at $\min(x_1^2 + x_2^2)$ subject to $|x - \beta|^2 = r_3^2$. This minimum is $(|\beta| - r_3)$, which is greater than 1, again because B_1 and B_3 are disjoint. Thus, $\Re\varphi > 1$ on B_3.

In addition, the sets $\varphi(B_1)$ and $\varphi(B_2)$ intersect at most at a point: We want to see that $|\varphi| \geq 1$ on B_2 and that only one value of φ on B_2 has modulus one. By the minimum principle, the modulus of φ assumes its minimum over B_2 on bB_2. Notice also that because $(z_1, z_2) \in bB_2$ implies $(z_1, e^{i\vartheta} z_2) \in bB_2$ for all $\vartheta \in \mathbb{R}$, it follows that the minimum of $|\varphi|$ on bB_2 occurs when $\arg z_2^2 = \arg z_1^2 + \pi$, which means that $z_2 = i\frac{z_1}{|z_1|}|z_2|$. Thus, the minimum of $|\varphi|$ on B_2 is the square root of the minimum of

$$\left| z_1^2 - \frac{z_1^2}{|z_1|^2}(r_2^2 - |z_1 - \gamma|^2) \right| = |z_1|^2 - r_2^2 + |z_1 - \gamma|^2$$

subject to $|z_1 - \gamma| \leq r_2$. If $F(z_1) = |z_1|^2 - r_2^2 + |z_1 - \gamma|^2$, then the unique critical point of F in the plane is at the point $\gamma/2$, which, as above, is not in the disk $|z_1 - \gamma| \leq r_2$. Thus, the minimum occurs on the boundary, and we want the minimum of $|z_1|^2$ on B_2. This minimum is $(|\gamma| - r_2)^2$, and it occurs at a unique point. It follows that, as desired, $\varphi(B_2)$ meets $\varphi(B_1)$ at most in a single point.

Theorem 1.6.19 applied to the polynomially convex sets $X_1 = B_1$ and $X_2 = B_2 \cup B_3$ implies that $\cup B_j$ is polynomially convex.

We must finally deduce the result in $\mathbb{C}^N$ from the result in $\mathbb{C}^2$. To do this, consider three mutually disjoint closed balls B_1, B_2, and B_3 in $\mathbb{C}^N$. We can choose coordinates such that B_1 is the closed unit ball in $\mathbb{C}^N$ and such that the radii of B_2 and B_3 are both not more than one. Moreover, we can suppose the centers of B_1 and B_2 to lie in the subspace $\Pi = \{z \in \mathbb{C}^N : z_3 = \cdots = z_N = 0\}$ of $\mathbb{C}^N$. If $B'_j = B_j \cap \Pi$, then $\cup_j B'_j$ is polynomially convex, because this union is the union of three disjoint closed balls in the copy Π of $\mathbb{C}^2$. If π is the orthogonal projection of $\mathbb{C}^N$ onto Π, then for every compact subset E of $\pi^{-1}(\cup B'_j)$, the hull $\widehat{E}$ is contained in $\pi^{-1}(\cup B'_j)$, whence $\cup B_j$ is polynomially convex.

Granted the polynomial convexity of $\cup B_j$, the equality of $\mathscr{P}(\cup B_j)$ and $A(\cup B_j)$ follows from the Oka–Weil theorem.

For further discussion of unions of convex sets see the work of Mueller [253–255].

An analysis rather like the one just used together with Theorem 1.6.19 implies that certain additional unions of balls are polynomially convex.

Theorem 1.6.21. *The union X of a finite number of closed balls with mutually disjoint interiors centered at points of $\mathbb{R}^N \subset \mathbb{C}^N$ is polynomially convex and satisfies $\mathscr{P}(X) = A(X)$.*

The polynomial convexity of X is a result of Khudaiberganov [205]. See also [336]. Notice that the closed balls of the theorem need not be disjoint, but because they have disjoint interiors, by pairs they intersect at most in singletons. No three of them can have a common point. In some cases their union is connected.

Proof. The proof is by induction on the number of balls. One ball is polynomially convex. Suppose, therefore that the union of n closed balls with disjoint interiors centered at points of $\mathbb{R}^N$ is polynomially convex. Given $n+1$ such balls, without loss of generality we can assume that one of them is $\mathbb{B}_N$ and that the others have radius not more than one. We will show that if $P(z)$ is the polynomial given by $P(z) = z_1^2 + \cdots + z_N^2$, if $a \in \mathbb{R}^N$ is of norm greater than one, and if $r \in (0, 1]$, then $\Re P \geq 1$ on the closed ball $\bar{\mathbb{B}}_N(a, r)$. Granted this, Kallin's lemma, Theorem 1.6.19, implies that the union, X, of our $n+1$ balls is polynomially convex.

Accordingly, let $z \in b\mathbb{B}_N(a, r)$ with $a \in \mathbb{R}^N$, $|a| > 1$, and $0 < r \leq 1$. Then, with $z = x + iy$, $x, y \in \mathbb{R}^N$, $\Re P(z) = |x|^2 - |y|^2 = |x|^2 + |x - a|^2 - r^2 = 2|x|^2 - 2x \cdot a + |a|^2 - r^2 \geq 2|x|^2 - 2|x||a| + |a|^2 - r^2$. The quadratic $\varphi(t) = 2t^2 - 2t|a| + |a|^2 - r^2$ takes its minimum when $t = |a|/2$. For our purposes, $t = |x|$ varies through the interval from $|a| - r$ to $|a| + r$. We have $|a|/2 \leq |a| - r$ because $|a| > 1$, $r \in (0, 1]$, and $|a| \geq r + 1$, so it follows that

$$\Re P(z) \geq 2(|a| - r)^2 - 2|a|(|a| - r) + |a|^2 - r^2 = (|a| - r)^2 \geq 1.$$

Thus X is seen to be polynomially convex. The final statement of the theorem is a consequence of two corollaries of the proof of Theorem 1.6.19:

Corollary 1.6.22. *With X_1, X_2, and p as in Theorem 1.6.19, if μ is a measure on $X_1 \cup X_2$ that annihilates $\mathscr{P}(X_1 \cup X_2)$, there is a decomposition $\mu = \mu_0 + \mu_1 + \mu_2$ with μ_0, μ_1, and μ_2 concentrated on $p^{-1}(0) \cap (X_1 \cup X_2)$, on $X_1 \setminus p^{-1}(0)$, and on $X_2 \setminus p^{-1}(0)$, respectively, and with each of μ_0, μ_1, and μ_2 orthogonal to the polynomials.*

Corollary 1.6.23. *With X_1, X_2, and p as in Theorem 1.6.19,*

$$\mathscr{P}(X_1 \cup X_2) = \{f \in \mathscr{C}(X_1 \cup X_2) : f|X_1 \in \mathscr{P}(X_1) \text{ and } f|X_2 \in \mathscr{P}(X_2)\}.$$

The former of these two corollaries implies the latter: If $f \in \mathscr{C}(X_1 \cup X_2)$ is not in $\mathscr{P}(X_1 \cup X_2)$, there is a measure μ carried by $X_1 \cup X_2$ that annihilates $\mathscr{P}(X_1 \cup X_2)$ but that does not annihilate f. Decompose μ as $\mu = \mu_0 + \mu_1 + \mu_2$ in accordance with the first corollary. The function f lies in both $\mathscr{P}(X_1)$ and $\mathscr{P}(X_2)$, so the measures μ_0, μ_1, and μ_2 all annihilate f, and we have a contradiction.

For the proof of Corollary 1.6.22, simply note that it is proved in the penultimate paragraph of the proof of Theorem 1.6.19.

To finish the proof of Theorem 1.6.21, we need to see that $\mathscr{P}(X) = A(X)$, which follows by induction, using Corollary 1.6.23.

1.6.5. The Effect of Proper Maps

It is sometimes useful to have information about the effect of holomorphic maps on the convexity properties of a set. It is surely not generally true that the holomorphic image of a polynomially convex set is again polynomially convex. However, for proper holomorphic maps, there is a useful result. (Recall that a map $f : X \to Y$ is proper if for each comapct subset K of Y, the set $f^{-1}(K)$ is a compact subset of X.)

Theorem 1.6.24. *If $F : \mathbb{C}^N \to \mathbb{C}^N$ is a proper holomorphic map, and if $X \subset \mathbb{C}^N$ is a compact set, then the set X is polynomially convex if and only if the set $F^{-1}(X)$ is polynomially convex, and $\mathscr{P}(X) = \mathscr{C}(X)$ if and only if $\mathscr{P}(F^{-1}(X)) = \mathscr{C}(F^{-1}(X))$.*

The proof of this depends on some information about proper holomorphic maps. If Ω and Ω' are domains in $\mathbb{C}^N$, and if $F : \Omega \to \Omega'$ is a proper holomorphic map, then there is a positive integer μ, the *multiplicity* of F, with the property that for every $z \in \Omega'$ outside some fixed analytic subvariety[7] $V \subset \Omega'$, the fiber $F^{-1}(z)$ consists of exactly μ distinct points. The algebra $\mathscr{O}(\Omega)$ is integral over $\mathscr{O}(\Omega')$ in the sense that for each $f \in \mathscr{O}(\Omega)$, there are functions $g_0, g_1, \ldots, g_{\mu-1}$ in $\mathscr{O}(\Omega')$ such that $f^\mu + \sum_{j=0}^{\mu-1} f^j g_j \circ F = 0$. The functions g_j are determined by the condition that if $z \in \Omega'$ is a point with μ distinct preimages under F, then $g_j(z)$ is, to within a sign, the $(\mu - j)$th elementary symmetric function of the values f assumes on the fiber $f^{-1}(z)$. (Thus, $g_0(z)$ is, to within sign, the product of all the values f assumes on the fiber $F^{-1}(z)$, and $g_{\mu-1}(z)$ is their sum.) This geometric situation is described by saying that the triple (Ω, F, Ω') is an *analytic cover*. The theory of these is developed in detail in books that discuss analytic varieties, e.g., [158].

Proof of the theorem. Suppose that X is polynomially convex. If $y \in \mathbb{C}^N \setminus F^{-1}(X)$, then $F(y) \notin X$, so there is a polynomial P with $P(F(y)) = 1$ but $|P|_X < 1$. The function $P \circ F$ shows that $y \notin \widehat{F^{-1}(X)}$.

[7] A brief summary of some basic facts about varieties is given at the beginning of Section 3.2 below.

Conversely, suppose the set $Y = F^{-1}(X)$ to be polynomially convex. If $y \in \mathbb{C}^N \setminus X$, then the set $S = F^{-1}(y)$ is disjoint from Y, and the set $S \cup Y$ is polynomially convex. The Oka–Weil theorem provides a polynomial P with $P = 1$ on S and $|P| < \frac{1}{2}$ on Y. As described in the paragraph above, P satisfies a monic polynomial equation with coefficients of the form $g_j \circ F$ for functions g_j holomorphic on $\mathbb{C}^N$. If the multiplicity of the map F is μ, then the coefficient of $P^{\mu-1}$ in this equation, call it $g_{\mu-1} \circ F$, satisfies $g_{\mu-1} \circ F(y) = -\mu$ and $|g_{\mu-1} \circ F| < \mu/2$ on X. Accordingly, X is polynomially convex.

If $\mathscr{P}(Y) = \mathscr{C}(Y)$, then for every $f \in \mathscr{C}(X)$, there is a sequence $\{P_k\}_{k=1,\dots}$ of polynomials that converges to $f \circ F$. Each of these polynomials is integral over $\mathscr{O}(\mathbb{C}^N)$. Denote by p_k the coefficient of $P_k^{\mu-1}$ in the monic equation over $\mathscr{O}(\mathbb{C}^N)$ satisfied by P_k, so that off a subvariety in $\mathbb{C}^N$, $p_k(z)$ is simply the sum of the values of P_k on $F^{-1}(z)$. Then $\{\frac{g_k}{\mu}\}_{k=1,\dots}$ is a sequence of holomorphic functions on $\mathbb{C}^N$—generally not polynomials—that converges uniformly to f. This implies that $\mathscr{P}(X) = \mathscr{C}(X)$.

Finally, suppose that $\mathscr{P}(X) = \mathscr{C}(X)$. In this case, the algebra $\mathscr{P}(Y)$ contains all the functions $g \circ F$, $g \in \mathscr{C}(X)$. From this point, argue by duality, as in previous situations. If $\mathscr{P}(Y) \neq \mathscr{C}(Y)$, then there are nonzero measures σ on Y such that $\int_Y P(z)\, d\sigma(z) = 0$ for all polynomials P. Choose such a σ that is an extreme point of the unit ball in the space of all such orthogonal measures. Because $\mathscr{P}(X) = \mathscr{C}(X)$, the measure σ has to be concentrated on a fiber $F^{-1}(x)$ for an $x \in X$. This fiber is a finite set. There are polynomials on $\mathbb{C}^N$ that take arbitrarily prescribed values at the points of a finite set, so it follows that the measure σ must be the zero measure. Consequently, $\mathscr{P}(Y) = \mathscr{C}(Y)$.

In the theorem just proved, we have a proper holomorphic map from $\mathbb{C}^N$ to $\mathbb{C}^N$. The parallel result in the case of proper holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^M$ with $M > N$ is correct, but the proof requires some deeper results from analytic geometry than have been used in the proof just given.

The following example of Wermer [369] complements the preceding result. Define $\Phi : \mathbb{C}^2 \to \mathbb{C}^3$ by

$$\Phi(z, w) = (z, zw, zw^2 - w).$$

One verifies without difficulty that Φ is injective, is regular, and is not proper. (That it is not proper can be seen by noting that $\Phi(\frac{1}{n}, n) = (\frac{1}{n}, 1, 0)$, which shows that the Φ-preimage of the (compact) closed unit tridisk in $\mathbb{C}^3$ is not compact.)

The image $\Phi(\bar{\mathbb{U}}^2)$ of the closed unit bicylinder, call it Δ, is not polynomially convex: The point $(0, 1, 0) \in \mathbb{C}^3$ does not lie in Δ, but Δ contains the points $\Phi(e^{i\vartheta}, e^{-i\vartheta}) = (e^{i\vartheta}, 1, 0)$, $\vartheta \in \mathbb{R}$, which constitute a circle, γ. The polynomially convex hull of γ is the disk $D = \{(z, 1, 0) : z \in \bar{\mathbb{U}}\}$, which passes through the point $(0, 1, 0)$, so Δ is not polynomially convex. In [369] Wermer determines the hull of Δ; it is the union $\Delta \cup D$.

This example turned out to be important in an entirely different context: In [117], Fornæss constructed a complex manifold $\mathscr{M}$ that is not a Stein manifold but that is a union $\mathscr{M} = \cup_{k=1,\dots}\Omega_k$ of domains with $\Omega_k \subset \Omega_{k+1}$ for all k and with each domain Ω_k a Stein manifold. This construction draws on the example just given. Another construction of such a manifold is given in [120]. The existence of such examples contrasts with the situation in $\mathbb{C}^N$, where a domain that is the union of an increasing sequence of domains of holomorphy is itself a domain of holomorphy.

1.7. Hulls with No Analytic Structure

If a subset X of $\mathbb{C}^N$ is not polynomially convex, it is important to understand the structure of the set $\widehat{X} \setminus X$. The only obvious explanation for the presence of points in this set is related to analytic structure.

Definition 1.7.1. *Let X be a compact subset of $\mathbb{C}^N$. If $x \in X$, an* analytic disk in X through *x is a nonconstant map $\psi : \mathbb{U} \to X$ with $x \in \psi(\mathbb{U})$ and with the property that for every polynomial P, the composition $P \circ \psi$ is holomorphic on $\mathbb{U}$.*

We shall say that a set $E \subset \mathbb{C}^N$ *contains an analytic disk* if there is a nonconstant analytic map $\varphi : \mathbb{U} \to \mathbb{C}^N$ with $\varphi(\mathbb{U}) \subset E$.

In all the simple examples that come to mind, the presence of points in $\widehat{X} \setminus X$ is associated with the existence of analytic disks. It was entirely natural to suppose that in the general case, the complementary set $\widehat{X} \setminus X$ should consist of a complex-analytic manifold or more generally of a complex-analytic variety, or at least that it should contain analytic disks in great plenitude. Stolzenberg [340] demonstrated the matter to be much more complicated than this by constructing a compact set X in $\mathbb{C}^2$ such that the nonempty set $\widehat{X} \setminus X$ contains *no* analytic disk. In the forty-odd years since its discovery, other, essentially simpler, examples of the same general phenomenon have been discovered: See the papers of Alexander [25], Basener [44], Duval and Levenberg [106], and Wermer [372, 374]. Levenberg's paper [222] contains a survey of the general subject

In this section we give the example of Duval and Levenberg:

Theorem 1.7.2. *If K is a compact, polynomially convex subset of $\mathbb{B}_N$, $N \geq 2$, then there is a compact subset X of $b\mathbb{B}_N$ such that $\widehat{X} \supset K$ and such that the set $\widehat{X} \setminus K$ contains no analytic disk.*

Corollary 1.7.3. *There are compact subsets X of $\mathbb{C}^N$, $N \geq 2$, such that $\widehat{X} \setminus X$ is not empty but contains no analytic disk.*

Granted the theorem, it follows immediately that *if Ω is an arbitrary bounded domain in $\mathbb{C}^N$, $N \geq 2$, and if K is a polynomially convex subset of Ω, then there exists a compact subset Y of $b\Omega$ such that $\widehat{Y} \supset K$ and $\widehat{Y} \setminus K$ contains no analytic disk.* To see this, let $R > 0$ be so large that the ball $\mathbb{B}_N(R)$ contains $\bar{\Omega}$. The theorem provides a compact set X contained in $b\mathbb{B}_N(R)$ such that $\widehat{X} \supset K$ and $\widehat{X} \setminus K$ contains no analytic disk. The local maximum modulus principle, which is proved below, Theorem 2.1.8, implies that for the set Y we can take $\widehat{X} \cap b\Omega$. Notice that there is absolutely no condition whatsoever imposed on $b\Omega$ in this construction.

The proof of Theorem 1.7.2 depends on two simple lemmas:

Lemma 1.7.4. *If $X \subset \mathbb{C}^N$ is a polynomially convex set, and if $E \subset X$ is polynomially convex, then for every holomorphic function f defined on a neighborhood of X, the set $E \cup (X \cap f^{-1}(0))$ is polynomially convex.*[8]

[8] The general fact is that *if V is any analytic variety in a neighborhood of X, then $E \cup (X \cap V)$ is polynomially convex.* This is immediate from the stated lemma, granted some standard information about varieties: If V is a subvariety of the neighborhood Ω of X, then by shrinking Ω if necessary, we can suppose it to be a polynomial polyhedron and so a domain of holomorphy. Every analytic subvariety of a domain of holomorphy

Proof. Consider a point $x \notin E \cup (X \cap f^{-1}(0))$. By replacing f by $f/f(x)$ if necessary, we can suppose $f(x) = 1$. The Oka–Weil theorem implies that $f|X \in \mathscr{P}(X)$. Also, because $x \notin E$ and E is polynomially convex, there is a polynomial p with $p(x) = 1$ and $\|p\|_E < \frac{1}{2}$. If for large integral n, $h = p^n f$, then $\|h\|_{E\cup(f^{-1}(0)\cap X)} < 1$ and $h(x) = 1$, whence $x \notin (E \cup (f^{-1}(0) \cap X))\widehat{}$. The lemma is proved.

Lemma 1.7.5. *If $K \subset \mathbb{B}_N$ is a polynomially convex set, and if $\{Q_j\}_{j=1,\ldots}$ is a sequence of polynomials such that all the sets $Z_j = \bar{\mathbb{B}}_N \cap Q_j^{-1}(0)$ are disjoint from K, then there is a compact subset X of $b\mathbb{B}_N$ such that $\widehat{X} \supset K$ and $\widehat{X} \cap Z_j = \emptyset$ for all $j = 1, \ldots$.*

Proof. The set X is constructed as the intersection $\cap_{k=1,\ldots} X_k$ of a decreasing sequence of compact subsets X_k of $b\mathbb{B}_N$ such that for each k, $K \subset \operatorname{int} \widehat{X}_k$ and $\widehat{X}_k \cap \left(\cup_{j=1}^k Z_j\right) = \emptyset$. The sets X_k are constructed inductively. To begin with, the set $K \cup Z_1$ is polynomially convex, so there is a polynomial F_1 with

$$\Re F_1 < -1 \text{ on } K \quad \text{and} \quad \Re F_1 > 1 \text{ on } Z_1.$$

Define X_1 to be $\{\Re F_1 \leq 0\} \cap \bar{\mathbb{B}}_N$.

For the inductive step, assume $X_1 \supset X_2 \supset \cdots \supset X_n$ to be compact subsets of $b\mathbb{B}_N$ such that for suitable polynomials $F_1, \ldots, F_n$, $X_k = \{\Re F_k \leq 0\} \cap \bar{\mathbb{B}}_N$ and such that $\widehat{X}_k$ is disjoint from Z_j for $1 \leq j \leq k \leq n$. We require also that $K \subset \{\Re F_k < 0\}$ for each k. With this arrangement, let $L_n = \{\Re F_n \geq 0\} \cap \bar{\mathbb{B}}_N$. This set is polynomially convex, and by Theorem 1.6.19, the set $K \cup L_n$ is also polynomially convex, for $\Re F_n < 0$ on K.

The set $K \cup L_n \cup Z_{n+1}$ is polynomially convex. Consequently, there is a polynomial F_{n+1} such that $\Re F_{n+1} < -1$ on K, and $\Re F_{n+1} > 1$ on $L_n \cup Z_{n+1}$. The set X_{n+1} is defined by $X_{n+1} = \{\Re F_{n+1} \leq 0\} \cap b\mathbb{B}_N$. We have that $\widehat{X}_{n+1} = \{\Re F_{n+1} \leq 0\} \cap \bar{\mathbb{B}}_N$.

This completes the construction of the sets X_n and the proof of the lemma.

Proof of Theorem 1.7.2. The theorem is proved by applying the preceding construction with a judiciously chosen sequence of polynomials Q_j.

For $x \in \mathbb{B}_N \setminus K$ choose a polynomial P_x with $P_x(x) = 0$ and $\Re P_x < -1$ on K. If $x = (x_1, \ldots, x_N)$, let $P_{x,j}(z) = P_x^2(z) + \varepsilon(z_j - x_j)$. For sufficiently small ε, the polynomials $P_{x,j}$ form a set of local coordinates in a ball B_x centered at x whose closure is disjoint from K. By making the balls sufficiently small, we can ensure that $P_{j,x}(B_x)$ is disjoint from $P_{j,x}(K)$.

Countably many of the balls B_x, say $\{B_k\}_{k=1,\ldots}$, cover $\mathbb{B}_N \setminus K$. Let the associated polynomials $P_{x,j}$ be $P_{k,1}, \ldots, P_{k,N}$. For each choice of k and j, let $\{\alpha_{m,k,j}\}_{m=1,\ldots}$ be a sequence dense in the open subset of $P_{k,j}(B_k)$ of the plane. Define $Q_{m,k}$ to be the polynomial

$$Q_{m,k}(z) = \prod_{j=1}^{N} [P_{k,j}(z) - \alpha_{m,k,j}].$$

The polynomial $Q_{m,k}$ vanishes on a variety $V_{m,k}$ that misses K. For the sake of notation, relabel the $Q_{m,k}$ as $Q_1, Q_2, \ldots$ and relabel the $V_{m,k}$ correspondingly. Let $Z_k = V_k \cap \bar{\mathbb{B}}_N$.

is the intersection of the zero loci of a family of functions holomorphic on the domain.

By the preceding lemma, there is a compact subset X of $b\mathbb{B}_N$ such that $\widehat{X} \supset K$ and $\widehat{X} \cap Z_n = \emptyset$ for all $n = 1, \ldots$.

There can be no nonconstant holomorphic map $\varphi : \mathbb{U} \to \mathbb{B}_N$ with $\varphi(\mathbb{U}) \subset \widehat{X} \setminus K$. Each point of $b\mathbb{B}_N$ is a peak point for the algebra $\mathscr{P}(X)$, so by the maximum principle any such function would have to carry $\mathbb{U}$ into $\widehat{X} \cap \mathbb{B}_N$. There would then be a nonconstant map, ψ, from a disk into one of the balls B_x; for ψ take the restriction of φ to a sufficiently small disk about the origin. Suppose Q_n to be one of the Q's associated with the ball B_x. Because $\widehat{X}$ is disjoint from the associated variety Z_k, it follows that for $j = 1, \ldots, N$, the functions $P_{x,j} \circ \psi$ have to omit all values in a dense subset of $P_{x,j}(B_x)$. Accordingly, they are constant. The $P_{x,j}$ constitute local holomorphic coordinates in B_x, so we have a contradiction to the assumption that φ is nonconstant.

The theorem is proved.

Chapter 2

SOME GENERAL PROPERTIES OF POLYNOMIALLY CONVEX SETS

Introduction. The main properties of polynomially convex sets discussed in this chapter are of a general character in that they do not depend on particular structural properties of the sets involved. Section 2.1 contains some of the information about polynomially convex sets that can be derived from the theory of the Cousin problems. Section 2.2 contains two characterizations of polynomially convex sets. Section 2.3 brings the geometric methods of Morse theory and algebraic topology to bear on polynomial convexity. Section 2.4 is devoted to some results for various classes of compacta in Stein manifolds that are parallel to results for polynomially convex subsets of $\mathbb{C}^N$.

2.1. Applications of the Cousin Problems

The study of the Cousin problems has played a central role in the development of the theory of holomorphic functions of several complex variables, and the solutions of these problems find many applications in function theory. The present section is devoted in the main to some applications of these problems in the theory of polynomial convexity.

A natural setting in which to study the Cousin problems is that of Stein manifolds. Recall that an M-dimensional complex manifold $\mathscr{M}$ is a *Stein manifold* if it possesses the following properties:

(a) Holomorphic functions separate points: If x and y are distinct points of $\mathscr{M}$, then there is an $f \in \mathscr{O}(\mathscr{M})$ with $f(x) \neq f(y)$.

(b) For each point x in $\mathscr{M}$, there are M functions $f_1, \dots, f_M \in \mathscr{O}(\mathscr{M})$ such that $df_1 \wedge \cdots \wedge df_M$ does not vanish at x. Thus, the globally defined functions $f_1, \dots, f_M$ provide local holomorphic coordinates near x.

(c) The manifold $\mathscr{M}$ is *holomorphically convex* in the sense that for each compact set $X \subset \mathscr{M}$, the $\mathscr{O}(\mathscr{M})$-convex hull $\mathscr{O}(\mathscr{M})$-hull X defined by

$$\mathscr{O}(\mathscr{M})\text{-hull } X = \{x \in \mathscr{M} : |f(x)| \leq \|f\|_X \text{ for all } f \in \mathscr{O}(\mathscr{M})\}$$

is compact.

Examples are the closed, complex submanifolds of $\mathbb{C}^N$. For such a manifold $\mathscr{M}$, the points (a) and (b) of the definition are evident; the holomorphic convexity, point (c), is not at all simple, though it follows from some standard results in function theory: It suffices to know that a closed complex submanifold $\mathscr{M}$ of $\mathbb{C}^N$ is the intersection of the zero loci of the functions in $\mathscr{O}(\mathbb{C}^N)$ that vanish on it, for then if X is a compact subset of $\mathscr{M}$, the set $\widehat{X}$ is contained in $\mathscr{M}$. It follows that the $\mathscr{O}(\mathscr{M})$-hull of X is contained in the compact set $\widehat{X}$, and so is compact.

By the fundamental embedding theorem for Stein manifolds [158], [180], the closed complex submanifolds of $\mathbb{C}^N$ are, to within biholomorphic equivalence, the only examples: *Every M-dimensional Stein manifold is biholomorphically equivalent to a closed complex submanifold of some* $\mathbb{C}^N$. (It suffices to take $N = 2M + 1$; better results are known.)

The formulation of the Cousin problems is as follows.[1] The context is a complex manifold $\mathscr{M}$, perhaps a domain in $\mathbb{C}^N$. For the *first Cousin problem*, or the *additive Cousin problem*, we are given an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathscr{M}$ and for each index $\alpha \in A$ a meromorphic function f_α. It is supposed that $f_\alpha - f_\beta$ is holomorphic on the intersection $U_{\alpha,\beta} = U_\alpha \cap U_\beta$, and it is required to find a function f meromorphic on $\mathscr{M}$ such that for all indices α, $f - f_\alpha$ is holomorphic on U_α. This is the problem of finding a meromorphic function with prescribed poles, i.e., the multivariate analogue of the problem solved by the classical Mittag-Leffler theorem. An alternative formulation is in terms of holomorphic functions only. For each pair α, β of indices in A, there is given a holomorphic function $g_{\alpha,\beta} \in \mathscr{O}(U_{\alpha,\beta})$; it is required to find holomorphic functions g_α on U_α such that $g_\alpha - g_\beta = g_{\alpha,\beta}$ on $U_{\alpha,\beta}$. For the desired g_α's to exist, it is plainly necessary that $g_{\alpha,\beta} - g_{\alpha,\gamma} + g_{\beta,\gamma} = 0$; this is the *cocycle condition*. The existence of the desired g_α's for every choice of functions $g_{\alpha,\beta}$ that satisfy the cocyle condition and for all choices of the open covering $\mathscr{U}$ is the vanishing of the first cohomology group $H^1(\mathscr{M}; \mathscr{O})$ of $\mathscr{M}$ with values in the sheaf $\mathscr{O}$ of germs of holomorphic functions on $\mathscr{M}$.

For the *second Cousin problem*, or *multiplicative Cousin problem*, we are again given an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ and for each index α a holomorphic function f_α. It is required that the quotient f_α / f_β be holomorphic and zero-free in $U_{\alpha,\beta}$, and we ask for a holomorphic function f such that for each α the quotient f/f_α is holomorphic and

[1]The name derives from the work of Pierre Cousin, who, in his thesis [92], studied these problems.

zero-free on U_α. This is the problem of finding a holomorphic function with prescribed zeros, which is the higher-dimensional analogue of the problem solved by the Weierstrass theorem in the theory of functions of one complex variable. An alternative formulation is the following. We are given zero-free holomorphic functions $g_{\alpha,\beta}$ on $U_{\alpha,\beta}$, and we ask for zero-free holomorphic functions g_α on U_α such that for all α, β, $g_\alpha/g_\beta = g_{\alpha,\beta}$ on $U_{\alpha,\beta}$. Again, for there to be a solution, the obvious *cocycle condition* $g_{\alpha,\beta}g_{\alpha,\gamma}^{-1}g_{\beta,\gamma} = 1$ must be satisfied. The existence of the desired g_α's for every choice of functions $g_{\alpha,\beta}$ that satisfy the cocyle condition is the vanishing of the first cohomology group $H^1(\mathscr{M}; \mathscr{O}^*)$ of $\mathscr{M}$ with values in the sheaf $\mathscr{O}^*$ of germs of zero-free holomorphic functions on $\mathscr{M}$.

These Cousin problems are not universally solvable. However, it is a standard result in the theory of functions of several complex variables that the first Cousin problem is solvable on every domain of holomorphy in $\mathbb{C}^N$ and, more generally, on every Stein manifold.

The second Cousin problem is not universally solvable on domains of holomorphy; there is a topological condition. If D is a domain of holomorphy in $\mathbb{C}^N$ or a Stein manifold, then the second problem of Cousin is universally solvable on D if and only if the second integral cohomology group $\check{H}^2(D; \mathbb{Z})$ vanishes.

The relation between the second Cousin problem and the vanishing of the second integral cohomology can be spelled out in concrete terms as follows. Given the open covering $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ for D and given functions $f_\alpha \in \mathscr{O}(U_\alpha)$ such that $f_\alpha f_\beta^{-1} = f_{\alpha,\beta}$ is holomorphic and zero-free on $U_{\alpha\beta}$, we pass to a refinement $\{V_\alpha\}_{\alpha\in A'}$. Thus, there is a map $\iota : A' \to A$ such that for all $\alpha \in A'$, $V_\alpha \subset U_{\iota\alpha}$. For each $\alpha \in A'$, we let f_α be $f_{\iota\alpha}|V_\alpha$. The refinement $\{V_\alpha\}_{\alpha\in A'}$ can be chosen fine enough that for all $\alpha, \beta \in A'$, there is a holomorphic determination of $\log(f_\alpha/f_\beta)$ on $V_{\alpha\beta}$. Set

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}\log(f_\alpha/f_\beta) + \log(f_\beta/f_\gamma) + \log(f_\gamma/f_\alpha).$$

The function $c_{\alpha,\beta,\gamma}$ is a locally constant $\mathbb{Z}$-valued function, and $2\pi i c_{\alpha,\beta,\gamma}$ is a determination of log 1. The collection $\{c_{\alpha,\beta,\gamma}\}_{\alpha,\beta,\gamma\in A'}$ is a 2-cocycle for the covering $\{V_\alpha\}_{\alpha\in A'}$ with values in $\mathbb{Z}$, for

$$c_{\beta,\gamma,\delta} - c_{\alpha,\gamma,\delta} + c_{\alpha,\beta,\delta} - c_{\alpha,\beta,\gamma} = 0$$

for all choices of $\alpha, \beta, \gamma, \delta \in A'$. It therefore determines an element of the cohomology group $\check{H}^2(D, \mathbb{Z})$. If this group is zero, then assuming we have chosen the covering $\{V_\alpha\}_{\alpha\in A'}$ to be sufficiently fine, this cocycle $\{c_{\alpha,\beta,\gamma}\}_{\alpha,\beta,\gamma\in A'}$ is a coboundary: There exist continuous $\mathbb{Z}$-valued functions $m_{\alpha,\beta}$ on $V_{\alpha\beta}$ with

$$c_{\alpha,\beta,\gamma} = m_{\beta,\gamma} - m_{\alpha,\gamma} + m_{\alpha,\beta}.$$

Define $h_{\alpha,\beta} \in \mathscr{O}(V_{\alpha,\beta})$ by

$$h_{\alpha,\beta} = \frac{1}{2\pi i}\log(f_\alpha/f_\beta) - m_{\alpha,\beta}.$$

Then $h_{\beta,\gamma} - h_{\alpha,\gamma} + h_{\alpha,\beta} = 0$. Thus, $\{h_{\alpha,\beta}\}_{\alpha,\beta\in A'}$ determines an element in the cohomology group $\check{H}^1(D, \mathscr{O})$, which, by assumption, vanishes, because $\mathscr{M}$ is a Stein manifold. Thus, there are functions $h_\alpha \in \mathscr{O}(V_\alpha)$ with

$$h_\alpha - h_\beta = h_{\alpha,\beta} - m_{\alpha,\beta}.$$

We now define $g_\alpha \in \mathscr{O}(V_\alpha)$ by

$$g_\alpha = f_\alpha e^{-2\pi i h_\alpha}.$$

On $V_{\alpha\beta}$ these functions satisfy

$$g_\alpha / g_\beta = (f_\alpha / f_\beta) e^{-2\pi i (h_\alpha - h_\beta)} = 1.$$

Therefore the function G defined by the condition that $G = g_\alpha$ on V_α is a well-defined holomorphic function on D with the property that G/f_α is holomorphic and zero-free on V_α. Thus, our multiplicative Cousin problem is solved.

Thorough treatments of the Cousin problems are given in [158] and [180]. The third memoir of Oka [273] is still well worth reading for its treatment of the second Cousin problem.

The first application of the Cousin problems in our study of polynomial convexity is to a characterization of polynomially convex sets in terms of certain families of analytic hypersurfaces, a result that goes back to Oka [272]. By an *analytic hypersurface* in a complex manifold we understand a closed set that is locally defined by the vanishing of a single holomorphic function. In general, an analytic hypersurface will not be defined globally by the vanishing of a single holomorphic function; as just noted, the second Cousin problem is generally not solvable. An analytic hypersurface Σ is termed *principal* if it is the zero locus of a holomorphic function. (Note: In this formulation it is not required that there be a holomorphic function F on the manifold whose zero locus is the hypersurface Σ and that satisfies, in addition, the condition that if g is a function holomorphic on an open set W in the manifold that vanishes on $W \cap \Sigma$, then $g = F\tilde{g}$ for a function $\tilde{g}$ holomorphic on W. Briefly put, F is not required to generate the ideal sheaf of functions vanishing on Σ.)

Definition 2.1.1. *A* continuous family of principal analytic hypersurfaces *in a complex manifold $\mathscr{M}$ is a family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in $\mathscr{M}$ such that there exists a continuous function $F : [0,1) \times \mathscr{M} \to \mathbb{C}$ with the property that for each $t \in [0,1)$, $F(t,\cdot)$ is a nowhere locally constant function holomorphic on $\mathscr{M}$ the zero locus of which is the hypersurface V_t.*

As an example, let $Y \subset \mathbb{C}^N$ be a polynomially convex set. If $z_o \in \mathbb{C}^N \setminus Y$, there is a polynomial P with $|P| < 1$ on Y and $P(z_o) = 1$. Define $F(t,z) = P(z) - 1 + \log(1-t)$ for $t \in [0,1)$. If V_t denotes the zero locus of $F(t,\cdot)$, then $\{V_t\}_{t\in[0,1)}$ is a continuous family of principal analytic, indeed, *algebraic*, hypersurfaces in $\mathbb{C}^N$ each member of which is disjoint from Y that contains the point z_o and that diverges to infinity in the sense of the following definition.

Definition 2.1.2. *A continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in a complex manifold $\mathscr{M}$* diverges to infinity *as $t \to 1^-$ if for each compact subset K of $\mathscr{M}$, there is a $t_K \in [0,1)$ large enough that for all $t \in [t_K, 1)$, the variety V_t is disjoint from K.*

There are two characterizations of polynomially convex sets in terms of continuous families of principal analytic hypersurfaces, one global, the other local. The former is the simpler.

Theorem 2.1.3. [272] (See also [341].) *Let $X \subset \mathbb{C}^N$ be a compact set. The point $z_o \in \mathbb{C}^N$ does not lie in the polynomially convex hull of X if and only if there is a continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in $\mathbb{C}^N$ such that $z_o \in V_o$, $V_t \cap X = \emptyset$ for all t, and V_t diverges to infinity as $t \to 1^-$.*

It is a simple matter to verify the following corollary.

Corollary 2.1.4. *Let $X \subset \mathbb{C}^N$ be a compact set. The point $z_o \in \mathbb{C}^N$ does not lie in the polynomially convex hull of X if and only if there is a continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in $\mathbb{C}^N$ such that $z_o \in V_o$, $V_t \cap X = \emptyset$ for all t, and, for some $t_o \in [0, 1)$, there is a polydisk Δ containing X that is disjoint from the variety V_{t_o}.*

Proof. Assume the theorem, and consider a compact subset X, which we assume to be contained in the open unit polydisk $\mathbb{U}^N$. Suppose there to be a continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in $\mathbb{C}^N$ defined by the continuous function $F : [0, 1) \times \mathbb{C}^N \to \mathbb{C}$ such that $V_t \cap X = \emptyset$ for all t, such that $z_o \in V_o$, and such that $V_{1/2} \cap \bar{\mathbb{U}}^N = \emptyset$. Define $\tilde{F} : [0, 1) \times \mathbb{C}^N \to \mathbb{C}$ by

$$\tilde{F}(t, z) = \begin{cases} F(t, z) & \text{if } t \in [0, \frac{1}{2}], \\ F(\frac{1}{2}, ze^{\frac{1-2t}{1-t}}) & \text{for } t \in [\frac{1}{2}, 1). \end{cases}$$

If $\{\tilde{V}_t\}_{t\in[0,1)}$ is the continuous family of principal analytic hypersurfaces defined by the function $\tilde{F}$, then it diverges to infinity in $\mathbb{C}^N$ and shows that z_o is not in $\widehat{X}$. The corollary is proved.

Proof of Theorem 2.1.3. By the discussion immediately after Definition 2.1.1, we need only verify the *if* direction of the theorem. To this end, let X be a compact subset of $\mathbb{C}^N$. Suppose that $z_o \in \mathbb{C}^N \setminus X$ and that $\{V_t\}_{t\in[0,1)}$ is a continuous family of principal analytic hypersurfaces in $\mathbb{C}^N$ that diverges to infinity for which each V_t is disjoint from X but with $z_o \in V_o$.

If some variety in the family $\{V_t\}_{t\in[0,1)}$ meets $\widehat{X}$, then compactness implies the existence of a $\tau_o \in [0, 1)$ such that V_{τ_o} meets $\widehat{X}$ but for no $t > \tau_o$ does V_t meet $\widehat{X}$. Let the continuous family $\{V_t\}_{t\in[0,1)}$ be defined by the function $F : [0, 1) \times \mathbb{C}^N \to \mathbb{C}$ as in Definition 2.1.1. Let S be the subset of $(\tau_o, 1)$ that consists of those t for which $\frac{1}{F(t,\cdot)}$ is approximable uniformly on $\widehat{X}$ by polynomials. The set S is not empty, for it contains all t sufficiently near 1: For t near 1, $\frac{1}{F(t,\cdot)}$ is holomorphic on a polydisk centered at the origin and large enough to contain $\widehat{X}$. On this polydisk the holomorphic function $\frac{1}{F(t,\cdot)}$ can be approximated uniformly on compact sets by polynomials. The set S is also closed in $(\tau_o, 1)$: If for $t_j \to t_o \in (\tau_o, 1)$, each $t_j \in S$, then because $F(t_j, \cdot)$ converges uniformly on $\widehat{X}$ to $F(t_{t_o}, \cdot)$, and because each $\frac{1}{F(t_j,\cdot)}$ is uniformly approximable on $\widehat{X}$ by polynomials, it follows that $\frac{1}{F(t_o,\cdot)}$ is uniformly approximable on $\widehat{X}$ by polynomials. Finally, S is open.

This is so, for if $t_o \in S$, then for t with $|t - t_o|$ small and for $z \in \widehat{X}$,

$$\frac{1}{F(t,z)} = \frac{1}{F(t_o,z)} \sum_{j=0,\dots} \left(\frac{F(t_o,z) - F(t,z)}{F(t_o,z)} \right)^j = \frac{1 + O(F(t,z) - F(t_o,z))}{F(t_o,z))}.$$

Because $t_o \in S$ and $|F(t,x) - F(t_o,x)|$ is uniformly small when $x \in \widehat{X}$ and $t - t_o$ is small, it follows that $t \in S$ when $t - t_o$ is small. That is, S is open. Consequently, $S = (\tau_o, 1)$.

If $t > \tau_o$ is close to τ_o, then the function $\frac{1}{F(t,\cdot)}$ is approximable uniformly on $\widehat{X}$ by polynomials. However, granted that t is sufficiently near τ_o, there will be points $y \in \widehat{X}$ at which $\frac{1}{|F(t,y)|} > \left\| \frac{1}{F(t,\cdot)} \right\|_X$. This leads to the existence of polynomials P with $|P(y)| > \|P\|_X$, a contradiction.

The conclusion is that none of the varieties V_t meet $\widehat{X}$. In particular, $z_o \notin \widehat{X}$.

There is a local version of the theorem stated above.

Theorem 2.1.5. [272] (See also [341].) *Let X be a compact subset of $\mathbb{C}^N$, let $z_o \in \mathbb{C}^N$, and let U be an open set in $\mathbb{C}^N$. If there is a continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in U with the properties that*

(a) $z_o \in V_o$,

(b) *there is a neighborhood Ω of X such that each V_t is disjoint from Ω,*

(c) $\{V_t\}_{t\in[0,1)}$ *diverges to infinity in U, and*

(d) *the set of $t \in [0,1)$ for which V_t meets $\widehat{X}$ is compact,*

then the point z_o does not lie in $\widehat{X}$.

A special case of this result is the following.

Corollary 2.1.6. *Let $X \subset \mathbb{C}^N$ be a compact set, let Ω be an open set that contains $\widehat{X}$, and let z_o be a point of Ω. The point z_o is not in $\widehat{X}$ if there exists a continuous family $\{V_t\}_{t\in[0,1)}$ of principal analytic hypersurfaces in Ω that diverges to infinity in Ω and that satisfies the conditions that $z_o \in V_o$ and $V_t \cap X = \emptyset$ for all $t \in [0,1)$.*

Corollary 2.1.7. *Let X be a compact subset of $\mathbb{C}^N$. If $x_o \in \mathbb{C}^N$, then $x_o \notin \widehat{X}$ if and only if there exist a polynomially convex set Y and an open set Ω containing Y such that $X \subset Y$ and for some continuous function $F : [0,1] \times \Omega \to \mathbb{C}$ that is holomorphic and nowhere locally constant in its second variable, $F(0,x) = 0$, $F^{-1}(0) \cap X = \emptyset$, and for t near* 1, $Y \cap \{z : F(t,z) = 0\} = \emptyset$.

Proof. The necessity of the condition is a consequence of Corollary 2.1.4. Its sufficiency follows from the local description of hulls, Theorem 2.1.5.

Proof of Theorem 2.1.5. The proof of this theorem is technically more involved than the rather elementary proof of Theorem 2.1.3, for it involves the solution of an additive Cousin problem.

Assume the statement false, so that $z_o \in \widehat{X}$.

The hypotheses of the theorem provide a continuous function $F : [0, 1) \times U \to \mathbb{C}$ that defines the family $\{V_t\}_{t\in[0,1)}$. We begin with a special case of the theorem, that in which the function F is defined and holomorphic on a neighborhood $\Delta \times U$ of $[0, 1) \times U$ in $\mathbb{C}^{N+1}$ for some connected, simply connected open set Δ in $\mathbb{C}$ that contains the half-open interval $[0, 1)$ as a closed subset. Let $W' = \Delta \times U \subset \mathbb{C}^{N+1}$, and put $W'' = \Delta \times \Omega$. Let $\tau = \sup\{t \in [0, 1) : V_t \cap \widehat{X} \neq \emptyset\}$, and fix $t_o \in (\tau, 1)$. Note that by (d), V_τ meets $\widehat{X}$. The set $W = W' \cup W''$ is a neighborhood of the polynomially convex set $[0, t_o] \times \widehat{X}$ in $\mathbb{C}^{N+1}$. Let $W_o \Subset W$ be a polynomial polyhedron that contains $[0, t_o] \times \widehat{X}$. Define a set of data for the additive Cousin problem on W_o by taking $f' = 0$ on $W_o \cap W'$ and $f'' = \frac{1}{F(\zeta,z)}$ for $(\zeta, z) \in W_o \cap W''$. There is a solution to this Cousin problem: There is a function f meromorphic on W_o such that $f - f'$ is holomorphic on $W_o \cap W'$ and $f - f''$ is holomorphic on $W_o \cap W''$.

For all $t > \tau$, the function $f(t, \cdot)$ is holomorphic on a neighborhood of $\widehat{X}$, but when t is sufficiently near τ, it assumes values at points of $\widehat{X}$ greater in modulus than the supremum of its modulus on X. By the Oka–Weil approximation theorem, this remark implies the existence of polynomials P such that $\|P\|_X < \|P\|_{\widehat{X}}$, a contradiction. Thus, z_o cannot lie in $\widehat{X}$; the theorem is proved under the assumption that the initial function F that defines the continuous family $\{V_t\}_{t\in[0,1)}$ is analytic in the parameter t.

In case it is merely continuous, there are two ways to finish the proof. The first, and simpler, is to remark simply that if U' is a large, relatively compact open subset of U and if $t_o \in [0, 1)$ is very near 1, then the function F can be approximated uniformly on $[0, t_o] \times \bar{U}$ by functions that are holomorphic on U' and are polynomials in t. This leads to a new continuous family of principal analytic hypersurfaces to which the preceding analysis applies.

A second, more difficult, approach to the continuous case is to invoke work of Narasimhan [258] to find solutions of Cousin I problems that depend continuously on parameters.

Above we have used the notion of continuous family of principal analytic hypersurfaces. It is possible to give a more general formulation of the notion of continuous family of hypersurfaces or divisors. Fix a complex manifold $\mathscr{M}$ that has complex dimension M. Recall that a *divisor* $\mathfrak{D}$ on $\mathscr{M}$ is a locally finite family $\{V_j\}_{j=1,\dots}$ of irreducible analytic hypersurfaces in $\mathscr{M}$ together with an assignment of an integer m_j to each V_j. Given a divisor $\mathfrak{D}$ and given a compactly supported smooth form β of bidegree $(M - 1, M - 1)$ on $\mathscr{M}$, a pairing $\langle \mathfrak{D}, \beta\rangle$ is defined by

$$\langle \mathfrak{D}, \beta\rangle = \sum_{j=1,\dots} m_j \int_{V_j} \beta.$$

(If V_j is a manifold, there is no difficulty about the definition of the integral; V_j can, however, have singular points in which case the integral is understood to be the integral over the open set of nonsingular points in V_j. The theory of integration on analytic sets was given by Lelong [218]. An exposition of this theory is given in [343].) If S is a topological space, a family $\{\mathfrak{D}_s\}_{s\in S}$ of divisors on $\mathscr{M}$ indexed by S is said to be *weakly continuous* if for each compactly supported, smooth form β of bidegree $(M - 1, M - 1)$ on $\mathscr{M}$, the function ψ defined on S by $\psi(s) = \langle \mathfrak{D}_s, \beta \rangle$ is continuous on S. The relation between the continuous families of analytic hypersurfaces considered above and the weakly continuous families of divisors just defined are explored in detail in the paper [231]. See also [232].

Our next result is a *local maximum principle for polynomially convex sets*, which is due to Rossi [303].

Theorem 2.1.8. *If X is a compact set in $\mathbb{C}^N$, if E is a compact subset of $\widehat{X}$, if U is an open subset of $\mathbb{C}^N$ that contains E, and if $f \in \mathscr{O}(U)$, then $\|f\|_E = \|f\|_{(E\cap X)\cup bE}$.*

Here b denotes the boundary operator with respect to the set $\widehat{X}$.

We present two proofs of this theorem. The first follows classical lines, the second is an argument of J.-P. Rosay [299].

First Proof of Theorem 2.1.8. Suppose the result false, so that there is a compact set $E \subset \widehat{X}$ for which there exists a function f holomorphic on a neighborhood of E that for some point $p \in E$ satisfies $1 = \|f\|_E = f(p) > \|f\|_{(E\cap X)\cup bE}$. Choose then a compact set $\tilde{E}$ in $\mathbb{C}^N$ with $\tilde{E} \cap \widehat{X} = E$ and $\operatorname{int} \tilde{E} \cap \widehat{X} = E \setminus bE$. The set $\tilde{E}$ can be chosen small enough that f is holomorphic on a neighborhood of it.

The function f is nonconstant on the component of $\operatorname{int} \tilde{E}$ that contains the point p, so the open set $f(\operatorname{int} \tilde{E})$ contains an interval $[1, 1+\delta]$ for some $\delta > 0$. For $(t, z) \in [1, 1+\delta] \times \operatorname{int} E$ define $F(t, z)$ to be $f(z) - t$. Let V_t be the zero locus of the holomorphic function $F(t, \cdot)$.

Write $\widehat{X} = \bigcap_{j=1,2,\ldots} \Omega_j$, where each Ω_j is an open set in $\mathbb{C}^N$ and where $\Omega_{j+1} \Subset \Omega_j$.

For all sufficiently large j the intersection $V_t \cap \Omega_j$ is a closed subset of Ω_j. To see this, suppose not. For all large j, let $q_j \in \Omega_j \setminus V_t$ be a limit point of $V_t \cap \Omega_j$. By compactness, we can suppose that $q_j \to q \in \tilde{E}$. The point q lies in $\tilde{E} \setminus \operatorname{int} \tilde{E}$, for each $V_t \cap \operatorname{int} \tilde{E}$ is closed. By continuity, $f(q_j) = t_j$, and, without loss of generality, $t_j \to t_o \in [1, 1+\delta]$. The point q lies in each Ω_j and so in $\widehat{X}$. Consequently, $q \in \widehat{X} \cap (\tilde{E} \setminus \operatorname{int} E) = bE$. This, however, contradicts the choice of p. Thus for sufficiently large j, the set $V_t \cap \Omega_j$ is a closed subset of Ω_j for all t.

Fix a large positive integer j_o. Let U_o be the component of $\operatorname{int} \tilde{E} \cap \Omega_{j_o}$ that contains the point p, and set $\tilde{V}_t = V_t \cap U_o$. This is a continuous family of principal analytic hypersurfaces in U_o. We have $p \in V_1$ and $V_t \cap \widehat{X} = \emptyset$ for all $t > 1$, because $|f| \leq 1$ on $E \cap \widehat{X}$, and $V_t \cap \widehat{X} \subset E$. It follows that V_t is disjoint from a neighborhood of $\widehat{X}$.

The preceding theorem implies that $p \notin \widehat{X}$, so the result is proved.

Second Proof of Theorem 2.1.8. If the theorem is false there is a function f holomorphic on U such that for some point $p \in \operatorname{int} E$ and for some $\delta \in (0,1)$, $f(p) = 1 > \delta > \|f\|_{(E\cap X)\cup bE}$. Define the plurisubharmonic function $\tilde{u}$ on a neighborhood V of $\widehat{X}$ by

$$\tilde{u} = \begin{cases} \max\{|f|, \delta\} & \text{near } E, \\ \delta & \text{off } E. \end{cases}$$

Let ρ be a plurisubharmonic function on $\mathbb{C}^N$ with $\rho = 0$ on X, $\rho > 0$ on $\mathbb{C}^N \setminus X$.

Theorem 1.3.8 provides such a ρ. Now define the plurisubharmonic function u on $\mathbb{C}^N$ by

$$u = \begin{cases} \max\{\tilde{u}, C\rho\} & \text{on } V, \\ C\rho & \text{off } V, \end{cases}$$

for a large positive constant C. If C is large enough, u is a well-defined function plurisubharmonic on all of $\mathbb{C}^N$ with $u(p) = 1$ and $|u| < 1$ on X. Thus, $p \notin$ Psh-hull X. This hull, however, is the same as $\widehat{X}$ by Theorem 1.3.11, so we have a contradiction. The proof is complete.

The second proof as presented is surely shorter than the first. However, it does depend on the identification of the hull with respect to plurisubharmonic functions with the polynomially convex hull, which in turn depends on some deep function-theoretic considerations. On balance, it is hard to say which of the proofs is actually the simpler when considered from the perspective of the structure of the entire theory.

Corollary 2.1.9. *With f and E as in the preceding theorem,*

$$b_{\mathbb{C}} f(E) \subset f(bE \cup (E \cap X)). \tag{2.1}$$

Proof. If there is a point $\zeta_o \in b_{\mathbb{C}}(f(E)) \setminus f(bE \cup (E \cap X))$, then by considering $\varphi \circ f$ where $\varphi(\zeta) = \frac{1}{\zeta - \zeta_1}$ for a ζ_1 near ζ_o but not in $f(E)$, we get a function that violates the maximum principle just proved.

Corollary 2.1.10. *If E is a compact subset of $\widehat{X}$, then $E \subset (bE \cup (E \cap X))\widehat{\ }$.*

The corollary follows from the theorem by taking the function f to be a polynomial.

There is a local characterization of rationally convex hulls somewhat akin to the local characterization of polynomially convex hulls, which was found by Stolzenberg [341].

Theorem 2.1.11. *If X is a compact subset of $\mathbb{C}^N$, if Ω is an open set that contains $\mathscr{R}$-*hull *X, and if $\check{H}^2(\mathscr{R}$-*hull *$X; \mathbb{Z}) = 0$, then every analytic hypersurface V in Ω that meets $\mathscr{R}$-*hull *X also meets X.*

Note that here we are dealing with a single analytic hypersurface, rather than with a family, as in the polynomially convex case, and note also that it is not assumed that the hypersurface is principal.

Proof. Let V be an analytic hypersurface in Ω, so that for each point $p \in V$ there is a neighborhood U of p on which there exists a holomorphic function f_p the zero locus of which is $U \cap V$. Suppose V disjoint from X. The hypothesis that $\check{H}^2(\mathscr{R}$-hull $X; \mathbb{Z}) = 0$ implies that there is an open set $\Omega_o \subset \Omega$ such that $V \cap \Omega_o$ is the zero locus of a holomorphic function f on Ω_o. Moreover, if we shrink Ω_o slightly, we can suppose that the function f can be approximated uniformly on Ω by rational functions. Accordingly, if $p \in V$, there is a rational function $r = P/Q$, P and Q polynomials that are relatively prime, such that $R(p) = 0$ but r is zero-free on a neighborhood of X. It follows that the polynomial P is zero-free on X but vanishes at the point p. Thus, p is not in $\mathscr{R}$-hull X.

Without the topological assumption that the group $\check{H}^2(\mathscr{R}$-hull $X; \mathbb{Z})$ vanishes, this result can fail. An example is given by Stolzenberg [341] based on an earlier example of Wermer [369].

Corollary 2.1.12. *If X is a compact subset of $\mathbb{C}^N$ with $\check{H}^2(\mathscr{R}\text{-hull } X; \mathbb{Z}) = 0$, and if E is a compact subset of $\mathscr{R}$-hull X, then $E \subset \mathscr{R}\text{-hull } (bE \cup (E \cap X))$.*

In this, b denotes the boundary operator with respect to $\mathscr{R}$-hull X.

Proof. If $z_o \in E \setminus \mathscr{R}\text{-hull } (bE \cup (E \cap X))$, then there is a polynomial P that is zero-free on $\mathscr{R}\text{-hull } (bE \cup (E \cap X))$ but that vanishes at z_o. Denote by Z the zero locus of P, an algebraic hypersurface in $\mathbb{C}^N$.

Write $\mathscr{R}\text{-hull } X = \cap_{j=1,\dots} \Omega_j$, where the sets Ω_j are open in $\mathbb{C}^N$ and satisfy $\Omega_j \Supset \Omega_{j+1}$ for all j. Denote by Z_j the component of the intersection $Z \cap \Omega_j$ that contains the point z_o. The set $Z_\infty = \cap_{j=1,\dots} Z_j$, as the intersection of a decreasing sequence of connected sets, is connected. It is contained in E, for it is connected, it meets E, it is contained in $\mathscr{R}$-hull X, and it is disjoint from bE. It is also disjoint from $E \cap X$. Consequently, $Z_\infty \subset E \setminus X$. By compactness there is an index j_o such that $Z_{j_o} \cap X = \emptyset$. This contradicts the preceding theorem.

Our next result concerns local peak points and sets in the sense introduced in Section 1.2.

Theorem 2.1.13. *Let X be a polynomially convex subset of $\mathbb{C}^N$. If the set E is a local peak set for the algebra $\mathscr{P}(X)$, it is a peak set for $\mathscr{P}(X)$.*

Recall that the hypothesis that E is a local peak set for $\mathscr{P}(X)$ means that there exist a neighborhood U of E in X and a function $f \in \mathscr{P}(X)$ such that $f|E = 1$ and $|f| < 1$ on $U \setminus E$. That it is a peak set means that there is $g \in \mathscr{P}(X)$ with $g|E = 1$ and $|g| < 1$ on $X \setminus E$.

This theorem is due to Rossi [303]; its proof was subsequently simplified by Stolzenberg [341].

Because of Theorem 1.2.11, the analogue of Theorem 2.1.13 obtained when the polynomially convex set X is replaced by a rationally convex set Y, and the algebra $\mathscr{P}(X)$ is replaced by the algebra $\mathscr{R}(Y)$, is also true.

A peak set E for the algebra $\mathscr{P}(X)$, X a polynomially convex set, is itself a polynomially convex set. The restriction algebra $\mathscr{P}(X)|E$ is the algebra $\mathscr{P}(E)$. (Recall Theorem 1.2.23.)

An extension of this theorem was found by Allan [30].

Theorem 2.1.14. *Let X be a polynomially convex set in $\mathbb{C}^N$, let E be a closed subset of X, let U be a relatively open subset of X that contains E, and let h be a continuous function on U such that $h = 1$ on E and $|h| < 1$ on $U \setminus E$. Assume h to be of the form $H \circ \mathbf{f}$, where $\mathbf{f} = (f_1, \dots, f_n) : X \to \mathbb{C}^n$ and, for each j, $f_j \in \mathscr{P}(X)$. The function H is to be a function holomorphic on a neighborhood in $\mathbb{C}^n$ of the compact set $\mathbf{f}(\overline{U})$. There is then a function $g \in \mathscr{P}(X)$ that peaks on the set E.*

Theorem 2.1.13 is an immediate corollary of Theorem 2.1.14. The corresponding result is correct also for the algebras $\mathscr{R}(X)$.

A very special case of this theorem, which was noted by Rossi [303], is this:

Corollary 2.1.15. *Let X, E, and U be as in the preceding theorem. If there is $f \in \mathscr{O}(U)$ with $f = 1$ on E and $|f| < 1$ on $U \setminus E$, then E is a peak set for $\mathscr{P}(X)$.*

The proof depends on the following elementary lemma.

Lemma 2.1.16. [303] *Let Y be a compact subset of $\mathbb{C}^N$, let S be a closed subset of Y, and let U be a neighborhood of S in $\mathbb{C}^N$. Let $h \in \mathscr{O}(U)$ satisfy $h = 0$ on S and $\Re h < 0$ on $(U \cap Y) \setminus S$. If there is a function f holomorphic on a neighborhood of Y that is zero-free on $Y \setminus S$ and such that for a suitable function g holomorphic on U, $f = he^{gh}$ on U, then there is $\tilde{f}$ holomorphic on a neighborhood of Y such that $\tilde{f}|S = 1$ and $|\tilde{f}| < 1$ on $Y \setminus S$.*

Proof. In the set U we have $h = fe^{-gh} = fe^{g_1 f} = f + f^2 g_2$ for suitable functions g_1 and g_2 holomorphic on U. This gives that on $(Y \setminus S) \cap U$,

$$0 > \Re h > \Re f - M|f|^2$$

if $M = \sup_U |g_2|$. (Should g_2 be unbounded on U initially, we can shrink U to a relatively compact open subset to ensure the finiteness of M.) This inequality implies that the range of f on the set U is contained in the exterior of the open disk with radius $\frac{1}{2M}$ centered at $\frac{1}{2M}$. The function f vanishes on S and is zero-free on $Y \setminus S$, so it follows that for the desired peaking function $\tilde{f}$ we can take $\frac{\varepsilon}{\varepsilon - f}$ for a small positive ε.

Proof of Theorem 2.1.14. Cover the set E by a finite collection $C_1, \ldots, C_\ell$, of open subsets of X, where for each k, $C_k \subset U$ is given by

$$C_k = \{x \in X : |f_j(x) - \alpha_{jk}| < r_{jk},\ 1 \le j \le n\}$$
$$\cap \{x \in X : |f_\mu(x)| < 1 \text{ for } \mu = \mu_k, \mu_k + 1, \ldots, \mu_{k+1}\}$$

with $\mu_1 = n + 1$ and $\mu_1 < \mu_2 < \cdots < \mu_{\ell+1}$. The α_{jk} are suitable complex numbers and the numbers r_{jk} are suitable positive numbers. Set $M = \mu_{\ell+1}$. For μ in the range $n+1, \ldots, \mu_{k+1}$, f_μ is an element of $\mathscr{P}(X)$. Define $\tilde{\mathbf{f}} : X \to \mathbb{C}^M$ by

$$\tilde{\mathbf{f}} = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_{\mu_{\ell+1}}).$$

The set W on which the function H is defined and holomorphic is open and contains the compact set $\overline{\mathbf{f}(U)}$, so the numbers r_{jk} can be chosen small enough that the closure of each of the polydisks

$$\Delta_j = \Delta(\alpha_j, r_j) = \{z \in \mathbb{C}^n : |z_k - \alpha_{jk}| < r_{jk} \text{ for } k = 1, \ldots, n\}$$

is contained in W. Put $B = 1 + \max_{1 \le j \le M} \|f_j\|_X$.

For $j = 1, \ldots, \ell$, let

$$V_j = \{z \in \mathbb{C}^M : (z_1, \ldots, z_n) \in \Delta_j, |z_i| < 1 \text{ for } \mu_{j-1} < i \le k_j \text{ and } |z_i| < B \text{ otherwise}\},$$

an open polydisk in $\mathbb{C}^M$. Put $V = \cup_{j=1}^{\ell} V_j$, and define $H_1 \in \mathscr{O}(V)$ by $H_1(z) = H(z_1, \ldots, z_n)$. On $\tilde{f}(X) \cap V$, $|H_1| \le 1$ with equality only on $\tilde{\mathbf{f}}(S)$, where $H_1 = 1$. The set $bV \cap \tilde{\mathbf{f}}(X)$ is a compact subset of $\tilde{\mathbf{f}}(U) \cap \{z \in V : |H_1|(z)| < 1\}$. Consequently, there is an open neighborhood V' of $\tilde{\mathbf{f}}(X) \setminus V$ such that $V' \cap V \subset \{z \in \mathbb{C}^n : |H_1(z)| < 1\}$. The set $\tilde{\mathbf{f}}(X)$ is polynomially convex, so there is a polynomial polyhedron Ω_o with $\tilde{\mathbf{f}}(X) \subset \Omega_o \Subset V \cup V'$.

Put $\Omega' = \Omega_o \cap V'$ and $\Omega = \Omega_o \cap V$. The function $\varphi = H_1 - 1$ is holomorphic on $\Omega \cap \Omega'$, and its real part is negative there. Accordingly, because the first Cousin problem is solvable on Ω_o, there are functions $\lambda \in \mathscr{O}(\Omega)$ and $\lambda' \in \mathscr{O}(\Omega')$ such that on $\Omega \cap \Omega'$, $\log \varphi = \lambda - \lambda'$. Thus, $\varphi = e^{\lambda}/e^{\lambda'}$ on $\Omega \cap \Omega'$. Next, set $g = e^{\lambda} \in \mathscr{O}(\Omega)$, $g' = e^{\lambda'} \in \mathscr{O}(\Omega')$. Solve Cousin I again to obtain $u \in \mathscr{O}(\Omega)$ and $u' \in \mathscr{O}(\Omega')$ with $(\log \varphi) g^{-1} = u - u'$. We then have $he^{g'u'\varphi} = e^{gu}$ on $\Omega \cap \Omega'$. Thus, the function $p \in \mathscr{O}(\Omega_o)$ defined by

$$p = \begin{cases} e^{gu} & \text{on } \Omega, \\ he^{g'u'h} & \text{on } \Omega', \end{cases}$$

is well defined. The preceding lemma provides an $F \in \mathscr{O}(\Omega)$ that peaks on $\tilde{\mathbf{f}}(E)$. The function $f = F \circ \tilde{\mathbf{f}}$ lies in $\mathscr{P}(X)$ and peaks on the set E. The theorem is proved.

Corollary 2.1.17. *Let X be a polynomially convex subset of $\mathbb{C}^N$, and let $f, g \in \mathscr{P}(X)$. If E is a compact subset of X and U is a neighborhood of E in X such that $f = g \neq 0$ on E and $|f| < |g|$ on $U \setminus E$, then E is a peak set for $\mathscr{P}(X)$.*

Proof. Apply Theorem 2.1.14 with the map $x \mapsto (f(x), g(x))$ from X to $\mathbb{C}^2$ and H the function given by $H(z_1, z_2) = z_1/z_2$.

Corollary 2.1.18. *Let X be a polynomially convex subset of $\mathbb{C}^N$, and let $f, g \in \mathscr{P}(X)$. Denote by $M(f, g)$ the set $\{x \in X : |f(x)| \geq |g(x)|\}$. If K is a component of $M(f, g)$ on which g is zero-free, then K meets the Shilov boundary for $\mathscr{P}(X)$.*

Proof. Suppose first that V is an open subset of X with the property that $V \cap M(f, g)$ is open and closed in $M(f, g)$ and contains the set K. Thus, for $x \in V \setminus M(f, g)$ we have $|f(x)| < |g(x)|$. By choosing V small enough we can ensure that g is zero-free on $\bar{V} \setminus M(f, g)$. Choose a point $x_o \in V \cap M(f, g)$ at which the function $|f/g|$ attains its supremum, say λ, over V. We can suppose that $f(x_o)$ and $g(x_o)$ are both positive. Then the function h defined on V by $h = \frac{1}{2}\big(1 + \frac{f}{\lambda g}\big)$ peaks on a set E in V that contains the point x_o. This set is a peak set for $\mathscr{P}(X)$ by Theorem 2.1.12, so it must meet the Shilov boundary for $\mathscr{P}(X)$. This is true for every choice of the open set V. It follows that K itself must meet the Shilov boundary for $\mathscr{P}(X)$, as we wished to prove.

Corollary 2.1.19. *Let X be a polynomially convex subset of $\mathbb{C}^N$, and let $f, g \in \mathscr{P}(X)$. If $|f| > |g|$ on the Shilov boundary for $\mathscr{P}(X)$, and if K is a component of $M(g, f) = \{x \in X : |g(x)| \geq |f(x)|\}$, then f vanishes at some point of K.*

Proof. If not, denote by U an open and closed subset of $M(g, f)$ that contains K, U chosen small enough that f is zero-free on $\bar{U}$. Then $|g/f| > 1$ on $M(g, f) \cap U$, but $|g/f| < 1$ on $U \setminus M(g, f)$. But then as above, U contains a peak set for the algebra $\mathscr{P}(X)$ and so meets the Shilov boundary for $\mathscr{P}(X)$. This is true for all choices of U, so the set K itself must meet the Shilov boundary for $\mathscr{P}(X)$.

We shall need below the local peak point theorem for general uniform algebras, which can be deduced easily from Theorem 2.1.14:

Theorem 2.1.20. [303] *Let A be a uniform algebra on the compact space X, which is the spectrum of A. Let $x_o \in X$. If there is $f_o \in A$ with $f_o(x_o) = 1$ and $|f_o| < 1$ on $U \setminus \{x_o\}$ for some neighborhood U of x_o, then there is $g \in A$ with $g(x_o) = 1$ and $|g| < 1$ on $X \setminus \{x_o\}$.*

Proof. By compactness and the hypothesis that the algebra A separates points on X, there exist functions $f_1, \dots, f_r$ in A such that for each j, $f_j(x_o) = 1$ and for all $x \in X \setminus U$, $\min_{j=1,\dots,r} |f(x)| < \frac{1}{2}$. Let V be a neighborhood of the point $(1, \dots, 1) \in \mathbb{C}^{r+1}$ such that if $\mathbf{f} = (f_o, \dots, f_r) : X \to \mathbb{C}^{r+1}$, then $\mathbf{f}^{-1}(V \cap \sigma_A(f_o, \dots, f_r)) \subset U$. Let V' be an open subset of $\mathbb{C}^{r+1}$ that contains $\sigma_A(f_o, \dots, f_r) \setminus U$ and is disjoint from a neighborhood of the point x_o. By Lemma 1.5.8, there are additional elements $f_{r+1}, \dots, f_s$ of A such that if $\pi : \mathbb{C}^{s+1} \to \mathbb{C}^{r+1}$ is the projection onto $\mathbb{C}^{r+1}$, then $\pi(\sigma_A(f_o, \dots, f_s)\widehat{\ }) \subset V \cup V'$. Define $H \in \mathscr{O}(\mathbb{C}^{s+1})$ by $H(z) = z_o$. Theorem 1.5.7 applied to H, thought of as being defined on the open set $\pi^{-1}(V)$, yields a function $g \in \mathscr{P}(\sigma(f_o, \dots, f_s)\widehat{\ })$ that peaks at the point $y_o = (f_o(x_o), \dots, f_s(x_o))$. The function $\tilde{g} \in \mathscr{C}(X)$ defined by $\tilde{g}(x) = g(f_o(x), \dots, f_s(x))$ lies in A and peaks at the point x_o.

The theorem is proved.

2.2. Two Characterizations of Polynomially Convex Sets

In the preceding section we saw characterizations of polynomially convex sets and rationally convex sets based on the theory of the Cousin problems, results that were obtained early in the study of polynomial convexity. The present section is devoted to two more recent characterizations of polynomially convex hulls, one obtained by Duval and Sibony [107], the other by Poletsky [280].

We begin with the characterization of Duval and Sibony, which is couched in terms of the theory of currents. It is convenient to begin by recalling the notion of current.

We shall not need very much of the highly developed theory of currents in this section, but it will be well to recall the general definition. If $\mathscr{M}$ is an n-dimensional $\mathscr{C}^\infty$ manifold, then $\mathscr{D}(\mathscr{M})$ is the subspace of $\mathscr{C}^\infty(\mathscr{M})$ that consists of the functions with compact support. For $p = 0, \dots,$ we have the spaces $\mathscr{E}^p(\mathscr{M})$ of smooth p-forms on $\mathscr{M}$. The space $\mathscr{E}^p(\mathscr{M})$ contains the subspace $\mathscr{D}^p(\mathscr{M})$ of the compactly supported p-forms on $\mathscr{M}$. Thus $\mathscr{D}(\mathscr{M}) = \mathscr{D}^0(\mathscr{M})$. If $\mathscr{M}$ is a complex manifold, we have also the spaces $\mathscr{E}^{p,q}(\mathscr{M})$ of all smooth forms of bidegree (p, q) and its subspace $\mathscr{D}^{p,q}(\mathscr{M})$, the space of compactly supported forms of bidegree (p, q). A *current of dimension p and of degree $n - p$* on $\mathscr{M}$ is a $\mathbb{C}$-linear functional T on the space $\mathscr{D}^p(\mathscr{M})$ that has the following continuity property: If $\{\alpha_j\}_{j=1,\dots}$ is a sequence in $\mathscr{D}^p(\mathscr{M})$ such that for some fixed compact set $K \subset \mathscr{M}$, $\operatorname{supp} \alpha_j \subset K$ for all j and if, in addition, α_j converges to 0 in the sense that for each set of local coordinates in $\mathscr{M}$, the sequences of coefficients of α_j in these coordinates, as well as the sequences of the derivatives of all fixed orders of these coefficients, converge to 0 uniformly on compacta in the domain of the coordinate system, then the sequence $\{T(\alpha_j)\}_{j=1,\dots}$ converges to zero.[2] The space of currents of dimension p (and degree $n - p$) on $\mathscr{M}$ is denoted by $\mathscr{D}^{p\prime}(\mathscr{M})$. When $\mathscr{M}$ is a complex manifold of complex dimension N, a current of bidimension (p, q) and bidegree $(N - p, N - q)$ is

[2]It is not evident from this description, but the space $\mathscr{D}^p$ does admit the structure of a locally convex topological vector space with respect to which $\mathscr{D}^{p\prime}$ is the space of continuous linear functionals. For the details, consult [320].

a $\mathbb{C}$-linear functional on $\mathscr{D}^{p,q}(\mathscr{M})$ with the indicated continuity property. Equivalently, a T in $\mathscr{D}^{p,q\prime}(\mathscr{M})$ can be thought of as an element of $\mathscr{D}^{p+q\prime}(\mathscr{M})$ that annihilates all the $(p+q)$-forms except those of bidegree (p,q). If T is a current of dimension p, then the support of T is the smallest closed subset K of $\mathscr{M}$ with the property that $T(\alpha)=0$ for all $\alpha \in \mathscr{D}^p(\mathscr{M})$ that vanish on a neighborhood of K.

The space $\mathscr{E}^p(\mathscr{M})$ is in a natural way a topological vector space. When $\mathscr{M}$ is a domain in $\mathbb{R}^N$, the topology is that of uniform convergence on compact sets for the coefficients of the forms as well as their derivatives of all orders. The topology in the case of a general manifold is the natural extension of this. With respect to this topology, $\mathscr{E}^p(\mathscr{M})$ is a Fréchet space. Its dual is the space $\mathscr{E}^{p\prime}(\mathscr{M})$ that consists of all compactly supported currents of dimension p. This is a natural extension of viewing the space of compactly supported distributions on $\mathbb{R}^n$ as the dual space of the space $\mathscr{C}^\infty(\mathbb{R}^n)$.

If $T \in \mathscr{D}^{p\prime}(\mathscr{M})$, then bT is the element of $\mathscr{D}^{p-1\prime}(\mathscr{M})$ defined by the equation $bT(\alpha)=T(d\alpha)$ for all $\alpha \in \mathscr{D}^{p-1}(\mathscr{M})$. The current T is *closed* if $bT=0$. The formalism of the exterior differential is extended to currents by way of the definition that for $T \in \mathscr{D}^{p\prime}(\mathscr{M})$, $dT=(-1)^{p+1}bT$. Thus, $d:\mathscr{D}^{p\prime}(\mathscr{M}) \to \mathscr{D}^{p-1\prime}(\mathscr{M})$. In addition, one has operators ∂ and $\bar\partial$ acting on currents: If $T \in \mathscr{D}^{p,q\prime}$, then $\partial T \in \mathscr{D}^{p-1,q\prime}(\mathscr{M})$ is defined by $\partial T(\alpha)=(-1)^{p+q+1}T(\partial\alpha)$ for all $\alpha \in \mathscr{D}^{p-1,q}(\mathscr{M})$. The operator $\bar\partial$ is defined in the analogous way. There is also the twisted differential d^c. It is evident from the definition that $d=\partial+\bar\partial$; the operator d^c is defined by $d^c=-i(\partial-\bar\partial)$. Then $dd^c=2i\partial\bar\partial$.

A standard example is the current of integration over $\mathscr{M}$, under the assumption that $\mathscr{M}$ is an oriented n-dimensional manifold. This is the current $[\mathscr{M}] \in \mathscr{D}^{n\prime}(\mathscr{M})$ defined by $[\mathscr{M}](\alpha)=\int_{\mathscr{M}}\alpha$ for all compactly supported smooth n-forms α on $\mathscr{M}$. Another example is provided by a differential form β of degree $n-p$ with locally integrable coefficients. The associated current is the functional T_β defined by $T_\beta(\alpha)=\int_{\mathscr{M}}\beta\wedge\alpha$.

The usual notation is that if $\mathscr{M}$ is an n-dimensional manifold, then

$$\mathscr{D}_p(\mathscr{M})=\mathscr{D}^{n-p\prime}(\mathscr{M}),$$

and that if $\mathscr{M}$ is an N-dimensional complex manifold, then

$$\mathscr{D}_{p,q}(\mathscr{M})=\mathscr{D}(\mathscr{M})^{N-p,N-q\prime}.$$

With this notation, $\bar\partial$ carries $\mathscr{D}_{p,q}(\mathscr{M})$ to $\mathscr{D}_{p,q+1}(\mathscr{M})$, and when we regard $\mathscr{E}^{p,q}(\mathscr{M})$ as a subspace of $\mathscr{D}_{p,q}(\mathscr{M})$ via the identification of the $(N-p,N-q)$-form β with the current T_β defined at the end of the preceding paragraph, the two possible interpretations of $\bar\partial\beta$ coincide.

There are notions of positive form and positive current on a complex manifold.

Definition 2.2.1. *If $\mathscr{M}$ is an N-dimensional complex manifold, an element $\varphi \in \mathscr{E}^{p,p}(\mathscr{M})$ is said to be* positive *if whenever local holomorphic coordinates $z_1,\dots,z_N$ are chosen in an open set and α_j, $j=1,\dots,N-p$, are* $(1,0)$*-forms defined in the domain of these coordinates with continuous compactly supported coefficients, then*

$$\int_{\mathscr{M}}\varphi\wedge i\alpha_1\wedge\bar\alpha_1\wedge\cdots\wedge i\alpha_{N-p}\wedge\bar\alpha_{N-p}\geq 0.$$

For $(1,1)$-forms, there is a particularly simple criterion: The $(1,1)$-form

$$\varphi = i \sum_{j,k=1,\dots,N} \varphi_{jk} dz_j \wedge d\bar{z}_k$$

is positive if and only if the matrix $[\varphi_{jk}]_{j,k=1,\dots,N}$ is positive semidefinite.

Definition 2.2.2. *A current T in $\mathscr{D}_{N-p,N-p}(\mathscr{M})$ is said to be* positive *if for all nonnegative functions $f \in \mathscr{D}(\mathscr{M})$ and for all forms $\alpha_1, \dots, \alpha_p \in \mathscr{D}^{1,0}(\Omega)$, the quantity $T(f\, i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_p \wedge \bar{\alpha}_p)$ is nonnegative.*

Perhaps the simplest example of a positive current is the current $[\lambda]$ of integration over the complex line λ in $\mathbb{C}^N$.

Among the positive currents are the positive forms: If $\beta \in \mathscr{E}^{N-p,N-p}(\mathscr{M})$ is a positive form, then the associated element $T_\beta \in \mathscr{D}_{N-p,N-p}(\mathscr{M})$ is positive.

If $T \in \mathscr{D}_{p,p}(\mathscr{M})$ is positive, and $\beta \in \mathscr{E}^{q,q}(\mathscr{M})$ is a positive form, then the exterior product $T \wedge \varphi \in \mathscr{D}_{N-p-q,N-p-q}(\mathscr{M})$, which is defined by $T \wedge \varphi(\alpha) = T(\varphi \wedge \alpha)$, is also positive.

Our application of positive currents will be through the following fact:

Lemma 2.2.3. *If $T \in \mathscr{D}_{N-1,N-1}(\mathscr{M})$ is a positive current with compact support that is not the zero current, and if the function u is of class $\mathscr{C}^2$ and strictly plurisubharmonic on the support of T, then $T(dd^c u) > 0$.*

Proof. Because T is not the zero current, there is a $(1, 1)$-form α with compact support such that $T(\alpha) > 0$. The function u is strictly plurisubharmonic, so if we write

$$dd^c u = 2i \sum_{j,k=1,\dots,N} u_{jk} dz_j \wedge d\bar{z}_k,$$

then the matrix $[u_{jk}]_{j,k=1,\dots,N}$ is positive definite on the support of T. Consequently, for a sufficiently large positive constant C, the form $Cdd^c u - \alpha$ is positive on a neighborhood of supp$T$, whence $T(Cdd^c u - \alpha) \geq 0$. This implies that $T(dd^c u)$ is positive, for $T(\alpha) > 0$.

Definition 2.2.4. *If Ω is a domain in $\mathbb{C}^N$ and $T \in \mathscr{D}_{N-p,N-p}(\Omega)$ is a positive current, then the* mass *of T is defined to be $|T| = T(\beta^p)$ with β the form $\frac{i}{2} \sum_{j=1,\dots,N} dz_j \wedge d\bar{z}_j$.*

Note that because T is by definition a functional on compactly supported forms, we must understand by $T(\beta^p)$ the supremum of the numbers $T(\chi\beta^p)$, where χ ranges over all smooth compactly supported functions on Ω with values in $[0, 1]$.

References for the theory of currents are [115], [290], and [320]. The theory of positive currents is developed in [219] and in [220].

We now turn to the characterization of polynomially convex sets found by Duval and Sibony. Denote by δ_x the positive measure of unit mass with support the singleton $\{x\}$.

Theorem 2.2.5. [107] *For a compact set X in $\mathbb{C}^N$ and a point $x \in \mathbb{C}^N$, the following are equivalent:*

(a) $x \in \widehat{X}$.

(b) *There is a positive current $T \in \mathscr{D}_{N-1,N-1}(\mathbb{C}^N)$ such that $dd^c T = \mu - \delta_x$ for a probability measure μ supported in X.*

The conclusion in part (b) is that for each $\mathscr{C}^\infty$ function φ on $\mathbb{C}^N$,

$$\int \varphi\, d\mu - \varphi(x) = T(dd^c\varphi).$$

That (b) implies (a) is a consequence of a more general result:

Theorem 2.2.6. [107] *If X is a compact subset of $\mathbb{C}^N$, if $T \in \mathscr{D}_{N-1,N-1}(\mathbb{C}^N \setminus X)$ is positive and has bounded support, and if dd^cT is negative in $\mathbb{C}^N \setminus X$, then the support of T is contained in $\widehat{X}$.*

Proof. If $x \in \operatorname{supp} T \setminus \widehat{X}$, then by Theorem 1.3.8, there is a nonnegative smooth plurisubharmonic function u on $\mathbb{C}^N$ that vanishes on a neighborhood of $\widehat{X}$ and that is strictly plurisubharmonic where it is positive, which includes a neighborhood of the point x. We then have that

$$0 < T(dd^cu) = (dd^cT)(u) \leq 0,$$

which is impossible.

We have $\operatorname{supp} dd^cT \subset \operatorname{supp} T$, so this result yields that (b) implies (a).

That (a) implies (b) is a consequence of a more precise statement:

Theorem 2.2.7. [107] *Let X be a compact subset of $\mathbb{C}^N$, let $x_o \in \widehat{X}$, and let μ be a Jensen measure for x_o supported in X. There is a positive current T of bidimension* $(1, 1)$ *and with bounded support such that $dd^cT = \mu - \delta_{x_o}$.*

Proof. Fix an $R > 0$ large enough that $\widehat{X} \subset \mathbb{B}_N(R)$.

By a *flat disk* contained in $\mathbb{B}_N(R)$ we shall understand a disk that is contained in the intersection of $\mathbb{B}_N(R)$ with a complex line in $\mathbb{C}^N$.

Introduce the class $\mathscr{K}_o$ of currents of bidimension $(1, 1)$ of the form $S = g_D[D]$, where D is a flat disk contained in $\mathbb{B}_N(R)$ and where g_D is the Green function for D, so that if c_D is the center of D, then g_D is nonnegative and harmonic on $D \setminus \{c_D\}$, g_D vanishes on bD, and, with Δ denoting the Laplacian in the complex line that contains D, $\Delta g_D = \delta_{c_D}$. (On the unit disk $\mathbb{U}$ in $\mathbb{C}$, the Green function is $-\log|z|$.) Thus, for a smooth two-form α on $\mathbb{C}^N$, $S(\alpha) = \int_D g_D\alpha$. This integral exists, for g_D has a logarithmic singularity at c_D.

Let $\mathscr{K}$ denote the cone generated by the set $\mathscr{K}_o$. We shall show that $\mu - \delta_{x_o}$ lies in the weak* closure of the cone $dd^c\mathscr{K} = \{dd^cS : S \in \mathscr{K}\}$ in the dual space of the space $\mathscr{E}^{1,1}(\mathbb{C}^N)$. In the contrary case, there is a weak* continuous linear functional on the dual space of $\mathscr{E}^{1,1}(\mathbb{C}^N)$ that separates $\mu - \delta_{x_o}$ from the cone $dd^c\mathscr{K}$. Weak* continuous linear functionals are point evaluations [309, p. 66], so there is a function $\varphi \in \mathscr{C}^\infty(\mathbb{C}^N)$ such that $\int \varphi\, d\mu - \varphi(x_o) < 0 \leq T(dd^c\varphi)$ for all $T \in \mathscr{K}$.

This condition implies that if D is a flat disk in $\mathbb{B}_N(R)$, then $\int_D g_D dd^c\varphi$ is nonnegative. Because this happens for all disks $\tilde{D}$ contained in the line λ that contains D and that are contained in $\mathbb{B}_N(R)$, it follows that the Laplacian of φ on $\lambda \cap \mathbb{B}_N(R)$ is nonnegative. Thus, φ is subharmonic on $\lambda \cap \mathbb{B}_N(R)$, and φ is plurisubharmonic on $\mathbb{B}_N(R)$. It satisfies

$$\int \tilde{\varphi}\, d\mu \ < \ \tilde{\varphi}(x_o),$$

which is impossible by Corollary 1.3.10, for μ is a Jensen measure for x_o. Thus, as claimed, $\mu - \delta_{x_o}$ lies in the weak* closure of the cone $dd^c\mathscr{K}$.

Consequently, there is a net $\{dd^cT_\gamma\}_{\gamma\in\Gamma}$ in $dd^c\mathscr{K}$ that converges in the weak* sense to $\mu-\delta_{x_o}$. For each $\varphi \in \mathscr{C}^\infty(\mathbb{C}^N)$, there are $\gamma_o \in \Gamma$ and $M > 0$ such that $|dd^cT_\gamma(\varphi)| \leq M$ for $\gamma > \gamma_o$. Apply this to the function $|z|^2$. Each T_γ is of the form

$$T_\gamma = \sum_{j=1,\dots} \lambda_j^\gamma g_j^\gamma [D_j^\gamma]$$

for some choice of positive numbers λ_j^γ and some choice of flat disks D_j^γ contained in $\mathbb{B}_N(R)$. For each γ and j, g_j^γ denotes the Green function associated with the disk D_j^γ. Thus,

$$dd^cT_\gamma(|z|^2) = \sum_{j=1,\dots} \lambda_j^\gamma \sum_{r=1,\dots,N} \frac{i}{2}\int_{D_j^\gamma} g_j^\gamma \, dz_r \wedge d\bar{z}_r.$$

It follows that if ν_r^γ is the positive measure defined by

$$\int f \, d\nu_r^\gamma = \sum_{j=1,\dots} \lambda_j^\gamma \frac{i}{2}\int_{D_j^\gamma} f g_j^\gamma \, dz_r \wedge d\bar{z}_r,$$

then for $\gamma > \gamma_o$, the measures ν_r^γ are uniformly bounded in norm. They are supported in $\mathbb{B}_N(R)$. By passing to a suitable subnet, we can suppose that each of the nets $\{\nu_r^\gamma\}_{\gamma\in\Gamma}$ converges in the weak* topology on the space of measures on $\overline{\mathbb{B}_N(R)}$, viewed as the dual space of the space $\mathscr{C}(\overline{\mathbb{B}_N(R)})$, to a measure ν_r. The measures ν_r are nonnegative.

We now have that the current T of bidimension (1, 1) given by

$$T\Bigg(\sum_{j,k=1,\dots,N} \alpha_{j,k} dz_j \wedge d\bar{z}_k\Bigg) = \sum_{r=1,\dots,N} \int \alpha_{r,r} \, d\nu_r$$

has support in $\overline{\mathbb{B}_N(R)}$, satisfies $T(dd^c\varphi) = \int \varphi \, d\mu - \varphi(x_o)$, and is positive.

This completes the proof of the theorem and with it the proof of Theorem 2.2.5.

Duval and Sibony give the following example. Let X be a compact subset of $\mathbb{C}^N$ for which there is a bounded holomorphic map $f : \mathbb{U} \to \mathbb{C}^N$ for which the radial limit $f^*(e^{i\vartheta})$ lies in X whenever it exists. Suppose that $f(0) = 0$ and that $0 \notin X$. The maximum principle implies that the image $f(\mathbb{U})$ is contained in $\widehat{X}$; it is an analytic disk in $\widehat{X}$ that passes through the origin. A current of the kind provided by the preceding theorem is defined by

$$T(\alpha) = \frac{i}{2}\int_{\mathbb{U}} \log|z| \, f^*\alpha$$

for all two-forms α on $\mathbb{C}^N$. In general, the integral $\int_{\mathbb{U}} f^*\alpha$ does not exist: If

$$\alpha = \sum_{r,s=1,\dots,N} a_{r,s} dz_r \wedge d\bar{z}_s,$$

then

$$\int_{\mathbb{U}} f^*\alpha = -2i \sum_{r,s=1,\dots,N} \int_{\mathbb{U}} a_{r,s}(f(z)) f_r'(z) \overline{f_s'(z)}\, d\mathscr{L}(z).$$

This integral typically does not exist, for the derivatives f_r' are generally not square-integrable.

However, the integral $\int_{\mathbb{U}} \log|z||g'(z)|^2\, d\mathscr{L}(z)$ does exist for an arbitrary bounded holomorphic function g on $\mathbb{U}$.[3]

A particular case of this situation is that in which $g \in H^\infty(\mathbb{U})$ vanishes at the origin and X is the set $\bar{\Gamma}_f \setminus \Gamma_f$ with Γ_f the graph of f, $\bar{\Gamma}_f$ its closure.

Next we take up the characterization of polynomially convex sets found by Poletsky:

Theorem 2.2.8. *If X is a compact subset of $\mathbb{C}^N$, then the point $x_o \in \mathbb{C}^N$ belongs to $\widehat{X}$ if and only if for each bounded pseudoconvex Runge domain $\Omega \subset \mathbb{C}^N$ that contains X, for every neighborhood W of X that is contained in Ω, and for every $\varepsilon > 0$, there is a holomorphic map f from a neighborhood of the closed unit disk in $\mathbb{C}$ to Ω with $f(0) = z_o$ and with $f(e^{i\vartheta}) \in W$ for all points $e^{i\vartheta} \in b\mathbb{U} \setminus E$ for a subset E of $b\mathbb{U}$ of measure less than ε.*

Note that the domain of the function f is allowed to vary.

Not every bounded Runge domain in $\mathbb{C}^N$ is pseudoconvex: $\mathbb{B}_N \setminus \{0\}$ is a Runge domain but is not pseudoconvex.

One direction in the proof of this theorem is based on a study of certain *disk functionals*. For the introduction of these, it is convenient to use the notation that for a complex manifold $\mathscr{M}$ and for each point $z_o \in \mathscr{M}$, $\mathscr{F}(\mathscr{M}, z_o)$ is the family of all holomorphic maps f to $\mathscr{M}$ that are defined on a neighborhood of $\bar{\mathbb{U}}$ and that satisfy $f(0) = z_o$. (The neighborhood is allowed to depend on the map.)

Definition 2.2.9. *If $\mathscr{M}$ is a complex manifold and u is an $\mathbb{R}$-valued Borel function on $\mathscr{M}$, then the associated* Poisson functional of $\mathscr{M}$ evaluated at u *is the function $\widehat{u} : \mathscr{M} \to \mathbb{R}$ defined by*

$$\widehat{u}(z) = \frac{1}{2\pi} \inf_{f \in \mathscr{F}(\mathscr{M}, z)} \int_{-\pi}^{\pi} u(f(e^{i\vartheta}))\, d\vartheta. \tag{2.2}$$

Two useful properties of $\widehat{u}$ are that $\widehat{u} \leq u$ and that if $u \geq 0$, then $\widehat{u} \geq 0$. These two combine to yield that if u is nonnegative, then $\widehat{u}$ vanishes wherever u does.

A fundamental result about the Poisson functional is this:

Theorem 2.2.10. *If u is an upper semicontinuous function on the complex manifold $\mathscr{M}$, then the Poisson functional $\widehat{u}$ is plurisubharmonic on $\mathscr{M}$.*

[3] Indeed, let g be a function in the Hardy class $H^2(\mathbb{U})$ with power series expansion $g(z) = \sum_{k=0,1,\dots} a_k z^k$, so that $\sum_{k=0,1,\dots} |a_k|^2 < \infty$. Then

$$\int_{\mathbb{U}} \log|z|\, |g'(z)|^2\, d\mathscr{L}(z) = 2\pi \sum_{k=1,\dots} k^2 |a_k|^2 \int_0^1 r^{2k-1} \log r\, dr = -\frac{\pi}{2} \sum_{k=1,\dots} |a_k|^2,$$

which is finite.

As stated, for arbitrary complex manifolds, this theorem is due to Rosay [298]. The result had previously been obtained by Lárusson and Sigurdsson [213] for certain manifolds. For domains in $\mathbb{C}^N$, the result was given by Poletsky [278]. It was Poletsky [279] who first recognized the relevance of such a result to polynomial convexity. See also the treatment of this point given in [213].

A notable aspect of Theorem 2.2.10 is that it is one of a very small number of known theorems valid on completely arbitrary complex manifolds.

A derivation of Theorem 2.2.8 from Theorem 2.2.10 is not long: First, let $X \subset \mathbb{C}^N$ be a compact set, let Ω be a bounded pseudoconvex Runge domain that contains X, and let $z_o \in \Omega$. Assume that for each neighborhood W of X and for each $\varepsilon > 0$ there is $\varphi \in \mathscr{F}(\Omega, z_o)$ with $f(e^{i\vartheta}) \in W$ for all $e^{i\vartheta}$ outside a subset of the unit circle of measure less than ε. If P is polynomial, then the function $|P \circ \varphi|$ is subharmonic on a neighborhood of $\bar{U}$, so

$$|P(z_o)| = |P(\varphi(0))| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\varphi(e^{i\vartheta})| \, d\vartheta \leq \|P\|_W + \varepsilon \|P\|_\Omega.$$

Because this is true for all choices of ε and all choices of the open set W, it follows that $|P(z)| \leq \|P\|_X$, i.e., $z_o \in \widehat{X}$.

For the opposite implication, suppose z_o to lie in $\widehat{X}$, and fix an open set $W \subset \Omega$ that contains X. Let u be the upper semicontinuous function on Ω that is 0 on W and 1 on $\Omega \setminus W$. The associated function $\widehat{u}$ is nonnegative and vanishes on X. Consequently, it vanishes at the point x, because of the identity of Psh-hull$_\Omega$ X and the hull of X with respect to the functions holomorphic on Ω. The domain Ω is a Runge domain, so this latter set coincides with the polynomially convex hull $\widehat{X}$. This means that if $\varepsilon > 0$, there is $\varphi \in \mathscr{F}(\Omega, x)$ with $\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\varphi(e^{i\vartheta})) \, d\vartheta < \varepsilon$. Because $u = 1$ on $\Omega \setminus W$, the point $\varphi(e^{i\vartheta})$ lies in W except when the point $e^{i\vartheta}$ lies in a set of measure at most ε.

We now turn to the proof of Theorem 2.2.10, following [278]. In fact, we shall give the proof only in the case that the manifold $\mathscr{M}$ in question is a domain in $\mathbb{C}^N$. By restricting to domains in $\mathbb{C}^N$, we avoid having to invoke some major results from analytic geometry on which the full argument of Rosay depends.

Fix a domain Ω in $\mathbb{C}^N$ and an upper semicontinuous function on Ω.

To prove that the Poisson functional $\widehat{u}$ is plurisubharmonic, it is necessary to show that it is upper semicontinuous, which can be established rather easily.

Lemma 2.2.11. *If u is an upper semicontinuous function on Ω, then $\widehat{u}$ is upper semicontinuous.*

Proof. Fix a point $z_o \in \Omega$ and an $\varepsilon > 0$. By definition, there is an $f \in \mathscr{F}(\Omega, z_o)$ with

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(f(e^{i\vartheta})) \, d\vartheta < \widehat{u}(z_0) + \varepsilon.$$

For $z \in \Omega$ near z_o, define $f_z \in \mathscr{F}(\Omega, z)$ by $f_z(\zeta) = f(\zeta) + z - z_o$. With this notation, $f_{z_o} = f$. We have

$$\widehat{u}(z) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(f_z(e^{i\vartheta})) \, d\vartheta.$$

As $z \to z_o$, the maps f_z converge uniformly on $\bar{\mathbb{U}}$ to f, and it follows from the upper semicontinuity of u and Fatou's lemma that when z is sufficiently near z_o,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(f_z(e^{i\vartheta}))\,d\vartheta < \frac{1}{2\pi}\int_{-\pi}^{\pi} u(f(e^{i\vartheta}))\,d\vartheta + \varepsilon.$$

Accordingly, $\limsup_{z\to z_o} \widehat{u}(z) \le \widehat{u}(z_o)$, and $\widehat{u}$ is seen to be upper semicontinuous.

To complete the proof of the theorem, it is necessary to establish the subaveraging property of $\widehat{u}$:

Lemma 2.2.12. *If $z \in \Omega$, and if $a \in \mathbb{C}^N$ is near the origin, then*

$$\widehat{u}(z) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(z + e^{i\vartheta}a)\,d\vartheta.$$

Proof. In this proof we consistently use the notation that $\mathbb{T}$ is the unit circle in the complex plane.

Fix $z \in \Omega$ and $a \in \mathbb{C}^N$, a so near the origin that for all $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$, the point $z + \zeta a$ is in Ω. Fix an $\varepsilon > 0$. Let $F_1 : \bar{\mathbb{U}} \times \mathbb{T} \to \mathbb{C}^N$ be a function with the property that for each $e^{i\vartheta} \in \mathbb{T}$, $F_1(\cdot, e^{i\vartheta}) \in \mathscr{F}(\Omega, z + e^{i\vartheta}a)$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_1(e^{i\psi}, e^{i\vartheta}a))\,d\psi < \widehat{u}(z + e^{i\vartheta}a) + \varepsilon.$$

Such a function exists: By the definition of $\widehat{u}$, for each $e^{i\vartheta} \in \mathbb{T}$ there is a corresponding element of $\mathscr{F}(\Omega, z + e^{i\vartheta}a)$. The function F_1 obtained in this way need have no particular regularity properties. A major part of the proof that follows consists in showing how to smooth F_1.

As a function of ϑ, the quantity $\widehat{u}(z + e^{i\vartheta}a)$ need not be continuous, but it is measurable. Accordingly, there is a function $v \in \mathscr{C}(\mathbb{T})$ such that $v(e^{i\vartheta}) \ge \widehat{u}(z + e^{i\vartheta}a)$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} v(e^{i\vartheta})\,d\vartheta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(z + e^{i\vartheta}a)\,d\vartheta + \varepsilon.$$

Let $|v| < A$. Having fixed v, choose $\tau_o > 0$ small enough that $|v(e^{i\vartheta}) - v(e^{i\vartheta'})| < \varepsilon$ if $|e^{i\vartheta} - e^{i\vartheta'}| < \tau_o$. Because the function u is upper semicontinuous, for each $e^{i\psi} \in \mathbb{T}$ there exist $\tau(e^{i\psi}) \in (0, \tau_o)$ and $r(e^{i\psi}) \in (0, 1)$ such that if

$$F_\psi(\zeta, e^{i\psi'}) = F_1(r(e^{i\psi})\zeta, e^{i\psi}) + (z + e^{i\psi'}a) - (z + e^{i\psi}a),$$

then $F_\psi(\cdot, e^{i\psi'}) \in \mathscr{F}(\Omega, z + e^{i\psi'}a)$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_\psi(e^{i\vartheta}, e^{i\psi'}))\,d\vartheta \le v(e^{i\psi}) + 2\varepsilon$$

when $|e^{i\psi} - e^{i\psi'}| < \tau(e^{i\psi})$.

Choose finitely many mutually disjoint closed circular arcs $I_j \subset \mathbb{T}$ with I_j centered at $e^{i\psi_j}$ and of length ℓ_j, $\ell_j < \tau(e^{i\psi_j})$. Require that the measure of $\mathbb{T} \setminus \cup_{j=1,\dots} I_j$ be less than $\varepsilon/(2A)$.

Define $F_2 : \bar{\mathbb{U}} \times \mathbb{T} \to \mathbb{C}$ by

$$F_2(\zeta, e^{i\psi}) = \begin{cases} F_{\psi_j}(\zeta, e^{i\psi}) & \text{when } \psi \in I_j, \\ z + e^{i\psi}a & \text{otherwise.} \end{cases}$$

Then $F_2(\cdot, e^{i\psi}) \in \mathscr{F}(\Omega, z + e^{i\psi}a)$, and for $\psi \in I_j$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_2(e^{i\vartheta}, e^{i\psi}))\, d\vartheta < v(e^{i\psi_j}) + 2\varepsilon < v(e^{i\psi}) + 3\varepsilon,$$

which implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_2(e^{i\vartheta}, e^{i\psi}))\, d\vartheta\right\} d\psi < \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(z + e^{i\vartheta}a)\, d\vartheta + 3\varepsilon.$$

Let the intervals I_j be indexed by the integers mod m for a suitable m and in such a way that I_j is between I_{j-1} and I_{j+1}. Denote by K_j the open interval between I_j and I_{j+1}. Let the endpoints of K_j be $e^{i\psi_j''}$ and $e^{i\psi_{j+1}'}$ with $e^{i\psi_j''}$ an endpoint of I_j and $e^{i\psi_{j+1}'}$ an endpoint of I_{j+1}. Divide K_j into three subintervals K_j', K_j'', and K_j'''. The interval K_j' is to abut I_j, K_j''' is to abut I_{j+1}, and K_j'' is to lie between K_j' and K_j'''. The intervals K_j' and K_j''' are to be short: For $e^{i\psi} \in K_j'$, we require that $|e^{i\psi} - e^{i\psi_j}| < \tau(e^{i\psi_j})$ and that for $e^{i\psi} \in K_j'''$, $|e^{i\psi} - e^{i\psi_{j+1}}| < \tau(e^{i\psi_{j+1}})$.

Let η be a nonnegative continuous function on $\mathbb{T}$ that is bounded by one, that is identically one on each I_j, and that vanishes on each K_j''.

Define $F_3 : \bar{\mathbb{U}} \times \mathbb{T} \to \mathbb{C}$ by

$$F_3(\zeta, e^{i\psi}) = \begin{cases} F_2(\zeta, e^{i\psi}), & e^{i\psi} \in \cup_j I_j, \\ F_2(\eta(e^{i\psi})\zeta, e^{i\psi_j'}) + (z + e^{i\psi}a) - (z + e^{i\psi_j'}a), & e^{i\psi} \in K_j', \\ z + e^{i\psi_j}a, & e^{i\psi} \in K_j'', \\ F_2(\eta(e^{i\psi})\zeta, e^{i\psi_{j+1}''}) + (z + e^{i\psi}a) - (z + e^{i\psi_{j+1}''}a), & e^{i\psi} \in K_j'''. \end{cases}$$

The function F_3 is continuous on $\bar{\mathbb{U}} \times \mathbb{T}$, and for all ψ, $F_3(\cdot, e^{i\psi}) \in \mathscr{F}(\Omega, z + e^{i\psi})$. Moreover, if the intervals K_j' and K_j''' are short enough, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_3(e^{i\vartheta}, e^{i\psi}))\, d\vartheta\right\} d\psi \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(z + e^{i\psi}a)\, d\psi + 5\varepsilon.$$

We want now to replace F_3 by a function F_4 that is holomorphic. For this, introduce, for each positive integer n, the function $f_n : \bar{\mathbb{U}} \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$ by

$$f_n(\zeta, \xi) = \frac{1}{n}\sum_{m=0}^{n-1}\sum_{k=-m}^{m}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\big\{F_3(\zeta, e^{is}) - (z + e^{is}a)\big\}e^{-iks}\, ds\right]\xi^k + z + \xi a.$$

For a fixed ζ and for $\xi = e^{i\psi} \in \mathbb{T}$, this is, aside from the term $z + \xi a$, the nth Cesàro mean of the Fourier series for the continuous function g given by $g(e^{i\psi}) = F_3(\zeta, e^{i\psi}a)$. It follows that as $n \to \infty$, $f_n(\zeta, e^{i\psi})$ converges to $F_3(\zeta, e^{i\psi})$ uniformly on $\bar{\mathbb{U}} \times \mathbb{T}$. Thus, for n sufficiently large,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(f_n(e^{i\vartheta}, e^{i\psi}))\,d\vartheta\right\}d\psi < \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{u}(z + e^{i\psi}a)\,d\psi + 6\varepsilon.$$

Let $n = n_o$ be large enough that this inequality holds.

The function $f_{n_o}(\zeta, \cdot)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a pole of order not more than n_o at the origin. We rid ourselves of this pole in the following way: Note that $f_{n_o}(0, \cdot)$ is holomorphic at the origin. If $k > n_o$, then the function $F_4 : \bar{\mathbb{U}} \times \bar{\mathbb{U}} \to \mathbb{C}$ given by $F_4(\zeta, \xi) = f_{n_o}(\xi^k\zeta, \xi)$ is holomorphic as a function of the two complex variables ζ and ξ, and $F_4(0, \xi) = z + \xi a$. Also, because the map $(e^{i\vartheta}, e^{i\psi}) \mapsto (e^{i(\psi+k\vartheta)}, e^{i\vartheta})$ is a bianalytic map of $\mathbb{T} \times \mathbb{T}$ onto itself that preserves area,

(2.3)
$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} u(F_4(e^{i\vartheta}, e^{i\psi}))\,d\vartheta d\psi = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} u(F_4(e^{i(\psi+\vartheta)}, e^{i\vartheta}))\,d\vartheta d\psi.$$

We have

(2.4)
$$\begin{aligned}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} u(F_4(e^{i\vartheta}, e^{i\psi}))\,d\vartheta d\psi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} u(f_{n_0}(e^{i\vartheta}, e^{i\psi}))\,d\vartheta\right\}d\psi\\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{u}(z + e^{i\psi}a)\,d\psi + 6\varepsilon,\end{aligned}$$

so it follows from (2.3) and (2.4) that for some choice of ψ,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_4(e^{i(\psi+\vartheta)}, e^{i\vartheta}))\,d\vartheta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{u}(z + e^{i\psi}a)\,d\psi + 6\varepsilon.$$

If now $h : \bar{\mathbb{U}} \to \bar{\mathbb{U}}^2$ is the holomorphic map $\xi \mapsto (e^{i\psi}\xi, \xi)$, then $F_4 \circ h \in \mathscr{F}(\Omega, z)$, and the inequality (2.4) can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(F_4 \circ h(e^{is}))\,ds \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{u}(z + e^{i\psi}a)\,d\psi + 6\varepsilon,$$

which implies that $\widehat{u}$ has the subaveraging property.

This completes the proof of Theorem 2.2.10.

Poletsky [281] has obtained an analogue of Theorem 2.2.10 on domains in Banach spaces.

A useful property of the function $\widehat{u}$ is that it is the greatest plurisubharmonic minorant of the function u:

Corollary 2.2.13. [278] *If u is an upper semicontinuous function on the domain Ω in $\mathbb{C}^N$, and if v is a plurisubharmonic function on Ω with $v \leq u$, then $v \leq \widehat{u}$.*

Proof. Let $z \in D$, and let $\varepsilon > 0$. There is an $f \in \mathscr{F}(\Omega, z)$ with the property that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(f(e^{i\vartheta}))\,d\vartheta < \widehat{u}(z) + \varepsilon.$$

It follows that

$$v(z) \leq \int_{-\pi}^{\pi} v(f(e^{i\vartheta}))\,d\vartheta \leq \int_{-\pi}^{\pi} u(f(e^{i\vartheta}))\,d\vartheta \leq \widehat{u}(z) + \varepsilon,$$

for v is plurisubharmonic. This inequality is correct for all $\varepsilon > 0$ and all $z \in \Omega$, so $v \leq \widehat{u}$.

As we have noted above, $\widehat{u} \leq u$, so the corollary is proved.

2.3. Applications of Morse Theory and Algebraic Topology

Our next results depend on well-known parts of Morse theory and algebraic topology. There is no possibility of our developing from the beginning the material we need from these disciplines; we refer the reader to standard sources for it. For Morse theory, one can consult the books of Milnor [247] and Hirsch [174], for algebraic topology, the books of Massey [241] and Spanier [338].

The property of polynomial convexity imposes significant restrictions on the topology of a subset of $\mathbb{C}^N$. The first step in this direction was taken by Browder [72], based on work of Serre [323] according to which if $\Omega \subset \mathbb{C}^N$ is a Runge domain of holomorphy then $H^p(\Omega; \mathbb{C}) = 0$ for $p \geq N$. Browder's result is that *if X is a compact, polynomially convex subset of $\mathbb{C}^N$, then the Čech cohomology groups $\check{H}^p(X; \mathbb{C})$ vanish when $p \geq N$.* This result implies that no compact orientable N-dimensional topological manifold in $\mathbb{C}^N$ is polynomially convex; it yields no information in the case of nonorientable manifolds. More recent contributions to the study of the topology of polynomially convex sets have been made by Alexander [23] and Forstnerič [127].

The first main theorem of this section concerns the homotopy structure of $\mathbb{C}^N \setminus X$ for a polynomially convex subset X of $\mathbb{C}^N$ and is due to Forstnerič [127].

Theorem 2.3.1. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, $N \geq 1$, then for $p = 0, \ldots, N-1$, the pth homotopy group $\pi_p(\mathbb{C}^N \setminus X)$ vanishes.*

The case $p = 0$ is trivial: The complement of a polynomially convex set is connected.

The case $p = 1$ is the statement that the complementary set $\mathbb{C}^N \setminus X$ is simply connected provided that $N \geq 2$.

The homotopy result Theorem 2.3.1 is the best possible result [127]: *For a polynomially convex subset X of $\mathbb{C}^N$, the homotopy groups $\pi_p(\mathbb{C}^N \setminus X)$ need not vanish when $p = N, \ldots, 2N-1$.* This is easily seen: Each compact subset of $\mathbb{R}^N$ is polynomially convex. If Y is a subset of $\mathbb{R}^N$ that consists of a disjoint union of spheres of dimensions $0, 1, \ldots, N-1$, then Y is a polynomially convex subset of $\mathbb{C}^N$ for which the homotopy groups $\pi_p(\mathbb{C}^N \setminus Y)$ are not zero in the range $p = N, \ldots, 2N-1$.

The proof of Theorem 2.3.1 depends on Morse theory.

The context for Morse theory is a smooth manifold $\mathscr{M}$, say of dimension d. On such a manifold a *Morse function* is a function u with only isolated critical points each of which is *nondegenerate* in the sense that at each point, the Hessian matrix $\left(u_{x_j x_k}\right)_{j,k=1,\ldots,d}$, computed with respect to some, and hence with respect to every, set of local coordinates is nonsingular. The *index* or *Morse index* of a nondegenerate critical point of u is the number of negative eigenvalues of this Hessian. This number is independent of the choice of local coordinates.

Morse functions exist in great abundance on every manifold. See [174]. A smooth real-valued function on a compact manifold can be approximated arbitrarily closely in the $\mathscr{C}^k$ topology by Morse functions. The appropriate result in the noncompact setting is approximation in what is called the *strong* topology, which involves asymptotically improving approximation at infinity. For the precise formulation, one can consult [174, p. 147].

We need the following lemma.

Lemma 2.3.2. *If u is a strictly plurisubharmonic function of class $\mathscr{C}^2$ on a complex manifold $\mathscr{M}$ of dimension N, then the index of each nondegenerate critical point of u is at most N.*

Proof. The problem is local, so we suppose $\mathscr{M}$ to be a domain in $\mathbb{C}^N$ that contains the origin. Let the origin be a nondegenerate critical point for u. If the Hessian has $N+1$ negative eigenvalues, then the real linear subspace of $\mathbb{C}^N$ on which the Hessian is negative definite contains a complex line, say λ. The restriction of u to λ is subharmonic, but because its Hessian at 0 is negative definite, 0 is a strict local maximum for $u|\lambda$. This contradicts the maximum principle for subharmonic functions.

Proof of Theorem 2.3.1. Fix a polynomially convex subset X of $\mathbb{C}^N$ and a neighborhood U of X. There is then a strictly plurisubharmonic function ϱ on $\mathbb{C}^N$ with the properties that $\varrho < 0$ on X, $\varrho > 0$ on $\mathbb{C}^N \setminus U$, and $\varrho(z) = |z|^2$ for large $z \in \mathbb{C}^N$. To construct ρ, recall that Theorem 1.3.8 provides a smooth nonnegative plurisubharmonic function v with $X = v^{-1}(0)$, with $v(z) = |z|^2$ for large z, and with v strictly plurisubharmonic off X. If u is a smooth, nonpositive function on $\mathbb{C}^N$ with compact support, u strictly plurisubharmonic on a neighborhood of X that is contained in U, then for a sufficiently small positive δ, the function $v + \delta u$ can be taken for ρ. Moreover, ρ can be assumed to be a Morse function, so that it has only finitely many critical points, each of which is nondegenerate. For each real t, let X_t be the set $\{z \in \mathbb{C}^N : -\rho(z) < -t\}$, so that X_t is a sublevel set of the Morse function $-\rho$. The index of each critical point of ρ is no more than N, so those of $-\rho$ are all at least N. Let the critical values of ρ be $t_1 < t_2 < \cdots < t_r$ with corresponding Morse indices $2N - m_k$, so that $2N - m_k \leq N$. The index of the critical value of $-t_k$ of $-\rho$ is then $m_k \geq N$.

A fundamental result of Morse theory [247] is that the homotopy type of X_s does not change as s ranges through one of the intervals $(-t_{k+1}, -t_k)$. If $t \in (-t_k, -t_{k-1})$ and $s \in (-t_{k+1}, -t_k)$, then X_t has the homotopy type of X_s to which a cell of dimension $m_k \geq N$ has been attached. The adjunction of a cell of dimension m_k to X_s has no effect on the homotopy groups π_j in the range $0 \leq j \leq N-1$. Thus, for all $t > 0$, the homotopy groups $\pi_j(X_t)$ in the range $0 \leq j \leq N-1$ are the same as the homotopy groups of X_R for large positive R. But by the choice of u, for large positive R, X_R is $\{z \in \mathbb{C}^N : |z| > R^2\}$, so that $\pi_j(X_R)$ vanishes in the range $0 \leq j \leq 2N-2$.

The conclusion is that the complementary set $\mathbb{C}^N \setminus X$ is a union $\cup_{k=1,\dots}\Omega_k$ of domains Ω_k with $\Omega_k \subset \Omega_{k+1}$ and with $\pi_j(\Omega_k) = 0$, $0 \leq j \leq N-1$. Consequently, $\pi_j(\mathbb{C}^N \setminus X)$ vanishes in the same range. The theorem is proved.

This kind of Morse-theoretic argument was used by Andreotti and Frankel [35] and by Andreotti and Narasimhan [37].

We now turn to some corollaries of Theorem 2.3.1.

Corollary 2.3.3. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, $N \geq 2$, then for $p = 1, \dots, N-1$, the pth singular homology group with integral coefficients, $H_p(\mathbb{C}^N \setminus X; Z)$, vanishes, and there is an isomorphism*

$$H_N(\mathbb{C}^N \setminus X; \mathbb{Z}) \simeq \pi_N(\mathbb{C}^N \setminus X; \mathbb{Z}).$$

Proof. Because $H_1(\mathbb{C}^N \setminus X; \mathbb{Z})$ is the abelianization of $\pi_1(\mathbb{C}^N \setminus X)$, it follows that the former group vanishes when the latter one does. The corollary then follows from the theorem because of the Hurewicz isomorphism theorem [338], which says that for a space E such that for a fixed $k \geq 2$, $\pi_{k-1}(E) = 0$, necessarily $\pi_k(E) = H_k(E; \mathbb{Z})$. This covers the cases $p = 2, 3, \dots, N$.

Corollary 2.3.4. *If X is a compact polynomially convex subset of $\mathbb{C}^N$, $N \geq 2$, and if G is an abelian group, then for $p = 1, \dots, N-1$, the pth singular homology group with coefficients from G, $H_p(\mathbb{C}^N \setminus X; G)$, vanishes, as does the pth singular cohomology group $H^p(\mathbb{C}^N \setminus X; G)$.*

Proof: The result for homology with coefficients in an arbitrary abelian group G follows from the result with coefficients in $\mathbb{Z}$ by the universal coefficients theorem for homology, the cohomology result from the corresponding universal coefficients theorem for cohomology. (For these universal coefficients theorems see [241] or [338].)

We need a form of the *Alexander duality theorem* [241] or [338]: *If A is a closed subset of $\mathbb{R}^n$, then $\tilde{H}_q(\mathbb{R}^n \setminus A; G) = \check{H}^{n-q-1}(A; G)$.* Here $\check{H}^*$ denotes Čech cohomology, and $\tilde{H}_*$ denotes the *reduced* homology group, so that $\tilde{H}_q = H_q$ if $q \geq 1$ and there is an exact sequence $0 \to \tilde{H}_0 \to H_0 \to G \to 0$. More generally, *if $\mathscr{M}$ is an n-dimensional orientable manifold and A and B are compact subsets of $\mathscr{M}$ with $A \supset B$, there is an isomorphism of relative groups*

$$H_q(\mathscr{M} \setminus B, \mathscr{M} \setminus A; G) = \check{H}^{n-q}(A, B; G).$$

With B the empty set, this gives $H_q(\mathscr{M}, \mathscr{M} \setminus A; G) = \check{H}^{n-q}(A; G)$. The long exact homology sequence of the pair $(\mathscr{M}, \mathscr{M} \setminus A)$ shows that if $H_q(\mathscr{M}; G) = H_{q+1}(\mathscr{M}; G) = 0$, e.g., as when $\mathscr{M} = \mathbb{R}^n$ or when $\mathscr{M}$ is contractible, then $H_q(\mathscr{M} \setminus A; G) = H_{q+1}(\mathscr{M}, \mathscr{M} \setminus A; G)$, and we reach

$$H_q(\mathscr{M} \setminus A; G) = \check{H}^{n-q-1}(A; G).$$

Corollary 2.3.5. *No N-dimensional compact topological manifold in $\mathbb{C}^N$, orientable or not, is polynomially convex.*

As noted above, in the orientable case this result was observed by Browder [72]. The nonorientable case was given by Duchamp and Stout [104].

Proof. If Σ is a compact, N-dimensional submanifold of $\mathbb{C}^N$, then by Alexander duality, $\check{H}^N(\Sigma;\mathbb{Z}) = H_{N-1}(\mathbb{C}^N \setminus \Sigma;\mathbb{Z})$. If Σ is polynomially convex, the latter group vanishes. However, the former group is $\mathbb{Z}$ or $\mathbb{Z}_2$, depending on whether Σ is orientable or not. That $H^N(\mathscr{M},\mathbb{Z}) = \mathbb{Z}$ in case $\mathscr{M}$ is orientable follows from the Poincaré duality theorem [241, p. 208] and the isomorphism $H_0(\mathscr{M},\mathbb{Z}) = \mathbb{Z}$. That for a connected, compact, nonorientable manifold $\mathscr{M}$ of dimension n, $H^n(\mathscr{M},\mathbb{Z}) = \mathbb{Z}_2$ is a standard result that seems not to be written explicitly in either [241] or [338]. It follows immediately from things found there: By compactness, all the homology groups of $\mathscr{M}$ are finitely generated. Because $\mathscr{M}$ is not orientable, $H_n(\mathscr{M},\mathbb{Z}) = 0$ [241, p. 213, Exercise 4.8]. The universal coefficients theorem in cohomology gives

$$0 \to \mathrm{Ext}(H_{n-1}(\mathscr{M},\mathbb{Z}),\mathbb{Z}) \to H^n(\mathscr{M},\mathbb{Z}) \to \mathrm{Hom}(H_n(\mathscr{M};\mathbb{Z}),\mathbb{Z}) \to 0. \tag{2.5}$$

Therefore $\mathrm{Ext}(H_{n-1}(\mathscr{M};\mathbb{Z}),\mathbb{Z}) = H^n(\mathscr{M};\mathbb{Z})$. The torsion subgroup of $H_{n-1}(\mathscr{M},\mathbb{Z})$ is $\mathbb{Z}_2$ [241, p. 214, Exercise 4.9], so for a suitable nonnegative integer r,

$$H^n(\mathscr{M};\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z}^r \oplus \mathbb{Z}_2,\mathbb{Z}).$$

The latter group is $\mathbb{Z}_2$.[4]

Corollary 2.3.6. *If X is a compact, polynomially convex subset of $\mathbb{C}^N$, then for every abelian group G, $\check{H}^p(X;G) = 0$ if $p \geq N$.*

Let us be clear about the *analytic*, as opposed to the *topological*, content of this statement. If $K \subset \mathbb{R}^n$, then $\check{H}^q(K;G) = 0$ for all $q \geq n$ [241, p. 222]. Thus the analytic content of the corollary lies in the cases $p = N, \dots, 2N-1$.

Proof: The space $\mathbb{C}^N \setminus X$ is arcwise connected, so $H_0(\mathbb{C}^N \setminus X;G) = G$, and $\tilde{H}_0(\mathbb{C}^N \setminus X;G) = 0$. Thus by Alexander duality, $\check{H}^{2N-q-1}(X;G) = 0$ when $q = 0,\dots,N-1$, which is $\check{H}^p(X;G) = 0$, $p \geq N$.

The preceding corollary has a consequence for polynomially convex subsets of the boundary of the ball or of certain more general domains:

[4] We need to use the universal coefficients theorem in cohomology, which involves the Ext functor, so it seems well to recall some of the properties of this functor. To every pair A and B of abelian groups is associated the abelian group $\mathrm{Ext}(A,B)$. It enjoys the following properties:

(a) $\mathrm{Ext}(\mathbb{Z}_p,\mathbb{Z}) = \mathbb{Z}_p$.

(b) $\mathrm{Ext}(F,\mathbb{Z}) = 0$ if F is a free abelian group.

(c) For abelian groups $A_1,\dots,A_n,B$, $\mathrm{Ext}(\oplus_{j=1}^n A_j,B) = \oplus_{j=1}^n \mathrm{Ext}(A_j,B)$.

(d) If B is a divisible abelian group, then $\mathrm{Ext}(A,B) = 0$ for all abelian groups A. (A group G is *divisible* if for each $g \in G$ and each positive integer n, there is $h \in G$ with $nh = g$.)

(e) If F is an abelian group of countable rank, then $\mathrm{Ext}(F,\mathbb{Z}) = 0$ implies that F is free.

Of these, (a), (b), and (c), (d) are among the very first facts in the theory of Ext. Point (e) is a rather substantial result. References are [241] and [173].

Corollary 2.3.7. *Let D be a bounded domain in $\mathbb{C}^N$, $N \geq 2$, such that bD is a connected topological manifold with $H_1(bD; \mathbb{Z}) = 0$. If X is a compact polynomially convex subset of bD, then $bD \setminus X$ is connected.*

Notice that there is no smoothness requirement whatsoever on the manifold bD.

Proof. The domain D is bounded, so bD is a *compact oriented* manifold, whence the exact homology sequence of the pair $(bD, bD \setminus A)$ and the Alexander duality theorem yield the equality $\tilde{H}_0(bD \setminus X; Z) = H^{2N-2}(X; \mathbb{Z})$. By the preceding corollary, the latter group vanishes, so the corollary is proved.

We return to the general theme suggested by this corollary below; see Theorem 2.4.8 and Theorem 2.4.11.

The paper [113] of Eroshkin contains some additional information about the homotopy structure of $bD \setminus K$, K a compact set, for certain domains D in $\mathbb{C}^2$.

These topological results for polynomially convex sets imply corresponding results, with a shift of indices, for rationally convex sets in $\mathbb{C}^N$, because, by Theorem 1.2.11, each rationally convex subset of $\mathbb{C}^N$ is homeomorphic to a polynomially convex subset of $\mathbb{C}^{N+1}$.

Corollary 2.3.8. *For a compact, rationally convex subset X of $\mathbb{C}^N$ and for every abelian group G, $\check{H}^p(X; G) = 0$ if $p \geq N + 1$.*

Proof. If X is homeomorphic to the compact polynomially convex subset Y of $\mathbb{C}^{N+1}$, then $\check{H}^p(X; G) = \check{H}^p(Y; G)$, and the latter group vanishes when $p \geq N+1$ by Corollary 2.3.6.

Corollary 2.3.9. *If X is a rationally convex set in $\mathbb{C}^N$, $N \geq 2$, and G is an abelian group, then for $p = 1, \ldots, N-2$, the singular homology and cohomology groups $H_p(\mathbb{C}^N \setminus X; G)$ and $H^p(\mathbb{C}^N \setminus X; G)$ vanish.*

Remark. For $N = 2$, the only assertion is that the groups in dimension 0 vanish, and this is just the statement that $\mathbb{C}^N \setminus X$ is connected, which we know it to be from other considerations.

Proof. By Alexander duality, $\check{H}^{2N-q-1}(X; G) \simeq \tilde{H}_q(\mathbb{C}^N \setminus X; G)$. Thus, the latter groups vanish when $N - 2 \geq q$. The cohomology result follows from the universal coefficients theorem.

Corollary 2.3.10. *No M-dimensional compact topological manifold in $\mathbb{C}^N$ with $M \geq N + 1$ is rationally convex.*

Compact manifolds of dimension N in $\mathbb{C}^N$ can be rationally convex, as the N-dimensional torus $\mathbb{T}^N$ shows.

In the preprint [379], Zeron invesigates the homotopy properties of sets of the form $M \setminus X$ with M a complex manifold and X a compact intersection of Stein domains.

The work above allows us to detect points in the polynomially convex hull of a compact subset of $\mathbb{C}^N$ in the following way. Consider a compact set X in $\mathbb{C}^N$ and a compact subset E in $\mathbb{C}^N \setminus X$. The question is, when does E intersect $\widehat{X}$? Various answers are available, depending on which homology or cohomology theory is used.

Denote by ι the inclusion $E \hookrightarrow \mathbb{C}^N \setminus X$, by ι'' the inclusion $\mathbb{C}^N \setminus \widehat{X} \hookrightarrow \mathbb{C}^N \setminus X$, and, provided $E \cap \widehat{X} = \emptyset$, by ι' the inclusion $E \hookrightarrow \mathbb{C}^N \setminus \widehat{X}$.

Fix an abelian group of coefficients.

Corollary 2.3.11. [127] *If E is a manifold, and if for some $q \in \{1, \dots, N-1\}$ the induced map $\iota_* : H_q(E; G) \to H_q(\mathbb{C}^N \setminus X; G)$ is not the zero map, then the set E meets $\widehat{X}$.*

Proof. If not, then because the induced map $\iota_* : H_q(E; G) \to H_q(\mathbb{C}^N \setminus X; G)$ between singular homology groups is the composition $\iota''_* \circ \iota'_*$, and $H_q(\mathbb{C}^N \setminus \widehat{X}; G) = 0$, we have a contradiction.

Corollary 2.3.12. *If for some $q \in \{1, \dots, N-1\}$ the induced map $\iota^* : \check{H}^q(\mathbb{C}^N \setminus X; G) \to \check{H}^q(E; G)$ is not the zero map, then E meets $\widehat{X}$.*

Proof: If not so that $E \subset \mathbb{C}^N \setminus \widehat{X}$, then because $\check{H}^q(\mathbb{C}^N \setminus \widehat{X}; G) = 0$ and $\iota^* = \iota'^* \circ \iota''^*$, we reach a contradiction.

The first result of this kind was given by Alexander [23] and was based on de Rham cohomology for manifolds.

Recall that the pth de Rham cohomology group of a smooth manifold $\mathscr{M}$ is the group $H^p_{\mathrm{deR}}(\mathscr{M})$ defined by

$$H^p_{\mathrm{deR}}(\mathscr{M}) = \frac{\{\text{closed smooth } p\text{-forms on } \mathscr{M}\}}{\{\text{exact smooth } p\text{-forms on } \mathscr{M}\}}.$$

De Rham's theorem asserts that *the group $H^p_{\mathrm{deR}}(\mathscr{M})$ is isomorphic to the singular cohomology group $H^p(\mathscr{M}; \mathbb{R})$*. (If we are working with complex-valued forms rather than real-valued ones, then $H^p_{\mathrm{deR}}(\mathscr{M})$ is isomorphic to $H^p(\mathscr{M}; \mathbb{C})$.)

Corollary 2.3.13. *If X and E are smooth compact manifolds such that for some $q \in \{1, \dots, N-1\}$ the induced map $\iota^* : H^q_{\mathrm{deR}}(\mathbb{C}^N \setminus X) \to H^q_{\mathrm{deR}}(E)$ is not the zero map, then E meets $\widehat{X}$.*

The condition of the last corollary can be rephrased as the requirement that there be a closed q-form on $\mathbb{C}^N \setminus X$ that when integrated over some q-cycle in the manifold E yields a nonzero result.

Because of Corollaries 2.3.8 and 2.3.9, the last three corollaries have evident analogues in which polynomially convex hulls are replaced by rationally convex hulls. The index q must then be restricted to the set $\{1, \dots, N-2\}$ rather than to $\{1, \dots, N-1\}$ as in the case of polynomial convexity.

The hypotheses of the last corollary imply restrictions on the dimensions of X and E. Denote by d the dimension of X and by e that of E. Because the map $\iota^* : H^q_{\mathrm{deR}}(\mathbb{C}^N \setminus X) \to H^q_{\mathrm{deR}}(E)$ is not the zero map, the latter group cannot be zero. Thus, $q \leq e$. Also, by de Rham's theorem, the group $H^q_{\mathrm{deR}}(\mathbb{C}^N \setminus X)$ is isomorphic to the singular cohomology group $H^q(\mathbb{C}^N \setminus X; \mathbb{C})$, which, because the coefficients in question are from a field, is isomorphic to the singular *homology* group $H_q(\mathbb{C}^N \setminus X; \mathbb{C})$. By Alexander duality, this group is isomorphic to the Čech cohomology group $\check{H}^{2N-q-1}(X; \mathbb{C})$. It follows that $2N - q - 1 \leq d$ or $2N - d - 1 \leq q$. The corollary requires that $q \leq N - 1$. The upshot is that the hypotheses of the last corollary can be satisfied only when the dimensions d and e satisfy $N \leq d$ and $e \geq q$. In particular, $2N - 1 \leq d + e$.

There is a notion of linked manifolds to which the preceding corollaries are related.

Definition 2.3.14. *Given mutually disjoint smooth, compact, oriented manifolds $\mathscr{M}$ and $\mathscr{N}$ in $\mathbb{R}^N$ with $\dim \mathscr{M} = p$, $\dim \mathscr{N} = q$, $p + q = N - 1$, the* linking number, *$\mathscr{L}(\mathscr{M}, \mathscr{N})$, of $\mathscr{M}$ and $\mathscr{N}$ is given by*

$$\mathscr{L}(\mathscr{M}, \mathscr{N}) = \int_{\mathscr{M}\times\mathscr{N}} \psi^*\theta,$$

where $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is the difference map $\psi(x, y) = x - y$ and where θ is some smooth closed $(N - 1)$-form on $\mathbb{R}^N \setminus \{0\}$ such that $\int_{\mathbb{S}^{N-1}} \theta = 1$.

Stokes's theorem implies that this linking number is independent of the choice of the form θ and also that the path $\mathscr{M} \times \mathscr{N}$ of integration in the definition can be replaced by any path homologous in $\mathbb{R}^N \times \mathbb{R}^N \setminus \psi^{-1}(0)$ to it.

The linking number satisfies $L(\mathscr{M}, \mathscr{N}) = (-1)^{pq+1}L(\mathscr{N}, \mathscr{M})$.

The linking number $L(\mathscr{M}, \mathscr{N})$ is the *degree* of the map $\psi : \mathscr{M} \times \mathscr{N} \to \mathbb{R}^N \setminus \{0\}$. For degree theory one can consult [96] or [88].

If the two manifolds $\mathscr{M}$ and $\mathscr{N}$, of dimensions p and q, respectively, are linked, i.e., if the linking number $\mathscr{L}(\mathscr{M}, \mathscr{N})$ does not vanish, then there are closed q-forms μ on $\mathbb{R}^N \setminus \mathscr{M}$ for which $\int_{\mathscr{N}} \mu \neq 0$, so that Corollary 2.3.13 applies. This is so, for the linking hypothesis implies that for some smooth closed $(N-1)$-form θ on $\mathbb{R}^N \setminus \{0\}$, the integral $\int_{\mathscr{M}\times\mathscr{N}} \psi^*\theta$ is nonzero, as we see in the following way. Write $\psi^*\theta = \sum_{|J|+|K|=N-1} A_{I,J}dx^I \wedge dy^J$ where the coefficients $A_{I,J}$ are smooth functions on $\mathbb{R}^N \times \mathbb{R}^N \setminus \psi^{-1}(0)$. If $\iota : \mathscr{M} \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ is the inclusion, then

$$\iota^*\psi^*\theta = \sum_{|I|=p,|J|=q} A_{I,J}dx^I \wedge dy^J,$$

as follows from degree considerations. Define $\mu \in \mathscr{E}^q(\mathbb{R}^N \setminus \mathscr{M})$ by

$$\mu = \sum_{|J|=q} \left(\int_{\mathscr{N}} \sum_{|I|=p} A_{I,J}\, dx^I \right) dy^J.$$

By Fubini's theorem,[5]

$$\int_{\mathscr{M}\times\mathscr{N}} \iota^*\psi^*\theta = \int_{\mathscr{N}} \mu.$$

Because $\mathscr{L}(\mathscr{M}, \mathscr{N}) \neq 0$, it follows that $\int_{\mathscr{N}} \mu \neq 0$. Also, the form μ is closed. This is so, for because $d\psi^*\theta = 0$, we have that on $\mathbb{R}^N \times \mathbb{R}^N \setminus \psi^{-1}(0)$,

$$\sum_{k=1}^{N} \sum_{|I|=p,|J|=q} \frac{\partial A_{I,J}}{\partial x_k} dx_k \wedge dx^I \wedge dy^J = (-1)^{p+1} \sum_{j=1}^{N} \sum_{|I|=p,|J|=q} \frac{\partial A_{I,J}}{\partial y_j} dx^I \wedge dy_j \wedge dy^J.$$

[5] The version of Fubini's theorem that we are using is stated carefully in the appendix to this section.

The form on the left is of degree $p+1$ in x and q in y, that on the right of degree p in x and $q+1$ in y. Accordingly, both are zero. We see that μ is closed as desired, for

$$d\mu = \sum_{|J|=q} \left(\int_{\mathcal{N}} \frac{\partial A_{I,J}}{\partial y_j} dx^I \right) dy_j \wedge dy^J = 0.$$

As an example of the foregoing, consider the case of two spheres in $\mathbb{C}^N$, $N \geq 2$. First identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ and then write M for $2N$ to simplify notation. Consider the spheres $\mathbb{S}'$ and $\mathbb{S}''$ given by

$$\mathbb{S}' = \{x = (x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^M : x_1^2 + \cdots + x_d^2 = 1\}$$

and

$$\mathbb{S}'' = \{x = (0, \ldots, 0, x_d, \ldots, x_M) \in \mathbb{R}^M : (x_d - 1)^2 + x_{d+1}^2 + \cdots + x_M^2 = 1\}.$$

The sphere $\mathbb{S}'$ has dimension $d-1$, $\mathbb{S}''$ dimension $M-d$. One of $d-1$ and $M-d$ is at least two; we suppose it to be $d-1$, whence $d \geq 3$.

These two spheres are linked. For this, it suffices to show that if ϑ denotes the closed form $\frac{1}{|x|^{2M}} \sum_{j=1,\ldots,M} (-1)^{j-1} x_j \omega_{[j]}(x)$ in which, for each j,

$$\omega_{[j]}(x) = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_M,$$

then the integral $\int_{\mathbb{S}' \times \mathbb{S}''} \psi^* \vartheta$ is nonzero. Here and below ψ denotes the difference map $\psi(x,y) = x - y$ from $\mathbb{R}^M \times \mathbb{R}^M$ to $\mathbb{R}^M$. The form ϑ does not satisfy $\int_{\mathbb{S}^{M-1}} \vartheta = 1$; the value of this integral is M times the volume of the unit ball in $\mathbb{R}^M$.

If $\mathbb{S}'_r$ denotes the sphere of radius r concentric with $\mathbb{S}'$, then the integral $\int_{\mathbb{S}'_r \times \mathbb{S}''} \psi^* \vartheta$ is independent of r, $r > 0$, by Stokes's theorem.

The desired result follows from Stokes's theorem and the observation that $\mathbb{S}' \times \mathbb{S}'' \subset \mathbb{R}^M \times \mathbb{R}^M$ is the boundary of the set $\mathbb{S}' \times \Delta$ if Δ denotes the open manifold

$$\Delta = \{(0, \ldots, 0, y_d, y_{d+1}, \ldots, y_M) \in \mathbb{R}^M : (y_d - 1)^2 + y_{d+1}^2 + \cdots + y_M^2 > 1\}.$$

The form $\psi^* \vartheta$ is given by

$$\psi^* \vartheta = |x - y|^{-M} \sum_{j=1}^{2M} (-1)^{j-1} (x_j - y_j) \omega_{[j]}(x - y).$$

On $\mathbb{S}' \times \Delta$ we have $x_{d+1} = \cdots = x_M = y_1 = \cdots = y_{d-1} = 0$. For large $s > 0$, let D_s denote the set of (x, y) for which $x \in \mathbb{S}'$, and $y = (0, \ldots, 0, y_d, \ldots, y_M) \in \Delta$ with $y_d^2 + \cdots + y_M^2 \leq s^2$. The form $\psi^* \vartheta$ is closed, so Stokes's theorem implies that $0 = \int_{bD_s} \psi^* \vartheta$. If Σ_s denotes the part of bD_s consisting of the points $(x, y) \in \mathbb{S}' \times \Delta$ with $|y| = s$, then on Σ_s we have $|x - y|^{-2M} \leq \frac{1}{2} s^{-2M}$. The x-integration is over the fixed sphere $\mathbb{S}'$, and for fixed $x \in \mathbb{S}'$, the y-integration is over part of a sphere of dimension

$M - d$ and radius s. Consequently, for each fixed $x \in \mathbb{S}'$, the corresponding y-integral is bounded by const $s^{-2M} s^{M-d}$, which tends to 0 as $s \to \infty$. It follows that if

$$\Pi_+ = \{(0, \dots, 0, y_{d+1}, \dots, y_M) : y_{d+1}^2 + \dots + y_M^2 \geq 1\} = b\Delta \cap \{y \in \mathbb{R}^M : y_d = 0\},$$

then

$$\int_{\mathbb{S}' \times \mathbb{S}''} \psi^* \vartheta = \int_{\mathbb{S}' \times \Pi_+} \psi^* \vartheta.$$

On $\mathbb{S}' \times \Pi_+$, $y_d = 0$, so that for $x \in \mathbb{S}'$ and $y \in \Pi_+$, $|x - y|^2 = (1 + |y|^2)$. Consequently,

$$\int_{\mathbb{S}' \times \mathbb{S}''} \psi^* \vartheta = (-1)^M \left\{ \int_{\mathbb{S}'} \sum_{j=1}^{d} (-1)^{j-1} x_j \, dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots \wedge dx_d \right\} \times \left\{ \int_{\Pi_+} \frac{dy_{d+1} \wedge \cdots \wedge dy_M}{(1 + |y|^2)^{M/2}} \right\}.$$

This number is not zero, so the linking number $\mathscr{L}(\mathbb{S}', \mathbb{S}'')$ is not zero.

Suppose now that $\dim \mathbb{S}' > \dim \mathbb{S}''$. The inclusion $\iota : \mathbb{S}'' \hookrightarrow \mathbb{C}^N \setminus \mathbb{S}'$ induces a nontrivial map from $H^{M-d}_{\text{deR}}(\mathbb{C}^N \setminus \mathbb{S}') \to H^{M-d}_{\text{deR}}(\mathbb{S}'')$. The same conclusion is correct if de Rham cohomology is replaced by singular cohomology. But then Corollary 2.3.12 implies that if Σ' is any *topological* $(d-1)$-sphere homologous to $\mathbb{S}'$ in $\mathbb{C}^N \setminus \mathbb{S}''$, then $\mathbb{S}''$ meets $\widehat{\Sigma}'$. And, having replaced $\mathbb{S}'$ by Σ', we can now replace $\mathbb{S}''$ by any *topological* $(2N-d)$-sphere Σ'' homologous in $\mathbb{C}^N \setminus \Sigma'$ to $\mathbb{S}''$.

As noted already, there are analogous conclusions concerning rationally convex hulls, once the dimensions have been restricted appropriately.

The next corollary can be derived from the preceding results on linking or, as we shall do, from Corollary 2.3.6.

Corollary 2.3.15. [23] *Let E be a real d-dimensional subspace of $\mathbb{C}^N$, $d \geq N + 1$, and let $E^\perp$ be the real orthogonal complement of E in $\mathbb{C}^N$, so that as a real vector space $\mathbb{C}^N = E \oplus E^\perp$. Let X be a compact subset of E, and let D be a bounded component of $E \setminus X$. If $f : X \to E^\perp$ is a continuous function, then the polynomially convex hull of the graph of f projects under the orthogonal projection $\pi : E \oplus E^\perp \to E$ onto a set that contains D. If $d \geq N + 2$, then the rationally convex hull of the graph of f projects onto a set in E that contains D.*

The dimension restriction in this proposition is necessary. If D is any bounded domain in C^N, and if $f : bD \to \mathbb{C}^N$ is the map $f(z) = \bar{z}$, then the graph of f is polynomially convex.

It is not true that the polynomially convex hull of a graph is necessarily a graph, as the following example [20] shows.

Let φ be the function defined on $b\mathbb{B}_2$ by $\varphi(z_1, z_2) = |z_1|$, and denote by Γ_φ its graph, the set $\{(z_1, z_2, w) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 = 1, \; w = |z_1|\}$. If $(z_1, z_2, |z_1|) \in \Gamma_\varphi$, then the point $(e^{i\vartheta} z_1, e^{i\vartheta} z_2, |z_1|)$ is also in Γ_φ. Thus, if P is a polynomial, then for each point

$(z_1, z_2) \in \mathbb{B}_2$ and for each $\zeta \in \mathbb{C}$ with $|\zeta| \le 1$, we have $|P(\zeta z_1, \zeta z_2, |z_1|)| \le \|P\|_{\Gamma_\varphi}$. Consequently, for each $t \in [0, 1]$, the point $(0, 0, t)$ lies in the polynomially convex hull of Γ_φ. It follows that $\widehat{\Gamma_\varphi}$ is not a graph over a set in $\mathbb{C}^2$.

To prove the corollary above, we need a simple fact about differential forms.

Lemma 2.3.16. *Let X be a compact subset of $\mathbb{R}^N$, let $\pi : \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}^N$ be the orthogonal projection, and let $Y \subset \mathbb{R}^N \times \mathbb{R}^k$ be a compact set carried homeomorphically onto X by π. If θ is a smooth p-form defined and closed on a neighborhood of X, and if $\pi^*\theta$ is exact on a neighborhood of Y, then θ is exact on a neighborhood of X.*

Proof. Let the form θ be defined, smooth, and closed on the neighborhood U of X in $\mathbb{R}^N$, and let Ω be a neighborhood of Y in $\mathbb{R}^N \times \mathbb{R}^k$ on which $\pi^*\theta$ is exact, say $\pi^*\theta = d\psi$ for a smooth $(p-1)$-form ψ. The map $\pi|Y$ is a homeomorphism, so there is a continuous map $f : X \to \mathbb{R}^k$ such that the map $\eta : X \to \mathbb{R}^N \times \mathbb{R}^k$ given by $\eta(x) = (x, f(x))$ is the inverse of $\pi|Y$. The Tietze extension theorem provides an extension, denoted again by f, of the map f to a map $f : \mathbb{R}^N \to \mathbb{R}^k$ such that if $\eta : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^k$ is the map $x \mapsto (x, f(x))$, then $\pi \circ \eta = \mathrm{id}$ on all of $\mathbb{R}^N$. The map f can be approximated by a smooth map f_1 in such a way that, with $\eta_1(x) = (x, f_1(x))$, which satisfies $\pi \circ \eta_1 = \mathrm{id}$ on $\mathbb{R}^N$, η_1 carries a neighborhood U_1 of X into Ω. Then $\theta = (\pi \circ \eta_1)^*\theta = \eta_1^*(\pi^*\theta) = d(\eta_1^*\psi)$. Thus θ is exact on some neighborhood of X.

Proof of Corollary 2.3.15. With notation as in the statement of the corollary, we suppose for convenience that $0 \in D$. Fix a smooth, closed form θ of degree $d-1$ on $E \setminus \{0\}$ that is not exact. Denote by $\pi : \mathbb{C}^N \to E$ the orthogonal projection. The form $\pi^*\theta$ is closed (and smooth) on $\mathbb{C}^N \setminus \pi^{-1}(0)$. Denote by Γ_f the graph of the function f. The polynomially convex set $\widehat{\Gamma}_f$ is contained in $\mathbb{C}^N \setminus \pi^{-1}(0)$, so θ is exact on some neighborhood of $\widehat{\Gamma}_f$. There are two ways to see this. One is to note that the set $\widehat{\Gamma}_f$ is the intersection of a decreasing sequence $\{W_j\}$ of polynomial polyhedra contained in $\mathbb{C}^N \setminus \pi^{-1}(0)$. As a polynomial polyhedron, each W_j is a Runge domain in $\mathbb{C}^N$ and so has vanishing cohomology with coefficients in $\mathbb{C}$ in degree greater than or equal to N by the theorem of Serre quoted at the beginning of this section. De Rham's theorem implies that θ is exact on each of the W_j. Alternatively, we know by Corollary 2.3.6 that $\check{H}^{d-1}(\widehat{\Gamma}_f; \mathbb{C})$ vanishes. It follows that if c is a cohomology class in $H^{d-1}(\Omega; \mathbb{C})$, Ω a neighborhood of $\widehat{\Gamma}_f$, then under the map $H^{d-1}(\Omega; \mathbb{C}) \to H^{d-1}(\widehat{\Gamma}_f; \mathbb{C})$ induced by the inclusion, the class c is taken to the zero element. This implies that it induces the zero element in $H^{d-1}(\Omega'; \mathbb{C})$ for all sufficiently small neighborhoods Ω' of $\widehat{\Gamma}_f$. Applied to the cohomology class in $H^{d-1}(\Omega; \mathbb{C})$ induced by the form $\pi^*(\theta)$, we again conclude that $\pi^*\theta$ is exact on some neighborhood of $\widehat{\Gamma}_f$. The preceding lemma implies that θ is exact on a neighborhood, say V, of X, for the projection π is a homeomorphism from Γ_f onto X. There is then an open set $D_o \Subset D$, $0 \in D_o$, with bD_o a smooth manifold that is contained in V. The form θ is exact on bD_o, so Stokes's theorem implies that $\int_{bD_o} \theta = 0$. However, this is impossible, for the hypothesis that θ is not exact implies that if $\Sigma \subset D_o$ is a small sphere centered at 0, then $\int_\Sigma \theta \neq 0$. The sphere Σ and the manifold bD_o are homologous in $E \setminus \{0\}$, so we have a contradiction. The part of the corollary concerning polynomial convexity is proved. The assertion concerning rationally convex hulls follows in a similar way.

Some particular cases of the last corollary are these.

A. Let f be a continuous function on a compact connected topological $(2N-1)$-manifold Σ, for example a sphere, in $\mathbb{C}^N$, $N \geq 2$. The manifold Σ splits $\mathbb{C}^N$ into two components D_i and D_e, the bounded and unbounded components of $\mathbb{C}^N \setminus \Sigma$, respectively, and the corollary implies that the polynomially convex hull of the graph of f projects onto a compact set in $\mathbb{C}^N$ that contains the component D_i.

B. If D is a relatively compact domain in $\mathbb{C}^N$ whose closure is polynomially convex, then given a continuous $\mathbb{C}$-valued function f on bD, the hull $\widehat{\Gamma}_f \subset \mathbb{C}^N \times \mathbb{C}$ projects onto $\bar{D}$.

C. For another example, take $\mathbb{S}^2$ to be the unit 2-sphere in $\mathbb{R}^3 \subset \mathbb{C}^2$ and let f be a real-valued function on $\mathbb{S}^3$. The polynomially convex hull of the graph of f projects onto the closed unit ball in $\mathbb{R}^3$.

Some further examples in this setting have been given by Jimbo and Sakai in their paper [189].

Under rather general conditions, it is possible to say something about the size of the complementary set $\widehat{X} \setminus X$. These results are due to Alexander [7].

Theorem 2.3.17. [7] *If $X \subset \mathbb{C}^N$ is a compact set with $\check{H}^p(X;G) \neq 0$ for some $p \geq N$ and some abelian group G, then $\check{H}_*^{p+1}(\widehat{X} \setminus X; G) \neq 0$.*

The space $\widehat{X} \setminus X$ is locally compact; $\check{H}_*^{p+1}(\widehat{X} \setminus X; G)$ denotes its cohomology *with compact support.*

Proof. The proof consists simply in writing the exact cohomology sequence for the pair $(\widehat{X}, X)$:

$$\cdots \to \check{H}^p(\widehat{X};G) \to \check{H}^p(X;G) \to \check{H}_*^{p+1}(\widehat{X} \setminus X; G) \to \cdots .$$

By Corollary 2.3.6, the group $\check{H}^p(\widehat{X};G)$ is zero, which implies that the nonzero group $\check{H}^p(X;G)$ injects into the group $\check{H}_*^{p+1}(\widehat{X} \setminus X; G)$, so the latter group is not zero. Done.

Corollary 2.3.18. *The topological dimension of $\widehat{X} \setminus X$ is at least $p+1$.*

For dimension theory see [185].

There are the expected rationally convex versions of the preceding results.

Theorem 2.3.19. *If $X \subset \mathbb{C}^N$ is a compact set with $\check{H}^p(X;G) \neq 0$ for some $p \geq N+1$ and some abelian group G, then $\check{H}_*^{p+1}(\mathscr{R}\text{-hull } X \setminus X; G) \neq 0$.*

Corollary 2.3.20. *The topological dimension of $\mathscr{R}$-hull $X \setminus X$ is at least $p+1$.*

In the case of manifolds, additional information is available.

Theorem 2.3.21. [7] *If Σ is a compact submanifold of $\mathbb{C}^N$ of dimension $p \geq N$, then the topological closure of $\widehat{\Sigma} \setminus \Sigma$ in $\mathbb{C}^N$ contains all of Σ.*

Proof. We assume, without loss of generality, that Σ is connected. Introduce the group G that is $\mathbb{Z}$ or $\mathbb{Z}_2$ according as Σ is or is not orientable. The cohomology groups in the following argument all have coefficients from G; we suppress the coefficient group. That

Σ is p-dimensional implies that $H^p(\Sigma) \neq 0$. Again, the proof of the theorem involves formal cohomology calculations involving the exact cohomology sequences of several pairs and excision.[6]

For the proof, put $A = (\widehat{\Sigma} \setminus \Sigma)^-$ and $Y = \Sigma \cap A$, and assume, for the sake of contradiction, that $Y \neq \Sigma$.

The hypothesis that $Y \neq \Sigma$ implies that $\check{H}^q(Y) = 0$ for $q \geq p$. The general fact here is that *a closed proper subset A of a compact connected n-dimensional manifold satisfies* $\check{H}^q(A; \mathbb{Z}) = 0$ *for all* $q \geq n$. See [241, p. 222] for the orientable case; the corresponding fact for nonorientable manifolds and coefficients from $\mathbb{Z}_2$ is indicated in [241, p. 223, Example 6.5].

In the exact cohomology sequence for the pair $(\widehat{\Sigma}, \Sigma)$ there is the segment

$$\cdots \to \check{H}^p(\widehat{\Sigma}) \to \check{H}^p(\Sigma) \to \check{H}^{p+1}(\widehat{\Sigma}, \Sigma) \to \check{H}^{p+1}(\widehat{\Sigma}) \to \cdots .$$

By Corollary 2.3.6, $\check{H}^p(\widehat{\Sigma}) = \check{H}^{p+1}(\widehat{\Sigma}) = 0$, so the group $\check{H}^{p+1}(\widehat{\Sigma}, \Sigma)$ is not zero; it is isomorphic to the nonzero group $\check{H}^p(\Sigma)$. Excision yields the isomorphism $\check{H}^{p+1}(\widehat{\Sigma}, \Sigma) = \check{H}^{p+1}(A, Y)$. The exact cohomology sequence for the pair (A, Y) contains the segment

$$\cdots \to \check{H}^p(Y) \to \check{H}^{p+1}(A, Y) \to \check{H}^{p+1}(A) \to \check{H}^{p+1}(Y) \to \cdots .$$

As already noted, the hypotheses imply that $\check{H}^p(Y) = \check{H}^{p+1}(Y) = 0$. Thus the two groups $\check{H}^{p+1}(A, Y)$ and $\check{H}^{p+1}(A)$ are isomorphic, and the latter group is found not to vanish. Next, consider the exact cohomology sequence of the pair $(\widehat{\Sigma}, A)$:

$$\cdots \to \check{H}^{p+1}(\widehat{\Sigma}) \to \check{H}^{p+1}(A) \to \check{H}^{p+2}(\widehat{\Sigma}, A) \to \check{H}^{p+2}(\widehat{\Sigma}) \to \cdots .$$

The extremities are zero, so $\check{H}^{p+2}(\widehat{\Sigma}, A) \neq 0$. Invoking excision yields $\check{H}^{p+2}(\widehat{\Sigma}, A) = \check{H}^{p+2}(\Sigma, Y)$ again, so the latter group is not zero. Finally, consider the exact cohomology sequence of the pair (Σ, Y):

$$\cdots \to \check{H}^{p+1}(Y) \to \check{H}^{p+2}(\Sigma, Y) \to \check{H}^{p+2}(\Sigma) \to \cdots .$$

Again, the extremities are zero, whence $\check{H}^{p+2}(\Sigma, Y) = 0$. We have reached a contradiction, so the result is proved.

The rationally convex version of the last result is correct; the proof follows the same lines as the proof just given:

Theorem 2.3.22. *If* Σ *is a compact submanifold of* $\mathbb{C}^N$ *of dimension* $p \geq N + 1$*, then the topological closure of* $\mathscr{R}$-hull $\Sigma \setminus \Sigma$ *in* $\mathbb{C}^N$ *contains all of* Σ.

A final corollary of Theorem 2.3.1: It implies the existence of Cantor sets and arcs in $\mathbb{C}^N$ that are not polynomially convex, for it implies that in $\mathbb{C}^N$, $N \geq 2$, if X is polynomially convex, then $\mathbb{C}^N \setminus X$ is necessarily simply connected. Examples have been given of sets E

[6]The excision axiom for cohomology is this: If U is open in X and $\bar{U}$ is contained in the interior of A, A a subset of X, then the inclusion of pairs $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism of relative cohomology groups $H^q(X, A; G) = H^q(X \setminus U, A \setminus U; G)$ for every coefficient group G.

in $\mathbb{R}^N$, $N \geq 3$, homeomorphic to the Cantor set or to an arc for which the complementary domain $\mathbb{C}^N \setminus E$ is not simply connected. Such *wild arcs* were found by Artin and Fox and are described in [177]. For the Cantor sets in question, see the paper of Blankinship [65]. See also the paper of Blankinship and Fox [64]. A comprehensive discussion of these matters is given in the book of Rushing [312]. Recall that an essentially simpler, analytic way of obtaining such examples of nonpolynomially convex arcs and Cantor sets is given in Section 1.6.

2.3.A. Appendix on Fubini's Theorem

We have used above and shall use in the sequel a general version of Fubini's theorem set in the context of smooth maps between manifolds. A careful treatment of the theorem is given in the book of Sulanke and Wintgen [353]. The result is as follows.

Theorem 2.3.23. *Let Y be an m-dimensional smooth, oriented manifold, and let X be an n-dimensional smooth, oriented manifold. Assume $m \geq n$. Let $\varphi : Y \to X$ be a mapping of class $\mathscr{C}^1$ with the property that the set of critical values of φ is a set of measure zero. Let Ω be an n-form on X, and let ω be an $(m-n)$-form on Y. Let the function f be defined and Lebesgue measurable on Y. If the integral $\int_Y f\omega \wedge \varphi^*\Omega$ exists as a Lebesgue integral, then for almost all $x \in X$, the integral $F(x) = \int_{\varphi^{-1}(x)} f\,\omega$ exists (we understand $F(x) = 0$ when $x \notin \varphi(Y)$), the function F defined on X in this way is measurable, and*

$$\int_Y f\omega \wedge \varphi^*\Omega = \int_X F\Omega.$$

Various comments are required about this statement.

Remark 1. The manifolds X and Y are assumed oriented; there is the issue of how the fibers $\varphi^{-1}(x)$ are to be oriented. In case the manifolds X and Y are complex manifolds and φ is a holomorphic map, then the orientation on the fibers $\varphi^{-1}(x)$ is the natural orientation on them as complex manifolds.

Remark 2. A special case is that in which X and Y have the same dimension, say n. In this case, if f is a measurable function on Y and Ω is a smooth form on X, then

$$\int_Y f\varphi^*\Omega = \int_X \Bigg[\sum_{y \in \varphi^{-1}(x)} e(y) f(y) \Bigg] \Omega,$$

in which the quantity $e(y)$ is 1 if φ preserves orientation at x, -1 if it reverses orientation. In particular, the sum $\sum_{y \in \varphi^{-1}(x)} e(y) f(y)$ defines a measurable function on the manifold X.

Remark 3. In the event that X and Y are manifolds of class $\mathscr{C}^\infty$ and φ is also of class $\mathscr{C}^\infty$, Sard's theorem implies that the set of critical values of φ is of measure zero. The same conclusion can be drawn under less-stringent regularity assumptions. For the sharp result, one can consult Federer's book [115].

2.4. Convexity in Stein Manifolds

Some of the results of the last section have analogues in the setting of arbitrary Stein manifolds. These developments were found in the papers of Alexander [23] and Forstnerič [127].

We shall use the basic result that *if $\mathscr{M}$ is a closed complex submanifold of $\mathbb{C}^N$, then for each compact subset X of $\mathscr{M}$, the sets $\mathscr{O}(\mathscr{M})$-hull X and $\widehat{X}$ coincide*. This is true, for as we have already noted, $\mathscr{O}(\mathscr{M})$-hull $X \subset \widehat{X}$. These two sets are equal, because each $f \in \mathscr{O}(\mathscr{M})$ is the restriction of a function holomorphic on all of $\mathbb{C}^N$.

The analogue of rational convexity in the general Stein manifold is less evident than is the analogue of polynomial convexity. The first notion natural in this setting is that of meromorphic convexity: A compact subset X of the Stein manifold $\mathscr{M}$ is *meromorphically convex* if for each point $z \in \mathscr{M} \setminus X$, there is a holomorphic function f on $\mathscr{M}$ with $f(z) = 0$ and with f zero-free on X. The second natural notion is that of convexity with respect to hypersurfaces: The compact subset X of $\mathscr{M}$ is *convex with respect to hypersurfaces* if for each point $z \in \mathscr{M} \setminus X$ there is an analytic hypersurface Z with $z \in Z$ and $Z \cap X = \emptyset$. If, as in $\mathbb{C}^N$, every hypersurface Z is the zero locus of a holomorphic function on $\mathscr{M}$, the two notions evidently coincide. A Stein manifold $\mathscr{M}$ satisfies the condition that each hypersurface is the zero locus of a holomorphic function if the integral cohomology group $\check{H}^2(\mathscr{M}; Z)$ vanishes or, more generally, if each element $\check{H}^2(\mathscr{M}; \mathbb{Z})$ is of finite order; these are purely topological conditions. For a discussion of these two analogues of rational convexity in the context of Stein manifolds, see the paper of Colţoiu [91] and those of Hirschowitz [175] and [176].

Basic results about the topology of Stein manifolds, which are based on Morse theory, were given by Andreotti and Frankel [35] and by Andreotti and Narasimhan [36]:[7]

Theorem 2.4.1. *If $\mathscr{M}$ is a Stein manifold of dimension N, and if $\Omega \subset \mathscr{M}$ is a Runge domain that is itself a Stein manifold, then*

(a) *$H_k(\mathscr{M}; \mathbb{Z}) = 0$ for $k > N$, and $H_N(\mathscr{M}; \mathbb{Z})$ is free.*

(b) *$H_k(\mathscr{M}, \Omega; \mathbb{Z}) = 0$ for $k > N$, and $H_N(\mathscr{M}, \Omega; \mathbb{Z})$ is torsion-free.*

(c) *The natural homomorphism $H_N(\Omega; \mathbb{Z}) \to H_N(\mathscr{M}; \mathbb{Z})$ is injective, whence the natural homomorphism $H^N(\mathscr{M}; \mathbb{R}) \to H^N(\Omega; \mathbb{R})$ is surjective.*

(d) *If $H_N(\mathscr{M}; \mathbb{Z}) = H_{N-1}(\mathscr{M}; \mathbb{Z}) = 0$, then $H_k(\Omega; \mathbb{Z}) = 0$ for all $k \geq N$, and $H_{N-1}(\Omega; \mathbb{Z})$ is torsion-free.*

As concerns statement (a), the example of $(\mathbb{C}^*)^N$, the N-fold Cartesian product of the punctured plane with itself, shows that in general, $H_N(\mathscr{M}; \mathbb{Z})$ does not vanish. Another example of this phenomenon is provided by the complexified N-sphere $\check{\mathbb{S}}^N = \{z \in \mathbb{C}^{N+1} : z_1^2 + \cdots + z_{N+1}^2 = 1\}$, which satisfies $H_N(\check{\mathbb{S}}^N; \mathbb{Z}) = \mathbb{Z} \neq 0$.

Proof. Part (a) is proved as follows. Let Q be a smooth, strictly plurisubharmonic exhaustion function that is a Morse function with the property that each critical value occurs

[7] In fact, the results of Andreotti and Frankel and of Andreotti and Narasimhan are somewhat more general than the following theorem in that some of them deal with complex spaces instead of manifolds.

at a single critical point. Recall that, as we saw in the preceding section, the index of each critical point of Q is not more than N. Thus, if m_k denotes the number of critical points of index k, then $m_k = 0$ for $k > N$. On the other hand, the *Morse inequalities* [174, p. 162] yield that for every field $\mathbb{F}$, $\dim_{\mathbb{F}} H_k(\mathscr{M}; \mathbb{F}) \le m_k$. Thus, for $k > N$, we have $H_k(\mathscr{M}; \mathbb{F}) = 0$. The universal coefficients theorem in homology[8] gives

$$H_k(\mathscr{M}; \mathbb{F}) = H_k(\mathscr{M}; \mathbb{Z}) \otimes \mathbb{F} \oplus \operatorname{Tor}(H_{k-1}(\mathscr{M}; \mathbb{Z}); \mathbb{F}),$$

whence $H_k(\mathscr{M}; \mathbb{Z}) \otimes \mathbb{F} = 0$. This is correct for every field $\mathbb{F}$, so $H_k(\mathscr{M}; \mathbb{Z}) = 0$ when $k = N+1, N+2, \ldots$. Also, the vanishing of $H_{N+1}(\mathscr{M}; \mathbb{Z}) \otimes F$ implies the vanishing of $\operatorname{Tor}(H_N(\mathscr{M}; \mathbb{Z}); \mathbb{F})$. Apply this with F the finite field $\mathbb{Z}_q = \mathbb{Z}/(q\mathbb{Z})$ for a prime q to find that $\operatorname{Tor}(H_N(\mathscr{M}; \mathbb{Z}); \mathbb{Z}_q) = 0$. Because for every group G, $\operatorname{Tor}(G, \mathbb{Z}_q)$ is the set of elements $g \in G$ with $qg = 0$, it follows that the group $H_N(\mathscr{M}; \mathbb{Z})$ is torsion-free. The proof that it is, in fact, free will be concluded after the proof of part (b) of the theorem.

We now look at the proof of assertion (b). Part (a) implies the first assertion of (b) for $k > N+1$, as follows from the exact homology sequence of the pair $(\mathscr{M}, \Omega)$, which contains the segment

$$\text{(2.6)} \qquad \cdots \to H_p(\mathscr{M}; \mathbb{Z}) \to H_p(\mathscr{M}, \Omega; \mathbb{Z}) \to H_{p-1}(\Omega; \mathbb{Z}) \to \cdots .$$

By the part of (a) that has been proved already, for $p \ge N+1$,

$$H_{p+1}(\mathscr{M}; \mathbb{Z}) = 0 = H_p(\Omega; \mathbb{Z}),$$

so for $p \ge N+2$, we find that $H_p(\mathscr{M}, \Omega; \mathbb{Z}) = 0$. The vanishing of $H_{N+1}(\mathscr{M}, \Omega; \mathbb{Z})$ seems to require more.

Let $K \subset \Omega$ be a compact set and $K' \subset \mathscr{M}$ be a compact neighborhood of $\bar{\Omega}$ in $\mathscr{M}$. The domain Ω is a pseudoconvex Runge domain in $\mathscr{M}$, so the $\mathscr{O}(\Omega)$-hull of K is a compact subset of Ω; it is the same as the $\mathscr{O}(\mathscr{M})$-hull of K. We suppose, therefore, that K is $\mathscr{O}(\mathscr{M})$-convex. Let V and V' be open subsets of $\mathscr{M}$ with $K \subset V \subset \Omega \subset K' \subset V'$.

