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An Alternative to the Metaplectic Representation

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Introduction

The n-dimensional metaplectic group $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ is the twofold cover of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, which is the group of linear transformations of $\mathcal{X}=\mathbb{R}^n\times\mathbb{R}^n$ that preserve the bilinear (alternate) form

$$[\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right),\left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right)]=-\langle x,\,\eta\rangle+\langle y,\,\xi\rangle. \tag{0.1}$$

There is a unitary representation of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R}^n)$, called the metaplectic representation, the image of which is the group of transformations generated by the following ones: the linear changes of variables, the operators of multiplication by exponentials with pure imaginary quadratic forms in the exponent, and the Fourier transformation; some normalization factor enters the definition of the operators of the first and third species. The metaplectic representation was introduced in a great generality in [28] – special cases had been considered before, mostly in papers of mathematical physics – and it is of such fundamental importance that the two concepts (the group and the representation) have become virtually indistinguishable. This is not going to be our point of view: indeed, the main point of this work is to show that a certain finite covering of the symplectic group (generally of degree n) has another interesting representation, which enjoys analogues of most of the nicer properties of the metaplectic representation. We shall call it the anaplectic representation – other coinages that may come to your mind sound too medical – and shall consider first the one-dimensional case, the main features of which can be described in quite elementary terms.

It may not be an exaggeration to claim that among the foundational objects of classical analysis, the one-dimensional Gaussian function $e^{-\pi x^2}$ occupies one of the foremost positions: it is central in Fourier analysis and special function theory, everywhere in probability and, through its appearance in theta functions, it is basic in modular form theory as well. With the help of some of its satellites – the Heisenberg representation and Bargmann–Fock transform, the metaplectic representation, the Weyl calculus – it lies again at the core of fundamental methods of harmonic analysis or partial differential equations; it is also the basis of some mathematical techniques used in quantum field theory.

A starting point of the present work might be the fact that there is an alternative to this function, leading to a different kind of analysis but with a possibly wide range of influence too: this is the Bessel function $|x|^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi\, x^2)$, which lies in the null space of the (formal) harmonic oscillator. It has at infinity the considerable growth of the more obvious function $|x|^{-\frac{1}{2}}\, e^{\pi x^2}$: therefore, it cannot, in general, occur in integrals on the real line of the usual type. Actually, the development of the present analysis requires that we stray away from the usual one in several aspects. Possibly the only mathematical object which will remain as it stands, at least formally, is the Heisenberg representation: but a new notion of integral – not destroying the invariance under translations – will be needed, and the Fourier transformation and associated Weyl calculus of operators will be replaced by some different, quite parallel objects; finally, the usual L^2 scalar product will have to be changed to an indefinite pseudoscalar product.

Turning to the n-dimensional case, let us first recall that the role of the homogeneous space $\mathrm{Sp}(n,\mathbb{R})/U(n)$ in analysis is well documented. On one hand, it is the set of *complex* polarizations of $\mathcal{X}$, *i.e.,* the set of complex structures on this space such that the symplectic form appears as the imaginary part of some (Hilbert) scalar product on $\mathcal{X}$; on the other hand, it is a Hermitian domain (Siegel's domain), a natural place for analysis in Bergman's style. What is more important here is that one may realize the space $L^2(\mathbb{R}^n)$ as a space of vector-valued functions on Siegel's domain, in a way that makes the metaplectic representation appear as quite natural. To introduce the anaplectic representation, we substitute for Siegel's domain a finite covering $\Sigma^{(n)}$ of the space $U(n)/O(n)$ of *real* polarizations of $\mathcal{X}$, *i.e.,* the space of Lagrangian subspaces of $\mathcal{X}$. Again, we consider a certain space of vector-valued functions on $\Sigma^{(n)}$, getting in a natural way a new representation of some covering of the symplectic group as a result. These functions can in turn be identified with scalar functions on $\mathbb{R}^n$: however, in contradiction to the metaplectic case, the class of functions on $\mathbb{R}^n$ which enter the new analysis consists only of functions which extend as entire functions on $\mathbb{C}^n$. The one–dimensional case of this analysis coincides with the one hinted at above. A common point of the metaplectic and anaplectic representations is that each of the two groups of operators normalizes the group of operators arising from the Heisenberg representation: the latter one is formally the same in both cases. The anaplectic representation (only) can be enriched by a rotation of ninety degrees in the complex coordinates on $\mathbb{C}^n$, an operation that corresponds to the matrix $\left(\begin{smallmatrix} -i\,I & 0 \\ 0 & i\,I \end{smallmatrix}\right)$.

The development of anaplectic analysis calls for mathematical techniques rather different from the usual ones, as it depends as much on elementary real algebraic geometry as on Hilbert space methods. Some of the main questions that have to be tackled concern the analytic continuation of functions, and depend on a careful examination of the singularities of certain fractional-linear transformations; homotopy considerations often play a role too.

Except in the one-dimensional case, it seems unlikely that one could define a space of functions on $\mathbb{R}^n$, invariant under the full anaplectic representation, and on which an invariant pseudoscalar product could be defined. However, anaplectic analysis is not concerned solely with representation by the same name. In anaplectic analysis, the spectrum of the harmonic oscillator L is $\mathbb{Z}$ rather than $\frac{n}{2} + \mathbb{N}$, and the usual creation and annihilation operators become raising and lowering operators; also, unless $n = 1$, all the eigenspaces of L are infinite-dimensional. Provided that $n \not\equiv 0 \mod 4$, one can build, in a way unique up to normalization, a pseudoscalar product on the space generated by the eigenfunctions of L just alluded to, with respect to which the infinitesimal generators of the Heisenberg representation are self-adjoint.

Despite its many similarities with the usual analysis, anaplectic analysis differs from it in two major respects. First, there is no natural embedding of, say, the group of one-dimensional anaplectic transformations into the group of two-dimensional ones, that would generalize what is obtained, in the usual analysis, by regarding one of a pair of variables as a parameter. On the other hand, there is in the usual analysis a class of quite simple functions, to wit the exponentials with a second-order polynomial (the real part of which has a positive-definite top-order part) in the exponent, which resists all operations taken from the Heisenberg representation or the metaplectic representation. No comparable class can be described in such simple terms in anaplectic analysis. This is why non-trivial identities can sometimes be obtained by calculations the analogues of which, in the usual analysis, would not produce anything interesting: examples will occur in Section 10.

In the last chapter, we imbed the one-dimensional anaplectic analysis into a one-parameter family of analyses. There is one such analysis for every complex number $\nu \mod 2$, $\nu \notin \mathbb{Z}$: the case when ν is an integer should be regarded as leading to the usual analysis, the case when $\nu = -\frac{1}{2} \mod 2$ is that considered in Section 1. In each case, there is a translation-invariant concept of integral, an associated Fourier transformation and ν-anaplectic representation. When ν is real, $\nu \notin \mathbb{Z}$, there is on the basic relevant space $\mathfrak{A}_\nu$ a pseudoscalar product, invariant both under the Heisenberg representation and under the ν-anaplectic representation: besides, this latter representation, when restricted to the space of even, or odd, functions on $\mathfrak{A}_\nu$ (this depends on whether $\nu \in]-1, 0[+2\,\mathbb{Z}$ or $\nu \in]0, 1[+2\,\mathbb{Z}$), is unitarily equivalent to one of the representations of the universal cover of $SL(2, \mathbb{R})$ as made explicit in [18]; not surprisingly, the series that occurs here is one which does not occur in the Plancherel theorem for the group under consideration.

It is our hope, and belief, that anaplectic analysis will prove useful in several domains: in quantum mechanics (especially in relativistic quantum mechanics), in partial differential equations, in special function theory. Let us only observe to start with that a mathematical analysis based on a harmonic oscillator unbounded from below cannot fail to help in questions in which we would like to have time circulate just as well in two directions. Also, the pseudoscalar product which occurs in the one-dimensional anaplectic analysis has a striking similarity to that which plays a role in the covariant formulation [5, p. 384] or [3, p. 68] of quantum electrodynamics. Concerning the possibility of using anaplectic analysis in partial differential equations, this only has, as yet, the status of wishful thinking. We have, however, initiated the study of the anaplectic Weyl calculus: though we have mostly dealt, up to now, with its more formal aspects only, one may expect that some kind of new pseudodifferential analysis will eventually emerge. Under the name of "Krein spaces", the subject of linear spaces with an indefinite metric is currently under much scrutiny, in particular in connection with spectral problems of an unusual type (*cf.* for instance [19]); such a kind of problems has also been considered by several authors [1, 2] for reasons having to do with PT-symmetry. Anaplectic analysis certainly provides a special domain of research related to this question, with a rich harmonic analysis of its own. Also, when it is completed, the anaplectic pseudodifferential analysis might be a useful tool for this kind of problems in general. Some possible connection between the one-dimensional anaplectic pseudodifferential analysis and a variant of the Lax–Phillips scattering theory for the automorphic wave equation has been briefly hinted at at the end of Section 10. Finally, but this goes beyond our current projects, there is the question whether some version of the anaplectic representation could be developed in the case of local fields such as the fields of p-adic numbers or their quadratic extensions, thus following in the steps of Weil's celebrated paper [28] on the metaplectic representation.

Let me apologize to M. Gell–Mann and Y. Ne'eman [8] for my choice of a title: I simply could not resist its poetic appeal. On the other hand, the first section of this volume will show that no other choice was possible.

Chapter 1

The One-dimensional Anaplectic Representation

In this chapter, we introduce one-dimensional anaplectic analysis in an elementary, though probably somewhat puzzling, way. The trick is to relate the functions u on the real line to be considered – they all extend as entire functions – to uniquely defined 4-tuples of functions. This is not as strange as it might seem, especially in connection with the study of the Fourier transformation: in mathematical tables dealing with this transform, functions are always split into their even and odd parts. Here, the introduction of the four functions f_0, f_1, $f_{i,0}$, $f_{i,1}$ (*cf.* Definition 1.1) is up to some point a matter of convenience, since the last two can be obtained from the first two by analytic continuation. The first ones are not exactly the even and odd parts of u: however, f_0 (*resp.* f_1) characterizes the even (*resp.* odd) part of u, while enjoying better estimates near $+\infty$. The first example, in Proposition 1.2, will make matters clear. A fundamental definition is that (Proposition 1.16) of the linear form Int which substitutes for the notion of integral: in connection with the Heisenberg representation – which is formally defined in the usual way – it makes it possible to define the anaplectic Fourier transformation, from which it is easy (Theorem 1.20) to obtain the anaplectic representation in general.

However, the proof of some major facts (including the characterization, given in Theorem 1.8, of the space $\mathfrak{A}$), requires that one should construct the anaplectic representation as the direct sum of two representations taken from the full non-unitary principal series of $SL(2,\mathbb{R})$. This is the object of Section 2: also, the decomposition (*cf.* Proposition 2.3) of analytic vectors of such a representation into their entire and ramified parts will play a role in several parts of this work. It is the characterization given in Theorem 1.8 that prepares the way for the definition of the anaplectic representation in the n-dimensional setting, to be developed in Chapter 2. We suggest that the reader satisfy himself with a look at the definition ((2.3) and (2.6)) of the representation $\pi_{\rho,\varepsilon}$, at the statements of Proposition

2.3 and of Theorems 2.9–2.11, otherwise jump directly from Section 1 to Section 3 or even Section 4, using the technical Section 2 mostly for reference. Another possibility is to continue the reading of Section 1 with that of Sections 11 and 12, coming back only later to the n-dimensional case.

Possibly the most specific feature of the one-dimensional anaplectic representation (which extends to the higher-dimensional case) is that it includes the complex rotation $\mathcal{R}$ such that $(\mathcal{R}\,u)(x) = u(ix)$: note that rotations by angles $\neq \frac{\pi n}{2}$, $n \in \mathbb{Z}$, are not permitted *in general*. Since the conjugate, under $\mathcal{R}$, of the operator $A = \pi^{\frac{1}{2}}\,(x + \frac{1}{2\pi}\,\frac{d}{dx})$ – also called the annihilation operator in the usual analysis because of its effect on the ground state $x \mapsto e^{-\pi x^2}$ of the harmonic oscillator – is the “creation” operator A^*, the distinction between A and A^*, usually so essential, blurs out, and the spectrum of the anaplectic harmonic oscillator is $\mathbb{Z}$ instead of $\frac{1}{2} + \mathbb{N}$.

1 The one-dimensional case

A representation π of a Lie group G in some complex linear space $\mathfrak{H}$ is a homomorphism π from G to the group of linear automorphisms of $\mathfrak{H}$: we shall usually concern ourselves with non-unitary representations.

Consider the Hilbert space $\mathfrak{H} = L^2(\mathbb{R})$. Given $u \in \mathfrak{H}$ and $(y,\eta) \in \mathbb{R}^2$, the function $\pi(y,\eta)\,u$ defined as

$$(\pi(y,\eta)\,u)(x) = u(x-y)\,e^{2i\pi(x-\frac{y}{2})\eta} \tag{1.1}$$

still lies in $L^2(\mathbb{R})$. An elementary calculation shows that one has

$$\pi(y,\eta)\,\pi(y',\eta') = \pi(y+y',\eta+\eta')\,e^{i\pi\,(-y\,\eta'+y'\,\eta)}. \tag{1.2}$$

Enlarging the group $\mathbb{R}^2$ to the so-called *Heisenberg group* which is the set-theoretic product $\mathbb{R}^2 \times S^1$ endowed with the law of composition defined as

$$(y,\eta\,;\,e^{i\theta}).\,(y',\eta'\,;\,e^{i\theta'}) = (y+y',\eta+\eta'\,;\,e^{i(\theta+\theta'-y\,\eta'+y'\,\eta)}), \tag{1.3}$$

one gets a unitary representation, the Heisenberg representation. Denoting as Q and P the (unbounded) self-adjoint operators on $L^2(\mathbb{R})$ that consist respectively in multiplying by x or taking $(2i\pi)^{-1}$ times the first-order derivative, one may also write, in the sense of Stone’s theorem relative to one-parameter groups of unitary operators,

$$\pi(y,\eta) = e^{2i\pi\,(\eta\,Q - y\,P)}\ : \tag{1.4}$$

we shall also use this notation later, outside the context of unitary operators, then *taking it as a definition of the operator on the right-hand side.*

Still with the same Hilbert space $\mathfrak{H} = L^2(\mathbb{R})$ as before, consider instead of $\mathbb{R}^2$ the group $SL(2,\mathbb{R})$: it is generated by the elements

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.5}$$

where a is an arbitrary positive number and c is an arbitrary real number. It is impossible to find a representation of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R})$ such that the automorphisms $\pi(g)$ associated to the three transformations above should be respectively:

(i) the transformation $u \mapsto v,\ v(x) = a^{-\frac{1}{2}}\, u(a^{-1}x)$;
(ii) the multiplication by the exponential $\exp(i\pi c x^2)$;
(iii) $e^{-\frac{i\pi}{4}}$ times the Fourier transformation $\mathcal{F}$, normalized as

$$(\mathcal{F}u)(\xi) = \int_{-\infty}^{\infty} u(x)\, e^{-2i\pi x\xi}\, dx. \tag{1.6}$$

To see this is immediate, since the fourth power of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the unit matrix, while $\mathcal{F}^4 = I$: despite appearances, dropping the factor $e^{-\frac{i\pi}{4}}$ in the definition of the transformation (iii) would only make matters worse, though it is a little bit harder to see. The difficulty is that if some matrix $g \in SL(2,\mathbb{R})$ can be written as $g = g_1 \dots g_k$, where all factors are of the special type described in (1.5), the product $\pi(g_1)\dots\pi(g_k)$ depends on the decomposition chosen, not only on g: however, the corresponding indeterminacy in such a definition is not that bad, since the *unordered* pair $\pm\,\pi(g_1)\dots\pi(g_k)$ depends only on g. To remedy it completely, one constructs a group "more precise" than $SL(2,\mathbb{R})$, namely the *metaplectic group* $\widetilde{SL}(2,\mathbb{R})$, a twofold covering of $SL(2,\mathbb{R})$: this means a connected Lie group together with a homomorphism: $\widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$, the kernel of which has two elements. That such a group exists is a consequence of the fact that the fundamental group, in the topological sense, of $SL(2,\mathbb{R})$, is $\mathbb{Z}$ (since $SL(2,\mathbb{R})$ has the homotopy type of its compact subgroup $SO(2)$), of which $\mathbb{Z}/2\mathbb{Z}$ is a quotient group: the two elements of $\widetilde{SL}(2,\mathbb{R})$ which are sent to some given $g \in SL(2,\mathbb{R})$ by the homomorphism in question are said to lie above g. One can then show that there exists a unitary representation Met of $\widetilde{SL}(2,\mathbb{R})$ in $L^2(\mathbb{R})$, the *metaplectic representation*, such that, given $g \in SL(2,\mathbb{R})$, the unordered pair $\pm\,\pi(g)$ as defined above should coincide with the pair $\{\mathrm{Met}(\gamma_1),\ \mathrm{Met}(\gamma_2)\}$, where $\{\gamma_1,\ \gamma_2\}$ is the pair of points in the metaplectic group lying above g.

To proceed towards the anaplectic representation, we may start from a complexification of the Heisenberg representation (1.1): that is, we want to substitute for the generic pair $(y,\eta) \in \mathbb{R}^2$ a pair of complex numbers; elements of the complexified Heisenberg group will then be triples $(y,\eta\,;\,\omega)$ with $(y,\eta) \in \mathbb{C}^2$ and $\omega \in \mathbb{C}^\times$. Of course, it is clear that, in this case, $\pi(y,\eta)$ can no longer operate within the space $L^2(\mathbb{R})$, and that we must substitute for this space an appropriate space $\mathfrak{A}$ of entire functions of one variable; also, it is impossible to preserve unitarity. So as to introduce the anaplectic representation, and above all to connect it to the Heisenberg representation, it is suitable to introduce first the definition of a certain space

$\mathfrak{A}$, which is to play the role of a set of analytic vectors of the anaplectic representation. It will be clearly explained in the remark following the proof of Theorem 2.9 why the use of analytic vectors, at least at this first stage, is essential.

Definition 1.1. Let us say that an entire function f of one variable is nice if on one hand $f(z)$ is bounded by a constant times some exponential $\exp(\pi R|z|^2)$, on the other hand the restriction of f to the *positive* half-line is bounded by a constant times some exponential $\exp(-\pi\varepsilon x^2)$: here, R and ε are assumed to be positive. The space $\mathfrak{A}$ consists of all entire functions u of one variable with the following properties:

(i) the even part u_{even} of u coincides with the even part of some nice function f_0 satisfying the property that the function $z \mapsto f_0(iz) + i\, f_0(-iz)$ is nice too;

(ii) the odd part u_{odd} of u coincides with the odd part of some nice function f_1 such that the function $z \mapsto f_1(iz) - i\, f_1(-iz)$ is nice as well.

It will be proven below (Corollary 1.7) that given $u \in \mathfrak{A}$, a pair $(f_0,\, f_1)$ satisfying the above properties is of necessity unique: for short, we shall refer to the pair $(f_0,\, f_1)$ as the $\mathbb{C}^2$-realization of u. We shall go one step further, associating with u the $\mathbb{C}^4$-valued function (indifferently written in line or column form), called the the $\mathbb{C}^4$-realization of u,

$$\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1}) \tag{1.7}$$

with

$$\begin{aligned} f_{i,0}(z) &= \frac{1-i}{2}\,(f_0(iz) + i\, f_0(-iz)),\\ f_{i,1}(z) &= \frac{1+i}{2}\,(f_1(iz) - i\, f_1(-iz)). \end{aligned} \tag{1.8}$$

All four components of $\boldsymbol{f}$ are thus nice functions in the sense of Definition 1.1.

Here is a basic example.

Proposition 1.2. *Set, for x real,*

$$\phi(x) = (\pi\,|x|)^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi\, x^2), \tag{1.9}$$

with [17, p. 66]

$$I_\nu(t) = \sum_{m\geq 0} \frac{(\frac{t}{2})^{\nu+2m}}{m\,!\,\Gamma(\nu+m+1)} \tag{1.10}$$

for $t > 0$. The function ϕ lies in $\mathfrak{A}$.

Proof. Clearly, ϕ extends as an entire even function. Note [17, p. 139] that it has the considerable growth of $|x|^{-\frac{1}{2}}\, e^{\pi x^2}$ as $|x| \to \infty$. Set, however, for $x > 0$,

$$\begin{aligned} \psi(x) &= 2^{\frac{1}{2}}\, \pi^{-\frac{1}{2}}\, x^{\frac{1}{2}}\, K_{\frac{1}{4}}(\pi\, x^2)\\ &= (\pi\, x)^{\frac{1}{2}}\, [I_{-\frac{1}{4}}(\pi\, x^2) - I_{\frac{1}{4}}(\pi\, x^2)]. \end{aligned} \tag{1.11}$$

From (1.10), this is the restriction to $(0,\infty)$ of an entire function, the even part of which coincides with ϕ: but now (*loc. cit.*), $\psi(x)$ goes to zero, as $x \to \infty$, just like $x^{-\frac{1}{2}}\, e^{-\pi x^2}$. On the other hand, for $x > 0$,

$$\psi(\pm ix) = (\pi\, x)^{\frac{1}{2}}\, [I_{-\frac{1}{4}}(\pi\, x^2) \mp I_{\frac{1}{4}}(\pi\, x^2)], \tag{1.12}$$

as can be seen from a careful use of (1.10), so that

$$\psi(ix) + i\,\psi(-ix) = (1+i)\,\psi(x). \tag{1.13}$$

Consequently $\phi \in \mathfrak{A}$: note that the $\mathbb{C}^4$-realization of ϕ is $(\psi,\, 0,\, \psi,\, 0)$. □

We shall prove presently that the map $(f_0, f_1) \mapsto u$ introduced in Definition 1.1 is one-to-one, and we take this opportunity to prove at the same time a few related lemmas which will be put to use later. All this is related to the Phragmén–Lindelöf lemma, an extension of the maximum principle to angular regions which can be found in many textbooks, including [26, p. 496]:

Lemma 1.3. *Let f be an entire function of one variable, let S be the sector defined by the inequality $|\mathrm{Arg}\, z| \leq \frac{\alpha\,\pi}{2}$ for some $\alpha \in]0,2[$, and let $\delta \in]0,\alpha^{-1}[$. Assume that one has $|f(z)| \leq \exp(|z|^\delta)$ if $z \in S$ and $|z|$ is sufficiently large. Then, if the restriction of f to the boundary of S is bounded, f is bounded in S. Moreover, if $f(z)$ goes to zero as z goes to infinity along any of the two sides of the sector, $f(z)$ goes to zero in a uniform way as z goes to infinity while staying in S.*

Lemma 1.4. *Let f be an entire function satisfying some estimate*

$$|f(z)| \leq C\, e^{\pi R |z|^2}, \quad z \in \mathbb{C}, \tag{1.14}$$

together with some estimate

$$|f(x)| \leq C\, e^{-2\pi\delta x^2}, \quad x > 0. \tag{1.15}$$

Then there exists $\theta_0 > 0$ such that

$$|f(xe^{i\theta})| \leq C\, e^{-\pi\delta x^2}, \quad x > 1,\ |\theta| \leq \theta_0. \tag{1.16}$$

Proof. With some $A > 0$ to be chosen later and an arbitrary $\gamma > 1$, set

$$\Phi(z) = \exp\left(2\pi(\delta + iA)e^{\frac{i\pi}{2\gamma}}\, z^2\right) f(z\, e^{\frac{i\pi}{4\gamma}}), \tag{1.17}$$

a function considered in the sector $|\mathrm{Arg}\ z| \leq \frac{\pi}{4\gamma}$ and satisfying the estimate $\log_+\, |\Phi(z)| \leq C\, |z|^2$ for z in this sector with $|z|$ large. When $z = |z|\, e^{-\frac{i\pi}{4\gamma}}$, one has

$$|\Phi(z)| \leq e^{2\pi\delta |z|^2}\, |f(|z|)| \leq C;$$

when $z = |z|\, e^{\frac{i\pi}{4\gamma}}$, one has

$$|\Phi(z)| \leq C\, \exp\left(2\pi |z|^2 \left(\delta\, \cos\frac{\pi}{\gamma} - A\, \sin\frac{\pi}{\gamma} + \frac{R}{2}\right)\right),$$

a bounded expression if A is chosen large enough. Then, by the Phragmén–Lindelöf lemma, Φ is bounded in the whole sector and, for $0 < \text{Arg } z < \frac{\pi}{2\gamma}$, one has

$$|f(z)| \leq C \exp\left(-2\pi \, \text{Re } ((\delta + iA)z^2)\right) \; : \tag{1.18}$$

when $z = |z|e^{i\theta}, \; 0 \leq \theta \leq \frac{\pi}{2\gamma}$, one has

$$\begin{aligned} \text{Re } ((\delta + iA)z^2) &= |z|^2 \, \text{Re } ((\delta + iA)e^{2i\theta}) \\ &= |z|^2 \, (\delta \cos 2\theta - A \sin 2\theta) \\ &\geq \frac{\delta}{2} \, |z|^2 \end{aligned} \tag{1.19}$$

if θ is small enough. The same holds if $-\frac{\pi}{2\gamma} \leq \theta \leq 0$, considering instead the function $z \mapsto \overline{f(\bar{z})}$. □

In a similar way, one can prove the following:

Lemma 1.5. *Let g be a function defined and holomorphic in some angular sector around the positive half-line, satisfying for some pair of positive constants C, R and every $z \in \mathbb{C}$ the estimate*

$$|g(z)| \leq C \, e^{2\pi R |z|}. \tag{1.20}$$

Assume that, for some $\delta > 0$, one has the inequality

$$|g(x)| \leq C \, e^{-2\pi\delta x}, \qquad x > 0 \; : \tag{1.21}$$

then, there exists $\theta_0 > 0$ such that

$$|g(x \, e^{i\theta})| \leq C \, e^{-\pi\delta x}, \qquad x > 1, \; \|\theta| \leq \theta_0. \tag{1.22}$$

Lemma 1.6. *Let f be an entire function such that, for some pair of positive constants C, R,*

$$|f(z)| \leq C \, e^{\pi R |z|^2}, \quad z \in \mathbb{C}. \tag{1.23}$$

If there exists $\delta > 0$, such that

$$|f(x)| + |f(ix)| \leq C \, e^{-\pi\delta x^2}, \qquad x \in \mathbb{R}, \tag{1.24}$$

the function f is identically zero.

Proof. By Lemma 1.4,

$$|f(x \, e^{i\theta})| \leq C \, e^{-2\pi\delta x^2}, \qquad x > 1, \; 0 \leq \theta \leq \theta_0 \; : \tag{1.25}$$

now the half-width of the sector $\theta_0 \leq \text{Arg}\, z \leq \frac{\pi}{2}$ is $< \frac{\pi}{4}$, so that the Phragmén–Lindelöf lemma applies and shows that $f(z)$ goes to zero, as $|z| \to \infty$, in a uniform way in the first quadrant. The same goes with the three other quadrants, so that the lemma is a consequence of Liouville's theorem. □

Corollary 1.7. *Let $u \in \mathfrak{A}$, the space of entire functions introduced in Definition* 1.1. *Then the pair of nice functions f_0, f_1 the existence of which is asserted there is unique.*

Proof. Taking the difference of any two such pairs, one remarks that if f_0 is nice, *odd* and if the function $z \mapsto f_0(iz) + i f_0(-iz) = (1-i)\, f_0(iz)$ is nice too, then $f_0 = 0$ according to the lemma that precedes; something similar goes with f_1. □

We now show how the vector $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ can be rebuilt from the knowledge of $u \in \mathfrak{A}$. We shall postpone to the next section the proof that, given an entire function u satisfying some global estimate $|u(z)| \leq C\, e^{\pi R |z|^2}$, the additional properties expressed below in terms of the pair (w_0, w_1) whose definition follows characterize the fact that u lies in $\mathfrak{A}$.