The manifold $\mathscr{M}$ embeds in a suitable $\mathbb{C}^d$ as a closed complex submanifold. The $\mathscr{O}(\mathscr{M})$-convex set K is polynomially convex in $\mathbb{C}^d$, so Theorem 1.3.8 provides a non-negative smooth function u on $\mathscr{M}$ that is strictly plurisubharmonic on $\mathscr{M} \setminus K$ and that

[8] We need to use the universal coefficients theorem in homology, which involves the Tor functor, so it seems well to recall some of the properties of this functor. To every pair A, and B of abelian groups is associated a third group $\operatorname{Tor}(A, B)$, which has the following properties:

(a) $\operatorname{Tor}(A, B)$ is naturally isomorphic to $\operatorname{Tor}(B, A)$.

(b) If A or B is torsion-free, then $\operatorname{Tor}(A, B) = 0$.

(c) For any abelian group A, $\operatorname{Tor}(A, \mathbb{Z}_n) = \{a \in A : na = 0\}$.

(d) If A', A'', and B are abelian groups, then $\operatorname{Tor}(A' \oplus A'', B) = \operatorname{Tor}(A', B) \oplus \operatorname{Tor}(A'', B)$.

(e) If $\{G_\gamma\}_{\gamma \in \Gamma}$ is a directed set of abelian groups with direct limit G and B is another abelian group, then $\operatorname{Tor}(G, B)$ is the direct limit of the groups $\operatorname{Tor}(G_\gamma, B)$.

References are [241] and [173].

satisfies $u^{-1}(0) = K$.[9]

Let v be a smooth, nonnegative, strictly plurisubharmonic exhaustion function for the domain Ω that satisfies $v < 1$ on V. Let U be a relatively compact open set in Ω with $\bar{V} \subset U$. There is a smooth function χ on $\mathscr{M}$ that takes values in the interval $[0, 1]$ and that satisfies $\chi = 0$ on $\mathscr{M} \setminus U$ and $\chi = 1$ on V. For every sufficiently large positive constant C, the function φ defined by $\varphi = Cu + \chi v$ is a strictly plurisubharmonic exhaustion function for $\mathscr{M}$. By replacing φ by a suitable smooth function that is sufficiently near it, we can suppose that φ is a Morse function for the manifold $\mathscr{M}$.

For $t \in \mathbb{R}$, let $\mathscr{M}_t^\varphi = \{z \in \mathscr{M} : \varphi(z) < t\}$, which is a relatively compact subset of $\mathscr{M}$. For sufficiently large values of C and for $s \in (0, 1)$ close enough to one, we have $K \subset \mathscr{M}_s^\varphi \Subset \Omega$. And if then S is large enough, we have $K' \subset \mathscr{M}_S^\varphi$. Without loss of generality, we can suppose that s and S are regular values for φ. The pair $(\mathscr{M}_S^\varphi, \mathscr{M}_s^\varphi)$ is an approximation to the pair $(\mathscr{M}, \Omega)$.

The continuity property of singular homology—see [241, Proposition 6.1, p. 62]—shows that to infer the vanishing of the relative group $H_k(\mathscr{M}, \Omega; \mathbb{Z})$ for any particular index k, it suffices to establish the vanishing of the relative group $H_k(\mathscr{M}_S^\varphi, \mathscr{M}_s^\varphi; \mathbb{Z})$ for all the choices of φ as above: The group $H_k(\mathscr{M}, \Omega; \mathbb{Z})$ is the direct limit of the groups $H_k(\mathscr{M}_S^\varphi, \mathscr{M}_s^\varphi; \mathbb{Z})$.

That the group $H_k(\mathscr{M}_S^\varphi, \mathscr{M}_s^\varphi; \mathbb{Z})$ vanishes in the range $k > N$ is proved as follows. (For the rest of this paragraph, we omit the superscript $^\varphi$ for simplicity of notation.) Let $\mathscr{M}_t = \{\varphi < t\}$, and fix regular values s and S for φ with $s < S$. We have the following exact homology sequence for the pair $(\mathscr{M}_S, \mathscr{M}_s)$:

$$\cdots \to H_{p+1}(\mathscr{M}_S; \mathbb{Z}) \to H_{p+1}(\mathscr{M}_S, \mathscr{M}_s; \mathbb{Z}) \to H_p(\mathscr{M}_s; \mathbb{Z}) \to H_p(\mathscr{M}_S; \mathbb{Z}) \to \cdots .$$

For $p \geq N + 1$, the groups $H_{p+1}(\mathscr{M}_S; \mathbb{Z})$ and $H_p(\mathscr{M}_s; \mathbb{Z})$ vanish, which implies that the relative groups $H_p(\mathscr{M}_S, \mathscr{M}_s; \mathbb{Z})$ vanish in the range $p \geq N + 2$. Also, for coefficients in a field $\mathbb{F}$, we have $H_N(\mathscr{M}_S; \mathbb{F}) = H_N(\mathscr{M}_s; \mathbb{F}) = 0$, so, by the exact homology sequence of the pair $(\mathscr{M}_S, \mathscr{M}_s)$ with coefficients in $\mathbb{F}$, $H_{N+1}(\mathscr{M}_S, \mathscr{M}_s; \mathbb{F}) = 0$. The universal coefficients theorem applied as it was applied in the proof of part (a) yields that $H_{N+1}(\mathscr{M}_S, \mathscr{M}_s; \mathbb{Z})$ vanishes and also that $H_N(\mathscr{M}_S, \mathscr{M}_s; \mathbb{Z})$ is torsion-free. The groups $H_{N+1}(\mathscr{M}, \Omega; \mathbb{Z})$ and $H_N(\mathscr{M}, \Omega; \mathbb{Z})$ are the direct limits of these groups, so the proof of part (b) is complete. (Recall the property (e) of Tor stated in the footnote above.)

The proof of assertion (a) of the theorem is now concluded as follows. The issue is to see that $H_N(\mathscr{M}; \mathbb{Z})$ is not only torsion-free but actually free. Let ρ be a strictly plurisubharmonic exhaustion function for $\mathscr{M}$ that is also a Morse function and for which every positive integer is a regular value. For each positive integer p let Ω_p be the sublevel set $\Omega_p = \{z \in \mathscr{M} : \rho(z) < p\}$. These are all Runge domains in $\mathscr{M}$, and each is a Runge domain in the next. The singular homology groups of the Ω's are all finitely generated. By the part of (a) that we have proved, we know that $H_N(\Omega_p; \mathbb{Z})$ is torsion-free, and by (b) the relative groups $H_N(\Omega_p, \Omega_{p-1}; \mathbb{Z})$ are torsion-free. Because these groups are finitely generated and torsion-free, they are free. The exact homology sequence for the

[9] Instead of invoking the embedding theorem at this point, we could instead rewrite the proof of Theorem 1.3.8 in the present context to obtain the function u.

pair (Ω_p, Ω_{p+1}) contains the segment

$$0 \to H_N(\Omega_{p-1}; \mathbb{Z}) \to H_N(\Omega_p; \mathbb{Z}) \to H_N(\Omega_p, \Omega_{p-1}; \mathbb{Z}) \to \cdots .$$

It follows that $H_N(\Omega_p, \Omega_{p-1}; \mathbb{Z})$ contains a subgroup F_{p-1}, which is necessarily free, for which there is an exact sequence

$$0 \to H_N(\Omega_{p-1}; \mathbb{Z}) \to H_N(\Omega_p; \mathbb{Z}) \to F_{p-1} \to 0,$$

so that

$$H_N(\Omega_p; \mathbb{Z}) = H_N(\Omega_{p-1}; \mathbb{Z}) \oplus F_{p-1}.$$

Iterating this process, starting with $p = 2$, gives

$$H_N(\Omega_p; \mathbb{Z}) = H_N(\Omega_1; \mathbb{Z}) \oplus F_1 \oplus F_2 \oplus \cdots \oplus F_{p-1}. \tag{2.7}$$

The group $H_N(\mathscr{M}; \mathbb{Z})$ is the direct limit of the groups $H_N(\Omega; \mathbb{Z})$; the preceding equation shows this direct limit to be a free abelian group. Thus, part (a) is proved.

For the proof of (c), use the exact homology sequence (2.6) again. By (b), $H_N(\Omega; \mathbb{Z})$ injects into $H_N(\mathscr{M}; \mathbb{Z})$. For the final assertion of (c), write the exact cohomology sequence of the pair $(\mathscr{M}, \Omega)$ with coefficients in $\mathbb{R}$:

$$\cdots \to H^N(\mathscr{M}; \mathbb{R}) \to H^N(\Omega; \mathbb{R}) \to H^{N+1}(\mathscr{M}, \Omega; \mathbb{R}) \to \cdots .$$

To see that $H^N(\mathscr{M}; \mathbb{R}) \to H^N(\Omega; \mathbb{Z})$ is surjective, it is enough to show that the group $H^{N+1}(\mathscr{M}, \Omega; \mathbb{R})$ vanishes. For this, use the universal coefficients theorem in cohomology:

$$0 \to \operatorname{Ext}(H_N(\mathscr{M}, \Omega; \mathbb{Z}); \mathbb{R}) \to H^{N+1}(\mathscr{M}, \Omega; \mathbb{R}) \to \operatorname{Hom}(H_{N+1}(\mathscr{M}, \Omega; \mathbb{Z}); \mathbb{R}) = 0.$$

The Ext group is zero because $\mathbb{R}$ is a divisible group, and the Hom group is zero because the relative group $H_{N+1}(\mathscr{M}, \Omega; \mathbb{Z})$ is zero. Thus, $H^{N+1}(\mathscr{M}, \Omega; \mathbb{R}) = 0$, as we wanted. Point (c) is proved.

Under the hypotheses of (d), the sequence (2.6) yields $H_N(\Omega; \mathbb{Z}) = 0$ and the isomorphism of the group $H_{N-1}(\Omega; \mathbb{Z})$ with the torsion-free group $H_N(\mathscr{M}, \Omega; \mathbb{Z})$, so (d) is established.

Theorem 2.4.2. [127] *Let $\mathscr{M}$ be a Stein manifold of dimension $N \geq 2$, and let X be a compact $\mathscr{O}(\mathscr{M})$-convex subset of $\mathscr{M}$.*

(a) *For every abelian group G, the inclusion $\mathscr{M} \setminus X \hookrightarrow \mathscr{M}$ induces isomorphisms*

$$H_k(\mathscr{M} \setminus X; G) = H_k(\mathscr{M}; G)$$

for $0 \leq k \leq N - 2$.

(b) *If $H_N(\mathscr{M}; \mathbb{R}) = 0$, then also $H_{N-1}(\mathscr{M}; \mathbb{R}) = H_{N-1}(\mathscr{M} \setminus X; \mathbb{R})$.*

(c) *If $H_{N-1}(\mathscr{M}; \mathbb{Z}) = H_N(\mathscr{M}; \mathbb{Z}) = 0$, then for every abelian group G,*

$$H_{N-1}(\mathscr{M} \setminus X; G) = H_{N-1}(\mathscr{M}; G).$$

Proof. Consider first the assertion (c), so that $H_{N-1}(\mathscr{M};\mathbb{Z}) = H_N(\mathscr{M};\mathbb{Z}) = 0$. The compact $\mathscr{O}(\mathscr{M})$-convex set X has a neighborhood basis of Stein Runge domains Ω such that $\bar{\Omega}$ is a compact manifold with boundary. All the homology groups of such an Ω are finitely generated. We fix attention on one of these domains. Theorem 2.4.1(d) gives that $H_k(\Omega;\mathbb{Z}) = 0$ for all $k \geq N$. Also $H_{N-1}(\Omega;\mathbb{Z})$ is torsion-free. It is free and so isomorphic to $\mathbb{Z}^r$ for some nonnegative integer r, for it is a finitely generated abelian group. Let G be an abelian group. The universal coefficients theorem in cohomology gives the sequence

$$0 \to \mathrm{Ext}(H_{N-1}(\Omega;\mathbb{Z}), G) \to H^N(\Omega;G) \to \mathrm{Hom}(H_N(\Omega;\mathbb{Z}), G) \to 0. \tag{2.8}$$

The vanishing of $H_{p-1}(\Omega;Z)$ and $H_p(\Omega;Z)$ for $p > N$ yields that $H^p(\Omega;G) = 0$ for $p \geq N+1$. The group $H^N(\Omega;G)$ also vanishes: The Hom term in the sequence (2.8) vanishes, because, by Theorem 2.4.1, $H_N(\Omega;\mathbb{Z}) = 0$. The Ext term also vanishes, because $H_{N-1}(\Omega;\mathbb{Z})$ is a free abelian group. The vanishing of all the groups $H^p(\Omega;G)$, $p \geq N$, implies the vanishing of the groups $H^p(X;G)$ for all $p \geq N$.

Alexander duality yields $H_k(\mathscr{M}, \mathscr{M}\setminus X; G) = H^{2N-k}(X;G) = 0$ in the range $k = N, \ldots, 2N$. The exact homology sequence of the pair $(\mathscr{M}, \mathscr{M}\setminus X)$ now yields the isomorphism of the groups $H_k(\mathscr{M};G)$ and $H_k(\mathscr{M}\setminus X;G)$ in the range $1 \leq k \leq N-1$.

For assertion (a), the argument just given, in the range $1 \leq k \leq N-2$, yields the result.

For assertion (b), we have that $H^N(\Omega;\mathbb{R}) = 0$, because of point (c) of Theorem 2.4.1. From this it follows that $H^N(X;\mathbb{R}) = 0$. Alexander duality then yields

$$H_N(\mathscr{M}, \mathscr{M}\setminus X;\mathbb{R}) = H^N(X;\mathbb{R}) = 0,$$

and then the result follows from the exact homology sequence of the pair $(\mathscr{M}, X)$ with coefficients in $\mathbb{R}$.

Corollary 2.4.3. *If $X \subset \mathscr{M}$ is $\mathscr{O}(\mathscr{M})$-convex, then $\mathscr{M}\setminus X$ is connected.*

Proof. Take $k = 0$ in assertion (a).

Corollary 2.4.4. *If X is a compact subset of $\mathscr{M}$, then $\mathscr{M}\setminus X$ has only one component that is not relatively compact in $\mathscr{M}$.*

Proof. Let $Y \subset \mathscr{M}$ be an $\mathscr{O}(\mathscr{M})$-convex subset of $\mathscr{M}$ that contains X. The set $\mathscr{M}\setminus Y$ has only one component, say W, and each unbounded component of $\mathscr{M}\setminus X$ must meet W.

There is a theorem about relative homology for a *pair* of $\mathscr{O}(\mathscr{M})$-convex sets in $\mathscr{M}$.

Theorem 2.4.5. [127] *Let $\mathscr{M}$ be a Stein manifold of dimension $N \geq 2$, and let X and Y be compact $\mathscr{O}(\mathscr{M})$-convex subsets of $\mathscr{M}$ with $Y \subset \mathrm{int}\, X$. For every abelian group G and for $0 \leq k \leq N-1$, the relative homology group $H_k(\mathscr{M}\setminus X, \mathscr{M}\setminus Y; G)$ vanishes.*

Proof. Let U be a neighborhood of Y with $\bar{U} \subset \mathrm{int}\, X$, and let V be a neighborhood of X.

We shall construct a smooth strictly plurisubharmonic exhaustion function ϱ on $\mathscr{M}$ that is a Morse function with the properties that (a) $\varrho < 0$ on Y and $\varrho > 0$ on $\mathscr{M}\setminus U$, (b) $\varrho < 1$ on X, $\varrho > 1$ on $\mathscr{M}\setminus V$, and (c) 0 and 1 are regular values for ϱ. Assume for the moment that we have such a ϱ.

For $t \in \mathbb{R}$, let $\mathscr{M}^t = \{z \in \mathscr{M} : \varrho(z) > t\}$. We then have that $\mathscr{M} \setminus U \subset \mathscr{M}^0 \subset \mathscr{M} \setminus Y$ and $\mathscr{M} \setminus V \subset \mathscr{M}^1 \subset \mathscr{M} \setminus X$. The Morse indices of $-\varrho$ are all at least N, so if t is a regular value for $-\varrho$, then for any field $\mathbb{F}$ of coefficients, $H_k(\mathscr{M}^t; F) = 0$ for k in the range $0 \leq k \leq N-1$. As before, the universal coefficients theorem yields that $H_k(\mathscr{M}^t; \mathbb{Z}) = 0$ and that $H_N(\mathscr{M}^t; \mathbb{Z})$ is torsion-free. The exact sequence for the pair $(\mathscr{M}^0, \mathscr{M}^1)$ yields that $H_p(\mathscr{M}^0, \mathscr{M}^t; \mathbb{Z}) = 0$ in the range $0 \leq p \leq N-1$.

Passing to the limit as U shrinks to Y and V to X now yields $H_k(\mathscr{M} \setminus X, \mathscr{M} \setminus Y; \mathbb{Z}) = 0$ as desired.

The construction of ϱ is similar to constructions we have already used. Choose smooth strictly plurisubharmonic functions φ and ψ on $\mathscr{M}$ with the properties that $\varphi < 0$ on Y, $\varphi > 0$ on $\mathscr{M} \setminus U$, and $\psi < 0$ on Y , $\psi > 0$ on $\mathscr{M} \setminus V$. Require also that both 0 and 1 be regular values for both φ and ψ. Let $\tilde{\psi} = C\psi + 1$ for a positive constant C large enough that $\tilde{\psi} < 0$ on U. Compose $\tilde{\psi}$ with a smooth function on $\mathbb{R}$ that is 0 on $(-\infty, 0]$ and is strictly convex on $(0, \infty)$. This gives ψ_1, which is smooth and plurisubharmonic on $\mathscr{M}$, that is 0 on $\bar{U}$, and that is strictly plurisubharmonic where it is positive. It also has property (b). Now let χ be a smooth function with values in [0, 1] that is identically 1 where $\psi_1 \leq \frac{1}{3}$ and is identically 0 where $\psi_1 \geq \frac{2}{3}$. The function $\varrho = \psi_1 + \varepsilon\chi\varphi$ has the desired properties if $\varepsilon > 0$ is sufficiently small.

The next theorem concerns strictly pseudoconvex domains. If $\mathscr{M}$ is a Stein manifold, the relatively compact domain $\Omega \subset \mathscr{M}$ is *strictly pseudoconvex* with boundary of class $\mathscr{C}^k$, $k = 2, 3, \dots, \infty$, or ω, if there is a strictly plurisubharmonic function Q defined and of class $\mathscr{C}^k$ on a neighborhood W of $\bar{\Omega}$ such that $dQ \neq 0$ on $b\Omega$ and such that $\Omega = \{z \in W : Q(z) < 0\}$. (As usual, the case $k = \omega$ is understood to be the real-analytic case.) These domains can be characterized locally along the boundary: The relatively compact domain Ω in $\mathscr{M}$ with boundary of class $\mathscr{C}^k$ is strictly pseudoconvex if for each point $p \in b\Omega$, there is a neighborhood W_p on which there is a strictly plurisubharmonic function Q_p of class $\mathscr{C}^k$ such that $dQ_p \neq 0$ on $b\Omega \cap W_p$ and such that $\Omega \cap W_p = \{z \in W_p : Q_p(z) < 0\}$. This characterization is given in [287, p. 61]. Another global characterization of strictly pseudoconvex domains is contained in results of Fornæss [118], Henkin [170], and Forstnerič [122]:

Theorem 2.4.6. *If Ω is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^k$, $2 \leq k \leq \infty$ or $k = \omega$, in the Stein manifold $\mathscr{M}$, then there is a domain W in $\mathbb{C}^N$ for suitably large N such that for some complex submanifold $\mathscr{N}$ of W, a neighborhood Ω' of $\bar{\Omega}$ in $\mathscr{M}$ is biholomorphically equivalent to $\mathscr{N}$ under a biholomorphic map Φ such that for some strictly convex subdomain Δ of $\mathbb{C}^N$ with boundary of class $\mathscr{C}^k$ contained in W, $\mathscr{N}$ is transverse to $b\Delta$, and Ω is carried by Φ onto $\mathscr{N} \cap \Delta$.*

The original treatments of this result by Fornæss and Henkin dealt with the cases $k = 2, 3, \dots, \infty$. The real-analytic case was deduced from the $\mathscr{C}^\infty$ case by Forstnerič.

An important fact is that each point of the boundary of a strictly pseudoconvex domain Ω is a peak point for the algebra $\mathscr{O}(\bar{\Omega})$ of functions holomorphic near $\bar{\Omega}$.

Corollary 2.4.3 implies a useful result for strictly pseudoconvex domains:

Corollary 2.4.7. *If Ω is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in a*

Stein manifold, then the boundary of Ω is connected.

Proof. The domain Ω is connected by hypothesis—by the convention that *domains* are connected open sets. Also, because Ω is strictly pseudoconvex, there is a strictly pseudoconvex function Q defined near $b\Omega$ such that $b\Omega = \{Q = 0\}$, dQ does not vanish at any point of $b\Omega$, and $Q < 0$ on the part of Ω on which it is defined.

For $\varepsilon > 0$ small let Ω_ε be the strictly pseudoconvex domain defined by $Q < \varepsilon$. This set is connected, because it is a small deformation of the domain Ω. If ε is small enough, then $\bar{\Omega}$ is $\mathscr{O}(\Omega_\varepsilon)$-convex. Also, by Corollary 2.4.3, $\Omega_\varepsilon \setminus \bar{\Omega}$ is connected, so the same is true of $\Omega_\varepsilon \setminus \Omega$. Because $b\Omega$ is the intersection of the connected, compact sets $\bar{\Omega}_\varepsilon \setminus \Omega$, it follows that $b\Omega$ is connected.

The following result was found by Forstnerič [127]. For balls, it is due to Alexander [7].

Theorem 2.4.8. *If Ω is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in a Stein manifold $\mathscr{M}$ of dimension $N \geq 2$, then for every abelian group G and for every compact subset X of $b\Omega$, if $X^\dagger = \mathscr{O}(\bar{\Omega})\text{-hull}\, X$, then the inclusion $b\Omega \setminus X \hookrightarrow \bar{\Omega} \setminus X^\dagger$ induces an isomorphism*

$$H_k(b\Omega \setminus X; G) \to H_k(\bar{\Omega} \setminus X^\dagger; G)$$

for $k = 0, \dots, N-2$.

Proof. Let Q be a strictly plurisubharmonic defining function of class $\mathscr{C}^2$ for Ω, so that for some neighborhood W of $\bar{\Omega}$, $\Omega = \{z \in W : Q(z) < 0\}$. Replacing W by a sublevel set $\{Q < \varepsilon\}$ for a small positive ε lets us suppose W to be a Stein domain in which Ω is a Runge domain such that $\bar{\Omega}$ is $\mathscr{O}(W)$-convex.

The domain $\bar{\Omega}$ is $\mathscr{O}(W)$-convex, which implies that $X^\dagger$ is the $\mathscr{O}(W)$-convex hull of X. Also, because each point of $b\Omega$ is a peak point for the algebra $\mathscr{O}(\bar{\Omega})$, we have that $X^\dagger \cap \bar{\Omega} = X$.

A neighborhood of $b\Omega$ in W can be identified with a neighborhood of the zero section of the normal bundle to the embedding $b\Omega \hookrightarrow W$. Let π be the projection. There is then a neighborhood E_o of $b\Omega$ such that each of the fibers $\pi^{-1}(z) \cap E_o$, $z \in b\Omega$, is a convex domain in the fiber of the normal bundle over the point z.

Let χ be a smooth nonnegative function on $b\Omega$ with $\chi^{-1}(0) = X$. Use the function χ to define a subset E of E_o by

$$E = \{z \in E_o : |Q(z)| < \chi(\pi(z))\}.$$

This is an open neighborhood of $b\Omega \setminus X$ and, granted the proper choice of χ, so that E is thin enough, we have $E \cap X^\dagger = \emptyset$.

Put $U = (\Omega \setminus X^\dagger) \cup E$ and $V = W \setminus (\bar{\Omega} \cup E)$. This gives $U \cup V = W \setminus X^\dagger$ and $U \cap V = E$.

The definition of V implies that V can be deformed into $W \setminus \bar{\Omega}$ by a homeomorphism of $W \setminus X^\dagger$. Thus for every group G of coefficients and for all nonnegative integers k,

$$H_k(W \setminus X^\dagger, V; G) = H_k(W \setminus X^\dagger, W \setminus \bar{\Omega}; G).$$

The latter group vanishes for k with $0 \le k \le N-1$, as follows from Theorem 2.4.5. (Note that, as stated, that theorem does not apply in this situation, because $X^\dagger$ is not contained in the interior of $\bar{\Omega}$. To circumvent this difficulty, replace $\bar{\Omega}$ by $\bar{\Omega}_\varepsilon$ with Ω_ε the sublevel set $\{Q < \varepsilon\}$ and then pass to the limit as ε decreases to 0.)

By excision,

$$H_k(W \setminus X^\dagger, V; G) = H_k(W \setminus (X^\dagger \cup (V \setminus E)), V \setminus (V \setminus E); G).$$

The definitions yield that $V \setminus (V \setminus E) = E$ and $W \setminus (X^\dagger \cup (V \setminus E)) = U$. Thus

$$0 = H_k(W \setminus X^\dagger, V; G) = H_k(U, E; G).$$

The exact homology sequence for the pair (U, E) contains the segment

$$\cdots \to H_{k+1}(U, E; G) \to H_k(E; G) \to H_k(U; G) \to H_k(U, E; G) \to \cdots,$$

so for $k = 0, 1, \ldots, N-2$, $H_k(E; G) = H_k(U; G)$. The domain E is homotopically equivalent to $b\Omega \setminus X$, and U to $\bar{\Omega} \setminus X^\dagger$, so for $0 \le k \le N-2$, we have the desired equality $H_k(b\Omega \setminus X; G) = H_k(\bar{\Omega} \setminus X^\dagger; G)$. The theorem is proved.

A simple example from [127] shows that the range of the last theorem cannot be extended to include the case $k = N-1$: If $X = b\mathbb{B}_N \cap \mathbb{R}^N = \mathbb{S}^{N-1}$, then, as a subset of $\mathbb{R}^N$, X is polynomially convex. We have that $H_{N-1}(\bar{\mathbb{B}}_N \setminus X; \mathbb{Z}) = 0$, but by Alexander duality, $H_{N-1}(b\mathbb{B}_N \setminus X; \mathbb{Z}) = H^{N-1}(X; \mathbb{Z}) = \mathbb{Z}$.

The first version of this result was given by Alexander [7]; it was set in the context of the ball $\mathbb{B}_N$ in $\mathbb{C}^N$. Recall Corollary 2.3.7. The case $k = 0$ for arbitrary strictly pseudoconvex domains in Stein manifolds is in [26].

Corollary 2.4.9. *With $\mathscr{M}$, Ω, and X as in the theorem, each component of $\bar{\Omega} \setminus \mathscr{O}(\mathscr{M})$-hull X contains exactly one component of $b\Omega \setminus X$.*

Corollary 2.4.10. *With $\mathscr{M}$, Ω, and X as in the theorem, if X is $\mathscr{O}(\mathscr{M})$-convex, then $b\Omega \setminus X$ is connected.*

A completely different proof of this result is contained in [301].

It will be important for us at one point below to have a version of the preceding theorem for certain domains that are not necessarily strictly pseudoconvex.

Theorem 2.4.11. *Let D be a bounded domain in $\mathbb{C}^N$ with bD a topological sphere. Let $X \subset bD$ be a compact set such that $\widehat{X} \cap bD = X$ and $\bar{D} \supset \widehat{X}$. Then for $0 \le p \le N-2$, $H_p(bD \setminus X; \mathbb{Z}) = H_p(D \setminus \widehat{X}; \mathbb{Z})$. In particular—the case $p = 0$—the boundary of each component of $D \setminus \widehat{X}$ contains precisely one component of $bD \setminus X$.*

It is to be emphasized that the sphere bD is here subject to no regularity condition. In particular, the domain D might be any bounded convex domain.

Proof. [7] Fix a p with $0 \le p \le N-2$. The boundary bD is a $(2N-1)$-dimensional manifold, so Alexander duality gives $H_p(bD \setminus X; \mathbb{Z}) = \check{H}^{2N-2-p}(X; \mathbb{Z})$. Denote by E the cone on the manifold bD, so that E is the space obtained from the product $bD \times [0, 1]$ by collapsing the subset $bD \times \{1\}$ to a point. If we identify each point $z \in bD$ with the corresponding point $\{z\} \times \{0\}$ in E, we obtain a topological sphere of dimension $2N$. By a small abuse of notation, we shall denote this sphere by $E \cup D$. Fix a $q \geq N$. There is an exact sequence in cohomology

$$\cdots \to \check{H}^q(\widehat{X};\mathbb{Z}) \to \check{H}^q(E \cup \widehat{X}, \widehat{X};\mathbb{Z}) \to \check{H}^{q+1}(E \cup \widehat{X};\mathbb{Z}) \to \check{H}^{q+1}(\widehat{X};\mathbb{Z}) \to \cdots .$$

The extremities are zero for $q \geq N$, so in this range, $\check{H}^q(E \cup \widehat{X}, \widehat{X};\mathbb{Z}) = \check{H}^{q+1}(E \cup \widehat{X};\mathbb{Z})$. By excision, $\check{H}^{q+1}(E \cup \widehat{X}, \widehat{X};\mathbb{Z}) = \check{H}^{q+1}(E, X;\mathbb{Z})$. There is also the exact sequence

$$\cdots \to \check{H}^q(E,\mathbb{Z}) \to \check{H}^q(X;\mathbb{Z}) \to \check{H}^{q+1}(E, X;\mathbb{Z}) \to \check{H}^{q+1}(E,\mathbb{Z}) \to \cdots .$$

Again, the extremities are trivial, because E is contractible, so

$$\check{H}^q(X,\mathbb{Z}) = \check{H}^{q+1}(E \cup \widehat{X};\mathbb{Z}).$$

If we concatenate the equalities we have obtained, we find that $\check{H}^q(X;\mathbb{Z}) = \check{H}^{q+1}E \cup \widehat{X};\mathbb{Z})$ in the range $q \geq N$. Alexander duality applied in the sphere $E \cup D$ yields

$$\check{H}^{q+1}(E \cup \widehat{X};\mathbb{Z}) = H_{2N-2-q}(E \cup D \setminus (E \cup \widehat{X});\mathbb{Z}) = H_{2N-2-q}(D \setminus \widehat{X};\mathbb{Z}).$$

In this, take $q = 2N-2-p$, which satisfies $q \geq N$ because $0 \leq p \leq N-2$, to find that $\check{H}^{2N-2-p}(X;\mathbb{Z}) = H_p(D \setminus \widehat{X};\mathbb{Z})$. The former group is $\check{H}_p(bD \setminus X;\mathbb{Z})$, as was noted at the outset, so the proof is concluded.

As a consequence of Corollary 2.4.9 there is a further result about the hulls of sets in the boundary of strongly pseudoconvex domains.

Theorem 2.4.12. *If Ω is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in a Stein manifold, and if X is a compact subset of $b\Omega$, then $(b(\mathscr{O}(\bar{\Omega})\text{-hull}\, X)) \cap \Omega \subset \mathscr{O}(\bar{\Omega})\text{-hull}\, b_{b\Omega}X$.*

The notation $b_{b\Omega}X$ used here is understood to indicate the relative boundary of X with respect to the manifold $b\Omega$, so that

$$b_{b\Omega}X = \{x \in b\Omega : \text{each neighborhood of } x \text{ contains points of } X \text{ and points of } b\Omega \setminus X\}.$$

This result was obtained by Basener [46], though he stated it only in the case of the ball.

Proof. If the theorem is false, there exist a compact subset X of $b\Omega$, a point $x \in (b(\mathscr{O}(\bar{\Omega})\text{-hull}\, X)) \cap \Omega$, and a polynomial p with $p(x) = 1$ and $\|p\|_{bX} \leq \frac{1}{2}$. Put $Y = \{z \in b\Omega : |p(z)| \leq \frac{3}{4}\}$. If V denotes the component of $\Omega \setminus \mathscr{O}(\bar{\Omega})\text{-hull}\, Y$ that contains the point x, then $\bar{V}$ contains a unique component, say U, of $b\Omega \setminus X$. The set U is connected and is a subset of $b\Omega \setminus b_{b\Omega}X$, so either $U \subset X$ or $U \subset b\Omega \setminus X$. Each of these alternatives leads to a contradiction.

Suppose first that $U \subset X$. Let $z \in V$, and let Z denote the algebraic variety $p^{-1}(p(z))$. The maximum principle implies that if q is a polynomial, then $|q(z)| \leq \|q\|_{Z \cap bV}$. Because $bV \subset \bar{U} \cup \{w \in \Omega : |p(w)| = \frac{3}{4}\}$, it follows that $|q(z)| \leq \|q\|_{\bar{U}}$.

Thus, $V \subset \mathscr{O}(\bar{\Omega})$-hull $\bar{U} \subset \mathscr{O}(\Omega)$-hull X. The set V is open and $x \in V$, so this contradicts $x \in b(\mathscr{O}(\bar{\Omega})\text{-hull}\, X)$.

Alternatively, U could be contained in $b\Omega \setminus X$. Let $L = (\mathscr{O}(\bar{\Omega})\text{-hull}\, X) \cap \bar{V}$. If $m = \|p\|_L$, then because $x \in L$, necessarily $m > \frac{3}{4}$. Fix a $y \in L$ that satisfies $p(y) = m$. Define the set T by $T = \{z \in L : |p(z)| = |p(y)|\}$. The set T is a subset of the set $(\mathscr{O}(\bar{\Omega})\text{-hull}\, X) \cap V$, because $|p| > \frac{3}{4}$ on T, but $bV \subset \bar{U} \cup \{z \in \Omega' : |p(z)| = \frac{3}{4}\}$ and $(\mathscr{O}(\bar{\Omega})\text{-hull}\, X) \cap U \subset b_{b\Omega}X$. We have $|p| \le \frac{1}{2}$ on the latter set. Thus T is a local peak set for $\mathscr{O}(\bar{\Omega})$. Consequently, T is a peak set for $\mathscr{O}(\bar{\Omega})$. This is impossible, for $T \subset V \subset \Omega$.

The theorem is proved.

Corollary 2.4.13. *With Ω as in the preceding theorem, if $\{U_j\}_{j=1,\dots}$ is a sequence of mutually disjoint open subsets of $b\Omega$, then*

$$\mathscr{O}(\bar{\Omega})\text{-hull}\,[b\Omega \setminus \cup_{j=1,\dots}U_j] = \cap_{j=1,\dots}\mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus U_j).$$

Proof. There are only countably many of the U_j. It suffices to consider the case in which there are only finitely many.

Plainly $\mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus \cup_{j=1,\dots,n}U_j) \subset \cap_{j=1,\dots,n}\mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus U_j)$.

For the opposite inclusion, let $x \in b\big(\cap_{j=1,\dots,n}\mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus \cup U_j)\big)$. We have $n < \infty$, so there is an index j with $x \in b\big(\mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus U_j) \cap \Omega\big)$. Thus by the preceding theorem, $x \in \mathscr{O}(\bar{\Omega})\text{-hull}\,(b_{b\Omega}(b\Omega \setminus U_j))$. That the sets U_j are disjoint implies that $b_{b\Omega}(b\Omega \setminus U_j) \subset b\Omega \cup_{j=1,\dots,n} U_j$, so $x \in \mathscr{O}(\bar{\Omega})\text{-hull}\,(b\Omega \setminus \cup_{j=1,\dots,n}U_j)$ as claimed.

As a particular example, suppose that Ω is the unit ball $\mathbb{B}_N$, and each U_j is a *spherical cap*, i.e., one of the two components of $b\mathbb{B}_N \setminus \Pi$, where Π is a real hyperplane in $\mathbb{C}^N$ that meets $\mathbb{B}_N$. If $\{U_j\}_{j=1,\dots}$ is a sequence of mutually disjoint caps in $b\mathbb{B}_N$, then the polynomially convex hull $\big(\cap_j\bar{\mathbb{B}}_N \setminus U_j\big)\widehat{\ }$ is the intersection of the convex sets $(b\mathbb{B}_N \setminus U_j)\widehat{\ }$. This implies that if $0 \in \big(\cap_j\bar{\mathbb{B}}_N \setminus U_j\big)\widehat{\ }$, then at least one of the caps U_j contains a hemisphere.

Corollary 2.4.14. *With Ω as in the preceding theorem, if $E \subset b\Omega$ is a compact, totally disconnected set, then for each $x \in \Omega$, there is a compact set $E_x \subset b\Omega \setminus E$ with $x \in \mathscr{O}(\bar{\Omega})$-hull E_x.*

Proof. The set E is compact and totally disconnected, so for each $\delta > 0$, there is a finite collection $U_1^\delta, \dots, U_n^\delta$ of mutually disjoint open subsets of $b\Omega$ each of which has diameter less than δ and whose union covers the set E. If δ is small enough, then $x \in \mathscr{O}(\bar{\Omega})\text{-hull}\,(\bar{\Omega} \setminus U_j^\delta)$ for each j, and so, by Corollary 2.4.13, $x \in \mathscr{O}(\bar{\Omega})\text{-hull}\,(\bar{\Omega} \setminus \cup_{j=1,\dots,n}U_j^\delta)$.

We have obtained some information about the cohomology, with constant coefficients, of the complements of polynomially convex sets. It is useful to have information about the Dolbeault cohomology[10] of these domains; via the Dolbeault isomorphism, this yields information about the corresponding sheaf cohomology with values in the sheaf of germs of holomorphic functions or holomorphic forms. Rather than the polynomially convex sets, we will deal with the more general class of Stein compacta:

[10] Recall that for a complex manifold $\mathscr{M}$, the Dolbeault group $H^{p,q}(\mathscr{M})$ is the quotient of the space of $\bar{\partial}$-closed (p, q)-forms by the subspace of $\bar{\partial}$-exact (p, q)-forms.

Definition 2.4.15. *A* Stein compactum *in a complex manifold $\mathscr{M}$ is a compact subset X of $\mathscr{M}$ such that there is a sequence $\{U_j\}_{j=1,\dots}$ of Stein domains in $\mathscr{M}$ with $X = \bigcap_{j=1,\dots} U_j$.*

Stein compacta have also been called S_δ's in the literature. Polynomially convex sets and rationally convex sets are the simplest examples in $\mathbb{C}^N$.

Our goal is to obtain an understanding of the Dolbeault cohomology of sets of the form $\mathscr{M} \setminus X$ with $\mathscr{M}$ a Stein manifold and X a Stein compactum in $\mathscr{M}$.

In addition to the most familiar Dolbeault cohomology groups $H^{p,q}(\mathscr{M})$ for a complex manifold $\mathscr{M}$, there are the groups $H_c^{p,q}(\mathscr{M})$ of Dolbeault cohomology with compact support. The group $H_c^{p,q}(\mathscr{M})$ is the quotient of the space of compactly supported $\bar\partial$-closed (p,q)-forms modulo the subspace of forms of the form $\bar\partial\beta$, where β is a smooth $(p,q-1)$-form *with compact support*. If X is a compact subset of $\mathscr{M}$, there is also the *bounded cohomology group* $H_\Phi^{p,q}(\mathscr{M} \setminus X)$, that is, the quotient of the space of smooth $\bar\partial$-closed (p,q)-forms α on $\mathscr{M} \setminus X$ whose support is a relatively compact subset of $\mathscr{M}$ modulo the subspace of forms of the form $\bar\partial\beta$, where the support of the smooth $(p,q-1)$-form β on $\mathscr{M} \setminus X$ is a relatively compact subset of $\mathscr{M}$. In the terminology of general sheaf cohomology, $H_\Phi^{p,q}(\mathscr{M} \setminus X)$ is the Dolbeault cohomology with supports in the family Φ of closed subsets of $\mathscr{M} \setminus X$ that have compact closure in $\mathscr{M}$. The family Φ has these properties: (1) Each element of Φ is a closed subset of $\mathscr{M} \setminus X$. (2) Each closed subset of each element of an element of Φ is an element of Φ. (3) Φ is closed under the formation of finite unions. (4) Each element of Φ has a closed neighborhood that belongs to Φ. The family Φ is a *paracompactifying family* of subsets of $\mathscr{M} \setminus X$ in the sense of general sheaf theory. See [150].

In the next of our results and at certain points in the rest of our work, it will be necessary to invoke a result that compares the cohomology of a manifold computed with smooth forms with the cohomology computed with currents. The context is this: Fix attention on a smooth manifold $\mathscr{N}$. We then have the space of forms $\mathscr{E}^p(\mathscr{N})$ and the space $\mathscr{D}_p(\mathscr{N})$ of currents, the dual space of $\mathscr{E}^{n-p}(\mathscr{N})$ if n is the dimension of $\mathscr{N}$. Similarly, if $\mathscr{M}$ is a complex manifold, we have the space $\mathscr{E}^{p,q}(\mathscr{M})$ of smooth forms and the corresponding space $\mathscr{D}_{p,q}(\mathscr{M})$. There are also the space $\mathscr{D}^p(\mathscr{N})$ of compactly supported p-forms on $\mathscr{N}$, the space $\mathscr{K}_p(\mathscr{N})$ dual to $\mathscr{E}^{n-p}(\mathscr{N})$, the space $\mathscr{D}^{p,q}(\mathscr{M})$ of compactly supported (p,q)-forms on $\mathscr{M}$ and the space $\mathscr{K}_{p,q}(\mathscr{M})$ of currents with compact supports, which is dual to the space $\mathscr{E}^{N-p,N-q}(\mathscr{M})$. In each of these settings, we have inclusions:

$$\begin{gathered}
\mathscr{E}^p(\mathscr{N}) \hookrightarrow \mathscr{D}_p(\mathscr{N}),\\
\mathscr{E}^{p,q}(\mathscr{M}) \hookrightarrow \mathscr{D}_{p,q}(\mathscr{M}),\\
\mathscr{D}^p(\mathscr{N}) \hookrightarrow \mathscr{K}_p(\mathscr{N}),\\
\mathscr{D}^{p,q}(\mathscr{M}) \hookrightarrow \mathscr{K}_{p,q}(\mathscr{M}).
\end{gathered}$$

The fact that is required is this:

Theorem 2.4.16. (Smoothing cohomology) *The maps at the cohomology level induced by the above inclusions are all isomorphisms.*

These results are exercises in abstract sheaf cohomology together with an analogue of the Poincaré lemma or the Dolbeault lemma for currents. Details are given in [155].

For Stein manifolds there are the following vanishing theorems:

Theorem 2.4.17. *If $\mathscr{M}$ is an N-dimensional Stein manifold, then*

(a) $H^{p,q}(\mathscr{M}) = 0$ *for all* $p = 0, 1, \dots$ *and* $q = 1, 2, \dots,$ *and*

(b) $H_c^{p,q}(\mathscr{M}) = 0$ *for all* $p = 0, 1, \dots$ *and* $q = 0, 1, \dots, N-1$.

Statement (a) is a consequence of Cartan's Theorem B and the Dolbeault isomorphism theorem, or, alternatively, it can be proved directly as in [180]. Statement (b) is a theorem of Cartan and Schwartz, and is closely related to the Serre duality theorem. See Serre's papers [322, 323]. The derivation of (b) from (a) is quite short; the main point is a lemma from functional analysis:

Lemma 2.4.18. [322] *If* $E \overset{u}{\to} F \overset{v}{\to} G$ *is an exact sequence of locally convex topological vector spaces and continuous maps, and if the range of* v *is closed, then the dual sequence* $G^* \overset{v^*}{\to} F^* \overset{u^*}{\to} E^*$ *is also an exact sequence.*

In this statement, E^*, F^*, and G^* denote the dual spaces of E, F, and G, and the maps u^* and v^* are the transposed maps.

Proof. It is to be shown that if $\varphi \in F^*$ satisfies $u^*\varphi = 0$, then $\varphi = v^*\psi$, for some $\psi \in G^*$. That $u^*\varphi = 0$ implies that φ induces a continuous linear functional on the quotient space $F/u(E)$. (Note that $u(E)$ is closed, for it is the kernel of the continuous linear map v.) The quotient space $F/u(E)$ is isomorphic to the closed subspace $v(F)$ of G. Thus, φ induces a continuous linear functional on $v(F)$. The Hahn–Banach theorem implies that this induced functional extends to an element, ψ of G^*. It satisfies $v^*\psi = \varphi$. The lemma is proved.

With this lemma in hand, the derivation of the statement about the vanishing of the compactly supported Dolbeault cohomology can be given as follows.

Fix a q with $1 \le q \le N-1$. The Dolbeault group $H^{p,q}(\mathscr{M})$ vanishes, because $\mathscr{M}$ is a Stein manifold, so the sequence

$$\mathscr{E}^{p,q-1} \overset{\bar\partial}{\to} \mathscr{E}^{p,q}(\mathscr{M}) \overset{\bar\partial}{\to} \mathscr{E}^{p,q+1}(\mathscr{M})$$

is exact. Also, the range of $\bar\partial : \mathscr{E}^{p,q}(\mathscr{M}) \to \mathscr{E}^{p,q+1}(\mathscr{M})$ is closed: It is the kernel of the continuous map $\bar\partial : \mathscr{E}^{p,q+1}(\mathscr{M}) \to \mathscr{E}^{p,q+2}(\mathscr{M})$. This implies that the corresponding sequence of dual spaces

$$\mathscr{E}^{p,q+1}(\mathscr{M})^* \overset{\bar\partial^*}{\to} \mathscr{E}^{p,q}(\mathscr{M})^* \overset{\bar\partial^*}{\to} \mathscr{E}^{p,q-1}(\mathscr{M})^*$$

is exact. The dual of the space $\mathscr{E}^{p,q}(\mathscr{M})$ is the subspace $\mathscr{K}_{p,q}$ of $\mathscr{D}_{p,q}$ that consists of all compactly supported currents of bidegree $(N-p, N-q)$. It follows that each $\bar\partial$-closed $(N-p, N-q)$-current is $\bar\partial$-exact, because the dual of the map

$$\bar\partial : \mathscr{E}^{p,q}(\mathscr{M}) \to \mathscr{E}^{p,q+1}(\mathscr{M})$$

is the map

$$(-1)^{p+q+1}\bar\partial : \mathscr{D}^{N-p,N-q-1}(\mathscr{M}) \to \mathscr{D}^{N-p,N-q}(\mathscr{M}).$$

By the result on smoothing cohomology, Theorem 2.4.16, the inclusion of $\mathscr{D}^{p,q}(\mathscr{M})$ in $\mathscr{K}_{p,q}(\mathscr{M})$ induces an isomorphism of cohomology groups with compact supports. This proves that the cohomology groups $H_c^{p,q}(\mathscr{M})$ vanish in the range $1 \le q \le N-1$. That $H_c^{p,0}(\mathscr{M}) = 0$ is evident: The only holomorphic form with compact support is the zero form.

It is also important to apply Lemma 2.4.18 to the sequence

$$\mathscr{O}(\mathscr{M}) \to \mathscr{E}^{0,0}(\mathscr{M}) \xrightarrow{\bar\partial} \mathscr{E}^{0,1}(\mathscr{M}).$$

The lemma implies the isomorphism

$$\mathscr{O}(\mathscr{M})^* = \mathscr{K}_{N,N}(\mathscr{M})/\bar\partial\mathscr{K}_{N,N-1}(\mathscr{M}). \tag{2.9}$$

Again, the cohomology group $\mathscr{K}_{N,N}(\mathscr{M})/\bar\partial\mathscr{K}_{N,N-1}(\mathscr{M})$ is isomorphic to the Dolbeault group $H_c^{N,N-1}(\mathscr{M})$ by the abstract de Rham theorem. The pairing that establishes this duality is that induced by the pairing of holomorphic functions f on $\mathscr{M}$ with compactly supported (N,N)-forms α given by $\langle f, \alpha\rangle = \int_{\mathscr{M}} f\,\alpha$. This argument applies, mutatis mutandis, to yield that if $\Omega^p(\mathscr{M})$ denotes the space of holomorphic p-forms on $\mathscr{M}$, then $\Omega^p(\mathscr{M})^*$ is the group $H_c^{N-p,N}(\mathscr{M})$.

Corollary 2.4.19. *A compactly supported $(N-p,N)$-form α on $\mathscr{M}$ is $\bar\partial\beta$ with β a compactly supported $(N-p,N-1)$-form if and only if $\int_{\mathscr{M}} f\,\alpha = 0$ for all $f \in \Omega^p(\mathscr{M})$.*

For complements of Stein compacta in Stein manifolds, there are the following vanishing results:

Theorem 2.4.20. *If X is a Stein compactum in the N-dimensional Stein manifold $\mathscr{M}$, then*

(a) *$H^{p,q}(\mathscr{M}\setminus X) = 0$ for all $p \ge 0$ and all q with $1 \le q \le N-2$.*

(b) *$H_c^{p,q}(\mathscr{M}\setminus X) = 0$ for all $p \ge 0$ and all p with $2 \le q \le N-1$.*

(c) *$H_\Phi^{p,q}(\mathscr{M}\setminus X) = 0$ for all $p \ge 0$ and all q with $0 \le q \le N-2$.*

(d) *If X is $\mathscr{O}(\mathscr{M})$-convex, then $H_\Phi^{p,N-1}(\mathscr{M}\setminus X) = 0$ for all $p \ge 0$.*

Information is also available about $H^{p,N-1}(\mathscr{M}\setminus X)$ and $H_c^{p,N}(\mathscr{M}\setminus X)$: *With respect to the natural locally convex topologies on them, these spaces are Hausdorff spaces.* We shall not go into the details of these latter results.

These results are well known; it is not clear what the original sources may be. The proof below is that in [87]. See also the paper of Rosay and Stout [301]. Lupacciolu [228] has obtained related sheaf-theoretic results.

In the proof of these vanishing results, we need to use a theorem that is a generalization to complex manifolds of a theorem of Hartogs in $\mathbb{C}^N$. The result was obtained in [322]. Ehrenpreis [109] gave a generalization of the result.

Theorem 2.4.21. *Let $\mathscr{M}$ be a complex manifold with $H_c^{p,1}(\mathscr{M}) = 0$, and let X be a compact subset of $\mathscr{M}$ such that $\mathscr{M} \setminus X$ is connected. If $\alpha \in \Omega^p(\mathscr{M} \setminus X)$, then there is an $\tilde{\alpha} \in \Omega^p(\mathscr{M})$ that agrees with α on $\mathscr{M} \setminus X$.*

This theorem applies in particular when $\mathscr{M}$ is a Stein manifold.

If $p = 0$ the result is that holomorphic functions extend through the compact set X. In $\mathbb{C}^N$ this can be proved using the Bochner–Martinelli integral.

Proof. Let U be a relatively compact neighborhood of X in $\mathscr{M}$, and let λ be a smooth function on $\mathscr{M}$ that is identically one on a neighborhood of $\mathscr{M} \setminus U$ and is identically zero on a neighborhood of X. Then $\bar{\partial}\lambda \wedge \alpha$, extended by zero through X, is a compactly supported, $\bar{\partial}$-closed $(p, 1)$-form on $\mathscr{M}$. The hypotheses of the theorem provide a compactly supported $(p, 0)$-form β with $\bar{\partial}\beta = \bar{\partial}\lambda \wedge \alpha$. If $\tilde{\alpha} = \lambda \wedge \alpha - \beta$, then $\bar{\partial}\tilde{\alpha} = 0$, so $\tilde{\alpha} \in \Omega^p(\mathscr{M})$, and $\tilde{\alpha}$ agrees with α near infinity. The set $\mathscr{M} \setminus X$ is connected, which implies that $\tilde{\alpha} = \alpha$ throughout $\mathscr{M} \setminus X$. The theorem is proved.

We now turn to the proof of Theorem 2.4.20.

Proof. We will use the notation that $\mathscr{Z}^{p,q}$ denotes the space of $\bar{\partial}$-closed (p, q)-forms, and $\mathscr{Z}_c^{p,q}$ the space of $\bar{\partial}$-closed (p, q)-forms with compact support. Similarly, in connection with statements (c) and (d) of the theorem, $\mathscr{Z}_\Phi^{p,q}(\mathscr{M} \setminus X)$ is the space of $\bar{\partial}$-closed (p, q)-forms on $\mathscr{M} \setminus X$ whose supports are relatively compact subsets of $\mathscr{M}$.

By hypothesis, we can write $X = \cap_{j=1,\dots} U_j$, where each U_j is a smoothly bounded, strictly pseudoconvex domain and U_{j+1} is relatively compact in U_j. For all j, let $\lambda_j \in \mathscr{C}^\infty(\mathscr{M})$ vanish identically on a neighborhood of $\mathscr{M} \setminus U_j$ and be identically one on a neighborhood of $\bar{U}_{j+1}$.

Fix an $\alpha \in \mathscr{Z}^{p,q}(\mathscr{M} \setminus X)$ for a q in the range $1 \le q \le N - 2$.

The form $\bar{\partial}\lambda_j \wedge \alpha$ extends by 0 through X to give a $\bar{\partial}$-closed (p, q)-form on $\mathscr{M}$ with support in U_j. It follows that there is a $(p, q-1)$-form β_j on $\mathscr{M}$ with support in U_j such that $\bar{\partial}\beta_j = \bar{\partial}\lambda_j \wedge \alpha$. Then the form $(1 - \lambda_j)\alpha$ extends by zero through X to a (p, q)-form on all of $\mathscr{M}$. The (p, q)-form $\beta_j - (1 - \lambda_j)\alpha$ on $\mathscr{M}$ has compact support, so there is a γ_j with $\bar{\partial}\gamma_j = \beta_j - (1 - \lambda_j)\alpha$, γ_j with compact support.

Suppose now that $q = 1$. The form $\gamma_j - \gamma_{j+1}$ is holomorphic on $\mathscr{M} \setminus U_j$, so because this set is connected—recall Corollary 2.4.3—there is a form h_j on $\mathscr{M}$ such that $\gamma_j - \gamma_{j+1} = h_j$ on $\mathscr{M} \setminus U_j$. We can define a form γ by

$$\gamma = \gamma_1 + \sum_{j=1}^{\infty} (\gamma_{j+1} - \gamma_j) + h_j.$$

This sum is locally finite, so there is no difficulty about its convergence. On $\mathscr{M} \setminus U_m$ we have

$$\bar{\partial}\gamma = \bar{\partial}\gamma_1 + \sum_{j=1}^{m} (\bar{\partial}\gamma_{j+1} - \bar{\partial}\gamma_j) = \bar{\partial}\gamma_{m+1} = \beta_{m+1} + (1 - \lambda_{m+1})\alpha = \alpha.$$

This proves the assertion (a) when $q = 1$.

For $N \geq 3$ and $q > 1$, the argument is inductive. We have $\bar{\partial}(\gamma_{j+1} - \gamma_j) = 0$ on $\mathscr{M} \setminus U_j$, so since $\bar{U}_j$ is a Stein compactum in $\mathscr{M}$, there is a $(p, q-1)$-form g_j on $\mathscr{M} \setminus \bar{U}_j$ with $\bar{\partial} g_j = \gamma_{j+1} - \gamma_j$. Then $(1 - \lambda_{j-1})g_j$ extends by zero through U_j as a smooth form, and we have $\gamma_{j+1} - \gamma_j = \bar{\partial}((1 - \lambda_{j-1})g_j)$ on $\mathscr{M} \setminus U_{j-1}$. The form γ defined by

$$\gamma = \gamma_1 + \sum_{j=1}^{\infty} \left[\gamma_{j+1} - \gamma_j - \bar{\partial}((1 - \lambda_{j-1} g_j)\right]$$

satisfies $\bar{\partial}\gamma = \alpha$.

The proof of (c) follows exactly the same lines: If we begin with a $\bar{\partial}$-closed (p, q)-form α on $\mathscr{M} \setminus X$ the support of which is a relatively compact subset of $\mathscr{M}$, then our construction yields a form γ with support a relatively compact subset of $\mathscr{M}$.

For (b), if $\alpha \in \mathscr{Z}^{p,q}(\mathscr{M} \setminus X)$, $1 < q < N$, extend α through X by zero. There is $\beta \in \mathscr{E}^{p,q}(\mathscr{M})$ with compact support that satisfies $\bar{\partial}\beta = \alpha$. The form β satisfies $\bar{\partial}\beta = 0$ in U_j for large j. Put $\beta' = \beta - \bar{\partial}(\lambda_j \gamma_j)$ with $\lambda_j \gamma_j$ extended by zero outside U_j. The form β' so defined is a smooth form with supp β' a relatively compact subset of $\mathscr{M} \setminus X$. It satisfies $\bar{\partial}\beta' = \bar{\partial}\beta = \alpha$, so (b) is proved.

It remains finally to prove (d). Given that X is $\mathscr{O}(\mathscr{M})$-convex, the sets U_j can be taken to be Runge domains in $\mathscr{M}$. Then the space $\Omega(\mathscr{M})|U_j$ is dense in $\mathscr{O}(U_j)$. The form $\bar{\partial}\lambda \wedge \alpha$, which is compactly supported in U_j, is orthogonal to $\Omega(U_j)$, and so is $\bar{\partial}\beta_j$ for an $(N, N-1)$-form β_j with compact support in U_j. The argument now continues as before.

Chapter 3

SETS OF FINITE LENGTH

Introduction. This chapter is devoted to a fairly self-contained discussion of the polynomially convex hull of a connected set of finite length or, more generally, a set that is contained in a connected set of finite length. One result that finally emerges in this chapter is that each rectifiable arc in $\mathbb{C}^N$ is polynomially convex. Much of the chapter is devoted to preliminaries from classical function theory and from real analysis. Section 3.1 contains a statement of one of the principal results of the chapter and some remarks about it. Section 3.2 assembles well-known information about one-dimensional analytic varieties. Section 3.3 contains geometric preliminaries concerning Hausdorff measures, integration, and sets of finite length. Section 3.4 is devoted to some essential results on conformal mapping and related issues. Section 3.5 establishes the subharmonicity of certain functions naturally associated with the polynomially convex hull of a compact set. Section 3.6 shows that the polynomially convex hull of a connected set of finite length is a one-dimensional variety. Section 3.7 shows that this hull has finite area. Section 3.8 applies the preceding theory to the continuation of one-dimensional varieties and, vice versa, this theory of continuation to the study of hulls.

3.1. Introduction

Simple examples show that some smooth curves in $\mathbb{C}^N$ are polynomially convex, while others are not, so two natural questions arise: One would like to characterize the polynomially convex curves, and for those that are not polynomially convex, one would like a description of the hulls. As the most immediate example, suppose Γ to be a simple closed curve in $\mathbb{C}^N$ that is the boundary of a Riemann surface $\mathscr{R}$ embedded as a bounded subset of $\mathbb{C}^N$. If P is a polynomial on $\mathbb{C}^N$, then $P|\mathscr{R}$ is holomorphic, so by the maximum principle, $\mathscr{R}$ is contained in the polynomially convex hull of Γ. Does $\widehat{\Gamma}$ contain anything other than $\mathscr{R}$? If we know only that Γ is not polynomially convex, is the hull a Riemann surface or, more generally, an analytic variety?

In this chapter and the next we study the hulls of smooth curves and, more generally, of sets of finite length and related questions. The theory is not simple but can now be said to be in nearly definitive form. The present state of the theory is the culmination of a long development by several mathematicians, a development that has spanned almost half a century. The centerpiece of the work is the following theorem, which motivates almost everything in the present chapter.

Theorem 3.1.1. *Let Y be a compact polynomially convex subset of $\mathbb{C}^N$, and let Γ be a subset of $\mathbb{C}^N$ contained in a compact connected set of finite length such that $\Gamma \cup Y$ is compact. The polynomially convex hull $\widehat{\Gamma \cup Y}$ has the property that the complementary set $(\widehat{\Gamma \cup Y}) \setminus (\Gamma \cup Y)$ either is empty or else is a purely one-dimensional analytic subvariety of $\mathbb{C}^N \setminus (\Gamma \cup Y)$. If the map $\check{H}^1(\Gamma \cup Y; \mathbb{Z}) \to \check{H}^1(Y; \mathbb{Z})$ induced by the inclusion $Y \hookrightarrow \Gamma \cup Y$ is an isomorphism, then the algebra $\mathscr{P}(\Gamma \cup Y)$ consists of all the continuous functions f on $\Gamma \cup Y$ with $f|Y \in \mathscr{P}(Y)$.*

It may seem unnatural to impose the hypothesis that the set Γ be contained in a set of finite length. Examples show, though, that it cannot be entirely abandoned: There are compact sets E of finite length for which $\widehat{E} \setminus E$ is not a variety. See [15].

Important contributions related to this theorem have been made by Wermer [368], Bishop [57], Royden [306], Stolzenberg [342], Alexander [6], Björk [62], Gamelin [137], Aupetit and Wermer [41], and Seničkin [321].

Corollary 3.1.2. *A rectifiable arc Γ in $\mathbb{C}^N$ is polynomially convex and satisfies $\mathscr{P}(\Gamma) = \mathscr{C}(\Gamma)$.*

This apparently simple case of the result is not at all easy to establish, even when the arc is required to be smooth.

Corollary 3.1.3. *If Γ is a rectifiable simple closed curve in $\mathbb{C}^N$, then $\widehat{\Gamma} \setminus \Gamma$, if not empty, is a purely one-dimensional subvariety of $\mathbb{C}^N \setminus \Gamma$.*

The corollary raises as many questions as it answers: What can be said about the variety $V = \widehat{\Gamma} \setminus \Gamma$? How many global branches does it have? Does it have finite area? Does it have finite topological type? Is there some version of Stokes's theorem in the setting of the corollary? These are all natural questions; we shall eventually see their answers.

In the context of Theorem 3.1.1, the variety $(\widehat{Y \cup \Gamma}) \setminus (Y \cup \Gamma)$ need not have finite area.

Example. Let B be an infinite Blaschke product with the point 1 as the sole limit point of its zeros. For example, $B(z)$ might be $\prod_{k=2}^{\infty} \frac{z-\alpha_k}{1-\alpha_k z}$ with $\alpha_k = 1 - \frac{1}{k^2}$. Let Y be the compact, polynomially convex subset $\{(e^{i\vartheta}, w) \in \mathbb{C}^2 : |\vartheta| \leq \frac{\pi}{4}, |w| \leq 1\}$ of $\mathbb{C}^2$. If Γ denotes the *analytic* arc $\{(e^{i\vartheta}, B(e^{i\vartheta})) : \frac{\pi}{4} \leq \vartheta \leq \frac{7\pi}{4}\}$, then by the theorem, $\widehat{Y \cup \Gamma} \setminus (Y \cup \Gamma)$ is a one-dimensional variety. It contains at least the graph $\{(z, B(z)) \in \mathbb{C}^2 : |z| < 1\}$. (In fact, these two sets are equal, but this is not yet evident.) The function B has infinitely many zeros, so every value in the unit disk $\mathbb{U}$ in $\mathbb{C}$ outside a set of zero area is assumed infinitely often. (This is a standard fact about infinite Blaschke products; in fact, the result is stronger: Every value outside a set of zero logarithmic capacity is assumed infinitely often. See [356, p. 324].) Accordingly, the area of the graph of B, which is the value of the integral $\int_{\mathbb{U}}(1+|B'|^2)$, is infinite.

Similar examples can be based on functions other than Blaschke products. For example, let $\phi : \mathbb{U} \to \mathbb{U} \setminus \{0\}$ be a uniformizing map, chosen to be analytic on $b\mathbb{U} \setminus \{1\}$. Let λ be a short open arc in $b\mathbb{U}$ that contains 1. If Y is the compact polynomially convex subset $(\bar{\lambda} \times \bar{\mathbb{U}}) \cup (\{0\} \times \bar{\mathbb{U}})$ of $\mathbb{C}^2$ and Γ is the set $\{(e^{i\vartheta}, \phi(e^{i\vartheta})) : e^{i\vartheta} \in b\mathbb{U} \setminus \lambda\}$, then the polynomially convex hull of $Y \cup \Gamma$ contains the graph of ϕ and so cannot have finite area. Note that the function ϕ is zero-free and so is surely not a Blaschke product.

Example. [217] Let B be the infinite Blaschke product considered in the preceding example with $\alpha_k = 1 - \frac{1}{k^2}$. Define F by $F(z) = \frac{1}{16}(1-z)^4 B(z)$, and let $V \subset \mathbb{U}^2$ be the analytic variety given by $V = \{(z_1, z_2) \in \mathbb{U}^2 : z_1^2 = F(z_2)\}$. This is an analytic subvariety of the unit bidisk, which, by the maximum principle, is contained in the polynomially convex hull of the set $\Gamma = \{(z_1, z_2) : |z_2| = 1,\ z_1^2 = F(z_2)\}$. The set Γ consists of two simple closed curves joined together at single point; topologically it is a figure-eight curve. It is smooth except at the point (0, 1), the point at which the two curves come together. The curve Γ also has finite length.

This is so, for logarithmic differentiation yields

$$(3.1) \qquad (1-z)^4 B'(z) = (1-z^2) \sum_{k=1}^{\infty} B_k(z)(1-\alpha_k^2) \frac{(1-z)^2}{(1-\alpha_k z)^2},$$

in which B_k is the Blaschke product given by $B_k(z) = B(z)/\frac{z-\alpha_k}{1-\alpha_k z}$. From $0 < \alpha_k < 1$ and $|z| < 1$, we conclude that

$$|(1-z)/(1-\alpha_k z)| \leq 1 + \sum_{n=1}^{\infty} |\alpha_k^n - \alpha_k^{n+1}| = 1 + \alpha_k < 2.$$

Consequently, the series (3.1) converges uniformly on the disk, and, because

$$F'(z) = \frac{1}{4}(1-z)^3 B(z) + \frac{1}{16}(1-z)^4 B'(z),$$

we see that F' is continuous on $\bar{\mathbb{U}}$, which implies that Γ has finite length.

According to Theorem 3.1.1, the polynomially convex hull of Γ is a one-dimensional variety that must contain V. Again, as we shall recognize later, the set $\widehat{\Gamma} \setminus \Gamma$ *is* the variety V. See Corollary 3.6.2. This variety has a branch point over each zero of B and so necessarily has infinite genus; it is not of finite topological type.

3.2. One-Dimensional Varieties

The main effort in this chapter is devoted to finding and determining the properties of one-dimensional analytic sets in polynomial hulls, so it seems well to begin by recalling precisely some facts about these varieties. By now there are many good sources for this material: [85], [157], [158], [199], [225], [259].

Analytic varieties are defined as follows.

Definition 3.2.1. *If Ω is an open set in $\mathbb{C}^N$, then an* analytic variety *in Ω (or* subvariety *of Ω) is a closed subset V of Ω with the property that for each point $z \in V$ there is a neighborhood W_z of z on which there is defined a collection $\mathscr{F}_{V,z}$ of holomorphic functions with the property that $V \cap W_z = \cap_{f \in \mathscr{F}_{V,z}} f^{-1}(0)$.*

It is one of the more immediate consequences of the Weierstrass preparation theorem that by shrinking the neighborhood W_z the family $\mathscr{F}_{V,z}$ can be replaced by a *finite* collection of functions.

Analytic varieties are also called *analytic sets*.

If X is an analytic variety in the open set U in $\mathbb{C}^N$, then a point $z \in V$ is said to be a *regular point* if there is a neighborhood W_z of z in the ambient $\mathbb{C}^N$ such that $X \cap W$ is a complex submanifold of W_z. The point z is a *singular point* of V if it is not a regular point. The set of singular points is closed and is an analytic subset of U. The set of regular points of X is denoted by X_{reg}, the set of singular points by X_{sing}. Note that, by definition, an isolated point of an analytic variety is a regular point.

The notion of analytic set is plainly a local one, so we can speak freely of analytic subsets of complex manifolds.

An analytic subset X of a complex manifold is one-dimensional if the set X_{reg} is a one-dimensional complex manifold together with the set, possibly empty, of the isolated points of X. The set of singular points of a one-dimensional analytic variety is discrete.

If X is an analytic subvariety of a complex manifold, then $\mathscr{O}(X)$ denotes the space of holomorphic functions on X. By definition, a function f defined on the variety X is holomorphic if for each $x \in X$, there is a neighborhood V_x of x in the ambient complex manifold on which there is defined a holomorphic function F_x with $F_x|(V_x \cap X) = f|(V_x \cap X)$. Such a function is necessarily continuous on X and holomorphic on the complex manifold X_{reg}. In general, a function g that is continuous on X and holomorphic on X_{reg} is not holomorphic on X. It is an important fact, which is not so easy to prove, that the space $\mathscr{O}(X)$ is closed under uniform convergence on compacta in X. That is to say, if $\{f_j\}_{j=1,\dots}$ is a sequence in $\mathscr{O}(X)$ that converges uniformly on compacta in X to the (necessarily continuous) function f, then f is holomorphic on X. A proof of this is given in [158]. The result is originally due to Grauert and Remmert [154]; Bungart and Rossi [77] gave a simpler proof.

We shall need a rather technical fact about one-dimensional varieties. The setting is this: Z is a compact subset of $\mathbb{C}^N$ with polynomially convex hull $\widehat{Z}$, and $\pi_1 : \mathbb{C}^N \to \mathbb{C}$ is the coordinate projection onto the z_1-axis in $\mathbb{C}^N$. Let Ω be a component of $\pi_1(\widehat{Z}) \setminus \pi_1(Z)$, and let Ω_n be an open subset of Ω such that $\Omega \setminus \Omega_n$ is a discrete subset of Ω. For each $z \in \Omega$ the fiber $\widehat{Z} \cap \pi_1^{-1}(z)$ contains at most n points, and for each $z \in \Omega_n$ this fiber contains exactly n points.

Lemma 3.2.2. *If $V_n = \widehat{Z} \cap \pi_1^{-1}(\Omega_n)$ is a one-dimensional complex submanifold of the open set $\pi_1^{-1}(\Omega_n)$, then the set $V = \widehat{Z} \cap \pi_1^{-1}(\Omega)$ is a one-dimensional subvariety of*

$\pi_1^{-1}(\Omega)$ *on which each element of the algebra* $\mathscr{P}(\widehat{Z})$ *is holomorphic.*

Proof. Start with the obvious remark that if W' is an analytic subvariety of an open set U in $\mathbb{C}^N$, and if W is a closed subset of U that differs from W' by a closed discrete subset of U, then W is also a subvariety of U.

By definition, we have to exhibit the set V as a local intersection of the zero loci of holomorphic functions.

For every function f holomorphic on $\mathbb{C}^N$, define $\tilde{F}_f : \pi_1^{-1}(\Omega) \to \mathbb{C}$ by

$$\tilde{F}_f(z) = \prod_{w_j \in \pi_1^{-1}(\pi_1(z)) \cap V} \{f(z) - f(w_j)\}.$$

The function $\tilde{F}_f$ is holomorphic on $\pi_1^{-1}(\Omega_n)$, as follows from the theory of symmetric functions. It is locally bounded at all points of $\pi_1^{-1}(\Omega)$, so it extends to be holomorphic on all of $\pi_1^{-1}(\Omega)$. Denote this extended function by F_f. The function F_f plainly vanishes on the closure in $\pi^{-1}(\Omega)$ of the set $\pi_1^{-1}(\Omega_n)$.

If p is a point in $\pi_1^{-1}(\Omega_n)$ not in $\widehat{Z}$, then there is a polynomial f with $f(p) = 1$ and $|f| < 1$ on $\pi_1^{-1}(\pi_1(p)) \cap \widehat{Z}$. For this f, the function F_f does not vanish at p. We have, therefore, that the variety $\tilde{V} = \cap_{f \in \mathscr{O}(\mathbb{C}^N)} F_f^{-1}(0)$ differs from the closure in $\pi_1^{-1}(\Omega)$ of the set $\widehat{Z} \cap \varphi^{-1}(\Omega_n)$ by at most a discrete subset of $\pi_1^{-1}(\Omega)$ contained in the set $\pi_1^{-1}(\Omega \setminus \Omega_n)$. Consequently, the set $\widehat{Z} \cap \pi_1^{-1}(\Omega)$ is a variety, as we wished to show.

That the elements of the algebra $\mathscr{P}(\widehat{Z})$ are holomorphic on the variety V is a consequence of the closure of the space $\mathscr{O}(V)$ under local uniform convergence and the fact that each polynomial on $\mathbb{C}^N$ is holomorphic on V. The polynomials are dense in $\mathscr{P}(\widehat{Z})$, so the assertion follows.

3.3. Geometric Preliminaries

This section is devoted to some properties of sets of finite length and to some further properties of Hausdorff measures that will be essential in the sequel.

To begin we present some results on the structure of continua of finite measure that were found by Besicovitch [51] in the course of his profound investigations of planar sets of finite length.

Recall that an *arc* is a homeomorphic image of the closed interval [0, 1]. An *open arc* is a homeomorphic image of the open interval (0, 1).

Definition 3.3.1. *A topological space Y is* arcwise connected *if for every pair of distinct points of Y there is an arc in Y that contains both of the points.*

We shall use the notation that in $\mathbb{R}^n$, $B(r)$ denotes the open ball of radius r centered at the origin and $B(x, r)$ the open ball of radius r centered at the point x.

Lemma 3.3.2. *If the connected subset E of $\mathbb{R}^n$ contains points x and y with $|x - y| = r$, then $\Lambda^1(E \cap B(r)) \geq r$.*

Proof. For fixed x, let $\chi : \mathbb{R}^n \to [0, \infty)$ be the map $\chi(y) = |y - x|$. If the connected subset E of $\mathbb{R}^n$ that contains x and y with $|x - y| = r$ satisfies $\Lambda^1(E \cap B(x, r)) < r$, then χ omits certain values in the interval $[0, r]$, for χ is a Lipschitz map with Lipschitz constant 1. Suppose χ to omit the value $r_o \in (0, r)$. Then there is a decomposition of E as the union of the two sets E' and E'', where E' is the subset of E consisting of the points at distance less than r_o from x, and E'' is the set of those points at distance greater than r_o from x. The sets E' and E'' are mutually disjoint open and closed subsets of E, in contradiction to the connectedness of E.

Theorem 3.3.3. [51] *A connected subset of $\mathbb{R}^n$ of finite length is arcwise connected.*

Proof. Let E be a connected subset of $\mathbb{R}^n$ that has finite length. The set E is connected, so if x and y are arbitrary points of E, then for every $\varepsilon > 0$ there is a finite sequence $\{x_j\}_{j=1,\dots,m(\varepsilon)}$ of points in E with $x_1 = x$, $x_{m(\varepsilon)} = y$ and with $|x_j - x_{j+1}| < 2\varepsilon$ for each j. This is so, because for fixed x and ϵ, the set of y's for which such a sequence exists is both open and closed in the connected set E.

Let $L = \Lambda^1(E)$. Fix points $a, b \in E$ and fix an $\eta > 0$. Let $\{x'_j\}_{j=1,\dots,m'(\eta)}$ be a finite sequence of points of E as just described: $x'_1 = a$, $x'_{m'(\eta)} = b$, and $|x'_j - x'_{j+1}| < 2\eta$. We will replace this sequence with a new one obtained by discarding some of the terms x'_j in the following way. Put $x_1 = x'_1 = a$. Then x_2 is the x'_j with largest index j that satisfies $\eta < |x'_j - x_1| < 2\eta$. Having constructed $x_1, \dots, x_d$, stop if $x_d = b$. If not, let x_{d+1} be the x'_j with largest index such that $\eta < |x'_j - x_d| < 2\eta$. We are led finally to a new sequence $\{x_j\}_{j=1,\dots,m(\eta)}$.

From this new sequence, construct the polygonal path P that consists of the segments $[x_j, x_{j+1}]$, $j = 1, \dots, m(\eta)$. This path P has all of its vertices in the set E. Also, it is an arc. Otherwise, two of its segments, say $[x_p, x_{p+1}]$ and $[x_q, x_{q+1}]$, would have to intersect at a point interior to both of these segments. The existence of such an intersection would contradict the construction of the points x_j. Thus, P is an arc. Moreover, the length of P does not exceed $4L$. This is so, for by Lemma 3.3.2, $\Lambda^1(E \cap B(x_j, \eta/2)) \geq \eta/2$. If $L(P)$ denotes the length of the arc P, then $L(P) \leq 2\eta m(\eta)$. The balls $B(x_j, \eta/2)$ are mutually disjoint, so $\Lambda^1(E) > \frac{1}{2}m(\eta)\eta$, and we have $L(P) \leq 4L$ as desired.

We perform this construction for a sequence of η's that decreases to zero, say for $\eta = 1/n$, $n = 1, 2, \dots$. Let the associated polygonal arcs be P_n. The arcs P_n have uniformly bounded lengths; let λ_n be the length of P_n. By passing to a subsequence if necessary, we can suppose that $\lambda_n \to \lambda_o$.

For each $n = 1, \dots,$ let $\psi_n : [0, 1] \to \mathbb{R}^n$ be the mapping given by $\psi(t) = \tilde{\psi}(t\lambda_n)$, where $\tilde{\psi} : [0, \lambda_n] \to \mathbb{R}^n$ is the parameterization of P_n by arc length fixed so that $\psi_n(0) = a$. The functions $\tilde{\psi}_n$ satisfy $|\tilde{\psi}_n(t) - \tilde{\psi}(t')| \leq |t - t'|$, so ψ_n satisfies $|\psi_n(t) - \psi_n(t')| \leq \lambda_n|t - t'| \leq 4L|t - t'|$. The Arzelà–Ascoli theorem implies that a subsequence of $\{\psi_n\}_{n=1,\dots}$ converges uniformly on $[0, 1]$, say to ψ. The function ψ is continuous, and its range is a subset of E. Moreover, $\psi(0) = a$, and $\psi(1) = b$.

To complete the proof, we have to show how to construct from ψ a *homeomorphism* from $[0, 1]$ into E that connects a and b. To do this, construct inductively a sequence $\{I_j\}_{j=1,\dots}$ of closed subintervals of $[0, 1]$ as follows. The interval I_1 is one of the longest closed subintervals of [0, 1] with the property that ψ identifies the endpoints of I_1. Let I_2 be one of the longest closed intervals contained in the complement $[0, 1] \setminus I_1$ with the property that ψ identifies the endpoints of I_2. Continue this process. Either it terminates at some step because the complementary set contains no closed interval with endpoints identified by ψ or else it generates an infinite sequence $\{I_j\}_{j=1,...}$ of mutually disjoint closed intervals the endpoints of each of which are identified by the map ψ. Let J be the space obtained from [0, 1] by collapsing each of the intervals I_j to a point. The space J with the quotient topology is homeomorphic to [0, 1]. We can define a map $\tilde{\psi} : J \to E$ by requiring that the value of $\tilde{\psi}$ at any point of J that corresponds to one of the intervals I_j be the common value assumed by ψ at the endpoints of I_j. At points of J corresponding to points of [0, 1] not in one of the intervals I_j, $\tilde{\psi}$ takes the value assumed there by ψ. The map constructed in this way is continuous and injective; it is an arc in E that connects a and b.

Theorem 3.3.4. *If λ is an arc in $\mathbb{R}^n$, then the length of λ is the same as its one-dimensional Hausdorff measure.*

By the *length* of λ, to be denoted by $\ell(\lambda)$, we understand the usual supremum of the lengths of inscribed polygonal paths.

Proof. If λ is rectifiable, let $\psi : [0, \ell(\lambda)] \to \lambda$ be a parameterization of λ by arc length. The mapping ψ satisfies $|\psi(s) - \psi(s')| \leq |s - s'|$ for all $s, s' \in [0, \ell(\lambda)]$: ψ is a Lipschitz map with Lipschitz constant one. Thus, ψ does not increase one-dimensional Hausdorff measure. Because $\Lambda^1([0, \ell(\lambda)]) = \ell(\lambda)$, we reach $\Lambda^1(\lambda) \leq \ell(\lambda)$.

For the reverse inequality, let the endpoints of λ be a and b. Let $a = p_1, p_2, \ldots, p_n = b$ be a partition of λ such that the points p_j all lie on λ and for each j, p_j is between p_{j-1} and p_{j+1}. Denote by λ_j the subarc of λ with endpoints p_j and p_{j+1}. By Lemma 3.3.2,

$$\sum_{j=1}^{n-1} |p_{j+1} - p_j| \leq \sum_{j=1}^{n-1} \Lambda^1(\lambda_j) = \Lambda^1(\lambda),$$

which implies that $\ell(\lambda) \leq \Lambda^1(\lambda)$. The theorem is proved.

Theorem 3.3.5. [51] *If E is a connected set of finite length in $\mathbb{R}^n$, then $E = F \cup \cup_{j=1,...}\lambda_j$, where the λ_j are mutually disjoint open rectifiable arcs and the set F has length zero and where each point of F is in the closure of the set $\cup_{j=1,...}\lambda_j$.*

Proof. For a subset S of E and a point $y \in E$, denote by $\delta_S(y)$ the infimum of the lengths of the arcs in E that connect y to a point of S.

To prove the theorem, construct a sequence $\{\lambda_j\}_{j=1...}$ of arcs in E as follows. Let λ_1 be any fixed arc in E. To construct λ_2, choose a point $y_2 \in E$ such that $\delta_{\lambda_1}(y_2) > \frac{3}{4}\sup_{z \in E}\delta_{\lambda_1}(z)$. Let λ_2 be an arc in E that connects y_2 to a point in the arc λ_1. The arc λ_2 is to have only one point in common with λ_1, that point an endpoint of λ_2. Next, let y_3 be a point in E such that $\delta_{\lambda_1 \cup \lambda_2}(y_3) > \frac{3}{4}\sup_{z \in E}\delta_{\lambda_1 \cup \lambda_2}(z)$. Let λ_3 be an arc in E that connects the point y_3 to the set $\lambda_1 \cup \lambda_2$. The arc λ_3 is to have only one point, an endpoint, in common with $\lambda_2 \cup \lambda_2$. We repeat this process. If it stops in a finite number of steps, we are done. If not, then because the interiors of the arcs λ_j are disjoint and $\Lambda^1(E) < \infty$, we see that $\Lambda^1(\lambda_j) \to 0$ as $j \to \infty$.

It is to be shown now that the set $F = E \setminus \cup_{j=1,\ldots}\lambda_j$ is a set of zero length. Set $E_o = \cup_{j=1,\ldots}\lambda_j$.

The construction of the λ's implies that $F \subset \bar{E}_o \setminus E_o$.

Define Γ_n and γ_n by $\Gamma_n = \lambda_1 \cup \cdots \cup \lambda_n$ and $\gamma_n = \lambda_{n+1} \cup \lambda_{n+2} \cup \cdots$. Then $\Lambda^1(\gamma_n) \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} \Lambda^1(\Gamma_n) \leq \Lambda^1(E)$.

Fix an n. Let x_1 be a point of $\bar{E}_o \setminus E_o$ at maximal distance from Γ_n, and let d_1 be this distance. Let x_2 be a point of $\bar{E}_o \setminus (E_o \cup B(x_1, 2d_1))$ at maximal distance, d_2, from Γ_n. Proceeding inductively, let x_{k+1} be a point of $\bar{E}_o \setminus (E_o \cup B(x_1, 2d_1) \cup \cdots \cup B(x_k, 2d_k))$ at maximal distance, d_{k+1}, from Γ_n. The balls $B(x_k, 2d_k)$ cover $\bar{E}_o \setminus E_o$, and the balls $B(x_k, d_k)$ are mutually disjoint and are disjoint from Γ_n. For each k, $\Lambda^1(\gamma_n \cap B(x_k, d_k)) \geq d_k$, as follows from Lemma 3.3.2. Thus $\sum_j d_j \leq \Lambda^1(\gamma_n)$. As $n \to \infty$, $\Lambda^1(\gamma_n) \to 0$, so because for every n, $\bar{E} \setminus E \subset \cup_{k=1,\ldots} B(x_k, 2d_k)$, we find that $\Lambda^1(\bar{E}_o \setminus E_o) = 0$. The theorem is proved.

We shall often need the following inequality, which is called *Eilenberg's inequality.*

Theorem 3.3.6. *If X is a metric space and $f : X \to \mathbb{R}$ is a Lipschitz map with Lipschitz constant K, then for $\alpha \geq 0$,*

$$\int_{\mathbb{R}}^{*} \Lambda^\alpha(f^{-1}(t))\, dt \leq K\Lambda^{\alpha+1}(X).$$

The integral is the *upper* integral; in fact, $t \mapsto \Lambda^\alpha(f^{-1}(t))$ is measurable, so that this upper integral is simply the usual integral, but we shall not enter into the proof of this.

This theorem is due to Eilenberg [110]. See also the paper by Eilenberg and Harrold [111].

Proof. Fix a doubly indexed family $\{E_{n,j}\}_{n,j=1,\ldots}$ of subsets of X such that for each n, $X = \cup_{j=1,\ldots} E_{n,j}$, such that for all n and j, the diameter, diam $E_{n,j}$, of $E_{n,j}$ is less than $1/n$, and such that

$$\lim_{n\to\infty} \gamma_{\alpha+1} \sum_{j=1,\ldots} (\text{diam } E_{n,j})^{\alpha+1} = \Lambda^{\alpha+1}(X).$$

(Recall the definition of the Hausdorff measures given in Definition 1.6.1.) By χ_S we denote the characteristic function of the set $S \subset \mathbb{R}$. The function $\chi_{\overline{f(E_{n,j})}}$ is a Borel measurable function. Define ψ by

$$\psi(t) = \liminf_{n\to\infty} \sum_{j=1,\ldots} (\text{diam } E_{n,j})^\alpha \chi_{\overline{f(E_{n,j})}}(t).$$

This also is a measurable function.

By the definition of Λ^α,

$$\Lambda^\alpha(f^{-1}(t)) \leq \liminf_{n\to\infty} \gamma_\alpha \sum_{j=1,\ldots} \big(\text{diam}\,(E_{n,j} \cap f^{-1}(t))\big)^\alpha.$$

We have $\big(\operatorname{diam}(E_{n,j}\cap f^{-1}(t))\big)^\alpha \le (\operatorname{diam} E_{n,j})^\alpha \chi_{\overline{f(E_{n,j})}}(t)$, so by Fatou's lemma,

$$\int_{\mathbb{R}} \psi(t)\,dt \le \liminf_{n\to\infty}\ \gamma_\alpha \sum_{j=1,\dots} (\operatorname{diam} E_{n,j})^\alpha \int_{\mathbb{R}} \chi_{\overline{f(E_{n,j})}}(t)\,dt.$$

Now $\int_{\mathbb{R}} \chi_{\overline{f(E_{n,j})}}(t)\,dt$ is the measure of the set $\overline{f(E_{n,j})}$, which is not more than $\operatorname{diam} f(E_{n,j})$. Thus

$$\int_{\mathbb{R}} \psi(t)\,dt \le (\gamma_\alpha/\gamma_{\alpha+1})K\Lambda^{\alpha+1}(X).$$

The result now follows from the following lemma:

Lemma 3.3.7. *For $p \ge 0$, $\gamma_p \le \gamma_{p+1}$.*

Proof. We have that $\gamma_p = \frac{2^p\Gamma(\frac{p}{2}+1)}{\pi^{p/2}}$, so $\gamma_p/\gamma_{p+1} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+\frac{3}{2})}$. Now apply the formula

$$\int_0^{\pi/2} \cos^{m-1} t \sin^{n-1} t\,dt = \frac{1}{2}\frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2}+\frac{n}{2})}$$

with $m = p+2$ and $n = 1$. (The quantity on the right side of the preceding equation is the beta function $B(\frac{m}{2},\frac{n}{2})$.) The evaluation of the integral on the left is a problem in elementary calculus when m and n are integers. For real values of these parameters, the evaluation is more involved and can be found in [375, p. 256]. We find that for $p \ge 0$, $\gamma_p/\gamma_{p+1} = \int_0^{\pi/2}\cos^{p+1} t\,dt$. The integral is a decreasing function of p, and for $p = 0$, its value is one. Thus, $\gamma_p/\gamma_{p+1} \le 1$ as desired.

Often the applications we make of Theorem 3.3.6 will occur when $\alpha = 0$. The 0-dimensional measure of a set is simply the number of points in the set.

A much more general theorem, with correspondingly elaborate proof, is available:

Theorem 3.3.8. *If X and Y are metric spaces and if $f : X \to Y$ is a Lipschitz map, then for $A \subset X$ and for $0 \le k < \infty$ and $0 \le m < \infty$,*

$$\int_Y^* \Lambda^k(A\cap f^{-1}(y))\,d\Lambda^m(y) \le C(m,k)(\operatorname{Lip}(f))^m\Lambda^{k+m}(A).$$

The constant $C(m,k)$ is explicitly given and depends only on m and k. $\operatorname{Lip}(f)$ denotes the Lipschitz constant of f. In [115, p. 188], this result is stated with additional hypotheses on X or on Y. These hypotheses were subsequently shown to be unnecessary in the paper [94].

In the sequel we will need to have estimates for the size of the set of k-planes that have certain properties. It is to this kind of estimate that we now turn.

Denote by $\mathbb{G}_{N,k} = \mathbb{G}_{N,k}(\mathbb{C})$ the Grassmannian of k-dimensional complex linear subspaces of $\mathbb{C}^N$ for $k = 0, 1, \dots, N$. Thus, $\mathbb{G}_{N,1}$ is the projective space $\mathbb{P}^{N-1}(\mathbb{C})$. In a natural way these Grassmannians have the structure of compact complex manifolds; they can be realized as complex submanifolds of complex projective spaces of sufficiently high dimension. For details in this direction one can consult [155].

For us it is important to consider the Grassmannians as homogeneous spaces of the unitary group. For each k there is a natural action of the unitary group $U(N)$ on $\mathbb{G}_{N,k}$ defined by the condition that for $g \in U(N)$ and $L \in \mathbb{G}_{N,k}$, gL is the k-plane $g(L)$ obtained by moving L with the transformation g. Using this action, we can define a map $\pi : U(N) \to \mathbb{G}_{N,k}$. Let $L_o \in \mathbb{G}_{N,k}$ be the copy of $\mathbb{C}^k$ contained in $\mathbb{C}^N$ that is defined by the vanishing of the coordinate functions $z_{k+1}, \ldots, z_N$. The map π is defined by $\pi g = gL_o$. The stability group of L_o is, by definition, the subgroup $U(N; L_o)$ of $U(N)$ consisting of all the $g \in U(N)$ that carry L_o to itself. If we identify $U(N)$ with the space of $N \times N$ complex matrices $u = (a_{jk})_{j,k=1,\ldots,N}$ such that $u^{-1} = \bar{u}^T$, then the element $u \in U(N)$ carries L_o to itself when and only when it is of the block form $\begin{bmatrix} u' & 0 \\ 0 & u'' \end{bmatrix}$ with $u' \in U(k)$ and $u'' \in U(N-k)$. Thus, if we identify $U(k)$ with the subgroup of $U(N)$ consisting of the matrices of the form $\begin{bmatrix} u' & 0 \\ 0 & I \end{bmatrix}$ and $U(N-k)$ with the subgroup of $U(N)$ consisting of the matrices of the form $\begin{bmatrix} I & 0 \\ 0 & u'' \end{bmatrix}$ with u' and u'' as above, then we have an identification of $\mathbb{G}_{N,k}$ with the space of cosets $U(N)/(U(k) \oplus U(N-k))$.

Denote by μ the Haar measure on $U(N)$ normalized so that $U(N)$ has total measure one. Thus, μ is the unique regular Borel measure on $U(N)$ that is invariant under translation by elements of the group and that satisfies $\mu(U(N)) = 1$. The group of $N \times N$ unitary matrices is a submanifold of $\mathbb{C}^{N \times N}$ of (real) dimension N^2. The restriction to $U(N)$ of the N^2-dimensional Hausdorff measure on $\mathbb{C}^{N \times N}$ induced by the Euclidean metric agrees, to within a constant factor, with the Haar measure μ on $U(N)$. The problem of determining the constant is precisely the problem of determining $\Lambda^{N^2}(U(N))$. This can be done; see [317]. We shall not require the result.

The measure μ on $U(N)$ enables us to construct a measure ν_k on $\mathbb{G}_{N,k}$ by the condition that for a Borel set E in $\mathbb{G}_{N,k}$, $\nu_k(E) = \mu(\pi^{-1}(E))$. The measure ν_k has the property of being invariant under the action of $U(N)$ on $\mathbb{G}_{N,k}$. The measure ν_k also has total mass one.

With the natural measure ν_k on $\mathbb{G}_{N,k}$, we can discuss the measure of sets of k-planes that have various geometric properties. This kind of problem is genuinely interesting; we refer the reader to the encyclopedia volume of Santalò [317] for a systematic treatment of this and related topics. Our needs are modest; we start with the following estimate.

Lemma 3.3.9. *There are constants $c(N,k)$ such that the measure (with respect to ν_k) of the set of k-planes in $\mathbb{G}_{N,k}$ that meet the ball $\mathbb{B}_N(z,r)$ centered at $z \in \mathbb{C}^N$ and of radius r is bounded by $c(N,k)\left(\frac{r}{|z|}\right)^{2(N-k)}$.*

Proof. For $z \in \mathbb{C}^N$ and $r > 0$ our problem is to estimate the quantity $\nu_k(\mathscr{I}(z,r))$, where $\mathscr{I}(z,r) = \{L \in \mathbb{G}_{N,k} : L \cap \mathbb{B}_N(z,r) \neq \emptyset\}$. This is the same as estimating $\mu(\tilde{\mathscr{I}}(z,r))$, where $\tilde{\mathscr{I}}(z,r) = \{g \in U(N) : gL_o \cap \mathbb{B}_N(z,r) \neq \emptyset\}$. Without loss of generality, $z = \mathbf{c} = (c, 0, \ldots, 0)$ with $c > 0$. The k-plane $g(L_o)$ meets $\mathbb{B}_N(\mathbf{c}, r)$ if and only if $\operatorname{dist}(g(L_o), \mathbf{c}) < r$, which is the same as the condition that $\operatorname{dist}(L_o, g^{-1}(\mathbf{c})) < r$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be the usual orthonormal basis for $\mathbb{C}^N$ so that $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is an orthonormal basis for L_o, and $\{\mathbf{e}_{k+1}, \ldots, \mathbf{e}_N\}$ is an orthonormal basis for the orthogonal complement $L_o^\perp$ of L_o. If g is identified with the unitary matrix $(a_{j,k})_{1 \le j,k \le N}$, then g^{-1} is identified with the matrix $(\bar{a}_{k,j})_{1\le j,k\le N}$. Thus, with $\langle\, ,\rangle$ the Hermitian inner product on $\mathbb{C}^N$,

$$\operatorname{dist}(g^{-1}(\mathbf{c}), L_o) = \left(\sum_{s=k+1}^{N} |\langle g^{-1}(\mathbf{c}), \mathbf{e}_s\rangle|^2\right)^{1/2} = \left(\sum_{s=k+1}^{N} c^2|a_{1,s}|^2\right)^{1/2}.$$

We have therefore to estimate the Haar measure of the set $E(\mathbf{c}, r)$ of unitary matrices $(a_{j,k})_{1\le j,k\le N}$ with $\sum_{s=k+1}^{N} |a_{1,s}|^2 < \left(\frac{r}{c}\right)^2$. The set $E(\mathbf{c}, r)$ is open and so is measurable. For the estimate, introduce the map $\eta : U(N) \to b\mathbb{B}_N$ defined by

$$\eta((a_{j,k})_{1\le j,k\le N}) = (a_{1,1}, \dots, a_{1,N}).$$

Under η, the set $E(\mathbf{c}, r)$ is carried onto the set

$$S(c, r) = \left\{\zeta \in b\mathbb{B}_N : |\zeta_{k+1}|^2 + \cdots + |\zeta_N|^2 < \left(\frac{r}{c}\right)^2\right\}.$$

The set $S(c, r)$ is the $\frac{r}{c}$-neighborhood of the sphere

$$\mathbb{S}^{2k-1} = \{\zeta \in b\mathbb{B}_N : \zeta_{k+1} = \cdots = \zeta_N = 0\}.$$

One can compute the volume of this neighborhood explicitly. However, because $S(c, r)$ is the product of $\mathbb{S}^{2k-1}$ and a $2(N-k)$-dimensional ball of radius less than $2(\frac{r}{c})$, the volume of $S(c, r)$ is bounded by $\operatorname{const}(\frac{r}{c})^{2(N-k)}$ with the constant depending only on N and k. It now follows that $\mu(\tilde{\mathscr{I}}(\mathbf{c}, r)) < \operatorname{const}(\frac{r}{c})^{2(N-k)}$, and the lemma is proved.

It is possible to determine precisely the measure of the set of k-planes in $\mathbb{C}^N$ that meet a given ball; it is a polynomial of degree $N-k$ in $\left(\frac{r}{|z|}\right)^2$ if r is the radius of the ball and z the center. This determination is a rather involved calculation with invariant forms on $U(N)$ and a fibration process together with Fubini's theorem. It can be found in the paper [349].

The following theorem was obtained by Shiffman [326]. See also [342] and [60].

Theorem 3.3.10. *There exist constants k_N, $N = 2, \dots,$ such that for every set $E \subset \mathbb{C}^N \setminus \mathbb{B}_N$,*

$$\int^{*}_{\mathbb{G}_{N,N-k}} \Lambda^{\alpha}(E \cap L)\, d\nu_{N-k}(L) \le k_N \Lambda^{2k+\alpha}(E)$$

for $\alpha \ge 0$ and for $0 < k < N$.

Again, the integral is the upper integral, as it must be since we permit the set E to be arbitrary.

Proof. As a preliminary, notice that for each set $S \subset \mathbb{C}^N \setminus \mathbb{B}_N$,

$$\int^{*}_{\mathbb{G}_{N,N-k}} (\operatorname{diam}(S \cap L))^{\alpha}\, d\nu_{N-k}(L) \le a_N (\operatorname{diam} S)^{2k+\alpha}$$

for a suitable constant a_N. This is so, for

$$\int^*_{\mathbb{G}_{N,N-k}} (\operatorname{diam}(S \cap L))^\alpha \, d\nu_{N-k}(L) \le (\operatorname{diam} S)^\alpha \nu_{N-k}(\{L \in \mathbb{G}_{N,N-k} : L \cap S \neq \emptyset\}).$$

The set S is contained in a ball of diameter twice the diameter of S, so the estimate in the preceding lemma yields

$$\int^*_{\mathbb{G}_{N,N-k}} (\operatorname{diam}(S \cap L)^\alpha \, d\nu_{N-k}(L) \le a_N \operatorname{diam} S^\alpha (\operatorname{diam} S)^{2k} = a_N (\operatorname{diam} S)^{2k+\alpha}.$$

Now to prove the lemma, use the set functions Λ^p_ε defined by

$$\Lambda^p_\varepsilon(S) = \inf \left\{ \gamma_p \sum_{j=1,\dots} (\operatorname{diam} S_j)^p : S \subset \cup_{j=1,\dots} S_j, \operatorname{diam} S_j < \varepsilon \right\},$$

so that $\Lambda^p(S) = \lim_{\varepsilon \to 0^+} \Lambda^p_\varepsilon(S) = \sup \Lambda^p_\varepsilon(S)$.

Consider a covering of the set E of the lemma by subsets S_j of $\mathbb{C}^N \setminus \mathbb{B}_N$ of diameter not more than ε such that

$$\gamma_{2k+\alpha} \sum_{j=1,\dots} (\operatorname{diam} S_j)^{2k+\alpha} \le \Lambda^{2k+\alpha}_\varepsilon(E) + \varepsilon.$$

We have, for suitable constants k'_N and k_N, that

$$\begin{aligned} \int^*_{\mathbb{G}_{N,N-k}} \Lambda^\alpha_\varepsilon(E \cap L) \, d\nu_{N-k}(L) &\le \sum_j \int^*_{\mathbb{G}_{N,N-k}} (\operatorname{diam}(S_j \cap L))^\alpha \, d\nu_{N-k} \\ &\le \sum_j (\operatorname{diam} S_j)^\alpha \nu_{N-k}(\{L \in \mathbb{G}_{N,N-k} : L \cap S_j) \neq \emptyset\} \\ &\le k'_N \sum_j (\operatorname{diam} S_j)^{2k+\alpha} \\ &\le k_N \Lambda^{2k+\alpha}(E) + \varepsilon. \end{aligned}$$

These inequalities are correct for every choice of ε, so the theorem is proved.

3.4. Function-Theoretic Preliminaries

In this section we assemble in a rather summary way certain function-theoretic results that are needed in the development below.

To begin with, we collect some results from H^p-theory. The definition is this.

Definition 3.4.1. *The function f holomorphic on the unit disk $\mathbb{U}$ belongs to the pth Hardy space on the disk, $f \in H^p(\mathbb{U})$, $0 < p < \infty$, if the quantity*

$$\|f\|_p = \lim_{r \to 1^-} \left\{ \int_{-\pi}^{\pi} |f(re^{i\vartheta})|^p \, d\vartheta \right\}^{1/p}$$

is finite.

By definition, $H^\infty(\mathbb{U})$ is the space of all bounded holomorphic functions on $\mathbb{U}$.

In the range $1 \leq p < \infty$, the functional $\| \cdot \|_p$ is a norm on $H^p(\mathbb{U})$ with respect to which this space is a Banach space. The norm on $H^\infty(\mathbb{U})$ is $\|f\|_\infty = \sup_{z \in \mathbb{U}} |f(z)|$; with respect to this norm, $H^\infty(\mathbb{U})$ is a commutative Banach algebra.

For the elementary parts of the theory of these Hardy spaces one can consult [311]. More extensive treatments are given in [140], [151], and [178].

We shall frequently need the result that for $f \in H^p(\mathbb{U})$, $0 < p \leq \infty$, the radial and nontangential limits $\lim_{r \to 1^-} f(re^{i\vartheta}) = f^*(e^{i\vartheta})$ exist for almost all ϑ. Moreover, for $0 < p < \infty$, if f_r is defined by $f_r(e^{i\vartheta}) = f(re^{i\vartheta})$, then the functions f_r converge, as $r \to 1^-$, in the L^p sense to the function f^*.

For a function $f \in H^p(\mathbb{U})$, the boundary function f^* is logarithmically integrable, as follows from *Jensen's inequality*

$$\log|f(0)| \leq \int_{-\pi}^{\pi} \log|f^*(e^{i\vartheta})| \, d\vartheta.$$

Thus for an $f \in H^p(\mathbb{U})$, $0 < p \leq \infty$, f not identically zero, the boundary function f^* vanishes only on a set of measure zero.

In discussing conformal maps we shall frequently encounter holomorphic functions with derivative in $H^1(\mathbb{U})$.

Theorem 3.4.2. [291] *If $f \in \mathscr{O}(\mathbb{U})$ and $f' \in H^1(\mathbb{U})$, then f extends continuously to $\bar{\mathbb{U}}$, and $f|b\mathbb{U}$ is absolutely continuous and of bounded variation. The total variation,* $\operatorname{var}_{b\mathbb{U}} f$, *of f around the boundary of the disk is given by*

$$\operatorname{var}_{b\mathbb{U}} f = \int_{-\pi}^{\pi} |f'(e^{i\vartheta})| \, d\vartheta.$$

Proof. The function f' belongs to $H^1(\mathbb{U})$, so the radial limits $f'(e^{i\vartheta})$ exist for almost all $\vartheta \in \mathbb{R}$. Fix a ϑ_o for which this limit exists. For $r \in (0, 1)$,

$$f(re^{i\vartheta}) = f(re^{i\vartheta_o}) + ir \int_{\vartheta_o}^{\vartheta} e^{is} f'(re^{is}) \, ds,$$

so for $r, r' \in (0, 1)$ we have

$$\begin{aligned} |f(re^{i\vartheta}) - f(r'e^{i\vartheta})| &\leq |f(re^{i\vartheta_o}) - f(r'e^{i\vartheta_o})| + \left| ir \int_{\vartheta_o}^{\vartheta} e^{is} \left[f'(re^{is}) - f'(r'e^{is}) \right] ds \right| \\ &\leq |f(re^{i\vartheta_o}) - f(r'e^{i\vartheta_o})| + r \int_{-\pi}^{\pi} |f'(re^{is}) - f'(r'e^{is})| \, ds. \end{aligned}$$

From this we find that $|f(re^{i\vartheta}) - f(r'e^{i\vartheta})| \to 0$ uniformly in ϑ as $r, r' \to 1^-$. Thus, f extends continuously to the closure of $\mathbb{U}$. Moreover,

$$f(e^{i\vartheta}) = f(e^{i\vartheta_o}) + i\int_{\vartheta_o}^{\vartheta} f'(e^{is})\,ds, \tag{3.2}$$

which implies that $f|b\mathbb{U}$ is absolutely continuous and of bounded variation.

To verify the formula for the variation of f on $b\mathbb{U}$, begin by recalling that

$$\operatorname{var}_{b\mathbb{U}} f = \sup \sum_{j=0}^{n-1} |f(e^{i\vartheta_{j+1}}) - f(e^{i\vartheta_j})|,$$

in which the supremum is extended over all $n = 1, \ldots$ and over all choices of ϑ_j with $0 = \vartheta_0 < \cdots < \vartheta_n = 2\pi$.

Fix $\varepsilon > 0$. For $n = 1, \ldots,$ there are positive numbers r_n that increase to 1 and numbers $z_{n,k}$, $k = 1, \ldots, K(n)$, with $|z_{n,k}| = r_n$ and with the property that for fixed n, the numbers $z_{n,k}$ are distributed cyclically around the circle $|z| = r_n$. We require, moreover, that

$$\operatorname{var}_{b\mathbb{U}} f \geq \sum_{k=1}^{K(n)-1} \left| f\left(\frac{z_{n,k+1}}{r_n}\right) - f\left(\frac{z_{n,k}}{r_n}\right) \right| > \operatorname{var}_{b\mathbb{U}} f - \varepsilon,$$

that, with L_n the length of the curve $f(\{|z| = r_n\})$,

$$L_n \geq \sum_{k=1}^{K(n)-1} |f(z_{n,k+1}) - f(z_{n,k})| > L_n - \varepsilon,$$

and that

$$\left| \sum_{k=1}^{K(n)-1} |f(z_{n,k+1}) - f(z_{n,k})| - \sum_{k=1}^{K(n)-1} \left| f\left(\frac{z_{n,k+1}}{r_n}\right) - f\left(\frac{z_{n,k}}{r_n}\right) \right| \right| < \varepsilon.$$

With these choices, we find that $\operatorname{var}_{\mathbb{U}} f \geq L_n - 3\varepsilon$, i.e., that

$$\operatorname{var}_{b\mathbb{U}} f \geq \int_{-\pi}^{\pi} |f'(r_n e^{i\vartheta})|\,d\vartheta - 3\varepsilon.$$

Let $n \to \infty$ and then $\varepsilon \to 0^+$ to find that

$$\operatorname{var}_{b\mathbb{U}} f \geq \int_{-\pi}^{\pi} |f'(e^{i\vartheta})|\,d\vartheta.$$

The reverse inequality is simpler:

$$\sum_{j=1}^{n-1} |f(e^{i\vartheta_{j+1}}) - f(e^{i\vartheta_j})| \leq \int_{-\pi}^{\pi} |f'(e^{i\vartheta})|\,d\vartheta,$$

whence

$$\operatorname{var}_{b\mathbb{U}} f \leq \int_{-\pi}^{\pi} |f'(e^{i\vartheta})|\, d\vartheta.$$

Consequently, the two quantities are equal.

Corollary 3.4.3. *With the notation of the preceding theorem,* $\frac{d}{d\vartheta} f(e^{i\vartheta}) = ie^{i\vartheta} f'(e^{i\vartheta})$.

Proof. Differentiate both sides of the equality (3.2).

This corollary implies that $\frac{d}{d\vartheta} f(e^{i\vartheta})$ vanishes at most on a set of measure zero.

For absolutely continuous functions and functions of bounded variation, one can consult [311].

A further result in the direction of Theorem 3.4.2 is this:

Theorem 3.4.4. [282] *If* f *is holomorphic on* $\mathbb{U}$ *and* $f' \in H^1(\mathbb{U})$*, then for* $E \subset b\mathbb{U}$*,* $\Lambda^1(E) = 0$ *if and only if* $\Lambda^1(f(E)) = 0$.

Proof. By the preceding theorem, f is absolutely continuous on $b\mathbb{U}$, so $\Lambda^1(E) = 0$ implies $\Lambda^1(f(E)) = 0$.

The proof in the opposite direction runs as follows. First, by regularity, it is enough to deal with sets E that are compact. We have that for almost all ϑ, $ie^{i\vartheta} f'(e^{i\vartheta}) = \frac{d}{d\vartheta} f(e^{i\vartheta})$, so if $f = u + iv$, then both u and v are absolutely continuous, and we can suppose that the quantity

$$\frac{d}{d\vartheta} u(e^{i\vartheta}) = -\Im(e^{i\vartheta} f'(e^{i\vartheta}))$$

is not zero at any point of E. (If this cannot be achieved with the initially given f, replace f by if.) Let $F = \{\vartheta \in [0, 2\pi] : e^{i\vartheta} \in E\}$, a compact set. Denote the characteristic function of F by χ_F. The functions $|u'|$ and $|\chi_F u'|$ are both integrable over $[0, 2\pi]$, so almost every point of $[0, 2\pi]$ is a Lebesgue point for each of them: For almost every ϑ,

$$\lim_{t\to 0} \frac{1}{t} \int_{\vartheta}^{\vartheta+t} |u'(t)|\, dt = |u'(\vartheta)|,$$

and

$$\lim_{t\to 0} \frac{1}{t} \int_{\vartheta}^{\vartheta+t} |\chi_F(t) u'(t)|\, dt = |\chi_F(\vartheta) u'(\vartheta)|.$$

Fix a $\vartheta_o \in F$ with $u'(e^{i\vartheta_o}) \neq 0$. Then

$$\lim_{t\to 0^+} \frac{1}{t} \int_{(\vartheta_o, \vartheta_o+t)\setminus F} |u'(t)|\, dt = \lim_{t\to 0^+} \frac{1}{t} \int_{\vartheta_o}^{\vartheta_o+t} |u'(t)|(1 - \chi_F(t))\, dt = 0.$$

The set $(\vartheta_o, \vartheta_o + t) \setminus F$ is a disjoint union of open intervals, so the last equations imply that

$$\Lambda^1(u((\vartheta_o, \vartheta_o + t) \setminus F)) = o(t) \text{ when } t \to 0^+.$$

Also, the function u is continuous and satisfies $u(\vartheta_o + t) - u(\vartheta_o) = u'(\vartheta_o) + o(t)$, so the interval $u((\vartheta_o, \vartheta_o + t))$ is of length $t|u'(\vartheta_o)| + o(t)$. Consequently, for small positive t,

$$\begin{aligned}\Lambda^1(u((\vartheta_o, \vartheta_o + t) \cap F)) &\geq \Lambda^1(u((\vartheta_o, \vartheta_o + t)) - \Lambda^1(u((\vartheta_o, \vartheta_o + t) \setminus F)) \\ &> t|u'(\vartheta_o)| - o(t).\end{aligned}$$

We have therefore that

$$\Lambda^1(f(E)) \geq \Lambda^1(\Re f(E)) = \Lambda^1(u(F)) > 0.$$

The theorem is proved.

From conformal mapping theory we shall need to use the Riemann mapping theorem and certain of its extensions. According to this theorem, *a simply connected domain in the plane with nonempty boundary (with respect to the plane) is biholomorphically equivalent to the open unit disk.* A Riemann map $\chi : \mathbb{U} \to D$ is uniquely determined by the value $\chi(0)$ and the value of Arg $\chi'(0)$. When the domain in question is bounded by a simple closed curve, there is an extension of the conformal map to the boundary due to Carathéodory.

Theorem 3.4.5. *If D is a bounded simply connected domain in $\mathbb{C}$ with bD a simple closed curve, and if $\chi : \mathbb{U} \to D$ is a conformal mapping as provided by the Riemann mapping theorem, then χ extends to a homeomorphism from $\bar{\mathbb{U}}$ to $\bar{D}$.*

For planar domains there are various elementary characterizations of simple connectivity. The definition of simple connectivity is that the fundamental group vanishes. For a domain D in $\mathbb{C}$, this is equivalent to the condition that the complement of D *in the Riemann sphere* be connected. Alternatively, the condition is equivalent to the condition that each zero-free continuous (or holomorphic) function have a continuous (or holomorphic) square root on D, which, in turn, is equivalent to the condition that each such function have a continuous (or holomorphic) logarithm on D. These various elementary notions of simple connectivity are discussed in detail in several modern books on the theory of functions of a complex variable. See, e.g., [311]. For a thoroughgoing discussion from the point of view of set-theoretic topology, see Newman's book [263].

A general principle is that the smoother the boundary of the domain D in Theorem 3.4.2, the smoother the boundary values of the Riemann mapping function χ. This is not a subject we shall pursue in detail, but we do need information about the case of rectifiable boundaries. The classical result in this case is due to F. and M. Riesz [292].

Theorem 3.4.6. *If D is a bounded simply connected domain in $\mathbb{C}$ and if bD is a rectifiable simple closed curve, then the derivative of a Riemann mapping function $\chi : \mathbb{U} \to D$ is in the Hardy space $H^1(\mathbb{U})$.*

Proof. The proof of this is quite short.

For $n = 1, 2, \ldots$, let $\eta_n = e^{i\frac{2\pi}{n}}$.

According to Carathéodory's extension of the Riemann mapping theorem, the function f extends continuously to $\bar{\mathbb{U}}$. Define $F_n : \bar{\mathbb{U}} \to \mathbb{R}$ by

$$F_n(z) = \sum_{j=1}^{n} |f(z\eta_n^{j-1}) - f(z\eta_n^j)|.$$

As a sum of subharmonic functions on $\mathbb{U}$, the function F_n attains its maximum on the boundary of the disk: There is $z_{n,o} \in b\mathbb{U}$ with $F_n(z) \leq F(z_{n,o})$ for all $z \in b\mathbb{U}$. The value $F(z_{n,o})$ is the length of a polygon inscribed in the rectifiable simple closed curve bD; it does not exceed the length, $\ell(bD)$, of bD. Thus, uniformly in n and z, $F_n(z) \leq \ell(bD)$. In particular, for $r \in (0, 1)$, $F_n(r) \leq \ell(bD)$. For fixed r, $F_n(r)$ is the length of a polygon inscribed in the simple closed curve $\{f(re^{\vartheta}) : \vartheta \in \mathbb{R}\}$, whose length is $\int_{\pi}^{\pi} |f'(re^{i\vartheta})|\, d\vartheta$. Because

$$\lim_{r \to 1^-} F_n(r) = \int_{-\pi}^{\pi} |f'(re^{i\vartheta})|\, d\vartheta,$$

the derivative f' is in the Hardy space $H^1(\mathbb{U})$ as claimed.

It is useful to notice that the proof just given yields another result:

Corollary 3.4.7. *If f is holomorphic on the unit disk and continuous on its boundary, and if f has bounded variation on the boundary of the disk, then $f' \in H^1(\mathbb{U})$.*

Another result on conformal mappings, which is a natural extension of the last theorem, is the following. See [282].

Theorem 3.4.8. *Let D be a bounded simply connected domain in the plane, and let $\chi : \mathbb{U} \to D$ be a Riemann map. The derivative χ' is in $H^1(\mathbb{U})$ if and only if the boundary of D has finite one-dimensional Hausdorff measure.*

Proof. If $f' \in H^1(\mathbb{U})$, then by Theorem 3.4.2 the function extends continuously to $b\mathbb{U}$ and is absolutely continuous and of bounded variation, which implies that bD has finite length.

The converse depends on the Riesz theorem above, Theorem 3.4.6. Assume that bD has finite length. Suppose, as we may with no loss of generality, that $0 \in D$, that $f(0) = 0$, and that $f'(0) > 0$.

Let K be a positive number greater than four times the length of bD. For $n = 1, \ldots,$ let $\Delta_{n,1}, \ldots, \Delta_{n,\nu(n)}$ be finitely many disks of radius not more than $1/n$ whose union, call it $\Delta(n)$, covers bD. Require also that the sum of the radii of $\Delta_{n,j}$, for fixed n, be not more than K. This is possible by the definition of the one-dimensional Hausdorff measure and the relation between Hausdorff measure and spherical measure. (Recall the remarks in Section 1.6.) The domain D is simply connected and bounded, so its boundary is connected. We can, therefore, assume $\Delta(n)$ to be connected, which implies that each bounded component of its complement is simply connected. Let D_n be the component of $D \setminus \Delta(n)$ that contains the origin. The $\Delta_{n,j}$'s can be chosen small enough that for each n, D_n is a relatively compact subset of D_{n+1}.

The boundary of the domain D_n consists of the union of a finite number of circular arcs. We modify D_n to obtain a new domain $D'_n \subset D_n$ as follows. For each n, let the finitely many points in bD_n at which two or more circular arcs meet be denoted by $v_{n,1}, \ldots, v_{n,\mu(n)}$. For each $j = 1, \ldots, \mu(n)$, let W_k be a small closed disk centered at $v_{n,j}$, and let $D'_n = D_n \setminus \cup_{j=1,\ldots,\mu(n)} W_j$. The W's are to be very small; in particular, W's corresponding to different subscripts are to be disjoint. The domain D'_n is a subdomain of D_n. If at each stage the W's are small enough, then D'_n is a relatively compact subdomain of D'_{n+1}. Moreover, its boundary is a simple closed curve that is the union of finitely many circular arcs. Its length is less than K, provided we make the W's small enough.

For each n let g_n be the conformal map from $\mathbb{U}$ to D'_n with $g_n(0) = 0$ and $g'_n(0) > 0$. The g_n's constitute a normal family; suppose $g_n \to g$ uniformly on compacta in $\mathbb{U}$. Then

$g(0) = 0$ and $g'(0) \geq 0$. The function g maps $\mathbb{U}$ onto D. This is so, for if E is a compact subset of D, then for $r \in (0, 1)$ sufficiently large, $E \subset g_n(r\mathbb{U})$ for all sufficiently large n. Thus, the image $g(rb\mathbb{U})$ of the circle $rb\mathbb{U}$ surrounds E, whence $g(\mathbb{U}) \supset E$. Not only is g surjective, it is injective: If $g(z) = g(z')$ for distinct points $z, z' \in \mathbb{U}$, then for sufficiently large values of n there will be points w_n near z and w_n' near z' with $g_n(w_n) = g_n(w_n')$, contradicting the injectivity of g_n.

The function g is therefore the conformal map f.

For $r \in (0, 1)$ let $\ell_n(r)$ be the length of the curve $g_n(rb\mathbb{U})$:

$$\ell_n(r) = r \int_{-\pi}^{\pi} |g_n'(re^{i\vartheta})| \, d\vartheta.$$

For fixed n, the quantity $\ell_n(r)$ increases with r to $\ell_n(1)$, which is the length of bD_n'. Therefore for all n and all r, $\ell_n(r) < K$. On the other hand, for fixed r we have

$$\lim_{n\to\infty} \ell_n(r) = \lim_{n\to\infty} r \int_{-\pi}^{\pi} |g_n'(re^{i\vartheta})| \, d\vartheta \to \int_{-\pi}^{\pi} |f'(re^{i\vartheta})| \, d\vartheta.$$

Thus $f' \in H^1(\mathbb{U})$, as we wished to show.

We shall need below the notion of sets of uniqueness for subharmonic functions:

Definition 3.4.9. *Let D be a domain in the plane. The subset E of $\bar{D}$ is a* set of uniqueness for subharmonic functions on D *if the function identically $-\infty$ is the only subharmonic function u on D such that* $\lim_{z\to e,\, z\in D} u(z) = -\infty$ *for every $e \in E$.*

Lemma 3.4.10. *If $E \subset \bar{D}$ is a set of uniqueness for subharmonic functions on D and if $E = \cup_{j=1,\ldots} E_j$, then one of the sets E_j is a set of uniqueness for subharmonic functions on D.*

Proof. If not, then for each $j = 1, \ldots$ there is a subharmonic function $\tilde{u}_j$ on D such that $\lim_{z\to e} \tilde{u}_j(z) = -\infty$ for each $e \in E_j$. Define u_j to be $\max\{\tilde{u}_j, 0\}$, which is also a subharmonic function. If $u = \sum_{j=1,\ldots} c_j u_j$ for a sufficiently rapidly decreasing sequence $\{c_j\}_{j=1,\ldots}$ of positive numbers, then u is subharmonic on D and has limit $-\infty$ at each point of E, contradicting the assumption that E is a set of uniqueness.

The next lemma provides an important class of examples.

Lemma 3.4.11. *Let D be a bounded simply connected domain in $\mathbb{C}$ such that the boundary bD has finite length. If E is a subset of bD that has positive length, then E is a set of uniqueness for subharmonic functions on D.*

Proof. The domain D is simply connected, so there is a Riemann map $\chi : \mathbb{U} \to D$. Theorem 3.4.8 implies that $\chi' \in H^1(\mathbb{U})$ and that $\chi|b\mathbb{U}$ is absolutely continuous. Consequently, χ maps subsets of $b\mathbb{U}$ that have zero measure into subsets of bD of zero length.

The set $E \subset bD$ has positive length, so the set $\chi^{-1}(E) = E^* \subset b\mathbb{U}$ has positive measure.

Let u be a subharmonic function on D such that $\lim_{z\to e, z\in D} u(z) = -\infty$ for all $e \in E$. The subharmonic function $u \circ \chi$ satisfies $u \circ \chi(z) \to -\infty$ as $z \in \mathbb{U}$ tends to

E^*. Because E^* has positive measure, this implies that $u \circ \chi \equiv -\infty$. Thus, E is a set of uniqueness, as required.

We shall also need a theorem of Lindelöf [224] about the boundary behavior of holomorphic functions:

Theorem 3.4.12. *Let $\gamma : [0, 1] \to \mathbb{U} \cup \{1\}$ be an arc with $\gamma(1) = 1$. If f is a bounded holomorphic function on $\mathbb{U}$ such that the limit $\lim_{t\to 1^-} f(\gamma(t))$ exists, call it L, then the radial limit $f^*(1) = \lim_{t\to 1^-} f(t)$ exists and is L.*

Recall that the existence of the radial limit for a bounded holomorphic function on $\mathbb{U}$ implies the existence of the nontangential limit.

There is no assumption in this theorem about how $f(\gamma(t))$ approaches the point L as $t \to 1^-$. The approach could be highly tangential or oscillatory.

Proof. It entails no loss of generality to suppose that the limit L is 0 and that $\|f\|_{\mathbb{U}} \leq 1$. Introduce the function $h \in H^\infty(\mathbb{U})$ by $h(z) = f(z)\bar{f}(\bar{z})$. We shall show that h has radial limit 0 at 1.

Suppose first that $\Im\gamma(t) > 0$ on an interval $(t_o, 1)$ for some $t_o \in (0, 1)$. Let Δ be the domain in the plane bounded above by the curve $\gamma|[t_o, 1)$, bounded below by the curve $\bar{\gamma}(t)|[t_o, 1]$, and bounded on the left by the vertical segment—which may be a single point—connecting $\gamma(t_o)$ to $\bar{\gamma}(t_o)$. The domain Δ is simply connected and is bounded by a simple closed curve. Let $\varphi : \mathbb{U} \to \Delta$ be a Riemann map. The map φ extends to a homeomorphism, still denoted by φ, of $\bar{\mathbb{U}}$ onto $\bar{\Delta}$. Without loss of generality, $\varphi(1) = 1$. Then the function $h \circ \varphi$ is bounded and holomorphic on $\mathbb{U}$, and it has continuous boundary values. Therefore $h \circ \varphi$ is continuous on $\bar{\mathbb{U}}$. (This is so, for by Fejér's theorem the Fourier series of $h \circ \varphi$ is uniformly Cesàro summable to $h \circ \varphi$. This provides a sequence of holomorphic polynomials that converges to $h \circ \varphi$ uniformly on $b\mathbb{U}$, whence $h \circ \varphi$ is in $A(\mathbb{U})$. Alternatively $h \circ \varphi$ is the Poisson integral of its continuous boundary values and so is continuous on the closed disk.) The function $h \circ \varphi$ is continuous on $\bar{\mathbb{U}}$, and φ is a homeomorphism on $\bar{\mathbb{U}}$, so the function h is continuous on $\bar{\Delta}$. Consequently, the radial limit $f^*(1)$ exists and is 0.

Consider now the case of a general arc γ. Suppose without loss of generality that $\gamma(0) = 0$. If $\gamma([0, 1])$ contains an interval $(t_o, 1)$ for some $t_o < 1$, we are done. In the contrary case, let $a_j, b_j \in (0, 1)$ be the points such that $[0, 1] \setminus \gamma([0, 1])$ is the union of the open intervals (a_j, b_j). For each j, let $a_j = \gamma(\alpha_j)$ and $b_j = \gamma(\beta_j)$. Thus, for each j, $\Im\gamma > 0$ on (α_j, β_j) or else $\Im\gamma < 0$ on this interval. In either case, we denote by Δ_j the relatively compact subdomain in $\mathbb{U}$ that contains the interval (a_j, b_j) and that is bounded by the simple closed curve that is the union of the two arcs $\gamma([\alpha_j, \beta_j])$ and $\bar{\gamma}([\alpha_j, \beta_j])$. The function h is bounded on Δ_j by $\max_{t\in[a_j,b_j]} |f(\gamma(t))|^2$. The function h therefore has the radial limit 0 at the point 1, for $f(\gamma(t)) \to 0$ when $t \to 1^-$. The proof is complete.

There is a result on the extension of a bounded holomorphic function to the closure of the disk that is based on a metric restriction of the total cluster set of the function at the boundary.

If f is a complex-valued continuous function defined on a domain D in $\mathbb{C}$, and if $z \in bD$, *then the cluster set of f at z*, denoted by $\mathscr{C}_z(f)$, is the set of all limits $\lim_{n\to\infty} f(z_n)$

along sequences $\{z_n\}_{n=1,\ldots}$ in D that converge to the point z. If f is bounded, its cluster set $\mathscr{C}_z(f)$ is a compact subset of the plane; in general, it is a compact subset of the Riemann sphere. If $\mathscr{C}_z(f)$ is a single point, then f extends continuously to z. There is a global notion of cluster set. If f is defined on the domain D in $\mathbb{C}$, the *total cluster set of f at bD*, denoted by $\mathscr{C}_{bD}(f)$, is the set of points $w \in \mathbb{C}$ that are limits $\lim_{n\to\infty} f(z_n)$ for some sequence $\{z_n\}_{n=1,\ldots}$ in D that approaches bD. If f is bounded, this is a compact subset of $\mathbb{C}$; if bD is connected, it is a connected set.

A theorem we shall need in the sequel was proved by Alexander [16] and, independently, by Pommerenke [283].

Theorem 3.4.13. *If f is a holomorphic function defined on $\mathbb{U}$ and if its total cluster set at $b\mathbb{U}$ has finite one-dimensional Hausdorff measure, then f extends continuously to $\bar{\mathbb{U}}$.*

Proof. [16] We shall show that for a fixed $z \in b\mathbb{U}$ the cluster set $\mathscr{C}_z(f)$ is a singleton, so that f extends continuously to the point z.

As a connected set of finite length, the global total cluster set $\mathscr{C}_{b\mathbb{U}}(f)$ is bounded, so the function f lies in $H^\infty(\mathbb{U})$. If f is not constant, then the set $\mathscr{C}_{b\mathbb{U}}(f)$ contains more than one point. Consequently, its projection on the x-axis is an interval or else its projection on the y-axis is an interval. Suppose $\mathscr{C}_{b\mathbb{U}}(f)$ projects onto the interval $[a, b]$, $a < b$, in the x-axis.

The hypothesis that $\mathscr{C}_{b\mathbb{U}}(f)$ has finite length implies that for almost all $c \in [a, b]$, the line L_c through c and perpendicular to the x-axis meets $\mathscr{C}_{b\mathbb{U}}(f)$ in a finite set. (Recall Theorem 3.3.6.) Fix $c \in (a, b)$ such that $L_c \cap \mathscr{C}_{b\mathbb{U}}(f)$ is a finite set, say $E = \{p_1, \ldots, p_n\}$, and such that f' vanishes at no point of $f^{-1}(L_c)$. The latter condition excludes at most countably infinitely many choices for c. We assume the points p_j to be indexed such that $\Im p_j < \Im p_{j+1}$.

For $j = 1, \ldots, n$, let λ_j^+ and λ_j^- be the open intervals in $L_c \setminus E$ that abut the point p_j from above and below, respectively. Thus $\lambda_j^+ = \lambda_{j+1}^-$.

Notice that if Ω is a component of $\mathbb{C} \setminus \mathscr{C}_{b\mathbb{U}}(f)$, then $f : f^{-1}(\Omega) \to \Omega$ is a proper holomorphic map, and each point in Ω is covered by f the same number of times as every other point, due account being taken of multiplicities. Consequently, for each j, there are multiplicities $m_j^\pm$ such that each point of $\lambda_j^\pm$ has $m_j^\pm$ preimages in $\mathbb{U}$ under the map f. (Some of these multiplicities may be zero.) We see now that each of the points p_j has at most finitely many preimages in $\mathbb{U}$ under f, for if $M_j = \max\{m_j^+, m_j^-\}$, and if $f^{-1}(p_j)$ contained more than M_j points, then because f' vanishes at none of these preimages, some points of λ_j^+ or λ_j^- would be covered more than M_j times, a contradiction. Denote by $\tilde{E}$ the set $f^{-1}(E) = f^{-1}(\{p_1, \ldots, p_n\})$, a finite subset of $\mathbb{U}$.

Fix j and let λ be a nonempty component of $f^{-1}(\lambda_j^+)$. This is a certain open arc in $\mathbb{U}$, for no critical value of f lies in the line L_c. Let S' and S'' be the cluster sets of λ at its two ends. These are compact, connected subsets of $\bar{\mathbb{U}}$. Initially there is no evident reason for them not to intersect or, perhaps, to coincide. Consider a $q \in S' \cap \mathbb{U}$. We claim that q lies in the set $\tilde{E}$. If not, then $f(q) \in \lambda_j^+$, which implies that $q \in \lambda$. This is impossible, for λ is an open arc carried by f injectively onto λ_j^+.

Thus each of the sets S' and S'' consists of one of the points of $\tilde{E}$ or else of a connected subset of $b\mathbb{U}$. If S' is a connected subset of $b\mathbb{U}$, then necessarily S' is a point. Otherwise, it is an arc at almost every point of which the bounded holomorphic function f has nontangential limit f^*. The geometry of the situation implies that these boundary limits can be only the points p_j or p_{j+1}. Thus f^* assumes some value on a set of positive measure in $b\mathbb{U}$. It is therefore a constant. This is a contradiction, so each of S' and S'' is a point, and $\bar{\lambda}$ is an arc in $\bar{\mathbb{U}}$.

For some choice of j, one endpoint of one of the λ's generated by the process above must be the point z. Otherwise, if W is the part of $\mathbb{U}$ in a disk of small radius centered at z, then $f(W)$ is a connected open set that can meet L_c only at the points p_j and so cannot meet it at all. That is, $f(W)$ lies on one side of the line L_c. This implies that the whole cluster set $\mathscr{C}_{b\mathbb{U}}(f)$ lies on one side of L_c, in contradiction to the way we have chosen c.

Thus we have produced a curve, say γ_c, in $\mathbb{U}$ that terminates at z and along which f has a limit, which lies in the line L_c.

We now repeat this whole construction with c replaced by a $c' \in (a, c)$. We get a new curve $\gamma_{c'}$ in $\mathbb{U}$ that terminates at z and along which the function f has a limit, which lies in the line $L_{c'}$.

We have reached a contradiction to the theorem of Lindelöf given above, Theorem 3.4.12. The theorem is proved.

This proof is very much in the geometric spirit of the present book. The proof given by Pommerenke is completely different.

Note that in this theorem there is no hypothesis that the function f is univalent.

The theorem does imply a result about conformal mapping: *If D is a simply connected domain with boundary of finite length, then each Riemann map from $\mathbb{U}$ to D extends continuously to $\bar{\mathbb{U}}$.* This result is also a corollary of Theorem 3.4.8.

To continue, we need a theorem of R.L. Moore [251] from plane topology.

Define a *triode* to be a compact space that is the union of three arcs that are disjoint except that there is one point x common to all of them, and it is an endpoint of each of them. Thus a triode is homeomorphic to the letter Y. The point x is called the *emanation point* of the triode, and the arcs that define the triode are called the *rays* of the triode.

Theorem 3.4.14. *A family of mutually disjoint triodes in the plane is countable.*

Moore proves a more general theorem: He uses a more general notion of triode—the arcs are replaced by irreducible continua, and his conclusion is stronger. The stated theorem suffices for our purposes.

The proof of this theorem depends on a lemma:

Lemma 3.4.15. *Let γ_1, γ_2, and γ_3 be disjoint arcs in the unit circle. Let T' and T'' be triodes contained in the closed unit disk, say $T' = \lambda_1' \cup \lambda_2' \cup \lambda_3'$ and $T'' = \lambda_1'' \cup \lambda_2'' \cup \lambda_3''$. Let the emanation points x' and x'' of T' and T'', respectively, lie in $\mathbb{U}$. Assume that one endpoint of each of λ_1' and λ_1'' to lie on γ_1, one endpoint of each of λ_2' and λ_2'' to lie on γ_2, and one endpoint of each of λ_3' and λ_3'' to lie on γ_3. Then T' and T'' have a point in common.*

Proof. Draw a picture! An analytic proof involves a careful argument based on the Jordan curve theorem, which we omit.

Proof of the theorem. Suppose $\mathscr{G}$ to be an uncountable family of mutually disjoint triodes in the plane. Without loss of generality, we can suppose that the diameter of each arc in each triode in $\mathscr{G}$ is at least 2. The set $\mathscr{G}$ is uncountable, so there is a point, say the origin, that is a cluster point for the set of emanation points for the triodes in $\mathscr{G}$ in the sense that every neighborhood of the origin contains the emanation point of uncountably many elements of $\mathscr{G}$. Thus, uncountably many elements of $\mathscr{G}$ have their emanation point in the unit disk $\mathbb{U}$. We suppose that all elements of $\mathscr{G}$ have this property, which implies that every ray of every triode in the collection $\mathscr{G}$ meets $b\mathbb{U}$. Replace $\mathscr{G}$ by the set of triodes, still denoted by $\mathscr{G}$, obtained as follows: Replace each $T = \lambda_1 \cup \lambda_2 \cup \lambda_3$ in $\mathscr{G}$ by the triode $T' = \lambda_1' \cup \lambda_2' \cup \lambda_3'$ in which each λ_j' is the minimal subarc of λ_j that connects the emanation point of T to $b\mathbb{U}$. After all these reductions, $\mathscr{G}$ consists of uncountably many disjoint triodes all with emanation point in the open disk $\mathbb{U}$ and all with their rays arcs that hit $b\mathbb{U}$ in a single point.

The set $\mathscr{G}$ is uncountable, so there are three disjoint arcs γ_1, γ_2, and γ_3 with the property that for uncountably many elements T of $\mathscr{G}$, T has an endpoint in each of the γ's. We have constructed a configuration whose existence contradicts the lemma, so the theorem is proved.

Corollary 3.4.16. [18] (a) *If E is a connected subset of $\mathbb{C}$ with finite length, then, with at most countably many exceptions, each point of E lies in the boundary of at most two of the components of the complement of E in the Riemann sphere.* (b) *If Ω is a simply connected domain in $\mathbb{C}$ with $b\Omega$ of finite length, and if $\chi : \bar{\mathbb{U}} \to \bar{U}$ is a Riemann map, then except for at most a countable set, each point $p \in b\Omega$ is the image of at most two points under the map χ.*

Proof. Let the components of $\mathbb{C}^* \setminus E$ be $\Omega_1, \ldots$. The set E is connected, so each Ω_j is simply connected. Let $\varphi_j : \mathbb{U} \to \Omega_j$ be a Riemann map. According to Theorems 3.4.8 and 3.4.2, the function φ_j extends continuously to $\bar{\mathbb{U}}$. Suppose $z_o \in E$ to lie in the boundary of each of $\Omega_p,\ \Omega_q$, and Ω_r for three distinct integers p, q, and r. Fix points $z_p,\ z_q,\ z_r \in b\mathbb{U}$ such that $\varphi_p(z_p) = \varphi_q(z_q) = \varphi_r(z_r) = z_o$. Let L_p be the closed radial segment in $\mathbb{U}$ that connects z_p to $\frac{1}{2}z_p$, and similarly for radial segments L_r and L_q. Then the sets $\varphi_p(L_p)$, $\varphi_q(L_q)$, and $\varphi_r(L_r)$ are disjoint except for the coincidence of one of their endpoints with the point z_o. Their union forms a triode T_{z_o}. Moreover, by their construction, different choices of the point z_o lead to *disjoint* triodes.

Thus the result (a) follows from Moore's theorem, Theorem 3.4.14.

Statement (b) of the theorem is proved by a similar argument.

In the discussion of analytic structures below, we shall need to draw on a theorem of Radó:

Theorem 3.4.17. [286] *If the continuous function f defined on a domain D in $\mathbb{C}$ is holomorphic on $D \setminus f^{-1}(0)$, then it is holomorphic on all of D.*

Proof. [198] The theorem is a local result, so it suffices to suppose that the function f is defined and continuous on the closed unit disk $\bar{\mathbb{U}}$. Let $\Delta = \mathbb{U} \setminus f^{-1}(0)$. We shall show that if g is holomorphic on a neighborhood of $\bar{\Delta}$, then on Δ, $|g| \leq \|g\|_{\bar{\Delta}\cap b\mathbb{U}}$. This is so, for if k is a positive integer, then because $f = 0$ on $b\Delta \cap \mathbb{U}$, if $z \in \Delta$, then

$|g^k(z)f(z)| \leq \|g^k f\|_{\bar{\Delta}\cap b\mathbb{U}}$. Take the kth root of both sides and let $k \to \infty$ to find the desired inequality. It follows that Δ is dense in $\mathbb{U}$. Otherwise, for some w's in $\mathbb{U} \setminus \bar{\Delta}$, the function g given by $g(z) = 1/(z - w)$ would be holomorphic on $\bar{\Delta}$ but would not satisfy $\|g\|_\Delta \leq \|g|_{\bar{\Delta}\cap b\mathbb{U}}$.

We can now see that $\Re f$ is harmonic in $\mathbb{U}$. Let $\{P_k\}_{k=1,\ldots}$ be a sequence of holomorphic polynomials with the property that on $b\mathbb{U}$, $|\Re(P_k - f)| \leq 1/k$ for $k = 1\ldots.$ Then on $b\mathbb{U}$ we have the inequalities $|e^{(P_k - f)}| \leq e^{1/k}$ and $|e^{-(P_k - f)}| \leq e^{1/k}$. These inequalities therefore persist throughout $\mathbb{U}$, so that $\Re P_k \to \Re f$ uniformly on $\mathbb{U}$. Thus, $\Re f$ is harmonic on $\mathbb{U}$. Similarly, $\Im f$ is harmonic on $\mathbb{U}$. Consequently, the function f is a complex-valued harmonic function on $\mathbb{U}$ that is holomorphic on the dense open set Δ. This implies that f is holomorphic throughout $\mathbb{U}$, and the theorem is proved.

The extension of this theorem to functions of several variables is immediate by considering one-dimensional slices and invoking Hartogs's theorem.[1]

3.5. Subharmonicity Results

Fix a compact polynomially convex subset X of $\mathbb{C}^N$. Denote by Γ the Shilov boundary for the algebra $\mathscr{P}(X)$, so that Γ is the smallest closed subset of X with the property that $\widehat{\Gamma} = X$. For a function $\varphi \in \mathscr{P}(X)$, we are going to study the set $\varphi^{-1}(\varphi(X) \setminus \varphi(\Gamma))$ and certain subharmonic functions naturally associated with it, the point being that in some circumstances this set has the structure of a one-dimensional analytic variety.

It will be convenient to introduce the sets $\gamma = \varphi(\Gamma)$ and $\Omega = \varphi(X) \setminus \gamma$. The set γ is compact, Ω open and bounded. The set Ω need not be connected; it could have infinitely many components. It is a union of components of $\mathbb{C} \setminus \gamma$. (The latter point is easily seen: If Ω_o is a component of $\mathbb{C} \setminus \gamma$ that contains a point of $\varphi(X)$ but is not contained in $\varphi(X)$, then for points $w \in \Omega_o \setminus \varphi(X)$ the function $1/(\varphi - w)$ lies in $\mathscr{P}(X)$, as follows from the Oka–Weil approximation theorem. If w is suitably chosen, this function attains its maximum at a point of $X \setminus \Gamma$. Contradiction.)

There are a few simple facts to be noted in this setting.

Lemma 3.5.1. *If $p \in b\Omega$, then $\varphi^{-1}(p)$ meets Γ.*

Proof. If not, then p would have to lie in the open set $\mathbb{C} \setminus \gamma$, which precludes its lying in $b\Omega$.

Lemma 3.5.2. *If $\{V_\alpha\}_{\alpha\in A}$ is a neighborhood basis for the point $p \in \varphi(X)$, then the sets $\varphi^{-1}(V_\alpha)$, $\alpha \in A$, constitute a neighborhood basis for the fiber $\varphi^{-1}(p)$.*

Proof. This is an immediate consequence of the compactness of our situation.

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[1] The appeal to Hartogs's theorem here is not precise. The theorem of Hartogs in question is the rather difficult result that an arbitrary, a priori not even measurable, function is holomorphic if it is holomorphic in each variable separately. This general result is rarely needed. What is needed much more frequently, as in the case at hand, is the result that a *continuous* function that is holomorphic in each variable separately is holomorphic. This is a very simple remark that has no standard name.

Lemma 3.5.3. *If E is a compact, polynomially convex subset of $\mathbb{C}$, then the set $\varphi^{-1}(E)$ is a compact polynomially convex subset of X, and $\mathscr{P}(\varphi^{-1}(E))$ is the closure in $\mathscr{C}(\varphi^{-1}(E))$ of the restriction algebra $\mathscr{P}(X)|\varphi^{-1}(E)$.*

Again, this is immediate.

The next result is the first of several subharmonicity results

Lemma 3.5.4. *If $f \in \mathscr{P}(X)$, then the function $\lambda_{\varphi,f}$ defined on $\varphi(X)$ by*

$$\lambda_{\varphi,f}(p) = \max_{\varphi^{-1}(p)} \ln|f|$$

is upper semicontinuous and is subharmonic on Ω.

Proof. For the semicontinuity of $\lambda_{\varphi,f}$, let c be a real number. It is to be shown that the sublevel set $\{p \in \varphi(X) : \lambda_{\varphi,f}(p) < c\}$ is open. This is correct, though, because f is continuous on X, and because of Lemma 3.5.2.

For the subharmonicity assertion, we are to show that if $a \in \Omega$, and if $r > 0$ is small enough that the disk $\mathbb{U}(a,r)$ is a relatively compact subset of Ω, then $\lambda_{\varphi,f}$ satisfies the inequality

$$\lambda_{\varphi,f}(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \lambda_{\varphi,f}(a + re^{i\vartheta})\,d\vartheta.$$

The local maximum modulus theorem implies that the Shilov boundary for the algebra $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}(a,r)))$ is contained in the set $\varphi^{-1}(b\mathbb{U}(a,r))$. Let $z_o \in \varphi^{-1}(a)$, and let ν be a Jensen measure for the point z_o with respect to the algebra $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}(a,r)))$ that is carried by $\varphi^{-1}(b\mathbb{U}(a,r))$. Define the measure ν_* on $b\mathbb{U}(a,r)$ by the condition that for each continuous function g on $\mathbb{C}$, $\int g\,d\nu_* = \int g \circ \varphi\,d\nu$.

The measure ν_* is a Jensen measure on $b\mathbb{U}(a,r)$ for the point a with respect to the algebra $\mathscr{P}(\bar{\mathbb{U}}(a,r))$, for if P is a polynomial on $\mathbb{C}$, then

$$\log|P(a)| = \log|P \circ \varphi(z_o)| \leq \int \log|P \circ \varphi|\,d\nu = \int \log|P|\,d\nu_*.$$

The measure ν_* satisfies $\int (z-a)^k\,d\nu_* = 0$ for $k = 1, \ldots$. Assume for notational convenience that $a = 0$ and $r = 1$. Thus $\int z^k\,d\nu_* = 0, k = 1, \ldots$. The measure ν_* is real, so $\int \bar{z}^k\,d\nu_* = \int \frac{1}{z^k}\,d\nu_* = 0, k = 1, \ldots$. If $d\vartheta$ denotes Lebesgue measure on $b\mathbb{U}$, then the measure μ given by $d\mu = d\nu_* - \frac{1}{2\pi}d\vartheta$ satisfies $\int z^k\,d\mu = 0$ for all integers k and so is the zero measure. Thus, $d\nu_* = \frac{1}{2\pi}d\vartheta$. This, combined with the inequality derived above, completes the proof that the function $\lambda_{\varphi,f}$ is subharmonic on Ω.

Fix now a point $a \in \Omega$, and fix an $r > 0$ small enough that the closed disk $\bar{\mathbb{U}}(a,r)$ is contained in Ω. Let z_o be a point of the fiber $\varphi^{-1}(a)$, and let σ be a representing measure for the point z_o with respect to the algebra $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}(a,r)))$, σ supported by the set $\varphi^{-1}(b\mathbb{U}(a,r))$. Such a measure exists, for the Shilov boundary of the algebra $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}(a,r)))$ is contained in the set $\varphi^{-1}(b\mathbb{U}(a,r))$.

Lemma 3.5.5. *If $f \in \mathscr{P}(X)$, then the function F defined on $\mathbb{U}(a,r)$ by*

$$F(\zeta) = \int_{\varphi^{-1}(b\mathbb{U}(a,r))} \frac{\varphi(z) - a}{\varphi(z) - \zeta} f(z)\, d\sigma(z)$$

is holomorphic on $\mathbb{U}(a,r)$, takes the value $f(z_o)$ at the point a, and satisfies

$$|F(\zeta)| \le \max_{\varphi^{-1}(b\mathbb{U}(a,r))} |f|.$$

Moreover, if $\zeta_o \in b\mathbb{U}(a,r)$ is a point at which the nontangential limit

$$F^*(\zeta_o) = \lim_{\zeta \to \zeta_o} F(\zeta)$$

exists, then $|F^*(\zeta_o)| \le \max_{z \in \varphi^{-1}(\zeta_o)} |f(z)|$.

The bounded holomorphic function F has nontangential limit at almost every point of $b\mathbb{U}(a,r)$ by Fatou's theorem.

Proof. The holomorphicity of F is evident. It is also clear that $F(a) = f(z_o)$.

To prove F bounded, it is a convenience to suppose that $a = 0$ and that $r = 1$, so that we are working on the open unit disk $\mathbb{U}$. Thus,

$$F(\zeta) = \int_{\varphi^{-1}(b\mathbb{U})} \frac{\varphi(z) f(z)}{\varphi(z) - \zeta}\, d\sigma(z).$$

The measure $\varphi\, d\sigma$ is orthogonal to $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}))$, because $\varphi(z_o) = 0$. Consequently, because for fixed $\zeta \in \mathbb{U}$, the function $f/(1 - \bar{\zeta}\varphi)$ is in $\mathscr{P}(\varphi^{-1}(\bar{\mathbb{U}}))$, we have

$$\begin{aligned} F(\zeta) &= \int_{\varphi^{-1}(b\mathbb{U})} \Big(\frac{1}{\varphi(z) - \zeta} + \frac{\bar{\zeta}}{1 - \bar{\zeta}\varphi(z)}\Big) \varphi(z) f(z)\, d\sigma(z) \\ &= \int_{\varphi^{-1}(b\mathbb{U})} \frac{1 - |\zeta|^2}{|\varphi(z) - \zeta|^2} f(z)\, d\sigma(z), \end{aligned}$$

for $|\varphi| = 1$ on $\varphi^{-1}(b\mathbb{U})$. Thus

$$|F(\zeta)| \le \|f\|_{\varphi^{-1}(b\mathbb{U})} \int_{\varphi^{-1}(b\mathbb{U})} \frac{1 - |\zeta|^2}{|\varphi(z) - \zeta|^2}\, d\sigma(z).$$

The analysis given in the last proof shows that the measure σ_* on $b\mathbb{U}$ given by $\int h\, d\sigma_* = \int h \circ \varphi\, d\sigma$ is normalized Lebesgue measure on the circle. Thus, the final integral above is the Poisson integral of the function identically one on $b\mathbb{U}$, so its value is one. This gives the desired bound that $|F| \le \|f\|_{\varphi^{-1}(b\mathbb{U})}$.

Finally, let $\zeta_o \in b\mathbb{U}$ be a point for which $F^*(\zeta_o)$ exists. By the semicontinuity property of the function $\lambda_{\varphi,f}$ established in the last lemma, if $\varepsilon > 0$, then there is a neighborhood U of ζ_o in $\varphi(X)$ such that for $\zeta \in U$, $|\lambda_{\varphi,f}(\zeta)| \le \lambda_{\varphi,f}(\zeta_o) + \varepsilon$. This implies that $|F^*(\zeta_o)| \le e^{\lambda_{\varphi,f}(\zeta_o)}$ as desired, for

$$|F(\zeta)| \le \frac{1}{2\pi} \int_{b\mathbb{U}} \frac{1 - |\zeta|^2}{|z - \zeta|^2} |f(z)|\, d\vartheta(z).$$

For ζ near ζ_o and $e^{i\vartheta} \in b\mathbb{U} \setminus U$, the integrand is small, and for $z \in b\mathbb{U} \cap U$, we have $|f(z)| \leq (1+\delta)\max_{\varphi^{-1}(\zeta_0)} |f|$ for a small $\delta > 0$.

Subharmonicity has been applied in the study of polynomially convex sets and, more generally, in the theory of Banach algebras by several authors. In this direction see the papers of Bishop [59], Vesentini [358], Wermer [373], Aupetit and Wermer [41], Seničkin [321], and the book of Alexander and Wermer [28].

The following lemmas and their corollaries are developed following Seničkin [321].

Consider the n-fold Cartesian product X^n of X with itself, which is a compact, polynomially convex subset of $\mathbb{C}^{Nn}$. It contains the compact set Y^n_φ defined by

$$Y^n_\varphi = \{(w_1, \ldots, w_n) \in X^n : \varphi(w_1) = \cdots = \varphi(w_n)\}.$$

Lemma 3.5.6. *The set Y^n_φ is polynomially convex.*

Proof. For $i, j = 1, \ldots, n$, define functions $L_{ij} \in \mathscr{P}(X^n)$ by

$$L_{ij}(w_1, \ldots, w_n) = \varphi(w_i) - \varphi(w_j).$$

Each of the zero sets $L^{-1}_{ij}(0)$ is polynomially convex, so their intersection, the set Y^n_φ, is also polynomially convex.

Define a map $\eta : Y^n_\varphi \to \mathbb{C}$ by $\eta(w_1, \ldots, w_n) = \varphi(w_1) = \cdots = \varphi(w_n)$. This map carries Y^n_φ onto $\varphi(X)$.

Lemma 3.5.7. *Under the map η, the Shilov boundary for the algebra $\mathscr{P}(Y^n_\varphi)$ is carried into the set γ.*

Recall that $\gamma = \varphi(\Gamma)$, Γ the Shilov boundary for $\mathscr{P}(X)$, and that $\Omega = \varphi(X) \setminus \gamma$.

Proof. We shall show that the set $Y^n_\varphi \setminus \eta^{-1}(\Omega)$ is a boundary for the algebra $\mathscr{P}(Y^n_\varphi)$.

Let $\pi_k : Y^n_\varphi \to X$ be the projection given by $\pi_k(z_1, \ldots, z_n) = z_k$. Let $a \in \Omega$ be $\eta(z^o)$ for a $z^o = (z^0_1, \ldots, z^o_n) \in Y^n_\varphi$. Let Δ be a disk centered at a that is a relatively compact subset of Ω.

The algebra $\mathscr{P}(Y^n_\varphi)$ is the uniform closure in $\mathscr{C}(Y^n_\varphi)$ of the set of functions h of the form

$$h = \sum_{j=1}^{L} \prod_{k=1}^{n} g_{jk} \circ \pi_k,$$

where the functions g_{jk} range through $\mathscr{P}(X)$, and $L = 1, \ldots$.

Each of the functions g_{jk} corresponds to a function $G_{jk} \in H^\infty(\Delta)$ by the process of Lemma 3.5.5, and h corresponds in the same way to a function H. For almost all points $\zeta \in b\Delta$, the radial limits $G^*_{jk}(\zeta)$ exist for each choice of j, k, as do the radial limits $H^*(\zeta)$. Fix a ζ at which all of these limits exist, and denote the limit $G^*_{jk}(\zeta)$ by c_{jk}.

Introduce the function h_1 given by

$$h_1 = \sum_{j=1}^{L} (\prod_{k=2}^{n} c_{jk}) g_{j1}.$$

Then by Lemma 3.5.5, h_1 corresponds to the function H_1 given by

$$H_1 = \sum_{j=1}^{L} \Big(\prod_{k=2}^{n} c_{jk}\Big) G_{j1},$$

and by the same lemma, $|H_1^*(\zeta)| \leq \max_{\varphi^{-1}(\zeta)} |h_1|$. Choose $z_1' \in \varphi^{-1}(\zeta)$ to satisfy $|h_1(z_1')| = \max_{\varphi^{-1}(\zeta)} |h_1|$.

Now set

$$h_2 = \sum_{j=1}^{L} \Big(\prod_{k=3}^{n} c_{jk}\Big) g_{j1}(z_1') g_{j2}.$$

The function h_2 corresponds to the function H_2 given by

$$H_2 = \sum_{j=1}^{L} \Big(\prod_{k=3}^{n} c_{jk}\Big) g_{j1}(z_1') G_{j2}.$$

We have that $H_2^*(\zeta) = h_1(z_1')$ and

$$|H_2^*(\zeta)| \leq \max_{\varphi^{-1}(\zeta)} |h_2|.$$

Choose $z_2' \in \varphi^{-1}(\zeta)$ such that $|h_2(z_2')| = \max_{\varphi^{-1}(\zeta)} |h_2|$. Thus, $|H_1^*(\zeta)| \leq |h_2(z_2')|$.

Iterate this process to find, finally, that $|H_1^*(\zeta)| \leq |h_n(z_n')|$ with

$$h_n = \sum_{j=1}^{L} \prod_{k=1}^{n-1} g_{jk}(z_k') g_{jn}$$

for some choice of points $z_1', \ldots, z_{n-1}' \in \varphi^{-1}(\zeta)$.

The function $\tilde{H} = \sum_{j=1}^{L} \prod_{k=1}^{n} G_{jk}$ has boundary limit $\tilde{H}^*(\zeta)$, which is $H_1^*(\zeta)$. Also, $h_n(z_n') = h(z')$ if $z' = (z_1', \ldots, z_n')$. It follows that $|\tilde{H}^*(\zeta)| \leq |h(z')|$. This is true for almost all $\zeta \in b\Delta$, so, by Lemma 3.5.5, $|h(a)| = |H(a)| \leq \max_{\varphi^{-1}(b\Delta)} |h| \leq \max_{\varphi^{-1}(\varphi(X)\setminus\Omega)} |h|$.

This analysis works for all $a \in \Omega$, so $\varphi^{-1}(\Omega)$ is disjoint from the Shilov boundary for $\mathscr{P}(Y_\varphi^n)$, as we wished to show.

If g is an arbitrary element of $\mathscr{P}(X)$, define $G \in \mathscr{P}(Y_\varphi^n)$ by

$$G(w) = G(w_1, \ldots, w_n) = \prod_{1 \leq i < j \leq n} [g(w_i) - g(w_j)].$$

For $n = 1, \ldots,$ define the function $\mathfrak{d}_{n,g} : \varphi(X) \to [0, \infty)$ by

$$\mathfrak{d}_{n,g}(x) = \max_{w_1, \ldots, w_n \in \varphi^{-1}(x)} |G(w)|.$$

Recall that $\pi_j : Y^n_\varphi \to X$ is the projection $(w_1, \dots, w_n) \mapsto w_j$. We have that $\varphi \circ \pi_1 \in \mathscr{P}(Y^n_\varphi)$. Because

$$\eth_{n,g}(x) = \max_{w \in \varphi \circ \pi_1^{-1}(x)} |G(w)|,$$

Lemma 3.5.4 implies the following statement.

Lemma 3.5.8. *The function* $\ln \eth_{n,g}$ *is upper semicontinuous on* $\varphi(X)$ *and subharmonic on* Ω.

Corollary 3.5.9. *The diameter of the set* $g(\varphi^{-1}(x))$ *is an upper semicontinuous function on* X *and is logarithmically subharmonic on* Ω.

Corollary 3.5.10. *The transfinite diameter of the set* $g(\varphi^{-1}(x))$ *is an upper semicontinuous function on* X *and is logarithmically subharmonic on* Ω.

Recall the definition of the transfinite diameter: If E is a compact subset of the complex plane, define

$$d_n(E) = \max_{z_1,\dots,z_n \in E} \Big\{ \prod_{1 \le j < k \le n} |z_j - z_k| \Big\}^{1/\binom{n}{2}}.$$

For fixed E, the functions $d_n(E)$ decrease; their limit is the transfinite diameter $\tau(E)$ of E. The transfinite diameter of a set coincides with its logarithmic capacity. (See [356].)

3.6. Analytic Structure in Hulls

The results of the preceding section imply the presence of one-dimensional varieties in certain polynomially convex sets.

Theorem 3.6.1. *Let* X *be a compact subset of* $\mathbb{C}^N$, $N > 1$, *let* $\varphi \in \mathscr{P}(X)$, *and let* Ω *be a component of* $\varphi(\widehat{X}) \setminus \varphi(X)$. *If there is* $E \subset \bar{\Omega}$ *that is a set of uniqueness for subharmonic functions on* Ω *such that for each* $e \in E$ *the fiber* $\varphi^{-1}(e) \cap \widehat{X}$ *is finite, then the set* $\varphi^{-1}(\Omega) \cap \widehat{X}$ *is a one-dimensional analytic subvariety of* $\mathbb{C}^N \setminus X$.

This theorem, as stated, is due to Seničkin [321], who was working in the formally more general context of general uniform algebras. A very similar result was obtained by Aupetit and Wermer [41].

Proof. For each $k = 1, \dots,$ let

$$E'_k = \{e \in E : \varphi^{-1}(e) \cap \widehat{X} \text{ consists of exactly } k \text{ points}\}.$$

By hypothesis $E' = \cup_{k=1,\dots} E'_k$ is a set of uniqueness for subharmonic functions on Ω, so, by Lemma 3.4.10, one of the sets E'_k is a set of uniqueness. Let n' be the smallest integer such that $E'_{n'}$ is a set of uniqueness. The function $\eth_{n',\varphi}$ is subharmonic on Ω and lower semicontinuous on $\bar{\Omega}$. It is identically $-\infty$ on $E'_{n'}$, and so is identically $-\infty$ on Ω. Thus, for each $z \in \Omega$, the fiber $\varphi^{-1}(z) \cap \widehat{X}$ consists of at most n' points.

For $k = 1, \dots, n'$, let

$$E_k = \{z \in \Omega : \varphi^{-1}(z) \cap \widehat{X} \text{ contains exactly } k \text{ points}\}.$$

Then $\cup_{k=1,\dots,n'} E_k = \Omega$, so one of the sets E_k is a set of uniqueness for subharmonic functions on Ω. Let n be the least integer such that E_n is a set of uniqueness. Thus, as above, each fiber $\varphi^{-1}(z) \cap \widehat{X}$, $z \in \Omega$, contains no more than n points.

We shall show that Ω_n, the set of $z \in \Omega$ for which φ^{-1} contains n points, is open and that the set $\Omega \setminus \Omega_n$ is a discrete subset of Ω.

To do this, begin by fixing a point $z_o \in \Omega_n$ and a disk Δ centered at z_o, Δ so small that $\bar{\Delta} \subset \Omega$. Fix $f \in \mathscr{P}(X)$ that assumes n distinct values on the fiber $\varphi^{-1}(z_o)$, say $f(\varphi^{-1}(z_o)) = \{p_1, \dots, p_n\}$. For a small $\delta > 0$, let $\Delta_{j,\delta}$ be the disk of radius δ centered at the point p_j. These disks have mutually disjoint closures, for δ is small. By Lemma 3.5.2, as W runs through a neighborhood basis of the point z_o, the sets $\varphi^{-1}(W)$ run through a neighborhood basis for the fiber $\varphi^{-1}(z_o)$. Consequently, if Δ is small enough, then $f(\varphi^{-1}(\Delta)) \subset \cup_{j=1,\dots,n} \Delta_{j,\delta}$.

For each j, the set $S_j = \varphi^{-1}(\bar{\Delta}) \cap f^{-1}(\bar{\Delta}_{j,\delta})$ is polynomially convex, and the Shilov boundary for the algebra $\mathscr{P}(S_j)$ is contained in the set $\varphi^{-1}(b\Delta) \cap f^{-1}(b\Delta_{j,\delta})$. The (restriction to S_j of the) function f lies in $\mathscr{P}(S_j)$, and the set $f(S_j)$ contains the point $p_j \in \Delta_{j,\delta}$, so $f(S_j) \supset \Delta_{j,\delta}$. Similarly, $\varphi(S_j) \supset \Delta$. That the sets S_j are disjoint implies that for each point $z \in \Delta$, the fiber $\varphi^{-1}(z)$ consists of exactly n points. This shows that the set Ω_n is open in $\mathbb{C}$.

For each $j = 1, \dots, n$, let $\psi_j : \Delta \to S_j$ be the map that satisfies $\varphi \circ \psi_j = \mathrm{id}$ on Δ. Each of the functions ψ_j is holomorphic in Δ. To see this, fix a disk $\Delta_{a,\varepsilon}$ centered at $a \in \Delta$ that is a relatively compact subset of Δ. Let μ be a representing measure with respect to the algebra $\mathscr{P}(S_j)$ for the point $\psi_j(a)$ carried on $\varphi^{-1}(b\Delta_{a,\varepsilon}) \cap S_j$. Denote its projection to $b\Delta_{a,\varepsilon}$ by μ_* so that for all continuous functions h on $b\Delta_{a,\varepsilon}$, $\int h\, d\mu_* = \int h \circ \varphi\, d\mu$. If $g \in \mathscr{P}(S_j)$, then for $r = 1, \dots$ we have

$$0 = \int (\varphi - a)^r g\, d\mu = \int (z - a)^r g \circ \psi_j(z)\, d\mu_*(z).$$

Applied with g the function identically one, we see by the reality of μ that μ_* is $\frac{1}{2\pi} d\vartheta$ if ϑ denotes the angular variable on the circle $|z - a| = \varepsilon$. We then see, by taking $r = 1$, that the function $g \circ \psi_j$ satisfies $\frac{1}{2\pi i} \int_{|z-a|=\varepsilon} g \circ \psi_j(z)\, dz = 0$. This is true for each choice of the disk $\Delta_{a,\varepsilon}$, so by Morera's theorem, $g \circ \psi_j$ is holomorphic on Δ. This is true in particular when g is one of the coordinate functions on $\mathbb{C}^N$. Thus ψ_j itself is holomorphic on Δ. The holomorphicity of the functions ψ_j implies that the open subset $\varphi^{-1}(\Omega_n)$ of $\widehat{X}$ is a one-dimensional complex submanifold of $\mathbb{C}^N \setminus (\widehat{X} \setminus \varphi^{-1}(\Omega_n))$.

With f the function considered above, define $\tilde{H} : \Omega_n \to \mathbb{C}$ by

$$\tilde{H}(z) = \prod_{1 \le j < k \le n} (f(\psi_j(z)) - f(\psi_k(z)))^2.$$

This function is holomorphic on Ω_n, and it tends to zero as z tends to a boundary point of Ω_n contained in Ω. Thus if we set $H(z) = \tilde{H}(z)$, $z \in \Omega_n$, and $H(z) = 0$, $z \in \Omega \setminus \Omega_n$, then

H is continuous on Ω and holomorphic off its zero set. Radó's theorem, Theorem 3.4.17, shows that H is holomorphic throughout Ω. The zero locus of H is a discrete subset of Ω, so $\Omega \setminus \Omega_n$ is a discrete subset of Ω.

That $\varphi^{-1}(\Omega)$ is a one-dimensional subvariety of $\mathbb{C}^N \setminus (\widehat{X} \setminus \varphi^{-1}(\Omega))$ is a consequence of Lemma 3.2.2 in the following way.

Define $\Phi : \widehat{X} \to \mathbb{C}^{N+1}$ by $\Phi(z) = (z_1, \ldots, z_N, \varphi(z))$. The set $\Phi(\widehat{X})$, which is the graph of φ, is polynomially convex because the set $\widehat{X}$ is. With $\pi_{N+1} : \mathbb{C}^{N+1} \to \mathbb{C}$ the projection $\pi_{N+1}(z_1, \ldots, z_{N+1}) = z_{N+1}$, Lemma 3.2.2 applies to show that $\Phi(\widehat{X}) \cap \pi_{N+1}^{-1}(\Omega)$ is a one-dimensional variety in $\mathbb{C}^{N+1} \setminus (\Phi(\widehat{X}) \setminus \pi_{N+1}^{-1}(\Omega))$ on which each element of $\mathscr{P}(\Phi(\widehat{X}))$ is holomorphic.

This implies that $\varphi^{-1}(\Omega) \cap \widehat{X}$ is an analytic subvariety of $\mathbb{C}^N \setminus (\widehat{X} \setminus \varphi^{-1}(\Omega))$ on which each element of $\mathscr{P}(\widehat{X})$ is holomorphic.

Theorem 3.6.1 is proved.

Corollary 3.6.2. *With X, φ, Ω, and E as in the preceding theorem, if for each point e of E, the fiber $\varphi^{-1}(e) \cap X$ consists of at most n points, then for each point $\zeta \in \Omega$, the fiber $\varphi^{-1}(\zeta) \cap \widehat{X}$ consists of at most n points.*

A second corollary to the preceding theorem concerns the second example of Section 3.1. There the polynomially convex hull of a certain figure-eight curve denoted by Γ was considered. It is evident that the hull $\widehat{\Gamma}$ contains the variety $V = \{(z_1, z_2) : z_1^2 = F(z_2),\ |z_1| < 1\}$. It is less evident that $\widehat{\Gamma} \setminus \Gamma$ is V. But this equality follows from Corollary 3.6.2: If $\pi : \mathbb{C}^2 \to \mathbb{C}$ is the projection onto the first coordinate, then the set Γ maps in a generically two-to-one way onto the unit circle. The same is therefore true of $\widehat{\Gamma} \setminus \pi^{-1}(b\mathbb{U})$. Because V maps in a generically two-to-one way onto the disk, it follows that $V = \widehat{\Gamma} \setminus \Gamma$.

As a further application of the criterion of Theorem 3.6.1, we consider hulls with finite area.

Theorem 3.6.3. *If X is a compact subset of $\mathbb{C}^N$ such that $\widehat{X} \setminus X$ has finite two-dimensional Hausdorff measure, then $\widehat{X} \setminus X$ is a one-dimensional analytic subvariety of $\mathbb{C}^N \setminus X$.*

This result was found independently by Alexander [10], Basener [45], and Sibony [331]. Alexander discusses also the situation in which $\widehat{X} \setminus X$ has σ-finite two-dimensional measure. Basener and Sibony give conditions under which a polynomial hull contains analytic varieties of dimension greater than one.