Theorem 1.8. *Let $u \in \mathfrak{A}$. Set, for σ real and large enough,*

$$w_0(\sigma) = \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, u(x\, e^{-\frac{i\pi}{4}})\, dx,$$

$$w_1(\sigma) = \frac{1-i}{2} \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, x\, u(x\, e^{-\frac{i\pi}{4}})\, dx. \tag{1.26}$$

On the one hand, each of these two functions extends as a holomorphic function, still denoted as w_0 (resp. w_1), in some strip $|\mathrm{Im}\ \sigma| < \varepsilon$. On the other hand, for $|\sigma|$ large enough, $w_0(\sigma)$ and $w_1(\sigma)$ admit the convergent expansions

$$w_0(\sigma) = \sum_{n \geq 0} a_n\, \sigma^{-n}\, |\sigma|^{-\frac{1}{2}}, \qquad w_1(\sigma) = \sum_{n \geq 0} b_n\, \sigma^{-n-1}\, |\sigma|^{-\frac{1}{2}} \tag{1.27}$$

so that, for R large enough, w_0 (resp. w_1) extends as a holomorphic function, denoted as $\tilde{w}_0$ (resp. $\tilde{w}_1$), in the part of the Riemann surface of the square root function lying above the set $|z| > R$: the two continuations of the two functions under consideration are related by the equations, valid for σ real and large,

$$\tilde{w}_0(\sigma\, e^{i\pi}) = -i\, w_0(-\sigma), \qquad \tilde{w}_1(\sigma\, e^{i\pi}) = -i\, w_1(-\sigma). \tag{1.28}$$

Finally, the $\mathbb{C}^4$-realization of u can be obtained, in terms of w_0 and w_1, by the formulas (involving semi-convergent only integrals in the first two cases), valid for $x > 0$ only,

$$f_0(x) = 2^{-\frac{1}{2}}\, x \int_{-\infty}^{\infty} w_0(\sigma)\, e^{i\pi\sigma x^2}\, d\sigma,$$

$$f_{i,0}(x) = 2^{-\frac{1}{2}}\, x \int_{-\infty}^{\infty} w_0(\sigma)\, e^{-i\pi\sigma x^2}\, d\sigma,$$

$$f_1(x) = \int_{-\infty}^{\infty} w_1(\sigma)\, e^{i\pi\sigma x^2}\, d\sigma,$$

$$f_{i,1}(x) = \int_{-\infty}^{\infty} w_1(\sigma)\, e^{-i\pi\sigma x^2}\, d\sigma. \tag{1.29}$$

Proof. Let $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ be the $\mathbb{C}^4$-realization of u. Since the even part of u coincides with that of f_0 and the odd part of u coincides with that of f_1, one can substitute f_0 (*resp.* f_1) for u in the integral defining w_0 (*resp.* w_1). Using Lemma 1.4 together with the global estimate of u, one sees that the integral

$$w_0^+(\sigma) = \int_0^\infty e^{-\pi\sigma x^2}\, f_0(x\, e^{-\frac{i\pi}{4}})\, dx \tag{1.30}$$

can also be written, for σ real and large, as

$$w_0^+(\sigma) = \frac{1+i}{2^{\frac{1}{2}}} \int_0^\infty e^{-i\pi\sigma x^2}\, f_0(x)\, dx\ : \tag{1.31}$$

this makes it possible to write

$$\begin{aligned} w_0(\sigma) - 2^{\frac{1}{2}} \int_0^\infty e^{-i\pi\sigma x^2}\, f_0(x)\, dx &= \int_0^\infty e^{-\pi\sigma x^2}\, [f_0(x\, e^{\frac{3i\pi}{4}}) + i\, f_0(x\, e^{-\frac{i\pi}{4}})]\, dx \\ &= (1+i) \int_0^\infty e^{-\pi\sigma x^2}\, f_{i,0}(x\, e^{\frac{i\pi}{4}})\, dx. \end{aligned} \tag{1.32}$$

With a new deformation of contour, made possible by a new application of Lemma 1.4, this time to the function $f_{i,0}$, one finds that, for large σ,

$$w_0(\sigma) = 2^{\frac{1}{2}} \int_0^\infty [e^{-i\pi\sigma x^2}\, f_0(x) + e^{i\pi\sigma x^2}\, f_{i,0}(x)]\, dx. \tag{1.33}$$

The same method, starting from the identity

$$\frac{1-i}{2} \int_0^\infty e^{-\pi\sigma x^2}\, x\, f_1(x\, e^{-\frac{i\pi}{4}})\, dx = -\frac{1+i}{2} \int_0^\infty e^{i\pi\sigma x^2}\, x\, f_1(-ix)\, dx, \tag{1.34}$$

shows that

$$w_1(\sigma) = \int_0^\infty x\, [e^{-i\pi\sigma x^2}\, f_1(x) + e^{i\pi\sigma x^2}\, f_{i,1}(x)]\, dx. \tag{1.35}$$

Since the four components of $\boldsymbol{f}$ are nice in the sense of Definition 1.1, the equations (1.33) and (1.35) show that w_0 and w_1 indeed extend as holomorphic functions in some open strip containing the real line.

The expansion of $w_0(\sigma)$ for large σ can be derived directly from (1.26): indeed, the estimate $|u(z)| \leq C \exp(\pi R |z|^2)$ and Cauchy's inequalities make it possible to write

$$u_{\text{even}}(x\, e^{-\frac{i\pi}{4}}) = \sum_{k \geq 0} c_k\, x^{2k} \tag{1.36}$$

with $|c_k| \leq C\, \frac{(2\pi R)^k}{k\,!}$: since $\int_{-\infty}^\infty e^{-\pi\sigma x^2}\, x^{2k}\, dx = \Gamma(k+\frac{1}{2})\, (\pi\sigma)^{-k-\frac{1}{2}}$, the expansion (1.36) can be integrated term-by-term against $e^{-\pi\sigma x^2}\, dx$, leading to the first series expansion (1.27) as soon as $\sigma > 2R$; the same goes with $w_1(\sigma)$ for large σ. We now prove (1.28), which will also imply the validity of these series expansions for $-\sigma$ large.

To do this, we go back to (1.26) and, for large σ, accompany, up to $\theta = \pi$, the change $\sigma \mapsto \sigma\, e^{i\theta}$ by the change of contour $x \mapsto x\, e^{-\frac{i\theta}{2}}$, ending up, with $u_i(x) = u(ix)$, with the pair of equations

$$\begin{aligned} \tilde{w}_0(\sigma\, e^{i\pi}) &= -i \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, u_i(x\, e^{-\frac{i\pi}{4}})\, dx, \\ \tilde{w}_1(\sigma\, e^{i\pi}) &= \frac{1-i}{2} \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, x\, u_i(x\, e^{-\frac{i\pi}{4}})\, dx. \end{aligned} \tag{1.37}$$

Now, if $u \in \mathfrak{A}$ is associated to the vector $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$, it is immediate to check (more about it in Proposition 1.13, which does not depend on any previous result) that u_i is associated to the vector $(f_{i,0},\, -i\, f_{i,1},\, f_0,\, -i\, f_1)$: using (1.33) and (1.35) and comparing the results obtained if one utilizes the $\mathbb{C}^4$-realization of u or that of u_i, one obtains the relation (1.28).

The inversion formulas (1.29) are obtained from (1.33) and (1.35), using the change of variable $y = \frac{x^2}{2}$ followed by the Fourier inversion formula. □

Examples. (i) Take for some non-negative integer n, and $x \in \mathbb{R}$,

$$u(x) = |x|^{2n+\frac{1}{2}}\, I_{n-\frac{1}{4}}(\pi x^2), \tag{1.38}$$

so that u extends as an entire even function. One finds if $\sigma > 0$, using [17, p. 66, 91],

$$\begin{aligned} w_0(\sigma) &= 2\,(-1)^n \int_0^{\infty} e^{-\pi\sigma x^2}\, x^{2n+\frac{1}{2}}\, J_{n-\frac{1}{4}}(\pi x^2)\, dx \\ &= (-1)^n\, 2^{n-\frac{1}{4}}\, \pi^{-n-\frac{5}{4}}\, \Gamma\left(n+\frac{1}{4}\right)\, (1+\sigma^2)^{-n-\frac{1}{4}}. \end{aligned} \tag{1.39}$$

Clearly, for $\sigma > 1$,

$$\tilde{w}_0(\sigma\, e^{i\pi}) = -i\, w_0(\sigma) = -i\, w_0(-\sigma), \tag{1.40}$$

which confirms (1.28). Using (1.29), one finds for $x > 0$

$$f_0(x) = (-1)^n\, 2^{\frac{1}{2}}\, \pi^{-1}\, x^{2n+\frac{1}{2}}\, K_{n-\frac{1}{4}}(\pi x^2), \tag{1.41}$$

and it is indeed immediate to check that the even part of the continuation of f_0 as an entire function coincides with u, and that the conditions of Definition 1.1 are satisfied. Thus $u \in \mathfrak{A}$: the particular case when $n = 0$ is the function $\pi^{-\frac{1}{2}}\phi$, where ϕ is the function introduced in Proposition 1.2.

(ii) More generally, with $n = 0, 1, \dots$ and $j \in \mathbb{Z}$, one defines two (disjoint) classes in $\mathfrak{A}$ by the consideration of the functions

$$|x|^{2(j+n)+\frac{1}{2}}\, I_{n-j-\frac{1}{4}}(\pi\, x^2) \qquad \text{and} \qquad |x|^{2(j+n)+\frac{1}{2}}\, I_{n-j+\frac{3}{4}}(\pi\, x^2). \tag{1.42}$$

Indeed, setting $u(x) = |x|^\lambda I_\rho(\pi x^2)$, assuming that $\rho + \frac{\lambda}{2} = 2n$ or $2n+1$ with $n = 0, 1, \dots$ so that u should be analytic and even, one finds [17, p. 91]

$$\begin{aligned} w_0(\sigma) &= 2\,(-i)^{\rho+\frac{\lambda}{2}} \int_0^\infty x^\lambda\, e^{-\pi\sigma x^2}\, J_\rho(\pi x^2)\, dx \\ &= (-i)^{\rho+\frac{\lambda}{2}}\, \Gamma\left(\rho + \frac{\lambda+1}{2}\right) \pi^{\frac{-1-\lambda}{2}}\, (1+\sigma^2)^{\frac{-1-\lambda}{4}}\, P^{-\rho}_{\frac{\lambda-1}{2}} \left(\frac{\sigma}{\sqrt{1+\sigma^2}}\right), \end{aligned} \tag{1.43}$$

where the Legendre function involved is even (*resp.* odd) in the case when $-\rho + \frac{\lambda-1}{2}$ is an even (*resp.* odd) integer [17, p. 170]. On the other hand, the continuation $\tilde{w}_0$ can be found from the expression [17, p. 47]

$$\begin{aligned} w_0(\sigma) = 2^{-\rho}\, (-i)^{\rho+\frac{\lambda}{2}}\, \pi^{\frac{-1-\lambda}{2}}\, \frac{\Gamma(\rho + \frac{\lambda+1}{2})}{\Gamma(1+\rho)}\, \sigma^{-\rho-\frac{1+\lambda}{2}} \left(1 + \frac{1}{\sigma^2}\right)^{-\frac{\rho}{2}-\frac{1+\lambda}{4}} \\ \times\, {}_2F_1\left(\frac{\lambda+1}{4} + \frac{\rho}{2}, \frac{1-\lambda}{4} + \frac{\rho}{2}; \rho+1; \frac{1}{1+\sigma^2}\right) \end{aligned} \tag{1.44}$$

since, if $\sigma > 1$, $\frac{1}{1+(\sigma\, e^{i\theta})^2}$ can never be a real number > 1 so that, in the continuation process, the argument of the hypergeometric function remains in a domain where this function is uniform: this makes it possible to conclude.

(iii) On the other hand, the function $u(x) = e^{-\pi x^2}$ does not belong to $\mathfrak{A}$. For w_0, as obtained by an application of (1.26), is given for $\sigma > 1$ as $w_0(\sigma) = (\sigma - i)^{-\frac{1}{2}}$: indeed, this function extends as an analytic function in the strip $|\mathrm{Im}\ \sigma| < 1$. However, following the determinations of the square root, one notices that, for $\sigma > 1$, one has the relation $\tilde{w}_0(\sigma\, e^{i\pi}) = -w_0(-\sigma)$ rather than the relation (1.28). A large class of entire functions not in $\mathfrak{A}$ is the class $\mathfrak{M}$ of multipliers of $\mathfrak{A}$ introduced in Proposition 1.15 below which, as proven there, only intersects $\mathfrak{A}$ trivially.

(iv) If the four components of the $\mathbb{C}^4$-realization of some function in $\mathfrak{A}$ are all less, on the positive half-line, than a multiple of $\exp(-\pi\varepsilon x^2)$ for some specific ε, it is clear that they can all be multiplied without harm by any even entire function of z globally less than a multiple of $\exp(\pi a |z|^2)$ for some $a < \varepsilon$: the same goes, as a consequence, for the function in $\mathfrak{A}$ we started out with.

As an explicit example of function in $\mathfrak{A}$ obtained in this way, take for some $\theta \in]0, \frac{\pi}{4}[$ the function defined for $x \in \mathbb{R}$ as

$$u(x) = \pi^{\frac{1}{2}}\, |x|\; I_{-\frac{1}{4}}(\pi x^2 \cos\theta)\, I_{-\frac{1}{4}}(\pi x^2 \sin\theta). \tag{1.45}$$

Note that this is the product of two factors, each of which is a rescaled version of the function ϕ from Proposition 1.2: also note that we explicitly discard the case when the two rescaling factors would be the same. Applying (1.26), one finds with the help of [10, p. 95] the equation

$$w_0(\sigma) = 2^{\frac{1}{2}}\, \pi^{-\frac{3}{2}}\, (\sin 2\theta)^{-\frac{1}{2}}\, \mathfrak{Q}_{-\frac{3}{4}} \left(\frac{\sigma^2+1}{\sin 2\theta}\right), \tag{1.46}$$

where the function $\mathfrak{Q}_{-\frac{3}{4}}$ is the Legendre function of the second species defined for $t > 1$ as

$$\mathfrak{Q}_{-\frac{3}{4}}(t) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2^{\frac{1}{4}}\,\Gamma(\frac{3}{4})}\; t^{-\frac{1}{4}}\; {}_2F_1\left(\frac{5}{8}, \frac{1}{8}; \frac{3}{4}; t^{-2}\right) \; : \tag{1.47}$$

it is analytic for $t > 1$. It is immediate that the equation (1.28) linking the two continuations of w_0 is satisfied. Using the equation (1.29) again, we find for $x > 0$, using [17, p. 194], that

$$\begin{aligned} f_0(x) &= 2\pi^{-\frac{3}{2}}\,(\sin\,2\theta)^{-\frac{1}{2}} \int_0^\infty \mathfrak{Q}_{-\frac{3}{4}}\left(\frac{\sigma^2+1}{\sin\,2\theta}\right) \cos\left(\pi\sigma x^2\right) d\sigma \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \; x\; K_{\frac{1}{4}}(\pi x^2\cos\theta)\, I_{-\frac{1}{4}}(\pi x^2 \sin\theta). \end{aligned} \tag{1.48}$$

Of course, this is the result we expected: but the proof above also shows that u is no longer in $\mathfrak{A}$ in the case when $\theta = \frac{\pi}{4}$, since then the function w_0 in (1.46) ceases to be analytic at $\sigma = 0$.

As a matter of fact, this example may be connected to the family in the example (i), since one has the so-called Neumann series [17, p. 125]

$$u(x) = \sum_{n\geq 0} \frac{(-1)^n}{n\,!\,\Gamma(\frac{3}{4}+n)} \; \left(\frac{\sin\,2\theta}{2}\right)^{2n-\frac{1}{4}} \; (\pi x^2)^{2n+\frac{1}{4}}\; I_{2n-\frac{1}{4}}(\pi x^2). \tag{1.49}$$

(v) Other examples of functions lying in $\mathfrak{A}$, or not lying in that space, will be given in Remark 1.2, at the end of this section.

One last pair of lemmas in the Phragmén–Lindelöf spirit will be useful later.

Lemma 1.9. *Let g be an entire function of one variable such that, for some pair of positive constants C, R, the estimate*

$$|g(z)| \leq C\, e^{2\pi R|z|}, \quad z \in \mathbb{C}, \tag{1.50}$$

holds. If $|g(x)|$ is less than $C\,e^{-\pi\delta|x|}$ for some $\delta > 0$ or if $|g(x)| + |g(ix)|$ goes to zero, as x is real and goes to $\pm\infty$, g is identically zero.

Proof. In the first case, we argue just as in the proof of Lemma 1.6, starting from Lemma 1.5 in place of Lemma 1.4, thus ending up with an application of the Phragmén–Lindelöf lemma in some angle of half-width $< \frac{\pi}{2}$. The second case is easier. □

Lemma 1.10. *Let f be an entire function satisfying some estimate*

$$|f(z)| \leq C\, e^{\pi R|z|^2}, \quad z \in \mathbb{C}, \tag{1.51}$$

together with some estimate

$$|f(x)| \leq C\, e^{-2\pi\delta x^2}, \quad x > 0. \tag{1.52}$$

Assume, moreover, that for every $\varepsilon > 0$*, there exists* $C > 0$ *such that*

$$|f(\pm ix)| \leq C\, e^{\pi\varepsilon x^2}, \quad x > 0. \tag{1.53}$$

Then, for every $\beta \in [0, \frac{\pi}{2}[$*,* $f(z)$ *goes to zero, as* $|z| \to \infty$ *and* $|\mathrm{Arg}\, z| \leq \beta$*, in a uniform way.*

Proof. Since the function $z \mapsto \overline{f(\bar{z})}$ satisfies the same assumptions as f, one may interest oneself in the sector $0 \leq \mathrm{Arg}\, z \leq \beta$ only. Set

$$\Phi_\varepsilon(z) = f(z)\, \exp\left(\pi\varepsilon\, z^2\, e^{-i\alpha}\right) \tag{1.54}$$

for some $\varepsilon\, \in]0, \delta[$ and some $\alpha \in [0, \frac{\pi}{2}[$ to be determined later. From Lemma 1.4, one gets

$$|\Phi_\varepsilon(x\, e^{i\theta})| \leq C\, e^{-\pi(\delta-\varepsilon)x^2}, \qquad x > 0, \qquad 0 \leq \theta \leq \theta_0. \tag{1.55}$$

On the other hand,

$$|\Phi_\varepsilon(ix)| = |f(ix)|\, e^{-\pi\varepsilon\, x^2\, \cos\alpha} \tag{1.56}$$

goes to zero as $x \to \infty$ so that, as an application of the Phragmén–Lindelöf lemma, $\Phi_\varepsilon(x\, e^{i\theta})$ goes to zero, as $x \to \infty$, uniformly for $\theta_0 \leq \theta \leq \frac{\pi}{2}$. Now, with $z = x + iy$,

$$\mathrm{Re}\, \left(z^2\, e^{-i\alpha}\right) = (x^2 - y^2)\, \cos\alpha + 2xy\, \sin\alpha \tag{1.57}$$

is ≥ 0 provided that $\frac{x}{y} \geq \frac{1-\sin\alpha}{\cos\alpha}$, an expression that is less than $\frac{\cos\beta}{\sin\beta}$ if α is chosen close enough from $\frac{\pi}{2}$. $\square$

Proposition 1.11. *For any* complex y, η*, the transformation*

$$\pi(y, \eta) = e^{2i\pi\, (\eta\, Q - y\, P)}$$

defined by the equation (1.1) *preserves the space* $\mathfrak{A}$*.*

Proof. Abbreviating $\pi(y, 0) = e^{-2i\pi y P}$ as τ_y, one may verify that $\tau_y u$ is given, in the $\mathbb{C}^2$-realization, as

$$(h_0, h_1) = \left(\frac{1}{2}(\tau_y\, f_0 + \tau_{-y}\, f_0 + \tau_y\, f_1 - \tau_{-y}\, f_1),\, \frac{1}{2}(\tau_y\, f_0 - \tau_{-y}\, f_0 + \tau_y\, f_1 + \tau_{-y}\, f_1)\right) : \tag{1.58}$$

then, the other two components of the $\mathbb{C}^4$-realization of the same function are

$$(h_{i,0}, h_{i,1}) = \left(\frac{1}{2}(\tau_{iy}\, f_{i,0} + \tau_{-iy}\, f_{i,0} + i\, \tau_{iy}\, f_{i,1} - i\, \tau_{-iy}\, f_{i,1}),\right.$$
$$\left.\frac{1}{2}(-i\, \tau_{iy}\, f_{i,0} + i\, \tau_{-iy}\, f_{i,0} + \tau_{iy}\, f_{i,1} + \tau_{-iy}\, f_{i,1})\right) \;: \tag{1.59}$$

as a consequence of Lemma 1.4, all the components are nice in the sense of Definition 1.1. We also need the explicit formulas relative to $\pi(0,\eta) = e^{2i\pi\eta Q}$ abbreviated as τ^η: the $\mathbb{C}^4$-realization $\boldsymbol{g}$ of $\tau^\eta\, u$ is given as

$$(g_0, g_1) = \left(\frac{1}{2}(\tau^\eta f_0 + \tau^{-\eta} f_0 + \tau^\eta f_1 - \tau^{-\eta} f_1),\, \frac{1}{2}(\tau^\eta f_0 - \tau^{-\eta} f_0 + \tau^\eta f_1 + \tau^{-\eta} f_1)\right) \tag{1.60}$$

and

$$(g_{i,0}, g_{i,1}) = \left(\frac{1}{2}(\tau^{i\eta} f_{i,0} + \tau^{-i\eta} f_{i,0} - i\,\tau^{i\eta} f_{i,1} + i\,\tau^{-i\eta} f_{i,1}),\right. \tag{1.61}$$
$$\left.\frac{1}{2}(i\,\tau^{i\eta} f_{i,0} - i\,\tau^{-i\eta} f_{i,0} + \tau^{i\eta} f_{i,1} + \tau^{-i\eta} f_{i,1})\right). \qquad \square$$

Proposition 1.12. *On analytic functions of x on the real line, define the operator Q as the operator of multiplication by x, and the operator P as $\frac{1}{2i\pi}\frac{d}{dx}$. The space $\mathfrak{A}$ is preserved under the action of the algebra generated by Q and P.*

Proof. In the $\mathbb{C}^4$-realization, the operator Q or P expresses itself as $\boldsymbol{f} \mapsto \boldsymbol{h}$ with

$$\boldsymbol{h}(z) = (z\, f_1(z),\ z\, f_0(z),\, z\, f_{i,1}(z),\, -z\, f_{i,0}(z)) \tag{1.62}$$

in the first case, and

$$\boldsymbol{h} = \frac{1}{2i\pi}\,(f_1',\ f_0',\, -f_{i,1}',\, f_{i,0}') \tag{1.63}$$

in the second one. $\square$

Obviously, the multiplication by z preserves the space of nice functions introduced in Definition 1.1, and the same holds for the operation of taking the derivative by virtue of Lemma 1.4 (together with Cauchy's integral formula for the derivative).

It is immediate to check how some basic symmetries on $\mathfrak{A}$ transfer to the $\mathbb{C}^4$-realization: the formulas below thus constitute a proof that the symmetries under examination do preserve $\mathfrak{A}$.

Proposition 1.13. *Define the linear operators $\mathcal{R}$ (for* rotation*) and $\mathcal{R}^2$ by the equations*

$$(\mathcal{R}^2\, u)(z) = u(-z), \qquad (\mathcal{R}\, u)(z) = u(iz)\,; \tag{1.64}$$

define the antilinear operator C (for conjugation*) by the equation*

$$(Cu)(x) = \bar{u}(x) \ \text{ if } x \in \mathbb{R} \quad \text{ or } \ (Cu)(z) = \overline{u(\bar{z})},\ z \in \mathbb{C}. \tag{1.65}$$

The operations $\mathcal{R}^2$, $\mathcal{R}$, C transfer respectively, in the $\mathbb{C}^4$-realization, to the operations

$$\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1}) \mapsto \boldsymbol{h} = (h_0,\, h_1,\, h_{i,0},\, h_{i,1}) \tag{1.66}$$

with

$$\begin{pmatrix} h_0 \\ h_1 \\ h_{i,0} \\ h_{i,1} \end{pmatrix} = \begin{pmatrix} f_0 \\ -f_1 \\ f_{i,0} \\ -f_{i,1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} f_{i,0} \\ -i\, f_{i,1} \\ f_0 \\ -i\, f_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} Cf_0 \\ Cf_1 \\ Cf_{i,0} \\ Cf_{i,1} \end{pmatrix}. \tag{1.67}$$

We can now define the scalar product on $\mathfrak{A}$, at the same time proving that it is non-degenerate.

Proposition 1.14. *Let* $(\ |\ )$ *be the scalar product on* $\mathfrak{A}$ *defined, in the* $\mathbb{C}^4$*-realization, as*

$$\begin{aligned} &(\boldsymbol{h}\,|\,\boldsymbol{f}) \\ &= 2^{\frac{1}{2}} \int_0^\infty \left(\bar{h}_0(x) f_0(x) + \bar{h}_1(x) f_1(x) + \bar{h}_{i,0}(x) f_{i,0}(x) - \bar{h}_{i,1}(x) f_{i,1}(x)\right) dx. \end{aligned} \tag{1.68}$$

This scalar product is non-degenerate.

Proof. Obviously, the subspaces of $\mathfrak{A}$ consisting of all even (*resp.* odd) functions are orthogonal with respect to $(\ |\ )$. On $\mathfrak{A}_{\text{even}}$, the scalar product is positive-definite. On the other hand, it follows from (1.62) and (1.68) that the operator Q is self-adjoint with respect to $(\ |\ )$: the non-degeneracy of the scalar product on the odd part of $\mathfrak{A}$ is then a consequence of its non-degeneracy on the even part together with the equation $(Q^2u|u) = (Qu|Qu)$. □

Proposition 1.15. *Let* $\mathfrak{M}$ *denote the space of all entire functions* m *satisfying for some pair* (R, C) *of positive numbers the estimate*

$$|m(z)| \leq C\, e^{\pi R |z|^2}, \qquad z \in \mathbb{C} \tag{1.69}$$

and the property that, for every $\varepsilon > 0$*, one has for some* $C > 0$ *the estimate*

$$|m(x)| + |m(ix)| \leq C\, e^{\pi \varepsilon x^2}, \qquad x \in \mathbb{R}. \tag{1.70}$$

Then, for every $u \in \mathfrak{A}$*, the product* mu *belongs to* $\mathfrak{A}$ *as well. The intersection* $\mathfrak{M} \cap \mathfrak{A}$ *reduces to zero.*

Proof. If $u \in \mathfrak{A}$ is associated to the vector $\boldsymbol{f}$ as before, the function mu is then associated to the vector $\boldsymbol{h}$, with

$$\begin{aligned} h_0(z) &= m_{\text{even}}(z)\, f_0(z) + m_{\text{odd}}(z)\, f_1(z), \\ h_1(z) &= m_{\text{odd}}(z)\, f_0(z) + m_{\text{even}}(z)\, f_1(z) \end{aligned} \tag{1.71}$$

and

$$\begin{aligned} h_{i,0}(z) &= m_{\text{even}}(iz)\, f_{i,0}(z) - i\, m_{\text{odd}}(iz)\, f_{i,1}(z), \\ h_{i,1}(z) &= i\, m_{\text{odd}}(iz)\, f_{i,0}(z) + m_{\text{even}}(iz)\, f_{i,1}(z), \end{aligned} \tag{1.72}$$

which proves the first part.

Since the assumptions relative to m are invariant under the symmetry $\mathcal{R}^2$ introduced in Proposition 1.13, one may, in the proof of the second part, deal separately with the even and odd parts of m; in view of Proposition 1.12, one may even consider only the case when m is even. Thus, assuming this to be the case, and that $m \in \mathfrak{M} \cap \mathfrak{A}$, let $\boldsymbol{f}$, reducing in this case to $(f_0,\, 0,\, f_{i,0},\, 0)$, be the vector associated with m. Since m is the even part of f_0, (1.8) yields the equations

$$\begin{aligned}(1-i)\, f_0(ix) &= (1+i)\, f_{i,0}(x) - 2i\, m(ix),\\ (1-i)\, f_0(-ix) &= 2\, m(ix) - (1+i)\, f_{i,0}(x),\end{aligned} \tag{1.73}$$

which show, since $m \in \mathfrak{M}$, that $|f_0(ix)|$ is, for every $\varepsilon > 0$, a $\mathrm{O}(e^{\pi\varepsilon x^2})$ as $x \to \pm\infty$. Lemma 1.10 thus shows that $f_0(z)$ goes to zero, as $|z| \to \infty$, in any closed sector contained in the half-plane $\mathrm{Re}\, z > 0$. Exchanging the roles of f_0 and $f_{i,0}$, and using the result already obtained for f_0, one finds that $f_0(z)$ also goes to zero, as $|z| \to \infty$, in any closed sector contained in the quadrant defined by $-\pi < \mathrm{Arg}\, z < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \mathrm{Arg}\, z < \pi$. One concludes with the help of the Phragmén–Lindelöf lemma together with Liouville's theorem. □

We now define a substitute for the notion of integral, to wit a translation-invariant linear form on $\mathfrak{A}$.

Proposition 1.16. *If $\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ is the $\mathbb{C}^4$-realization of some function $u \in \mathfrak{A}$, set*

$$\mathrm{Int}\,[u] = 2^{\frac{1}{2}} \int_0^\infty (f_0(x) + f_{i,0}(x))\, dx. \tag{1.74}$$

For every $y \in \mathbb{C}$, with $(e^{-2i\pi\, yP}\, u)(z) = u(z-y)$, one has

$$\mathrm{Int}\,[e^{-2i\pi\, yP}\, u] = \mathrm{Int}\,[u]. \tag{1.75}$$

Proof. Set $v = e^{-2i\pi yP}\, u$. From the proof of Proposition 1.11, one has

$$\begin{aligned}2^{\frac{1}{2}}\, \mathrm{Int}\,[v] = &\int_0^\infty [f_0(x-y) + f_0(x+y) + f_{i,0}(x-iy) + f_{i,0}(x+iy)]\, dx \\ &+ \int_0^\infty [f_1(x-y) - f_1(x+y) + i\, f_{i,1}(x-iy) - i\, f_{i,1}(x+iy)]\, dx.\end{aligned} \tag{1.76}$$

The second line is

$$\begin{aligned}\int_{-y}^{y} f_1(z)\, dz + i \int_{-iy}^{iy} f_{i,1}(z)\, dz &= \int_{-y}^{y} f_1(z)\, dz - \int_{-y}^{y} f_{i,1}(iz)\, dz \\ &= \int_{-y}^{y} \frac{1+i}{2}\, (f_1(z) - f_1(-z))\, dz = 0.\end{aligned} \tag{1.77}$$

Now that Int $[v]$ reduces to the first line of the equation above, one can write

$$2^{\frac{1}{2}}\,\mathrm{Int}\,[v-u] = \int_{-y}^{0} f_0(z)\,dz - \int_0^{y} f_0(z)\,dz + \int_{-iy}^{0} f_{i,0}(z)\,dz - \int_0^{iy} f_{i,0}(z)\,dz$$
$$= \int_0^{y} [f_0(-z) - f_0(z) + i\,f_{i,0}(-iz) - i\,f_{i,0}(iz)]\,dz, \tag{1.78}$$

which reduces to zero since

$$f_{i,0}(iz) - f_{i,0}(-iz) = i\,(f_0(z) - f_0(-z)). \tag{1.79}$$

□

From Proposition 1.15, an entire function such as $y \mapsto e^{-2i\pi xy}$ lies in $\mathfrak{M}$, thus is never in $\mathfrak{A}$: however, it may serve as a multiplier of such a function.