Proof. Having fixed a point $x_o \in \widehat{X} \setminus X$, we show that $\widehat{X} \setminus X$ has the structure of a one-dimensional variety near x_o. Suppose coordinates to have been chosen such that x_o is the origin. Let $r_o > 0$ be small enough that the closed ball $\bar{\mathbb{B}}_N(r_o)$ is disjoint from the set X. The local maximum principle implies that if $E = \widehat{X} \cap b\mathbb{B}_N(r_o)$, then $\widehat{E} = \widehat{X} \cap \bar{\mathbb{B}}_N(r_o)$. Call the latter set E_1. The function ϱ defined on $\mathbb{C}^N$ by $\varrho(z) = |z|$ satisfies a Lipschitz condition on all of $\mathbb{C}^N$, so Eilenberg's inequality, Theorem 3.3.6, implies that if r_o is chosen properly, the set $E_1 = \varrho^{-1}(r_o) \cap \widehat{X}$ has finite one-dimensional measure.

Because $\Lambda^2(E_1) = 0$, the set E_1 is rationally convex, which implies the existence of a polynomial P with $P(0) = 0$ and with P zero-free on E_1. Let Ω be the component of $\mathbb{C} \setminus P(E_1)$ that contains the origin.

We shall show that Ω contains a set S of positive measure with the property that for each point s of S, the set $P^{-1}(s)$ meets $\mathbb{B}_N(r_o) \cap \widehat{X}$ in a finite set. To do this, set $u = \Re P$. Eilenberg's inequality implies that for almost all $t \in \mathbb{R}$ the fiber $F_t = u^{-1}(t) \cap \widehat{X} \cap \mathbb{B}_N(r_o)$ has finite length. Choose a t such that F_t has finite length and such that the line $L_t = \{t + i\varsigma : \varsigma \in \mathbb{R}\}$ meets the domain Ω. Eilenberg's inequality applied again yields that for almost every point $t + i\varsigma \in L_t \cap \Omega$, the fiber $P^{-1}(L_t + i\varsigma) \cap \mathbb{B}_N(r_o) \cap \widehat{X}$ is a finite set.

Let Ω^* be the set of $z \in \Omega$ for which the fiber $\Omega^z = P^{-1}(z) \cap \mathbb{B}_N(r_o) \cap \widehat{X}$ is finite. If Ω_k^* is the subset of Ω^* for which this fiber consists of k distinct points, then, since the Lebesgue outer measure, which is defined on the family of all subsets of the plane, is countably subadditive, one of the sets Ω_k^*, say Ω_n^*, has positive outer measure.

The subharmonic function $\ln \mathfrak{d}_{n+1}$ introduced at the end of the preceding section assumes the value $-\infty$ on the set Ω_n^*. The set on which it takes the value $-\infty$ is measurable, for $\ln \mathfrak{d}_{n+1}$ is an upper semicontinous, hence measurable, function. This set has positive area, so $\ln \mathfrak{d}_{n+1}$ takes the value $-\infty$ identically. This implies that every one of the fibers Ω^z contains at most n points.

That the set $\mathbb{B}_N \cap \widehat{X}$ is a one-dimensional variety is now seen to be a consequence of Theorem 3.6.1.

Note that for the preceding argument to work, it suffices to assume not that $\widehat{X} \setminus X$ has *finite* two-dimensional measure but only that this set has *locally finite* two-dimensional measure. On the other hand, granted the local maximum principle, the latter result is a consequence of the result we proved.

We can now prove the fundamental Theorem 3.1.1.

Proof of Theorem 3.1.1. Set $X = Y \cup \Gamma$, a set that is rationally convex by Corollary 1.6.8.

Assume to begin with that the set Γ is connected.

Fix a point $p \in \widehat{X} \setminus X$. We are to show that $\widehat{X}$ has the structure of an analytic variety near p. Because Y is polynomially convex and Γ has finite length, there is a polynomial P with $\Re P < -1$ on Y and with $P(p) = 1$. The polynomial P can be chosen so that $P(p) \notin P(\Gamma)$. To construct such a P, choose first a polynomial Q such that $\Re Q < -2$ on Y and $Q(p) = 0$. The set X is rationally convex, so there is a polynomial R such that $0 \notin R(X)$ and $R(p) = 0$. If M is sufficiently large and positive, then $\Re(MQ + \zeta R) < -2$ on Y for all choices of $\zeta \in \mathbb{C}$ with $|\zeta| < 1$. No matter what the choice of M and ζ, $(MQ + \zeta R)(p) = 0$. The function $H = (MQ)/R$ is holomorphic on a neighborhood of X, so the set $H(\Gamma)$ has zero area in $\mathbb{C}$. If $-\zeta \in \mathbb{C} \setminus H(\Gamma)$, $|\zeta| < 1$ but $\zeta \neq 0$, then the polynomial $P_\zeta = MQ + \zeta R$ vanishes at p but at no point of X. For the desired polynomial P, take $1 + P_\zeta$.

Let Π^+ be the open right half-plane $\{\zeta \in \mathbb{C} : \Re\zeta > 0\}$.

Fix a large positive constant K, K so large that $P(X) \subset \{\zeta \in \mathbb{C} : |\zeta| < K\}$. Let I_K denote the interval $[-iK, iK]$ in the imaginary axis. The set $S = I_K \cup (\Pi^+ \cap P(\Gamma))$ is compact. It is also connected. Otherwise, $P(\Gamma) \cap \Pi^+$ would have a compact component, which would imply that Γ is not connected. Consequently, each bounded component of $\mathbb{C} \setminus S$ is simply connected. Let Ω be the component of $\mathbb{C} \setminus S$ that contains the point 1.

The boundary of Ω has finite length, and it is connected and not disjoint from Π^+. Consequently, by Theorem 3.3.6, Eilenberg's inequality, there is a set $E \subset b\Omega \cap \Pi^+$ that has positive length and that has the property that for every $e \in E$, the fiber $P^{-1}(e) \cap \Gamma$ is a finite set.

Because of Lemma 3.4.11, Theorem 3.6.1 implies that the set $\widehat{X} \setminus X$ has the structure of a one-dimensional variety near the point p. The point $p \in \widehat{X} \setminus X$ was chosen arbitrarily, so the theorem is proved, under the supplementary hypothesis that the set Γ is connected.

In case Γ is not connected, we proceed as follows. Let $p \in \widehat{Y \cup \Gamma} \setminus (Y \cup \Gamma)$. Choose a connected set Γ' of finite length that contains Γ and such that $Y \cup \Gamma'$ is compact. If $p \notin \Gamma'$, we are done. If, on the other hand, $p \in \Gamma'$, replace Γ' by the set Γ'' obtained in this way. Let $\delta > 0$ be a small positive number, so small that the ball $\mathbb{B}_N(p, \delta)$ is disjoint from $Y \cup \Gamma$. Let q be some point of $\mathbb{B}_N(p, \delta) \setminus \Gamma'$. Then Γ'' is the set that agrees with Γ' outside the ball $\mathbb{B}_N(p, \delta)$ and that in $\bar{\mathbb{B}}_N(p, \delta)$ is obtained by projecting $\mathbb{B}_N(p, \delta) \cap \Gamma'$ radially onto $b\mathbb{B}_N(p, \delta)$ from the point q. The set Γ'' obtained in this way contains Γ and is connected. Moreover, $Y \cup \Gamma''$ is compact.

It remains only to establish the last assertion of Theorem 3.1.1. The map

$$H^1(Y \cup \Gamma; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \tag{3.3}$$

is assumed to be an isomorphism, and we are to prove that $\mathscr{P}(Y \cup \Gamma) = \{f \in \mathscr{C}(Y \cup \Gamma) : f|Y \in \mathscr{P}(Y)\}$. What has to be seen is that $Y \cup \Gamma$ is polynomially convex, for granted this, $\mathscr{P}(Y \cup \Gamma) = \mathscr{R}(Y \cup \Gamma)$, so that the result follows from Corollary 1.6.8.

The hypothesis that the map (3.3) is an isomorphism means that each zero-free continuous function on $Y \cup \Gamma$ that has a continuous logarithm on Y necessarily has a continuous logarithm on $Y \cup \Gamma$. (Recall the discussion at the end of Section 1.2.)

If $Y \cup \Gamma$ is not polynomially convex, then $\widehat{Y \cup \Gamma} \setminus (Y \cup \Gamma)$ is a one-dimensional variety. Denote it by V.

The set Y is polynomially convex, so there is a polynomial P such that $\Re P < -1$ on Y and $0 \in P(V)$. The set $P(\Gamma)$ has zero area, so the polynomial can be chosen so that $0 \notin P(Y)$. Then P is a zero-free continuous function on $Y \cup \Gamma$ that has a continuous logarithm on Y. Accordingly, it has a continuous logarithm on $Y \cup \Gamma$. Then it has a continuous logarithm on an open set Ω containing $Y \cup \Gamma$. This logarithm must be holomorphic on Ω because P is holomorphic there.

The boundary of the variety V lies in the set Ω, which implies a contradiction to the argument principle: The function $P|V \in \mathscr{O}(V)$ vanishes at some points of V, but it has a logarithm off a sufficiently large compact subset of V.

This completes the proof.

3.7. Finite Area

The main goal of the present section is the result that if the connected set Γ has finite length, then the variety $\widehat{\Gamma} \setminus \Gamma$ has finite area. To establish this, we need some preliminary information about the area of varieties.

The area of a complex submanifold of a domain in $\mathbb{C}^N$ is given by a classical formula of Wirtinger.

Introduce the (1, 1)-form $\omega = \frac{1}{2i}\sum_{j=1}^{N} d\bar{z}_j \wedge dz_j$.

To begin with, let $\varphi : D \to \mathbb{C}^N$ be a holomorphic map from the domain D in $\mathbb{C}$ into $\mathbb{C}^N$. Assume φ' zero-free and φ injective. The map φ has coordinates $(\varphi_1, \dots, \varphi_N)$ for certain functions φ_j holomorphic on D.

Lemma 3.7.1. *The area of the image $\varphi(D)$ is $\int_D \varphi^*\omega$.*

Proof. To prove this formula, it suffices to show that for every $\zeta_o \in D$,

$$\lim_{r\to 0^+} \frac{1}{A(r)} \int_{\{|\zeta-\zeta_o|<r\}} \varphi^*\omega = 1,$$

in which $A(r)$ the area of the set $\varphi(\mathbb{U}(\zeta_o, r))$.

For this we can suppose coordinates chosen in $\mathbb{C}$ such that $\zeta_o = 0$ and in $\mathbb{C}^N$ such that $\varphi(0)$ is the origin. Then by a unitary change of coordinates in $\mathbb{C}^N$, we can suppose that $\varphi(D)$ is tangent at 0 to the z_1-axis, i.e., that $\varphi_1' \neq 0$ but that $\varphi_2'(0) = \cdots = \varphi_N'(0) = 0$. A final change of coordinates in $\mathbb{C}$ lets us suppose that $\varphi_1'(0) = 1$.

Then

$$\int_{\{|\zeta|<r\}} \varphi^*\omega = \frac{1}{2i}\int_{\{|\zeta|<r\}} (1 + O(r))\, d\bar{\zeta} \wedge d\zeta = \pi r^2 + O(r^3).$$

Also, if $\varphi_j = u_{2j-1} + iu_{2j}$, and if $\zeta = \xi + i\eta$, then

$$A(r) = \int_{\{\xi^2+\eta^2<r^2\}} \left\{ \sum_{1\le\mu<\nu\le 2N} D_{\mu\nu}^2 \right\}^{1/2} d\xi\, d\eta,$$

in which $D_{\mu\nu}$ is the Jacobian determinant $\frac{\partial(u_\mu, u_\nu)}{\partial(\xi,\eta)}$. Thus, with our normalizations, for ζ near 0, $D_{12}(\zeta) = 1 + O(|\zeta|)$, and $D_{\mu\nu}(\zeta) = O(|\zeta|)$ when $(\mu, \nu) \neq (1, 2)$. Consequently, $A(r) = \pi r^2 + O(r^3)$. From this the lemma follows.

Corollary 3.7.2. *If V is a one-dimensional subvariety of an open set in $\mathbb{C}^N$, then the area of the set of regular points of V is $\int_{V_{\mathrm{reg}}} \omega$.*

We have remarked that the area of a k-dimensional submanifold of $\mathbb{R}^N$ coincides with its k-dimensional Hausdorff measure. For a one-dimensional subvariety of (an open set in) $\mathbb{C}^N$, the set of singular points is discrete and so has vanishing 2-dimensional measure. Consequently, for such a variety V, $\Lambda^2(V) = \int_{V_{\mathrm{reg}}} \omega$. This last integral we define to be $\int_V \omega$.

The integral $\int_{V_{\mathrm{reg}}} \omega$ is the sum $\sum_{j=1}^{N} \frac{1}{2i}\int_{V_{\mathrm{reg}}} d\bar{z}_j \wedge dz_j$. If π_j denotes the projection of $\mathbb{C}^N$ onto the jth coordinate axis, then the jth of these integrals is $\int_{\pi_j(V)} \Lambda^o(\pi_j^{-1}(\zeta))\, d\bar{\zeta} \wedge d\zeta$.

Corollary 3.7.3. *The area of a 1-dimensional subvariety V of an open set in $\mathbb{C}^N$ is the sum of the areas of the projections, counted with multiplicities, of V onto the coordinate axes.*

There is no analogue of this result in the geometry of real submanifolds.

Notice that the form ω is exact: $\omega = d\vartheta$ with $\vartheta = \frac{i}{2}\sum_{j=1}^{N} \bar{z}_j dz_j$. This has the surprising, if simple, consequence that if V is a one-dimensional variety and if Ω is a domain in V for which Stokes's theorem holds, then the area of Ω is given by a boundary integral: $\Lambda^2(\Omega) = \int_{b\Omega} \vartheta$. Again, there is no analogue of this in real geometry.

We have presented these results on areas in the case of one-dimensional varieties, which is the case used in the sequel. There are natural analogues for k-dimensional subvarieties. A full exposition of this theory is given in [343].

It is now possible to prove the finiteness-of-area result stated above. There is an apparently more general statement that does not refer to polynomial convexity.

Theorem 3.7.4. *If V is a one-dimensional subvariety of a bounded open set in $\mathbb{C}^N$ such that the set $\bar{V} \setminus V$ is contained in a connected set of finite one-dimensional measure, then V has finite area.*

Alexander [17] proved this result under the assumption that $\bar{V} \setminus V$ is a rectifiable simple closed curve; the general case was established by Lawrence [217]. A generalization is given in the next chapter.

Proof. Let Γ be a connected set of finite length that contains $\bar{V} \setminus V$. Let F be a function holomorphic on a neighborhood of $\widehat{\Gamma}$. The set $F(\Gamma)$ is a connected subset, say γ, of $\mathbb{C}$. The complementary set $\mathbb{C} \setminus \gamma$ consists of one unbounded component, Ω_0, and of countably many bounded components Ω_j, $j = 1, \dots$, each of which is simply connected. For each $j = 1, \dots$, let m_j be the multiplicity of the map $F|(V \cap F^{-1}(\Omega_j))$. We are going to show that $\sum_{j=1,\dots} m_j \operatorname{area} \Omega_j < \infty$.

The proof of this depends on a lemma, which goes back to the work of Alexander [7].

Lemma 3.7.5. *If $S \subset b\Omega_j \cap b\Omega_k$ is a set of positive length such that $\Lambda^0(F^{-1}(\zeta) \cap \Gamma) = n$ for all $\zeta \in S$, then $|m_j - m_k| \le n$.*

Proof. We shall show that if $\zeta \in S$, then $F^{-1}(\zeta) \cap (V \setminus \Gamma)$ contains at most m_j points. If not, let this set contain distinct points $w_1, \dots, w_q$ with $q > m_j$. Associated with each point w_k is a neighborhood W_k in $V \setminus \Gamma$ that is mapped properly onto a neighborhood of ζ in $\mathbb{C}$. This implies that certain points in Ω_j are covered at least q times, which is impossible, for the multiplicity of F over C_j is m_j. Thus, for each $\zeta \in S$, $F^{-1}(\zeta) \cap (\Gamma \cup V)$ contains at most $n + m_j$ points. The set S is a set of uniqueness for subharmonic functions on the simply connected domain Ω_k, because it has positive length, so it follows from Corollary 3.6.2 that the multiplicity of F over Ω_k is not more than $n + m_j$. Thus, $m_k - m_j \le n$. Symmetrically, $m_j - m_k \le n$, so, as claimed, $|m_j - m_k| \le n$.

We continue with the proof of the theorem as follows.

Consider a positive number ρ, and set

$$A(\rho) = \sum_{j=1,\dots} \min(m_j, \rho) \operatorname{area} \Omega_j.$$

This sum is finite.

Let $\eta : \mathbb{C} \to \mathbb{R}$ be the projection $\eta(s + it) = s$. There is a set $S \subset \mathbb{R}$ of zero measure such that for all $s \in \mathbb{R} \setminus S$, the fiber $\eta^{-1}(s) \cap \gamma$ is a finite set, for the set γ has finite length.

By Fubini's theorem,

$$\operatorname{area}\Omega_j = \int_{\Omega_j} d\mathscr{L}(w) = \int_{\mathbb{R}\setminus S}\int_{\eta^{-1}(s)\cap\Omega_j} dt\, ds.$$

For all $s \in \mathbb{R}\setminus S$, the set $\eta^{-1}(s)\cap\Omega_j$ is a finite union of mutually disjoint intervals. Let these be $(s+i\alpha_{j,1}(s), s+i\beta_{j,1}(s)), \dots, (s+i\alpha_{j,\nu_j(s)}(s), s+i\beta_{j,\nu_j(s)}(s))$ with each α strictly less than the corresponding β. Then

$$\int_{\eta^{-1}(s)\cap\Omega_j} dt = \sum_{j=1}^{\nu_j(s)}\{\beta_{j,k}(s)-\alpha_{j,k}(s)\}.$$

Fubini's theorem assures us that this sum is a measurable function of s. We have now that

$$A(\rho) = \int_{\mathbb{R}\setminus S}\min(m_j,\rho)\left\{\sum_{j=1,\dots}\left\{\sum_{k=1}^{\nu_j(s)}\{\beta_{j,k}(s)-\alpha_{j,k}(s)\}\right\}\right\}ds. \tag{3.4}$$

Introduce the notation that for $\zeta\in\gamma$, j'_ζ and j''_ζ are the indices j for which $\zeta\in b\Omega_j$. By Lemma 3.4.16, almost all $[d\Lambda^1]$ $\zeta\in\gamma$ belong to at most two of the boundaries $b\Omega_j$. (For this count, we have to include $b\Omega_0$.) It follows that the integrand in (3.4) is

$$\sum_{\zeta\in\eta^{-1}(s)} \pm(\min(m_{j'(\zeta)},\rho)-\min(m_{j''(\zeta)},\rho))\zeta.$$

By the lemma,

$$|\min(m_{j'(\zeta)},\rho)-\min(m_{j''(\zeta)},\rho)| \le \Lambda^0(F^{-1}(\zeta)\cap\Gamma)$$

for almost all ζ. Also, on the set γ, $|\zeta|$ is bounded, say by K. This leads to

$$\begin{aligned} A(\rho) &\le K\int_{\mathbb{R}\setminus S}\sum_{\zeta\in\eta^{-1}(s)}\Lambda^0(F^{-1}(\zeta)\cap\Gamma)\,ds \\ &= K\int_{\mathbb{R}\setminus S}\Lambda^0((\eta\circ F)^{-1}(s)\cap\Gamma)\,ds \\ &\le KC\Lambda^1(\Gamma) \end{aligned}$$

if C is the Lipschitz constant of the map $\eta\circ F$ on Γ.

The right-hand side of this is independent of ρ, so by letting $\rho\to\infty$, we find that $\sum_{j=1,\dots} m_j \operatorname{area}\Omega_j$ is finite.

If we apply the result concerning F in the N cases that F runs through the projections onto the coordinate axes of $\mathbb{C}^N$, we find that the areas of the projections of the variety V, counted with multiplicities, have finite sum, so the area of V itself is finite by Corollary 3.7.3. The theorem is proved.

Notice that the proof implies the bound that area $V \le N(\operatorname{diam}\Gamma)\Lambda^1(\Gamma)$, for without loss of generality, the set Γ can be supposed to contain the origin, in which case it is contained within the ball centered at the origin of radius $\operatorname{diam}\Gamma$.

Corollary 3.7.6. *If $\Gamma \subset \mathbb{C}^N$ is a compact subset of a compact connected set of finite length, then $\widehat{\Gamma}$ has finite two-dimensional measure.*

Proof. If Γ_o is a connected set of finite length that contains Γ, then $\widehat{\Gamma}_o \supset \widehat{\Gamma}$.

In particular, if γ is a rectifiable simple closed curve, its hull has finite area.

Implicit in what we have done is the result that a one-dimensional variety has locally finite area. There are much simpler, direct routes to this fact. The corresponding statement for higher-dimensional varieties is also correct. For these results, see, e.g., [343].

3.8. The Continuation of Varieties

In this section we establish some results on the continuation of one-dimensional analytic varieties. In part this work draws on the results of the preceding sections concerning hulls, and in part it is used to obtain further results about polynomial hulls of sets of finite length.

For this discussion, a standard result about currents on $\mathbb{R}^n$ or on manifolds will be needed.

Theorem 3.8.1. *If T is a current of degree n on $\mathbb{R}^n$ that is closed, then T is of the form $T(\alpha) = c\int_{\mathbb{R}^n} \alpha$ for some constant c.*

Proof. The current T is a continuous linear functional acting on compactly supported smooth forms of degree n on $\mathbb{R}^n$ with the property that for each compactly supported smooth $(n-1)$-form β, $T(d\beta) = 0$. This means that there is a distribution ϕ on $\mathbb{R}^n$ such that for each n-form $\alpha = A\,dx_1 \wedge \cdots \wedge dx_n$, $T(\alpha) = \phi(A)$. The condition that T be closed is the condition that for each compactly supported $(n-1)$-form $\beta = \sum_{j=1}^n B_j \omega_{[j]}(x)$, $\phi\big(\sum_{j=1}^N (-1)^{j-1} \frac{\partial B_j}{\partial x_j}\big) = 0$. (As usual, we are using the notation that $\omega_{[j]}(x) = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge [j] \wedge dx_{j+1} \wedge \cdots \wedge dx_n$.)

Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be a smooth approximate identity with the property that $\operatorname{supp}\chi_\varepsilon$ is contained in $\{x \in \mathbb{R}^n : |x| < \varepsilon\}$. Let g_ε be the convolution $\phi * \chi_\varepsilon$. Thus g_ε is a function of class $\mathscr{C}^\infty$ with the property that for each smooth function A on $\mathbb{R}^n$ with compact support,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} A(x) g_\varepsilon(x) dx = \phi(A).$$

Moreover, for each derivative, D^α, $D^\alpha(\phi * \chi_\varepsilon) = (D^\alpha \phi) * \chi_\varepsilon = \phi * (D^\alpha \chi_\varepsilon)$.

The condition $bT = 0$ implies that if $\beta = \sum_{j=1}^n B_j \omega_{[j]}(x)$ is a compactly supported $(n-1)$-form on $\mathbb{R}^n$, then

$$0 = \int_{\mathbb{R}^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial B_j}{\partial x_j}(x) g_\varepsilon(x) dx.$$

This equation is correct for each choice of β and so implies, by an integration by parts, that

$$\int_{\mathbb{R}^n} H(x) \frac{\partial g_\varepsilon}{\partial x_j}(x) dx = 0$$

for each compactly supported function H and for each j. Thus, grad $g_\varepsilon = 0$, whence g_ε is a constant, say c_ε. We therefore reach the equality

$$\phi(H) = \lim_{\varepsilon \to 0^+} c_\varepsilon \int_{\mathbb{R}^n} H\, dx$$

for all H. Therefore the limit $\lim_{\varepsilon\to 0^+} c_\varepsilon$ exists; call it c. We then have that $T(\alpha) = c\int_{\mathbb{R}^n} \alpha$ for all compactly supported n-forms. The theorem is proved.

The extension to manifolds is important and immediate.

Corollary 3.8.2. *If $\mathscr{M}$ is an orientable n-dimensional smooth manifold and if T is a current of degree n on $\mathscr{M}$ that is closed, then there is a locally constant function g on $\mathscr{M}$ such that for all compactly supported smooth n-forms α on $\mathscr{M}$, $T(\alpha) = \int_{\mathscr{M}} g\alpha$.*

We now turn to the main subject of this section. The first result is a very special case of a general theorem proved by King [206].

Theorem 3.8.3. *Let Ω be a bounded domain in $\mathbb{C}^N$, let $E \subset \Omega$ be a relatively closed set with $\Lambda^2(E) = 0$, and let $V_1, \ldots, V_q$ be one-dimensional analytic subvarieties of $\Omega \setminus E$ each of which has finite area. If there are positive integers $m_1, \ldots, m_q$ such that $b(m_1[V_1] + \cdots + m_q[V_q]) = 0$, then the relative closure $\overline{V_1 \cup \cdots \cup V_q} \cap \Omega$ of the union $V_1 \cup \cdots \cup V_q$ is a one-dimensional subvariety of Ω.*

Explicitly written, the condition on the boundary is that for each compactly supported 1-form α defined on Ω,

$$m_1 \int_{V_1} d\alpha + \cdots + m_q \int_{V_q} d\alpha = 0.$$

The proof of this result depends on some preparatory lemmas. The first is an elementary fact from the theory of distributions.

Lemma 3.8.4. *Let W be an open set in the plane, and let h be a locally integrable function on W. If for every compactly supported function g of class $\mathscr{C}^\infty$ on W the integral $\int_W \frac{\partial g}{\partial \bar{z}}(z)h(z)\, d\bar{z} \wedge dz$ vanishes, then there is a holomorphic function $\tilde{h}$ on W that agrees with h almost everywhere.*

Proof. Suppose first that h is smooth. Then for a compactly supported smooth function g on W, we have

$$0 = \int_W d(ghdz) = \int_W \left\{ \frac{\partial g}{\partial \bar{z}}(z)h(z) + g(z)\frac{\partial h}{\partial \bar{z}}(z) \right\} d\bar{z} \wedge dz.$$

We have then that $\int_W g(z) \frac{\partial h}{\partial \bar{z}}(z)\, d\bar{z} \wedge dz = 0$. This is true for every choice of g, so the function h satisfies the Cauchy–Riemann equations and so is holomorphic.

When h is only locally integrable, we apply the preceding observation to a regularization of h. Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be a smooth approximate identity on $\mathbb{C}$, say $\chi_\varepsilon(z) = \varepsilon^{-2}\chi(z/\varepsilon)$, where χ is a compactly supported, even, nonnegative function of class $\mathscr{C}^\infty$ with $\int_{\mathbb{C}} \chi\, d\bar{z} \wedge dz = 1$. Fix an $\varepsilon_o > 0$ and define h_ε for $\varepsilon < \varepsilon_o$ on the domain

$$W_{2\varepsilon_o} = \{z \in W : \operatorname{dist}(z, bW) < 2\varepsilon_o\}$$

by

$$h_\varepsilon(z) = h * \chi_\varepsilon(z) = \int_W h(z-\zeta)\chi_\varepsilon(\zeta)\, d\bar\zeta \wedge d\zeta = \int_W h(\zeta)\chi_\varepsilon(z-\zeta)\, d\bar\zeta \wedge d\zeta.$$

The function h_ε is defined and smooth on $W_{2\varepsilon}$, and, by Fubini's theorem, it satisfies the condition that $\int_{W_{2\varepsilon}} \frac{\partial g}{\partial \bar z} h_\varepsilon = 0$ for every smooth function g with compact support in $W_{2\varepsilon}$. Therefore h_ε is holomorphic in $W_{2\varepsilon}$. Moreover, $\int_{W_{2\varepsilon}} |h_\varepsilon - h| \to 0$ as $\varepsilon \to 0^+$. The L^1 convergence of $\{h_\varepsilon\}$ on $W_{2\varepsilon}$ implies uniform convergence on compacta contained in $W_{2\varepsilon}$. Thus, h does indeed agree almost everywhere with a holomorphic function, as we wished to show.

The next lemma contains the main step in the proof of the theorem.

Lemma 3.8.5. *Let Ω be an open set in $\mathbb{C}^N$ of the form $\Omega = W \times W'$, where W is an open set in the plane and W' is an open set in $\mathbb{C}^{N-1}$. Let $E \subset \Omega$ be a relatively closed set with $\Lambda^2(E) = 0$. Let $V_1, \dots, V_q$ be one-dimensional analytic subvarieties of $\Omega \setminus E$ each of which has finite area. If the projection $\pi : \mathbb{C}^N \to \mathbb{C}$ onto the z_1-plane carries $V_1 \cup \dots \cup V_q \cup E$ properly onto W, and if for some positive integers $m_1, \dots, m_q$, we have $b(m_1[V_1] + \dots + m_q[V_q]) = 0$, then the relative closure of $V_1 \cup \dots \cup V_q$ in Ω is contained in a one-dimensional subvariety of Ω.*

Proof. Set $V = V_1 \cup \dots \cup V_q$. The map π carries $V \cup E$ properly onto W, so it carries the set E onto a closed subset of W, which has zero area. Then π carries the variety $V \setminus \pi^{-1}(\pi(E))$ properly onto $W \setminus \pi(E)$, and for each j, it carries $V_j \setminus \pi^{-1}(\pi(E))$ properly onto $W \setminus \pi(E)$. Consequently, for some positive integer μ_j, the fibers $\pi^{-1}(z_1) \cap V_j$ for $z_1 \in W \setminus E$ contain μ_j points, if multiplicities are counted appropriately.

Fix a j, and consider the current $\pi_*[V_j]$ on W defined by

$$\pi_*[V_j](\beta) = [V_j](\pi^*\beta) = \int_{V_j} \pi^*\beta.$$

Because $V_j \cap \pi^{-1}(\pi(E))$ is a set of vanishing two-dimensional measure, we can invoke Fubini's theorem to write

$$\int_{V_j} \pi^*\beta = \int_{V_j \setminus \pi^{-1}(\pi(E))} \pi^*\beta = \mu_j \int_{W \setminus \pi(E)} \beta,$$

so $\pi_*[V_j] = \mu_j[W]$, and we find that

$$\pi_*(m_1[V_1] + \dots + m_q[V_q]) = (m_1\mu_1 + \dots + m_q\mu_q)[W].$$

Denote the integer $m_1\mu_1 + \dots + m_q\mu_q$ by ν.

For $j = 2, \dots, N$, we will now produce a function $P_j \in \mathscr{O}(\Omega)$ of the form

$$P_j(z) = z_j^\nu + a_{j,1}(z_1)z_j^{\nu-1} + \dots + a_{j,\nu-1}(z_1)z_j + a_{j,\nu}(z_1) \tag{3.5}$$

with coefficients $a_{j,k}$ holomorphic on W such that $P_j^{-1}(0) \supset V$. To do this, define a polynomial $P_j(z_1, X)$ in the indeterminate X for z_1 in the open set $W \setminus \pi(E)$ by the condition

$$P_j(z_1, X) = \prod_{r=1,\dots,q} P_j^{(r)}(z_1, X),$$

where for each $r = 1, \dots, q$, the polynomial $P_j^{(r)}(z_1, X)$ is associated with the variety V_r and is defined by

$$P_j^{(r)}(z_1, X) = \prod [(X - (w^{(r,k)}(z_1))_j)]^{\mu_j}.$$

In the latter product, the product extends over all the points $w^{(r,k)}(z_1)$ in the fiber $\pi^{-1}(z_1)$ that lie in the variety V_r, each included according to its multiplicity, and $(w^{(r,k)}(z_1))_j$ denotes the jth coordinate of the point $w^{(r,k)}(z_1)$. Thus, $P_j^{(r)}(z_1, X)$ is a polynomial of degree $m_r\mu_r$, and the total degree of $P_j(z_1, X)$ is ν. The polynomial $P_j(z_1, X)$ can be expanded into the form (3.5). The coefficients are then the elementary symmetric functions of the numbers $(w^{(r,k)}(z_1))_j$, taken with the indicated multiplicities. These are well-defined functions on the domain W; we claim they they are holomorphic. To establish the holomorphicity of the coefficients, we need to recall that the elementary symmetric functions can be expressed as polynomials with integral coefficients in the power sums of the $(w^{(r,k)}(z_1))_j$, i.e., as polynomials in the functions σ_p given, for $p = 0, 1, \dots,$ by

$$\sigma_p(z_1) = \sum \left[(w^{(r,k)}(z_1))_j\right]^p.$$

These power sums are holomorphic on $W \setminus \pi(E)$. That is to say, if we define $\psi_{j,p} : \mathbb{C}^N \to \mathbb{C}$ by $\psi_{j,p}(z) = z_j^p$, then $\sum_{z\in\pi^{-1}(z_1)\cap V_r} \psi_{j,p}(z)$ is a holomorphic function on $W \setminus \pi(E)$. That this function is holomorphic on $W \setminus \pi(E)$ is proved by the standard argument involving the Cauchy integral formula and the residue theorem in one variable that is used in treating the Weierstrass preparation theorem and the Weierstrass division theorem. That it continues holomorphically through all of W is contained in the next lemma.

Lemma 3.8.6. *If f is holomorphic on $\pi^{-1}(W)$, and if $s_{f,p}$ is defined on $W \setminus \pi(E)$ by $s_{p,f}(z_1) = \sum_{w\in\pi^{-1}(z_1)\cap V} m(w) f^p(w)$, where $m(w)$ is m_j if $w \in V_j$, then $s_{f,p}$, which is holomorphic on $W \setminus \pi(E)$, extends holomorphically through all of W.*

Note that the value $m(w)$ is well defined outside a subset of $V_1 \cup \cdots \cup V_q$ of measure zero.

Proof. It is in this lemma that the hypothesis that $b(m_1[V_1] + \cdots + m_q[V_q]) = 0$ is used. The function $s_{f,p}$ is defined and holomorphic on $W \setminus \pi(E)$, and it is locally bounded on W, and so a fortiori is locally integrable. To prove that it extends holomorphically through all of W, we need only show that $\int_W s_{f,p}\bar{\partial} g \wedge dz_1 = 0$ for all smooth functions g on W that have compact support. But this is immediate: If g is a smooth function on W with compact support, then by hypothesis we have

$$\begin{aligned} 0 &= \sum_{j=1}^{q} m_j \int_{V_j} f^p \, d((g \circ \pi) dz_1) = \sum_{j=1}^{q} m_j \int_{V_j} f^p \pi^* \bar{\partial}(g dz_1) \\ &= \int_W \sum_{w\in\pi^{-1}(z)} m_j(w) f^p(w) \, \bar{\partial} g \wedge dz_1 = \int_W s_{f,p} \, \bar{\partial} g \wedge dz_1, \end{aligned}$$

so that, as desired, the function $s_{f,p}$ continues holomorphically into all of W.

We now know that the coefficients $a_{j,k}$ in the polynomial (3.5) extend to be holomorphic in the domain W. We shall denote these extensions again by $a_{j,k}$. If we set, for $j = 2, \dots, N$, $P_j(z) = P_j(z_1, z_j)$, then $P_j \in \mathscr{O}(\Omega)$, and the intersection $P_2^{-1}(0) \cap \cdots \cap P_N^{-1}(0)$ is a one-dimensional subvariety of Ω that contains V. This completes the proof of Lemma 3.8.5.

To prove the theorem we need the notion of clear coordinate system.

Definition 3.8.7. *If S is a subset of $\mathbb{C}^N$ and p is a point of S, then a coordinate system with linear coordinates $\zeta_1, \dots, \zeta_N$ centered at p is said to be a* clear coordinate system *with respect to the set S if for each $j = 1, \dots, N$, there is a neighborhood W_j of p such that the function ζ_j carries $W_j \cap S$ properly into a neighborhood of* $0 \in \mathbb{C}$.

Lemma 3.8.8. *If $E \subset \mathbb{C}^N$ is a closed set that satisfies $\Lambda^3(E) = 0$, and if $p \in E$, then there is a clear set of coordinates at p with respect to E.*

The proof we give will establish more: There are many such sets of clear coordinates.

Proof. Without loss of generality, we can suppose p to be the origin. Theorem 3.3.10 implies that for almost all $L \in \mathbb{G}_{N,N-1}$, the set $E \cap L$ has zero length. Consequently, there are linear coordinates $\zeta_1, \dots, \zeta_N$ centered at the origin such that for each $j = 1, \dots, N$, the set $L_j = \zeta_j^{-1}(0)$ meets E in a set of zero length.

We fix our attention on a particular choice of j, say $j = 1$. The set L_j meets E in a set of length zero, so there is a small sphere S of dimension $2N - 3$ centered at the origin and contained in L_1 that is disjoint from E. The sphere S is defined as the set $\{(0, \zeta_2, \dots, \zeta_N) \in \mathbb{C}^N : |\zeta_2|^2 + \cdots + |\zeta_N|^2 = r^2\}$ for some small positive r. The set E is closed. Consequently, if $W_1 = \{(\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N : |\zeta_1| < \delta, \text{ and } |\zeta_2|^2 + \cdots + |\zeta_N|^2 = r^2\}$ for a small positive δ, then W_1 is a neighborhood of 0 with the property that the coordinate function ζ_1 carries $E \cap W_1$ properly to the disk of radius δ in $\mathbb{C}$. The other coordinates are treated similarly; the lemma is proved.

Proof Theorem 3.8.3. Let $V = V_1 \cup \cdots \cup V_q$. Fix a point $z \in \bar{V} \cap \Omega$.

Because $\Lambda^3(\bar{V} \cap \Omega) = 0$, there is a set of clear coordinates for the set $\bar{V} \cap \Omega$ centered at z. Let these coordinates be $z_1, \dots, z_N$. Apply Lemma 3.8.5 to find a one-dimensional subvariety W in a neighborhood $\Omega(z)$ of z with $V \cap \Omega(z) \subset W$.

Consider the current T defined on W_{reg} to be the restriction of the current $\sum_{j=1}^q m_j[V_j]$, so that for a 2-form α defined on W_{reg} and having compact support in this surface, $T(\alpha) = \sum_{j=1}^n m_j \int_{V_j} \alpha$. This current is closed: $dT = 0$, so by Corollary 3.8.2 it is of the form $T(\alpha) = \sum_k c_k[W_{\text{reg},k}]$, where the $W_{\text{reg},k}$ are the components of the complex manifold W_{reg} and the c_k are suitable constants. The only k for which c_k is not zero are those for which the associated $W_{\text{reg},k}$ contains a branch of one of the V_j, and, granted that $W_{\text{reg},k}$ does contain a branch of $V_j \cap \Omega$, it follows that the constant c_k has to be the associated integer m_j. If we let W'_{reg} be the union of the components of W_{reg} for which the associated c_k is not zero, then we have $W' \supset \cup_j V_j$, and the equality of the currents $\sum_k c_k[W_{\text{reg},k}]$ and $\sum_j m_j[V_j \cap \Omega]$ implies that the sets W'_{reg} and $\cup_j V_j \cap \Omega(z)$ coincide. Consequently, the set $\overline{\cup_j V_j} \cap \Omega(z)$ coincides with the union of the global branches of W

that contain branches of some $V_j \cap \Omega$. We now see that, as claimed, the set $\overline{\cup_j V_j} \cap \Omega$ is a variety.

The theorem is proved.

The next result concerns curves that are invariant under complex conjugation. It was found by Alexander [4]. Let $\tau : \mathbb{C}^N \to \mathbb{C}^N$ be the conjugation operator given by

$$\tau(z_1, \ldots, z_N) = (\bar{z}_1, \ldots, \bar{z}_N),$$

an antiholomorphic involution of $\mathbb{C}^N$.

Theorem 3.8.9. *If $\Omega \subset \mathbb{C}^N$ is a domain that is invariant under the action of τ, and if V is a one-dimensional subvariety of $\Omega \setminus \mathbb{R}^N$ that is invariant under the action of τ, then $\bar{V} \cap \Omega$ is a one-dimensional subvariety of Ω.*

With the additional assumption that V has finite area, the result was established earlier by Shiffman [328].

Proof. The problem is to prove that near a point x of $\bar{V} \cap \Omega \cap \mathbb{R}^N$, the set $\bar{V}$ has the structure of a one-dimensional variety. Fix such a point x, and choose an $r > 0$ small enough that the closed ball $\bar{B}(r)$ centered at x and of radius r is contained in Ω. By choosing r properly, we shall have that $bB(r) \cap V \setminus \mathbb{R}^N$ is a locally finite family of open analytic arcs and simple closed curves. Set $Y = \bar{V} \cap \bar{B}(r)$, and denote by Γ the Shilov boundary of the algebra $\mathscr{P}(Y)$, so that Γ is the smallest closed subset of Y whose polynomially convex hull coincides with the polynomially convex hull of Y.

We shall show that $\Gamma = Y \cap bB(r) = \bar{V} \cap bB(r)$. By the maximum principle, $\Gamma \cap V \cap B(r) = \emptyset$. Further, no point of Γ can lie in $\mathbb{R}^N \cap B(r)$, as we see in the following way. Suppose Γ to meet $\mathbb{R}^N \cap B(r)$. There is then a polynomial P such that $\| P\|_{\Gamma \cap bB(r)} < \frac{1}{2}$ but $1 = \|P\|_\Gamma = P(y)$ for some point $y \in Y \cap \mathbb{R}^N \cap B(r)$. Introduce the polynomial Q given by $Q(z) = P(z)\overline{P(\bar{z})}$, a polynomial real on $\mathbb{R}^N$ that satisfies $Q(y) = 1$ and $\|Q\|_{\Gamma \cap bB(r)} < \frac{1}{4}$. The maximum principle implies that $Q(\bar{V} \cap \bar{B}(r))$ is contained in the set $\{\zeta \in \mathbb{C} : |\zeta| \leq \frac{1}{4}\} \cup [0, 1]$. The point y lies in the closure of the set $V \cap B(r)$, so there are points z in V near the point y. For such a point z, we have that $Q(z)$ is in the interval $(\frac{1}{4}, 1]$, so there is a polynomial in one variable, say p, such that $p(Q(z)) = 1$ but $|p \circ Q| < 1$ on the set $\bar{V} \cap Q^{-1}(Q(z))$, in contradiction to the maximum principle. Thus $\Gamma = Y \cap bB(r) = \bar{V} \cap bB(r)$.

Next, for $n = n_o, n_o + 1, \ldots$ for some sufficiently large n_o, its choice depends on the value of r, let

$$X_n = \Gamma \cup \{z \in \mathbb{C}^N : \operatorname{dist}(z, \Gamma \cap \mathbb{R}^N) < 1/n\}\widehat{\ }.$$

By Theorem 3.1.1, the polynomially convex hull $\widehat{X}_n$ has the property that the set $W_n = \widehat{X}_n \setminus X_n$ is either empty or else is an analytic variety. The set W_n contains

$$B(r) \cap V \setminus \{z \in \mathbb{C}^N | \operatorname{dist}(z, \Gamma \cap \mathbb{R}^N) < 1/n\}\widehat{\ },$$

and so it is not empty. The intersection of an arbitrary family of varieties is itself a variety, so the set of points $z \in \mathbb{C}^N$ that lie in all but finitely many of the sets W_n is a one-dimensional analytic subvariety, W, of $B(r)$ that contains $B(r) \cap V$ and consequently $B(r) \cap \bar{V}$.

We have now shown that $B(r) \cap \bar{V}$ is contained in the subvariety W of $B(r)$. We have to show that $B(r) \cap \bar{V}$ actually *is* a subvariety of $B(r)$. Each of the sets X_n is invariant under the reflection τ, so the same is true of the variety W. The set of singular points of the variety W is discrete, so if $B(r')$ is a ball centered at the point x of W and of sufficiently small radius, then $B(r') \Subset B(r)$, and the point x is the only possible singular point of the variety W in $B(r')$. In addition, $W \cap B(r')$ has a finite number of global branches, say $W \cap B(r') = W_1 \cup \cdots \cup W_q$, each of which is irreducible at the point x. We show that $V \cap B(r')$ is a union of some, perhaps all, of the W_j. For each j, $V \cap W_j$ is either empty or else is open in W_j. In the latter case, let $y \in W_j$ be a point of $\bar{V} \setminus V$. The point y lies in $\mathbb{R}^N$. Near y, $\mathbb{R}^N$ splits W_j into two pieces. We have that W_j is invariant under the reflection τ, for W is τ-invariant, and the various W_j's meet only at x. The set V is invariant under τ, so it follows that near x, $V \cap W_j$ agrees with $W_j \setminus \mathbb{R}^N$. Consequently, $V \cap W_j$ is dense in W_j, so $W_j \setminus \mathbb{R}^N \subset V$. Thus $V \cap B(r')$ is a union of some of the sets W_j. The theorem is proved.

Corollary 3.8.10. [149] *Let Ω be an open subset of $\mathbb{C}^N$ that contains the open analytic arc λ as a closed subset. If S_1 and S_2 are irreducible one-dimensional subvarieties of $\Omega \setminus \lambda$ such that $\lambda \subset \bar{S}_1$ and $\lambda \subset \bar{S}_2$, then either S_1 and S_2 coincide or else $S_1 \cup \lambda \cup S_2$ is a subvariety of Ω.*

Proof. The arc λ is an open analytic arc and a closed subset of Ω. It follows that there is a domain Ω' that contains λ and that is a subdomain of Ω in which there are holomorphic coordinates $\zeta = (\zeta_1, \dots, \zeta_N)$ such that λ is the $\Re\, \zeta_1$ coordinate axis. Denote by R_ζ the component of the submanifold of Ω' on which $\zeta_1, \dots, \zeta_N$ are all real-valued that contains λ. There is an antiholomorphic reflection, say ϱ, of a neighborhood of R_ζ that leaves R_ζ fixed pointwise. By shrinking Ω' we can suppose ϱ to be defined on all of Ω' and that Ω' is invariant under the reflection ϱ. Let $S_1' = S_1 \cap \Omega'$ and similarly for S_2'. Let V_1 be the variety $S_1' \cup \varrho(S_1')$, V_2 the variety $S_2' \cup \varrho(S_2')$. According to Theorem 3.8.9, V_1 and V_2 extend to analytic subvarieties of Ω'. They are necessarily irreducible, for S_1 and S_2 are supposed to be irreducible. Their intersection contains the arc λ, so they coincide. At this point, there are two possibilities. It may happen that $S_1 = S_2$, and we are done. Alternatively, $S_1 = \varrho(S_2)$, and then, $S_2 = \varrho(S_1)$. In this case, we have that $S_1 \cup \lambda \cup S_2$ is a variety. The proof is complete.

We will indicate also the proofs of the following two corollaries, though they draw on some points in algebraic and analytic geometry whose development lies beyond the scope of the present work.

Corollary 3.8.11. (Tornehave—see [327].) *If V is a one-dimensional subvariety of the unit polydisk $\mathbb{U}^N$ such that $\bar{V} \setminus V \subset \mathbb{T}^N$, then there is an algebraic curve Σ in $\mathbb{C}^N$ with $V = \Sigma \cap \mathbb{U}^N$.*

Here, as usual, $\mathbb{T}^N$ is the distinguished boundary of the polydisk.

Proof. We shall use $\mathbb{C}^{*N}$ to denote the N-fold product of the Riemann sphere with itself. The proof of the corollary uses the antiholomorphic reflection of $\rho : \mathbb{C}^{*N} \to \mathbb{C}^{*N}$ given by $\rho(z_1, \dots, z_N) = (1/\bar{z}_1, \dots, 1/\bar{z}_N)$ with the conventions that $1/0 = \infty$ and $1/\infty = 0$. The involution ρ has the torus $\mathbb{T}^N$ as its fixed-point set. Under the exponential mapping a neighborhood of a point $p \in \mathbb{T}^N$ is equivalent to a neighborhood of a point $\tilde{p} \in \mathbb{R}^N$ with a piece of $\mathbb{T}^N$ corresponding to a piece of $\mathbb{R}^N$.

Let $V \subset \mathbb{U}^N$ be a one-dimensional variety as in the statement of the corollary. The set ρV is a one-dimensional subvariety of $\mathbb{C}^{*N} \setminus \mathbb{T}^N$, for ρ is antiholomorphic. Theorem 3.8.9 implies that the set $\overline{V \cup \rho(V)}$ is a one-dimensional subvariety of $\mathbb{C}^{*N}$.

The result now follows from a theorem of Chow [158, p. 170], according to which subvarieties of the projective spaces $\mathbb{P}^M(\mathbb{C})$ are algebraic. (To be sure, for $N > 1$, $\mathbb{C}^{*N}$ is not a projective space, but it embeds into $\mathbb{P}^{2^N-1}(\mathbb{C})$ under the Segre embedding, which embeds a product of projective spaces into a projective space. The Segre embedding is discussed in [256].)

Corollary 3.8.12. [327] *If $\mathscr{R}$ is a connected Riemann surface and f_1 and f_2 are meromorphic functions on $\mathscr{R}$ such that $|f_j(p)| \to 1$ as $p \to \infty$, $j = 1, 2$, then f_1 and f_2 are algebraically related.*

Explicitly, there is a nonzero polynomial P in two variables such that $P(f_1, f_2) = 0$ on $\mathscr{R}$.

Proof. The hypothesis implies that each of the functions f_1, f_2 has at most finitely many poles in $\mathscr{R}$. Accordingly, there is a neighborhood Ω of $\mathbb{T}^2$ in $\mathbb{C}^2$, which we can take to be invariant under the antiholomorphic involution ρ used in the preceding corollary, such that under the map $F : \mathscr{R} \to \mathbb{C}^{*2}$ given by $F(p) = (f_1(p), f_2(p))$, the Riemann surface $\mathscr{R} \setminus F^{-1}(\Omega)$ is carried properly into $\Omega \setminus \mathbb{T}^2$. By the proper mapping theorem for varieties (see [158, p. 162]), the image $F(F^{-1}(\Omega))$ is a subvariety of $\Omega \setminus \mathbb{T}^2$. Call this variety V. Then $V \cup \rho(V)$ is a subvariety of $\Omega \setminus \mathbb{T}^2$, and it is invariant under the action of the involution ρ used in the preceding lemma. Consequently, $\overline{V \cup \rho(V)} \cap \Omega$ is a subvariety of Ω. It follows that the closure of the set $F(\mathscr{R}) \cup F(\rho(\mathscr{R}))$ in $\mathbb{C}^{*2}$ is a subvariety of $\mathbb{C}^{*2}$, and the corollary follows again from Chow's theorem.

Next we apply the Bochner–Martinelli kernel to study the boundaries of bounded varieties.

If $w \in \mathbb{C}^N$, then the Bochner–Martinelli kernel

$$k_{\mathrm{BM}}(z, w) = |z - w|^{-2N} \omega'(\overline{z - w}) \wedge \omega(z)$$

is a closed form of degree $(2N - 1)$ on the manifold $\mathbb{C}^N \setminus \{w\}$, and it is not exact, for

$$1 = c_N \int_{\{z:\ |z-w|=1\}} k_{\mathrm{BM}}(z, w).$$

Thus, it determines a nonzero element $[k_{\mathrm{BM}}(\cdot, w)]$ in the de Rham cohomology group $H^{2N-1}_{\mathrm{deR}}(\mathbb{C}^N \setminus \{w\})$. By de Rham's theorem, this group is isomorphic to the singular group $H^{2N-1}(\mathbb{C}^N \setminus \{w\}; \mathbb{C})$, which is isomorphic to $H^{2N-1}(\mathbb{S}^{2N-1}; \mathbb{C}) \simeq \mathbb{C}$, so each element of $H^{2N-1}_{\mathrm{deR}}(\mathbb{C}^N \setminus \{w\})$ is a complex multiple of the class determined by the Bochner–Martinelli kernel $K_{\mathrm{BM}}(\cdot, w)$.

Definition 3.8.13. *If V is an analytic variety in an open set Ω in $\mathbb{C}^N$ (or in a complex manifold), the* boundary *of V is the set $bV = \bar{V} \setminus V$.*

The set bV is closed; in the event that the open set Ω is bounded, the boundary bV is compact. In the latter case, V is contained in the polynomially convex hull $\widehat{bV}$.

Recall that a compact subset X of $\mathbb{C}^N$ is said to be convex with respect to varieties of dimension p if for each point $z_o \in \mathbb{C}^N \setminus X$ there is a purely p-dimensional subvariety V of $\mathbb{C}^N$ that is disjoint from X but that contains z_o. A related notion is that of convexity with respect to p-dimensional set-theoretic complete intersections:

Definition 3.8.14. *A compact subset X of $\mathbb{C}^N$ is* convex with respect to p-dimensional set-theoretic complete intersections *if for each point $z_o \in \mathbb{C}^N \setminus X$ there is a holomorphic map $F : \mathbb{C}^N \to \mathbb{C}^{N-p}$ such that the fiber $F^{-1}(0)$ is purely p-dimensional and contains z_o but is disjoint from X.*

Convexity with respect to p-dimensional set-theoretic complete intersections is the same as convexity with respect to p-dimensional *manifolds* that are set-theoretic complete intersections.

One case of these notions is familiar: In $\mathbb{C}^N$ convexity with respect to varieties of dimension $N-1$ is simply the notion of rational convexity, as we have noted above. The condition of convexity with respect to set-theoretic complete intersections of dimension $N-1$ is equivalent in $\mathbb{C}^N$ to the condition of convexity with respect to $(N-1)$-dimensional varieties, because the second Cousin problem is universally solvable on $\mathbb{C}^N$. Whether this equivalence persists in all dimensions is not clear.

However, convexity with respect to one-dimensional varieties is equivalent to the condition of convexity with respect to one-dimensional complete intersections. This is so, for it is known that every one-dimensional complex submanifold of $\mathbb{C}^N$ is a set-theoretic (and, indeed, an ideal-theoretic) complete intersection. This is a deep result found in [121]; we cannot enter into the details here, nor shall we use the result below.

Theorem 3.8.15. *If V is a purely k-dimensional subvariety of a bounded domain in $\mathbb{C}^N$ with $\Lambda^{2k}(bV) = 0$, then $\check{H}^{2k-1}(bV; \mathbb{Z}) \neq 0$.*

This theorem, with the coefficients $\mathbb{Z}$ replaced by coefficients in $\mathbb{C}$, was obtained in [148].

Proof. We shall prove the theorem only in the case that V is a manifold. The proof in the general case runs along precisely the same lines but requires a less-standard version of Stokes's theorem than the one we use, which can be found, e.g., in [343]

Thus, we suppose V to be a purely k-dimensional submanifold of the bounded open set Ω in $\mathbb{C}^N$ with $\Lambda^{2k}(bV) = 0$. We will construct a holomorphic map $f : \mathbb{C}^N \to \mathbb{C}^k$ such that $0 \in f(V)$ and $0 \notin f(bV)$. To do this, suppose $0 \in V$ and that the tangent space T_0V is $\mathbb{C}^k$. Let $\pi : \mathbb{C}^N \to \mathbb{C}^k$ be the orthogonal projection. The set $\pi(bV)$ has measure zero in $\mathbb{C}^k$, so there is a point $a \in \pi(V)$ near 0 such that $a \notin (\pi)(bV)$. In this case, we take $f = \pi - a$.

Write $f = (f_1, \dots, f_k)$, so that the functions $f_1, \dots, f_k$ are holomorphic on $\mathbb{C}^N$, they have a common zero in V, and they have no common zero near bV. We shall show that the map $f/|f|$ from bV to $\mathbb{S}^{2k-1}$ is not homotopic to a constant map.

With $\omega(f) = df_1 \wedge \cdots \wedge df_k$ and $\omega'(\bar{f}) = \sum_{j=1}^{k} (-1)^{j-1} \bar{f}_j d\bar{f}_1 \wedge \cdots \wedge \widehat{d\bar{f}_j} \wedge \cdots \wedge d\bar{f}_k$, we let θ be the form given by $\theta = |f|^{-2k} \omega'(\bar{f}) \wedge \omega(f)$. That is to say,

$\theta = f^* k_{\mathrm{BM}}(z, 0)$, where k_{BM} is the Bochner–Martinelli kernel on $\mathbb{C}^k$.

Choose a sequence $\{\Omega_j\}_{j=1,\ldots}$ of domains in $\mathbb{C}^N$ with $\Omega_j \Supset \Omega_{j+1}$, with $\cap\Omega_j = bV$, and such that $V \cap \Omega_j$ is a domain on which Stokes's theorem is applicable. (For this, it suffices that $b\Omega_j$ be smooth and transversal to the manifold V.) Let $\Gamma_j = b(\Omega_j \cap V)$. For large j,

$$\int_{\Gamma j} \theta = \int_{\Gamma j+1} \theta$$

by Stokes's theorem, for θ is a closed form. For large j, these integrals are nonzero.

This is so, for because $f^{-1}(0)$ avoids a neighborhood of bV, the set

$$S = \{z \in V : f(z) = 0\}$$

is a compact variety in $\mathbb{C}^N$ and so is finite. Let S= $\{p_1, \ldots, p_s\}$. For small $\varepsilon > 0$ and each $j = 1, \ldots, s$, let W_j be the component of the set where $|f| < \varepsilon$ that contains p_j. We have by Stokes's theorem that when j is large,

$$\int_{\Gamma j} \theta = \sum_{j=1}^{s} \int_{bW_j} \theta = \sum_{j=1}^{s} \varepsilon^{-2k} \int_{W_j} \omega(\bar{f}) \wedge \omega(f),$$

and this quantity is not zero.

It follows that the map f from Γ_j to $\mathbb{C}^k \setminus \{0\}$ is not homotopic to a constant. The same is therefore true of the map $f/|f|$ from Γ_j to $\mathbb{S}^{2k-1}$ for large j: The map $f/|f|$ from $\mathbb{C}^N \setminus f^{-1}(0)$ to $\mathbb{S}^{2k-1}$ is continuous. If it were homotopic to a constant on bV, then it would be homotopic to a constant on a neighborhood of bV in $\mathbb{C}^N$ and so would be homotopic to a constant on Γ_j for large j. The preceding paragraph implies, therefore, that $f/|f|$ is not homotopic to a constant on V.

To conclude the argument, we must invoke some dimension theory. The standard reference is the classic of Hurewicz and Wallman [185]. The hypothesis that $\Lambda^{2k}(bV) = 0$ implies that the topological dimension of bV is not more than $2k - 1$, by [185, Theorem VII.3, p. 164]. Since the map $f/|f|$ from bV to $\mathbb{S}^{2k-1}$ is not homotopic to a constant, it follows that $\check{H}^{2k-1}(bV; \mathbb{Z}) \neq 0$ by [185, Corollary, p. 150].

The theorem is proved.

In the case of one-dimensional varieties, which we shall use below, there is an essentially simpler derivation of the result of Theorem 3.8.15 than the one given above: We are to show that if V is a one-dimensional subvariety of a bounded domain in $\mathbb{C}^N$ such that $\Lambda^2(bV) = 0$, then the group $\check{H}^1(bV; \mathbb{Z})$ does not vanish. As in the proof of Theorem 3.8.15 given above, we have a holomorphic function f on $\mathbb{C}^N$ that assumes the value zero at some points of V but that is zero-free on bV. If $\check{H}^1(bV, \mathbb{Z}) = 0$, then there is a branch of $\log f$ defined on a neighborhood of bV. Then f has a logarithm on the curves Γ_j for large j. By the argument principle, this is incompatible with f's vanishing at some points of V.

Remark. There is an extension of a weakened version of the preceding result to a slightly different class of boundaries: *If in Theorem 3.8.15 it is assumed that bV is convex with* *respect to* $(N-k)$*-dimensional complete intersections, then* $\check{H}^{2k-1}(bV;\mathbb{C})=0$. The convexity hypothesis implies the existence of a map $f:\mathbb{C}^N\to\mathbb{C}^k$ that vanishes at some point of V but at no point of V. We now use the domains Ω_j, the manifolds Γ_j, and the form θ constructed above.

That $\int_{\Gamma_j}\theta\neq 0$ implies that the form θ represents a nonzero element $[\theta]_j$ in the de Rham cohomology class $H^{2k-1}_{\text{deR}}(\Omega_j)$. The map $H^{2k-1}_{\text{deR}}(\Omega_j)\to H^{2k-1}_{\text{deR}}(\Omega_{j+1})$ induced by the inclusion $\Omega_{j+1}\hookrightarrow\Omega_j$ carries $[\theta]_j$ to $[\theta]_{j+1}$.

Now $H^{2k-1}_{\text{deR}}(\Omega_j)\simeq\check{H}^{2k-1}(\Omega_j;\mathbb{C})\simeq\check{H}^{2k-1}(\bar{\Omega}_j;\mathbb{C})$. Thus, the nonzero element $[\theta]_j\in H^{2k-1}_{\text{deR}}(\Omega_j)$ corresponds to a nonzero element $\check{[\theta]_j}\in\check{H}^{2k-1}(\bar{\Omega}_j,\mathbb{C})$, and under the map $\iota_{j,j+1}:\check{H}^{2k-1}(\bar{\Omega}_j,\mathbb{C})\to\check{H}^{2k-1}(\bar{\Omega}_{j+1},\mathbb{C})$ induced by the inclusion $\bar{\Omega}_{j+1}\hookrightarrow\bar{\Omega}_j$, $\check{[\theta]_j}$ goes to $[\theta]_{j+1}$. The compact space bV is the *inverse* limit of the system of compact spaces $\bar{\Omega}_j$, $j=1,\dots,$ with the natural inclusions, so $\check{H}^{2k-1}(bV;\mathbb{C})$ is the *direct* limit of the groups $\check{H}^{2k-1}(\Omega_j;\mathbb{C})$. The elements $\check{[\theta]_{j+1}}$ taken together give a nonzero element of this group, so $\check{H}^{2k-1}(bV;\mathbb{C})\neq 0$, as we claimed.

These results may be thought of as a geometric version of a simple observation in classical function theory: Let g be a bounded holomorphic function on the unit disk. If the global cluster set of g at $b\mathbb{U}$ is nowhere dense, then it separates the plane.

Remark. The relation between the condition that $\Lambda^{2k}(bV)=0$ and the condition that bV be convex with respect to $(N-k)$-dimensional complete intersections cases is not clear. We know that a compact set that has vanishing two-dimensional measure is rationally convex and so convex with respect to $(N-1)$-dimensional complete intersections. More seems not to be known.

Examples. We have already seen examples of the phenomena of the last two theorems. In Section 1.6.2 we have exhibited arcs and Cantor sets that are of the form bV for one-dimensional varieties in bounded domains of $\mathbb{C}^3$. If E is an arc or a Cantor set that is bV for a one-dimensional subvariety of a bounded domain in $\mathbb{C}^N$, then the cohomology group $\check{H}^1(bV;\mathbb{C})$ vanishes, so the set E cannot be rationally convex; in particular, it cannot have vanishing two-dimensional measure.

We will next establish a result on the continuation of one-dimensional varieties. This work must be preceded by a lemma about logarithms on thin subsets of $\mathbb{R}^N$.

Lemma 3.8.16. *If* X *is a compact subset of* $\mathbb{R}^N$ *with zero two-dimensional Hausdorff measure and if* E *is a compact subset of* X*, then every zero-free function on* E *extends to a zero-free function on* X.

Proof. If $f\in\mathscr{C}(E)$ is zero-free, then for some $\delta>0$, $|f(x)|\geq\delta$ for all $x\in E$. There is a function f' that is continuous on all of $\mathbb{R}^N$ that agrees with f on E and that is of class $\mathscr{C}^\infty$ on $\{x\in\mathbb{R}^N:|f'(x)|<\delta/2\}$. Then the set $f'(X)\cap\mathbb{U}(\delta/2)$ is a set with area zero in $\mathbb{C}$, so there are α's in $\mathbb{C}$ with $|\alpha|$ arbitrarily near zero such that $f'-\alpha$ is zero-free on X. If α is sufficiently small then $|f/(f'-\alpha)|<\frac{1}{2}$ on E. Thus, for some continuous function h on E, $f(f'-\alpha)^{-1}=e^h$ on the set E. If $h'\in\mathscr{C}(X)$ agrees on E with f, then the function $(f'-\alpha)e^{h'}$ is a continuous, zero-free function on X that agrees on E with h. The lemma is proved.

Corollary 3.8.17. *If X is a compact subset of $\mathbb{R}^N$ with zero two-dimensional measure, then for each compact subset E of X, the natural map $\check{H}^1(X;\mathbb{Z}) \to \check{H}^1(E;\mathbb{Z})$ is surjective.*

The corollary follows from the lemma because of the theorem of Bruschlinsky discussed at the end of Section 1.2.

Remark. The preceding lemma and its corollary are special cases of much more general results in dimension theory. See [185].

Theorem 3.8.18. *Let D be a bounded domain in $\mathbb{C}^N$, let E be a compact subset of $\bar{D}$, and let V be a one-dimensional subvariety of $D \setminus E$. If $\Lambda^2(E) = 0$, if $\check{H}^1(E;\mathbb{Z}) = 0$, and if $E \cap bD$ is a single point, then $\bar{V} \cap D$ is a one-dimensional variety.*

Note that in this result there is no hypothesis of smoothness on the boundary bD nor is there a pseudoconvexity hypothesis on D. For $D = \mathbb{B}_N$, the result was given Y. Xu [376].

An example of this result occurs when D is the unit ball in $\mathbb{C}^N$ and E is an arc with zero two-dimensional measure that meets the boundary of the ball in a single point.

Corollary 3.8.19. *Let D be a bounded domain in $\mathbb{C}^N$, and let λ be a rectifiable arc with endpoints in bD, interior in D, and let V be a purely one-dimensional subvariety of $D \setminus \lambda$. If there is a point $p \in \lambda$ such that for some neighborhood U of p in D, $(V \cup \lambda) \cap U$ is a one-dimensional subvariety of U, then $V \cup \lambda$ is a one-dimensional subvariety of D.*

Corollary 3.8.20. *If D is a strictly pseudoconvex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$, and if $V \subset D$ is a purely one-dimensional variety with $\Lambda^2(bV) = 0$, then no open subset of bV is totally disconnected.*

Proof. Suppose $E \subset bV$ to be an open set of bV that is totally disconnected. Fix $p \in E$. There is then an open connected set Ω in $\mathbb{C}^N$ with $p \in \Omega$ and such that $b\Omega \cap E = \emptyset$. The intersection $V \cap \Omega$ is not empty, for $p \in bV$. Theorem 3.8.18 implies that $W = \bar{V} \cap \Omega$ is an analytic subvariety of Ω. This is impossible, for there is a function f holomorphic on a neighborhood of $\bar{D}$ that satisfies $f(p) = 1 > |f(z)|$ for all $z \in \bar{D} \setminus \{p\}$. The corollary is proved.

Proof of Theorem 3.8.18. The set $E \cap bD$ is a point, which we assume to be the origin, so there is a domain $\Omega \subset D$ such that $E \subset \Omega$, such that $0 \in b\Omega$, such that $b\Omega \setminus \{0\}$ is a real-analytic hypersurface, say Σ, that meets V in a subset of V_{reg} that is a one-dimensional real-analytic submanifold of Σ, and such that $E \cap b\Omega = \{0\}$. Let $V_o = V \cap \Omega$. We have that $(\bar{V} \setminus V) \cap \Omega = (\bar{V}_o \setminus V_o) \cap \Omega$.

For a small $\varepsilon > 0$, let $\bar{\mathbb{B}}(\varepsilon)$ be the closed ball of radius ε centered at 0, and let $\Gamma_\varepsilon = (V \cap b\Omega) \setminus \bar{\mathbb{B}}(\varepsilon)$. Also, we let $\Gamma = \bar{V}_o \cap b\Omega$, a compact set.

Lemma 3.8.21. $V_o \subset \widehat{\Gamma}$.

Proof. Suppose not, so that there is a point, z_o, in $V_o \setminus \widehat{\Gamma}$. Thus, there is a polynomial P with $\Re P < -1$ on $\widehat{\Gamma}$ but with $P(z_o) = 0$. The set $P(E)$ in $\mathbb{C}$ also has measure zero, for $\Lambda^2(E) = 0$. Accordingly, if we replace P by $P - \alpha$ for a suitably chosen small $\alpha \notin P(E)$, we can suppose that P is also zero-free on E. Then P has a logarithm on E because

$\check{H}^1(E;\mathbb{Z})$ is assumed to be zero, and it also has a logarithm on Γ, for $\Re P < 0$ there. The set E meets Γ at a single point, so P has a logarithm on the set $E \cup \Gamma$. This implies that for a sufficiently large compact subset K of V_o that contains z_o, the polynomial P has a logarithm on $V_o \setminus K$. But P vanishes at z_o, and we have reached a contradiction to the argument principle. The lemma is proved.

We now continue with the proof of the theorem.

The set $\widehat{\Gamma_\varepsilon \cup \overline{\mathbb{B}(\varepsilon)}} \setminus (\Gamma_\varepsilon \cup \overline{\mathbb{B}(\varepsilon)})$, which we shall denote by W_ε, contains $\widehat{\Gamma}$, which, in turn, contains V_o. Accordingly, W_ε is a variety. Also, for $\delta < \varepsilon$, $W_\varepsilon = W_\delta \setminus \mathbb{B}(\varepsilon)$. From $\Gamma_\varepsilon \cup \mathbb{B}(\varepsilon) \supset \Gamma_\delta \cup \mathbb{B}(\delta)$ follows $W_\varepsilon \supset W_\delta \setminus \overline{\mathbb{B}(\varepsilon)}$. Denote by W the subset of Ω that consists of all the points that are contained in W_ε for all sufficiently small $\varepsilon > 0$. The set W is locally the intersection of a sequence, and so is a one-dimensional subvariety of Ω. It contains the subvariety V_o of $\Omega \setminus E$ and so it contains $\bar{V}_o \cap \Omega$.

The reverse inclusion also holds: $W \subset \bar{V}_o \cap \Omega$. To establish this, it is enough to show that if W' is a branch of the subvariety $W \setminus E$ of $\Omega \setminus E$, then $W' \subset V_o$. Given such a branch W', we see that the set $bW' = \overline{W'} \setminus W'$ has to contain a simple closed curve contained in Γ. This is so, for, because $bW' \subset \Gamma \cup E$, necessarily $\Lambda^2(bW') = 0$, whence, by Theorem 3.8.15, $\check{H}^1(bW';\mathbb{Z}) \neq 0$. The set E satisfies $\check{H}^1(E;\mathbb{Z}) = 0$, and it meets Γ only at the point 0. Because Γ is a real-analytic one-dimensional submanifold of Σ, the only way for $\check{H}^1(bW';\mathbb{Z})$ not to vanish is for bW' to contain a simple closed curve. Let $\gamma \subset \bar{\Gamma}$ be a simple closed curve contained in bW'.

There is a point $z_1 \in \gamma \setminus \{0\}$ such that for some neighborhood U of z_1 in $\mathbb{C}^N$, $\gamma \cap U$ is an analytic arc that is contained in a connected one-dimensional complex submanifold, Δ, of U. The variety V meets Δ along the arc λ, so Δ and V coincide near λ. In addition, bW' abuts λ, so necessarily, near λ, W' is contained in V_o, as follows from Theorem 3.8.14.

We have that each global branch of $W \setminus E$ is contained in V_o and, a fortiori, in $\bar{V}_o$. Consequently, $W \subset \bar{V}_o$. Thus, the two sets W and $\bar{V}_o \cap \Omega$ coincide, and the latter set is found to be a variety. The proof is complete.

Corollary 3.8.22. *Let D be a bounded domain in $\mathbb{C}^N$, let $E \subset \Omega$ be a compact set with $\Lambda^2(E) = 0$ and $\check{H}^1(E;\mathbb{Z}) = 0$. If V is a one-dimensional subvariety of $\Omega \setminus E$, then $\bar{V} \cap \Omega$ is a one-dimensional subvariety of Ω.*

A special case of this corollary was found by Alexander [17]:

Corollary 3.8.23. *Let D be a domain in $\mathbb{C}^N$, and let $E \subset D$ be a compact set of finite length that is totally disconnected. If $V \subset D \setminus E$ is a one-dimensional variety, then $\bar{V} \cap D$ is a one-dimensional subvariety of D.*

Chapter 4

SETS OF CLASS $\mathscr{A}_1$

Introduction. In this chapter we discuss the hulls of a class of sets with finite length more general than those contained in connected sets. Section 4.1 is introductory. Section 4.2 assembles some results from geometric measure theory for use in subsequent sections. Section 4.3 introduces the class $\mathscr{A}_1$ of sets that are the main object of study in this chapter. Section 4.4 establishes the finiteness of the area of certain one-dimensional varieties. Section 4.5 contains a version of Stokes's theorem. Section 4.6 introduces a useful multiplicity function. Section 4.7 contains a bound on the number of global branches of a hull.

4.1. Introductory Remarks

In our discussion of the hull of a set of finite length there has so far always been the hypothesis that the set in question lies in a connected set of finite length. Examples show that for general sets of finite length, the hull can be very complicated and need not be a variety. It has turned out, though, that the hypothesis is unduly restrictive. Dinh and Lawrence have shown how to deal with the hulls of certain more general sets. This chapter is mainly an exposition of these ideas. One easily appreciated result that emerges from this work is that *if X is* any *compact subset of $b\mathbb{B}_N$ of finite length, then $\widehat{X} \setminus X$ is a one-dimensional variety*. (See the very end of Chapter 7.)

Much of this work seems to require some decidedly nontrivial parts of geometric measure theory. The next section is devoted to a résumé of what is needed—the statement of the relevant definitions and statements of results.

4.2. Measure-Theoretic Preliminaries

The context is that of a class $\mathscr{A}_1$ of sets, which properly includes the connected sets of finite length. The definition requires some preliminary notions. References for most of this material are Federer's book [115] and the volume [114] of Falconer.

Definition 4.2.1. *A set $E \subset \mathbb{R}^N$ is* k-rectifiable, $k = 1, \ldots,$ *if it is of the form $E = f(S)$ with $S \subset \mathbb{R}^k$ a bounded set and $f : S \to \mathbb{R}^N$ a Lipschitz map.*

Definition 4.2.2. *A set $E \subset \mathbb{R}^N$ is* countably (Λ^k, k)-rectifiable *if it is measurable and if there is a countable family of k-rectifiable sets whose union contains almost all $[d\Lambda^k]$ of E.*

Definition 4.2.3. *A set $E \subset \mathbb{R}^N$ is* (Λ^k, k)-rectifiable *if $\Lambda^k(E) < \infty$ and E is of the form $E = E_o \cup \cup_{j=1,\ldots} E_j$ with E_o a set of zero k-dimensional measure and with each E_j, $j = 1, \ldots,$ a k-rectifiable set.*

Thus, a set $E \subset \mathbb{R}^N$ is (Λ^k, k)-rectifiable if it is countably (Λ^k, k)-rectifiable and if $\Lambda^k(E)$ is finite.

The definition of (Λ^k, k)-rectifiable set is equivalent to the condition that $E = E_o \cup \cup_{j=1,\ldots} E_j$, where E_o is a set of zero k-dimensional measure and each E_j, $j = 1, \ldots,$ is a subset of a $\mathscr{C}^1$-submanifold of $\mathbb{R}^N$. This equivalence is not simple; it is established in [115, Theorem 3.2.29].

We will have to invoke the *structure theorem* at certain points below. A subset P of $\mathbb{R}^N$ with $\Lambda^k(S) < \infty$ is said to be *purely* (Λ^k, k)*-unrectifiable* if it contains no (Λ^k, k)-rectifiable set T with $\Lambda^k(T) > 0$. The condition of being purely (Λ^k, k)-unrectifiable is equivalent to the condition that for almost every orthogonal projection $\pi : \mathbb{R}^N \to \mathbb{R}^N$ of rank k, the set $\pi(T)$ satisfies $\Lambda^k(\pi(T)) = 0$.

There is the following fundamental result:

Theorem 4.2.4. *If $S \subset \mathbb{R}^N$ has finite k-dimensional measure, then there is a decomposition $S = R \cup P$ with R a countably (Λ^k, k)-rectifiable set and P a purely (Λ^k, k)-unrectifiable set.*

This is a very deep result, due in the case that $k = 1$ for sets in the plane to Besicovitch, and in the general case to Federer. For the result of Besicovitch, one can consult [114, p. 289]. The general case is in [115, Theorem 3.3.13].

We need the notion of density.

Definition 4.2.5. *For a subset E of $\mathbb{R}^N$, the* k-dimensional density of *E at a point $x \in \mathbb{R}^N$ is the number*

$$\Theta^k(\Lambda^k \llcorner E, x) = \lim_{r \to 0^+} \alpha(k)^{-1} r^{-k} \Lambda^k(E \cap B(x, r))$$

if this limit exists.

In this and below, $B(x, r)$ denotes the ball of radius r centered at x in the ambient Euclidean space, and $\alpha(k)$ is the volume of the unit ball in $\mathbb{R}^k$. Also, $\Lambda^k \llcorner E$ is the measure defined by $(\Lambda^k \llcorner E)(S) = \Lambda^k(E \cap S)$.

Definition 4.2.6. *For a subset E of $\mathbb{R}^N$, the upper k*-dimensional density of *E at a point $x \in \mathbb{R}^N$ is the number*

$$\Theta^{*k}(\Lambda^k \llcorner E, x) = \limsup_{r \to 0^+} \alpha(k)^{-1} r^{-k} \Lambda^k(E \cap B(x, r)).$$

If in the last definition we replace *lim sup* by *lim inf*, we obtain the *lower k-dimensional density of E* at x, which is denoted by $\Theta_*^k(\Lambda^k \llcorner E, x)$.

There are two notions of tangent cone that will be important in the sequel. For their definition it is convenient to fix some notation involving cones. If L is a real line passing through a point $a \in \mathbb{R}^N$, and if τ is a positive real number, then by $C(a, L, \tau)$ we understand the solid cone with axis L, vertex a, and aperture τ. If $a, v \in \mathbb{R}^N$, and $\tau \in \mathbb{R}$, $\tau > 0$, then we put

$$C(a, v, \tau) = \{y \in \mathbb{R}^N : \text{for an } r > 0, |r(y - a) - v| < \tau\}.$$

Thus, if L is the line that contains v, then $C(a, v, \tau)$ is one nappe of the cone $C(a, L, \tau)$.

Definition 4.2.7. *If E is a subset of $\mathbb{R}^N$ and $x \in E$, then the* tangent cone of E at x *is the set*

$$\operatorname{Tan}(E, x) = \{v \in \mathbb{R}^N : \text{for all } \varepsilon > 0 \text{ there are } r > 0 \text{ and } y \in E \text{ with } |x - y| < \varepsilon \text{ and } |r(y - x) - v| < \varepsilon\}.$$

The (Λ^k, k)-tangent cone of E at x *is the set*

$$\operatorname{Tan}^k(\Lambda^k \llcorner E, x) = \cap\{\operatorname{Tan}(F, x) : F \subset E,\ \Theta^k(\Lambda^k \llcorner (E \setminus F), x) = 0\}.$$

In particular, $\operatorname{Tan}^k(\Lambda^k \llcorner E, x) \subset \operatorname{Tan}(E, x)$.

By definition,

$$\operatorname{Tan}(E, x) = \{v \in \mathbb{R}^N : \text{for all } \varepsilon > 0 \text{ there exist } y \in E \cap C(x, v, \varepsilon) \text{ with } |x - y| < \varepsilon\}.$$

Introduce the notation that for a set $E \subset \mathbb{R}^N$, a point $a \in \mathbb{R}^N$, a vector $v \in \mathbb{R}^N$, and an $\varepsilon > 0$,

$$E(a, v, \varepsilon) = E \cap C(a, v, \varepsilon).$$

The following lemma makes it possible to describe the spaces Tan^1:

Lemma 4.2.8.[115, p. 252] *The vector v lies in* $\operatorname{Tan}^1(\Lambda^1 \llcorner E, x)$ *if and only if*

$$\Theta^{*1}(\Lambda^1 \llcorner E(x, v, \varepsilon), x) > 0 \tag{4.1}$$

for every $\varepsilon > 0$.

Proof. Assume the inequality (4.1) to hold. Suppose that $F \subset E$ is a set such that

$$\Theta^1(\Lambda^1 \llcorner (E \setminus F), x) = 0. \tag{4.2}$$

Then there is $c > 0$ so small that for all small $r > 0$, $E(x, v, \varepsilon) \cap B(x, r)$ is a set of length at least cr. Thus F must meet $C(x, v, \varepsilon) \cap B(x, r)$ for all small $r > 0$, so $v \in \mathrm{Tan}(F, x)$. This is so for all F of the kind considered, so $x \in \mathrm{Tan}^1(\Lambda \llcorner E, x)$.

Conversely, suppose $v \in \mathrm{Tan}^1(\Lambda^1 \llcorner E, x)$. Thus, $v \in \mathrm{Tan}(F)$ if $\Theta^1(E \setminus F, x) = 0$. If ^{c} denotes the operator of complementation, then $E \cap C(x, v, \varepsilon) = E \setminus C(x, v, \varepsilon)^c$, so we have $v \in \mathrm{Tan}(E \setminus C(x, v, \varepsilon), x)$. This is a contradiction. The proof is complete.

The following is a basic theorem in the subject.

Theorem 4.2.9. [115, Theorem 3.2.19] *If $E \subset \mathbb{R}^N$ is (Λ^k, k)-rectifiable and Λ^k-measurable, then for almost all $[d\Lambda^k]$ points $x \in E$,*

$$\Theta^k(\Lambda^k \llcorner E, x) = 1,$$

and the (Λ^k, k)-tangent cone $\mathrm{Tan}^k(\Lambda^k \llcorner E, x)$ is a k-dimensional vector subspace of $\mathbb{R}^N$.

Theorem 4.2.10. *If $E \subset \mathbb{R}^N$ satisfies $\Lambda^1(E) < \infty$, then the upper density satisfies*

$$\Theta^{*1}(\Lambda^1 \llcorner E, x) \leq 1$$

for almost all $[d\Lambda^1]$ $x \in E$.

In the plane this is a theorem of Besicovitch [50]. The proof Besicovitch gives applies equally well to sets in $\mathbb{R}^N$ for any $N \geq 2$.

Theorem 4.2.11. *If $E \subset \mathbb{R}^N$ is a set with $\Lambda^1(E) < \infty$, then E is countably $(\Lambda^1, 1)$-rectifiable if and only if for almost all $[d\Lambda^1]$ x in E, $\Theta^{*1}(\Lambda^1 \llcorner E, x) = \Theta^1_*(\Lambda^1 \llcorner E, x)$.*

In the plane, this also is due to Besicovitch [50]. The case $N > 2$, which requires an approach essentially different from Besicovitch's, is due to E.F. Moore [250].

4.3. Sets of Class $\mathscr{A}_1$

By definition, sets of class $\mathscr{A}_1$ are sets with simple geometric tangent cones:

Definition 4.3.1. *A closed subset E of an open subset of $\mathbb{R}^N$ is of class $\mathscr{A}_1$ if it is locally $(\Lambda^1, 1)$-rectifiable and if for almost every $[d\Lambda^1]$ point $x \in E$, the tangent cone $\mathrm{Tan}(E, x)$ is a one-dimensional real vector subspace of $\mathbb{R}^N$.*

The notion of sets of class $\mathscr{A}_1$ extends immediately to subsets of manifolds. If $\mathscr{M}$ is a Riemannian manifold and E a closed subset of an open subset of $\mathscr{M}$, then E is of class $\mathscr{A}_1$ if for each $x \in E$, there is a neighborhood U of x on which there are coordinates with respect to which $E \cap U$ is of class $\mathscr{A}_1$. One verifies that this condition does not depend on the choice of local coordinates.

A compact set of class $\mathscr{A}_1$ has finite length.

Sets of class $\mathscr{A}_1$ are also called *geometrically* 1-*rectifiable*.

There is the following extension of Theorem 3.1.1, which is one of the main reasons for introducing sets of class $\mathscr{A}_1$.

Theorem 4.3.2. *Let Y be a compact polynomially convex subset of $\mathbb{C}^N$, and let Γ be a bounded closed subset of $\mathbb{C}^N \setminus Y$ that is of class $\mathscr{A}_1$. The set $\widehat{(\Gamma \cup Y)} \setminus (\Gamma \cup Y)$ either is empty or else is a purely one-dimensional subvariety of $\mathbb{C}^N \setminus (\Gamma \cup Y)$. If the map*

$$\check{H}^1(\Gamma \cup Y; \mathbb{Z}) \to \check{H}^1(Y; \mathbb{Z})$$

induced by the inclusion $Y \hookrightarrow \Gamma \cup Y$ is an isomorphism, then the algebra $\mathscr{P}(\Gamma \cup Y)$ consists of all the continuous functions f on $\Gamma \cup Y$ with $f|Y \in \mathscr{P}(Y)$.

There is an extension of the preceding statement, the first version of which was given by Alexander [15]. The present version was given by Dinh [100] and by Lawrence [215].

Corollary 4.3.3. *Let $Y \subset \mathbb{C}^N$ be a polynomially convex set, and for each $j = 1, \ldots,$ let Γ_j be a closed and bounded subset of $\mathbb{C}^N \setminus Y$ of class $\mathscr{A}_1$. Let Γ be a closed and bounded subset of $\mathbb{C}^N \setminus Y$ that is contained in $\cup_{j=1,\ldots}\Gamma_j$. If $z \in \widehat{(Y \cup \Gamma)} \setminus (Y \cup \Gamma)$, then there exist an open subset U of $\mathbb{C}^N$ and in U a purely one-dimensional subvariety Z with $z \in Z \subset \widehat{(Y \cup \Gamma)}$.*

The proof of this corollary is based on the observation that if z is a point in $\widehat{X}$ for some compact subset X of $\mathbb{C}^N$, then there are compact subsets Y of X with the property that $z \in \widehat{Y}$ and that Y is *minimal* with respect to this property, so that if Y' is a proper, closed subset of Y, then $z \notin \widehat{Y'}$.

Proof. Let $K \subset Y \cup \Gamma$ be a compact set minimal with respect to the condition that $z \in \widehat{K}$. Let $\Gamma' = K \setminus Y$. According to the Baire category theorem, there is a k such that Γ_k contains an open subset W of Γ'. Let $Y'' = \widehat{K \setminus W}$, which is polynomially convex, and let $\Gamma'' = W \setminus Y''$. Then $Y'' \cup \Gamma''$ contains $K \cup W$ and so $z \in \widehat{(Y'' \cup \Gamma'')}$. By the choice of K, $z \notin Y''$. Theorem 4.3.2 implies that $\widehat{(Y'' \cup \Gamma'')}$ has the structure of a one-dimensional variety. The corollary is proved.

A simple example of the phenomenon contemplated in the preceding corollary is the following. Take Y to be the empty set, and define Γ_k, $k = 1, \ldots, \infty$, to be the circle $\{(e^{i\vartheta}, \frac{1}{k}) : \vartheta \in \mathbb{R}\}$ when $k < \infty$ and Γ_∞ to be the limit circle $\{(e^{i\vartheta}, 0) : \vartheta \in \mathbb{R}\}$. The hull of the set $X = \cup_{k=1,\ldots,\infty}\Gamma_k$ is the union of the disks bounded by the circles Γ_k. Neither Theorem 3.1.1 nor Theorem 4.3.2 applies in this situation. The corollary just given provides analytic varieties through each point of $\widehat{X} \setminus X$.

We begin the study of sets of class $\mathscr{A}_1$ with two simple observations.

Lemma 4.3.4. *A closed subset of a set of class $\mathscr{A}_1$ is itself of class $\mathscr{A}_1$.*

Proof. Let the closed subset E of the open set Ω in $\mathbb{R}^N$ be $(\Lambda^1, 1)$-rectifiable and have the property that for almost every $[d\Lambda^1]$ $x \in E$ the tangent $\mathrm{Tan}(E, x)$ is a real line.

Let K be a closed subset of E. Plainly K is locally $(\Lambda^1, 1)$-rectifiable. And for all $x \in K$, $\mathrm{Tan}(K, x) \subset \mathrm{Tan}(E, x)$, so for almost all $x \in K$, the tangent $\mathrm{Tan}(K, x)$ is contained in a real line. The result follows now from Theorem 4.2.9.

Lemma 4.3.5. *If $E \subset \mathbb{R}^N$ is a compact set of class $\mathscr{A}_1$ and Φ is a diffeomorphism from a neighborhood W of E onto an open subset of a closed submanifold of an open set in $\mathbb{R}^M$, then the set $\Phi(E)$ is a set of class $\mathscr{A}_1$.*

Proof. The set $\Phi(E)$ is locally $(\Lambda^1, 1)$-rectifiable. The differential of Φ carries $\mathrm{Tan}(E, x)$ onto $\mathrm{Tan}(\Phi(E), \Phi(x))$, and $d\Phi_x$ is injective. If x is a point for which $\mathrm{Tan}(E, x)$ is a line, the tangent $\mathrm{Tan}(\Phi(E), \Phi(x))$ is $d\Phi_x(\mathrm{Tan}(E, x))$, so the lemma is proved.

We can now exhibit some nontrivial sets that belong to the class $\mathscr{A}_1$.

Theorem 4.3.6. *A connected set of finite length in $\mathbb{R}^N$ is a set of class $\mathscr{A}_1$.*

This was proved by Alexander [22] and by Dinh [101]. It implies that Theorem 4.3.2 is an extension of Theorem 3.1.1.

Proof. Let X denote the set in question. We have proved in Theorem 3.3.5 that X is $(\Lambda^1, 1)$-rectifiable. It is also measurable. Thus, Theorem 4.2.9 implies that for almost all $[d\Lambda^1]$ points $x \in X$, the $(\Lambda^1, 1)$-tangent cone $\mathrm{Tan}^1(\Lambda^1 \llcorner X, x)$ is a one-dimensional vector subspace of $\mathbb{R}^N$. We are to prove that for almost all $[d\Lambda^1]$ $x \in X$, the tangent cone $\mathrm{Tan}(X, x)$ is a one-dimensional vector subspace of $\mathbb{R}^N$. The space $\mathrm{Tan}^1(\Lambda^1 \llcorner X, x)$ is almost surely a real line, and by definition, $\mathrm{Tan}^1(\Lambda^1 \llcorner X, x) \subset \mathrm{Tan}(X, x)$. Thus, what is to be proved is that if $v \in \mathrm{Tan}(X, x)$, then $v \in \mathrm{Tan}^1(\Lambda^1 \llcorner X, x)$. Without loss of generality, suppose $|v| = 1$.

By Lemma 4.2.8, it is enough to prove that for every $\varepsilon > 0$,

$$\Theta^{*1}(\Lambda^1 \llcorner E(X, v, \varepsilon), x) > 0. \tag{4.3}$$

This is simple to verify because of connectivity. Fix an $\varepsilon \in (0, 1)$, and choose a sequence $\{x_k\}_{k=1,\ldots}$ in $E(X, v, x)$ with $x_k = x + r_k v_k$, where $v_k \to v$, $|v_k| = 1$, and $r_k \to 0$ as $k \to \infty$, $r_k > 0$ for all k. Because X is connected and contains both x and x_k, the total length of the part of X that lies in the truncated set $E(X, v, x) \cap B(x, 2r_k)$ is at least const r_k with the constant depending on ε but not on k. This implies (4.3), and the result is proved.

Another class of examples of sets of class $\mathscr{A}_1$ was found by Lawrence [215]. See also [99].

Theorem 4.3.7. *If $D \subset \mathbb{C}^N$ is a bounded strictly convex domain with boundary of class $\mathscr{C}^2$, if $z \in D$, and if $E \subset bD$ is compact a set of finite length with $z \in \widehat{E}$, then any subset E' of E that is minimal with respect to the property that $z \in \widehat{E}'$ is a set of class $\mathscr{A}_1$.*

In this statement the condition of strict convexity is the condition that the real Hessian of a defining function for D be positive definite at each point of bD.

To prepare for the proof of this result, let us establish some notation. We suppose that $0 \in bD$ and that Q is a strictly convex function of class $\mathscr{C}^2$ that is a defining function for D so that $D = \{Q < 0\}$ and $dQ \neq 0$ on bD.

Under the assumption that $0 \in E$, we are going to show that $\Theta^1_*(\Lambda^1 \llcorner E, 0) = 1$.

Lemma 4.3.8. *There are holomorphic linear coordinates on $\mathbb{C}^N$ with respect to which*

$$Q(z) = \Re P(z) + \frac{1}{2}(|z|^2 + \rho(z)) \tag{4.4}$$

with $P(z)$ a quadratic polynomial and with the remainder ρ a function of class $\mathscr{C}^2$ that satisfies $\rho(z) = o(|z|^2)$ for $z \to 0$.

Proof. The Taylor expansion of Q about the origin gives an expansion

$$Q(z) = \Re P(z) + \frac{1}{2}L(z) + r(z)$$

with a holomorphic quadratic polynomial P, with a remainder term $r(z)$ that is $o(|z|^2)$, and with L the Levi form of Q at 0:

$$L(z) = \bar{z}Mz^t$$

with the matrix M the positive definite Hermitian symmetric matrix $M = \left[\frac{\partial^2 Q(0)}{\partial z_j \partial \bar{z}_k}\right]$. If we set $z = U\zeta$ for a suitable unitary transformation U, then the corresponding expansion of Q in the ζ-coordinates will be

$$Q(U\zeta) = \Re\tilde{P}(\zeta) + \frac{1}{2}\tilde{L}(\zeta) + \tilde{r}(\zeta),$$

and the Levi form $\tilde{L}(\zeta)$ will be a diagonal matrix with positive diagonal entries. Then a further transformation of the form $\zeta_j = d_j\eta_j$ will bring Q, in terms of the η coordinates, into the desired form.

Polynomial convexity enters the proof of the theorem by way of the following lemma. Notice that according to the expansion (4.4), $\Re P(z) < 0$ when $z \in bD$, z near 0.

Lemma 4.3.9. *If $c > 0$ is small, then the surface H with equation $\Re P = -c$ meets E in at least two points.*

Proof. If H meets E in at most one point, then set

$$E^+ = E \cap \{z \in \mathbb{C}^N : \Re P(z) \geq -c\}$$

and

$$E^- = E \cap \{z \in \mathbb{C}^N : \Re P(z) \leq -c\}.$$

The polynomially convex sets $\Re\widehat{P(E^+)}$ and $\Re\widehat{P(E^-)}$ meet at most at a single point, say p, which is $-c + it$ for some real t. Kallin's lemma, Theorem 1.6.19, implies that the set $\widehat{E}$ is the union of the sets $\widehat{E^+}$ and $\widehat{E^-}$. Then the point z must lie in the hull of one of E^+ and E^-, which contradicts the minimality assumption. The lemma is proved.

Proof of Theorem 4.3.7. Introduce the function ψ given by

$$\psi(z) = \sqrt{|z|^2 + \rho(z)}.$$

If $\eta > 0$, then ψ satisfies a Lipschitz condition with Lipschitz constant $1 + \eta$ on a small neighborhood of the origin. This is so, for a short calculation shows that if W is a small neighborhood of the origin, then on $W \setminus \{0\}$, $|\text{grad}\, \rho| < 1 + \eta$. For this compute as follows: Except at 0,

$$|\text{grad}\, \psi(z)|^2 = \frac{|\text{grad}\, (|z|^2 + \rho(z))|^2}{4(|z|^2 + \rho(z))}.$$

Now grad $|z|^2 = 2z$. The function ρ is of class $\mathscr{C}^2$, because all the other terms in the equation (4.4) are of class $\mathscr{C}^2$. Moreover, $\rho(z) = o(|z|^2)$, so grad $\rho(0) = 0$. Therefore no matter how small $\eta > 0$ may be, if $z \in W \setminus \{0\}$ and W is sufficiently small, we have $|\text{grad}\, \rho(z)| < \eta|z|$, and then

$$|\text{grad}\,(|z|^2 + \rho(z))|^2 \leq 4|z|^2(1+\eta)^2,$$

and we get

$$|\text{grad}\, \psi(z)|^2 \leq \frac{4|z|^2(1+\eta)^2}{4(|z|^2 + \rho(z))} \leq (1+\eta)^2.$$

Eilenberg's inequality, Theorem 3.3.6, now implies that, for small $r > 0$,

$$\int_0^r \Lambda^0(E \cap \psi^{-1}(t))\, dt \leq (1+\eta)\Lambda^1(E \cap \{z : \psi(z) < r\}). \tag{4.5}$$

The function Q vanishes on E, so for $z \in E$, $\psi(z) = t$ when $\Re P(z) = -t^2/2$. Thus, for small r, the integrand in the integral (4.5) is at least two by Lemma 4.3.9. We have, therefore, that when $r > 0$ is small,

$$1 \leq \frac{1+\eta}{2r}\Lambda^1(E \cap \{z : \psi(z) < r\}). \tag{4.6}$$

Because $\eta(z) = o(|z|^2)$ we can write $\psi(z) = |z| + o(|z|)$. Thus, if $m > 0$ is small, then for small $r > 0$, we have

$$E \cap \{z : \psi(z) < r\} \subset E \cap \mathbb{B}_N(r(1+m)),$$

and (4.6) gives

$$1 \leq \frac{(1+\eta)}{2r}\Lambda^1(E \cap \mathbb{B}_N(r(1+m))) = (1+\eta)(1+m)\frac{1}{2r(1+m)}\Lambda^1(E \cap \mathbb{B}_N(r(1+m))).$$

By letting $r \to 0^+$ we find that

$$\frac{1}{(1+\eta)(1+m)} \leq \Theta^1_*(E, 0).$$

This is correct for all small m and all small η, so, as claimed, $\Theta^1_*(E, 0) \geq 1$. The theorem is proved.

For future reference, we note that the analysis of the preceding proof is localized at a point of the set E. Thus, there is a formally stronger result: *Let $E \subset bD$, $D \subset \mathbb{C}^N$ a strictly convex domain with* $0 \in \widehat{E}$. *If E is minimal with respect to the latter condition, and if $p \in E$ has a closed neighborhood B in E that has finite length, then B is of class $\mathscr{A}_1$.*

The next lemma shows that the generic projection of a set of class $\mathscr{A}_1$ is again of class $\mathscr{A}_1$.

Lemma 4.3.10. [101] *If $E \subset \mathbb{C}^N$ is a compact set of class $\mathscr{A}_1$, then for almost every linear functional φ on $\mathbb{C}^N$, the subset $\varphi(E)$ of $\mathbb{C}$ is a set of class $\mathscr{A}_1$.*

Proof. Almost every linear functional on $\mathbb{C}^N$ is of the form $z \mapsto \alpha_1 z_1 + \cdots + \alpha_N z_N$ for some choice of $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_1 \neq 0$. Consequently, it suffices to show that for almost every linear functional φ of the form

$$\varphi_\alpha(z) = z_1 + \alpha_2 z_2 + \cdots + \alpha_N z_N, \tag{4.7}$$

the set $\varphi(E)$ is a set of class $\mathscr{A}_1$ in $\mathbb{C}$. (*Almost every* is here understood with respect to the Lebesgue measure on the $\mathbb{C}^{N-1}$ in which $\alpha = (\alpha_2, \dots, \alpha_N)$ lies.)

From the definition, it is clear that no matter what the choice of α, the set $\varphi_\alpha(E)$ is $(\Lambda^1, 1)$-rectifiable. What is to be established is that at almost every point in $\varphi_\alpha(E)$, the tangent to $\varphi_\alpha(E)$ is a line. As noted immediately after Definition 4.2.3, the set E is of the form $E = E_o \cup \cup_{j=1,\dots} E_j$ with E_o a set of zero length and with each of the other sets E_j a subset of a $\mathscr{C}^1$ curve. Thus, it suffices to prove the lemma under the assumption that the set E is an arc of class $\mathscr{C}^1$. In this case, each tangent $\operatorname{Tan}(E, x)$ is a line (except for the case that x is an endpoint of E).

Introduce the set S of all points $(x, \alpha) \in E \times \mathbb{C}^{N-1}$ for which the tangent $\operatorname{Tan}(E, x)$ is a line transversal to the $\mathbb{C}$-affine hyperplane $\varphi_\alpha^{-1}(\varphi_\alpha(x))$. We shall show that if $\Lambda = \Lambda^1 \times \Lambda^{2N-2}$ is the product measure, then the set S is of full measure with respect to Λ. What is immediate is that for a fixed $x \in E$ other than an endpoint, the set

$$S_x = \{\alpha \in \mathbb{C}^{N-1} : \operatorname{Tan}(E, x) \text{ is transversal to } \varphi_\alpha^{-1}(\varphi_\alpha(x))\}$$

is of full measure in $\mathbb{C}^{N-1}$. Fubini's theorem implies then that the set S is of full measure in $E \times \mathbb{C}^{N-1}$. Fubini's theorem applied once more implies that for almost every α, the $\mathbb{C}$-affine hyperplane $\varphi_\alpha^{-1}(\varphi_\alpha(x))$ is transversal to almost every one of the tangent spaces $\operatorname{Tan}(E, x)$. Thus for almost every $\alpha \in \mathbb{C}^{N-1}$, the tangent $\operatorname{Tan}(\varphi_\alpha(E), y)$ is a line for almost every $y \in \varphi_\alpha(E)$.

The lemma is proved.

The next lemma shows that the generic linear functional on $\mathbb{C}^N$ carries a compact set of class $\mathscr{A}_1$ almost injectively into the plane.

Lemma 4.3.11. [101] *If $E \subset \mathbb{C}^N$ is a $(\Lambda^1, 1)$-rectifiable set, then for almost every linear functional φ on $\mathbb{C}^N$, there is a subset S_φ of $\mathbb{C}$ of zero length such that $\varphi|(E \setminus \varphi^{-1}(S))$ is one-to-one.*

Proof. A $(\Lambda^1, 1)$-rectifiable set is a union of a set of zero length and a countable family of sets each contained in a one-dimensional $\mathscr{C}^1$ submanifold of $\mathbb{C}^N$, so we can suppose that the set E is itself a $\mathscr{C}^1$ submanifold of $\mathbb{C}^N$.

Because almost every linear functional φ on $\mathbb{C}^N$ is of the form $\varphi(z) = \sum_{j=1}^N \beta_j z_j$ with $\beta_1 \neq 0$, it is sufficient to consider linear functionals of the form $\varphi = \varphi_\alpha$ for $\alpha \in \mathbb{C}^{N-1}$ and φ_α as in (4.7). For a fixed α, let

$$T_\alpha = \{\zeta \in \mathbb{C} : \varphi_\alpha^{-1}(\zeta) \text{ contains at least two points}\}.$$

We are to prove that for almost all α, the set T_α is a set of zero length. To do this, introduce the set $P_\alpha = \{(x, y) \in E \times E : x \neq y \text{ and } \varphi_\alpha(x) = \varphi_\alpha(y)\}$, and notice that $\Lambda^1(T_\alpha) > 0$ implies that $\Lambda^1(P_\alpha) > 0$. Define the set Q by

$$Q = \{(x, y, \alpha) : x, y \in E,\ x \neq y,\ \text{and } \varphi_\alpha(x) = \varphi_\alpha(y)\}.$$

We shall prove that $\Lambda^{2N-1}(Q) = 0$. To do this, let the projection $\eta : Q \to E$ be given by $\eta(x, y, \alpha) = x$. The range of η is the set E, which has σ-finite one-dimensional measure. Each fiber $\eta^{-1}(x)$ for $x \in E$ is the set

$$Q_x = \{(x, y, \alpha) : y \in E \setminus \{x\} \text{ and } \varphi_\alpha(y) = \varphi_\alpha(x)\},$$

which has the structure of a $\mathscr{C}^1$ manifold of dimension $2N-3$. Therefore $\Lambda^{2N-1}(Q) = 0$. Define the projection $\mu : Q \to \mathbb{C}^{N-1}$ by $\mu(x, y, \alpha) = \alpha$. We have $\Lambda^{2N-3}(Q) = 0$, so Eilenberg's inequality implies that for almost all $\alpha \in \mathbb{C}^{N-1}$, the 1-dimensional measure of the fiber $\mu^{-1}(\alpha)$ is zero. That is, the set P_α has zero length, whence so also does the set T_α. The result is proved.

Theorem 4.3.12. [101]. See also [216, 217]. *If D is a bounded domain in $\mathbb{C}$ with bD a $(\Lambda^1, 1)$-rectifiable set, and if $E \subset bD$ is a set of positive length at each point x of which the tangent space* $\operatorname{Tan}(bD, x)$ *is a real line, then there is a domain $\Omega \subset D$ such that $b\Omega$ is a rectifiable simple closed curve that contains a subset E' of E of positive length.*

In the proof of this, we shall use Stolz angles.

Definition 4.3.13. *For $\vartheta \in \mathbb{R}$ and $\eta \in (0, 1)$, the* Stolz angle with vertex $e^{i\vartheta}$ and aperture *η is the open convex hull of the disk of radius η centered at the origin of $\mathbb{C}$ together with the point $e^{i\vartheta}$. It will be denoted by $K(e^{i\vartheta}, \eta)$.*

Thus, $K(e^{i\vartheta}, \eta)$ is a figure with the shape of an ice cream cone.

Given a Stolz angle $K(e^{i\vartheta}, \eta)$, the *truncated Stolz angle* $K(e^{i\vartheta}, \eta, r)$, which is defined for $r \in (\eta, 1)$, is the part of $K(e^{i\vartheta}, \eta)$ that lies outside the circle centered at the origin and of radius r.

Proof. By hypothesis, bD is $(\Lambda^1, 1)$-rectifiable, so there are rectifiable simple closed curves $\gamma_1, \ldots$ in $\mathbb{C}$ such that $\Lambda^1(bD \setminus \cup_{j=1,\ldots}\gamma_j) = 0$. Without loss of generality, one of them, say γ, contains the set E. We suppose that D is contained in the bounded component of γ: If necessary, we can replace D by a suitable component of $D \setminus \gamma$. If no such component lies in the interior of γ and contains in its boundary a subset of E of positive length, then some component of $D \setminus \gamma$ that lies in the exterior of γ can be used; the analysis given below would require only technical alterations to suit this case. Denote by W the interior of γ.

The domain W is bounded and simply connected with a rectifiable simple closed curve as boundary, so if $\varphi : \mathbb{U} \to W$ is a Riemann map, then φ extends continuously to a homeomorphism from $\bar{\mathbb{U}}$ to $W \cup \gamma = \bar{W}$. The derivative φ' belongs to the Hardy space $H^1(\mathbb{U})$.

The rectifiable simple closed curve γ has a tangent at almost every point; at almost every $[d\Lambda^1]$ point x of E, the tangents $\operatorname{Tan}(bD, x)$ and $\operatorname{Tan}(\gamma, x)$ coincide.

With φ'^* the boundary value function of φ', the function $\log|\varphi'^*|$ is integrable because the derivative φ' is in $H^1(\mathbb{U})$, so that φ'^* vanishes only on a set of measure zero. Also, by Theorem 3.4.4, a subset F of $b\mathbb{U}$ satisfies $\Lambda^1(\varphi(F)) = 0$ if and only if $\Lambda^1(F) = 0$.

Fix a compact subset S of $b\mathbb{U}$ of positive length. We require that $\varphi(S) \subset E$, that φ'^* vanish at no point of S, and that for all points $e^{i\vartheta} \in S$, the tangents $\mathrm{Tan}(bD, \varphi(e^{i\vartheta}))$ and $\mathrm{Tan}(E, \varphi(e^{i\vartheta}))$ coincide. The set $\varphi(S)$ has positive length. Because the function φ' is of class H^1, if we replace S by a suitable compact subset of it that still has positive length, then we can suppose that $|\varphi'|$ is bounded on the domain $\cup_{e^{i\vartheta}\in S} K(e^{i\vartheta}, \eta)$ for any given small η.

The function φ satisfies $ie^{i\vartheta}\varphi'(e^{i\vartheta}) = \frac{d}{d\vartheta}\varphi(e^{i\vartheta})$—recall Corollary 3.4.3, which implies that if $e^{i\vartheta} \in S$, then the radial curve $r \mapsto \varphi(re^{i\vartheta})$ approaches the point $\varphi(e^{i\vartheta})$ along a trajectory that is orthogonal to the tangent $\mathrm{Tan}(bD, \varphi(e^{i\vartheta}))$. Consequently, there are $\eta > 0$ and $r > 0$ small enough that for a subset S_1 of S of positive length, the open subset V of $\mathbb{U}$ given by

$$V = \cup_{e^{i\vartheta}\in S_1} K(e^{i\vartheta}, \eta, r)$$

is contained in D. Let V' be a component of V such that bV' contains a subset S_2 of S of positive length. By its construction, V' is a simply connected domain in $\mathbb{U}$ on which φ' is bounded and with boundary a rectifiable simple closed curve. Thus, $\varphi(V')$ is a simply connected domain in D with $b(\varphi(V'))$ a rectifiable simple closed curve in $D \cup E$ that meets the set E in a set of positive length.

The theorem is proved.

Corollary 4.3.14. *The set E is a set of uniqueness for subharmonic functions on D.*

This is a consequence of Corollary 3.4.11.

Corollary 4.3.15. *If E is a subset of $\mathbb{C}$ of class $\mathscr{A}_1$, then almost every* $[d\Lambda^1]$ *point of E belongs to the boundary of at most two components of $\mathbb{C} \setminus E$.*

Proof. In the contrary case there is a subset E_o of E with positive length such that for distinct components V, V', and V'' of $\mathbb{C} \setminus E$, each of bV, bV', and bV'' contains E_o. By Theorem 4.3.13 there is a domain $D \subset V$ that is bounded by a rectifiable simple closed curve that meets E in a set E_o of positive length. Then there is a domain $D' \subset V'$ that is bounded by a rectifiable simple closed curve that meets E_o in a set E_o' of positive length. Finally, there is a domain $D'' \subset V''$ that is bounded by a rectifiable simple closed curve that meets E_o' in a set of positive length. Thus, the triple intersection $S = bD \cap bD' \cap bD''$ is a set of positive length. Each point p of S is accessible by an arc in each of $D \cup \{p\}$, $D' \cup \{p\}$, and $D'' \cup \{p\}$. This implies to the existence of uncountably many mutually disjoint triodes in $\mathbb{C}$, and so contradicts Moore's theorem, Theorem 3.4.14.

Lemma 4.3.16. *If the closed set E of the open set Ω in $\mathbb{C}^N$ is of class $\mathscr{A}_1$, and if E_o is a subset of E with positive length, then almost every real-linear functional $\varphi : \mathbb{C}^N \to \mathbb{R}$ satisfies $\Lambda^1(\varphi(E_o)) > 0$.*

Proof. The set E_o has positive length, so there is an arc γ of class $\mathscr{C}^1$ that contains a subset E_1 of E_o of positive length. Thus, it suffices to prove the result when E is an arc of class $\mathscr{C}^1$.

Fix a point $p \in E_1$ at which the lower density is one. Almost every real-linear functional φ on $\mathbb{C}^N$ has the property that $d\varphi = \varphi$ carries the tangent line to γ at p to a line in $\mathbb{C}$. Fix such a φ. Suppose coordinates to have been chosen so that p is the origin of $\mathbb{C}^N$. There is a $\mathscr{C}^1$ arc γ' contained in γ and containing p as an interior point that is carried diffeomorphically by φ onto a neighborhood of $0 \in \mathbb{R}$. The set $\varphi(E_1 \cap \gamma')$ has positive length. The lemma is proved.

Lemma 4.3.17. *If $E \subset \mathbb{C}$ is a compact set of class $\mathscr{A}_1$, then almost every $[d\Lambda^1]$ point of E lies in the boundary of a component of $\mathbb{C} \setminus E$.*

Proof. Let $\pi : \mathbb{C} \to L$ be the orthogonal projection of $\mathbb{C}$ onto the real line L in $\mathbb{C}$. The set E has finite length, so the integral $\int_L \Lambda^0(\pi^{-1}(s) \cap E)\, d\Lambda^1(s)$ is finite, which implies that for almost all $s \in L$, the fiber $E_s = E \cap \pi^{-1}(s)$ is finite. If $p \in E_s$ and E_s is finite, then p lies in the boundary of at least one component of $\mathbb{C} \setminus E$.

Thus, if $E_o \subset E$ is the set of points contained in the boundary of no component of $\mathbb{C} \setminus E$, then for all projections $\pi : \mathbb{C} \to \mathbb{C}$ with one-dimensional range, the set $\pi(E)$ has zero length.

Because E is a set of class $\mathscr{A}_1$, for any subset Y of E of positive length there is a projection from $\mathbb{C}$ to a real line in $\mathbb{C}$ that carries Y to a set of positive length, so we conclude that the set E must have length zero.

The lemma is proved.

Corollary 4.3.15 and Lemma 4.3.17 will be used in the study of the multiplicity function introduced in Section 4.6 below.

We can now prove the result stated above about hulls.

Proof of Theorem 4.3.2. We place ourselves in the context of Theorem 4.3.2, so that Y is a compact polynomially convex subset of $\mathbb{C}^N$, and Γ is a closed, bounded subset of $\mathbb{C}^N \setminus Y$ that is of class $\mathscr{A}_1$. Set $X = \Gamma \cup Y$. We are to show that $\widehat{X} \setminus X$ is a purely one-dimensional analytic set.

For this, fix a point $p \in \widehat{X} \setminus X$. We shall show that in the vicinity of p, the set $\widehat{X}$ has the structure of a purely one-dimensional variety.

Let W be a compact polynomial polyhedron that is a neighborhood of Y and that does not contain p. Let $\Gamma' = \overline{\Gamma \setminus W}$, a compact set of class $\mathscr{A}_1$. The argument at the beginning of the proof of Theorem 3.1.1 provides a polynomial P such that $\Re P < 0$ on W, $P(p) = 1$, and $1 \notin P(\Gamma')$.

Lemma 4.3.18. *The polynomial P can be chosen so that the subset $P(\Gamma')$ of $\mathbb{C}$ is of class $\mathscr{A}_1$.*

Proof. Introduce the graph map $\Phi : \mathbb{C}^N \to \mathbb{C}^{N+1}$ by $\Phi(z) = (z, P(z))$. By Lemma 4.3.5, the set $\Phi(\Gamma')$ is of class $\mathscr{A}_1$. Consequently, by Theorem 4.3.10, for almost every linear functional φ on $\mathbb{C}^{N+1}$, the subset $\varphi(\Phi(\Gamma'))$ of $\mathbb{C}$ is of class $\mathscr{A}_1$. The functional φ is of the form $\varphi(z_1, \ldots, z_{N+1}) = \sum_{j=1}^{N+1} c_j z_j$. If we take $c_1, \ldots, c_N$ very small and c_{N+1} very near 1, then $\varphi \circ \Phi$ is a polynomial that is very near to P and that has the properties we desire, except for assuming the value 1 at the point p. A small modification of $\varphi \circ \Phi$—multiply it by $1/c_{N+1}$—yields the polynomial we seek. The lemma is proved.

Denote by Ω the component of $\mathbb{C} \setminus P(X)$ that contains the point 1. This is a bounded set, and $\Lambda^1(b\Omega \cap \{\zeta \in \mathbb{C} : \Re\zeta > 0\})$ is positive. Theorem 4.3.12 and Lemma 3.4.11 imply that $b\Omega \cap \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$ contains a set E of positive length that is a set of uniqueness for subharmonic functions. We have $P^{-1}(E) \subset \Gamma'$ and $\Lambda^1(\Gamma')$ is finite, so it follows that for almost all $[d\Lambda^1]$ $\zeta \in E$, the fiber $P^{-1}(\zeta) \cap \Gamma'$ is finite. Consequently, by Theorem 3.6.1, the set $\widehat{W \cup \Gamma'} \setminus (W \cup \Gamma')$ is a one-dimensional variety.

Let $r > 0$ be so small that $\mathbb{B}_N(p, 2r) \cap (W \cup \Gamma') = \emptyset$, whence $\mathbb{B}_N(p, 2r) \cap \widehat{W \cup \Gamma'}$ is a one-dimensional variety. Because $\widehat{X} \subset \widehat{W \cup \Gamma'}$, we have that $\mathbb{B}_N(p, 2r) \cap \widehat{X}$ is contained in a one-dimensional variety, so $\bar{\mathbb{B}}_N(p, r) \cap \widehat{X}$ has finite area. By the local maximum principle, this set is contained in the set $b\mathbb{B}_N(p, r) \cap \widehat{X}$, so by Theorem 3.6.3, $\mathbb{B}_N(p, r) \cap \widehat{X}$ is a variety.

The final assertion of the theorem is proved as is the final assertion of Theorem 3.1.1.

4.4. Finite Area

We know that the hull of a compact set X of class $\mathscr{A}_1$ consists of X together with a possibly empty one-dimensional variety. We have seen in Chapter 3 that if the set X is contained in a connected set of finite length, then the variety in question has finite area. There is a corresponding result when X is only supposed to be of class $\mathscr{A}_1$. This is contained in a more general fact about varieties with boundary a set of class $\mathscr{A}_1$ that was found by Dinh [101] and, in a slightly less general form, by Lawrence [215]; it is an extension of Theorem 3.7.4.

Theorem 4.4.1. *If X is a closed subset of class $\mathscr{A}_1$ of an open set Ω in $\mathbb{C}^N$, and if V is a closed one-dimensional subvariety of $\Omega \setminus X$, then the area of V is locally finite in Ω.*

This is a local result, so the corresponding statement with $\mathbb{C}^N$ replaced by an arbitrary complex manifold endowed with a Hermitian metric is also correct.

The following lemma is a small variation of Lemma 3.7.5.

Lemma 4.4.2. *Let E be a closed subset of the open subset Ω of $\mathbb{C}^N$ that is of class $\mathscr{A}_1$. Let V be a one-dimensional analytic subvariety of $\Omega \setminus E$, and let $p \in E$. Let P be a polynomial such that for some bounded neighborhood U of p, P carries $U \cap (E \cup V)$ properly into the open disk Δ in $\mathbb{C}$ and such that the set $P(E \cap \bar{U})$ is of class $\mathscr{A}_1$. Let C and C' be two components of $\Delta \setminus P(E \cap U)$ with $\Lambda^1(bC \cap bC') > 0$. Let m and m' be, respectively, the multiplicities of the map $P : P^{-1}(C) \cap V \to C$ and $P : P^{-1}(C') \cap V \to C'$. If there are a subset S of $bC \cap bC'$ of positive one-dimensional measure and a positive integer n such that for each point $\zeta \in S$, the fiber $P^{-1}(\zeta) \cap E$, which we shall denote by E_ζ, consists of n points, then $|m - m'| \leq n$.*

Proof. The polynomial P is constant on no branch of $V \cap U$, and the compact set $P(E \cap U)$ is a set of class $\mathscr{A}_1$. By Theorem 4.3.12, there are rectifiable simple closed curves γ and γ' contained in $C \cup S$ and $C' \cup S$, respectively, such that $\gamma \cap \gamma'$ is a subset S_o of S of positive length. We shall show that if $\zeta \in S_o$, then $V \cap (P^{-1}(\zeta) \cap V)$ contains at most m points. Suppose, to the contrary, that the set $V \cap (P^{-1}(\zeta) \cap V)$ contains $m + 1$ distinct points, say $a_1, \dots, a_{m+1}$. There are then disjoint neighborhoods W_j of the points a_j each of which is mapped properly onto a fixed neighborhood W_ζ of ζ. This implies that for some points ζ' in C, the fiber $P^{-1}(\zeta') \cap V$ contains more than m points, which contradicts the choice of m as the multiplicity of P on V over C. Consider now the Jordan domain C'' bounded by the curve γ', a subdomain of C'. The set $(V \cup E) \cap P^{-1}(\gamma')$ is compact, and its polynomially convex hull contains the variety $P^{-1}(C'') \cap V$. Moreover, for each $\zeta \in \gamma' \cap S_o$, the set $P^{-1}(\zeta)$ meets $(V \cup E) \cap P^{-1}(\gamma')$ in a set containing not more than $m+n$ points. The set $\gamma' \cap S_o$ has positive length, so Corollary 3.6.2 implies that the set

$$[(V \cup E) \cap P^{-1}(\gamma')]\widehat{\ } \setminus (V \cup E) \cap P^{-1}(\gamma')$$

is a one-dimensional analytic variety mapped properly onto C'' with multiplicity not more than $m+n$ by P. Consequently, $m' \leq n+m$, so that $m' - m \leq n$. Symmetrically, $m - m' \leq n$, so $|m - m'| \leq n$ as desired .

Proof of Theorem 4.4.1. We suppose the origin to lie in X and show that near 0, the variety V has finite area.

To do this, start by recalling that by invoking Lemma 3.8.8, we can suppose the coordinates $z_1, \dots, z_N$ on $\mathbb{C}^N$ are clear at the origin with respect to the set $V \cup X$. Thus, if $\pi_j : \mathbb{C}^N \to \mathbb{C}$ is the projection $\pi_j(z) = z_j$, then for each j, there is a neighborhood W_j of the origin with the property that π_j carries $W_j \cap (V \cup X)$ properly onto its image, a subset of the unit disk $\mathbb{U}$ in $\mathbb{C}$. The proof of Lemma 3.8.8 shows that almost all coordinate systems have this property, so by Lemma 4.3.11, we can suppose, in addition, that each of the sets $\pi_j(X \cap \bar{\mathbb{B}}_N(\delta))$, for a small, positive δ, is of class $\mathscr{A}_1$.

Fix attention on π_1, which we shall denote by π. Fix the holomorphic coordinate $w = s + it$ on $\mathbb{C}$. The set $\pi(W_1 \cap X)$ is a closed subset of $\mathbb{U}$ of finite length. Let the components of $\mathbb{U} \setminus \pi(W \cap X)$ be Ω_j, $j = 1, \dots,$ and let m_j be the multiplicity of the map $\pi : V \cap \pi^{-1}(\Omega_j) \to \Omega$. We shall show that if K is a compact subset of $\mathbb{U}$, then $\sum_{j=1,\dots} m_j \operatorname{area}(K \cap \Omega_j)$ is finite. Once we have this, then because the other coordinate projections $\pi_2, \dots, \pi_N$ can be treated in a similar way, it will follow that near the origin, V has finite area.

Thus, fix a compact subset K of $\mathbb{U}$. Choose $r \in (0, 1)$ large enough that $K \subset r\mathbb{U}$. The number r can be chosen such that there are points $p \in \mathbb{U}$ of modulus r that do not lie in $\pi(X)$, for the set $\pi(X)$ has finite length. Denote by S_r^+ the closed half of the circle $rb\mathbb{U}$ that lies in the closed half-plane $\Re w \geq 0$. We assume this semicircle to be disjoint from $\pi(W_1 \cap X)$. Consequently, S_r^+ is contained entirely in one of the components Ω_j, say Ω_0.

Let Δ_r^+ be the half-disk $\{w : |w| < r, \ \Re w > 0\}$. Fix a positive number ρ, and consider the sum

$$A(\rho) = \min(m_0, \rho) \operatorname{area} \Omega_0 + \sum_{j=1,\dots} \min(m_j, \rho) \operatorname{area}(\Omega_j \cap \Delta_r^+). \tag{4.8}$$

We have

$$\operatorname{area}(\Omega_j \cap \Delta_r^+) = \int_{\Omega_j \cap \Delta_r^+} d\mathscr{L}(w).$$

Let $\eta : \mathbb{C} \to \mathbb{R}$ be the projection $\eta(w) = s$. For all $s > 0$ the set $\cup_j b\Omega_j \cap \{\Re w > s\}$ has finite length, so there is a set $S \subset \mathbb{R}$ of zero length such that for all $s > 0$, $s \in \mathbb{R} \setminus S$, the set $\eta^{-1}(s) \cap \cup_j b\Omega_j$ is finite.

Then

$$A(\rho) = \min(m_0, \rho) \operatorname{area} \Omega_0 + \sum_{j=1,\dots} \min(m_j, \rho) \int_{\mathbb{R}\setminus S} \left\{ \int_{\eta^{-1}(s)\cap\Omega_j} dt \right\} ds.$$

An argument like the one in the proof of Theorem 3.7.4 shows this sum to be bounded uniformly in ρ, so $\sum_{j=1,\dots} m_j \operatorname{area}(\Omega_j \cap \Delta_r^+) < \infty$, whence

$$\sum_{j=1,\dots} \operatorname{area}(\Omega_j \cap K \cap \Delta_r^+) < \infty.$$

This bound has been derived under the assumption that the semicircle S_r^+ is disjoint from $\pi(W_1 \cap X)$. If this condition is not satisfied, we fix a point p of modulus r that is not in the set $\pi(W_1 \cap X)$. There is a diffeomorphism ψ of the plane onto itself that leaves a neighborhood of K fixed pointwise, that carries $\bar{\mathbb{U}}$ onto itself, and that carries an arc in the circle $\{w : |w| = r\}$ that contains p onto the semicircle S_r^+. Having ψ, we replace π by $\psi \circ \pi$ in the analysis just given to obtain the finiteness of $\sum m_j \operatorname{area}(\Omega \cap K \cap \Delta_r^+)$. In the same way, $\sum m_j \operatorname{area}(\Omega_j \cap K \cap \Delta_r^-)$ is finite if Δ_r^- is $\{w : |w| < r, \Re w < 0\}$.

The theorem is proved.

Corollary 4.4.3. *If $X \subset \mathbb{C}^N$ is a compact set of class $\mathscr{A}_1$, then $\Lambda^2(\widehat{X}) < \infty$.*

4.5. Stokes's Theorem

We now take up the question of Stokes's theorem. Let Ω be a bounded domain in $\mathbb{C}^N$, let Γ be a closed subset of class $\mathscr{A}_1$ of the domain Ω, and let V be a purely one-dimensional subvariety of $\Omega \setminus \Gamma$. By Theorem 4.4.1, we know that the variety V has locally finite area in Ω, so the current $[V]$ of integration over V is defined as a functional on smooth 2-forms with compact support in Ω. If α is such a form, then $[V](\alpha) = \int_V \alpha$. The boundary $b[V]$ of the current $[V]$ is the current acting on smooth 1-forms with compact support in Ω by the condition that if α is such a form, then $b[V](\alpha) = [V](d\alpha)$. The support of $b[V]$ is a closed subset of $(\bar{V} \setminus V) \cap \Omega$; it can be a proper subset of this set, as, for example, when $\bar{V} \setminus V$ contains isolated points. The problem that Stokes's theorem solves is that of understanding the nature of the current $b[V]$ in more or less concrete terms.

The following theorem can be regarded as a provisional version of Stokes's theorem. We continue with the notation we have just established.

Theorem 4.5.1. *There are measurable functions B_j, $j = 1, \dots, 2N$, defined and bounded by one in absolute value on Γ such that for every smooth 1-form $\alpha = \sum_{j=1,\dots,2N} a_j(z)dx_j$ on Ω with compact support,*

$$b[V](\alpha) = \int_\Gamma \Big\{ \sum_{j=1,\dots,2N} a_j(z) B_j(z) \Big\}\, d\Lambda^1(z). \tag{4.9}$$

Alternatively phrased, if $\vec{B}$ is the measurable vector $(B_1, \dots, B_{2N})$ and $\langle \vec{B}(x), \alpha(x) \rangle$ is the value of the one-form $\alpha(x)$ on the vector $\vec{B}(x)$, then

$$b[V](\alpha) = \int_\Gamma \langle \vec{B}(x), \alpha(x) \rangle \, d\Lambda^1(x).$$

A version of Stokes's theorem in this general setting was first obtained by Lawrence [216, 217]. A result in the context of sets of class $\mathscr{A}_1$ was given by Dinh [101]. Their formulations are more refined than the result just stated; we will discuss them at the end of this section.

We consider in Theorem 4.5.10 below the special case that $\bar{V} \setminus V$ is a rectifiable simple closed curve.

Proof. The problem is local, and it is independent of the choice of holomorphic coordinates. Accordingly, we suppose $0 \in \Gamma$, and we suppose $z_1, \dots, z_N$ to be a clear set of coordinates at 0 for the set $V \cup \Gamma$. (Recall Lemma 3.8.8.) We can suppose that no z_j is constant on any branch of V. Choose neighborhoods $W_1, \dots, W_N$ of the origin with the property that if $\pi_j : \mathbb{C}^N \to \mathbb{C}$ is the jth coordinate projection, then π_j carries $W_j \cap (V \cup \Gamma)$ properly onto its image, which is contained in a disk in the plane, which, after an additional change of coordinates, we can suppose to be the unit disk. As in the proof of Theorem 4.4.1, we suppose $\pi_j(\Gamma \cap \bar{W}_j)$ to be a set of class $\mathscr{A}_1$.

We consider the projection π_1, which, as a notational convenience, we rename π. Let $\phi = \Re\pi$, so that $\phi(z) = x_1$.

Fix attention on a form $a\, dx_1$ with coefficient a a smooth function on $\mathbb{C}^N$ with support in W_1.

The set of singularities V_{sing} of V is at most countably infinite, and almost every point in $\phi(V_{\text{reg}})$ is a regular value of $\phi|V_{\text{reg}} \cap W_1$. Thus, for a subset E of $\mathbb{R}$ of measure zero, if $t \in \mathbb{R} \setminus E$, then $\phi^{-1}(t) \cap V \cap W_1$ is a disjoint union of open analytic arcs. By enlarging E with at most a countable set, we can suppose that E contains all the points $\phi(z)$ for $z \in V_{\text{reg}} \cap W_1$ at which the differential $d(\pi|V_{\text{reg}})$ vanishes.

Fubini's theorem implies that

$$b[V](a\, dx_1) = \int_V d(a\, dx_1) = \int_{\mathbb{R}} \left\{ \int_{\phi^{-1}(t) \cap V \cap W_1} da \right\} dt.$$

We must specify the orientation on the fibers $\phi^{-1}(t)$ for which this is correct. The manifold V_{reg}, as a complex manifold, has a natural orientation, and the holomorphic projection $\pi|V_{\text{reg}}$ is orientation-preserving. Thus, the orientation on $\phi^{-1}(t)$ in the integral formula just given is the orientation that makes $\pi|\phi^{-1}(t)$ orientation-preserving as a map to the line $L_t = \{t + is : s \in \mathbb{R}\}$ when the positive direction on this line is declared to be the direction of increasing s. Note that for almost all $t \in \mathbb{R}$, the fiber $\phi^{-1}(t)$ has finite length: By Eilenberg's inequality, Theorem 3.3.6, $\int_R^* \Lambda^1(\phi^{-1}(t) \cap V \cap W_1)\, dt \le \Lambda^2(V \cap W_1)$.

Write $\mathbb{U} \setminus \pi(\Gamma \cap W_1) = \Omega_1 \cup \cdots$, where the Ω's are the components of the set $\mathbb{U} \setminus \pi(\Gamma \cap W_1)$. They are bounded domains with $b\Omega_j \subset \pi(\Gamma \cap W_1 \cup b\mathbb{U})$; they need not be simply connected. For each j, let m_j be the multiplicity of $\pi|(V \cap W_1)$ over Ω_j. Some of the m_j may be zero.

For almost every $t \in \mathbb{R}$, the vertical line L_t introduced above meets $\pi(\Gamma \cap W_1)$ in a finite set. (Eilenberg's inequality gives $\int_{\mathbb{R}} \Lambda^0(L_t \cap \pi(\Gamma \cap W_1))\, dt \leq \Lambda^1(\pi(\Gamma \cap W_1)) < \infty$.) Thus, by enlarging the set E by at most a null set, we can suppose that for each $t \in \mathbb{R} \setminus E$, the line L_t meets $\pi(\Gamma \cap W_1)$ at most in a finite set.

For each j, the set $L_t \cap \Omega_j$ is a finite union $\cup_{k=1,\dots,\nu(t,j)} L_{t;j,k}$ of mutually disjoint open intervals.

By hypothesis, the projection $\pi|V$ is unbranched over L_t for $t \notin E$. For such t, $\pi^{-1}(L_{t;j,k})$ consists of m_j open arcs, each of finite length, say

$$\pi^{-1}(L_{t;j,k}) = \cup_{\kappa=1,\dots,m_j} \lambda_{t;j,k,\kappa}.$$

Thus, for $t \in \mathbb{R} \setminus E$,

$$\int_{\phi^{-1}(t)\cap V \cap W_1} da = \sum_{j,k,\kappa} \int_{\lambda_{t;j,k,\kappa}} da.$$

The arc $\lambda_{t;j,k,\kappa}$ has finite length, so it has definite endpoints, say $e^+_{t;j,k,\kappa}$ and $e^-_{t;j,k,\kappa}$, with the labeling chosen so that the point $\pi(e^+_{t;j,k,\kappa})$ lies above the point $\pi(e^-_{t;j,k,\kappa})$ in the line L_t. Accordingly,

$$\int_{\lambda_{t;j,k,\kappa}} da = a(e^+_{t;j,k,\kappa}) - a(e^-_{t;j,k,\kappa}).$$

We have therefore reached the equality

$$b[V](a\,dx_1) = \int_{\mathbb{R}} \sum_{j,k,\kappa} \left\{ a(e^+_{t;j,k,\kappa}) - a(e^-_{t;j,k,\kappa}) \right\} dt. \tag{4.10}$$

The endpoints $e^\pm_{t;j,k,\kappa}$ at which the function a does not vanish all project into $\pi(\Gamma \cap W_1)$ under the projection π, though they may not all lie in Γ.

We need the observation that for each j and k and for almost every fixed t, the endpoints $e^+_{t;j,k,\kappa}$ are distinct for distinct values of κ and that a similar observation holds for the endpoints e^-. It suffices to treat the e^+. To do this, suppose, for the sake of contradiction, that there are a set $S_o \subset \mathbb{R}$ of positive measure and a choice of j and k such that for each $t \in S_o$, two of the points $e^+_{t;j,k,\kappa}$ coincide. By Theorem 4.3.12 there is a rectifiable simple closed curve J that bounds a simply connected domain $\Omega(S_o)$ contained in Ω_j such that $J \cap b\Omega_j$ has positive length and for each point $\zeta \in J \cap b\Omega_j$, two of the endpoints $e^+_{t;j,k,\kappa}$ with $t = \Re\zeta$ coincide, and moreover, the segment $L_{t;j,k}$ approaches ζ through the domain $\Omega(S_o)$.

Let P be a polynomial that separates points in the fiber $\pi^{-1}(\zeta) \cap V$ for some $\zeta \in \Omega_j$ for which this fiber consists of m_j distinct points. The Riemann removable singularity theorem implies that there is a function Q bounded and holomorphic on Ω_j such that for all $\zeta \in \Omega_j$ for which the fiber $\pi^{-1}(\zeta) \cap V$ consists of m_j distinct points

$$Q(\zeta) = \prod [P(z_j) - P(z_k)]^2$$

in which the product is extended over all points $z_j, z_k \in \pi^{-1}(\zeta)$ with $z_j \neq z_k$.

If, for a fixed t, j, and k, two of the points $e^+_{t;j,k,\kappa}$ coincide, then $\lim Q(\zeta) = 0$, where the limit is taken as $\zeta \in L_{t;j,k}$ approaches the top endpoint of $L_{t;j,k}$. Thus, the top half of $L_{t;j,k}$ is an open arc in Ω_j along which the bounded holomorphic function Q approaches 0. The theorem of Lindelöf, Theorem 3.4.12, together with a conformal mapping of Ω_j to the unit disk shows that Q necessarily has nontangential limit 0 at the top endpoint of $L_{t;j,k}$. If this happens for a set of t's that constitute a set of positive length, then Q must vanish identically, in contradiction to the choice of P.

Thus, by enlarging E yet again by a null set if necessary, we can suppose that for all $t \notin E$ and for all j, k, the endpoints $e^+_{t;j,k,\kappa}$ are distinct, as are the endpoints $e^-_{t;j,k,\kappa}$.

Consider a point $w \in \bar{V}$ with $\pi(w) \in \pi(\Gamma) \cap L_t$ for $t \notin E$. There are three possibilities. 1^o: No arc $\lambda_{t;j,k,\kappa}$ terminates at w. 2^o: Exactly two of the arcs $\lambda_{t;j,k,\kappa}$ terminate at w. 3^o. Exactly one of the arcs $\lambda_{t;j,k,\kappa}$ terminates at w. In the first case, no $e^\pm_{t;j,k,\kappa}$ is the point w, so w does not enter into the sum on the right of equation (4.10). In the second case, at least for almost all $t \in \mathbb{R}$, the projections under π of the two arcs approach the point $\pi(w)$ from above and below (or below and above), respectively through the line L_t, for the points $e^+(t; j, k, \kappa)$ for fixed t, j, and k are distinct for distinct κ's. In either case, w is $e^+_{t;j,k,\kappa}$ and $e^-_{t;j',k',\kappa'}$ for some choice of j, k, κ and j', k', κ'. In this case, the corresponding terms in the sum under the integral in equation (4.10) cancel. We are left with

$$\sum_{j,k,\kappa} a(e^+_{t;j,k,\kappa}) - a(e^-_{t;j,k,\kappa}) = \sum_{z\in\phi^{-1}(t)\cap W_1\cap\Gamma} \varepsilon(z)a(z), \tag{4.11}$$

where for each z, the number $\varepsilon(z)$ is -1, 0, or 1. More precisely, $\varepsilon(z) = 1$ if z is an e^+, and $\varepsilon(z) = -1$ if z is an e^-. If z is neither an e^+ nor an e^- or if it is both an e^+ and an e^-, then $\varepsilon(z) = 0$. This equation is correct for almost all $t \in \mathbb{R}$. Consequently,

$$\begin{aligned} |b[V](a\,dx_1)| &\leq \sup\nolimits_{\Gamma\cap W_1}|a| \int_{\mathbb{R}} \Lambda^0(\phi^{-1}(t)\cap\Gamma\cap W_1)\,dt \\ &\leq \|a\|_{\Gamma\cap W_1}\Lambda^1(\Gamma\cap W_1). \end{aligned} \tag{4.12}$$

The preceding inequality implies that the current $b[V]$ extends to a functional acting on forms of the form $a\,dx_1$ with the function a merely continuous.

The Riesz representation theorem applied to the functional $a \mapsto b[V](a\,dx_1)$ on the space of continuous functions on $\Gamma \cap W_1$ with compact support yields a locally finite regular Borel measure μ_1 on $\Gamma \cap W_1$ such that

$$b[V](a\,dx_1) = \int_{\Gamma\cap W_1} a(z)\,d\mu_1(z)$$

for all forms of the kind we are considering.

We now show that $d\mu_1 = B_1 d\Lambda^1 \llcorner (\Gamma \cap W_1)$ with B_1 a function satisfying $|B_1| \leq 1$ a.e. $[d\Lambda^1]$. For this, fix $p \in \Gamma \cap W_1$, and let $\rho > 0$ be small enough that the closure of the ball $b\mathbb{B}_N(p,\rho)$ is contained in W_1. We have that

$$|\mu_1(\mathbb{B}_N(p,\rho))| \leq \left| \int_{(\phi(p)-\rho,\phi(p)+\rho)} \left\{ \sum_{\phi^{-1}(t)\cap\mathbb{B}_N(p,\rho)} \varepsilon(z) \right\} dt \right| \leq \Lambda^1(\mathbb{B}_N(p,\rho)\cap\Gamma).$$

This is correct for all $\rho > 0$ and all choices of $p \in \Gamma \cap W_1$, so the measure μ_1 is absolutely continuous with respect to $\Lambda^1 \llcorner (\Gamma \cap W_1)$, and the Radon–Nikodym derivative $\frac{d\mu_1}{d\Lambda^1 \llcorner \Gamma}$ is bounded in modulus by one. We have therefore that $\mu_1 = B_1 \Lambda^1 \llcorner (\Gamma \cap W_1)$ with $|B_1| \leq 1$.

We have been working with the projection $\pi = \pi_1$; the other projections π_j are handled in a similar way.

We have proved that when acting on forms with support near the origin, the current $b[V]$ has the stated structure. The theorem is proved.

Let us fix a compact set Γ of class $\mathscr{A}_1$ in the domain Ω and a purely one-dimensional subvariety V of Ω as in Theorem 4.5.1, so that there are bounded measurable functions B_j on Γ for which, with $\vec{B}$ the vector with entries B_j,

$$b[V](\alpha) = \int_\Gamma \langle \vec{B}(x), \alpha(x) \rangle \, d\Lambda^1(x)$$

for all smooth 1-forms α on Ω with compact support. If u and f are smooth functions on $\mathbb{C}^N$ with $f = 0$ on Γ, then

$$0 = b[V](d(uf)) = b[V](udf + fdu) = \int_\Gamma u(x) \langle \vec{B}(x), df(x) \rangle \, d\Lambda^1(x).$$

This is correct for every choice of u, so $\langle \vec{B}(x), df(x) \rangle$, as a function of x, vanishes a.e. $[d\Lambda^1]$ on Γ.

In the case that Γ is a simple closed curve of class $\mathscr{C}^1$, the vector $\vec{B}(x)$ is almost everywhere tangent to Γ. Otherwise, by a theorem of Lusin there is a compact subset E of Γ of positive length on which $\vec{B}$ is continuous. There is then a fixed vector $\beta \in \mathbb{C}^N$ such that for all x in a subset E_β of positive length, $\vec{B}(x)$ is very near β. There is a $\mathscr{C}^1$ function f that vanishes on Γ and whose gradient is not orthogonal to β. For this function f, the quantity $\langle \vec{B}(x), df(x) \rangle$ does not vanish a.e. $[d\Lambda^1]$. Thus, when Γ is a simple closed curve of class $\mathscr{C}^1$, the vector $\vec{B}(x)$ of Theorem 4.5.1 is a.e. $[d\Lambda^1]$-tangent to Γ.

We now turn our attention to the version of Stokes's theorem for varieties bounded by rectifiable simple closed curves. To reach an understanding of this theorem requires considerable further effort; several preliminaries are required.

In [6], Alexander introduced the useful notion of *ample adjacency*:

Definition 4.5.2. *If X is a closed subset of $\mathbb{C}$, the components Ω and Ω' of $\mathbb{C} \setminus X$ are* amply adjacent *if there are an interval $I = [a,b] \subset \mathbb{R}$, a compact subset K of I of positive length, and positive numbers $c < c'$ such that $I \times \{c\} \subset \Omega$, $I \times \{c'\} \subset \Omega'$, and the set $E = (K \times [c,c']) \cap X$ is a subset of $b\Omega \cap b\Omega'$ that is mapped homeomorphically onto the set K by the projection of $\mathbb{C}$ onto the x-axis.*

The condition that Ω and Ω' be amply adjacent should be thought of as a generalization of the condition that their boundaries share an arc that is not vertical. In this description, the x-axis has a preferred role. One could consider a similar condition in which one allows any real line to replace the x-axis. As we shall see, the notion formalized in the definition suffices for our purposes.

Let $\pi : \mathbb{C} \to \mathbb{R}$ be the projection onto the x-axis.

Theorem 4.5.3. [6] *If $X \subset \mathbb{C}$ is a compact set such that $\int_{\mathbb{R}} \Lambda^0(\pi^{-1}(x) \cap X)\, dx < \infty$, then given two components Ω and Ω' of $\mathbb{C} \setminus X$, there exists a finite sequence $\Omega_1, \ldots, \Omega_s$ of components of $\mathbb{C} \setminus X$ with $\Omega_1 = \Omega$, $\Omega_s = \Omega'$ and with Ω_j and Ω_{j+1} amply adjacent for each $j = 1, \ldots, s-1$.*

For the proof of this result, a measure-theoretic lemma is needed:

Lemma 4.5.4. *Let X be a second-countable topological space, let Y be a set, let $f : X \to Y$ be a function, and let μ be a nonnegative measure on a σ-algebra of subsets of Y. If for every open set V in X, the set $f(V)$ is measurable with respect to μ, then for almost every $[d\mu]$ $y \in Y$, the image under f of each neighborhood of every point of $f^{-1}(y)$ has positive measure.*

Proof. Let U be the open subset of X that consists of all the points in X that have neighborhoods V such that $f(V)$ has measure zero. Because the space X is second-countable, there is a *countable* family $\{V_j : j = 1, \ldots\}$ of open sets with union U and with the property that for each j, $f(V_j)$ is a μ-null set. Then $f(U) = \cup_{j=1,\ldots} f(V_j)$ is a null set, and the lemma is proved.

Proof of Theorem 4.5.3. With no loss of generality we suppose Ω' to be the unbounded component of $\mathbb{C} \setminus X$. Choose $a, b, c \in \mathbb{R}$ such that the interval $[a, b] \times \{c\}$ is contained in Ω and such that for some compact subset K of $[a, b]$ of positive length, $\Lambda^0(\pi^{-1}(x) \cap X) \leq s$ for all $x \in K$.

The proof is by induction: We prove the following proposition for $s = 0, 1, \ldots$. $\mathscr{P}(s)$: If $Y \subset \mathbb{C}$ is a compact set, if Ω is a component of $\mathbb{C} \setminus Y$, if $a, b, c \in \mathbb{R}$ satisfy $[a, b] \times \{c\} \subset \Omega$, and if there is a compact subset K of $[a, b]$ of positive length such that for all $x \in K$, $\Lambda^0(\pi^{-1}(x)) \cap Y \leq s$, then Ω can be connected to the unbounded component of $\mathbb{C} \setminus Y$ by a finite chain of amply adjacent components.

The statement $\mathscr{P}(0)$ is true; in this case, Ω is the unbounded component of $\mathbb{C} \setminus Y$. We shall show that $\mathscr{P}(s-1)$ implies $\mathscr{P}(s)$. To do this, assume the statement $\mathscr{P}(s-1)$ to be true.

Let Y, a, b, c, and K be as in the statement $\mathscr{P}(s)$.

It could be that for a subset K_o of K of positive length, $\Lambda^0(\pi^{-1}(x) \cap Y) \leq s - 1$ for all $s \in K_o$, in which case the validity of $\mathscr{P}(s-1)$ implies that, as desired, Ω can be connected to the unbounded component of $\mathbb{C} \setminus Y$ with a finite chain of amply adjacent components of $\mathbb{C} \setminus Y$.

Thus, we suppose that for every $x \in K$, the fiber $Y_x = Y \cap \pi^{-1}(x)$ has cardinality s. By Lemma 4.5.4, there is a point $x_o \in K$ such that the fiber Y_{x_o} consists of s distinct points each of which has a neighborhood that projects under π onto a set of positive length. Let $Y_{x_o} = \{(x_o, y_1), \ldots, (x_o, y_s)\}$ with $y_1 < \cdots < y_s$. Choose numbers $c_1, \ldots, c_{s+1}$ with

$c_1 < y_1 < c_2 \cdots < c_s < y_s < c_{s+1}$ and with the number c one of the c_j. By shrinking the interval $[a, b]$ if necessary, we can suppose that each of the intervals $[a, b] \times \{c_j\}$ lies in $\mathbb{C} \setminus Y$.

Let W_i be the component of $\mathbb{C} \setminus Y$ that contains $[a, b] \times \{c_i\}$, and let R_i be the rectangle $R_i = [a, b] \times [c_i, c_{i+1}]$. If for each i, $\pi(R_i \cap Y) \supset K$, then $\pi|(K \times [c_i, c_{i+1}] \cap Y)$ is injective and hence a homeomorphism. Thus, each pair W_i, W_{i+1} is amply adjacent, and we can take the appropriate W's for the desired chain of amply adjacent domains.

The remaining case is that for some $j \in \{1, \ldots, s\}$, the set $\pi(R_j \cap Y) \cap K$ is a proper subset of K. Then $W_j = W_{j+1}$. Put $K' = \pi(R_i \cap Y) \cap K$. If $Y' = Y \setminus \operatorname{int} W_j$, then π is at most $(s-1)$-to-one on Y' over points of K'. Let Ω' be the component of $\mathbb{C} \setminus Y'$ that contains Ω. Apply $\mathscr{P}(s-1)$: There is a sequence $\Omega'_1, \ldots, \Omega'_q$ of amply adjacent components of $\mathbb{C} \setminus Y'$ that connects Ω' to the unbounded component of $\mathbb{C} \setminus Y'$. We can delete any repetitions that occur in this sequence. Let W be the component of $\mathbb{C} \setminus Y'$ that contains $\operatorname{int} R_i$. If no Ω'_j is W, then the Ω'_j's are components of $\mathbb{C} \setminus Y$, and we are done. If Ω'_k is W, then the sequence $\Omega'_1, \ldots, \Omega'_{k-1}, W, \Omega'_{k+1}, \ldots, \Omega$ works.

The theorem is proved.

An important fact is the irreducibility of the variety $\widehat{\Gamma} \setminus \Gamma$ when Γ is a rectifiable simple closed curve:

Theorem 4.5.5. [17] *If Γ is a rectifiable simple closed curve in $\mathbb{C}^N$, then the analytic variety $\widehat{\Gamma} \setminus \Gamma$ has only one global branch.*

A generalization of this result is given in Section 4.7 below.

Corollary 4.5.6. *If γ is a rectifiable simple closed curve in $\mathbb{C}^N$, and if V is a bounded, purely one-dimensional subvariety of $\mathbb{C}^N \setminus \gamma$, then V has a single global branch, and $\gamma \cup V$ is polynomially convex.*

Proof of Theorem 4.5.5. Let $\varphi : \mathbb{C}^N \to \mathbb{C}$ be a linear functional that is constant on no global branch of the variety V. Denote by Ω_j, $j = 0 \ldots$, the components of $\mathbb{C} \setminus \varphi(\Gamma)$ with Ω_0 the unbounded component.

Let K be a subset of $b\Omega_0$ of positive length with the properties that for each point $p \in K$, the fiber $\varphi^{-1}(p) \cap \Gamma$ contains exactly s points and that each point of K lies in the boundary of a fixed Ω_j with $j > 0$, say Ω_1.

Each point of K is covered by s points of Γ, so φ maps $V \cap \varphi^{-1}(\Omega_1)$ in an at most s-to-one way onto Ω_1.

On the other hand, if V_o is a global branch of V, then $\bar{V}_o \setminus V_o$ contains Γ, so if $p \in K$ and $\varphi^{-1}(p) \cap \Gamma = \{w_1, \ldots, w_s\}$, then near each of the points w_j there is a point of V_o. This implies that the points in Ω_1 near K are covered s times under φ by V_o and hence that all of Ω is covered s times by V_o. The set Ω_1 is covered at most s times by $V \cap \varphi^{-1}(\Omega)$. Consequently, V must coincide with V_o over Ω_1, and therefore V must be V_o, whence V is irreducible.

The theorem is proved.

A major ingredient of the proof of Stokes's theorem for rectifiable simple closed curves is the following result.

Theorem 4.5.7. [17] *Let Γ be a rectifiable simple closed curve in $\mathbb{C}^N$, and let V be a bounded purely one-dimensional subvariety of $\mathbb{C}^N \setminus \Gamma$. There is an orientation on Γ with the following property: If P is a polynomial on $\mathbb{C}^N$ and if for every component Ω_k of $\mathbb{C} \setminus P(\Gamma)$, m_k denotes the multiplicity of the proper holomorphic map $P|(V \cap P^{-1}(\Omega_k)) \to \Omega_k$, then for every k and for every point $w \in \Omega_k$, m_k is the index of the curve $P(\Gamma)$ about the point w.*

If we regard Γ as a map from the unit circle $\mathbb{T}$ to $\mathbb{C}^N$, so that $P \circ \Gamma : \mathbb{T} \to \mathbb{C} \setminus \{w\}$ is a continuous map, then the index in question is the integer $\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{d(P \circ \Gamma)}{P \circ \Gamma}$, which will be denoted by μ_k. According to the theorem, this index is always nonnegative.

Proof. One special case of the assertion of Theorem 4.5.7 is evident: If Ω_o denotes the unbounded component of $\mathbb{C} \setminus \Gamma$, then m_o and μ_o both vanish.

For an oriented closed curve γ in $\mathbb{C}$, we denote by $\mathrm{Ind}(\gamma, z)$ the index of γ about the point $z \in \mathbb{C} \setminus \gamma$, which is the integer $\frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$.

The theorem is a statement about all polynomials. It is plainly true of polynomials that are constant on the variety $V = \widehat{\Gamma} \setminus \Gamma$, so we restrict our attention from this point on to polynomials that are nonconstant on V.

There is a simple argument that shows that if for every polynomial P there is an orientation on Γ for which for every component Ω_k of $\mathbb{C} \setminus P(\Gamma)$ the multiplicity of $P|V$ over Ω_k agrees with the index of the curve $P \circ \Gamma$ about every point of Ω_k, then the statement of the theorem is correct. The argument runs as follows. Fix an orientation on Γ that works for the fixed polynomial P. Let P_1 be a second polynomial that is not constant on V. It is enough to show that when Γ is given the orientation that works for the polynomial P, we have $\mathrm{Ind}(P_1 \circ \Gamma, z) \geq 0$ for all $z \in \mathbb{C} \setminus P_1(\Gamma)$. To this end, suppose there to be a point $q \in \mathbb{C} \setminus P_1(\Gamma)$ for which $\mathrm{Ind}(P_1 \circ \Gamma, q) < 0$. Let $-\mu$ be this index, so that $\mu > 0$. Choose a p such that $\nu = \mathrm{Ind}(P \circ \Gamma, p) > 0$, and introduce the function $F = (P - p)^\mu (P_1 - q)^\nu$. Integrate over Γ:

$$\frac{1}{2\pi i} \int_\Gamma \frac{dF}{F} = \mu \mathrm{Ind}(P \circ \Gamma, p) + \nu \mathrm{Ind}(P_1 \circ \Gamma, q) = \mu \mathrm{Ind}(P \circ \Gamma, p) + \nu(-\mu) = 0.$$

Thus, F has a logarithm on Γ. Consequently, by the argument principle, the number of zeros of F in V coincides with the number of poles of F. There are no poles, so there are no zeros. This is a contradiction, for F vanishes at the point p because $\mu > 0$.

Thus, to prove the theorem, we have only to show that for each polynomial nonconstant on V, there is an orientation of Γ for which the multiplicities coincide with the winding numbers.

A further reduction is possible. Fix a polynomial P that is nonconstant on V, and suppose there to be an orientation on Γ and a component Ω of $\mathbb{C} \setminus P(\Gamma)$ such that the multiplicity of F over Ω coincides with the index $\mathrm{Ind}(P \circ \Gamma, z)$ for each $z \in \Omega$, and that these numbers are greater than zero. Then over every component of $\mathbb{C} \setminus P(\Gamma)$ the multiplicity agrees with the index. For this, let Ω_j be some other component of $\mathbb{C} \setminus P(\Gamma)$. Choose points $p \in \Omega$ and $q \in \Omega_j$ such that the fibers $V \cap P^{-1}(p)$ and $V \cap P^{-1}(q)$ are contained in the set of regular points of V and such that both p and q are regular values of $P|V$. Let m be the multiplicity of P over Ω, and m_j that over Ω_j. Let $\nu = \mathrm{Ind}(P \circ \Gamma, p)$,

$\nu_j = \operatorname{Ind}(P \circ \Gamma, q)$, and define F by $f = \frac{(P-p)^{\nu_j}}{(P-q)^{\nu}}$. Again integrate over Γ:

$$\frac{1}{2\pi i}\int_\Gamma \frac{dF}{F} = \nu_j \operatorname{Ind}(P \circ \Gamma, p) - \nu \operatorname{Ind}(P \circ \Gamma, q) = \nu_j \nu - \nu \nu_j = 0.$$

Thus, F has a logarithm on Γ, and the number of zeros of F on V is the same as the number of poles. The function F has m zeros of order ν_j and m_j poles of order ν. By hypothesis $m = \nu$, so $\nu_j = m_j$, and we are done.

To conclude the proof, it is enough to prove the following lemma.

Lemma 4.5.8. *If P is a polynomial nonconstant on V, and if Ω is a component of $\mathbb{C} \setminus P(\Gamma)$ that is amply adjacent to the unbounded component, Ω_o, of $\mathbb{C} \setminus P(\Gamma)$, then the multiplicity of P over Ω agrees with the index* $\operatorname{Ind}(P \circ \Gamma, z)$ *for each $z \in \Omega$.*

The proof of this lemma is based on a further simple observation:

Fix an oriented closed curve γ in $\mathbb{C}$, which we regard as a continuous map from the unit circle $\mathbb{T}$ into $\mathbb{C}$, though we shall also speak of γ as a subset of $\mathbb{C}$. Let R be a closed rectangle in $\mathbb{C}$ with interior R^o. We assume that γ is disjoint from both the top and the bottom edges of R and that it meets the vertical sides of R. We also assume the bottom edge of R to lie in the unbounded component of $\mathbb{C} \setminus \gamma$. Let $L_1, \dots, L_s$ be the open arcs in $\mathbb{T}$ with the property that each L_j is carried by the map γ to a curve in R^o in such a way that γ takes the endpoints of L_j to opposite vertical edges of R. For each j, let γ_j be the restriction of γ to the closure of L_j, so that the set γ_j is a subset of R. Several of the sets γ_j can coincide. That γ is oriented implies that it is meaningful to say that γ_j goes from the left side of R to the right side or from the right side to the left. With this configuration, we have the following fact.

Lemma 4.5.9. *Let γ_k go from the left side of R to the right side of R for $k = 1, \dots, s_1$, and from the right side of R to the left side of R for $k = s_1 + 1, \dots, s$. If z is a point in the top edge of R, and if $s_2 = s - s_1$ then* $\operatorname{Ind}(\gamma, z) = s_1 - s_2$.

We give an analytic proof of this assertion, but a drawing of the situation makes the result clear.

Proof. Assume without loss of generality that z is the origin. Let T be the cut in the plane from the origin to the point at infinity obtained by connecting z to the center of the bottom side of R and proceeding from there to the point at infinity along the ray parallel to the negative imaginary axis.

Let λ be the branch of the logarithm function defined on $\mathbb{C} \setminus T$ that is real on the positive real axis. Let $\tilde{\lambda}$ be the branch of the logarithm function defined on $R \setminus \{0\}$ that agrees with λ on the left side of R and so satisfies $\tilde{\lambda} = \lambda + 2\pi i$ on the right side of R.

For $k = 1, \dots, s$, let σ_k be an arc in $\mathbb{C} \setminus (R^o \cup T)$ that connects the endpoints of γ_k and that is oriented so that the curve $\alpha_k = \gamma_k + \sigma_k$ is closed. Then $\operatorname{Ind}(\alpha_k, 0) = 1$ or -1 according as γ_k goes from the left edge of R to the right edge or the other way round. We have that $\gamma = \alpha_1 + \cdots + \alpha_s + \gamma_o$ for a closed curve γ_o in $\mathbb{C} \setminus (R^o \cup T)$. Because γ_o lies in the simply connected domain $\mathbb{C} \setminus T$, its index around the origin is zero, whence $\operatorname{Ind}(\gamma, 0) = \operatorname{Ind}(\alpha_1, 0) + \cdots + \operatorname{Ind}(\alpha_s, 0)$. This quantity is $s_1 - s_2$, so the lemma is proved.

Proof of Lemma 4.5.8. Let Ω_o be the unbounded component of $\mathbb{C} \setminus P(\Gamma)$, and let Ω be a component of $\mathbb{C} \setminus P(\Gamma)$ that is amply adjacent to Ω_o.

Ample adjacency provides a rectangle $R = [a, b] \times [c, d]$ with interior R^o such that the bottom edge of R lies in Ω_o, the top edge in Ω, and that has the property that for a certain compact set $K_1 \subset [a, b]$ of positive length that contains both the points a and b as limit points, each vertical line $x = t$ for $t \in K_1$ meets $P(\Gamma) \cap R$ in a single point. These points of intersection constitute a compact set K that projects homeomorphically onto K_1 under the projection onto the x-axis. Moreover, there is an integer s such that for each point $p \in K$, the fiber $P^{-1}(p) \cap \Gamma$ consists of s distinct points.

Construct two rectifiable simple closed curves $J_o \subset \Omega_o \cup K$ and $J \subset \Omega \cup K$ as follows. The set $[a, b] \setminus K_1$ consists of mutually disjoint open intervals (x'_j, x''_j). For each j, R'_j is the smallest rectangle of the form $[x'_j, x''_j] \times [\delta'_j, \delta''_j]$ such that $R'_j \subset R$, and $R'_j \supset P(\Gamma) \cap ([x'_j, x''_j] \times [c, d])$. Then R'_j consists of two points in K together with a bottom arc contained in Ω_o and a top one contained in Ω. Form the two simple closed curves $J_o \subset \bar{\Omega}_o$ and $J \subset \bar{\Omega}$ as follows: J_o is the union of K, the bottom arcs just constructed, and the bottom arc of R. The curve J is formed similarly, using the top arcs. The rectifiability of Γ implies that of J_o and J. We have $J_o \cap J = K$ and $J_o \cup J \supset bR$. Let W be the closure of the domain bounded by the curve J.

If we shrink R appropriately, we shall have that the set $P^{-1}(R) \cap \widehat{\Gamma}$ consists of s mutually disjoint, closed, relatively open subsets $A_1, \ldots, A_s$ of $P^{-1}(R) \cap \widehat{\Gamma}$ each of which is mapped homeomorphically onto $\bar{W}$ by P. To see this, note that if we shrink R to a point p of K, then the preimage $P^{-1}(R) \cap \widehat{\Gamma}$ shrinks to the fiber $P^{-1}(p) \cap \widehat{\Gamma}$, which consists of s points. Thus, if R is small enough, then $P^{-1}(R) \cap \widehat{\Gamma}$ will be the union of s mutually disjoint sets, each open and closed in $P^{-1}(R) \cap \Gamma$ and each of which maps injectively onto $\bar{W}$.

Fix a point $p \in K$ with $a < \Re p < b$. The set $P^{-1}(p) \cap \widehat{\Gamma}$ consists of s points, $p_1, \ldots, p_s$, all of which are in Γ.

Let Γ_k be the shortest open subarc of Γ that contains p_k and both of whose endpoints lie over bR. By construction, the endpoints of Γ_k lie over the vertical edges of R. If the indexing is chosen so that $p_k \in A_k$, then by connectedness, $\Gamma_k \subset A_k$.

With this configuration, $\Gamma \cap A_k \cap P^{-1}(R^o) = \Gamma_k$. Otherwise, there is a point $q \in (A_k \setminus \Gamma_k) \cap \Gamma$ that lies over R^o. Let γ be the shortest open arc in Γ through q with endpoints over bR. The existence of γ together with the hypothesis that both a and b are limit points of K implies the existence of points in K that are covered by at least $(s + 1)$ points in Γ under the map P. This is a contradiction.

Next, each of the arcs Γ_k has an endpoint over each of the horizontal edges of R. Suppose that Γ_k does not. Then the curve $P(\Gamma_k)$ does not separate J_o from J. The local maximum principle implies that if

$$Z = (P^{-1}(bR) \cup \Gamma) \cap A_k,$$

then $A_k \subset \widehat{Z}$. Because J_o and J are in the same component of $\mathbb{C} \setminus P(Z)$, and J_o is in the unbounded component of $\mathbb{C} \setminus P(Z)$, $\widehat{Z}$ can have no points over J. This contradicts $A_k \subset \widehat{Z}$.

Thus, our situation is that $P^{-1}(R^o) \cap \Gamma$ is the union of s open subarcs $\Gamma_1, \dots, \Gamma_s$ of Γ each with endpoints lying in opposite vertical sides of R with, say s_1 going from left to right and s_2 going from right to left.

By Lemma 4.5.9, $\operatorname{Ind}(P \circ \Gamma, z) = s_1 - s_2$ for each $z \in \Omega$. We shall show that, for a suitable orientation on Γ, $s_2 = 0$, whence $s = s_1 = \operatorname{Ind}(P \circ \Gamma, z)$. The multiplicity of $P|\widehat{\Gamma} \setminus \Gamma$ over Ω is s, so Lemma 4.5.8, and with it Theorem 4.5.7, is proved.

That there is an orientation with respect to which $s_2 = 0$ is seen in the following way. First, note that changing the orientation of Γ has the effect of interchanging s_1 and s_2. Thus, it is sufficient to show that not both s_1 and s_2 can be positive. Therefore, for the sake of deriving a contradiction, suppose that $P(\Gamma_1)$ goes from left to right, $P(\Gamma_2)$ from right to left. Let β_1 be the open arc $J \cap R^o$ and let β_2 be the open arc obtained by removing the endpoints from the arc $bR \cap J$. The endpoints of β_1 (and of β_2) lie in bR.

Let $\Gamma'_k = P^{-1}(\beta_1) \cap A_k$, an open arc in $\mathbb{C}^N$ that has the same endpoints as Γ_k. Orient it so that the first point of Γ'_k is the first point of Γ_k, whence the last points coincide, too. Form a simple closed curve Γ' by replacing the arc Γ_k in Γ by Γ'_k for $k = 3, \dots, s$. Thus,

$$\Gamma' = (\Gamma \setminus \cup\{\Gamma_k : k = 3, \dots, s\}) \cup \cup\{\Gamma'_k : k = 3, \dots, s\}.$$

We shall show that

$$\widehat{\Gamma'} \setminus \Gamma' = \widehat{\Gamma} \setminus (\cup\{A_k : k = 3, \dots, s\} \cup \Gamma). \tag{4.13}$$

To do this, introduce, for small $\varepsilon > 0$, the rectangle $R_\varepsilon = [a + \varepsilon, b - \varepsilon] \times [c, d]$. The rectangle R_ε plays for Γ' the same role as R does for Γ: The bottom edge of R lies in the unbounded component of $\mathbb{C} \setminus P(\Gamma')$, and if $K_\varepsilon = K \cap R_\varepsilon$ and $K_{\varepsilon,1} = K_1 \cap [a+\varepsilon, b-\varepsilon]$, then the vertical line $x = t$ for $t \in K_{\varepsilon,1}$ meets K_ε exactly once at a point with two P-preimages in Γ', one in Γ_1, one in Γ_2. In this case, $s = 2$, $s_1 = s_2 = 1$.

The set $\widehat{\Gamma'} \setminus \Gamma'$ is empty or else has a single analytic branch, as shown in Theorem 4.5.5.

If V is the right-hand side of (4.13), then $\bar{V} \setminus V \subset \Gamma'$, so $\bar{V} \setminus V$ is a one-dimensional analytic subset of $\widehat{\Gamma'} \setminus \Gamma'$, whence $V = \widehat{\Gamma'} \setminus \Gamma'$. The equality (4.13) is established.

Now let p be a point in the top edge of R_ε, and put $P_o = P - p$. We have

$$\frac{1}{2\pi i} \int_{P \circ \Gamma'} \frac{dP_o}{P_o} = \operatorname{Ind}(P \circ \Gamma', p) = s_1 - s_2 = 0.$$

This integral vanishes, so the polynomial P_o has a logarithm on Γ', so that the number of zeros on V of P_o is the same as the number of poles. The function P_o has no poles, so it has no zeros. However, $\widehat{\Gamma'}$ contains two points over p, one in A_1, one in A_2, so P_o has two zeros. Contradiction.

We have shown that Γ can be oriented so that s_2 is zero, so the proof of Lemma 4.5.8 is complete, and Theorem 4.5.7 is proved.

The version of Stokes's theorem for varieties bounded by rectifiable simple closed curves is this:

Theorem 4.5.10. *Let Γ be a rectifiable simple closed curve in $\mathbb{C}^N$, and let V be a bounded, one-dimensional subvariety in $\mathbb{C}^N \setminus \Gamma$. There is an orientation on Γ such that $b[V] = [\Gamma]$.*

Explicitly, if α is a smooth 1-form on $\mathbb{C}^N$, then $\int_V d\alpha = \int_\Gamma \alpha$, in which, on the right, Γ is given the orientation-prescribed by the theorem.

As we see below, the orientation in question is that given by the last theorem.

Notice that since $\bar{V} \setminus V$ is a subset of Γ, the variety V is known to have finite area by Theorem 4.4.1.

Recall that the case of a simple closed curve of class $\mathscr{C}^1$ has been discussed in the remarks immediately following the proof of Theorem 4.5.1.