Definition 1.17. Given $x \in \mathbb{R}$, define the function $\boldsymbol{e}_x \in \mathfrak{M}$ as $\boldsymbol{e}_x(y) = e^{-2i\pi xy}$. For any $u \in \mathfrak{A}$, the anaplectic Fourier transform $\mathcal{F}_{\mathrm{ana}}\,u$ of u is defined as

$$(\mathcal{F}_{\mathrm{ana}}\,u)(x) = \mathrm{Int}\,[\boldsymbol{e}_x\,u]. \tag{1.80}$$

In view of (1.71), (1.72), a fully developed version of the preceding definition is

$$(\mathcal{F}_{\mathrm{ana}}\,u)(x) = 2^{\frac{1}{2}} \int_0^{\infty} f_0(y)\,\cos 2\pi xy\;dy - 2^{\frac{1}{2}}\,i \int_0^{\infty} f_1(y)\,\sin 2\pi xy\;dy \tag{1.81}$$
$$+\,2^{\frac{1}{2}} \int_0^{\infty} f_{i,0}(y)\,\cosh 2\pi xy\;dy - 2^{\frac{1}{2}}\,i \int_0^{\infty} f_{i,1}(y)\,\sinh 2\pi xy\;dy.$$

It will be shown later that the transformation $\mathcal{F}_{\mathrm{ana}}$ preserves the space $\mathfrak{A}$. In anaplectic analysis, the function ϕ introduced in Proposition 1.2 is just as basic as the Gaussian function in usual analysis. In particular,

Proposition 1.18. *The function ϕ is normalized,* i.e., $(\phi\,|\,\phi) = 1$. *One has $\mathcal{F}_{\mathrm{ana}}\,\phi = \phi$.*

Proof. The first part is a consequence of (1.68), (1.11), (1.13) and of the formula [17, p. 101]

$$\int_0^{\infty} (K_{\frac{1}{4}}(\pi t))^2\,dt = 2^{-\frac{3}{2}}\,\pi. \tag{1.82}$$

Using (1.81) and (1.13), one finds

$$(\mathcal{F}_{\mathrm{ana}}\,\phi)(x) = 2\,\pi^{-\frac{1}{2}} \int_0^{\infty} y^{\frac{1}{2}}\,K_{\frac{1}{4}}(\pi\,y^2)\,(\cos\,2\pi xy + \cosh\,2\pi xy)\,dy. \tag{1.83}$$

This integral can be computed, even though it could hardly be found from the inspection of books devoted to special functions. Instead, consider the harmonic oscillator, acting on functions of x,

$$L = \pi\,x^2 - \frac{1}{4\pi}\,\frac{d^2}{dx^2}, \tag{1.84}$$

where the first term on the right-hand side stands for the operator of multiplication by πx^2. Using the differential equation of Bessel functions

$$K_\nu''(t) + \frac{1}{t} K_\nu'(t) - \left(1 + \frac{\nu^2}{t^2}\right) K_\nu(t) = 0, \tag{1.85}$$

it is a routine matter to check that $L\phi = 0$. Using an integration by parts and this equation, one sees, splitting the integrand of the right-hand side of (1.83) as a sum of two terms, that the first of the two integrals thus obtained satisfies the differential equation

$$\left(\pi x^2 - \frac{1}{4\pi}\frac{d^2}{dx^2}\right)\left(\int_0^\infty y^{\frac{1}{2}} K_{\frac{1}{4}}(\pi y^2)\cos 2\pi x y\, dy\right) = \frac{1}{4\pi}\frac{d}{dy}[y^{\frac{1}{2}} K_{\frac{1}{4}}(\pi y^2)](y=0), \tag{1.86}$$

a non-zero constant; the same integration by parts, with cosh in place of cos, would lead to the negative of that constant. Adding the two equations, one sees that the function $\mathcal{F}_{\text{ana}}\,\phi$ is also a (generalized) eigenfunction of the harmonic oscillator, corresponding to the eigenvalue 0. Since both ϕ and $\mathcal{F}_{\text{ana}}\,\phi$ are even functions, they are proportional. From (1.10), one finds

$$\phi(0) = \frac{2^{\frac{1}{4}}\,\pi^{\frac{1}{4}}}{\Gamma(\frac{3}{4})} \tag{1.87}$$

and from (1.83) and [17, p. 91],

$$(\mathcal{F}_{\text{ana}}\,\phi)(0) = 2^{-\frac{1}{4}}\,\pi^{-\frac{3}{4}}\,\Gamma\left(\frac{1}{4}\right). \tag{1.88}$$

The two results agree as a consequence of the duplication formula of the Gamma function [17, p. 3]

$$\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) = (2\pi)^{\frac{1}{2}}\,2^{\frac{1}{2}-2z}\,\Gamma(2z). \tag{1.89}$$

□

We shall see in the next section that the spectrum of the harmonic oscillator in the space $\mathfrak{A}$ is $\mathbb{Z}$, and that all its eigenfunctions can be built from ϕ with the help of a pair of *raising* and *lowering* operators, which substitute for the usual creation and annihilation operators.

We are now in a position to give a characterization of the *anaplectic representation*, first giving the definition of the *anaplectic group* itself.

Definition 1.19. The anaplectic group $SL_i(2,\mathbb{R})$ is the subgroup of $SL(2,\mathbb{C})$ generated by $SL(2,\mathbb{R})$ together with the element $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. In other words, a matrix in $SL(2,\mathbb{C})$ will lie in $SL_i(2,\mathbb{R})$ if and only if all its entries are real, or all its entries are pure imaginary.

Remark 1.1. A maximal compact subgroup of $SL_i(2,\mathbb{R})$ consists of the union of two disjoint circles: the subgroup $SO(2)$, and the class $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} SO(2)$ consisting of all matrices of the kind $\begin{pmatrix} -i\cos\theta & i\sin\theta \\ i\sin\theta & i\cos\theta \end{pmatrix}$ with $\theta \in \mathbb{R}$. Now, if one considers the canonical embedding of these two circles in $SU(2)$, a maximal compact subgroup of $SL(2,\mathbb{C})$ homeomorphic to the 3-sphere S^3, one finds that they are linked in a non-trivial way (just like two fibers of the Hopf fibration, or two adjoining elements of a chain). Indeed, consider the set of matrices

$$\begin{pmatrix} a - (1-a^2-b^2)^{\frac{1}{2}}\, i & -b \\ b & a + (1-a^2-b^2)^{\frac{1}{2}}\, i \end{pmatrix} \in SU(2), \qquad a, b \in \mathbb{R},\ a^2+b^2 \leq 1, \tag{1.90}$$

a homeomorphic image of the unit disk. The boundary of this set is just the first circle $SO(2)$, while the intersection of this set with the second circle reduces to the point $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ corresponding to the center of the disk. Thus, topologically speaking, even though the anaplectic group is simpler than the metaplectic group (it is an honest linear group, while the definition of the latter one involves some covering space), there is still some non-trivial topology remaining in its embedding into $SL(2,\mathbb{C})$.

The following theorem, which sums up the main properties of the one-dimensional anaplectic representation, will be proved in the next section.

Theorem 1.20. *There is a unique representation* Ana *of the anaplectic group in the space* $\mathfrak{A}$ *with the following properties:*

(i) *if* $g \in SL(2,\mathbb{R})$ *is one of the first two matrices from the set of generators given in* (1.5)*, and if* $u \in \mathfrak{A}$ *is given in its realization as a scalar-valued function, the associated transformation* $\mathrm{Ana}(g)$ *is given on* u *by the same formula as in the case of the metaplectic representation;*

(ii) *one has* $\mathrm{Ana}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \mathcal{F}_{\mathrm{ana}}$*;*

(iii) *one has* $\mathrm{Ana}\left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) = \mathcal{R}$*, as introduced in Proposition* 1.13*.*

When restricted to the subgroup $SL(2,\mathbb{R})$ *of the anaplectic group, this representation is pseudo-unitary,* i.e., *it preserves the scalar product introduced in Proposition* 1.14*.*

Finally, given $(y,\eta) \in \mathbb{C}^2$*, and with* $e^{2i\pi\,(\eta Q - yP)} = \pi(y,\eta)$ *as defined in* (1.1) *and* (1.4)*, one has*

$$\mathrm{Ana}(g)\, e^{2i\pi\,(\eta Q - yP)}\, \mathrm{Ana}(g^{-1}) = e^{2i\pi\,(\eta' Q - y' P)} \tag{1.91}$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_i(2,\mathbb{R})$ *and* $g \begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} y' \\ \eta' \end{pmatrix}$*. The transformations* $e^{2i\pi\,(\eta Q - yP)}$ *with* $(y,\eta) \in \mathbb{R}^2$ *also preserve the scalar product.*

Observe, from Proposition 1.13, that the transformation $\mathcal{R}$ is not pseudo-unitary: instead, given v and $u \in \mathfrak{A}$, one has $(\mathcal{R}v \,|\, \mathcal{R}u) = (\mathcal{R}^2\, v \,|\, u)$.

Remark 1.2. The anaplectic representation does not extend as a holomorphic representation of the group $SL(2,\mathbb{C})$: however, given $u \in \mathfrak{A}$, one can always define

$\mathrm{Ana}(g)\,u$ for g in some complex neighborhood of $SL(2,\mathbb{R})$ in $SL(2,\mathbb{C})$, depending on u. For instance, considering the basic function ϕ used throughout this section, one can define, as an element of $\mathfrak{A}$,

$$(\mathrm{Ana}(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right))\,\phi)(x) = a^{-\frac{1}{2}}\,\phi(a^{-1}x) \qquad \text{or} \qquad (\mathrm{Ana}(\left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right))\,\phi)(x) = \phi(x)\,e^{i\pi c x^2} \tag{1.92}$$

not only for $a > 0$ or $c \in \mathbb{R}$, but also for $a = e^{i\theta}$, $\theta \in \mathbb{R}$, $|\theta| < \frac{\pi}{4}$ or $|\mathrm{Im}\; c| < 1$. Indeed, the function w_0 associated with ϕ by means of (1.26) is the function $\sigma \mapsto C\,(1+\sigma^2)^{-\frac{1}{4}}$ (with $C = 2^{-\frac{1}{4}}\,\pi^{-\frac{3}{4}}\,\Gamma(\frac{1}{4})$); that associated with the first (*resp.* the second) of the two functions above is

$$\sigma \mapsto C\,e^{-\frac{i\theta}{2}}\,(\sigma^2 + e^{-4i\theta})^{-\frac{1}{4}} \qquad \Big(\text{resp.} \qquad \sigma \mapsto C\,[1 + (\sigma - c)^2]^{-\frac{1}{4}}\Big)\ : \tag{1.93}$$

that each of these two functions is associated to some element of $\mathfrak{A}$ is a consequence of Theorem 1.8. The same shows (taking $\theta = \frac{\pi}{4}$ or $c = i$) that, on the contrary, the entire function defined for $x > 0$ as $x \mapsto (\pi\,x)^{\frac{1}{2}}\,J_{-\frac{1}{4}}(\pi x^2)$ or the function $x \mapsto \phi(x)\,e^{-\pi x^2}$ does not lie in $\mathfrak{A}$.

Remark 1.3. Let us anticipate a point which will be of importance only later, but which may already be helpful for a good comprehension. There exist two theories of the anaplectic representation and related concepts: the one we are developing, and another one, based on the introduction of a space $\mathfrak{A}^{\natural}$, analogous to the one in Definition 1.1 but for which the properties of f_0 and f_1 are exchanged. Though certainly similar, the two theories, as will be shown at the end of Section 7, are not equivalent in the representation-theoretic sense.

In particular, in this context, the *median* state ϕ (introduced in Proposition 1.2) of the harmonic oscillator (1.84) would have to be replaced by the odd (analytic) function

$$\phi^{\natural}(x) = -(\pi\,|x|)^{\frac{1}{2}}\,(\mathrm{sign}\,x)\,I_{\frac{1}{4}}(\pi\,x^2)\ : \tag{1.94}$$

observe that this function does not lie in $\mathfrak{A}$ (it lies in $\mathfrak{A}^{\natural}$). On the other hand, it is still an eigenfunction, corresponding to the eigenvalue 0, of the harmonic oscillator. The function $\phi^{\natural}$ will show up again – without our having done anything to invite it – at the end of Section 10.

2 Analytic vectors of representations of $SL(2,\mathbb{R})$

This rather technical section is needed at various places in this work, to start with the proof of the converse of Theorem 1.8: it will also make it possible to complete the proof of Theorem 1.20, and to show that the one-dimensional anaplectic representation decomposes as a sum of two well-known irreducible representations.

Remark 2.1. The representation $\pi_{\rho,\varepsilon}$ to be introduced presently will be needed, in the present work, only for the values of ρ which are integers or half-integers: it is the pairs $(-\frac{1}{2},0)$ and $(\frac{1}{2},1)$ that are needed for the understanding of the one-dimensional anaplectic representation; the other quoted values of the parameter ρ will be needed in Sections 3 and 6. It is possible, to some extent, to develop some analogue of the one-dimensional anaplectic representation based on the use of the pairs (ρ,ε) and $(\rho+1,1-\varepsilon)$ for general values of ρ. However, unless $\rho=-\frac{1}{2}$, the operators that substitute for the operators Q and P of position and momentum do not generate a finite-dimensional Lie algebra. Still, some of the theory subsists, and though we have decided against taking this more general case here, we have refrained, in some lemmas, from specializing the parameter ρ when this did not lead to any significant simplification: this may save space on another occasion. Let us also mention that if, instead of the two pairs $(-\frac{1}{2},0)$ and $(\frac{1}{2},1)$, one considers the two pairs $(-\frac{1}{2},1)$ and $(\frac{1}{2},0)$, one ends up with the analysis briefly reported about in Remark 1.3.

For every real number $\sigma\neq 0$, and $\alpha\in\mathbb{C}$, let us set

$$\langle\sigma\rangle^{\alpha}=|s|^{\alpha}\operatorname{sign}s. \tag{2.1}$$

It will be convenient, too, to set

$$|\sigma|_{\varepsilon}^{\alpha}=\begin{cases}|\sigma|^{\alpha} & \text{if } \varepsilon=0,\\ \langle\sigma\rangle^{\alpha} & \text{if } \varepsilon=1.\end{cases} \tag{2.2}$$

Let us recall the definition of the full non-unitary principal series [14, p. 38]:

$$(\hat{\pi}_{\rho,\varepsilon}(\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right))\,w)(\sigma)=|-b\sigma+d|_{\varepsilon}^{-1-\rho}\,w\left(\frac{a\sigma-c}{-b\sigma+d}\right),\qquad \rho\in\mathbb{C},\ \varepsilon=0,1\ , \tag{2.3}$$

in which w is a function on the real line: we also abbreviate $\hat{\pi}_{\rho,0}$ as $\hat{\pi}_{\rho}$. Recall that it is unitary in the case when $\mathrm{Re}\,\rho=0$, in which case the Hilbert space to use is $L^2(\mathbb{R})$ (this is the case of the two principal series). So far as questions related to unitarity are concerned, we shall be more interested, however, in the case of the complementary series, *i.e.,* ρ real $\neq 0$, $-1<\rho<1$, and $\varepsilon=0$: the representation is then unitary on the Hilbert space defined by the norm $|||\ \ |||_{\rho}$ such that

$$|||w|||_{\rho}^2=\int_{-\infty}^{\infty}|(|D|^{-\frac{\rho}{2}}\,w)(\sigma)|^2\,d\sigma, \tag{2.4}$$

with $D = \frac{1}{2i\pi}\frac{\partial}{\partial\sigma}$: here, $|D|^{-\frac{\rho}{2}}$ stands for the operator of convolution by the Fourier transform of the function $s \mapsto |s|^{-\frac{\rho}{2}}$.

We prefer to consider instead of the Hilbert space just defined its image under the inverse Fourier transform: we shall thus set $w = \hat{v} = \mathcal{F}v$, where our convention regarding the normalization of $\mathcal{F}$ has been made in (1.6). The new Hilbert space is then the space $\mathcal{H}_\rho$ consisting of all (classes of) measurable functions v on $\mathbb{R}$ such that

$$\|v\|_\rho^2 = \int_{-\infty}^{\infty} |s|^{-\rho}\, |v(s)|^2\, ds < \infty, \tag{2.5}$$

and we denote as π_ρ the unitary representation on $\mathcal{H}_\rho$ defined as

$$\pi_\rho(g) = \mathcal{F}^{-1}\, \hat{\pi}_\rho(g)\, \mathcal{F}, \qquad g \in SL(2,\mathbb{R}). \tag{2.6}$$

Let us remind the reader, at this point, that the operator of multiplication by the function $s \mapsto |s|^{-\rho}$ is an intertwining operator from the representation π_ρ to the representation $\pi_{-\rho}$. On the Fourier side, it transfers to the operator (intertwining the representation $\hat{\pi}_\rho$ and the representation $\hat{\pi}_{-\rho}$) of convolution by a distribution which coincides, for $\sigma \neq 0$, with the function

$$\sigma \mapsto \pi^{\rho-\frac{1}{2}}\, \frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(\frac{\rho}{2})}\, |\sigma|^{-1+\rho} : \tag{2.7}$$

when $0 < \rho < 1$, this locally summable function is just the distribution we are looking for; when $-1 < \rho < 0$, the latter should be understood as the derivative, in the distribution sense, of the (locally summable) function

$$\sigma \mapsto \frac{1}{2}\, \pi^{\rho-\frac{1}{2}}\, \frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(\frac{2+\rho}{2})}\, \langle\sigma\rangle^{\rho}. \tag{2.8}$$

Let us first make the space of C^∞-vectors of the representation $\hat{\pi}_\rho$ explicit. Assuming that w lies in this space, set $w_1 = |D|^{-\frac{\rho}{2}} w$. Since $\hat{\pi}_\rho((\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}))$ is just the operator of translation by the vector $c \in \mathbb{R}$, and the operator $|D|^{-\frac{\rho}{2}}$ commutes with translations, the function w_1 must lie in the intersection of all Sobolev spaces on the real line. Applying the Cauchy–Schwarz inequality to the integral giving w in terms of $\mathcal{F}^{-1}w = |s|^{\frac{\rho}{2}}\, \mathcal{F}^{-1}w_1$, where the function $s \mapsto (1+s^2)^k\, (\mathcal{F}^{-1}w_1)(s)$ lies in $L^2(\mathbb{R})$ for all k, shows that w is a C^∞ function on $\mathbb{R}$ in the usual sense. Next, since the space of C^∞-vectors of the representation $\hat{\pi}_\rho$ is preserved under the transformation associated with the matrix $(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$, the function

$$\sigma \mapsto |\sigma|^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \tag{2.9}$$

extends as a C^∞ function near 0 as well. Actually, it is easily seen that the two conditions just found, namely that both functions w and $\hat{\pi}_\rho((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}))\, w$ lie in $C^\infty(\mathbb{R})$, characterize the sought-after space.

We now characterize the operator L_ρ which is the generator, in the sense of Stone's theorem, of the one-parameter unitary group associated with the restriction of the representation π_ρ to the subgroup $K = SO(2)$ of $SL(2,\mathbb{R})$, in other words the operator such that

$$\exp(it\,L_\rho) = \pi_\rho\left(\begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}\right). \tag{2.10}$$

Thus, $L_\rho = \mathcal{F}^{-1}\,\hat{L}_\rho\,\mathcal{F}$, with

$$\hat{L}_\rho\,w = \frac{1}{i}\,\frac{d}{dt}\bigg|_{t=0}\,\hat{\pi}_\rho\left(\begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}\right). \tag{2.11}$$

From (2.3), one immediately gets, in the case when $w \in C_0^\infty(\mathbb{R})$, the equation

$$\hat{L}_\rho\,w = -\frac{1}{2i}\left[(\sigma^2+1)\,\frac{d}{d\sigma} + (\rho+1)\,\sigma\right] w \tag{2.12}$$

so that, when acting on some appropriate subspace of $\mathcal{H}_\rho$,

$$L_\rho = -\frac{1}{2\pi}\,s\frac{d^2}{ds^2} + \frac{\rho-1}{4\pi}\,\frac{d}{ds} + \pi s. \tag{2.13}$$

The generalized eigenfunctions, corresponding to real (generalized) eigenvalues, of the first-order differential operator $\hat{L}_\rho$, are given as

$$\begin{aligned} w_\rho^k(\sigma) &= (1+\sigma^2)^{-\frac{\rho+1}{2}-k}\,(1-i\sigma)^{2k} \\ &= (1+\sigma^2)^{-\frac{\rho+1}{2}}\,\left(\frac{1-i\sigma}{1+i\sigma}\right)^k, \end{aligned} \tag{2.14}$$

where the second fractional power is a principal determination on the plane cut along the negative real half-line; the corresponding eigenvalue is any real number k.

On the other hand, using (2.3), one finds

$$\left(\hat{\pi}_\rho\left(\begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}\right) w_\rho^k\right)(\sigma) = (1+\sigma^2)^{-\frac{\rho+1}{2}-k}(1-i\sigma)^{2k}\left(\frac{e^{\frac{it}{2}}\,|\sigma\sin\frac{t}{2}+\cos\frac{t}{2}|}{\sigma\sin\frac{t}{2}+\cos\frac{t}{2}}\right)^{2k} : \tag{2.15}$$

this is the same as $e^{itk}\,w_\rho^k(\sigma)$ if and only if k is an integer. This shows that the self-adjoint extension of $\hat{L}_\rho$ given by Stone's theorem is that for which a complete orthogonal set of eigenfunctions is the sequence $(w_\rho^k)_{k\in\mathbb{Z}}$ above, the k^{th}-eigenfunction corresponding to the eigenvalue k.

The corresponding sequence of eigenfunctions of L_ρ, denoted as $(v_\rho^k)_{k\in\mathbb{Z}}$, is defined by the equation $v_\rho^k = \mathcal{F}^{-1}\,w_\rho^k$.

We now come back to the general case of the full non-unitary principal series $(\hat{\pi}_{\rho,\varepsilon})$ as recalled in (2.3), in which, for the time being, ρ could be an arbitrary complex number: we still set

$$\pi_{\rho,\varepsilon}(g) = \mathcal{F}^{-1}\,\hat{\pi}_{\rho,\varepsilon}(g)\,\mathcal{F}, \qquad g \in SL(2,\mathbb{R}). \tag{2.16}$$

Despite the fact that the representation $\hat{\pi}_{\rho,\varepsilon}$ is generally non-unitarizable, one may nevertheless introduce the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ of all functions $w \in C^{\infty}(\mathbb{R})$ which satisfy the property that the function

$$\sigma \mapsto |\sigma|_{\varepsilon}^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \qquad \text{is } C^{\infty} \text{ near } 0 \tag{2.17}$$

as well. In the case when ρ is real, $0 < |\rho| < 1$, and $\varepsilon = 0$, it coincides, as remarked in (2.9), with the space of C^{∞}-vectors of the representation $\hat{\pi}_{\rho}$ discussed above. Under the inverse Fourier transformation, the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ transfers to a space of distributions $v = v(s)$ of course denoted as $C^{\infty}_{\rho,\varepsilon}$.

The operator $\hat{\pi}_{\rho,\varepsilon}(g)$ preserves the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ for every $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$: that it does so when $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in which case we set $\hat{\pi}_{\rho,\varepsilon}(g)\,w = w_1$, is just the condition (2.17). Since

$$\begin{aligned}(\hat{\pi}_{\rho,\varepsilon}(g)\,w)(\sigma) &= |-b\sigma + d|_{\varepsilon}^{-1-\rho}\, w\left(\frac{a\sigma - c}{-b\sigma + d}\right) \\ &= |-a\sigma + c|_{\varepsilon}^{-1-\rho}\, w_1\left(\frac{-b\sigma + d}{-a\sigma + c}\right),\end{aligned} \tag{2.18}$$

the inequality

$$\begin{aligned}(-b\sigma + d)^2 + (-a\sigma + c)^2 &= (a^2 + b^2)\left(\sigma - \frac{ac + bd}{a^2 + b^2}\right)^2 + \frac{1}{a^2 + b^2} \\ &\geq \frac{1}{a^2 + b^2}\end{aligned} \tag{2.19}$$

and the condition (2.17) show that if $w \in \widehat{C}^{\infty}_{\rho,\varepsilon}$, the function $\hat{\pi}_{\rho,\varepsilon}(g)\,w$ is C^{∞} on the real line for every $g \in SL(2,\mathbb{R})$: that it lies in the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ too is found after applying $\hat{\pi}_{\rho,\varepsilon}(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ again.

One still defines the operator $\hat{L}_{\rho,\varepsilon}$ by the same formula as (2.11), after substituting $\hat{\pi}_{\rho,\varepsilon}$ for $\hat{\pi}_{\rho}$: only it is to be understood that, on the σ-side, all operators are to act on the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ (no Hilbert space is present in general in the picture). The result of the computation is that $\hat{L}_{\rho,\varepsilon}$ is always given by the formula (2.12), whether $\varepsilon = 0$ or 1. One may check that, indeed, $\hat{L}_{\rho,\varepsilon}$ preserves the space $\widehat{C}^{\infty}_{\rho,\varepsilon}$ by observing that this operator commutes with the operator $\hat{\pi}_{\rho,\varepsilon}(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ involved in (2.17): only, do not forget that, when $\varepsilon = 1$, the square of the transformation just alluded to is not the identity, but minus the identity.

For every integer $k \in \mathbb{Z}$, one sets

$$w^k_{\rho,0}(\sigma) = (1+\sigma^2)^{-\frac{\rho+1}{2}-k}\,(1-i\sigma)^{2k}$$
$$w^k_{\rho,1}(\sigma) = (1+\sigma^2)^{-\frac{\rho+2}{2}-k}\,(1-i\sigma)^{2k+1}\,: \tag{2.20}$$

in both cases, one gets a function in the space $\widehat{C}^\infty_{\rho,\varepsilon}$.

Under the map $\hat{\pi}_{\rho,\varepsilon}\left(\begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix}\right)$, the function $w^k_{\rho,0}$ transforms to $e^{ikt}\,w^k_{\rho,0}$ and the function $w^k_{\rho,1}$ transforms to $e^{i(k+\frac{1}{2})t}\,w^k_{\rho,1}$. It is easily seen that the sequence $(w^k_{\rho,\varepsilon})_{k\in\mathbb{Z}}$ constitutes a sequence of eigenfunctions of the operator $\hat{L}_{\rho,\varepsilon}$ acting within the space $\widehat{C}^\infty_{\rho,\varepsilon}$, the k^{th}-eigenfunction so defined corresponding to the eigenvalue k (*resp.* $k+\frac{1}{2}$) in the case when $\varepsilon = 0$ (*resp.* 1); there is no other eigenvalue, and each eigenspace is one-dimensional.

The image under $\mathcal{F}^{-1}$ of the space $\widehat{C}^\infty_{\rho,\varepsilon}$ is denoted as $C^\infty_{\rho,\varepsilon}$; also, one sets

$$L_{\rho,\varepsilon} = \mathcal{F}^{-1}\,\hat{L}_{\rho,\varepsilon}\,\mathcal{F} \qquad \text{and} \qquad v^k_{\rho,\varepsilon} = \mathcal{F}^{-1}\,w^k_{\rho,\varepsilon}. \tag{2.21}$$

Provided that $\operatorname{Re}\rho > -1$, $v^k_{\rho,\varepsilon}$ is a locally summable function: in particular,

$$v^0_{\rho,0}(s) = \frac{2\,\pi^{\frac{\rho+1}{2}}}{\Gamma(\frac{\rho+1}{2})}\,|s|^{\frac{\rho}{2}}\,K_{\frac{\rho}{2}}(2\pi|s|) \tag{2.22}$$

and

$$\begin{aligned} v^0_{\rho,1}(s) &= \left(1 - \frac{1}{2\pi}\,\frac{d}{ds}\right)\,v^0_{\rho+1,0}(s) \\ &= \frac{2\,\pi^{\frac{\rho+2}{2}}}{\Gamma(\frac{\rho+2}{2})}\,\left[|s|^{\frac{\rho+1}{2}}\,K_{\frac{\rho+1}{2}}(2\pi|s|) + \langle s\rangle^{\frac{\rho+1}{2}}\,K_{\frac{\rho-1}{2}}(2\pi|s|)\right]. \end{aligned} \tag{2.23}$$

In the next section, we shall indicate how, imitating the procedure well-known in the case of the harmonic oscillator, one can inductively build the sequences $(v^k_{\rho,0})_{k\in\mathbb{Z}}$ and $(v^k_{\rho,1})_{k\in\mathbb{Z}}$ with the help of creation operators. This can only be done if one tackles two such sequences simultaneously, which is another reason why piecing together two irreducible representations, as we have done in Section 3 and as we shall do again in the next one, is essential to our purpose.

In view of the important role played by the spaces $\widehat{C}^\infty_{\rho,\varepsilon}$ and $C^\infty_{\rho,\varepsilon}$, the following characterization of the latter space is useful.