This theorem was first obtained by Lawrence [216, 217] as a consequence of his general version of Stokes's theorem for polynomial hulls of connected sets of finite measure. Lawrence's development draws more heavily on the geometric theory of currents than does the present one.

As a corollary, we have the following elegant criterion:

Corollary 4.5.11. *The rectifiable simple closed curve γ in $\mathbb{C}^N$ is polynomially convex if and only if there is a holomorphic 1-form α on $\mathbb{C}^N$ such that $\int_\gamma \alpha \neq 0$.*

The 1-forms contemplated in this statement are of the form $\alpha = \sum_{j=1,\dots,N} a_j(z) dz_j$ with coefficients holomorphic on the whole of $\mathbb{C}^N$.

Proof of Corollary 4.5.11. If γ is not polynomially convex, then $\widehat{\gamma} \setminus \gamma$ is a one-dimensional subvariety of $\mathbb{C}^N \setminus \gamma$ that satisfies the current equation $b[V] = [\gamma]$ for a suitable orientation of γ. If α is a holomorphic 1-form on $\mathbb{C}^N$, then $\int_\gamma \alpha = b[V](\alpha) = [V](d\alpha) = \int_V d\alpha$, and this quantity vanishes, because the holomorphic 2-form $d\alpha$ on the one-dimensional variety V vanishes.

Conversely, if γ is polynomially convex, fix a parameterization $h : [0, L] \to \gamma$ by arc length, so that $h'(t)$, which exists for almost all $t \in [0, 1]$, satisfies $|h'| = 1$ a.e. $[dt]$ on $[0, L]$. Let g be a homomeorphism of γ onto the unit circle in $\mathbb{C}$. Thus the winding number $\frac{1}{2\pi i} \Delta_\gamma \arg g$ is ± 1. Suppose it to be 1. There is a polynomial P with $\|P - g\|_\gamma < \frac{1}{2}$, for the curve γ is polynomially convex. This polynomial satisfies $\frac{1}{2\pi i} \int_\gamma \frac{dP}{P} = 1$. Let $W \subset \mathbb{C}^N \setminus P^{-1}(0)$ be a compact polynomially convex set that contains γ in its interior. Let Q be a polynomial such that $\|Q - 1/P\|_W < \frac{1}{2}\left\{\|\frac{d(P \circ h)}{dt}\|_{[0,L]} \Lambda^1(\gamma)\right\}^{-1}$. We then have that

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{dP}{P} = \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{P} - Q\right) dP + \frac{1}{2\pi i} \int_\gamma Q\, dP.$$

The first integral on the right is less than 1, so the second integral on the right is not zero: $Q\, dP$ is a holomorphic 1-form on $\mathbb{C}^N$ with nonvanishing integral around γ.

The corollary is proved.

This corollary extends immediately to rectifiable simple closed curves in Stein manifolds: *If $\mathscr{M}$ is a Stein manifold and γ is a rectifiable simple closed curve in $\mathscr{M}$, then γ is $\mathscr{O}(\mathscr{M})$-convex if and only if there is a holomorphic 1-form α on $\mathscr{M}$ such that $\int_\gamma \alpha \neq 0$.* This is evident, because each Stein manifold is biholomorphically equivalent to a complex submanifold of $\mathbb{C}^N$ for a suitable N, and holomorphic 1-forms on complex submanifolds of

$\mathbb{C}^N$ extend to holomorphic 1-forms on $\mathbb{C}^N$. Much more complicated is the corresponding problem in $\mathbb{P}^N$.

A natural regularity problem presents itself at this point: *If the polynomially convex hull of the rectifiable simple closed curve γ of $\mathbb{C}^N$ is $\gamma \cup V$ for a necessarily irreducible subvariety V of $\mathbb{C}^N \setminus \gamma$, and if some further smoothness condition is imposed on γ, for example, if γ is of class $\mathscr{C}^p$ or $\mathscr{C}^{p,\alpha}$, how smooth is the pair (V, γ)?* Good results of this kind are known. Some can be found in the papers of Harvey [165] and Chirka [84]. It should be noted also that the theory of minimal surfaces contains many boundary regularity results that apply in the present setting. See in this connection the treatise of Nitsche [265]. Certain boundary regularity results are given in Section 7.2 below.

One particularly useful special case of our theory is that in which γ is a real-analytic simple closed curve and V a one-dimensional subvariety of $\mathbb{C}^N \setminus \gamma$ with $\bar{V} \setminus V = \gamma$. In this case, V continues through γ as a one-dimensional variety that contains γ. This is easily seen: The curve γ is real-analytic and so admits a real-analytic parameterization: There is a real-analytic map φ from the unit circle $b\mathbb{U}$ onto γ with γ' nowhere vanishing on $b\mathbb{U}$. The map φ extends to a holomorphic map from an annular domain R in $\mathbb{C}$ into $\mathbb{C}^N$. Call the extended map φ. If R is thin enough, φ will be one-to-one on R and its derivative will not vanish on R: If R is small enough, φ will embed R as a one-dimensional complex submanifold—call it $\tilde{R}$—of an open set Ω in $\mathbb{C}^N$. The variety V abuts the manifold $\tilde{R}$ along γ, so near γ, V is contained in $\tilde{R}$, as follows from Corollary 3.8.10. This provides the desired analytic continuation of V. In this situation, γ may not be a subset of the set of regular points of the extended variety, which we call call V'. However, if $p \in V'$ lies in γ, then the irreducible branch of the germ of V' at p that contains the germ at p of γ is nonsingular.

The next lemma is a preparatory step for the proof of Theorem 4.5.10.

Lemma 4.5.12. *If Γ is oriented as in Theorem 4.5.10 and if P is a polynomial on $\mathbb{C}^N$, then for every smooth 1-form ϑ on $\mathbb{C}$,*

$$[\Gamma](P^*\vartheta) = b[V](P^*\vartheta). \tag{4.14}$$

Proof. Let Ω_j, $j = 0, \ldots,$ be the components of $\mathbb{C} \setminus P(\Gamma)$, with Ω_0 the unbounded one. Let the integer m_j be the multiplicity of the map $P|(P^{-1}(\Omega_j) \cap V)$, which, by Theorem 4.5.7, is the index of the curve $P \circ \Gamma$ about each point $w \in \Omega_j$.

There is no loss in generality in assuming the form ϑ to have compact support. Thus, let

$$\vartheta = \alpha(\zeta)d\zeta + \beta(\zeta)d\bar{\zeta}$$

with α and β smooth, compactly supported functions on $\mathbb{C}$. We treat first the form $\vartheta' = \alpha\, d\zeta$. To do this, apply the generalized Cauchy integral formula to write that

$$\alpha(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\tilde{\alpha}(\eta)}{\eta - \zeta}\, d\eta \wedge d\bar{\eta}$$

with $\tilde{\alpha} = \frac{\partial \alpha}{\partial \bar{\eta}}$.

We then compute

$$\begin{aligned}\int_\Gamma P^*(\alpha d\zeta) &= \int_\Gamma P^* \left\{\left(\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\tilde{\alpha}(\eta)}{\eta-\zeta}\, d\eta \wedge d\bar{\eta}\right) d\zeta\right\} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \tilde{\alpha}(\eta)\left\{\int_\Gamma P^*\left(\frac{d\zeta}{\eta-\zeta}\right)\right\} d\eta\wedge d\bar{\eta} \\ &= -\int_{\mathbb{C}} \tilde{\alpha}(\eta)\,\mathrm{Ind}(P\circ\Gamma,\eta)\, d\eta\wedge d\bar{\eta} \\ &= -\sum_{j=0,\dots} m_j \int_{\Omega_j} \tilde{\alpha}(\eta)\, d\eta\wedge d\bar{\eta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}b[V](P^*(\alpha d\zeta)) &= \int_V d(P^*(\alpha d\zeta)) \\ &= \int_V P^*\left(\frac{\partial\alpha}{\partial\bar{\zeta}} d\bar{\zeta}\wedge d\zeta\right) \\ &= \int_V P^*(\tilde{\alpha} d\bar{\zeta}\wedge d\zeta) \\ &= -\sum_{j=0,\dots} m_j \int_{\Omega_j} \tilde{\alpha}\; d\zeta\wedge d\bar{\zeta}.\end{aligned}$$

We have found that $[\Gamma](P^*\vartheta') = b[V](P^*\vartheta')$. A similar calculation with $\vartheta'' = \beta(\zeta)d\bar{\zeta}$ based on the representation

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\tilde{u}(\zeta)\, d\zeta\wedge d\bar{\zeta}}{\bar{\zeta}-\bar{z}}$$

for a solution of the equation $\bar{\partial} u = \tilde{u}$ (with $\tilde{u}$ smooth and compactly supported) shows that $[\Gamma](P^*\vartheta'') = b[V](P^*\vartheta'')$, whence

$$[\Gamma](P^*\vartheta) = b[V](P^*\vartheta). \tag{4.15}$$

Equation (4.15) holds for all choices of the polynomial P and all choices of the smooth 1-forms ϑ on $\mathbb{C}$, so the lemma is proved.

Proof of Theorem 4.5.10. By Theorem 4.5.1, there are bounded Borel functions B'_j and B''_j on Γ such that for every smooth 1-form α, if $\alpha = \sum_{j=1,\dots,N} \alpha'_j dz_j + \alpha''_j d\bar{z}_j$, then

$$b[V] = \int_\Gamma \left(\sum_{j=1,\dots,N} \alpha'_j B'_j + \alpha''_j B''_j\right) d\Lambda^1. \tag{4.16}$$

Also, we know by Lemma 4.5.12 that for every polynomial P and every smooth 1-form α on $\mathbb{C}$,

$$\int_\Gamma P^*\alpha = b[V](P^*\alpha).$$

The functional I defined on smooth 1-forms by

$$I(\alpha) = \int_\Gamma \alpha$$

extends to a functional on the space $\mathscr{B}^1$ of 1-forms on $\mathbb{C}^N$ with bounded measurable coefficients. Call this extended functional $\tilde{I}$. Similarly, the functional B on 1-forms defined by

$$B(\alpha) = \int_\Gamma \left(\sum_{j=1,\ldots,N} \alpha'_j B'_j + \alpha''_j B''_j \right) d\Lambda^1$$

extends to a functional, denoted by $\tilde{B}$, on $\mathscr{B}^1$.

Lemma 4.5.13. *If $E_o \subset \Gamma$ is a set with $\Lambda^1(E_o) > 0$, there is a subset E of E_o with $\Lambda^1(E) > 0$ such that if $\alpha \in \mathscr{B}^1$ vanishes a.e. $[d\Lambda^1]$ on $\Gamma \setminus E$, then $\tilde{I}(\alpha) = \tilde{B}(\alpha)$.*

Proof. The curve Γ is rectifiable and $\Lambda^1(E) > 0$, so there is an arc λ of class $\mathscr{C}^1$ such that $\Lambda^1(\lambda \cap E_o) > 0$. Set $E_1 = \lambda \cap \Gamma$. Let $g : [0, L] \to \lambda$ be an arc-length parameterization of Γ, so that $\left|\frac{dg}{dt}\right| = 1$ everywhere. Set $F_o = g^{-1}(\lambda \cap E_o)$, a compact subset of $[0, L]$ with positive length. Let $t_o \in F_o$ be a point of metric density for F_o, so that

$$\lim_{\delta \to 0} \frac{\Lambda^1(F_o \cap (t_o - \delta, t_o + \delta))}{2\delta} = 1.$$

Introduce new unitary coordinates $w_1, \ldots, w_N$ chosen such that for each j, $\frac{d}{dt} w_j \circ g(t_o) = r_j > 0$. Put $r = \min r_j,\ \ j = 1, \ldots, N$. Then for all $\delta > 0$,

$$\Lambda^1(\{w_j \circ g(t) : t \in F_o \cap (t_o - \delta, t_o + \delta)\}) > 0,$$

and there is a $\delta_o > 0$ so small that each coordinate function w_j is injective on the set $\{g(t) : t \in F_o \cap (t_o - \delta_o, t_o + \delta_o)\}$.

Next, invoke Theorem 4.3.11 to find a linear functional ϕ_j on $\mathbb{C}^N$ and a small positive η with $|\phi_j(w) - w_j| < \eta|w|$ for all $w \in \mathbb{C}^N$ and such that for some subset S_j of the plane of zero length, ϕ_j is injective on $\Gamma \setminus \phi_j^{-1}(S_j)$. If η is small enough, then for sufficiently small $\delta_1 > 0$, $\frac{d}{dt}\Re(\phi_j \circ g(t) > r/4$ for $t \in [t_o - \delta_o, t_o + \delta_o]$. Note that

$$(t_o - \delta_1, t_o + \delta_1) \cap g^{-1}(\phi^{-1}(S_j) \cap \Gamma)$$

is a null set contained in F_o.

Thus, if we set $F = [t_o - \delta_1, t_o + \delta_1] \cap F_o$ and $E = g(F)$, then (a) $E \subset E_o$, (b) ϕ_j is injective on a subset of Γ that contains E, (c) E is compact, and (d) $\Lambda^1(E) > 0$.

There are bounded measurable functions C'_j and C''_j such that for a smooth 1-form $\beta = \sum_{j=1,\ldots,N} \beta'_j dw_j + \beta''_j d\bar{w}_j$,

$$\tilde{B}(\beta) = \int_\Gamma \left(\sum_{j=1,\ldots,N} \beta'_j C'_j + \beta''_j C''_j \right) d\Lambda^1.$$

Apply the formula $\int_\Gamma P^*\vartheta = b[V](P^*\vartheta)$, valid for all polynomials P and all smooth 1-forms ϑ on $\mathbb{C}$ in the case that $P = \phi_j$ and $\vartheta = f(\zeta)d\zeta$ for a smooth function f on $\mathbb{C}$. If $\phi_j(w) = \sum_{k=1,\ldots N} b_{jk}w_k$, it yields

$$\int_\Gamma \left(\sum_{k=1,\ldots N} f \circ \phi_j b_{jk} \right) dw_k = \int_\Gamma \left(\sum_{j=1,\ldots,N} f \circ \phi_j b_{jk} C'_k \right) d\Lambda^1.$$

This equation is true for *every* choice of smooth or continuous function f on $\mathbb{C}$. Let $\tilde{h}$ be a function continuous on the set E. The function ϕ_j is injective on $\Gamma \setminus \phi^{-1}(S)$ for a subset S of the plane of zero length, and we have $E \subset (\Gamma \setminus \phi^{-1}(S))$, so there is a uniformly bounded sequence $\{f_n\}$ of smooth functions on the plane with $f_n \to 0$ off $\phi_j(E)$ and with $f_n \to \tilde{h}$ on E. Thus for every bounded measurable function h on E,

$$\int_E h \sum_{k=1,\ldots,N} b_{jk}\, dw_k = \int_E h \sum_{k=1,\ldots,N} b_{jk} C'_k \, d\Lambda^1.$$

Apply this formula with h replaced by hc_{ij}, where $[c_{ij}]$ is the inverse of the matrix $[b_{jk}]$. (By its construction, $[b_{jk}]$ is a small perturbation of the identity.) Because

$$\sum_{j=1,\ldots,N} c_{ij} b_{jk} = \delta_{ik},$$

we get

$$\int_E h\, dw_k = \int_E hC'_k \, d\Lambda^1$$

for each k. Similarly for each k,

$$\int_E h\, d\bar{w}_k = \int_E hC''_k \, d\Lambda^1.$$

This completes the proof of the lemma.

In the context of the lemma, we shall write that $\int_E \alpha = (b[V] \llcorner E)(\alpha)$.

Proof of Theorem 4.5.10 concluded. Let m be the supremum of the numbers $\Lambda^1(E)$, the supremum taken over all measurable subsets E of Γ such that for all 1-forms α, $\int_E \alpha = (b[V] \llcorner E)(\alpha)$. Let $\{F_k\}$ be a sequence of measurable subsets of Γ with $\Lambda^1(F_k) > m - \frac{1}{k}$ and with $\int_{F_k} \alpha = (b[V] \llcorner F_k)(\alpha)$ for all α. The set $F_o = \cup_k F_k$ is a measurable subset of Γ with $\int_{F_o} \alpha = (b[V] \llcorner F_o)(\alpha)$ for all α. If $m = \Lambda^1(\Gamma)$ the theorem is correct. If not, then the lemma provides a measurable set $F \subset \Gamma \setminus F_o$ with $\Lambda^1(F) > 0$ and with $\int_F \alpha = (b[V] \llcorner F)(\alpha)$ for all α. We have reached a contradiction to the definition of m, for $\Lambda^1(F \cup F_o) > m$. The theorem is proved.

Lawrence [216, 217] and Dinh [101] established very general versions of Stokes's theorem. Their formulations are in terms of currents; the more general version is that of Dinh:

Theorem 4.5.14. *Let Ω be a domain in $\mathbb{C}^N$ and let the closed subset X of Ω be of class $\mathscr{A}_1$. If $V \subset \Omega \setminus X$ is a purely one-dimensional analytic subvariety, then the current $b[V]$ is locally rectifiable with multiplicity* 0 *or* 1 *almost everywhere* $[d\Lambda^1]$.

Stated without the terminology of currents, this means the following: There exists a measurable field $\vec{T}$ of unit vectors defined almost everywhere $[d\Lambda^1]$ on X with the properties that for almost every $[d\Lambda^1]$ $x \in \Gamma$, the vector $\vec{T}(x)$ lies in the tangent line $\operatorname{Tan}(X, x)$ and for every smooth 1-form α on Ω with compact support,

$$b[V](\alpha) = \int_X \mu(x)\langle \vec{T}(x), \alpha(x)\rangle \, d\Lambda^1(x) \tag{4.17}$$

for a measurable function μ that takes the values 0 and 1.

A proof of this result, essentially that given by Lawrence [215–217], can be executed as follows; the essential point, granted the development given above, is an appeal to a deep theorem about currents.

First, we need to formalize the notion of *rectifiable current*. Doing so depends on a few preliminaries from multilinear algebra.

Let V be an N-dimensional real inner product space. For each $p = 0, 1, \ldots,$ there is the pth exterior power $\bigwedge_p V$ of V. The elements of $\bigwedge_p V$ are the *p-vectors* of V. By definition, $\bigwedge_0 V$ is the field $\mathbb{R}$, and for $p > N$, $\bigwedge_p V = 0$. In the direct sum $\bigwedge V = \bigoplus_{p=0,1,\ldots} \bigwedge_p V$ there is an associative multiplication such that if $v \in \bigwedge_p V$ and $w \in \bigwedge_q V$, the product $v \wedge w$ lies in $\bigwedge_{p+q} V$. This exterior multiplication has the properties that $c(v \wedge w) = (cv \wedge w) = v \wedge (cw)$ for all $c \in \mathbb{R}$ and $v \wedge w = (-1)^{pq} w \wedge v$ all $v \in \bigwedge_p V$, and all $w \in \bigwedge V_q$ for all p and q. The space $\bigwedge_p V$ is endowed with the inner product with the property that if $u_1, \ldots, u_N$ is an orthonormal basis for V, then the products $u_{i_1 i_2 \ldots i_p} = u_{i_1} \wedge \cdots \wedge u_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq N$ constitute an orthonormal basis for $\bigwedge_p V$. Note that $\dim_{\mathbb{R}} \bigwedge_p V = \binom{N}{p}$. A p-vector $v \in \bigwedge_p V$ is *simple* if $v = v_1 \wedge \cdots \wedge v_p$ for some choice of vectors $v_1, \ldots, v_p \in V$. The simple vectors are also called *decomposable* vectors. For example, the 2-vector $u_{13} + u_{23} - u_{12}$ is simple: it is the exterior product $(u_1 + u_2) \wedge (u_1 + u_3)$. The *oriented* p-dimensional vector subspaces of V are in one-to-one correspondence with the simple unit p-vectors of V. The correspondence is established as follows. Let L be an oriented p-dimensional subspace of V, and let $v_1, \ldots, v_p$, taken in that order, be a positively oriented orthonormal basis for L. Then L corresponds to the p-vector $v_1 \wedge \cdots \wedge v_p$. One verifies that this is a one-to-one correspondence, and that it is independent of the choices.

If we start this process with V replaced by its dual space V^*, we obtain the exterior algebra $\bigwedge V^*$. The elements of $\bigwedge_p V^*$ are the *p-covectors* of V. They are linear functionals on $\bigwedge_p V$, and we have in a natural way that $\bigwedge_p(V^*) = (\bigwedge_p V)^*$. There is a natural bilinear pairing $\langle , \rangle : \bigwedge_p V \oplus \bigwedge_p V^* \to \mathbb{R}$ given by $\langle v, \alpha \rangle = \alpha(v)$.

We can now define the rectifiable k-currents. For this, fix a measurable set $E \subset \mathbb{R}^N$ that is (Λ^k, k)-rectifiable. At almost every $[d\Lambda^k]$ point this set has a tangent plane, $T_x(E)$. Choose a measurable orientation on E, by which we understand an orientation on each of the tangent planes $T_x(E)$ that varies measurably in the point x. To each $T_x(E)$ is then associated a unit k-vector, to be denoted by $\vec{T}(x)$, by the process described above. If, finally, μ is a measurable function on E with values in the nonnegative integers, then we can define a current S by the condition that for every smooth k-form α,

$$S(\alpha) = \int_E \mu(x)\langle \vec{T}(x), \alpha(x)\rangle \, d\Lambda^k(x).$$

A current of this form is said to be a *rectifiable k-current.*

Another class of currents is that of integral currents: The space I_k of *integral k-currents* is the space of all rectifiable k-currents T such that bT is a rectifiable $(k-1)$-current. There is an alternative characterization of the integral currents as follows.

The definition depends on certain seminorms. Given $\varphi \in \bigwedge_p^*$, the quantity $\|\varphi\|^*$ is defined by

$$\|\varphi\|^* = \sup\{\varphi(v) : v \text{ is a simple } p\text{-vector of unit length}\}.$$

Given a p-form α on a domain D in $\mathbb{R}^N$, for each $x \in D$, the value $\alpha(x)$ is an element of $\bigwedge_p T_x(\mathbb{R}^N)^*$, and so has a norm $\|\alpha(x)\|^*$ in accordance with the preceding definition.

If T is a k-current on a domain D, then the *mass norm* $\boldsymbol{M}(T)$ of T is given by

$$\boldsymbol{M}(T) = \sup_\alpha |T(\alpha)|,$$

where the supremum is extended over all k-forms α on D with $\sup_{x \in D} \|\alpha(x)\|^* < 1$.

The following theorem characterizes the integral currents:

Theorem 4.5.15. [115, Theorem 4.2.16] *The compactly supported k-current on $\mathbb{R}^N$ is an integral current if and only if T is rectifiable and $\boldsymbol{M}(bT)$ is finite.*

This is a complicated result the proof of which will not be given here.

As an example, if D is a bounded domain in $\mathbb{R}^2$ bounded by a rectifiable simple closed curve, then the current $[D]$ of integration over D is an integral 2-current. If D is a bounded domain in $\mathbb{R}^2$ bounded by a nonrectifiable simple closed curve, then $[D]$ is not an integral current.

For our immediate purposes, the example of interest is that of integration over a one-dimensional analytic variety. Let Ω be a domain in $\mathbb{C}^N$, and let Σ be a one-dimensional subvariety of Ω with finite area. If V_{reg} denotes the set of regular points of V, then for a smooth two-form α on $\mathbb{C}^N$, $[V](\alpha) = \int_{V_{\text{reg}}} \alpha$. With $\omega = \frac{1}{2i}\sum_{j=1}^N dz_j \wedge d\bar{z}_j$, we have seen in Lemma 3.7.1 that for a function f on Σ,

$$[V](f\omega) = \int_{V_{\text{reg}}} f\omega = \int_{V_{\text{reg}}} f(z)\, d\Lambda^2(z).$$

If α is a smooth two-form on V_{reg}, we have that

$$\int_{V_{\text{reg}}} \alpha = \int_{V_{\text{reg}}} (\alpha/\omega)\, \omega.$$

A comment about the quotient α/ω may be in order. The form ω vanishes at no point of V_{reg}, so there is a function g such that $\alpha = g\,\omega$. This function g is denoted by the quotient α/ω.

Given a two-form $\alpha = \sum_{r,s=1}^{N} a_{rs} dz_r \wedge d\bar{z}_s$, we can determine the function α/ω on V_{reg} by working with a biholomorphic parameterization $\varphi : \mathbb{U} \to V$ of a disk in V_{reg}. Given such a φ, we have

$$\begin{aligned}\int_{\varphi(\mathbb{U})} \alpha &= \int_{\mathbb{U}} \varphi^* \alpha \\ &= \int_{\mathbb{U}} \left(\sum_{r,s} a_{rs}(\varphi(\zeta)) \varphi_r'(\zeta) \overline{\varphi_s'(\zeta)} \right) d\zeta \wedge d\bar{\zeta} \\ &= \int_{\mathbb{U}} \left\{ \frac{\sum_{r,s} a_{rs}(\varphi(\zeta)) \varphi_r'(\zeta) \overline{\varphi_s'(\zeta)}}{|\varphi'(\zeta)|^2} \right\} |\varphi'(\zeta)|^2 d\zeta \wedge d\bar{\zeta}.\end{aligned} \tag{4.18}$$

Note that $\varphi^*\omega = \frac{1}{2i} \sum_{j=1}^{N} \varphi_j'(\zeta) \overline{\varphi_j'}(\zeta)\, d\zeta \wedge d\bar{\zeta} = \frac{1}{2i} |\varphi'|^2 d\zeta \wedge d\bar{\zeta}$. The vector $\varphi'(\zeta)/|\varphi'(\zeta)|$ is tangent to V at $\varphi(\zeta)$ as is the vector $i\varphi'(\zeta)/|\varphi'(\zeta)|$, and these two vectors, in that order, give a positively oriented orthogonal basis for the tangent space of V at $\varphi(\zeta)$. The quantity in braces in the last integral is $2i\big\langle \frac{\varphi'(\zeta)}{|\varphi'(\zeta)|} \wedge \frac{i\varphi'(\zeta)}{|\varphi'(\zeta)|}, \alpha(\varphi(\zeta))\big\rangle$. We have thus exhibited the current $[V]$ as a rectifiable 2-current.

With these preliminaries in hand, we now turn to the proof of Theorem 4.5.14. First we consider a bounded, one-dimensional subvariety V of $\mathbb{C}^N \setminus \Gamma$ with the compact set Γ of class $\mathscr{A}_1$. The current $[V]$ of integration over V is a rectifiable 2-current, as we have just seen. We will study the current boundary $b[V]$, which is supported in Γ.

According to Theorem 4.5.1, there is a measurable vector $\vec{B} = (B_1, \ldots, B_{2N})$ on Γ such that for every smooth 1-form $\alpha = \sum_{j=1}^{2N} a_j dx_j$ on $\mathbb{C}^N$,

$$b[V](\alpha) = \int_\Gamma \sum_{j=1}^{2N} B_j(z) a_j(z)\, d\Lambda^1(z). \tag{4.19}$$

The functions B_j are all bounded by one. This does not by itself exhibit $b[V]$ as a rectifiable current; we do not see immediately that $\vec{B}$ is of the form $\mu\vec{T}$ with the multiplicity function μ a measurable integer-valued function on Γ and with $\vec{T}$ a unit vector tangent almost everywhere $[d\Lambda^1]$ to Γ. The proof of Theorem 4.5.1 as it stands does not yield this refined information.

However, the representation (4.19) for $b[V]$ does let us show $[V]$ to be an integral current, by way of Theorem 4.5.15, for it implies easily that the seminorm $\boldsymbol{M}(b[V])$ is finite, whence $[V]$ is an integral 2-current, which implies that $b[V]$ is rectifiable. By what we have above, the mass of $b[V]$ is given by

$$\boldsymbol{M}(b[V]) = \sup \left| \int_\Gamma \sum_{j=1}^{2N} B_j(x) \alpha_j(x)\, d\Lambda^1(x) \right|,$$

in which the supremum is taken over all 1-forms $\alpha = \sum_{j=1}^{2N} \alpha_j\, dx_j$ satisfying $|\langle v, \alpha(x)\rangle| \le 1$ for all simple 1-vectors v of unit length and for all $x \in \mathbb{C}^N$. Every 1-vector, i.e., every vector, is simple, so duality shows that the forms in question are simply those for which $\{\sum_{j=1}^{2n} \alpha_j(x)^2\}^{1/2} \le 1$ for all x. Consequently, the integral on the right of the last equation is bounded by $\max_{x\in\Gamma}\{\sum_{j=1}^{2N} B_j(x)^2\}^{1/2}\Lambda^1(\Gamma)$. The functions B_j are bounded by one, so we reach

$$\boldsymbol{M}(b[V]) \le \sqrt{2N}\Lambda^1(\Gamma).$$

We have proved that the current $[V]$ is an integral current, so $b[V]$ is a rectifiable current.

We need to see that the multiplicity function associated with this current assumes a.e. $[d\Lambda^1]$ the value 0 or 1. Suppose this is false, so that there is a compact set $E \subset \Gamma$ of positive length on which $\mu \ge 2$. We can suppose that μ assumes the constant value m on E. Then because the vector $\vec{T}$ is measurable, we can suppose that it is continuous on E by Lusin's theorem. By passing to a smaller E, still of positive length, we can suppose that if $x, y \in E$, then $|\vec{T}(x) - \vec{T}(y)| < \frac{1}{3m}$. Fix a point $x_o \in E$. Let $\varphi : \mathbb{C}^N \to \mathbb{R}$ be the real-linear functional given by $\varphi(x) = x \cdot \vec{T}(x_o)$. Choose holomorphic coordinates $z_1, \ldots, z_N$ on $\mathbb{C}^N$ with $z_j = x_{2j-1} + ix_{2j}$, and with $x_1 = \varphi(z)$. There is a measurable vector $\vec{B}$ on Γ such that

$$b[V]\left(\sum_{j=1}^{2N} a_j(x)dx_j\right) = \int_\Gamma \sum_{j=1}^{2N} B_j(x)a_j(x)\,d\Lambda^1(x).$$

Apply this with the form $g\,d\varphi$ for a smooth function g on $\mathbb{C}^N$ to get

$$\int_\Gamma \mu(x)\langle\vec{T}(x), g(x)d\varphi\rangle\,d\Lambda^1(x) = \int_\Gamma B_1(x)g(x)\,d\Lambda^1(x).$$

This equation holds for all choices of the smooth function g on $\mathbb{C}^N$. If we apply it to a sequence of such functions that decreases monotonically to the characteristic function of the set E, we find that

$$m\int_E \langle\vec{T}(x), \vec{T}(x_o)\rangle\,d\Lambda^1(x) = \int_E B_1(x)\,d\Lambda^1(x).$$

We have $|B_1| \le 1$ a.e. $[d\Lambda^1]$, $m \ge 2$, $|\vec{T}(x_o)| = 1$, and $|\vec{T}(x) - \vec{T}(x_o)| < \frac{1}{3m}$ a.e. $[d\Lambda^1]$ on E, so we have a contradiction. Thus, as desired, $\mu(x) \le 1$ a.e. $[d\Lambda^1]$.

We have so far been considering the case of a *compact* set Γ of class $\mathscr{A}_1$ that contains bV for a bounded, purely one-dimensional variety V. The more general case in which we have a domain D and in D a closed set Γ of class $\mathscr{A}_1$ that contains the set bV for a subvariety V of $D \setminus \Gamma$ follows immediately. In this case, we consider the current $[V]$ as acting on 2-forms α on D with $\operatorname{supp}\alpha \Subset D$. We are to show that there is a measurable field $\vec{T}$ of unit vectors on Γ with $\vec{T}(x)$-tangent to Γ a.e. $[d\Lambda^1]$ and a measurable multiplicity function μ on Γ that assumes only the values 0 and 1 such that

$$b[V](\alpha) = \int_\Gamma \mu(x)\langle\vec{T}(x), \alpha(x)\rangle\,d\Lambda^1(x)$$

for every smooth 1-form α with $\operatorname{supp}\alpha \Subset D$.

This follows from the version of the theorem already established. By using a partition of unity, we need only consider forms α with small support. If $x \in \Gamma$, then for almost all small positive r, the set $V \cap b\mathbb{B}_N(x, r)$ is a real-analytic one-dimensional submanifold of $b\mathbb{B}_N(x, r) \setminus \Gamma$ that has finite length. Fix such an r. The set $bV \cap \bar{\mathbb{B}}_N(x, r)$ is a compact set of class $\mathscr{A}_1$. When restricted to forms α with support a compact subset of $\mathbb{B}_N(x, r)$, the current $b[V]$ has the stated form, as follows from the version of Stokes's theorem already established.

If we restrict attention to the case that Γ is a rectifiable simple closed curve that is the boundary of the bounded one-dimensional subvariety V of $\mathbb{C}^N \setminus \Gamma$, the results we have been discussing now do not immediately imply the earlier result, Theorem 4.5.10. It is possible to derive that earlier result from Theorem 4.5.14, but such a proof seems to require a further excursus through nontrivial parts of the geometric theory of currents. Such a derivation was given in [216, 217]. This development also requires the result of Theorem 4.5.7.

4.6. The Multiplicity Function

In [216, 217] Lawrence introduced a multiplicity function that is useful in the study of the boundaries of one-dimensional varieties.

Definition 4.6.1. *Let E be a closed subset of an open set Ω in $\mathbb{C}^N$, and let V be a one-dimensional subvariety of $\Omega \setminus E$. For $r > 0$ and for $z \in E$, let $m_r(z; V)$ be the number of irreducible branches of the variety $V \cap (\mathbb{B}_N(z, r) \setminus E)$ each of whose closures contain the point z.*

Thus, $m_r(z; V)$ is a nonnegative integer or else ∞. As a function of r, $m_r(z; V)$ is nonincreasing.

Definition 4.6.2. *With the notation of the preceding definition,*

$$m(z; V) = \lim_{r \to 0^+} m_r(z; V).$$

The multiplicity $m(z; V)$ will usually be denoted simply by $m(z)$; in practice it will be clear which variety is under discussion

The function m is measurable on the set E as is seen from the following description of it. For each k, fix a locally finite covering of the set E by balls of radius $1/k$, and partition E into a family of mutually disjoint measurable subsets $E_{k,j}$ each of which is contained in one of the balls. Define a function μ_k on E by the condition that $\mu_k(z)$ be the number of branches of the set $B \cap V$ that contain z in their boundary, where B is the one of the balls that contains the set $E_{k,j}$ in which x lies. The function μ_k is measurable, and $m(z) = \lim_{k \to \infty} \mu_k(z)$, so m is a measurable function.

A basic fact about the multiplicity function m is that there is a universal almost everywhere bound on it when the set X is restricted to be of class $\mathscr{A}_1$.

Theorem 4.6.3. [216, 217] *If Ω is an open subset of $\mathbb{C}^N$, if X is a closed subset of Ω that is of class $\mathscr{A}_1$, and if V is a one-dimensional subvariety of $\Omega \setminus X$, then for almost every $[d\Lambda^1]$ $x \in X$, $m(x) \leq 2$.*

Proof. Start by recalling that, according to Theorem 4.4.1, the area of V is locally finite in Ω. The problem is local, so we can suppose that Ω is bounded, that V has finite area, and that the set $\bar{X}$ has finite length and is of class $\mathscr{A}_1$. By Lemma 4.3.10, almost every linear functional ϕ on $\mathbb{C}^N$ carries $\bar{X}$ to a set of class $\mathscr{A}_1$; moreover, almost every such ϕ is constant on no global branch of the variety V. Fix a ϕ with these two properties.

Let the components of $\mathbb{C} \setminus \phi(\bar{X})$ be $\Omega_0, \ldots$. By Lemma 4.3.15, almost every point of $\phi(\bar{X})$ lies in $b\Omega_j$ for at most two values of j. In addition, by Corollary 4.3.17, almost every point of $\phi(\bar{X})$ lies in $b\Omega_j$ for some choice of j.

Let $Y \subset \phi(X)$ be a set with $\Lambda^1(Y) > 0$, with the property that every point $y \in Y$ belongs to $b\Omega_j$ for some j, and with the property that for all $y \in Y$, the fiber $\phi^{-1}(y) \cap \bar{X}$ is finite. The latter condition can be achieved because $\bar{X}$ has finite length.

If $Y_j = Y \cap b\Omega_j$, then by Theorem 4.3.12 there are a compact subset $Y_j' \subset Y_j$ such that $\Lambda^1(Y_j') > 0$ and a domain D_j contained in Ω_j with bD_j a rectifiable simple closed curve that contains Y_j'. Denote by λ_j the multiplicity of ϕ over D_j, so that for all points $\zeta \in D_j$, the fiber $\phi^{-1}(\zeta) \cap V$ contains λ_j points taking multiplicities into account; for all but countably many points $\zeta \in D_j$, the fiber $\phi^{-1}(\zeta) \cap V$ contains λ_j distinct points.

For almost all points $\zeta \in bD_j$, the fiber $\phi^{-1}(\zeta) \cap \overline{\phi^{-1}(D_j) \cap V}$ contains λ_j points. Let $y \in Y_j'$ be such a point, and let $\phi^{-1}(y) \cap X = \{w_1, \ldots, w_s\}$. Let $\delta > 0$ satisfy $\delta < \frac{1}{2} \min_{1 \leq j < k \leq s} |w_j - w_k|$.

If $r > 0$ is small enough, then each of the balls $\mathbb{B}_N(w_j, \delta)$ meets at most one branch of the variety $\phi^{-1}(D_j) \cap V$.

Fix a small $r > 0$. The sets $\phi^{-1}(\mathbb{B}_1(y, r)) \cap \mathbb{B}_N(w_j, \varepsilon)$ for varying $\varepsilon > 0$ constitute a neighborhood basis for the point w_j, and $V \cap \phi^{-1}(\mathbb{B}_1(y, r)) \cap \mathbb{B}_N(w_j, \varepsilon)$ meets only one sheet of $V \cap \phi^{-1}(D_j)$.

If $w_1 \in \bar{V} \setminus V$, let V_r' be a branch of $\phi^{-1}(\mathbb{B}_1(y, r)) \cap \mathbb{B}_N(w_1, \varepsilon) \cap \phi^{-1}(D_j)$. Then $\phi(V_r')$ is an open connected subset of $\mathbb{B}_1(y, r)$ whose boundary contains y. It has to be a component of $\mathbb{B}_1(y, r) \cap D_j$: Only one component, C, of $\mathbb{B}_1(y, r) \cap D_j$ has y in its boundary, so $\phi(V_r') = C$. The set $\phi^{-1}(\mathbb{B}_1(y, r)) \cap \mathbb{B}_N(w_1, \delta)$ meets only one sheet of $\phi^{-1}(D_j) \cap V$, so there is only one branch of $\phi^{-1}(\mathbb{B}_1(y, r)) \cap \mathbb{B}_N(w_1, \delta) \cap \phi^{-1}(D_j)$. The point y is in bD_j for at most two distinct values of j, so for all choices of the indices j and k, almost every point w of $X \cap \phi^{-1}(bD_j \cap bD_k)$ satisfies $m(w) \leq 2$.

Denote by S the set of points $x \in X$ at which $m(x) > 2$. If $\Lambda^1(S) > 0$, then because the set X belongs to the class $\mathscr{A}_1$, there is a ϕ of the kind we have been using such that $\Lambda^1(\phi(S)) > 0$. If we take for the Y of the discussion above a compact subset of $\phi(S)$ of positive length, we get a contradiction to what we have done, so the theorem is proved.

There is an analytic continuation result phrased in terms of the multiplicity m:

Theorem 4.6.4. [217] *If Ω is a domain in $\mathbb{C}^N$, if λ is a rectifiable arc in $\bar{\Omega}$ with interior contained in Ω, and if V is a one-dimensional subvariety of $\Omega \setminus \lambda$ such that $m(z; V) = 2$ a.e.$[d\Lambda^1]$ on λ, then $\Omega \cap \bar{V}$ is a subvariety of Ω.*

The proof of this theorem depends on a preliminary lemma:

Lemma 4.6.5. *If V' and V'' are irreducible analytic subvarieties of $\mathbb{B}_N$ and if $\lambda \subset b\mathbb{B}_N$ is an open analytic arc that is an open subset of both bV' and bV'', then $V' = V''$.*

Proof. The open arc λ is real-analytic, so there is a real-analytic parameterization $g : (0, 1) \to \lambda$. By shrinking λ, we can suppose that g is real-analytic on an open interval that contains $[0, 1]$, that g' is zero-free on this larger interval, and that g extends to be holomorphic on the open subset U of $\mathbb{C}$. Shrinking U a little allows us to suppose that bU is a smooth simple closed curve and also that g is holomorphic on a neighborhood of $\bar{U}$. The open arc λ splits $g(U)$ into two components, of which, by the maximum principle, at most one lies in $\mathbb{B}_N$. A priori, both might lie outside $\mathbb{B}_N$, though we will see that this does not happen.

Then $g(U)$ is an analytic subvariety of $\mathbb{C}^N \setminus g(bU)$. Fix a point $p \in \lambda$. There is a neighborhood Ω of p in $\mathbb{C}^N$ on which there are holomorphic functions f_j, $j = 1, \dots, r$, for which $g(U) \cap \Omega = \cap_{j=1,\dots,r} f^{-1}(0)$.

We will show that if the function $f \in \mathscr{O}(\Omega)$ vanishes on $g(U)$, then it vanishes on the part of V' in Ω near λ. For this purpose, use the notation that Π^+ is the open right half-plane. Because λ is open in bV', there is a linear functional φ on $\mathbb{C}^N$ with $\Re\varphi < 0$ on $bV' \setminus \lambda$ and with $\varphi(\lambda) \cap \Pi^+$ not empty. The functional φ carries $(bV' \setminus \lambda)\widehat{}$ into the left half-plane. Consequently, by Theorem 4.3.2, the hull $\widehat{bV'}$ has the structure of a one-dimensional variety near the points of $\lambda \cap \varphi^{-1}(\Pi^+)$. For the generic choice of φ, there is a finite set $E \subset \lambda$ with the property that $\varphi|(\lambda \setminus E)$ is one-to-one and regular in that $d(\varphi|(\lambda \setminus E))$ does not vanish anywhere. Given a φ with these properties, the part of $\varphi(\lambda \setminus E)$ that lies in Π^+ consists of a finite number of open real-analytic arcs. The map $\varphi|(V' \cap \varphi^{-1}(\Pi^+ \setminus \varphi(bV'))$ is proper over each component of $\Pi^+ \setminus \varphi(bV')$. Let W_o be one of the components of $\Pi^+ \setminus \varphi(\lambda)$ that is contained in $\varphi(V')$ and that abuts the unbounded component of $\Pi^+ \setminus \varphi(\lambda)$ along an open arc, γ. The map φ is injective on $\lambda \cap \varphi^{-1}(\gamma)$, so by Theorem 3.6.1 is injective over W_o. Let $\psi : W_o \to V' \cap \varphi^{-1}(W_o)$ be the map inverse to φ.

By hypothesis the function f vanishes on $g(U)$. Consequently, $f \circ \psi$ tends continuously to zero at the arc $\gamma \subset bW_o$. Accordingly, it vanishes identically on W_o. The set $g(U)$ thus contains an open subset of V'. Thus near λ, $g(U)$ coincides with V'. Similarly, near λ, the set $g(U)$ coincides with V''. Consequently, V' and V'' meet in an open set; the irreducibility hypothesis implies that they coincide.

The lemma is proved.

Proof of Theorem 4.6.4. Because of Theorem 3.8.18, the only case of interest is that in which both endpoints of λ lie in $b\Omega$. Also, because of the same result, it suffices to prove that V continues analytically through one point of λ. The problem is a local one, so we can suppose that $\Omega = \mathbb{B}_N(2)$.

Notice to begin with that for almost every $r \in (0, 2)$, the set $Y_r = \lambda \cup (b\mathbb{B}_N(r) \cap V)$ is a set of class $\mathscr{A}_1$. This is so, for, first of all, no matter what the value of $r \in (0, 2)$, the set Y_r is compact. Eilenberg's inequality implies that for almost all r, the set $\lambda \cap b\mathbb{B}_N(r)$ is finite and that the set Y_r has finite length, because by Theorem 4.4.1, the variety V has locally finite area in Ω. Because the variety V has only countably many singularities, Sard's theorem yields that for almost all r, the intersection $V \cap b\mathbb{B}_N(r)$ is a real-analytic, one-dimensional closed submanifold of $b\mathbb{B}_N(r) \setminus \lambda$. If we choose r to satisfy both of these conditions, then the set Y_r is a $(\Lambda^1, 1)$-rectifiable set and at every point of $Y_r \setminus \lambda$ the tangent is a real line. Thus, we can suppose that we are working on the unit ball $\mathbb{B}_N$ and that the set $X = \bar{V} \cap b\mathbb{B}_N$ is of class $\mathscr{A}_1$. Note, though, that with these reductions, the set $\lambda \cap \mathbb{B}_N$ may not be an arc; it may be a union of at most finitely many subarcs of the originally given arc. We now change notation and denote the closure of the set $\lambda \cap \mathbb{B}_N$ by λ.

Lemma 4.6.6. *The set $X \cup \lambda \cup V$ is polynomially convex.*

Proof. What is to be proved is that $(X \cup \lambda)\widehat{\ }$ is $X \cup \lambda \cup V$. If not, then because $X \cup \lambda$ is of class $\mathscr{A}_1$, the set $(X \cup \lambda)\widehat{\ } \setminus (X \cup \lambda)$ is in any case an analytic subvariety, W, of $\mathbb{C}^N \setminus (X \cup \lambda)$ that contains V. Let W_o be a global branch of W not contained in V. By Theorem 3.8.15, $\bar{W}_o \setminus W_o$ must meet X in an analytic arc that is an open subset of X. Lemma 4.6.5 implies that W must be contained in V. Contradiction.

Proof of Theorem 4.6.4 concluded. We are going to show that $V \subset \widehat{X}$. Plainly $V \subset \{X \cup \lambda\}\widehat{\ }$. Thus, if V is not contained in $\widehat{X}$, then λ is not contained in $\widehat{X}$.

If λ is not contained in $\widehat{X}$, there is a polynomial Q with $\Re Q < 0$ on X that satisfies $\Re Q > 0$ at some points of λ. Theorem 4.3.11 implies the existence of a subset E_o of $\mathbb{C}$ that has zero length and that has the further property that Q is injective from $(X \cup \lambda) \setminus Q^{-1}(E_o)$ to $Q(X \cup \lambda) \setminus E_o$.

Let W be a component of $\mathbb{C} \setminus Q(X \cup \lambda)$ that is contained in $Q(V)$ and that meets the open right half-plane, and let $J = \{\zeta = \xi + i\eta_o : a \leq \xi \leq b\}$ be a horizontal interval contained in W. There is a subset S of J of positive length with the property that if L_ζ is the vertical line in $\mathbb{C}$ through the point $\zeta \in S$, then L_ζ meets $Q(\lambda)$ in a finite set. This finite set partitions L_ζ into a finite number of open intervals, two infinite in length, the others finite. For each $\zeta \in S$, let I_ζ be the topmost of these finitely many intervals that is contained in the set $Q(V)$, and let W_ζ be the component of $Q(V)$ that contains I_ζ. The set $Q(V) \setminus Q(X \cup \lambda)$ has only countably many components, so there is a subset of S, which we shall denote by S, with positive length and with the property that for a fixed component W_o of $Q(V) \setminus Q(X \cup \lambda)$ and a fixed component W_o^+ of $\mathbb{C} \setminus Q(X \cup \lambda)$ that is not contained in $Q(V)$, each interval I_ζ for $\zeta \in S$ is contained in W_o and the top endpoint of I_ζ is contained in bW_o^+. If necessary, we can shrink S further in such a way that for each $\zeta \in S$, the top endpoint of I_ζ lies outside the set E_o. In addition, by shrinking the set S by at most a countably infinite set, we can suppose that each of these top endpoints lies in the boundary of no component of $\mathbb{C} \setminus Q(X \cup \lambda)$ other than W_o and W_o^+. (Recall Theorem 3.4.16.)

To continue, let $x \in \lambda$ be a point with $m(x, V) = 2$ taken by Q to one of the top endpoints considered in the last paragraph. Let B be a small connected open set that contains x, B small enough that $Q(B)$ is contained in the half-plane $\Re\zeta > 0$ and $B \cap V$ has two components. We suppose $\gamma = bB \cap V$ to be a smooth one-dimensional set with finite total length and $\lambda \cap B$ to be the interior of the subarc λ_o of λ.

Under the map Q, the set $V \cap bB \setminus \lambda$ is taken into $Q(V)$. Let W_1 be the component of $\mathbb{C} \setminus Q(\lambda_o \cup \gamma)$ contained in W_o that contains $Q(x)$ in its boundary. The map

$$Q : Q^{-1}(W_1) \cap V \to W_1$$

is proper and so is proper on each branch of the variety $V \cap Q^{-1}(W_1)$. There are at least two of these branches by the choice of B. Accordingly, some points in W_1 are covered at least twice by $Q|V$.

By construction, there is a set of positive length in bW_o over which $Q|(X \cup \lambda)$ is injective. This set of positive length is a set of uniqueness for subharmonic functions on W_o. Consequently, by Corollary 3.6.2, Q is injective on $V \cap Q^{-1}(W_o)$.

We have reached a contradiction, so, as claimed, $\lambda \subset \widehat{X}$.

The set $\widehat{X} \setminus X$ is a purely one-dimensional subvariety of $\mathbb{C}^N \setminus X$, and we have shown it to be $V \cup \operatorname{int} \lambda$. The lemma is proved.

Corollary 4.6.7. *If Γ is a rectifiable simple closed curve in $\mathbb{C}^N$, then the variety $\widehat{\Gamma} \setminus \Gamma$ has at most one global branch.*

Proof. Let V_1 and V_2 be distinct global branches of $\widehat{\Gamma} \setminus \Gamma$. By Theorem 3.8.15, each of the sets $\bar{V}_1 \setminus V_1$ and $\bar{V}_2 \setminus V_2$ contains Γ, so $m(z; V_1 \cup V_2) \geq 2$ for every $z \in \Gamma$. By Lemma 4.6.3, $m = 2$ a.e. $[d\Lambda^1]$ on γ. Theorem 4.6.4 implies that $V_1 \cup V_2 \cup \Gamma$ is a subvariety of $\mathbb{C}^N$. The space $\mathbb{C}^N$ contains no compact subvarieties of positive dimension, so we have a contradiction. The corollary is proved.

This corollary was established above in Theorem 4.5.5 by entirely different methods. A more general result in the same vein is given in Theorem 4.7.1 below.

There is a local version of this corollary:

Lemma 4.6.8. *If Γ is a rectifiable simple closed curve and $\widehat{\Gamma} \setminus \Gamma$ is not empty, then for each $p \in \Gamma$ and for every neighborhood U of p, there is one and only one component V_o of the variety $V \cap U$ such that $\bar{V}_o$ contains an arc λ in Γ that contains the point p in its interior. Every other branch of $V \cap U$ meets λ in a set of length zero.*

Proof. If V_1 and V_2 are branches of $U \cap V$ such that both $\bar{V}_1$ and $\bar{V}_2$ contain an arc $\lambda \subset \Gamma$ with p an interior point of λ, then the multiplicity function m is 2 a.e. $[d\Lambda^1]$ on λ, so $V_1 \cup V_2$ continues through the interior of the arc γ to form a one-dimensional variety W. By the maximum principle, $\widehat{\Gamma}$ is contained in the hull of a proper subset of Γ, which is a contradiction, for rectifiable arcs are polynomially convex. This establishes the first assertion of the lemma.

For the second assertion, there are two cases. If there is a global branch V_o of $U \cap V$ that contains an arc λ in Γ that contains p in its interior, then every other branch W of $U \cap V$ meets λ in a totally disconnected set, possibly the empty set. Otherwise, there is a branch W such that $\bar{W}$ meets λ in an arc. Then $V_o \cup W$ continues through this arc, and we again have a contradiction. Thus, for every W, $\bar{W}$ meets λ at most in a totally disconnected set. Call this set T. Theorem 3.8.23 implies that W continues analytically through $\bar{W} \cap \lambda$ as a one-dimensional subvariety. Let λ be a rectifiable arc in the variety W that contains a subset T' of T of positive length. The set $\lambda \cup \gamma$ is a set of class $\mathscr{A}_1$. If $q \in T'$ is a set of metric density for T', and if δ is sufficiently small, then the subvariety $(V_o \cup (W \setminus \lambda)) \cap \mathbb{B}_N(q, \delta)$ has multiplicity at least three on the set T' of positive length contained in $\gamma \cup \lambda$. This contradicts Theorem 4.6.2.

The theory of the multiplicity function implies a vector-valued analogue of a classical theorem of F. and M. Riesz, Theorem 3.4.6.

Theorem 4.6.9. *If E is a compact subset of $\mathbb{C}^N$ of finite length and $f : \mathbb{U} \to \mathbb{C}^N \setminus E$ is a proper, holomorphic map onto a bounded subvariety V of $\mathbb{C}^N \setminus E$, then f' belongs to the Hardy class $H^1(\mathbb{U})$.*

In this statement we are implicitly using the convention that a $\mathbb{C}^N$-valued holomorphic function belongs to the Hardy class $H^1(\mathbb{U})$ when each of its components belongs to this class in the usual sense.

The hypotheses of the theorem are redundant in that granted that $f : U \to \mathbb{C}^N \setminus E$ is a proper holomorphic map, the image $f(\mathbb{U})$ is automatically an analytic variety, as follows from Remmert's proper mapping theorem in analytic geometry [259].

This result was obtained by Lawrence [216, 217]; the case that the cluster set of f at $b\mathbb{U}$ is contained in a rectifiable simple closed curve was found earlier by Globevnik and Stout [149].

The following lemma implies that we need only consider the case that the map f is essentially injective. Recall that a *finite Blaschke product* is a function h on $\mathbb{C}$ of the form

$$h(z) = e^{i\vartheta} \prod_{j=1}^{m} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$$

with $\alpha_j \in \mathbb{U}$. It is a standard exercise in function theory to show that a proper holomorphic map from $\mathbb{U}$ to itself is a finite Blaschke product.

Lemma 4.6.10. [347] *With f as in Theorem 4.6.9, there is a factorization $f = \eta \circ b$ with b a finite Blaschke product and with $\eta : \mathbb{U} \to \mathbb{C}^N \setminus E$ a proper holomorphic map that is one-to-one off a discrete subset of $\mathbb{U}$.*

Proof. This lemma depends on the fact, which we shall not prove here, that there exist a Riemann surface $\mathscr{R}$, a proper holomorphic map $\psi : \mathbb{U} \to \mathscr{R}$, and a holomorphic map $\eta : \mathscr{R} \to V$ with these properties: ψ is proper, there is a discrete subset S of V such that η carries $\mathscr{R} \setminus \eta^{-1}(S)$ biholomorphically onto $V \setminus E$, and $f = \eta \circ \psi$. For the pair $(\mathscr{R}, \eta)$, one takes the *normalization* of the variety V. Normalizations are treated, for example, in [259]. In the case at hand that V is one-dimensional, one can give a relatively simple ad hoc construction of $\mathscr{R}$ using the theory of fractional power series. The existence of $\mathscr{R}$ and ψ shows that to prove the lemma, it is sufficient to prove it in the case that V is nonsingular.

Let us therefore assume V to be a one-dimensional manifold. The map f is proper, so there is an integer μ, the multiplicity of f, such that, due attention being paid to multiplicities, each point of V has μ preimages in U under f. Define the function F on $\mathscr{R}$ by $F(\zeta) = \Pi\{z \in \mathbb{U} : f(z) = \zeta\}$ in which each factor z is taken as many times as indicated by the multiplicity of f at z. This function F is holomorphic and bounded on $\mathscr{R}$.

The map $b = F \circ f : \mathbb{U} \to \mathbb{U}$ is proper and so is a finite Blaschke product. The derivative b' has a finite number of zeros in $\mathbb{U}$, so there is an annulus $A = \{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1 + \varepsilon\}$ on such that

$$b : b^{-1}(\mathbb{U} \cap A) \to \mathbb{U} \cap A$$

is a covering map. Consequently, $F^{-1}(\mathbb{U} \cap A)$ is a finite union of annuli. We can therefore form the double, V^*, of V, which is a compact Riemann surface V together with an anticonformal involution ρ such that $V^* = V \cup \rho(V)$ together with a finite number of analytic simple closed curves. The function f extends by reflection to a holomorphic function, still denoted by f, from the Riemann sphere, $\mathbb{C}^*$, onto V^* that satisfies the functional equation $f \circ \kappa = \rho \circ f$, where κ denotes the anticonformal involution $z \mapsto 1/\bar{z}$ of $\mathbb{C}^*$.

The map F gives a covering of $\mathbb{U} \cap A$ by the set $F^{-1}(\mathbb{U} \cap A)$, so F extends by reflection to a meromorphic function, denoted by F, on the compact surface V^*. The extended map F satisfies the functional equation $F \circ \rho = \kappa \circ F$.

To see that V is a disk, it suffices to see that V^* is a sphere. If V^{**} is the universal covering space of V^* with projection h, then $f : \mathbb{C}^* \to V^*$ lifts to $f^* : \mathbb{C}^* \to V^{**}$, so V^{**} is compact. The only compact simply connected Riemann surface is, to within conformal equivalence, the Riemann sphere. Uniformization theory shows that the only Riemann surface with the sphere as its universal covering space is the sphere itself: V^* is the sphere.

Thus, the map f is a proper holomorphic map from $\mathbb{U}$ to itself; such a map is a finite Blaschke product.

The lemma is proved.

The proof of Theorem 4.6.9 depends on a result from the theory of functions of a real variable:

Theorem 4.6.11. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The integer-valued function $N(\cdot, f)$ defined on $\mathbb{R}$ by $N(t, f) = \Lambda^0(f^{-1}(t))$ is measurable, and the integral $\int_{\mathbb{R}} N(t, f)\, dt$ is the total variation of f on the interval* $[a, b]$.

In this context, the function $N(\cdot, f)$ is called the *Banach indicatrix* of the function f. For the theorem one can consult [171, p. 270].

Proof of Theorem 4.6.9. Without loss of generality, we can suppose that the set E is the global cluster set of the map f. As such, it is connected. (Recall the discussion of cluster sets immediately preceding Theorem 3.4.13.) It has finite length, so the theorem of Alexander and Pommerenke, Theorem 3.4.13, implies that f extends to be continuous on $\bar{\mathbb{U}}$. We shall denote this extended map by f. What is to be proved is that the total variation of the function f on $b\mathbb{U}$ is finite. (Recall Corollary 3.4.7.)

According to Lemma 4.6.3, the multiplicity function $m(z; V)$ is almost everywhere not more than 2. If $z \in E$ satisfies $m(z; V) = 1$, then $f^{-1}(z)$ is a singleton. Suppose not, so that for two distinct points ζ_1 and ζ_2 of $b\mathbb{U}$, $f(\zeta_1) = f(\zeta_2)$. If Δ_j is a small disk in $\mathbb{C}$ about the point ζ_j, then because f is one-to-one off a discrete set, the sets $f(\Delta_1)$ and $f(\Delta_2)$ show $m(z; V)$ to be at least two, contrary to hypothesis. Similarly, if $m(z; V) = 2$, then $f^{-1}(z)$ consists of at most two points. That is to say, the fibers $f^{-1}(z)$ for $z \in E$ have almost surely $[d\Lambda^1]$ cardinality at most two.

Let $\varphi : \mathbb{C}^N \to \mathbb{R}$ be a *real*-linear functional. The function φ satisfies a Lipschitz condition with Lipschitz constant, say K_φ, so by Eilenberg's inequality,

$$\int_{\mathbb{R}} \Lambda^0(\varphi^{-1}(t) \cap E)\, dt \le K_\varphi \Lambda^1(E).$$

Consider now $\varphi \circ f : b\mathbb{U} \to E$. Because the fibers $f^{-1}(z)$ have, for $z \in E$, almost surely cardinality no more than two, it follows that the fibers $(\varphi \circ f)^{-1}(t)$, for $t \in \mathbb{R}$, almost surely satisfy $\Lambda^0((\varphi \circ f)^{-1}(t \cap b\mathbb{U})) \le 2\Lambda^0(\varphi^{-1}(t))$, whence for $t \in \varphi(E)$, $\Lambda^0((\varphi \circ f)^{-1}(t) \cap b\mathbb{U}) \le 2\Lambda^0(\varphi^{-1}(t))$. Accordingly,

$$\int_{\mathbb{R}} \Lambda^0((\varphi \circ f)^{-1}(t) \cap b\mathbb{U})\, dt \le 2 \int_R \Lambda^0(\varphi^{-1}(t) \cap E)\, dt < \infty.$$

Banach's theorem on the indicatrix quoted above implies that the function $\varphi \circ f$ is of bounded variation. Apply this with φ running through the $2N$ real-orthogonal projections of $\mathbb{C}^N$ onto its real cooridnate axes to reach the conclusion that f is of bounded variation, as we wished to show.

For a map from the disk to $\mathbb{C}^N$ with boundary values in smooth simple closed curves, more can be said than is given by Theorem 4.6.9. Let Γ be a rectifiable simple closed curve in $\mathbb{C}^N$, and let $f : \mathbb{U} \to \mathbb{C}^N \setminus \Gamma$ be a bounded proper holomorphic map from $\mathbb{U}$ onto the variety V. By the preceding theorem, the map f extends to a continuous map, also denoted by f, from $\bar{\mathbb{U}}$ to $\mathbb{C}^N$, and the extended map is of bounded variation on $b\mathbb{U}$.

Theorem 4.6.12. [149] *If the multiplicity of f is one, then $f : b\mathbb{U} \to \Gamma$ is a homeomorphism.*

The multiplicity in question is the cardinality of the generic fiber $f^{-1}(f(\zeta))$ for $\zeta \in \mathbb{U}$.

This theorem has the following corollary:

Corollary 4.6.13. *Let $g : \mathbb{U} \to \mathbb{C}^N \setminus \Gamma$ be a proper holomorphic map. If g has multiplicity p, then $g : b\mathbb{U} \to \Gamma$ is a p-fold covering map.*

Proof. Factor g as above: $g = f \circ \varphi$, where $\varphi : \mathbb{U} \to \mathbb{U}$ is a finite Blaschke product, and f is a normalization map. Thus $f : b\mathbb{U} \to b\mathbb{U}$ is a homeomoprhism by the theorem. The map $\varphi : b\mathbb{U} \to b\mathbb{U}$ is a p-fold covering map, so the corollary is proved.

To prove the theorem, we need two lemmas. In them we shall use $\mathbb{T}$ to denote $b\mathbb{U}$.

Lemma 4.6.14. *Let Γ be a simple closed curve, and let $f : \mathbb{T} \to \Gamma$ be a continuous surjective map. Assume f to be constant on no interval.*

(a) *If there is a dense set $K \subset \Gamma$ such that for all $p \in K$, the fiber $f^{-1}(p)$ contains at most two points, and if there is a point $p \in \Gamma$ such that the fiber $f^{-1}(p)$ is a single point, then f is a homeomorphism from $\mathbb{T}$ onto Γ.*

(b) *If there is a dense subset K of Γ such that for all points $p \in K$, the fiber $f^{-1}(p)$ consists of two points, then f is two-to-one.*

Proof. Consider first case (a). Let the point $p_1 \in \Gamma$ be a point such that the fiber $f^{-1}(p_1) = \{1\} \in \mathbb{T}$. Suppose that $f(q_1) = f(q_2)$ for distinct points $q_1, q_2 \in \mathbb{T}$. Let $\varphi : [0, 1] \to \mathbb{T}$ be the exponential map given by $\varphi(t) = e^{2\pi i t}$, and let $\psi : [0, 1] \to \Gamma$

be a map that satisfies $\psi(0) = \psi(1) = p_1$ and that is a homeomorphism of $(0, 1)$ onto $\Gamma \setminus \{p_1\}$. There is then a unique map $\tilde{f} : [0, 1] \to [0, 1]$ with $\psi \circ \tilde{f} = f \circ \varphi$. We have $\tilde{f}(0) = 0$ and $\tilde{f}(1) = 1$ or else $\tilde{f}(0) = 1$ and $\tilde{f}(1) = 0$. Assume the former case to obtain. Let $\varphi(t_1) = q_1$, $\varphi(t_2) = q_2$ for $t_1, t_2 \in (0, 1)$, $t_1 < t_2$. Then $\tilde{f}(t_1) = \tilde{f}(t_2)$. The map $\tilde{f}$ has the property that for a dense set S of points in the interval $(0, 1)$ the fiber $\tilde{f}^{-1}(s)$ consists of at most two points for each $s \in S$. The graph of $\tilde{f}$ meets the horizontal line $y = \tilde{f}(t_1) = \tilde{f}(t_2)$ in two points. The function $\tilde{f}$ is constant on no interval, so there are points $s \in (t_1, t_2)$ such that $\tilde{f}(s) > \tilde{f}(t_1) = \tilde{f}(t_2)$ or else such that $\tilde{f}(s) < \tilde{f}(t_1) = \tilde{f}(t_2)$. The density hypothesis yields a y_1 in the interval between $\tilde{f}(t_1)$ and s such that $\tilde{f}^{-1}(y_1)$ consists of at most two points. By continuity, if $\tilde{f}(s) < \tilde{f}(t_1)$, then the value y_1 is taken by $\tilde{f}$ in each of the intervals $(0, t_1)$, (t_1, s), and (s, t_2). If $\tilde{f}(s) > \tilde{f}(t_1)$, the value y_1 is taken by $\tilde{f}$ in each of the intervals (t_1, s), (s, t_2), and $(t_2, 1)$. This contradicts the assumption that the fiber of $\tilde{f}$ over y_1 consists of only two points. Case (a) is proved.

For case (b) a very similar argument works. Suppose for the sake of deriving a contradiction that there is a point $p_3 \in \Gamma$ such that the fiber $f^{-1}(p_3)$ contains three distinct points q_2, q_2, and q_3. Let $p_2 \in \Gamma$ be a point such that the fiber f^{-1} consists of two points, say 1 and -1. Let λ^+ and λ^- be, respectively, the top and bottom halves of $\mathbb{T}$. There are two cases: Either all three of q_1, q_2, and q_3 lie in λ^+, for else two of them, say q_2 and q_2, lie in one of $\lambda^\pm$, and q_3 lies in the other. In the case that q_1 and q_2 lie in λ^+, an analysis strictly parallel to the proof of case (a) shows that in fact, some of the values in the dense set K are assumed three times, contrary to hypothesis. In the case that all three of the q's lie in λ^+, we see that some values in K are assumed at least four times.

The lemma is proved.

Lemma 4.6.15. *Let Γ be a simple closed curve. If $f : \mathbb{T} \to \Gamma$ is a continuous two-to-one map, then either f is homotopic to a constant or else it is a (two-sheeted) covering map.*

Both cases can occur.

Proof. It is a convenience to suppose that $\Gamma = \mathbb{T}$.

The map f is two-to-one: For each $p \in \mathbb{T}$, the fiber $f^{-1}(p)$ consists of two distinct points.

If f is not surjective, it is homotopic to a constant, so from here on we suppose f to be surjective.

Suppose $f^{-1}(1) = \{-1, 1\}$. Let λ^+ and λ^- be, respectively, the top and bottom halves of $\mathbb{T}$. We consider two cases: It may be that $f(\lambda^+) = \mathbb{T}$ or it may be that $f(\lambda^+)$ is a proper subset of $\mathbb{T}$.

We shall show that in the case that $f(\lambda^+)$ is a proper subset of $\mathbb{T}$, the map f is homotopic to a constant. If $f(\lambda^+) \neq \mathbb{T}$, then $f(\lambda^+)$ is an arc contained in $\mathbb{T}$. Let its endpoints be p' and p''. One of p' and p'' must be the point 1. Suppose not. Let q' and q'' be points of λ^+ that satisfy $f(q') = p'$ and $f(q'') = p''$. Because the point 1 is not one of the endpoints of $f(\lambda^+)$, it is an interior point, and it is necessarily contained in the image under f of the arc from q' to q'' in λ^+. But this implies that $f^{-1}(1)$ has cardinality at least three. Thus 1 is an endpoint of $f(\lambda^+)$.

The continuous map f maps λ^+ onto an arc L in $\mathbb{T}$, and it takes both endpoints of λ^+ to one endpoint, 1, of L. Let the other endpoint of L be p_0.

Every point of the interior of L is covered at least twice (and therefore exactly twice) by f. The map f is two-to-one, so (λ^-) can meet $f(\lambda^+)$ only at the endpoints of L, at 1 and p_0.

We conclude that the change in the argument Arg f of f over λ^+ is zero, as is that over λ^-. Consequently, the change in Arg f over the whole of $\mathbb{T}$ is zero, and f is found to be homotopic to a constant.

Thus, if f is not homotopic to a constant, then $f(\lambda^+) = f(\lambda^-) = \mathbb{T}$. The hypothesis that f is two-to-one implies that f carries the interior of λ^+ homeomorphically onto $\mathbb{T}\setminus\{1\}$ and also carries the interior of λ^- homeomorphically onto $\mathbb{T}\setminus\{1\}$.

Thus the change $\Delta_{\lambda^+}\operatorname{Arg} f$ along λ^+ (moving from 1 to -1) is $2\pi\varepsilon_+$ with ε_+ either 1 or -1. Similarly, $\Delta_{\lambda^-}\operatorname{Arg} f$ (moving from -1 to 1) is $2\pi\varepsilon_-$ with ε_- either 1 or -1. If $\varepsilon_+\varepsilon_- = -1$, then $\Delta_{\mathbb{T}}\operatorname{Arg} f = 0$, so there is a branch of $\log f$ defined on $\mathbb{T}$, and f is homotopic to a constant. If $\varepsilon_+\varepsilon_- = 1$, so that $\varepsilon_+ = \varepsilon_-$, then f is a two-sheeted covering map.

The lemma is proved.

Proof of Theorem 4.6.12. The map $f : b\mathbb{U} \to \Gamma$ has the property that for almost all $p \in \Gamma$, the fiber $f^{-1}(p)$ contains at most two points. Also, it is constant on no interval. If there is one point $p \in \Gamma$ such that the fiber $f^{-1}(p)$ consists of a single point, then by Lemma 4.6.14, f is a homeomorphism, and we are done.

Assume, therefore, that each fiber $f^{-1}(p)$, $p \in \Gamma$, contains at least two points. Then by Lemma 4.6.14, f is exactly two-to-one, and by Lemma 2, $f|b\mathbb{U}$ is either homotopic to a constant or else is a two-sheeted covering map.

The map $f|b\mathbb{U}$ cannot be homotopic to a constant: The curve Γ is rectifiable and so rationally convex. There is, therefore, a polynomial, P, that is zero-free on Γ but that vanishes at some point, perhaps several, of V. The function $P \circ f$ is continuous on $\bar{\mathbb{U}}$ and has a zero in $\mathbb{U}$. Consequently, $\Delta_{b\mathbb{U}}\operatorname{Arg} P \circ f$ is not zero. However, if the map $f|b\mathbb{U}$ is homotopic to a constant, so is $P \circ f$, whence the variation $\Delta_{b\mathbb{U}} P \circ f$ is zero. Contradiction. Thus, $f|b\mathbb{U}$ is not homotopic to a constant.

We have now reached the conclusion that if $f|b\mathbb{U}$ is not one-to-one, then it is a two-sheeted covering map. The latter case is untenable. For in this case, as above, there is a polynomial P such that $P^{-1}(0) \cap \Gamma = \emptyset$ but $P^{-1}(0) \cap V \neq \emptyset$. We can suppose the zeros of P on V to lie in V_{reg}. Moreover, because f is assumed to have multiplicity one, we can suppose that each point of $P^{-1}(0) \cap V$ is covered only once by f and that df is zero-free on $f^{-1}(P^{-1}(0))$. Stokes's theorem implies that the integral $\frac{1}{2\pi i}\int_\Gamma \frac{dP}{P}$ is the number of zeros of $P|V$.

On the other hand, because $f|b\mathbb{U}$ is a two-sheeted covering and is of bounded variation.

$$\frac{1}{2\pi i}\int_\Gamma \frac{dP}{P} = 2 \cdot \frac{1}{2\pi i}\int_{b\mathbb{U}} \frac{d(P \circ f)}{P \circ f}.$$

The quantity on the right is 2ν if ν is the total number of zeros of $P \circ f$ in $\mathbb{U}$. By the choice of the polynomial P, the function $P \circ f$ has the same number of zeros on $\mathbb{U}$ as P has on V.

We have reached a contradiction, and the theorem is proved.

4.7. Counting the Branches

The question we are going to consider next concerns the number of global branches of the variety $\widehat{X} \setminus X$. Alexander [17] proved that if Γ is a rectifiable simple closed curve, then the variety $\widehat{\Gamma} \setminus \Gamma$ is irreducible: It has a single global branch. Recall Theorem 4.5.5.

For certain choices of X, the variety $\widehat{X} \setminus X$ will have infinitely many branches, as in the case of a closed rectifiable curve in the plane that crosses itself infinitely often.

There is a sharp general result in this direction:

Theorem 4.7.1. *If X is a compact subset of $\mathbb{C}^N$ with finite length and if V is a bounded purely one-dimensional subvariety of $\mathbb{C}^N \setminus X$, then the number of global branches of V does not exceed the rank of the group $\check{H}^1(X; \mathbb{Z})$.*

This statement, with the additional hypothesis that X is connected, was obtained by Lawrence [217].

Note that in the statement of the theorem there is no hypothesis of regularity for the set X beyond the condition that it be of finite length.

The analysis given below will show the rank in question to be the number of simple closed curves contained in X.

Corollary 4.7.2. *If $X \subset \mathbb{C}^N$ is a compact set of class $\mathscr{A}_1$, then the number of global branches of $\widehat{X} \setminus X$ does not exceed the rank of the cohomology group $\check{H}^1(X; \mathbb{Z})$.*

The proof of the theorem depends on a preliminary analysis of sets with finite length that satisfy $\check{H}^1(X; \mathbb{Z}) \neq 0$.

Lemma 4.7.3. *If X is a compact set every component, K, of which satisfies $\check{H}^1(K; \mathbb{Z}) = 0$, then $\check{H}^1(X; \mathbb{Z}) = 0$.*

Proof. We assume that $\check{H}^1(K; \mathbb{Z}) = 0$ for every component K of X, and we show that $\check{H}^1(X; \mathbb{Z}) = 0$ by showing that every zero-free continuous function f is an exponential. To do this it suffices to show that each component of K is contained in an open and closed subset of X on which f has a logarithm. Let K be a component of X. There is a continuous function g_K on X such that on K, $f = e^{g_K}$. Let $U_K = \{x \in X : |fe^{-g_K} - 1| < \frac{1}{2}\}$, an open neighborhood of K.

The set K is the intersection of all the open and closed subsets of X that contain it, because it is a component of X. Let $V_K \subset U_K$ be an open and closed subset of X that contains K. On V_K, fe^{-g_K} is an exponential, and so the same is true of f. Thus f is an exponential on all of X.

Lemma 4.7.4. *If X is a compact subset of $\mathbb{R}^N$ with finite length and with $\check{H}^1(X; \mathbb{Z}) \neq 0$, then X contains a simple closed curve.*

Proof. By Lemma 4.7.3, we can suppose X to be connected. It is therefore arcwise connected by Theorem 3.3.3. We shall prove that if X contains no simple closed curve, then every zero-free continuous function on X is of the form e^g for a function g continuous on X.

Suppose, therefore, that X contains no simple closed curve. The set X is arcwise connected, but more is true: Because it contains no simple closed curve, for every pair x, y

of distinct points in X, there is a *unique* arc in X with endpoints x and y. Fix a base point x_o in X, and for every $y \in X \setminus \{x_o\}$, let λ_y be the arc with endpoints x_o and y.

Fix a zero-free continuous function f on X that is normalized to satisfy $f(x_o) = 1$. For each $y \in X$, let g_y be the continuous function on λ_y with $g_y(x_o) = 0$ and with $f = e^{g_y}$ on λ_y. Define the function $g : X \to \mathbb{C}$ by $g(x_o) = 0$ and $g(y) = g_y(y)$ for all $y \in X \setminus \{x_o\}$. The function g is well defined and satisfies $f = e^g$ on X.

The continuity of g is elementary, albeit a bit fussy, to establish. Let the sequence $\{y_n\}_{n=1,\dots}$ in X converge to y_o. We shall prove that $g(y_n) \to g(y_o)$. Notice that $g(y_n) = g(y_o) + \varepsilon_n + 2\pi i k_n$ with $\varepsilon_n \to 0$, and with k_n an integer.

Fix $\delta > 0$ so small that f has a continuous logarithm, L_f, on the set $X \cap \bar{\mathbb{B}}(y_o, 2\delta)$ and so small that, in addition, $\delta < |x_o - y_o|$. We normalize L_f so that $L_f(y_o) = g_{y_o}(y_o)$. The finite-length hypothesis implies that only finitely many of the components of $X \cap \bar{\mathbb{B}}(y_o, 2\delta)$ meet the concentric ball $\bar{\mathbb{B}}(y_o, \delta)$. We have therefore that for all large n, y_n lies in the component Y of $X \cap \bar{\mathbb{B}}(y_o, 2\delta)$ that contains y_o. Suppose that Y contains all of the points y_n. The set Y is compact, connected, and has finite length, so it is arcwise connected. For each n, let ℓ_n be the arc in X that connects y_o and y_n. This arc is contained in Y. The assumption that X contains no simple closed curves implies that the arcs ℓ_n and λ_{y_o} meet either at a single point or else in an arc. In either case, let x_n be the first point in λ_{y_o}, traveling from x_o toward y_o, that lies in ℓ_n. The subarc $[x_o, x_n]$ of λ_{y_o} taken together with the subarc $[x_n, y_n]$ is an arc in X connecting x_o and y_n, and so is λ_{y_n}. We have that $g(x_n) = L_f(x_n)$, so $g = L_f$ on the subarc $[x_o, x_n]$ of λ_{y_o}. It then follows that $g = L_f$ on the subarc $[x_n, y_n]$ of ℓ_n. Consequently, $g(y_n) = L_f(y_n)$. The proof is complete, for $L_f(y_n) \to L_f(y_o) = g(y_o)$.

Lemma 4.7.5. *If the compact subset X of $\mathbb{R}^N$ satisfies $\Lambda^1(X) < \infty$ and if X contains m simple closed curves, then the rank of $\check{H}^1(X; \mathbb{Z})$ is at least m.*

Proof. Let the compact subset C of X be the union of m distinct simple closed curves. The map $\check{H}^1(X; \mathbb{Z}) \to \check{H}^1(C; \mathbb{Z})$ induced by the inclusion of C in X is surjective—recall Corollary 3.8.17, so the rank of $\check{H}^1(X; \mathbb{Z})$ is at least as large as the rank of $\check{H}^1(C; \mathbb{Z})$, which is at least m.

Lemma 4.7.6. *If X is a compact subset of $\mathbb{C}^N$ of finite length and if $\mathbb{C}^N \setminus X$ contains a bounded, one-dimensional variety, then X contains a simple closed curve.*

Proof. If $V \subset \mathbb{C}^N \setminus X$ is a bounded, one-dimensional variety, then by Theorem 3.8.15, $\check{H}^1(bV; \mathbb{Z}) \neq 0$. The result follows from Lemma 4.7.4, because $\bar{V} \setminus V \subset X$, and the natural map $\check{H}^1(X; \mathbb{Z}) \to \check{H}^1(\bar{V} \setminus V; \mathbb{Z})$ is surjective.

If the X of this lemma is not required to have finite length, it may not contain a simple closed curve, as shown in Section 1.6.2.

The proof of the theorem will use the following fact.

Lemma 4.7.7. *Let Ω be a bounded domain in $\mathbb{C}^N$, and let $E \subset \Omega$ be a compact, connected subset of Ω of finite length that contains only finitely many simple closed curves whose union, denoted by E_o, is connected. If $V \subset \Omega \setminus E$ is a purely one-dimensional subvariety, then $\bar{V} \cap \Omega \setminus E_o$ is a one-dimensional subvariety of $\Omega \setminus E_o$ and each global branch of the extended variety contains a unique branch of the variety V.*

Proof. If λ is a component of $E \setminus E_o$, then $\bar{\lambda}$ meets E_o in only one point. The set $E \setminus \lambda$ is compact, and $\Lambda^1(\bar{\lambda}) = 0$. Moreover, $\check{H}^1(\bar{\lambda}; \mathbb{Z}) = 0$, so Theorem 3.8.18 implies that V extends through λ. This is true for every choice of λ, so V continues analytically as a one-dimensional variety through $E \setminus E_o$. Denote this continued variety by V'.

The set V is the complement in V' of the closed subset $E \setminus E_o$ of V' and

$$\check{H}^1(\overline{E \setminus E_o}; \mathbb{Z}) = 0,$$

so each global branch of V' remains connected when the set $E \setminus E_o$ is removed. Thus, V' can contain at most one global branch of V.

Proof of Theorem 4.7.1. We can suppose that $X = bV$, because by Lemma 3.8.17, the natural map $H^1(X, \mathbb{Z}) \to H^1(bV, \mathbb{Z})$ is surjective. Let the rank of $\check{H}^1(X; \mathbb{Z})$ be r, which we take to be finite. Then by Lemma 4.7.5, X contains at most r simple closed curves. Accordingly, there are only finitely many components of X that contain a simple closed curve; denote them by $X_1, \ldots, X_s$. Thus, $s \le r$. Put $Y = \cup_{j=1,\ldots,s} X_j$.

To begin with, we are going to show that the variety V continues analytically through the set $X \setminus Y$. This is so, for if U is an open subset of $\mathbb{C}^N$ that contains Y, then there is an open and closed subset of X, say Y', with $Y \subset Y' \subset U$. The set $X \setminus Y'$ is open and closed in X, and it satisfies $\check{H}^1(X \setminus Y'; \mathbb{Z}) = 0$, so by Theorem 3.8.18, the variety V continues through $X \setminus Y'$. This is correct for every choice of U and every choice of $Y' \subset U$, so V continues holomorphically to a variety V' in $\mathbb{C}^N \setminus Y$. Each global branch of V' contains a unique global branch of V, because the set $X \setminus Y$ cannot disconnect any branch of V'.

We have that $\widehat{Y} \supset V'$. Moreover, as a set of finite length with only finitely many components, Y is contained in a connected set of finite length, which implies that the set $\widehat{Y} \setminus Y$ is a bounded analytic subvariety of $\mathbb{C}^N \setminus Y$. Call it W.

Denote by Γ the union of the simple closed curves contained in Y, which is the same as the union of the simple closed curves contained in X. The variety W continues analytically through $Y \setminus \Gamma$. To see this, denote by F_α, $\alpha \in A$, the components of $Y \setminus \Gamma$. None of the F_α's is a point. Each therefore has positive length, so there are only countably many of them. Also, for each $\alpha \in A$, the set $\cup_{\beta \in (A \setminus \{\alpha\})} F_\beta$ is closed in $\mathbb{C}^N \setminus Y$. Moreover, because each F_α is disjoint from Γ, each set $\bar{F}_\alpha$ meets Γ in at most one point. The group $\check{H}^1(\bar{F}_\alpha; \mathbb{Z})$ vanishes, so the variety W continues holomorphically through F_α. Denote by W_1 the variety obtained by continuing W through $Y \cap \Gamma$. Again, the global branches of W_1 each contain a unique global branch of the variety W, so $W = \widehat{Y} \setminus Y \subset \widehat{\Gamma} \setminus \Gamma$.

The proof concludes by showing that the variety $\Sigma = \widehat{\Gamma} \setminus \Gamma$, which contains W', has at most r global branches.

Let Γ_o be the minimal closed subset of Γ whose polynomially convex hull contains that of Γ. Thus, Γ_o is a compact subset of Γ. It is the union of some of the simple closed curves that make up Γ. This is seen to be so as follows. The set Γ_o is contained in a compact connected set of finite length, so $\widehat{\Gamma}_o \setminus \Gamma_o$ is a variety, which we denote by Σ_o. The variety Σ_o contains the variety Σ. If Γ_o is not a union of simple closed curves, let $\Gamma_1 \subset \Gamma_o$ be the union of all the simple closed curves in Γ_o. The analysis given above applies, mutatis mutandis, to show that Σ_o continues holomorphically through $\Gamma_o \setminus \Gamma_1$, which contradicts the minimality of Γ_o. Thus, as claimed, Γ_o is a union of a finite number, not more than r, of simple closed curves.

If the variety V has more than r global branches, then so does the variety Σ_o. If T is a global branch of Σ_o, then $\bar{T} \setminus T$ contains a simple closed curve. Thus, there are two distinct branches, say T_1 and T_2, of Σ_o such that $\bar{T}_1 \cap \bar{T}_2$ contains one of the simple closed curves, say C, that constitute Γ_o.

Theorem 4.6.4 implies that $T_1 \cup T_2 \cup C$ is a variety. From this we see that if Γ'_o is the union of the simple closed curves other than C that constitute Γ_o, then $C \subset \widehat{\Gamma}'_o$, so $\widehat{\Gamma}'_o$ is $\widehat{\Gamma}$. This contradicts the minimality of Γ_o. Thus, the variety Σ has at most r branches.

The theorem is proved.

Chapter 5

FURTHER RESULTS

Introduction. This chapter treats some further results in the theory of polynomial convexity, which are rather loosely related but draw substantially on the work of the preceding chapters. Section 5.1 discusses isoperimetric questions in the context of polynomial convexity. Section 5.2 considers some questions in the theory of removable singularities for holomorphic functions and their boundary values. Section 5.3 treats certain convexity problems for two-dimensional surfaces in three-dimensional strictly pseudoconvex boundaries.

5.1. Isoperimetry

The motivation for most of the work in the present section stems from the classical isoperimetric inequality. If γ is a simple closed curve of length $L(\gamma)$ in the plane that bounds a domain Ω of area $\mathrm{A}(\Omega)$, then $4\pi\mathrm{A}(\Omega) \leq L(\gamma)^2$. A much more general result is true: If X is a compact connected subset of the plane, then

$$4\pi\mathscr{L}(X) \leq \left[\Lambda^1(bX)\right]^2. \tag{5.1}$$

Here $\mathscr{L}$ denotes planar Lebesgue measure; it is the same as Λ^2, the two-dimensional Hausdorff measure in the plane. The inequality (5.1), and various generalizations of it, can be found in [115].

There are extensions of the isoperimetric inequality (5.1) in many directions. The present section is devoted to some results of this general flavor in the context of polynomial convexity.

A proof of the inequality (5.1) can be given rather quickly. The proof of the general result depends on the classical case of a domain D bounded by a rectifiable simple closed curve. The following simple proof of that result was given by Komatu [209].

Denote by $\mathbb{U}^*$ the complement in the Riemann sphere of the closed unit disk. Let φ be a conformal map from the domain $\mathbb{U}^*$ to the complement in the Riemann sphere of $\bar{D}$

that satisfies $\varphi(\infty) = \infty$, so that φ has a Laurent expansion of the form

$$\varphi(z) = \alpha z + \sum_{j=0}^{\infty} \alpha_j z^{-j}.$$

For $R > 1$ denote by D_R the set $\mathbb{C}^* \setminus \varphi(R\mathbb{U}^*)$ and by Γ_R the boundary of D_R, which is the image under φ of the circle of radius R centered at the origin. As $R \to 1^+$, the area of D_R approaches the area of D, and the length of Γ_R, which is $\int_{\{|z|=R\}} |\varphi'(z)|\,|dz|$, approaches $\int_{-\pi}^{\pi} |\varphi'(e^{i\vartheta})|\,d\vartheta$—recall Theorem 3.4.8, which is the length of bD. Consequently, we can suppose that bD is real-analytic.

Recall that if C is a smooth simple closed curve in the plane, then the area bounded by C is given by the integral $\frac{1}{2i}\int_C \bar{z}\,dz$. Apply this to the domain D and use the parameterization φ of bD to compute the integral. A direct calculation gives

$$A = \pi\left[|\alpha|^2 - \sum_{j=0}^{\infty} j|\alpha_j|^2\right].$$

We can also use the parameterization φ to compute the length of bD. The map φ is conformal, so φ' is zero-free, whence $\varphi' = g^2$ for some function g holomorphic in $\mathbb{U}^*$. The function g has the Laurent expansion $g(z) = \sum_{j=0}^{\infty} \beta_j z^{-j}$, so

$$L = \int_{-\pi}^{\pi} |\varphi'(e^{i\vartheta})|\,d\vartheta = \int_{-\pi}^{\pi} |g(e^{i\vartheta})|^2\,d\vartheta = 2\pi\sum_{j=0}^{\infty} |\beta_j|^2. \tag{5.2}$$

The theorem is proved, for $\beta_o = \alpha$ because $g^2 = \varphi'$. We therefore have that $L^2 \geq 4\pi^2|\alpha|^2 \geq 4\pi A$. Note that equality holds when $\alpha_1 = \alpha_2 = \cdots = 0$, i.e., when D is a disk.

Having this result for domains bounded by simple closed curves, we can deduce the general inequality (5.1).

We can suppose X not to separate the plane: If X separates the plane, then adjoining to X the bounded components of $\mathbb{C} \setminus X$ neither decreases the area of X nor increases the length of bX.

If $\mathscr{L}(X) = 0$ we are done. And if $\mathscr{L}(X) > 0$ and the interior, $\operatorname{int} X$, of X is empty we are also done, for in this case, $X = bX$, and $\mathscr{L}(X) > 0$ implies $\Lambda^1(bX) = \infty$.

Accordingly, we assume that $\operatorname{int} X$ is not empty and that $\Lambda^1(bX) < \infty$. Let Ω_j, $j = 1, \ldots,$ be the countably many components of $\operatorname{int} X$. Each Ω_j is simply connected by the next lemma.

Lemma 5.1.1. *If Y is a polynomially convex subset of the plane, then each component of* $\operatorname{int} Y$ *is simply connected.*

Proof. Suppose the component Ω of $\operatorname{int} Y$ is not simply connected, so that there is a simple closed curve γ in Ω that is not null-homotopic in Ω. By the Jordan curve theorem, γ splits the plane into two components; let D be the bounded one. Then $D \subset \widehat{\gamma}$, so the polynomial convexity of Y implies that $D \subset Y$. The domain D meets Ω, but γ is not null-homotopic in Ω so D cannot be contained in Ω. We have reached a contradiction to the assumption that Ω is a component of $\operatorname{int} Y$.

We now complete the proof of (5.1). For each $j = 1, \ldots,$ let $\phi_j : \mathbb{U} \to \Omega_j$ be a conformal mapping as provided by the Riemann mapping theorem. For each j, $b\Omega_j \subset bX$, so $\Lambda^1(b\Omega_j) < \infty$. It follows that ϕ_j extends continuously to $\bar{\mathbb{U}}$, that $\phi_j|bU$ is of bounded variation, and that ϕ_j' belongs to the Hardy space $H^1(\mathbb{U})$.

We will prove that $\phi_j|b\mathbb{U}$ is one-to-one and so a homeomorphism. If it is not, then there is a point p in $b\Omega_j$ such that $p = \phi_j(q') = \phi_j(q'')$ for some choice of $q', q'' \in b\mathbb{U}$, $q' \neq q''$. With no loss of generality, $q' = -1$, $q'' = 1$. The set $\phi_j([-1, 1])$ is a simple closed curve, call it γ, contained in $\Omega_j \cup \{p\}$. Let D be the bounded component of $\mathbb{C} \setminus \gamma$. By the maximum principle, $D \subset X$. The map ϕ_j is a homeomorphism of $\mathbb{U}$ onto Ω_j, so it carries one component of $\mathbb{U} \setminus [-1, 1]$, say $\mathbb{U}^+$, the open upper half of $\mathbb{U}$, onto D. If $\vartheta \in (0, \pi)$, then as $r \to 1^-$, the point $\phi_j(re^{i\vartheta})$ approaches $b\Omega_j$ through D. This implies that $\lim_{r\to 1^-} \phi_j(re^{i\vartheta}) = p$. This happens for all $\vartheta \in (0, \pi)$, so ϕ_j must be constant. This is impossible, so $\phi|b\mathbb{U}$ is one-to-one as claimed.

Next, if $j \neq k$, then $\bar{\Omega}_j \cap \bar{\Omega}_k$, if not empty, is a singleton. Suppose, to the contrary, that there are two distinct points, p and q, in $\bar{\Omega}_j \cap \bar{\Omega}_k = b\Omega_j \cap b\Omega_k$. Let $\phi_j(-1) = p = \phi_k(-1)$ and $\phi_j(1) = q = \phi_k(1)$. The set $\phi_j([-1, 1]) \cup \phi_k([-1, 1])$ is a simple closed curve, which we shall denote by λ. Let D be the bounded component of $\mathbb{C} \setminus \lambda$. Again, $D \subset X$ by polynomial convexity, and D meets both Ω_j and Ω_k. But this is impossible, for Ω_j and Ω_k are components of the interior of X. Thus, the set $\bar{\Omega}_j \cap \bar{\Omega}_k$ contains at most one point.

For each j, $\phi_j|b\mathbb{U}$ is a homeomorphism of $b\mathbb{U}$ onto $b\Omega_j$, so the latter set is a simple closed curve, which is rectifiable, for $\Lambda^1(bX) < \infty$. The classical isoperimetric inequality yields $4\pi \mathscr{L}(\bar{\Omega}_j) \leq [\Lambda^1(b\Omega_j)]^2$. Sum over j to get

$$4\pi \sum_{j=1,\ldots} \mathscr{L}(\bar{\Omega}_j) \leq \sum_{j=1,\ldots} [\Lambda^1(b\Omega_j)]^2 \leq \left[\sum_{j=1,\ldots} \Lambda^1(b\Omega_j) \right]^2.$$

By pairs, the sets $\bar{\Omega}_j$ meet at most in singletons, so

$$\sum_{j=1,\ldots} \mathscr{L}(\bar{\Omega}_j) = \mathscr{L}(\cup_{j=1,\ldots} \bar{\Omega}_j).$$

This quantity is $\mathscr{L}(X)$, for if not, the set $X \setminus \cup_{j=1,\ldots} \Omega_j$ has positive area and consists entirely of boundary points of X. This is impossible: The set bX is assumed to have finite length. Also, the sets $b\Omega_j$ meet, by pairs, only in singletons, so $\sum_{j=1,\ldots} \Lambda^1(b\Omega_j) = \Lambda^1(\cup_{j=1,\ldots} b\Omega_j) \leq \Lambda^1(bX)$. The theorem is proved.

The main results of the present section begin with a theorem of isoperimetric type for hulls due to Alexander [8]. Let π_j, $j = 1, \ldots, N$, denote the projection of $\mathbb{C}^N$ onto the jth coordinate axis.

Theorem 5.1.2. *If X is a compact subset of $b\mathbb{B}_N$ with $0 \in \widehat{X}$, then*

$$\sum_{j=1}^{N} \mathscr{L}(\pi_j(\widehat{X} \cap \mathbb{B}_N)) \geq \pi.$$

The sets $\pi_i(\widehat{X} \cap \mathbb{B}_N)$ are σ-compact and so measurable.

In [8] Alexander applies this result to obtain a new proof of Hartogs's theorem about the holomorphicity of separately holomorphic functions.

The proof of Theorem 5.1.2 depends on a quantitative version of the Hartogs–Rosenthal theorem. Recall Theorem 1.6.4: *If E is a compact subset of the plane with area zero, then $\mathscr{R}(E) = \mathscr{C}(E)$*. Because of the Stone–Weierstrass theorem, the equality $\mathscr{R}(E) = \mathscr{C}(E)$ holds if and only if $\bar{z} \in \mathscr{R}(E)$.

Lemma 5.1.3. *If E is a compact subset of $\mathbb{C}$, then*

$$\inf_{f \in \mathscr{R}(E)} \sup_{z \in E} |\bar{z} - f(z)| \leq \left(\mathscr{L}(E)/\pi\right)^{1/2}.$$

The quantity on the left of this inequality is the distance in $\mathscr{C}(E)$ from the function $\bar{z}$ to $\mathscr{R}(E)$.

In case E has zero area this inequality gives the Hartogs–Rosenthal theorem.

Proof. Let φ be a function of class $\mathscr{C}^\infty$ on the plane with compact support that on a neighborhood of the set E satisfies $\varphi(z) = \bar{z}$. The generalized Cauchy integral formula yields that for $z \in E$,

$$\bar{z} = -\frac{1}{\pi} \int_E \frac{d\mathscr{L}(\zeta)}{\zeta - z} - \frac{1}{\pi} \int_{\mathbb{C} \setminus E} \frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta) \frac{d\mathscr{L}(\zeta)}{\zeta - z}.$$

The second integral, as a function of z, is an element of $\mathscr{R}(E)$. The lemma follows therefore from an estimate of Ahlfors and Beurling [1]:

Lemma 5.1.4. *If E is a compact subset of $\mathbb{C}$, then*

$$\left| \int_E \frac{d\mathscr{L}(\zeta)}{\zeta - z} \right| \leq \left(\pi \mathscr{L}(E)\right)^{1/2}.$$

Proof. By a rigid motion in the ζ-plane, we can suppose that $z = 0$ and that the integral $J = \int_E \frac{d\mathscr{L}(\zeta)}{\zeta}$ is positive. With this normalization of the problem, denote by E^+ the part of the set E that lies in the right half-plane. By passing to polar coordinates $\zeta = \rho e^{i\vartheta}$, we see that

$$J \leq \int_{E^+} \cos \vartheta \, d\rho \, d\vartheta.$$

Let $\ell(r, \vartheta)$ be the length of the set of points $\zeta \in E^+$ with $\operatorname{Arg}\zeta = \vartheta$ and $|\zeta| \le r$. If $\ell(\vartheta) = \ell(\infty, \vartheta)$, then

$$\begin{aligned}\int_{E^+} \cos\vartheta\, d\rho\, d\vartheta &= \int_{-\pi/2}^{\pi/2} \ell(r, \vartheta) \cos\vartheta\, d\vartheta \\ &\le \left(\int_{-\pi/2}^{\pi/2} \cos^2\vartheta\, d\vartheta\right)^{1/2} \left(\int_{-\pi/2}^{\pi/2} \ell^2(\vartheta)\, d\vartheta\right)^{1/2} \\ &= \left(\frac{\pi}{2}\int_{-\pi/2}^{\pi/2} \ell^2(\vartheta)\, d\vartheta\right)^{1/2}.\end{aligned}$$

Also, for fixed ϑ, $\ell(r, \vartheta) \le r$, so

$$\int \rho\, d\rho \ge \int \ell(r, \vartheta)\, d\ell(r, \vartheta) \ge \frac{\ell^2(\vartheta)}{2}.$$

Thus

$$\mathscr{L}(E) \ge \mathscr{L}(E^+) = \int_{E^+} \rho\, d\rho d\vartheta \ge \frac{1}{2}\int_{-\pi/2}^{\pi/2} \ell^2(\vartheta)\, d\vartheta.$$

We reach therefore the desired inequality

$$J \le \left(\frac{\pi}{2}\int_{-\pi/2}^{\pi/2} \ell^2(\vartheta)\, d\vartheta\right)^{1/2} \le \big(\pi\mathscr{L}(E)\big)^{1/2}.$$

The lemma is proved.

Proof of Theorem 5.1.2. It is enough to prove the inequality with $\mathbb{B}_N$ replaced by $\bar{\mathbb{B}}_N$. For each $\varepsilon > 0$ and each j, let r_j be a rational function on the plane that is holomorphic on the set $\pi_j(\widehat{X})$ and that satisfies

$$|\bar{z} - r_j(z)| \le \big(\mathscr{L}(\pi_j(\widehat{X}) + \varepsilon)/\pi\big)^{1/2}$$

for all $z \in \pi_j(\widehat{X} \cap \bar{\mathbb{B}}_N)$. If $f_j = r_j \circ \pi_j$, then f_j is holomorphic on a neighborhood of $\widehat{X}$. Consequently, by the Oka–Weil theorem, it lies in the algebra $\mathscr{P}(X)$. The function $g = \sum_{j=1}^N z_j f_j$ also lies in $\mathscr{P}(X)$, and, because it vanishes at the origin, a point of $\widehat{X}$, it is not invertible in this algebra. For $z \in X$, the function h given by $h(z) = \sum_{j=1}^N z_j(\bar{z}_j - f_j)$ satisfies $h(z) = 1 - g(z)$. The Cauchy–Bunyakowski–Schwarz inequality and the choice of f_j give

$$\|1 - g\|_X \le \left(\sum_{j=1}^N (\mathscr{L}(\pi_j(\widehat{X}) + \varepsilon))/\pi\right)^{1/2}.$$

Because the function $1 - g$ is not invertible in $\mathscr{P}(X)$, the left side of the last inequality must be at least one (recall the proof of Lemma 1.2.3), so the theorem follows.

The following result, which is due to Alexander [9], gives some information about representing measures.

Theorem 5.1.5. *Let X be a compact, polynomially convex subset of $\mathbb{C}^N$, let x_o be a point of X, and let f be an element of the algebra $\mathscr{P}(X)$ that vanishes at x_o. If μ is a representing measure for x_o with support in X, then for $t \geq 0$,*

$$\mu\{x \in X : |f(x)| \geq t\} \leq \frac{1}{2\pi t}\Lambda^1(C_t \cap f(X))$$

if C_t denotes the circle of radius t centered at the origin.

Proof. Fix a $t > 0$ and an $\varepsilon > 0$. By the regularity of the measure Λ^1 on C_t, there is a function u defined and of class $\mathscr{C}^\infty$ on the circle C_t that satisfies $0 < u \leq 1$ there, that is identically one on an open set V of C_t that contains the intersection $C_t \cap f(X)$, and, finally, that satisfies

$$\int_{-\pi}^{\pi} u(te^{i\vartheta})\, d\vartheta < \frac{1}{t}\{\Lambda^1(C_t \cap f(X)) + \varepsilon\}.$$

The function u extends through the disk $t\mathbb{U}$ as a harmonic function, which will also be denoted by u. Let v be the conjugate harmonic function that satisfies $v(0) = 0$. This function has smooth boundary values. The function $F = u + iv$ is holomorphic in the disk $t\mathbb{U}$, and along the set V it assumes values in the line $\{1 + is : s \in \mathbb{R}\}$, so the reflection principle implies that F extends to be holomorphic in $\mathbb{C} \setminus (C_t \setminus V)$. The continued function, again denoted by F, is holomorphic on $f(X)$. The composition $F \circ f$ is therefore an element of $\mathscr{P}(X)$.

The measure μ is a representing measure for the point x_o, so

$$F(0) = F(f(x_o)) = \int F \circ f\, d\mu.$$

But also,

$$F(0) = u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(te^{i\vartheta})\, d\vartheta < \frac{1}{2\pi t}\{\Lambda^1(C_t \cap f(X)) + \varepsilon\}.$$

The function F satisfies the functional equation $F(z) = 2 - \overline{F(\frac{t^2}{\bar{z}})}$ because of the reflection process used to extend it beyond the disk $t\mathbb{U}$. We have that $\Re F > 0$ on the disk $t\mathbb{U}$. Because $\Re F \circ f \geq 1$ on $\{x \in X : |f(x)| \geq t\}$, we therefore reach

$$\int \Re F \circ f\, d\mu \geq \mu\{x \in X : |f(x)| \geq t\}.$$

We now have that for every $\varepsilon > 0$,

$$\mu\{x \in X : |f(x)| \geq t\} \leq \frac{1}{2\pi t}\{\Lambda^1(C_t \cap f(X)) + \varepsilon\},$$

which implies the result.

A useful corollary of the preceding theorem depends on an integration by parts process from abstract measure theory. Its formulation requires the notion of the distribution function of a measurable function.

Definition 5.1.6. *If μ is a σ-finite measure defined on a σ-field $\mathscr{F}$ of subsets of a set X, and if f is a nonnegative $\mathscr{F}$-measurable function on X, then the function M defined on $[0, \infty]$ defined by $M(t) = \mu\{x \in X : f(x) \geq t\}$ is the* distribution function *of f.*

In the context of this definition, one has the integration formula that for every real-valued increasing function φ of class $\mathscr{C}^1$ on $[0, \infty)$ with $\varphi(0) = 0$,

$$\int_X \varphi \circ f \, d\mu = \int_0^\infty \varphi'(t) M(t) \, dt. \tag{5.3}$$

This is a standard result in integration theory; its proof is given in [171, p. 421].

We apply this integration by parts formula as follows.

Corollary 5.1.7. *With X, μ, and f as in the preceding theorem,*

$$\pi \int |f|^2 \, d\mu \leq \mathscr{L}(f(X)).$$

Proof. Theorem 5.1.5 says that the distribution function M of $|f|$ satisfies

$$2\pi t M(t) \leq \Lambda^1(C_t \cap f(X)).$$

The corollary results by integrating this inequality over the interval $[0, \infty)$, invoking the formula (5.3) with $\varphi(t) = t^2$, and, finally, using Eilenberg's inequality.

A particular case of this corollary is the statement that *if f is in the disk algebra $A(\mathbb{U})$, then $\pi \int_{-\pi}^{\pi} |f(e^{i\vartheta})|^2 \, d\vartheta \leq \mathscr{L}(f(\bar{\mathbb{U}}))$*, a result of Alexander, Taylor, and Ullman [27].

To continue, we need the *projection lemma*, Lemma 5.1.9 below, which is based on some further preliminaries from real-variable theory.

Lemma 5.1.8. [19] *If f is a real-valued absolutely continuous function on an interval (a, b) in $\mathbb{R}$, and if K is a compact subset of (a, b), then $f' = 0$ almost everywhere on K if the set $f(K)$ has measure zero.*

Proof. Assume not. By shrinking K if necessary, and perhaps replacing f by $-f$, we can suppose that f' exists and is positive at every point of K. Let χ_K be the characteristic function of the set K. Almost every point of K is a Lebesgue point for both $|f'|$ and $f'\chi_K$; fix such a point, say c. For $\varepsilon > 0$, let J_ε denote the interval $(c - \varepsilon, c + \varepsilon)$. As $\varepsilon \to 0$, the quantity $Q(\varepsilon)$ satisfies

$$\begin{aligned} Q(\varepsilon) &= \frac{1}{2\varepsilon} \int_{J_\varepsilon \setminus K} |f'(t)| \, dt \\ &= \frac{1}{2\varepsilon} \int_{J_\varepsilon} |f'| \, dt - \frac{1}{2\varepsilon} \int_K |f'| \chi_K \, dt \\ &\to f'(c) - f'(c) = 0. \end{aligned}$$

Write $f(t) = f(c) + f'(c)(t-c)(1+\rho(t))$ with $\rho(t) = o(t-c)$ as $t \to c$. Choose $\varepsilon > 0$ small enough that $|\rho(t)| < \frac{1}{3}$ if $|t| \le \varepsilon$. Then

$$\begin{aligned} f(J_\varepsilon) &\supset (f(c-\varepsilon), f(c+\varepsilon)) \\ &\supset \left(f(c) - \varepsilon f'(c)\left(1 - \frac{1}{3}\right), f(c) + \varepsilon f'(c)\left(1 + \frac{1}{3}\right)\right), \end{aligned}$$

so that the length of the interval $f(J_\varepsilon)$ is at least $2\varepsilon f'(c)\big(1-\frac{1}{3}\big)$. Also, the length of the set $f(J_\varepsilon \setminus K)$ is bounded by $\int_{J_\varepsilon \setminus K} |f'(t)|\, dt$, which is not more than $2\varepsilon Q(\varepsilon)$. However, $f(K \cap J_\varepsilon) \supset f(J_\varepsilon) \setminus f(J_\varepsilon \setminus K)$, so the length of $f(K \cap J_\varepsilon)$ is at least the measure of $f(J_\varepsilon)$ minus the measure of $f(J_\varepsilon \setminus K)$. This difference is bounded below by $\frac{4\varepsilon f'(c)}{3} - 2\varepsilon Q(\varepsilon) = 2\varepsilon\big(\frac{2f'(c)}{3} - Q(\varepsilon)\big)$. The quantity $Q(\varepsilon)$ is small when ε is small, so if ε is small enough that $Q(\varepsilon) < \frac{f'(c)}{3}$, we have that the measure of $f(K \cap J_\varepsilon)$ is at least $\frac{2\varepsilon}{3} f'(c)$, which is greater than 0. This contradicts the assumption that $f(K)$ has measure zero. The lemma is proved.

Besicovitch [51, 52] has constructed sets of positive length in the plane that project onto every line in the plane as a set of measure zero. Alexander's *projection lemma* implies that such sets are not contained in continua of finite length:

Lemma 5.1.9. [19] *If K is a connected set of finite length in the plane, then for every compact subset E of K of positive length, the orthogonal projection of E onto one of the coordinate axes is a set of positive measure.*

Proof. Let π_1 and π_2 denote the orthogonal projections of the plane onto the coordinate axes. The set K is connected, so Theorem 3.3.5 implies that there is a rectifiable arc in K that contains a compact subset of E of positive length. That is to say, it suffices to suppose K to be a rectifiable arc. If $\varphi = (\varphi_1, \varphi_2) : [0, L] \to \mathbb{R}^2$ is a parameterization of K by arc length, then $|\varphi(t) - \varphi(t')| = |t - t'|$ for all $t, t' \in [0, L]$. Thus φ' exists and has modulus one almost everywhere. Moreover, each of the coordinate functions φ_1 and φ_2 is absolutely continuous.

The set $E' = \varphi^{-1}(E)$ is a compact subset of $[0, L]$ of positive measure.

If the projections of the set E on both coordinate axes are of measure zero, then the sets $\varphi_1(E')$ and $\varphi_2(E')$ are both sets of zero length. The preceding lemma implies then that φ_1' and φ_2' both vanish almost everywhere on E'. But then φ' vanishes almost everywhere on E', which contradicts $|\varphi'| = 1$ almost everywhere on $[0, L]$. The lemma is proved.

The next result is a *quasi-isoperimetric inequality* for arbitrary polynomially convex sets that was found by Alexander [23].

By a *unitary coordinate system* for $\mathbb{C}^N$ we shall understand a set of $\mathbb{C}$-linear functionals on $\mathbb{C}^N$ that constitute an orthogonal basis for the dual space $\mathbb{C}^{N*}$ when that space is given the Hermitian inner product dual to the standard inner product on $\mathbb{C}^N$.

By duality, the choice of a unitary coordinate system on $\mathbb{C}^N$ is equivalent to the choice of a *unitary frame*, i.e., to the choice of an N-tuple $\{v_1, \dots, v_N\}$ of mutually orthogonal unit vectors in $\mathbb{C}^N$. The collection of all such unitary frames is denoted by $\mathbb{V}_N(\mathbb{C})$, or, because we will deal only with complex vector spaces, by $\mathbb{V}_N$. There is a natural transitive action of the unitary group $U(N)$ on $\mathbb{V}_N$; $\mathbb{V}_N$ is a homogeneous space of $U(N)$. This action is given by the condition that if $\gamma \in U(N)$ and $v = (v_1, \dots, v_N) \in \mathbb{V}_N$, then $\gamma v = (\gamma v_1, \dots, \gamma v_N)$. If we fix an element v_o of $\mathbb{V}_N$, there is a map $\eta : U(N) \to \mathbb{V}_N$ given by $\eta(\gamma) = \gamma v_o$. In a natural way, $\mathbb{V}_N$ has the structure of a real-analytic manifold; this structure will not be necessary for what follows. It will be important, though, to notice that $\mathbb{V}_N$ is a closed subset of the N-fold product of the unit sphere $\mathbb{S}^{2N-1}$ in $\mathbb{C}^N$ with itself and so is a compact space. The map η is continuous. A measure β is defined in $\mathbb{V}_N$ by the condition that for each Borel set $E \subset \mathbb{V}_N$, $\beta(E) = \mu(\eta^{-1}(E))$, in which μ denotes the normalized Haar measure on $U(N)$. The measure β is a positive measure of total mass one and is invariant under the action of $U(N)$ described above.

There is a map $\eta' : \mathbb{V}_N \to \mathbb{S}^{2N-1}$ defined by $\eta'((v_1, \dots, v_N)) = v_N$. If σ denotes the unitarily invariant positive measure of total mass one on $\mathbb{S}^{2N-1}$, then for a Borel subset F of $\mathbb{S}^{2N-1}$, $\sigma(F) = \beta({\eta'}^{-1}(F))$. A subset E of $\mathbb{V}_N$ has measure zero with respect to β exactly when the image set $\eta'(E)$ has measure zero with respect to σ.

Denote by J the element of $U(N)$ that multiplies each vector in $\mathbb{C}^N$ by i.

Lemma 5.1.10. *If E is a subset of $\mathbb{S}^{2N-1}$ of measure zero, then for almost every $v = \{v_1, \dots, v_N\} \in \mathbb{V}_N$, $v_k \notin E$ and $Jv_k \notin E$ for all $k = 1, \dots, N$.*

Proof. What is to be shown is that if A_k denotes the set of $v \in \mathbb{V}_N$ for which $v_k \in E$, then $\beta(A_k) = 0$, and that if A'_k is the set of $v \in \mathbb{V}_N$ for which $v_k \in J^{-1}E$, then also $\beta(A'_k) = 0$. It is enough to show that $\beta(A_N) = 0$. That this quantity vanishes is evident, for because $A_N = {\eta'}^{-1}(E)$, it follows that $0 = \sigma(E) = \beta({\eta'}^{-1}(E)) = \beta(A_N)$.

Theorem 5.1.11. *If X is a compact subset of $\mathbb{C}^N$, then for almost all unitary coordinate systems $\{\phi_1, \dots, \phi_N\}$ on $\mathbb{C}^N$,*

$$4\pi \sum_{k=1}^{N} \Lambda^2(\phi_k(\widehat{X})) \le (\Lambda^1(X))^2.$$

The case of the unit circle in $\mathbb{C}$ shows the constant to be best possible.

For the proof we require a lemma about projections in $\mathbb{R}^n$. For this, fix a collection $L_1, \dots, L_r$ of mutually orthogonal subspaces of $\mathbb{R}^n$ of dimensions $d_1, \dots, d_r$ with $d_1 + \cdots + d_r \le n$. Let R be the orthogonal complement in $\mathbb{R}^n$ of the sum of the L_j's, so that there is a direct sum decomposition $\mathbb{R}^n = L_1 \oplus \cdots \oplus L_r \oplus R$. For $j = 1, \dots, r$, let η_j be the orthogonal projection from $\mathbb{R}^n$ onto L_j, and let η_{r+1} be the orthogonal projection onto R.

Lemma 5.1.12. *If E is a $(\Lambda^1, 1)$-rectifiable subset of $\mathbb{R}^n$, then*

$$\sum_{j=1}^{r} [\Lambda^1(\eta_j(E))]^2 \le [\Lambda^1(E)]^2.$$

Proof. Assume first that the set E is contained in a rectifiable arc, γ. Let $g : [0, L] \to \gamma$ be the arc-length parameterization of γ. Write $g = (g_1, \dots, g_r, g_{r+1})$, where for each j, $g_j = \eta_j \circ g$. Let the set $S \subset [0, L]$ correspond to the set E under the parameterization g.

Then $\Lambda^1(E) = \int_S |g'(t)|dt$. For each j we have $\int_S |g'_j(t)|dt \geq \Lambda^1(\eta_j(E))$. The Cauchy–Bunyakowski–Schwarz inequality yields that for every vector $c = (c_1, \dots, c_r) \in \mathbb{R}^r$,

$$\sum_{j=1}^{r} c_j \int_S |g'_j(t)|\, dt \leq \Bigl(\sum_{j=1}^{r} |c_j|^2\Bigr)^{1/2} \Bigl(\sum_{j=1}^{r} \Bigl(\int_S |g'_j(t)|\, dt\Bigr)^2\Bigr)^{1/2} \leq \|c\|\Lambda^1(E).$$

The inequality holds for all choices of the vector $c \in \mathbb{R}^n$, so by duality,

$$\left\{\sum_{j=1}^{r}\left\{\int_S |g'_j(t)|\, dt\right\}^2\right\}^{1/2} \leq \Lambda^1(E).$$

The last inequality implies the statement of the lemma, for

$$\int_S |g'_j(t)|\, dt \geq \Lambda^1(\eta_j(E)).$$

Thus, we have the lemma when the set E is contained in a rectifiable arc. However, this implies the general case immediately, for in the general case, the hypothesis that E is $(\Lambda^1, 1)$-rectifiable implies that almost all $[d\Lambda^1]$ of E is contained in the union of a countable family of rectifiable arcs. Thus for every $\varepsilon > 0$, there is a subset E_ε of E such that $\Lambda^1(E \setminus E_\varepsilon) < \varepsilon$ and E_ε is contained in a rectifiable arc. Consequently, the inequality of the lemma holds for the set E_ε, and the inequality necessarily holds for the set E itself.

Much more general results of this kind are known. See [24] and [115].

Proof of Theorem 5.1.11. The only case of interest is that in which $\Lambda^1(X)$ is finite, so we assume this condition from here on.

According to the structure theorem, Theorem 4.2.4, there is a representation $X = R \cup P$ in which R is $(\Lambda^1, 1)$-rectifiable, and P is purely unrectifiable, so that the orthogonal projection of P onto almost every real line in $\mathbb{C}^N$ has length zero.

The set P is purely unrectifiable, so Lemma 5.1.10 implies that for almost all unitary N-frames $v = \{v_1, \dots, v_N\}$ in $\mathbb{C}^N$, if $\phi_k \in \mathbb{C}^{N*}$ is given by $\phi_k(z) = \langle z, v_k \rangle$ with $\langle\, , \rangle$ the standard Hermitian product on $\mathbb{C}^N$, then for each k, $\Re\phi_k$ and $\Im\phi_k$ take P to sets of measure zero in the real line. We shall show that the conclusion of the theorem holds for such a choice of unitary coordinate system.

The main step in the proof is contained in the following lemma.

Lemma 5.1.13. *If Z is a component of $\widehat{X}$ and if $Y = Z \cap X$, then*

$$4\pi \sum_{k=1}^{N} \Lambda^2(\phi_k(Z)) \leq [\Lambda^1(Y)]^2.$$

Proof. That the set Z is connected implies that $\widehat{\phi_k(Z)}$ is connected, whence $\mathbb{C} \setminus \widehat{\phi_k(Z)}$ is connected, and, finally, $b\widehat{\phi_k(Z)}$ is connected.

We have that $Y = Z \cap X$, so Corollary 1.5.6 implies that $Z = \widehat{Y}$. Thus $b\widehat{\phi_k(Z)} \subset \phi_k(Y)$.

Now

$$(5.4) \qquad b\widehat{\phi_k(Z)} = b\widehat{\phi_k(Z)}) \cap \phi_k(Y) \subset [b\widehat{\phi_k(Z)}) \cap \phi_k(Y \cap R)] \cup [b\widehat{\phi_k(Z)}) \cap \phi_k(Y \cap P)].$$

The projections of $\phi_k(Y \cap P)$ to the x- and y-axes both have measure zero by the choice of the unitary coordinate system we are working with. Consequently, by the projection lemma, Lemma 5.1.9, $\Lambda^1(\phi_k(Y \cap P)) = 0$. Thus, $\Lambda^1(b\widehat{\phi_k(Z)}) \leq \Lambda^1(\phi_k(Y \cap R))$. The set $Y \cap R$ is countably $(\Lambda^1, 1)$-rectifiable, because the set R is. Lemma 5.1.12 implies, with $m = 1$ and with the orthogonal decomposition of $\mathbb{C}^N$ induced by the ϕ's, that

$$[\Lambda^1(Y)]^2 \geq [\Lambda^1(Y \cap R)]^2 \geq \sum_{k=1}^{N} [\Lambda^1(\phi_k(Y \cap R))]^2.$$

The classical isoperimetric inequality, inequality (5.1), gives

$$[\Lambda^1(b\widehat{\phi_k(Z)})]^2 \geq 4\pi\,\Lambda^2(\phi_k(Z)),$$

so, as desired,

$$[\Lambda^1(Y)]^2 \geq \sum_{k=1}^{N} \Lambda^1(b\widehat{\phi_k(Z)}) \geq 4\pi \sum_{k=1}^{N} \Lambda^2(\phi_k(Z)).$$

The proof of the theorem concludes in the following way. Let $\{Z_\alpha\}_{\alpha \in A}$ be the collection of components of $\widehat{X}$, and for each α, let $Y_\alpha = Z_\alpha \cap X$. Denote by A' the countable subset of A that consists of those α for which $\Lambda^1(Y_\alpha) > 0$. Plainly $\sum_{\alpha \in A'} \Lambda^1(Y_\alpha) \leq \Lambda^1(X)$.

For each $\alpha \in A'$ we have

$$4\pi \sum_{k=1}^{N} \Lambda^2(\phi_k(Z_\alpha)) \leq [\Lambda^1(Y_\alpha)]^2.$$

Sum over $\alpha \in A'$ and interchange the order of summation to find that

$$4\pi \sum_{k=1}^{N} \sum_{\alpha \in A'} \Lambda^2(\phi_k(Z_\alpha)) \leq \sum_{\alpha \in A'} [\Lambda^1(Y_\alpha)]^2.$$

With $Z = \cup_{\alpha \in A \setminus A'} Z_\alpha$, we have

$$\widehat{X} = Z \cup \cup_{\alpha \in A'} Z_\alpha.$$

The set Z has two-dimensional measure zero. This is so, for if $\alpha \in A \setminus A'$, then $\Lambda^1(Y_\alpha) = 0$, so by Theorem 1.6.2 the set Y_α is polynomially convex. This implies that $Z_\alpha \subset X$, whence $Z \subset X$. Necessarily $\Lambda^2(Z) = 0$, because $\Lambda^1(X) < \infty$.

Thus for each $k = 1, \ldots, N$,

$$4\pi\,\Lambda^2(\phi_k(\widehat{X})) \leq \sum_{\alpha\in A'} \Lambda^2(\phi_k(Z_\alpha)).$$

Sum over k to get

$$(5.5) \qquad 4\pi \sum_{k=1}^{N} \Lambda^2(\phi_k(\widehat{X})) \leq \sum_{k=1}^{N}\sum_{\alpha\in A'} \Lambda^2(\phi_k(Z_\alpha)) \leq [\Lambda^1(X)]^2.$$

The theorem is proved.

The next theorem was established independently by several mathematicians. See [24], [126], [215], and [280].[1]

Theorem 5.1.14. *If E is a compact subset of the sphere $b\mathbb{B}_N(r)$ such that $0 \in \widehat{E}$, then $\Lambda^1(E) \geq 2\pi r$.*

The constant is best possible, as shown by the example of a slice of the ball $\mathbb{B}_N$ by a complex line that passes through the origin.

In [280] Poletsky considers the corresponding problem in which the ball is replaced by a cube.

Proof. [24] It suffices to treat the case that $r = 1$. Fix a compact subset X of $b\mathbb{B}_N$ with the property that $\widehat{X}$ contains the origin. Without loss of generality we suppose the standard unitary coordinate system to be one for which the inequality of Theorem 5.1.11 is correct. Let μ be a representing measure for the origin with support contained in the set X. Corollary 5.1.7 implies that for $j = 1, \ldots, N$,

$$\pi \int |z_j|^2\, d\mu(z) \;\leq\; \Lambda^2(\pi_j(\widehat{X})).$$

Sum on j, note that $\sum_{j=1}^N |z_j|^2 = 1$ on the support of μ, and invoke Theorem 5.1.11 to obtain the desired inequality.

There is an alternative proof of Theorem 5.1.14 that stems from [126] and [215]. It depends on some integral-geometric considerations. We will not derive the required result from integral geometry, so to that extent the following discussion remains incomplete. This proof is based on a simple lemma.

Lemma 5.1.15. *Almost every real hyperplane through the origin meets E in at least two points.*

Proof. The length of S is finite, so almost every *complex* hyperplane through the origin of $\mathbb{C}^N$ misses S, as follows from the theorem of Shiffman, Theorem 3.3.10. Denote by Σ the collection of all such complex hyperplanes, a subset of the Grassmannian $\mathbb{G}_{N,N-1}(\mathbb{C})$.

[1] It is a commonplace that mathematical discoveries frequently are made independently and simultaneously by more than one person. Four independent simultaneous discoveries is an extreme case, however. The explanation in the present case is that these four solutions were offered at about the same time for a problem proposed by the author of this monograph in a web-based list of problems in several complex variables.

Denote by $\tilde{\Sigma}$ the collection of all real hyperplanes in $\mathbb{C}^N$ through the origin such that $L \cap iL$ is in Σ. The set $\tilde{\Sigma}$ is of full measure in the real Grassmannian $\mathbb{G}_{2N,2N-1}(\mathbb{R})$ of real hyperplanes through the origin, as is seen by considering the map $\mathbb{G}_{2N,2N-1}(\mathbb{R}) \to \mathbb{G}_{N,N-1}(\mathbb{C})$ given by $L \mapsto L \cap iL$. This is a smooth surjective map of constant rank; Fubini's theorem implies that $\tilde{\Sigma}$ is of full measure because the set Σ is of full measure in $\mathbb{G}_{N,N-1}(\mathbb{C})$.

Suppose now that the real hyperplane L belongs to $\tilde{\Sigma}$ and meets E in only one point. Let L_o be the *complex* hyperplane through the origin that is contained in L. Thus L_o divides L into two disjoint half-spaces, only one of which meets E. We can therefore move the complex hyperplane L_o continuously to infinity through the half-space of L that contains no point of $\widehat{E}$. This contradicts Oka's characterization of the polynomially convex hull, given that $0 \in \widehat{E}$.

Second proof of Theorem 5.1.14. The integral-geometric result we need is this: Denote by $\mathbb{O}(n+1)$ the group of real orthogonal transformations acting on $\mathbb{R}^{n+1}$. These transformations leave the unit sphere $\mathbb{S}^n$ invariant. Let dg denote the Haar measure on $\mathbb{O}(n+1)$ normalized so that the volume of $\mathbb{O}(n+1)$ is one. *If γ is a rectifiable arc contained in $\mathbb{S}^n$, then for an explicitly given constant c_n, which depends only on n,*

$$\Lambda^1(\gamma) = c_n \int_{\mathbb{O}(n+1)} \Lambda^0(\gamma \cap g(\Sigma))\, dg$$

if Σ denotes the $(n-1)$-sphere $\mathbb{S}^n \cap \{x_{n+1} = 0\}$. The same result holds if γ is a $(\Lambda^1, 1)$-rectifiable set that is measurable. In particular, it is correct when γ is a compact $(\Lambda^1, 1)$-rectifiable set. This Crofton-like formula can be found, in a much more general form, in [115, Theorem 3.2.48]. It is also contained in generality sufficient to treat the case that X is a rectifiable simple closed curve in the more accessible work of Santalò [317].

If we apply this integral-geometric result to the set E, we obtain Theorem 5.1.14 as follows. Assume that E is minimal with respect to the property that $0 \in \widehat{E}$. Then by Theorem 4.3.7, E is $(\Lambda^1, 1)$-rectifiable. With S now the $(2N-2)$-sphere $b\mathbb{B}_N \cap \{y_N = 0\}$, we have for each $g \in \mathbb{O}(2N)$ that $\Lambda^0(E \cap g(S)) \geq 2$, so the formula just given yields $\Lambda^1(E) \geq 2c_{2N}$. We have to evaluate the constant c_{2N}. This can be done by choosing for the set E the intersection of $b\mathbb{B}_N$ with the z_1-axis. In this case, $\Lambda^1(E) = 2\pi$, and for almost all $g \in \mathbb{O}(2N)$, the number $\Lambda^0(E \cap g(S))$ equals 2. Consequently, $c_{2N} = \pi$, and the proof is complete.

Corollary 5.1.16. *If E is a compact subset of $b\mathbb{B}_N(r)$ with $0 \in \widehat{E}$, then $\Lambda^2(\widehat{E} \cap \mathbb{B}_N(r)) \geq \pi r^2$.*

Proof. By the local maximum principle, if $s \in (0, r)$, then $0 \in [b\mathbb{B}_N(s) \cap \widehat{E}]\widehat{\ }$, which implies that $\Lambda^1(b\mathbb{B}_N(s) \cap \widehat{E}) \geq 2\pi s$. The result now follows from Eilenberg's inequality, Theorem 3.3.6.

In particular, if V is a one-dimensional variety in $\mathbb{B}_N(r)$ that passes through the origin, then $\Lambda^2(V) \geq \pi r^2$.

This last result goes back, for nonsingular V, to Rutishauser [313]. See also the paper of Bishop [60].

We conclude this section with an estimate for hulls due to Sibony [332]. In connection with Sibony's result, it is convenient to use the notation that if E is a compact subset of $\mathbb{C}^N$ and z is a point of $\mathbb{C}^N$, then $\chi(z, E) = \max\{|z - e| : e \in E\}$.

Theorem 5.1.17. *Let E be a compact subset of $\mathbb{C}^N$ with $0 \notin E$. If there is a Jensen measure for the origin with E as its support, then $\Lambda^1(E) \geq 2\chi(0, E)$.*

Explicitly, the hypothesis is that every relatively open subset of E has positive measure with respect to the Jensen measure.

Applied to the case of a set $E \subset b\mathbb{B}_N$ with $0 \in \widehat{E}$, Sibony's result gives that $\Lambda^1(E) \geq 2$. This bound is not sharp; see Theorem 5.1.14 above. A simple example shows the constant in Sibony's theorem to be best possible in general: Let T be the triangle in $\mathbb{C}$ with vertices at the points i, $\delta - i\delta$, and $-\delta - i\delta$ for a small positive δ. If ν is harmonic measure on bT for the origin, then ν is a Jensen measure for the origin the support of which has length only a little more than 2 when δ is small. In this case $\chi(0, bT) = 1$.

Proof. Fix a Jensen measure μ for the origin that has E as its support.

We suppose $\Lambda^1(E) < \infty$. By Theorem 3.3.10, almost every complex hyperplane through the origin of $\mathbb{C}^N$ misses E. Fix such a hyperplane, say L. Let $L^\perp$ be the orthogonal complement of L in $\mathbb{C}^N$, and let $\pi : \mathbb{C}^N \to L^\perp$ be the orthogonal projection.

Let ν denote the measure $\pi_*\mu$, so that ν is the measure on $L^\perp$ defined by $\int g\, d\nu = \int g \circ \pi\, d\mu$ for all continuous functions on $L^\perp$. This is a Jensen measure for the origin in $L^\perp$ that has $\pi(E)$ as its support. Let $a \in L^\perp$ be a point with $|a| = \sup\{|\pi(x)| : x \in E\}$. We identify $L^\perp$ with $\mathbb{C}$ by identifying the point $\zeta \in \mathbb{C}$ with the point ζa.

Let ℓ be the real line in $L^\perp$ through the origin and the point a, and for x in the segment $[0, a]$, let $\ell_x^\perp$ be the real line in $L^\perp$ through the point x and orthogonal to ℓ.

For each x in the *open* segment $(0, a)$ the line $\ell_x^\perp$ meets $\widehat{\pi(E)}$ in at least two points. To see this, suppose $\ell_x^\perp$ to meet $\widehat{\pi(E)}$ in only one point, say αa, $\alpha \in \mathbb{C}$. Define the function g on $\widehat{\pi(E)}$ by the condition that g be zero on the part of $\widehat{\pi(E)}$ on one side (in $L^\perp$) of $\ell_x^\perp$ and that $g(\zeta a)$ be $\zeta - \alpha$ when ζa is in $\widehat{\pi(E)}$ but on the other side of $\ell_x^\perp$. This function lies in the algebra $\mathscr{P}(\widehat{\pi(E)})$ by Mergelyan's theorem. Its value at the origin is $-\alpha$. The measure ν is a Jensen measure for the origin supported by $\pi(E)$, so we have

$$-\infty < \log|\alpha| \leq \int \log|g|\, d\nu,$$

which is impossible, for g vanishes on an open subset of $\pi(E)$, and $\pi(E)$ is the support of ν, so that this open set has positive measure with respect to ν.

Let η be the orthogonal projection of L onto ℓ. Thus, for each $x \in (0, a)$ we have $\Lambda^0((\eta \circ \pi)^{-1}(x)) \geq 2$. Theorem 3.3.6 yields

$$2|a| \leq \int_{[0,a]} \Lambda^0((\eta \circ \pi)^{-1}(x))\, dx \leq \Lambda^1(E).$$

Thus, as desired, $\Lambda^1(E) \geq 2\chi(0, E)$.

5.2. Removable Singularities

In this section we bring the theory of polynomial convexity to bear on the study of removable singularities. For holomorphic functions, the general question can be phrased in this way: *If* Ω *is a domain in* $\mathbb{C}^N$*, and if* X *is a closed subset of* Ω*, what conditions on* X *guarantee that every function holomorphic on* $\Omega \setminus X$ *continues holomorphically into all of* Ω*?* In the plane, every domain is a domain of holomorphy, so the question is without interest there. In $\mathbb{C}^N$ with $N \geq 2$, Hartogs's theorem ensures that whenever X is compact and $\Omega \setminus X$ is connected, each $f \in \mathscr{O}(\Omega \setminus X)$ extends holomorphically through X. Thus, the interest in the problem lies in the case of closed sets X without compact components.

There is a boundary version of this problem that concerns the boundary values of holomorphic functions. For its treatment some preliminaries on CR-functions are required; we begin with them. We then formulate and study the notion of removable singularity for CR-functions. The section ends with some results about removable singularities for holomorphic functions.

Consider to begin with an open set Ω in $\mathbb{C}^N$, $N \geq 2$, and in Ω an orientable real hypersurface Σ of class $\mathscr{C}^1$.

Definition 5.2.1. *The function* f *of class* $\mathscr{C}^1$ *on the surface* Σ *satisfies the* tangential Cauchy–Riemann equations*, in symbols* $\bar\partial_\Sigma f = 0$*, if* $Xf = 0$ *for every vector field* X *defined and of class* $\mathscr{C}^1$ *on* Ω *that is tangent to* Σ *and that is of the form* $X = \sum_{j=1}^N \alpha_j(z)\frac{\partial}{\partial \bar z_j}$.

When Σ is the boundary of a domain, we will generally write $\bar\partial_b$ rather than $\bar\partial_\Sigma$.

The tangency condition in this definition is that $Xh = 0$ for each smooth function h that vanishes on Σ. Vector fields of the form $X = \sum_{j=1}^N \alpha_j(z)\frac{\partial}{\partial \bar z_j}$ are said to be of *type* (0,1).

In the context of the preceding definition, it is easy to give reformulations of the condition that a smooth function satisfy the tangential Cauchy–Riemann equations. We give two: one a differential condition, one an integral condition.

Theorem 5.2.2. *With* Σ *and* f *as in Definition* 5.2.1*, each of the following conditions is equivalent to the condition that* $\bar\partial_\Sigma f = 0$:

(a) $df \wedge \iota^*\omega(z) = 0$ *with* $\omega(z)$ *the form* $dz_1 \wedge \cdots \wedge dz_N$*, and with* ι^* *the map on forms induced by the inclusion of* Σ *in* $\mathbb{C}^N$.

(b) $\int_\Sigma f\bar\partial\alpha = 0$ *for every smooth* $(N, N-2)$*-form* α *on* $\mathbb{C}^N$ *whose support meets* Σ *in a compact set.*

Both of these are local conditions.

Proof. The equivalence of the conditions (a) and (b) is immediate: For every smooth $(N-2)$-form ϑ whose support meets Σ in a compact set, we have

$$0 = \int_\Sigma f\bar\partial(\omega(z) \wedge \vartheta) = \int_\Sigma df \wedge \omega(z) \wedge \vartheta,$$

which implies that (a) and (b) are equivalent.

Next, suppose $\bar\partial_\Sigma f$ to vanish. Fix a point p in Σ and choose holomorphic coordinates on $\mathbb{C}^N$ with respect to which p is the origin and such that the tangent plane to Σ is the real hyperplane with equation $y_N = 0$. At 0 we have

$$df = \sum_{j=1,\dots,N-1} a_j\, dz_j + \sum_{j=1,\dots,N-1} b_j\, d\bar z_j + c\, dx_N$$

for suitable constants a_j, b_j, and c. For $k = 1, \dots, N-1$, the vector $\frac{\partial}{\partial \bar z_k}$ is tangent to Σ at 0, so each of the b's must be zero. The form $df \wedge \iota^*\omega(z)$ therefore vanishes at 0, so that the condition (a) is seen to hold.

Conversely, if $df \wedge \iota^*\omega(z) = 0$, then computing again at 0 in the coordinates used above, we find that

$$\iota^*\Bigg(\sum_{j=1,\dots,N-1} b_j d\bar z_j\Bigg) \wedge \iota^*\omega(z) = 0,$$

which implies the existence of a linear relation among the forms $\sum_{j=1,\dots,N-1} b_j \iota^* d\bar z_j$ and $\iota^* dz_j$, $j = 1, \dots, N-1$. However, at 0, the form $\iota^*\big(d\bar z_1 \wedge \cdots \wedge d\bar z_{N-1} \wedge dz_1 \wedge \cdots \wedge dz_{N-1} \wedge dx_N\big)$ is, to within a nonzero constant multiple, the volume form on Σ, so the 1-forms $\iota^* dz_1, \dots, \iota^* dz_{N-1}, \iota^* d\bar z_1, \dots, \iota^* d\bar z_{N-1}, \iota^* dx_N$ are linearly independent. Thus the b_j's are all zero, and this implies that $\bar\partial_\Sigma f = 0$.

The theorem is proved.

The preceding result serves to motivate the notion of *continuous* CR-function.

Definition 5.2.3. *The continuous function f on Σ satisfies the* weak tangential Cauchy–Riemann equations *on Σ, in symbols $\bar\partial_\Sigma f = 0$, if $\int_\Sigma f \bar\partial\alpha = 0$ for every smooth form α of bidegree $(N, N-2)$ on $\mathbb{C}^N$ whose support meets Σ in a compact set.*

An analogous definition can be given for functions that are only locally integrable with respect to surface area measure on Σ.

It is evident that there are some CR-functions on Σ: If F is holomorphic on a neighborhood of Σ, then $F|\Sigma$ is a CR-function. Another class of CR-functions consists of the continuous functions on Σ that can be approximated uniformly on compact subsets of Σ by holomorphic functions. In fact, because the integral condition above is local, it suffices that the approximation be possible only locally, i.e., on a neighborhood of each point of Σ. A theorem of Baouendi and Trèves [43] implies that these are the only CR-functions on Σ. An approximation theorem obtained by Chirka [83] also yields this fact.

It is a useful fact that certain restrictions of CR-functions are CR-functions:

Lemma 5.2.4. *Let Σ be a $\mathscr{C}^1$ real hypersurface in the domain Ω in $\mathbb{C}^N$, and let $\mathscr{M}$ be a complex submanifold of Ω that is transversal to Σ. If f is a continuous CR-function on Σ, then $f|(\Sigma \cap \mathscr{M})$ is a CR-function on $\Sigma \cap \mathscr{M}$.*

Proof. The problem is entirely local, and, by induction, it suffices to suppose that $\mathscr{M}$ is a complex hypersurface. Thus, we can suppose that $\mathscr{M}$ is defined by the vanishing of a single holomorphic function: $\mathscr{M} = \{z \in \Omega : F(z) = 0\}$ for some $F \in \mathscr{O}(\Omega)$ with dF zero-free along $\mathscr{M}$.

Fix a point $p_o \in \Sigma \cap \mathscr{M}$, and let μ be a smooth $(N-1, N-3)$-form on $\mathbb{C}^N$ with support contained in a small neighborhood of p_o.

Let φ be a nonnegative function on the plane with compact support, with integral one, and with the property that it is constant on a neighborhood of the origin. Define χ_ε by $\chi_\varepsilon(z) = \varepsilon^2\varphi(\frac{1}{\varepsilon}|z|)$, so that $\{\chi_\varepsilon\}_{\varepsilon>0}$ is a smooth approximate identity on the plane. By hypothesis, $F^{-1}(0)$ is transversal to Σ, so $F^{-1}(\zeta)$ is transversal to Σ near p_o when ζ is sufficiently near the origin. Then

$$\begin{aligned} 0 &= \int_\Sigma f\bar\partial\left(\mu \wedge F^*\left(\chi_\varepsilon(\zeta) \wedge \frac{i}{2} d\bar\zeta \wedge d\zeta\right)\right) \\ &= \int_{\mathbb{C}} \left\{\int_{F^{-1}(\zeta)} f\bar\partial\mu\right\} \chi_\varepsilon(\zeta)\frac{i}{2} d\bar\zeta \wedge d\zeta \\ &\to \int_{\Sigma\cap\mathscr{M}} f\bar\partial\mu. \end{aligned}$$

This is true for every choice of μ, so the restriction of f to $\Sigma \cap \mathscr{M}$ is a CR-function as claimed.

We should note that this restriction result is also an immediate consequence of the approximation theorems mentioned above.

The CR-functions arise naturally as the boundary values of holomorphic functions:

Theorem 5.2.5. *If Ω is a bounded domain in $\mathbb{C}^N$ with connected boundary of class $\mathscr{C}^1$, then the continuous function f on $b\Omega$ is of the form $F|b\Omega$ for a function $F \in A(\Omega)$ if and only if f satisfies the weak tangential Cauchy–Riemann equations on $b\Omega$.*

Recall that $A(\Omega)$ is the algebra of functions continuous on $\bar\Omega$, holomorphic in Ω.

The history of Theorem 5.2.5 is complicated, and we shall not go into it in detail. A result of this kind, in the case of real-analytic data, was published in the early 1930s by Severi [324]. For strictly pseudoconvex domains in $\mathbb{C}^2$, it is contained in the paper of Kneser [208]. The result as stated in Theorem 5.2.5 seems to have first been proved by Weinstock [363]. For surveys of the history of this subject see the papers of Range [288] and Fichera [116].

The hypothesis that the boundary is connected is necessary in the last theorem. There is, however, a result, due to Weinstock [362], for domains with disconnected boundary that is in the spirit of the theorem just stated:

Theorem 5.2.6. *If Ω is a bounded domain in $\mathbb{C}^N$ with $b\Omega$ of class $\mathscr{C}^1$, then $f \in \mathscr{C}(b\Omega)$ is of the form $F|b\Omega$ for a function $F \in A(\Omega)$ if and only if $\int_{b\Omega} f\beta = 0$ for each smooth form β of bidegree $(N, N-1)$ that is defined and $\bar\partial$-closed on a neighborhood of $\bar\Omega$.*

It is not difficult to derive Theorem 5.2.6 from Theorem 5.2.5. For this, let Ω and f be as in Theorem 5.2.6. Denote by $\Gamma_0, \ldots, \Gamma_s$ the components of $b\Omega$, indexed in such a way that each Γ_j, $j \geq 1$, is in the bounded component of $\mathbb{C}^N\setminus\Gamma_0$. Then each Γ_k, for $k \neq j$, is in the unbounded component of $\mathbb{C}^N\setminus\Gamma_j$, provided $j \neq 0$. For each j, let Ω_j denote the bounded component of $\mathbb{C}^N\setminus\Gamma_j$. The hypotheses of Theorem 5.2.6 imply that $f \mid \Gamma_j$ is a CR-function, so by Theorem 5.2.5, there is $F_j \in A(\Omega_j)$ with $F_j = f$ on Γ_j.

We will use the Bochner–Martinelli integral to show that $F_j = F_0$ on Ω_j. This is so, for if $\zeta \in \Omega_j$, and $k \neq 0, j$, then

$$\int_{\Gamma_k} f(z) k_{\mathrm{BM}}(z, \zeta) = 0,$$

which implies, because $\int_{b\Omega} f(z) k_{\mathrm{BM}}(z,\zeta) = 0$, that

$$F_0(\zeta) = c_N \int_{\Gamma_0} f(z) k_{\mathrm{BM}}(z,\zeta) = c_N \int_{\Gamma_j} f(z) k_{\mathrm{BM}}(z,\zeta) = F_j(\zeta).$$

In the planar case, $N = 1$, Theorem 5.2.6 says that if D is a bounded domain in the plane with smooth boundary, then a continuous function f on bD extends holomorphically through D if $\int_{bD} f\alpha = 0$ for every holomorphic 1-form α on a neighborhood of $\bar{D}$. This is an entirely classical result. The corresponding result for relatively compact domains D with smooth boundary in Riemann surfaces was established by Royden [305]. Formally the criterion is the same: *If D is a relatively compact domain in a Riemann surface $\mathscr{R}$ with bD of class $\mathscr{C}^1$, then $f \in \mathscr{C}(bD)$ extends holomorphically through D if and only if $\int_{bD} f\alpha = 0$ for every holomorphic 1-form α on a neighborhood of $\bar{D}$ in $\mathscr{R}$.*

Proof of Theorem 5.2.5. That the condition is necessary is quickly seen. It is sufficient to show that if $F \in A(\Omega)$, then for smooth $(N, N-2)$-forms φ on $\mathbb{C}^N$ *with small support*, $\int_{b\Omega} F\bar{\partial}\varphi = 0$. For this purpose, fix a point $p \in b\Omega$ and let U_p be a small neighborhood of p in $\mathbb{C}^N$. Let $\varphi \in \mathscr{E}^{N,N-2}(\mathbb{C}^N)$ have support a compact subset of U_p. Denote by Γ_ε the surface obtained by translating $b\Omega$ ε units along the inner normal to $b\Omega$ at p. Then $\int_{b\Omega} F\bar{\partial}\varphi = \lim_{\varepsilon\to 0^+} \int_{\Gamma_\varepsilon} F\bar{\partial}\varphi$, and for each $\varepsilon > 0$, the latter integral is zero, for F is holomorphic on a neighborhood of $\Gamma_\varepsilon \cap \operatorname{supp}\varphi$ and so is a smooth CR-function on a neighborhood in Γ_ε of $\Gamma_\varepsilon \cap \operatorname{supp}\varphi$. Thus, the necessity of the condition is proved.

The scheme for proving that a continuous CR-function f extends to an element of $A(\Omega)$ is to show that the Bochner–Martinelli integral

$$F(\zeta) = c_N \int_{b\Omega} f(z) k_{\mathrm{BM}}(z,\zeta)$$

is holomorphic on $\mathbb{C}^N \backslash b\Omega$ and vanishes on $\mathbb{C}^N \backslash \bar{\Omega}$. Granted this, a result on the jump behavior of the Bochner–Martinelli integral implies that $F \in A(\Omega)$ with $F = f$ on $b\Omega$.

To prove that F is holomorphic, define an $(N, N-2)$-form L on $\mathbb{C}^N \backslash \{z : z_N = \zeta_N\}$, for fixed ζ_N, by

$$(5.6) \qquad L = \frac{(-1)^{N-1}}{(N-1)(z_N - \zeta_N)} \frac{1}{|z-\zeta|^{2N-2}} \sum_{j=1}^{N-1} (-1)^{j-1} (\bar{z}_j - \bar{\zeta}_j) \omega_{[j,N]}(\bar{z}) \wedge \omega(z),$$

in which $\omega_{[j,N]}(\bar{z}) = d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_{N-1}$. By type, $dL = \bar{\partial} L$, and we verify by direct calculation that off the complex hyperplane $\zeta_N = z_N$,

$$dL = k_{\mathrm{BM}}(z,\zeta);$$

L is a primitive for k_{BM}. (Martinelli [239] introduced this primitive.) We now compute $\frac{\partial F}{\partial \bar{\zeta}_N}$ on $\mathbb{C}^N \backslash b\Omega$. It is

$$\frac{\partial F}{\partial \bar{\zeta}_N}(\zeta) = \frac{\partial}{\partial \bar{\zeta}_N} c_N \int_{b\Omega} f(z) k_{\mathrm{BM}}(z, \zeta) = c_N \int_{b\Omega} f(z) \frac{\partial}{\partial \bar{\zeta}_N} k_{\mathrm{BM}}(z, \zeta). \tag{5.7}$$

In the last integral, we understand by $\frac{\partial}{\partial \bar{\zeta}_N} k_{\mathrm{BM}}(z, \zeta)$ the form obtained by differentiating the coefficients of $k_{\mathrm{BM}}(z, \zeta)$ with respect to $\bar{\zeta}_N$.

Off the set where $z_N = \zeta_N$,

$$\frac{\partial}{\partial \bar{\zeta}_N} k_{\mathrm{BM}}(z, \zeta) = \frac{\partial}{\partial \bar{\zeta}_N} dL = d \frac{\partial}{\partial \bar{\zeta}_N} L = d_z \left\{ (-1)^{N-1} \frac{\omega'_{[N]}(\bar{z} - \bar{\zeta}) \wedge \omega(z)}{|z - \zeta|^{2N}} \right\} \tag{5.8}$$

if we write $\omega'_{[N]}(\bar{z} - \bar{\zeta})$ for the form $\sum_{j=1}^{N-1} (-1)^{j-1} (\bar{z}_j - \bar{\zeta}_j) d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_{N-1}$. The equation (5.8) has been derived under the assumption that $z_N \neq \zeta_N$, but it is meaningful and correct on $\mathbb{C}^N \setminus \{\zeta\}$. Insert (5.8) into (5.7) to get

$$\frac{\partial F}{\partial \bar{\zeta}_N}(\zeta) = c_N \int_{b\Omega} f(z)\, d \left\{ \frac{(-1)^{N-1} \omega'_{[N]}(\bar{z} - \bar{\zeta}) \wedge \omega(z)}{|z - \zeta|^{2N}} \right\}. \tag{5.9}$$

The integral vanishes, for the d can be replaced by $\bar{\partial}$ by type, and the function f is assumed to be a CR-function.

Thus, F is seen to be holomorphic on $\mathbb{C}^N \setminus b\Omega$. By Hartogs's theorem, the restriction of F to the unbounded component of $\mathbb{C}^N \setminus \bar{\Omega}$ extends holomorphically to all of $\mathbb{C}^N$. Because $F \to 0$ at infinity, F vanishes on the unbounded component of $\mathbb{C}^N \setminus \bar{\Omega}$.

We have finally to show that $F|\Omega$ assumes continuously the boundary values f. That this is so is a consequence of the following *jump formula*:

Theorem 5.2.7. *Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with boundary of class $\mathscr{C}^1$. If $f \in \mathscr{C}(b\Omega)$, and if*

$$F(\zeta) = c_N \int_{b\Omega} f(z) k_{\mathrm{BM}}(z, \zeta),$$

then, with ν_ζ the outer unit normal to $b\Omega$ at $\zeta \in b\Omega$,

$$\lim_{s \to 0^+} [F(\zeta - s\nu_\zeta) - F(\zeta + s\nu_\zeta)] = f(\zeta)$$

uniformly in ζ.

Proof. Fix $\zeta \in b\Omega$, and fix $c \in (0, 1)$. There is an $\eta > 0$ small enough that if $z, z' \in b\Omega$ satisfy $|z - \zeta| < \eta$ and $|z' - \zeta| < \eta$, then $|\langle z - z', \nu_\zeta \rangle| < c|z - z'|$ if we denote by $\langle\, ,\rangle$ the Hermitian inner product on $\mathbb{C}^N$.

Fix $f \in \mathscr{C}(b\Omega)$, and fix an $\varepsilon > 0$. Because $c_N \int_{b\Omega} k_{\mathrm{BM}}(z,\zeta) = 1$ if $\zeta \in \Omega$ and $= 0$ if $\zeta \in \mathbb{C}^N \setminus \bar{\Omega}$, we have that for $\zeta \in b\Omega$,

(5.10)

$$\begin{aligned} &F(\zeta - s\nu_\zeta) - F(\zeta + s\nu_\zeta) - f(\zeta) \\ &\quad = c_N \int_{b\Omega} \{f(z) - f(\zeta)\}\left[\frac{\omega'(\bar{z} - (\bar{\zeta} - s\bar{\nu}_\zeta))}{|z - (\zeta - s\nu_\zeta)|^{2N}} - \frac{\omega'(\bar{z} - (\bar{\zeta} + s\bar{\nu}_\zeta))}{|z - (\zeta + s\nu_\zeta)|^{2N}}\right] \wedge \omega(z). \end{aligned}$$

If we write $\nu_\zeta = (\nu_{\zeta,1}, \ldots \nu_{\zeta,N})$, then

$$\begin{aligned} \omega'(\bar{z} - (\bar{\zeta} \pm s\bar{\nu}_\zeta))) &= \sum_{j=1}^{N} (-1)^{j-1}((\bar{z}_j - \bar{\zeta}_j) \pm s\nu_{\zeta,j}) d\bar{z}_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge d\bar{z}_N \\ &= \omega'(\bar{z} - \bar{\zeta}) \pm s \sum_{j=1}^{N} (-1)^{j-1} \bar{\nu}_{\zeta,j} d\bar{z}_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge d\bar{z}_N. \end{aligned}$$

Call the second form on the right of the last equation $s\vartheta_\zeta$.

Choose $\eta' \in (0, \eta)$ small enough that $|f(z) - f(z')| < \varepsilon$ when $z, z' \in b\Omega$ satisfy $|z - z'| < \eta'$.

Decompose the integral on the right of (5.10) as

$$\int_{b\Omega} = \int_{b\Omega \setminus \mathbb{B}_N(\zeta,\eta')} + \int_{b\Omega \cap \mathbb{B}_N(\zeta,\eta')} = I'(s) + I''(s).$$

It is clear that $I'(s)$ tends to 0 as $s \to 0^+$, uniformly in $\zeta \in b\Omega$; the main analysis must focus on $I''(s)$. For this, write

$$I''(s) = J'(s) + J''(s),$$

in which

$$J''(s) = c_N \int_{b\Omega \cap \mathbb{B}_N(\zeta,\eta')} \{f(z) - f(\zeta)\}\left[\frac{-s\theta_\zeta}{|z - (\zeta - s\nu_\zeta)|^{2N}} - \frac{s\theta_\zeta}{|z - (\zeta + s\nu_\zeta)|^{2N}}\right] \wedge \omega(z),$$

and

$$J'(s) = c_N \int_{b\Omega \cap \mathbb{B}_N(\zeta,\eta')} \{f(z) - f(\zeta)\}\left[\frac{\omega'(\bar{z} - \bar{\zeta})}{|z - (\zeta - s\nu_\zeta)|^{2N}} - \frac{\omega'(\bar{z} - \bar{\zeta})}{|z - (\zeta + s\nu_\zeta)|^{2N}}\right] \wedge \omega(z).$$

On $b\Omega \cap \mathbb{B}_N(\zeta, \eta')$, $|\langle z - \zeta, \nu_\zeta \rangle| < c|z - \zeta|$, so

(5.11) $$|z - \zeta \pm s\nu_\zeta|^2 > (1 - c)(|z - \zeta|^2 + s^2).$$

Consequently, the $J''(s)$ is majorized by

$$\mathrm{const}\, \varepsilon \int_{b\Omega \cap \mathbb{B}_N(\zeta,\eta')} \frac{s}{(|z - \zeta|^2 + s^2)^N}\, dS(z)$$

with a constant that is independent of η', provided η' is sufficiently small. The term $J'(s)$ is majorized by

$$\text{const}\,\varepsilon \int_{b\Omega\cap\mathbb{B}_N(\zeta,\eta')} \frac{|z-\zeta|\big||z-\zeta-s\nu_\zeta|^{2N}-|z-\zeta+s\nu_\zeta|^{2N}\big|}{|z-\zeta-s\nu_\zeta|^{2N}|z-\zeta+s\nu_\zeta|^{2N}}\,dS(z). \tag{5.12}$$

The integrand here is bounded by const $\frac{s}{(|z-\zeta|^2+s^2|)^N}$, as we see in the following way: Write

$$\begin{aligned} |z-\zeta\pm s\nu_z|^{2N} &= [\langle z-\zeta\pm s\nu_z, z-\zeta\pm s\nu_z\rangle]^N \\ &= [|z-\zeta|^2 \pm 2\Re s\langle z-\zeta,\nu_z\rangle + s^2]^N. \end{aligned} \tag{5.13}$$

Expand this by the multinomial theorem to get a sum of terms of the form

$$|z-\zeta|^{2p}(\pm s\Re\langle z-\zeta,\nu_\zeta\rangle)^q s^{2r}$$

with $0\le p,q,r$, and $p+q+r=N$. In the numerator of (5.12) the $|z-\zeta|^{2N}$ occurs with a plus and with a minus sign and so cancels. It follows that the numerator in (5.12) is a sum of finitely many terms each of which is bounded by a term of the form

$$\text{const}\,|z-\zeta|^{2p+1}(s|z-\zeta|)^q s^{2r} = \text{const}\,|z-\zeta|^{2p+q+1}s^{q+2r}$$

with $p+q+r=N$ and with $p\le N-1$. The latter inequality entails $q+r\ge 1$. It follows that $J'(s)$ is bounded by a finite sum of terms of the form

$$\begin{aligned} &\text{const}\,\varepsilon \int_{b\Omega\cap\mathbb{B}_N(\zeta,\eta')} \frac{s|z-\zeta|^{2p+q+1}s^{q+2r-1}}{(|z-\zeta|^2+s^2)^{2N}}\,dS \\ &\quad\le \text{const}\,\varepsilon \int_{b\Omega\cap\mathbb{B}_N(\zeta,\eta')} \frac{s}{(|z-\zeta|^2+s^2)^N}\,dS. \end{aligned}$$

Thus, what is to be proved is that the quantity

$$J(s) = \int_{b\Omega\cap\mathbb{B}_N(\zeta,\eta')} \frac{s}{(|z-\zeta|^2+s^2|)^N}\,dS(z)$$

is bounded uniformly in s. The quantity η' is small, so the path of integration is essentially a $(2N-1)$-ball of radius η'. Thus,

$$J(s) < \text{const} \int_{x\in\mathbb{R}^{2N-1}} \frac{s}{(|x|^2+s^2)^N}\,d\mathscr{L}(x).$$

If we replace x by sy in this integral, we find that

$$J(s) < \text{const} \int_{\mathbb{R}^{2N-1}} \frac{d\mathscr{L}(y)}{(|y|^2+1)^N} < \infty.$$

The proof is complete.

Remark. It should be emphasized that the analysis in this proof is entirely local, so there is a local statement parallel to the statement of the theorem.

For some remarks on the history of this kind of result, see Fichera [116].

At this point, we have a complete proof of Theorem 5.2.5.

Corollary 5.2.8. *Let Ω be a bounded domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^1$. If f is continuous on $b\Omega$, then with $F : \mathbb{C}^N \setminus b\Omega \to \mathbb{C}$ the Bochner–Martinelli integral of f, the function $F|\Omega$ extends continuously to $\bar{\Omega}$ if and only if $F|(\mathbb{C}^N \setminus \bar{\Omega})$ extends continuously to $\mathbb{C}^N \setminus \Omega$.*

We shall need some information on the local behavior of solutions of $\bar\partial f = u$. The setting we have to consider is this: Let Σ be an oriented real hypersurface in $\mathbb{B}_N$ of class $\mathscr{C}^1$, and let f be a continuous CR-function on Σ. Suppose that Σ divides $\mathbb{B}_N$ into two domains, Δ^+ and Δ^-. Consider a solution F of the current equation $\bar\partial F = f[\Sigma]$. (Here $[\Sigma]$ denotes the current of integration over the manifold Σ, which we take to be oriented as the boundary of the domain Δ^-.) That there is such a solution follows from the assumption that f satisfies the tangential CR-equations: If α is a smooth compactly supported $(N, N-2)$-form on $\mathbb{B}_N$, then because f is a CR-function, $\int_\Sigma f\bar\partial\alpha = 0$. This is the assertion that the current $f[\Sigma]$ is $\bar\partial$-closed, so F does indeed exist, by Theorem 2.4.17. The current F satisfies the condition that $\bar\partial F = 0$ in $\mathbb{B}_N \setminus \Sigma$. Accordingly, on $\mathbb{B}_N \setminus \Sigma$, F is a holomorphic function.[2]

We shall establish the following regularity result:

Theorem 5.2.9. *If $F = 0$ in Δ^+, then the holomorphic function $F|\Delta^-$ extends continuously to $\Delta^- \cup \Sigma$ and assumes the values f continuously along Σ.*

Proof. [83] The proof of this result depends on comparing the solution F with a particular solution U of the equation $\bar\partial S = f[\Sigma]$ that we construct explicitly enough that its behavior can be analyzed.

Assume that the hypersurface Σ is defined in the somewhat larger ball $\mathbb{B}_N(R)$ for an $R > 1$.

The construction of U is based on the solution of $\bar\partial$ given in the appendix to the present section. It is shown there, equation (5.28), that if u is a smooth $\bar\partial$-closed (0, 1)-form on a neighborhood of $\bar{\mathbb{B}}_N(R)$, then the function U defined by

$$U(z) = c_N \int_{\mathbb{B}_N(R)} u(\zeta) \wedge K(\zeta, z)$$

satisfies $\bar\partial U = u$ in $\mathbb{B}_N(R)$ if by $K(\zeta, z)$ we denote the kernel $dT(\zeta, z) - \frac{\omega'(\overline{\zeta - z})\wedge\omega(\zeta)}{|\zeta - z|^{2N}}$ with T as in the appendix to this section, so that $T(\zeta, z)$ is smooth in $(\zeta, z) \in \bar{\mathbb{B}}_N \times \mathbb{B}_N$.

We will prove that the function U defined by

$$U(z) = c_N \int_{\Sigma\cap\mathbb{B}_N} f(\zeta)K(\zeta, z) \tag{5.14}$$

[2]The assertion that F is a holomorphic function in $\mathbb{B}_N \setminus \Sigma$ is understood in this sense. We have that $\bar\partial F = f[\Sigma]$. Define the functional χ on $\mathscr{D}(\mathbb{B}_N \setminus \Sigma)$ by $\chi(\varphi) = F(\varphi\big(\frac{i}{2}\big)^N \omega(\bar z) \wedge \omega(z))$. Then because supp $\bar\partial F = \Sigma$,

$$\chi\left(\frac{\partial\varphi}{\partial\bar z_j}\right) = F\left(\frac{\partial\varphi}{\partial\bar z_j}\left(\frac{i}{2}\right)^N \omega(\bar z)\wedge\omega(z)\right) = F\left(\bar\partial\left(\left(\frac{i}{2}\right)^N (-1)^{j-1}\varphi\omega_{[j]}(\bar z)\wedge\omega(z)\right)\right) = 0.$$

Thus, the *distribution* χ satisfies $\frac{\partial\chi}{\partial\bar z_j} = 0$ on $\mathbb{B}_N \setminus \Sigma$ for all $j = 1, \ldots, N$. By the regularity theorem for distribution solutions of the Cauchy–Riemann equations, there is a holomorphic function F_o on $\mathbb{B}_N \setminus \Sigma$ with $\chi(g) = \int gF_o$ for all $g \in \mathscr{D}(\mathbb{B}_N \setminus \Sigma)$.

satisfies the current equation $\bar\partial U = f[\Sigma]$ in $\mathbb{B}_N$. Observe that U is integrable on $\mathbb{B}_N$.

Let us denote the current $f[\Sigma]$ by u; $u \in \mathscr{D}_{0,1}(\mathbb{B}_N(R))$.

Denote by u^ε the regularization of u that is constructed as follows. Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be a smooth approximate identity on $\mathbb{C}^N$ centered at the origin, so that for each ε, χ_ε is a smooth function on $\mathbb{C}^N$ with support in the ball $\mathbb{B}_N(\varepsilon)$, with integral one, and with $\chi_\varepsilon(-\zeta) = \chi_\varepsilon(\zeta)$. The action of the current u on forms $\alpha \in \mathscr{D}^{N,N-1}$ is this. There are distributions φ_j on $\mathbb{B}_N(R)$ such that $u = \sum_{j=1}^N \varphi_j d\bar z_j$ in the sense that for each $\alpha = \sum_{j=1}^N a_j\omega_{[j]}(\bar z) \wedge \omega(z) \in \mathscr{D}^{N,N-1}(\mathbb{B}_N(R))$,

$$u(\alpha) = \sum_{j=1}^N (-1)^{j-1}\varphi_j(a_j).$$

For each j, let φ_j^ε be the convolution $\varphi_j * \chi_\varepsilon$, which is a smooth function on a neighborhood of $\bar{\mathbb{B}}_N$ as soon as ε is sufficiently small. For all small $\varepsilon > 0$, the form $u^\varepsilon = \sum_{j=1}^N \varphi_j^\varepsilon d\bar z_j$ is $\bar\partial$-closed on a neighborhood of $\bar{\mathbb{B}}_N$, because convolution and differentiation commute. It follows that the equation $\bar\partial S = u^\varepsilon$ has in $\mathbb{B}_N$ the solution U^ε given by

$$U^\varepsilon(z) = c_N \int_{\mathbb{B}_N(R)} u^\varepsilon(\zeta) \wedge K(\zeta, z).$$

For an $\alpha \in \mathscr{D}^{N,N-1}(\mathbb{B}_N)$, we have that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{B}_N} u^\varepsilon(z) \wedge \alpha(z) = \int_{\Sigma\cap\mathbb{B}_N} f(z)\alpha(z). \tag{5.15}$$

We also have $u^\varepsilon(z) = \bar\partial_z \int_{\mathbb{B}_N} u^\varepsilon(\zeta) \wedge K(\zeta, z)$, so that by Stokes's theorem,

$$\begin{aligned}\int_{\mathbb{B}_N} u^\varepsilon(z) \wedge \alpha(z) &= \int_{\mathbb{B}_N} \bar\partial_z \left\{\int_{\mathbb{B}_N} u^\varepsilon(\zeta) \wedge K(\zeta, z)\right\} \wedge \alpha(z)\\ &= \int_{\mathbb{B}_N}\int_{\mathbb{B}_N} u^\varepsilon(\zeta) \wedge K(\zeta, z) \wedge \bar\partial\alpha(z)\\ &= \int_{\mathbb{B}_N} u^\varepsilon(\zeta) \int_{\mathbb{B}_N} K(\zeta, z) \wedge \bar\partial_z\alpha(z)\\ &\overset{\varepsilon\to 0^+}{\to} \int_\Sigma f(\zeta)K(\zeta, z)\bar\partial_z\alpha(z).\end{aligned} \tag{5.16}$$

The equations (5.15) and (5.16) show that, as desired, the function U defined in equation (5.14) satisfies the current equation $\bar\partial U = f[\Sigma]$.

The proof of the theorem concludes as follows. We have the given solution F of $\bar\partial F = f[\Sigma]$, which satisfies $F = 0$ on Δ^+, and we have the solution U given by equation (5.14), which satisfies the same equation. Then $\bar\partial(F - U) = 0$, so $F = U + H$ for a holomorphic function H. We also know that the jump of U across Σ is f: Recall the remark immediately following the proof of Theorem 5.2.7. We have therefore that the jump of F across Σ is f, and that F assumes continuously the boundary values f from Δ^-.

Having established some of the theory of CR-functions on hypersurfaces, we can introduce the notion of removable singularity for CR-functions.

Definition 5.2.10. *Let Ω be a domain in the complex manifold $\mathscr{M}$ such that $b\Omega = X \cup \Gamma$, where X is a closed subset of $b\Omega$ and Γ is a closed $\mathscr{C}^1$-submanifold of $\mathscr{M} \setminus X$. The set X is said to be a* removable set for CR-functions *if for each continuous CR-function on Γ there is a function F continuous on $\Gamma \cup \Omega$ and holomorphic on Ω with $F|\Gamma = f$.*

In this definition no growth condition is imposed on the function f near the exceptional set X, nor is the manifold Γ supposed to have finite area.

Convexity enters the theory of removable sets as follows.

Theorem 5.2.11. [87, 226, 348] *Let X be a compact, $\mathscr{O}(\mathscr{M})$-convex subset of the Stein manifold $\mathscr{M}$, and let Ω be a bounded domain in $\mathscr{M}$. Assume that $\Gamma = b\Omega \setminus X$ is a real hypersurface of class $\mathscr{C}^1$, that $\Omega \setminus X$ is connected, and that $\mathscr{M} \setminus (\bar{\Omega} \cup X)$ is connected. If f is a continuous CR-function on Γ, then there is a function F holomorphic on $\Omega \setminus X$ and continuous on $\Gamma \cup (\Omega \setminus X)$ that agrees on with f on Γ.*

Proof. Consider the current $f[\Gamma]$ defined on $\mathscr{M} \setminus X$ by $f[\Gamma](\beta) = \int_\Gamma f\beta$ for all $\beta \in \mathscr{D}^{N,N-1}(\mathscr{M} \setminus X)$. The condition that f be a CR-function is the condition that the current $f[\Gamma]$ be $\bar\partial$-closed. The support of this current is a bounded subset of $\mathscr{M}$. By Theorem 2.4.20, $H^{0,1}_\Phi(\mathscr{M} \setminus X) = 0$, so the result on smoothing cohomology, Theorem 2.4.16, provides a current u on $\mathscr{M} \setminus X$ with support a bounded subset of $\mathscr{M}$ such that $\bar\partial u = f[\Gamma]$. (Note that at this point, rather than invoking the general result on smoothing cohomology, it is sufficient to observe that the proofs of parts (c) and (d) of Theorem 2.4.20 apply equally well when the given data are taken to be currents of the appropriate degrees rather than smooth forms.)

We have $\bar\partial u = 0$ off Γ, so there is a holomorphic function F on $\mathscr{M} \setminus (\Gamma \cup X)$ with the property that $u(\alpha) = \int F\alpha$ for all (N, N)-forms on $\mathscr{M}$ whose support is a compact subset of $\mathscr{M} \setminus (\Gamma \cup X)$. The support of u is relatively compact in $\mathscr{M}$, so the holomorphic function F necessarily vanishes on $\mathscr{M} \setminus (\bar\Omega \cup X)$. The function $F|(\Omega \setminus X)$ is the function we seek: Theorem 5.2.9 implies that it assumes the boundary values f along Γ.

The theorem is proved.

For domains in $\mathbb{C}^N$, there is a relatively explicit formula for the extension provided by Theorem 5.2.11 in terms of the Bochner–Martinelli integral. For this, let Ω, X, and f be as in the statement of Theorem 5.2.11 with $\mathscr{M} = \mathbb{C}^N$. Let F be the holomorphic extension of f into $\Omega \setminus \widehat{X}$. Fix a point $w \in \Omega \setminus \widehat{X}$. There is a polynomial P such that $P(w) = 1$ and $\|P\|_X < 1$. For every s less than but sufficiently near one, the function F can be represented in the domain $\Delta_s = \Omega \cap \{z : |P(z)| > s\}$ by the Bochner–Martinelli formula:

$$F(w) = c_N \int_{b\Delta_s} F(z) k_{\mathrm{BM}}(z, w) \tag{5.17}$$
$$= c_N \int_{b\Delta_s \cap b\Omega} f(z) k_{\mathrm{BM}}(z, w) + c_N \int_{b\Delta_s \cap \Omega} F(z) k_{\mathrm{BM}}(z, w).$$

The Bochner–Martinelli kernel is $\bar\partial$-exact in $\mathbb{C}^N \setminus V_{P,w}$ if

$$V_{P,w} = \{z : P(z) = P(w)\}.$$

We can give an explicit $(N, N-2)$-form $\vartheta_{P,w}$ on $\mathbb{C}^N \setminus V_{P,w}$ that satisfies

$$\bar{\partial}_z \vartheta_{P,w} = k_{\mathrm{BM}}(z, w)$$

on $\mathbb{C}^N \setminus V_{P,w}$. For this, recall that Lemma 1.5.3 provides polynomials p_r, $r = 1, \ldots, N$, in two N-dimensional variables such that $P(z) - P(w) = \sum_{j=1}^{N} (z_j - w_j) p_j(z, w)$.

This decomposition gives rise to the desired $\bar{\partial}$-primitive by the following purely algebraic construction.

We have constructed above, equation (5.6), a primitive for the Bochner–Martinelli kernel. A permutation of the variables in that formula leads to the result that if ϵ_j is suitably chosen, 1 or -1 for each $j = 1, \ldots, N$, then the form

(5.18) $$M_k(z, w) = \frac{\epsilon_k}{|z - w|^{2N-2}} \sum_{\substack{j=1 \\ j \neq k}}^{N} (-1)^{j-1} (z_j - w_j) \omega_{[j,k]}(\bar{z}) \wedge \omega(z)$$

satisfies $\bar{\partial}_z M_k(z, w) = (z_k - w_k) k_{\mathrm{BM}}(z, w)$. If we multiply both sides of equation (5.18) by $p_k(z, w)$ and sum on k, we discover that if

$$\vartheta_{P,w} = \frac{1}{P(z, w)} \sum_{k=1}^{N} p_k(z, w) M_k(z, w),$$

then $\bar{\partial}_z \vartheta_{P,w} = k_{\mathrm{BM}}(z, w)$, so that $\vartheta_{P,w}$ is a $\bar{\partial}$- and d-primitive for k_{BM} in $\mathbb{C}^N \setminus V_{P,w}$.

The primitive $\vartheta_{P,w}$ leads to the sought formula, because we can use Stokes's theorem in equation (5.17) to write

$$c_N \int_{b\Delta_s \cap \Omega} F(z) k_{\mathrm{BM}}(z, w) = c_N \int_{b\Omega \cap \{z : |P(z)| = s\}} f(z) \vartheta_{P,w},$$

so that

(5.19) $$F(w) = c_N \int_{b\Delta_s} F(z) k_{\mathrm{BM}}(z, w) + c_N \int_{b\Omega \cap \{z : |P(z)| = s\}} f(z) \vartheta_{P,w}.$$

In fact, this argument requires somewhat more attention. First, since we want to integrate over the locus $\Omega \cap \{|P| = s\}$, the number s needs to be chosen to be a regular value of the function $|P|$ on Ω. Almost all values of s will satisfy this condition. More serious is the condition that $b\Omega$ and the level set $|P| = s$ meet transversely. If we impose the condition that $b\Omega \setminus X$ be a manifold of class $\mathscr{C}^2$, rather than of class $\mathscr{C}^1$, then the sharp version of Sard's theorem provides the necessary transversality condition for almost all values of s. With the hypothesis that $b\Omega \setminus X$ is merely of class $\mathscr{C}^1$, there are technical complications to be dealt with in the indicated line of argument. We will not proceed further in this direction.

A special case of Theorem 5.2.11 concerns strictly pseudoconvex domains.

Corollary 5.2.12. *If Ω is a relatively compact, strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and if $X \subset b\Omega$ is a compact set that is $\mathscr{O}(\bar{\Omega})$-convex, then X is removable for CR-functions.*

Proof. The set X is $\mathscr{O}(\bar{\Omega})$-convex, so $b\Omega \setminus X$ is connected by Corollary 2.4.10. Thus, the corollary follows from Theorem 5.2.11.

In particular, a subset of $b\mathbb{B}_N$ is removable for CR-functions if it is polynomially convex.

More generally, there is the following result.

Corollary 5.2.13. *If Ω is a relatively compact strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and if $X \subset b\Omega$ is a compact set, then every CR-function on $b\Omega \setminus X$ extends holomorphically into $\Omega \setminus \mathscr{O}(\bar{\Omega})$-hull X.*

Proof. Let Ω' be a strictly pseudoconvex domain in $\mathbb{C}^N$ that contains Ω in such a way that $\bar{\Omega}$ is $\mathscr{O}(\Omega')$-convex. If Δ is a component of $\Omega \setminus \mathscr{O}(\bar{\Omega})$-hull X, then by Corollary 2.4.9, $b\Delta \cap b\Omega$ is connected. Consequently, $\Omega \setminus (\bar{\Delta} \cup \mathscr{O}(\bar{\Omega})$-hull X is connected. Theorem 5.2.11 implies that $f|(b\Delta \cap b\Omega)$ extends holomorphically into Δ. The corollary is proved.

Corollary 5.2.14. *Let Ω be a domain in $\mathbb{C}^N$, and let $p \in b\Omega$ be a point near which $b\Omega$ is a manifold of class $\mathscr{C}^2$ that is strictly pseudoconvex. If f is a continuous CR-function defined on the part of $b\Omega$ near p, then there is a neighborhood W of p in $\mathbb{C}^N$ such that there is a holomorphic function F defined on $\Omega \cap W$ that assumes continuously the boundary values f near p.*

Proof. Let W' be a neighborhood of p in $\mathbb{C}^N$ on which there is a strictly plurisubharmonic function Q that defines $b\Omega \cap W'$ so that $Q = 0$ on $b\Omega \cap W'$, $dQ \neq 0$ there, and Q is negative on $W' \cap \Omega$. Denote by P the Levi polynomial associated with Q at the point p, so that

$$P(z) = \sum_{j=1}^{N} \frac{\partial Q}{\partial z_j}(p)(z_j - p_j) - \frac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 Q}{\partial z_j \partial z_k}(p)(z_j - p_j)(z_k - p_k).$$

The Taylor expansion of Q about the point p is

$$Q(z) = 2\Re P(z) + \mathscr{L}_Q(p; z-p) + o(|z-p|^2), \tag{5.20}$$

in which $\mathscr{L}_Q(p;\cdot)$ denotes the Levi form of Q at p. The strict pseudoconvexity of Q implies that $\mathscr{L}_Q(p; z-p) \geq \text{const}\,|z-p|^2$. Thus, for $z \in \Omega$ near p, $\Re P(z) > 0$. It follows from Theorem 5.2.11 that if we let $E_\varepsilon = \{z : \Re P(z) = \varepsilon\}$, then for small, negative values of ε, the function f continues from the component of $b\Omega \setminus E_\varepsilon$ that contains p, call it Γ_ε, into the component of $\Omega \setminus E_\varepsilon$ that abuts Γ_ε.

As a consequence of this remark, we see that *if Σ is an orientable strictly pseudoconvex hypersurface in a complex manifold, and if f is a continuous CR-function on Σ, then there is a one-sided neighborhood of Σ in the ambient manifold into which f continues as a holomorphic function. In particular, if Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and if X is a compact subset of $b\Omega$, then there is a neighborhood W of $b\Omega \setminus X$ in $\bar{\Omega} \setminus X$ such that f continues holomorphically into $W \cap \Omega$.*

To be sure, there are more-direct routes to the results of Corollary 5.2.14 and the last paragraph than the one we have followed.

It is useful to notice that the analysis above entails the following convexity observation:

Lemma 5.2.15. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$. If $p \in b\Omega$, then there are holomorphic coordinates defined on a neighborhood of p with respect to which $b\Omega$ is strictly convex near p.*

Proof. Let Q be a strictly plurisubharmonic defining function of class $\mathscr{C}^2$ for Ω. Near p, Q has the expansion (5.20). With coordinates $w_1, \ldots, w_N$ defined near p by $w_1 = z_1, \ldots, w_{N-1} = z_{N-1}, w_N = P(z)$, we have

$$Q(w) = 2\Re w_N + \mathscr{L}_Q(p; w - w(p)) + o(|w - w(p)|)^2.$$

The Levi form is positive definite, so with respect to the w-coordinates, Q is strictly convex, whence the lemma.

In general, the converse of Theorem 5.2.11 fails. There is, however, a converse, for two-dimensional strictly pseudoconvex domains.

Theorem 5.2.16. [351] *If Ω is a bounded, strictly pseudoconvex domain in $\mathbb{C}^2$ with boundary of class $\mathscr{C}^2$, the compact subset X of $b\Omega$ is removable if and only if it is convex with respect to the algebra of functions holomorphic on a neighborhood of $\bar{\Omega}$.*

The theorem applies in particular to the case of the ball:

Corollary 5.2.17. *A compact subset X of $b\mathbb{B}_2$ is a removable set if and only if it is polynomially convex.*

The proof of Theorem 5.2.16 is based on a result of Słodkowski [335]:

Theorem 5.2.18. *If Ω is a pseudoconvex domain in a two-dimensional Stein manifold $\mathscr{M}$, and if X is a compact subset of $\mathscr{M} \setminus \Omega$, then $\Omega \setminus \mathscr{O}(\mathscr{M})$-hull X is pseudoconvex.*

Proof. [301] If $\Omega \setminus \mathscr{O}(\mathscr{M})$-hull X is not pseudoconvex, then by a suitable choice of coordinates on an open set in Ω, we can suppose that $\bar{\mathbb{U}}^2 \subset \Omega$, that $\bar{\mathbb{U}}^2$ is $\mathscr{O}(\mathscr{M})$-convex, that the origin is in $\mathscr{O}(\mathscr{M})$-hull X, that for some choice of functions φ_j holomorphic on the closed unit disk $\bar{\mathbb{U}}$ each of which is bounded by $\frac{1}{2}$, the set

$$E = \overline{\{\cup_{j=1,\ldots}(\zeta_1, \varphi_j(\zeta_1)) : |\zeta_1| = 1\}}$$

is a compact subset of $\Omega \setminus \mathscr{O}(\mathscr{M})$-hull X, and that $\lim_{j=1,\ldots}(0, \varphi_j(0))$ exists and is the origin.

For large j, the function F_j defined by $F_j(\zeta_1, \zeta_2) = \frac{1}{\zeta_2 - \varphi_j(\zeta_1)}$ is holomorphic on a neighborhood of $\bar{\mathbb{U}}^2 \cap \mathscr{O}(\mathscr{M})$-hull X and satisfies

$$|F_j(0,0)| > \|F_j\|_{b\mathbb{U}^2 \cap \mathscr{O}(\mathscr{M})\text{-hull } X}. \tag{5.21}$$

The set $\bar{\mathbb{U}}^2 \cap \mathscr{O}(\mathscr{M})$-hull X is $\mathscr{O}(\mathscr{M})$-convex, so F_j can be approximated uniformly there by functions holomorphic on $\mathscr{M}$. If the approximation is good enough, the approximating functions will satisfy an inequality like (5.21), which violates the local maximum principle.

Simple examples show that the analogous statement in $\mathbb{C}^3$ is false. This proof does show that the polynomially convex hull $\widehat{X}$ can be replaced by the rationally convex hull of X. That rationally convex hulls enjoy this property was noted by Lupacciolu [228], who also considered the possibility of replacing the polynomially convex hull $\widehat{X}$ by the compact set Y that is the intersection of all the Stein domains in $\mathbb{C}^2$ that contain X. Lupacciolu's paper establishes, by sheaf-theoretic methods, a natural analogue of the theorem in the N-dimensional case.

Corollary 5.2.19. *If Ω is a strictly pseudoconvex domain in $\mathbb{C}^2$ with $b\Omega$ of class $\mathscr{C}^2$, and if X is a compact subset of $b\Omega$, then $\Omega \setminus \mathscr{O}(\bar{\Omega})$-hull X is pseudoconvex.*

Proof of Theorem 5.2.16. We know that if X is $\mathscr{O}(\bar{\Omega})$-convex, then it is removable.

Conversely, suppose that the compact subset X of $b\Omega$ is removable. We are to prove that X is $\mathscr{O}(\bar{\Omega})$-convex. Assume it is not. Let $\tilde{\Omega}$ be a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ obtained by pushing the boundary of Ω out slightly along $b\Omega \setminus X$ and leaving $b\Omega$ fixed at each point of the set X. If the domain $\tilde{\Omega}$ is sufficiently near Ω, the sets $\mathscr{O}(\bar{\Omega})$-hull X and $\mathscr{O}(\bar{\tilde{\Omega}})$-hull X coincide, so the set $W = \tilde{\Omega} \setminus \mathscr{O}(\bar{\Omega})$-hull X is pseudoconvex, whence there is a function f holomorphic on W that continues into no larger domain. The function $f|(b\Omega \setminus X)$ is a CR-function on $b\Omega \setminus X$ that does not continue through all of Ω. This contradicts the assumed removability of X.

Theorem 5.2.11 lets us exhibit some examples of removable sets.

Theorem 5.2.20. [351] *If $\varepsilon > 0$, there is a removable set $X \subset b\mathbb{B}_N$ such that the area of $b\mathbb{B}_N \setminus X$ is less than ε.*

Proof. For a small $\eta > 0$, let S be a compact totally disconnected subset of the unit circle in the plane whose complement in the circle has measure less than η. Let

$$Y = \{re^{i\vartheta} : r \in [0, 1],\ e^{i\vartheta} \in S\},$$

a compact subset of the closed unit disk. Mergelyan's theorem implies that $\mathscr{P}(Y) = \mathscr{C}(Y)$. If Y^N denotes the N-fold Cartesian product of Y with itself, then $\mathscr{P}(Y^N) = \mathscr{C}(Y^N)$, so every compact subset of Y^N is polynomially convex. If η is chosen sufficiently small at the outset, then the subset $b\mathbb{B}_N \setminus Y^N$ of $b\mathbb{B}_N$ has measure less than ε, and the set $b\mathbb{B}_N \cap Y^N$ is polynomially convex and so removable.

The preceding construction was suggested to the author by N. Sibony. A different construction of sets with the desired property is given in [351].

There is a metric criterion for removability:

Theorem 5.2.21. *If D is a bounded strictly pseudoconvex domain in $\mathbb{C}^N$, $N \geq 2$, with boundary of class $\mathscr{C}^2$, and if $E \subset bD$ is a compact set with $\Lambda^{2N-3}(E) = 0$, then E is removable.*

This result was stated in [230]. Results of this kind are known for more general domains; see [227] and [87].

The proof of this depends on a lemma from the theory of Hartogs series:

Lemma 5.2.22. *Let $R > 1$ and let $f : \mathbb{B}_N(R) \to \mathbb{C}$ be defined and satisfy*

(a) $f|\mathbb{B}_N \in \mathscr{O}(\mathbb{B}_N)$, *and*

(b) *for almost all $z \in \mathbb{B}_N$, the function g_z defined by $g_z(\xi) = f(\xi z)$ is holomorphic in $|\xi| < R/|z|$.*

Then $f \in \mathscr{O}(\mathbb{B}_N(R))$.

Proof. Let the power series expansion of f in $\mathbb{B}_N$ be $f(z) = \sum_\alpha c_\alpha z^\alpha$. Define $F : \mathbb{B}_N \times \mathbb{U} \to \mathbb{C}$ by

$$F(z, \xi) = f(\xi z) = \sum_\alpha c_\alpha z^\alpha \xi^{|\alpha|} = \sum_{k=0,1,\dots} b_k(z)\xi^k$$

with $b_k \in \mathscr{O}(\mathbb{B}_N)$ the function given by $b_k(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$.

By the theory of Hartogs series [360, p. 120], the function F extends to be holomorphic in the domain

$$\{(z, \xi) \in \mathbb{B}_N \times \mathbb{C} : |\xi| < \tilde{R}(z)\}$$

with the function $\tilde{R}$ determined by

$$-\ln \tilde{R}(z) = \limsup_{z' \to z} \limsup_{k \to \infty} \frac{1}{k} \ln |b_k(z')|,$$

and $-\ln \tilde{R}(z)$ is a plurisubharmonic function of z. The series defining F converges in $|\xi| < 1$ for all $z \in \mathbb{B}_N$, so $\tilde{R}(z) \geq 1$ for all $z \in \mathbb{B}_N$. Moreover, for almost all $z \in \mathbb{B}_N$, the series converges in $|\xi| < R/|z|$. Thus, for almost all $z \in \mathbb{B}_N$, $\tilde{R}(z) \geq R$. The upper semicontinuity of $\tilde{R}(z)$ as a function of z implies that $\tilde{R}(z) \geq R$ for all z.

Thus, $F \in \mathscr{O}(\mathbb{B}_N \times R\mathbb{U})$, which implies that the power series expansion of f about the origin converges in $\mathbb{B}_N(R)$, and the lemma is proved.

With the preceding lemma, we can establish an analogue of Theorem 5.2.21 in the case of certain convex domains.

Lemma 5.2.23. *Let D be a bounded convex domain in $\mathbb{C}^N$ such that $bD = \Gamma \cup \Gamma'$ with Γ' a closed convex subset of a real hyperplane H in $\mathbb{C}^N$ and with Γ a strictly convex closed submanifold of class $\mathscr{C}^2$ of $\mathbb{C}^N \setminus H$. Let E be a closed subset of Γ of $(2N-3)$-dimensional measure zero. If f is a continuous CR-function on $\Gamma \setminus E$, then f continues holomorphically through D.*

Proof. To begin with, note that the function f continues holomorphically into the set $D \setminus \widehat{\Gamma' \cup E}$ by Theorem 5.2.11. Call this extension $\tilde{F}$. Note that $\Gamma \cap \widehat{\Gamma' \cup E} = E$.

The proof of the corollary is an induction: If $N = 2$, the corollary is a consequence of Theorem 5.2.11, for because Γ' is convex and E has zero length, the union $\Gamma' \cup E$ is polynomially convex by Corollary 1.6.3. Assume now that the corollary is true in dimension N and consider the $(N+1)$-dimensional case. Let $z \in D$. There are complex affine hyperplanes Π in $\mathbb{C}^N$ passing through z that meet E in a set of zero $(2N-3)$-dimensional measure; indeed, by Theorem 3.3.10 almost all the hyperplanes through z

have this property. Fix such a Π. The N-dimensional domain $\Pi \cap D$ is a domain of the kind contemplated in the statement of the lemma, so by the inductive hypothesis, the function $f|(\Pi \cap (bD \setminus E))$ continues holomorphically through $\Pi \cap D$, say as F_Π. The value $F_\Pi(z)$ is independent of the choice of Π. This is so, for any two complex affine hyperplanes, Π and Π', through z meet in a complex affine plane L of codimension two through z. If $\Lambda^{2N-3}(\Pi \cap E) = \Lambda^{2N-3}(\Pi' \cap E) = 0$, then the extensions F_Π and $F_{\Pi'}$ agree almost everywhere on $L \cap \Gamma$, and so they agree on $L \cap D$, whence their values at z coincide. Thus we have obtained a well-defined function, call it F, on D.

The function F agrees with the function $\tilde{F}$ near $\Gamma \setminus E$ and so is holomorphic on an open subset of D. In fact, F is holomorphic on all of D. In the proof of this we suppose, as a notational convenience, that $\mathbb{B}_N \subset D$ and that $F \in \mathscr{O}(\mathbb{B}_N)$. We shall show that if $p \in D$, then F is holomorphic on a neighborhood of p. For this, let $[0, p]$ be the straight line segment connecting the origin and the point p. Let

$$\tau_o = \sup\{\tau \in (0,1) : F \text{ is holomorphic on a neighborhood of the interval } [0, \tau p]\}.$$

We have $\tau_o > 0$, since $f \in \mathscr{O}(\mathbb{B}_N)$. Let $\tau_1 \in (0, \tau_o)$ be a point with

$$|\tau_o - \tau_1| < \operatorname{dist}(\tau_1 p, bD).$$

Set $p_1 = \tau_1 p$.

The function F is holomorphic in a ball $\mathbb{B}_N(p_1, r)$ for some $r > 0$, and if $R = \operatorname{dist}(p_1, bD)$, then for almost every $\mathbb{C}$-affine hyperplane Π through p_1, F is holomorphic on the $(N-1)$-dimensional ball $\Pi \cap \mathbb{B}_N(p_1, R)$. The preceding lemma implies that F is holomorphic on $\mathbb{B}_N(p_1, R)$, which is a neighborhood of the point $\tau_o p$. It follows that $\tau_o = 1$, and then that F is holomorphic on a neighborhood of p.

The lemma is proved, for F assumes the boundary values f along $\Gamma \setminus E$.

Proof of Theorem 5.2.21. Consider a point $p \in bD$. With respect to suitable local coordinates, bD is strictly convex in a neighborhood of p, so the lemma provides a ball $\mathbb{B}_N(p, \varepsilon)$ such that the function f continues holomorphically into the intersection $D \cap \mathbb{B}_N(p, \varepsilon)$. Finitely many of these balls cover bD, so in this way, we obtain a continuation of f into $D \setminus K$ for a sufficiently large compact subset K of D. It follows that f continues into all of D, and the theorem is proved.

Theorem 5.2.16 provides a characterization of the removable sets in the boundary of a two-dimensional strictly pseudoconvex domain. This characterization is valid only in the two-dimensional case, but there is a characterization in the higher-dimensional case, a characterization due to Lupacciolu [229]. See also [87].

Theorem 5.2.24. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$, $N \geq 3$, with boundary of class $\mathscr{C}^2$. A compact subset X of $b\Omega$ is removable if and only if the Dolbeault cohomology group $H^{0,1}(\mathbb{C}^N \setminus X)$ vanishes.*

Lupacciolu also gives intrinsic characterizations of removable sets, characterizations that are phrased in terms of the cohomology of the set X itself rather than in terms of its complement.

There is the following noteworthy difference between the characterization in dimension two and that in the higher-dimensional case. In $\mathbb{C}^2$, the question of removability for a given set X depends on which strictly pseudoconvex boundary it is considered to lie in. In the higher-dimensional case, removability is an absolute notion; it is independent of which particular strictly pseudoconvex boundary may contain the set.

A simple example shows that in the two-dimensional case the removability of a set in the boundary depends on which boundary is considered. Let $Q(z, w) = (|z|^2 - 1)^2 + |w|^2$. Then $Q^{-1}(0)$ is the circle $K = \{(e^{i\vartheta}, 0) : \vartheta \in \mathbb{R}\}$. A short calculation shows that near K, dQ vanishes only on K and that the Levi form of Q is positive definite on a neighborhood of K. Thus, if for small positive r we put $D_r = \{(z, w) : Q(z, w) < r^2\}$, then D_r is a strictly pseudoconvex domain with smooth boundary. Its boundary contains the circle $C_r = \{(\sqrt{1 + r}e^{i\vartheta}, 0) : \vartheta \in \mathbb{R}\}$. The circle C_r is convex with respect to the algebra $\mathscr{O}(\bar{D}_r)$, for the latter algebra contains all polynomials in z and $1/z$, and the linear span of these functions is dense in $\mathscr{C}(C_r)$. Thus, C_r is removable for D_R. However, C_r is also contained in the boundary of the ball $\mathbb{B}_N(\sqrt{1 + r})$, and it is not polynomially convex. Consequently, it is not a removable set for this ball.

We continue our discussion of CR-functions with a version of Radó's theorem valid in the context of CR-functions on strictly pseudoconvex hypersurfaces. Recall—Theorem 3.4.17—that a function f continuous on a planar domain and holomorphic off $f^{-1}(0)$ is holomorphic on the whole domain. There is an extension of this in which $f^{-1}(0)$ is replaced by a larger set:

Theorem 5.2.25. [344] *Let f be a bounded function defined and holomorphic on the open set $\mathbb{U} \setminus X$, where $X \subset \mathbb{U}$ is a closed set. Also assume f to be nonconstant on some component of $\mathbb{U} \setminus X$. If the global cluster set $\mathscr{C}_X(f)$ at X is contained in a subset E of $\mathbb{C}$ that has zero logarithmic capacity, then f continues holomorphically into all of $\mathbb{U}$.*

Proof. [66] The cluster set of f at X is compact, so we can suppose E to be compact. The set E is a compact set of zero logarithmic capacity, so there is an *Evans function* for E, which is a function u harmonic on the complement in the Riemann sphere of E that tends to $+\infty$ at E. (For the Evans function, one can consult [356].) The function $u \circ f$ is harmonic on $\mathbb{U} \setminus f^{-1}(E)$ and tends continuously to $+\infty$ at the points of $f^{-1}(E)$. Thus, $f^{-1}(E)$ is a set of capacity zero and so a removable set for bounded harmonic functions. Consequently, f continues holomorphically through the set X.

If E consists of a single point, this argument provides a different proof of the classical theorem of Radó.

A version of Radó's theorem for CR-functions is the following.

Theorem 5.2.26. *Let Σ be a connected strictly pseudoconvex hypersurface of class $\mathscr{C}^2$ in an open set in $\mathbb{C}^N$, $N \geq 2$. Let X be a proper, closed subset of Σ, and let $E \subset \mathbb{C}$ be a set of zero logarithmic capacity. Let $f \in \mathscr{C}(\Sigma \setminus X)$ be bounded and be nonconstant on some component of $\Sigma \setminus X$, and let it satisfy the tangential Cauchy–Riemann equations on $\Sigma \setminus X$. If the global cluster set $\mathscr{C}_X(f)$ is contained in the set E, then the function f extends uniquely to all of Σ as a continuous CR-function.*

This theorem, for E a singleton, was proved by Rosay and Stout [301], where a parallel result is obtained in which the strict pseudoconvexity hypothesis is weakened. The result as stated was given by Alexander [23], who, in the same paper, also gives a result of this general flavor for bounded measurable CR-functions.

Proof. The cluster set $\mathscr{C}_X(f)$ is compact, so there is no loss of generality in assuming that E is compact, so that it has an Evans function, say u. We replace the set X with the set $X \cup f^{-1}(E)$, which is again a proper closed subset of Σ and which has the property that f is constant on no component of $\Sigma \setminus X$.

Consider initially the two-dimensional case, $N = 2$. Suppose to begin with that X has no interior in Σ.

The problem is local and Σ is strictly pseudoconvex, so we work near a point $p \in X$ near which holomorphic coordinates have been chosen such that for a strictly convex domain D contained in $\{\Im z_N > 0\}$, $bD = \Delta \cup \Gamma$, where Δ is a compact subset of the real hyperplane $\{\Im z_N = 0\}$, and $\Gamma \subset \Sigma$ is a convex surface, which is a neighborhood in Σ of p.

With this arrangement, set $K = \Delta \cup (X \cap \Gamma)$. That X has no interior in Σ implies that $D \setminus \widehat{K}$ is not empty. The function f is defined and continuous on $D \setminus K$ and satisfies the tangential Cauchy–Riemann equations there. If W is a component of $D \setminus \widehat{K}$, then by Theorem 2.4.11, it abuts a unique component C_W of $bD \setminus K$. Theorem 5.2.11 implies that $f|C_W$ extends holomorphically through W. In this way, we obtain a function F defined and holomorphic on the set $\Omega = D \setminus \widehat{K}$ that assumes continuously the boundary values f along $bD \setminus K$. We shall see that Radó's theorem, which, as noted after Theorem 3.4.17, applies in $\mathbb{C}^N$, implies that F continues holomorphically through D and then that it assumes continuous boundary values along Γ. This will imply that f extends continuously to all of Γ, so that the theorem follows in the two-dimensional case.

Notice that $F(\Omega) \cap E = \emptyset$: If $z_o \in \Omega$ satisfies $F(z_o) \in E$, introduce the variety $V = \{z \in \Omega : F(z) = F(z_o)\}$. We have that $bV \subset bD \cup \widehat{K}$. On $bD \setminus \widehat{K}$, f omits the values in E, so $bV \subset \widehat{K}$. This implies that $V \subset \widehat{K}$, contradicting $z_o \in D \setminus \widehat{K}$. The function $u \circ F$ is pluriharmonic on Ω, and if $z \in X \cap b\Omega$ satisfies $\Im z_N > 0$, then $u \circ F(\zeta) \to \infty$ as $\zeta \in \Omega$ tends to z. This is correct, because we know that $u \circ f(\zeta) \to \infty$ as $\zeta \to z$, $\zeta \in \Gamma \setminus X$. Choose a large positive M, and let g be a function holomorphic on $\bar{D}$ with $g(z) = 1$ and $|g| < 1$ on $\bar{D} \setminus \{z\}$. If $v(\zeta) = M + m(\Re g(\zeta) - 1)$, then $v \leq u \circ F$ on $bD \setminus (\Gamma \setminus X)$, if $m > 0$ is big enough, because $u \circ F \to \infty$ at the points of X in Γ. Then $v \leq u \circ F$ on Ω, too. To see this, let $z_o \in \Omega$. There is a polynomial P with $P(z_o) = 1 > \|P\|_{\widehat{K}}$. Consequently, the boundary of the subvariety $\{\zeta : P(\zeta) = 1\} \cap \Omega$ lies in $bD \cap (\Gamma \setminus X)$, on which $v \leq u \circ F$. Therefore,

$$M = \liminf_{\zeta \to z, \zeta \in \Omega} v(\zeta) \leq \liminf_{\zeta \to z, \zeta \in \Omega} u \circ F.$$

In this, M can be arbitrarily large, so, as claimed, $u \circ F(\zeta) \to \infty$ as $\zeta \to z$, $z \in b\Omega$.

The rest of the analysis depends on a lemma from [301]:

Lemma 5.2.27. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^2$. Let $K \subset bD$ be a compact set with $\widehat{K} \cap bD = K$. Set $D' = D \setminus \widehat{K}$. If U is a neighborhood of K in $\mathbb{C}^2$, then there is an open set $\Omega \subset D'$ that contains the intersection of D' with some neighborhood of $\widehat{K}$ and such that if v is a plurisubharmonic function on D', then* $\sup_\Omega v \leq \sup_{D' \cap U} v$.

Proof. Let W_1 be a neighborhood of $\widehat{K}$, and let W_2 be a neighborhood of $\overline{bD \setminus K}$, these neighborhoods chosen so that $W_1 \cap W_2 \subset U$. Because the domain D is pseudoconvex, the same is true of the open set $D' = D \setminus \widehat{K}$. Accordingly, D' can be exhausted by smoothly bounded strictly pseudoconvex domains. In particular, there is such a domain, D_1, a relatively compact subset of D', with $D_1 \subset D' \setminus (\overline{W_1 \cup W_2})$. Set $\Omega = (D' \setminus \bar{D}_1) \cap W_1$. This is the intersection of D' with a neighborhood of $\widehat{K}$. At points of $b\Omega \setminus (\widehat{K} \cup U)$, $b\Omega$ is strictly pseudoconcave *as seen from* Ω. We show that if v is plurisubharmonic on D', then $\|v\|_\Omega \leq \|v\|_{U \cap D'}$. To do this, fix $p \in \Omega$. Let W_3 be a small strictly pseudoconvex neighborhood of $\widehat{K}$ with $p \notin W_3$ and with $\bar{W}_3 \cap \bar{D}_1 = \emptyset$. Put $\Omega_p = \Omega \setminus W_p$. The boundary of Ω_p consists of points in U, of strictly pseudoconcave points (as seen from Ω_p) contained in bW_p, and of strictly pseudoconcave points in $b\Omega$. A plurisubharmonic function on D' cannot assume a local maximum, with respect to the set Ω_p, at any strictly pseudoconcave point of $b\Omega_p$, so $v(p) \leq \|v\|_{D' \cap U}$. The lemma is proved.

Corollary 5.2.28. *If* $\limsup_{\zeta \to z, \zeta \in \Omega} v(\zeta) \leq M$ *for all* $z \in b\Omega \cap K$, *then this inequality is correct for all* $z \in b\Omega \cap D$.

We now resume the proof of the theorem itself. We will show that

$$\lim_{\zeta \to z_o, \zeta \in \Omega} -u \circ F(\zeta) = -\infty$$

for all $z_o \in b\Omega \cap D$. To do this, fix a large positive constant A. The function v_1 given by $v_1(z) = -u \circ F(z) + A\Re z_N$ is plurisubharmonic on Ω. For $z \in b\Omega \cap \Delta$, $\limsup_{\zeta \to z} v_1(\zeta) \leq 0$, and for $z \in bD \cap K \cap \{z_N > 0\}$, $u \circ F(\zeta) \to \infty$ as $\zeta \to z$, $\zeta \in \Omega$. We therefore have that $\limsup_{\zeta \to z, \zeta \in \Omega} v_1(\zeta) = -\infty$. The last corollary yields $\limsup_{\zeta \to z, \zeta \in \Omega} v_1(\zeta) \leq 0$, whence

$$\limsup_{\zeta \to z, \zeta \in \Omega} -u \circ F(\zeta) \leq -A\Re z_N.$$

If we let $A \to \infty$, we obtain that $h \circ F \to -\infty$ at the points of $b\Omega \cap D$. Thus, the cluster set of F along the set $b\Omega \cap D$ is contained in the set E. Radó's theorem implies that F continues holomorphically through all of D. Moreover, we find that $\widehat{K} \cap D$ has no interior and that $F(\widehat{K} \cap D) \subset E$.

We have proceeded under the assumption that $X \cap bD$ has no interior in bD. This hypothesis can be removed in the following way. If the set $X \cap bD$ has interior, let p be a boundary point of this interior. The only place in the argument given above at which the assumption that $X \cap bD$ had no interior was used, was to be sure that Ω is nonempty. If, though, p is in the boundary of the interior of X, the corresponding set Ω is still nonempty, and the analysis can proceed as above. We find that F extends to be holomorphic in all of D and that $\widehat{X} \cap D$ has no interior. We now have a contradiction, for if X has interior, then $\widehat{X}$ has interior, too.

We finally see that F has continuous boundary values. This is so, for if $z \in X$, and if $\mathscr{C}'_z(f)$ denotes the cluster set of $f|(\Sigma \setminus X)$ at z and $\mathscr{C}''_z(F)$ denotes the cluster set of $F|D$ at z, then $\mathscr{C}''_z(F)$ is a connected set contained in the totally disconnected set E. Because $\mathscr{C}'_z(f) \subset \mathscr{C}_z(F)$, f extends continuously to the point z, and we are done with the two-dimensional case.

The result in the higher-dimensional case is deduced by a slicing argument using induction on the dimension. We assume, therefore, that the theorem is correct in $\mathbb{C}^N$, $N \geq 2$. Consider the $(N+1)$-dimensional case. Thus, Σ is a strictly pseudoconvex hypersurface of class $\mathscr{C}^2$ in an open set in $\mathbb{C}^{N+1}$, and $X \subset \Sigma$ is a closed set. The function f is a continuous CR-function on $\Sigma \setminus X$ that is not constant on some component of $\Sigma \setminus X$. We work near a point $p \in X$ that lies in the boundary of a component of $\Sigma \setminus X$ on which f is nonconstant. Fix coordinates in $\mathbb{C}^N$ near p so that with respect to these coordinates, Σ is strictly convex near p. Let $D \subset \mathbb{C}^{N+1}$ be a convex domain with $bD = \Gamma \cup \Delta$, where Δ is a compact, convex set in a real hyperplane in $\mathbb{C}^{N+1}$ and Γ is a neighborhood of p in Σ. We are going to define a holomorphic function F on D. To do this, fix a point $z \in D$. Choose a $\mathbb{C}$-affine N-plane L in $\mathbb{C}^{N+1}$ through z that meets Γ in a set that has a component on which f is nonconstant. Because p lies in the boundary of a component of $\Gamma \setminus X$ on which f is not constant, and because we can find an affine copy of $\mathbb{C}^N$ in $\mathbb{C}^{N+1}$ that passes through three prescribed points, it is possible to find the desired L. The inductive hypothesis implies that f extends continuously to all of $\Gamma \cap L$ as a CR-function and that the set $L \cap X$ is nowhere dense in $\Gamma \cap L$. As a continuous CR-function on $L \cap \Gamma$, $f|(L \cap \Gamma)$ extends holomorphically through the slice $L \cap D$; we denote this extension by F_L. Then we define $F(z)$ to be $F_L(z)$. Because the value of $F_L(z)$ is independent of L, the function F is well defined on D. It is bounded by the supremum of $|f|$ on $\Sigma \setminus X$. It is also holomorphic. It assumes continuously the boundary values f along $\Gamma \setminus X$. We have to see that F has continuous boundary values. To do this, it is enough to show that at any point $q \in \Sigma \setminus X$, the cluster set of F, as a function on D, is contained in the cluster set of f, a function on $\Sigma \setminus X$, at the point q. (This containment will imply the equality of the two sets.) Accordingly, let $\{z_k\}_{k=1,\dots}$ be a sequence in D that converges to q such that the limit $\lim_{k=1,\dots} F(z_k)$ exists. Call this limit w. For each k, let L_k be an affine copy of $\mathbb{C}^N$ contained in $\mathbb{C}^{N+1}$ that passes through the point z_k and that is nearly parallel to the complex tangent plane $T_q^{\mathbb{C}}$. When $k \to \infty$, the diameter of the intersection $L_k \cap D$ shrinks to zero. By rotating L_k slightly about the point z_k if necessary, we can suppose that for large k, the set $L_k \cap \Sigma$ is not contained in X. Set $c_k = F(z_k)$. The level set $V_k = \{z \in L_k : F(z) = c_k\}$ meets $L \cap \Gamma$. By hypothesis, the set X is nowhere dense in Γ, so there is a point z'_k in $\Gamma \setminus X$ near enough to V_k that $|f(z'_k) - c_k| < 2^{-k}$. Thus, $\{z'_k\}_{k=1,\dots}$ is a sequence in $\Gamma \setminus X$ that converges to q along which f tends to w: The number w lies in the cluster set of f at q. We have, therefore, that the cluster set of f at q contains the cluster set of F at q. As in the two-dimensional case, this implies that F has continuous boundary values along Γ.

To conclude the proof, let Ω be a maximal connected open subset of Σ into which f continues as a continuous CR-function. The argument we have just used implies that Ω can have no boundary points in Σ, so $\Omega = \Sigma$, and the theorem is proved.

There is a rather extensive literature concerning removable sets for CR-functions. For this, we refer to the papers of Chirka and Stout [87], Stout [351], and to the references cited in them. More recent developments are given in the Habilitationsschrift [284] of Porten.

There is a relation between removable sets for CR-functions and removable sets for holomorphic functions:

Theorem 5.2.29. [350] *A compact subset X of $b\Omega$, Ω a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, is removable (for CR-functions) if and only if the set $X^\dagger = \Omega \cap \mathscr{O}(\bar{\Omega})$-hull X is removable (for holomorphic functions).*

Proof. Assume first that the set $X^\dagger$ is removable for holomorphic functions. If f is a CR-function on $b\Omega \setminus X$, then by Theorem 5.2.13, there is a function F holomorphic on $\Omega \setminus X^\dagger$ that assumes continuously the boundary values f along $b\Omega \setminus X^\dagger$. Because the set $X^\dagger$ is removable for holomorphic functions, F continues holomorphically into all of Ω, and the set X is seen to be removable for CR-functions.

Conversely, let X be removable for CR-functions, and let $f \in \mathscr{O}(\Omega \setminus X^\dagger)$. We will prove that f continues holomorphically into all of Ω. (Note that f is not assumed to have boundary values in any sense along $b\Omega \setminus X$.)

Construct two strictly pseudoconvex domains Ω^+ and Ω^- with $\Omega^+ \supset \Omega \supset \Omega^-$, Ω^+ obtained by deforming $b\Omega$ outward a little bit along $b\Omega \setminus X$, Ω^- obtained by deforming $b\Omega$ inward a little bit along $b\Omega \setminus X$. The set X itself is to be left fixed under these deformations, so that $b\Omega^+ \cap b\Omega = X = b\Omega \cap b\Omega^-$. Moreover, the domain Ω^- is to contain the set $X^\dagger$.

There is an open cover of Ω^+ that consists of the sets Ω and $\Omega^+ \setminus \bar{\Omega}^-$. The function f is holomorphic on the intersection of these two sets, so there are functions $f^- \in \mathscr{O}(\Omega)$ and $f^+ \in \mathscr{O}(\Omega^+ \setminus \bar{\Omega}^-)$ such that[3] $f = f^+ - f^-$. The function $f^+|(b\Omega \setminus X)$ is a CR-function and so, because X is removable, continues holomorphically, say as F^+, to an element of $\mathscr{O}(\Omega)$. We now have that on Ω, $f = F^+ - f^-$. The function $F^+ - f^-$ is holomorphic in Ω, so the set $X^\dagger$ is seen to be removable for holomorphic functions. This completes the proof.

The theory of removable sets in the boundary can be applied to show that certain sets are removable for holomorphic functions.

Theorem 5.2.30. *If X is a closed subset of $\mathbb{C}^N$, $N > 1$, that is homeomorphic to the real line, then every $f \in \mathscr{O}(\mathbb{C}^N \setminus X)$ continues through X as a function holomorphic on all of $\mathbb{C}^N$.*

It is to be emphasized that in this statement no smoothness at all is imposed on the homeomorphic copy X of $\mathbb{R}$; it may have locally positive $2N$-dimensional measure at every point.

The result, in case $N \geq 3$, was obtained in [87] by methods that do not yield the case $N = 2$. That case was settled by Santillán [318], whose proof, which works in all dimensions, is given below.

The proof is entirely local, so the theorem remains correct if $\mathbb{C}^N$ is replaced by any other N-dimensional complex manifold.

The proof depends on a topological fact about Euclidean spaces and certain of their subdomains:

Definition 5.2.31. *A topological space Y is* unicoherent *if whenever $Y = T \cup T'$ with T and T' connected, closed subsets of Y, the intersection $T \cap T'$ is connected.*

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[3] We are using here that the additive Cousin problem is solvable on strictly pseudoconvex domains.

Theorem 5.2.32. *If Σ is a connected manifold such that each continuous map from Σ to a circle is homotopic to a constant, then Σ is unicoherent.*

As a consequence of this theorem, every Euclidean space is unicoherent, as is each space of the form $\mathbb{R}^N \setminus E$ for a finite subset E of $\mathbb{R}^N$, $N \geq 3$.

Proof. [89] If Σ is not unicoherent, then there exist connected, closed sets A and B in Σ such that $\Sigma = A \cup B$ but $A \cap B$ is not connected. Write $A \cap B = C \cup C'$ with C and C' disjoint closed subsets of Σ. Denote by q a nonnegative continuous function on Σ that vanishes identically on C and is constantly one on C'. Then define $f_A : A \to \mathbb{T}$ and $f_B : B \to \mathbb{T}$ by

$$f_A(x) = e^{\pi i q(x)} \quad \text{and} \quad f_B(x) = e^{-\pi i q(x)},$$

respectively. These two functions agree on the set $A \cap B$. We can, therefore, define a function $g : \Sigma \to \mathbb{T}$ by $g(x) = f_A(x)$ when $x \in A$ and $g(x) = f_B(x)$ when $x \in B$. This map is well defined and continuous. By hypothesis, it is homotopic to a constant map, so there is a continuous map $\tilde{g} : \Sigma \to \mathbb{R}$ with $g = e^{\pi i \tilde{g}}$. This is impossible, though: On C, $q = 0$, so there is an integer μ such that on C, $\tilde{g} = 2\mu$. Because the set A is connected and $e^{\pi i \tilde{g}} = f_A$ there, it follows by continuity that on all A, $\tilde{g} = q + 2\mu$. Similarly, on the set B, $\tilde{g} = q + 2\nu$ for an integer ν. Thus, we have that on C, $\tilde{g} = 2\mu$, on C', $\tilde{g} = 1 + 2\nu$: An even integer is odd. This is impossible, so $A \cap B$ is connected, as desired.

Proof of Theorem 5.2.30. [318] Let $x \in X$, and let $\lambda \subset X$ be an arc that contains x in its interior. Let the endpoints of λ be a and b.

Choose a nonnegative function Q defined and of class $\mathscr{C}^\infty$ on $\mathbb{C}^N \setminus \{a, b\}$ that vanishes identically on the interior of λ, that is greater than one on $X \setminus \lambda$, and that satisfies $Q(z) \to \infty$ when $z \to \infty$. Denote by r, $r \in (0, 1)$, a regular value of Q. Thus the set Σ defined by $\Sigma = Q^{-1}(r)$ is a closed $(2N-1)$-dimensional submanifold of $\mathbb{C}^N \setminus \{a, b\}$. It is also a bounded subset of $\mathbb{C}^N$ with closure $\bar{\Sigma} = \Sigma \cup \{a, b\}$. There is no reason for Σ to be connected.

Let W be the component of $\mathbb{C}^N \setminus \bar{\Sigma}$ that contains the interior of λ. The set W is a bounded, open, connected subset of $\mathbb{C}^N$ that contains the point x. Let W_∞ be the unbounded component of $\mathbb{C}^N \setminus \bar{W}$, and then let $W_o = \mathbb{C}^N \setminus \bar{W}_\infty$. The set W_o is bounded and open and contains the point x. Also, $bW_o \subset \bar{W}_\infty \setminus W_\infty = bW_\infty$. Finally, $\bar{W}_\infty \cap \bar{W}_o = bW_o$.

The set W_o is connected. To see this, let V be a component of W_o. Then $bV \subset bW_o = bW_\infty$. Let $y \in bV \setminus \{a, b\}$. Near y, bV is a smooth manifold, so if $\mathbb{B}$ is a sufficiently small ball centered at y, then bV splits $\mathbb{B}$ into two components, one of which lies in W_∞, the other in W. Each component of W_o meets the connected set W, so W_o is connected.

More is true: bW_o is connected, for we have $\bar{W}_o \cup \bar{W}_\infty = \mathbb{C}^N$ and $\bar{W}_o \cap \bar{W}_\infty = bW_o = bW_\infty$, so that unicoherence implies that bW_o is connected. In the same way, $W_o \setminus \{a, b\}$ is connected.

Theorem 5.2.11 implies that the restriction $f|bW_o \setminus \{a, b\}$, which is a CR-function, extends holomorphically through all of W_o and so, in particular, through a neighborhood of x. The theorem is proved.

Whether the analogous statement for topological copies of $\mathbb{R}^2$ in $\mathbb{C}^3$ is true remains an open question.

The "Λ^{2N-3} theorem," Theorem 5.2.21, implies an analogous theorem for holomorphic functions, which was proved by Shiffman [326]: *If X is a closed subset of $\mathbb{C}^N$ and $\Lambda^{2N-2}(X) = 0$, then X is removable for holomorphic functions.* Alternatively, the envelope of holomorphy of $\mathbb{C}^N \setminus X$ is $\mathbb{C}^N$. To see this, fix a function $f \in \mathscr{O}(\mathbb{C}^N \setminus X)$. By Eilenberg's inequality, $\Lambda^{2N-3}(X \cap b\mathbb{B}_N(R)) = 0$ for almost all $R > 0$. Fix such an R. Theorem 5.2.21 implies that the CR-function $f|(b\mathbb{B}_N(R) \setminus X)$ continues holomorphically into $\mathbb{B}_N(R)$. This is true for arbitrarily large R, so f continues holomorphically into all of $\mathbb{C}^N$, which was to be shown. (An earlier version of this result was found by Caccioppoli [79].) More is true:

Theorem 5.2.33. *If X is a closed subset of $\mathbb{C}^N$ such that*

$$\liminf_{R\to\infty} \frac{\Lambda^{2N-2}(X \cap \mathbb{B}_N(R))}{\frac{\pi^{N-1}}{(N-1)!} R^{2N-2}} < 1, \tag{5.22}$$

then X is removable for holomorphic functions.

This result in the case $N = 2$ was obtained in [349]. The case of arbitrary N was obtained in [216].

For the case $N = 2$ of the theorem, we can give a complete proof based on the tools at our disposal. The case $N > 2$ requires methods beyond the scope of what we have developed; we will only sketch this argument.

In treating the case $N = 2$, we will use the following lemma.

Lemma 5.2.34. *If X is a closed subset of $\mathbb{C}^2$ such that*

$$\liminf_{R\to\infty} \frac{\Lambda^{2}(X \cap \mathbb{B}_2(R))}{\pi R^2} < 1,$$

then for each $z \in \mathbb{C}^2$,

$$\liminf_{R\to\infty} \frac{\Lambda^{2}(X \cap \mathbb{B}_2(z, R))}{\pi R^2} < 1.$$

Proof. This follows immediately from the inclusion $\mathbb{B}_2(z, R) \subset \mathbb{B}_2(R + |z|)$.

Proof of Theorem 5.2.33 ($N = 2$). The set X has locally finite two-dimensional measure, so the set $\mathbb{C}^2 \setminus X$ is connected and dense in $\mathbb{C}^2$. The lemma implies that to prove the theorem, it suffices to show that each $f \in \mathscr{O}(\mathbb{C}^2 \setminus X)$ continues holomorphically into a neighborhood of the origin. Set $\alpha = \liminf_{R\to\infty} \frac{\Lambda^2(X\cap\mathbb{B}_2(R))}{\pi R^2}$. Eilenberg's inequality, Theorem 3.3.6, shows that for $R > 0$,

$$\alpha R^2\, \Lambda^2(X \cap \mathbb{B}_2(R)) \geq \int^*_{[0,R]} \Lambda^1(X \cap \{z : |z| = r\})\, dr \tag{5.23}$$

for infinitely many R's. For such R's there are $r \in (0, R)$ with $\Lambda^1(X\cap\{z : |z| = r\}) < 2\pi r$. Fix such an r. Theorem 5.1.14 implies that the origin is not in the polynomially convex hull of the set $X \cap b\mathbb{B}_2(r)$. Consequently, by Theorem 5.2.11, the CR-function $f|(b\mathbb{B}_2(r) \setminus X)$

continues holomorphically into a neighborhood of the origin. The theorem in the case $N = 2$ follows from this.

An outline of the proof of the case $N > 2$ of the theorem is as follows. It depends on a general fact about envelopes of holomorphy, which is due to Lawrence [216, 217].

Lemma 5.2.35. *Let D be a domain of holomorphy in $\mathbb{C}^N$, and let $X \subset D$ be a closed subset of D with the property that $(*)$ if $W \subset D$ is a connected open set, then $W \setminus X$ is connected. The envelope of holomorphy of the domain $D \setminus X$ is one-sheeted.*

The closed subsets of D that have the property $(*)$ are the closed sets that have topological dimension not more than $2N - 2$. That subsets of topological dimension not more than $2N-2$ have the property is contained in [185, Theorem IV.4, p. 48]. That sets of dimension $2N - 1$ do not have the property seems not to be contained in the standard books on dimension theory. It is in [29, Kor. I, p. 208]; see also the earlier paper [129, p. 78]. In particular, if the set X has finite or locally finite $(2N-2)$-dimensional Hausdorff measure, then it has the property $(*)$. This is easily seen directly, without appeal to dimension theory.

Proof. If the result is incorrect, then because D is a domain of holomorphy, there is a finite sequence $\{B_j\}_{j=1,\dots,r}$ of balls *each contained in D* with $B_r = B_1$ and with the property that each B_j meets its predecessor in a nonempty set and the further property that for each j there is $f_j \in \mathscr{O}(B_j)$ such that $f_j = f_{j+1}$ on $B_j \cap B_{j+1}$ but $f_1 \neq f_r$ and, moreover, $f_1|(B_1 \setminus X) = F|(B_1 \setminus X)$ for some $F \in \mathscr{O}(D \setminus X)$.

This is impossible: By induction, $f_j|(B_j \setminus X) = F|(B_j \setminus X)$. This is true for $j = 1$. If it is true for a given j, then because $B_{j+1} \setminus X$ is connected and $f_{j+1}|(B_j \cap B_{j+1}) = f_j|(B_j \cap B_{j+1})$ and $f_j = F$ on $B_j \setminus X$, it follows that $f_{j+1} = F$ on $B_{j+1} \setminus X$. In particular, f_r agrees with $F = f_1$ on B_r. Contradiction.

Thus, if $\Omega = D \setminus X$, the envelope of holomorphy $\tilde{\Omega}$ of Ω is a domain in $\mathbb{C}^N$ that satisfies $\Omega \subset \tilde{\Omega} \subset D$.

There is then the following result, found by Lawrence [216]:

Theorem 5.2.36. *If D is a domain of holomorphy in $\mathbb{C}^N$, and if X is a closed subset of D of locally finite $(2N - 2)$-dimensional Hausdorff measure, then the envelope of holomorphy of $D \setminus X$ is the domain $D \setminus E$, where E denotes the union of all the $(N - 1)$-dimensional analytic subvarieties of D contained in X.*

We cannot give the full details of the proof of this result. The general lines of the argument are as follows. Set $\Omega = D \setminus X$, and let $\tilde{\Omega}$ be the envelope of holomorphy of Ω. By the preceding lemma, $\tilde{\Omega}$ is a domain in $\mathbb{C}^N$ that satisfies $\Omega \subset \tilde{\Omega} \subset D$. The set $E = D \setminus \tilde{\Omega}$ is a closed subset of D that is a *pseudoconcave set* in the sense of Oka. It follows from work of Oka that, because E has locally finite $(2N-2)$-dimensional measure, it is necessarily a variety. For details about Oka's work on pseudoconcave sets, we refer to Nishino's volume [264], which is devoted mainly to an exposition of Oka's ideas.

Theorem 5.2.36 yields Theorem 5.2.33 in the case $N \geq 2$, because if $V \subset \mathbb{C}^N$ is a variety of codimension $(N-1)$, then for every $\varepsilon > 0$ and all large R, $\Lambda^{2N-2}(V \cap \mathbb{B}_N(R)) > (1 - \varepsilon)\frac{\pi^{N-1}}{(N-1)!}R^{2N-2}$. This is a result of Bishop [60, 85].

A theorem in the spirit of this result but set in the context of compact manifolds was obtained by Lawrence [216]. Its statement depends on a simple observation: Let $\mathscr{M}$ be a compact complex manifold endowed with a smooth Hermitian metric, g. Let Λ_g^p denote the p-dimensional Hausdorff measure derived from the distance function associated with g. There is a constant $c_p > 0$ such that $\Lambda_g^p(Z) \geq c_p$ if Z is a p-dimensional subvariety of $\mathscr{M}$.

Theorem 5.2.37. *Let $\mathscr{M}$ be an N-dimensional compact complex manifold endowed with a Hermitian metric, g. Let Λ_g^p denote the p-dimensional Hausdorff measure derived from the distance function associated with g. If X is a closed subset of $\mathscr{M}$ such that $\Lambda_g^{2N-2}(X)$ is strictly less than the greatest lower bound of the areas of complex hypersurfaces in $\mathscr{M}$, then X is removable for meromorphic functions on $\mathscr{M}$.*

That is, if F is a function meromorphic on $\mathscr{M} \setminus X$, then F extends to a function meromorphic on the whole of $\mathscr{M}$.

Example. In the complex projective space $\mathbb{P}^N(\mathbb{C})$ endowed with the Fubini–Study metric, the complex hypersurfaces with minimal area are the copies of $\mathbb{P}^{N-1}(\mathbb{C})$ contained in $\mathbb{P}^N(\mathbb{C})$. If $[z_0 : \cdots : z_N]$ are homogeneous coordinates on $\mathbb{P}^N(\mathbb{C})$, the surfaces in question are the linear varieties defined by the vanishing of some linear form ϕ in the coordinates $z_0, \ldots, z_N$. Their volume is $\pi^{(N-1)}/(N-1)!$. Thus, if X is a subset of $\mathbb{P}^N(\mathbb{C})$ with $\Lambda_{FS}^{2N-2}(X) < \pi^{(N-1)}/(N-1)!$, then X is removable for meromorphic functions. (For a discussion of the Fubini–Study metric on projective spaces one can consult [256].)

5.2.A. Appendix: A solution of $\bar{\partial} f = u$

In this brief appendix, we bring the formalism of Cauchy–Fantappiè forms to bear on the problem of solving $\bar{\partial} f = u$ for a $\bar{\partial}$-closed $(0, 1)$-form. There is a vast literature of this problem; everything we need lies very near the surface.

We know that for any compactly supported smooth $\bar{\partial}$-closed form u of bidegree $(0, 1)$ on $\mathbb{C}^N$, $N \geq 2$, the equation $\bar{\partial} f = u$ has a smooth, compactly supported solution. If we invoke the integral formula (1.21), we find that a solution is given by

$$f(w) = c_N \int_{\mathbb{C}^N} u(z) \wedge \frac{\omega'(\overline{z-w}) \wedge \omega(z)}{|z-w|^{2N}}.$$

In writing this formula, we have used the a priori information that a compactly supported solution of $\bar{\partial} f = u$ exists; one can verify directly that the given function f does satisfy this equation.

The solution of $\bar{\partial} f = u$ in case u does not have compact support is more complicated than the case of compact support. Consider a bounded domain D in $\mathbb{C}^N$ with smooth boundary and a $(0, 1)$-form u defined, smooth, and $\bar{\partial}$-closed on a neighborhood of $\bar{D}$. Suppose f to be a smooth function that satisfies $\bar{\partial} f = u$ on a neighborhood of $\bar{D}$. The function f admits a representation based on the Bochner–Martinelli formula.

$$f(z) = c_N \left\{ \int_{bD} f(\zeta) \frac{\omega'(\overline{\zeta - z}) \wedge \omega(\zeta)}{|\zeta - z|^{2N}} - \int_D \bar{\partial} f(\zeta) \wedge \frac{\omega'(\overline{\zeta - z}) \wedge \omega(\zeta)}{|\zeta - z|^{2N}} \right\}.$$

The form $\frac{\omega'(\overline{\zeta-z})\wedge\omega(\zeta)}{|\zeta-z|^{2N}}$ is the Cauchy–Fantappiè form $\Omega_{\bar\zeta-\bar z;\zeta-z}$. If φ is a smooth map from a neighborhood of $\bar D$ to $\mathbb{C}^N$ such that $\varphi(\zeta)\cdot(\zeta-z)$ does not vanish when ζ is near bD, then the difference $\Omega_{\varphi;\zeta-z}-\Omega_{\bar\zeta-\bar z;\zeta-z}$ is $\bar\partial$- and d-exact where it is defined: For a form Θ of bidegree $(N, N-2)$, $\Omega_{\varphi;\zeta-z} - \Omega_{\bar\zeta-\bar z;\zeta-z} = d\Theta = \bar\partial\Theta$. It follows that

(5.24)
$$f(z) = c_N\left\{\int_{bD} f(\zeta)\Omega_{\varphi;\zeta-z} - \int_{bD} f(\zeta)\,d\Theta - \int_D \bar\partial f(\zeta)\wedge\frac{\omega'(\overline{\zeta-z})\wedge\omega(\zeta)}{|\zeta-z|^{2N}}\right\}.$$

By Stokes's theorem, $\int_{bD} f(\zeta)\,d\Theta = -\int_{bD} u(\zeta)\wedge\Theta$, so the representation (5.24) is

(5.25)
$$f(z) = c_N\left\{\int_{bD} f(\zeta)\Omega_{\varphi;\zeta-z} + \int_{bD} u(\zeta)\wedge\Theta - \int_D u(\zeta)\wedge\frac{\omega'(\overline{\zeta-z})\wedge\omega(\zeta)}{|\zeta-z|^{2N}}\right\}.$$

If $\Omega_{\varphi;\zeta-z}$ happens to depend holomorphically on z, so that the first integral in (5.25) is holomorphic, then the function U defined by

$$U(z) = c_N\left\{\int_{bD} u(\zeta)\wedge\Theta - \int_D u(\zeta)\wedge\frac{\omega'(\overline{\zeta-z})\wedge\omega(\zeta)}{|\zeta-z|^{2N}}\right\} \tag{5.26}$$

satisfies $\bar\partial U = u$.

We are going to look at this general process in the particular case of forms defined on balls.

Fix an $R>0$ and a smooth $\bar\partial$-closed $(0,1)$-form on a neighborhood of $\bar{\mathbb{B}}_N(R)$. With $\varphi:\mathbb{C}^N\to\mathbb{C}^N$ the map $\varphi(\zeta)=\bar\zeta$, we have that $\varphi(\zeta)\cdot(\zeta-z)=|\zeta|^2-\langle z,\zeta\rangle$. In this case, the first integral in (5.25) is the Cauchy integral

$$\int_{b\mathbb{B}_N(R)} f(\zeta)\frac{\omega'(\bar\zeta)\wedge\omega(\zeta)}{(R^2-\langle z,\zeta\rangle)^N},$$

which is holomorphic in $\mathbb{B}_N(R)$. Thus in this case, we have the representation (5.26) for a solution U of $\bar\partial f = u$.

We have to obtain some information about the form Θ that appears in this case. The mechanism of the Cauchy–Fantappiè forms established in Chapter 1 would permit us to determine Θ explicitly with a bit of calculation. For our purposes, there is no need to perform these calculations. The construction given in Section 1.4 shows Θ to be a smooth $(N, N-2)$-form defined on the manifold

$$\mathscr{W} = (\mathbb{C}^N_\zeta\times\mathbb{C}^N_z)\setminus\{(\zeta,z)\in\mathbb{C}^N\times\mathbb{C}^N : \zeta=z \text{ or } |\zeta|^2=\langle z,\zeta\rangle\}$$

that depends smoothly on $z\in\mathbb{B}_N(R)$ when $\zeta\in b\mathbb{B}_N(R)$. Let us denote this form by $\Theta_{\mathbb{B}}$, so that we have the representation

$$U(z) = c_N\left\{\int_{b\mathbb{B}_N(R)} u(\zeta)\wedge\Theta_{\mathbb{B}} - \int_{\mathbb{B}_N(R)} u(\zeta)\wedge\frac{\omega'(\overline{\zeta-z})\wedge\omega(\zeta)}{|\zeta-z|^{2N}}\right\} \tag{5.27}$$

for a function U that satisfies $\bar\partial U = u$ on $\mathbb{B}_N(R)$. Given that the initial form u is smooth on a neighborhood of $\bar{\mathbb{B}}_N(R)$, the function U is smooth in $\mathbb{B}_N(R)$.

We are interested in one further transformation of this formula. The form $\Theta_{\mathbb{B}}$ is of the form

$$\Theta_{\mathbb{B}} = \sum_{1\le j<k\le N} \theta_{j,k}\omega_{j,k}(\bar\zeta) \wedge \omega(\zeta)$$

with functions $\theta_{j,k}$ that are smooth in $(\zeta, z) \in b\mathbb{B}_N(R) \times \mathbb{B}_N(R)$. They can be extended to be smooth functions on $\bar{\mathbb{B}}_N(R) \times \mathbb{B}_N(R)$. These extensions yield an extension of the form Θ to a smooth form, which we shall denote by T, on $\bar{\mathbb{B}}_N(R) \times \mathbb{B}_N(R)$. Stokes's theorem yields

$$\int_{b\mathbb{B}_N(R)} u(\zeta) \wedge \Theta_{\mathbb{B}_N(R)} = \int_{\mathbb{B}_N(R)} u(\zeta) \wedge dT.$$

We therefore have that the solution U is given by

$$U(z) = c_N \left\{ \int_{\mathbb{B}_N(R)} u(\zeta) \wedge dT - \int_{\mathbb{B}_N(R)} u(\zeta) \wedge \frac{\omega'(\overline{\zeta - z}) \wedge \omega(z)}{|\zeta - z|^{2N}} \right\}. \tag{5.28}$$

5.3. Surfaces in Strictly Pseudoconvex Boundaries

In our work on removable singularities we have seen the importance of considering subsets of the boundary of the ball that are polynomially convex. We are now going to show that certain surfaces contained in the boundary of the two-dimensional ball, or, more generally, in the boundary of a two-dimensional strictly pseudoconvex domain, Ω, are convex with respect to the algebra $\mathscr{O}(\bar\Omega)$. This line of investigation was initiated by Jöricke [190], who proved that a totally real disk in the boundary of $\mathbb{B}_2$ is a removable set for CR-functions. It is therefore polynomially convex by Theorem 5.2.16. As she noted, her proof yielded a corresponding result for totally real disks in two-dimensional strictly pseudoconvex boundaries.

Fix a relatively compact strictly pseudoconvex domain Ω with boundary of class $\mathscr{C}^2$ in a two-dimensional Stein manifold $\mathscr{M}$. We consider compact two-dimensional submanifolds with boundary in $b\Omega$, and we seek conditions under which they are $\mathscr{O}(\bar\Omega)$-convex.

Theorem 5.3.1. *A compact totally real disk Δ of class $\mathscr{C}^2$ in $b\Omega$ is $\mathscr{O}(\bar\Omega)$-convex.*

Explicitly, the hypothesis on Δ is that there is a closed two-dimensional totally real submanifold Σ of class $\mathscr{C}^2$ of an open neighborhood of $b\Omega$ for which there exists a $\mathscr{C}^2$ diffeomorphism ψ from a neighborhood of $\bar{\mathbb{U}}$ in $\mathbb{C}$ onto Σ that carries $\bar{\mathbb{U}}$ onto Δ.

Corollary 5.3.2. *A compact totally real disk of class $\mathscr{C}^2$ contained in $b\mathbb{B}_2$ is polynomially convex.*

If a totally real manifold $\Sigma \subset b\Omega$ is $\mathscr{O}(\bar\Omega)$-convex, then, because each continuous function on Σ can be approximated uniformly on Σ by functions holomorphic on a neighborhood of Σ, each continuous f on Σ can be approximated uniformly on Σ by functions holomorphic on a neighborhood of $\bar\Omega$. We have not yet proved this approximation result;

it is in Corollary 6.5.8 below. When $\bar{\Omega}$ is polynomially convex, e.g., in the case of the ball, we get polynomial approximation.

It should be observed that Theorem 5.3.1 is a result in two dimensions: There are simple examples of nonpolynomially convex totally real disks in the boundary of $b\mathbb{B}_3$. One such example is this: Start by noticing that the two-sphere $S = \mathbb{R}^3 \cap b\mathbb{B}_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, \Im z_2 = 0\}$ is totally real except at the points on S where $z_1 = 0$, i.e., at the points $(0, \pm 1)$. Introduce the function ψ given by $\psi(z) = \left[\frac{\bar{z}-2}{z-2}\right]^{1-|z|^2}$. This function is smooth on a neighborhood of $|z| \leq 1$, and it satisfies $\bar{\partial}\psi(0) \neq 0$. In addition, $|\psi|$ is identically one, and $\psi = 1$ on $|z| = 1$. Thus the surface $\Sigma = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 = 1, \Im z_2 = 0,$ and $z_3 = \psi(z_1)\}$ is a totally real two-sphere contained in $b\mathbb{B}_3(\sqrt{2})$. The sphere Σ contains the circle $\gamma = \{(e^{i\vartheta}, 0, 1) : \vartheta \in \mathbb{R}\}$, which bounds the disk $D = \{(z_1, 0, 1) : |z_1| \leq 1\}$. The circle γ splits Σ into two totally real disks; denote one of them by Δ. Thus, Δ is a totally real, smooth disk contained in $b\mathbb{B}_3(\sqrt{2})$ with $b\Delta = \gamma$; it is not polynomially convex.[4]

There are now three proofs for Theorem 5.3.1: The original argument of Jöricke combined with Theorem 5.2.16, a subsequent one of Forstnerič and the author [128], and a slightly more recent one found by Duval [105]. None of these proofs is very simple. In the developments of this section, we follow Duval's approach; it seems to be the most accessible, though it does draw essentially on some nontrivial results from the classical theory of ordinary differential equations.

Our context will be a bit more general than that of Theorem 5.3.1 in that we shall admit a finite number of nontotally real points. Thus, we have a relatively compact strictly pseudoconvex domain Ω with boundary of class $\mathscr{C}^2$ contained in the two-dimensional Stein manifold $\mathscr{M}$. In $b\Omega$, we fix a compact two-dimensional submanifold Σ with (possibly empty) boundary that we assume to be totally real except for a finite number of complex tangents. Denote by Σ^* the set of totally real points of Σ. On Σ^* we define a field of lines by the condition that for every $p \in \Sigma$,

$$L_p = T_p\Sigma \cap T_p^{\mathbb{C}}(b\Omega), \tag{5.29}$$

in which $T_p(\Sigma)$ denotes the tangent space of Σ at p, and $T_p^{\mathbb{C}}(b\Omega)$ denotes the maximal complex subspace of $T_p(b\Omega)$ that passes through p. Thus, $T_p^{\mathbb{C}}(b\Omega)$ is the unique complex line through p that is tangent to $b\Omega$. This line field defines a foliation of Σ^* the leaves of which are those curves that are everywhere tangent to the lines L_p. That is, a curve $x : (a, b) \to b\Omega$ is contained in a leaf of the foliation if and only if for all $t \in (a, b)$, the derivative $x'(t)$ lies in the line $L_{x(t)}$. This gives a foliation of class $\mathscr{C}^1$ of Σ^*, which is called the *characteristic foliation* of Σ. The analysis below depends on studying the characteristic foliation.

A neighborhood of a totally real point of Σ is foliated smoothly by the leaves of the characteristic foliation; points at which Σ has a complex tangent are singular points for it.

[4]The totally real disk Δ and the holomorphic disk D have the same boundary but are otherwise disjoint. This kind of configuration does not exist in $\mathbb{C}^2$ as shown by Duchamp and Forstnerič [103].

Given a set $X \subset b\Omega$, the *essential hull* of X is the set

$$\text{esshull-}X = \overline{\mathscr{O}(\bar{\Omega})\text{-hull}\, X \setminus X},$$

which is a compact subset of $\mathscr{O}(\bar{\Omega})$-hull X. The local maximum principle implies that the essential hull of X is contained in the $\mathscr{O}(\bar{\Omega})$-hull of the set $X \cap$ esshull-X.

The main theorem here is the following, due to Duval [105].

Theorem 5.3.3. *The essential hull of Σ cannot intersect a leaf of the characteristic foliation at a point at which Σ is totally real without crossing it.*

We shall see later—Theorem 5.3.13—what happens near certain points where Σ is not totally real.

The proof of this theorem depends on the characterization of hulls given by Oka. There is a technicality we have to deal with because we are working on a general two-dimensional Stein manifold rather than in $\mathbb{C}^2$. Recall that if X is a compact subset of $\mathbb{C}^N$ and $x \in \mathbb{C}^N \setminus X$, then $x \notin \widehat{X}$ if and only if there is a continuous function $F : [0,1] \times \mathbb{C}^N \to \mathbb{C}$ such that if for each $t \in [0,1]$, V_t denotes the analytic variety $\{z : F(t,z) = 0\}$, then $x \in V_0$, $V_t \cap X = \emptyset$ for all $t \in [0,1]$, and, finally, for an $R > 0$ so large that $X \subset \mathbb{B}_N(R)$, $V_1 \cap \mathbb{B}_N(R) = \emptyset$. We shall need the analogous statement in our context.

Lemma 5.3.4. *Let $\mathscr{R}$ be an N-dimensional Stein manifold, $N \geq 1$, and let X be a compact subset of $\mathscr{R}$. The point x does not belong to $\mathscr{O}(\mathscr{R})$-hull X if and only if there are a compact, $\mathscr{O}(\mathscr{R})$-convex set $Y \subset \mathscr{R}$ with $X \subset Y$ and a continuous function $f : [0,1] \times \mathscr{R} \to \mathbb{C}$ such that for all $t \in [0,1]$ the function $f(t,\cdot)$ is holomorphic and, with $W_t = \{z \in \mathscr{R} : f(t,z) = 0\}$, we have that $x \in W_0$, $W_t \cap X = \emptyset$ for all t, and $W_1 \cap Y = \emptyset$.*

We obtain this result as a consequence of the Oka result in $\mathbb{C}^N$ by using a very special case of the *Michael selection theorem*:

Theorem 5.3.5. *If E and F are Fréchet spaces and $u : E \to F$ is a continuous linear surjective map, then there is a continuous map $\varsigma : F \to E$ such that $u \circ \varsigma$ is the identity on F.*[5]

The general Michael selection theorem is a theorem in topology; it has nothing to do with linear spaces. See the paper [245]. The version just stated, for which a short proof has been given in [308], is a considerably easier result than the general version. The theorem is also contained in [70, Proposition 12, p. II.35].

Proof of Lemma 5.3.4. We invoke the embedding theorem for Stein manifolds to suppose that $\mathscr{R}$ is a closed complex submanifold of $\mathbb{C}^M$ for some M. With this understanding, a compact subset of $\mathscr{R}$ is $\mathscr{O}(\mathscr{R})$-convex if and only if it is polynomially convex.

Thus, suppose that $X \subset \mathscr{R}$ is convex with respect to $\mathscr{O}(\mathscr{R})$ and so polynomially convex. If $x \notin \mathscr{O}(\mathscr{R})$-hull X, then by the result of Oka there is a continuous function $F : [0,1] \times \mathbb{C}^M$ that is holomorphic in the second variable for a fixed value of the first variable and such that with V_t the zero locus of $F(t,\cdot)$, we have $x \in V_0$, $V_t \cap X = \emptyset$

[5] The question of when ς can be chosen to be linear has been investigated. In general, it is impossible to find a ς that is linear; under special circumstances a linear ς does exist. See [248]. In general, the selection ς cannot even be chosen to be homogeneous: See [308].

for all t, and $F^{-1}(1,\cdot)(0) \cap \mathbb{B}_M(R) = \emptyset$ for a large $R > 0$. The function f defined by $f = F|([0,1] \times \mathscr{R})$ is the function required by the lemma, with $Y = \mathscr{R} \cap \bar{\mathbb{B}}_M(R)$.

Conversely, we must show that if the function f of the statement of the lemma exists, then $x \notin \widehat{X}$. For this, let us change our point of view slightly. Consider f as a continuous map $\varphi : [0,1] \to \mathscr{O}(\mathscr{R})$, so that for all $t \in [0,1]$ and all $z \in \mathscr{R}$, $\varphi(t)(z) = f(t,z)$. The restriction map $\varrho : \mathscr{O}(\mathbb{C}^M) \to \mathscr{O}(\mathscr{R})$ is a continuous, surjective linear map between Fréchet spaces. According to the Michael selection theorem, it admits a continuous section: There is a continuous map $\varsigma : \mathscr{O}(\mathscr{R}) \to \mathscr{O}(\mathbb{C}^M)$ such that for all $g \in \mathscr{O}(\mathscr{R})$, $\varsigma g|\mathscr{R} = g$. If we define $F : [0,1] \times \mathbb{C}^M \to \mathbb{C}$ by $F(t,z) = (\varsigma\varphi(t))(z)$, then F, used in connection with Corollary 2.1.7 and the result that the polynomially convex hull of X is contained in $\mathscr{R}$, shows that x is not in the polynomially convex hull of X. The lemma is proved.

Below it will be necessary to have the notion of positive intersection for a totally real surface and a complex line. For this, start by considering a totally real two-plane π in $\mathbb{C}^2$ and a complex line λ, both passing through the origin, and meeting transversally there. The complex line has a natural orientation induced on it from the complex structure on $\mathbb{C}^2$. If an orientation is established on π, we can speak of π and λ intersecting positively or negatively in the following way. Let e_1 and e_2, taken in that order, be a positively oriented basis for λ over $\mathbb{R}$, and let f_1 and f_2, in that order, be one for π. If then e_1, e_2, f_1, f_2, with the vectors taken in that order, is a positively oriented basis for $\mathbb{C}^2 = \mathbb{R}^4$ over $\mathbb{R}$, then λ and π intersect positively. In the contrary case, they intersect negatively. The corresponding definition is meaningful for transversal intersections of *oriented* totally real manifolds and complex manifolds in $\mathbb{C}^2$; one considers the intersections of the associated tangent spaces.

There are simple examples: Let π be the real subspace $\{(x_1, 0, x_3, 0) : x_1, x_3 \in \mathbb{R}\}$, and let $\lambda_1 = \{(z, iz) : z \in \mathbb{C}\} = \{(x_1, x_2, -x_2, x_1) \in \mathbb{R}^4\}$, $\lambda_2 = \{(z, -iz) : z \in \mathbb{C}\} = \{(x_1, x_2, x_2, -x_1) \in \mathbb{R}^4\}$. The planes π and λ_1 meet positively at the origin and π and λ_2 meet negatively there.

Lemma 5.3.6. *Let D be a strictly convex domain in $\mathbb{C}^2$ with boundary of class $\mathscr{C}^2$ that is contained in the half-space $\Im z_2 > 0$ and whose boundary contains the origin. Let Σ be a totally real surface of class $\mathscr{C}^2$ contained in bD and passing through the origin. If α and β are small complex numbers, then the complex line $L_{\alpha\beta}$ with equation $z_2 = \alpha z_1 + \beta$ meets Σ in at most two points.*

Proof. The tangent $T_0\Sigma$ is a totally real two-plane that meets the z_1-axis in a real line. By a rotation about the origin in the z_1-plane, we can suppose that this line is the real x_1-axis. When α and β are small, $L_{\alpha,\beta}$ meets bD in a point or in a small convex curve, $C_{\alpha,\beta}$, or else in the empty set. If this intersection is a convex curve, denote by $C'_{\alpha,\beta}$ its projection into the z_1-plane, so that $C_{\alpha,\beta}$ is a graph over $C'_{\alpha,\beta}$.

Near the origin, the surface Σ is given by an equation $z_2 = g(z_1)$ with a function g of class $\mathscr{C}^2$ that vanishes at the origin and satisfies $g_{\bar{z}_1}(0) \neq 0$. Write $g = h + ik$ with real functions h and k. Because the x_1-axis is tangent to Σ and $g_{\bar{z}_1}(0) \neq 0$, we have that $g(z_1) = (A + iB)x_2 + O(|z_1|^2)$ with $A = h_{x_2}(0)$ and $B = k_{x_2}(0)$, where one of A and B is not zero. An intersection of the line $L_{\alpha,\beta}$ with Σ corresponds to a solution of the equation $g(z_1) = \alpha z_1 + \beta$ that lies on the curve $C'_{\alpha,\beta}$. That is, we are considering solutions of the equation $(A + iB)x_2 + O(|z_1|^2) = \alpha z_1 + \beta$. When α and β are small, there are at most two solutions to this equation on the curve $C'_{\alpha,\beta}$.

Let D be a strictly convex domain in $\mathbb{C}^2$ with boundary of class $\mathscr{C}^2$ contained in the half-space $\Im z_2 > 0$. Let Σ be a totally real two-dimensional surface of class $\mathscr{C}^2$ contained in bD that passes through the origin. Assume Σ to be oriented. This orientation induces a natural orientation on the leaves of the characteristic foliation: Given a point $p \in \Sigma$, let v_1 and v_2 lie in $T_p(\Sigma)$ and give, in that order, a positively oriented basis for $T_p(\Sigma)$. A nonzero vector v-tangent to the leaf of the characteristic foliation though p is positive if the triple v_1, v_2, Jv is a positively oriented basis for $T_p(bD)$. Here we denote by J the complex structure on $\mathbb{C}^2$, so that Jv can be identified with iv.

Let x and y be points of Σ that lie near the origin and that lie on the same leaf of the characteristic foliation. Assume Σ to be oriented in such a way that the direction along the leaf in question from x to y is the positive direction. By the preceding lemma, the complex line $\lambda_{x,y}$ through the points x and y meets Σ only at x and y. The total reality of Σ implies that this intersection is transversal.

Lemma 5.3.7. *The intersection of $\lambda_{x,y}$ with Σ at x is positive, that at y is negative.*

As an example, consider the torus $\mathbb{T}^2 = \{(e^{i\vartheta_1}, e^{i\vartheta_2}) : \vartheta_1, \vartheta_2 \in \mathbb{R}\}$, which is contained in the boundary of the ball $\mathbb{B}_2(\sqrt{2})$. The characteristic foliation consists of the subgroup $\Gamma = \{(e^{i\vartheta}, e^{-i\vartheta}) : \vartheta \in \mathbb{R}\}$ and its cosets. If we denote by λ the complex line through the points $(1, 1)$ and $(i, -i)$, then a calculation shows that λ meets the torus $\mathbb{T}^2$ at no other points and that its intersection with $\mathbb{T}^2$ at $(1, 1)$ is of opposite sign from its intersection at $(i, -i)$. (Without specifying an orientation on $\mathbb{T}^2$, we cannot say which is positive, which negative.)

Proof. Fix a strictly pseudoconvex defining function Q of class $\mathscr{C}^2$ for the domain D, so that $D = \{Q < 0\}$.

Let γ defined on $[0, 1]$ be a $\mathscr{C}^1$ parameterization of the arc of the characteristic leaf connecting x and y. The curve γ satisfies a differential equation, which is obtained as follows. For all t, $\gamma'(t)$ lies in $T^{\mathbb{C}}_{\gamma(t)}(bD)$. This line is orthogonal to the complex line spanned by the normal to bD. Thus $\gamma'(t) \cdot \operatorname{grad} Q(\gamma(t)) = 0 = \gamma'(t) \cdot J \operatorname{grad} Q(\gamma(t))$, in which $\cdot$ is the real inner product on $\mathbb{C}^N$.

The curve γ was assumed to be of class $\mathscr{C}^1$; in fact, it is of class $\mathscr{C}^2$: The function Q is of class $\mathscr{C}^2$, so the vector fields grad Q and Jgrad Q are of class $\mathscr{C}^1$. If we choose local coordinates $x = (x_1, x_2)$ in the surface Σ, then the equation satisfied by γ can be written in the form $x' = g(x)$ for a function g of class $\mathscr{C}^1$. Consequently, x' is of class $\mathscr{C}^1$, so x is of class $\mathscr{C}^2$.

Choose a vector $v(t)$ along γ such that for each t, the pair $v(t)$ and $\gamma'(t)$ taken in that order constitute a positively oriented basis for the tangent space $T_{\gamma(t)}\Sigma$. We choose $v(t)$ to be orthogonal to $\gamma'(t)$ for each t. By definition, $v(t)$, $\gamma'(t)$, and $J\gamma'(t)$ constitute a positively oriented basis for $T_{\gamma(t)}bD$. This means that if $N_{\gamma(t)}$ is the outer unit normal to bD at $\gamma(t)$, then the vectors $v(t)$, $\gamma'(t)$, $J\gamma'(t)$, and $N_{\gamma(t)}$ form a positively oriented basis for $T_{\gamma(t)}\mathbb{C}^2$.

The signs of intersection are defined independently of the choice of holomorphic coordinates. We choose new orthogonal coordinates for $\mathbb{C}^2$ such that x is the origin, such that $-e_4 = (0, 0, 0, -1)$ is the outer unit normal to bD at x, and such that $\gamma'(0) = e_1$. Then $J\gamma'(0) = e_2$. The vector $v(0)$ is tangent to bD but orthogonal to $\gamma'(0)$, so it is a linear combination of e_2 and e_3: $v(0) = (0, v_2(0), v_3(0), 0)$. Because the ordered quadruple $v(0), e_1, e_2, -e_4$ is a positively oriented basis for the tangent space $T_0\mathbb{C}^2$, the determinant $\det[v(0), e_1, e_2, -e_4]$, by which we understand the 4×4 determinant with the indicated columns, is positive. This implies that $v_3(0) < 0$.

We now show that the intersection of Σ and λ at the origin is positive.

The point x is the origin, so the vectors y and Jy constitute a positive basis for the tangent space of the complex line $\lambda_{x,y}$ at every point. The positivity assertion we are to prove is that $\det[v(0), e_1, y, Jy] > 0$, i.e., that

$$\det \begin{bmatrix} 0 & 1 & y_1 & -y_2 \\ v_2 & 0 & y_2 & y_1 \\ v_3 & 0 & y_3 & -y_4 \\ 0 & 0 & y_4 & y_3 \end{bmatrix} > 0. \tag{5.30}$$

In this determinant, we are writing v_2 and v_3 for $v_2(0)$ and $v_3(0)$, respectively; we retain this notation for the rest of the proof. By transversality, this quantity is not zero. Moreover, it changes continuously when y moves along the characteristic leaf in question. Thus, to establish the inequality (5.30), it is enough to establish the corresponding equality when the point y is replaced by a point $\gamma(t)$ with t a little larger than 0. By hypothesis, $\gamma'(0) = e_1$, so

$$\gamma(t) = (t + o(t), \gamma_2(t), \gamma_3(t), \gamma_4(t)),$$

and the coordinates $\gamma_2(t)$, $\gamma_3(t)$, and $\gamma_4(t)$ are all $O(t^2)$ as $t \to 0$, because γ is of class $\mathscr{C}^2$. What must be shown then is that

$$v_3[\gamma_2(t)\gamma_3(t) - (t + o(t))\gamma_4(t)] - v_2[\gamma_3^2(t) + \gamma_4^2(t)] > 0. \tag{5.31}$$

Because $v_3(0)$ is negative, and $\gamma_2(t)$ and $\gamma_3(t)$ are $O(t^2)$ as $t \to 0$, the inequality (5.31) will be proved if we can show that $\gamma_4(t) > \text{const}\, t^2$ for a positive constant and for t positive and small.

To do this, use the differential equation satisfied by γ. As we noted above, γ satisfies $\gamma'(t) \cdot \operatorname{grad} Q(\gamma(t)) = 0 = \gamma'(t) \cdot J \operatorname{grad} Q(\gamma(t))$. Explicitly written, this is the system of equations

$$\begin{aligned} \gamma_1' Q_1 + \gamma_2' Q_2 + \gamma_3' Q_3 + \gamma_4' Q_4 &= 0, \\ -\gamma_1' Q_2 + \gamma_2' Q_1 - \gamma_3' Q_4 + \gamma_4' Q_3 &= 0. \end{aligned}$$

In this system, all the derivatives of Q are evaluated at $\gamma(t)$. Solve these equations for γ_4' in terms of γ_1' and γ_2':

$$\gamma_4' = \frac{-\gamma_1'(Q_1Q_4 - Q_2Q_3) - \gamma_2'(Q_2Q_4 + Q_1Q_3)}{Q_3^2 + Q_4^2}. \tag{5.32}$$

Because $D \subset \{\Im z_2 > 0\}$, Q_4 is negative at the origin and so is bounded above by a negative quantity on a neighborhood of the origin. We need to estimate $Q_1(\gamma(t))$ for t small and positive. Write

$$Q_1(\gamma(t)) = Q_1(t + o(t), \gamma_2(t), \gamma_3(t), \gamma_4(t)) = Q_1(t, 0, 0, 0) + o(t).$$

We have that $Q_{11}(0) > 0$, because the function Q is strictly convex, so from $Q_1(t, 0, 0, 0) = \int_0^t Q_{11}(\tau, 0, 0, 0)\, d\tau$, it follows that for small positive t, $Q_1(t, 0, 0, 0) > \text{const}\, t$ for a positive constant. Thus, from (5.32), we deduce that $\gamma_4'(t) > \text{const}\, t$—recall that $\gamma_1'(t) = 1 + o(1)$, that Q_4 is negative, and that the derivatives Q_1, Q_2, and Q_3 vanish at the origin, as does γ_2'.

We have now shown that the intersection of $\lambda_{x,y}$ with Σ at x is positive. A parallel argument, now moving x toward y along γ, shows the intersection at y to be negative. The lemma is proved.

Return to the general setting: $\mathscr{M}$ is a two-dimensional Stein manifold, Ω is a relatively compact, strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in $\mathscr{M}$, and Σ is a compact totally real two-dimensional submanifold with boundary of class $\mathscr{C}^2$ of $b\Omega$.

The following lemma depends on Oka's characterization of hulls.

Lemma 5.3.8. *The point p of Σ does not lie in the essential hull of Σ if there exist two continuous families $\{V_t\}_{t\in[0,1)}$ and $\{W_t\}_{t\in[0,1)}$ of analytic curves in a neighborhood Ω' of $\bar{\Omega}$ with these properties:*

(a) *V_0 and W_0 meet Σ transversally at the point p and with opposite signs of intersection.*

(b) *For $t > 0$, the varieties V_t and W_t are disjoint from the intersection of the essential hull of Σ with Σ.*

Proof. Let $F, G : [0, 1) \times \Omega' \to \mathbb{C}$ be the functions that define the curves $\{V_t\}_{t\in[0,1)}$ and $\{W_t\}_{t\in[0,1)}$, respectively. For ease of notation, let $f_t \in \mathscr{O}(\Omega')$ be the function $F(t, \cdot)$, g_t the function $G(t, \cdot)$.

That V_0 and W_0 intersect Σ transversally at p and with opposite signs of intersection implies that f_0 and g_0 provide local holomorphic coordinates on a neighborhood of p and that, near p, the surface Σ can be described by an equation of the form $g_0 = h \circ f_0$ with h a diffeomorphism of class $\mathscr{C}^2$ defined near the origin in $\mathbb{C}$ that reverses orientation at the origin and so satisfies $|h_{\bar{z}}(0)| > |h_z(0)|$.

We show that if $\varepsilon > 0$ is small and if $b = \varepsilon h_{\bar{z}}(0)$, then the curves C_a defined by the equation

$$(f_0 - a)(g_0 - h(a)) + b = 0$$

as a runs through a small neighborhood of $0 \in \mathbb{C}$ fill out an open set of the form $U \setminus \Sigma$ for a suitable neighborhood U of p in $\mathscr{M}$.

We must see that for a near the origin in the complex plane, the curve C_a avoids Σ near the point p. In doing this, it is convenient to set $\xi = f_0$, $\eta = g_0$, so that ξ and η are holomorphic coordinates near the point p. On Σ, $\eta = h(\xi)$. Using the linear approximation to h at a leads to

$$\frac{\eta - h(a)}{h_{\bar{z}}(a)}(\xi - a) = |\xi - a|^2 + \frac{h_z(a)}{h_{\bar{z}}(a)}(\xi - a)^2 + o((\xi - a)^2).$$

The quantity on the right of the preceding equation lies in the right half-plane when a and ξ are near zero, because $|h_{\bar{z}}| > |h_z|$ near the origin. This implies that in a neighborhood of the point p, the curves C_a miss Σ, as desired.

Having this observation, we can conclude the proof as follows. To show that p is not in the set esshull-Σ, we must find a neighborhood U of p such that no point of $U \setminus \Sigma$ is in the hull of Σ. To do this, it is enough to show that each of the curves C_a of the last paragraph can be swept out of Ω through a continuous family of analytic curves in accordance with the criterion of Oka's characterization of hulls. Construct explicitly a family $\{W_t\}$: W_0 is the curve C_a. The family $\{W_t\}$ continues from here with the curves with equation $(f_0 - ta)(g_0 - h(ta)) + b = 0$ with t going from 1 to 0. These curves are followed by the curves with equation $f_t g_t + b = 0$ with t going from 0 to a small positive u, u small enough that these curves all avoid Σ near p. Next come the curves with equation $f_u g_u + tb = 0$ as t varies from 1 to 0. Finally, take the curves with equation $f_t g_t = 0$ as t goes from u to 1. The continuous family $\{W_t\}$ of curves described in this way shows that the essential hull of Σ avoids a small neighborhood of p, so the lemma is proved.

Proof of Theorem 5.3.3. The analysis is local. Fix a point $p \in \Sigma$ that lies in a leaf γ of the characteristic foliation but with the property that, locally, the essential hull of Σ does not meet both sides of the leaf.

The domain Ω is strictly pseudoconvex, so there are local holomorphic coordinates on a neighborhood of p with respect to which Ω is convex. Thus, for distinct points x and y in $b\Omega$ near p we denote by $\Delta(x, y)$ the domain in the complex line through x and y that is the intersection of this line with a slightly bigger, fixed convex domain. The domain $\Delta(x, y)$ is holomorphically equivalent to a disk. As x and y tend to p, $\Delta(x, y)$ tends to a domain in the complex line through p and tangent there to $b\Omega$.

Suppose Σ to have been assigned an orientation. In the event that Σ is not orientable, we assume that we are given an orientation of a neighborhood in Σ of the point p. Orient the leaves of the foliation as described above.

Denote by γ' a leaf of the foliation near γ and parallel to it. By hypothesis, γ' can be chosen to be disjoint from the essential hull of Σ in the vicinity of p. Let $\alpha : [0, 1] \to \Sigma$ be a short $\mathscr{C}^1$ arc connecting p to a point $p' \in \gamma'$, taking $\alpha(0) = p,\ \alpha(1) = p'$, and that lies entirely, except for its endpoint p, on the same side of γ as does γ'.

Fix a point $x \in \gamma$ that precedes p in the order along γ, and fix a corresponding point x' in γ' that precedes p'. Let $\beta : [0, 1] \to \Sigma$ be a $\mathscr{C}^1$ arc of γ' from x' to p', with $\beta(0) = x'$.

We are now going to define a continuous family $\{V_t\}$ of analytic curves. We start with the family $\Delta(x', \alpha(t))$ with $0 \le t \le 1$. The family continues with the disks $\Delta(\beta(t), p')$ starting at $t = 0$. At $t = 1$, we have arrived at the complex tangent $\Delta(p', p')$. The final segment of the family $\{V_t\}$ is obtained by translating $\Delta(p', p')$ into the complement of Ω along the normal to $b\Omega$ at p'.

Next construct a similar family $\{W_t\}$ of analytic curves of the same kind, starting this time from a point $y \in \gamma$ that succeeds p and a corresponding point y' in γ' that succeeds p' along γ'. In this construction, we work with a parameterization $\tilde{\beta}$ of the arc from y' to p' in γ' that starts from p'.

The curves V_o and W_o meet transversally at p with opposite signs of intersection, and the curves V_t and W_t are disjoint from the essential hull of Σ when $t > 0$.

Lemma 5.3.8 implies that the point p is not in the essential hull of Σ, and the theorem is proved.

The first application of Theorem 5.3.3 is to the polynomial convexity of totally real disks. Consider a compact two-manifold with boundary that is a compact disk, Δ, of class $\mathscr{C}^2$, in $b\Omega$, Ω a strictly pseudoconvex domain in a two-dimensional Stein manifold. That Δ is a compact manifold with boundary yields a slightly larger, open disk $\tilde{\Delta}$ that is a $\mathscr{C}^2$-submanifold of an open subset of $b\Omega$ in which Δ lies.

The leaves of the characteristic foliation of $\tilde{\Delta}$ are curves γ such that for all t, $\gamma'(t) \in T_{\gamma(t)}\Delta \cap T^{\mathbb{C}}_{\gamma(t)} b\Omega$. On the disk $\tilde{\Delta}$ there are everywhere-nonzero vector fields X such that for all points $y \in \tilde{\Delta}$, the vector X_y is tangent to the leaf through y of the characteristic foliation. Thus, the solution curves of the differential equation $y' = X_y$ are contained in leaves of the foliation. The Poincaré–Bendixson theory from ordinary differential equations implies that all of the integral curves for this foliation are *arcs*, homeomorphs of the interval [0, 1], with endpoints on $b\tilde{\Delta}$.[6] If, in this case, the hull of Δ is not empty, then its essential hull, esshull-Δ, is a nonempty compact subset of Δ. Then there are leaves of the characteristic foliation of $\tilde{\Delta}$ that miss this essential hull, and it is possible to find at least one such leaf, say γ, with the property that γ meets the essential hull of Δ but the essential hull lies on one side of γ. According to Theorem 5.3.3 this does not happen. The conclusion is that, as claimed, Δ is $\mathscr{O}(\bar{\Omega})$-convex. Theorem 5.3.1 is proved.

The preceding argument can be formulated more generally.

Corollary 5.3.9. *If Σ is a totally real submanifold of an open subset of $b\Omega$ on which the characteristic foliation can be defined as the level curves of a real-valued function, then compact subsets of Σ are polynomially convex.*

Proof. Let the level sets of the real-valued function φ on Σ be the leaves of the characteristic foliation. If $X \subset \Sigma$ is compact but not $\mathscr{O}(\bar{\Omega})$-convex, then φ attains its maximum over the compact set esshull-X at some point $x \in$ esshull-X. The set esshull-X then meets the leaf $\varphi^{-1}(\varphi(x))$ and lies on one side of it. This is a contradiction.

In general, totally real compact annuli contained in strictly pseudoconvex boundaries are not convex. As a simple example, let $C \subset b\mathbb{B}_2$ be the circle on which $z_2 = 0$. The polynomially convex hull of C is the closed unit disk in the z_1-axis. If the annulus A is chosen to be a thin ribbon in $b\mathbb{B}_2$ centered along C, then A cannot be polynomially convex. Such a ribbon can be chosen to be totally real.

[6]The precise formulation of the required result from the theory of Poincaré and Bendixson is this: *Consider the first-order ordinary differential equation $y' = f(y)$ on the simply connected planar domain D. Assume f does not vanish on D. If $y = y(t)$ is a solution of the equation defined on its maximal interval of definition, then the curve $y(t)$ does not remain in any compact subset of D when t approaches either endpoint of the interval of definition of $y(t)$.* For the details of this result, one can consult Hartman's book [160, p. 156].

Notice that the totally real annulus A in the preceding example is not contained in any totally real disk contained in $b\mathbb{B}_2$; such a disk could not be polynomially convex, contradicting Theorem 5.3.1.

Alexander [21] has given some nontrivial examples of polynomially convex annuli in strictly pseudoconvex boundaries. As Duval [105] remarked, these can be exhibited in the context of Theorem 5.3.3. For relatively prime positive integers p and q denote by $E_{p,q}$ the ellipsoid in $\mathbb{C}^2$ given by

$$E_{p,q} = \{z : p|z_1|^2 + q|z_2|^2 < 1\}.$$

These domains are strictly pseudoconvex, and each contains the torus

$$T = \left\{\left(\frac{e^{i\varphi}}{\sqrt{2p}}, \frac{e^{i\psi}}{\sqrt{2q}}\right) : \varphi, \psi \in \mathbb{R}\right\}$$

in its boundary. The leaves of the characteristic foliation in this case are the circles

$$\left\{\left(\frac{e^{ip\vartheta}}{\sqrt{2p}}, \frac{e^{-iq\vartheta}}{\sqrt{2q}}\right) : \vartheta \in [0, 2\pi]\right\}$$

in the torus T and their translates. That is to say, if $\pi(z, w) = z^p w^q$, then the characteristic foliation of T corresponds to the fibration of T in circles defined by the map $\pi : T \to K$ if K is the circle $\{\zeta \in \mathbb{C} : |\zeta| = (2p)^{-p/2}(2q)^{-q/2}\}$. If λ is an arc in K, which is necessarily a proper subset of K, then $\pi^{-1}(\lambda) \cap T$ is an annulus in T.

The integers p and q are relatively prime, so the fibers $\pi^{-1}(e^{i\alpha}) \cap T$ are connected. If now λ is an arc in K, then the annulus $A = \pi^{-1}(\lambda) \cap T$ is polynomially convex, for the leaves of the characteristic foliation are in this case defined by the real function $\log \pi$ on A.

For $\zeta \in K$, the fibers $F_\zeta = \pi^{-1}(\zeta)$, are polynomially convex and satisfy $\mathscr{P}(F_\zeta) = \mathscr{C}(F_\zeta)$, so by Theorem 1.2.16 and the remarks following its proof, we find that $\mathscr{P}(A) = \mathscr{C}(A)$. In fact, that earlier result can be invoked to give a proof of the polynomial convexity of the annulus A that is essentially simpler than the one just outlined, which depends on Theorem 5.3.3. This approach is near that of Alexander.

Alexander's paper [21] also contains a polynomial convexity result for totally real disks contained in cylinders: With $\mathbb{T}$ the unit circle in the plane, denote by M the cylinder $\mathbb{T} \times \mathbb{C} \subset \mathbb{C}^2$, a smooth real hypersurface that is not strictly pseudoconvex.

Theorem 5.3.10. *Every smooth, totally real compact disk Δ in M is polynomially convex.*

Proof. Define $\pi : M \to \mathbb{C}$ to be the projection onto the first factor: $\pi(e^{i\vartheta}, w) = e^{i\vartheta}$.

The integral curves for the characteristic foliation for M are the curves contained in the fibers $\pi^{-1}(e^{i\beta})$ for some real β. This is easily seen. With real coordinates (x, y, u, v) on $\mathbb{C}^2 = \mathbb{R}^4$, M is defined by the equation $x^2 + y^2 = 1$. At a point $(x, y, u, v) \in M$, the unit normal to M is the vector $(x, y, 0, 0)$. The condition that the curve γ in M given by $\gamma(t) = (x(t), y(t), u(t), v(t))$ be complex-tangential is that the vector $\gamma'(t)$ be Hermitian orthogonal to the normal to M at $\gamma(t)$ for all t. With $\cdot$ the real inner product on $\mathbb{R}^2$, this orthogonality condition is expressed by the equations

$$(x'(t), y'(t)) \cdot (x(t), y(t)) = 0 = (x'(t), y'(t)) \cdot (-y(t), x(t)),$$

which together imply that $x'(t) = y'(t) = 0$. It follows that the integral curves of the characteristic foliation are contained in the fibers of π.

Each fiber $F_\alpha = \pi^{-1}(e^{i\alpha}) \cap \Delta$ is polynomially convex. If it is not, then it separates the plane $\lambda_\alpha = \{e^{i\alpha}\} \times \mathbb{C}$ into several components, whence, by Alexander duality, the cohomology group $\check{H}^1(F_\alpha, \mathbb{Z})$ does not vanish. The compact disk Δ is contained in a larger open, totally real disk $\tilde{\Delta}$, which contains the fiber F_α. The set $\tilde{\Delta} \setminus F_\alpha$ is not connected, since $\check{H}^1(F_\alpha, \mathbb{Z}) \neq 0$. However, by the Poincaré–Bendixson theorem, each integral curve of the characteristic foliation of $\tilde{\Delta}$ is an arc with endpoints in $b\tilde{\Delta}$. Moreover, each such characteristic curve is contained in a plane $\pi^{-1}(e^{i\beta})$ for some β. Thus, if $p \in \Delta \setminus F_\alpha$, then p can be connected to $b\tilde{\Delta}$ by a curve that misses F_α. This is impossible under the assumption that F_α disconnects the disk $\tilde{\Delta}$. Consequently, F_α is polynomially convex.

Now introduce the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\Phi(\zeta_1, \zeta_2) = (e^{i\zeta_1}, \zeta_2)$. This map exhibits $\mathbb{R} \times \mathbb{C}$ as the universal covering space of $\mathbb{T} \times \mathbb{C}$. The set Δ is contractible, so $\Phi^{-1}(\Delta)$ is a disjoint union of subsets of $\mathbb{R} \times \mathbb{C}$ each of which is carried diffeomorphically onto Δ by Φ. Denote by Δ' one of these sets. Fix a positive integer m large enough that Δ' is contained in the product $(-2m\pi, 2m\pi) \times \mathbb{C}$. If $\Phi_m(\zeta_1, \zeta_2) = \Phi(\zeta_1/(4m), \zeta_2)$, and if $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ is the map given by $\Psi(z_1, z_2) = (z_1^{4m}, z_2)$, then $\Psi \circ \Phi_m(\zeta_1, \zeta_2) = \Phi(\zeta_1, \zeta_2)$. The map Φ_m carries the set Δ' diffeomorphically onto a subset Δ'' of $\mathbb{T} \times \mathbb{C}$, which, under the projection of $\mathbb{T} \times \mathbb{C}$ onto $\mathbb{T}$, is carried onto a proper subset of $\mathbb{T}$. The map Ψ is a proper holomorphic map from $\mathbb{C}^2$ to itself, so Δ is polynomially convex if and only if Δ'' is polynomially convex. That Δ'' is polynomially convex follows directly, though, for under the projection onto $\mathbb{T}$, the set Δ is carried into a compact subset S of $\mathbb{T}$ that is polynomially convex and that satisfies $\mathscr{P}(S) = \mathscr{C}(S)$. Consequently—recall Theorem 1.2.16—Δ'' is polynomially convex if and only if each of the fibers $S_{z_1} = \{(z_1, z_2) \in \Delta'' : z_1 \in S\}$ is a polynomially convex set that satisfies $\mathscr{P}(S_{z_1}) = \mathscr{C}(S_{z_1})$. These fibers do not separate the plane $\{z_1\} \times \mathbb{C}$, as follows from the analysis of the fibers F_α given above. Thus Δ'' is polynomially convex and satisfies $\mathscr{P}(\Delta'') = \mathscr{C}(\Delta'')$. We know then that the disk Δ possesses the same properties. Done.

We have examined above the behavior of the essential hull near totally real points. We will consider next the situation near points that are not totally real. This analysis begins most naturally with a study of the geometry of a surface near a point at which the surface is not totally real. This study is entirely local, so we consider a two-dimensional surface Σ in $\mathbb{C}^2$. Precisely, we suppose that Σ is a two-dimensional closed submanifold of an open set in $\mathbb{C}^2$, Σ of class $\mathscr{C}^1$. If $p \in \Sigma$, then there are only two choices for the tangent plane $T_p(\Sigma)$, which we view as a two-dimensional real-affine subspace of $\mathbb{C}^2$ that passes through p: $T_p(\Sigma)$ can be a complex line, in which case we say that Σ has a complex tangent at p, or else $T_p(\Sigma)$ is totally real, and we say that Σ is totally real at p. If Σ has a complex tangent at p, we say that Σ is complex at p.

It is a classical theorem of Levi-Civita, which is proved below—see Theorem 6.1.12—

that surfaces that have complex tangents at every point are complex manifolds.

In the Grassmannian $\mathbb{G}_{4,2}(\mathbb{R})$ of real two-dimensional subspaces of $\mathbb{C}^2$, the complex one-dimensional subspaces of $\mathbb{C}^2$ constitute a real-analytic subset of dimension two; in particular, they are nowhere dense: The totally real subspaces of $\mathbb{C}^2$ constitute a dense, open subset in $\mathbb{G}_{4,2}(\mathbb{R})$. We have therefore that the set of totally real points in our manifold Σ, if not empty, is an open subset of Σ. That the set of complex lines through the origin in $\mathbb{C}^2$ is a real-analytic set of dimension two in $\mathbb{G}_{4,2}(\mathbb{R})$ is immediate. Identify $\mathbb{C}^2$ with coordinates z_1, z_2 with $\mathbb{R}^4$ coordinates $x_1, \dots, x_4$ where $z_1 = x_1 + ix_2,\ z_2 = x_3 + ix_4$. Define the operator $J : \mathbb{R}^4 \to \mathbb{R}^4$ to be the complex structure on $\mathbb{C}^2$, so that $J(x_1, \dots, x_4) = (x_2, -x_1, x_4, -x_3)$. This is a real-linear transformation; it induces a real-analytic isomorphism, denoted by $\tilde{J}$, of $\mathbb{G}_{4,2}(\mathbb{R})$ onto itself. The elements of $\mathbb{G}_{4,2}(\mathbb{R})$ left fixed by $\tilde{J}$ are the complex lines in $\mathbb{G}_{4,2}(\mathbb{R})$. The equation $\tilde{J}(\pi) = \pi$ is an analytic equation the solutions of which are the complex lines.

The only case we will deal with below is that of *isolated* complex tangents. We suppose from here on that Σ is of class $\mathscr{C}^2$. There is a classification of isolated complex tangents in the following terms. Let $p \in \Sigma$ be a point at which Σ is complex. We can choose coordinates on $\mathbb{C}^2$ such that p is the origin and then write Σ as a graph over its tangent plane, at least locally near the origin: For some $\mathbb{C}$-valued function h of class $\mathscr{C}^2$ defined on a neighborhood of $0 \in \mathbb{C}$ and vanishing at the origin, Σ is given near the origin by an equation $z_2 = h(z_1)$. The condition that Σ be tangent at the origin to the z_1-axis is the condition that the differential dh vanish at the origin. We consider only certain nondegenerate cases: Suppose that not all the second-order derivatives of h vanish at the origin, so that near the origin, Σ is given by the equation

$$z_2 = az_1^2 + b\bar{z}_1^2 + cz_1\bar{z}_1 + o(z_1^2) \tag{5.33}$$

for some constants a, b, and c not all of which are zero. If in equation (5.33), the quantities b and c are zero, then by hypothesis the quantity a is not zero, and if $\alpha^2 = a$, then with $z_1' = \alpha z_1$ and writing z_1 for z_1', the equation (5.33) is

$$z_2 = z_1^2 + o(z_1^2). \tag{5.34}$$

Similarly, if $c = 0$ and $a = 0$, then equation (5.33) is equivalent to

$$z_2 = \bar{z}_1^2 + o(z_1^2). \tag{5.35}$$

If $c = 0$ and neither a nor b is zero, then replacing z_1 by $e^{i\vartheta} z_1$ for ϑ such that $e^{-2\vartheta} b > 0$ and then replacing z_2 by $z_2 - (ae^{2i\vartheta} - be^{-2i\vartheta})z_1^2$ brings the equation (5.33) into the form $z_2 = \gamma(z_1^2 + \bar{z}_1^2) + o(z_1^2)$ with $\gamma > 0$. The further change of variables $z_1 = z_1/\gamma$ then transforms the equation into the form

$$z_2 = (z_1^2 + \bar{z}_1^2) + o(z_1^2). \tag{5.36}$$

When $c \neq 0$, if we replace z_2 by cz_2, then the equation is brought into the form $z_2 = az_1^2 + b\bar{z}_1^2 + z\bar{z}_1 + o(z_1^2)$. With a change of variable that replaces z_1 by $e^{i\vartheta} z_1$ as before, we can arrange for b to be positive. Then replacing z_2 by $z_2 - (ae^{2i\vartheta} - be^{-2i\vartheta})z_1^2$ brings the equation into the form

$$z_2 = \gamma(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 + o(z_1^2) \tag{5.37}$$

with $\gamma > 0$.

We can now formulate the following basic notions. Fix a point $p \in \Sigma$ at which Σ is complex and that is isolated among the complex points of Σ.

Definition 5.3.11. *If near p with respect to suitable local coordinates, Σ is given by the equation (5.37) with $\gamma \geq 0$, then p is said to be an* elliptic point, *a* parabolic point, *or a* hyperbolic point *according as $\gamma \in [0, \frac{1}{2})$, $\gamma = \frac{1}{2}$, or $\gamma \in (\frac{1}{2}, \infty)$.*

This classification was introduced in the present context by Bishop [61].

Elliptic points are not polynomially convex; parabolic points may or may not be polynomially convex; and hyperbolic points are polynomially convex. More precisely, if $p \in \Sigma$ is an elliptic point, then no neighborhood of p in Σ is polynomially convex; if p is a hyperbolic point, then small neighborhoods of p in Σ are polynomially convex. As for parabolic points, some have polynomially convex neighborhoods in Σ; some do not.

It is easy to see that the quadratic model of an elliptic point does not have polynomially convex neighborhoods. In this case, the surface is defined by the equation $z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$, and the surface is contained in the three-dimensional space $\mathbb{C} \times \mathbb{R}$. It is a convex surface that contains the boundaries of the ellipses

$$\Delta_\tau = \{(z_1, \tau) : (1 + 2\gamma)x_1^2 + (1 - 2\gamma)x_2^2 = \tau\} \subset \mathbb{C} \times \{\tau\}$$

for $\tau > 0$. Thus, no neighborhood of $0 \in \Sigma$ is polynomially convex. The general elliptic point is a small perturbation of this quadratic model. *If p is such a point, then there is a continuous map $F : [0, 1] \times \bar{\mathbb{U}} \to \mathbb{C}^2$ such that for every $t \in (0, 1]$ the partial function $F(t, \cdot)$ is holomorphic and nonconstant in $\mathbb{U}$, such that for every $z \in \mathbb{U}$, $F(0, z) = p$, and such that $F([0, 1] \times b\mathbb{U}) \subset \Sigma$.* Thus, for small t the disks $F(\{t\} \times \mathbb{U})$ lie in the polynomially convex hull of small neighborhoods of p in Σ, and we see that p has no polynomially convex neighborhoods in Σ. This result was established by Bishop [61] and elaborated on by Kenig and Webster [203, 204]. See also the work of Forstnerič [124].

The situation for parabolic points can be illustrated by two simple examples.

For a nonpolynomially convex parabolic point, consider the surface Σ given by the equation $z_2 = x_1^2 + x_2^4$ in which the origin is a parabolic point. If $\varepsilon > 0$, then Σ contains the curve $\gamma_\varepsilon = \{(z_1, \varepsilon) : x_1^2 + x_2^4 = \varepsilon\}$ and so contains its polynomially convex hull $\widehat{\gamma_\varepsilon}$, which is the closure of a domain in the complex line with equation $z_2 = \varepsilon$. Thus, no small neighborhood of $0 \in \Sigma$ is polynomially convex.

The example given at the very end of Section 1.6.3 is an example of a polynomially convex parabolic point.

We shall say nothing more about parabolic points. For a penetrating analysis of them, which includes a theory of onions, the reader can consult the paper [191] of Jöricke.

Hyperbolic points are polynomially convex:

Theorem 5.3.12. [128] *If p is a hyperbolic point in the $\mathscr{C}^2$ surface Σ in $\mathbb{C}^2$, then there is a compact subset X of Σ that contains p in its interior such that $\mathscr{P}(X) = \mathscr{C}(X)$, whence X is polynomially convex.*

Proof. Choose coordinates such that p is the origin and the z_1-axis is the tangent plane to Σ at 0. Thus the part of Σ near the origin is given by an equation

$$z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + r(z_1) \tag{5.38}$$

for a $\gamma \in (\frac{1}{2}, \infty)$ and for a remainder term $r(z_1)$ that is $o(z_1^2)$ as $z_1 \to 0$.

Define a map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\Phi(z_1, z_2) = (z_1, z_1z_2 + \gamma(z_1^2 + z_2^2))$. This is a proper holomorphic map of $\mathbb{C}^2$ onto itself, the generic fiber $\Phi^{-1}(z_1, z_2)$ of which consists of two distinct points. It carries the two totally real planes defined by

$$V_1 = \{(\zeta, \bar{\zeta}) : \zeta \in \mathbb{C}\},$$
$$V_2 = \{(\zeta, -\gamma^{-1}\zeta - \bar{\zeta}) : \zeta \in \mathbb{C}\},$$

injectively onto the surface Σ_o given by the equation $z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$, though for what follows we do not need to verify this.

We will construct surfaces S_1 and S_2 that are tangent to V_1 and V_2 at the origin and that are carried onto neighborhoods of the origin in Σ by Φ. To construct S_1, which is to be a small perturbation of V_1 near the origin, we let

$$S_1 = \{(\zeta, \bar{\zeta} + f(\zeta)) : \zeta \in \mathbb{C}\}$$

with f a function to be determined that vanishes at the origin. The condition that Φ carry a neighborhood of $0 \in S_1$ into a neighborhood of $0 \in \Sigma$ is the condition that f satisfy the quadratic equation

$$\gamma f(\zeta)^2 + (\zeta + 2\gamma\bar{\zeta})f(\zeta) - r(\zeta) = 0. \tag{5.39}$$

The roots of this equation are given by

$$f(\zeta) = \frac{1}{2\gamma}\left\{-(\zeta + 2\gamma\bar{\zeta}) \pm \sqrt{(\zeta + 2\gamma\bar{\zeta})^2 + 4\gamma r(\zeta)}\right\}.$$

We want $f(\zeta) = o(\zeta)$ when $\zeta \to 0$, so we choose the plus sign in this equation. In this way, we obtain a function f that is of class $\mathscr{C}^1$ away from the origin and that satisfies $f(0) = 0$ and $df(0) = 0$. The function f is also of class $\mathscr{C}^1$ near the origin. To see this, write

$$\sqrt{(\zeta + 2\gamma\bar{\zeta})^2 + 4\gamma r(\zeta)} = (\zeta + 2\gamma\bar{\zeta}) + q(\zeta).$$

The function q satisfies the equation

$$q^2 + 2(\zeta + \gamma\bar{\zeta})q = 4\gamma r.$$

Differentiation of this equation with respect to ζ leads to

$$2(\zeta + \gamma\bar{\zeta} + q)q_\zeta = 4\gamma r_\zeta - 2q.$$

The right-hand side of this is $o(\zeta)$, so $q_\zeta \to 0$ as $\zeta \to 0$. Thus, q_ζ is found to be continuous at the origin. In the same way the derivative $q_{\bar\zeta}$ is also continuous there. The function f therefore is of class $\mathscr{C}^1$.

The construction of the surface S_2 follows the same lines. It is to be given by

$$S_2 = \{(\zeta, -\gamma^{-1}\zeta - \bar\zeta + g(\zeta)) : \zeta \in \mathbb{C}\}$$

with $g(\zeta) = o(\zeta)$, $\zeta \to 0$. The condition that $\Phi(S_2) \subset \Sigma$ is the condition that g satisfy the quadratic equation

$$\gamma g^2(\zeta) - (2\gamma\bar\zeta + \zeta)g(\zeta) - r(\zeta) = 0. \tag{5.40}$$

The quadratic formula yields

$$g(\zeta) = \frac{1}{2\gamma}\left\{(2\gamma\bar\zeta + \zeta) \pm \sqrt{(2\gamma\bar\zeta + \zeta)^2 + 4\gamma r(\zeta)}\right\},$$

in which we want the minus sign to obtain so that $g(\zeta) = o(\zeta)$, $\zeta \to 0$. As before, g is plainly of class $\mathscr{C}^1$ away from the origin, and its derivatives are found to be continuous at the origin by an argument like that used to prove f to be of class $\mathscr{C}^1$.

For small $\delta > 0$, the disks $S_1(\delta)$ and $S_2(\delta)$ defined by

$$S_j(\delta) = S_j \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le \delta\},\ j = 1, 2,$$

are polynomially convex and satisfy $\mathscr{P}(S_j(\delta)) = \mathscr{C}(S_j(\delta))$, for the surfaces S_1 and S_2 are totally real and therefore locally polynomially convex. Recall Corollary 1.6.15.

The surfaces $S_1(\delta)$ and $S_2(\delta)$ meet only at the origin, as follows from the condition that $\gamma > \frac{1}{2}$.

Define now a function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\psi(z) = \frac{1}{4}(z_1^2 - z_2^2) + \varepsilon z_1 z_2.$$

We are going to show that ψ takes $S_1(\delta)$ and $S_2(\delta)$ into sectors in the plane that meet only at the origin. For this purpose, notice that if $\zeta = \xi + i\eta$, then because $f(\zeta) = o(\zeta)$ when $\zeta \to 0$,

$$\psi(\zeta, \bar\zeta + f(\zeta)) = i\xi\eta + \varepsilon(\xi^2 + \eta^2) + o(\zeta^2).$$

This implies that

$$\psi(S_1) \subset \{u + iv \in \mathbb{C} : |v| \le Cu\}$$

for some positive constant C. In particular, the set $\psi(S_1(\delta))$ is contained in the right half-plane. The analysis of $\psi(S_2(\delta))$ is similar but a little more involved algebraically. Because $g(\zeta) = o(\zeta)$ when $\zeta \to 0$, we have that

$$\begin{aligned}&\psi(\zeta, -\gamma^{-1}\zeta - \bar\zeta + g(\zeta)) \\ &\quad = \frac{1}{4}\left\{-(\frac{1}{\gamma^2} + \frac{2}{\gamma})\xi^2 - (\frac{2}{\gamma} - \frac{1}{\gamma^2})\eta^2 + 2(2 - \frac{1}{\gamma^2})\xi\eta i\right\} - \varepsilon\zeta\left\{\frac{1}{\gamma}\zeta + \bar\zeta\right\} + o(\zeta^2).\end{aligned}$$

Because $\gamma > \frac{1}{2}$, the values of ψ, for small ζ and sufficiently small positive ε, are seen to lie in the closed left half-plane.

Because $\psi^{-1}(0) = \{0\}$, a polynomially convex set in $\mathbb{C}^2$, Theorem 1.6.19 implies that the union $S_1(\delta) \cup S_2(\delta)$ is polynomially convex and that

$$\mathscr{P}(S_1(\delta) \cup S_2(\delta)) = \mathscr{C}(S_1(\delta) \cup S_2(\delta)),$$

provided δ is small enough.

Now suppose that $E \subset \Sigma$ is a small neighborhood of the origin, small enough that the set $E^* = \Phi^{-1}(E)$ is contained in $S_1(\delta) \cup S_2(\delta)$. Thus, the set E^* is polynomially convex and satisfies $\mathscr{P}(E^*) = \mathscr{C}(E^*)$. The result now follows from Theorem 1.6.24.

We shall now examine what happens in the vicinity of a hyperbolic point in a surface contained in a strictly pseudoconvex boundary.

To begin with, we examine the characteristic foliation near an isolated complex tangent of a two-dimensional surface contained in a strictly pseudoconvex boundary. To this end we consider a two-dimensional surface Σ in $\mathbb{C}^2$ that passes through the origin, that is tangent to the z_1-axis there, and that is of class $\mathscr{C}^2$. We suppose Σ to be contained in $b\Omega$ for a strictly pseudoconvex domain Ω with boundary of class $\mathscr{C}^2$. Let Q be a strictly plurisubharmonic defining function for $b\Omega$. Let Σ be described near the origin by the equation

$$z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + r(z_1),$$

in which $r(z_1) = o(z_1^2)$, $z_1 \to 0$. We are interested in curves in Σ lying near the origin and tangent to the line field L_p that we have defined above. (For $p \in \Sigma$, the line L_p is the intersection of $T_p\Sigma$ and $T_p^{\mathbb{C}}(b\Omega)$.) Any curve in Σ that lies near the origin is the lift to Σ of a curve in the z_1-axis. Thus, consider a curve C in the z_1-axis with real parametric representation $x(t) = (x_1(t), x_2(t))$. If the remainder function r is given by $r(z) = s_1(z) + is_2(z)$ with real-valued functions s_1 and s_2, then the curve C lifts to the curve $\tilde{C}$ in Σ with the real parametric representation

$$\text{(5.41)}\quad \tilde{x}(t) = \big(x_1(t), x_2(t), (1+2\gamma)x_1^2 + (1-2\gamma)x_2^2 + s_1(x_1(t), x_2(t)), s_2(x_1(t), x_2(t))\big).$$

For $\tilde{x}(t)$ to lie in $b\Omega$ we must have $Q(\tilde{x}(t)) = 0$. The condition that the tangent of the curve $\tilde{C}$ lie in the complex subspace of the tangent space to $b\Omega$ is that the $\tilde{x}'(t)$ be orthogonal to the vector $J\,\mathrm{grad}\, Q(\tilde{x}(t))$, which is the condition that

(5.42)

$$-Q_2x_1' + Q_1x_2' - Q_4[2(1+2\gamma)x_1x_1' + 2(1-2\gamma)x_2x_2' + s_1(x_1, x_2)'] + Q_3s_2(x_1, x_2)' = 0,$$

in which we are writing Q_j for the derivative $\frac{\partial Q}{\partial x_j}$, and the derivatives Q_j are understood to be evaluated at the point $\tilde{x}(t)$. If for $j = 1, 2$ we use the Taylor expansion of first order for Q_j, i.e., if we write

$$Q_j(x) = \sum_{k=1}^{4} Q_{jk}(0)x_k + \tilde{q}_j(x)$$

with $\tilde{q}_j(x) = o(|x|)$, we find that the equation (5.42) can be rewritten as

$$\begin{aligned} -[Q_{21}(0)x_1 + Q_{22}(0)x_2 + Q_{23}(0)x_3 + Q_{24}(0)x_4 + \tilde{q}_1(x)]x_1' \\ + [Q_{11}(0)x_1 + Q_{12}(0)x_2 + Q_{13}(0)x_3 + Q_{14}(0)x_4 + \tilde{q}_2(x)]x_2' \\ - Q_4(0)[(2(1+2\gamma)x_1x_1' + 2(1-2\gamma)x_2x_2' + s_1(x_1,x_2)'] \\ + Q_3(0)s_2(x_1,x_2)' = 0. \end{aligned} \tag{5.43}$$

The second-order Taylor expansion about the origin of the function Q is

$$Q(x) = \sum_{j=1}^{4} Q_j(0)x_j + \frac{1}{2}\sum_{j,k=1}^{4} Q_{jk}(0)x_jx_k + \tilde{p}(x)$$

with $\tilde{p}(x) = o(|x|^2)$. In this, $Q_1(0) = Q_2(0) = 0$ because the complex line $z_2 = 0$ is tangent at the origin to $b\Omega$. The equation $Q(\tilde{x}) = 0$ now implies that

$$\begin{aligned} 0 = Q_3(0)((1+2\gamma)x_1^2 + (1-2\gamma)x_2^2 + s_1(x_1,x_2)) + Q_4(0)s_2(x_1,x_2) \\ + \frac{1}{2}\sum_{j,k=1}^{4} Q_{jk}(0)x_jx_k + p(x_1,x_2), \end{aligned} \tag{5.44}$$

in which $p(x_1, x_2) = \tilde{p}(x)$, and x_3 and x_4 are expressed in terms of x_1 and x_2 by means of the parameterization given in (5.41). From (5.44) we deduce that $Q_{11}(0) = -Q_3(0)(2(1+2\gamma))$ and $Q_{22}(0) = -Q_3(0)(2(1-2\gamma))$, whence $Q_{11}(0) + Q_{22}(0) = -2Q_3(0)$, which implies that $Q_3(0) \neq 0$, for the strict plurisubharmonicity of Q implies that Q is strictly subharmonic on the line $z_2 = 0$ near the origin, so that the partial Laplacian $Q_{11} + Q_{22}$ is positive at the origin. Equation (5.44) also implies that the partial derivatives $Q_{jk}(0)$ vanish if either of j and k is bigger than two.

It follows that the equation (5.43) can be rewritten as

$$\begin{aligned} 0 = [Q_3(0)(2(1-2\gamma))x_2 - Q_4(0)(2(1+2\gamma))x_1 + \varphi_1(x_1,x_2)]x_1' \\ + [-Q_3(0)(2(1+2\gamma))x_1 - Q_4(0)(2(1-2\gamma))x_2 + \varphi_2(x_1,x_2)]x_2', \end{aligned} \tag{5.45}$$

in which

$$\varphi_1(x_1,x_2) = q_1(x_1,x_2) - Q_4(0)s_{1,1}(x_1,x_2) + Q_3(0)s_{2,1}(x_1,x_2)$$

and

$$\varphi_2(x_1,x_2) = q_2(x_1,x_2) - Q_4(0)s_{1,2}(x_1,x_2) + Q_3(0)s_{2,2}(x_1,x_2)$$

with $s_{j,k} = \frac{\partial s_j}{\partial x_k}$. Also, $q_j(x_1, x_2) = \tilde{q}_j(x_1, x_2, x_3, x_4)$ with x_3 and x_4 expressed in terms of x_1 and x_2 by way of the parameterization of $\tilde{C}$.

The solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of this equation is the solution of the system

$$x' = Ax + E(x) \tag{5.46}$$

with A the matrix given by

$$\begin{bmatrix} Q_3(0)(2(1+2\gamma)) & Q_4(0)(2(1-2\gamma)) \\ -Q_4(0)(2(1+2\gamma) & Q_3(0)(2(1-2\gamma)) \end{bmatrix}$$

and

$$E(x) = \begin{bmatrix} -\varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \end{bmatrix}.$$

The determinant of the matrix A is $4(Q_3(0)^2 + Q_4(0)^2)(1-4\gamma^2)$, the sign of which is that of the quantity $1 - 4\gamma^2$, because as we noted above, $Q_3(0) \neq 0$.

From here on we restrict ourselves to the hyperbolic case, the case that $\gamma \in (\frac{1}{2}, \infty)$, which implies that the matrix A has distinct real eigenvalues of opposite sign. It follows that the origin is a saddle point for the equation $x' = Ax$. In this case, there are two integral curves that pass through the origin; they are straight lines and are the *separatrices* for the equation. They divide the plane into four domains each of which is foliated by the other integral curves of the equation, which are branches of hyperbolas.

The equation (5.46) is a small perturbation of the equation $x' = Ax$, so the geometry of its integral curves is qualitatively similar to that of the integral curves for the latter equation. Specifically, the vector $E(x)$ is of class $\mathscr{C}^1$ near the origin, and it and its derivatives of first order vanish at the origin. In Hartman's book [160, Theorem 7.1, p. 244], it is implied that there is a homeomorphism (which may not be of class $\mathscr{C}^1$) of the origin that leaves the origin fixed and that carries the integral curves of the equation (5.46) onto the integral curves of the equation $x' = Ax$.

The separatarices of $x' = Ax$ correspond under this homeomorphism to the two integral curves of equation (5.46) that pass though the origin; these are the separatrices for (5.46).

By lifting this local geometry of the equation (5.46) back to the surface Σ, we obtain in the vicinity of the origin two characteristic curves that pass through the origin; these are the separatrices at the origin. They divide a small neighborhood of the origin in Σ into four domains, each of which is foliated by characteristic curves. Near the origin in Σ the geometry of the characteristic curves is topologically the same as that of the integral curves of the constant-coefficient equation $x' = Ax$.

With this local analysis of the geometry of the characteristic foliation near a hyperbolic point in mind, we return to the setting of Theorem 5.3.3, so that Ω is a strictly pseudoconvex domain in a two-dimensional Stein manifold $\mathscr{M}$, $b\Omega$ is of class $\mathscr{C}^2$, and Σ is a compact two-dimensional manifold with boundary contained in $b\Omega$.

Theorem 5.3.13. [105] *If $p \in \Sigma$ is a hyperbolic point that lies in the essential hull of Σ, then this essential hull must meet at least two of the sectors of Σ determined by the separatrix at p near the point p.*

Proof. Let $p \in \Sigma$ be a hyperbolic point, and suppose the essential hull of Σ to meet only one of the sectors determined by the separatrix at p in the vicinity of p. Denote this sector by S; we take it to be defined only on a neighborhood of p.

The problem is a local one; we can choose coordinates near p in $\mathscr{M}$ with respect to which $b\Omega$ is strictly convex.

The plane tangent to Σ at p is a complex line; denote it by λ. As a complex line, λ has a natural orientation. We orient Σ so that the orientation on it is consistent with the orientation on λ. Having oriented Σ, we have an induced orientation on the leaves of the characteristic foliation as in the proof of Theorem 5.3.3. The tangent plane $T_p(b\Omega)$ is disjoint from the (open) sector S, and the same will be true of any real hyperplane obtained by moving it by a small rotation about the point p. Fix such a nearby hyperplane, say L. The complex line in L that passes through p contains a small analytic disk Δ that contains p and that meets $b\Omega$ in a convex curve. The intersection of Δ with Σ at p is positive, for the intersection of two complex lines is positive. If we fix a point q in $\Sigma \cap \Delta$, then there is a neighborhood V of p in Σ such that for every $p' \in \bar{V}$, the intersection of the complex line $\lambda_{q,p'}$ with Σ at p is positive.

By Theorem 5.3.3, the essential hull of Σ does not touch the separatrices in the boundary of the sector S near p, so if V is small enough, it will not meet $\bar{V} \cap bS$.

Consequently, if γ is a leaf of the foliation that passes through the point p, then γ meets the essential hull of Σ at most at the point p (within V).

In fact, $\bar{V} \cap \bar{S}$ is disjoint from the essential hull except possibly at the point p.

Let us assume this for the moment. The local maximum principle then implies that the essential hull of Σ is contained in the essential hull of $(\Sigma \setminus V) \cup \{p\}$. Consequently, if Δ' is a small totally real disk in $b\Omega$ that avoids $\Sigma \setminus V$ and the boundary of which contains an arc of the characteristic foliation, then it follows from Theorem 5.3.3 that the essential hull of $(\Sigma \setminus V) \cup \Delta'$ does not contain p.

Thus, to complete the proof, it suffices to establish the assertion above that $\bar{V} \cap \bar{S}$ is disjoint from the essential hull except possibly at the point p. To do this, we argue by contradiction, using Lemma 5.3.8. Let γ be a leaf of the characteristic foliation that passes near p in the sector S, and let q be the last point, in the sense of the natural orientation on γ, of the intersection of γ with the essential hull of Σ.

Introduce a curve $\alpha(t)$, $0 \le t \le 1$, connecting q through V to a point p_o near p but outside the sector S. Also, let $\beta(t)$, $0 \le t \le 1$, be a curve that lies in γ and connects the point q to a point q' just a little farther along γ than q.

Associate with these two curves two continuous families $\{V_t\}_{0 \le t \le 2}$ and $\{W_t\}_{0 \le t \le 2}$ of analytic curves as follows. For $t \in [0, 1]$, V_t is the disk $\Delta(p_o, \alpha(t))$ for the point p_o. Thus, V_1 is the complex tangent to $b\Omega$ at p_o. We then obtain the curves V_t, $1 \le t \le 2$, by translating V_1 into the complement of Ω. Similarly, W_t, $0 \le t \le 1$, is the disk $\Delta(q, \beta(t))$. Thus, W_1 is the complex tangent to $b\Omega$ at q'. In the range $1 \le t \le 2$, W_t is obtained by translating W_1 into the complement of Ω. These curves show, by way of Lemma 5.3.8, that the point q does not lie in the essential hull of Σ, and the theorem is proved.

As a corollary of the preceding result, we have the convexity of disks totally real except for a finite number of hyperbolic points:

Corollary 5.3.14. [128] *If Δ is a compact $\mathscr{C}^2$ disk with boundary in $b\Omega$ that is totally real except for a finite number of hyperbolic points, then Δ is $\mathscr{O}(\bar{\Omega})$-convex.*

Proof. [105] The proof depends on a study of the separatrices at the hyperbolic points. According to the theorem of Poincaré and Bendixson, a separatrix at a given hyperbolic point can terminate at another hyperbolic point or, alternatively, must go to the boundary of the disk. They cannot create a cycle. Accordingly, the union of all the separatrices is formed of a finite number of trees. Among these there are *terminal* trees, which are defined as those with the property that their complement, which is a union of simply connected domains, has at most one component that is not totally real.

Terminal trees exist, as can be seen in the following way. Given a tree, T, associate to it the set bT, a certain finite subset of the boundary of Δ. Then (a) the components of bT correspond bijectively to the components of $\Delta \setminus T$, (b) the totally real components of $\Delta \setminus T$ correspond to components of $b\Delta \setminus bT$ that contain boundary points of no other tree, and (c) if T and T' are two trees, then a component of $b\Delta \setminus bT$ cannot meet a component of bT' without containing it. Consider now a tree T in the union of the separatrices in Δ. If it is not terminal, then one of the components of $\Delta \setminus bT$ contains the boundary of another tree, T_1. If T_1 is terminal, done. Otherwise, one of the components of $b\Delta \setminus T_1$ contains the boundary of a further tree, T_2. In a finite number of steps, we arrive at a terminal tree.

Now fix a terminal tree T. Each component of $\Delta \setminus T$ is a topological disk that is totally real or else admits at most one hyperbolic point. On the totally real disks in the complement of T, the characteristic foliation is defined by the level sets of a function, so it follows that the essential hull does not meet any totally real component of $\Delta \setminus T$. Consequently, it can meet Δ at the hyperbolic points or at points of the nontotally real component, Δ_1, of $\Delta \setminus T$. Then the essential hull meets only one of the sectors determined by the hyperbolic points. Consequently, by Theorem 5.3.13, the essential hull of Δ meets Δ only in $\Delta_1 \setminus (V_1 \cup \cdots \cup V_s)$, where the V_j are small disks about the hyperbolic points. The local maximum principle implies that the essential hull of Δ is contained in that of Δ_1. In this way, the problem is reduced to a disk with at least one fewer hyperbolic point than Δ. Iterating the process leads finally to the conclusion that the essential hull of Δ misses Δ, whence it must be empty. The corollary is proved.

Our work so far in this section has shown that certain disks are convex. We have not yet established that they have the approximation property that each continuous function on them can be approximated uniformly by holomorphic functions. This issue will be settled in Section 6.5 of the next chapter.

A corollary that emerges from the work above is that not all smooth simple closed curves in $b\Omega$, Ω a strictly pseudoconvex domain in $\mathbb{C}^2$, bound totally real disks that are contained in $b\Omega$. If $\gamma \subset b\Omega$ is a smooth curve that is $b\Delta$ for a smooth totally real disk in $b\Omega$, then because Δ is $A(\Omega)$-convex, γ cannot be the boundary of a variety in Ω.

Chapter 6

APPROXIMATION

Introduction. The present chapter is devoted in the main to some approximation theorems for continuous functions defined on totally real sets. The approximation results established here are of a global nature. Section 6.1 contains preparatory material on totally real sets and manifolds. Section 6.2 introduces holomorphically convex compacta and develops some of their main properties. Section 6.3 contains a result on uniform approximation on compacta in totally real sets. Section 6.4 presents some material on the algebras $\mathscr{R}(X)$ for planar compacta X for use in the following section. Section 6.5 considers algebras on smooth manifolds and analytic varieties. Section 6.6 contains results on tangential approximation.

6.1. Totally Real Manifolds

Before proceeding to the study of approximation per se, it is important to obtain some information about totally real manifolds and certain more general sets.

A useful generalization of the notion of totally real manifold is that of totally real set, defined as follows.

Definition 6.1.1. *A closed subset X of a complex manifold $\mathscr{M}$ is said to be a* totally real set *if there is a neighborhood U of X in $\mathscr{M}$ on which is defined a nonnegative strictly plurisubharmonic function Q of class $\mathscr{C}^2$ such that $X = \{p \in U : Q(p) = 0\}$.*

One of the important facts about totally real sets is that they have fundamental neighborhood bases consisting of Stein domains. That this is so depends on the solution of the Levi problem: *A complex manifold is a Stein manifold if and only if there is a strictly plurisubharmonic exhaustion function for it*. That is, the complex manifold $\mathscr{M}$ is a Stein manifold if and only if there is a $\mathscr{C}^2$ strictly plurisubharmonic function Q on $\mathscr{M}$ such that for each $t \in \mathbb{R}$, the sublevel set $\mathscr{M}_t = \{p \in \mathscr{M} : Q(p) \leq t\}$ is compact. For this we refer to [180].

The following result was obtained by Harvey and Wells [167].

Theorem 6.1.2. *Let $\mathscr{M}$ be a complex manifold, and let $X \subset \mathscr{M}$ be a totally real set. If U is a neighborhood of X in $\mathscr{M}$, there exists a neighborhood V of X, $V \subset U$, that is a Stein manifold. Moreover, there exists a fundamental set $\mathscr{B}$ of neighborhoods of X each member of which is a Stein domain and which has the properties that* (a) *if $U, V \in \mathscr{B}$ and $U \subset V$, then $\mathscr{O}(V)$ is dense in $\mathscr{O}(U)$ and* (b) *each compact subset of X is $\mathscr{O}(V)$-convex for every $V \in \mathscr{B}$.*

Proof. Let Q be a $\mathscr{C}^2$ strictly plurisubharmonic function on the neighborhood W_0 of X in $\mathscr{M}$ such that $X = \{p \in W_0 : Q(p) = 0\}$. Denote by $\mathscr{E}$ the set of all positive functions ε of class $\mathscr{C}^2$ on W_0 that tend to zero at infinity, i.e., that are small off compact sets in W_o, and such that $Q - \varepsilon$ is strictly plurisubharmonic. The set $\mathscr{E}$ is nonempty, for if the derivatives of ε of first and second order are sufficiently small, then $Q-\varepsilon$ is strictly plurisubharmonic. For $\varepsilon \in \mathscr{E}$, let $V_\varepsilon = \{p \in W_0 : Q(p) < \varepsilon(p)\}$. We shall see that $\mathscr{B} = \{V_\varepsilon\}_{\varepsilon\in\mathscr{E}}$ has the properties we seek.

First of all, each V_ε is a neighborhood of the set X. Also, each V_ε is a Stein manifold, for the function χ_ε given on V_ε by $\chi_\varepsilon(p) = \big(\varepsilon(p) - Q(p)\big)^{-1}$ is a $\mathscr{C}^2$ plurisubharmonic exhaustion function for V_ε. It is plainly of class $\mathscr{C}^2$, and that it is an exhaustion function follows immediately, for χ_ε tends to $+\infty$ at infinity in V_ε, i.e., is large off compact subsets of V_ε. Finally, to see that V_ε is a Stein domain, we verify that χ_ε is plurisubharmonic. This is so, for χ_ε is the composition of the plurisubharmonic function $Q - \varepsilon$ and the function $-1/t$, which is increasing and convex on the set $(Q - \varepsilon)(V_\varepsilon)$.

That the family $\mathscr{B}$ is a neighborhood basis for the set X can be seen as follows. Fix a neighborhood V_0 of X in W_0, and let $\{U_j\}_{j=1,\dots}$ be a locally finite covering of W_0 by open sets U_j with $U_j \Subset W_0$. Choose $\mathscr{C}^\infty$ functions φ_j on W_0 such that $\varphi_j \geq 0$, $\sum_{j=1,\dots}\varphi_j = 1$, and $\operatorname{supp}\varphi_j \Subset U_j$. Let $\{c_j\}_{j=1,\dots}$ be a sequence of positive numbers, and set $\varepsilon = \sum_{j=1,\dots} c_j\varphi_j$. If the c_j's are small enough, then $Q-\varepsilon$ is strictly plurisubharmonic on W_0 and is positive on $W_0\backslash V_0$. Thus V_ε, which is $\{p \in W_0 : Q(p) < \varepsilon(p)\}$, is contained in V_0. The family $\mathscr{B}$ is seen to be a neighborhood basis for the set X.

To complete the proof of the theorem, we have to show that our family $\mathscr{B}$ enjoys the properties (a) and (b).

To prove (a), we are to show that if $V_{\varepsilon_1}, V_{\varepsilon_2} \in \mathscr{B}$ with $V_{\varepsilon_1} \subset V_{\varepsilon_2}$ and if K is a compact subset of V_{ε_1}, then every function $f \in \mathscr{O}(V_{\varepsilon_1})$ can be approximated uniformly on K by functions in $\mathscr{O}(V_{\varepsilon_2})$. To do this, it is enough to find an $\mathscr{O}(V_{\varepsilon_2})$-convex compact set K_1 with $K \subset K_1 \subset V_{\varepsilon_1}$. We are using here that if $\mathscr{N}$ is a Stein manifold and $S \subset \mathscr{N}$ is a compact, $\mathscr{O}(\mathscr{N})$-convex set, then each function g holomorphic on a neighborhood of S can be approximated uniformly on S by functions holomorphic on all of $\mathscr{N}$.[1] Denote by χ a strictly plurisubharmonic exhaustion function for the domain V_{ε_2} with $\chi < 0$ on K. We have that $V_{\varepsilon_1} = \{p \in W_0 : Q(z) < \varepsilon_1(z)\}$ for some $\varepsilon_1 \in \mathscr{E}$. The set K is compact,

[1] This is an extension of the Oka–Weil theorem and can be derived from it as follows. The Stein manifold $\mathscr{N}$ can be supposed to be a closed submanifold of a $\mathbb{C}^N$ for some N. The set S is then polynomially convex. If f is holomorphic on the neighborhood Ω of S in $\mathscr{N}$, then there is a polynomial polyhedron Δ in $\mathbb{C}^N$ such that $\Delta \cap \mathscr{N} \subset \Omega$. The function $f|(\Delta \cap \mathscr{N})$ extends as a holomorphic function, say F on Δ. The Oka–Weil theorem provides uniform approximation of F on S by polynomials.

so for $c \in (0,1)$ sufficiently near 1, $K \subset \{p \in W_0 : Q(p) < c\varepsilon_1(p)\}$. The function $Q - c\varepsilon_1$ is positive on the set $Y = \{p \in V_{\varepsilon_2} : \chi(p) \le 0 \text{ and } Q(p) \ge \varepsilon_1(p)\}$. Thus, for large $C > 0$, the function $\widetilde{\chi} = \chi + C(Q - c\varepsilon_1)$ is positive on Y. The function $\widetilde{\chi}$ is strictly plurisubharmonic on V_{ε_2} and is an exhaustion function for V_{ε_2}, for χ is an exhaustion function for V_{ε_2} and Q is bounded on V_{ε_2}. Then $K \subset \{p \in V_{\varepsilon_2} : \chi(p) \le 0\} \cap \{p \in V_{\varepsilon_2} : \widetilde{\chi}(p) \le 0\}$. The set on the right is the intersection of two $\mathscr{O}(V_{\varepsilon_2})$-convex sets and so is $\mathscr{O}(V_{\varepsilon_2})$-convex. This establishes (a).

For the proof of (b), let $K \subset X$ be compact. Notice that if $V_{\varepsilon_1} \in \mathscr{B}$, then $V_{\varepsilon_1} \supset V_\varepsilon$ for certain functions ε with $0 < \varepsilon \le c$, c a positive constant, that satisfy $K = \{p \in W_0 : \varepsilon(p) = c\}$. That this is so is evident if we recall the proof that $\mathscr{B}$ is a neighborhood basis for X. Now let $V_{\varepsilon_1} \in \mathscr{B}$. There is a function ε of the special kind just described such that $V_{\varepsilon_1} \supset V_\varepsilon$ and K is $\mathscr{O}(V_\varepsilon)$-convex. Because $\mathscr{O}(V_{\varepsilon_1})$ is dense in $\mathscr{O}(V_\varepsilon)$ by (a), this implies that K is $\mathscr{O}(V_{\varepsilon_1})$-convex: With ε a function chosen as above, the set K is

$$\{p \in V_\varepsilon : c + Q(p) \le \varepsilon(p)\} = \{p \in V_\varepsilon : \big(\varepsilon(p) - Q(p)\big)^{-1} \le c^{-1}\}.$$

The function $(\varepsilon - Q)^{-1}$ is, as we saw above, a strictly plurisubharmonic exhaustion function for V_ε, so its sublevel sets, e.g., K, are $\mathscr{O}(V_\varepsilon)$-convex.

The theorem is proved.

Next we show totally real manifolds to be totally real sets. To do this, we first show that a set is totally real if it is *locally* totally real.

Lemma 6.1.3. *Let $\mathscr{M}$ be a complex manifold and X a closed subset of $\mathscr{M}$. The set X is totally real provided that for each $p \in X$, there is a neighborhood U_p of p in $\mathscr{M}$ on which there is a nonnegative strictly plurisubharmonic function Q_p of class $\mathscr{C}^2$ such that $X \cap U_p = \{q \in U_p : Q_p(q) = 0\}$.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite collection of open sets in $\mathscr{M}$ with $X \subset \cup_{\alpha\in A} U_\alpha = U$ such that for each α there is a nonnegative strictly plurisubharmonic function Q_α of class $\mathscr{C}^2$ defined on U_α with $X \cap U_\alpha = \{p \in U_\alpha : Q_\alpha(p) = 0\}$. Let $\{\chi_\alpha\}_{\alpha\in A}$ be a partition of unity on U with $\chi_\alpha \ge 0$, supp $\chi_\alpha \Subset U_\alpha$, χ_α of class $\mathscr{C}^\infty$. If we understand $\chi_\alpha Q_\alpha$ to be zero on $U \backslash U_\alpha$, then $Q = \sum_{\alpha\in A} \chi_\alpha Q_\alpha$ is a well-defined nonnegative function of class $\mathscr{C}^2$ on U, and its zero set is X. If we compute in local holomorphic coordinates $z = (z_1, \ldots, z_N)$ in U, we find that for $p \in X$ and $w \in \mathbb{C}^N \backslash \{0\}$,

$$\sum_{r,s=1}^{N} \frac{\partial^2 Q}{\partial z_r \partial \bar{z}_s}(p) w_r \bar{w}_s > \sum_{\alpha\in A} \chi_\alpha(p) \sum_{r,s=1}^{N} \frac{\partial^2 Q_\alpha}{\partial z_r \partial \bar{z}_s}(p) w_r \bar{w}_s > 0.$$

Thus, if $V \subset U$ is a sufficiently small neighborhood of X, then Q is strictly plurisubharmonic on V. The set X is thus seen to be a totally real set.

It is now easy to prove that a closed totally real submanifold Σ of dimension k and class $\mathscr{C}^p$, $p \ge 2$, of a complex manifold $\mathscr{M}$ is a totally real set. By the foregoing lemma, it suffices to show that Σ is locally a totally real set. For this purpose, fix $p \in \Sigma$, and choose local holomorphic coordinates $(z_1, \ldots, z_N)$ on a neighborhood of p in $\mathscr{M}$ in which p is the origin and in which $T_0\Sigma = \{z : y_1 = \cdots = y_k = z_{k+1} = \cdots = z_N = 0\}$. Thus, near

0, Σ has a parameterization

$$y_r = \varphi_r(x') \qquad r = 1, \dots, k,$$
$$z_s = \varphi_s(x') \qquad s = k+1, \dots, N,$$

where we write x' for $(x_1, \dots, x_k) \in \mathbb{R}^k$. The functions φ_j vanish at the origin as do their gradients. Define Q near the origin by

$$Q(z) = \sum_{r=1}^{k} |y_r - \varphi_r(x')|^2 + \sum_{s=k+1}^{N} |z_s - \varphi_s(x')|^2,$$

where $z = x + iy$ with $x, y \in \mathbb{R}^N$. The function Q is nonnegative, and its zero locus is the manifold Σ. We have $\varphi_j(0) = 0$ and grad $\varphi_j(0) = 0$, so the Taylor expansion of φ_j about the origin is $\varphi_j(x') = q_j(x') + o(|x'|^2)$ for a suitable homogeneous quadratic polynomial q_j. Thus, the Taylor expansion of Q about the origin is

$$Q(z) = y_1^2 + \cdots + y_k^2 + |z_{k+1}|^2 + \cdots + |z_N|^2 + o(|z|^2).$$

The function Q is strictly plurisubharmonic near the origin, for the quadratic polynomial $y_1^2 + \cdots + y_k^2 + |z_{k+1}|^2 + \cdots + |z_N|^2$ is strictly plurisubharmonic. Consequently, Σ is a totally real set.

The argument just given does depend on having Σ be of class at least $\mathscr{C}^2$, and it establishes somewhat more than has been stated: *If the manifold is of class $\mathscr{C}^p$, $2 \le p \le \infty$, then there is a nonnegative strictly plurisubharmonic function Q of class $\mathscr{C}^p$ on a neighborhood of Σ in $\mathscr{M}$ such that Σ is the zero locus of Q. Moreover, if Σ is real-analytic, then Q can be taken to be real-analytic.*

Nothing we have done so far bears on the case of totally real submanifolds of class $\mathscr{C}^1$, and, as usual, the analysis required in the $\mathscr{C}^1$ case is more involved than that required in the case of smoother manifolds. We are going to treat a somewhat more general result than simply the case of $\mathscr{C}^1$ totally real manifolds. There are two reasons for doing this. First, the result we give seems to be the definitive result of its kind, and second, the proof we give is not essentially more complicated than is the result in the case of $\mathscr{C}^1$ totally real manifolds. The argument below is not long, but it is also not self-contained, for it draws on the *Whitney extension theorem*. The formulation of this result is recalled at the very end of this section. We shall work with *q-convex functions* in the sense of the following definition.

Definition 6.1.4. *If Q is a function of class $\mathscr{C}^2$ on the domain D in $\mathbb{C}^N$ such that the matrix $\left[\frac{\partial^2 Q}{\partial z_r \partial \bar{z}_s}\right]_{r,s=1,\dots,N}$ has at each point at least $N - q + 1$ positive eigenvalues, then Q is called q-convex.*

Thus, a 1-convex function is strictly plurisubharmonic, and every $\mathscr{C}^2$ function is $(N+1)$-convex. The condition of being q-convex is independent of the choice of local holomorphic coordinates and so is well defined for functions defined on complex manifolds.

It is well to issue a warning at this point: The terminology concerning q-convex functions is not absolutely fixed, and one must approach the literature with this in mind.

Definition 6.1.5. *If Σ is a submanifold of class $\mathscr{C}^1$ of an open set in $\mathbb{C}^N$, the CR-*dimension *of Σ at $p \in \Sigma$ is the dimension over $\mathbb{C}$ of the largest complex affine subspace of $\mathbb{C}^N$ through p and tangent at p to Σ.*

Thus, if Σ is totally real at p, its CR-dimension at p is zero. If Σ is a d-dimensional complex submanifold of $\mathbb{C}^N$, then at each of its points, its CR-dimension is d.

The following theorem was formulated and proved by Chirka [86]. It is an extension of earlier results of several authors. See in particular [289] and [186].

Theorem 6.1.6. *Let $\mathscr{M}$ be a complex manifold of dimension N, let Σ be a closed submanifold of $\mathscr{M}$ of class $\mathscr{C}^s$, $s = 1, 2, \dots, \infty$, and let $E \subset \Sigma$ be a closed subset of Σ at each point of which the CR-dimension of Σ is not more than n. There then exists a function Q on $\mathscr{M}$ with the following properties:* (a) *Q is nonnegative and has E as its zero set.* (b) *Q is of class $\mathscr{C}^{s+1}$ on $\mathscr{M}$ and of class $\mathscr{C}^\infty$ on $\mathscr{M}\setminus\Sigma$.* (c) *At each point of E, the Levi form of Q has at least $N - n$ positive eigenvalues.*

Thus, the function Q is $(n+1)$-convex on a neighborhood of the set E. In particular, if Σ is a totally real submanifold of class $\mathscr{C}^1$, there is a strictly plurisubharmonic function of class $\mathscr{C}^2$ on a neighborhood of Σ that is nonnegative and that has Σ as its zero locus.

Proof. We deal first with the local problem. Let $p \in \Sigma$. There is a neighborhood W_p of p in $\mathscr{M}$ in which $\mathscr{M}$ is defined by the vanishing of k, $k = 2N - \dim \Sigma$, real-valued functions of class $\mathscr{C}^s$ defined on $W_p : \Sigma \cap W_p = \{q \in W_p : \varphi_1(q) = \cdots = \varphi_k(q) = 0\}$ with $d\varphi_1 \wedge \cdots \wedge d\varphi_k$ nonzero at each point of W_p. We denote by $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_k$ functions in $\mathscr{C}^s(\mathscr{M})$ and of class $\mathscr{C}^\infty(\mathscr{M}\setminus\Sigma)$ that vanish on Σ, that satisfy $d\widetilde{\varphi}_j = d\varphi_j$, $1 \le j \le k$, *at the point* p, and that satisfy the further condition that their derivatives of order $s+1$, $D^\alpha \widetilde{\varphi}_j$ with $|\alpha| = s+1$, are estimated by $|D^\alpha \widetilde{\varphi}_j(q)| = o\big(\mathrm{dist}(q, \Sigma)^{-1}\big)$. The Whitney extension theorem provides the functions $\widetilde{\varphi}_j$ when it is applied to the jets $\{f_\alpha^{(j)}\}_{|\alpha| \le s}$ given by $f_\alpha^{(j)} = D^\alpha \varphi_j|\Sigma$.

Once we have the functions $\widetilde{\varphi}_j$, the argument proceeds as follows. Consider the function Q_p defined by $Q_p = \sum_{j=1}^k \widetilde{\varphi}_j^2$. This function is of class $\mathscr{C}^{s+1}$ on $\mathscr{M}$ and of class $\mathscr{C}^\infty$ on $\mathscr{M}\setminus\Sigma$. (That it is of class $\mathscr{C}^{s+1}$ on $\mathscr{M}$ is contained in the lemma given immediately below.) It is nonnegative, and, in a sufficiently small neighborhood W_p' of the point p, the zero locus of Q_p coincides with Σ.

We can compute the Levi form of Q_p at points of Σ: If $q \in \Sigma$, then $\widetilde{\varphi}_j(q) = 0$, so, in terms of local coordinates on $\mathscr{M}$,

$$\frac{\partial^2 \widetilde{\varphi}_j^2}{\partial z_\mu \partial \bar{z}_\nu}(q) = 2 \frac{\partial \widetilde{\varphi}_j}{\partial z_\mu}(q) \frac{\partial \widetilde{\varphi}_j}{\partial \bar{z}_\nu}(q),$$

which implies that

$$\sum_{\mu,\nu} \frac{\partial^2 Q_p}{\partial z_\mu \partial \bar{z}_\nu}(q) w_\mu \bar{w}_\nu = 2 \sum_{j=1}^k \Big| \sum_{\mu=1}^N \frac{\partial \widetilde{\varphi}_j}{\partial z_\mu}(q) w_\mu \Big|^2,$$

a nonnegative quantity. That is, Q_p is plurisubharmonic in W'_p if W'_p is sufficiently small. In addition, when $p \in E$, the Levi form of Q_p has at least $N - n$ positive eigenvalues: By hypothesis, the CR-dimension of Σ is not more than n at any point of Σ. Now $T_q^{\mathbb{C}}(\Sigma)$ is orthogonal, in the Hermitian sense, to the gradients of the defining functions $\widetilde{\varphi}_j$, so the complex linear span of these gradients has dimension at least $N - n$ at points $q \in \Sigma$ near p. The explicit computation of the Levi form of Q_p at points of Σ given above implies it to be positive definite on the complex linear span of the gradients of the functions $\widetilde{\varphi}_j$. Thus, granted that $p \in E$, the function Q_p is $(n+1)$-convex on a neighborhood of p.

It remains now only to glue these local solutions of our problem together with a partition of unity to obtain a nonnegative function Q' on $\mathscr{M}$ that is of class $\mathscr{C}^\infty$ on $\mathscr{M} \setminus \Sigma$, of class $\mathscr{C}^{s+1}$ on $\mathscr{M}$, and that is plurisubharmonic near Σ with zero locus Σ.

To do this, for each $p \in \Sigma$, let Q_p be a function as constructed above, and let V_p be a neighborhood of p in $\mathscr{M}$ on which Q_p is plurisubharmonic and on which its Levi form is positive definite on the orthogonal complement of $T_q^{\mathbb{C}}(\Sigma)$ when $q \in E \cap V_p$. Let $\{W_\alpha\}_{\alpha \in A}$ be a locally finite family of open sets in $\mathscr{M}$ with union that covers Σ. Assume that for each α, $W_\alpha \subset V_{p(\alpha)}$ for some $p(\alpha) \in \Sigma$; let Q_α be the associated function $Q_{p(\alpha)}$. Let $W_0 = \mathscr{M} \setminus \Sigma$. There is a family $\{\chi_\alpha\}_{\alpha \in A \cup \{0\}}$ of nonnegative functions of class $\mathscr{C}^\infty$ with supp χ_α a closed subset of W_α for each α such that $\sum_{\alpha \in A \cup \{0\}} \chi_\alpha = 1$ on $\mathscr{M}$. Define $Q_0 \in \mathscr{C}^\infty(\mathscr{M})$ to be the function identically 1. Then $Q' = \sum_{\alpha \in A \cup \{0\}} \chi_\alpha Q_\alpha$ is a nonnegative function on $\mathscr{M}$ that is of class $\mathscr{C}^\infty$ on $\mathscr{M} \setminus \Sigma$ and of class $\mathscr{C}^{s+1}$ on $\mathscr{M}$. Its zero locus is Σ, and it is plurisubharmonic on a neighborhood of Σ. Moreover, at points $p \in E$, the Levi form is positive on the orthogonal complement of $T_p^{\mathbb{C}}(\Sigma)$ and so Q is $(n+1)$-convex on a neighborhood of the set E. If, finally, Q'' is a nonnegative function of class $\mathscr{C}^\infty$ on $\mathscr{M}$ that has E as its zero locus and that vanishes to sufficiently high order on E, then the function $Q = Q' + Q''$ will serve as the function we seek, and the theorem is proved subject only to the proof of the following fact from calculus:

Lemma 6.1.7. *Let E be a closed subset of the open subset U of $\mathbb{R}^N$ that has finite $(N-1)$-dimensional measure. Let ψ be a function of class $\mathscr{C}^s$, s a positive integer, on U that is of class $\mathscr{C}^{s+1}$ on $U \setminus E$. If ψ vanishes on the set E and satisfies the condition that for $x \in U \setminus E$,*

$$D^\alpha \psi(x) = o(\operatorname{dist}(x, E)^{-1})$$

for all multi-indices α with $|\alpha| = s + 1$, then ψ^2 is of class $\mathscr{C}^{s+1}$.

Proof. Set $g = \psi^2$. We first show that if α is a multi-index with $|\alpha| = s + 1$, then $D^\alpha g$, which is defined on $U \setminus E$, extends continuously through E. For this, write

$$\begin{aligned} D^\alpha \psi^2 &= \sum_{\beta+\gamma=\alpha} c_{\beta\gamma} D^\beta \psi D^\gamma \psi \\ &= \sum_{\beta+\gamma=\alpha, \beta\neq 0,\alpha} c_{\beta\gamma} D^\beta \psi D^\gamma \psi + (c_{0\alpha} + c_{\alpha 0}) \psi D^\alpha \psi \\ &= I + (c_{0\alpha} + c_{\alpha 0}) \psi D^\alpha \psi. \end{aligned}$$

In this the term I is continuous on all of U, for, by hypothesis, ψ is of class $\mathscr{C}^s$ on U, and the summands of I involve derivatives of order not more than s. Also, the term $\psi D^\alpha \psi$

tends continuously to 0 at E, since $D^\alpha = o(\mathrm{dist}(x, E))^{-1}$, and, because ψ vanishes on E, $\psi(x) = O(\mathrm{dist}(x, E)$.

We have thus shown the existence of a continuous function G_α on U that agrees with $D^\alpha g$ on $U \setminus E$. It has to be shown that the derivative $D^\alpha g$ exists at points of E and that its values coincide on E with those of G_α. We first treat the one-dimensional case of this point. In this case, the set E is a finite set; it suffices to treat the case that it consists only of the origin. Because g^{s+1} extends continuously through the origin, we can write

$$g^s(x) - g^s(0) = \int_0^x g^{s+1}(\tau)\, d\tau = xg^{s+1}(\tau(x))$$

for a function τ that satisfies $|\tau(x)| \leq |x|$. It follows that

$$g^{s+1}(0) = \lim_{x\to 0}(g^s(x) - g^s(0))/x = \lim_{x\to 0} g^{s+1}(\tau(x)) = \lim_{x\to 0} g^{s+1}(x),$$

so the one-dimensional case of the lemma is proved.

We now consider the N-dimensional version of the result. Consider a point $y \in E$, which, without loss of generality, we suppose to be the origin. Consider also a multi-index α with $|\alpha| = s + 1$. We suppose that $D^\alpha g = \frac{\partial}{\partial x_1} D^\beta g$ for a multi-index β with $|\beta| = s$. Let $\eta : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the projection given by $\eta(x_1, \ldots, x_N) = (x_2, \ldots, x_N)$. By hypothesis, $\Lambda^{N-1}(E)$ is finite, so for almost every $x' \in \mathbb{R}^{N-1}$, the set $E \cap \eta^{-1}(x')$ is finite. Consequently, we can choose a sequence of points $\{y^{(n)}\}_{n=1,\ldots}$ in $\mathbb{R}^{N-1}$ that converges to the origin such that the line $\eta^{-1}(x')$ meets E in a finite set. (Recall Eilenberg's theorem, Theorem 3.3.6.) We then have that if e_1 is the unit vector $(1, 0, \ldots, 0)$, then

$$D^\beta g(y^{(n)} + he_1) - D^\beta g(y^{(n)}) = \int_0^h G_\alpha(y^{(n)} + \tau e_1)\, d\tau.$$

Take the limit of this as $n \to \infty$ to find that

$$D^\beta g(h, 0, \ldots, 0) - D^\beta(0, \ldots, 0) = \int_0^h G_\alpha(\tau, 0, \ldots, 0)\, d\tau,$$

whence $\frac{\partial}{\partial x_1} D^\beta g(0)$ exists and has the value $G_\alpha(0)$ as is shown by an analysis parallel to the analysis given for the one-dimensional case. The lemma is proved.

We have seen that totally real submanifolds are the zero loci of strictly plurisubharmonic functions, i.e., are totally real sets. There is a result in the reverse direction: Totally real sets are contained locally in totally real manifolds. This is a result of Harvey and Wells [168]. Precisely, they prove the following fact.