Proposition 2.1. *Assume that* $\operatorname{Re}\rho > -1$ *and* $\rho \notin \mathbb{Z}$. *Then* $v \in C^\infty_{\rho,\varepsilon}$ *if and only if* v *is a function on the real line with the following properties:*

(i) v *is* C^∞ *outside* 0;

(ii) v *and its derivatives are rapidly decreasing at infinity;*

(iii) *near* 0, v *admits an expansion*

$$v(s) \sim a_0 + a_1 s + a_2 s^2 + \cdots \\ + |s|_\varepsilon^\rho \, (b_0 + b_1 s + b_2 s^2 + \cdots) \qquad (2.24)$$

where any specified derivative of the remainder shall be $O(|s|^N)$ *with* N *as large as one pleases provided the expansion is pushed far enough.*

Proof. The case when $\varepsilon = 0$ was treated (for a rather different purpose) in [24, Prop. 2.1]: for the sake of completeness, let us give a proof of the case when $\varepsilon = 1$. First, from the expansion

$$2\,|s|^{\frac{\rho}{2}}\, K_{\frac{\rho}{2}}(2\pi\,|s|) = \pi^{-\frac{\rho}{2}}\, \Gamma\left(-\frac{\rho}{2}\right) \Gamma\left(\frac{2+\rho}{2}\right) \\ \times \left[-\sum_{m\geq 0} \frac{(\pi s)^{2m}}{m!\,\Gamma(-\frac{\rho}{2}+m+1)} + |\pi s|^{\,\rho} \sum_{m\geq 0} \frac{(\pi s)^{2m}}{m!\,\Gamma(\frac{\rho}{2}+m+1)} \right], \qquad (2.25)$$

and from (2.23), one sees, writing $|s|^{\rho+1} = \langle\, s\rangle^\rho\, s$ and $s\,|s|^{\rho-1} = \langle\, s\rangle^\rho$, that each of the functions $v^0_{\rho+2j,1}$ with $j = 0, 1, \ldots$ satisfies near $s = 0$ the expansion (2.24). Take now an arbitrary function v in the space $C^\infty_{\rho,1}$, and let $w = \mathcal{F}\, v$. The Taylor expansion near $\sigma = 0$ of the function (*cf.* (2.17))

$$(\hat{\pi}_{\rho,1}\, \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\, w)(\sigma) = \langle\, \sigma\,\rangle^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \qquad (2.26)$$

can be written if so wished as

$$\langle\, \sigma\,\rangle^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \sim (1 - i\sigma)\,(1+\sigma^2)^{\frac{-\rho-2}{2}}\, \left(\beta_0 + \beta_1\,\sigma^2\,(1+\sigma^2)^{-1} + \cdots\right) \\ -\sigma\,(1 - i\sigma)\,(1+\sigma^2)^{\frac{-\rho-4}{2}}\, \left(\gamma_0 + \gamma_1\,\sigma^2\,(1+\sigma^2)^{-1} + \cdots\right) \quad (2.27)$$

(expand the even and odd parts of the product of the left-hand side by $(1 - i\sigma)^{-1}$ in the manner indicated), from which one gets the expansion at infinity

$$i\, w(\sigma) \sim (1 - i\,\sigma)\,(1+\sigma^2)^{\frac{-\rho-2}{2}}\, \left(\beta_0 + \beta_1\,(1+\sigma^2)^{-1} + \cdots\right) \\ +\sigma\,(1 - i\,\sigma)\,(1+\sigma^2)^{\frac{-\rho-4}{2}}\, \left(\gamma_0 + \gamma_1\,(1+\sigma^2)^{-1} + \cdots\right). \qquad (2.28)$$

Then, using the relation $(\mathcal{F}\, v^0_{\rho,1})(\sigma) = (1 - i\sigma)\,(1+\sigma^2)^{\frac{-\rho-2}{2}}$, one gets the expansion

$$i\, v(s) \sim \beta_0\; v^0_{\rho,1}(s) + \beta_1\; v^0_{\rho+2,1}(s) + \cdots \\ + \frac{1}{2i\pi}\, \frac{d}{ds}\, \left(\gamma_0\; v^0_{\rho+2,1}(s) + \gamma_1\; v^0_{\rho+4,1}(s) + \cdots\right), \qquad (2.29)$$

which leads to (i) and (iii) since a remainder term from the expansion (2.28) of w at infinity, on the other hand, can only contribute to v some extra term differentiable on the real line as many times as one pleases. That v is rapidly decreasing at infinity can also be seen from (2.28) since, apart from the first one, all the terms in this expansion of w are summable, even the more so after one has taken their derivative any number of times; setting apart a finite number of terms will work in the same way in connection with a derivative of v.

In the reverse direction, we observe that the non-smooth part of the expansion near $s = 0$ of the function $v^0_{\rho,1}$ starts with a term like $\langle\, s \,\rangle^{\rho}$ and the expansion of $\frac{d}{ds}\, v^0_{\rho+2,1}$ starts with a term like $\langle\, s \,\rangle^{\rho}\, s$, in both cases with non-zero coefficients; also, in both cases, the terms that follow are the same, multiplied by some even power of s. Assuming (i),(ii) and (iii) to hold, this makes it possible to successively choose the coefficients $\beta_0,\, \beta_1, \dots, \gamma_0,\, \gamma_1, \dots$ so as to recover an expansion near $s = 0$ of the kind (2.29), where the remainder, if pushed far enough, will be as many times differentiable on the real line as one pleases, while still being rapidly decreasing at infinity. The argument above linking the behaviors of v near 0 and that of w near ∞ can then be reversed. □

Even though the representation $\pi_{\rho,1}$ is not unitarizable, one may consider on $C^\infty_{\rho,\varepsilon}$, in the case when ρ is real, $0 < |\rho| < 1$, the Hermitian form defined as

$$(v_1 \,|\, v_2)_{\rho,\varepsilon} = \int_{-\infty}^{\infty} |s|_\varepsilon^{-\rho}\, \bar{v}_1(s)\, v_2(s)\, ds : \tag{2.30}$$

it is positive-definite only in the unitarizable case, in which it is just the scalar product associated with (2.5). It transfers to the space $\widehat{C}^\infty_{\rho,\varepsilon}$ as

$$(w_1 \,|\, w_2)^\wedge_{\rho,\varepsilon} = (-i)^\varepsilon \pi^{\frac{1}{2}+\rho}\, \frac{\Gamma(\frac{1-\rho+\varepsilon}{2})}{\Gamma(\frac{\rho+\varepsilon}{2})} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{w}_1(\sigma)\, w_2(\tau)}{|\sigma-\tau|_\varepsilon^{1-\rho}}\, d\sigma\, d\tau, \tag{2.31}$$

a genuine integral only when $0 < \rho < 1$, but an expression that can still be given a meaning when $0 < |\rho| < 1$, and is easily shown to be invariant under the representation $\hat{\pi}_{\rho,\varepsilon}$.

We now introduce a space obviously related to the notion of analytic vectors, though it also makes sense in the absence of any useful Banach space structure.

Definition 2.2. Given $\rho \in \mathbb{C}$ and $\varepsilon = 0$ or 1, we denote as $\widehat{C}^\omega_{\rho,\varepsilon}$ the space of real-analytic (complex-valued) functions $w = w(\sigma)$ on $\mathbb{R}$ such that the function (defined for $\sigma \neq 0$)

$$w_1(\sigma) = |\sigma|_\varepsilon^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \tag{2.32}$$

extends as an analytic function on $\mathbb{R}$ too.

Proposition 2.3. *Assume that $\rho \in \mathbb{C}$, ρ is not an integer, and* $\mathrm{Re}\,\rho > -1$. *The image of the space $\widehat{C}^{\omega}_{\rho,\varepsilon}$ under the inverse Fourier transformation $\mathcal{F}^{-1}$ is the space $C^{\omega}_{\rho,\varepsilon}$ of functions v on $\mathbb{R}\backslash\{0\}$ with the following properties:*

(i) *there exist two entire functions v^{ent} and v^{ram} (corresponding to the entire part and the ramified part of v) such that*

$$v(s) = v^{\mathrm{ent}}(s) + |s|^{\rho}_{\varepsilon}\, v^{\mathrm{ram}}(s), \quad s \in \mathbb{R}; \tag{2.33}$$

(ii) *there exists $R > 0$ and $C > 0$ such that*

$$|v^{\mathrm{ent}}(s)| + |v^{\mathrm{ram}}(s)| \le C\, e^{2\pi R|s|}, \qquad s \in \mathbb{C}; \tag{2.34}$$

(iii) *there exist $\delta > 0$ and $C > 0$ such that*

$$|v(s)| \le C\, e^{-2\pi\delta|s|}, \qquad s \text{ real},\ |s| > 1. \tag{2.35}$$

Proof. The map $w \mapsto (1+i\sigma)w$, is an isomorphism from $\widehat{C}^{\omega}_{\rho,\varepsilon}$ onto $\widehat{C}^{\omega}_{\rho-1,1-\varepsilon}$. On the other hand, using Lemma 1.5 and Cauchy's integral formula to evaluate $v'(s)$, the conditions (2.33)–(2.35) associated to the pair (ρ,ε) transfer to the same conditions, in association to the pair $(\rho-1, 1-\varepsilon)$, under the map $v \mapsto (1+\frac{1}{2\pi}\frac{d}{ds})\,v$. It is thus no loss of generality to assume that $\varepsilon = 0$ and that $\mathrm{Re}\,\rho$ is as large as one pleases: this will be tacitly assumed so as to let us feel more secure, as it will sometimes transform semi-convergent only integrals into genuine ones.

Next, if w lies in $\widehat{C}^{\omega}_{\rho,\varepsilon}$, so do its even and odd parts though, of course, the representation $\hat{\pi}_{\rho,\varepsilon}$ does not preserve the corresponding decomposition of this space: we may thus prove the *direct* part (properties of $v = \mathcal{F}^{-1}w \in C^{\omega}_{\rho,\varepsilon}$ in terms of those of w) under the additional assumption that w has a definite parity indexed by η ($\eta = 0$ if w is even, 1 if w is odd). Then, we shall first prove that there exist two entire functions v^{ent} and v^{ram} such that

$$v(s) = v^{\mathrm{ent}}(s) + s^{\rho}\ v^{\mathrm{ram}}(s) \tag{2.36}$$

for $s > 0$ and that v satisfies the estimate

$$|v(s)| \le C\, e^{-2\pi\,\delta s}, \qquad s > 1: \tag{2.37}$$

observe that, of necessity (since ρ is not an integer), v^{ent} must then have the parity associated with η, and v^{ram} that associated with $\varepsilon+\eta$ mod 2. Next, we shall show that v, initially regarded as a function on $]0,\infty[$, admits an extension $\tilde{v}$ as a function on the Riemann surface of the logarithm; finally, we shall prove that there exist positive constants C, R such that

$$|\tilde{v}(se^{i\theta})| \le C\, e^{2\pi\,R\,s} \tag{2.38}$$

for all $s > 1$ and $\theta \in \mathbb{R}$.

Thus, assuming that $\mathrm{Re}\ \rho$ is large, start from a given function $w \in \widehat{C}^{\omega}_{\rho,0}$, with the parity associated with η, so that there exists a function h holomorphic in the complement of some disk centered at zero with a radius $< R$, with a finite limit at infinity, such that

$$w(\sigma) = |\sigma|^{-1-\rho} h(\sigma), \qquad \sigma \in \mathbb{R},\ |\sigma| \geq R : \tag{2.39}$$

observe that the conditions just stated are equivalent to the condition (2.32) of Definition 2.2 as seen with the help of the Laurent expansion of w. We then split the integral defining $v = \mathcal{F}^{-1} w$ as the sum of three terms,

$$v(s) = \int_{-\infty}^{-R} e^{2i\pi s\sigma}\, w(\sigma)\, d\sigma + \int_{-R}^{R} e^{2i\pi s\sigma}\, w(\sigma)\, d\sigma + \int_{R}^{\infty} e^{2i\pi s\sigma}\, w(\sigma)\, d\sigma, \tag{2.40}$$

denoted as $I_1(s)$, $I_2(s)$, $I_3(s)$: obviously, $I_2(s)$ extends as an entire function of s, satisfying an estimate with the same right-hand side as (2.38).

Lemma 2.4. *Assume that* $\mathrm{Re}\ \rho > -1$, $\rho \notin \mathbb{N}$, *and let* $h(\sigma) = \sum_{n\geq 0} a_n \sigma^{-n}$, *an absolutely convergent series for* $\sigma \geq R'$, *some positive number* $< R$. *Consider the (possibly improper) integral*

$$F(s) = \int_{R}^{\infty} e^{2i\pi s\sigma}\, \sigma^{-1-\rho} h(\sigma)\, d\sigma. \tag{2.41}$$

Set, for $s \in \mathbb{C}$,

$$\chi(s) = \sum_{n\geq 0} a_n \frac{\Gamma(\rho+1)}{\Gamma(\rho+n+1)}\, (2i\pi s)^n,$$

$$\psi(s) = \sum_{n\geq 0} a_n \sum_{j=1}^{n} \frac{\Gamma(\rho+j)}{\Gamma(\rho+n+1)}\, R^{-\rho-j}\, (2i\pi s)^{n-j}, \tag{2.42}$$

and, for $s \in \mathbb{R}\backslash\{0\}$,

$$B_0(s) = \int_{R}^{\infty} e^{2i\pi s\sigma}\, \sigma^{-1-\rho}\, d\sigma, \tag{2.43}$$

an improper integral if $\mathrm{Re}\ \rho \leq 0$ (*recall that* $\mathrm{Re}\ \rho > -1$). *Then,* χ *and* ψ *are both entire functions, and satisfy the estimate*

$$|\chi(s)| + |\psi(s)| \leq C\, e^{2\pi R''|s|}, \qquad s \in \mathbb{C} \tag{2.44}$$

for some R'' *with* $R' < R''$, *which may be assumed to be* $< R$; *next, one has*

$$B_0(s) = B_0^{\infty}(s) - B_0^0(s), \tag{2.45}$$

where the function B_0^0 *extends as an entire function satisfying, given any* $R'' < R$, *the estimate*

$$|B_0^0(s)| \leq C\, e^{2\pi(2R-R'')|s|}, \qquad s \in \mathbb{C}, \tag{2.46}$$

and, for $s \in \mathbb{R}\backslash\{0\}$*,*

$$B_0^\infty(s) = \frac{1}{2}\,\pi^{\frac{1}{2}+\rho}\,\left[\frac{\Gamma(-\frac{\rho}{2})}{\Gamma(\frac{1+\rho}{2})}\,|s|^\rho + i\,\frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(\frac{2+\rho}{2})}\,\langle s\rangle^\rho\right]. \tag{2.47}$$

Finally,

$$F(s) = \chi(s)B_0(s) + \psi(s)\; e^{2i\pi Rs} \tag{2.48}$$

for $s \in \mathbb{R}\backslash\{0\}$*.*

Proof. Since $|a_n| \leq C\,{R'}^n$, one may write, whenever $R'' > R'$,

$$\begin{aligned}\sum_{n\geq 0} |a_n|\,\frac{\Gamma(\rho+1)}{\Gamma(\rho+n+1)}\,(2\pi|s|)^n &\leq C\sum_{n\geq 0} {R'}^n\,\frac{(2\pi|s|)^n}{(\rho+1)\cdots(\rho+n)}\\ &\leq C\left(1+\sum_{n\geq 0}\frac{(2\pi R'|s|)^n}{(n-1)\,!}\right)\\ &\leq C\,e^{2\pi R''|s|},\end{aligned} \tag{2.49}$$

which yields the estimate regarding χ. In the same way, with $R'' = (RR')^{\frac{1}{2}}$,

$$\begin{aligned}&\sum_{n\geq 0}|a_n|\sum_{j=1}^n\frac{\Gamma(\rho+j)}{\Gamma(\rho+n+1)}\,(2\pi|s|)^{n-j}\,R^{-\rho-j}\\ &\quad\leq C\sum_{n\geq 0}{R'}^n\sum_{j=1}^n\frac{(2\pi|s|)^{n-j}\,R^{-\rho-j}}{(\rho+j)\cdots(\rho+n)}\\ &\quad\leq C\,R^{-\rho}\sum_{n\geq 0}(\frac{R''}{R})^n\sum_{j=1}^n\frac{(2\pi|s|R'')^{n-j}}{(n-j)\,!}\\ &\quad\leq C\,e^{2\pi R''|s|}\,R^{1-\rho}(R-R'')^{-1}.\end{aligned} \tag{2.50}$$

This completes the estimate concerning ψ.