Theorem 6.1.8. *Let Q be a nonnegative strictly plurisubharmonic function of class $\mathscr{C}^{k+1}$, $k \geq 1$, on the open set U in $\mathbb{C}^N$, and let E be its zero set. For each $z_0 \in E$ there is a neighborhood U_{z_0} of z_0 in which there is a totally real submanifold Σ of class $\mathscr{C}^k$ with $\Sigma \supset E \cap U_0$.*

Proof. We may suppose that $z_0 = 0$. The nonnegative function Q vanishes on the set E, so grad Q also vanishes there. It follows that the Taylor expansion of Q about the origin is

$$Q(z) = \frac{1}{2}\left\{ \sum_{r,s=1}^{N} \frac{\partial^2 Q}{\partial z_r \partial \bar{z}_s}(0) z_r \bar{z}_s + \sum_{r,s=1}^{N} \frac{\partial^2 Q}{\partial z_r \partial z_s}(0) z_r z_s + \sum_{r,s=1}^{N} \frac{\partial^2 Q}{\partial \bar{z}_r \partial \bar{z}_s}(0) \bar{z}_r \bar{z}_s \right\}$$
$$+ o(|z|^2). \tag{6.1}$$

The matrix $\left[\frac{\partial^2 Q}{\partial z_r \partial \bar{z}_s}(0)\right]$ is positive definite, so, by making a nonsingular linear change of coordinates on $\mathbb{C}^N$, we can suppose that $\frac{\partial^2 Q}{\partial z_r \partial \bar{z}_s}(0) = 2\delta_{rs}$, where δ_{rs} is the Kronecker delta. With this choice of coordinates,

$$Q(z) = |z|^2 + \Re(z^t S z) + o(|z|^2)$$

with S the matrix $\left[\frac{\partial^2 Q}{\partial z_r \partial z_s}(0)\right]$ and with z taken as a column and with z^t as its transpose.

We shall construct below a nonsingular linear transformation $A : \mathbb{C}^N \to \mathbb{C}^N$ such that if $\zeta = Az$ with $\zeta = \xi + i\eta, \xi, \eta \in \mathbb{R}^N$, then the Hessian matrix $\left[\frac{\partial^2 Q}{\partial \xi_r \partial \xi_s}(0)\right]$ is positive definite at 0.

With this we are done: Near zero, the functions $\frac{\partial Q}{\partial \xi_r}$, $1 \leq r \leq N$, have linearly independent gradients, so the set Σ of their common zeros is a manifold of class $\mathscr{C}^k$ near 0. The tangent space $T_0\Sigma$ is the totally real space spanned by the vectors $\frac{\partial}{\partial \eta_1}, \ldots, \frac{\partial}{\partial \eta_N}$, so Σ is totally real near 0. Finally, Σ contains E, for grad Q vanishes on E.

Thus, it remains to find the linear transformation A. Let $z = x + iy \in \mathbb{C}^N$ with $x, y \in \mathbb{R}^N$, and let τ be the column vector $\tau = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^{2N}$. Express Q in terms of τ:

$$Q(\tau) = |\tau|^2 + \tau^t T \tau + o(|\tau|^2) = \tau^t (I + T)\tau + o(|\tau|^2), \tag{6.2}$$

in which the real symmetric $2N \times 2N$ matrix T is defined by the equality $\tau^t T \tau = \Re z^t S z$. If S is given by $S = A + iB$ with real matrices, then T has the block form $\left[\begin{smallmatrix} A & -B \\ -B & -A \end{smallmatrix}\right]$. Let J denote the $2N \times 2N$ real orthogonal matrix that corresponds to the real linear transformation on $\mathbb{C}^N$ defined by multiplication by i, so that $J = \left[\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right]$.

If $v = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ is an eigenvector of T with eigenvalue λ, so that $Tv = \lambda v$, then $T(Jv) = T\left[\begin{smallmatrix} -b \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} -Ab & Ba \\ -Ba & Ab \end{smallmatrix}\right] = -\lambda\left[\begin{smallmatrix} -b \\ a \end{smallmatrix}\right] = -\lambda(Jv)$, so Jv is also an eigenvector of T, with eigenvalue $-\lambda$. We can therefore choose an orthonormal family in $\mathbb{R}^{2N}$ that consists of eigenvectors of T, say $v_1, \ldots, v_N$, that correspond to *positive* eigenvalues $\lambda_1, \ldots, \lambda_N$, respectively. Denote by C the matrix with respect to the standard basis $\{e_1, \ldots, e_{2N}\}$ of $\mathbb{R}^{2N}$ of the linear transformation $\widetilde{C} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ determined by $\widetilde{C}(e_j) = v_j$, $j = 1, \ldots, N$, and $\widetilde{C}(e_{N+j}) = Jv_j$, $j = 1, \ldots, N$. We have $J(e_j) = e_{N+j}$, so $\widetilde{C}$ is complex linear from $\mathbb{C}^N$ to itself. The new coordinates σ we want are determined by $\sigma = C^{-1}\tau$. Because the matrix C is orthogonal, $C^{-1} = C^t$. If $\widetilde{Q}(\sigma) = Q(C\tau)$, then

$$\begin{aligned} \widetilde{Q}(\sigma) &= (C\sigma)^t (I + T)(C\sigma) + o(|\sigma|^2) \\ &= \sigma^t \sigma + \sigma^t (C^t T C)\sigma + o(|\sigma|^2) \\ &= \sigma^t \sigma + \sigma^t D\sigma + o(|\sigma|^2), \end{aligned} \tag{6.3}$$

provided D denotes the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_N, -\lambda_1, \ldots, -\lambda_N$. Take $\sigma = \xi + i\eta$, $\xi, \eta \in \mathbb{R}^N$. On the real space $\eta = 0$, the Taylor expansion of $\widetilde{Q}$ is

$$\widetilde{Q}(\xi) = \frac{1}{2} \sum_{r,s=1}^{N} \frac{\partial^2 \widetilde{Q}}{\partial \xi_r \partial \xi_s}(0)\xi_r \xi_s + o(|\xi|^2),$$

and by (6.3), this is

$$\sum_{r=1}^{N} (1 + \lambda_r)\xi_j^2 + o(|\xi_r|^2),$$

which is positive for small ξ. Thus, as we wished, the matrix $\left[\frac{\partial^2 \widetilde{Q}}{\partial \xi_r \partial \xi_s}\right]_{r,s=1,\ldots,N}$ is positive definite.

There is a result analogous to this theorem valid for q-convex functions. For this we refer to Iordan [186] and to Chirka [86].

The theorem just proved shows that totally real sets cannot be too big:

Corollary 6.1.9. *If E is a totally real set in a complex manifold $\mathscr{M}$ of dimension N, then the N-dimensional measure of E is locally finite.*

The theorem also implies that certain sets cannot be totally real. For example, every $\mathscr{C}^1$ arc in the complex plane is a totally real set, but a rectifiable curve that has a corner, e.g., the graph $\{(t, |t|) : -1 \leq t \leq 1\}$, is not a totally real set.

In Theorem 6.1.8 we have seen that totally real sets are locally contained in totally real manifolds. In some contexts it would be useful if there were a global result of the same kind. There is no such result, as an example given by Chaumat and Chollet [81] shows. This example depends on a criterion for totally real immersions.

Lemma 6.1.10. *If $\mathscr{M}$ is an N-dimensional $\mathscr{C}^1$-manifold, then a map $F = (f_1, \ldots, f_N) : \mathscr{M} \to \mathbb{C}^N$ of class $\mathscr{C}^1$ immerses $\mathscr{M}$ as a totally real immersed submanifold of $\mathbb{C}^N$ if and only if $F^*\omega = df_1 \wedge \cdots \wedge df_N$ is nowhere zero on $\mathscr{M}$.*

Here, as usual, ω denotes the form $dz_1 \wedge \cdots \wedge dz_N$ on $\mathbb{C}^N$.

Proof. Assume first that $F^*\omega$ is not zero at $p \in \mathscr{M}$. This implies that the smallest complex subspace of the tangent space $T_{F(p)}\mathbb{C}^N$ containing $dF(T_p\mathscr{M})$ is the whole space $T_{F(p)}\mathbb{C}^N$ itself. The image $dF(T_p\mathscr{M})$ is a real subspace of dimension no more than N, so that necessarily, $\dim_{\mathbb{R}} dF(T_p\mathscr{M}) = N$ and $dF(T_p\mathscr{M})$ is totally real. Thus, F is a totally real immersion.

The converse is clear: If F is a totally real immersion, then for each p, $dF(T_p\mathscr{M})$ is an N-dimensional totally real subspace and ω does not vanish on it.

Example 6.1.11. [81] Define $\varphi : \mathbb{R}^3 \to \mathbb{C}^3$ by

$$\varphi(\xi_1, \xi_2, \xi_3) = (\xi_1 \cos \xi_3, \xi_1 \sin \xi_3, \xi_2 e^{i\xi_3/2}).$$

This is a real-analytic map from $\mathbb{R}^3$ into the real subspace $\{z \in \mathbb{C}^3 : y_1 = y_2 = 0\}$ of $\mathbb{C}^3$, and it is an immersion on $\mathbb{R}^3 \backslash \{\xi : \xi_1 = 0\}$. The immersion is totally real, by the previous lemma, for

$$\varphi^*\omega = -\xi_1 e^{\frac{i\xi_3}{2}} d\xi_1 \wedge d\xi_2 \wedge d\xi_3,$$

which is nowhere zero on $\mathbb{R}^3 \setminus \{\xi : \xi_1 = 0\}$. The map φ exhibits the domain $D = \{\xi \in \mathbb{R}^3 : \xi_1 \in (0,2), \xi_2 \in (-1,1), \xi_3 \in \mathbb{R}\}$ as the universal cover of its image Σ, which is a totally real, three-dimensional submanifold Σ of an open set in $\mathbb{C}^3$. The domain D contains the infinite strip $S = \{\xi \in \mathbb{R}^3 : \xi_1 = 1, -\frac{1}{2} \leq \xi_2 \leq \frac{1}{2}, \xi_3 \in \mathbb{R}\}$, which is carried by φ onto a compact set M that is a Möbius band. To see that the image is a Möbius band, consider φ on the rectangle $R = \{\xi \in S : 0 \leq \xi_3 \leq 2\pi\}$. On R, φ is one-to-one except that the images of the intervals $\xi_1 = 1$, $-\frac{1}{2} \leq \xi_2 \leq \frac{1}{2}$, $\xi_3 = 0$, and $\xi_1 = 1$, $-\frac{1}{2} \leq \xi_2 \leq \frac{1}{2}$, $\xi_3 = 2\pi$, coincide but are traced out in opposite directions. Thus, M is a Möbius band. Let Δ be the disk $\{z \in \mathbb{C}^3 : x_1^2 + x_2^2 \leq 1, y_1 = y_2 = z_3 = 0\}$, and define X to be the compact set $M \cup \Delta$. The disk Δ and the Möbius band M meet in the circle $\Gamma = \{z \in \mathbb{C}^3 : x_1^2 + x_2^2 = 1, y_1 = y_2 = z_3 = 0\}$. We have that $X \setminus \{0\} \subset \Sigma$, that Δ is contained in the totally real manifold $\{y_1 = y_2 = y_3 = 0\}$, and that a neighborhood of Γ in the real plane $\{z \in \mathbb{C}^3 : y_1 = y_2 = z_3 = 0\}$ is contained in Σ. It follows that X is locally a totally real set, and so, by Lemma 6.1.3, it is a totally real set: There is a neighborhood U of X on which there is a nonnegative strictly plurisubharmonic function Q of class $\mathscr{C}^2$ with X as its zero locus.

However, the set X is not contained in a totally real submanifold of an open set in $\mathbb{C}^3$: It is not contained in any three-dimensional $\mathscr{C}^1$-manifold. Suppose that Σ_o is a three-dimensional $\mathscr{C}^1$ submanifold that contains X. The tangent bundle $T\Sigma_o$, when restricted to the disk Δ, is trivial, because Δ is contractible, and thus, in particular, $T\Sigma_o|\Gamma$ is trivial. At each point $p \in \Gamma$, $T_p\Sigma_o$ contains both T_pL and T_pM, where L denotes the real two-plane that contains the disk Δ. This is impossible, because M is not orientable. The bundle $T\Sigma_o$ is trivial near the circle Γ, so there is a neighborhood of Γ in Σ on which there exists a zero-free three-form Θ of class $\mathscr{C}^1$. Denote by ξ a $\mathscr{C}^1$ vector field on a neighborhood of Γ such that at each point $p \in \Gamma$, ξ_p is the inner normal to the circle Γ in (the real two-plane containing) the disk Δ. If the two-form Θ_ξ is the contraction $\Theta \llcorner \xi$, then Θ_ξ gives an orientation on the Möbius band M, at least near the circle Γ. Contradiction.

Sakai [316] has exhibited a broad class of totally real sets: *If D_1 and D_2 are disjoint strictly pseudoconvex domains in a complex manifold, then the intersection of their closures, if not empty, is a totally real set.*

We have been considering totally real manifolds. In contrast to these, there are submanifolds of $\mathbb{C}^N$ that provisionally may be called *totally complex*. These are the submanifolds $\mathscr{M}$ for which the tangent spaces $T_x\mathscr{M}$ are all complex-affine subspaces of $\mathbb{C}^N$. Totally complex submanifolds are, in fact, complex submanifolds:

Theorem 6.1.12. *If $\mathscr{M}$ is a $\mathscr{C}^1$ submanifold of a domain in $\mathbb{C}^N$ such that for each $p \in \mathscr{M}$, the tangent plane $T_p\mathscr{M}$ is a complex affine subspace of $\mathbb{C}^N$, then $\mathscr{M}$ a complex submanifold.*

This result is due to Levi-Civita [223]. The following proof follows [132].

Proof. We recognize immediately that $\mathscr{M}$ must be of even (real) dimension, for its tangent spaces are of even (real) dimension. The proof consists in exhibiting $\mathscr{M}$ locally as the graph of a holomorphic map.

To this end, fix a point $p_0 \in \mathscr{M}$. By choosing coordinates on $\mathbb{C}^N$ appropriately, we can suppose that p_0 is the origin in $\mathbb{C}^N$ and that $T_0\mathscr{M}$ is $\mathbb{C}^k = \{z \in \mathbb{C}^N : z_{k+1} = \cdots = z_N = 0\}$. We can represent $\mathscr{M}$ near 0 as the graph of a $\mathscr{C}^1$ map: There is a neighborhood V of 0 in $\mathbb{C}^k$ such that for some $f = (f_{k+1}, \ldots, f_N) : V \to \mathbb{C}^{N-k}$, $\mathscr{M}$ is, near 0, the graph of f, i.e., locally $\mathscr{M}$ is $\{z = (z', z'') \in \mathbb{C}^k \times \mathbb{C}^{N-k} : z'' = f(z')\}$. The map f is of class $\mathscr{C}^1$. For $z' \in V$, denote by $df_{z'}$ the linear map $\mathbb{C}^k \to \mathbb{C}^{N-k}$ that is tangent to f. The tangent plane $T_{(z', f(z'))}\mathscr{M}$ is then the set $\{(z', f(z')) + (z, df_{z'}(z)) : z \in \mathbb{C}^k\}$, the translate by $\big(z', f(z')\big)$ of the graph of $df_{z'}$. By hypothesis, $T_{(z', f(z'))}\mathscr{M}$ is a complex-affine subspace of $\mathbb{C}^N$, so the graph of the linear map $df_{z'}$ is a complex-linear subspace of $\mathbb{C}^N$. We show in the lemma below that this implies that $df_{z'}$ is complex linear. But the complex linearity of $df_{z'}$ for $z' \in V$ is precisely the condition that f satisfy the Cauchy–Riemann equations. It follows that f is holomorphic in V, as we wished.

Lemma 6.1.13. *If $L : \mathbb{C}^k \to \mathbb{C}^m$ is a real-linear map, then L is complex linear if and only if the graph $\mathscr{G} = \{\big(z, L(z)\big) : z \in \mathbb{C}^k\}$ is a complex subspace of $\mathbb{C}^k \times \mathbb{C}^m$.*

Proof. Granted that L is $\mathbb{R}$-linear, $\mathscr{G}$ is a real-linear subspace of $\mathbb{C}^k \times \mathbb{C}^m$, and if L is complex-linear, then $\mathscr{G}$ is a complex subspace.

Assume, conversely, that $\mathscr{G}$ is complex-linear. Then, if $z \in \mathbb{C}^k$, because $\mathscr{G}$ is a complex subspace, both $i\big(z, L(z)\big)$ and $\big(iz, L(iz)\big)$ belong to $\mathscr{G}$, and we have $iL(z) = L(iz)$, whence L is $\mathbb{C}$-linear, as desired.

The theorem can be regarded as a regularity theorem: Under the assumption that a $\mathscr{C}^1$ manifold has complex tangents, it is necessarily complex-analytic.

There is a relatively simple extension of what we have just done to the case of manifolds that are locally the graphs of Lipschitz mappings. Thus, consider an open set U in $\mathbb{C}^k$ and on U a $\mathbb{C}^m$-valued function f that satisfies $|f(z) - f(z')| \leq \lambda|z - z'|$ for some constant λ and all $z, z' \in U$. Let Γ be the graph of f, a closed submanifold of $U \times \mathbb{C}^m$. As a Lipschitz function, f is differentiable at almost every point of U by Rademacher's theorem [337]. If $z \in U$ is a point at which f has a differential, then Γ has a tangent plane at $\big(z, f(z)\big)$. *If at almost every point $z \in U$ the tangent plane $T_{(z, f(z))}\Gamma$ is a complex affine subspace of $\mathbb{C}^N$, then Γ is a complex manifold.* As above, we find that at the points $z \in U$ such that $T_{(z, f(z))}\Gamma$ is complex, the differential df is complex linear.

Thus, our assertion comes to establishing the following assertion: *Let D be a domain in $\mathbb{C}^k$ and let $g : D \to \mathbb{C}$ be a Lipschitz function such that for almost all $z \in D$, the differential $dg_z : \mathbb{C}^N \to \mathbb{C}$ is complex linear. Then g is holomorphic.* That the function g satisfies a Lipschitz condition implies that its partial derivatives are bounded. By a standard regularity theorem, to prove that g is holomorphic, it suffices to show that if ϕ is a smooth function on D with compact support, then

$$\int_D g \frac{\partial \phi}{\partial \bar{z}_j} \omega(\bar{z}) \wedge \omega(z) = 0 \tag{6.4}$$

for each $j = 1, \ldots, N$.

That this suffices is easily seen. First, let h be a *smooth* function on D such that for all ϕ smooth and compactly supported on D,

$$\int_D h \frac{\partial \phi}{\partial \bar{z}_j} \omega(\bar{z}) \wedge \omega(z) = 0. \tag{6.5}$$

By Stokes's theorem we deduce that $\int_D \frac{\partial h}{\partial \bar{z}_j} \phi \omega(\bar{z}) \wedge \omega(z) = 0$. This happens for all ϕ, so $\frac{\partial h}{\partial \bar{z}_j}$ must be the zero function. If now h is bounded and measurable and satisfies (6.5), extend h to all of $\mathbb{C}^N$ by taking $h = 0$ on $\mathbb{C}^N \setminus D$. The extended function is still bounded and measurable. Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be a smooth approximate identity on $\mathbb{C}^N$, so that $\chi_\varepsilon \in \mathscr{C}^\infty(\mathbb{C}^N)$, $\chi_\varepsilon \geq 0$, $\operatorname{supp}\chi_\varepsilon \subset \mathbb{B}_N(0, \varepsilon)$, and $\int_{\mathbb{C}^N} \chi_\varepsilon = 1$. Put $h_\varepsilon(z) = \int_{\mathbb{C}^N} h(z - \zeta)\chi_\varepsilon(\zeta) d\mathscr{L}(\zeta)$. This function is smooth, and it satisfies (6.5) and so is holomorphic on D. Moreover, $h_\varepsilon(z) \to h(z)$ for almost every $z \in \mathbb{C}^N$. Because $|h_\varepsilon(z)|$ is bounded by the supremum of h on $\mathbb{C}^N$, h must be holomorphic.

We revert to our function g and show that it satisfies the condition (6.4). Consider the case $j = 1$. The integral is $\int_D g d\phi \wedge \omega_{[1]}(\bar{z}) \wedge \omega(z)$, and by Stokes's theorem[2] this is $-\int_D \phi dg \wedge \omega_{[1]}(\bar{z}) \wedge \omega(z)$. By hypothesis, dg is complex linear almost everywhere and so is of the form

$$dg = \sum_{j=1}^N \frac{\partial g}{\partial z_j} dz_j.$$

This means that $dg \wedge \omega(z) = 0$ almost everywhere by type considerations, whence the integral vanishes.

This result can be summarized by saying that a manifold that is locally a Lipschitz graph and almost all of whose tangents are complex planes is necessarily a complex manifold. A much more general result of this kind has been given by King [206]; it characterizes complex-analytic varieties within the space of rectifiable currents in terms of complex tangents.

6.1.A. Appendix on the Whitney Extension Theorem

In this short appendix we recall the formulation of the Whitney extension theorem.

The formulation is in terms of jets. Let X be a closed set in the open set Ω of $\mathbb{R}^N$. A *jet of order k* on X is a collection $\{f_\alpha\}_{|\alpha|\leq k}$ of continuous functions on X indexed by the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$, each α_j a nonnegative integer, subject to the condition that $|\alpha| = |\alpha_1| + \cdots + |\alpha_N| \leq k$. The case that $k = \infty$ is admitted. The question answered by the Whitney extension theorem is, *given a jet* $\{f_\alpha\}_{|\alpha|\leq k}$ *of order k on X, is there a function F of class* $\mathscr{C}^k$ *on a neighborhood of X with* $D^\alpha F|X = f_\alpha$ *for each* α *with* $|\alpha| \leq k$? The result is the following.

[2] Our use of Stokes's theorem is not quite standard, but is easily justified. What is at issue is that if f is a Lipschitz function on $\mathbb{R}^n$ and ψ is a smooth compactly supported function on $\mathbb{R}^n$, then $\int_{\mathbb{R}^n} f(x) \frac{\partial \psi}{\partial x_1}(x) dx = -\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x)\psi(x)dx$. By Fubini's theorem the left integral is $\int_{\mathbb{R}^{n-1}} \{\int_R f(x) \frac{\partial \psi}{\partial x_1}(x) dx_1\} dx_2 \cdots dx_n$. By the Lebesgue theory of integration by parts, [171], the inner integral is $\int_R \frac{\partial f}{\partial x_1}(x)\psi(x)dx_1$, so another application of Fubini's theorem yields the result.

Theorem 6.1.14. *Given a closed set X in the open set Ω in $\mathbb{R}^N$ and given a jet $\{f_\alpha\}_{|\alpha|\le k}$ of order $k<\infty$, there is $F\in\mathscr{C}^k(\Omega)$ that is of class $\mathscr{C}^\infty$ on $\Omega\setminus X$ with $D^\alpha F|X=f_\alpha$ for each α with $|\alpha|\le k$ if and only if for each α,*

$$f_\alpha(x)=\sum_{|\beta|\le k-|\alpha|}\frac{1}{\beta!}f_{\alpha+\beta}(y)(x-y)^\beta+o(|x-y|^{k-|\alpha|}),$$

where the o-term is uniform when y is restricted to any compact set in X. If the jet is of infinite order, the necessary and sufficient condition for F to exist is that for every integer $m\ge 0$,

$$f_\alpha(x)=\sum_{|\beta|\le m}\frac{1}{\beta!}f_{\alpha+\beta}(y)(x-y)^\beta+o(|x-y|^m),$$

where for given m, the o-term is uniform when y is restricted to any compact subset of X.

The proof of the theorem yields a bit more than stated in the case of finitely smooth functions: *If β is a multi-index with $|\beta|>k$, then $|D^\beta F(x)|=o(dist(x,X))dist(x,X)^{|\beta|-k}$ for $x\in\Omega\setminus X$ with the o-term uniform when x is restricted to any compact set in Ω.*

If we are in $\mathbb{C}^N$, it is natural to deal with derivatives $\partial/\partial z_j$ and $\partial/\partial\bar{z}_k$. Given a closed subset X of the open set Ω in $\mathbb{C}^N$, consider a family $\{f_{\alpha\beta}\}$ of continuous functions indexed by pairs of multi-indices α,β with $|\alpha|+|\beta|\le k<\infty$. Then there is $F\in\mathscr{C}^k(\Omega)$ with $\partial^{|\alpha|+|\beta|}F/\partial z^\alpha\partial\bar{z}^\beta|X=f_{\alpha\beta}$ if and only if for each α,β,

$$f_{\alpha\beta}(z)=\sum_{|\mu|+|\nu|\le k-|\alpha|-|\beta|}\frac{1}{\mu!\nu!}f_{\alpha+\mu,\beta+\nu}(w)(z-w)^\mu(\overline{z-w})^\nu+o(|z-w|)^{k-|\alpha|-|\beta|}$$

with the o-term uniform when w is restricted to any compact set in X. There is a similar formulation for $\mathscr{C}^\infty$ functions.

This complex version of the theorem is a purely formal consequence of the real version.

The Whitney extension theorem is proved in Malgrange's monograph [233] on differentiable functions and in Federer's work [115]. See also [260].

6.2. Holomorphically Convex Sets

In addition to the notions of polynomially convex and rationally convex set, there is a notion of holomorphically convex set that is sometimes useful.

For the definition of these sets, a preliminary definition is required. For a set X in a complex manifold $\mathscr{M}$ the *algebra of germs of functions holomorphic on X* is the set of equivalence classes in the union $\cup\mathscr{O}(V)$, in which the union is extended over all open subsets V of $\mathscr{M}$ that contain the set X, defined by the equivalence relation $\sim_X$ given by the condition that if $f\in\mathscr{O}(V)$ and $f'\in\mathscr{O}(V')$, then $f\sim_X f'$ if there is a neighborhood V'' of X with $V''\subset V\cap V'$ such that $f|V''=f'|V''$. The equivalence classes are called *germs of functions holomorphic on X*. The set of these germs is a $\mathbb{C}$-algebra with the evident operations of addition and multiplication; it is denoted by $\boldsymbol{\mathscr{O}}(X)$.

Definition 6.2.1. *A compact subset X of a complex manifold $\mathscr{M}$ is* holomorphically convex *if every nonzero $\mathbb{C}$-algebra homomorphism $\chi : \mathscr{O}(X) \to \mathbb{C}$ is of the form $\chi([f]) = f(p)$ for some $p \in X$.*

In this definition, we use the notation that $[f]$ is the element of the algebra $\mathscr{O}(X)$ corresponding to the *function* f defined on a neighborhood of X. *The elements of $\mathscr{O}(X)$ are not, properly speaking, functions.* Nonetheless, one can assign a value $[f](x)$ to a germ $[f] \in \mathscr{O}(X)$ for any $x \in X$, for if g is holomorphic on a neighborhood of X and satisfies $[g] = [f]$, then the functional value $g(x)$ depends only on the equivalence class $[f]$.

The homomorphism χ of the definition is called a *character* of the algebra $\mathscr{O}(X)$.

Other notions of holomorphically convex set appear in the literature; one must take care when reading.

For any compact set X in $\mathbb{C}^N$, we denote by $\operatorname{spec}\mathscr{O}(X)$ the space of characters of $\mathscr{O}(X)$ endowed with the weak* topology. This is a compact space.

Lemma 6.2.2. *If X is a compact subset of $\mathbb{C}^N$, then* $\operatorname{spec}\mathscr{O}(X)$ *is the spectrum of the uniform algebra A of functions uniformly approximable on X by functions holomorphic on varying neighborhoods of X.*

Proof. If $\rho : \mathscr{O}(X) \to A$ is the restriction map, then by definition ρ has dense range, so that the dual map $\rho^* : \operatorname{spec} A \to \operatorname{spec}\mathscr{O}(X)$ is injective. It is also surjective. To see this, note that for each character χ of $\mathscr{O}(X)$, $\chi([f]) \in f(X)$ for each f holomorphic on a neighborhood: Otherwise, the element $[f - \chi([f])]$ of $\mathscr{O}(X)$ is invertible, which is impossible, because χ annihilates it. It follows that $|\chi([f])| \leq \|f\|_X$ for every function f holomorphic on a neighborhood of X. This bound implies that χ extends to a character $\tilde{\chi}$ of A. We have $\rho^*\tilde{\chi} = \chi$. Thus, ρ^* is surjective. It is also continuous, as follows from the definitions of the topologies on $\operatorname{spec}\mathscr{O}(X)$ and $\operatorname{spec} A$. Therefore, by compactness, it is a homeomorphism, and the lemma is proved.

Corollary 6.2.3. *If X is a compact connected set in $\mathbb{C}^N$, then* $\operatorname{spec}\mathscr{O}(X)$ *is connected.*

Proof. If $\operatorname{spec}\mathscr{O}(X) = \operatorname{spec} A$ is not connected, then the uniform algebra A contains a function that assumes the values 0 and 1 and no others, as follows from the Shilov idempotent theorem. (Recall Theorem 1.5.9.) It follows that the same is true of the algebra $\mathscr{O}(X)$, which is impossible given that X is connected.

A simple class of holomorphically convex sets is the class of compact, totally disconnected subsets of $\mathbb{C}^N$. That these sets are holomorphically convex is easily seen. If X is any compact totally disconnected subset of $\mathbb{C}^N$, then the algebra A is all of $\mathscr{C}(X)$. This is so, for if $X = X' \cup X''$ is a decomposition of X into a union of mutually disjoint open (and therefore closed) subsets, then the function g that is one on X', zero on X'' extends to be holomorphic on a neighborhood of X. Thus, A contains enough real-valued functions to separate the points of X, and the Stone–Weierstrass theorem implies that $A = \mathscr{C}(X)$. Consequently, if X is totally disconnected, then for each character χ of $\mathscr{O}(X)$, there is a point $x \in X$ for which $\chi([f]) = f(x)$ for all f holomorphic on a neighborhood of X. That is, X is holomorphically convex. We know that not every compact, totally disconnected set is rationally convex.

That for an arbitrary compact set $X \subset \mathscr{M}$ and that for every character χ of $\mathscr{O}(X)$,

$|\chi(f)| \le \|f\|_X$, implies that each χ has the following continuity property: If Ω is an open set containing X, then $f \mapsto \chi([f])$ is a continuous linear functional on $\mathscr{O}(\Omega)$.

A compact set in $\mathbb{C}^N$ that is an intersection of domains of holomorphy is holomorphically convex, but the converse of this statement is not true. (Examples are given in [63], [54], [275], and [339].) There is a result of this general flavor, which involves Riemann domains and was stated by Birtel [53, 54].

To recall the notion of Riemann domain, fix a complex manifold $\mathscr{M}$. A *Riemann domain* over $\mathscr{M}$ is a pair $(\Omega, \ \pi)$ consisting of a complex manifold Ω and a locally biholomorphic map $\pi : \Omega \to \mathscr{M}$. (The map π is not required to be surjective.)

If $\mathscr{M}$ is a Stein manifold, the envelope of holomorphy $(\tilde{\Omega}, \pi)$ of a domain Ω in $\mathscr{M}$ is a Riemann domain over $\mathscr{M}$ for which $\tilde{\Omega}$ is a Stein manifold. More precisely, if Ω is a domain in $\mathscr{M}$, there exists a Riemann domain $(\tilde{\Omega}, \pi)$ over $\mathscr{M}$ with the properties that (a) $\tilde{\Omega}$ is a Stein manifold and (b) there is a biholomorphic map ι from Ω onto a domain in $\tilde{\Omega}$ such that $\pi \circ \iota$ is the identity map on Ω and such that for each $f \in \mathscr{O}(\Omega)$, there is a unique $\tilde{f} \in \mathscr{O}(\tilde{\Omega})$ such that $f = \tilde{f} \circ \iota$. The Riemann domain $(\tilde{\Omega}, \pi)$ is the envelope of holomorphy of Ω. For domains in Stein manifolds, the envelope of holomorphy exists; it is unique to within isomorphisms of Riemann domains. One of the standard constructions for the envelope of holomorphy of a domain Ω is that as a set, $\tilde{\Omega}$ is the set of all characters of the algebra $\mathscr{O}(\Omega)$, i.e., the set of all continuous, nonzero $\mathbb{C}$-linear maps $\mathscr{O}(\Omega) \to \mathbb{C}$ that are multiplicative. (Note that $\mathscr{O}(\Omega)$ is not a uniform algebra, so the previously given definition of character of a uniform algebra is not applicable here.) The space $\tilde{\Omega}$ is endowed with the weak* topology, so that a net $\{\chi_\alpha\}_{\alpha \in A}$ converges to χ_o if and only if for each $f \in \mathscr{O}(\Omega)$, $\chi_\alpha(f) \to \chi_o(f)$. We shall denote by spec $\mathscr{O}(\Omega)$ the space of all characters of $\mathscr{O}(\Omega)$ endowed with the weak* topology. The inclusion map $\iota : \Omega \to \tilde{\Omega}$ is the map that takes the point $z \in \Omega$ to the evaluation functional ε_z given by $\varepsilon_z(f) = f(z)$. That the space $\tilde{\Omega}$ constructed in this way is indeed a Stein manifold is a decidedly nontrivial point in the theory of functions. This theory is developed in detail in the book of Gunning and Rossi [158]. In the case that $\mathscr{M}$ is $\mathbb{C}^N$, the associated map $\pi : \tilde{\Omega} \to \mathbb{C}^N$ is the map $\chi \mapsto (\chi(z_1|\Omega), \dots, \chi(z_N|\Omega))$ if we understand $z_1, \dots, z_N$ to be the coordinate functions on $\mathbb{C}^N$. When working over a general Stein manifold, the description of the projection is a little more complicated than in the case of $\mathbb{C}^N$. Because $\mathscr{M}$ is a Stein manifold, every character of the algebra $\mathscr{O}(\mathscr{M})$ is of the form $f \mapsto f(p)$ for a fixed $p \in \mathscr{M}$. The restriction map $\rho : \mathscr{O}(\mathscr{M}) \to \mathscr{O}(\Omega)$ given by $\rho(f) = f|\Omega$ therefore induces a continuous map $\rho^* : \tilde{\Omega} \to \mathscr{M}$. This map is the projection $\pi : \tilde{\Omega} \to \mathscr{M}$ of the definition of the envelope holomorphy. It must be and can be verified that the projection π defined in this way is locally biholomorphic. For $f \in \mathscr{O}(\Omega)$, the function $\tilde{f} \in \mathscr{O}(\tilde{\Omega})$ with $f = \tilde{f} \circ \iota$ is defined by $\tilde{f}(\chi) = \chi(f)$ for each character χ of $\mathscr{O}(\Omega)$.

The papers [53, 54] contain a characterization of holomorphically convex sets in terms of envelopes of holomorphy as follows. In this connection, one should also consult the paper [166], which is closely related to the characterization.

Theorem 6.2.4. *The compact connected subset X of $\mathbb{C}^N$ is holomorphically convex if and only if for every sequence $\{\Omega_j\}_{j=1,\dots}$ of bounded domains in $\mathbb{C}^N$ with $\Omega_j \supset \bar{\Omega}_{j+1}$ for all j and with $\bigcap_{j=1,\dots}\Omega_j = X$, we have $X = \bigcap_{j=1,\dots}\pi_j(\tilde{\Omega}_j)$.*

Proof. Fix a neighborhood basis $\{\Omega_j\}_{j=1,\dots}$ for X as in the statement of the theorem. Because $X = \cap_{j=1,\dots}\Omega_j$ and $\Omega_j \subset \pi_j(\tilde{\Omega}_j)$ for all j, we have for all choices of the set X that $X \subset \cap_{j=1,\dots}\pi_j(\tilde{\Omega}_j)$.

Suppose the set X to be holomorphically convex, and let $p \in \cap_{j=1,\dots}\pi_j(\tilde{\Omega}_j)$. We will show that $p \in X$.

To this end, introduce the uniform algebra A_j for each j defined to be the uniform closure on $\bar{\Omega}_j$ of the algebra of functions $f|\bar{\Omega}_j$, where f runs through all the functions holomorphic on varying neighborhoods of $\bar{\Omega}_j$. For each pair of integers j and k with $j \leq k$, there is the restriction homomorphism $\rho_{j,k} : A_j \to A_k$ given by $\rho_{j,k} f = f|\bar{\Omega}_k$ for each $f \in A_j$. There are also maps $\rho_j : A_j \to \boldsymbol{\mathscr{O}}(X)$ defined by $\rho_j(f) = [f]$, in which $[f]$ denotes the germ on X of the function f, which is holomorphic on Ω_j.

For each j, let $\tilde{Z}_j : \operatorname{spec} A_j \to \mathbb{C}^N$ be the joint spectrum map given by

$$\tilde{Z}_j(\chi) = (\chi(z_1|\bar{\Omega}_j), \dots, \chi(z_N|\bar{\Omega}_j)).$$

For $m = 1, \dots,$ define the set T_m by

$$T_m = \Big\{\{\chi_j\}_{j=1,\dots} : \chi_j \in \operatorname{spec} A_j, \text{ and for } 1 \leq j \leq m,\ \rho^*_{j,j+1}\chi_{j+1} = \chi_j \text{ and } \tilde{Z}_j(\chi_j) = p\Big\}.$$

The set T_m is a closed and hence compact subset of the product space $\prod_{j=1,\dots} \operatorname{spec} A_j$. It is not empty: By hypothesis there is $\psi \in \tilde{\Omega}_{m+1}$ with $\pi_{m+1}(\psi) = p$. If $\sigma_{j,k} : A_j \to \mathscr{O}(\Omega_k)$ is given for $k > j$ by $\sigma_{j,k}(g) = g|\Omega_k$, then $\sigma^*_{m,m+1}(\psi) = \chi_m \in \operatorname{spec} A_m$. (This χ_m is not the zero functional, for $1 = \psi(1) = \psi(\sigma_{m,m+1})(1) = \sigma^*_{m,m+1}(\psi)(1)$.) Then $\tilde{Z}_m(\chi_m) = p$. And if for $k \leq m$, $\chi_k = \rho^*_{k,k+1}\chi_{k+1}$, then for all r, $1 \leq r \leq m$, $\tilde{Z}_r(\chi_r) = \tilde{Z}_r(\rho_{r,r+1}\chi_{r+1}) = p$. Thus, T_m is not empty.

The sets T_m are nested: $T_m \supset T_{m+1}$. Let $\{\chi_j\}_{j=1,\dots} \in \cap_{m=1,\dots}T_m$; compactness implies that this intersection is not empty.

Define the functional χ on $\boldsymbol{\mathscr{O}}(X)$ by $\chi([f]) = \chi_j(f_j)$ if $f_j \in \mathscr{O}(\Omega_j)$ and the germ of f_j on X is the germ $[f]$. This functional is well defined: If $f_j \in \Omega_j$, $f_k \in \mathscr{O}(\Omega_k)$ and $[f_j] = [f_k] = [f] \in \boldsymbol{\mathscr{O}}(X)$, then there is an n such that $\Omega_n \subset \Omega_j \cap \Omega_k$ and $f_j|\Omega_n = f_k|\Omega_n$. Then $\chi_n(f_j|\Omega_n) = \chi_n(\rho_{j,n} f_j) = (\rho^*_{j,n}\chi_n)(f_j) = \chi_j f_j$. Consequently, χ is well defined.

The functional χ of the last paragraph is a character on $\boldsymbol{\mathscr{O}}(X)$, so, because X is assumed to be holomorphically convex, there is $q \in X$ such that $\chi([f]) = f(q)$ for all q. That is, $\chi_j(f_j) = f_j(q)$ if $f_j \in \boldsymbol{\mathscr{O}}(\Omega_j)$. However, this means that $\tilde{Z}_j(\chi_j) = q$. We have $p = q$, for $p = \tilde{Z}(\chi_j)$.

Thus, we have shown that if X is holomorphically convex, then $X = \cap_{j=1,\dots}\pi_j(\tilde{\Omega}_j)$.

Conversely, suppose $X = \cap_{j=1\dots}\pi_j(\tilde{\Omega}_j)$, and let χ be a character of $\boldsymbol{\mathscr{O}}(X)$. Define $\chi_j \in \operatorname{spec} \mathscr{O}(\Omega_j)$ by $\chi_j(f) = \chi([f])$ if $f \in \mathscr{O}(\Omega_j)$. Let $r_{j,k} : \mathscr{O}(\Omega_j) \to \mathscr{O}(\Omega_k)$ be the restriction map defined for $j \leq k$. The dual map $r^*_{j,k}$ carries $\tilde{\Omega}_k$ to $\tilde{\Omega}_j$, and $r^*_{j,k}(\chi_k) = \chi_j$, so that $\tilde{Z}_j(\chi_j) = \tilde{Z}_k(\chi_k)$. Thus, the point $p = \tilde{Z}_k(\chi_k)$ lies in $\cap_{j=1\dots}\pi_j(\tilde{\Omega}_j)$. Note that if $[f] \in \boldsymbol{\mathscr{O}}(X)$, then $\chi([f]) = \chi_j([f_j])$ if the germ on X of the element f_j of $\mathscr{O}(\Omega_j)$ is $[f]$. In particular, $\tilde{Z}(\chi) = \chi_j(\tilde{Z}_j) = p$.

From this fact follows that for each polynomial P, $\chi(P) = P(p)$, but to see that for every $[f] \in \mathscr{O}(X)$ we have $\chi([f]) = f(p)$ requires further discussion. Suppose, for the sake of deriving a contradiction, that χ is not the point evaluation ε_p at the point p. Let r_k^* : spec $\mathscr{O}(X) \to \tilde{\Omega}_k$ be the map dual to the restriction map. The space spec $\mathscr{O}(X)$ is connected since X is, so the set $r_k^*(\text{spec}\,\mathscr{O}(X))$ is also connected. There is a k such that $r_k^*(\varepsilon_p) \neq r_k^*(\chi)$, for if not, $\chi = \varepsilon_p$, contrary to the hypothesis. The set $r_k^*(\text{spec}\,\mathscr{O}(X))$ cannot be contained in Ω_k—we are here identifying Ω_k with its canonical image in $\tilde{\Omega}_k$, because the projection $\pi_k : \tilde{\Omega}_k \to \mathbb{C}^N$ is the identity map on Ω. Thus, because $r_k^*(\text{spec}\,\mathscr{O}(X))$ is connected and contains the point $r_k^*(\chi)$, which is not in Ω_k, the set $r_k^*(\text{spec}\,\mathscr{O}(X))$ has to contain a point of $b\Omega_k$. This is impossible, for $\pi_k(r_k^*(\text{spec}\,\mathscr{O}(X))) \subset \Omega_k$. Contradiction, and χ is found to be ε_p as desired.

The theorem is proved.

Compact connected holomorphically convex sets in $\mathbb{C}^N$ are also characterized as the compact connected sets that have neighborhood bases of a particularly useful form:

Theorem 6.2.5. *The compact connected subset X of $\mathbb{C}^N$ is holomorphically convex if and only if for every open set U in $\mathbb{C}^N$ that contains C, there is a domain Ω with envelope of holomorphy $(\tilde{\Omega}, \pi)$ such that $X \subset \Omega \subset \pi(\tilde{\Omega}) \subset U$.*

In the terminology introduced by Stensønes [339], connected holomorphically convex sets are those compact connected sets that have *nonschlicht Stein neighborhood bases.*

Proof. Suppose first that X is holomorphically convex and that U is an open subset of $\mathbb{C}^N$ that contains X. Choose a sequence of domains $\Omega_1 \supset \Omega_2 \supset \cdots$ with $\Omega_1 \subset U$, with Ω_{j+1} relatively compact in Ω_j, and with $\cap_{k=1,\ldots}\Omega_k = X$. That X is holomorphically convex implies that if $(\tilde{\Omega}_k, \pi_k)$ is the envelope of holomorphy of Ω_k, then $\cap_{k=1,\ldots}\pi_k(\Omega_k) = X$. We shall show that for all sufficiently large k, the set $\pi_k(\tilde{\Omega}_k)$ is contained in U.

Without loss of generality, we assume U to be bounded, so that bU is compact. It is sufficient to prove that for some k, the set $\pi_k(\tilde{\Omega}_k) \cap bU$ is empty. Assume, for the sake of contradiction, that for each k there is a point p_k in $\pi_k(\tilde{\Omega}_k) \cap bU$. Let $\tilde{p}_k \in \tilde{\Omega}_k$ satisfy $\pi_k(\tilde{p}_k) = p_k$. By passing to a subsequence if necessary, we can assume that the sequence $\{p_k\}_{k=1,\ldots}$ converges to a point $p \in bU$.

Each $f \in \mathscr{O}(\Omega_k)$ has a unique extension to a function $\tilde{f} \in \mathscr{O}(\tilde{\Omega}_k)$. The map $f \mapsto \tilde{f}(\tilde{p}_k)$ is a continuous linear functional, which is multiplicative, on the space $\mathscr{O}(\Omega_k)$, so by the Hahn–Banach and Riesz representation theorems, there is a finite regular Borel measure μ_k with compact support on Ω_k such that for all $f \in \mathscr{O}(\Omega_k)$,

$$\tilde{f}(\tilde{p}_k) = \int f\, d\mu_k.$$

The measure μ_k is not unique; it can be chosen to have total mass one, in which case it is positive.

The collection of measures μ_k, all thought of as measures on Ω_1, has weak* limit points; denote by μ such a limit point. We can suppose that in fact, the sequence $\{\mu_k\}_{k=1,\ldots}$ converges in the weak* sense to μ. Thus if g is any continuous function on Ω_1, then

$$\int g\, d\mu = \lim_k \int g\, d\mu_k.$$

The support of μ is contained in the set X.

The measure μ is not the zero measure, for $1 = \int 1\, d\mu_k \to \int d\mu$. It is also multiplicative on the space $\mathscr{O}(X)$: If f and g are holomorphic on a neighborhood of X, then $f, g \in \mathscr{O}(\Omega_k)$ for all large k, and we have that

$$\int fg\, d\mu_k = \int f\, d\mu_k \int g\, d\mu_k,$$

which implies that

$$\int fg\, d\mu = \int f\, d\mu \int g\, d\mu.$$

The assumed holomorphic convexity of X implies the existence of a point $q \in X$ such that for all $f \in \mathscr{O}(X)$,

$$\int f\, d\mu = f(q).$$

On the other hand, if we denote by z_j the jth coordinate function on $\mathbb{C}^N$, and if $p = (p^{(1)}, \dots, p^{(N)})$ and $q = (q^{(1)}, \dots, q^{(N)})$, then we have that

$$p^{(j)} = \lim_k \int z_j\, d\mu_k = \int z_j\, d\mu = q^{(j)},$$

so that $p = q$. This is impossible, though, since $p \in bU$ and $q \in X$.

Thus, for all large k, we have $\pi_k(\tilde{\Omega}) \subset U$.

For the other direction of the proof, it suffices to argue nearly verbatim as in the last paragraph of the proof of Theorem 6.2.4.

It should be noted that the first part of the proof just given is essentially analytic and not merely topological, as one might expect. This is necessary, for one can perfectly well have a compact set Y in $\mathbb{R}^N$ that is of the form $Y = \cap_{j=1,\dots} W_j$ with $\{W_j\}_{j=1,\dots}$ a decreasing sequence of open connected sets such that for some neighborhood U of X, each W_j meets $\mathbb{R}^N \setminus U$.

Corollary 6.2.6. *The compact connected subset X of $\mathbb{C}^N$ is holomorphically convex if and only if there is a neighborhood V of X such that for each $z \in V \setminus X$, there is a function holomorphic on a neighborhood of X that cannot be continued holomorphically into a neighborhood of the point z.*

By using some standard results on Stein manifolds, the preceding result can be extended to the case that the set X is contained in a Stein manifold. In addition to the embedding theorem for Stein manifolds, it will be necessary to use a theorem about holomorphic retractions:

Theorem 6.2.7. *If $\mathscr{M}$ is a closed submanifold of the Stein manifold $\mathscr{M}'$, then there is a neighborhood Ω of $\mathscr{M}$ in $\mathscr{M}'$ on which is defined a holomorphic retraction $\varrho : \Omega \to \mathscr{M}$.*

That is to say, ϱ is a holomorphic map from Ω to $\mathscr{M}$ that satisfies $\varrho(p) = p$ for all $p \in \mathscr{M}$. The existence of this holomorphic retraction is a theorem of Docquier and Grauert [102]. See also [158]. We shall not give its proof here.

Consider a complex submanifold $\mathscr{M}$ of $\mathbb{C}^N$; $\mathscr{M}$ is a Stein manifold. If X is a compact subset of $\mathscr{M}$, there are the algebras $\boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$ of germs on X of functions holomorphic on $\mathscr{M}$, and $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$ of germs on X of functions holomorphic on $\mathbb{C}^N$. These are not in general isomorphic algebras. One must ask therefore about the relation between the condition that X be holomorphically convex with respect to $\mathscr{M}$ and the condition that it be holomorphically convex with respect to $\mathbb{C}^N$. The relation is as expected:

Theorem 6.2.8. *The spectrum of $\boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$ is X if and only if the spectrum of $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$ is X.*

Alternatively put, *the set X is holomorphically convex with respect to $\mathscr{M}$ when and only when it is holomorphically convex with respect to $\mathbb{C}^N$.*

Proof. The gist of the matter is contained in an algebraic decomposition of $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$ induced by the holomorphic retraction $\varrho : \Omega \to \mathscr{M}$ of a neighborhood Ω of $\mathscr{M}$ in $\mathbb{C}^N$ onto $\mathscr{M}$. For this decomposition, consider a function f holomorphic on a neighborhood of X in $\mathbb{C}^N$. On a possibly smaller open set in $\mathbb{C}^N$ we can write $f = f \circ \varrho + (f - f \circ \varrho)$, which gives the equation $[f] = [f \circ \varrho] + [f - f \circ \varrho]$ in $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$. The germ $[f - f \circ \varrho]$ lies in the ideal $\boldsymbol{\mathscr{I}}_{\mathscr{M}}(X)$ that consists of the germs of functions f such that if f is defined on the open set V in $\mathbb{C}^N$ that contains X, then $f|(V \cap \mathscr{M}) = 0$. We define an operator $R : \boldsymbol{\mathscr{O}}_{\mathscr{M}}(X) \to \boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$ by $R[f] = [f \circ \varrho]$. There is then the decomposition

$$\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X) = R\boldsymbol{\mathscr{O}}_{\mathscr{M}}(X) \oplus \boldsymbol{\mathscr{I}}_{\mathscr{M}}(X). \tag{6.6}$$

With this mechanism in hand, we can see that if X is holomorphically convex with respect to $\mathscr{M}$, then it is holomorphically convex with respect to $\mathbb{C}^N$. To do this, consider a character χ of $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$. Then χ' given by $\chi'([f]) = \chi([f \circ \varrho])$ is a character of $\boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$. The set X is supposed to be holomorphically convex with respect to $\mathscr{M}$, so there is a point $x_o \in X$ with $\chi'([f]) = f(x_o)$ for all $[f] \in \boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$. If now $g \in \boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$, decompose it in accordance with the decomposition (6.6), so that $[g] = [g \circ \varrho] + [g - g \circ \varrho]$. Because the second summand vanishes on X, $\chi([g]) = \chi[g \circ \varrho] = \chi'([g|\mathscr{M}]) = g(x_o)$: For every $[g] \in \boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$, $\chi([g]) = g(x_o)$, so X is holomorphically convex with respect to $\mathbb{C}^N$.

In the opposite direction, suppose X to be holomorphically convex with respect to $\mathbb{C}^N$.

There is an algebra homomorphism $r : \boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X) \to \boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$ defined by the condition that if f is holomorphic on a neighborhood U in $\mathbb{C}^N$ of X, then $r([f]) = [f|(M \cap U)]$. This is a well-defined homomorphism with kernel the ideal $\boldsymbol{\mathscr{I}}_{\mathscr{M}}(X)$. The restriction map is surjective, because if $[f] \in \boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$, then $[f] = r([f \circ \varrho])$.

Suppose we are given a character χ of $\boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$. The composition χ' given by $\chi'([f]) = \chi(r[f])$ is a character of $\boldsymbol{\mathscr{O}}_{\mathbb{C}^N}(X)$, and so is of the form $\chi'([f]) = f(x_o)$ for some $x_o \in X$. Then we have that for $[g] \in \boldsymbol{\mathscr{O}}_{\mathscr{M}}(X)$, $\chi([g]) = \chi'[g \circ \varrho] = g(x_o)$, so as claimed, X is holomorphically convex with respect to $\mathscr{M}$

It is now possible to deduce the expected extension of Theorem 6.2.4 to the case of compacta in Stein manifolds:

Theorem 6.2.9. *The compact connected subset X of the Stein manifold $\mathscr{M}$ is holomorphically convex if and only if there is a sequence Ω_j of domains in $\mathscr{M}$ with $\Omega_j \supset \Omega_k$ when*

$j \le k$ and with $\cap_j \Omega_j = X$ such that if for each j, $(\tilde{\Omega}, \pi_j)$ is the envelope of holomorphy of Ω_j, then $\cap_j \pi_j(\tilde{\Omega}_j) = X$.

Proof. Without loss of generality, let $\mathscr{M}$ be a closed submanifold of $\mathbb{C}^N$ for a suitable N. Let X be a compact subset of $\mathscr{M}$ that is holomorphically convex with respect to $\mathscr{M}$. It is therefore holomorphically convex with respect to $\mathbb{C}^N$, so by Theorem 6.2.4, there is a decreasing sequence $\{\Omega_j\}_{j=1,\dots}$ of domains in $\mathbb{C}^N$ with intersection X such that if $(\tilde{\Omega}_j, \pi_j)$ is the envelope of holomorphy of Ω_j, then $\cap_{j=1,\dots}\pi_j(\tilde{\Omega}_j) = X$.

Let W_j be the component of $\pi_j^{-1}(\mathscr{M})$ that contains the set $\iota_j(X)$ if $\iota_j : \Omega_j \to \tilde{\Omega}_j$ is the holomorphic injection provided by the definition of envelope of holomorphy. Then W_j is a Stein manifold, and $(W_j, \pi_j | W_j)$ is a Riemann domain spread over the manifold $\mathscr{M}$. Let Ω'_j be the component of $\Omega_j \cap \mathscr{M}$ that contains X. Under ι_j, Ω'_j is carried biholomorphically onto a domain Ω''_j in W_j. The domain Ω''_j has an envelope of holomorphy $(\tilde{\Omega}''_j, \pi''_j)$, which is a Riemann domain over W'. Then $(\Omega''_j, \pi_j \circ \pi''_j)$ is the envelope of holomorphy of Ω'_j, and $\pi_j \circ \pi''_j(\tilde{\Omega}''_j) \subset \pi_j(\tilde{\Omega}_j)$, so the intersection of the domains $\pi_j \circ \pi''_j(\tilde{\Omega}''_j)$ is X.

For the converse: If there is a decreasing sequence Ω_j of domains in $\mathscr{M}$ with intersection X such that the intersection of the projections of the envelopes of holomorphy of the Ω_j's is X, then the argument in the last paragraph of the proof of Theorem 6.2.4, suitably modified, yields that each character of $\mathscr{O}_{\mathscr{M}}(X)$ is evaluation at a point of X.

The theorem is proved.

We have considered *connected* holomorphically convex sets. This is essentially all that is necessary because of the following simple fact:

Theorem 6.2.10. *The compact subset X of the Stein manifold $\mathscr{M}$ is holomorphically convex if and only if each of its components is holomorphically convex.*

Proof. First, X is holomorphically convex if and only if each open and closed subset of X is holomorphically convex. For this, suppose $X = X' \cup X''$ to be a decomposition of X as a union of disjoint open and closed subsets. There is then a decomposition $\mathscr{O}(X) = \mathscr{O}(X') \oplus \mathscr{O}(X'')$ with the operations in the direct sum taken coordinatewise. If χ is a character of $\mathscr{O}(X)$, then one of the maps $[f] \mapsto \chi(([f], 0))$ and $[f] \mapsto \chi((0, [f]))$ from $\mathscr{O}(X')$ and $\mathscr{O}(X'')$, respectively, to $\mathbb{C}$ is not the zero map. Suppose the first not to be zero, and denote it by χ'. It is a character of $\mathscr{O}(X')$. If X' is holomorphically convex, then χ' is of the form $\chi'([f]) = f(x')$ for some $x' \in X'$. Thus, if both X' and X'' are holomorphically convex, then so is X. Conversely, if X is holomorphically convex, and if χ' is a character of $\mathscr{O}(X')$, then $[f] \mapsto \chi([f|X'])$ is a character of $\mathscr{O}(X)$ and so must be of the form $[f] \mapsto f(x_o)$ for some point $x_o \in X$. The point x_o must lie in X', so X' is holomorphically convex.

Next, if $\{X_j\}_{j=1,\dots}$ is a sequence of compact sets each of which is holomorphically convex, then the intersection $X = \cap_{j=1,\dots}X_j$ is also holomorphically convex: Let χ be a character of $\mathscr{O}(X)$. For each j define $\chi_j : \mathscr{O}(X_j) \to \mathbb{C}$ by $\chi_j([f]) = \chi([f|X])$. The functional χ_j so defined is a character of $\mathscr{O}(X_j)$ and so is of the form $\chi_j([f]) = f(x_j)$ for some point x_j in X_j. In fact, the points x_j all coincide and all lie in X. It follows that χ is evaluation at the point x_j, so X is holomorphically convex.

If X is holomorphically convex, then each of its open and closed subsets is holomorphically convex, and each component Y of X is an intersection $\cap_{j=1,\ldots}X_j$ of open and closed subsets X_j of X as noted in the discussion of Theorem 1.5.5. It follows that Y is holomorphically convex.

Conversely, suppose each component of X to be holomorphically convex and χ to be a character of $\mathscr{O}(X)$. For every decomposition $X = X' \cup X''$ with X' and X'' open, closed, and disjoint, χ induces the zero functional on one of $\mathscr{O}(X')$, $\mathscr{O}(X'')$ and a character on the other. Let $\{X_\alpha\}_{\alpha\in A}$ be the family of all open and closed subsets of X such that χ induces a character on $\mathscr{O}(X_\alpha)$. The intersection $X_o = \cap_{\alpha\in A}X_\alpha$ is not empty, for if it were, then by compactness there would be a finite number of the X_α's, say X_{α_j}, $j = 1, \ldots, s$, with empty intersection. If $Y_\alpha = X \setminus X_\alpha$, then Y_α is open and closed, and $\cup_{j=1,\ldots,r}Y_{\alpha_j} = X$. If we define sets Z_j, $j = 1, \ldots, s$, by $Z_1 = Y_{\alpha_1}$, and $Z_j = Y_{\alpha_j} \setminus \cup_{k=1,\ldots,j-1}Z_k$, then the sets Z_j are mutually disjoint open and closed subsets of X with $\cup_{j=1,\ldots,r}Z_j = X$. The functional χ induces the zero functional on each $\mathscr{O}(Y_j)$, so it induces the zero functional on each $\mathscr{O}(Z_j)$, and so is the zero functional. Contradiction. Thus the set X_o is not empty.

The set X_o is connected. If not, there is a decomposition $X = X' \cup X''$ with X' and X'' disjoint open and closed subsets of X with both of the sets $X_o \cap X'$ and $X_o \cap X''$ nonempty. The character χ induces a character on one of $\mathscr{O}(X')$, $\mathscr{O}(X'')$, say on the former. Then X' is an X_α, so $X_o \subset X'$, contradicting $X_o \cap X'' \neq \emptyset$. Thus, X_o is connected.

Moreover, the set X_o is a component of X: If X_1 is a connected subset of X that properly contains X_o, then for suitable α, we have $X = X_\alpha \cup Y_\alpha$ with X_α and Y_α both open and closed in X, and with $X \subset X_\alpha$ and $X_1 \cap Y_\alpha \neq \emptyset$, for X_o is the intersection of the X_α's. This implies the decomposition $X_1 = (X_1 \cap X_\alpha) \cap (X_1 \cap Y_\alpha)$ of X into the union of two nonempty open and closed subsets, contradicting the assumed connectedness of X_1. Thus, X_o is a component of X.

The character χ induces a character χ_o on $\mathscr{O}(X_o)$ by the following condition: If f is holomorphic on a neighborhood W of X_o, then there is an X_α with $X_o \subset X_\alpha \subset W$. Denote by χ_α the character on $\mathscr{O}(X_\alpha)$ induced by χ, and define $\chi_o([f])$ to be the number $\chi_\alpha([f]_\alpha)$ with $[f]_\alpha$ denoting the germ of f on X_α. The functional χ_o is well defined: Its definition does not depend on the choice of X_α. The set X_o is a component of X, so it is holomorphically convex by hypothesis, whence there is $x_o \in X_o$ such that $\chi_o([f]) = f(x_o)$ for all $f \in \mathscr{O}(X_o)$.

It follows that the original character χ of $\mathscr{O}(X)$ is given by $\chi([f]) = f(x_o)$ for all f holomorphic on a neighborhood of X. The theorem is proved.

Note that this theorem gives a (rather involved) alternative proof of the observation above that each compact totally disconnected set is holomorphically convex.

Harvey and Wells [166] have given a sheaf-theoretic characterization of holomorphically convex sets:

Theorem 6.2.11. *For a compact subset X of the Stein manifold $\mathscr{M}$, the following three conditions are equivalent:*

(a) *X is holomorphically convex.*

(b) *For each coherent analytic sheaf $\mathscr{F}$ on each neighborhood V of X, the cohomology groups $\check{H}^p(X;\mathscr{F})$ vanish for $p = 1, \ldots$.*

(c) *For $p = 1, \dots,$ the cohomology groups $\check{H}^p(X; \mathscr{R})$ vanish whenever the sheaf $\mathscr{R}$ is a sheaf of relations on a neighborhood, V, of X, i.e., whenever $\mathscr{R}$ is the kernel of a sheaf homomorphism $\varphi : \mathscr{O}^p \to \mathscr{O}^q$ on V for some choice of positive integers p and p.*

We shall not prove this result here, nor shall we use it below.

C. Laurent-Thiébaut [214] has given a characterization of holomorphically convex sets in terms of Dolbeault cohomology:

Theorem 6.2.12. *The compact subset X of a Stein manifold is holomorphically convex if and only if the groups $H^{(0,q)}(X)$ vanish in the range $1 \leq q$.*

The condition that $H^{(0,q)}(X)$ vanish is the condition is that each smooth $(0, q)$-form that is defined and $\bar\partial$-closed on a neighborhood of X be $\bar\partial$-exact on a possibly smaller neighborhood of X.

It has been seen above that polynomially convex sets, rationally convex sets, and Stein compacta are subject to various topological conditions. Holomorphically convex sets are subject to similar restrictions:

Theorem 6.2.13. [166] *If X is a compact holomorphically convex subset of the N-dimensional Stein manifold $\mathscr{M}$, then $\check{H}^p(X; \mathbb{Z}) = 0$ for $p > N$.*

Proof. This depends on the result that for an N-dimensional Stein manifold $\mathscr{N}$, the cohomology groups $\check{H}^p(\mathscr{N}; \mathbb{Z})$ vanish in the range $N + 1 \leq p$. These groups vanish, for according to Theorem 2.4.1, $H_p(\mathscr{N}, \mathbb{Z}) = 0$ when $p > N$, and $H_N(\mathscr{N}, \mathbb{Z})$ is free. The universal coefficients theorem for cohomology implies that $\check{H}^N(\mathscr{N}, \mathbb{Z}) = 0$. It follows that if the compact subset Y of $\mathscr{N}$ is $\mathscr{O}(\mathscr{N})$-convex, then $\check{H}^p(Y; \mathbb{Z}) = 0$ when $N+1 \leq p$, for Y is the intersection of Stein domains in the fixed manifold $\mathscr{N}$.

A compact holomorphically convex set is not necessarily an intersection of Stein domains, but Theorem 6.2.9 in essence shows that such a set, if connected, is the inverse limit of a sequence of $\mathscr{O}(\mathscr{N})$-convex compacta that lie in varying Stein manifolds. This was the idea used by Harvey and Wells in proving Theorem 6.2.13.

The proof of the theorem goes as follows. Suppose first that X is connected. By hypothesis, X is holomorphically convex, so there is a decreasing sequence Ω_j, $j = 1, \dots,$ of open sets in $\mathscr{M}$ with intersection X such that if $(\tilde\Omega_j, \pi_j)$ is the envelope of holomorphy of Ω_j, then $\cap_j \pi_j(\tilde\Omega_j) = X$. For each j, let $\iota_j : \Omega_j \to \tilde\Omega_j$ be the canonical injection, and let $Y'_j \subset \tilde\Omega_j$ be the compact set $\iota_j(X)$. Let Y_j be the $\mathscr{O}(\tilde\Omega_j)$-hull of Y'_j. Because $\tilde\Omega_j$ is a Stein manifold, Y_j is compact.

For each $j = N + 1, \dots,$ $\check{H}^p(Y_j; \mathbb{Z}) = 0$; from this it is to be deduced that $\check{H}^p(X; \mathbb{Z}) = 0$.

For each pair of positive integers j and k with $j \leq k$, there is the restriction map $\rho_{j,k} : \mathscr{O}(\Omega_j) \to \mathscr{O}(\Omega_k)$. It induces a dual map $\rho^*_{j,k} : \tilde\Omega_k \to \tilde\Omega_j$. These dual maps satisfy $\rho^*_{j,k} = \rho^*_{j,m} \circ \rho^*_{m,k}$ if $j \leq m \leq k$ and the condition that $\iota_j = \rho^*_{j,k} \circ \iota_k$. They also satisfy $\pi_k = \pi_j \circ \rho^*_{j,k}$, for the projections $\pi_j : \tilde\Omega_j \to \mathscr{M}$ are simply the duals of the restriction maps $r_j : \mathscr{O}(\mathscr{M}) \to \mathscr{O}(\Omega_j)$. Thus, if $\chi \in \tilde\Omega_k$, then for $f \in \mathscr{O}(\mathscr{M})$, $\pi_k(\chi)(f) = f(\pi_k(\chi)) =$

$\chi(f|\Omega_k)$, and $\pi_j\rho^*_{j,k}(\chi)(f) = f(\pi_j(\rho^*_{j,k}(\chi))) = \rho^*_{j,k}(\chi)(f|\Omega_j) = \chi((f|\Omega_j)|\Omega_k) = \chi(f|\Omega_k)$.

For $k \geq j$, the map $\rho^*_{j,k}$ carries Y_k into Y_j, for $\rho_{j,k}(Y'_k) = Y'_j$ because $Y'_j = \iota_j(X) = \rho^*_{j,k} \circ \iota_k(X) = \rho^*_{j,k}(Y'_k)$.

Introduce the product space $Y = \prod_{j=1,\ldots} Y_j$, a compact Hausdorff space. Let L be the closed subset of Y that consists of sequences $\{y_j\}_{j=1,\ldots}$ with the property that for $k \geq j$, $\rho^*_{j,k}(y_k) = y_j$. This nonempty set is the inverse limit of the inverse system $\{Y_j, \rho^*_{j,k}\}$ of compact spaces. By the continuity property of Čech cohomology, for all nonnegative integers q, $\check{H}^q(L;\mathbb{Z})$ is the direct limit of the system $\check{H}^q(Y_j;\mathbb{Z})$. These groups vanish when $q \geq N+1$, so $\check{H}^p(L;\mathbb{Z}) = 0$ when $p \geq N+1$.

The space L is homeomorphic to X. To prove this, introduce the continuous map $\eta : X \to Y$ by the prescription that for $x \in X$, $\eta(x) \in Y$ is the sequence $\{\iota_j x\}_{j=1,\ldots}$. As noted above, $\iota_j|\Omega_k = \rho^*_{j,k} \circ \iota_k$, so $\eta(x)$ does lie in L. The map η is continuous, as follows from the definition of the product topology. It carries X onto L: If $\{y_j\}_{j=1,\ldots} \in L$, then the point $x = \pi_j(y_j)$ is independent of j, and it lies in X for $X = \cap_j \pi_j(\tilde{\Omega}_j)$. We have $\eta(x) = \{y_j\}_{j=1,\ldots}$. Finally, η is injective: If $\eta(x) = \eta(x')$ for $x, x' \in X$, then for every function f holomorphic on a neighborhood of X, $f(x) = f(x')$, which implies that $x = x'$. Consequently, η is a homeomorphism, so the topology of X is the same as that of L. In particular, the integral cohomology groups of the two spaces coincide.

The theorem is proved in the case that X is connected.

The case of general, not necessarily connected, sets X follows from cohomological formalities. If $X = X' \cup X''$ is a decomposition of X into disjoint open, closed subsets, then there is the corresponding direct sum decomposition $\check{H}^p(X;\mathbb{Z}) = \check{H}^p(X';\mathbb{Z}) \oplus \check{H}^p(X'',\mathbb{Z})$. Thus, $\check{H}^p(X,\mathbb{Z})$ vanishes if and only $\check{H}^p(Y;\mathbb{Z})$ vanishes for every open and closed subset Y of X.

A component Y of X is the intersection of a decreasing sequence of open and closed subsets of X, so it follows that if $\check{H}^p(X;\mathbb{Z}) = 0$, then $\check{H}^p(Y;\mathbb{Z}) = 0$.

Conversely, it is to be shown that if each component Y of X satisfies $\check{H}(Y;\mathbb{Z}) = 0$, then $\check{H}(X;\mathbb{Z}) = 0$. Denote by $\{Y_\alpha\}_{\alpha\in A}$ the collection of components of X, and assume that for each $\alpha \in A$, $\check{H}^p(Y_\alpha;\mathbb{Z}) = 0$. For each pair E, F of subsets of X with $E \subset F$, let $\iota_{E,F} : E \to F$ be the inclusion and let $\iota^*_{E,F} : \check{H}^p(F;\mathbb{Z}) \to \check{H}^p(E,\mathbb{Z})$ be the induced map of cohomology. By hypothesis, each of the maps $\check{H}^p(X;\mathbb{Z}) \to \check{H}^p(Y_\alpha;\mathbb{Z})$ is the zero map. Consequently, if $c \in \check{H}^p(X;\mathbb{Z})$, then for all α, $\iota^*_{X,Y_\alpha}(c) = 0$. It follows that for some sufficiently small open and closed subset V of X that contains Y_α, we have $\iota^*_{X,V}(c) = 0$. A finite number of these open and closed subsets V, which can be taken to be mutually disjoint, cover the compact set X, say $V_j,\ j = 1,\ldots,s$. The group $\check{H}^p(X;\mathbb{Z})$ is equal to $\oplus_{j=1,\ldots,s}\check{H}^p(V_j;\mathbb{Z})$, so the cohomology class c is zero: $\check{H}^p(X;\mathbb{Z}) = 0$. Note: It is not claimed here that the groups $\check{H}^p(V_j;\mathbb{Z})$ all vanish, only that the elements $\iota^*_{Z,V_j}(c)$ are zero.

The theorem is proved.

Corollary 6.2.14. *If X is a compact holomorphically convex subset of the Stein manifold $\mathscr{M}$ with $\dim\mathscr{M} = N$, then $\check{H}^p(\mathscr{M};\mathbb{Z}) = \check{H}^p(\mathscr{M} \setminus X;\mathbb{Z})$ when $p < N-1$.*

Proof. Alexander duality gives that for $p < N$, $\check{H}^p(\mathscr{M}; \mathscr{M} \setminus X; \mathbb{Z}) = 0$. The exact cohomology sequence for the pair $(\mathscr{M}, \mathscr{M} \setminus X)$ then yields the result.

It would be reasonable to expect at this point a homotopy result in the spirit of Theorems 2.3.1 and to the effect that certain of the homotopy groups of the complementary set $\mathbb{C}^N \setminus X$ vanish when X is holomorphically convex. It seems that nothing is known in this direction, and there is the following example: No matter how large N is, for a compact holomorphically convex subset X of $\mathbb{C}^N$, the fundamental group $\pi_1(\mathbb{C}^N \setminus X)$ need not vanish. This is immediate from the fact that, as noted in Section 2.3, there are Cantor sets, i.e., compact, perfect, totally disconnected subsets of $\mathbb{R}^N$ for all $N \geq 2$ whose complement is not simply connected although every compact, totally disconnected set is holomorphically convex.

6.3. Approximation on Totally Real Manifolds

In the two preceding sections, we have discussed the two general classes of sets that will be central to the approximation theorems of this section. We now turn to the approximation results themselves.

The first result goes back to a paper of Hörmander and Wermer [183]. The present form of the result was found by Harvey and Wells [167].

Theorem 6.3.1. *If Σ is a closed totally real submanifold of class $\mathscr{C}^1$ of an open set in $\mathbb{C}^N$, and if $f \in \mathscr{C}(\Sigma)$, then f can be approximated uniformly on compacta in Σ by functions holomorphic on a neighborhood of Σ.*

An extension of this result was given by O'Farrell, Preskenis, and Walsh [270] based on a method of Berndtsson [49].

Theorem 6.3.2. *Let X be a compact holomorphically convex set in $\mathbb{C}^N$, and let X_o be a closed subset of X for which the complementary set $X \setminus X_o$ is a totally real subset of the manifold $\mathbb{C}^N \setminus X_o$. A function $f \in \mathscr{C}(X)$ can be approximated uniformly on X by functions holomorphic on a neighborhood of X if and only if $f|X_o$ can be approximated uniformly on X_o by functions holomorphic on X.*

It is not assumed in this statement that the totally real set $X \setminus X_o$ has finite area in any dimension, though, by Corollary 6.1.9, it is known to have locally finite N-dimensional measure.

Corollary 6.3.3. *If X is a polynomially convex subset of $\mathbb{C}^N$ and X_o is a closed subset of X_o that is also polynomially convex and for which the complementary set $X \setminus X_o$ is contained in a totally real submanifold of class $\mathscr{C}^1$ of $\mathbb{C}^N \setminus X_o$, then $\mathscr{P}(X) = \{f \in \mathscr{C}(X) : f|X_o \in \mathscr{P}(X_o)\}$.*

A concrete application of the theorem is given in the following example from [270]. Let $Y = \{(re^{i\vartheta}, re^{3i\vartheta}) : \vartheta \in \mathbb{R},\ r \in [\frac{1}{2}, 2]\}$, a certain smooth totally real annulus in $\mathbb{C}^2$, and let $X = Y \cup \mathbb{T}^2$ with $\mathbb{T}^2$ the unit 2-torus. The intersection $Y \cap \mathbb{T}^2$ is the circle $\gamma = \{(e^{i\vartheta}, e^{3i\vartheta}) : \vartheta \in \mathbb{R}\}$. The set $X \setminus \gamma$ is a totally real manifold, and one verifies that the set X is rationally convex. For this point, verify that if $p_o = (z_o, w_o) \in \mathbb{C}^2 \setminus X$, then one of the polynomials $P_1, \dots, P_4$ defined by $P_1(z, w) = z - z_o$, $P_2(z, w) = w - w_o$, $P_3(z) = zw_o - z_o w$, and $P_4(z) = z^3 w_o - z_o^3 w$ does not vanish on X. The polynomials P_1 and P_2 suffice to show that $\mathscr{R}$-hull X is contained in the product S of two copies of the closed annulus $\{\zeta \in \mathbb{C} : \frac{1}{2} \le |\zeta| \le 2\}$. If $p_o \in S \setminus X$ and $|z_o| \ne |w_o|$, then P_3 vanishes at p_o but not on X. If $p_o \in S \setminus X$ and $|z_o| = |w_o|$, then P_4 vanishes at p_o but not on X. According to Theorem 6.3.2, every continuous function g on X for which $g|\gamma$ is approximable uniformly by functions holomorphic on X can itself be approximated uniformly on the whole of X by functions holomorphic on X, and so, by the rational convexity of X, by rational functions. It follows that $\mathscr{R}(X) = \mathscr{C}(X)$, for *every* function continuous on γ can be approximated uniformly by polynomials in z_1 and $1/z_1$. These polynomials are holomorphic on a neighborhood of X.

The following preliminary lemma shows that in the proof of Theorem 6.3.2 it suffices to suppose that the set X is connected.

Lemma 6.3.4. *Let X be a compact subset of $\mathbb{C}^N$, and let Y be a closed subset of X. Let $\{X_\alpha\}_{\alpha \in A}$ be the set of components of X, and let $Y_\alpha = X_\alpha \cap Y$ for each $\alpha \in A$. Assume (*) that for each α, every $f \in \mathscr{C}(X_\alpha)$ that can be approximated uniformly on Y_α by functions holomorphic on X_α can be approximated uniformly on all of X_α by functions holomorphic on X_α. Then every $f \in \mathscr{C}(X)$ that can be approximated uniformly on Y by functions holomorphic on X can be approximated uniformly on the whole of X by functions holomorphic on X.*

Proof. Assume $f \in \mathscr{C}(X)$ to be approximable uniformly on Y by functions holomorphic on X. Then for each α, $f_\alpha = f|X_\alpha$ is uniformly approximable on Y_α by functions holomorphic on X_α, and so by (*), f_α is uniformly approximable on X_α by functions holomorphic on X_α.

Let $\varepsilon > 0$. For all α, let g_α be holomorphic on X_α and satisfy $\|g_\alpha - f_\alpha\|_{X_\alpha} < \varepsilon/2$. Let U_α be an open subset of $\mathbb{C}^N$ containing X_α on which g_α is defined. The set

$$\{z \in U_\alpha \cap X : |g_\alpha(z) - f(z)| < \varepsilon\}$$

contains an open and closed subset, say Z_α, of X that contains the component X_α. A finite number of the sets Z_α cover X. If we define the function g by the condition that $g = g_\alpha$ near Z_α, then g is holomorphic on a neighborhood of X and satisfies $|f - g| < \varepsilon$ on X.

The lemma is proved.

The proof of Theorem 6.3.2 depends on the apparatus of the Cauchy–Fantappiè theory.

Recall that for a domain D in $\mathbb{C}^N$ and smooth maps $\varphi, f : D \to \mathbb{C}^N$ with f holomorphic, the Cauchy–Fantappiè kernel $\Omega_{\varphi;f}$ is defined by $\Omega_{\varphi;f} = [\varphi \cdot f]^{-N} \omega'(\varphi) \wedge \omega(f)$ with $\omega(f) = df_1 \wedge \cdots \wedge df_N$ and $\omega'(\varphi) = \sum_{j=1}^N (-1)^{j-1} \bar\partial \varphi_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge \bar\partial \varphi_N$. The dot notation is here used in the sense that if a, b are points in $\mathbb{C}^N$, then $a \cdot b = \sum_{j=1}^N a_j b_j$. In the development below, the Cauchy–Fantappiè theory will be applied in particular with $f(z) = z - \zeta$ and with φ replaced by a smooth map $s : D \to \mathbb{C}^N$ that depends in a $\mathscr{C}^1$ way on a parameter ζ. We begin with a lemma about the forms $\Omega_{\varphi;f}$:

Lemma 6.3.5. *Let U and D be open sets in $\mathbb{C}^N$, and let $s : U \times D \to \mathbb{C}^N$ be a function of class $\mathscr{C}^1$ that satisfies $s(\zeta, z) \cdot (z - \zeta) \ne 0$ when $z \ne \zeta$. Assume that*

(a) *for each compact subset K of $U \times D$, there is a constant C_K such that for $(\zeta, z) \in K$*

$$|s(\zeta, z) \cdot (z - \zeta)| \geq C_K |z - \zeta|^2,$$

and

(b) $|s(\zeta, z)| \leq C_K |z - \zeta|$.

Then for all $g \in \mathscr{C}^\infty(\mathbb{C}^N)$ with supp g *a compact subset of U, and for all $z \in D$,*

$$g(z) = -c_N \int_{\mathbb{C}^N} \bar{\partial} g(\zeta) \wedge \Omega_{s(\zeta,z);\zeta-z}.$$

Note that if $z \in D \setminus U$, then both sides are zero. The constant c_N is the constant that appeared in Section 1.4: $c_N = \frac{(-1)^{\frac{1}{2}N(N-1)}(N-1)!}{(2\pi i)^N}$.

Proof. This is a simple calculation. We can assume $z \in U$. Then

$$\begin{aligned} \int_{\mathbb{C}^N} \bar{\partial} g(\zeta) \wedge \Omega_{s(\zeta,z);z-\zeta} &= \lim_{\varepsilon \to 0^+} \int_{U \cap \{\zeta : |z-\zeta| > \varepsilon\}} \bar{\partial} g(\zeta) \wedge \Omega_{s(\zeta,z);z-\zeta} \\ &= -\lim_{\varepsilon \to 0^+} \int_{b\mathbb{B}_N(z,\varepsilon)} g(\zeta)\, \Omega_{s(\zeta,z);\zeta-z} \\ &= -g(z) \lim_{\varepsilon \to 0^+} \int_{b\mathbb{B}_N(z,\varepsilon)} \Omega_{s(\zeta,z);\zeta-z} \\ &= -c_N^{-1} g(z). \end{aligned}$$

(The integral that appears in the next-to-last line is, except for the factor c_N, the Cauchy–Fantappiè integral of the holomorphic function identically one, and so has the value c_N^{-1} independently of ε.)

Proof of Theorem 6.3.2. By Lemma 6.3.4 it is enough to treat the case in which the set X is connected. Recall that by Theorem 6.2.10, a component of a compact holomorphically convex set is holomorphically convex.

The argument is by duality: It is enough to show that if μ is a measure on X that annihilates the algebra $\mathscr{O}(X)$,[3] then supp $\mu \subset X_o$. The first step in the proof is to remark that the problem can be localized away from the set X_o. We show that if μ is orthogonal to $\mathscr{O}(X)$, then each point $x_o \in X \setminus X_o$ has a neighborhood U with $|\mu|(U) = 0$.

Fix a point $x_o \in X \setminus X_o$. By hypothesis, the set $X \setminus X_o$ is totally real, so by Theorem 6.1.8 there is a neighborhood U' of x_o such that for some totally real submanifold W' of U' of class $\mathscr{C}^1$, which we take to be disjoint from X_o, $X \cap U' \subset M'$. By shrinking U' a bit if necessary, we can suppose as a matter of convenience that dim $W' = N$. The manifold W' is of class $\mathscr{C}^1$, so we can choose holomorphic linear coordinates on $\mathbb{C}^N$ with respect to which x_o is the origin and with respect to which the tangent space of W' at 0 is

[3]There is an abuse of notation here: One integrates functions, not germs; the elements of $\mathscr{O}(X)$ are germs. It seems unlikely that confusion will arise from this abuse of language.

given by $y_1 = \cdots = y_N = 0$. Thus, by shrinking U' further to obtain the open set U, we can suppose that $W = W' \cap U$ is given by an equation $y = g(x)$ with g an $\mathbb{R}^N$-valued function on a neighborhood of $0 \in \mathbb{R}^N$ that is of class $\mathscr{C}^1$ and that satisfies $g(0) = 0$ and $dg(0) = 0$.

Following Berndtsson [49] we find neighborhoods $U_o \subset U$ of 0 and U_1 of X in $\mathbb{C}^N$ and maps $s, t : U_o \times U_1 \to \mathbb{C}^N$ with the following properties:

(a) s and t are of class $\mathscr{C}^1$,

(b) for each compact subset K of $U_o \times U_1$ there is a constant C_K such that

$$|s(\zeta, z) \cdot (\zeta - z)| \geq |\zeta - z|^2$$

when $(\zeta, z) \in U_o \times U_1$,

(c) $|s(\zeta, z)| \leq C_K |\zeta - z|$ when $(\zeta, z) \in K$,

(d) for $\zeta \in U_o$ and $z \in X$, $s(\zeta, z) = t(\zeta, z)$, and

(e) for $\zeta \in U_o$, the partial function $t(\zeta, \cdot)$ is holomorphic on a neighborhood of X.

The utility of these maps is shown by the following conclusion of the proof, granted the existence of maps s and t with properties (a)–(e). We show that if g is a smooth function on $\mathbb{C}^N$ with support contained in U_o, then $\int_X g(z)\, d\mu(z) = 0$: By Lemma 6.3.5,

$$\begin{aligned}\int_X g(z)\, d\mu(z) &= -c_N \int_X \Big\{ \int_{\mathbb{C}^N} \bar{\partial} g(\zeta) \wedge \Omega_{s(\zeta,z);\zeta - z} \Big\}\, d\mu(z) \\ &= -c_N \int_X \Big\{ \int_{U_o} \bar{\partial} g(\zeta) \wedge \Omega_{t(\zeta,z);\zeta - z} \Big\}\, d\mu(z) \\ &= 0,\end{aligned}$$

because the inner integral in the next-to-last equation is holomorphic on a neighborhood of X by property (e). The support of the measure μ must lie outside U_o, because $\int_{\mathbb{C}^N} g(z)\, d\mu(z) = 0$ for all choices of g. It follows that supp $\mu \subset X_o$ as desired.

The main effort of the proof is to construct the maps s and t with the properties (a)–(e).

We are to construct maps $s, t : U_o \times U_1 \to \mathbb{C}^N$. We begin with a more modest construction: We construct a map $s^0 : U_o \times U_o \to \mathbb{C}^N$, and then we will use the solution of a suitable Cousin I problem to extend it to a larger set. We take $z = x + iy$ and $\zeta = \xi + i\eta$ both initially restricted to the domain U_o.

Define $s^o : U_o \times U_o \to \mathbb{C}^N$ by

$$s^o(\zeta, z) = \bar{\zeta} + 2ig(\xi) - (\bar{z} + 2ig(x)).$$

Then

$$s^o(\zeta, z) \cdot (\zeta - z) = |\zeta - z|^2 + 2i(g(\xi) - g(x)) \cdot (\zeta - z),$$

and therefore, because g is of class $\mathscr{C}^1$ and $dg(0) = 0$,

$$|s^o(\zeta, z) \cdot (\zeta - z)| > \frac{1}{2}|\zeta - z|^2$$

when ζ and z are near the origin. If $z \in W$, then $s^o(\zeta, z) = \bar{\zeta} + 2ig(\xi) - z$. Define $t^o : U_o \times U_o \to \mathbb{C}^N$ by

$$t^o(\zeta, z) = \bar{\zeta} + 2ig(\xi) - z,$$

which is holomorphic in z.

Introduce the function $H_1 : U_o \times U_o \to \mathbb{C}$ by

$$H_1(\zeta, z) = (\bar{\zeta} + 2ig(\xi) - z) \cdot (\zeta - z).$$

If A_r is the annular domain given by $A_r = \{z \in \mathbb{C}^N : r \leq |z| \leq 2r\}$, then if $r > 0$ is small, we have $\Re H_1(\zeta, z) > 0$ when $|\zeta|$ is small and z lies in a neighborhood, which we shall call V, of $A_r \cap W$.

By hypothesis, the set X is holomorphically convex and connected, so by Theorem 6.2.4, there is a decreasing sequence $\{\Delta_j\}_{j=1,\dots}$ of domains $\mathbb{C}^N$ with $\cap_{j=1,\dots}\Delta_j = X$ and, if $(\tilde{\Delta}_j, \pi_j)$ denotes the envelope of holomorphy of Δ_j, with $\cap_{j=1,\dots}\pi_j(\tilde{\Delta}_j) = X$. We can suppose the Δ's to satisfy the condition that $(A_r \setminus V) \cap \pi_j(\tilde{\Delta}_j) = \emptyset$. We do not need the entire sequence; the main point is that there is a domain Δ with envelope of holomorphy $(\tilde{\Delta}, \pi)$ such that $\pi(\tilde{\Delta}) \cap (A_r \setminus V) = \emptyset$ but $\pi(\tilde{\Delta}) \supset X$.

Write $\tilde{\Delta} = \Delta^+ \cup \Delta^-$ with $\Delta^+ = \{p \in \tilde{\Delta} : |\pi(p)| > r\}$ and $\Delta^- = \{p \in \tilde{\Delta} : |\pi(p)| < 2r\}$.

For each ζ near 0, there are functions $\tilde{h}_\zeta^+ \in \mathscr{O}(\Delta^+)$ and $\tilde{h}_\zeta \in \mathscr{O}(\Delta^-)$ such that for $p \in \tilde{\Delta}$ with $\pi(p) \in A_r$,

$$\log H_1(\zeta, \pi(p)) = h_\zeta^+(p) - h_\zeta^-(p),$$

because Cousin I problems are solvable on the Stein manifold $\tilde{\Delta}$. Moreover, the functions h_ζ^- and h_ζ^+ can be chosen to depend in a $\mathscr{C}^1$ way on the parameter ζ near zero. For this additional point, see the appendix to the present section.

With $\iota : \Delta \to \tilde{\Delta}$ the inclusion map that exhibits $\tilde{\Delta}$ as the envelope of holomorphy of Δ, define domains Ω^+ and Ω^- in $\mathbb{C}^N$ by $\Omega^+ = \iota^{-1}(\Delta^+)$ and $\Omega^- = \iota^{-1}(\Delta^-)$, and define functions h^+ and h^- on Ω^+ and Ω^-, respectively, by $h^+(\zeta, z) = h_\zeta^+(\iota(z))$ and $h^-(\zeta, z) = h_\xi^-(\iota(z))$. If $U_o' \subset U_o$ is a sufficiently small neighborhood of the origin, then we define a function on $U_o' \times (\Omega^+ \cup \Omega^-)$,

$$H(\zeta, z) = \begin{cases} H_1 e^{h^-} & \text{on } U_o' \times \Omega^-, \\ e^{h^+} & \text{on } U_o' \times \Omega^+. \end{cases}$$

The function H is of class $\mathscr{C}^1$ in ζ near zero and is holomorphic in z, $z \in \Omega^- \cup \Omega^+$.

There is a decomposition

$$H(\zeta, z) = \sum_{j=1}^{N} t_j(\zeta, z)(\zeta_j - z_j)$$

with each t_j of class $\mathscr{C}^1$ in ζ, ζ near zero, and holomorphic in z, z near X.

The possibility of this decomposition is seen in the following way. Let Δ_o be a domain in $\mathbb{C}^N$ with $X \subset \Delta_o \subset \Omega^- \cup \Omega^+$ and with the further property that if $(\tilde{\Delta}_o, \eta)$ is the envelope of holomorphy of Δ_o, then $\eta(\tilde{\Delta}_o) \subset \Omega^- \cup \Omega^+$. The manifold $U_o' \times \tilde{\Delta}_o$ is a Stein manifold; it is the envelope of holomorphy of $U_o' \times \Delta_o$. Define functions $\zeta_1, \dots, \zeta_N$ on U_o' to be the coordinate functions and $\eta_1, \dots \eta_N$ to be the coordinates of the map η.

For each $\zeta \in U_o'$, the partial function $H(\zeta, \cdot)$ is holomorphic on Δ_o and so extends holomorphically to $\tilde{\Delta}_o$. Denote this extended function by $\tilde{H}$. The function $\zeta_j - \eta_j$ is holomorphic on $U_o' \times \tilde{\Delta}_o$. Because $H(\zeta, z) = e^{h_\zeta^-}(\bar{\zeta} - 2ig(\xi) - z) \cdot (\zeta - z)$ on $U_o' \times U_o'$, we have

$$\tilde{H}(\zeta, p) = e^{h_\zeta^-(p)}(\bar{\zeta} - 2ig(\xi) - \eta(p)) \cdot (\zeta - \eta(p))$$

on $U_o' \times \eta^{-1}(U_o') \subset U_o' \times \tilde{\Delta}_o$. And on the set $(U_o' \times \tilde{\Delta}_o) \setminus (U_o' \times \eta^{-1}(U_o'))$, the functions $\zeta_j - \eta_j$ have no common zero. Thus, if $(\zeta_o, p_o) \in (U_o' \times \tilde{\Delta}_o) \setminus (U_o' \times \eta^{-1}(U_o'))$, then there is a neighborhood W_o of (ζ_o, p_o) on which there are holomorphic functions c_i with $1 = \sum_{j=1,\dots,N} c_j(\zeta, p)(\zeta_j - \eta_j(p))$. Then in W_o,

$$\tilde{H}(\zeta, p) = \sum_{j=1,\dots,N} \tilde{H}(\zeta, p) c_j(\zeta, p)(\zeta_j - \eta_j(p)).$$

Thus, $\zeta_j - \eta_j$ are holomorphic functions on the Stein manifold $U_o' \times \tilde{\Delta}_o$ with the property that there exists an open cover $\{V_j\}_{j=1,\dots}$ of $U_o' \times \tilde{\Delta}_o$ such that in each V_j there are holomorphic $\mathscr{C}^1(U_o')$-valued functions $G_{j,1}, \dots, G_{j,N}$ with

$$\tilde{H} = \sum_{k=1,\dots,N} G_{j,k}(\zeta - p)(\zeta_k - \eta_k(p)).$$

Theorem 6.3.13 of the appendix to this section provides holomorphic $\mathscr{C}^1(U_o')$-valued functions $G_1, \dots, G_N$ on $U_o' \times \tilde{\Delta}_o$ that satisfy

$$\tilde{H}(\zeta, p) = \sum_{k=1,\dots,N} G_j(\zeta, p)(\zeta_j - \eta_j(p)).$$

Restricted to $U_o' \times \Delta_o$, this gives the desired decomposition on a neighborhood of $U_o' \times X$.

Denote by $t(\zeta, z)$ the vector $(t_1(\zeta, z), \dots, t_N(\zeta, z))$.

For ζ and z both near the origin, define the function ψ by

$$\psi(\zeta, z) = t(\zeta, z) - He^{h^-}(\zeta + 2ig(\xi) - z).$$

With this definition, we have that $\psi(\zeta, z) \cdot (\zeta - z) = 0$.

For ζ and z both near the origin, define the map s^1 by

$$s^1(\zeta, z) = e^{h^-}(\bar{\zeta} + 2ig\xi) - \bar{z} - 2ig(x) + \psi(\zeta, z). \tag{6.7}$$

For ζ and z both near the origin, we have

$$\begin{aligned} s^1(\zeta, z) \cdot (\zeta - z) &= e^{-h^-}(|\zeta - z|^2 + 2i(g(\xi) - g(x)) \cdot (\zeta - z) + \psi(\zeta, z) \cdot (\zeta - z) \\ &> \text{const}\, |\zeta - z|^2. \end{aligned}$$

Now let α be a smooth function on $\mathbb{C}^N$ with support contained in a small neighborhood of the origin and that is identically one on a neighborhood of the origin, and define s by

$$s(\zeta, z) = \alpha(z)s^1(\zeta, z) + (1 - \alpha(z))t(\zeta, z).$$

Then $s(\zeta, z) \cdot (\zeta - z) > \text{const}\, |\zeta - z|^2$ when ζ is near the origin and z is near X. Also, $s(\zeta, z) < \text{const}\, |\zeta - z|$ on compacta in the domain of definition of s, as follows from $\psi(\zeta, \zeta) = 0$. That this is so is seen without difficulty: Fix ζ, and set $z = -\varepsilon\bar{\psi}(\zeta, \zeta) + \zeta$ with ε small. We have $\psi(\zeta, z) \cdot (\zeta - z) = 0$, so

$$\psi(\zeta, -\varepsilon\bar{\psi}(\zeta, \zeta) + \zeta) \cdot \bar{\psi}(\zeta, \zeta) = 0,$$

which gives $0 = \psi(\zeta, \zeta)(\varepsilon\bar{\psi}(\zeta, \zeta) + O(\varepsilon^2)$. Letting $\varepsilon \to 0$ shows that $\psi(\zeta, \zeta) = 0$.

The maps s and t have the properties (a)–(e) that we require, and the theorem is proved.

Example 6.3.6. The set X of Example 6.1.11 is a compact totally real set. It is not contained in a totally real manifold. However, by Theorem 6.3.2, every continuous function on X can be approximated uniformly by functions holomorphic on varying neighborhoods of X. This approximation is not implied in any obvious way by Theorem 6.3.1.

6.3.A. Appendix on Certain Vector-Valued Function-Theoretic Problems

Our object in this appendix is to discuss two results used in the proof of Theorem 6.3.2 concerning the dependence on parameters of solutions of certain function-theoretic problems.

We begin with Cousin I problems depending on parameters. This kind of problem can be considered in the context of solving $\bar{\partial}$ with parameters; the approach followed here, which was developed by Bishop and others, is to regard the problem as one concerning vector-valued functions. This approach seems to offer the most direct route to the particular results we need. However, we will not give full details for the development. In a certain sense they are elementary, depending, as they do, only on some relatively simple parts of Banach space theory. Nonetheless, the full details of what we need are rather elaborate, and we will not give them.

One of the principal results of this appendix is the following statement, which was an essential tool in the proof of the approximation theorem Theorem 6.3.2.

Theorem 6.3.7. [56] *Let $\mathscr{M}$ be a Stein manifold, let Ω be a domain in $\mathbb{R}^M$, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $\mathscr{M}$. For all $\alpha, \beta \in A$ let the function*[4] $f_{\alpha,\beta} : U_{\alpha,\beta} \times \Omega \to \mathbb{C}$ *be of class $\mathscr{C}^p$, $p \in \{1, \dots, \infty\}$, and for each $x \in \Omega$, let the partial function $f_{\alpha,\beta}(\cdot, x)$ be holomorphic. If the functions $f_{\alpha,\beta}$ satisfy the cocycle condition that $f_{\alpha,\beta}+f_{\beta,\gamma}+f_{\gamma,\alpha} = 0$, then there are functions $F_\alpha : U_\alpha \times \Omega \to \mathbb{C}$ of class $\mathscr{C}^p$ such that $F_\alpha(\cdot, x)$ is holomorphic for all $x \in \Omega$ and such that $f_{\alpha,\beta} = F_\alpha|U_{\alpha,\beta} - F_\beta|U_{\alpha,\beta}$.*

That is, the cocycle $\{f_{\alpha,\beta}\}_{\alpha,\beta\in A}$ is a coboundary.

We are going to exhibit this result as a consequence of a more general theorem of Bishop [56]. The formulation of this theorem requires a preliminary definition.

Definition 6.3.8. *If E is a Fréchet space and $\{e_n\}_{n=1,\dots}$ is a sequence in E, then the series $\sum_{n=1,\dots} e_n$* converges absolutely *if for every continuous seminorm $\|\cdot\|$ on E the series $\sum_{n=1,\dots} \|e_n\|$ converges.*

Theorem 6.3.9. *Let F be a Fréchet space, and let $\{\mathscr{M}_i\}_{i=1,\dots}$ be a sequence of complex manifolds. For each i let $\varphi_i : \mathscr{M}_i \to F$ be a holomorphic map. There exist a sequence $\{b_n\}_{n=1,\dots}$ in F and a sequence $\{P_n\}_{n=1,\dots}$ of rank-one mutually orthogonal projections of F to itself with these properties:*

(i) *For all i, $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ converges on $\mathscr{M}_i$ to φ_i.*

(ii) *For all n, $P_n b_n = b_n$, so that $P_n \circ \varphi_i = \varphi_i^{(n)} b_n$ for a $\varphi_i^{(n)} \in \mathscr{O}(\mathscr{M}_i)$.*

(iii) *$\sum_{n=1}^{\infty} \varphi_i^{(n)}$ converges absolutely in $\mathscr{O}(\mathscr{M}_i)$.*

(iv) *For every continuous seminorm $\|\cdot\|$ on F, the sequence $\{\|b_n\|\}_{n=1,\dots}$ is bounded.*

That the P_n are orthogonal means that $P_n \circ P_m = 0$ when $m \neq n$.

For a Fréchet space E there are various equivalent ways of defining E-valued holomorphic functions defined on a complex manifold. A convenient one is this:

Definition 6.3.10. *A function $f : \mathbb{U}^N \to E$ is holomorphic if there is an expansion*

$$f(z) = \sum_\alpha e_\alpha z^\alpha$$

in which the summation is over all N-tuples α of nonnegative integers, in which each coefficient e_α is an element of E, and in which z^α is the usual monomial $z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. It is assumed, moreover, that for every continuous seminorm $\|\cdot\|$ on E, the series $\sum_\alpha \|e_\alpha\| z^\alpha$ converges absolutely for each $z \in \mathbb{U}^N$.

With this local definition in mind, the notion of holomorphic E-valued function on a complex manifold is immediate: If $\mathscr{M}$ is a complex manifold, then $f : \mathscr{M} \to E$ is holomorphic if its restriction to every coordinate polydisk in $\mathscr{M}$ is holomorphic in the sense of the definition just given. Grothendieck [156] has given a thorough discussion of vector-valued holomorphic functions, which includes examples.

[4] Recall that in this kind of situation, $U_{\alpha,\beta}$ denotes the intersection $U_\alpha \cap U_\beta$.

We are particularly interested in the special case that the Fréchet space E is the space of $\mathscr{C}^p$ functions on a domain in $\mathbb{R}^N$. Here the seminorms that define the topology are the seminorms $\|\cdot\|_{p,K}$ defined for each compact subset K of Ω by

$$\|g\|_{p,K} = \max \|D_\beta g\|_K,$$

where $D_\beta g$ is the derivative of order β of g, and the maximum is extended over all the derivatives of order not more than p.

The following is a simply verified fact.

Lemma 6.3.11. *If $\mathscr{M}$ is a complex manifold and Ω is a domain in $\mathbb{R}^N$, then $f : \mathscr{M} \to \mathscr{C}^p(\Omega)$ is holomorphic if and only if the function $F : \mathscr{M} \times \Omega \to \mathbb{C}$ given by $F(z,x) = f(z)(x)$ is of class $\mathscr{C}^p$ and for each $x \in \Omega$, the partial function $f(\cdot, x)$ is holomorphic on $\mathscr{M}$.*

The derivation of Theorem 6.3.7 from Theorem 6.3.9 depends on a functional-analytic lemma.

Lemma 6.3.12. [56] *If E' and E'' are Fréchet spaces and $\varphi : E' \to E''$ is a surjective continuous linear mapping, then for every sequence $\{e''\}_{n=1,\dots}$ in E'' such that the series $\sum_{n=1,\dots} e''_n$ converges absolutely, there is a sequence $\{e'_n\}_{n=1,\dots}$ in E' such that for all n, $\varphi(e'_n) = e''_n$, and such that the series $\sum_{n=1,\dots} e'_n$ converges absolutely.*

Proof. The space E' is a Fréchet space, so its topology is defined by a sequence of seminorms. Accordingly, there is a sequence $\{\|\cdot\|_k\}_{k=1,\dots}$ of seminorms that defines the topology and that is increasing in that $\|\cdot\|_k \le \|\cdot\|_{k+1}$ for all $k = 1, \dots$. Such a sequence is called a *defining sequence* of seminorms for E'.

The map φ is surjective and so open. Consequently, for each k, the set $\{\varphi(y) : y \in E', \|y\|_k < 1\}$ contains the set $\{z \in E'' : \|z\|''_k < 1\}$ for some seminorm $\|\cdot\|''_k$ on E''. For each k, choose $j(k)$ with $\sum_{n=j(k)}^{\infty} \|e''_n\|''_k < 2^{-k}$. Thus, $\sum_{k=1}^{\infty}\sum_{n=j(k)}^{\infty} \|e''_n\|''_k < \infty$. Without loss of generality, the $j(k)$ can be assumed to be strictly increasing in k.

If n satisfies $j(k) \le n < j(k+1)$, choose $e'_n \in E'$ with $\varphi(e'_n) = e''_n$ and $\|e'_n\|_k \le \|e''_n\|''_k$. With $k(n)$ the least value of k for which $n < j(k+1)$, we have $\sum_{n=1}^{\infty} \|e'_n\|_{k(n)} < \infty$. For all p, $\|e'_n\|_p \le \|e_n\|'_k$ when $k > p$, so it follows that for all p the series $\sum_{n=1}^{\infty} \|e'_n\|_p$ converges, whence the series $\sum_{n=1}^{\infty} e'_n$ converges absolutely in E'. The lemma is proved.

Theorem 6.3.7 can be deduced from Theorem 6.3.9 as follows. We are given an open covering of $\mathscr{M}$ and an associated set of Cousin I data with values in $\mathscr{C}^p(\Omega)$. There is no loss of generality in assuming the open cover to be countable and indexed by the positive integers. Thus $\{U_i\}_{i\in\mathbb{N}}$ is an open cover of $\mathscr{M}$, and $f_{i,j}$ is a holomorphic function on $U_{i,j}$ with values in $\mathscr{C}^p(\Omega)$. The functions $f_{i,j}$ satisfy the cocycle condition. Choose elements $b_n \in \mathscr{C}^p(\Omega)$ and mutually orthogonal rank-one projections P_n of $\mathscr{C}^p(\Omega)$ such that for all i and j, $\sum_n P_n \circ f_{i,j}$ converges on $U_{i,j}$ to $f_{i,j}$; for all n, $P_n \circ f_{i,j} = f_{i,j}^{(n)} b_n$ with $f_{i,j}^{(n)} \in \mathscr{O}(U_{i,j})$; for all i and j, the series $\sum_{n=1}^{\infty} f_{i,j}^{(n)}$ converges absolutely in $\mathscr{O}(U_{i,j})$; and, finally, the sequence $\|b_n\|$ is bounded for every continuous seminorm $\|\cdot\|$ on $\mathscr{C}^p(\Omega)$.

By hypothesis, $\mathscr{M}$ is a Stein manifold, so $\mathbb{C}$-valued Cousin I problems are solvable on it. In particular, the range of the continuous linear mapping $\delta : \oplus_{i\in\mathbb{N}}\mathscr{O}(U_i) \to$

$\oplus_{i,j\in\mathbb{N}}\mathscr{O}(U_{i,j})$ given by $\delta(\{g_i\}_{i\in\mathbb{B}_N}) = \{g_i|U_{i,j} - g_j|U_{i,j}\}_{i,j\in\mathbb{N}}$ is the space, which we shall denote by $\mathscr{Z}$, of all $\{h_{i,j}\}_{i,j\in\mathbb{N}}$ with $h_{i,j} \in \mathscr{O}(U_{i,j})$ that satisfy the cocycle condition. This is a closed subspace of the product $\oplus_{i,j\in\mathbb{N}}\mathscr{O}(U_{i,j})$.

Because the functions $f_{i,j}^{(n)}$ satisfy $\sum_{i,j=1}^{\infty} \| f_{i,j}^{(n)} \|_C < \infty$ for every compact subset C of $U_{i,j}$, the sequence $\{f_{i,j}^{(n)}\}$ in $\oplus_{i,j}\mathscr{O}(U_{i,j})$ is absolutely convergent and so lifts: There is $\{f_j^{(n)}\}_{j\in\mathbb{N}} \in \delta^{-1}(f_{i,j}^{(n)})$ with $\sum \| f_j^{(n)} \| < \infty$ for every continuous seminorm $\| \cdot \|$ on the space $\oplus_{i\in\mathbb{N}}\mathscr{O}(U_i)$.

If now $F_j^{(n)} \in \mathscr{O}(U_j, \mathscr{C}^p(\Omega))$ is given by $F_j^{(n)} = f_j^{(n)} b_n$, we have that

$$f_{i,j} = \sum_n f_{i,j}^{(n)} = \sum_n (f_i^{(n)} - f_j^{(n)}) b_n = \sum_n F_i^{(n)} - \sum_n F_j^{(n)}.$$

Theorem 6.3.7 is therefore proved, assuming Theorem 6.3.9.

We are not going to prove Theorem 6.3.9 but instead refer to Bishop's paper [56] for the details.

We also need to discuss the solution of a decomposition problem. A general result sufficient for our purposes is this:

Theorem 6.3.13. *Let $\mathscr{M}$ be a Stein manifold and Ω a domain in $\mathbb{R}^M$, let $h_1, \dots, h_r \in \mathscr{O}(\mathscr{M})$, let $F \in \mathscr{O}(\mathscr{M}, \mathscr{C}^p(\Omega))$, and let $\{U_i\}_{=1,\dots}$ be an open cover of $\mathscr{M}$. If for each i there are $t_i = (t_{i,1}, \dots, t_{i,r}) \in \mathscr{O}^r(U_i, \mathscr{C}^p(\Omega))$ such that in U_i, $F = \sum_{k=1,\dots,r} t_{i,k} h_k$, then there are $t \in \mathscr{O}^r(\mathscr{M}, \mathscr{C}^p(\Omega))$ such that on all of $\mathscr{M}$, $F = \sum_{k=1,\dots,r} t_k h_k$.*

First, we notice that the corresponding statement is true if the space $\mathscr{C}^p(\Omega)$ is replaced by $\mathbb{C}$ so that we are dealing with $\mathbb{C}$-valued holomorphic functions rather than vector-valued functions. To see this, we suppose that $F \in \mathscr{O}(\mathscr{M})$ and that for each i, there are $g_{i,k} \in \mathscr{O}(U_i)$ such that $\sum_{k=1,\dots,r} g_{i,k} h_k = F$ in U_i. Define $g_{i,j;k} \in \mathscr{O}(U_{ij})$ by

$$g_{i,j;k} = g_{i,k}|U_{ij} - g_{j,k}|U_{ij}.$$

Then $\sum_{k=1,\dots,r} g_{i,j;k} h_k = 0$ on U_{ij}. Thus, if $\mathscr{R}_h$ is the sheaf on $\mathscr{M}$ defined by the condition that for every open set $V \subset \mathscr{M}$,

$$\mathscr{R}(V) = \{g \in \mathscr{O}^r(V) : g_1 h_1 + \cdots + g_r h_r = 0\},$$

then the holomorphic vectors $g_{i,j} = (g_{i,j;1}, \dots, g_{i,j;r})$ are elements of $\mathscr{R}_h(U_{ij})$. They satisfy the cocycle condition $g_{i,j} + g_{j,k} + g_{k,i} = 0$.

The sheaf $\mathscr{R}_h$, which is the *sheaf of relations among the functions* $h_1, \dots, h_r$, is coherent by one of Oka's theorems, see [157], so the cohomology group $H^1(\mathscr{M}, \mathscr{R}_h)$ vanishes: There are $\tilde{g}_j \in \mathscr{R}_h(U_j)$ with $\tilde{g}_i|U_{ij} - \tilde{g}_j|U_{ij} = g_{ij}$ for all i, j. On U_{ij}, $g_i - \tilde{g}_i = g_j - \tilde{g}_j$, so if $G = (G_1, \dots, G_r)$ with $G_k = g_k - \tilde{g}_k$ on U_k, then G is a well-defined element of $\mathscr{O}^r(\mathscr{M})$ that satisfies $\sum_{j=1,\dots,r} G_k h_k = F$.

The crux of this argument is the vanishing of the cohomology group $H^1(\mathscr{M}, \mathscr{R}_h)$.

The case of $\mathscr{C}^p(\Omega)$-valued holomorphic functions follows the same general route. Introduce the sheaf $\mathscr{R}_{h;p}$ defined by

$$\mathscr{R}_{h;p}(V) = \left\{ g \in \mathscr{O}^r(V, \mathscr{C}^p(\Omega)) : \sum_{k=1,\dots,r} g_k h_k = 0 \right\}.$$

An argument entirely parallel to the one just given shows that if the cohomology group $H^1(\mathscr{M}, \mathscr{R}_{h;p})$ vanishes, then there exists $G \in \mathscr{O}^p(\mathscr{M}, \mathscr{C}^p(\Omega))$ such that $\sum_{k=1,\dots,r} G_k h_k = F$.

That the group $H^1(\mathscr{M}, \mathscr{R}_{h;p})$ vanishes is proved in Bishop's paper [56]. The key is again Theorem 6.3.9 together with some further functional analysis. We refer to that paper for the details.

Mazzilli [242] has given a discussion of division problems that is based on integral formulas.

6.4. Some Tools from Rational Approximation

We are now going to establish some tools from the theory the algebras $\mathscr{R}(X)$ on planar compacta X, tools that will be used below in our further discussion of approximation results.

We start with a theorem of Bishop [55].

Theorem 6.4.1. *If X is a compact subset of $\mathbb{C}$, then $\mathscr{C}(X) = \mathscr{R}(X)$ if and only if almost every point of X is a peak point for $\mathscr{R}(X)$.*

The *almost every* of the statement is in the sense of planar Lebesgue measure.

A corollary of this theorem is the Hartogs–Rosenthal theorem, Theorem 1.6.4, which has already been proved by other means.

Proof. It is plain that if $\mathscr{C}(X) = \mathscr{R}(X)$, then *every* point of X is a peak point for $\mathscr{R}(X)$.

For the opposite implication, let P denote the set of peak points for $\mathscr{R}(X)$, and suppose that $\mathscr{R}(X) \neq \mathscr{C}(X)$. There is then a nonzero measure μ supported on X that is orthogonal to $\mathscr{R}(X)$. The measure μ annihilates $\mathscr{R}(X)$, so the Cauchy transform $\hat{\mu}$ vanishes on $\mathbb{C} \setminus X$. Also, because μ is not the zero measure, the function $\hat{\mu}$ is not almost everywhere zero, as follows from Lemma 1.6.5. Consequently, there is a set $E \subset X$ of positive Lebesgue measure such that $\hat{\mu}$ vanishes at no point of E. Because the measure μ is finite and so assigns nonzero mass to at most countably many points, the set E can be supposed to contain no point to which μ assigns nonzero mass. We shall show that $E \subset X \setminus P$, which contradicts the hypotheses of the theorem.

Suppose for the sake of contradiction that there is a point $z_o \in P \cap E$. Let δ denote the unit point mass concentrated at the point z_o. Put $c = \hat{\mu}(z_o)$, a nonzero quantity. We show that the measure ν defined by $\nu = (z - z_o)^{-1}\mu - c\delta$ is orthogonal to $\mathscr{R}(X)$. This is equivalent to showing that if f is a rational function on $\mathbb{C}$ without poles on X, then

$$\int \frac{f(z)}{z - z_o} d\mu(z) = cf(z_o). \tag{6.8}$$

To establish this equality, write $f(z) = \{(f(z) - f(z_o))/(z - z_o)\}(z - z_o) + f(z_o)$ to find that

$$\int \frac{f(z)d\mu(z)}{z - z_o} = \int \frac{f(z) - f(z_o)}{z - z_o} d\mu(z) + f(z_o) \int \frac{d\mu(z)}{z - z_o}.$$

Because $(f(z) - f(z_o))/(z - z_o)$, as a function of z, is in $\mathscr{R}(X)$, the first integral vanishes, so we have the desired formula (6.8). There is $f \in \mathscr{R}(X)$ with $1 = f(z_o) > |f(z)|$ for all $z \in X \setminus \{z_o\}$, for by hypothesis, $z_o \in P$. For all $n = 1, \dots,$

$$c = \int \frac{f^n(z)}{z - z_o} d\mu(z).$$

As $n \to \infty$, the integrals on the right tend to zero, so necessarily $c = 0$. But $c \neq 0$, and the proof is complete.

The next result, due to Alexander [5], is in the spirit of the observation made in Section 1.6 that the algebras $\mathscr{R}(X)$ on planar compacta are locally determined algebras.

Theorem 6.4.2. *If X is a compact subset of the plane with $X = \cup_{j=1,\dots} X_j$ where each X_j, $j = 1, \dots,$ is a compact set with $\mathscr{C}(X_j) = \mathscr{R}(X_j)$, then $\mathscr{C}(X) = \mathscr{R}(X)$.*

Proof. Because for each j, $\mathscr{C}(X_j) = \mathscr{R}(X_j)$, each X_j has empty interior, and the set X itself has empty interior by the Baire category theorem.

Suppose that $\mathscr{C}(X) \neq \mathscr{R}(X)$. There is then a nonzero finite measure μ with (minimal, closed) support $S \subset X$ that is orthogonal to $\mathscr{R}(X)$: That the measure μ is orthogonal to $\mathscr{R}(X)$ implies that the Cauchy transform $\hat{\mu}$ vanishes on $\mathbb{C} \setminus X$ and so on $\mathbb{C} \setminus S$, for every point of $\mathbb{C} \setminus S$ is a limit point of $\mathbb{C} \setminus X$. The Baire category theorem implies that one of the sets $S \cap X_j$ contains an open subset U of S. By the decomposition property introduced in Section 1.6.1, the orthogonal measure μ can be decomposed as the sum $\mu = \mu' + \mu''$ with μ' and μ'' both nonzero measures on S orthogonal to $\mathscr{R}(S)$, μ'' supported in U, and with $\operatorname{supp} \mu' \setminus \operatorname{supp} \mu''$ nonempty. Because $\mathscr{C}(X_j) = \mathscr{R}(X_j)$, μ'' is necessarily the zero measure. Thus, $\mu = \mu'$, which contradicts the choice of S as the support of μ.

Theorem 6.4.3. *There is a compact subset X of the plane with the properties that $\mathscr{R}(X) \neq \mathscr{C}(X)$ but the only Jensen measures for $\mathscr{R}(X)$ are point masses.*

It seems that the first example of this phenomenon was found by McKissick in his thesis [243, 244]. The details of this example can be found in the book [345]. They are complicated; a considerable simplification has been effected by Körner [211]. The example of McKissick goes far beyond the result stated in the theorem in that it is an example of a compact set X in the plane for which $\mathscr{R}(X) \neq \mathscr{C}(X)$, *but $\mathscr{R}(X)$ is* normal *in the sense that if E and F are mutually disjoint closed subsets of X, then there is $f \in \mathscr{R}(X)$ with $f|E = 1$ and $f|F = 0$.*

The construction contained in the following proof of the theorem was given by Browder [73].

Proof. The set X will be exhibited as an intersection $\cap_{n=0,\dots} X_n$ of an inductively constructed sequence of compact subsets of the plane. For X_0 take the unit square $\{z = x + iy : 0 \leq x, y \leq 1\}$. If $X_0, \dots, X_{n-1}$ have been constructed, denote by S_n the collection of all Gaussian integers $p = \alpha + i\beta$, $\alpha, \beta \in \mathbb{Z}$, for which the point $2^{-n}p$ lies in the interior of

X_{n-1}. Then let G_n be the union of all the open squares with centers $2^{-n}p$ for a $p \in S_n$ and with sides of length 2^{-3n+1}. The set X_n is defined to be $X_{n-1} \setminus G_n$.

The sets X_n are compact and constitute a decreasing sequence. The set X is compact, nonempty, and of positive area. That the area is positive is a consequence of the observation that the sum of the areas of the sets G_n is less than one: The set G_n is the union of not more than 2^{2n} squares each of which has area 2^{-6n+2}. The set X also has the property that $\mathscr{R}(X) \neq \mathscr{C}(X)$. For each n the set X_n is a certain closed domain with piecewise smooth boundary: It is obtained from the square X_0 by removing a finite number of smaller squares. Thus, if the measure ν_n is defined on X_n by $\int g\, d\nu_n = \int_{bX_n} g(z)\, dz$, then because the sum of the lengths of the perimeters of squares that constitute all the G_n's is finite, the measures ν_n are bounded uniformly in norm. Let ν be a weak* limit point of this sequence of measures. The measure ν is supported in X, and it is not the zero measure:

$$\int \bar{z}\, d\nu(z) = \lim_{n\to\infty} \int \bar{z}\, d\nu_n(z) = \lim_{n\to\infty} \int_{X_n} d\bar{z} \wedge dz = 2i \text{ Area } X \neq 0.$$

Thus, as claimed, ν is not the zero measure. It does, however, annihilate the algebra $\mathscr{R}(X)$, for if the rational function R has no poles on X, then it has no poles on some X_n, and for all $m \geq n$, $\int R\, d\nu_m = 0$.

What has to be shown to complete the proof is that the only Jensen measures for the algebra $\mathscr{R}(X)$ are the point masses at points of X.

This depends on an arithmetic fact:

Lemma 6.4.4. *For all $x \in X$, there exist infinitely many n's for which there are $p_n \in S_n$ such that*

$$\max\{|\Re(x - 2^{-n}p_n)|,\ |\Im(x - 2^{-n}p_n)|\} < 2^{1-n}.$$

We assume this lemma for the moment and complete the proof of the theorem. Consider a point $x \in X$, and define the function u_n by

$$u_n(z) = c_n(\log 2 - \log|z - q_n|) = c_n \log\left|\frac{2}{z - q_n}\right|$$

with $c_n = \left[\log 2 - \log|x - q_n|\right]^{-1}$ for a sequence q_n of points such that for each n, $q_n = p_n 2^{-n}$ for a $p_n \in S_n$, and $|x - q_n| < 2^{1-n}$. This function is harmonic on a neighborhood of the set X. Let μ be a measure on X that is a Jensen measure for the point x with respect to the algebra $\mathscr{R}(X)$. We have $|z - q_n| \geq 2^{-3n}$ for all n and all $z \in X$ and $|x - q_n| < 2^{1-n}$, so

$$0 \leq u_n(z) \leq \frac{\log 2 + \log 2^{3n}}{\log 2 + \log 2^{n-1}} < 4$$

for all $z \in X$. Also, $u_n(x) = 1$, and $u_n(z) \to 0$ for all $z \in X \setminus \{x\}$. For the latter point, note that $q_n \to x$, so $|z - q_n|$ is bounded away from zero. Thus,

$$1 = u_n(x) \leq \int u_n(z)\, d\mu(z) \to \mu(\{x\}),$$

and μ is seen to be the point mass concentrated on the point x. The theorem is proved, subject only to the verification of the lemma.

Remark. The proof just given did not use the full hypothesis that the measure μ is a Jensen measure: It suffices for μ to satisfy the weaker condition that $\log |\int g\, d\mu| \leq \int |\log g|\, d\mu$ for all *invertible* elements of the algebra $\mathscr{R}(X)$. Such measures are called *Arens–Singer* measures. They satisfy $\log |\int g\, d\mu| = \int |\log g|\, d\mu$ for all invertible elements g of $\mathscr{R}(X)$

Proof of Lemma 6.4.4. It will be a notational convenience to use in this proof the notation that $\| \cdot \|_\infty$ is the norm on $\mathbb{C}$ defined by $\|x + iy\|_\infty = \max\{|x|, |y|\}$. Thus for $z_o \in \mathbb{C}$ and $r > 0$, the ball $\{z \in \mathbb{C} : \|z - z_o\|_\infty < r\}$ is the open square in $\mathbb{C}$ centered at the point z_o and of side $2r$. Denote this ball by $Q(z_o, r)$.

The proof of the lemma is executed by showing that if $x \in X$, then for any $n > 3$ that is not divisible by 3, the desired p_n exists. It is sufficient to show that for such n, $x \in \bar{Q}(p2^{-n}, 2^{-n})$.

Fix an $n > 3$ that is not divisible by 3.

We consider two cases. First it may be that

$$\bar{Q}(x, 2^{-n}) \cap Q(p2^{-m}, 2^{-2m}) = \emptyset$$

for all $m < n$ and all $p \in S_m$. In this case, any Gaussian integer p with $p2^{-n} \in \bar{Q}(x, 2^{-n})$ will work.

In the contrary case that there is a Gaussian integer p such that

$$2^{-3m} < \|p2^{-n} - q2^{-m}\|_\infty < 2^{-3m} + 2^{-n}$$

for some $m < n,\ q \in S_m$, and $\|p2^{-n} - x\|_\infty \leq 2^{-n}$, we show that $p \in S_n$. If not, there are $k < n$ and $r \in S_m$ such that

$$\|p2^{-n} - r2^{-k}\|_\infty < 2^{-3k}.$$

Then

$$\begin{aligned} 0 &\leq \|r2^{-n} - q2^{-m}\|_\infty \\ &\leq \|r2^{-k} - p2^{-n}\|_\infty + \|p2^{-n} - q2^{-m}\|_\infty \\ &\leq 2^{-3k} + 2^{-3m} + 2^{-n} \end{aligned} \tag{6.9}$$

with $r \in S_k,\ q \in S_m$, and $k, m < n$. Without loss of generality, let $k \leq m$. Then

$$\|q2^{-m} - r2^{-k}\|_\infty > 2^{-3k}$$

because $q \in S_m$, so

$$\|q - r2^{m-k}\|_\infty > 2^{m-3k}. \tag{6.10}$$

If $m > 3k$, then because we are computing the distance between Gaussian integers in the supremum norm $\| \cdot \|_\infty$, (6.10) implies

$$\|q - r2^{m-k}\|_\infty \geq 1 + 2^{m-3k}.$$

Then (6.9) and (6.10) imply $1 < 2^{-2m} + 2^{m-n}$, which is impossible, since $n > m \geq 1$.

If, on the other hand, $m \leq 3k$, then $\|q - r^{m-k}\|_\infty > 0$, so as above,

$$\|q - r^{m-k}\|_\infty \geq 1.$$

In this case, it follows that

$$1 \leq 2^{-m} + 2^{m-3k} + 2^{m-n}.$$

There are now two cases. First, it may be that 2^{m-3k} and 2^{n-m} are both $\frac{1}{2}$. In this case, $m - 3k = 1 = n - m$, which implies that n is divisible by 3, which contradicts the hypothesis about n. Alternatively, 2^{-m} and 2^{m-3k} are both $\frac{1}{2}$. In this case, $k = m = 1$, which is impossible, for S_1 contains a single element.

This completes the proof of the lemma and of the theorem.

6.5. Algebras on Surfaces

There are some striking results about approximation on compact two-dimensional manifolds, possibly with boundary. With a little extra effort, this work can be executed in the setting of uniform algebras on surfaces generated by smooth functions rather than in that of polynomial approximation on embedded surfaces, so it is this more general context in which we shall work initially.

Consider a compact two-dimensional manifold Σ, perhaps with boundary, of class $\mathscr{C}^1$. Let A be a uniform algebra on Σ that is generated by functions of class $\mathscr{C}^1$, so that the subalgebra A_1 of A that consists of all the functions in A of class $\mathscr{C}^1$ is dense in A.

Definition 6.5.1. *The* exceptional set *of Σ is the complement of the set of points p with the property that there exist functions f_1 and f_2 in A_1 such that the differential $df_1 \wedge df_2$ does not vanish at p.*

We shall use consistently the notation that E is the exceptional set.

Lemma 6.5.2. [130] *If Σ is the spectrum of A, then every point of $\Sigma \setminus E$ is a peak point for A.*

Proof. The proof depends on the result that local peak points are peak points—recall Theorem 2.1.20.

Let p be a point of $\Sigma \setminus E$. Choose local real coordinates x_1, x_2 on a neighborhood of the point p with respect to which p is the origin. The point p is not in the essential set, so there are f_1, f_2 in A_1 such that the form $df_1 \wedge df_2$ does not vanish at the point p. The f's can be chosen to vanish at p. Let F be the column vector $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. If J denotes the Jacobian matrix $[\partial f_j / \partial x_k]$ evaluated at p, then

$$F(q) = Jx(q) + o(x(q)).$$

The matrix J is invertible, so this equation can be solved approximately for the vector x. Using this solution, compute the norm $\|x(q)\|^2$: With t denoting transpose,

$$\|x(q)\|^2 = x(q)^t x(q) = F(q)^t (J^{-1})^t J^{-1} F(q) + o(\|x(q)\|). \tag{6.11}$$

The quantity $F(q)^t(J^{-1})^t J^{-1}F(q)$ is a polynomial in the functions f_1 and f_2, and so lies in the algebra A_1. Call this element g. The equation (6.11) implies that there is a neighborhood U of p so small that $g(U \setminus \{p\})$ is contained in the right half-plane. We have $g(p) = 0$. If h is an entire function that takes the value one at the origin and is strictly less than one in modulus at every other point of the closed right half-plane, then the function $h \circ g$ lies in A and achieves a local maximum at the point p. We are to conclude from this that there is a function in the algebra A that peaks at the point p. Because Σ is assumed to be the spectrum of the algebra A, the local peak point theorem, Theorem 2.1.20, yields this conclusion.

Given a measure μ on Σ and given a continuous function g on Σ, the measure $g_*\mu$ is the measure on the complex plane, which has compact support, defined by $\int \varphi \, d(g_*\mu) = \int_\Sigma \varphi \circ g \, d\mu$.

The following theorem of Freeman [130] is basic for much of what follows.

Theorem 6.5.3. *If Σ is the spectrum of A, then for every $f \in A_1$ and every $\mu \in A^\perp$, the Cauchy transform of the measure $f_*\mu$ vanishes almost everywhere $[d\mathscr{L}]$ on $\mathbb{C} \setminus f(E)$, whence* supp $f_*\mu \subset f(E)$.

This result is not simple. There is, though, a relatively simple result of the same flavor in the context of general uniform algebras.

Lemma 6.5.4. *If B is a uniform algebra on the compact space X that is its spectrum, and if μ is a measure on X that annihilates B, then for every $g \in B$, $g_*\mu$ annihilates the algebra $\mathscr{R}(g(X))$.*

Proof. The set $g(X)$ is the spectrum $\sigma_B g$, so $r \circ g \in B$ for every rational function r on the plane that has no poles on $g(X)$. Thus

$$0 = \int_X r \circ g(x) \, d\mu(x) = \int_{g(X)} r(z) \, d(g_*\mu)(z).$$

Consequently, $g_*\mu$ annihilates $\mathscr{R}(g(X))$.

Thus, the essential content of Theorem 6.5.3 is the vanishing of the Cauchy transform of $f_*\mu$ a.e. $[d\mathscr{L}]$ on $f(X) \setminus f(E)$.

Proof of Theorem 6.5.3. We are to show that

$$\int \frac{1}{\zeta - z} \, d(f_*\mu)(\zeta) = 0 \quad \text{a.e. } [d\mathscr{L}] \text{ off } f(E). \tag{6.12}$$

The integral on the left is $\int_\Sigma \frac{1}{f(x)-z} \, d\mu(x)$. The measure $f_*\mu$ is a finite measure, so $|f_*\mu| = f_*|\mu|$ is a finite measure on $\mathbb{C}$, and, by Lemma 1.6.5,

$$\int \frac{d|\mu(x)|}{|f(x) - z|} = \int_{\mathbb{C}} \frac{d|f_*\mu|(\zeta)}{|\zeta - z|} < \infty \quad \text{a.e. } [d\mathscr{L}].$$

We restrict attention to those $z \in \mathbb{C}$ for which this integral is finite. Also, for almost every point $z \in \mathbb{C}$, the fiber $f^{-1}(z)$ is a finite set, as follows from Eilenberg's inequality and the finiteness of the area of Σ, and Sard's theorem implies that almost every value of f

is a regular value for f. Thus, to prove the theorem, it is enough to deal only with points $z \in \mathbb{C} \setminus f(E)$ for which $f^{-1}(z)$ is a finite set and for which, with $f = u + iv$, the functions u and v give real coordinates at each point of the fiber $f^{-1}(z)$.

Fix a $z \in \mathbb{C} \setminus f(E)$ with these properties. We are going to construct a sequence $\{f_n\}$ in A such that $\lim_{n\to\infty} f_n = \frac{1}{f-z}$ and $|f_n| \leq \frac{2}{|f-z|}$ a.e. $[d|\mu|]$. The construction of this sequence is based on a lemma that is easily stated but whose proof is rather involved:

Lemma 6.5.5. *There are functions h and h_1 in A such that $h = (f - z)h_1$ and $h(\Sigma) \subset \{0\} \cup \{\zeta \in \mathbb{C} : |\zeta - 1| > 1\}$.*

Let us assume the lemma for the moment and proceed to the construction of the functions f_n. For $n = 1, \ldots,$ let

$$\phi_n(\zeta) = \frac{1}{\zeta}\left(1 - \frac{1}{(\zeta - 1)^{2n}}\right).$$

The function ϕ_n is holomorphic on the whole plane except for a pole at the point $\zeta = 1$. If $|\zeta - 1| > 1$, then $\phi_n(\zeta) \to \frac{1}{\zeta}$, and $|\phi_n(\zeta)| < \frac{2}{\zeta}$. Theorem 1.2.25 implies that the function $f_n = (\phi_n \circ h)h_1$ lies in A. For all n we have $|f_n| \leq \frac{2}{|f-z|}$, and as $n \to \infty$, $f_n \to \frac{1}{h}h_1 = \frac{1}{f-z}$.

With these functions, the proof of the theorem concludes with an application of the Lebesgue dominated convergence theorem: Because the function $\left|\frac{2}{f-z}\right|$ is integrable with respect to the measure μ, we can write

$$\int \frac{d(f_*\mu)(\zeta)}{\zeta - z} = \int \frac{d\mu(x)}{f(x) - z} = \lim_{n\to\infty} \int f_n(x)\, d\mu(x) = 0.$$

The proof of Lemma 6.5.5 depends on a result about finding a holomorphic function with prescribed zeros, the solution of a multiplicative Cousin problem, which is a small variation of a result given by Rossi [303, Theorem 2.4]:

Lemma 6.5.6. *Let X be a compact, polynomially convex set in $\mathbb{C}^N$, let U be an open set in $\mathbb{C}^N$, and let $g \in \mathscr{O}(U)$. Assume that the set $\{z \in X \cap U : \Re g(z) \geq 0\}$ is closed in $\mathbb{C}^N$ (or, equivalently, in X). There is then an open set W in $\mathbb{C}^N$ that contains X on which there is defined a holomorphic function ψ that is zero-free on $X \setminus U$ and such that ψ/g is holomorphic and zero-free on $W \cap U$.*

Proof. Let U' be a relatively compact neighborhood of $\{z : \Re g(z) \geq 0\} \cap X$ whose closure is contained in U. Denote by W' a neighborhood of $X \setminus U'$ such that $W' \cap \bar{U}' \subset \{z : \Re g(z) > 0\}$. Let W'' be a neighborhood of $X \setminus W'$ with $\bar{W}'' \subset U'$. Then $W' \cup W''$ is an open set that contains X. The set X is polynomially convex, so there is a polynomial polyhedron Ω with $X \subset \Omega \subset W' \cup W''$. As a polynomial polyhedron, Ω is a domain of holomorphy. We have that $\Omega \cap U' \subset \{z : \Re g(z) > 0\}$, because $\Omega \cap bU' \subset W' \cap \bar{U}'$. Also, $\Omega \cap \{g = 0\}$ is closed in Ω.

Choose a locally finite cover $\{V_j\}_{j=0,1,\ldots}$ of Ω as follows. Take $V_0 = U' \cap \Omega$, $V_j \cap \{z : g(z) = 0\} = \emptyset$ for all $j \neq 0$, and $V_0 \cap V_j \subset \{z : \Re g(z) > 0\}$ for all $j \neq 0$. Then $\cup_{j=1,\ldots} V_j \cap V_0 \subset \{z : \Re g(z) > 0\}$, so the function $\log g$ has a well-defined determination on this set. Denote by h a branch of $\log g$ defined on this set. Define a set of Cousin II data associated with the cover $\{V_j\}_{j=0,1,\dots}$ by $f_0 = g$, and $f_j = 1$ if $j > 0$. Then $f_j/f_k \in \mathscr{O}^*(V_{jk})$ for all choices of j and k. Define branches h_{jk} of $\frac{1}{2\pi i}\log f_j/f_k$ as follows: $h_{jk} = 0$ unless one of j, k is zero, and $h_{j0} = -h,\ h_{0j} = h$. With these determinations the quantities $c_{ijk} = h_{ij} + h_{jk} + h_{ki}$ all vanish. Thus—recall the discussion of the second Cousin problem at the beginning of Section 2.1—there is a function $\psi \in \mathscr{O}(\Omega)$ such that for each index $j = 0, 1, \dots,$ ψ/f_j is holomorphic and zero-free on V_j. This completes the proof.

Proof of Lemma 6.5.5. For the proof of this lemma, it is convenient to suppose Σ to be a $\mathscr{C}^1$ submanifold of a Euclidean space and to use the distance function on the ambient space to compute distances on Σ.

Fix attention on a $p \in f^{-1}(z)$. There are functions $g_1, g_2 \in A_1$ such that $dg_1 \wedge dg_2$ does not vanish at p. Accordingly, near p in Σ,

$$dg_1 \wedge dg_2 = A df \wedge d\bar{f} \text{ with } A = \frac{\partial g_1}{\partial f}\frac{\partial g_2}{\partial \bar{f}} - \frac{\partial g_1}{\partial \bar{f}}\frac{\partial g_2}{\partial f}. \tag{6.13}$$

That this quantity does not vanish implies that one of the derivatives with respect to $\bar{f}$ does not vanish. Suppose $\partial g_2/\partial\bar{f}(p) \neq 0$. The first-order Taylor expansion of g_2 about p in terms of f and $\bar{f}$ is

$$g_2(q) = g_2(p) + \frac{\partial g_2}{\partial f}(p)(f(q) - z) + \frac{\partial g_2}{\partial \bar{f}}(p)(\overline{f(q) - z}) + r(q), \tag{6.14}$$

in which the remainder term $r(q)$ is $o(1)|f(q) - z| = o(1)\text{dist}\,(p, q)$. Define the polynomial $G(z_1, z_2)$ by

$$G(z_1, z_2) = -\left[\frac{\partial g_2}{\partial \bar{f}}(p)\right]^{-1}(z_1 - z)\left[(z_2 - g_2(p)) - \frac{\partial g_2}{\partial f}(p)(z_1 - z)\right], \tag{6.15}$$

so that

$$G(f(q), g_2(q)) = -|f(q) - z|^2 - \left[\frac{\partial g_2}{\partial \bar{f}}(p)\right]^{-1}(f(q) - z)r(q),$$

as follows from (6.14). The function $G(f(q), g_2(q))$ is divisible by $f(q) - z$ in A, and, because $r(q) = o(\text{dist}\,(q, p))$, we have that $\Re G(f(q), g_2(q)) < 0$ when q is confined to a suitable deleted neighborhood of p in Σ. Thus it is a solution to our problem on a neighborhood of the point p.

The fiber $f^{-1}(z)$ is finite; let its points be $p_1, \dots, p_r$. Perform the preceding construction at each of the points p_j to obtain $G_1, \dots, G_r$ with G_j constructed using functions $g_{1,j}, g_{2,j}$. Set $\varphi_j = g_{2,j}$. For $j = 1, \dots, r$, let W_j be a neighborhood of p_j small enough that $\Re G_j(f(q), \varphi_j(q)) < 0$ for $q \in W_j \setminus \{p_j\}$. By making W_j small enough, we can suppose that $f|W_j$ is a homeomorphism. For each j, let W_j' be a compact neighborhood of p_j contained in the interior of W_j.

There are finitely many functions $\varphi_{r+1}, \ldots, \varphi_s$ in A such that if $\mathbf{g} = (f, \varphi_1, \ldots, \varphi_s) : \Sigma \to \mathbb{C}^{s+1}$, then $\mathbf{g}(\Sigma \setminus W_j) \cap \mathbf{g}(W_j') = \emptyset$.

The set $\mathbf{g}(\Sigma)$ is the joint spectrum $\sigma_A(f, \varphi_1, \ldots, \varphi_s)$. In the event that A admits a finite number of generators, we can suppose that among the φ's there is a set of generators, in which case this joint spectrum is polynomially convex. In general, it will not be polynomially convex.

For each $j = 1, \ldots, r$, let V_j be an open set in $\mathbb{C}^{s+1}$ that is disjoint from $\mathbf{g}(\Sigma \setminus W_j)$ and with $V_j \supset \mathbf{g}(W_j')$. The sets $\mathbf{g}(W_j')$ are mutually disjoint, so we can suppose the sets V_j to be mutually disjoint. We have $\mathbf{g}(p_j) \in V_j$ and $\Re G_j < 0$ on $V_j \cap \mathbf{g}(\Sigma \setminus \{p_j\})$.

Define a holomorphic function G_o on $V = \cup_{j=1,\ldots,s} V_j$ by the condition that on V_j,

$$G_o(z_0, \ldots, z_s) = G_j(z_o, z_j), \tag{6.16}$$

where G_j is the polynomial associated with φ_j as $G(z_1, z_2)$ in (6.15) is associated with the construction of the initial function g_2. The function G_o is well defined and holomorphic in V. It satisfies $G_o(f^{-1}(z)) \subset V$ and $V \cap G_o(\Sigma \setminus f^{-1}(z) \subset \{\Re G_o < 0\}$.

If the joint spectrum $\sigma_A(f, \varphi_1, \ldots, \varphi_s)$ were polynomially convex, we could invoke Lemma 6.5.6. However, it typically is not, and a further step, based on the Arens–Calderón approximation process of Lemma 1.5.8, has to be used. Let Ω be a relatively compact open subset of $\mathbb{C}^{s+1}$ with $f^{-1}(z) \subset \Omega$ and $\bar{\Omega} \subset V$. The set $\mathbf{g}(\Sigma) \setminus \Omega$ is compact, and the sets $\mathbf{g}(\Sigma) \setminus \Omega$ and $\Omega \cap \{\Re G_o \geq 0\}$ are disjoint. Let D' be a neighborhood of $\mathbf{g}(\Sigma) \setminus \Omega$ that is disjoint from $\Omega \setminus \{\Re G_o \geq 0\}$. The set $D' \cup \Omega$ is a neighborhood, D, of $\mathbf{g}(\Sigma)$.

There are elements $\varphi_{s+1}, \ldots, \varphi_t$ of A such that under the orthogonal projection $\varphi : \mathbb{C}^{t+1} \to \mathbb{C}^{r+1}$, the polynomially convex hull $\sigma_A(f, \varphi_1, \ldots, \varphi_t)\widehat{}$ is carried into D. The set

$$\varphi(\sigma_A(f, \varphi_1, \ldots, \varphi_t)\widehat{}) \cap \Omega \cap \{g_o \geq 0\}$$

is closed in $\mathbb{C}^{r+1}$, and the function $G_o \circ \varphi$ is holomorphic on $\varphi^{-1}(\Omega)$. The set

$$\sigma_A(f, \varphi_1, \ldots, \varphi_t)\widehat{} \cap \{\zeta \in \varphi^{-1}(\Omega) : \Re G_o \circ \varphi(\zeta) \geq 0\}$$

is closed in $\mathbb{C}^{t+1}$.

Let $\Phi : \Sigma \to \mathbb{C}^{t+1}$ be the joint spectrum map given by

$$\Phi(p) = (f(p), \varphi_1(p), \ldots, \varphi_t(p)).$$

By Lemma 6.5.6 there exists a function ψ holomorphic on a neighborhood Ω' of the joint spectrum $\sigma_A(f, \varphi_1, \ldots, \varphi_t)$ without zeros on $\Omega' \setminus \pi^{-1}(\Omega)$ such that $\psi / G_o \circ \varphi$ is zero-free on $\pi^{-1}(\Omega) \cap \sigma_A(f, \varphi_1, \ldots, \varphi_t)$.

The function h_o defined on Σ by

$$h_o(p) = \psi(f(p), \varphi_1(p), \ldots, \varphi_t(p))$$

is in A because of Theorem 1.5.7. We obtain the desired function h of the lemma by modifying h_o slightly. First, we can choose ψ such that h_o takes the value one at each point of the fiber $f^{-1}(z)$: If the initially chosen ψ does not yield this, replace it by $\eta\psi$

with η a suitable zero-free entire function. Because of (6.16), arg $G_o \circ \varphi \circ \Phi(q) \to \pi$ as $q \to f^{-1}(z)$, so the same is true of arg $\psi \circ \Phi$.

The function f gives local coordinates near each point of the fiber $f^{-1}(z)$, so Φ carries a neighborhood of f^{-1} homeomorphically into $\mathbb{C}^{t+1}$. If we shrink Ω, then

$$\Phi(\Sigma \setminus f^{-1}(z)) \cap \varphi^{-1}(\Omega) \subset \{\Re G_o < 0\}.$$

Because ψ has no zeros on $\Phi(\Sigma) \setminus \varphi^{-1}(\Omega)$, if we multiply ψ by a sufficiently large positive constant, we shall have $|\psi| > 2$ on $\Phi(\Sigma) \setminus \Omega$. Thus ψ maps $\Phi(\Sigma)$ into $\{\zeta \in \mathbb{C} : |\zeta - 1| > 1 \cup \{0\}$. By shrinking Ω' a little if necessary, we can suppose that $\Omega' \setminus \varphi^{-1}(\Omega)$ is disjoint from the complex hyperplane $\{(z_o, \ldots, z_t) \in \mathbb{C}^{t+1} : z_o = z\}$, so ψ is divisible by $z_o - z$, for $G_o \circ \varphi$ has this property in $\varphi^{-1}(\Omega)$. That is,

$$\psi(z_o, \ldots, z_t) = (z_o - z)\psi_1(z_o, \ldots, z_t)$$

for a function ψ_1 holomorphic on Ω'. If $h_1 = \psi_1 \circ \Phi$ and $h = (f - z)h_1$, then h and h_1 lie in A and are as required by the lemma.

The proof of Theorem 6.5.3 is now complete

In the following theorem we continue the notation that A is a uniform algebra on the compact surface Σ, possibly with boundary, of class $\mathscr{C}^1$ that is generated by functions of class $\mathscr{C}^1$. It is supposed that Σ is the spectrum of A. The set E is the exceptional set.

Theorem 6.5.7. [130] *The algebra A contains the ideal $\mathscr{I}_E$ of all functions continuous on Σ and vanishing on E.*

Proof. For this proof, denote by B the algebra generated by the functions of the form $\varphi \circ f$, where f runs through A and, for fixed f, φ runs through the collection of functions continuous on $\mathbb{C}$ that vanish on $f(E)$. We take B *not* to contain the identity. The algebra B is closed under complex conjugation and is contained in the ideal $\mathscr{I}_E$ of $\mathscr{C}(\Sigma)$. It also separates points on the set $\Sigma \setminus E$, as follows from Lemma 6.5.2. The Stone–Weierstrass theorem implies that the algebra B is dense in the algebra $\mathscr{I}_E$.

Suppose now that $\mu \in A^\perp$. We shall show that necessarily $\mu \in B^\perp$. If f and g are in A, then $g\mu \in A^\perp$, so by Theorem 6.5.3, supp $f_*(g\mu) \subset f(E)$. Consequently, if φ is continuous on the plane and vanishes on $f(E)$, then

$$0 = \int_C \varphi \, d(f_*(g\mu)) = \int_\Sigma \varphi \circ f \, d(g\mu) = \int_\Sigma (\varphi \circ f) g \, d\mu = \int_\Sigma g \, d((\varphi \circ f)\mu).$$

This equation holds for all choices of g, so the measure $(\varphi \circ f)\mu$ is orthogonal to A. Thus each measure $(\varphi \circ f)\mu$ is orthogonal to A, so the measure μ lies in $B^\perp$, and the theorem is proved.

As an application, we have the following corollary, which settles a point left open in Section 5.3.

Corollary 6.5.8. [128] *If Ω is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in a Stein manifold of dimension two, and if Δ is a compact $\mathscr{C}^2$ disk contained in $b\Omega$ that is totally real except for a finite number of hyperbolic points, then $\mathscr{O}(\Omega)|\Delta$ is dense in $\mathscr{C}(\Delta)$. In particular, if $\Omega \subset \mathbb{C}^2$ and $\bar{\Omega}$ is polynomially convex, then $\mathscr{P}(\Delta) = \mathscr{C}(\Delta)$.*

A special case is that in which Δ is contained in $b\mathbb{B}_2$, when $\mathscr{P}(\Delta) = \mathscr{C}(\Delta)$.

Proof. According to Theorem 5.3.1, the disk Δ is $\mathscr{O}(\bar{\Omega})$-convex. The exceptional set is the finite set H of hyperbolic points, and the theorem just proved shows that $\mathscr{O}(\Delta)$ contains the ideal $\mathscr{I}_H$ of all continuous functions on Δ that vanish at each point of H. It follows that $\mathscr{O}(\Delta) = \mathscr{C}(\Delta)$.

The next lemma will be used below.

Lemma 6.5.9. *Let A be a uniform algebra on the compact space X. If there is a dense subalgebra A_o of A such that for each $f \in A_o$ and each measure μ that annihilates A, $f_*\mu = 0$, then $A = \mathscr{C}(X)$.*

Proof. The hypotheses imply that for $f \in A_o$,

$$0 = \int \bar{z}\, d(f_*\mu)(z) = \int \bar{f}(x)\, d\mu(x).$$

Thus, for all $f \in A_o$, $\bar{f} \in A$. Consequently, the algebra A is closed under conjugation, whence, by the Stone–Weierstrass theorem, $A = \mathscr{C}(X)$.

Freeman gave some further corollaries of Theorem 6.5.3.

Corollary 6.5.10. [130] *If the essential set is totally disconnected or if it has zero area, then $A = \mathscr{C}(\Sigma)$.*

Proof. Suppose first that E has measure zero. Fix a measure $\mu \in A^\perp$. If $f \in A$ is of class $\mathscr{C}^1$, then the Cauchy transform of the measure $f_*\mu$ vanishes off the set $f(E) \subset \mathbb{C}$, which has zero planar measure. Consequently, the measure $f_*\mu$ is the zero measure by Lemma 1.6.5. The result follows from the preceding lemma.

If the set E is totally disconnected, again fix a measure $\mu \in \mathscr{O}(\Sigma)^\perp$. The algebra $\mathscr{O}(\Sigma)$ contains the ideal $\mathscr{I}_E$, so the measure μ must be concentrated on the set E, and it is orthogonal to the uniformly closed subalgebra B of $\mathscr{C}(E)$ generated by the restrictions $f|E$, $f \in A$. The spectrum of B is the set E: If χ is a character of B, then $f \mapsto \chi(f|E)$ is a character of A, so there is a unique point $s \in \Sigma$ such that $\chi(f) = f(s)$ for all $f \in A$. Because B contains the ideal $\mathscr{I}_E$, the point s must lie in E, so E is indeed the spectrum of B.

The set E is totally disconnected, so it admits decompositions of the form $E = E' \cup E''$ with E' and E'' disjoint open and closed subsets of E. By Lemma 1.5.9, there are finitely many functions $f_1, \ldots, f_r$ in A such that if $F = (f_1, \ldots, f_r)$, then $F(E') \cap F(E'') = \emptyset$. If U' and U'' are disjoint open sets in $\mathbb{C}^r$ with $U' \supset F(E')$ and $U'' \supset F(E'')$, then the function g that is 1 on U' and 0 on U'' is holomorphic on the joint spectrum $\sigma_B(f_1, \ldots, f_r)$, and so $g \circ F \in B$. It follows that B contains enough real-valued functions to separate points on E so that $B = \mathscr{C}(E)$. Then μ must be the zero measure, for it annihilates B.

The corollary is proved.

Freeman in [130] gives the following example. Let T_o be a compact totally disconnected subset of the interval $[-\frac{1}{2}, \frac{1}{2}]$ of positive length. Let $T \subset \bar{\mathbb{U}}$ be the set of all points $z = x + iy$ with $x \in T_o$. The set E is neither totally disconnected nor of zero area. Let g_o be a smooth function on $\mathbb{R}$ with $g_o' = 0$ on the set T_o and nowhere else. Define $g \in \mathscr{C}(\bar{\mathbb{U}})$

by $g(x + iy) = g_o(x)$, and let A be the closed subalgebra of $\mathscr{C}(\bar{\mathbb{U}})$ generated by z and g. The exceptional set for the algebra A is the set T: $\frac{\partial g}{\partial \bar{z}}$ vanishes on T and nowhere else on $\bar{\mathbb{U}}$. The set $\Gamma = \{(z, g(z)) : z \in \bar{\mathbb{U}}\}$ is polynomially convex, as follows from Theorem 1.2.16, and, by the same theorem, $\mathscr{P}(\Gamma) = \mathscr{C}(\Gamma)$, so that $A = \mathscr{C}(\bar{\mathbb{U}})$.

Thus, the exceptional set of an algebra of the kind contemplated in Theorem 6.5.3 need not be totally disconnected or of measure zero for the algebra to coincide with the algebra of all continuous functions on the surface.

The following corollary is implicit in Freeman's paper [130], though it is not stated there.

Corollary 6.5.11. *If Σ is a real-analytic, two-dimensional compact submanifold of $\mathbb{C}^N$ that is polynomially convex, then $\mathscr{P}(\Sigma) = \mathscr{C}(\Sigma)$.*

Proof. Suppose without loss of generality that Σ is connected. The set Σ_c of points at which Σ is not totally real is a real-analytic subset of Σ: With ι the inclusion $\Sigma \hookrightarrow \mathbb{C}^N$, it is the set on which all the two-forms $\iota^*\vartheta$, ϑ a holomorphic two-form on $\mathbb{C}^N$, vanish. Accordingly, either it is of dimension not more than one or else it is the whole of Σ. In the former case, it has two-dimensional measure zero, and Freeman's result, Corollary 6.5.10, yields that $\mathscr{P}(\Sigma) = \mathscr{C}(\Sigma)$. (This conclusion also follows, albeit less directly, by invoking Theorem 6.5.12 below.)

In the latter case, each tangent space to Σ is a complex line, and so by the theorem of Levi-Civita, Theorem 6.1.12, Σ is a *complex* submanifold of $\mathbb{C}^N$. The maximum principle precludes the existence of compact, complex submanifolds of $\mathbb{C}^N$, so the corollary is proved.

There is a further theorem about approximation on two-dimensional manifolds, a result whose interest lies in part in the absence, which we are going to establish eventually, of an analogue on manifolds of higher dimension; it is special to the two-dimensional case. The result was found by Anderson and Izzo [34].

Theorem 6.5.12. *If Σ is a compact two-dimensional manifold of class $\mathscr{C}^1$, if A is a uniform algebra on Σ generated by a family of functions of class $\mathscr{C}^1$ with Σ as its spectrum and for which almost every point is a peak point for the algebra A, then $A = \mathscr{C}(X)$.*

Here *almost every* is understood with respect to the surface area measure derived from a fixed Riemannian metric on Σ.

We shall see in Theorem 6.5.20 below that one cannot prove such a result for three-dimensional manifolds! The first such example seems to be that of Basener [44], which is a polynomially convex 3-sphere Σ in $\mathbb{C}^6$ every point of which is a peak point for $\mathscr{P}(\Sigma)$ but for which $\mathscr{P}(\Sigma) \neq \mathscr{C}(\Sigma)$.

It is well to understand something about the context of Theorem 6.5.12. It was a conjecture current in the 1950s and early 1960s that if A is a uniform algebra on a compact space X, if X is the spectrum of A, and if every point of X is a peak point for A, then A is $\mathscr{C}(X)$. This conjecture, called the *peak point conjecture*, was shown to be false by Cole in his thesis [90]. (See also [345].) A more concrete counterexample was constructed a bit later by Basener [44], whose example is a rationally convex subset of $b\mathbb{B}_2$. Bishop's theorem, Theorem 6.4.1, shows that the peak point conjecture is correct for the algebras

$\mathscr{R}(X)$ on planar compacta, and Theorem 6.5.12 gives some further examples in which the peak point conjecture is valid.

Corollary 6.5.13. *Let Ω be an open set in $\mathbb{C}^N$, let Σ be a two-dimensional closed submanifold of Ω of class $\mathscr{C}^1$, and let X be a compact subset of Σ. If X is polynomially convex and if almost every point of X, with respect to the two-dimensional Hausdorff measure on $\mathbb{C}^N$, is a peak point for $\mathscr{P}(X)$, then $\mathscr{P}(X) = \mathscr{C}(X)$.*

A simple argument yields the following result, which contains this corollary as a special case.

Corollary 6.5.14. *Let X be a compact subset of a two-dimensional manifold Σ of class $\mathscr{C}^1$. If A is a uniform algebra on X generated by a family of $\mathscr{C}^1$ functions on Σ, if X is the spectrum of the algebra A, and if almost every point of X is a peak point for the algebra A, then $A = \mathscr{C}(X)$.*

Proof. We may suppose Σ to be compact (and without boundary): If Σ is not compact, the compact subset X of Σ is contained in a relatively compact domain Ω in Σ, and Ω is diffeomorphic under a $\mathscr{C}^1$ diffeomorphism, say ψ, to a domain Ω_o in a compact manifold Σ_o that is without boundary. Set $X_o = \psi(X)$. Denote by $\mathscr{F}$ a family of $\mathscr{C}^1$ functions on Σ that generates A. Let $\mathscr{F}_o$ be the collection of all $\mathscr{C}^1$ functions f_o on Σ_o with the property that for some $f \in \mathscr{F}$, $f_o \circ \psi$ agrees on a neighborhood of X with f. Denote by A_o the uniform algebra on X_o generated by the family $\mathscr{F}_o$. Finally, let B be the uniform algebra on Σ_o consisting of all the continuous functions g on Σ_o such that $g \circ \psi \in A$.

By Lemma 6.5.15 below, the spectrum of B is Σ_o, and because almost every point of X is a peak point for A, almost every point of Σ_o is a peak point for B. The algebra B is $\mathscr{C}(\Sigma_o)$ if and only if the algebra A is $\mathscr{C}(X)$. The corollary now follows by applying Theorem 6.5.12 to the algebra B.

The following general result about uniform algebras was found by H.S. Bear [47]:

Lemma 6.5.15. *Let X be a compact space and let Y be a closed subset of X. Suppose B to be a uniform algebra on Y for which each character is of the form $g \mapsto g(y)$ for some $y \in Y$. If $\tilde{B}$ denotes the subalgebra of $\mathscr{C}(X)$ that consists of all the functions f with $f|Y \in B$, then every character of $\tilde{B}$ is of the form $g \mapsto g(x)$ for some $x \in X$.*

Proof. Let χ be a character of $\tilde{B}$. If $\tilde{B}_E$ denotes the ideal in $\tilde{B}$ consisting of all the functions that vanish on E, then $\chi|\tilde{B}_E$ is a character of the algebra $\tilde{B}_E$, and so is either the zero functional or else is the character $g \mapsto g(x)$ for some point $x \in X \setminus Y$.[5] In the former case, χ induces a nonzero complex homomorphism of the quotient algebra $\tilde{B}/B_E$, which is naturally identified with the algebra B: The restriction $\rho : \tilde{B} \to B$ given by $\rho g = g|E$ is a surjective homomorphism with kernel $\mathscr{I}_E$, the ideal of continuous functions on X that

[5]We established in Section 1.2 that all the characters of the algebra $\mathscr{C}(Z)$ for a compact space Z are point evaluations at points of Z. In the case at hand, we are dealing with the algebra $\mathscr{C}_o(X \setminus Y)$ of all continuous functions that vanish at the point at infinity on the locally compact space $X \setminus Y$. Every character on this algebra extends to a character of the algebra $\mathscr{C}_o^\dagger(X \setminus Y)$ obtained by formally adjoining an identity to $\mathscr{C}_o(X \setminus Y)$, which can be identified in a natural way with the algebra $\mathscr{C}((X \setminus Y)^*)$ of all continuous functions on the one-point compacitification $(X \setminus Y)^*$ of $X \setminus Y$. From this it follows, as we are using above, that each nonzero character of $\mathscr{C}_o(X \setminus Y)$ is a point evaluation at a point of $X \setminus Y$.

vanish on E. Thus, χ induces a character on the algebra A and so is evaluation at a point of Y by hypothesis.

The proof of Theorem 6.5.12 requires a preliminary analysis of algebras on disks:

Theorem 6.5.16. *Let A be a uniform algebra on the closed unit disk $\bar{\mathbb{U}}$ in $\mathbb{C}$ generated by a family $\mathscr{F}$ of functions of class $\mathscr{C}^1$. If A contains the function z and if the spectrum of A is $\bar{\mathbb{U}}$, then with*

$$E = \left\{ p \in \bar{U} : \frac{\partial f}{\partial \bar{\zeta}}(p) = 0 \text{ for all } f \in \mathscr{F} \right\},$$

$A = \{g \in \mathscr{C}(\bar{U}) : g|E \in \mathscr{R}(E)\}$.

In the case that $\mathscr{F}$ is a singleton, this result is due to Wermer [371]; the general case is in [34].

That A contains the function z implies that A contains the disk algebra $A(\mathbb{U})$.

Proof. Let μ be a measure on $\mathbb{U}$ that is orthogonal to A. According to Theorem 6.5.3, the Cauchy transform $\hat{\mu}$ of the measure μ vanishes a.e. $[d\mathscr{L}]$ on $\mathbb{C} \setminus E$. It follows that $\operatorname{supp} \mu \subset E$. The function $\hat{\mu}$ vanishes almost everywhere off E, so μ is orthogonal to the algebra $\mathscr{R}(E)$, and, consequently, $A \supset \{f \in \mathscr{C}(\bar{\mathbb{U}}) : f|E \in \mathscr{R}(E)\}$.

If the function $f \in \mathscr{C}^1(\bar{\mathbb{U}})$ satisfies $\frac{\partial f}{\partial \bar{\zeta}} = 0$ on E, then $f|E \in \mathscr{R}(E)$, as follows from the generalized Cauchy integral formula: Without loss of generality, f is defined on all of $\mathbb{C}$ and has compact support. Then there is the representation

$$f(w) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - w} = \frac{1}{2\pi i} \int_{\mathbb{C}\setminus E} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - w}, \tag{6.17}$$

which implies that f is uniformly approximable on E by rational functions, e.g., by Riemann sums for the integral.

This completes the proof of Theorem 6.5.16.

The proof of Theorem 6.5.12 requires a further lemma.

Lemma 6.5.17. *If X is a compact subset of $\mathbb{C}^N$ with the properties that* (a) *X is polynomially convex, and* (b) *each point of X is a peak point for the algebra $\mathscr{P}(X)$, then every closed subset Y of X has properties* (a) *and* (b).

Proof. Let Y be a closed subset of X. Plainly each point of Y is a peak point for $\mathscr{P}(Y)$. What needs to be shown is that Y is polynomially convex. The set X is polynomially convex, so $\widehat{Y} \subset X$. If $x \in X \setminus Y$, then, because x is a peak point for $\mathscr{P}(X)$, it does not lie in $\widehat{Y}$. Done.

Proof of Theorem 6.5.12. Because of the argument in the proof of Corollary 6.5.14, it is enough to suppose that the manifold Σ is without boundary. Denote by A_1 the subalgebra of A that consists of functions of class $\mathscr{C}^1$. By hypothesis, A_1 is dense in A. The theorem is proved by showing that for each $f \in A_1$ and for each measure μ on Σ that annihilates A, the measure $f_*\mu$ is the zero measure, whence the result by Lemma 6.5.9.

With E the exceptional set of the algebra A, Theorem 6.5.3 implies that the Cauchy transform of the measure $f_*\mu$ vanishes on $\mathbb{C} \setminus f(E)$, so that $\operatorname{supp} f_*\mu \subset E$. It suffices therefore to show that $\mathscr{R}(f(E)) = \mathscr{C}(f(E))$.

Let C_f denote the critical set of the function f. By Sard's theorem the set $f(C_f)$ has zero area and so satisfies $\mathscr{R}(C_f) = \mathscr{C}(C_f)$ and, a fortiori, $\mathscr{R}(C_f \cap E) = \mathscr{R}(C_f \cap E)$.

Now consider a point $p \in \Sigma$ that is not a critical point for f. There is a compact neighborhood V_p of p in Σ' that is mapped diffeomorphically by f onto a compact disk Δ_p in the plane centered at $f(p)$, V_p so small that a neighborhood of V_p is mapped diffeomorphically by f onto a neighborhood of Δ_p. Denote by $\psi : \Delta_p \to V_p$ the inverse of the map $F|V_p$, and let B_p be the subalgebra of $\mathscr{C}(\Delta_p)$ generated by functions of the form $g \circ \psi$ with $g \in A$. The algebra B_p contains the function z, and we shall show that the spectrum of B_p is Δ_p.

To prove the latter statement, observe that B_p is isomorphic to the uniform algebra A_p on V_p generated by the restrictions $g|V_p$ as g runs through A. Proving that Δ_p is the spectrum of B_p is equivalent to proving that V_p is the spectrum of A_p. If χ is a character of A_p, then $g \mapsto \chi(g|V_p)$ is a character of A, so, because Σ is the spectrum of A, there is a unique point $q_\chi \in \Sigma$ such that for all $g \in A$, $\chi(g) = g(q_\chi)$. For every $\alpha \in \mathbb{C} \setminus \Delta_p$, the function $z - \alpha$ is invertible in B_p, so for every such α, $f - \alpha$ is invertible in A_p. This implies that $f(q_\chi) \in \Delta_p$ or, equivalently, that $q_\chi \in f^{-1}(\Delta_p)$. It is also evident that q_χ cannot be a peak point for A that lies outside V_p: If it is, then for some $h \in A$, $h(q_\chi) = 1 > \|h\|_{V_p}$, which is impossible, because χ is of norm one as a linear functional on A_p.

Thus, we have that the spectrum of A_p can be identified with the union of V_p and a subset K of Σ, necessarily compact, that consists entirely of nonpeak points for A. That K is compact follows from the observations that the spectrum of A_p is compact and that V_p is open in this spectrum, for f carries a neighborhood of V_p diffeomorphically onto a neighborhood of Δ_p. Then Lemma 1.5.10 implies the existence of a finite family of functions $g_1, \ldots, g_r$ in A such that the joint spectrum $\sigma_{A_p}(g_1, \ldots, g_r)$ is the union of two compact sets Z' and Z'' with Z' the image of V_p under the map $q \mapsto (g_1(q), \ldots, g_r(q))$ from the spectrum of A_p to $\mathbb{C}^r$ and Z'' the image of the complement of V_p in the spectrum under the same map. By Theorem 1.5.7 there is a function in A_p that is zero on V_p and greater than one in modulus on the rest of the spectrum of A_p. This is impossible, so the spectrum of A_p is, as claimed, V_p, and that of the algebra B_p is Δ_p.

Theorem 6.5.16 now implies that the algebra B_p consists of all of $g \in \mathscr{C}(\Delta_p)$ for which $g|f(V_p \cap E) \in \mathscr{R}(f(V_p \cap E))$. Because almost every point of Σ is a peak point for the algebra A, almost every $[d\mathscr{L}]$ point of Δ_p is a peak point for the algebra B_p. This implies that almost every $[d\mathscr{L}]$ point of $f(V_p \cap E)$ is a peak point for the algebra $\mathscr{R}(f(V_p) \cap E)$, so that by Bishop's criterion, Theorem 6.4.1, $\mathscr{R}(f(V_p \cap E)) = \mathscr{C}(f(B_p \cap E))$. Countably many of the sets V_p cover $\Sigma \setminus (C_f \cap E)$. Thus, we have exhibited the compact set $f(E)$ as a countable union of closed sets S_j each of which satisfies $\mathscr{R}(S_j) = \mathscr{C}(S_j)$. Theorem 6.4.2 implies that $\mathscr{R}(f(E)) = \mathscr{C}(f(E))$, and the theorem is proved.

One situation to which Theorem 6.5.12 applies is that of two-dimensional surfaces in strictly pseudoconvex hypersurfaces.

Theorem 6.5.18. [34] *Let Ω be an open set in $\mathbb{C}^N$, and let Γ be a closed, strictly pseudoconvex hypersurface of class $\mathscr{C}^2$ contained in Ω. If Σ is a compact two-dimensional* *submanifold of class $\mathscr{C}^2$, possibly with boundary, of Γ, then the set of points of Σ at which Σ is not totally real has zero two-dimensional measure. If Σ is polynomially convex, then $\mathscr{P}(\Sigma) = \mathscr{C}(\Sigma)$.*

It is not assumed in this theorem that Γ is compact.

The proof of this requires a general fact about differentiable functions.

Lemma 6.5.19. [34] *Let $\mathscr{M}$ be an n-dimensional manifold of class $\mathscr{C}^1$, and let $E \subset \mathscr{M}$ be a compact set. Define the subset E^* of $\mathscr{M}$ by*

$$E^* = \{x \in \mathscr{M} : df(x) = 0 \ for\ all\ f \in \mathscr{C}^1(\mathscr{M})\ with\ f|E = 0\}.$$

The set $E \setminus E^*$ has n-dimensional measure zero.

Proof. Without loss of generality, assume the manifold $\mathscr{M}$ to be an open subset of $\mathbb{R}^n$.

Denote by $\mathscr{I}_E^1$ the set of those $f \in \mathscr{C}^1(\mathscr{M})$ that vanish on E. For each $j = 1, \ldots, n$, let E_j^* be the set of those points $x \in E$ for which $D_j f(x) \neq 0$ for each $f \in \mathscr{I}_E^1$. (Here D_j denotes the operator $\partial/\partial x_j$.) The theorem will be proved if it can be shown that for each j, the set E_j^* has measure zero. We treat the case $j = 1$.

Take coordinates (t, y) on $\mathbb{R}^n$ with $t \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. For each $y \in \mathbb{R}^{n-1}$, let $E(y)$ be the set $\{t \in \mathbb{R} : (t, y) \in E\}$, and let $E_1^*(y)$ be the set of those $t \in \mathbb{R}$ for which $(t, y) \in E(y)$ and the partial derivative $D_1 f(t, y) = 0$. The set $E_1^*(y)$ contains every limit point of $E(y)$. The points in $E(y)$ that are not limit points of $E(y)$ are at most countable in number, so for each y, the set $E(y) \setminus E_1^*(y)$ is countable, whence, by Fubini's theorem, the set $E \setminus E_1$ has measure zero. The lemma is proved.

Proof of Theorem 6.5.18. Suppose the set E of points at which Σ is not totally real to have positive two-dimensional measure. By the last lemma, there is then a point $p \in \Sigma \setminus b\Sigma$ such that $df(p) = 0$ for all $f \in \mathscr{C}^1(\Sigma)$ with $f|E = 0$. Let M be the set of all such points. The point p can be chosen to be a point of metric density for the set M, i.e., a point at which the two-dimensional density of M is one. For this, one can invoke Theorem 4.2.9. A much simpler approach is to appeal to the result that almost every point is a Lebesgue point for the characteristic function of M. (See [311].)

The surface Γ is strictly pseudoconvex, so there is a neighborhood Ω_o of p in $\mathbb{C}^N$ on which there is a strictly plurisubharmonic function Q of class $\mathscr{C}^2$ such that $\Gamma \cap \Omega_o = \{z \in \Omega_o : Q(z) = 0\}$ and such that dQ does not vanish on $\Gamma \cap \Omega_o$.

Choose holomorphic linear coordinates $z_1, \ldots, z_N$ on $\mathbb{C}^N$ such that the point p is the origin and such that the tangent to Σ at the origin is the z_1-axis. By shrinking Ω_o if necessary, we can suppose that the part of Σ in Ω_o is a graph over the domain D in the z_1-axis: There are functions $g_2, \ldots, g_N$ of class $\mathscr{C}^2$ defined on D such that

$$\Sigma \cap \Omega_o = \{(z, g_2(z), \ldots, g_N(z)) : z \in D\}$$

and such that $dg_j(0) = 0$ for each j.

If the function q is defined on D by

$$q(z) = Q(z_1, g_2(z_1), \ldots, g_N(z_1)),$$

then q vanishes identically.

Use the chain rule to compute from this the value of $\frac{\partial^2 q}{\partial z_1 \partial \bar{z}_1}(0)$. The result is the equation

$$0 = \frac{\partial^2 q}{\partial z_1 \partial \bar{z}_1}(0) = \frac{\partial^2 Q}{\partial z_1 \partial \bar{z}_1}(0) + \sum_{j=2}^{N} \frac{\partial Q}{\partial z_j}(0) \frac{\partial^2 q_j}{\partial z_1 \partial \bar{z}_1}(0).$$

Each of the functions $\frac{\partial g_j}{\partial \bar{z}_1}$ vanishes on the set in D that corresponds to the set of points of $\Sigma \cap \Omega$ at which Σ is not totally real. Consequently, $d\frac{\partial q_j}{\partial \bar{z}_1}$ vanishes at p. This implies that

$$0 = \frac{\partial^2 q}{\partial z_1 \partial \bar{z}_1}(0) = \frac{\partial^2 Q}{\partial z_1 \partial \bar{z}_1}(0),$$

which violates the assumed strict plurisubharmonicity of the function Q.

The first part of the theorem is now proved. The second assertion follows from Theorem 6.5.12.

As was stated immediately after the formulation of Theorem 6.5.12, there is no direct analogue of that result for three-dimensional manifolds. This is shown by a construction of Izzo [187]. The following example is a minor modification of one of Izzo's.

Theorem 6.5.20. *There is a submanifold X of $b\mathbb{B}_6 = \mathbb{S}^{11}$ of class $\mathscr{C}^\infty$ that is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, that is polynomially convex, and that satisfies $\mathscr{P}(X) \neq \mathscr{C}(X)$.*

Note that because $X \subset b\mathbb{B}_6$, every point of X is a peak point for the algebra $\mathscr{P}(X)$. And by Theorem 6.5.12, if Σ is a compact two-dimensional submanifold of X of class $\mathscr{C}^1$, then $\mathscr{P}(\Sigma) = \mathscr{C}(\Sigma)$. (By Lemma 6.5.17, every compact subset of X is polynomially convex.) This example also shows that *a polynomially convex subset X of $b\mathbb{B}_N$ need not satisfy $\mathscr{P}(X) = \mathscr{C}(X)$*. Theorem 6.5.23 below implies that there can be no example of this kind with X a real-analytic manifold.

In [187] Izzo gave an example of a smooth submanifold Σ with boundary in $b\mathbb{B}_5$ that is diffeomorphic to $b\mathbb{U} \times \bar{\mathbb{U}}$—and so is a solid three-dimensional doughnut—that is polynomially convex and that does not satisfy $\mathscr{P}(\Sigma) = \mathscr{C}(\Sigma)$. He also gives an example of a compact polynomially convex subset E of $b\mathbb{B}_3$ for which $\mathscr{P}(E) \neq \mathscr{C}(E)$.

Proof. Denote by $\mathbb{S}^*$ the Riemann sphere. Let $E \subset \mathbb{C}$ be a compact set contained in a small neighborhood of the origin for which $\mathscr{R}(E) \neq \mathscr{C}(E)$ but for which each Jensen measure is a point mass at a point of E. (Such an example was constructed in Theorem 6.4.3.) Let ϱ be a nonnegative $\mathscr{C}^\infty$ function on $\mathbb{C}^*$ that vanishes, together with all its derivatives, on the set E and is otherwise zero-free. By Theorem 1.2.11, there is a function $g \in \mathscr{C}^\infty(\mathbb{C}^*)$ such that $\mathscr{R}(E)$ is generated by g and the function z. Let $\varphi \in \mathscr{C}^\infty(\mathbb{C}^*)$ satisfy $\varphi(z) = z$ on a neighborhood U_o of E. Choose a neighborhood U_1 of E that satisfies $\bar{U}_1 \subset U_o$. We can suppose that the function g satisfies $g(z) = 1/z$ on $\mathbb{C}^* \setminus \bar{U}_1$. Fix two distinct points α and β of E, and define $\psi(z) = \frac{z-\alpha}{z-\beta}$ for $z \in \mathbb{C}^*$.

Let $\mathscr{A}$ be the subalgebra of $\mathscr{C}(\mathbb{C}^*)$ that consists of all continuous functions f with $f|E \in \mathscr{R}(E)$ and by A the subalgebra of $\mathscr{A}$ generated by the five functions φ, g, $\varrho\psi$, $\varrho\bar{\psi}$, and ϱ. In fact, $A = \mathscr{A}$. To establish this, note first that A separates the points of $\mathbb{C}^*$: If z, z' are distinct points of $\mathbb{C}^*$, there are three cases: If $z, z' \in E$, then $\varphi(z) \neq \varphi(z')$. If $z \in E$ and $z' \in \mathbb{C}^* \setminus E$, then $\varrho(z) \neq \varrho(z')$. Finally, if $z, z' \notin E$, then $\varrho(z)$ and $\varrho(z')$ are both nonzero. If they are equal, then $\varrho(z)\psi(z) \neq \varrho(z')\psi(z')$. Thus, A is seen to separate points on $\mathbb{C}^*$. The algebra A contains the three real-valued functions ϱ, $\varrho\Re\psi$, and $\varrho\Im\psi$, which separate points on $\mathbb{C}^* \setminus E$, so by the Stone–Weierstrass theorem $A \supset \{g \in \mathscr{C}(\mathbb{C}^*) : g|E = 0\}$. Now, finally, $A = \mathscr{A}$. If not, there is a nonzero measure μ on $\mathbb{C}^*$ with $\int g\, d\mu = 0$ for all $g \in A$ but for which $\int f\, d\mu \neq 0$ for some $f \in \mathscr{A}$. Because A contains all continuous functions on $\mathbb{C}^*$ that vanish on E, μ is supported by E. The measure μ annihilates the subalgebra of A generated by g and z, which, restricted to E, is $\mathscr{R}(E)$. Thus, μ annihilates $\mathscr{A}$, contrary to hypothesis.

Next, the characters of the uniform algebra $\mathscr{A}$ are all of the form $f \mapsto f(z)$ for some necessarily unique $z \in \mathbb{C}^*$, as follows from Lemma 6.5.15.

If we suppose E to be contained in a sufficiently small neighborhood of the origin, and if we replace φ, g, and ϱ by $c\varphi$, cg, and $c\varrho$ for suitably small positive c, we can suppose that $\frac{1}{2} < 1-(|\varphi|^2+\varrho^2|\psi|+\varrho^2|\bar{\psi}|^2+\varrho^2)$ on all of $\mathbb{C}^*$ and, in addition, that $0 < 2|g| < \frac{1}{2}$ on $\mathbb{C}$. This implies the existence of a positive function r of class $\mathscr{C}^\infty$ on $\mathbb{C}^*$ such that

$$|g(z)|^2 r^2(z) + r^{-2}(z) = 1 - (|\varphi|^2 + \varrho^2|\psi|^2 + \varrho^2|\bar{\psi}|^2 + \varrho^2). \tag{6.18}$$

(Regard this equation as a quadratic equation for r^2. It has two solutions, both positive on $\mathbb{C}$. With $A = 1 - (|\varphi|^2 + \varrho^2|\psi|^2 + \varrho^2|\bar{\psi}|^2 + \varrho^2)$, the smaller of the two solutions is $\frac{1+O(|z|^{-2})}{1-A}$ for $z \to \infty$. The positive square root of this solution therefore gives the desired function r. This process does yield a function of class $\mathscr{C}^\infty$ on all of $\mathbb{C}^*$.)

Now define a map $F : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^6$ by

$$F(z, w) = \left(\varphi(z), g(z)r(z)w, \frac{\bar{w}}{r(z)}, \varrho(z)\psi(z), \varrho(z)\bar{\psi}(z), \varrho(z)\right).$$

By the definition of F, $|F(z, e^{i\vartheta})| = 1$ for all $z \in \mathbb{C}^*$ and all $\vartheta \in \mathbb{R}$, so F carries $\mathbb{C}^* \times b\mathbb{U}$ into $b\mathbb{B}_6$. By construction, F is injective. It is also of constant real rank three: on U_0 $d\varphi \neq 0$, and on $\mathbb{C}^* \setminus \bar{U}_1$, $dg \neq 0$. Thus, $F(\mathbb{C}^* \times b\mathbb{U})$ is a submanifold of $b\mathbb{B}_6$ of class $\mathscr{C}^\infty$. Call this manifold Σ. It is diffeomorphic to the product $\mathbb{S}^2 \times \mathbb{S}^1$ of spheres.

The manifold Σ is polynomially convex. We prove this by showing that every character of $\mathscr{P}(\Sigma)$ is point evaluation at a point of Σ, for which it suffices to show that every Jensen measure for $\mathscr{P}(\Sigma)$ is a point mass.

For this, take coordinates $\zeta_1, \ldots, \zeta_6$ on $\mathbb{C}^6$ and introduce the closed subalgebra $\mathscr{Q}$ of $\mathscr{P}(\Sigma)$ generated by the functions $\zeta_1, \zeta_4, \zeta_5, \zeta_6$ and the product $\zeta_2\zeta_3$. This algebra is isomorphic to the algebra $\mathscr{A}$ under the isomorphism $\Phi : \mathscr{A} \to \mathscr{Q}$ given by the condition that for all $f \in \mathscr{A}$,

$$(\Phi f)(\varphi(z), g(z)r(z)e^{i\vartheta}, \frac{e^{-i\vartheta}}{r(z)}, \varrho(z)\psi(z), \varrho(z)\bar{\psi}(z), \varrho(z)) = f(z)$$

for all $z \in \mathbb{C}^*$ and all $\vartheta \in \mathbb{R}$. Thus, for $\zeta \in \Sigma$, $\Phi(f(\zeta)) = f(F^{-1}(\zeta))$. The map $\Phi : \mathscr{A} \to \mathscr{P}(\Sigma)$ induces a map Φ^* from the spectrum of $\mathscr{P}(\Sigma)$ to the spectrum of $\mathscr{A}$

given by $\Phi^*\chi(f) = \chi(\Phi f)$ for all characters χ on $\mathscr{P}(\Sigma)$ and all $f \in \mathscr{A}$. The restriction of Φ^* to Σ, which is naturally a subset of the spectrum of $\mathscr{P}(\Sigma)$, is given by

$$\Phi^* F(z, e^{i\vartheta}) = z.$$

Suppose now that μ is a Jensen measure for the algebra $\mathscr{P}(\Sigma)$, and define μ_*, a measure on $\mathbb{C}$, by

$$\int g \, d\mu_* = \int g \circ \Phi^* \, d\mu.$$

On Σ, the map Φ^* is given by $\Phi^*(F(z, e^{i\vartheta})) = z$, so for $f \in \mathscr{A}$, we have that

$$\begin{aligned}\int \log|f| \, d\mu_* &= \int \log|f \circ \Phi^*| \, d\mu \\ &\geq \log|\chi(f \circ \Phi^*)| \\ &= \log|\chi(\Phi f)|.\end{aligned}$$

The map $f \mapsto \chi(\Phi f)$ is a character of $\mathscr{A}$, and the last equations show μ_* to be a Jensen measure for it. Thus, μ_* is a point mass, say concentrated on the point $z \in \mathbb{C}^*$. It follows that μ is concentrated on the circle $\gamma_z = \{F(z, e^{i\vartheta}) : \vartheta \in \mathbb{R}\}$. It is multiplicative on $\mathscr{P}(\Sigma)$, and so on the restriction algebra $\mathscr{P}(\Sigma)|\gamma_z$. This restriction algebra is dense in $\mathscr{C}(\gamma_z)$, because it contains the functions $\zeta_2|\gamma_z$ and $\frac{1}{\zeta_2}|\gamma_z$. Consequently, μ must be a point mass concentrated on some point of γ_z.

Every Jensen measure for $\mathscr{P}(\Sigma)$ is a point mass at some point of Σ, so every character of this algebra is point evaluation at some point of Σ. Consequently, as desired, Σ is polynomially convex.

Finally, $\mathscr{P}(\Sigma) \neq \mathscr{C}(\Sigma)$. To see this, choose a measure ν on $\mathbb{C}^*$ that annihilates $\mathscr{A}$. The measure ν is concentrated on the set E. Define a measure $\tilde{\nu}$ on Σ by

$$\int h \, d\tilde{\nu} = \int_E \int_{-\pi}^{\pi} h(F(z, e^{i\vartheta})) \, d\vartheta d\nu. \tag{6.19}$$

The measure $\tilde{\nu}$ is not the zero measure. It is supported in the set $F(E)$, and it annihilates the algebra $\mathscr{P}(X)$: If $M(\zeta) = \zeta_1^{k_1} \cdots \zeta_6^{k_6}$ is a monomial, then M vanishes on the support of $\tilde{\nu}$ unless k_4, k_5, k_6 are all zero. And because $M|\Sigma$ contains a factor of $e^{i(k_2-k_3)\vartheta}$, the integral on the right of (6.19), with h replaced by M, vanishes because of the ϑ integration unless $k_2 = k_3$. Finally, when $k_2 = k_3$, and $k_4 = k_5 = k_6 = 0$, the integral on the right of (6.19) is

$$2\pi \int_E \varphi^{k_1}(z) g^{k_2}(z) \, d\nu(z),$$

which vanishes because ν is orthogonal to $\mathscr{A}$.

The theorem is proved.

There is a strong approximation result on compact real-analytic subsets of $\mathbb{C}^N$ to which we now turn our attention. Before proceeding to the statement, it seems well to recall a few ideas from the theory of real-analytic sets. This theory is largely parallel to the theory of complex-analytic sets, but it does have its own twists. The books [259] and [115] develop the theory of real-analytic sets.

To begin with, a real-analytic set in an open set Ω in $\mathbb{R}^N$ is a closed subset X of Ω with the property that for each $x \in X$, there is a neighborhood V_x of x contained in Ω on which there are finitely many real-analytic functions $f_1, \ldots, f_r$ such that

$$V_x \cap X = \{y \in V_x : f_1(y) = \cdots = f_r(y) = 0\}.$$

Important examples are the real-analytic submanifolds of open subsets of $\mathbb{R}^N$ and the complex-analytic subvarieties of open sets in $\mathbb{C}^N$. The notion of real-analytic set is a local notion, so the idea extends immediately to subsets of real-analytic manifolds. If X is a real-analytic subset of an open set Ω, then a point $x \in X$ is said to be a *regular point* (of *dimension* d) if there is a neighborhood V_x of X contained in Ω such that $V_x \cap X$ is a real-analytic submanifold of V_x (of dimension d). The set of regular points of X is denoted by X_{reg}. A point $x \in X$ is a *singular* point of X if it is not a regular point. The set of singular points is denoted by X_{sing}. The set X_{reg} is plainly open in X; X_{sing} closed. The *dimension* of a real-analytic set X is the largest integer d such that X has regular points of dimension d. A real-analytic subset of a real-analytic manifold is *irreducible* if it is not a union of two distinct real-analytic subsets of the ambient manifold.

The metric properties of real-analytic sets are similar to those of complex-analytic sets. If V is a real-analytic variety of dimension d in an open set in $\mathbb{R}^N$, then the measure of V in dimension d is locally finite: Each compact subset E of V satisfies $\Lambda^d(E) < \infty$ and $\Lambda^{d-1}(V_{\text{sing}} \cap E) < \infty$. A source for these estimates is [115, Section 3.4.10].

A fundamental complicating fact about X_{sing} is that it is not necessarily an analytic set. This contrasts with the situation for complex-analytic sets. For an example, see [259, pp. 106–107].

We shall need below the following result, which serves in our context to compensate for the possible nonanalyticity of the set X_{sing}. We do not give the proof here but refer instead to the work of Narasimhan [259, Section 1 of Chapter 2, especially Proposition 5] for it. The result is essentially related to the possibility of realizing a pure-dimensional analytic set locally as an analytic cover.

Lemma 6.5.21. *Let X be an irreducible real-analytic subvariety of dimension d in the open set Ω in $\mathbb{R}^n$. If $p \in X_{\text{sing}}$, there exists a neighborhood V of p in $\mathbb{R}^n$ on which are defined real-analytic functions δ and $f_1, \ldots, f_{n-d}$ such that δ does not vanish identically on V but does vanish on $X_{\text{sing}} \cap \Omega$, the functions $f_1, \ldots, f_{n-d}$ vanish on $V \cap \Omega$, and the differentials $df_1, \ldots, df_{n-d}$ are linearly independent at each point of $X \setminus \{x \in X : \delta(x) = 0\}$.*

The principal result we obtain about approximation in the context of real-analytic sets is the following:

Theorem 6.5.22. [352] *If X is a compact, real-analytic subvariety of $\mathbb{C}^N$ that is holomorphically convex, then every continuous $\mathbb{C}$-valued function on X can be approximated uniformly by functions holomorphic on (varying) neighborhoods of X.*

Corollary 6.5.23. *If X is a compact, polynomially convex, real-analytic subset of $\mathbb{C}^N$, then $\mathscr{P}(X) = \mathscr{C}(X)$.*

This corollary was proved, under the additional hypothesis that each point of X is a peak point for $\mathscr{P}(X)$, by Anderson, Izzo, and Wermer [33].

In contrast with Theorem 6.5.20, there is the following particular case:

Corollary 6.5.24. *If X is a compact polynomially convex real-analytic subset of $\mathbb{C}^N$, then $\mathscr{P}(X) = \mathscr{C}(X)$.*

In this connection, recall Corollary 6.5.11.

Corollary 6.5.25. *If X is a compact rationally convex real-analytic subset of $\mathbb{C}^N$ then $\mathscr{R}(X) = \mathscr{C}(X)$.*

In connection with Theorem 6.5.22, Freeman's result from [131] should be recalled. It implies that *if X is a compact, real-analytic submanifold of $\mathbb{C}^N$ that is holomorphically convex, then every $f \in \mathscr{C}(X)$ that vanishes on the set of nontotally real points of X can be approximated uniformly by elements of $\boldsymbol{\mathscr{O}}(X)$*. This statement is a special case of Freeman's general result.

The proof of Theorem 6.5.22 depends on two results that we will state here and prove in appendices at the end of this section.

The first of these is due to Diederich and Fornæss [98].

Theorem 6.5.26. *A compact real-analytic subvariety of $\mathbb{C}^N$ contains the germ of no complex-analytic variety of positive dimension.*

The proof of this theorem is given in Appendix 6.5.A below.

The second is a detail, sufficient for our purposes and with a relatively simple proof, in a more extensive theory of analytic disks with boundary in a prescribed manifold. For details of the general theory we refer to [172, 285].

Theorem 6.5.27. *If Σ is a closed real-analytic submanifold of an open set in $\mathbb{C}^N$ and if the CR-dimension of Σ is constant and greater than zero at each point, then given a point $p \in \Sigma$, there is a continuous map $\varphi : \bar{\mathbb{U}} \times [0, 1] \to \mathbb{C}^N$ with the properties that for each value of $t \in [0, 1]$, the function $\varphi(\cdot, t)$ is holomorphic on $\mathbb{U}$, that φ carries $b\mathbb{U} \times [0, 1]$ into Σ, that $\varphi(\cdot, t)$ is not constant for some $t \in [0, 1]$, that for all ϑ, the partial function $\varphi(e^{i\vartheta}, \cdot)$ extends to be holomorphic in $\mathbb{U}$, and, finally, that the partial function $\varphi(\cdot, 0)$ takes only the value p.*

Appendix 6.5.B contains a proof of this fact.

We now begin the preparations for the proof of Theorem 6.5.23. We start by establishing some notation that will be used consistently. Fix a real-analytic subset X of an open set Ω in $\mathbb{C}^N$. The symbol X_c will be used to denote the subset of X_{reg} at which this manifold is not totally real. Thus, X_c is a closed real-analytic subset of X_{reg}. The set $X^* = X_{\text{sing}} \cup X_c$ is a closed subset of X. Because X_{sing} is not necessarily a real-analytic set, the set X^* may not be a real-analytic set.

We need the following observation. Recall that the CR-dimension of a real submanifold Σ of an open set in $\mathbb{C}^N$ at a point $p \in \Sigma$ is the dimension (over $\mathbb{C}$) of $T_p^{\mathbb{C}}\Sigma$. The manifold Σ is a CR-manifold if this dimension is constant.

Lemma 6.5.28. *If Σ is a closed submanifold of class $\mathscr{C}^1$ of an open subset of $\mathbb{C}^N$, then there is an open subset of Σ on which the CR-dimension of Σ is constant.*

Proof. Let the real dimension of Σ be d, and for the integers $j = 1, 2, \ldots, [\frac{d}{2}]$, let Σ^j be the set of points p in Σ at which $\dim T_p^{\mathbb{C}}(\Sigma) \geq j$. Each of these is a closed set, because for a given positive integer q, the Grassmannian of all q-dimensional vector subspaces of $\mathbb{C}^N$ is compact. We have $\Sigma = \Sigma^0 \supset \Sigma^1 \supset \cdots \supset \Sigma^{[\frac{d}{2}]}$. Let k be the largest integer for which $\Sigma^k = \Sigma$. If $k = [\frac{d}{2}]$, we are done, for then $\dim T_p^{\mathbb{C}}\Sigma$ is constant on all of Σ. If $k < [\frac{d}{2}]$, then the CR-dimension of Σ is constant on the open subset $\Sigma \setminus \Sigma^{k-1}$ of Σ.

Lemma 6.5.29. *If X is a compact, holomorphically convex, real-analytic subset of $\mathbb{C}^N$, and if Σ is a component of X_{reg}, then the set Σ_o of points of Σ at which the CR-dimension is zero is everywhere dense in Σ.*

Granted that Σ_o is dense in Σ, the complement $\Sigma \setminus \Sigma_o$ is a nowhere dense, real-analytic subset of Σ.

Proof. To see that Σ_0 is dense, denote by Σ_k the subset of Σ at which the CR-dimension is k. If Σ_0 is not dense in Σ, then, by Lemma 6.5.28, there is a positive integer k such that Σ_k contains an open subset of Σ. The set Σ_k is a real-analytic submanifold of an open set Ω in $\mathbb{C}^N$, and it has constant positive CR-dimension. Fix a point $p \in \Sigma_k$ and let $\varphi : \bar{\mathbb{U}} \times [0, 1] \to \mathbb{C}^N$ be the continuous function provided by Theorem 6.5.27. We will prove that the set $\varphi(\bar{\mathbb{U}} \times [0, 1])$ is contained in Σ.

For the latter point define, for each $(z, t) \in \mathbb{U} \times [0, 1]$, the linear functional $L_{z,t}$ on the space $\mathscr{O}(X)$ of germs of holomorphic functions on X by

$$L_{z,t}(\mathbf{f}) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\mathbf{f}(\varphi(\zeta, t))}{\zeta - z}\, d\zeta$$

for all $\mathbf{f} \in \mathscr{O}(X)$. These functionals are plainly linear.

Also, for each choice of $z \in \mathbb{U}$ and each choice of $\mathbf{f}$, the quantity $L_{z,t}(\mathbf{f})$ depends real-analytically on $t \in (0, 1)$ because the function $\varphi(\zeta, \cdot)$ is, for fixed $\zeta \in b\mathbb{U}$, holomorphic in $\mathbb{U}$. They are also multiplicative. To see this, let $\mathbf{f}, \mathbf{g} \in \mathscr{O}(X)$. There is then a ball B centered at the point p small enough that there are representatives f and g of the germs $\mathbf{f}$ and $\mathbf{g}$, respectively, that are defined and holomorphic in B. By continuity, if $\delta > 0$ is sufficiently small, then $\varphi(\bar{\mathbb{U}} \times [0, \delta])$ is contained in B. For such values of z and t, the Cauchy integral formula implies that $L_{z,t}(\mathbf{fg}) = L_{z,t}(\mathbf{f})L_{z,t}(\mathbf{g})$. Each $\varphi(\zeta, \cdot)$ is holomorphic in $\mathbb{U}$ when $\zeta \in b\mathbb{U}$, so this equality must persist for all values of $(z, t) \in \mathbb{U} \times [0, 1]$. That is to say, each of the functionals $L_{z,t}$ is a character of the algebra $\mathscr{O}(X)$. By hypothesis, X is holomorphically convex, so the set $\varphi(\bar{\mathbb{U}} \times [0, 1])$ is found to lie in X.

If $t \in [0, 1]$ is a value for which $\varphi(\cdot, t)$ is nonconstant, then the set $\varphi(\bar{\mathbb{U}} \times \{t\})$ contains the germ of a one-dimensional complex-analytic variety. Theorem 6.5.26 implies that X can contain no germ of a complex-analytic variety; we have a contradiction, so, as claimed, the set Σ_0 of totally real points of Σ_0 is dense in Σ_0.

Note that the argument given above shows that the dimension of no compact holomorphically convex real-analytic subset of $\mathbb{C}^N$ exceeds N; this is the best possible bound.

Lemma 6.5.30. [33] *If X is a real-analytic subset of dimension d of the open set Ω in $\mathbb{R}^N$, and if* $\dim X_c < d$, *then for each $x \in X$, there is an open set U contained in Ω such that for some real-analytic subset Y of Ω of dimension less than d, $(X^* \cap U) \subset Y$.*

Proof. If $x \notin X^*$, take U to be disjoint from X^* and take Y to be the empty set.

If $x \in X^*$, choose a small neighborhood U of x such that $X \cap U$ has the decomposition $X \cap U = Z_1 \cap \cdots \cap Z_s$ into irreducible components, Z_j of dimension d_j. Suppose $d_j = d$ when $j = 1, \dots, r$ and $d_{r+1}, \dots, d_s$ to be less than d. We can suppose U to be small enough that for each j, Lemma 6.5.21 applies to each of the sets Z_j at the point x. Thus, there are real-analytic functions δ_j and $f_{j,1}, \dots, f_{j,n-d_j}$ defined on U associated with each Z_j. If Δ_j is the zero set of δ_j, then $(Z_j)_{\text{sing}} \subset \Delta_j \cap Z_j$, and, for fixed j, the differentials of the functions $f_{j,1}, \dots, f_{j,n-d_j}$ are linearly independent at each point of $Z_j \setminus \Delta_j$. By the definition of the set $(Z_j)_c$, there is a finite family Φ_j of real-analytic functions on U such that $(Z_j)_c \subset \cap_{g \in \Phi_j} g^{-1}(0)$. Let $Y_j = \Delta_j \cup Y_j'$, where Y_j' is the variety defined by the functions in Φ_j together with the functions $f_{j,k}$, $k = 1, \dots, n - d_j$. For the set Y take the union of all the Y_j' together with all the pairwise intersections $Z_j \cap Z_{j'}$ for distinct Z_j and $Z_{j'}$ with $j, j' \leq r$ together, finally, with the union of all Z_j with $r + 1 \leq j \leq s$. The set Y so defined is an analytic set of dimension less than d that satisfies $X^* \cap U \subset Y$.

An inductive argument yields the next lemma.

Lemma 6.5.31. [33] *Let X be a d-dimensional real-analytic subset of the open set Ω in $\mathbb{R}^N$. If $x \in X$, there exist $r > 0$ and real-analytic varieties $Y_1, \dots, Y_d$ in the ball $B(x, r)$ such that with $Y_o = X \cap B(x, r)$, we have* (a) $Y_{j-1}^* \subset Y_j \subset Y_{j-1}$ *and* (b) $\dim\ Y_j \leq d - j$.

Proof. The set X_c has no interior in X by Lemma 6.5.29, so $\dim\ X_c < d$. Let $x \in X$. If $x \in X \setminus X^*$, let $r > 0$ be small enough that $\mathbb{B}_N(x, r) \cap X^* = \emptyset$, and take $Y_1 = \emptyset$. If $x \in X^*$, invoke the preceding lemma to find an $r_1 > 0$ small enough that there is an analytic set Y_1 in the ball $\mathbb{B}_N(x, r_1)$ with $\dim\ Y_1 < d$ and $\mathbb{B}_N(r_1, x) \cap X^* \subset Y_1 \subset \mathbb{B}_N(x, r_1) \cap X$. With this choice of Y_1, both the properties (a) and (b) of the lemma hold. Now proceed by induction. Assume that $Y_1, \dots, Y_k$ have been constructed as in the statement of the lemma, Y_j a subvariety of the ball $\mathbb{B}_N(x, r_j)$ for positive, decreasing radii r_j. If $x \in Y_k \setminus Y_k^*$, choose $r_{k+1} > 0$ small enough that $\mathbb{B}_N(x, r_{k+1}) \cap Y_j^* = \emptyset$, and take $Y_{k+1} = \emptyset$. If $x \in Y_k^*$, choose $r_{k+1} > 0$ and Y_{k+1} as in the previous lemma such that $\mathbb{B}_N(x, r_{k+1}) \cap Y_k^* \subset Y_{k+1} \subset \mathbb{B}_N(x, r_{k+1}) \cap Y_k$, $\dim\ Y_{k+1} < d - k$. Then (a) and (b) hold for $j \leq k + 1$. For the r of the lemma, take the smallest of the r_j generated in this process. The lemma is proved.

Proof of Theorem 6.5.22. In the rest of the proof, it will be helpful to use the notation that if Z is a real-analytic set with regular set Z_{reg} and singular set Z_{sing}, then Z_{reg}^c is the subset of Z_{reg} at which Z_{reg} is not totally real, and Z^* is the set $Z_{\text{reg}}^c \cup Z_{\text{sing}}$. The set Z^* is a closed subset of Z and so is compact if Z is compact.

Theorem 6.3.2 implies that if X is a compact, real-analytic, holomorphically convex subset of $\mathbb{C}^N$, then each continuous function on X that vanishes on X^* can be approximated uniformly on X by elements of $\mathscr{O}(X)$, whence, if μ is a measure on X orthogonal to $\mathscr{O}(X)$, then $\text{supp}\mu \subset X^*$.

Let the dimension of X be d, which we assume to be positive.

If $x \in X$, then by Lemma 6.5.31 there is a ball $\mathbb{B}_N(x, r)$ centered at x and small enough that for a sequence $Y_0, \dots, Y_d$ of real-analytic subvarieties of $\mathbb{B}_N(x, r)$, we have $Y_0 = \mathbb{B}_N(x, r) \cap X$, and $Y_{k-1}^* \subset Y_k \subset Y_{k-1}$ for each $k = 1, \dots, d$. Define Z_k to be the compact set $(X \setminus \mathbb{B}_N(x, r)) \cup Y_k$.

The sets Z_k are holomorphically convex. This assertion is proved inductively. It is true when $k = 0$, for then $Z_k = X$. We assume, therefore, that Z_k is holomorphically convex and show under this hypothesis that Z_{k+1} is holomorphically convex. That is, we show that if χ is a character of $\boldsymbol{\mathscr{O}}(Z_{k+1})$, then χ is evaluation at some point of Z_{K+1}. The character χ of $\boldsymbol{\mathscr{O}}(Z_{k+1})$ induces a character of $\boldsymbol{\mathscr{O}}(Z_k)$ by restriction, so there is a unique point $x \in Z_k$ such that for all $\mathbf{f} \in \boldsymbol{\mathscr{O}}(Z_k)$, $\chi(\mathbf{f}) = \mathbf{f}(x)$. We show the point x to lie in Z_{k+1}.

This is based on the inequality $|\psi(\mathbf{f})| \leq \|\mathbf{f}\|_{Z_{k+1}}$, valid for all characters ψ of $\boldsymbol{\mathscr{O}}(Z_{k+1})$ and all $\mathbf{f} \in \boldsymbol{\mathscr{O}}(Z_{k+1})$.

An argument parallel to one used above shows that $Y^c_{k,\text{reg}}$ is a nowhere dense, analytic subvariety of $Y_{k,\text{reg}}$. The approximation theorem, Theorem 6.3.2, implies that each function continuous on Z_k that vanishes on $(X \setminus \mathbb{B}_N(x, r)) \cup Y^*_k$ can be approximated uniformly on Z_k by functions holomorphic on a neighborhood of Z_k. It follows that if $q \in Z_k \setminus Z_{k+1} = Y_k \setminus Y_{k+1}$, then there is an $\mathbf{h} \in \boldsymbol{\mathscr{O}}(Z_k)$ with $\mathbf{h}(q) = 1 > \|\mathbf{h}\|_{Z_{k+1}}$. Thus, as claimed, the point x corresponding to the character χ lies in Z_{k+1}, whence Z_{k+1} is holomorphically convex.

We now have the sequence $X = Z_o \supset Z_1 \supset \cdots \supset Z_d$ in which each Z_k is holomorphically convex, and, moreover, as the preceding argument shows, each Z_k is convex with respect to $\boldsymbol{\mathscr{O}}(Z_{k-1})$ in the sense that for each point $p \in Z_{k-1} \setminus Z_k$, there is $\mathbf{h} \in \boldsymbol{\mathscr{O}}(Z_{k-1})$ with $\mathbf{h}(p) = 1 > \|\mathbf{h}\|_{Z_k}$.

The following lemma applied inductively implies that for each k, $\boldsymbol{\mathscr{O}}(X)$ is dense in $\boldsymbol{\mathscr{O}}(Z_k)$ in the sense that if g is holomorphic on a neighborhood of Z_k, then g can be approximated uniformly in Z_k by functions holomorphic on X.[6]

Lemma 6.5.32. *Let A and B be compact holomorphically convex subsets of $\mathbb{C}^N$ with $B \subset A$ and with B convex with respect to $\boldsymbol{\mathscr{O}}(A)$ in the sense that if $p \in A \setminus B$, there is a function f holomorphic on a neighborhood of A such that $f(p) = 1 > \|f\|_B$. Then each function g holomorphic on B can be approximated uniformly on B by functions holomorphic on A.*

This lemma is essentially a version of the Oka–Weil theorem.

Proof. Let $\mathscr{O}(A)$ be the closure in $\mathscr{C}(A)$ of the algebra of restrictions $f|A$ with f holomorphic on A, and let $\mathscr{B}$ be the closure in $\mathscr{C}(B)$ of the algebra of restrictions $f|B$ with $f \in \mathscr{O}(A)$. That A is holomorphically convex implies that the spectrum of the algebra $\mathscr{O}(A)$ is the set A itself, and because B is $\boldsymbol{\mathscr{O}}(A)$-convex, the spectrum of the algebra $\mathscr{B}$ is the set B. If we denote by π_j the jth coordinate function on $\mathbb{C}^N$ thought of as an element of the algebra $\mathscr{B}$, then the joint spectrum $\sigma_{\mathscr{B}}(\pi_1, \ldots, \pi_N)$ is the set B.

A function g holomorphic on a neighborhood of B is thus holomorphic on the joint spectrum $\sigma_{\mathscr{B}}(\pi_1, \ldots, \pi_N)$, so by the holomorphic functional calculus—recall Theorem 1.5.7, the function g lies in $\mathscr{B}$, which is the statement of the lemma.

Proof of Theorem 6.5.22 concluded. Consider a measure μ on X that is orthogonal to $\boldsymbol{\mathscr{O}}(X)$. We know it to be supported in X^*. We show that if $\operatorname{supp}\mu \subset Z_k$, then $\operatorname{supp}\mu \subset Z_{k+1}$. But because μ is orthogonal to $\boldsymbol{\mathscr{O}}(Z_k)$, the support of μ must be contained in the

[6]Note that in this work we have never introduced a topology on the space of germs $\boldsymbol{\mathscr{O}}(Y)$ for a compact set $Y \subset \mathbb{C}^N$, so the preceding assertion must not be read as the assertion that the topological space $\boldsymbol{\mathscr{O}}(X)$ is dense in the topological space $\boldsymbol{\mathscr{O}}(Z_k)$. These spaces of germs do admit natural topologies, and with respect to these topologies, the approximation assertion is *not* the assertion of the density of $\boldsymbol{\mathscr{O}}(X)$ in $\boldsymbol{\mathscr{O}}(Z_k)$.

set $(X \setminus \mathbb{B}_N(x,r)) \cup Y^*$, which is a subset of Z_k. Iteration of this process leads to the conclusion that $\operatorname{supp}\mu$ is contained in the union of $X \setminus \mathbb{B}_N(x,r)$ and a finite set and so must be contained, in fact, in $X \setminus \mathbb{B}_N(x,r)$. For each $x \in X$, there is an $r > 0$ for which this conclusion holds, so μ must be supported by the empty set, i.e., μ must be the zero measure.

The theorem is proved.

6.5.A. Appendix: Holomorphic Varieties in Compact Real Varieties

This appendix is devoted to the proof of the theorem of Diederich and Fornæss quoted above, Theorem 6.5.26

Proof of Theorem 6.5.26. This result is a consequence of the maximum principle. The proof breaks naturally into two parts, the first local, the second global.

Let X be a compact, real-analytic subvariety of $\mathbb{C}^N$ that contains the germ of a complex-analytic variety of positive dimension. Then X contains the germ of a complex-analytic manifold, call it M, of positive dimension. Let $p \in M$. Suppose, for the sake of notation, that p is the origin. There is then a polydisk $\mathbb{U}^N(r)$ centered at the origin on which is defined a real-analytic function χ such that $X \cap \Omega = \{z \in U : \chi(z) = 0\}$. (Locally X is defined by the vanishing of a finite number of real-valued, real-analytic functions. For χ take the sum of their squares.) We can choose r small enough that in $\mathbb{U}^N(r)$, the function χ has a power series expansion

$$\chi(z) = \sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha \bar{z}^\beta$$

with coefficients $c_{\alpha\beta}$ that satisfy $c_{\beta\alpha} = \overline{c_{\alpha\beta}}$.

We shall show that X contains a complex-analytic subvariety of $\mathbb{U}^N(r)$ that contains the germ at 0 of M.

Introduce the function $\tilde{\chi} : \mathbb{U}^N(r) \times \mathbb{U}^N(r) \to \mathbb{C}$ defined by

$$\tilde{\chi}(z,w) = \sum_{\alpha,\beta} c_\alpha z^\alpha \bar{w}^\beta, \tag{6.20}$$

which is real-analytic, holomorphic in z, and antiholomorphic in w.

Let the dimension of M at 0 be d. There is a polydisk $\mathbb{U}^N(r') \subset \mathbb{U}^N(r)$ on which are defined holomorphic functions $\varphi_1, \dots, \varphi_n$, $n = N - d$, that define $M \cap \mathbb{U}^N(r')$ as a complex submanifold of $\mathbb{U}^N(r')$. In particular, $d\varphi_1 \wedge \cdots \wedge d\varphi_n$ vanishes at no point of $\mathbb{U}^N(r')$.

The function $\tilde{\chi}$ vanishes on the diagonal of $X \times X$, and the functions Φ_j given by $\Phi_j(z,w) = \varphi_i(z) - \varphi_j(w)$ vanish on the part of the diagonal of $M \times M$ in $\mathbb{U}^N(r') \times \mathbb{U}^N(r')$. Moreover, these functions have nonvanishing differential. Consequently, we can write near $M \times M$ that

$$\tilde{\chi}(z,w) = \sum_{j=1,\dots,n} \left\{c_j(z,w)\Phi_i(z,w) + \bar{c}_j(z,w)\bar{\Phi}_j(z,w\right\})$$

with real-analytic coefficients c_j. This implies that when $w \in M$,

$$\tilde{\chi}(z, w) = \sum_{j=1,\dots,n} c_j(z, w)\varphi_j(z) + \bar{c}_j(z, w)\bar{\varphi}_j(z),$$

whence $\tilde{\chi}(\cdot, w) = 0$ on $M \cap \mathbb{U}^N(r)$ for every $w \in M$. Define V_1, a complex-analytic subvariety of $\mathbb{U}^N(r)$, by

$$V_1 = \cap_{w \in M \cap \mathbb{U}^N(r')} \{z \in \mathbb{U}^N(r) : \tilde{\chi}(z, w) = 0\}.$$

Symmetry implies that because $\tilde{\chi} = 0$ on $V_1 \times (M \cap \mathbb{U}^N(r'))$, necessarily $\tilde{\chi} = 0$ on $(M \cap \mathbb{U}^N(r')) \times V_1$. Thus, if V_2 is the complex subvariety

$$V_2 = \{z \in \mathbb{U}^N(r) : \tilde{\chi}(z, w) = 0 \text{ for all } w \in V_1\}$$

of $\mathbb{U}^N(r)$, then V_2 is contained in X and contains $M \cap \mathbb{U}^N(r')$.

This completes the local part of the proof.

For the global part of the proof, let W be the union of all the germs of complex-analytic subvarieties of positive dimension contained in X. If W is nonempty, let p be a point of the closure of W at maximal distance from the origin. Let $\mathbb{U}^N(p, r)$ be a polydisk centered at the point p throughout which the defining function χ admits a power series expansion as above. Choose a point $q \in W$ at distance ε from p with ε small in comparison with r. The power series expansion of χ about p can be rearranged to yield a power series expansion of χ about the point q. This rearranged power series converges in $\mathbb{U}^N(q, r-\varepsilon)$. By the local part of the argument, there is an analytic variety V in $\mathbb{U}^N(q, r-\varepsilon)$ that is contained in X and so in W. There is a function f holomorphic on $\mathbb{C}^N$ such that $|f|$ assumes its maximum over $\bar{W}$ at the point p. If ε is small in comparison with r, then $|f|$ will assume its maximum over V at an interior point of V, contradicting the maximum principle.

Theorem 6.5.26 is proved.

6.5.B. Appendix on Lifting Disks

The proof of Theorem 6.5.27 requires some preliminaries.

To begin with, we introduce certain spaces that will be convenient for our purposes.

Definition 6.5.33. $\mathbf{A}_1$ *is the space of all* $\mathbb{C}$*-valued functions on the unit circle that have absolutely convergent Fourier series.*

Thus, $u \in \mathbf{A}_1$ if

$$u(e^{i\vartheta}) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ik\vartheta} \qquad (6.21)$$

with $\|u\|_1 = \sum_k |\alpha_k|$ finite.

Definition 6.5.34. $\mathbf{A}_{1,0}$ *is the subspace of* $\mathbf{A}_1$ *comprising the functions of the form* (6.21) *with* $\alpha_0 = 0$*, and* $\mathbf{A}_1^+$ *is the subspace of* $\mathbf{A}_1$ *that consists of the functions of the form* (6.21) *with* $\alpha_k = 0$ *when* $k < 0$.

Alternatively, $\mathbf{A}_{1,0}$ consists of the functions in $\mathbf{A}_1$ whose Poisson integrals vanish at the origin, and $\mathbf{A}_1^+$ is the set of functions in $\mathbf{A}_1$ that continue holomorphically into the unit disk. We set $\mathbf{A}_{1,0}^+ = \mathbf{A}_1^+ \cap \mathbf{A}_{1,0}$, so that $\mathbf{A}_{1,0}^+$ can be identified with the space of functions holomorphic on $\mathbb{U}$ whose power series expansions about the origin converge absolutely on $\bar{U}$ and that vanish at the origin.

The space $\mathbf{A}_1$ is a Banach space with the norm $\|\cdot\|_1$ and is, moreover, a commutative Banach algebra, for if $u, v \in \mathbf{A}_1$, then $uv \in \mathbf{A}_1$ and $\|uv\|_1 \leq \|u\|_1\|v\|_1$, as a short calculation verifies. It is not difficult to verify that each character of the Banach algebra $\mathbf{A}_1$ is of the form $u \mapsto u(e^{i\vartheta_o})$ for a fixed point $e^{i\vartheta_o} \in \mathbb{T}$, so the spectrum of $\mathbf{A}_1$ is naturally identified with $\mathbb{T}$. (As we have done before, we use $\mathbb{T}$ to denote the unit circle.) Note that $\mathbf{A}_1$ is not a uniform algebra; it is a proper dense subalgebra of $\mathscr{C}(\mathbb{T})$.

A fact basic for our purposes is that multiple power series operate on the algebra $\mathbf{A}_1$:

Lemma 6.5.35. *Let* $\sum_\alpha c_\alpha z^\alpha$ *be a power series with complex coefficients that is convergent in the polydisk* $\mathbb{U}^N(R)$ *for some* $R > 0$*. If* $f_1, \ldots, f_N$ *are elements of* $\mathbf{A}_1$ *that satisfy* $\|f_j\|_1 < R$ *for* $j = 1, \ldots, N$*, then the series* $\sum_\alpha c_\alpha f^\alpha$ *converges in the sense of the norm* $\|\cdot\|_1$ *in* $\mathbf{A}_1$.[7]

Here $f^\alpha = f_1^{\alpha_1} \cdots f_N^{\alpha_N}$.

Proof. For such a choice of $f_1, \ldots, f_N$, the series $\sum_\alpha |c_\alpha| \|f_1\|_1^{\alpha_1} \cdots \|f_N\|_1^{\alpha_N}$ converges, which implies the convergence of the series $\sum_\alpha c_\alpha f^\alpha$ in $\mathbf{A}_1$.

It will be convenient to use the notation that if B is a Banach space with norm $\|\cdot\|$ and if $\varepsilon > 0$, then $\varepsilon - \text{ball}\, B$ is the open ball in B centered at the origin and of radius ε. Also, if n is a positive integer, B^n is the direct sum of n copies of B with itself normed with the norm $\|\cdot\|_n$ given by

$$\|(b_1, \ldots, b_n)\|_n = \max_{1 \leq j \leq n} \|b_j\|.$$

Consider now a function $F \in \mathscr{O}(\mathbb{U}^N(r))$ with power series expansion $\sum_\alpha c_\alpha z^\alpha$ about the origin. Lemma 6.5.35 allows us to define a function $\mathbf{F}$ from the product

$$r - \text{ball}\,\mathbf{A}_1^N = r - \text{ball}\,\mathbf{A}_1 \times \cdots \times r - \text{ball}\,\mathbf{A}_1$$

(N factors) into $\mathbf{A}_1$ by

$$\mathbf{F}(f_1, \ldots, f_N) = \sum_\alpha c_\alpha f^\alpha.$$

Thus, the map $\mathbf{F}$ is analytic as a map from a neighborhood of the origin in $\mathbf{A}_1^N$ to $\mathbf{A}_1$.

[7]This is a rather weak result, though it is enough for our purposes. It suffices to suppose not that each $\|f_j\|_1$ is less than R but only that each of the supremum norms $\|f_j\|_{\mathbb{T}}$ is less than R. This is a consequence of the lemma and the *spectral radius formula*, according to which $\|f\|_{\mathbb{T}} = \lim_{n\to\infty} \|f^n\|_1^{1/n}$ for each $f \in \mathbf{A}_1$. This kind of formula, which we have not proved and will not prove, holds in an arbitrary Banach algebra.

Definition 6.5.36. *The linear operator $T : \mathbf{A}_1 \to \mathbf{A}_1$ is defined by the condition that if $u = \sum_k \alpha_k e^{ik\vartheta} \in \mathbf{A}_1$, then $Tu(e^{i\vartheta}) = \sum_k \beta_k e^{ik\vartheta}$ with β_k given by $\beta_k = 0$ if $k = 0$, $\beta_k = -i\alpha_k$ if $k > 0$, and $\beta_k = i\alpha_k$ if $k < 0$.*

The map T is continuous with norm one. If $u = \sum_k \alpha_k e^{ik\vartheta} \in \mathbf{A}_1$, then

$$u + iTu = \alpha_o + 2 \sum_{k=1,\dots} \alpha_k e^{ik\vartheta}$$

is the element of $\mathbf{A}_1^+$ that is the boundary value of the function f given by $f(z) = \alpha_o + 2\sum_{k=1,\dots} \alpha_k z^k$, which is holomorphic on the unit disk. Note that $f(0) = \alpha_0$. The operator T restricts to the space $\mathbf{A}_{1,0}$ as an isometry of $\mathbf{A}_{1,0}$ onto itself.

The operator T is called the *conjugation* operator, and Tu the *conjugate* of u.

We now turn to the problem that is our main concern,

Fix a closed real-analytic CR-submanifold Σ of an open set Ω in $\mathbb{C}^N$. Assume Σ to be of dimension d and suppose the CR-dimension of Σ to be $p > 0$. Thus, for each $x \in \Sigma$, $\dim T_x^{\mathbb{C}}\Sigma = p$. Consider a point $x_o \in \Sigma$.

We begin by considering a special case: Suppose that $d = N + p$, the *generic case*. Choose coordinates $z_1, \dots, z_N$ with $z_j = x_j + iy_j$ on $\mathbb{C}^N$ such that x_o is the origin and such that

$$T_0\Sigma = \mathbb{C}^p_{z_1,\dots,z_p} \times \mathbb{R}^{N-p}_{x_{p+1},\dots,x_N}. \tag{6.22}$$

Thus, near the origin, Σ is given as a graph: For $j = p + 1, \dots, N$,

$$y_j = g_j(z_1, \dots, z_p, x_{p+1}, \dots, x_N) \tag{6.23}$$

with the g_j real-analytic functions that satisfy $g_j(0) = 0$ and $dg_j(0) = 0$.

We consider the following *lifting problem*: Given a continuous map $\varphi : \bar{\mathbb{U}} \times [0, 1] \to \mathbb{C}^p$ for which $\varphi(\cdot, t)$ is holomorphic on $\mathbb{U}$ for all $t \in [0, 1]$ with range sufficiently near the origin and with $\varphi(z, 0) = 0$ for all $z \in \bar{\mathbb{U}}$, find a continuous map $\tilde{\varphi} : \bar{\mathbb{U}} \times [0, 1] \to \mathbb{C}^N$ that, for each $t \in [0, 1]$, is holomorphic in the first variable on $\mathbb{U}$ and that satisfies $\pi \circ \tilde{\varphi} = \varphi$ if $\pi : \mathbb{C}^N \to \mathbb{C}^p$ is the natural orthogonal projection and the further condition that $\tilde{\varphi}$ carry $b\mathbb{U} \times [0, 1]$ into Σ. Using the space $\mathbf{A}_1$ introduced above, we shall prove the existence of such a map under certain conditions.

Fix $\varphi = (\varphi_1, \dots, \varphi_p) \in \mathbf{A}_1^{+p}$. We seek $\psi = (\psi_1, \dots, \psi_{N-p}) \in \mathbf{A}_1^{N-p}$ such that for all $\vartheta \in \mathbb{R}$, the point $(\varphi(e^{i\vartheta}), \psi(e^{i\vartheta}))$ lies in Σ. If we set $\psi = u + iv$ with $v(0) = 0$, so that $\psi_j = u_j + iv_j$ with real-valued functions u_j and v_j, we want a ψ that satisfies the conditions that for $j = p + 1, \dots, N$,

$$v_j(e^{i\vartheta}) = g_j(\varphi_1(e^{i\vartheta}), \dots, \varphi_p(e^{i\vartheta}), u_{p+1}(e^{i\vartheta}), \dots, u_N(e^{i\vartheta})) \tag{6.24}$$

for all $\vartheta \in \mathbb{R}$. For the functions ψ to extend holomorphically through $\mathbb{U}$, they must satisfy in addition the conditions that $v_j = Tu_j$ with T the conjugation operator T. Because $T^2u = -u + u(0)$, equation (6.24) leads to the following nonlinear equation for u_j, $j = p + 1, \dots, N$:

$$-u_j + u_j(0) = T[g_j(\varphi_1, \dots, \varphi_p, u_{p+1}, \dots, u_N)], \tag{6.25}$$

which can be solved using the implicit function theorem.

The functions g_j, $j = 1, N-p$, are real-analytic by hypothesis, so they admit power series expansions

$$g_j(z,x) = \sum_{\alpha,\beta\gamma} c_{j;\alpha\beta\gamma} z^\alpha \bar{z}^\beta x^\gamma$$

with α and β running through the p-tuples of nonnegative integers and γ running through the $(N-p)$-tuples of nonnegative integers. Because the function g_j is real-valued, the coefficients satisfy $c_{j;\alpha\beta\gamma} = \overline{c_{j;\alpha\beta\gamma}}$. Let $R_o > 0$ be small enough that the series converge when

$$\max\{|z_1|, \ldots, |z_p|, |x_{p+1}|, \ldots, |x_N|\} < R_o.$$

Then for any choice of functions $h \in \mathbf{A}_1^p$, $h' \in \mathbf{A}_1^{n-p}$ with $\max\{\|h\|_{1,p}, \|h'\|_{1,n-p}\} < R_0$ and for any choice of $j = 1, \ldots, n-p$, the function defined by

$$\Phi_j(h,h') = \sum_{\alpha,\beta,\gamma} c_{j:\alpha\beta\gamma} h^\alpha \bar{h}^\beta h'^\gamma$$

is in $\mathbf{A}_1$. In this way, we obtain a function

$$\Phi = (\Phi_{p+1}, \ldots, \Phi_N) : (R_o - \text{ball}\,\mathbf{A}_1^p) \times (R_0 - \text{ball}\,\mathbf{A}_1^{N-p}) \to \mathbf{A}_1^{N-p},$$

which is analytic. The function Φ carries $(R_o - \text{ball}\,A_{1,0}^p) \times (R_0 - \text{ball}\,A_{1,0}^{N-p})$ into $A_{1,0}^{N-p}$. Also, if h' is real, $\Phi(h,h')$ is also real, i.e., is a vector of real-valued functions.

We define $\Psi : (R_o - \text{ball}\,\mathbf{A}_{1,0}^p) \times (R_0 - \text{ball}\,A_{1,0}^{N-p}) \to A_{1,0}^{N-p}$ by

$$\Psi(h,u) = u + T\Phi(h,u).$$

As a composition of analytic maps, the function Ψ is analytic.

By $\Re\mathbf{A}_1^n$ we understand the real subspace of $\mathbf{A}_1^n$ that consists of n-vectors of real-valued functions. We shall use similar notation in varying situations below.

Theorem 6.5.37. *For $R_1 > 0$ sufficiently small, there is a unique continuous map*

$$u : (R_1 - \text{ball}\,A_{1,0}^p) \to R_0 - \text{ball}\,\Re A_{1,0}^{N-p}$$

such that $u(0,0) = 0$ and such that $\Psi(f, u(f)) = 0$. The function u is analytic.

Proof. To establish this, notice first that Ψ does take $(R_o - \text{ball}\,\mathbf{A}_{1,0}^p) \times (R_0 - \text{ball}\,\Re A_{1,0}^{N-p})$ to $\Re A_{1,0}^{N-p}$. To invoke the implicit function theorem we verify that the partial derivative $D_u\Psi(0,0) : \Re\mathbf{A}_{1,0}^{N-p} \to \Re\mathbf{A}_{1,0}^{N-p}$ is a linear topological isomorphism. This derivative is a continuous linear map; what is to be verified is that it is an isomorphism—that it is injective and surjective. We have that

$$(D_u\Psi)(0,0) = \text{Id} + (D_uT\Phi)(0,0).$$

Thus, to verify that $(D_u\Psi)(0,0)$ is an automorphism, it suffices to show that the norm of $(D_uT\Phi)(0,0)$ is less than one. The operator T has norm one, so it is enough to show that

$(D_u\Phi)(0,0)$ has norm less than one, which is immediate, for this derivative is the zero map. To verify this claim, we have to show that for each j there is a bound

$$\Phi_j(0,u) = o(\|u\|_1).$$

For this, note that

$$\Phi_j(0,u) = \sum_\gamma c_{j;00\gamma} u^\gamma$$

and that, because $c_{j;\alpha\beta\gamma} = 0$ unless $|\alpha| + |\beta| + |\gamma| > 1$, this series yields the bound

$$\|\Phi(0,u)\|_1 \leq \text{const } \|u\|_1^2$$

for some positive constant, the inequality valid when $\|u\|_1$ is small. Thus, $(D_u\Psi)(0,0)$ is an isomorphism.

The existence and uniqueness of the continuous function u together with the fact that it is analytic follows from the implicit function theorem in Banach spaces, which can be found in [96, p. 151].

We can now give the proof of the main result about lifting disks:

Proof of Theorem 6.5.27. We work initially in the generic case so that the machinery assembled above applies. The nondegenerate case will be handled by a simple reduction to the generic case. We have the real-analytic CR-manifold Σ of an open subset of $\mathbb{C}^N$ that passes through the origin and that is defined near the origin by the system of equations (6.23) so that the tangent space at the origin is given by (6.22). Define $f : \bar{\mathbb{U}} \to \mathbb{C}^p$ by $f(w) = (w, w, \dots, w)$. This is holomorphic in $\mathbb{U}$. Let $F : b\mathbb{U} \to \mathbb{C}^p$ be $f|b\mathbb{U}$, so that $F(e^{i\vartheta}) = (e^{i\vartheta}, \dots, e^{i\vartheta})$. The map F lies in $\mathbf{A}_{1,0}^{+p}$. For sufficiently small R_1, we have the analytic map

$$u : (R_1\text{-ball}\,\mathbf{A}_{1,0}^p) \to \Re\mathbf{A}_{1,0}^{N-p}$$

constructed above that satisfies $\Psi(h, u(h)) = 0$ and $u(0,0) = 0$. Fix $\delta > 0$ small enough that $\delta F \in R_1\text{-ball}\,\mathbf{A}_{1,0}^p$. Thus, for all $t \in (-1,1)$, the function $u(t\delta F) \in \mathbf{A}_{1,0}^{N-p}$ is defined. We can therefore define a map

$$\varphi : \bar{\mathbb{U}} \times (-1,1) \to \mathbb{C}^p \times \mathbb{C}^{N-p}$$

by

$$\varphi(w,t) = (t\delta f(w), H(w,t))$$

in which the $\mathbb{C}^{N-p}$-valued function H is specified as follows: For fixed $t \in (-1,1)$ the partial function $H(\cdot,t)$ is the holomorphic function $U(\cdot,t) + iV(\cdot,t)$ with real part U that satisfies

$$U(e^{i\vartheta},t) = u(t\delta F)(e^{i\vartheta}),$$

$V(\cdot,t)$ is the $\mathbb{R}^N$-valued harmonic conjugate of $U(\cdot,t)$ that vanishes at the origin and so satisfies

$$V(e^{i\vartheta},t) = T(u(t\delta F))(e^{i\vartheta}).$$

For each $t \in (-1, 1)$, $\varphi(\cdot, t)$ is holomorphic in $\mathbb{U}$, and $\varphi(\cdot, 0)$ takes the entire disk to the origin. We also have that $\tilde{\varphi}(e^{i\vartheta}, t) \in \Sigma$. This is by construction: The function u satisfies the equation $\Psi(\delta t F, u(\delta t F)) = 0$ in $\mathbf{A}^p_{1,0} \times \mathbf{A}^{N-p}_{1,0}$, so that

$$T[u(\delta t F, \delta t F)] = \Phi(\delta t F, u(\delta t F)).$$

From this it follows that if $V = (V_{p+1}, \ldots, V_N)$, then

$$V_j(e^{i\vartheta}, t) = \sum_{\alpha,\beta,\gamma} c_{j;\alpha\beta\gamma}(\delta t F(e^{i\vartheta}))^\alpha (\overline{\delta t F(e^{i\vartheta}})^\beta [u(\delta t F, \overline{\delta t F})(e^{i\vartheta})]^\gamma.$$

Thus, we have that for t real and near zero,

$$\varphi(e^{i\vartheta}, t) = \big(\delta t F(e^{i\vartheta}), u(\delta t F, \delta t F)(e^{i\vartheta}) + ig(\delta t F(e^{i\vartheta}), u(\delta t F, \delta t F)(e^{i\vartheta})\big),$$

which means that $\tilde{\varphi}(\cdot, t) \in \Sigma$.

Finally, $\varphi(e^{i\vartheta}, \cdot)$ is real-analytic on $(-1, 1)$, so it extends holomorphically into a neighborhood of the origin.

The theorem is proved in the generic case.

The nongeneric case is handled as follows. (See [285].) Consider a real-analytic CR-submanifold of $\mathbb{C}^N$ of positive CR-dimension. Let the real dimension of Σ be d and let the CR-dimension be p. Suppose Σ to contain the origin. Let Π be the minimal $\mathbb{C}$-subspace of $\mathbb{C}^N$ that contains the tangent space $T_0\Sigma$. The dimension of Π is $p + d = n$. By a suitable choice of coordinates, we can suppose $\Pi = \mathbb{C}^n_{z_1,\ldots,z_n}$. Let $\pi : \mathbb{C}^N \to \mathbb{C}^n$ be the orthogonal projection. Under π, a sufficiently small neighborhood of the origin in Σ is carried bianalytically onto a generic CR-submanifold, Σ', of a neighborhood of the origin in $\mathbb{C}^n$.

The map $\pi|\Sigma$ is bianalytic on a neighborhood W of the origin onto Σ'. It therefore has an inverse map $\eta : \Sigma' \to W$, which is real-analytic. Moreover, π is holomorphic, so its differential $d\pi = \pi$ is $\mathbb{C}$-linear on the complex tangent spaces $T^{\mathbb{C}}_z(\Sigma)$ for $z \in \Sigma$ near the origin. That is to say, $\pi|W$ is a CR-map. Consequently, $d\eta$ is complex linear on the complex tangent spaces $T^{\mathbb{C}}_z(\Sigma')$, $z \in \Sigma$: If $\eta = (\eta_1, \ldots, \eta_N)$, then each of the functions η_j satisfies the tangential Cauchy–Riemann equations on Σ'.

It follows that the map η extends to a holomorphic map H defined on a neighborhood of Σ' and taking values in $\mathbb{C}^N$.

If we have this holomorphic map H, then because by the generic case of the theorem, which has already been established, there is a map $\varphi : \bar{\mathbb{U}} \times [0, 1] \to \Sigma'$ with the desired properties, the composition $H \circ \varphi$ is a map of the required kind.

The existence of the holomorphic extension H of η is a consequence of a theorem of Tomassini [355]. In fact, we need only a local version: For our purposes, it is sufficient to have an extension of η into a neighborhood of the origin. This extension is provided immediately by the Cauchy–Kovalevsky theorem [181]: It is desired to solve the differential equation $\bar{\partial} H_j = 0$ near the origin in $\mathbb{C}^n$ subject to the Cauchy condition $H_j = \eta_j$ on Σ'. The surface Σ' is not noncharacteristic for the operator $\bar{\partial}$, because it is a CR-manifold of positive CR-dimension. However, the condition that η_j satisfy the tangential CR-equations on Σ' is the compatibility condition required to guarantee the existence of the desired function H_j. For full details, see the paper of Tomassini.

6.6. Tangential Approximation

It is a classical result of Carleman [80] that *if* η *and* f *are continuous functions on the real axis in the complex plane with* η *positive, then there is an entire function* F *such that* $|F(x) - f(x)| < \eta(x)$ *for all* $x \in \mathbb{R}$. It is striking that in this theorem, the function η can decay to zero arbitrarily rapidly at infinity, while the function F can increase arbitrarily rapidly or oscillate wildly at infinity. Carleman noted that in his result one can replace the real line by a locally rectifiable curve in $\mathbb{C}$ that goes to infinity in both directions or, indeed, by certain systems of such curves.

The present section is devoted to some more-recent results in the direction of Carleman's work. We begin with a result in the plane in which the real line in Carleman's theorem is replaced by more general sets. Then it will be shown that in $\mathbb{C}^N$, one can obtain the same kind of tangentially improving approximation on $\mathbb{R}^N$. Finally, an analogue of Carleman's theorem will be obtained for certain unbounded locally rectifiable curves in $\mathbb{C}^N$.

It will be a convenience to have the following terminology.

Definition 6.6.1. *A closed subset* X *of* $\mathbb{C}^N$ admits tangential approximation *if for every* $f \in \mathscr{C}(X)$ *and for every positive continuous function* η *on* X*, there is an entire function* F *such that* $|f(x) - F(x)| < \eta(x)$ *for all* $x \in X$.

Connected sets in the plane that admit tangential approximation are often called *Carleman continua*.

There is a general principle that applies in the discussion of tangential approximation:

Lemma 6.6.2. *The closed subset* X *of* $\mathbb{C}^N$ *admits tangential approximation if and only if for each* $f \in \mathscr{C}(X)$ *there is an entire function* F *such that* $|F(x) - f(x)| < 1$ *for all* $x \in X$.

Proof. Suppose that every $f \in \mathscr{C}(X)$ can be approximated to within one by an entire function. Given $f \in \mathscr{C}(X)$ and a positive continuous function η on X, there is an entire function g such that $\Re g(x) < \ln \eta(x)$. There is then an entire function h such that $|h(x) - f(x)e^{-g(x)}| < 1$ for all $x \in X$. Thus, the entire function F defined by $F(x) = e^{g(x)}h(x)$ satisfies $|F(x) - f(x)| < |e^{g(x)}| < \eta(x)$.

Keldych and Lavrentieff [202] characterized the planar sets that admit tangential approximation. To state their characterization, it is useful to introduce the following notion.

Definition 6.6.3. *An open subset* W *of the plane such that* $W \cup \{\infty\}$ *is a connected subset of the Riemann sphere is* locally connected at infinity *if there is a function* $r : (0, \infty) \to (0, \infty)$ *with* $\lim_{t\to\infty} r(t) = \infty$ *and with the property that for every* $w \in W$ *there is an arc in* $W \cup \{\infty\}$ *that connects* w *to the point at infinity and that lies in* $\{\zeta \in \mathbb{C} : |\zeta| > r(|w|)\} \cup \{\infty\}$.

This condition is equivalent to the condition that the subset $W^* = W \cup \{\infty\}$ of the Riemann sphere be locally connected at the point $\{\infty\}$ in the usual sense of set-theoretic topology: For each neighborhood V of $\{\infty\}$ in W^*, there is a connected neighborhood V' of ∞ in W^* with $V' \subset V$.

It is evident that if X is a closed subset of $\mathbb{C}$ such that the set $(\mathbb{C} \setminus X) \cup \{\infty\}$ is connected, then $\mathbb{C} \setminus X$ has no bounded components. In the event that $\mathbb{C} \setminus X$ has no bounded components, the set $\mathbb{C} \setminus X$ is locally connected at infinity if and only if for every closed disk Δ in $\mathbb{C}$, the union of the bounded components of the complement of $\Delta \cup X$ is bounded.

Theorem 6.6.4. [202] *A closed subset of $\mathbb{C}$ admits tangential approximation if and only if it has no interior, $\mathbb{C} \setminus X$ has no bounded components, and $\mathbb{C} \setminus X$ is locally connected at infinity.*

There is a related theorem of Arakelyan [38] that concerns *uniform* approximation on sets that may have interior.

Theorem 6.6.5. *If X is a closed subset of $\mathbb{C}$, every continuous function on X that is holomorphic on* int X *is uniformly approximable on X if and only if the complement in the Riemann sphere of the set X is connected, and $\mathbb{C} \setminus X$ is locally connected at infinity.*

There are simple examples of sets in the plane that do not admit tangential approximation.

Example 6.6.6. Let E be the closed subset of the plane defined by

$$E = \left\{x + iy \in \mathbb{C} : x \in \mathbb{R},\ y = \frac{1}{n},\ n = 1, \dots\right\} \cup \mathbb{R},$$

a union of a sequence of horizontal lines. The complement of this set is locally connected at infinity, so E admits tangential approximation. If we adjoin to the set E the intervals $I_n = \{n + iy \in \mathbb{C} : \frac{1}{n} \le y \le \frac{1}{n+1}\}$, $n = 1, \dots$, we obtain a closed, nowhere dense subset, X, of the plane whose complement is *not* locally connected at infinity, so that, according to Theorem 6.6.4, X does not admit tangential approximation.

In light of Lemma 6.6.2, the theorem of Arakelyan contains that of Keldych and Lavrentieff.

As an example in the direction of Theorem 6.6.4, there is a result of Roth [304].

Corollary 6.6.7. *If the closed, nowhere dense subset X of $\mathbb{C}$ is a union of rays of the form $L_\vartheta = \{\rho e^{i\vartheta} : \rho \ge r_\vartheta\}$ with each $r_\vartheta \ge 0$, then X admits tangential approximation.*

Such a set can perfectly well have infinite area.

Fuchs [135] has given an exposition of the classical approach to Arakelyan's theorem. A remarkably simple proof of the sufficiency of the hypotheses of the theorem was found by Rosay and Rudin [300]:

Proof of Theorem 6.6.5. If the complement of X is locally connected at infinity, there is an increasing sequence $\{\Delta_k\}_{k=1,\dots}$ of closed disks in $\mathbb{C}$ with union the entire plane and with the property that Δ_{k+1} contains all the bounded components of $\mathbb{C} \setminus (X \cup \Delta_k)$. Define a sequence $\{X_k\}_{k=1,\dots}$ of subsets of $\mathbb{C}$ by the prescription that $X_0 = X$, and $X_k = \Delta_k \cup X \cup \bar{H}_k$, $k = 1, \dots$, in which H_k denotes the union of the bounded components of $\mathbb{C} \setminus (\Delta_k \cup X)$. By definition, the set $\mathbb{C} \setminus X_k$ has no bounded components.

Let $f \in \mathscr{C}(X)$ be holomorphic on the interior of X, and let ε be a positive real number. We will define a sequence of functions h_n inductively. Take $h_0 = f$. Suppose $h_0, \dots, h_n$ have been defined with h_k continuous on X_k, holomorphic on its interior.

Let D_n be a closed disk with $\Delta_{n+1} \cup \bar{H}_{n+1} \subset D_n \subset \Delta_{n+2}$. Denote by ψ a smooth function on $\mathbb{C}$ with $0 \le \psi \le 1$, $\psi = 1$ on D_n and $\psi = 0$ in $\mathbb{C} \setminus \Delta_{n+2}$. By Mergelyan's theorem, there is a polynomial P with

$$(6.26) \qquad |h_n - P| < \varepsilon 2^{-n}$$

on $X_{n+1} \cap \Delta_{n+1}$, and

$$\frac{1}{\pi} \int_{X_n} \Big|(h_n - P)(\zeta) \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta)\Big| \frac{d\mathscr{L}(\zeta)}{|\zeta - z|} < \varepsilon 2^{-n}$$

for all $z \in \mathbb{C}$. (Note that the integrand has compact support; its support is contained in the annular set $\Delta_{n+2} \setminus D_n$.)

Define r_n on $\mathbb{C}$ by

$$r_n(z) = \frac{1}{\pi} \int_{X_n} (h_n - P)(\zeta) \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{d\mathscr{L}(\zeta)}{\zeta - z}$$

and h_{n+1} on $D_n \cup X_{n+1}$ by

$$h_{n+1} = \psi P + (1 - \psi) h_n + r_n.$$

The function r_n is holomorphic in D_n, because the integrand vanishes on $X_n \cap D_n$. The function h_{n+1} is well defined, since $1 - \psi = 0$ in D_n. It is continuous on X_{n+1} and holomorphic on the interior of this set. As for the holomorphy of h_{n+1}, this function is plainly holomorphic at points of the interior of X_{n+1} that lie in D_n. For the points in $\operatorname{int} X_{n+1} \setminus D_n$, use the fact that there, $\bar{\partial} r_n = (h_n - P)\bar{\partial}\psi$.

Equation (6.26) and the bound $|r_n| < \varepsilon 2^{-n}$ combine to yield that on X_n,

$$|h_{n+1} - h_n| = |(P - h)\psi + r| < \varepsilon 2^{-n+1}.$$

Thus the limit $F(z) = \lim_{n\to\infty} h_n(z)$ exists uniformly on compacta in $\mathbb{C}$ and satisfies

$$|F(z) - f(z)| < \varepsilon$$

on X.

That the conditions of Arakelyan's theorem are necessary is easily proved. Fix a closed subset X of $\mathbb{C}$ on which each function $f \in A(X)$ can be approximated uniformly by entire functions. If $\mathbb{C} \setminus X$ has a bounded component, say V, let w be a point of V, and let $f(z) = \frac{1}{z-w}$. This function is in $A(X)$, but it cannot be approximated uniformly on X by entire functions. Suppose, indeed, that there are entire functions F_k with $F_k \to f$ uniformly on X. By the maximum principle, the uniform convergence of the sequence $\{F_k\}_{k=1,\dots}$ on bV implies the uniform convergence on $\bar{V}$. From this it follows that there is a function $g \in A(\bar{V})$ with $g(z) = \frac{1}{z-w}$ on bV. Then $(z - w)g(z)$ is in $A(\bar{V})$, equals one on bV, and is zero at w. Contradiction. Suppose now that $\mathbb{C} \setminus X$ has no bounded components but that it is not locally connected at infinity. Thus there exist $r > 0$ and a sequence $\{z_n\}_{n=1,\dots}$ in $\mathbb{C}^N$ with $|z_n| \to \infty$, and with the property that any arc in the Riemann sphere that misses X and that connects z_n to the point at infinity must meet the circle $K_r = \{z \in \mathbb{C} : |z| = r\}$. For each n, let D_n be the component of $\mathbb{C} \setminus (X \cup K_r)$ that contains z_n. Each D_n is bounded. By passing to a subsequence of the z_n's if necessary, we can suppose the D_n's to be mutually disjoint. We require that $r < |z_1| < |z_2| < |z_3| < \cdots$ and that $|z_n| > \frac{1}{2}\sup_{z \in D_n} |z|$. Mittag-Leffler's theorem provides a meromorphic function g on the plane with simple poles at the points z_n, with no other poles, and with residue z_n^2 at z_n. The function $g|X$ is continuous on X and holomorphic on its interior. Suppose F to be an entire function that is uniformly within 1 of g on X. Set $M = \sup_{|z|=r}\{1+|F(z)|+|g(z)|\}$. If $h(z) = (z - z_n)(F(z) - g(z))$, then h is continuous on X, holomorphic on its interior. On bD_n, we have $|h(z)| < 2|z_n|M$, and the value of h at z_n is z_n^2. Thus, by the maximum principle $|z_n^2| \le 2|z_n|M$. Because $z_n \to \infty$, this is impossible for large n, so the function g cannot be approximated uniformly on X by entire functions. This construction was given by Fuchs [135]; it is somewhat simpler than the corresponding construction of Arakelyan.

We now take up the problem of tangential approximation in $\mathbb{C}^N$. The desideratum is a characterization of the closed subsets of $\mathbb{C}^N$ that admit tangential approximation. Such a characterization is far beyond our grasp, but there are some results in this direction. We begin by considering $\mathbb{R}^N$ contained in $\mathbb{C}^N$.

Theorem 6.6.8. *The subset $\mathbb{R}^N$ of $\mathbb{C}^N$ admits tangential approximation.*

It seems that this theorem was first given by Hoischen [179], who proved somewhat more: *If the given function f is of class $\mathscr{C}^p$ on $\mathbb{R}^N$, then f and all its derivatives of order not more than p can be approximated tangentially on $\mathbb{R}^N$.* Hoischen's proof makes use of integral kernels. Theorem 6.6.8 was found independently by Scheinberg [319], whose method was also to use integral kernels. Scheinberg noted in his paper that the theorem entails the (previously known) solution of certain nonclassical Dirichlet problems:

Corollary 6.6.9. *If f is a continuous function on $\mathbb{R}$, there is a function u harmonic on the upper half-plane that assumes continuously the boundary values f.*

Proof. Given $f \in \mathscr{C}(\mathbb{R})$, the theorem provides an entire function F such that $|F(x) - f(x)| < 1$ for all $x \in \mathbb{R}$. The Poisson integral of the bounded function $F - f$, call it u, is a harmonic function on the upper half-plane with boundary values $F - f$. Thus, $F - u$ solves the Dirichlet problem on the upper half-plane with boundary values f.

Our proof of Theorem 6.6.8 is based on the methods of polynomial convexity and Oka–Weil approximation; in spirit it is entirely classical and very close to the process used by Carleman.

We will need to use certain Saturn-like sets: For $\rho > 1$, let

$$X_\rho = \bar{\mathbb{B}}_N \cup \{x \in \mathbb{R}^N : |x| \le \rho\},$$

a compact set in $\mathbb{C}^N$. That X_ρ is polynomially convex follows from Kallin's lemma, Theorem 1.6.19: Let P be the polynomial given by $P(z) = z_1^2 + \cdots + z_n^2$. Then $P(X_\rho) = \bar{\mathbb{U}} \cup [0, \rho]$, and Kallin's lemma can be invoked to show that the set $X_\rho = \bar{\mathbb{B}}_N \cup \setminus (X_\rho \setminus \mathbb{B}_N)$

is polynomially convex. The approximation result Theorem 6.3.2 yields that $\mathscr{P}(X_\rho) = \mathscr{C}(X_\rho) \cap \mathscr{P}(\bar{\mathbb{B}}_n)$. A much simpler route to the latter conclusion is provided by Corollary 8.1.27 below.

Proof of Theorem 6.6.8. We prove that if $f \in \mathscr{C}(\mathbb{R}^N)$, then there is an entire function F on $\mathbb{C}^N$ such that $|F(x) - f(x)| < 1$ for all $x \in \mathbb{R}^N$.

To do this, introduce the sequence $\{S_k\}_{k=1,\dots}$ of sets defined by

$$S_k = \bar{\mathbb{B}}_N(k) \cup (\bar{\mathbb{B}}_N(k+1) \cap \mathbb{R}^N).$$

These sets are polynomially convex and satisfy $\mathscr{P}(S_k) = A(S_k)$.

Fix a small positive c. The Weierstrass approximation theorem provides a polynomial P_1 with $|P_1(x) - f(x)| < c$ when $x \in \mathbb{R}^N$, $|x| \le 1$. By Tietze's extension theorem, there is $\psi_1 \in \mathscr{C}(\mathbb{R}^N)$ with $|\psi_1| < c$ and with

$$\psi_1(x) = P_1(x) - f(x) \text{ for } x \in \mathbb{R}^N,\ |x| = 1.$$

Define $Q_1 \in \mathscr{P}(S_1)$ by

$$Q_1(z) = \begin{cases} P_1(z), & z \in \bar{\mathbb{B}}_N, \\ f(z) + \psi_1(z), & z \in \mathbb{R}^N,\ 1 \le |z| \le 2. \end{cases}$$

There is a polynomial P_2 such that $|P_2 - Q_1| < c/2$ on S_1.

If $z \in \bar{\mathbb{B}}_N$, then $|P_2(z) - P_1(z)| < c/2$; if $x \in \mathbb{R}^N$, $|x| \le 1$, then

$$|P_2(x) - f(x)| \le |P_2(x) - P_1(x)| + |P_1(x) - f(x)| < c/2 + c;$$

and if $x \in \mathbb{R}^N$, $1 \le |x| \le 2$, then

$$|P_2(x) - f(x)| \le |P_2(x) - f(x) - \psi_1(x)| + |\psi_1(x) < c/2 + c.$$

Inductively, suppose that polynomials $P_1, \dots, P_n$ have been chosen with the properties that

$$\begin{aligned} &|P_k - P_{k-1}| < c/2^k && \text{on } \mathbb{B}_N(k-1), \\ &|P_k(x) - f(x)| < c + c/2 + \cdots + c/2^{k-1}, && x \in \mathbb{R}^N,\ |x| \le k, \end{aligned}$$

and

$$|P_k(x) - f(x)| < c/2^{k-1}, \qquad x \in \mathbb{R}^N,\ |x| = k.$$

Let then $\psi_n \in \mathscr{C}(\mathbb{R}^N)$ satisfy $\psi_n = P_n - f$ on $|x| = n$ and $|\psi_n| < c/2^{n-1}$ everywhere. Require also that the support of ψ_n be in $|x| < n + \frac{1}{2}$. Define $Q_n \in \mathscr{P}(S_n)$ by

$$Q_n = \begin{cases} P_n & \text{on } \mathbb{B}_N(n), \\ f + \psi_n & \text{on } \{x \in \mathbb{R}^N : n \le |x| \le n+1\}. \end{cases}$$

Let P_{n+1} be a polynomial with $|P_{n+1} - Q_n| < c/2^{n+1}$ on S_n.

With this construction, we have that on $\mathbb{B}_N(n)$, $|P_{n+1} - P_n| = |P_{n+1} - Q_n| < c/2^{n+1}$. If $x \in \mathbb{R}^N$ with $|x| \leq n$, then

$$|P_{n+1}(x) - f(x)| < c + c/2 + \cdots + c/2^n.$$

For $x \in \mathbb{R}^N$ with $n \leq |x| \leq n+1$,

$$|P_{n+1}(x) - f(x)| \leq |P_{n+1}(x) - f(x) - \psi_n(x)| + |\psi_n(x)| \leq c/2^n + c/2^{n-1}.$$

We now define the function F to be the sum of the series $P_1 + \sum_{n=1}^{\infty}\{P_{n+1} - P_n\}$. This series converges uniformly on compacta in $\mathbb{C}^N$, so F is an entire function. And if $x \in \mathbb{R}^N$ and $n_o > |x|$, then

$$|F(x) - f(x)| \leq |P_{n_o}(x) - f(x)| + \sum_{n=n_o}^{\infty} |P_{n+1}(x) - P_n(x)| < 3c.$$

The quantity c can be taken less that $\frac{1}{3}$, so we have constructed an entire function that is uniformly to within one of f on $\mathbb{R}^N$. The conclusion of the theorem now follows from Lemma 6.6.2.

Another generalization of Carleman's theorem goes back to a theorem of Alexander [11], according to which a closed subset of $\mathbb{C}^N$ that is homeomorphic to $\mathbb{R}$ and that has locally finite one-dimensional measure admits tangential approximation. In his paper, Alexander indicated that certain stronger results could be obtained by his methods. We shall work in the context of locally rectifiable dendrites.

Definition 6.6.10. *A subset X of $\mathbb{R}^n$ is a* locally rectifiable dendrite *if it is closed, connected, has locally finite one-dimensional measure, and satisfies $\check{H}^1(X, \mathbb{Z}) = 0$.*

We shall need some information about about the geometry of locally rectifiable dendrites.

Lemma 6.6.11. *If X is a locally rectifiable dendrite in $\mathbb{C}^N$, then every compact subset of X is contained in a compact connected subset of X.*

Proof. First of all, X is arcwise connected. Note that if X is compact, then this result is contained in Theorem 3.3.3. We shall use this result. We consider in the following the case that X is not compact, i.e., not bounded. Let $x \in X$, and denote by $A(x)$ the subset of X consisting of all the points $y \in X$ to which x can be connected by an arc. The set $A(x)$ is open: Suppose $y \in A(x)$. By Eilenberg's theorem there are $r > 0$ such that $X \cap b\mathbb{B}_N(y, r)$ is a finite set. The set X is connected, so it has no compact components. Thus, each component of $\mathbb{B}_N(y, r)$ must approach $b\mathbb{B}_N(y, r)$. Because this sphere meets X in a finite set, there can be only finitely many components of $X \cap \bar{\mathbb{B}}_N(y, r)$, and each must be open in $X \cap \bar{\mathbb{B}}_N(y, r)$. Each of these components is arcwise connected, by the earlier result cited above. It follows that $A(x)$ is open in X. A similar argument proves it to be closed. Thus, X is arcwise connected as claimed.

If now K is a compact subset of X, let Δ be a closed ball that contains K and with the property that $b\Delta \cap X$ is a finite set. As in the last paragraph, $X \cap \Delta$ has only finitely many components, say $K_1, \dots, K_s$. If q_j is a point in K_j, and if λ_j is an arc in X that contains q_j and q_{j+1} for $j = 1, \dots, s_1$, then the union of the K_j's and the λ_j's is a compact, connected subset of X that contains K.

It is not difficult to characterize the locally rectifiable dendrites:

Lemma 6.6.12. *A closed, connected subset X of $\mathbb{R}^N$ with locally finite length satisfies $\check{H}^1(X, \mathbb{Z}) = 0$ if and only if it contains no simple closed curves.*

Proof. If the closed connected subset X of $\mathbb{R}^N$ satisfies $\check{H}^1(X, \mathbb{Z}) = 0$, then by Corollary 3.8.17, $\check{H}^1(C, \mathbb{Z}) = 0$ for every compact subset C of X. Thus, X can contain no simple closed curve

The proof of Lemma 4.7.4 provides the reverse implication.

The main result to be proved in this context is the following.

Theorem 6.6.13. *For $j = 1, \dots,$ let X_j be a locally rectifiable dendrite in $\mathbb{C}^N$. If the collection $\{X_1, \dots\}$ is locally finite, and if the X_j are mutually disjoint, then the union $Y = \cup_{j=1,\dots} X_j$ admits tangential approximation.*

The locally finite condition is the condition that no compact subset of $\mathbb{C}^N$ meet more than finitely many of the sets X_j.

There may be finitely many or at most countably infinitely many of the X_j.

As stated, the theorem applies when X is a connected set of finite length with zero first integral cohomology, a special case of Theorem 3.1.1. That earlier result will be used in the present developments.

The arrangement of the proof that follows is very similar to the treatment found in the paper of Gauthier and Santillan [141]; it depends essentially on the work of Alexander [11].

We begin with a lemma:

Lemma 6.6.14. *Let X be a closed subset of $\mathbb{C}^N$ with zero two-dimensional measure such that for every compact subset K of $\mathbb{C}^N$, the union*

$$\cup\{\widehat{S} \setminus X : S \text{ a compact subset of } K \cup X\}$$

is bounded. If E is a polynomially convex subset of $\mathbb{C}^N$ with the property that for every compact subset K of $X \cup E$, the hull $\widehat{K}$ is contained in $X \cup E$, then for every $f \in \mathscr{C}(X)$ that can be approximated uniformly on $X \cap E$ by polynomials, and for every positive continuous function η on X, there is an entire function F such that $|F(x) - f(x)| < \eta(x)$ for every $x \in X$.

Example 6.6.6 provides an example in which the condition imposed on X is not satisfied.

Corollary 6.6.15. *If X is a closed subset of $\mathbb{C}^N$ of two-dimensional measure zero that satisfies $\widehat{K} \subset X$ for every compact subset K of X and with the property that for every compact set T of $\mathbb{C}^N$, the set $\cup\{\widehat{S} \setminus X : S$ is a compact subset of $T \cup X\}$ is a compact set, then the set X admits tangential approximation.*

The corollary follows by taking E to be the empty set in the lemma.

Proof of Lemma 6.6.14 . Construct a sequence $\{\Delta_k\}_{k=1,\dots}$ of closed balls centered at the origin as follows. For all k, the interior of Δ_{k+1} contains Δ_k, and the radius of Δ_k is at least k. Moreover, Δ_1 contains $\widehat{S} \setminus S$ for every compact subset S of X. Having chosen $\Delta_1, \dots, \Delta_n$, we choose the ball Δ_{n+1} large enough that it contains $\widehat{S}$ whenever S is a compact subset of $\Delta_n \cup X$.

By hypothesis, there is a polynomial P_0 with $|P_0 - f| < \frac{1}{2}\eta$ on $E \cap X$. Let $h_0 \in \mathscr{C}(E \cup X)$ satisfy $h_0 = P_0$ on E and $|h_0 - f| < \frac{1}{2}\eta$ on $\Delta_1 \cap X$. The set $(\Delta_1 \cap X) \setminus E$ has two-dimensional measure zero, so $E \cup (\Delta_1 \cap X)$ is rationally convex by Corollary 1.6.8. Moreover, $h_0 \in \mathscr{P}(E \cup (\Delta_1 \cap X))$, because E is polynomially convex. Thus, there is a polynomial P_1 such that

$$|P_1 - P_0| < \frac{1}{2} \min\{\eta(x) : x \in \Delta_1 \cap X\}$$

on $E \cup (\Delta_1 \cap X)$.

Having chosen polynomials $P_1, \dots, P_n$ and functions $h_1, \dots, h_{n-1}$ with $h_k \in A(\Delta_k \cup (\Delta_{k+1} \cap X))$, let $h_{n+1} \in A(\Delta_n \cup (\Delta_{n+1} \cap X))$ satisfy $h_{n+1} = P_n$ on Δ_n and $|h_{n+1} - f| < \frac{1}{2^{n+2}}\eta$ on $\Delta_{n+1} \cap X$. Let P_{n+1} be a polynomial such that

$$|P_{n+1} - P_n| < \frac{1}{2^{n+1}} \min\{\eta(x) : x \in \Delta_n \cap X\} \tag{6.27}$$

on Δ_n and

$$|P_{n+1} - P_n| < \frac{1}{2^{n+3}}\eta$$

on $\Delta_{N+1} \cap X$. Equation (6.27) implies that the sequence $\{P_n\}_{n=1,\dots}$ converges uniformly on compacta in $\mathbb{C}^N$ to an entire function F. If $x \in X$, then for sufficiently large n, the point x lies in Δ_n, and we have the estimate

$$\begin{aligned} |F(x) - f(x)| &\le |F(x) - P_n(x)| + |P_n(x) - h_n(x)| + |h_n(x) - f(x)| \\ &\le |F(x) - P_n(x)|_{\Delta_n} + \frac{1}{2^{n+1}}\eta(x) + \frac{1}{2^{n+2}}\eta(x). \end{aligned}$$

If n is large enough, the final sum is less than $\eta(x)$. The lemma is proved.

Proof of Theorem 6.6.13. We are given a locally finite collection $\{X_j\}_{j=1,\dots}$ of mutually disjoint locally rectifiable dendrites in $\mathbb{C}^N$, and we are to prove that their union, X, admits tangential approximation. If K is any compact subset of X, it meets only finitely many of the X_j's and so is contained in a connected set of finite length. It satisfies $\check{H}^1(K, \mathbb{Z}) = 0$, so by Theorem 3.1.1, it is polynomially convex.

To prove the theorem, it suffices, by Corollary 6.6.15, to show that if Δ is a closed ball in $\mathbb{C}^N$, then the set

$$H(\Delta, X) = \cup\{\widehat{S} \setminus X : \text{ is a compact subset of } \Delta \cup X\}$$

is bounded.

The set X contains no simple closed curves, but $\Delta \cup X$ can contain simple closed curves not contained in Δ. Such a simple closed curve will be called a Δ-*loop*. If $R \in \mathbb{R}$

is sufficiently large, then $\mathbb{B}_N(R)$ contains all the Δ-loops. To see this, choose an R_o such that $\mathbb{B}_N(R_o)$ contains the ball Δ and such that $b\mathbb{B}_N(R_o)$ contains only finitely many points of X, say $p_1, \ldots, p_s$. Each Δ-loop L that meets $\mathbb{C}^N \setminus \bar{\mathbb{B}}_N(R_o)$ gives rise to a collection of open arcs, the components of $L \setminus \mathbb{B}_N(r_o)$. These arcs are mutually disjoint, and their endpoints lie among the points p_j. Because the set X contains no simple closed curve, two such arcs can share at most one endpoint. There are only finitely many possible endpoints, so there can be only finitely many of the arcs. In the same way, we see that only a finite number of Δ-loops can meet $\mathbb{C}^N \setminus \bar{\mathbb{B}}_N(R_o)$. Because Δ-loops are compact, a sufficiently large ball centered at the origin will contain them all.

That the set $H(\Delta, X)$ is bounded is now a simple consequence of the following lemma of Alexander [11]:

Lemma 6.6.16. *If $\mathbb{B}_N(R_o)$ contains all the Δ-loops, then for every $R > R_o$ and for every compact set $K \subset \Delta \cup (X \cap \mathbb{B}_N(R))$, the hull $\widehat{K}$ is contained in $\bar{\mathbb{B}}_N(R_o) \cup (X \cap \bar{\mathbb{B}}_N(R)$.*

Proof. For $t \geq R_o$, put

$$Y_t = \Delta \cup (X \cap \bar{\mathbb{B}}_N(t))$$

and

$$A_t = \overline{Y_t \setminus Y_{R_o}}.$$

Then $Y_t = Y_{R_o} \cup A_t$.

We shall show that for $t > R_o$, $\widehat{Y_t} = \widehat{Y}_{R_o} \cup A_t$. For this, notice that

$$\widehat{Y}_{R_o} \subset \widehat{Y_t} \subset [\widehat{Y_{R_o} \cup A_t}],$$

so what is to proved is that the set $Z_t = \widehat{Y}_{R_o} \cup A_t$ is polynomially convex.

According to Theorem 3.1.1, $\widehat{Z_t} \setminus Z_t$, if not empty, is a one-dimensional analytic subvariety of $\mathbb{C}^N \setminus Z_t$. Suppose this variety not to be empty, and let V be one of its global branches. We will show that $\bar{V} \setminus Z_t$ is an analytic subvariety of $\mathbb{C}^N \setminus Y_t$. (Note that V *is* a subset of $\mathbb{C}^N \setminus Y_t$ so that $\bar{V} \setminus Z_t$ is contained in $\mathbb{C}^N \setminus Y_t$.) Set $Q = \widehat{Y}_{R_o} \setminus Y_{R_o}$. The set Q is a one-dimensional variety. We show that near a point $x \in Q \cap \bar{V}$, the set V has the structure of a one-dimensional variety. Because $Y_{R_o} \subset Y_t$, whence $\widehat{Y}_{R_o} \subset \widehat{Y_t}$, our x lies in the set $\widehat{Y} \setminus Y_t$, which is a variety. We also have that $\widehat{Y_t} \setminus Y_t \supset Q$, so near x, the germ at x of the set Q is a union of the branches of the germ at x of the variety $\widehat{Y} \setminus Y_t$. Because $V \cap Q = \emptyset$, the germ at x of the set $\bar{V}$ is a union of some other branches of the germ at x of $\widehat{Y} \setminus Y_t$. Consequently, near x, $\bar{V} = V \cup \{x\}$, and we have that near $\widehat{Y}_{R_o} \setminus Y_t$, $\bar{V}$ is a variety.

Set $W = \bar{V} \setminus Y_t$. Then $\bar{W} \setminus W \subset Y_t$. If $z \in V \subset W$, then $z \notin Z_t$, and so $z \notin \widehat{Y}_{R_o}$. Consequently, there is a polynomial P with $P(z) = 0$ and $\Re P < 0$ on $\widehat{Y}_{R_o}$. If P vanishes identically on W, then it vanishes on $\bar{W}$, whence $P^{-1}(0) \cap Y_{R_o} = \emptyset$, and then $\bar{W} \subset A_t$. This is impossible, for A_t has area zero. Thus, P is nonconstant on W, which implies that $P(W)$ is a neighborhood of $0 \in \mathbb{C}$. The set $P(X)$ has measure zero in the plane, so there is $\alpha \in P(W) \setminus P(X)$ arbitrarily near the origin.

Set $P' = P - \alpha$. If α is chosen correctly, we have that (a) $\Re P' < 0$ on $\widehat{Y}_{R_o}$, (b) $P'(z') = 0$ for a $z' \in W$, and (c) $0 \notin P(A_t)$. By (a), P' has a continuous logarithm on $\widehat{Y}_{R_o}$ and so on Y_{R_o}. By (c), this logarithm extends to a continuous logarithm on all of Y_t, because all the Δ-loops are contained in $\mathbb{B}_N(R_o)$. Because P' vanishes at a point of W but has a logarithm on the set Y_t, which contains $\bar{W} \setminus W$, we have reached a contradiction to the argument principle, and the lemma is proved.

This concludes the proof of Theorem 6.6.13

There are immediate extensions of some of the results obtained above to results on tangential approximations on sets in Stein manifolds, results that follow simply by invoking the embedding theorem. Thus, for example, *if the closed subset X of the Stein manifold $\mathscr{M}$ is a locally rectifiable dendrite, then for each pair of continuous functions f and η on X with η positive, there is a function $F \in \mathscr{O}(\mathscr{M})$ with $|F(x) - f(x)| < \eta(x)$ for all $x \in X$.*

In a similar vein, it is evident that there is an analogue of Theorem 6.6.8 in which $\mathbb{C}^N$ is replaced by $\mathbb{B}_N$ and $\mathbb{R}^N$ by $\mathbb{R}^N \cap \mathbb{B}_N$.

In addition to the results on asymptotic approximation given above, Frih and Gauthier [133] have discussed asymptotic approximation on products of dendrites that are piecewise $\mathscr{C}^1$, and Kasten and Schmieder [197] have given some other results in this direction.

Some other results on tangential approximation have been obtained, but we shall not present their details, for the methods involved in their derivation depend on methods completely different from those developed here.

Nunemacher [268, 269] obtains tangential approximation on totally real $\mathscr{C}^1$ manifolds in domains in $\mathbb{C}^N$. In this work, the domain of definition of the approximating function varies.

Manne [234] obtains tangential approximation of a function and its derivatives of order not more that p on a smooth totally real submanifold of $\mathbb{C}^N$. The approximating functions are here defined on a fixed domain. In [236], he extends this work to the context of Whitney functions defined on totally real sets in $\mathbb{C}^N$. Finally, in [235], he discusses tangential approximation on the union of two totally real, real-linear subspaces of $\mathbb{C}^N$.

Chapter 7

VARIETIES IN STRICTLY PSEUDOCONVEX DOMAINS

Introduction. This chapter is devoted to the study of one-dimensional subvarieties of strictly pseudoconvex domains. The motivation comes in good measure from the highly developed theory of the boundary behavior of holomorphic functions; the present chapter may be regarded as presenting some results toward an analogous geometric theory for varieties. Section 7.1 contains work on interpolation, which serves as a tool in the subsequent sections. Section 7.2 treats boundary regularity questions. Section 7.3 considers boundary uniqueness results.

7.1. Interpolation

This section is devoted to the subject of interpolation, not with the intention of giving a detailed treatment of the subject but rather with the goal of developing certain tools essential in the next section. The initial steps in the subject are best taken in the context of general uniform algebras. The notions considered here are given in the following definition.

Definition 7.1.1. *Let A be a uniform algebra on the compact space X, and let $E \subset X$ be a compact subset.*

(a) *E is an* interpolation set *if given $\varphi \in \mathscr{C}(E)$, there is $f \in A$ with $f|E = \varphi$.*

(b) *E is a* peak-interpolation set *if given $\varphi \in \mathscr{C}(E)$, φ not identically zero, there is $f \in A$ with $f|E = \varphi$ and $|f(x)| < \sup_E |\varphi|$ for all $x \in X \setminus E$.*

There is a considerable theory concerning interpolation sets. Some of this theory in the general setting of uniform algebras can be found in [345]. For results concerning interpolation on the ball in $\mathbb{C}^N$ we refer to [310].

Arguments involving interpolation often proceed by duality and involve the study of measures orthogonal to the algebra A (annihilating measures), as the following theorem suggests.

Theorem 7.1.2. *Let A be a uniform algebra on the compact metrizable space X, and let $E \subset X$ be a compact set.*

(a) *If for every neighborhood U of E, there is $f_U \in A$ with $f_U|E = 1$, $\|f_U\|_X < \frac{4}{3}$, $\|f_U\|_{X \setminus U} < \frac{1}{3}$, then E is a peak set.*

(b) *If E is a peak set, then $A|E$ is closed in $\mathscr{C}(E)$.*

(c) *If E is a peak set and an interpolation set, then E is a peak-interpolation set.*

(d) *The set E is a peak-interpolation set if and only if $\mu_E = 0$ for all measures orthogonal to A.*

(e) *If E is an interpolation set and if each point of E is a peak set, then E is a peak-interpolation set.*

The results (a)–(d) are due to Bishop [55, 58], and the result (e) is a theorem of Varopoulos [357].

Proof of (a). We construct inductively a sequence $\{V_i\}_{i=1}^{\infty}$ of neighborhoods of E and a corresponding sequence $\{f_i\}_{i=1}^{\infty}$ in A. For V_1 we take an arbitrary neighborhood of E, and for the associated f_1 we take an element of A with $f_1 = 1$ on E, $\|f_1\|_X < \frac{4}{3}$ and $\|f_1\|_{X \setminus V_1} < \frac{1}{3}$. Having constructed $V_1, \ldots, V_{n-1}$ and corresponding $f_1, \ldots, f_{n-1} \in A$, let

$$V_n = \left\{ x \in V_{n-1} : |f_j(x)| < 1 + \frac{1}{2^n 3} \text{ for } j = 1, \ldots, n-1 \right\},$$

and let $f_n \in A$ satisfy $f_n = 1$ on E, $\|f_n\|_X < \frac{4}{3}$, and $\|f_n\|_{X \setminus V_n} < \frac{1}{3}$. Put $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. This function belongs to A and satisfies $f = 1$ on E. If $x \notin V_1$, then $|f_n(x)| < \frac{1}{3}$ for all n, so $|f(x)| < \frac{1}{3}$. If $x \in V_n \setminus V_{n+1}$, then $|f_j(x)| < \frac{1}{3}$ for $j > n$, and $|f_j(x)| < 1 + \frac{1}{2^n 3}$ for $j = 1, \ldots, n-1$. This gives

$$\begin{aligned} |f(x)| &\leq \sum_{j=1}^{n-1} 2^{-j} \left(1 + \frac{1}{2^n 3}\right) + \frac{4}{2^n 3} + \frac{1}{3} \sum_{j=n+1}^{\infty} \frac{1}{2^j} \\ &= \left(1 - \frac{1}{2^{n-1}}\right)\left(1 + \frac{1}{2^n 3}\right) + \frac{4}{2^n 3} + \frac{1}{2^n 3} < 1. \end{aligned}$$

If $x \in V_n$ for every n, then $|f_n(x)| \leq 1$, so $|f(x)| \leq 1$. It follows that the function f is bounded by 1 and assumes values of modulus one only on a subset of V_1. The function $\frac{1}{2}(1 + f)$ is bounded in modulus by 1 and peaks on a subset of V_1 that contains E.

The neighborhood V_1 of E was chosen arbitrarily, so we have proved that E is an intersection of peak sets. Because X is metrizable and compact, the set E is a *countable* intersection of peak sets. If now $g_j \in A$ peaks on the set E_j and if $\bigcap_{j=1,\dots} E_j = E$, then $g = \sum_{j=1}^{\infty} 2^{-j} g_j \in A$ peaks on E.

This completes the proof of (a).

Proof of (b). Assertion (b) is contained in Theorem 1.2.23.

Proof of (c). For (c), recall that if E is a peak set and an interpolation set, then for each $f \in \mathscr{C}(E)$, Theorem 1.2.23 provides $g \in A$ with $g|E = f$ and with $|f| \leq \|g\|_E$. If $\varphi \in A$ peaks on E, then φg interpolates f on A and satisfies $|\varphi g|(x) < \|f\|_E$ when $x \in X \setminus E$.

Proof of (d). If E is a peak-interpolation set and if μ is a measure orthogonal to A, we are to see that for each closed set $E_0 \subset E$, $\mu(E_0) = 0$. The set E_0 is closed and E is a peak-interpolation set, so there is $f \in A$ with $f|E_0 = 1$, $|f| < 1$ on $X \setminus E_0$. The measure μ is orthogonal to A, so for each $n = 1, 2, \dots,$

$$0 = \int f^n d\mu.$$

By the dominated convergence theorem this yields $\mu(E_0) = 0$, as we wished to prove.

Conversely, assume the closed subset E of X to have the property that for every annihilating measure μ of A, $\mu_E = 0$. We are to see that E is a peak-interpolation set.

The first step of the proof is to show that given $f \in \mathscr{C}(E)$ with $\|f\|_E = r < 1$, there is $\widetilde{f} \in A$ with $\|\widetilde{f}\|_X < 1$ and $\widetilde{f} = f$ on E.

Denote by $\rho : A \to \mathscr{C}(E)$ the restriction map given by $\rho g = g|E$. If by $A(s)$ we denote the open ball of radius s, center 0 in A, we have to see that $f \in \rho A(1)$. We first show that $f \in \overline{\rho A(r)}$, the closure of $\rho A(r)$ in $\mathscr{C}(E)$. If not, then by the Hahn–Banach and Riesz representation theorems, there is a finite regular Borel measure μ_1 on E with $\int f \, d\mu_1 > 1$ and $|\int g \, d\mu_1| < 1$ for all $g \in \overline{A(r)}$.

Define a linear functional ψ on A by $\psi(g) = \int g \, d\mu_1$. We have $|\psi(g)| < 1$ when $\|g\|_X < r$, so the function ψ has norm no more than $\frac{1}{r}$, whence, by the Hahn–Banach and Riesz representation theorems, there is a finite regular Borel measure μ_2 on X with $\|\mu_2\| \leq \frac{1}{r}$ and with $\psi(g) = \int g \, d\mu_2$ for $g \in A$.

The measure $\mu = \mu_1 - \mu_2$ is orthogonal to A, and so, by hypothesis, $\mu_E = 0$. However, this is impossible, for

$$\left| \int_E f d\mu \right| \geq \int_E f d\mu_1 - r\|\mu_2\| > 1 - rr^{-1} = 0.$$

Thus, $f \in \overline{\rho A(r)}$ is claimed.

Consequently, there is a function $f_1 \in A$ with $\|f_1\|_X < r$ and $\|f - f_1\|_E < (1-r)/2$. By the same argument, there is $f_2 \in A$ with $\|f_2\|_X < (1-r)/2$ and $\|f_1 - f_2\|_E < (1-r)/2^2$. We iterate this process to find a sequence $\{f_k\}_{k=1}^{\infty}$ in A with $\|f_k\|_X < (1-r)/2^{k-1}$ and $|f - \sum_{k=1}^{n} f_k| < (1-r)/2^n$. It follows that we can define $\tilde{f} \in A$ by $\tilde{f} = \sum_{k=1}^{\infty} f_k$ and that the function so defined satisfies $\rho \tilde{f} = f$ and $\|\tilde{f}\|_X < r + \sum_{k=2}^{\infty}(1-r)/2^{k-1} = 1$.

It is worth noting that this argument depends only on the linear structure of A and not on its multiplicative structure. Consequently, it is valid for any closed *subspace* of $\mathscr{C}(X)$.

If $g \in \mathscr{C}(E)$ and Δ is a positive continuous function on X with $\Delta > g$ on E, then there is $g^* \in A$ with $g^*|E = g$ and $|g^*| < \Delta$ on X. For this consider the closed subspace $B = \{g \in \mathscr{C}(X) : \Delta g \in A\}$ of $\mathscr{C}(X)$. This subspace *is* closed because Δ is bounded away from zero. We have that $B^\perp = \{\Delta\mu : \mu \in A^\perp\}$, which implies that each $\nu \in B^\perp$ satisfies $\nu_E = 0$. Apply the first part of the proof to B to conclude that there is $g_0 \in B$ with $\|g_0\|_X < 1$ and $g_0 = g/\Delta$ on E. Then the function $g^* = g_0\Delta$ is in A, $g_0\Delta = g$ on E, and $\|g_0\Delta\|_X < \Delta$.

We have therefore that if U is a neighborhood of E, there is $f \in A$ with $f = 1$ on E, $\|f\|_X < \frac{4}{3}$, and $\|f\|_{X\setminus U} < \frac{1}{3}$, whence, by (a), E is a peak set.

We now know that E is an interpolation set and a peak set, so by (c), it is a peak-interpolation set; (d) is proved.

Proof of (e). For this, it suffices to prove that, with the hypotheses of (e), for every measure μ orthogonal to A, the measure μ_E is also orthogonal to A, for then, because E is an interpolation set, $\mu_E = 0$, whence the result by (d). The argument we give for this is a simplification due to S.J. Sidney of the proof of Varopoulos. (See [145].)

We will use the terminology that a compact set $S \subset X$ is a set of *type* $\varepsilon(c)$ if it is an interpolation set and if for each $f \in \mathscr{C}(S)$, there is $f^* \in A$ with $f^*|S = f$ and $\|f\|_X \leq c\|f\|_S$. Banach's open mapping theorem implies that each interpolation set is of type $\epsilon(c)$ for some c. A peak-interpolation set is a set of type $\epsilon(1)$.

Fix $\mu \in A^\perp$, and let $\eta > 0$ be given. By the regularity of μ, there is a compact set $K \subset X \setminus E$ with $|\mu|(X \setminus (E \cup K)) < \eta$. By assumption, each point of E is a peak point for A, so if $x \in E$ and V is a neighborhood of x, then there is $f \in A$ such that f peaks at x and $|f| < \eta$ on $X \setminus V$. Compactness yields a finite family $f_1, \dots, f_n$ of elements of A and a corresponding family of measurable sets $E'_1, \dots, E'_n$ such that $E = \cup_{j=1}^n E'_j$ and such that $|f_j - 1| < \eta$ on $E'j$, and $|f_j| < \eta$ on K. For each j, let $E_j \subset E'_j$ be a compact set such that

$$|\mu|(\cup_{j=1}^n E'_j \setminus E_j) < \eta.$$

The set E is of type $\varepsilon(c)$, so the same is true of the union $E_1 \cup \dots \cup E_n$. Accordingly, there exist functions $\varphi_j \in A$ with $\|\varphi_j\|_X \leq c$ and with $\varphi_j = e^{2\pi ijk/n}$ on X.

Define the function Φ_r by

$$\Phi_r = \left(\frac{1}{n}\sum_{j=1}^n e^{-2\pi ijr/n}\varphi_j\right)^2.$$

On the set E_r we have that $\Phi_j = \delta_{j,r}$, and therefore on E_r,

$$\left|\sum_{j=1'}^n \Phi_j f_j - 1\right| = |f_r - 1| < \eta.$$

This is correct for all r, and so the inequality is correct on the union $E_1 \cup \cdots \cup E_n$. Also,

$$\begin{aligned}\sum_{j=1}^{n} |\Phi_j| &= \frac{1}{n^2} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} e^{2\pi ijk/n} \varphi_k \sum_{r=1}^{n} e^{-2\pi ijr/n} \bar{\varphi}_r \right) \\ &= \frac{1}{n^2} \sum_{k,r=1}^{n} \sum_{j=1}^{n} e^{2\pi ij(k-r)/n} \varphi_k \bar{\varphi}_r \\ &= \frac{1}{n} \sum_{k=1}^{n} |\varphi_k|^2 \leq c^2.\end{aligned}$$

Finally, set $g = \sum_{j=1}^{n} \Phi_j f_j$, which is an element of A with the properties that $\|g\|_X \leq c^2$, that $|g - 1| \leq \eta$ on $E_1 \cup \cdots \cup E_n$, and that $|g| \leq \eta c^2$ on K.

In the preceding construction, $\eta > 0$ is at our disposal. By performing the construction for a sequence of η's that converges to zero, we obtain a corresponding sequence $\{g_n\}_{n=1,\ldots}$ of elements in A that is uniformly bounded and that tends pointwise a.e. $[d\mu]$ to the characteristic function of the set E. Each of the measures $g_n\mu$ lies in $A^\perp$, so the measure μ_E is also orthogonal to A, as we wished to show.

This completes the proof of (e) and the proof of Theorem 7.1.2.

The following corollary of the preceding theorem will be used below.

Corollary 7.1.3. *If the compact subset E of X is the union of a sequence $\{E_j\}_{j=1}^{\infty}$ of peak-interpolation sets, then E is a peak-interpolation set.*

Proof. If $\mu \in A^\perp$, then $|\mu|(E_j) = 0$ for each j, whence $|\mu|(E) = 0$.

Having the general notion of peak-interpolation set, one would like to have characterizations of peak-interpolation sets in various explicit situations. This is possible in the case of the unit disk in the complex plane, where the *Rudin–Carleson theorem* gives a characterization of the peak-interpolation sets for the disk algebra: *The compact subset E of the unit circle is a peak-interpolation set for the disk algebra $A(\mathbb{U})$ if and only E has zero length.* This result can be found, for example, in [178]. There are no nontrivial domains D in $\mathbb{C}^N$ with $N \geq 2$ for which there is known a characterization of the peak-interpolation sets for the algebra $A(D)$. In spite of this, there are some substantial results for interpolation on polydisks and on strictly pseudoconvex domains. For the former, we refer to [308]. For interpolation results for the ball algebra $A(\mathbb{B}_N)$, one can consult [310]. Results on strongly pseudoconvex domains generally parallel those on the ball.

In the work below on one-dimensional subvarieties of strictly pseudoconvex domains, we shall need to use a particular class of peak-interpolation sets on strictly pseudoconvex domains, a class determined by metric conditions. The result in question was found by Davie and Øksendal [93] and is the most general metric condition known to be sufficient for interpolation on strictly pseudoconvex domains.

Let $D \subset \mathbb{C}^N$ be a domain with boundary of class $\mathscr{C}^1$. For each point $p \in bD$, there is the complex tangent space $T_p^{\mathbb{C}}(bD)$, which is identified with the complex affine hyperplane through p that is contained in the real affine hyperplane through p and tangent to bD at p. This real hyperplane is identified with $T_p(bD)$. In $T_p(bD)$ there is the *real* orthogonal complement $T_p(bD) \ominus T_p^{\mathbb{C}}(bD)$ of $T_p^{\mathbb{C}}(bD)$. If N_p denotes the real line that is real orthogonal to bD at p, and if $N_p^{\mathbb{C}}$ is the complex line that contains it, then $T_p(bD) \ominus T_p^{\mathbb{C}}(bD)$ can be identified with $N_p^{\mathbb{C}} \cap T_p(bD)$. This space may be denoted by JN_p with J the complex structure on $\mathbb{C}^N$. There is then the real orthogonal decomposition

$$T_p(bD) = T_p^{\mathbb{C}}(bD) \oplus JN_p.$$

Let π_p'' and π_p' be the orthogonal projections of $\mathbb{C}^N$ onto $T_p^{\mathbb{C}}(bD)$ and JN_p, respectively. For $E \subset \mathbb{C}^N$, we introduce two numbers:

Definition 7.1.4. *The quantities $d_p''(E)$ and $d_p'(E)$ are defined by*

$$d_p''(E) = \text{diameter}\,\pi_p''(E) = \sup\{|w - w'|, w, w' \in \pi''(E)\}$$

and

$$d_p'(E) = \text{diameter}\,\pi_p'(E) = \sup\{|w - w'| : w, w' \in \pi'(E)\},$$

respectively.

Definition 7.1.5. *A subset E of bD is* null in the sense of Davie and Øksendal *if given $\epsilon > 0$, there is a sequence $\{V_j\}_{j=1}^{\infty}$ of open sets each of diameter less than ϵ such that for some $p_j \in V_j \cap E$,*

$$\sum_j d_{p_j}'(V_j) < \epsilon, \qquad \sum_j \left(d_{p_j}''(V_j)\right)^2 < \epsilon, \quad \text{and} \quad E \subset \bigcup_j V_j.$$

Thus, the condition is that the set E admit fine coverings with certain metric properties.

We have formulated the notion of being null in the sense of Davie and Øksendal for domains in $\mathbb{C}^N$. It is evident, though, that the condition is local and is independent of the particular choice of coordinates. Accordingly, the notion is meaningful for sets in the boundary of a domain in a complex manifold.

The theorem of Davie and Øksendal is the following.

Theorem 7.1.6. *If D is a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in a Stein manifold and if the compact set $E \subset bD$ is null in the sense of Davie and Øksendal, then it is a peak-interpolation set for $A(D)$.*

The gist of the matter is contained in the following construction.

Lemma 7.1.7. *Let $D \subset \mathbb{C}^N$ be a bounded, strictly convex domain with bD of class $\mathscr{C}^2$. If the compact subset E of bD is null in the sense of Davie and Øksendal, then given $\epsilon > 0$ and a neighborhood U of E, there is $f \in A(D)$ with $\|f\|_D \leq 2$, $|f| < \epsilon$ on $bD \setminus U$, and $f \mid E = 1$.*

Proof. Let Q be a strictly convex defining function of class $\mathscr{C}^2$ for D, so that $D = \{Q < 0\}$ and $dQ \neq 0$ on bD.

Define $H : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ by

$$H(z, w) = \sum_{j=1}^{N} \frac{\partial Q}{\partial z_j}(z)(z_j - w_j).$$

The function H is of class $\mathscr{C}^1$, and convexity yields the estimates that for some positive constants m and M,

$$m|z - w|^2 < \Re H(z, w) < M|z - w|^2$$

for $z \in bD$, $w \in \overline{D}$.

If $z = (z_1, \ldots, z_N)$ with $z_j = x_{2j-1} + ix_{2j}$, and if $w_j = u_{2j-1} + iu_{2j}$, then

$$\begin{aligned}\Im H(z, w) &= \frac{1}{2} \sum_{j=1}^{N} \left\{ -\frac{\partial Q}{\partial x_{2j}}(z)(x_{2j-1} - u_{2j-1}) + \frac{\partial Q}{\partial x_{2j-1}}(z)(x_{2j} - u_{2j}) \right\} \\ &= (J \operatorname{grad} Q(z), x - u),\end{aligned}$$

where $x = (x_1, \ldots, x_{2N})$, $u = (u_1, \ldots, u_{2N})$, J is the complex structure on $\mathbb{C}^N$, and $(\, , \,)$ denotes the real inner product on $\mathbb{R}^{2N} = \mathbb{C}^N$.

For $z \in bD$ and W_z a sufficiently small neighborhood of z, we have that for any subset V of W_z, if $z \in V$ and if $w \in V \cap bD$, then

$$|H(z, w)| \le |\Re H(z, w)| + |\Im H(z, w)| \le M|z - w|^2 + |(J \operatorname{grad} Q(z), x - u)|.$$

In this,

$$|z - w|^2 \le (d_z'(V))^2 + (d_z''(V))^2.$$

Also, we can write

$$x - u = \pi_z'(z - w) + \pi_z''(z - w) + O(|z - w|^2),$$

because $z, w \in bD$. The vector J grad Q is orthogonal to $T_z^{\mathbb{C}}(bD)$, so

$$\begin{aligned}|H(z, w)| &\le M[\big(d_z'(V)\big)^2 + \big(d_z''(V)\big)^2] + |\operatorname{grad} Q(z)| d_z'(V) \\ &\le \widetilde{M}[(d_z'(V))^2 + (d_z''(V))^2]\end{aligned}$$

for a suitable constant $\widetilde{M}$.

For $n > \widetilde{M}$, set $E_n = \left\{z \in E : \mathbb{B}_N(z, \tfrac{1}{n}) \subset W_z,\ m\,[\operatorname{dist}(z, bU)]^2 > \tfrac{1}{n}\right\}$. Then $E = \cup E_n$.

For every n, let $\{V_j^{(n)}\}_{j=1}^{\infty}$ be a sequence of open sets of diameter less than $\frac{1}{n}$ such that every point of E_n is contained in infinitely many of the $V_j^{(n)}$ and such that for a suitable choice of $z_{n,j} \in V_j^{(n)}$,

$$\sum_{j=1}^{\infty} \left\{ d'_{z_{n,j}}(V_j^{(n)}) + \left[d''_{z_{n,j}}(V_j^{(n)}) \right]^2 \right\} < \epsilon n^{-2} 2^{-n-2}.$$

Relabel the $V_j^{(n)}$ for $n > \widetilde{M}$, $j = 1, 2, \ldots,$ as $V_1, V_2, \ldots,$ and for all j, choose n_j such that $V_j = V_i^{n(j)}$ for some i, and set then $z_j = z_{n_{j,i}}$.

With $c_j = d'_{z_j}(V_j) + \left[d''_{z_j}(V_j)\right]^2$, set

$$B_r(w) = \prod_{j=1}^{r} \frac{H(z_j, w)}{2n_j c_j + H(z_j, w)}.$$

For $w \in \overline{D}$, we have $|B_r(w)| \leq 1$. If $w \in \overline{D} \setminus V$, then

$$\Re H(z_j, w) > m|z_j - w|^2 > \frac{1}{n_j},$$

so

$$\sum_{j=1}^{\infty} \left| 1 - \frac{H(z_j, w)}{2n_j c_j + H(z_j, w)} \right| \leq \sum_{j=1}^{\infty} \frac{2n_j c_j}{2n_j c_j + \frac{1}{n_j}} \leq \sum_{j=1}^{\infty} 2n_j^2 c_j < \frac{\epsilon}{2}.$$

This estimate implies that if $B(w) = \lim_{r\to\infty} B_r(w)$, then $|B(w) - 1| < \epsilon$ on $\overline{D} \setminus V$.[1]

Also, if $w \in \overline{D} \setminus E$, then

$$\sum_{j=1}^{\infty} \frac{2n_j c_j}{|2n_j c_j + H(c_j, w)|} \leq \frac{1}{m[\text{dist}\,(E, w)]^2} \sum_{j=1}^{\infty} 2n_j c_j.$$

The series on the right converges, so $\lim_{r\to\infty} B_r(w)$ exists at every point of $\overline{D} \setminus E$, and $B \in \mathscr{C}(\overline{D} \setminus E)$. Finally, $\lim_{r\to\infty} B_r(w) = 0$ if $w \in E$, for w lies in infinitely many of the V_j. If $z \in V_j$, then because $V_j \subset W_{z_{n,j}}$, and therefore

$$\frac{|H(z_j, w)|}{|2n_j c_j + H(z_j, w)|} \leq \widetilde{M} \frac{c_j}{2n_j c_j} < \frac{1}{2},$$

we have $\lim_{r\to\infty} B_r(w) = 0$ as claimed. For the function f of the lemma, we take $1 - B$. The lemma is proved.

We now show how to deduce Theorem 7.1.6 from the result on convex domains that we have established. This deduction involves the solution of a Cousin I problem.

Proof of Theorem 7.1.6. Let $\mathscr{M}$ be a Stein manifold, let $D \subset \mathscr{M}$ be a relatively compact strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$, and let $E \subset bD$ be a compact subset that is null in the sense of Davie and Øksendal. To prove that E is a peak-interpolation set for $A(D)$, we shall show that if $p \in E$, then there is a neighborhood U of p in bD with the property that if μ is a measure on D orthogonal to $A(D)$, then for every compact subset K of U, $\mu(E) = 0$. This means that the variation of $|\mu|$ over U is zero, so the

[1] Given $\sum_{j=1}^{\infty} |1 - b_j| < \frac{\epsilon}{2}$, write $b_j = 1 + \delta_j$, so that $\Sigma|\delta_j| < \frac{\epsilon}{2}$. Then,

$$\left| \left(\prod_{j=1}^{\infty} b_j \right) - 1 \right| = \left| \left(\prod_{j=1}^{\infty} 1 + \delta_j \right) - 1 \right| \leq e^{\Sigma \delta_j} - 1 < e^{\epsilon/2} - 1 < \epsilon$$

when ϵ is small.

support of μ is disjoint from U. Finitely many of the neighborhoods U cover the set E, so $|\mu|(E) = 0$, and E is recognized to be a peak-interpolation set.

Let Q be a strictly plurisubharmonic defining function for D, which we can take to be defined on all of $\mathscr{M}$, so that $bD = \{Q = 0\}$, $dQ \neq 0$ on bD, and $D = \{Q < 0\}$. For small $\varepsilon > 0$, we let D_ε be the domain $\{Q < \varepsilon\}$, which is also a Stein domain.

Fix attention on a point $p \in E$. The strict pseudoconvexity of D implies the existence of an open set Ω in $\mathscr{M}$ that contains p and on which there are holomorphic coordinates $z_1, \dots, z_N$ with respect to which the domain $\Delta = \Omega \cap D$ is strictly convex with boundary of class $\mathscr{C}^2$.

Let B_p and B'_p be concentric balls in the z-coordinate system centered at p with $\bar{B}_p \subset B'_p$ and with $B'_p \cap bD \subset bD \cap b\Omega$.

Let Δ' be a strictly convex domain with $b\Delta' \supset bD \cap \bar{B}'_p$ and with $\Delta' \supset b\Delta \setminus (bD \cap \bar{B}_p)$. Thus, Δ' is obtained from Δ by pushing the boundary of Δ outward a little along $b\Delta \setminus \bar{B}_p$.

The set E is null in the sense of Davie and Øksendal, so the same is true of the subset $E \cap \bar{B}'_p$. Consequently, if $K \subset E \cap B'_p$ is a compact set, there is a function $g \in A(\Delta')$ with $g = 0$ on K and $\Re g > 0$ on $\bar{\Delta}' \setminus K$. A branch of $\ln g$ is defined and continuous on $\bar{\Delta}' \setminus K$ and satisfies the condition that $\Re \ln g \to -\infty$ at points of K.

If $\varepsilon > 0$ is small enough, then $\ln g$ is defined and holomorphic on $D_\varepsilon \cap (B'_p \setminus \bar{B}_p)$. The sets $D^-_\varepsilon = D_\varepsilon \cap B'_p$ and $D^+_\varepsilon = D_\varepsilon \setminus \bar{B}_p$ constitute an open cover for D_ε. Because the additive Cousin problem is solvable on the domain D_ε, there are functions $h^+ \in \mathscr{O}(D^+_\varepsilon)$ and $h^- \in \mathscr{O}(D^-_\varepsilon)$ with $\ln g = h^+|(B'_p \setminus \bar{B}_p) - h^-|(B'_p \setminus \bar{B}_p)$. Accordingly, we obtain a well-defined function g_1 on D by setting

$$g_1 = \begin{cases} \ln g + h^- & \text{on } D \cap B'_p, \\ h^+ & \text{on } D \setminus \bar{B}_p. \end{cases} \tag{7.1}$$

This function g_1 is holomorphic on D and is continuous on $\bar{D} \setminus K$. In addition, $\Re g_1 \to -\infty$ at points of K. Moreover, $\Re g_1$ is bounded above, say $\Re g_1 < M$ for some $M > 0$. The function $g_2 = g_1/(g_1 - M)$ is holomorphic on D, is continuous on $bD \setminus K$, tends continuously to the value 1 at K, and has modulus strictly less than 1 at all points of $\bar{D} \setminus K$. Thus, the function g_2 is in $A(D)$ and peaks on K.

Consequently, if μ is a measure orthogonal to $A(D)$, then

$$0 = \int g_2^k \, d\mu \underset{k \to \infty}{\to} \mu(K).$$

We thus have that for every compact subset K of $B_p \cap bD$ and for every measure μ orthogonal to $A(D)$, $\mu(K) = 0$. It follows that the set E is a peak-interpolation set, as we were to prove.

As an example, we have the following.

Corollary 7.1.8. *If the compact subset E of bD satisfies $\Lambda^1(E) = 0$, then E is a peak-interpolation set.*

Proof. To say that E is a set of zero length means that given $\delta, \epsilon > 0$, it is possible to cover E by a sequence of balls $B_1, \ldots$, each of which has diameter less than δ and the sum of whose diameters is no more than ϵ. Such a set is plainly null in the sense of Davie and Øksendal.

A classical theorem of F. and M. Riesz states that *if the finite regular Borel measure μ on the unit circle is orthogonal to the disk algebra, then μ is absolutely continuous with respect to the arc-length measure on the circle.* For this, one can consult [178]. The theorem of Davie and Øksendal implies corresponding results on strictly pseudoconvex domains:

Theorem 7.1.9. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$, and let μ be a finite regular Borel measure with support in bD. The measure μ is absolutely continuous with respect to the one-dimensional Hausdorff measure Λ^1 on $\mathbb{C}^N$ if μ is orthogonal to the algebra $A(D)$ or is a representing measure for a point in D.*

Proof. By regularity it suffices to show that for compact sets E of zero length, $\mu(E) = 0$. For such an E, Theorem 7.1.6 provides a $\varphi \in A(D)$ that peaks on E. Because $\mu \in A(D)^{\perp}$, we have that $0 = \int \varphi^n d\mu \to \mu(E)$. Similarly, if μ is a representing measure for $p_0 \in D$, so that $\int g\, d\mu = g(p_0)$ for all $g \in A(D)$, then $\mu(E) = 0$: We can choose a peak function g for E with $g(p_0) = 0$. Then again $0 = \int g^n\, d\mu \to \mu(E)$.

Certain curves in bD are peak-interpolation sets. In this connection it is convenient to introduce the following definition.

Definition 7.1.10. *If D is a domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^1$, the submanifold Σ of bD is said to be* complex-tangential *if it is of class $\mathscr{C}^1$ and if, in addition, at each point $p \in \Sigma$, the tangent space $T_p(\Sigma)$ is contained in the complex tangent space $T_p^{\mathbb{C}}(bD)$. A submanifold Σ of bD is said to be* complex-transverse *if for no point $p \in \Sigma$ is $T_p(\Sigma)$ contained in $T_p^{\mathbb{C}}(bD)$.*

Theorem 7.1.9 is sufficient to show that smooth curves in strictly pseudoconvex boundaries that are complex-tangential are peak-interpolation sets, for such a curve is easily seen to be null in the sense of Davie and Øksendal.

Theorem 7.1.11. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$, and let $\gamma : \mathbb{R} \to bD$ be a curve of class $\mathscr{C}^1$. If $K \subset \mathbb{R}$ is a compact set with $\gamma'(t) \in T^{\mathbb{C}}_{\gamma(t)}(bD)$ for almost every $[d\Lambda^1]$ $t \in K$, then $\gamma(K)$ is a peak-interpolation set for $A(D)$.*

In particular, if γ' is always complex tangential, then every compact subset of $\gamma(\mathbb{R})$ is a peak-interpolation set.

Proof. We will show that $\gamma(K)$ is null in the sense of Davie and Øksendal, which will imply the result. Fix $\epsilon > 0$. For $t_0 \in K$,

$$\gamma(t) = \gamma(t_0) + \gamma'(t_0)(t - t_0) + o(|t - t_0|)$$

with the o-term uniform in t and t_0 (when both are restricted to lie in K). Consequently, if L is a short interval centered at $t_0 \in K$, then the projection of $\gamma(L \cap K)$ into $T^{\mathbb{C}}_{\gamma(t_0)}$ has diameter bounded by C.length L for some constant C. Also, the diameter of the projection of $\gamma(L \cap K)$ into $JN_{\gamma(t_0)}$ is $o(\text{length}\, L)$, for $\gamma'(t_0) \in T^{\mathbb{C}}_{\gamma(t_0)}$. Thus if $K = (L_1 \cap K) \cup \cdots \cup (L_s \cap K)$, where the L_j are short intervals with mutually disjoint interiors, then for points $t_j \in L_j \cap K$,

$$\sum_j \{d'_{\gamma(t_j)}\gamma(L_j \cap K) + (d''_{\gamma(t_j)}\gamma(L_j \cap K))^2\}$$
$$\leq \sum_j \{\epsilon \cdot \text{const}(\text{length}\, L_j) + \text{const}(\text{length}\, L_j)^2\}$$

with constants that are independent of the partition. The inequality

$$\sum_j (\text{length}\, L_j) < \Lambda^1(K) + \delta,$$

which is correct for arbitrarily small positive δ if the intervals L_j are chosen appropriately, implies that the quantity $\sum_j \{\cdots\}$ can be made as small as we wish by taking the L's short.

The curve $\gamma = \{(e^{it}, 0) : t \in [-\pi, \pi]\} \subset b\mathbb{B}_2$ is not complex-tangential, and is not a peak-interpolation set. (It is the boundary of the analytic disk $\Delta = \{(z, 0) : |z| < 1\}$ in $b\mathbb{B}_2$ and so is not the zero set of any $f \in A(\mathbb{B}_2)$.) One can verify directly that γ is not null in the sense of Davie and Øksendal.

There is a general result that shows complex-tangential manifolds of whatever dimension to be peak-interpolation sets.

Theorem 7.1.12. *If D is a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$ and if Σ is a closed submanifold of bD of class $\mathscr{C}^1$ that is complex-tangential, then Σ is a peak-interpolation set for $A(D)$.*

We will not prove this theorem here; it and the general circle of ideas around it are discussed in [310]. The converse is also true, i.e., interpolation manifolds of class $\mathscr{C}^1$ are necessarily complex tangential.

The following convexity observation arises naturally in the present context. Suppose D to be a strictly pseudoconvex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$. There is then a neighborhood W of $\bar{D}$ on which is defined a strictly plurisubharmonic function Q that is a defining function for D in that $D = \{z \in W : Q(z) < 1\}$ and $dQ \neq 0$ on bD. For $c, c' \in \mathbb{R}$ with $1 \leq c < c' \leq c_o$ for a sufficiently small c_o, the domain $D_c = \{z \in W : Q(z) < c\}$ is a Runge domain in D_{c_o}. If $E \subset bD$ is a peak-interpolation set for $A(D)$, then the set E is convex with respect to the algebra $\mathscr{O}(D_c)$ for c slightly larger than 1. This is so, for granted that E is a peak-interpolation set for $A(D)$, it is, in particular, a peak set for this algebra. Accordingly, it is convex with respect to the algebra $A(D)$: By hypothesis, there is a function $g \in A(D)$ with $g|E = 1$ and $|g| < 1$ on $\bar{D} \setminus E$. There exist such peak functions that take values only in the half-plane $\Re\zeta > \frac{1}{3}$, for if g is any peak function for E, then so is $\frac{1}{3}(2 + g)$. For such a g, the function $1/g$ lies in $A(D)$, assumes the value 1 on E, and assumes values of modulus greater than 1 on $\bar{D} \setminus E$. Thus, E is convex with respect to the algebra $A(D)$ and so with respect to the algebra $\mathscr{O}(\bar{D})$ of functions holomorphic on neighborhoods of D. (The latter conclusion depends on the approximation result, which we do not prove here, that for a strictly pseudoconvex domain D, $\mathscr{O}(\bar{D})$ is uniformly dense in $A(D)$. For this result one can consult the book of Range [287].) A function holomorphic on a neighborhood of $\bar{D}$ can be approximated by functions holomorphic in D_{c_o}, so, as claimed, the set is $\mathscr{O}(D_{c_o})$-convex. In particular, we have the following:

Corollary 7.1.13. *If E is a peak-interpolation set in $b\mathbb{B}_N$ or in the boundary of a bounded, strictly convex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, then E is polynomially convex.*

As a special case, we consider rectifiable curves in the boundary.

Corollary 7.1.14. *If γ is a rectifiable curve in bD, D a bounded, strictly pseudoconvex domain with bD of class $\mathscr{C}^2$, and if at almost every $[d\Lambda^1]$ point p of γ the tangent to γ is complex-tangential, then γ is a peak-interpolation set for $A(D)$ and is convex with respect to the algebra of functions holomorphic on $\bar{D}$.*

Proof. To see that γ is a peak-interpolation set, consider a measure μ on bD that is orthogonal to $A(D)$. We shall show that the variation of μ over γ is zero. We know μ to be absolutely continuous with respect to Λ^1.

Let $T \subset \gamma$ be the set of points at which γ does not have a tangent or at which the tangent is not complex tangential. The set T is of zero length, so $|\mu|(T) = 0$.

The rectifiability of the curve γ implies that there is a sequence $\{\gamma_j\}_{j=1,\dots}$ of $\mathscr{C}^1$ arcs with the property that the set $S = \gamma \setminus \cup_{j=1,\dots}\gamma_j$ has length zero. Thus, $|\mu|(S) = 0$. Fix a j. If $K \subset \gamma_j \setminus T \cap \gamma$ is a compact set with no isolated points, then at each point of K, the tangent to γ_j is the same as the tangent to γ and so is complex tangential there. Thus, by Theorem 7.1.11, K is a peak-interpolation set, whence $|\mu|(K) = 0$.

Thus, as desired, the variation of the measure μ on γ is zero, so γ is a peak-interpolation set, and the corollary is proved.

A stronger convexity result is proved below in Corollary 7.1.18.

As we have remarked, with D strictly pseudoconvex with boundary of class $\mathscr{C}^2$, each compact submanifold Σ of bD that is complex-tangential is a peak-interpolation set, and so, by Corollary 7.1.13, convex with respect of the algebra $\mathscr{O}(\bar{D})$. In the ball case, complex-tangential submanifolds of the boundary are polynomially convex. For curves of class $\mathscr{C}^2$, a single point of complex tangency suffices to guarantee convexity, as shown by Forstnerič [125]:

Theorem 7.1.15. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and let γ be a simple closed curve of class $\mathscr{C}^2$ in bD. If there is a point in γ at which γ is complex-tangential, then γ is convex with respect to $\mathscr{O}(\bar{D})$.*

This is a theorem about curves of class $\mathscr{C}^2$: Rosay [296] has given an example to show that if γ is assumed to be only of class $\mathscr{C}^1$, the conclusion of the theorem can fail.

Proof. [296] By the embedding theorem of Fornæss and Henkin, Theorem 2.4.6, we can suppose D to be a bounded convex domain. Let $\gamma \subset bD$ be a simple closed curve of class $\mathscr{C}^2$ with the property that for some point $p \in \gamma$ the tangent $T_p\gamma$ is contained in the space $T_p^{\mathbb{C}}(bD)$. We shall show γ to be polynomially convex.

For this purpose, let $g : [-1, 1] \to bD$ be a $\mathscr{C}^2$ parameterization of γ with $g(0) = p$ and with $g(-1) = g(1)$.

As a matter of notational convenience, we suppose that $D \subset \mathbb{B}_N$ and that the point p is the point $(1, 0, \dots, 0)$. We can also assume that $g'(0) = (0, 1, 0, \dots, 0) \in T^{\mathbb{C}}_{(1,0,\dots,0)}(bD)$. Let $g = (g_1, \dots, g_N)$.

For $\varepsilon \in \mathbb{R}$ define $\varphi_\varepsilon : [-1, 1] \to \mathbb{C}$ by $\varphi_\varepsilon(t) = g_1(t) + i\varepsilon g_2(t)$, a planar curve of class $\mathscr{C}^2$.

Lemma 7.1.16. *There are arbitrarily small* $\varepsilon \in \mathbb{R}$, *points* $t_o \in (0, 1)$, *and* $h > 0$ *such that*

(a) $\Re\varphi_\varepsilon(t) \leq 1 - h$ *when* $t_o \leq |t| \leq 1$,

(b) $\frac{d}{dt}\Re\varphi_\varepsilon(t) > 0$ *when* $-t_o \leq t < 0$,

(c) $\frac{d}{dt}\Re\varphi_\varepsilon(t) < 0$ *when* $0 < t \leq t_o$, *and*

(d) *the set* $\varphi_\varepsilon([-1, 1])$ *meets the line* $\Re\zeta = 1 - h$ *in only one point.*

Proof. Write the Taylor expansion of φ_ε about $0 \in \mathbb{R}$:

$$\varphi_\varepsilon(t) = \varphi_\varepsilon(0) + \varphi'_\varepsilon(0)t + \frac{1}{2}\varphi''_\varepsilon(0)t^2 + o(t^2).$$

We have that $\varphi_\varepsilon(0) = 1$ and $\varphi'_\varepsilon(0) = i\varepsilon$. We can obtain some information about the second-order derivatives as follows. Introduce real coordinates on $\mathbb{C}^N$ by $z_j = x_{2j-1} + ix_{2j}$, and let $g_j(t) = \beta_{2j-1}(t) + i\beta_{2j}(t)$.

Let Q be a strictly convex defining function of class $\mathscr{C}^2$ for the domain D normalized so that $Q_{x_1}(1, 0, \dots, 0) = 1$. Differentiation of the equation $Q(g(t)) = 0$ twice at $t = 0$ leads to $\beta''_1(0) = -q$ with q the number $Q_{x_3x_3}(g(0))$, which is positive, because Q is strictly convex. Thus, the Taylor expansion of φ_ε about 0 is

$$\varphi_\varepsilon(t) = \left\{1 - \frac{1}{2}qt^2 + o(t^2) + \varepsilon O(t^2)\right\} + i\varepsilon\big\{t + O(t^2)\big\}.$$

We now see that there are $\varepsilon_o > 0$ and $t_o \in (0, 1)$ such that for $|\varepsilon| < \varepsilon_o$ and $|t| < t_o$, $\Re\varphi_\varepsilon$ is strictly increasing on $(-t_o, 0)$ and strictly decreasing on $(0, t_o)$. Fix $h_o > 0$ small enough that $\Re g_1(t) < 1 - 2h_o$ if $|t| > t_o$. If $|\varepsilon| < h_o$, then

$$\Re\varphi_\varepsilon(t) = \Re\{g_1(t) + i\varepsilon g_2(t)\} < 1 - h_o.$$

Put $\varepsilon_1 = \min(\varepsilon_o, h_o)$. For $h \in (0, h_o)$ and $|\varepsilon| < \varepsilon_o$, each of the curves $\varphi_\varepsilon([-t_o, 0))$ and $\varphi_\varepsilon((0, t_o])$ meets the line $\Re\zeta = 1 - h$ in one and only one point, say $1 - h + ip^-(t)$ and $1 - h + ip^+(t)$, respectively. We have $\frac{d}{dt}\Im\varphi_\varepsilon(t)\big|_{t=0} = \varepsilon > 0$, so $p^-(\varepsilon_1) < 0 < p^+(\varepsilon_1)$, and $p^-(-\varepsilon_1) > 0 > p^+(-\varepsilon_1)$. Thus, by continuity, there is an $\varepsilon_2 \in (-\varepsilon, \varepsilon)$ such that $p^+(\varepsilon_2) = p^-(\varepsilon_2)$.

The lemma is proved.

Proof of Theorem 7.1.15 concluded. Fix ε and h such that the curve φ_ε meets the line $\Re\zeta = 1 - h$ in exactly one point, say ζ_o. Let γ^+ be $\gamma \cap \varphi_\varepsilon^{-1}(\{\Re\zeta \geq 1 - h\})$ and

$\gamma^- = \gamma \cap \varphi_\varepsilon^{-1}(\{\Re\zeta \leq 1 - h\})$. The arcs γ^+ and γ^- are polynomially convex and satisfy $\mathscr{P}(\gamma^\pm) = \mathscr{C}(\gamma^\pm)$. Kallin's lemma, Theorem 1.6.19, implies that γ is polynomially convex and satisfies $\mathscr{P}(\gamma) = \mathscr{C}(\gamma)$. The theorem is proved.

Recall that for a subvariety V of a domain in $\mathbb{C}^N$, bV denotes the set $\bar{V} \setminus V$.

Theorem 7.1.17. [22] *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and let V be an irreducible one-dimensional subvariety of D. If B is an open subset of bV with $\Lambda^1(B) < 0$, then at almost every point of B, the tangent to bV is complex transverse.*

Note that for a point $z \in V$, the set bV is minimal with respect to the condition $z \in \widehat{bV}$, so the set B is of class $\mathscr{A}_1$ as remarked immediately after the proof of Theorem 4.3.7, and at almost every point its tangent is a real line.

Proof. Suppose the statement is false. As a set of class $\mathscr{A}_1$, the set B is the union of a set B_o of length zero and a countable collection $B_1, \ldots$ of sets B_j each of which is contained in an arc of class $\mathscr{C}^1$. Thus, the assumption implies the existence of an arc of class $\mathscr{C}^1$, say λ, such that the set $S_o = \lambda \cap B$ has positive length and at each point p of this intersection the tangent to λ is transverse to $T_p^{\mathbb{C}}(bD)$. Each compact subset of S_o is, therefore, a peak-interpolation set. Let S be a compact subset of S_o of positive length. Because the set S is a peak-interpolation set for the algebra $A(D)$, there exists a function $F \in A(D)$ with $F = 0$ on S and F zero-free on $\bar{D} \setminus S$. This leads to the following contradiction.

Fix a small ball $\mathbb{B}_N(p, \delta)$ in $\mathbb{C}^N$ centered at a point $p \in S$ of metric density for the restriction of Λ^1 to S. By Lemmas 4.3.10 and 4.3.11 there is a linear functional π on $\mathbb{C}^N$ that carries the set $S \cap \mathbb{B}_N(p, \delta)$ to a subset of the plane of class $\mathscr{A}_1$, that carries the set $S \cap \mathbb{B}_N(p, \delta)$ onto a set of positive length, and that is constant on no branch of $V \cap \mathbb{B}_N(p, \delta)$. By Theorem 4.3.14 there is a component Δ of $\pi(V \cap \mathbb{B}_N(p, \delta))$ for which the set $T = b\Delta \cap \pi S$ is a set of uniqueness for subharmonic functions on Δ. The projection π carries $\pi^{-1}(\Delta) \cap V$ properly onto Δ as a q-fold branched cover for some positive integer q.

There is a unique bounded holomorphic function h on Δ with the property that for all $\zeta \in \Delta$, $h(\zeta) = \prod F(w)$, where the product extends over all the points $w \in \pi^{-1}(\Delta \cap V)$, each w counted according to its multiplicity. The function h tends to zero at the points of T, and so h is identically zero. This implies that the function F vanishes at some points of V, a contradiction to the construction of F as a function that vanishes only on the set S, a subset of bD.

The theorem is proved.

Corollary 7.1.18. *If γ is a rectifiable simple closed curve in bD, D a strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$, and if γ is complex tangential at all points of a set of positive length, then γ is $\mathscr{O}(\bar{D})$ convex.*

Proof. Suppose D to be strictly convex, in which case we are to prove γ to be polynomially convex. If γ is not polynomially convex, then $\widehat{\gamma} \setminus \gamma$ is a purely one-dimensional subvariety of D. If V is an irreducible branch of $\widehat{\gamma} \setminus \gamma$, then $bV = \gamma$, and the preceding theorem implies that at almost every $[d\Lambda^1]$ point of γ, γ is complex transverse. Contradiction. (Note that $\widehat{\gamma} \setminus \gamma$ has but one branch by Theorem 4.5.5. That rather involved result is not needed here.) If D is only strictly pseudoconvex rather than convex, the proof can be concluded by invoking the Fornæss–Henkin embedding theorem.

7.2. Boundary Regularity

This section is concerned with the regularity at the boundary of one-dimensional varieties in strictly pseudoconvex domains.

We begin with certain regularity questions for hulls of *smooth* curves. This investigation depends ultimately on boundary regularity results for functions of one variable. In particular, the following theorem is essential:

Theorem 7.2.1. *Let D be a domain in $\mathbb{C}$, let $p \geq 1$ be an integer, and let λ be an open arc of class $\mathscr{C}^p$ in bD that is an open subset of bD. Let g be holomorphic in D and assume continuous boundary values along λ. If $g|\lambda \in \mathscr{C}^p(\lambda)$, then the derivatives of g of order not more than p extend to be continuous on $D \cup \lambda$.*

Proof. To begin with, we can suppose that bD is of class $\mathscr{C}^p$ and that $g \in A(D)$. This is so, for the problem is local along λ. If the condition is not satisfied, then fix a compact subset K of λ, and let $D' \subset D$ be a domain with boundary of class $\mathscr{C}^p$ such that $bD \cap bD'$ contains a neighborhood in λ of the set K. If we prove the theorem for $g|D'$ and with λ replaced by the interior of the set $\lambda \cap bD'$, we shall be done.

Thus, the function g has the representation as a Cauchy integral: If G is the Cauchy integral of g, then

$$G(z) = \frac{1}{2\pi i} \int_{bD} \frac{g(\zeta)d\zeta}{\zeta - z} = \begin{cases} g(z) & \text{if } z \in D, \\ 0 & \text{if } z \in \mathbb{C} \setminus \bar{D}. \end{cases}$$

Write

$$G(z) = G_1(z) + \frac{1}{2\pi i} \int_{\lambda} \frac{g(\zeta)d\zeta}{\zeta - z},$$

so that the function G_1 is holomorphic on $\mathbb{C} \setminus (bD \setminus \lambda)$. Consequently,

$$G'(z) = G_1'(z) + \frac{1}{2\pi i} \int_{\lambda} \frac{g(\zeta)d\zeta}{(\zeta - z)^2}. \tag{7.2}$$

Integration by parts shows that for $z \notin bD$,

$$G'(z) = G_1'(z) + \frac{1}{2\pi i} \int_{\lambda} \frac{g_1(\zeta)d\zeta}{(\zeta - z)} + \frac{1}{2\pi i} \frac{g(e^+)}{(e^+ - z)} - \frac{1}{2\pi i} \frac{g(e^-)}{(e^- - z)}, \tag{7.3}$$

in which the points e^+ and e^- are the endpoints of λ and g_1 is a function continuous on λ. Denote the sum of these two boundary terms by $G_2(z)$, so that G_2 is holomorphic on the Riemann sphere except for poles at e^+ and e^-. We now have that

$$G_1'(z) + G_2(z) + \frac{1}{2\pi i} \int_{\lambda} \frac{g_1(\zeta)d\zeta}{\zeta - z} = \begin{cases} g'(z) & \text{if } z \in D, \\ 0 & \text{if } z \in \mathbb{C} \setminus \bar{D}. \end{cases}$$

It is a result from classical function theory, which is contained in Theorem 5.2.7, that the jump of the integral $G_3(z) = \frac{1}{2\pi i}\int_\lambda \frac{g_1(\zeta)d\zeta}{\zeta - z}$ across λ is the function g_1, uniformly on compact subsets of λ. Now if p is a point in λ and p_ε^+ and p_ε^- are the points at distance ε from p along the normal at p to λ with $p_\varepsilon^- \in D$ and $p_\varepsilon^+ \notin D$, then for small $\varepsilon > 0$,

$$g'(p_\varepsilon^-) = \{G_1'(p_\varepsilon^-) + G_2(p_\varepsilon^-) + G_3(p_\varepsilon^-)\} - \{G_1'(p_\varepsilon^+) + G_2(p_\varepsilon^+) + G_3(p_\varepsilon^+)\}. \tag{7.4}$$

Thus $\lim_{\varepsilon\to 0^+} g'(p_\varepsilon^-) = g_1(p)$, uniformly on compacta in λ, and g' is seen to extend to be continuous on $D \cup \lambda$. If $p = 1$, we are done. Otherwise, iterate the process p times to find that the pth derivative of g extends continuously to λ.

The theorem is proved.

A geometric version of the preceding theorem was given by Chirka [84].

Theorem 7.2.2. *Let D be a domain in $\mathbb{C}^N$, and let λ' be an arc of class $\mathscr{C}^p$, $p \geq 1$, in $\mathbb{C}^N$ with endpoints in bD and interior, λ, in D. If V is a purely one-dimensional subvariety of $\Omega \setminus \lambda$, then either $\bar{V} \cap D$ is a purely one-dimensional subvariety of D or else there is a set E contained in λ with $\Lambda^1(E) = 0$ and with the property that near points of $\lambda \setminus E$, the set $V \cup (\lambda \setminus E)$ has the structure of a two-dimensional manifold of class $\mathscr{C}^p$ with boundary. In the latter case, V is nonsingular near $\lambda \setminus E$.*

A simple example shows the necessity of considering the exceptional set E. Let f and g be polynomials in one variable such that $f(0) = f(1) = g(0) = g(1) = 0$, such that df is zero-free on $|z| = 1$, and such that f and g jointly separate points on $|z| = 1$. If γ is the image of the unit circle under the map $\varphi = (f, g)$, then γ is a real-analytic simple closed curve in $\mathbb{C}^2$ with hull $\varphi(\bar{\mathbb{U}})$. This set is neither a variety nor a two-dimensional manifold with boundary in a neighborhood of the origin in $\mathbb{C}^2$.

Proof. The problem is entirely local, so we suppose $0 \in \lambda$. Because the set $V \cup \lambda$ has vanishing 3-dimensional measure, by Theorem 3.8.8 there are many sets of clear coordinates at the origin for the set $V \cup \lambda$. Choose such a set of coordinates that has the property that if π is the orthogonal projection of $\mathbb{C}^N$ onto the z_1-axis, then π carries a neighborhood of 0 in λ diffeomorphically onto a curve γ of class $\mathscr{C}^p$ through the origin. Require also that π be constant on no global branch of V. We are working with clear coordinates with respect to the set $V \cup \lambda$, so there is a neighborhood U of the origin with the property that π carries $\bar{V} \cap U$ properly onto the subset $\pi(\bar{V} \cap U)$ of the disk U_1 centered at the origin. By shrinking further if necessary, we can suppose that the curve γ separates the disk U_1 into two domains U_1' and U_1''. Let $V' = V \cap U \cap \pi^{-1}(U_1')$ and $V'' = V \cap U \cap \pi^{-1}(U_1'')$. Then V' and V'' are analytic covers over U_1' and U_1'', respectively, say of multiplicities m' and m''. The labeling can be chosen so that $m' \geq m''$. Not both of m' and m'' can be zero, because π is not constant on any global branch of V. For every $p \in \lambda$, $\pi^{-1}(p) \cap V$ is a discrete, possibly empty, subset of V, again because π is constant on no branch of V. Thus, if $p \in \lambda$ and $a \in \pi^{-1}(p) \cap V$, there is a neighborhood W_a of a in V such that π exhibits W_a as an analytic cover over the set $\pi(W_a)$.

Assume now that $N = 2$. The variety V' is then the zero locus of a Weierstrass polynomial with coefficients holomorphic on U_1'. That is to say, there is a function P_1

holomorphic on $U_1' \times \mathbb{C}$ of the form

$$P_1(z_1, z_2) = z_2^{m'} + b_1(z_1)z_2^{m'-1} + \cdots + b_{m'-1}z_2 + b_{m'}(z_1)$$

such that V' is the variety on which P_1 vanishes. If V'' is not empty, there is a similar polynomial P_2 of degree m'' whose zero locus is V''. The coefficients b_k of P_1 are bounded holomorphic functions on the domain U_1', and as such, they have nontangential limits at almost every point of the curve λ. If Δ' is the discriminant of the polynomial P_1, then Δ' is a bounded holomorphic function on U_1' that vanishes on the set of points over which there are fewer than m' points in V. The function Δ' does not vanish identically, so it has nonzero nontangential limits at almost every point of λ. Denote by E' the subset of λ consisting of the endpoints of λ together with the points at which Δ' fails to have a nontangential limit or else where this limit is zero. In case that $m'' \neq 0$, we perform a similar construction over U_1' and get an associated subset E'' of λ. Let $E = E' \cup E''$, again a set of zero length.

Fix attention on a point $p \in \lambda \setminus E$. There are points $p_j \in U_1'$ such that $p_j \to p$ and $\Delta'(p_j) \to \Delta'(p) \neq 0$. By passing to a subsequence of the p_j if necessary, we can suppose that the polynomials $P_1(p_j, z_2)$, each of which is a monic polynomial of degree m' in z_2, converge to a polynomial $P_1(p, z_2)$.

The zeros of the polynomial $P_1(p, \cdot)$ are the z_2-coordinates of the points lying over p in $\bar{V} \cap U$. There are m' such points. Let $\pi^{-1}(p) \cap \bar{V} \cap U = \{p_1, \ldots, p_{m'}\}$. Choose mutually disjoint neighborhoods W_j of the points p_j contained in U such that π carries $W_j \cap V'$ properly onto the open set $\pi(W_j \cap V')$ in the plane. Those points p_j that do not lie in λ lie in V; they do not concern us. The multiplicity of $\pi|(W_j \cap V')$ must be one. If the point p_j lies in λ, it is the limit of a sequence in V'. The domain $\pi(W_j \cap V')$ abuts λ along an open interval J containing p, and $W_j \cap V'$ is the graph of a function f_j holomorphic on this domain that assumes continuous boundary values along J. These boundary values are of class $\mathscr{C}^p$, for the graph of f_1 over the boundary is of class $\mathscr{C}^p$. It follows that near p the set $\lambda \cup V'$ is a $\mathscr{C}^p$ manifold with boundary.

If the multiplicity m'' is not zero, a similar analysis can be carried out to show that for points $p \in \gamma \setminus E$, if $p \in \pi^{-1}(p) \cap U \cap \bar{V}$, then either $p \in V$, or else the set $\lambda \cup V''$ is a $\mathscr{C}^p$ manifold with boundary near p.

Consider now a point q in λ such that both $\lambda \cup V'$ and $\lambda \cup V''$ are $\mathscr{C}^p$ manifolds with boundary. By construction there is a neighborhood W of $\pi(q)$ in $\mathbb{C}$ on which there is a function h of class $\mathscr{C}^p$ that is holomorphic on $W \cap (U_1' \cup U_1'')$ and whose graph is a neighborhood in $\bar{V}$ of q. The function h continues holomorphically into all of W by Morera's theorem, so in this case, q is a regular point of V.

The theorem is proved in the case $N = 2$.

The N-dimensional case is deduced from the 2-dimensional case as follows. We have the projection π as in the first paragraph of the proof and the sets U, U_1', U_1'' and the varieties V' and V'' with the associated multiplicities m' and m'', $m' \geq m''$, not both zero.

Let ℓ be a linear functional on $\mathbb{C}^N$ with the property that for a choice of $p' \in U_1'$ and $p'' \in U_1''$, if $m'' \neq 0$, the map ℓ carries the set $E = (\pi^{-1}(p') \cap V') \cup (\pi^{-1}(p'') \cap V'')$ injectively into $\mathbb{C}$.

Consider the map $\phi : \mathbb{C}^N \to \mathbb{C}^2$ given by

$$\phi(z) = (\pi(z), \ell(z)).$$

There is a neighborhood W of $0 \in \mathbb{C}^N$ small enough that $\lambda \cap W$ is an arc carried diffeomorphically onto an open arc λ' of class $\mathscr{C}^p$ in a neighborhood W_0 of $0 \in \mathbb{C}^2$ and $V \cap W$ is carried by ϕ onto a subvariety of W_o. Moreover, because ℓ separates points of the set E, ϕ is biholomorphic off a discrete subset of $V \cap W$.

The two-dimensional case applies to λ' and $\phi(V \cap W)$: Either the variety $\phi(V \cap W)$ continues through λ' as a variety, or else there is a set $S \subset \lambda'$ of length zero such that $(\phi(V \cap W) \cup \lambda') \setminus S$ is a $\mathscr{C}^p$ manifold with boundary. In the former case, V continues holomorphically through λ, and in the latter, $(V \cup \lambda) \setminus \phi^{-1}(S)) \cap W$ is a $\mathscr{C}^p$ manifold with boundary.

The theorem is proved.

Corollary 7.2.3. *If $\gamma \subset \mathbb{C}^N$ is a simple closed curve of class $\mathscr{C}^p$, $p \geq 1$, that is not polynomially convex, then there is a subset E of γ with $\Lambda^1(E) = 0$ and with the property that the set $\widehat{\gamma} \setminus E$ is a $\mathscr{C}^p$ manifold with boundary near $\gamma \setminus E$.*

When V is a subvariety of a strictly pseudoconvex domain, $\bar{V}$ cannot be a variety at points in the boundary because of the maximum principle, so in this case by the theorem just proved, if bV is a simple curve of class $\mathscr{C}^p$, $p \geq 1$, then $\bar{V}$ is a manifold with boundary near most points of bV. In this case, at least when $p \geq 2$, there can be no exceptional set, as a result of Forstnerič [125] shows:

Theorem 7.2.4. *Let D be a bounded strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in $\mathbb{C}^N$, let V be an irreducible one-dimensional subvariety of D, and let bV, which is contained in bD, be a simple closed curve of class $\mathscr{C}^p$, $p \geq 2$. There is neighborhood W of bV in $\bar{V}$ that is a two-dimensional $\mathscr{C}^p$ manifold with boundary such that $W \cap V$ is a one-dimensional complex manifold.*

Proof. Let γ be the curve bV so that $\widehat{\gamma} = \bar{V}$. According to Theorem 7.1.15, γ is complex transverse at each of its points.

Fix a point $p \in \gamma$. The problem is local, so we can suppose that bD is strictly convex at p, that p is the origin, that $T_0 bD = \{\Re z_1 = 0\}$, and that near the origin D is contained in the half-space $\Re z_1 \geq 0$. With this configuration, $T_0^{\mathbb{C}} bD = \{z_1 = 0\}$, and the tangent space $T_0\gamma$ is transverse to $\{z_1 = 0\}$. Accordingly, if π is the orthogonal projection of $\mathbb{C}^N$ onto the z_1-axis, π carries $T_0\gamma$ onto a real line in the z_1-plane. There are a disk U in the z_1-plane and a polydisk U' in $\mathbb{C}^{N-1}_{z_2,\ldots,z_N}$ such that π carries $\bar{D} \cap (U \times U')$ properly to U. Under π, if U and U' are small enough, $\gamma \cap (U \times U')$ is carried diffeomorphically (of class $\mathscr{C}^p$) onto an open arc λ in U that is tangent at the origin to the x_1-axis ($x_1 = \Re z_1$) and that separates U into two domains U^+ and U^- with U^+ contained in $\{\Re z_1 \geq 0\}$. The map π carries $(V \cup \gamma) \cap (U \times U')$ properly to U, and $\pi(V \cap (U \times U'))$ is contained in U^+. Therefore $\pi : V \cap (U \times U') \to U^+$ is an analytic cover. This map is injective on γ (near the origin), so the multiplicity of the analytic cover is one, whence $V \cap (U \times U')$ is a graph over U^+, say the graph of the function f. The function f is holomorphic, because its graph is an analytic set. It follows that $V \cap (U \times U')$ is nonsingular, and it has continuous boundary values along the $\mathscr{C}^p$ curve $\pi(\gamma \cap (U \times U'))$, and these boundary values are of class $\mathscr{C}^p$—they are the function inverse to the $\mathscr{C}^p$ function $\pi|\gamma$. Also, by the regularity theorem for holomorphic functions proved above, the derivatives of order not more than p of f extend continuously to the boundary near $\pi(p)$, so near p, V is found to be a manifold with boundary of class $\mathscr{C}^p$, as claimed.

The theorem is proved.

Corollary 7.2.5. *If V is an irreducible one-dimensional subvariety of a strictly pseudoconvex domain D with bD of class $\mathscr{C}^2$, and if bV is a simple closed curve of class $\mathscr{C}^2$, then V has only finitely many singular points.*

Given a planar domain Δ and given a boundary point p of Δ, there may or may not be an arc contained in $\Delta \cup \{p\}$ that has p as an endpoint. If there is such an arc, then p is said to be *arcwise accessible* from Δ. If Δ is a bounded, simply connected domain bounded by a simple closed curve, then a Riemann map $\mathbb{U} \to \Delta$ extends to a homeomorphism between $\bar{\mathbb{U}}$ and $\bar{\Delta}$, so every point of $b\Delta$ is arcwise accessible from Δ.

We shall consider the question of arcwise accessibility of points of bV from V when V is a one-dimensional subvariety of $\mathbb{B}_N$ or more generally of a bounded strictly pseudoconvex domain. The precise definition is this:

Definition 7.2.6. *If D is a domain in $\mathbb{C}^N$ and V is a subvariety of D, then the point $p \in bD$ is* arcwise accessible from V *if there is an arc in $V \cup \{p\}$ with p as an endpoint.*

We shall also refer to a point that is arcwise accessible from V as a point that is *accessible from V* for the sake of brevity.

It is a simple remark of Alexander's that if V is a one-dimensional variety in an arbitrary bounded domain, then some points in bV are accessible. If V is nonsingular, and so a Riemann surface, the uniformization theorem provides a uniformizing map $\varphi : \mathbb{U} \to V$. The map φ is bounded and holomorphic and so has radial limits at almost every point of $b\mathbb{U}$. If $e^{i\vartheta}$ is a point of the unit circle at which the radial limit $\varphi^*(e^{i\vartheta}) = p$ exists, then the radial image $I_\vartheta = \{\varphi(re^{i\vartheta}) : r \in [0, 1)\} \cup \{p\}$ contains an arc that exhibits p as a point accessible from V. The point is that I_ϑ is the image of the interval [0, 1] under a continuous map and so is locally connected and thus arcwise connected. For an accessible but thorough discussion of the set-theoretic topology involved here, one can consult [159]. If V has singular points, then there are a Riemann surface $\mathscr{R}$ and a surjective holomorphic map $\eta : \mathscr{R} \to V$ that carries $\mathscr{R} \setminus \eta^{-1}(V_{\text{sing}})$ biholomorphically onto V_{reg}. (Here $\mathscr{R}$ is the normalization of V.) If $\varphi : \mathbb{U} \to \mathscr{R}$ is a uniformizing map, then $\eta \circ \varphi$ is a bounded holomorphic map on $\mathbb{U}$ with limits almost surely in bV, and the argument finishes as in the nonsingular case.

When $\Lambda^1(bV)$ is finite, every point is accessible from V, as shown by Alexander [22]. In fact, his result is stronger in that it is local:

Theorem 7.2.7. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with bD of class $\mathscr{C}^2$, and let V be an irreducible one-dimensional subvariety of D. If $p \in bV$ is a point such that for some ball $\mathbb{B}_N(p, r)$ about p, the set $(bV) \cap \mathbb{B}_N(p, r)$ has finite length, then p is accessible from V.*

Proof. The domain D is strictly pseudoconvex and the problem local, so we can suppose D

to be strictly convex. Let $p_o \in bD$ be a point such that for some ball $\mathbb{B}_N(p_o, \eta_o)$ centered at p_o, the set $bV \cap \mathbb{B}_N(p_o, \eta_o)$ has finite length.

Introduce the affine function g on $\mathbb{C}^N$ given by $g(z) = \langle z - p_o, \nu\rangle$, where ν is the outer unit normal to bD at p_o. Then $\Im g(z) > 0$ when $z \in \bar{D} \setminus \{p_o\}$. For a real number t, let L_t be the horizontal line in $\mathbb{C}$ that passes through the point it. Choose a point q_o in V, choose a $c > 0$, and choose an $\eta_1 \in (0, \eta_o)$ such that g carries $bV \setminus \mathbb{B}_N(p_o, \eta_1)$ into the half-plane above the line L_c and $\Im g(q_o) < c$.

For almost all $t \in (0, c)$, $\Lambda^1(bV \cap \mathbb{B}_N(p_o, \eta_o)) < \infty$, so the set $g^{-1}(L_t)$ meets bV in a finite set. Choose a strictly decreasing sequence $\{\delta_n\}_{n=1,\dots}$ of positive numbers with limit zero and with $\Im g(q_o) > \delta_1$. Require that for all n, $V \cap g^{-1}(L_{\delta_n}) \subset V_{\mathrm{reg}}$, that dg vanish at no point of $V \cap g^{-1}(L_{\delta_n})$, and that the set $F_n = g(bV \cap \mathbb{B}_N(p_o, \eta_1)) \cap L_{\delta_n}$ be finite.

For each n, let V_n be the part of the variety V taken by g into the half-plane below the line L_{δ_n}. The point p_o lies in bV_n for all n.

Each V_n has only finitely many branches. To see this, denote by λ_j, $j = 1, \dots,$ the finitely many mutually disjoint intervals that compose the set $L_{\delta_n} \setminus F_n$. If λ is one of these, and if Ω is the component of $g(V) \setminus g(\mathbb{B}_N(p_o, \eta_1))$ that contains λ, then over Ω the variety V is an analytic cover with projection g, and so is finitely sheeted. Moreover, g carries each component of $V \cap g^{-1}(\lambda)$ onto λ as a covering map, and so must be injective there. It follows that $V \cap g^{-1}(\lambda)$ has finitely many components. These components are analytic arcs, so no two distinct global branches of V_n can meet along any component of $V \cap g^{-1}(\lambda)$. The upshot of this is that each V_n has a finite number of branches. Moreover, the closure of each such branch, say $V_{n,k}$, of V_n meets one of the components of $g^{-1}(\lambda)$ for some choice of λ: Pick a point z_o in some global branch of V_n. The point p_o can be joined to our initial point q_o with an arc in V. This arc has to stay in $\bar{V}_{n,k}$ until it comes to the first of its points projected into L_{δ_n} by g. Thus, $\bar{V}_{n,k}$ meets $g^{-1}(\lambda)$ for a suitable choice of λ. Define now W_1 to be some branch of V_{δ_1} with $p_o \in bW_1$. This is possible, for V_1 has only finitely many branches and $p_o \in bV$. Let q_1 be some point in W_1. Then, let W_2 be some branch of the part of W_1 carried by g into the half-plane below L_{δ_2} with $p_o \in bW_2$, and let q_2 be a point of W_2. We iterate this procedure to obtain a decreasing sequence $W_1 \supset W_2 \supset \cdots$ and points q_k in W_k.

Let γ_1 be an arc in V that connects q_o to q_1, and for $k = 1, \dots,$ let γ_k be an arc in W_k that connects q_k to q_{k+1}. The union Γ of the γ_k's together with the point p_o contains an arc in $V \cup \{p_o\}$ that connects q_o to p_o.

The point p_o is thus seen to be accessible from V, and the theorem is proved.

A notion stronger than accessibility is that of nontangential accessibility:

Definition 7.2.8. *If D is a bounded domain in $\mathbb{C}^N$ and V is a subvariety of D, then the point p is* nontangentially accessible from V *if bD has a tangent plane at p and if there is an open cone K in $\mathbb{C}^N$ with vertex p and axis the normal to bD at p such that for some $r > 0$, the intersection $K \cap \mathbb{B}_N(p, r)$ is contained in D and there is an arc λ in $\{p\} \cup (V \cap K)$ with p as an endpoint.*

The question of nontangential accessibility is more delicate than that of accessibility. For it there is the following result:

Theorem 7.2.9. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$. Let $V \subset D$ be an irreducible one-dimensional subvariety. If E is an open subset of bV that is a continuum of finite length, then almost every $[d\Lambda^1]$ point of E is nontangentially accessible from V.*

This is a result of Alexander's [22].

7.3. Uniqueness

The following uniqueness theorem was established by Lawrence [215], generalizing earlier versions due to Alexander [22] and Globevnik and Stout [147]. It can be regarded as a geometric analogue of the result from classical function theory to the effect that two bounded holomorphic functions on the unit disk with boundary values coincident on a subset of the unit circle of positive length necessarily coincide throughout the disk.

Theorem 7.3.1. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$. Let V_1 and V_2 be irreducible one-dimensional subvarieties of D. Assume there to be an open set $\Omega \subset bD$ such that $\Lambda^1(bV_j \cap \Omega) < \infty$ for $j = 1, 2$. If $\Lambda^1(bV_1 \cap bV_2 \cap \Omega) > 0$, then $V_1 = V_2$.*

The hypotheses of this theorem cannot be essentially weakened. Globevnik and Stout [147] constructed distinct subvarieties V' and V'' of the ball $\mathbb{B}_N$, $N \geq 2$, each of which is an immersed copy of the open unit disk and each of which has finite area, indeed arbitrarily small area, and that satisfy $\bar{V}' \setminus V' = b\mathbb{B}_N = \bar{V}'' \setminus V''$.

Another example is this:

Example. Alexander [22] constructed proper holomorphic maps φ_1 and φ_2 from $\mathbb{U}$ to $\mathbb{B}_2$ such that with $V_j = \varphi_j(\mathbb{U})$, the set bV_2 is a connected set of finite measure, the set bV_1 has σ-finite linear measure, the sets bV_1 and bV_2 meet in a set of positive length, but the varieties V_1 and V_2 are distinct. This construction is the following.

The variety V_2 is the disk given by

$$V_2 = \{(\zeta, 0) : \zeta \in \mathbb{C}, \ |\zeta| < 1\}.$$

The variety V_1 is more complicated. Let S be the spiral $\{r(\vartheta)e^{\vartheta} : 0 \leq \vartheta\}$ with r the function given by $r(\vartheta) = \vartheta/(1+\vartheta)$. The domain $\mathbb{U} \setminus S$ is simply connected. Let $f : \mathbb{U} \to \mathbb{U} \setminus S$ be a Riemann map chosen so as to extend continuously to $\bar{\mathbb{U}} \setminus \{1\}$; the absolute value $|f|$ extends continuously to $\bar{\mathbb{U}}$ and satisfies $|f(1)| = 1$. There is a holomorphic function g on $\mathbb{U}$ such that $|g|$ extends continuously to $\bar{\mathbb{U}}$ and satisfies $|g| = \sqrt{1 - |f|^2}$ on $b\mathbb{U}$. For g one can take

$$g(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\vartheta} + z}{e^{i\vartheta} - z} \log\sqrt{1 - |f(e^{i\vartheta})|^2}\, d\vartheta\right\},$$

so that $g(z) = e^{u+iv}$ with u the Poisson integral of $\log\sqrt{1 - |f(e^{i\vartheta})|^2}$.

With this choice for f and g, define $\varphi_1 : \mathbb{U} \to \mathbb{B}_2$ by $\varphi_1(z) = (f(z), g(z))$, a proper holomorphic map. Denote its range by V_1. The set $\bar{V}_1 \setminus V_1$ is the set $\varphi_1(b\mathbb{U} \setminus \{1\}) \cup (b\mathbb{U} \times \{0\})$, which has σ-finite one-dimensional measure. The sets $\bar{V}_1 \setminus V_1$ and $\bar{V}_2 \setminus V_2$ meet in the circle $b\mathbb{U} \times \{0\}$, which is a connected set of positive length, but the varieties V_1 and V_2 do not coincide.

To justify the preceding analysis, it is necessary to show that the integral

$$\int_{-\pi}^{\pi} \log\sqrt{1 - |f(e^{i\vartheta})|^2}\, d\vartheta$$

exists. This can be done, but the details are not perfectly simple.

A simpler alternative construction is the following. Consider the function f defined by

$$f(z) = \exp\left\{ i \left\{ \left(\frac{\alpha}{\pi}\right) \log\left(i\frac{1+z}{1-z} \right) + \beta i \right\}^{1/2} \right\},$$

in which $\alpha = 2(c - c')$ and $\beta = 2c'$ for positive numbers c and c' with $c' < c$. The logarithm is taken to be defined on the plane cut down the negative imaginary axis and to be normalized by $\log 1 = 0$. The square root function is defined on the same slit plane and satisfies $\sqrt{1} = 1$. The geometry of this mapping from the disk to the plane can be understood by considering it as a composition. The linear fractional map $z \mapsto i\frac{1+z}{1-z}$ carries $\mathbb{U}$ onto the upper half-plane with the point $1 \in b\mathbb{U}$ going to the point at infinity. The logarithm carries the upper half-plane to the infinite strip of width π bounded below by the x-axis. The effect of the constants α and β is to shift this infinite strip upward and make its boundary lines, lines parallel to the x-axis, be the lines through the points $c'i$ and ci. Denote this strip by $\mathscr{S}$. The square root function carries $\mathscr{S}$ onto the domain in the first quadrant bounded above by a branch of the hyperbola $xy = c$ and below by the corresponding branch of the hyperbola $xy = c'$. Call this curvilinear strip $\mathscr{S}'$. (To confirm this, it is simplest to note that the map $z \mapsto z^2$ carries the hyperbola $xy = k$ into the horizontal line through the point $2ki$.) Finally, the exponential map $z \mapsto e^{iz}$ carries the strip $\mathscr{S}'$ into $\mathbb{U}$, in such a way that as $z \in \mathscr{S}'$ approaches ∞ through the first quadrant, the point e^{iz} circles infinitely often around the origin and approaches $b\mathbb{U}$. In particular, the modulus $|f|$ is continuous on $\bar{\mathbb{U}}$. The argument of f is not.

We will show now that the continuous function $\sqrt{1 - |f|^2}$ on $b\mathbb{U}$ is $|g|$ for a function g holomorphic and bounded on $\mathbb{U}$ whose modulous extends continuously to $\bar{\mathbb{U}}$. Any such function g satisfies $g(1) = 0$. To verify the existence of g, it is necessary to verify that

$$\int_{-\pi}^{\pi} \log(1 - |f|^2(e^{i\vartheta}))\, d\vartheta < \infty. \tag{7.5}$$

(If this integral is finite, a choice for the function g is given by $g = e^{u+iv}$, where the harmonic function u is the Poisson integral of $\frac{1}{2}\log(1 - |f|^2)$ and v is some harmonic conjugate of u. The function g is uniquely determined to within a factor of $e^{i\theta}$, which is determined by the value of $v(0)$.)

That the integral in (7.5) exists is easily seen. Note first that

$$\log\left(i\frac{1+e^{i\vartheta}}{1-e^{i\vartheta}} \right) = \log\frac{-\sin\vartheta}{1-\cos\vartheta}, \tag{7.6}$$

which is $\log\left|\frac{\sin\vartheta}{1-\cos\vartheta}\right|$ if $-\pi < \vartheta < 0$ and is $\log\left|\frac{\sin\vartheta}{1-\cos\vartheta}\right| + \pi i$ if $0 < \vartheta < \pi$. Thus

$$\int_{-\pi}^{0} \log(1-|f|^2(e^{i\vartheta}))\,d\vartheta = \int_{-\pi}^{0} \log\left(1 - \exp\left\{2\Re i\left\{\frac{2\alpha}{\pi}\log\frac{-\sin\vartheta}{1-\cos\vartheta} + \beta i\right\}^{1/2}\right\}\right) d\vartheta.$$

Now $\left\{\frac{2\alpha}{\pi}\log\frac{-\sin\vartheta}{1-\cos\vartheta} + \beta i\right\}^{1/2} = a + ib$ with $ab = c'$, so

$$\Im\left\{\frac{2\alpha}{\pi}\log\frac{-\sin\vartheta}{1-\cos\vartheta} + \beta i\right\}^{1/2} = b = c'/a = c'\Re\left\{\frac{2\alpha}{\pi}\log\frac{-\sin\vartheta}{1-\cos\vartheta} + \beta i\right\}^{-1/2},$$

which, for ϑ negative but close to 0, is about const $\left\{\alpha\log\frac{-\sin\vartheta}{1-\cos\vartheta}\right\}^{-1/2}$, so that for these values of ϑ, $|f|^2(e^{-\vartheta}) \sim \exp\{-\operatorname{const}\{\log\left|\frac{\sin\vartheta}{1-\cos\vartheta}\right|\}^{-1/2}$ with a positive constant.

For ϑ near 0,

$$\log\left(1 - \exp\left\{-\operatorname{const}\left(\log\left|\frac{\sin\vartheta}{1-\cos\vartheta}\right|\right)^{-1/2}\right\}\right) \sim \log\frac{\operatorname{const}}{\log\frac{1}{|\vartheta|}}.$$

Because the singularity at 0 of $\log\log\frac{1}{|\vartheta|}$ is integrable, we have that $\log(1-|f|^2)$ is integrable on $(-\pi, 0)$.

A similar analysis gives its integrability on $(0, \pi)$.

This completes the discussion of the example.

Notice that the points of the circle $b\mathbb{U} \times \{\beta\}$ are not accessible from the variety $V = \{(f(z), g(z)) : |z| < 1\}$.

Theorem 7.3.1 requires some convexity condition on the domain D, as an example given in [147] shows.

The uniqueness result, Theorem 7.3.1, is an immediate consequence of a result, which is due to Lawrence [215], on the multiplicity function introduced in Section 4.6.

Theorem 7.3.2. *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$. If V is a one-dimensional subvariety of D with $\Lambda^1(bV) < \infty$, then for almost every $[d\Lambda^1]$ point z of bV, the multiplicity $m(V; z)$ is one.*

Proof. By definition $m(z; V) \geq 1$ for every $z \in bV$, and we know from Theorem 4.6.3 that $m(z; V) \leq 2$ a.e. $[d\Lambda^1]$. We suppose therefore, for the sake of deriving a contradiction, that $m(z; V) = 2$ on a subset of bV of positive length.

Without loss of generality, we can suppose that the domain D is strictly convex.

Fix attention on a compact set $E \subset bV$ of positive length with $m(z; V) = 2$ for all $z \in E$. The set bV is minimal with respect to the condition that $p \in \widehat{bV}$ for all $p \in V$ near bD, so bV is a set of class $\mathscr{A}_1$ by Theorem 4.3.7. Thus, we can suppose that at every point of E, the tangent $\operatorname{Tan}(bV, z)$ is a real line. Moreover, by Theorem 7.1.17 we can suppose that this line is complex transverse. By Lemmas 4.3.4, 4.3.10, 4.3.11, and 4.3.16, we know that by shrinking E a bit if necessary, we can find a $\mathbb{C}$-linear functional φ on $\mathbb{C}^N$ that is injective on E and for which $\Lambda^1(\varphi(E)) > 0$ and $\varphi(E)$ is of class $\mathscr{A}_1$. Let the components of $\mathbb{C} \setminus \varphi(E)$ be Ω_j, $j = 0, \ldots,$ with Ω_0 the unbounded component.

Lemma 7.3.3. *If Ω_p is a component of $\mathbb{C} \setminus \varphi(E)$ such that $b\Omega_p \cap \varphi(E)$ has positive length, then $V \cap \varphi^{-1}(\Omega_p)$ has one and only one branch, say W_p, such that bW_p meets E in a set of positive length.*

Proof. Suppose W' and W'' to be distinct branches of $V \cap \varphi^{-1}(\Omega_p)$ such that $\Lambda^1(bW' \cap E)$ and $\Lambda^1(bW'' \cap E)$ are both positive. We have $\varphi(W') = \Omega_p = \varphi(W'')$, so

$$\Lambda^1(b\varphi(W') \cap b\varphi(W'') \cap \varphi(E)) > 0,$$

and then, because φ is injective on E,

$$\Lambda^1(bW' \cap bW'' \cap E) > 0.$$

The union $W' \cup W''$ is a bounded subvariety of $\mathbb{C}^N \setminus \varphi^{-1}(\varphi(bV))$, and $\varphi|(W' \cup W'') \to \Omega_p$ is a proper holomorphic map. Because $W' \neq W''$, the generic fiber $\varphi^{-1}(\zeta) \cap (W' \cup W'')$ contains at least two points. This is impossible, for $\varphi(E) \cap b\Omega_p$ is a set of positive length and so is a set of uniqueness for subharmonic functions on Ω_p. (Recall Corollary 3.4.11.)

Corollary 7.3.4. *If W is a branch of $V \cap \varphi^{-1}(\Omega_p)$ with $\Lambda^1(\varphi(bW) \cap \varphi(E)) > 0$, then $\varphi|W$ is biholomorphic onto Ω_p.*

As a consequence, W is nonsingular.

Proof. We know from the proof of the lemma that $\varphi|W$ is injective and proper as a map to Ω_p. Thus there is a holomorphic map $\chi : \Omega_p \setminus \varphi(W_{\text{sing}}) \to W$ inverse to $\varphi|W_{\text{reg}}$. The map χ continues through $\varphi(W_{\text{sing}})$ by the Riemann removable singularity theorem, for it is bounded. The extended map is the inverse of $\varphi|W$, so the latter map is biholomorphic as claimed.

Proof of the theorem concluded. The lemma implies that there are components, say Ω_1 and Ω_2, of $\mathbb{C} \setminus \varphi(E)$ such that branches W_1 and W_2 of $\varphi^{-1}(\Omega_1) \cap V$ and $\varphi^{-1}(\Omega_2) \cap V$, respectively, are mapped biholomorphically onto Ω_1 and Ω_2 and such that $b\Omega_1 \cap b\Omega_2 \cap \varphi(E)$ contains a set S_0 of positive length.

By Theorem 4.3.12 there are rectifiable simple closed curves γ_1 and γ_2 that bound simply connected domains D_1 and D_2, respectively, contained in Ω_1 and Ω_2 with the property that $\gamma_1 \cap \gamma_2$ contains a subset S_1 of S_0 of positive length. There are conformal maps $\psi_j : \mathbb{U} \to D_j$, $j = 1, 2$, as provided by the Riemann mapping theorem. By Theorem 3.4.6, the derivatives ψ_j' lie in the Hardy class $H^1(\mathbb{U})$. The domain D_j is the image of a domain W_j' in W_j under the map φ, and there are biholomorphic maps $\eta_j : \mathbb{U} \to W_j'$ with $\varphi \circ \eta_j = \psi_j$.

Under the map ψ_j, the set S_1 corresponds to a set of positive length in the circle $b\mathbb{U}$. At almost every point $e^{i\vartheta}$ of $\psi^{-1}(S_1)$ the derivative ψ_j' has nontangential limit $\psi_j'(e^{i\vartheta})$. For such a point $e^{i\vartheta}$, the curve γ_j has a tangent at the point $\psi_j(e^{i\vartheta})$.

The variety V has finite area, because bV is of class $\mathscr{A}_1$. Consequently, the domains W_j' have finite area. It follows that the derivatives η_j' satisfy

$$\int_{\mathbb{U}} |\eta_j'(\zeta)|\, d\mathscr{L}(\zeta) = \int_0^{2\pi} \int_0^1 |\eta_j'(re^{i\vartheta})|\, r\, dr\, d\vartheta < \infty, \tag{7.7}$$

so for almost all values of ϑ, the integral $\int_0^1 |\eta_j'(re^{i\vartheta})|\, rdr$ is finite. Thus, for almost all values of ϑ the radial image $\eta_j([0, e^{i\vartheta}])$ has finite length. Moreover, for almost all values of ϑ, the limit $\lim_{r\to 1^-} \eta_j'(re^{i\vartheta}) = \eta_j'(e^{i\vartheta})$ is not zero, as follows by differentiating the equation $\psi_j(\zeta) = \varphi \circ \eta_j(\zeta)$ and passing to the radial limit and then recalling that the boundary values $\psi_j'(e^{i\vartheta})$ are almost everywhere nonzero.

Consider now a point $z_0 \in E$ with $\varphi(z_0) = \psi_1(e^{i\vartheta_1}) = \psi(e^{i\vartheta_2})$ for points $e^{i\vartheta_j}$ at which the limits $\psi_j'(e^{i\vartheta_j})$ exist and are nonzero. By changing coordinates suitably, we can suppose that z_0 is the origin in $\mathbb{C}^N$, whence $\varphi(z_0) = 0$, and also that the points $e^{i\vartheta_j}$ are both the point $1 \in b\mathbb{U}$. By the last paragraph, we can also assume that the radial limits $\eta_j'(1)$ both exist and are zero-free. Under φ, the tangent $\mathrm{Tan}(V, 0)$ is carried to the tangent $T_0\gamma_1$, which must coincide with the tangent $T_0\gamma_2$. Call this line ℓ.

The curves $L_j = \psi_j([0, 1])$ lie in D_j, so they approach the origin in $\mathbb{C}$ from opposite sides of the line ℓ, and the approach is tangent to the normal to the line ℓ.

The curves L_j lift to the curves $\tilde{L}_j = \eta_j([0, 1])$ in $W_j' \cup \{0\}$. That is, $\varphi(\tilde{L}_j) = L_j$.

Let λ be the complex line in $\mathbb{C}^N$ that contains the line $\mathrm{Tan}(V, 0)$.

Let χ be a linear functional on $\mathbb{C}^N$ with $\chi^{-1}(0) = T_0^{\mathbb{C}}(bD)$ and with $\Re\chi > 0$ on the domain D. Thus, the function χ is constant on neither of the curves $\tilde{L}_j$. It carries them into curves in the plane that approach the origin from the right half-plane.

The curves $\tilde{L}_j$ project orthogonally into the complex line λ as curves $\tilde{L}_j^*$. We can write that for $t \in [0, 1]$,

$$\eta_j(t) = \eta_j(1) + (1 - r)\eta_j'(1) + o(1 - r) = (1 - r)\eta_j'(1) + o(1 - r).$$

The line λ is the line $\mathbb{C}\eta_j'(1)$, so the curve L_j^* is the curve $(1 - t)\eta_h'(1) + o(1 - r)$ for a suitable remainder term. Consequently, the curves L_j^* are carried by the linear functional φ to curves that approach the origin in $\mathbb{C}$ from opposite sides of the line ℓ.

Let $\sigma : \mathbb{C} \to \lambda$ be the map inverse to $\varphi|\lambda$. (Note that φ is not constant on λ, because $\mathrm{Tan}(bV, 0)$ is complex transversal. The map $h = \chi \circ \sigma$ is a nonsingular linear map of $\mathbb{C}$ onto itself. It carries the curves $\varphi(L_j^*)$, which approach the origin from opposite sides of the line ℓ, onto the curves $\chi(L_j^*)$, which approach the origin from the same side of the line $h(\ell) = \chi(\mathrm{Tan}(V, 0))$. The map h is conformal, so we have a contradiction, and the theorem is proved.

The next result was found by Lawrence [215] and depends essentially on the preceding result. See also [99].

Theorem 7.3.5. *If D is a domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$ that is strictly convex, and if $X \subset bD$ is a compact set with $\Lambda^1(X) < \infty$, then $\widehat{X} \setminus X$ is an analytic subvariety of $\mathbb{C}^N \setminus X$.*

By virtue of the embedding theorem of Fornæss and Henkin, Theorem 2.4.6, there is an analogue of this theorem in which the domain D is taken to be strictly pseudoconvex and polynomial convexity is replaced by convexity with respect to the algebra of functions holomorphic on a neighborhood of $\bar{D}$.

Proof. Fix a point $x \in \widehat{X} \setminus X$. It is to be shown that near x, the set $\widehat{X} \setminus X$ has the structure of a one-dimensional variety. To begin with, let $E \subset X$ be a compact set with $x \in \widehat{E}$ and minimal with respect to this property. The set E is of class $\mathscr{A}_1$ by Theorem 4.3.7, so by Theorem 4.3.2 the set $\widehat{E} \setminus E$ is a one-dimensional variety. Consequently, $\widehat{X} \setminus X$ contains varieties through x.

Let U be a small open set that contains x and that is relatively compact in D. Only finitely many irreducible subvarieties of D contained in $\widehat{X} \setminus X$ can meet U. To see this, denote by d a positive number less than the distance from U to bD. If V is an irreducible subvariety of D contained in $\widehat{X} \setminus X$ that meets U, then $\widehat{bV}$ meets U, whence $\Lambda^1(bV) > d$ by the result of Sibony, Theorem 5.1.17.

Now suppose there are infinitely many distinct irreducible subvarieties V_j, $j = 1, \dots,$ of D contained in $\widehat{X} \setminus X$ that meet the set U. The preceding paragraph implies that for each j, $\Lambda^1(\bar{V}_j \setminus V_j)$ is a least d. The finiteness of $\Lambda^1(X) < \infty$ implies that two of the sets $\bar{V}_j \setminus V_j$ must meet in a set of positive measure. Theorem 7.3.1 implies that these two varieties must coincide, contradicting their choice as distinct subvarieties. Thus, the set $\widehat{X} \setminus X$ contains only finitely many irreducible subvarieties of D that meet the set U. Let W be their union.

Consider now the set $(\widehat{X} \setminus W) \cap U$. If this set is not empty, let y be one of its points. By the first paragraph of the proof, $\widehat{X} \setminus X$ contains a subvariety that passes through the point y. This contradicts the choice of W. Consequently, $W \cap U$ is a neighborhood of the initially chosen point x that has the structure of a variety.

The theorem is proved.

The following corollary was announced in the remarks that introduce Chapter 4.

Corollary 7.3.6. *If X is a compact subset of $b\mathbb{B}_N$ with $\Lambda^1(X) < \infty$, then $\widehat{X} \setminus X$ is a one-dimensional subvariety of $\mathbb{B}_N$.*

Corollary 7.3.7. *If D is a bounded, strictly pseudoconvex domain in $\mathbb{C}^N$ that is polynomially convex and that has boundary of class $\mathscr{C}^2$, and if $E \subset bD$ is a compact set of finite length that satisfies $\check{H}^1(E, \mathbb{Z}) = 0$, then E is polynomially convex.*

The corollary applies in particular when the set E is totally disconnected.

Proof. If not, $\widehat{E} \setminus E$ is a variety, V. Because $\Lambda^1(E)$ is finite, we must have $\check{H}^1(bV, \mathbb{Z}) \neq 0$ by Theorem 3.8.15. However $H^1(E, \mathbb{Z}) = 0$ implies that $\check{H}^1(bV, \mathbb{Z}) = 0$, and we have a contradiction.

Chapter 8

EXAMPLES AND COUNTEREXAMPLES

Introduction. In this chapter we discuss some additional questions related to polynomial convexity, topics that are concerned with polynomial convexity per se and also topics that depend on the application of the ideas of polynomial convexity. Section 8.1 discusses the polynomial convexity of unions of linear spaces passing through the origin. Section 8.2 is devoted to the study of pluripolar graphs. Section 8.3 considers certain deformations of polynomially convex sets. Section 8.4 concerns sets with symmetries.

8.1. Unions of Planes and Balls

The question broached here is, under what conditions does the union of a family of linear—perhaps $\mathbb{R}$-linear—subspaces of $\mathbb{C}^N$ have the property that its intersection with $\bar{\mathbb{B}}_N$ is polynomially convex?

We begin with the complex case. A finite union of $\mathbb{C}$-linear subspaces of $\mathbb{C}^N$ is an analytic variety, so its intersection with any closed ball or, more generally, with any polynomially convex set is polynomially convex. Certain nontrivial infinite unions of complex lines through the origin are known to be polynomially convex as well. This seems first to have been noticed by Sibony and Wong [334]. Alexander [13] introduced a notion of *projective capacity* that is well suited to the discussion of such unions. The definition of this capacity requires some preliminaries.

Consider the projection $\pi : \mathbb{C}^N \setminus \{0\} \to \mathbb{P}^{N-1}$. For a set $S \subset \mathbb{P}^{N-1}$ we denote by Y_S the set $\bar{\mathbb{B}}_N \cap \pi^{-1}(S) \cup \{0\}$, a certain subset of the closed unit ball, which is *circled* in the sense that if $z \in Y_S$, then for all $\vartheta \in \mathbb{R}$, the point $e^{i\vartheta} z$ is also in Y_S. Indeed, if $\zeta \in \mathbb{C}$, $|\zeta| \leq 1$, then $\zeta z \in Y_S$. The set Y_S is called the *truncated cone over S*. Conversely, if $Y \subset \bar{\mathbb{B}}_N$ is a compact subset with the property that for each $z \in Y$ and each $\zeta \in \mathbb{C}$,

$|\zeta| \le 1$, we have $\zeta z \in Y$, then $Y = Y_S$ if $S = \pi(Y \setminus \{0\})$. The set Y_S is compact if and only if the set S is compact.

The definition of projective capacity depends on the notion of normalized homogeneous polynomial and on some simple facts about them.

Definition 8.1.1. *A homogeneous polynomial p of degree d on $\mathbb{C}^N$ is said to be* normalized *if*

$$\int_{\mathbb{S}^{2N-1}} \log|p(z)|\, d\sigma(z) = d \int_{\mathbb{S}^{2N-1}} \log|z_1|\, d\sigma(z)$$

with $d\sigma$ the element of normalized surface area on the sphere $\mathbb{S}^{2N-1} = b\mathbb{B}_N$.

The motivation for this rather arbitrary-appearing definition comes from the consideration of the two-dimensional case. Homogeneous polynomials of two variables factor into linear factors: If $q(z_1, z_2)$ is a homogeneous polynomial of degree d, then there is a representation

$$q(z_1, z_2) = C \prod_{j=1}^{d} (\alpha_j z_1 + \beta_j z_2). \tag{8.1}$$

If we impose the conditions that $|\alpha_j|^2 + |\beta_j|^2 = 1$ and that $C > 0$, then C is uniquely determined. We will say that q is *C-normalized* if $C = 1$. A homogeneous polynomial of two variables is C-normalized exactly when it is normalized: With $q(z_1, z_2)$ given by (8.1),

$$\begin{aligned}\int_{\mathbb{S}^3} \log|q(z)|\, d\sigma(z) &= \log C + \sum_{j=1}^{d} \int_{\mathbb{S}^3} \log|\alpha_j z_1 + \beta_j z_2|\, d\sigma(z) \\ &= \log C + d \int_{\mathbb{S}^3} \log|z_1|\, d\sigma(z).\end{aligned}$$

(The second equality follows from the unitary invariance of the measure σ.) Thus, for homogeneous polynomials of two variables, being C-normalized is the same as being normalized. It is not generally true that homogeneous polynomials of more than two variables factor into linear factors, so the notion of C-normality is not meaningful for them.

Observe that the product of normalized polynomials is a normalized polynomial.

The integral $\int_{\mathbb{S}^{2N-1}} \log|z_1|\, d\sigma(z)$ can be evaluated without difficulty:

Lemma 8.1.2. *For $N \ge 2$,*

$$\int_{\mathbb{S}^{2N-1}} \log|z_1|\, d\sigma(z) = -\frac{1}{2}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{N-1}\right\}.$$

Proof. Denote the integral by I_N; its evaluation is based on the general integration formula that for $g \in \mathscr{C}(\mathbb{S}^{2N-1})$,

$$\int_{\mathbb{S}^{2N-1}} g(z)\, d\sigma(z) = \frac{(N-1)!}{\pi^{N-1}} \int_{\mathbb{B}_{N-1}} \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} g(z', \sqrt{1-|z'|^2} e^{i\vartheta})\, d\vartheta\right\} d\mathscr{L}(z'),$$

which can be proved using the usual formula for surface integrals together with the parameterization $\mathbb{B}_{N-1} \times (-\pi, \pi) \ni (z', \vartheta) \mapsto (z', \sqrt{1 - |z'|^2} e^{i\vartheta}) \in b\mathbb{B}_N$ for (almost all of) $b\mathbb{B}_N$. A different derivation of the formula is given in [310, p. 15]. Applied with the integrand $\log|z_N|$, this formula gives

$$I_N = \frac{(N-1)!}{2\pi^{N-1}} \int_{\mathbb{B}_{N-1}} \log(1 - |z'|^2)\, d\mathscr{L}(z').$$

If we pass to spherical coordinates in $\mathbb{C}^N$ and use the expansion $\log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$ valid for $-1 \le x < 1$, we find that

$$I_N = -\frac{1}{2}(N-1) \sum_{k=1}^{\infty} \frac{1}{k(k+N-1)} = -\frac{1}{2} \sum_{k=1}^{N-1} \frac{1}{k}.$$

We shall need the following estimate.

Lemma 8.1.3. *If p is a homogeneous polynomial of degree d with $\|p\|_{\mathbb{B}_N} = 1$, then*

$$\int_{\mathbb{S}^{2N-1}} \log|p(z)|\, d\sigma(z) \ge d \int_{\mathbb{S}^{2N-1}} \log|z_N|\, d\sigma(z).$$

Proof. Denote by $\eta : \mathbb{S}^{2N-1} \to \mathbb{C}$ the projection given by $\eta(z) = z_N$. There is an integral formula[1] that, with $z' = (z_1, \ldots, z_{N-1})$ and with σ_ζ the normalized surface area measure on the sphere $\eta^{-1}(\zeta)$ when $|\zeta| < 1$, is given by

$$(8.2) \qquad \int_{\mathbb{S}^{2N-1}} g(z)\, d\sigma(z) = \int_{|\zeta|<1} \left\{ \int_{\eta^{-1}(\zeta)} g(z', \zeta)\, d\sigma_\zeta(z') \right\} A(\zeta)\, d\mathscr{L}(\zeta)$$

for each continuous function g on $\mathbb{S}^{2N-1}$. Here A is a fixed smooth positive function on the open unit disk in the ζ-plane.

The statement of the lemma is invariant under unitary changes of variables, so without loss of generality, we can suppose that $p(z) = z_N^d + q(z)$ with $q(z)$ a homogeneous polynomial of degree d each summand of which is divisible by one of z_j, $1 \le j \le N-1$.

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[1]This formula is a consequence of Fubini's theorem: If

$$\begin{aligned} \Theta &= \bar{z}_1 d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_N \wedge dz_1 \cdots \wedge dz_N \\ &= \pm \bar{z}_1 d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_{N-1} \wedge dz_1 \wedge \cdots \wedge dz_{N-1} \wedge d\bar{z}_N \wedge dz_N \\ &= \Theta' \wedge d\bar{z}_N \wedge dz_N, \end{aligned}$$

then, to within a constant factor, integrating a function g on $\mathbb{S}^{2N-1}$ against the form Θ is the same as integrating against the measure $d\sigma$. Fubini's theorem yields the formula

$$\int_{\mathbb{S}^{2N-1}} g\Theta = \int_{|\zeta|<1} \left\{ \int_{\eta^{-1}(\zeta)} g\Theta' \right\} d\bar{\zeta} \wedge d\zeta,$$

from which the formula (8.2) follows.

Thus, because σ_ζ is a Jensen measure for the point $(0, \dots, 0, \zeta)$, we have

$$\int \log|p(z)|\, d\sigma(z) = \int_{|\zeta|<1} \left\{ \int_{\eta^{-1}(\zeta)} \log|\zeta^d + q(z', \zeta)|\, d\sigma_\zeta(z') \right\} A(\zeta)\, d\mathscr{L}(\zeta)$$
$$\geq d \int_{|\zeta|<1} \log|\zeta| A(\zeta)\, d\mathscr{L}(\zeta).$$

But also,

$$\int_{\mathbb{S}^{2N-1}} \log|z_N^d|\, d\sigma(z) = d \int_{|\zeta|<1} \left\{ \int_{\eta^{-1}(\zeta)} \log|\zeta|\, d\sigma_\zeta(z') \right\} A(\zeta)\, d\mathscr{L}(\zeta)$$
$$= d \int_{|\zeta|<1} \log|\zeta| A(\zeta)\, d\mathscr{L}(\zeta),$$

so the lemma is correct.

Corollary 8.1.4. *If p is a homogeneous polynomial of degree d, then*

$$\int_{\mathbb{S}^{2N-1}} \log|p(z)|\, d\sigma(z) \geq \log\|p\|_{\mathbb{S}^{2N-1}} + d \int_{\mathbb{S}^{2N-1}} \log|z_N|\, d\sigma(z).$$

Corollary 8.1.5. *If p is a normalized homogeneous polynomial, then*

$$\|p\|_{\mathbb{S}^{2N-1}} \leq 1.$$

Definition 8.1.6. *If $E \subset \mathbb{S}^{2N-1}$ is compact, then*

$$m_k(E) = \inf\{\|p\|_E : p \text{ is a normalized polynomial of degree } k\}. \tag{8.3}$$

The numbers m_k satisfy

$$0 \leq m_k(E) \leq \|z_1^k\|_{\mathbb{B}_N} = 1 \text{ and } m_{k+k'}(E) \leq m_k(E) m_{k'}(E).$$

The following elementary lemma will enable us to understand the asymptotic behavior of the numbers $m_k(E)$.

Lemma 8.1.7. *If $\{\alpha_k\}_{k=1,\dots}$ is a sequence of nonnegative numbers that satisfy $\alpha_{k+k'} \leq \alpha_k + \alpha_{k'}$, then* $\lim_{k\to\infty} \frac{\alpha_k}{k}$ *exists and is* $\inf_k \frac{\alpha_k}{k}$.

Proof. Let $c = \inf_k \frac{\alpha_k}{k}$. Choose an $\varepsilon > 0$ and then a k_o such that $c \leq \frac{\alpha_{k_o}}{k_o} < c + \varepsilon$. The α's satisfy $\alpha_{pk_o} \leq p\alpha_{k_o}$, so for all positive integers p we have $\frac{\alpha_{pk_o}}{pk_o} \leq \frac{\alpha_{k_o}}{k_o} \leq c + \varepsilon$. For large n, write $n = pk_o + r$ with $0 \leq r < k_o$. Then $\frac{\alpha_n}{n} \leq \frac{\alpha_{pk_o}}{n} + \frac{\alpha_r}{n}$, and when n is large, this is not more than $c + \varepsilon$. The lemma follows.

Corollary 8.1.8. *The limit* $\lim_{k\to\infty}[m_k(E)]^{1/k}$ *exists.*

Definition 8.1.9. *The* projective capacity *of a compact subset E of $\mathbb{S}^{2N-1}$ is the number* $\lim_{k\to\infty}[m_k(E)]^{1/k}$.

The projective capacity of a compact subset E of $\mathbb{S}^{2N-1}$ will be denoted by $\mathsf{pcap}\,E$.

The notion of projective capacity of subsets of projective space is defined in terms of the projective capacity on $\mathbb{S}^{2N-1}$.

Definition 8.1.10. *If $S \subset \mathbb{P}^{N-1}$ is a compact set, then*

$$\mathsf{pcap}\,S = \mathsf{pcap}(\pi^{-1}(S) \cap \mathbb{S}^{2N-1}).$$

These projective capacities are monotone set functions that are invariant under the action of the unitary group and that satisfy $\mathsf{pcap}\,E \leq 1$.

A simple estimate shows some sets to have positive projective capacity.

Lemma 8.1.11. *If $E \subset \mathbb{S}^{2N-1}$ is compact, then*

$$\mathsf{pcap}\,E \geq \exp\left\{\frac{1}{\sigma(E)}\int_{\mathbb{S}^{2N-1}} \log|z_N|\,d\sigma(z)\right\}. \tag{8.4}$$

Proof. If p is a normalized polynomial of degree d, then because $|p| \leq 1$ on $\mathbb{S}^{N-1}$,

$$\begin{aligned} d\int_{\mathbb{S}^{2N-1}} \log|z_N|\,d\sigma(z) &= \int_{S^{2N-1}} \log|p(z)|\,d\sigma(z) \leq \int_E \log|p(z)|\,d\sigma(z) \\ &\leq \sigma(E)\log\|p\|_E. \end{aligned}$$

It follows that

$$\int_{\mathbb{S}^{2N-1}} \log|z_N|\,d\sigma(z) \leq \sigma(E)\log[m_d(E)]^{1/d},$$

which implies the inequality of the lemma.

Thus, subsets of $\mathbb{S}^{2N-1}$ of positive area have positive projective capacity. In particular, we have the following estimate:

Corollary 8.1.12. *The projective capacity of $\mathbb{P}^{N-1}$ satisfies*

$$\mathsf{pcap}\,\mathbb{P}^{N-1} \geq \exp\int_{\mathbb{S}^{2N-1}} \log|z_N|\,d\sigma(z) = \exp\left\{-\frac{1}{2}\sum_{k=1}^{N-1}\frac{1}{k}\right\}.$$

The sum in the last equation satisfies $\sum_{k=1}^{N-1}\frac{1}{k} = \log N + \gamma + o(1)$ with γ Euler's constant, which is $0.5772\ldots$. Thus, $\mathsf{pcap}\,\mathbb{P}^{N-1} \geq \frac{1}{\sqrt{N}}e^{-\frac{\gamma}{2}+o(1)}$. Also, the polynomial $q(z) = (z_1 \cdots z_N)^s$, which is of degree Ns, satisfies $\|q\|_{\mathbb{S}^{2N-1}} = N^{-Ns/2}$, so $m_{Ns}(\mathbb{P}^{N-1}) \leq N^{-Ns/2}$, whence $\mathsf{pcap}\,\mathbb{P}^{N-1} \leq \frac{1}{\sqrt{N}}$, and we have that

$$\frac{1}{\sqrt{N}}e^{-\frac{\gamma}{2}+o(1)} \leq \mathsf{pcap}\,\mathbb{P}^{N-1} \leq \frac{1}{\sqrt{N}}. \tag{8.5}$$

By identifying the projective capacity on $\mathbb{P}^1$ with the elliptic capacity of Tsuji [356], Alexander showed the projective capacity of $\mathbb{P}^1$ to be $e^{-1/2}$. The value of $\mathsf{pcap}\,\mathbb{P}^N$ for other values of N was determined to be $\exp\bigl(-\frac{1}{2}\sum_{j=1}^{N-1}\frac{1}{j}\bigr)$ in the paper [188].

The following result of Alexander is the main theorem relating projective capacity and polynomial convexity.

Theorem 8.1.13. [13] *Let S be a compact subset of $\mathbb{P}^{N-1}$. If* pcap $S = \beta > 0$, *then* $\widehat{Y_S} \supset \bar{\mathbb{B}}_N(\beta)$. *Conversely, if $Y_S \supset \bar{\mathbb{B}}_N(r)$, then* pcap $S \geq r(\text{pcap}\,\mathbb{P}^{N-1})$.

There is a gap between the sizes of the balls considered in the last two statements of the theorem. Whether this gap exists in nature is not clear; its appearance may be an idiosyncrasy of the proof.

It is important to recognize that this theorem *does not* characterize the subsets S of $\mathbb{P}^{N-1}$ for which the truncated cone is polynomially convex.

The proof of Theorem 8.1.13 requires two lemmas.

Lemma 8.1.14. *If $E \subset \mathbb{S}^{2N-1}$ is a compact set and p is a homogeneous polynomial of degree d, then $\|p\|_{\mathbb{S}^{2N-1}} \leq \frac{\|p\|_E}{[\text{pcap}\,E]^d}$.*

Proof. It is enough to prove the inequality when p is normalized, so that by Corollary 8.1.5, $\|p\|_{\mathbb{S}^{2N-1}} \leq 1$. By definition, pcap $E = \inf[m_k(E)]^{1/k}$, so the definition of $m_k(E)$ yields

$$1 \leq \frac{\|p\|_E}{m_d(E)} \leq \frac{\|p\|_E}{[\text{pcap}\,E]^d},$$

whence the result.

Lemma 8.1.15. *If $E \subset \mathbb{S}^{2N-1}$ is a compact, circled set and $p = p_0 + \cdots + p_d$ is a polynomial of degree d with each p_k homogeneous of degree k, then $\|p_k\|_E \leq \|p\|_E$.*

Proof. If $z \in E$ and $\lambda \in \mathbb{C}$, then $p(\lambda z) = \sum_{k=0}^{d} \lambda^k p_k(z)$, so

$$|p_k(z)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\vartheta} p(e^{i\vartheta} z)\, d\vartheta \right| \leq \|p\|_E.$$

Proof of the theorem. We first show that if $|z| \leq$ pcap S, then $z \in \widehat{Y_S}$. Let $E = Y_S \cap \mathbb{S}^{2N-1}$. If $p = p_o + \cdots + p_d$ is a polynomial of degree d with the p_k homogeneous of degree k, then, for $z \neq 0$,

$$|p(z)| \leq \sum_{k=0}^{d} |z|^k \left| p_k\left(\frac{z}{|z|}\right) \right| \leq \sum_{k=0}^{d} |z|^k \frac{\|p\|_E}{[\text{pcap}\,E]^k} \leq \|p\|_E \frac{1}{1 - \frac{|z|}{\text{pcap}\,E}}.$$

Apply this with p^n in place of p, take the nth root of both sides, and let $n \to \infty$ to find that $|p(z)| \leq \|p\|_E$. Thus, $z \in \widehat{Y_S}$.

Conversely, suppose $\widehat{E} \supset \mathbb{B}_N(r)$. If p is a homogeneous polynomial of degree d, then

$$\|p\|_E \geq \|p\|_{\mathbb{B}_N(r))} = r^k \|p\|_{\mathbb{B}_N},$$

whence, for each positive integer k,

$$m_k(E)^{1/k} \geq r[m_k(\mathbb{P}^{N-1})]^{1/k},$$

which yields

$$r \leq \frac{\text{pcap}\,E}{\text{pcap}\,\mathbb{P}^{N-1}}$$

as desired.

Sibony and Wong [334] were the first to obtain a result in the spirit of Theorem 8.1.13. They based their analysis on a notion of capacity due to Ronkin, the Γ-*capacity*, which is discussed in [294]. Among their results is the following: *With homogeneous coordinates* $[z_1 : z_2]$ *on* $\mathbb{P}^1$, *let the compact set E be contained in the set* $\{[1 : z_2] : z_2 \in \mathbb{C}\} \subset \mathbb{P}^1$. *Then* $\widehat{Y}_E \supset \{(z_1, z_2) \in \mathbb{C}^2 : M|z_1| + |z_2| \leq \frac{\mathsf{cap}\, E}{\sqrt{1+M^2}}\}$, *where* $M = \max\{|z_2| : [1 : z_2] \in E\}$ *and* cap *denotes the logarithmic capacity on the plane.*

Theorem 8.1.13 shows the importance of projective capacity for the theory of polynomial convexity. In applications it will be important to understand more clearly the nature of the sets of vanishing projective capacity. A first result in this direction is that the class of these sets is stable under the formation of countable unions:

Theorem 8.1.16. [13] *If* $S_1, \dots$ *are compact subsets of* $\mathbb{P}^{N-1}$ *with union the compact set S, and if* $\mathsf{pcap}\, S_j = 0$ *for each j, then* $\mathsf{pcap}\, S = 0$.

Proof. Suppose for the sake of contradiction that $\mathsf{pcap}\, S > 0$. For $k = 1, \dots,$ let $E_k = \pi^{-1}(S_k) \cap \mathbb{S}^{2N-1}$, and let $E = \pi^{-1}(S) \cap \mathbb{S}^{2N-1}$. The set S, and therefore the set E, has positive projective capacity, so $\widehat{E}$ contains a closed ball $\bar{\mathbb{B}}_N(r)$ for some $r > 0$. Denote by σ_r the normalized surface area measure on $b\mathbb{B}_N(r)$. Remark 1.2.18 applied to the algebra $\mathscr{P}(Y_S)$, which has Shilov boundary E, yields a Jensen measure μ for the measure σ_r: μ is a positive measure of total mass one carried by the set E that satisfies

$$\int \log|f(z)| d\sigma_r(z) \leq \int \log|f(z)|\, d\mu(z)$$

for all $f \in \mathscr{P}(Y_S)$. For some k, $\mu(E_k) > 0$. We shall show that this E_k has positive projective capacity.

If p is a homogeneous polynomial of degree n, then with $\theta = \mu(E_k)$,

$$\int \log|p(z)|\, d\mu(z) \leq \theta \log\|p\|_{E_k} + (1-\theta)\log\|p\|_{\mathbb{S}^{2N-1}}.$$

We also have

$$\int \log|p(z)|\, d\sigma_r(z) = \int \log|p(rz)|\, d\sigma(z) = \int \log|p(z)|\, d\sigma(z) + n\log r,$$

so

$$\int \log|p(z)|\, d\sigma(z) \leq -n\log r + \theta\log\|p\|_{E_k} + (1-\theta)\log\|p\|_{\mathbb{S}^{2N-1}}. \tag{8.6}$$

Recall—Corollary 8.1.4—that

$$\log\|p\|_{\mathbb{S}^{2N-1}} + n\int \log|z_N| d\sigma(z) \leq \int \log|p(z)|\, d\sigma(z). \tag{8.7}$$

It follows from (8.7) and (8.6) that if $c = \exp\big\{(-\log r - \int \log|z_N|\, d\sigma(z))/\theta\big\}$, then

$$\|p\|_{\mathbb{S}^{2N-1}} \leq c^n \|p\|_{E_k}.$$

This inequality, used as in the proof of Theorem 8.1.13, yields that E_k has positive projective capacity.

There is a useful characterization of sets of vanishing projective capacity, found by Alexander [13].

Theorem 8.1.17. *A compact subset of* $\mathbb{P}^N$ *has vanishing projective capacity if and only if it is locally a pluripolar set.*

Recall that a subset E of a complex manifold is locally a pluripolar set if for each point $p \in E$ there is a neighborhood V of p on which there is a plurisubharmonic function u with $E \cap V \subset u^{-1}(-\infty)$, u not identically $-\infty$ near p. If there is a plurisubharmonic function u defined on the whole manifold and not identically $-\infty$ with $E \subset u^{-1}(-\infty)$, then E is a pluripolar set. It is a theorem of Josefson [193] that *in* $\mathbb{C}^N$ *a locally pluripolar set is pluripolar*. There can be no such theorem on $\mathbb{P}^N$ or on any compact complex manifold, because, by the maximum principle, there are no nonconstant plurisubharmonic functions on a compact manifold. In the plane the compact pluripolar sets are the compact sets of zero logarithmic capacity. An alternative proof of Josefson's theorem is given in [207].

The proof of Theorem 8.1.17 requires a lemma.

Lemma 8.1.18. [14] *If E is a nonpluripolar compact subset of the connected complex manifold $\mathscr{M}$, then for every compact subset X of $\mathscr{M}$ there is a constant* $\theta \in (0, 1)$ *such that for every $f \in \mathscr{O}(\mathscr{M})$,*

$$\|f\|_X \le \|f\|_E^{\theta} \|f\|_{\mathscr{M}}^{1-\theta}. \tag{8.8}$$

Proof. If the lemma is false, there exists a sequence $\{f_n\}_{n=1,\dots}$ in $\mathscr{O}(\mathscr{M})$ with $\|f_n\|_{\mathscr{M}} = 1$ and $\|f_n\|_X \ge \|f_n\|_E^{1/n}$. With the functions f_n we construct a plurisubharmonic function u on $\mathscr{M}$ with $u = -\infty$ on E but u not identically $-\infty$, contradicting the assumption that E is nonpluripolar.

To construct u, note that $\log|f_n|$ is negative, since $|f_n| < 1$. Thus, there are positive constants c_n such that $\max_X c_n \log|f_n| = -1$. Let $\varphi_n = c_n \log|f_n|$ and $\varphi = \limsup_{n=1,\dots} \varphi_n$. The function φ is not identically $-\infty$, because if it were, the result of Hartogs, Theorem 1.3.2, and the remark immediately after its proof would yield that $c_n \log|f_n| \to -\infty$ uniformly on compacta, contradicting $\max_X c_n \log|f_n| = -1$. Consequently, there are a point $x \in X$ and a number $q > -\infty$ such that for a sequence $\{n_j\}_{j=1,\dots}$ that increases to ∞, $\varphi_{n_j}(x) > q$. The sum $u = \sum_{j=1,\dots} \varphi_{n_j}/j^2$ is therefore a nonconstant plurisubharmonic function on $\mathscr{M}$ that assumes the value $-\infty$ on the set E. (The function u is plurisubharmonic, because it is the limit of a decreasing sequence of plurisubharmonic functions.) This contradicts the assumption that E is nonpluripolar.

Proof of the theorem. There are two proofs that a compact locally pluripolar set in $\mathbb{P}^N$ has zero projective capacity, both due to Alexander [13, 14]. One of them is quite short, but invokes the theorem of Josefson quoted above. It is as follows. With $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ the usual projection, if S is a compact locally pluripolar subset of $\mathbb{P}^N$, then $\pi^{-1}(E)$ is a locally pluripolar subset of $\mathbb{C}^{N+1}$ and so, by Josephson's theorem, there is a plurisubharmonic function u on $\mathbb{C}^{N+1}$ with $u^{-1}(-\infty) \supset \pi^{-1}(S)$, u not identically $-\infty$. Because the polynomially convex hull of the compact set Y_S coincides with its hull with respect to plurisubharmonic functions by Theorem 1.3.11, it follows that $\widehat{Y_S}$ does not contain an open ball centered at the origin: If it did, then because u is identically $-\infty$ on an open set, it would have to be identically $-\infty$. Theorem 8.1.13 implies now that $\pi^{-1}(S)$ has zero projective capacity and that S itself therefore has zero projective capacity. This completes the first proof. The second proof is longer and invokes some of the ideas of Josefson's proof. We refer to [13] for it.

For the proof in the opposite direction, Alexander again gave two proofs, one in each of the cited papers. The proof given in [14] is this. We have a compact subset S of $\mathbb{P}^N$ that we assume not to be locally pluripolar. Consequently, the set $X_S = Y_S \cap \mathbb{S}^{2N+1}$ is not locally pluripolar in $\mathbb{C}^{N+1} \setminus \{0\}$. This is contained in Lemma 8.1.21, proved, in somewhat more generality, below. It follows that the set X_S is not a pluripolar subset of $\mathbb{B}_{N+1}(2)$. Lemma 8.1.18 provides a $\theta \in (0, 1)$ such that for all $f \in \mathscr{O}(\mathbb{B}_{N+1}(2))$,

$$\|f\|_{\mathbb{B}_{N+1}} \leq \|f\|_{X_S}^{\theta} \|f\|_{\mathbb{B}_{N+1}(2)}^{1-\theta}.$$

If f is a homogeneous polynomial of degree k, this yields

$$\|f\|_{\mathbb{S}^{2N+1}} \leq 2^{k/\vartheta} \|f\|_{X_S}.$$

This inequality, with an argument like that used in the proof of Theorem 8.1.13, implies that $\widehat{X}_S$ contains a ball centered at the origin. The proof is complete.

The notion of plurisubharmonic function extends to functions defined on analytic varieties:

Definition 8.1.19. *If V is an irreducible analytic subvariety of a complex manifold $\mathscr{M}$, the function $u : V \to [-\infty, \infty)$ is* plurisubharmonic *if for each $z \in V$, there is a neighborhood Ω_z of z in $\mathscr{M}$ on which there is defined a plurisubharmonic function $\tilde{u}$ such that $\tilde{u}|(V \cap \Omega_z) = u|(V \cap \Omega_z)$ and u is not identically $-\infty$ on any neighborhood in V of z.*

Thus, a plurisubharmonic function on an analytic variety is plurisubharmonic on the set of manifold points of V.

There is the corresponding notion of pluripolar subset of a variety:

Definition 8.1.20. *If V is an analytic variety in a complex manifold, the subset E of V is* locally pluripolar *if for every $z \in E$, there is a neighborhood V_z of z in V such that for some function u plurisubharmonic on V_z and not identically $-\infty$ near z, $E \subset u^{-1}(-\infty)$. If $E \subset u^{-1}(-\infty)$ for a plurisubharmonic function u defined on all of V and nowhere locally identically $-\infty$, then E is* pluripolar

Lemma 8.1.21. [249] *Let V be an irreducible analytic subvariety of the open set Ω of $\mathbb{P}^{N-1}$. If E is a compact, circled subset of the variety $\pi^{-1}(V)$, then E is locally pluripolar in $\pi^{-1}(V)$ if and only if the compact set $\pi(E)$ is locally pluripolar in V.*

Note that $\pi^{-1}(V)$ is an analytic subvariety of the open set $\pi^{-1}(\Omega)$.

Proof. It is clear that if $\pi(E)$ is a locally pluripolar subset of V, then E is locally pluripolar in $\pi^{-1}(V)$.

For the opposite implication assume E to be locally pluripolar in V, and let E^* be the cone $\{\lambda z : \lambda \in \mathbb{C} \setminus \{0\},\ z \in E\}$. The set E^* is locally pluripolar in $\pi^{-1}(V)$, because E is. Fix a $p \in \pi(E)$, which we take to have projective coordinates $[1 : 0 : \cdots : 0]$. Let L_t be the $\mathbb{C}$-affine hyperplane $\{t\} \times \mathbb{C}^{N-1}$ in $\mathbb{C}^N$. Fix a point $q \in L_1$ with $\pi(q) = p$. The set $L_1 \cap E^*$ is pluripolar in $L_1 \cap \pi^{-1}(V)$ in a neighborhood of q, because E is locally pluripolar in V.

Let now u be a plurisubharmonic function on a neighborhood W_q of q in L_1 that assumes the value $-\infty$ on $E^* \cap W_q$ but that is not identically $-\infty$ near q in L_1. With homogeneous coordinates $[z_1 : \cdots : z_N]$ in $\mathbb{P}^{N-1}$, define $\tilde{u}$ near q by

$$\tilde{u}\left(\left[1 : \frac{z_2}{z_1} : \cdots : \frac{z_N}{z_2}\right]\right) = u\left(1, \frac{z_2}{z_1}, \ldots, \frac{z_N}{z_2}\right).$$

The function $\tilde{u}$ is plurisubharmonic near p in V, vanishes on the part of $\pi(E)$ near p, and is not constantly $-\infty$.

Thus, $\pi(E)$ is locally pluripolar in V.

Theorem 8.1.22. [249] (See also [14].) *Let V be an irreducible analytic subvariety of $\mathbb{P}^{N-1}$. If $S \subset V$ is a compact subset that is not locally pluripolar in V, then $\widehat{Y}_S$ contains a neighborhood of* 0 *in $\pi^{-1}(V \cup \{0\})$.*

Proof. By Lemma 8.1.21, the compact set $X_S = \pi^{-1}(S) \cap \mathbb{S}^{2N-1}$ is nonpluripolar in the subvariety $\pi^{-1}(V)$ of $\mathbb{C}^N \setminus \{0\}$. It is therefore nonpluripolar in $\mathbb{C}^N$ and therefore in the ball $\mathbb{B}_N(2)$. Thus, by Lemma 8.1.18 there is a $\theta \in (0, 1)$ such that for all $f \in \mathscr{O}(\mathbb{B}_N(2))$, $\|f\|_{\pi^{-1}(V)\cap\mathbb{S}^{2N-1}} \leq \|f\|^{\theta}_{X_S}\|f\|^{1-\theta}_{\pi^{-1}(V)\cap\mathbb{B}_N(2)}$. If f is a homogeneous polynomial of degree d, this yields

$$\|f\|_{\pi^{-1}(V)\cap\mathbb{S}^{2N-1}} \leq c^d\|f\|_{X_S}$$

if $c = 2^{1/\theta}$. It follows that if $z \in \pi^{-1}(V)$ satisfies $|z| < 1/c$, then for a polynomial $p = p_0 + \cdots + p_d$ with each p_j homogeneous of degree j,

$$|p(z)| \leq \sum_{k=0}^{d} |z|^k \left|p_k\left(\frac{z}{|z|}\right)\right| \leq \sum_{k=0}^{d} |z|^k c^k \|p\|_{X_S} \leq \|p\|_{X_S}(1 - |z|c)^{-1}.$$

As in the proof of Theorem 8.1.13, this implies that $z \in \widehat{X}_S$.

Theorem 8.1.17 makes it possible to give examples of Theorem 8.1.13. If $S \subset \mathbb{P}^{N-1}$ is an analytic variety, then $\pi^{-1}(S) \cup \{0\}$ is an analytic variety in $\mathbb{C}^N$, and consequently, the truncated cone Y_S is polynomially convex in $\bar{\mathbb{B}}_N$. Varieties are locally pluripolar sets.

With one exception, every complex line through the origin in $\mathbb{C}^2$ can be given by an equation $z_2 = \alpha z_1$ for an $\alpha \in \mathbb{C}$. The number α is the *slope* of the line. If $E \subset \mathbb{P}^1$ is a set of lines whose slopes compose a compact set of positive logarithmic capacity, then the polynomially convex hull $\widehat{Y}_E$ contains a ball centered at the origin. As a particular example, if E is the set of all complex lines in $\mathbb{C}^2$ with slope in the interval $[1, 2]$, then $\widehat{Y}_E$ contains a ball centered at the origin. The same is true if E consists of all lines with slope in the set K obtained from the usual Cantor middle-third set by shifting it one unit to the right (in order to get a compact subset of $\mathbb{C} \setminus \{0\}$), for the Cantor middle-third set formed on the interval [0, 1] is known to have positive logarithmic capacity [262, pp. 145–148].

To pursue this example a little further, introduce the notion of the polynomial envelope of a closed, not necessarily bounded subset of $\mathbb{C}^N$.

Definition 8.1.23. *If X is a closed subset of $\mathbb{C}^N$, then the* polynomial envelope *of X is the set* $\cup\{\widehat{E} : E \subset X,\ E \text{ compact}\}$.

For compact X, this is just the ordinary polynomially convex hull $\widehat{X}$ that we have worked with. In case X is closed but not compact, its polynomial envelope is not compact; it may or may not be closed in $\mathbb{C}^N$.

As an example, let Σ be the cubic surface in $\mathbb{C}^2$ with equation $z_2 = z_1^2\bar{z}_1$, a two-dimensional real submanifold of $\mathbb{C}^2$. The pertinent observation, which is attributed to J. Wiegerinck in [354], is that if $r > 0$, then the disks $\Delta_r = \{(r\zeta, r^3\zeta) : |\zeta| \leq 1\}$ satisfy $b\Delta_r \subset \Sigma$, so that $\cup_{r>0}\Delta_r$ is contained in the polynomial envelope of Σ. For each $r \in \mathbb{R}$, let λ_r be the complex line in $\mathbb{C}^2$ that contains Δ_r. The set $\widehat{\Sigma}$ contains Δ_r and therefore contains the disk $\lambda_r \cap \bar{\mathbb{B}}_2$, provided $r \geq 1$. The subset $S = \{\lambda_r\}_{r\geq 1}$ of $\mathbb{P}^1$ is a set of positive projective capacity, so by Theorem 8.1.13, the truncated cone Y_S contains a neighborhood of the origin. The set Σ is invariant under the transformation $(z_1, z_2) \mapsto (\rho^3 z_1, \rho z_2)$ for each $\rho > 0$, so the same is true of its polynomial envelope. This set contains an open set; it must be the whole of $\mathbb{C}^2$.

As shown above, if $X \subset \mathbb{C}^2$ is a union of complex lines through the origin and if the set of slopes of these lines constitute a set of positive capacity, then the polynomial envelope of X is all of $\mathbb{C}^2$.

We now turn to unions of real planes. A special case of this topic is contained in the discussion of isolated complex tangents of surfaces earlier, where the polynomial convexity of hyperbolic points derives finally from the local polynomial convexity of a certain associated union of totally real planes. The general question is this: Given two $\mathbb{R}$-linear subspaces of $\mathbb{C}^N$, are compact subsets of their union polynomially convex?

In the case of the union of two totally real subspaces of maximal dimension that meet only at the origin, the situation is completely understood from work of Weinstock [366].

We say that a totally real $\mathbb{R}$-affine subspace of $\mathbb{C}^N$ is *maximally real* if its real dimension is N. Each totally real $\mathbb{R}$-affine subspace of $\mathbb{C}^N$ of dimension less than N is contained in many maximally real subspaces. Every maximally real $\mathbb{R}$-linear subspace of $\mathbb{C}^N$ is obtained from $\mathbb{R}^N$ by applying a $\mathbb{C}$-linear automorphisms of $\mathbb{C}^N$ to $\mathbb{R}^N$. In general, this automorphism cannot be chosen to be unitary.

It is immediate that every compact subset of every totally real $\mathbb{R}$-linear subspace E of $\mathbb{C}^N$ is polynomially convex and satisfies $\mathscr{P}(X) = \mathscr{C}(X)$: Without loss of generality E is maximally real. Then there is a nonsingular complex $N \times N$ matrix A such that $E = A\mathbb{R}^N$. The set $Y = A^{-1}X$ is polynomially convex and satisfies $\mathscr{P}(Y) = \mathscr{C}(Y)$, for it is contained in $\mathbb{R}^N$. Consequently, the set X is polynomially convex and satisfies $\mathscr{P}(X) = \mathscr{C}(X)$.

Consider two maximally real $\mathbb{R}$-linear subspaces of $\mathbb{C}^N$, say E and E', with the property that $E \cap E' = \{0\}$. There is a $\mathbb{C}$-linear automorphism of $\mathbb{C}^N$ that carries E' to

$\mathbb{R}^N$; it preserves polynomial convexity. Thus, the general question to be considered is this: *Under what conditions on the maximally real subspace E of $\mathbb{C}^N$ is it true that every compact subset of $\mathbb{R}^N \cup E$ is polynomially convex?*

Let us fix a maximally real $\mathbb{R}$-linear subspace E of $\mathbb{C}^N$ with the property that $\mathbb{R}^N \cap E = \{0\}$. There is then a direct sum decomposition of *real* vector spaces $\mathbb{C}^N = \mathbb{R}^N \oplus E$. Let $v_1, \dots, v_N$ be vectors in E that constitute a basis for E over $\mathbb{R}$. If we write $v_k = v_k' + iv_k''$ with vectors $v_k', v_k'' \in \mathbb{R}^N$, then $\{v_1'', \dots, v_N''\}$ is a basis for $\mathbb{R}^N$, because $\mathbb{R}^N$ and E jointly span $\mathbb{C}^N$ over $\mathbb{R}$. It follows that there is an $N \times N$ real matrix A with $Av_k'' = v_k'$ for $k = 1, \dots, N$. Consequently, E is expressible in the form $E = (A + iI)\mathbb{R}^N$. In this representation E determines the matrix A uniquely.

For an $N \times N$ real matrix A, we denote by M_A the $\mathbb{R}$-linear subspace $(A + iI)\mathbb{R}^N$ of $\mathbb{C}^N$. The space M_A is totally real if and only if i is not an eigenvalue of A: If $Av = iv$ for some nonzero vector in $\mathbb{R}^N$, then v and iv belong to $(A + iI)\mathbb{R}^N$, and $(A + iI)\mathbb{R}^N$ is not totally real. Conversely, if for some $v = v' + iv''$ both v and iv lie in $(A + iI)\mathbb{R}^N$, so that $v = Ax + ix$ and $iv = Ay + iy$ for some choice of x and y, then necessarily $x = -Ay$, and $y = Ax$, so $A(x - iy) = i(x - iy)$, and i is found to be an eigenvalue of A.

The local polynomial convexity of the union $\mathbb{R}^N \cup M_A$ at the origin depends on the spectrum of the matrix A. The analysis will be carried out in terms of the structure of the real Jordan canonical form for A. That this is reasonable may be suggested by the following observation. If S is a nonsingular real $N \times N$ matrix, then $S^{-1}(A + iI)\mathbb{R}^N = (S^{-1}AS + iI)\mathbb{R}^N$, so every compact subset of $\mathbb{R}^N \cup M_A$ is polynomially convex if and only if every compact subset of $\mathbb{R}^N \cup M_{S^{-1}AS}$ is polynomially convex for every nonsingular real $N \times N$ matrix S.

The principal result is this:

Theorem 8.1.24. [366] *If A is a real $N \times N$ matrix, then every compact subset of $\mathbb{R}^N \cup M_A$ is polynomially convex if and only if A has no purely imaginary eigenvalue of modulus greater than one. If all compact subsets of $\mathbb{R}^N \cup M_A$ are polynomially convex, then for every compact subset $X$of $\mathbb{R}^N \cup M_A$, $\mathscr{P}(X) = \mathscr{C}(X)$.*

Thus, generically in A, the compact subsets of the configuration $\mathbb{R}^N \cup M_A$ are polynomially convex.

Corollary 8.1.25. *If $\|A\| < 1$, then compact subsets X of $\mathbb{R}^N \cup M_A$ are polynomially convex and satisfy $\mathscr{P}(X) = \mathscr{C}(X)$.*

Proof of the theorem. Suppose A to have no purely imaginary eigenvalues of modulus greater than one.

Because all the compact subsets of $\mathbb{R}^N \cup M_A$ are polynomially convex when and only when all the compact subsets of $\mathbb{R}^N \cup M_{S^{-1}AS}$ are for every nonsingular S, it is sufficient to consider the case in which the matrix A is in real Jordan canonical form [329]: For every real $N \times N$ matrix A there is a real nonsingular $N \times N$ matrix S such that $S^{-1}AS$ is in real Jordan form. That is, $S^{-1}AS$ is of block diagonal form

$$S^{-1}AS = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_p \end{bmatrix}, \tag{8.9}$$

in which the D_j are $m_j \times m_j$ matrices and all entries in $S^{-1}AS$ outside the D_j are zero. We shall denote by $[D_1, \ldots, D_p]$ the block diagonal matrix on the right of equation (8.9). Each D_j is of one of two forms: A_λ or $C_{s,t}$. The matrices A_λ are defined for $\lambda \in \mathbb{R}$ and are of the form

$$A_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

The λ's that occur are the real eigenvalues of A.

The matrices $C_{s,t}$ are defined for $s, t \in \mathbb{R}$; the ones that occur in our situation are those with $s + it$ an eigenvalue of A, which implies $|t| \le 1$ if $s = 0$, because of the restriction that A have no purely imaginary eigenvalue with modulus greater than one. They are of block form $C_{s,t} = \left[A_{j,k}\right]$, where each $A_{j,k}$ is a 2×2 block. For each j,

$$A_{j,j} = \begin{bmatrix} s & -t \\ t & s \end{bmatrix} \quad \text{and} \quad A_{j,j+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

All the other blocks $A_{i,j}$ are zero.

For a compact subset X of $\mathbb{R}^N \cup M_A$, there are compact sets $X' \subset \mathbb{R}^N$ and $X'' \subset M_A$ such that $X = X' \cup X''$. Both X' and X'' are polynomially convex; the problem is to see that their union is. This is accomplished by considering some special cases and then using the Jordan form to put them together to obtain the general result.

We show first that if A is in the Jordan canonical form A_λ, $\lambda \in \mathbb{R}$, then the conclusions of the theorem are true. To see this, consider first the case $\lambda > 0$ and proceed by induction. Consider the polynomial p defined by $p(z) = z_N^2$. For all N, $p(\mathbb{R}^N) \subset [0, \infty)$. If $N = 2$, then for $z \in (A + iI)\mathbb{R}^2$, $z = (A_\lambda + iI)y$ for some $y \in \mathbb{R}^2$. Then $p(z) = ((\lambda + i)y_2)^2$, so $\Im p(z) = 2\lambda y_2^2$, which is strictly positive unless $y_2 = 0$. We have that $p^{-1}(0) \cap \mathbb{R}^2 = \{0\}$ and that $p^{-1}(0) \cap M_A$ is the real line $\{(\lambda + i)y_1 : y_1 \in \mathbb{R}\}$. Consequently, $p^{-1}(0) \cap X$ is polynomially convex and satisfies $\mathscr{P}(p^{-1}(0) \cap X) = \mathscr{C}(p^{-1}(0) \cap X)$. By Kallin's lemma, Theorem 1.6.19, the set X is polynomially convex and satisfies $\mathscr{P}(X) = \mathscr{C}(X)$. That is, the theorem is true when $N = 2$ and A is of the form A_λ with $\lambda > 0$. Suppose now that the result is correct in $\mathbb{C}^{N-1}$ with matrices $A = A_\lambda$ with $\lambda > 0$. Let $p(z)$ be the polynomial z_N^2. Then $p(X') \subset \mathbb{R}$. If $z = (A + iI)y$ with $y \in \mathbb{R}^N$, then again $\Im p(z) \ge 0$ with equality only when $y_N = 0$. We have that

$$p^{-1}(0) \cap X = X \cap (((\mathbb{R}^N \times \{0\}) \cup M_A) \cap \{y_N = 0\}),$$

an $(N-1)$-dimensional configuration to which the induction hypothesis applies: $p^{-1}(0) \cap X$ is polynomially convex and satisfies $\mathscr{P}(p^{-1}(0) \cap X) = \mathscr{C}(p^{-1}(0) \cap X)$. Consequently, Kallin's lemma applies again to yield the polynomial convexity of X together with the equality $\mathscr{P}(X) = \mathscr{C}(X)$. The case $\lambda < 0$ follows the above lines if we use $-p$ instead of p.

If $\lambda = 0$ a different argument is necessary. In this case, we set $q(z) = (A_0 - iI)z \cdot z$, in which we are using the notation that for $w, w' \in \mathbb{C}^N$, $w \cdot w' = w_1 w_1' + \cdots + w_N w_N'$. For $x \in \mathbb{R}^N$, we have

$$q(x) = (A_0 - iI)x \cdot x = -ix \cdot x + A_0 x \cdot x,$$

so that $\Im q(x) < 0$ unless $x = 0$. Also $q^{-1}(0) \cap X = \{0\}$. If $z \in M_A$, say $z = (A_0 + iI)y$ with $y \in \mathbb{R}^N$, then

$$\Im q(z) = \Im\big[(A_0^2 + I)y \cdot (A_0 + iI)y\big] = (A_0^2 + Iy) \cdot y,$$

so

$$\begin{aligned}\Im q(z) &= \sum_{j=1,\dots,N-2} (y_j + y_{j+2})y_j + y_{N-1}^2 + y_N^2 \\ &= \frac{1}{2} \sum_{j=1,\dots,N-2} (y_j + y_{j+2})^2 + \frac{1}{2}(y_1^2 + y_2^2 + y_{N-1}^2 + y_N^2),\end{aligned}$$

which is positive unless $y = 0$. Kallin's lemma again yields that X is polynomially convex and satisfies $\mathscr{P}(X) = \mathscr{C}(X)$.

Now we consider the case that $A = C_{s,t}$. First, suppose that $s \neq 0$. If $N = 2$, let $p(z) = s(z_1^2 + z_2^2)$. We have $p(\mathbb{R}^N) \subset \mathbb{R}$, and for $z \in M_A$, say $z = (C_{s,t} + iI)y$ with $y \in \mathbb{R}^2$, we find that $\Im p(z) = 2s(y_1^2 + y_2^2)$, which is not zero unless $y = 0$. Again X is found to be polynomially convex and to satisfy $\mathscr{P}(X) = \mathscr{C}(X)$. Assume now that $N = 2k$ and that the result holds in $\mathbb{C}^{2k-2}$. With $p(z) = s(z_{N-1}^2 + z_N^2)$, we have $p(X') \subset \mathbb{R}$,

$$p(X'') \subset \{0\} \cup \{w \in \mathbb{C} : \Im w > 0\},$$

and

$$p^{-1}(0) \cap (\mathbb{R}^N \cup M_A) \subset (\mathbb{R}^{N-2} \times \{0\}) \cup M_{A'}$$

with A' the matrix obtained from A by suppressing the bottom two rows and the rightmost two columns of A. The union on the right of the last inclusion is a configuration in $\mathbb{C}^{N-2}$ to which the induction hypothesis applies, so the result we seek again follows.

If $s = 0$, then necessarily $|t| \leq 1$, and the argument is parallel to the one just given, using $p(z) = z_1^2 + z_2^2$ in the case that $N = 2$ and $p(z) = z_{N-1}^2 + z_N^2$ when $N = 2k > 2$.

We now consider the general case: The matrix A is in the real Jordan canonical form $A = [D_1, \dots, D_p]$ with each of the blocks D_k of size $n_k \times n_k$ and $n_1 + \cdots + n_p = N$. There is a corresponding orthogonal decomposition $\mathbb{C}^N = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_p}$. For each $k = 1, \dots, p$, let $\eta_k : \mathbb{C}^N \to \mathbb{C}^{n_k}$ be the orthogonal projection. If $X \subset \mathbb{R}^N \cup M_A$ is a compact set, then $\eta_k(X) \subset \mathbb{R}^{n_k} \cup (A + iI)\mathbb{R}^{n_k}$ is a compact subset that is polynomially convex and that satisfies $\mathscr{P}(\eta_k(X)) = \mathscr{C}(\eta_k(X))$. It follows that $\mathscr{P}(X)$ contains enough real-valued functions to separate the points of X, whence, by the Stone–Weierstrass theorem, $\mathscr{P}(X) = \mathscr{C}(X)$, and then X is polynomially convex. (Recall Theorem 1.2.10.)

We must show, conversely, that if the matrix A has an eigenvalue that is purely imaginary and of modulus greater than one, then $\mathbb{R}^N \cup M_A$ is not polynomially convex. To this end, suppose $i\sigma$ with $\sigma > 1$ to be an eigenvalue for A. Set $r = r(\sigma) = \sqrt{\frac{\sigma+1}{\sigma-1}}$, a certain positive number. Let $v \in \mathbb{C}^N$ be an eigenvector for A associated with the eigenvalue $i\sigma$, and define $\varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C}^N$ by

$$\varphi(\zeta) = \zeta v + \zeta^{-1}\bar{v}.$$

This map carries the unit circle in the ζ-plane into $\mathbb{R}^N$, for $\varphi(e^{i\vartheta})$ is the real vector $e^{i\vartheta} v + e^{-i\vartheta}\bar{v}$. It is more complicated to verify but true that φ carries the circle K_r of radius r centered at the origin of the ζ-plane into the $\mathbb{R}$-linear subspace M_A. To establish this, write $v = v' + iv''$ with v' and v'' real vectors. Then $\varphi(re^{i\vartheta})$ is found to be the vector

$$\left(r + \frac{1}{r}\right)(\cos\vartheta\ v' - \sin\vartheta\ v'') + i\left(r - \frac{1}{r}\right)(\sin\vartheta\ v' + \cos\vartheta\ v'') = T' + iT''.$$

It is then to be verified that $T' = AT''$. This follows after a brief calculation, using the equations $Av' = -\sigma v''$ and $Av'' = \sigma v'$, which follow from $Av = i\sigma v$. Thus if $\Delta(r)$ denotes the annulus $\{\zeta \in \mathbb{C} : 1 \le |\zeta| \le r\}$, then the boundary of the annulus $\varphi(\Delta(r))$ is contained in $\mathbb{R}^N \cup M_A$. The full annulus $\varphi(\Delta(r))$ is not contained in $\mathbb{R}^N \cup M_A$, but its boundary is, so the union $\mathbb{R}^N \cup M_A$ is not polynomially convex.

Weinstock in [366] goes somewhat further than the results of this theorem in that he describes the polynomial envelope of the noncompact set $\mathbb{R}^N \cup (A + iI)\mathbb{R}^N$ when the matrix A has purely imaginary eigenvalues of modulus greater than one. It is a set fibered by annuli like the ones used above or by higher-dimensional analogues of them. Its dimension depends on the number of purely imaginary eigenvalues of modulus greater than one. For the details, we refer to Weinstock's paper. A somewhat different treatment of this hull in the two-dimensional case is given in [354].

The two-dimensional case of Theorem 8.1.24 is the simplest. There the matrix A of the theorem is a real 2×2 matrix; its Jordan canonical form is $\left[\begin{smallmatrix} s & -t \\ t & s \end{smallmatrix}\right]$. The configuration $\mathbb{R}^2 \cup M_A$ is polynomially convex unless trace $A = 0$ and det $A > 1$, i.e., unless $s = 0$ and $|t| > 1$.

The study of unions of more than two totally real planes gives rise to new phenomena: The polynomial envelope of the union of a finite number of totally real planes can be all of the ambient $\mathbb{C}^N$. Such unions were first discussed by Thomas [354], who showed that *in* $\mathbb{C}^2$ *there are three totally real planes that pair-by-pair meet only at the origin and are locally polynomially convex but such that the polynomial envelope of their union is all of* $\mathbb{C}^2$. Thomas also obtained similar results in $\mathbb{C}^N$. Gourlay [153] found that *in* $\mathbb{C}^N$ *there are* $(N+1)$ *maximally real planes whose union has all of* $\mathbb{C}^N$ *as its polynomial envelope.* Gourlay [152] also exhibited some examples of countably infinite unions of totally real planes through the origin in $\mathbb{C}^2$ that are locally polynomially convex.

We now turn our attention to the union of a ball and a real plane. Such a configuration may or may not be polynomially convex. If the plane is $\mathbb{R}^N$, we can deal with sets more general than balls.

The union of a compact subset of $\mathbb{R}^N$ and a *symmetric* polynomially convex subset of $\mathbb{C}^N$ is polynomially convex:

Theorem 8.1.26. *If X is a compact subset of $\mathbb{R}^N$ and if Y is a polynomially convex subset of $\mathbb{C}^N$ that is symmetric with respect to $\mathbb{R}^N$, then $X \cup Y$ is polynomially convex, and*

$$\mathscr{P}(X \cup Y) = \{f \in \mathscr{C}(X \cup Y) : f|Y \in \mathscr{P}(Y)\}.$$

That the set $X \cup Y$ is polynomially convex was established by Chirka and Smirnov [336].

The symmetry condition of the theorem is the condition that if $z = x + iy \in Y$ with $x, y \in \mathbb{R}^N$, then $\bar{z} = x - iy \in Y$. This symmetry condition cannot be entirely abandoned in the theorem: If $X = [-1, 1] \subset \mathbb{R}$ and Y is the upper half of the unit circle—a set that is not symmetric, then the union $X \cup Y$ is not polynomially convex.

Proof. Fix a $w \in \mathbb{C}^N \setminus (X \cup Y)$. We have $w \notin Y$, so $\bar{w} \notin Y$, and there is therefore a polynomial P with $|P| < \frac{1}{4}$ on Y and with $P(w) = 1 = P(\bar{w})$. Then the polynomial Q given by $Q(z) = \frac{1}{2}\{P(z) + \overline{P(\bar{z})}\}$ is real on $\mathbb{R}^N$ and satisfies $Q(w) = 1$, $|Q| < \frac{1}{4}$ on Y. Every compact subset of $\mathbb{R}^N$ is polynomially convex, so there is a polynomial R such that $R(w) = 1$ and $|R| < \frac{1}{2}$ on X.

Two cases are to be considered. First, it may be that $1 \notin Q(X \cup Y)$. In this case, Runge's theorem is enough to show that $w \notin \widehat{X \cup Y}$. If $1 \in Q(X \cup Y)$, then there is a polynomial in one variable, say $p(t)$, such that $p(Q(w)) = 1$ and $|p(Q(z))| < \frac{1}{2\max_Y |R|+1}$ on $(X \cup Y)$. Then the polynomial $(p \circ Q)R$ takes the value 1 at w and is less than one in modulus at the points of $X \cup Y$. Thus, $X \cup Y$ is polynomially convex as claimed.

To show that the algebra $\mathscr{P}(X \cup Y)$ is as stated in the theorem, it is sufficient to show that if μ is a measure on $X \cup Y$ that is orthogonal to all polynomials, then the variation of μ on the set $X \setminus Y$ is zero.

To do this, it is enough to show that if $x_o \in X \setminus Y$, then there is a neighborhood of x_o on which the total variation of μ is zero. The point x_o is a peak point for the algebra $\mathscr{P}(X \cup Y)$, as follows from Rossi's local peak point theorem, Theorem 2.1.13, and the observation that if $F(z) = \exp\bigl\{\sum_{j=1}^N (z_j - z_j^o)^2\bigr\}$, then $F|(X \cup Y) \in \mathscr{P}(X \cup Y)$, and F has the point $x_o = (x_1^o, \dots x_N^o)$ as a local peak point. Consequently, there is $g \in \mathscr{P}(X \cup Y)$ with $g(x_o) = 0$ and $|g| < 1$ on $(X \cup Y) \setminus \{x_o\}$. By the assumed symmetry of Y, the function h defined by $h(z) = \overline{g(\bar{z})}$ is in $\mathscr{P}(X \cup Y)$. The function $f = \frac{1}{2}(g + h)$, which lies in $\mathscr{P}(X \cup Y)$, peaks at x_o, and is real on X. We have that $f(X \cup Y) \subset r\bar{\mathbb{U}} \cup [-1, 1]$ for a suitable $r \in (0, 1)$.

Mergelyan's theorem provides $\varphi \in \mathscr{P}(f(X \cup Y))$ that vanishes on $f(X \cup Y) \setminus (r, 1]$ and that is strictly positive on $(r, 1]$. The function $p = \varphi \circ f$ is in $\mathscr{P}(X \cup Y)$.

If μ is a measure supported by $X \cup Y$ that is orthogonal to all polynomials, then for every polynomial P,

$$\int Pp\, d\mu = 0.$$

The support of the measure $p\,d\mu$ is a compact subset of $\mathbb{R}^N$, and by the Weierstrass approximation theorem, polynomials approximate all continuous functions on compact subsets of $\mathbb{R}^N$. Consequently, $p\,d\mu$ is the zero measure, which implies that the variation of the measure μ over the set where $p \neq 0$ is zero.

We have shown that for every measure on $X \cup Y$ orthogonal to $\mathscr{P}(X \cup Y)$, and for every point $x \in X \setminus Y$, there is a neighborhood of x on which the total variation of μ vanishes. Thus, μ is supported by Y: $\mu \in \mathscr{P}(Y)^\perp$, and the theorem is proved.

Corollary 8.1.27. *If $B_1, \dots, B_r$ are closed balls in $\mathbb{C}^N$ with mutually disjoint interiors and with centers in $\mathbb{R}^N$, and if X is a compact subset of $\mathbb{R}^N$, then the set $Y = X \cup \cup_{j=1}^r B_j$ is polynomially convex and satisfies*

$$\mathscr{P}(Y) = \{f \in \mathscr{C}(Y) | f \text{ is holomorphic on the interior of each } B_j\}.$$

Proof. We need only recall that, by Theorem 1.6.21, the union of the balls B_j is polynomially convex.

In particular, if X is a compact subset of $\mathbb{R}^N$ and $T : \mathbb{C}^N \to \mathbb{C}^N$ is a nonsingular, $\mathbb{C}$-linear transformation, then $T(\bar{\mathbb{B}}_N) \cup T(X)$ is polynomially convex. When T is unitary, so that $T(\bar{\mathbb{B}}_N) = \bar{\mathbb{B}}_N$, the conclusion is that $\bar{\mathbb{B}}_N \cup T(X)$ is polynomially convex. It would be natural to ask whether, for an arbitrary nonsingular linear transformation $T : \mathbb{C}^N \to \mathbb{C}^N$, the set $\bar{\mathbb{B}}_N \cup T(X)$ is polynomially convex. It turns out that this set need not be polynomially convex. When T runs through the group of linear automorphisms of $\mathbb{C}^N$, the $\mathbb{R}$-linear subspace $T(\mathbb{R}^N)$ runs through the space $\mathbb{G}^{\mathrm{TR}}_{2N,N}$ of N-dimensional totally real $\mathbb{R}$-linear subspaces of $\mathbb{C}^N$. The subspaces $T(\mathbb{R}^N)$ with T unitary are special: They are the *Lagrangian* subspaces of $\mathbb{C}^N$, i.e., the N-dimensional real subspaces L of $\mathbb{C}^N$ with the property that if Ω is the form $dz_1 \wedge d\bar{z}_1 + \cdots + dz_N \wedge d\bar{z}_N$ and if $\iota : L \hookrightarrow \mathbb{C}^N$ is the inclusion, then $\iota^*\Omega = 0$. The form Ω is invariant under the action of the unitary group, so because it induces the zero form on $\mathbb{R}^N$, it induces the zero form on $T(\mathbb{R}^N)$ for every unitary transformation T. That conversely, a Lagrangian subspace of $\mathbb{C}^N$ is $T(\mathbb{R}^N)$ for some unitary T can be seen from a calculation: If $v_1, \dots, v_N$ is a basis over $\mathbb{R}$ for L that is orthonormal with respect to the real inner product on $\mathbb{C}^N = \mathbb{R}^{2N}$, then the condition that Ω induce the zero form on L implies that the vectors $v_1, \dots, v_N$ are orthogonal with respect to the Hermitian inner product on $\mathbb{C}^N$. This implies that there is a unitary transformation of $\mathbb{C}^N$ that carries $\mathbb{R}^N$ onto L.

The two-dimensional case is the easiest to examine. Fix a real 2-plane L in $\mathbb{C}^2$ that passes through the origin. By a unitary transformation it can be moved to a real 2-plane L_λ that is given by the equation $z_2 = \lambda\bar{z}_1$ with $\lambda \in [0, 1]$; λ is uniquely determined. That this is so is an exercise in linear algebra: L is given by an equation of the form $A'z_1 + B'z_2 + A''\bar{z}_1 + B''\bar{z}_2 = 0$. A unitary change of coordinates yields coordinates, again denoted by z_1 and z_2, with respect to which the equation of L is

$$z_2 = A\bar{z}_1 + B\bar{z}_2. \tag{8.10}$$

If $A = 0$, then we have the equation $z_2 = B\bar{z}_2$, which defines a three-dimensional real subspace of $\mathbb{C}^2$, contrary to hypothesis. Accordingly, we assume from here that $A \neq 0$.

If $B = 0$, we reach the form $z_2 = \mu\bar{z}_1$ with $\mu \geq 0$ by a rotation in the z_1-plane. Thus, we suppose L to be given by the equation (8.10) in which neither A nor B is zero. Then a rotation in the z_1-axis allows the assumption that $A \geq 0$. We seek a real number λ and a unitary matrix $U = \begin{bmatrix} a & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$, $|\alpha|^2 + |\beta|^2 = 1$, such that if L' is the two-dimensional plane $\{(z, \lambda\bar{z}) : z \in \mathbb{C}\}$, then U carries L' onto L. This leads to the homogeneous system of linear equations for α and $\bar{\beta}$

$$(8.11) \qquad \begin{aligned} (A - \lambda)\alpha - \bar{B}\bar{\beta} &= 0, \\ B\lambda\alpha + (A\lambda + 1)\bar{\beta} &= 0, \end{aligned}$$

which has a nonzero solution if and only if $\det \begin{bmatrix} (A-\lambda) & -\bar{B} \\ B\lambda & A\lambda+1 \end{bmatrix} = 0$. This is a quadratic in λ with real coefficients and with real, positive discriminant. Thus, there are two real solutions, one positive, one negative. For the positive solution, there is a vector $(\alpha, \bar{\beta})$ of length one that solves the system (8.11). This solution leads to a unitary matrix U that carries the 2-plane L' onto L. If the λ we have lies in $(0, 1]$, we are done. If it is in $(1, \infty)$, we interchange z_1 and z_2. The uniqueness assertion is immediate.

If $\lambda = 0$, then L is a complex line, and if $\lambda = 1$, then L is a Lagrangian plane. In these cases, for every compact subset Y of L, the set $\bar{\mathbb{B}}_2 \cup Y$ is polynomially convex. For $\lambda \in (0, 1)$, Chirka and Smirnov [336] have observed that compacta in $\bar{\mathbb{B}}_2 \cup Y$ need not be polynomially convex. To see this, consider the map $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}^2$ given by $g(\zeta) = (\zeta, \lambda/\zeta)$. We have that $g(e^{i\vartheta}) = (e^{i\vartheta}, \lambda e^{-i\vartheta}) \in L$, and that $\|g(e^{i\vartheta})\| = \sqrt{1 + \lambda^2}$. Also, $\|g(\sqrt{\lambda}e^{i\vartheta})\| = \sqrt{2\lambda}$. Because $\lambda \in (0, 1)$, we have $\sqrt{2\lambda} < \sqrt{1 + \lambda^2}$. It follows that the set

$$\bar{\mathbb{B}}_2(\sqrt{2\lambda}) \cup \{(e^{i\vartheta}, \lambda e^{-i\vartheta}) : \vartheta \in \mathbb{R})\}$$

is a compact subset of $\bar{\mathbb{B}}_2(\sqrt{2\lambda}) \cup L$ that is not polynomially convex.

8.2. Pluripolar Graphs

The result to be proved in this section is that holomorphic functions are characterized as those functions that have pluripolar graphs. (Recall the discussion of pluripolar sets immediately after the formulation of Theorem 8.1.17.)

The first result of this section is due to Shcherbina [325]:

Theorem 8.2.1. *If f is a continuous function on a domain in $\mathbb{C}^N$, then f is holomorphic if and only if its graph is pluripolar.*

It is a simple remark that *if the function φ is subharmonic and not identically $-\infty$ on a domain D in $\mathbb{C}$, then φ does not vanish identically on any arc contained in D*. At one point below, we will need a stronger result: *A polar set in $\mathbb{C}$ has zero length.* A sharper result is that a polar set in $\mathbb{C}$ has Hausdorff dimension zero, i.e., that if E is a polar set in $\mathbb{C}$, and $\alpha > 0$, then $\Lambda^\alpha(E) = 0$. This result can be found in Tsuji's book [356]. In $\mathbb{R}^N$, there is an analogous statement: *A polar set in $\mathbb{R}^N$ has Hausdorff dimension not more that*

$N - 2$. (For this, one can consult [40, Theorem 5.9.6].) In particular, for a pluripolar set E in $\mathbb{C}^N$, $\Lambda^\alpha(E) = 0$ if $\alpha > 2N - 2$.

Lemma 8.2.2. *If $X \subset \mathbb{C}^N$ is a compact, polynomially convex set and $E \subset \mathbb{C}^N$ is a compact, pluripolar set, then $\widehat{\{X \cup E\}} \setminus X$ is a pluripolar set.*

Proof. In fact, the set $\widehat{\{X \cup E\}} \setminus X$ is contained in $\varphi^{-1}(-\infty)$ if φ is a plurisubharmonic function on $\mathbb{C}^N$ with $E \subset \varphi^{-1}(-\infty)$. Suppose not for the sake of deriving a contradiction. Then there is a point $p \in \widehat{\{X \cup E\}} \setminus X$ with $\varphi(p) > -\infty$. There is a plurisubharmonic function v with $v(p) > \sup_X v$, because $p \notin X$. For small $\varepsilon > 0$, we have $v(p)+\varepsilon\varphi(p) > \sup_X(v + \varepsilon\varphi)$. Having fixed ε, $v(p) + \varepsilon\varphi(p) > \sup_E(v + \varepsilon\varphi)$, because $\varphi = -\infty$ on E. It follows that the point p is not in the hull of $X \cup E$ with respect to plurisubharmonic functions, whence the lemma, by Theorem 1.3.11.

Lemma 8.2.3. *If D is a domain in $\mathbb{C}$ and $f = u + iv$ is a function with both u and v real-valued and harmonic, then the graph of f is a pluripolar set if and only if f is holomorphic.*

For the rest of the present discussion, we shall use the notation that for a function g, Γ_g denotes the graph of g.

Proof. The problem is local, so it is enough to treat the case that D is the open unit disk $\mathbb{U}$. If f is holomorphic, then $\log|w - f(z)|$ is a plurisubharmonic function on $D \times \mathbb{C}$ that assumes the value $-\infty$ on Γ_f, so this graph is pluripolar.

Conversely, suppose Γ_f to be pluripolar, say $\Gamma_f \subset \varphi^{-1}(-\infty)$ for a plurisubharmonic function φ on $\mathbb{C}^2$ that is not identically $-\infty$.

Let $\tilde{v}$ be the harmonic conjugate of u on $\mathbb{U}$ with $\tilde{v}(z_o) = v(z_o)$ for some fixed point $z_0 \in \mathbb{U}$.

For small $t \in \mathbb{R}$, let

$$\Sigma_t = \{z \in \mathbb{U} : \tilde{v}(z) + t = v(z)\}.$$

For small values of t outside a set of zero length, the set Σ_t is the disjoint union of a locally finite family of analytic curves. Thus, the graph $\Gamma_{u+i(\tilde{v}+t)}$, which is a Riemann surface, meets the graph Γ_f in a system of analytic curves. The function φ is $-\infty$ on Γ_f, so it assumes the value $-\infty$ on an arc in Σ_t and so on all of Σ_t, because an arc in the Riemann surface Σ_t is a nonpluripolar set in that surface. This is true for a set of t's of positive length. Consequently, φ assumes the value $-\infty$ on a set of positive three-dimensional measure in $\mathbb{C}^2$. Such a set is nonpluripolar in $\mathbb{C}^2$, so φ is identically $-\infty$. Contradiction.

Proof of Theorem 8.2.1. As above, if f is holomorphic, its graph is pluripolar.

To prove the opposite implication, begin with the case $N = 1$. The result is local, so we suppose that D is the open unit disk in $\mathbb{C}$, that f is continuous on $\bar{\mathbb{U}}$, and that Γ_f is pluripolar.

Suppose that f is not holomorphic. Under this hypothesis, by Lemma 8.2.3 if $f = u + iv$, then one of u and v is not harmonic. Suppose u not to be harmonic. Let $\tilde{u}$ be continuous on $\bar{\mathbb{U}}$, harmonic on $\mathbb{U}$, and equal to u on $b\mathbb{U}$. Because u is not harmonic, there is a point $z_o \in \mathbb{U}$ at which u and $\tilde{u}$ differ; suppose $u(z_o) < \tilde{u}(z_o)$. Put

$$C = \max\{\sup_{\mathbb{U}}|u(z)|\,,\, \sup_{\mathbb{U}}|v(z)|\}$$

and

$$K = \{(z, w) \in \bar{\mathbb{U}} \times \mathbb{C} : \tilde{u} \leq \Re w \leq 3C, \text{ and } |v(z)| \leq C\}.$$

This set is polynomially convex, for by its definition, it coincides with its hull with respect to plurisubharmonic functions on $\mathbb{C}^2$. To see this, consider a point $p_o = (z_o, w_o) \in \mathbb{C}^2 \setminus K$. If $|z_o| > 1$, then $p_o \notin$ Psh-hull K. If $|z_o| \leq 1$ and $|\Re w_o| > 3C$ or $|\Im w_o| > C$, then again $w_o \notin$ Psh-hull K. Finally, if $|z_o| \leq 1$, $|\Im w_o| \leq C$, and $\Re w_o < \tilde{u}(z_o)$, choose a function u^* harmonic on all of $\mathbb{C}$ such that

$$\sup_{z \in \bar{\mathbb{U}}} |u^*(z) - \tilde{u}(z)| < \frac{1}{2}(\tilde{u}(z_o) - \Re w_o).$$

Then $u^*(z_o) - \Re(z_o) > \frac{1}{2}(\tilde{u}(z_o) - \Re w_o) > 0$, so again p_o is found not to lie in the hull Psh-hull K, and K is found to be polynomially convex as claimed.

Define the domain U by

$$U = \{(z, w) \in \mathbb{U} \times \mathbb{R} : \tilde{u}(z) \leq u \leq u(z) + 2C\},$$

in which we are taking $w = u + iv$. Define $g(z, u) = v(z)$ for $z \in \mathbb{U}$ and $u \in \mathbb{R}$. Then $\sup_{\mathbb{U}} |u| \leq C$, so $\tilde{u}(z) \leq u(z) + 2C \leq 3C$ on U. The graph of g is therefore contained in $\Gamma_f \cup K$, whence $\widehat{\Gamma_g} \subset \widehat{\Gamma_f \cup K}$. Corollary 2.3.15 implies that, with π the projection of $\mathbb{C}^2$ onto $\mathbb{C} \times \mathbb{R}$, $\pi(\Gamma \cup K) \supset U$. By the choice of $\tilde{u}$, the set

$$\tilde{U} = \{(z, u) \in \mathbb{U} \times \mathbb{R} : u(z) < u < \tilde{u}(z)\}$$

is not empty. Also, $K \cap \tilde{U}$ is empty, so

$$\pi\big(\widehat{\Gamma_f \cup K}) \setminus K\big) \supset \tilde{U}.$$

The set on the left is pluripolar by Lemma 8.2.2. Let φ be a plurisubharmonic function on $\mathbb{C}^2$ that takes the value $-\infty$ identically on this set.

Denote by V a neighborhood of z_o in $\mathbb{C}$ such that $u(z) < \tilde{u}(z)$ for all $z \in V$.

For each $a \in \mathbb{C}$, consider the line $\lambda_a = \{(z, w) \in \mathbb{C}^2 : z = a\}$, and let

$$E_a = ((\widehat{\Gamma_f \cup K}) \setminus K) \cap \lambda_a.$$

For each $a \in V$, the projection of E_a on the line $\lambda_a \cap \{v = 0\}$ contains a segment, so for all $a \in V$, $\Lambda^1(E_a) > 0$. A polar set in $\mathbb{C}$ has length zero, so E_a is not polar. This implies that $\varphi = -\infty$ on all of λ_a. Consequently, $\varphi = -\infty$ on an open subset of $\mathbb{C}^2$, whence it is identically $-\infty$ on all of $\mathbb{C}^2$.

This contradiction establishes the result in the case $N = 1$.

The deduction of the general case from the case $N = 1$ is a standard slicing argument. Assume the graph of the continuous function $f : D \to \mathbb{C}$ to be pluripolar for the domain D in $\mathbb{C}^N$. Let the plurisubharmonic function φ on $\mathbb{C}^{N+1}$ assume the value $-\infty$ on $\Gamma(f)$. For almost all points $\zeta' = (\zeta_2, \ldots, \zeta_{N+1})$ in $\mathbb{C}^N$, the function φ_1 defined by $\varphi_1(\eta) = \varphi(\eta, \zeta_1, \ldots, \zeta_N)$ is subharmonic and not identically $-\infty$ on its domain of definition. For any such choice of ζ', the partial function $\eta \mapsto f(\eta, \zeta_1, \dots, \zeta_N)$ is holomorphic on its domain of definition by the case $N = 1$ of the theorem. Continuity implies that this partial function is holomorphic on its domain of definition for *every* choice of $\zeta' \in \mathbb{C}^N$: f is holomorphic in z_1. Similarly, it is holomorphic on the other variables, so by Hartogs's theorem f is holomorphic.

The theorem is proved.

The theorem of Shcherbina just proved should be compared with the much earlier theorem of Hartogs [162], according to which *if f is a continuous function on a domain D in the plane with the property that the domain $(D \times \mathbb{C}) \setminus \Gamma_f$ is pseudoconvex, then f is holomorphic.* In fact, the theorem of Hartogs is somewhat more general than this. It has burgeoned into the theory of *pseudoconcave sets*, for which we refer to the book of Nishino [264].

An extension of Theorem 8.2.1 to multifunctions has been given by Shcherbina; an alternative derivation has been given by Edigarian [108]. The formulation is this:

Theorem 8.2.4. *Let D be a domain in $\mathbb{C}^N$, let $a_1, \dots, a_m$ be continuous $\mathbb{C}$-valued functions on D, and let*

$$P(z, w) = z^m + a_1(w)z^{m-1} + \cdots + a_{m-1}(w)z + a_m(w).$$

The set

$$\Gamma = \{(z, w) \in \mathbb{C} \times D : P(z, w) = 0\}$$

is pluripolar if and only if each of the functions $a_1, \dots, a_m$ is holomorphic.

Proof. [108] If the functions a_j are holomorphic, then Γ is pluripolar, for P is holomorphic on $\mathbb{C} \times D$.

Suppose, conversely, that Γ is pluripolar. The proof that the coefficients a_j are holomorphic is by induction on the degree m of the polynomial P. The case $m = 1$ is correct by Theorem 8.2.1. Assume therefore that the theorem is correct for all degrees strictly less than m. We fix a point $w_o \in D$ and prove the coefficients a_j to be holomorphic near w_o. There are two cases: It may be that the polynomial $P(\cdot, w_o)$ has at least two distinct roots, or it may be that the polynomial $P(\cdot, w_o)$ has a single root of degree m.

In the former case, let $z_1, \dots, z_r$ be the distinct roots of $P(\cdot, w_o)$, z_k a zero of multiplicity m_k, so that $m_1 + \cdots + m_k = m$. Let Ω_1 be a small disk in the z-plane centered at z_1 and containing no other zero of $P(\cdot, w_o)$. There is a neighborhood W_1 of w_o in D so small that for each $w \in W_1$, the polynomial $P(\cdot, w)$ has m_1 zeros in Ω_1. Let these zeros be $\zeta_{1,k}$, $k = 1, \dots, m_1$. Define P_1 by

$$P_1(z, w) = \prod_{k=1,\dots,m_1} (z - \zeta_{1,k}) = z^{m_1} + a_{1,1}(w)z^{m_1-1} + \cdots + a_{1,m_1}(w).$$

The zero locus of P_1 is a pluripolar subset of $\mathbb{C} \times W_1$, and so the coefficients $a_{1,1}, \dots, a_{1,m_1}$ are holomorphic in W_1 by induction.

We treat the other zeros $z_2, \dots, z_r$ in the same way and obtain for $j = 1, \dots, r$, polynomials $P_j(\cdot, w)$ that are defined and holomorphic on $\mathbb{C} \times D_{w_o}$ for a neighborhood

D_{w_o} of w_o contained in D. We have that $\Gamma \cap (\mathbb{C} \times D_{w_o})$ is the set on which the product $P_1 \cdots P_r$ vanishes. Because the coefficients of each P_j are holomorphic, and because the P_j and also P are monic, it follows that P is the product of the P_j, and then that the coefficients a_j of P are polynomials in the coefficients of the P_j and are, therefore, holomorphic near w_o.

We have now to treat the case that the polynomial $P(\cdot, w_o)$ has a unique zero, perforce of degree m. Let $E \subset D$ be the closed subset of D consisting of the points w at which $P(\cdot, w)$ has a root of order m. We know the coefficients a_j of P are holomorphic off the set E. If E is a pluripolar set, we are done, for the functions a_j automatically continue through E. Suppose, therefore, that E is not pluripolar.

Introduce the functions φ_k, $k = 0, \dots, m$, by

$$\varphi_k(z, w) = \frac{\partial^k P}{\partial z^k}(z, w),$$

a monic polynomial in z of degree $m - k$.

We have that

$$\varphi_{m-2}(z, w) = \frac{m!}{2} z^2 + (m-1)! a_1(w) z + (m-2)! a_2(w).$$

Let $\Delta(w)$ be the discriminant of this quadratic:

$$\Delta(w) = [(m-1)! a_1(w)]^2 - 4 \frac{m!}{2} (m-2)! a_2(w).$$

The function Δ is continuous, it vanishes on the set E, and is holomorphic off E. Consequently, Radó's theorem, Theorem 3.4.17, implies that Δ is holomorphic. Then because E is not pluripolar, Δ is identically zero, which implies that

$$a_2(w) = \frac{(m-1)}{2m} a_1(w)^2,$$

which yields

$$\varphi_{m-2}(z, w) = \frac{m!}{2} \left(z + \frac{a_1(w)}{m} \right)^2 .$$

We now prove by downward induction that

$$\varphi_p(z, w) = \frac{m!}{(m-p)!} \left(z + \frac{a_1(w)}{m} \right)^{m-p} . \tag{8.12}$$

For $p = m, m-1$, and $m-2$, we have this fact. Suppose the formula to be true when p is replaced by $p+1$. Then, because φ_{p+1} is the derivative of φ_p with respect to z, we have by the inductive hypothesis that for some function $c_p(w)$ that is independent of z,

$$\varphi_p(z, w) = \frac{m!}{(m-p)!} \left(z + \frac{a_1(w)}{m} \right)^{m-p} + c_p(w).$$

By evaluating φ_p at $(0, w)$, we find that

$$c_p(w) = -\frac{m!}{(m-p)!}\left(\frac{a_1(w)}{m}\right)^{m-p} + p!a_{m-p}(w),$$

so that

$$\varphi_p(z, w) = \frac{m!}{(m-p)!}\left(z + \frac{a_1(w)}{m}\right)^{m-p} - \frac{m!}{(m-p)!}\left(\frac{a_1(w)}{m}\right)^{m-p} + p!a_{m-p}(w).$$

The function c_p is continuous on D and vanishes on the set E. Moreover, it is holomorphic off its zero set. Consequently, by Radó's theorem, it is holomorphic on D, and, because E is not pluripolar, it vanishes identically.

We have therefore established the equation (8.12). From it and Theorem 8.2.1 we find that the function a_1 is holomorphic. The form established for c_p and the vanishing of c_p now imply that a_p is holomorphic for each p.

The theorem is proved.

8.3. Deformations

The operation of passing to the polynomially convex hull exhibits certain continuity properties, which we will treat in this section.

To begin with, there is the simple semicontinuity observation: *If X is a compact set and if U is a neighborhood of $\widehat{X}$, then there is a neighborhood V of X such that if $Y \subset V$, Y compact, then $\widehat{Y} \subset U$.* This is evident: For V take any polynomial polyhedron contained in U that contains X (and hence $\widehat{X}$).

Simple examples show that a small perturbation of a set does not necessarily lead to a small perturbation of the hull.

Example. Let γ be a smooth function defined on the unit circle in the plane that takes the value 1 at the point -1 and that vanishes on the right half of the circle. For $t \geq 0$, let γ_t be the graph of the function $t\gamma$, so that $\gamma_t = \{(e^{i\vartheta}, t\gamma(e^{i\vartheta})) : \vartheta \in \mathbb{R}\}$. This is a family of smooth deformations of the unit circle in the z_1-plane, i.e., of the curve γ_0. The curve γ_0 is not polynomially convex; its hull is the closed unit disk in the z_1-plane. All the curves γ_t for $t > 0$ are polynomially convex: They are smooth, so the hull $\widehat{\gamma_t}$, if it differs from γ_t, is the union of γ_t and a purely one-dimensional variety, V_t. We must have $\bar{V}_t \supset \gamma_t$, because smooth arcs are polynomially convex. Thus, the closures of the two varieties V_t and V_0 share an open analytic arc, the part of the unit circle in the open right half-plane. According to Corollary 3.8.10, the varieties V_0 and V_t coincide or else are contained in a one-dimensional variety. Neither case can occur in the present situation, so each V_t is polynomially convex. A small deformation can destroy the hull.

A second example was given by Forstnerič [123]:

Example. Let $X_\varepsilon = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = \varepsilon|z_1|^2 \text{ and } |z_1| \leq 1\}$. The set X_0 is polynomially convex: It is the closed unit disk in the z_1-plane. However, for $\varepsilon > 0$, the set X_ε is not polynomially convex, for it contains the boundary of the analytic disk $\{(z_1, \delta^2\varepsilon) : |z_1| \leq \delta\}$.

In the opposite direction, there is the following simple observation: If we have a sequence $\{X_j\}_{j=0,\dots}$ of compact sets such that for every neighborhood Ω of X_0, $\widehat{X_j} \subset \Omega$ for all sufficiently large j, and if $\{x_j\}_{j=1,\dots}$ is a sequence of points with $x_j \in \widehat{X}_j$ that converges to a point x_0, then $x_0 \in \widehat{X}_0$. To see this, note that the set $E = \cup_{j=0,\dots} X_j$ is compact. For each $j = 1, \dots,$ let μ_j be a representing measure for the point x_j that is supported in the set X_j. The set of all measures of total mass not more than one carried by the compact set E is compact when it is endowed with the weak* topology. If μ_0 is a weak* limit point of the sequence $\{\mu_j\}_{j=1,\dots}$, then μ_0 is a representing measure for x_0, and it is supported in the set X_0. Thus, $x_0 \in \widehat{X}_0$.

There are positive results about smooth deformations of certain polynomially convex sets.

The context is this. We fix a compact set Y in $\mathbb{C}^N$, and two neighborhoods U and U_1 of Y with $U_1 \Subset U$. Fix a function $\chi \in \mathscr{C}^2(\mathbb{C}^N)$ with $\chi = 0$ on a neighborhood of $\mathbb{C}^N \setminus U$, $\chi = 1$ on a neighborhood of $\bar{U}_1$. Let $\varphi : U \to \mathbb{C}^N$ be a map of class $\mathscr{C}^2$. We consider then the map $\Phi : \mathbb{C}^N \to \mathbb{C}^N$ defined by

$$\Phi(z) = z + \chi(z)\varphi(z). \tag{8.13}$$

The $\mathscr{C}^2$ norm of the function $\chi\varphi$ is understood to be

$$\|\chi\varphi\|_2 = \max \|D^\alpha(\chi\varphi)\|_U,$$

in which the maximum is extended over all derivatives D^α with respect to the underlying real coordinates of order not more than two. If the $\mathscr{C}^2$ norm of $\chi\varphi$ is sufficiently small, then Φ is a diffeomorphism of $\mathbb{C}^N$ onto itself: It is plainly a proper map, and, granted that $\chi\varphi$ is small in the $\mathscr{C}^2$ sense, it is regular. Consequently, it is one-to-one. It follows that it is a $\mathscr{C}^2$ diffeomorphism.

That Φ is a diffeomorphism of class $\mathscr{C}^2$ requires a short argument. First, since Φ is regular and proper, the set $\Phi(\mathbb{C}^N)$ is both open and closed in $\mathbb{C}^N$, so Φ is surjective. We have only to verify that it is injective. For this, notice that because Φ is regular, the determinant of its Jacobian is never zero and so must be either everywhere positive or else everywhere negative. That Φ is the identity near infinity implies that this determinant is everywhere positive. Thus, Φ is orientation-preserving. Now fix a point $w \in \mathbb{C}^N$. We will show that the fiber $\Phi^{-1}(w)$ is a singleton. Suppose not, and let $\{w_1, \dots, w_q\} = \Phi^{-1}(w)$ for some $q \geq 2$. (The regularity implies that this fiber is discrete, and properness then implies that it is finite.) Choose $R > 0$ so large that for each j, $|w_j| < R$. Consider the integral

$$I = c_N \int_{\{z \in \mathbb{C}^N : |z| = R\}} \Phi^* k_{\mathrm{BM}}(z, w)$$

with k_{BM} the Bochner–Martinelli kernel and c_N the constant that appears in the Bochner–Martinelli integral formula. The map Φ is the identity map near the sphere $\{z \in \mathbb{C}^N : |z| = R\}$, so the integral simply evaluates the function identically one at the point w: $I = 1$. On the other hand, Stokes's theorem yields that for small $r > 0$,

$$I = \sum_{j=1}^{q} c_N \int_{b\mathbb{B}_N(w_j,r)} \Phi^* k_{\mathrm{BM}}(z, w).$$

The map Φ is regular, so if r is small enough, Φ carries a neighborhood of the closed ball $\bar{\mathbb{B}}_N(w_j, r)$ onto a neighborhood of the point w. Let Φ_j^{-1} be the inverse of Φ defined on a neighborhood of $\bar{\mathbb{B}}_N(w.r)$ that satisfies $\Phi_j^{-1}(w) = w_j$. We can use Φ_j^{-1} to change variables in each of the integrals on the right of the preceding sum to find that the jth of them is equal to

$$c_N \int_{\Phi(b\mathbb{B}_N(w_j,r))} k_{\mathrm{BM}}(z, w) = 1.$$

Thus $I = q$. However, we already know that $I = 1$, so $q = 1$, and Φ is found to be injective and so a diffeomorphism (of class $\mathscr{C}^2$) as desired.

Theorem 8.3.1. [123] *Let X be a polynomially convex subset of $\mathbb{C}^N$ that is totally real. If $\Phi : \mathbb{C}^N \to \mathbb{C}^N$ is a map of the form* (8.13) *with the $\mathscr{C}^2$ norm of $\chi\varphi$ sufficiently small, then the set $\Phi(X)$ is polynomially convex.*

We shall abbreviate this statement by saying that all sufficiently small compactly supported $\mathscr{C}^2$ perturbations of X are polynomially convex.

Corollary 8.3.2. *If X is a polynomially convex subset of a totally real submanifold of an open subset of $\mathbb{C}^N$, then all sufficiently small compactly supported $\mathscr{C}^2$ perturbations of X are polynomially convex.*

In particular, a small $\mathscr{C}^2$ perturbation of any compact subset of $\mathbb{R}^N$ in $\mathbb{C}^N$ is polynomially convex.

The main point of the theorem is the following observation.

Lemma 8.3.3. *If the compact subset Y of $\mathbb{C}^N$ is totally real and polynomially convex, then there is a nonnegative strictly plurisubharmonic function of class $\mathscr{C}^2$ on $\mathbb{C}^N$ whose zero locus is Y.*

Proof. The set Y is totally real, so there is a bounded open set V on which is defined a nonnegative strictly plurisubharmonic function ρ' of class $\mathscr{C}^2$ with Y as its zero locus. The polynomial convexity of the set Y implies the existence of a polynomial polyhedron W with $Y \subset W \subset \bar{W} \subset V$. The set $\bar{W}$ is polynomially convex, so by Theorem 1.3.8 there is a nonnegative plurisubharmonic function ρ'' on $\mathbb{C}^N$ with $\bar{W}$ as its zero locus. Moreover, ρ'' can be chosen to be strictly plurisubharmonic on $\mathbb{C}^N \setminus Y$.

Given the functions ρ' and ρ'', define ρ by $\rho = C\rho'' + \psi\rho'$, where C is a large positive constant and ψ is a nonnegative smooth function on $\mathbb{C}^N$ that is identically one on W and vanishes on a neighborhood of $\mathbb{C}^N \setminus V$. We understand the function $\psi\rho'$ to be extended by zero outside V. For all choices of C and ψ, the function ρ is nonnegative, of class $\mathscr{C}^2$ on $\mathbb{C}^N$, and vanishes exactly on the set Y. If ψ is chosen first and then C is chosen sufficiently large, then ρ will also be strictly plurisubharmonic on $\mathbb{C}^N$.

Proof of the theorem. The proof is very short. The given set X is polynomially convex and is totally real, so by the lemma, there is a nonnegative strictly plurisubharmonic function on $\mathbb{C}^N$ with X as its zero locus. If the $\mathscr{C}^2$ norm of the map $\chi\varphi$ in (8.13) is small enough, then $\rho \circ \Phi^{-1}$ is a strictly plurisubharmonic function on $\mathbb{C}^N$ with the set $\Phi(X)$ as its zero locus. Thus, $\Phi(X)$ is polynomially convex.

There is a perturbation result for strictly pseudoconvex domains: *If D is a bounded, strictly pseudoconvex domain with boundary of class $\mathscr{C}^2$ in $\mathbb{C}^N$ such that $\bar{D}$ is polynomially convex, then sufficiently small $\mathscr{C}^2$ perturbations of $\bar{D}$ are also polynomially convex.* The proof of this follows precisely the lines of the proof just given, once we recall that there is a $\mathscr{C}^2$ defining function for D that is defined and strictly pseudoconvex on a neighborhood of $\bar{D}$.

In [123] Forstnerič also obtains a result about the polynomial convexity of perturbations of certain other more general sets.

In [12] Alexander discusses the hulls of deformations of bidisks and spheres, and Bedford in [48] treats the stability of the hulls of deformations of the two-dimensional torus $\mathbb{T}^2$.

8.4. Sets with Symmetry

If a compact subset X of $\mathbb{C}^N$ has symmetry properties, it is to be expected that the hull $\widehat{X}$ will exhibit analogous symmetry properties and that an a priori knowledge of this symmetry will be of some assistance in determining $\widehat{X}$. In the present section some particular examples of this kind will be exhibited.

One kind of symmetry that a compact set can have is invariance under a closed, and so compact, subgroup of the unitary group $U(N)$. Any compact group Γ of linear automorphisms of $\mathbb{C}^N$ is conjugate in $GL(N, \mathbb{C})$ to a subgroup of $U(N)$. To see this, let G be a compact subgroup $GL(N, \mathbb{C})$. Because G is compact, there is a G-invariant inner product $\langle\, ,\rangle_G$ on $\mathbb{C}^N$. (If $\langle\, ,\rangle$ is the standard inner product on $\mathbb{C}^N$, let $\langle z, w\rangle_G = \int_G \langle gz, gw\rangle\, dg$, where dg is the Haar measure on G.) Let $u_1, \dots, u_N$ be an orthonormal basis for $\mathbb{C}^N$ with respect to the inner product $\langle\, ,\rangle_G$. The matrix of any $U \in G$ with respect to the basis $u_1, \dots, u_N$ is unitary. Denote by P the change of basis matrix from the basis $u_1, \dots, u_N$ to the standard basis $e_1, \dots, e_N$ for $\mathbb{C}^N$, so that if T is a linear transformation of $\mathbb{C}^N$ with matrix A with respect to the u-basis and matrix B with respect to the standard basis, then $A = P^{-1}BP$. The conjugated group $G' = P^{-1}GP$ is a subgroup of $U(N)$.

The first observation is this:

Lemma 8.4.1. *If the compact subset X of $\mathbb{C}^N$ is invariant under the action of the compact subgroup Γ of $U(N)$, then so are the hulls $\widehat{X}$ and $\mathscr{R}$-*hull *X.*

Proof. It is easiest to remark that the sets complementary to these sets are invariant under the action of Γ. For this, suppose that $x \notin \widehat{X}$, so that there is a polynomial p with $p(x) = 1 > \|p\|_X$. If $x' = \gamma(x)$, $\gamma \in \Gamma$, then $p' = p \circ \gamma^{-1}$ is a polynomial with $p'(x') = 1 > \|p'\|_X$, so $x' \notin \widehat{X}$. If $y \notin \mathscr{R}$-hull X, there is a polynomial q with $q(y) = 0 \notin X$. If $y' = \gamma(y)$ for a $\gamma \in \Gamma$, then $q' = q \circ \gamma^{-1}$ is a polynomial that vanishes at y' but not on $\gamma^{-1}(X) = X$.

We can determine the polynomially convex hull of the unitary group $U(N)$, which we identify with the set of unitary matrices contained in the space $\mathbb{C}^{N\times N}$ of $N \times N$ matrices with complex entries.

The group $U(N)$ is an N^2-dimensional real-analytic submanifold of the $2N^2$-dimensional real manifold $\mathbb{C}^{N\times N}$. It is, moreover, totally real: Because multiplication by a unitary matrix induces a complex-linear automorphism of $\mathbb{C}^{N\times N}$, it is enough to verify the total reality of $U(N)$ at a single point. For this point, we choose the identity I_N. The exponential map $E : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ is given by the usual exponential series: $E(A) = \sum_{k=0,\dots} \frac{A^k}{k!}$, which is a holomorphic map that carries $0 \in \mathbb{C}^{N\times N}$ to I_N, and a short calculation shows that the differential of this map at the origin is the identity map. Thus, E effects a biholomorphism of a neighborhood of $0 \in \mathbb{C}^{N\times N}$ onto a neighborhood of $I_N \in \mathbb{C}^{N\times N}$. The map carries the space $\mathfrak{u}(N)$ of skew Hermitian matrices into $U(N)$. The space $\mathfrak{u}(N)$ is a totally real, real-linear subspace of $\mathbb{C}^{N\times N}$ of real dimension N^2, so a neighborhood of $0 \in \mathfrak{u}(N)$ is carried onto a neighborhood of $I_N \in U(N)$. The total reality of $U(N)$ follows.

We know that no N^2-dimensional compact submanifold of $\mathbb{C}^{N\times N}$ is polynomially convex, so $U(N)$ has nontrivial hull, which we know to have topological dimension at least $N^2 + 1$. We shall see that, in fact, $\widehat{U(N)}$ is the closure of an open, convex subset of $\mathbb{C}^{N\times N}$.

If A is an $N \times N$ matrix, we understand the notation $A \geq 0$ to mean that the eigenvalues of A are all nonnegative. With this notation, there is a simple description of the hull. In this connection, see [184, 377].

Theorem 8.4.2. *For* $N = 1, \dots,$

$$\widehat{U(N)} = \{A \in \mathbb{C}^{N\times N} : I_N - \bar{A}^* A \geq 0\}.$$

Denote by $SU(N)$ the special unitary group, which consists of all the unitary matrices A with $\det A = 1$, a certain closed subgroup of $U(N)$.

Corollary 8.4.3. *For* $N = 1, \dots$

$$\widehat{SU(N)} = \{A \in \mathbb{C}^{N\times N} : I_N - \bar{A}^* A \geq 0 \textit{ and } \det A = 1\}.$$

The corollary is immediate: The function det is holomorphic on $\mathbb{C}^{N\times N}$, so the locus on which it is identically one is an analytic variety, in fact, a complex submanifold of $\mathbb{C}^N$. The group $SU(N)$ is the intersection of $U(N)$ with this variety, so the result follows.

The cases of $N = 1$ are particularly simple: $U(1)$ is the circle group $\mathbb{T}$ with hull the closed unit disk, and $SU(1)$ is just the point 1.

Proof of the theorem. The group $U(N)$ contains the torus Θ that consists of the diagonal $N \times N$ matrices with diagonal entries $e^{i\vartheta_1}, \dots, e^{i\vartheta_N}$ with $\vartheta_1, \dots, \vartheta_N \in \mathbb{R}$. Necessarily $\widehat{U(N)}$ contains $\widehat{\Theta}$. The latter set is the set of all diagonal $(N \times N)$-matrices with diagonal entries of modulus not more than one. Fix attention on an $A \in \mathbb{C}^{N\times N}$ such that $I_N - \bar{A}^* A \geq 0$. Thus the eigenvalues of $\bar{A}^* A$ are no more than one, so the eigenvalues of A are no more than one in modulus. There are unitary matrices U and V such that $A = UDV$ with D a diagonal matrix. The group $U(N)$ is invariant under left and right multiplication by unitary matrices, so the same is true of its hull. Thus, because the diagonal matrix $D = U^{-1}AV^{-1}$ has diagonal entries of modulus not more than one and so lies in $\widehat{U(N)}$, the matrix A also lies in $\widehat{U(N)}$.

Conversely, if $I_N - \bar{A}^* A \geq 0$ fails, then A has an eigenvalue, say λ_1, of modulus greater than one. There is a unitary matrix U such that $U^{-1}AU = T$ with $T = [t_{jk}]_{j,k=1,\dots,N}$ an upper triangular matrix with $t_{11} = \lambda_1$. Let U be the matrix $[u_{jk}]_{j,k=1,\dots,N}$ and let $U^{-1} = [v_{jk}]_{j,k=1,\dots,N}$. Define the linear polynomial P on $\mathbb{C}^{N\times N}$ by the condition that if Z is the matrix $[z_{jk}]_{j,k=1,\dots,N}$ then

$$P(Z) = \sum_{s,t=1,\dots,N} u_{s1} z_{st} v_{1t}.$$

Then $P(A) = \lambda_1$, but $|P(Z)| \leq 1$ for every unitary matrix Z. Thus, $A \notin \widehat{U(N)}$, and the theorem is proved.

The proof just given shows $\widehat{U(N)}$ to be convex with respect to linear polynomials, so it is the closure of a convex domain. Alternatively, the convexity of $\widehat{U(N)}$ follows from the simple observation that

$$I_N - (\tau \bar{A}^* A + (1-\tau)\bar{B}^* B) = \tau(I_N - \bar{A}^* A) + (1-\tau)(I_N - \bar{B}^* B).$$

Theorem 8.4.4. *If Γ is a compact subgroup of $U(N)$, then $\widehat{\Gamma}$ is a multiplicative semigroup of $\mathbb{C}^{N\times N}$.*

We know that $\widehat{U(N)}$ is the set of matrices $\{A \in \mathbb{C}^{N\times N} : I - \bar{A}^* A \geq 0\}$. The hull of Γ is a subset of this set, and the assertion of the theorem implies that this hull is closed under matrix multiplication. In particular, $\widehat{U(N)}$ is closed under matrix multiplication.

This is a special case of more general results concerning the maximal ideal space of an invariant uniform algebra on a compact group, which can be found in Gichev's paper [143] and the references given there.

Corollary 8.4.5. *If $\widehat{\Gamma}$ contains a matrix A with $\|A\| < 1$, then $0 \in \widehat{\Gamma}$.*

For the proof of the theorem, we will evaluate $\mathbb{C}$-valued polynomials in N^2 variables on $N \times N$ matrices. This requires a word of explanation. If we label the N^2 complex variables on which the polynomial P depends as $z_{j,k}$, $1 \leq j, k \leq N$, and let the $N \times N$ matrix A have entries $a_{j,k}$, then $P(A)$ is understood to be the number obtained by replacing in $P(z)$ the variable $z_{j,k}$ by the entry $a_{j,k}$ of A.

Proof of Theorem 8.4.4. We work with the characters of the algebra $\mathscr{P}(\Gamma)$. Each such character χ is determined by the condition that there be a unique matrix $A_\chi \in \widehat{\Gamma}$ such that for every polynomial P on $\mathbb{C}^{N\times N}$, $\chi(P) = P(A_\chi)$. Let φ and ψ be characters of $\mathscr{P}(\Gamma)$ with associated matrices A_φ and A_ψ. Define a functional $\varphi * \psi$ by $\varphi * \psi(P) = P(A_\varphi A_\psi)$. This is a well-defined linear functional on the space of polynomials. It extends to a character of $\mathscr{P}(\Gamma)$. To see this, let μ_φ and μ_ψ be representing measures for φ and ψ with support in Γ. We have that for every polynomial P,

$$\begin{aligned}\int_\Gamma \int_\Gamma P(AB)\, d\mu_\varphi(A)\, d\mu_\psi(B) &= \int_\Gamma \left\{ \int_\Gamma P(AB)\, d\mu_\varphi(A) \right\} d\mu_\psi(B) \\ &= \int_\Gamma P(A_\varphi B)\, d\mu_\psi(B)\end{aligned}$$

$$= P(A_\varphi A_\psi)$$
$$= \varphi * \psi(P).$$

This equality implies that $|\varphi * \psi(P)| \leq \|P\|_\Gamma$, so the functional $\varphi * \psi$ extends to a continuous linear functional on $\mathscr{P}(\Gamma)$. The functional $P \mapsto P(A_\varphi A_\psi)$ is multiplicative on polynomials, so the extended functional $\varphi * \psi$ is multiplicative on $\mathscr{P}(\Gamma)$. It follows that if $A, B \in \widehat{\Gamma}$, then $AB \in \widehat{\Gamma}$, i.e., that $\widehat{\Gamma}$ is a multiplicative semigroup.

As an example, the unitary group $U(N)$ contains the torus Θ considered above that consists of the diagonal matrices with unimodular entries on the diagonal. The polynomially convex hull of this torus is the set of diagonal matrices with diagonal entries of modulus no more than one, which is a semigroup under matrix multiplication.

A compact subgroup of $U(N)$ may be polynomially convex. Each finite subgroup is an example. A continuous example is the group Γ of 2×2 matrices with diagonal entries $e^{i\vartheta}$ and $e^{-i\vartheta}$. If $Z = [z_{j,k}]_{j,k=1,2}$ is the generic 2×2 matrix and if the polynomial P is defined on $\mathbb{C}^{2\times 2}$ by $P(Z) = z_{1,1}z_{2,2}$, then the variety $V = \{Z \in \mathbb{C}^{2\times 2} : P(Z) = 1\}$ meets $\widehat{U}(2)$ along the subgroup Γ and nowhere else. It follows that Γ is polynomially convex.

If the compact subset X of $\mathbb{C}^N$ is invariant under the action of the *finite* group Γ, there is not much to be said about the hull, beyond the remark that it, too, is invariant under the action of Γ. There is, though, a procedure that, in principle, can be applied. Denote by $\mathbb{C}_\Gamma[z]$ the ring of polynomials on $\mathbb{C}^N$ that are invariant under the action of the group Γ. This ring is finitely generated as an algebra over $\mathbb{C}$; let $g_1, \dots, g_r$ be a set of generators for it. The map $G : \mathbb{C}^N \to \mathbb{C}^r$ determined by $G(z) = (g_1(z), \dots, g_r(z))$ carries X onto a compact subset Y of $\mathbb{C}^r$. The compact subset $G^{-1}(\widehat{Y})$ of $\mathbb{C}^N$ is the polynomially convex hull of X.

For infinite groups, we begin with Hartogs sets.

Definition 8.4.6. *A compact subset Y of $\mathbb{C}^{N+1} = \mathbb{C}^N \times \mathbb{C}$ is a* Hartogs set *if it has the property that for each point $(z, \zeta) \in Y$, the points $(z, e^{i\vartheta}\zeta)$ are in Y for all $\vartheta \in \mathbb{R}$. It is a* complete Hartogs set *if for each point $(z, \zeta) \in Y$, the points $(z, \eta\zeta)$ are in Y for every choice of $\eta \in \mathbb{C}$ with $|\eta| \leq 1$.*

A complete Hartogs set Y is of the form $\{(z, \zeta) : z \in X, |\zeta| \leq r(z)\}$ for some compact subset X in $\mathbb{C}^N$ and some $\mathbb{R}$-valued function r on X. In this situation, Y is said to be the complete Hartogs set *over X defined by the function r*. The condition that Y be compact implies that the function r satisfies $\limsup_{x \to x_o} r(x) \leq r(x_o)$ for all $x_o \in X$, i.e., that r is upper semicontinuous.

The maximum principle in one variable implies that if $Y \subset \mathbb{C}^{N+1}$ is a Hartogs set, then $\widehat{Y}$ is a complete Hartogs set.

More is true: *If Y is a complete Hartogs set over the compact subset X in $\mathbb{C}^N$, then $\widehat{Y}$ is a complete Hartogs set over $\widehat{X}$.* To see this, note that $\widehat{Y} \subset \widehat{X} \times \mathbb{C}$: If $(z, \zeta) \in \widehat{Y}$, but $z \notin \widehat{X}$, then there is a polynomial P in N variables such that $P(z) = 1$, and $\|P\|_X < 1$. Then P, considered as a polynomial on $\mathbb{C}^{N+1}$, satisfies $P(z, \zeta) = 1 > \|P\|_Y$. Thus, $\widehat{Y} \subset \widehat{X} \times \mathbb{C}$. It follows that $\widehat{Y}$ is a complete Hartogs set over $\widehat{X}$.

There is a characterization of the polynomially convex hulls of Hartogs sets over polynomially convex sets:

Theorem 8.4.7. *Let the compact subset Y of $\mathbb{C}^{N+1}$ be the complete Hartogs set over the compact polynomially convex subset X of $\mathbb{C}^N$ defined by the function r on X. The polynomially convex hull $\widehat{Y}$ is the complete Hartogs set Z over X defined by the function $\tilde{\rho}$ given by $\tilde{\rho}(x) = \inf \rho(z)$, where ρ runs through the family of functions defined on a neighborhood of X that satisfy $\rho(x) > r(x)$ when $x \in X$ and the condition that $-\log \rho$ be plurisubharmonic on a neighborhood of X.*

The analysis of Hartogs sets is based on the theory of Hartogs domains. Recall that a *Hartogs domain* is a domain D in $\mathbb{C}^{N+1}$ such that if $(z, \zeta) \in D$ with $z \in \mathbb{C}^N$ and $\zeta \in \mathbb{C}$, then for all $\vartheta \in \mathbb{R}$, the point $(z, e^{i\vartheta}\zeta)$ also belongs to D. The domain D is a *complete Hartogs domain* if $(z, \zeta) \in D$ implies $(z, \eta\zeta) \in D$ for all $\eta \in \mathbb{C}$, $|\eta| \leq 1$. If D is a complete Hartogs domain in $\mathbb{C}^{N+1}$, there is a domain $\Omega \subset \mathbb{C}^N$ on which there is defined a positive lower semicontinuous function ρ such that

$$D = D_{\Omega;\rho} = \{(z, \zeta) \in \Omega \times \mathbb{C} : |\zeta| < \rho(z)\}.$$

If $D = D_{\Omega;\rho}$ is a complete Hartogs domain in $\mathbb{C}^{N+1}$, then each function f holomorphic on D admits an expansion $f(z, \zeta) = \sum_{\nu=0}^{\infty} f_\nu(z)\zeta^\nu$ in which the coefficients f_ν are holomorphic on Ω. This has the consequence that if Ω is a Runge domain in $\mathbb{C}^N$, then $D_{\Omega;\rho}$ is a Runge domain in $\mathbb{C}^{N+1}$. The basic fact we use concerning Hartogs domains is that *if Ω is a domain of holomorphy in $\mathbb{C}^N$, then the Hartogs domain $D_{\Omega;\rho}$ is a domain of holomorphy if and only if $-\log \rho$ is plurisubharmonic.* The theory of Hartogs domains is developed in the standard texts on several complex variables, for example, [287].

Proof of Theorem 8.4.7. Let $Y \subset \mathbb{C}^{N+1}$ be the complete Hartogs set over the polynomially convex set X in $\mathbb{C}^N$ defined by the function r.

Suppose, to begin with, that $\Omega \subset \mathbb{C}^N$ is a Runge domain of holomorphy that contains X and that ρ is a function on Ω that satisfies $\rho > r$ on X and for which $-\log \rho$ is plurisubharmonic. The Hartogs domain $D_{\Omega;\rho}$ is then a domain of holomorphy and so is holomorphically convex: If $E \subset D_{\Omega;\rho}$ is compact, then its hull with respect to the algebra $\mathscr{O}(D_{\Omega;\rho})$ is a compact subset of $D_{\Omega;\rho}$. This shows that $\widehat{X}$ is contained in the set Z of the statement of the theorem, because $D_{\Omega;\rho}$ is a Runge domain, so that the $\mathscr{O}(D_{\Omega;\rho})$-hull of E is the polynomially convex hull $\widehat{E}$.

For the opposite inclusion, notice that the polynomially convex hull $\widehat{Y}$ is an intersection $\cap_{j=1,\dots} W_j$ of Hartogs domains of holomorphy each of which is a Runge domain. We suppose $W_j \Supset W_{j+1}$. Let the point (z_o, ζ_o) be in $X \times \mathbb{C}$ and satisfy $\rho(z_o) < |\zeta_o|$ for some function ρ defined on a neighborhood of X for which $-\log \rho$ is plurisubharmonic and for which $\rho(z_o) > r(z_o)$. Thus, for large j, the point (z_o, ζ_o) is outside Ω_j. Fix a sufficiently large index j, and, having fixed j, fix a function $f \in \mathscr{O}(\Omega_j)$ that has Ω_j as its domain of holomorphy. The function f admits an expansion of the form

$$f(z, \zeta) = \sum_{\nu=0,\dots} f_\nu(z)\zeta^\nu$$

with coefficients holomorphic in the domain $W_j' = W_j \cap (\mathbb{C}^N \times \{0\})$. Denote by $R(z)$ the radius of convergence of this series. For fixed z, Hadamard's formula gives

$$\frac{1}{R(z)} = \limsup_{\nu\to\infty} |f_\nu(z)|^{1/\nu}.$$

If $R(z_o)$ were as large as $|\zeta_o|$, then, by Hartogs's theorem, Theorem 1.3.2, we would have that for large values of ν, $|f_\nu(z)|^{1/\nu} \leq \frac{1}{R(z_o)} + \varepsilon$ for all z near z_o. This would imply that f extends holomorphically outside of Ω_j, a contradiction. The function $-\log R$ is plurisubharmonic on Ω_j, and $R \geq r$ on X. Thus, the point (z_o, ζ_o) does not lie in the set Z, and the theorem is proved.

Another group action that we consider is the action of the N-dimensional torus $\mathbb{T}^N$ on $\mathbb{C}^N$ given by the condition that if $\gamma_\vartheta = (e^{i\vartheta_1}, \ldots, e^{i\vartheta_N}) \in \mathbb{T}^N$, then

$$\gamma_\vartheta(z) = (e^{i\vartheta_1} z_1, \ldots, e^{i\vartheta_N} z_N)$$

for all $z \in \mathbb{C}^N$.

Definition 8.4.8. *A subset X of $\mathbb{C}^N$ is a* Reinhardt set *if it is invariant under the action of the group $\mathbb{T}^N$ on $\mathbb{C}^N$. It is a* complete Reinhardt *set if for each point $z \in X$ and for each $\zeta \in \bar{\mathbb{U}}^n$, the point $(\zeta_1 z_1, \ldots, \zeta_N z_N)$ is in X.*

Again by the maximum principle in one variable, the polynomially convex hull of a compact Reinhardt set is a complete Reinhardt set.

The discussion of hulls of Reinhardt sets involves the notion of convexity with respect to monomials. We shall understand by *holomorphic monomial* a function of the form $M(z) = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ with $\alpha_1, \ldots, \alpha_N$ nonnegative integers.

Definition 8.4.9. *A compact subset X of $\mathbb{C}^N$ is* monomially convex *if*

$$X = \{z \in \mathbb{C}^N : \text{for all holomorphic monomials } M,\ |M(z)| \leq \|M\|_X\}. \tag{8.14}$$

The description of the hull of a Reinhardt set depends on certain *logarithmic representations* of the set. To introduce these, denote by $\mathscr{K}$ the collection of all nonempty subsets of the set $\{1, \ldots, N\}$. For $K \in \mathscr{K}$, let $\mathbb{C}(K) \subset \mathbb{C}^N$ consist of all the points z with $z_p \neq 0$ if and only if $p \in K$. Thus, for example, $\mathbb{C}(\{2, 3\}) = \{(0, z_2, z_3, 0, \ldots, 0) : z_2 z_3 \neq 0\}$. For $K = \{r_1, \ldots, r_k\} \in \mathscr{K}$, define $\ell_K : \mathbb{C}(K) \to \mathbb{R}^k$ by

$$\ell_K(z) = (\log|z_{r_1}|, \ldots, \log|z_{r_k}|).$$

The main fact about the polynomial hulls of Reinhardt sets was obtained in [97] by deLeeuw:

Theorem 8.4.10. *If $X \subset \mathbb{C}^N$ is a compact complete Reinhardt set, then the following are equivalent:*

(a) *X is polynomially convex,*

(b) *X is convex with respect to holomorphic monomials, and*

(c) *for each $K \in \mathscr{K}$, the set $\ell_K(X)$ is convex if not empty.*

Proof. We begin by showing that (c) implies (b). For this, note that for each $K \in \mathscr{K}$, the set $\ell_K(\mathbb{C}(K))$ is a linear subspace of $\mathbb{R}^k$: If $x = \ell_K(z)$ and $x' = \ell_K(z')$ with $z, z' \in \mathbb{C}(K)$, then $x + x' = \ell_K(w)$ if w is the point in $\mathbb{C}(K)$ with coordinates $z_1 z'_1, \dots, z_N z'_N$, and, for $t \in \mathbb{R}$, $tx = \ell_K(z_1^t, \dots, z_N^t)$ for any determination at all of the powers.

That X is a complete Reinhardt set implies that if $x \in \ell_K(X)$ and if $y \in \mathbb{R}^k$ satisfies $y_j \le x_j$ for each j, then $y \in \ell_K(X)$.

We now suppose that for each $K \in \mathscr{K}$, the set $\ell_K(X)$ is convex and show that X is monomially convex. Thus, suppose we are given $z_o \in \mathbb{C}^N \setminus X$, so that $\ell_K(z_o) \notin \ell_K(\mathbb{C}(K) \cap X)$. Call the latter set Y_K. The set Y_K is convex by hypothesis, so it can be separated from the point $\ell_K(z_o)$ by a real linear functional: There is a real linear functional on $\mathbb{R}^k$ such that $\varphi(\ell_K(z_o)) = 1$ and $\varphi < 1$ on $\ell_K(\mathbb{C}(K) \cap X)$. The functional φ is necessarily nonpositive on the negative cone $\{a \in \mathbb{R}^k : a_1, \dots, a_k < 0\}$. Consequently, with $\cdot$ the standard inner product on $\mathbb{R}^k$, there is an $\alpha \in \mathbb{R}^k$ with nonnegative coefficients such that $\varphi(x) = x \cdot \alpha$. If $z \in \mathbb{C}(K)$ and $K = \{n_1, \dots, n_k\}$, then

$$\varphi \circ \ell_K(z) = \alpha_1 \log|z_{n_1}| + \cdots + \alpha_k \log|z_{n_k}|.$$

Thus, if $g(z) = |z_{n_1}|^{\alpha_1} \cdots |z_{n_k}|^{\alpha_k}$, then $g(z_o) > \|g\|_{Y_K}$. This inequality remains correct if we decrease the α's a little, so we can suppose that they are all rational with the same denominator, say $\alpha_j = \mu_j/d$ for positive integers μ_j and d, d independent of j. If $M(z) = z_{n_1}^{\mu_1} \cdots z_{n_k}^{\mu_k}$, then M is a monomial that is bigger in modulus at z_o than on $\mathbb{C}(K) \cap X$. We have that $\|M\|_{\mathbb{C}(K)\cap X} = \|M\|_X$. To see this, suppose $z \in X$ has nonzero coordinates $z_{n_1}, \dots, z_{n_k}$. If z' is the orthogonal projection of z into $\mathbb{C}(K)$, then $z' \in \mathbb{C}(K) \cap X$, and because $M(z) = M(z')$, it follows that $\|M\|_{\mathbb{C}(K)\cap X} \ge |M(z')| = |M(z)|$. Thus $\|M\|_{\mathbb{C}(K)\cap X} \ge \|M\|_X$, so the two must be equal. Thus, X is monomially convex.

If X is monomially convex, it is, a fortiori, polynomially convex. That is, the implication (b) implies (a) is immediate.

It remains to show that polynomial convexity implies the convexity properties (c). Assume that X is polynomially convex. Fix $K \in \mathscr{K}$ and $z, z' \in \mathbb{C}(K) \cap X$. We are to prove that the segment $[\ell_K(z), \ell_K(z')]$ is contained in $\ell_K(X)$. The domain X is a Reinhardt domain, so there is no loss in assuming that the nonzero coordinates of z and of z' are positive. Put $\lambda_j = \log\frac{z_j}{z'_j}$ if $j \in K$. Otherwise, let $\lambda_j = 0$. For $t = \sigma + i\tau \in \mathbb{C}$, if $\sigma < 0$, then the point $z(t) = (z'_1 e^{\lambda_1 t}, \dots, z'_N e^{\lambda_N t})$ is in X. If p is a polynomial, define $p^*(t) = p(z(t))$, a certain exponential polynomial, which is bounded in the strip S in the t-plane defined by $0 \le \sigma \le 1$. The Phragmén–Lindelöf principle implies that $\|p^*\|_S = \|p^*\|_{bS}$. If $\sigma \in [0, 1]$, then $z(t) \in X$, so $\|p^*\|_S \le \|p\|_X$. In particular, if $\sigma = 0$ or 1, then $z(\sigma) \in X$. The set $\ell_K(\mathbb{C}(K) \cap X)$ is seen to be convex because $\ell_K(z(\sigma)) = \sigma\ell_K(z) + (1 - \sigma)\ell_K(z')$.

The theorem is proved.

It is a classical result that complete logarithmically convex Reinhardt domains are the domains of convergence of power series. There is a corresponding result for Reinhardt sets:

Theorem 8.4.11. *If $\sum_\alpha c_\alpha z^\alpha$ is a power series in N variables that converges absolutely precisely on the compact subset X of $\mathbb{C}^N$, then X is a polynomially convex complete Reinhardt set. Conversely, if the compact subset X of $\mathbb{C}^N$ is a polynomially convex complete Reinhardt set, then there is a power series that converges absolutely at the points of X and that converges nowhere else.*

This result was found by Almer [31] and rediscovered by deLeeuw [97].

Proof. First, suppose X to be a polynomially convex complete Reinhardt set in $\mathbb{C}^N$. It is therefore monomially convex. Let $m_1, m_2, \ldots$ be an enumeration of the monomials such that if m_k is a monomial of degree d, then $k \leq Cd^N$, C a constant independent of k and d. For each k, let $r_k = \|m_k\|_X$. The series $\sum_k \frac{1}{k^2}\frac{m_k(z)}{r_k}$ converges absolutely on X. If $z \in \mathbb{C}^N \setminus X$, then for some k_o, $|m_{k_o}(z)| > r_{k_o}$. The series then contains the subseries $\sum_{j=1}^\infty \frac{1}{\mu(k_o,j)^2}\left[\frac{m_{k_o}(z)}{r_{k_o}}\right]^j$, in which $\mu(k_o, j)$ denotes the index of the monomial $m_{k_o}(z)^j$ in the given enumeration. Thus, $\mu(k_o, j) \leq C(k_o j)^N$, and the terms of the subseries are seen not to tend to zero. Consequently, if we rearrange the series $\sum_k \frac{1}{k^2}\frac{m_k(z)}{r_k}$ into a power series, we obtain a power series whose set of absolute convergence is the set X.

The proof in the opposite direction is a minor variant of the standard proof that the domain of convergence of a power series is a logarithmically convex Reinhardt domain. Let $\sum_\alpha c_\alpha z^\alpha$ be a power series in N variables that has the compact subset X of $\mathbb{C}^N$ as its set of absolute convergence. The set X is plainly a Reinhardt set. It is also monomially convex and so polynomially convex. To see this, we use the criterion of Theorem 8.4.10. Using the observation that a closed set E in $\mathbb{R}^N$ is convex if and only if it is *midpoint convex* in the sense that if x, y lie in E then so does the midpoint of the segment $[x, y]$, we see that what has to be proved is that if the series converges absolutely at z and at w, then it converges at the point $u = (u_1, \ldots, u_N)$ that has coordinates u_j satisfying $|u_j|^2 = |z_j||w_j|$. Because $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$, this is clear, for given any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$,

$$\left|u_1^{\alpha_1} \cdots u_N^{\alpha_N}\right| \leq \frac{1}{2}\left(|z_1^{\alpha_1} \cdots z_N^{\alpha_N}| + |w_1^{\alpha_1} \cdots w_N^{\alpha_N}|\right).$$

The theorem is proved.

The final notion of symmetry that we shall consider is that possessed by stars.

Definition 8.4.12. A star with center the origin *in $\mathbb{R}^n$ is a set carried into itself by the homothety $x \mapsto rx$ for every $r \in [0, 1]$.* A star domain *is a star that is an open set.*

In the case of stars, there need not be a group of symmetries, but the homotheties in question do constitute a semigroup.

A star domain need not be bounded.

Every convex domain that contains the origin is a star domain with star center the origin, and every compact convex set that contains the origin is a star with center the origin. For example, in $\mathbb{C}^N$ a closed $2N$-dimensional cube with one vertex at the origin is a compact star with star center the origin.

If $X \subset \mathbb{C}^N$ is a compact star with star center the origin, then the polynomially convex hull $\widehat{X}$ is another such set. Also, if $K_1, \ldots, K_m$ are compact convex subsets of $\mathbb{C}^N$ each of which contains the origin, then $K = \cup_{j=1,\dots,m} K_j$ is a star with star center the origin.

Star domains are Runge domains. This observation goes back at least to Almer [31]; it has been rediscovered several times. See [82]. The proof we give is one given by El Kasimi [112].

Theorem 8.4.13. *If $D \subset \mathbb{C}^N$ is a star domain with star center the origin, then D is a Runge domain.*

Proof. Let D be a star domain with star center the origin. We are to prove that if $f \in \mathscr{O}(D)$, then f is uniformly approximable on compacta in D by polynomials. If not, there are a compact set $K \subset D$ and a function $f \in \mathscr{O}(D)$ such that for some finite regular Borel measure μ carried by K, $\int f\, d\mu \neq 0$, but $\int P\, d\mu = 0$ for every polynomial P. Introduce the function φ defined on a neighborhood of the interval [0, 1] in $\mathbb{C}$ by

$$\varphi(\zeta) = \int f(\zeta x)\, d\mu(x).$$

This function is holomorphic on a neighborhood of the closed interval [0, 1], it vanishes when $|\zeta|$ is small, but, by hypothesis, it does not vanish at the point 1. This contradiction establishes the theorem.

It is easy to determine the envelope of holomorphy of a star domain. To this end, let $D \subset \mathbb{C}^N$ be a star domain with star center the origin. Let $\{X_k\}_{k=1,\dots}$ be a sequence of compact stars with star center the origin, with each X_k a subset of the interior of X_{k+1}, and with $\cup_{k=1,\dots} X_k = D$.

Theorem 8.4.14. [112] *The set $\tilde{D} = \cup_{j=1,\dots} \widehat{X}_k$ is a pseudoconvex star domain with star center the origin into which every $f \in \mathscr{O}(D)$ extends holomorphically.*

Thus, $\tilde{D}$ is the envelope of holomorphy of D. In particular, the envelope of holomorphy of a star domain is single-sheeted.

Proof. That X_k is contained in the interior of X_{k+1} implies that $\tilde{D}$ is a domain. To see that $\tilde{D}$ is an open set, argue as follows. Consider a point $z_o \in \tilde{D}$. Thus, $z_o \in \widehat{D}_k$ for some k. There is a representing measure μ for z_o with support in X_k. By hypothesis, $X_k \subset \operatorname{int} X_{k+1}$, so for $w \in \mathbb{C}^N$, w near 0, the measure μ_w defined on $\mathbb{C}^N$ by

$$\int g(z)\, d\mu_w(z) = \int g(z+w)\, d\mu(z)$$

is supported in int X_{k+1} and satisfies

$$\int P(z)\, d\mu_w(z) = P(z_o + w)$$

for all polynomials P. This implies that $z_o + w \in \widehat{X}_{k+1}$. The set $\tilde{D}$ thus contains a neighborhood of z_o, and $\tilde{D}$ is found to be open. It is plainly a star with star center the origin. It is, moreover, holomorphically convex: Let X be a compact subset of $\tilde{D}$. We want to show that the $\mathscr{O}(\tilde{D})$-hull of X is compact. For this, note that the hull in question is the polynomially convex hull $\widehat{X}$, because $\tilde{D}$ is a Runge domain, and that X is contained in the interior of $\widehat{X}_j$ for sufficiently large j. Thus, $\tilde{D}$ is holomorphically convex.

We have to see that each $f \in \mathscr{O}(D)$ extends holomorphically into the domain $\tilde{D}$. To do this, fix a $j > 1$ and choose a sequence $\{P_k\}_{k=1,\dots}$ of polynomials that converges uniformly on X_j to f. This sequence necessarily converges uniformly on $\widehat{X}_j$, which is a neighborhood of $\widehat{X}_{j-1}$. In this way, we obtain an extension of f to a holomorphic function on $\tilde{D}$. The theorem is proved.

If D is a bounded star domain with star center the origin that is *proper* in the sense that each real ray emanating from the origin meets bD in a single point, then $A(\bar{D}) = \mathscr{P}(\bar{D})$, i.e., each $f \in \mathscr{C}(\bar{D})$ that is holomorphic on D is uniformly approximable on $\bar{D}$ by polynomials. This can be seen by the following short argument. Given $f \in A(\bar{D})$, define f_r by $f_r(z) = f(rz)$, $r \in [0, 1)$. The domain D is proper, so $r\bar{D} \subset D$ when $r \in [0, 1)$. This implies that $f \in \mathscr{P}(\bar{D})$, for D is a Runge domain.

Without the hypothesis that D is a proper star domain, the conclusion of the previous paragraph can fail, as an example of El Kasimi [112] shows.

We finish this discussion of star sets with a result on the polynomial convexity of $\bar{D}$.

Theorem 8.4.15[112] *If D is a bounded, proper star domain with star center* 0, *and if D is a domain of holomorphy, then $\bar{D}$ is polynomially convex.*

Proof. If $\bar{D}$ is not polynomially convex, then for $r \in [0, 1)$ sufficiently near 1, the hull $(r\bar{D})\widehat{\ }$ is not contained in $\bar{D}$. However, D is its own envelope of holomorphy, and we constructed the envelope of holomorphy of D in Theorem 8.4.14 as $\cup_{j=1,\dots}\widehat{X}_j$ in which $\{X_j\}_{j=1,\dots}$ is a suitable sequence of compacta in D that exhausts D. Thus, for each j, $\widehat{X}_j \subset D$. Because $(r\bar{D})\widehat{\ } \subset X_j$ for large j, the set $(r\bar{D})\widehat{\ }$ is contained in D after all. Contradiction. It follows that $\bar{D}$ is polynomially convex.

El Kasimi [112] pointed out the following consequence:

Corollary 8.4.16. *If D is a domain of holomorphy in $\mathbb{C}^N$ with bD of class $\mathscr{C}^1$, then for every $p \in bD$, there is a ball $\mathbb{B}$ such that $p \in b\mathbb{B}$ and $\overline{B \cap D}$ is a polynomially convex set.*

Proof. For $\mathbb{B}$ take a ball centered on the inner normal to bD at p that contains p. If bB is small enough, then $\mathbb{B} \cap D$ is a bounded, proper star domain of holomorphy, so its closure is polynomially convex.

Example. The truncated cone. For $\alpha \in (0, \infty)$, let $\tilde{Y}_\alpha = \{z \in \mathbb{C}^N : |y|^2 \le \alpha|x|^2\}$. That is, with $z = (z_1, \dots, z_N)$ and $z_j = x_j + iy_j$, $z \in \tilde{Y}_\alpha$ if and only if $y_1^2 + \cdots + y_N^2 \le \alpha(x_1^2 + \cdots + x_N^2)$. Thus, the set $\tilde{Y}_\alpha$ is a star with star center the origin; it is a conical neighborhood of $\mathbb{R}^N \setminus \{0\}$. These cones are nested: $\tilde{Y}_\alpha \subset \tilde{Y}_\beta$ if $\alpha \le \beta$. Let Y_α be the cone $\tilde{Y}_\alpha$ truncated by $b\mathbb{B}_N$, so that

$$Y_\alpha = \tilde{Y}_\alpha \cap \bar{\mathbb{B}}_N,$$

which is a compact set. We shall show that *the truncated cone Y_α is polynomially convex if and only if $\alpha \le 1$. If $\alpha > 1$, then $\widehat{Y_\alpha}$ contains a ball $\mathbb{B}_N(r)$ for some positive r.*

To see that Y_α is polynomially convex when $\alpha \in (0, 1]$, introduce the polynomial P given by $P(z) = z_1^2 + \cdots + z_N^2$. Thus, with $\cdot$ the real inner product on $\mathbb{C}^N = \mathbb{R}^{2N}$,

$P(z) = |x|^2 - |y|^2 - 2ix \cdot y$. If $z \in Y_\alpha$, then $|y|^2 \le \alpha|x|^2$, so $\Re P(z) \ge (1-\alpha)|x|^2 \ge 0$, and if $z \in \mathbb{C}^N \setminus Y_\alpha$, then $|y|^2 > \alpha|x|^2$, so that $\Re P(z) < (1-\alpha)x^2$. Fix a $z \in \mathbb{C}^N \setminus Y_\alpha$, say $z = x + iy$, and let ℓ_z be the ray $x + i\tau y$ for $\tau \in [1, \infty)$. We have that, when $\tau \in (1, \infty)$, $\Re P(x + i\tau y) \le (1 - \tau^2\alpha)|x|^2 < (1-\alpha)|x|^2$. Thus, $P(\ell_z)$ is a curve in $\mathbb{C} \setminus P(Y_\alpha)$ that connects $P(z)$ to infinity. Consequently, $z \notin \widehat{Y_\alpha}$. We have shown Y_α to be polynomially convex when $\alpha \in (0, 1]$.

We now have to see that when $\alpha > 1$, the hull $\widehat{Y_\alpha}$ contains a ball centered at the origin. As we have noted, $Y_\alpha \supset Y_1$. If $\varphi(\zeta) = (\zeta, i\zeta)$, then $\varphi(\mathbb{C}) \subset b\tilde{Y}_1$. We have that for every polynomial P,

$$P(0) = \frac{1}{2\pi i} \int_{|\zeta| = \frac{1}{2}} P(\zeta, i\zeta) \frac{dz}{\zeta},$$

so that $0 \in (Y_1 \cap b\mathbb{B}_N(1/2))\widehat{\ }$. Let γ be the circle $\{(\frac{1}{2}e^{\vartheta}, \frac{i}{2}e^{i\vartheta}) : \vartheta \in \mathbb{R}\}$. If $w = (w_1, w_2) \in \mathbb{C}^N$ lies near the origin, then the measure μ_w defined by

$$\int g \, d\mu_w = \frac{1}{2\pi i} \int_{|\zeta| = \frac{1}{2}} g(\zeta + w_1, i\zeta + w_2) \frac{dz}{\zeta}$$

for $g \in \mathscr{C}(\mathbb{C}^N)$ is supported on the translate, γ_w, by w of the circle γ, and it satisfies

$$\int P \, d\mu_w = P(w)$$

for all polynomials P. Since $\alpha > 1$, it follows that for some sufficiently small $r > 0$, each of the curves γ_w for w with $|w| < r$ is contained in Y_α. Consequently, the polynomially convex hull of Y_α contains $\mathbb{B}_N(r)$, as we claimed.

Now let us consider the *open* cone Ω_α, which is defined to be the interior of the closed cone $\tilde{Y}_\alpha$. This domain is not a star domain in the sense of our definition, for it does not contain the origin. It is, however, invariant under the multiplicative group of homotheties $z \mapsto rz$ for $r \in (0, \infty)$. It is also invariant under the natural action of the real orthogonal group $\mathbb{O}(N)$ on $\mathbb{C}^N$. We propose to compute the envelope of holomorphy of Ω_α.

For $\alpha \in (0, 1)$, the boundary of Ω_α is strictly pseudoconvex except at the origin, and $Q(z) = |y|^2 - \alpha|x|^2$ is a strictly plurisubharmonic defining function for Ω_α in that $\Omega_\alpha = \{z : Q(z) < 0\}$, so for α in this range, Ω_α is a domain of holomorphy. Because we have $\Omega_1 = \cup_{\alpha \in (0,1)} \Omega_\alpha$, the domain Ω_1 is also a domain of holomorphy. (The union of a monotonically increasing sequence of domains of holomorphy in $\mathbb{C}^N$ is itself a domain of holomorphy.)

For $\alpha \in (1, \infty)$, the domain Ω_α is not a domain of holomorphy. Its envelope of holomorphy is $\mathbb{C}^N$. To see this, fix a $\beta \in (1, \alpha)$. The function Q defined by $Q(z) = \beta|x|^2 - |y|^2$ is strictly plurisubharmonic on $b\Omega_\beta \setminus \{0\}$, and $\mathbb{C}^N \setminus \Omega_\beta$ is the set $\{z \in \mathbb{C}^N : Q(z) \le 0\}$: The boundary of the domain $\mathbb{C}^N \setminus \bar{\Omega}_\beta$ is strictly pseudoconvex except at the origin.

Consider an $f \in \mathscr{O}(\Omega_\alpha)$. The restriction $f|b\Omega_\beta$ satisfies the tangential Cauchy–Riemann equations on $b\Omega_\beta \setminus \{0\}$. This restriction continues holomorphically through all of $\mathbb{C}^N \setminus \bar{\Omega}_\beta$, as we see in the following way. Let D be the domain $(\mathbb{C}^N \setminus \bar{\Omega}_\beta) \cap \mathbb{B}_N$. We have

$$bD = (b\Omega_\beta \cap \mathbb{B}_N) \cup ((\mathbb{C}^N \setminus \Omega_\beta) \cap b\mathbb{B}_N).$$

Let $\Sigma = (\mathbb{C}^N \setminus \bar{\Omega}_\beta) \cap b\mathbb{B}_N$. If $E(z) = e^{z_1^2+\cdots+z_N^2}$, then $E(0) = 1$, and $|E(z)| < 1$ on Σ. Accordingly, the polynomially convex hull $\widehat{\Sigma}$ does not contain the origin. The set $\widehat{\Sigma} \cup \{0\}$ is polynomially convex. The function $f|b\Omega_\beta \cap \mathbb{B}_N$ continues holomorphically into $D \setminus \widehat{\Sigma}$ by Theorem 5.2.11. This means that each $f \in \mathscr{O}(\Omega_\alpha)$ continues holomorphically into $\mathbb{B}_N(\delta) \setminus \{0\}$ for some positive δ, and so, because of the invariance of Ω_α under the maps $z \mapsto rz$ for $r > 0$, it follows that each $f \in \mathscr{O}(\Omega_\alpha)$ continues holomorphically into all of $\mathbb{C}^N$: $\mathbb{C}^N$ is the envelope of holomorphy of Ω_α when $\alpha \in (1, \infty)$.

We will pursue the study of these cones Ω_α a little further. We denote by T the homothety of $\mathbb{C}^N$ given by $T(z) = 2z$ and the group of homotheties generated by T by Γ. Denote by $\mathscr{H}$ the quotient manifold $(\mathbb{C}^N \setminus \{0\})/\Gamma$. Under the quotient map $\pi : \mathbb{C}^N \setminus \{0\} \to \mathscr{H}$, the domain Ω_α goes onto a domain Ω'_α in $\mathscr{H}$. For $\alpha \in (0, 1)$, Ω'_α is a strictly pseudoconvex domain Ω'_α in the quotient manifold, $\mathscr{H}$, the domain Ω_1 is a domain of holomorphy in $\mathscr{H}$ with Levi flat boundary, and for $\alpha > 1$, the domain Ω_α has Levi concave boundary. The quotient manifold $\mathscr{H}$ is a *Hopf manifold*; it is nonalgebraic. See [199].

For $\alpha \in (0, 1)$, the domain Ω'_α is a Stein manifold and so has a rich supply of holomorphic functions. In the range $\alpha > 1$, the domain Ω'_α has no nonconstant holomorphic functions, and the only meromorphic functions on it are the restrictions of meromorphic functions on $\mathscr{H}$. This we can see as follows. Suppose f to be a meromorphic function on Ω'_α. Then $F = f \circ \pi$ is a nonconstant meromorphic function on the cone Ω_α. We have remarked above that the envelope of holomorphy of Ω_α is all of $\mathbb{C}^N$, so, if f is holomorphic, rather than meromorphic, then F continues holomorphically to all of $\mathbb{C}^N$, say as the function $\tilde{F}$. The function $\tilde{F}$ is invariant under the action of the group Γ, so $\tilde{F}$ is of the form $\tilde{f} \circ \pi$ for a function $\tilde{f}$ holomorphic on the compact manifold $\mathscr{H}$. This is impossible, and we see that Ω_α, for $\alpha \in (1, \infty)$, admits no nonconstant holomorphic functions. The situation for meromorphic functions is similar: If f is meromorphic, then the function F continues to all of $\mathbb{C}^N$ as a meromorphic function, which we denote by $\tilde{F}$, because for a domain in $\mathbb{C}^N$, the envelope of holomorphy coincides with the envelope of meromorphy. The extended function $\tilde{F}$ is invariant under the action of the group Γ, and so is of the form $\tilde{F} = \tilde{f} \circ \pi$ for some function $\tilde{f}$ meromorphic on $\mathscr{H}$. Note, finally, that there are some nonconstant meromorphic functions on $\mathscr{H}$, for if P and Q are homogeneous polynomials of degree d, $d > 0$, on $\mathbb{C}^N$, then the quotient $R = P/Q$ is constant on the orbits of the group Γ and so descends to a function meromorphic on $\mathscr{H}$.

For more on the hulls of cones (and wedges) see the paper [297] of Rosay.

We conclude this section with some brief descriptions of other work that has been done on hulls of sets that admit symmetries.

A. In [373], Wermer considered the hull of a compact set X invariant under the action of the circle group given by $(z_1, z_2) \mapsto (e^{i\vartheta} z_1, e^{-i\vartheta} z_2)$.

B. Gamelin in [138] describes the polynomially convex hull of a compact set X in $\mathbb{C}^2$ that is invariant under the group of automorphisms of the form $(z, w) \mapsto (e^{i\vartheta} z, e^{i\alpha\vartheta} w)$ for various choices of $\alpha < 0$.

C. Debiard and Gaveau [95] consider the action of $SU(2)$ on $\mathbb{C}^3$ defined as follows. Identify $\mathbb{C}^3$ with the vector subspace of the space $\mathbb{C}^{2\times 2}$ of 2×2 complex matrices that consists of the matrices Z of the form $Z = \left[\begin{smallmatrix} z_1 & z_3 \\ z_3 & z_2 \end{smallmatrix}\right]$. With $SU(2)$ the group of all 2×2 complex matrices with determinant one, $SU(2)$ acts on $\mathbb{C}^3$ by taking $U \in SU(2)$ and $Z \in \mathbb{C}^3$ to the matrix UZU^t with U^t the transpose of Z.

D. Anderson [32] considers the action of $SU(2)$ on $\mathbb{C}^3$ introduced by Debiard and Gaveau in **C** above and determines the polynomially convex hulls of compact subsets X of $\mathbb{C}^3$ that are invariant under this action.

E. Sacré [314] studies the N-dimensional analogues of the problems considered by Debiard and Gaveau and by Anderson mentioned in **C** and **D**.

F. Gichev and Latypov [142] discuss the orbits $\{\gamma(x) : \gamma \in \Gamma\}$ for Γ a closed subgroup of $U(N)$ for a fixed point $x \in \mathbb{C}^N$. In particular, a characterization is given of the case in which this orbit is polynomially convex.

G. Gichev [143] studies the maximal ideal spaces of subalgebras of $\mathscr{C}(G)$, G a compact group, that are G-invariant. This involves the study of the polynomially convex hulls of compact subgroups of the unitary group.

H. Bou Attour and Faraut [68] study the hulls of compact G-invariant sets in $\mathbb{C}^N$ when G is the isotropy subgroup for the origin in the group of automorphisms of a bounded symmetric domain of tube type.

I. Kaup and Zaitsev [201] study the orbits of points under certain compact groups acting on bounded symmetric domains.

J. Kaup [200] determines the G-invariant compact polynomially convex sets contaianed in a bounded symmetric domain G/K.

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