The integral $B_0(s)$, for a given value of ρ, can be connected, if $\text{Re}\;\rho > 0$, to the same integral where $\rho - 1$ is substituted for ρ, by means of an obvious integration by parts: in this way, we can satisfy ourselves with proving all facts *concerning this function only* in the case when $-1 < \text{Re}\;\rho < 1$. Then, $B_0^\infty(s)$, as expressed in (2.47), is just the finite part of the improper integral $\int_0^\infty e^{2i\pi s\sigma}\,\sigma^{-1-\rho}\,d\sigma$, *i.e.,*

$$B_0^\infty(s) = \lim\nolimits_{\varepsilon=0}\left[\frac{\varepsilon^{-\rho}}{\rho} + \int_\varepsilon^\infty e^{2i\pi s\sigma}\,\sigma^{-1-\rho}\,d\sigma\right] \tag{2.51}$$

(still involving an improper integral), and $B_0^0(s)$, which is the finite part of the same integral taken from 0 to R, clearly extends as an entire function of s satisfying the estimate

$$|B_0^0(s)| \leq C\,e^{2\pi R|s|}, \qquad s \in \mathbb{C}. \tag{2.52}$$

Setting

$$B_n(s) = \int_R^\infty e^{2i\pi s\sigma}\, \sigma^{-1-\rho-n}\, d\sigma, \tag{2.53}$$

one has

$$B_n(s) = e^{2i\pi Rs} \sum_{j=1}^{n} \frac{\Gamma(\rho+j)}{\Gamma(\rho+n+1)}\, (2i\pi s)^{n-j}\, R^{-\rho-j} + (2i\pi s)^n\, \frac{\Gamma(\rho+1)}{\Gamma(\rho+n+1)}\, B_0(s) \tag{2.54}$$

for all $n \geq 0$, as one can see by induction; next, if s is real and > 0,

$$F(s) = \sum_{n\geq 0} a_n\, B_n(s), \tag{2.55}$$

which yields the sought-after decomposition of $F(s)$. □

End of Proof of Proposition 2.3. We apply Lemma 2.4 to the function $I_3(s)$ and to the function

$$I_1(s) = (-1)^\eta \int_R^\infty e^{-2i\pi s\sigma}\, \sigma^{-1-\rho}\, h(\sigma)\, d\sigma, \tag{2.56}$$

both only defined for $s > 0$. With the notations of Lemma 2.4, remarking that

$$\chi(-s) = (-1)^\eta\, \chi(s), \tag{2.57}$$

one thus has

$$\begin{aligned} I_3(s) &= \chi(s)\, (B_0^\infty(s) - B_0^0(s)) + \psi(s)\, e^{2i\pi Rs}, \\ I_1(s) &= \chi(s)\, (B_0^\infty(-s) - B_0^0(-s)) + (-1)^\eta\, \psi(-s)\, e^{-2i\pi Rs}, \end{aligned} \tag{2.58}$$

and the only functions of s which fail to be entire on the right-hand sides of the two equations are $B_0^\infty(\pm s)$: however,

$$e^{\frac{i\pi\rho}{2}}\, B_0^\infty(s) + e^{-\frac{i\pi\rho}{2}}\, B_0^\infty(-s) = 0, \tag{2.59}$$

as it follows from (2.47) together with the duplication formula (1.89) of the Gamma function. Adding the two equations that precede, one finds that (2.33) is satisfied for $s > 0$ provided one defines

$$\begin{aligned} v^{\mathrm{ram}}(s) &= (1 - e^{i\pi\rho})\, (s^{-\rho} B_0^\infty(s))\, \chi(s), \qquad (2.60)\\ v^{\mathrm{ent}}(s) &= I_2(s) - \chi(s)\, (B_0^0(s) + B_0^0(-s)) + \psi(s)e^{2i\pi Rs} + (-1)^\eta \psi(-s) e^{-2i\pi Rs}. \end{aligned}$$

Since $s^{-\rho} B_0^\infty(s)$ is a constant for $s > 0$, it is clear that v^{ram} and v^{ent}, initially defined for $s > 0$, extend as entire functions of s, with the parity prescribed in (iv). Also, the estimate (2.38) follows from the estimates, established in lemma 2.4, concerning the functions χ and ψ. In this direction, only the estimate (2.37), regarding the exponential decrease of $\tilde{v}$ near the real positive half-line, remains to be proved. Choose $\delta > 0$ such that $R-\delta > R'$ and small enough so that $\sigma + i\delta'$ should remain, when $0 \leq \delta' \leq \delta$ and $-R \leq \sigma \leq R$, in a domain where w extends as a holomorphic function: we can then write

$$\tilde{v}(s) = \int_{-\infty}^{\infty} e^{2i\pi s(\sigma+i\delta)}\, w(\sigma+i\delta)\, d\sigma, \tag{2.61}$$

which leads to (2.37).

In the reverse direction, we now establish the properties of

$$w(\sigma) = \int_{-\infty}^{\infty} e^{-2i\pi\, s\sigma}\, v(s)\, ds \tag{2.62}$$

as a consequence of the properties of v stated in Proposition 2.3. Again, one may assume that $\mathrm{Re}\,\rho$ is large: however, in view of a formula which will have subsequent use, it is better not to assume that $\varepsilon = 0$. We first note that the pair of functions v^{ent} and v^{ram} whose existence is asserted in (2.33) is of necessity unique. Next, if $\check{v}(s) = v(-s)$, the function $\check{v}$ has a decomposition similar to (2.33) with

$$(\check{v})^{\mathrm{ent}} = (v^{\mathrm{ent}})^{\vee} \qquad \text{and} \qquad (\check{v})^{\mathrm{ram}} = (-1)^{\varepsilon}\,(v^{\mathrm{ram}})^{\vee} : \tag{2.63}$$

consequently, so do the even and odd parts of v. On the other hand, the estimate (2.35) also holds for v_{even} and v_{odd} if it does for v. We may thus work under the additional assumption that v (or w) has the parity associated with η: then, again, v^{ent} (*resp.* v^{ram}) has the parity associated with η (*resp.* $\varepsilon+\eta \bmod 2$).

From (2.62), rewritten as

$$w(\sigma) = \int_0^{\infty} [(-1)^{\eta} e^{2i\pi\, s\sigma} + e^{-2i\pi\, s\sigma}]\, v(s)\, ds, \tag{2.64}$$

one sees that, to start with, w is indeed analytic on the real line. We still have to show that there exists some function $h = h(\sigma)$ holomorphic for large $|\sigma|$, with a finite limit at infinity, such that

$$w(\sigma) = |-\sigma|_{\varepsilon}^{-1-\rho}\, h(\sigma) \qquad \text{for } \sigma \in \mathbb{R},\ |\sigma| \text{ large}. \tag{2.65}$$

We shall show presently that the function w, considered on $]0,\infty[$, admits a holomorphic continuation $\tilde{w}$ to the part of the Riemann surface of the logarithm lying above the complement of some disk centered at zero: setting $\tilde{h}(\sigma) = (-1)^{\varepsilon}\sigma^{1+\rho}\tilde{w}(\sigma)$, the remaining problems will be to show that, for large $\sigma > 0$, $\tilde{h}(\sigma\, e^{i\pi}) = \sigma^{1+\rho}\, w(-\sigma)$, in other words that

$$\tilde{w}(\sigma\, e^{i\pi}) = (-1)^{\varepsilon+\eta+1} e^{-i\pi\rho} w(\sigma), \qquad \sigma > 0 \ \text{ and large}, \tag{2.66}$$

and that, as $\sigma > 0$ goes to ∞, $\tilde{h}(\sigma\, e^{i\phi})$ admits a finite limit independent of ϕ. Starting from (2.64), we take advantage of Lemma 1.5 to write

$$\begin{aligned} w(\sigma) = & \int_0^{\infty} (-1)^{\eta} e^{2i\pi s e^{i\theta_0}\sigma}\, \tilde{v}(se^{i\theta_0})\, e^{i\theta_0}\, ds \\ & + \int_0^{\infty} e^{-2i\pi s e^{-i\theta_0}\sigma}\, \tilde{v}(se^{-i\theta_0})\, e^{-i\theta_0}\, ds, \end{aligned} \tag{2.67}$$

then of (2.34) to write, for $\sigma > 0$ and large,

$$w(\sigma) = i \int_0^\infty e^{-2\pi s\sigma} \left[(-1)^\eta \tilde{v}(s\, e^{\frac{i\pi}{2}}) - \tilde{v}(s\, e^{-\frac{i\pi}{2}})\right] ds. \tag{2.68}$$

Using (2.33) together with parity considerations, one remarks that

$$\begin{aligned}(-1)^\eta \tilde{v}(s\, e^{\frac{i\pi}{2}}) - \tilde{v}(s\, e^{-\frac{i\pi}{2}}) &= (-1)^\eta \left[e^{\frac{i\pi\rho}{2}} - (-1)^\varepsilon e^{-\frac{i\pi\rho}{2}}\right] s^\rho\, v^{\mathrm{ram}}(is) \\ &= (-1)^\eta \frac{2\pi i^{1-\varepsilon}}{\Gamma(\frac{\varepsilon+\rho}{2})\,\Gamma(\frac{2-\varepsilon-\rho}{2})}\, s^\rho\, v^{\mathrm{ram}}(is)\end{aligned} \tag{2.69}$$

so that

$$w(\sigma) = -\frac{2\pi\, i^\varepsilon}{\Gamma(\frac{\varepsilon+\rho}{2})\,\Gamma(\frac{2-\varepsilon-\rho}{2})} \int_0^\infty s^\rho\, e^{-2\pi s\sigma}\, v^{\mathrm{ram}}(-is)\, ds, \tag{2.70}$$

an equation obviously valid even in the case when v has no definite parity. A contour deformation using (2.34) yields for $\sigma > 0$ large and arbitrary $\phi \in [0, \pi]$ the equation

$$\tilde{w}(\sigma\, e^{i\phi}) = -\frac{2\pi\, i^\varepsilon}{\Gamma(\frac{\varepsilon+\rho}{2})\,\Gamma(\frac{2-\varepsilon-\rho}{2})} \int_0^\infty s^\rho\, e^{-2\pi s\sigma}\, v^{\mathrm{ram}}(-is\, e^{-i\phi})\, e^{-i(1+\rho)\phi}\, ds : \tag{2.71}$$

indeed, if ϕ_0 is the least upper bound of all $\phi \in [0, \pi]$ such that (2.71) holds for some given $\sigma > 2R$, then it holds also when $\phi = \phi_0$ in view of the estimate (2.34). Next, another deformation of contour makes it possible to write also

$$\tilde{w}(\sigma e^{i\phi_0}) = -\frac{2\pi i^\varepsilon}{\Gamma(\frac{\varepsilon+\rho}{2})\Gamma(\frac{2-\varepsilon-\rho}{2})} \int_0^\infty s^\rho\, e^{-2\pi\sigma s e^{-i(\phi_1-\phi_0)}}\, v^{\mathrm{ram}}(-is e^{-i\phi_1}) e^{-i(1+\rho)\phi_1}\, ds \tag{2.72}$$

provided that $\phi_0 \leq \phi_1 < \phi_0 + \frac{\pi}{2}$, which implies (using complex continuation in order to substitute $\sigma\, e^{i(\phi_1-\phi_0)}$ for σ in the last equation) that (2.71) holds with ϕ_1 substituted for ϕ_0: thus $\phi_0 = \pi$, in which case, comparing (2.71) to (2.70), one just finds (2.66). Also, $(\sigma\, e^{i\phi})^{1+\rho}\, \tilde{w}(\sigma\, e^{i\phi})$ goes to some finite constant independent of ϕ as $\sigma \to \infty$, as the change of variable $s \mapsto \frac{s}{\sigma}$ shows. □

Observe that any $v \in C^\omega_{\rho,\varepsilon}$ is characterized by any of the two terms v^{ent} and $s \mapsto |s|^\rho_\varepsilon\, v^{\mathrm{ram}}(s)$ from the decomposition (2.33) or, which amounts to the same, a non-zero $v \in C^\omega_{\rho,\varepsilon}$ cannot reduce to either one of the two terms above. For, in the first case, the entire function v^{ent} has to be exponentially decreasing at $\pm\infty$ on the real line (a consequence of (2.35)), thus $v^{\mathrm{ent}} = 0$ according to Lemma 1.9. The second case is proved in the same way.

Proposition 2.5. *Assume that* $\rho \in \mathbb{C}$, *ρ is not an integer, and* $\mathrm{Re}\,\rho > -1$; *let* $\varepsilon = 0$ *or* 1. *Let* $v^\bullet$ *be an entire function, satisfying for some pair* C, R *of positive constants the estimate*

$$|v^\bullet(s)| \leq C\, e^{4\pi R|s|}, \quad s \in \mathbb{C}. \tag{2.73}$$

Consider the function w *defined for large* $\sigma > 0$ *by the equation*

$$w(\sigma) = -\frac{2\pi\, i^{\varepsilon}}{\Gamma(\frac{\varepsilon+\rho}{2})\,\Gamma(\frac{2-\varepsilon-\rho}{2})} \int_0^{\infty} s^{\rho}\, e^{-2\pi s\sigma}\, v^{\bullet}(-is)\, ds : \tag{2.74}$$

it extends as a holomorphic function $\tilde{w}$ *on the part of the Riemann surface of the logarithm lying above the complement of some disk centered at zero, satisfying the equation*

$$\tilde{w}(\sigma\, e^{2i\pi}) = e^{-2i\pi\rho}\, \tilde{w}(\sigma). \tag{2.75}$$

In order that there should exist a function $v \in C^{\omega}_{\rho,\varepsilon}$ *such that* $v^{\mathrm{ram}} = v^{\bullet}$, *it is necessary and sufficient that* w (*defined for large* $\sigma > 0$) *should extend as an analytic function* (*still denoted as* w) *on the real line, satisfying for large* $\sigma > 0$ *the equation*

$$\tilde{w}(\sigma\, e^{i\pi}) = (-1)^{\varepsilon+1}\, e^{-i\pi\rho}\, w(-\sigma). \tag{2.76}$$

If such is the case, the function v *is given as* $v = \mathcal{F}^{-1} w$.

Proof. All that needs being done is showing that w satisfies the condition (2.65): it suffices to rework the very last part of the proof of Proposition 2.3, that begins at the equation (2.70), after one has broken down, for simplicity, $v^{\bullet}$ into its even and odd parts. □

Examples. 1) If we apply (2.74) with $\varepsilon = 0$ and

$$v^{\bullet}(s) = -\pi^{\frac{\rho+1}{2}}\, \frac{\Gamma(\frac{\rho}{2})\Gamma(\frac{2-\rho}{2})}{\Gamma(\frac{\rho+1}{2})}\, s^{-\frac{\rho}{2}} I_{\frac{\rho}{2}}(2\pi s) \qquad s > 0, \tag{2.77}$$

we find [17, p. 91]

$$\begin{aligned} w(\sigma) &= \frac{2\,\pi^{\frac{\rho+3}{2}}}{\Gamma(\frac{\rho+1}{2})} \int_0^{\infty} s^{\frac{\rho}{2}}\, e^{-2\pi s\sigma} J_{\frac{\rho}{2}}(2\pi s)\, ds \\ &= (1+\sigma^2)^{-\frac{\rho+1}{2}}. \end{aligned} \tag{2.78}$$

2) If we apply (2.74) with $\varepsilon = 1$ and

$$v^{\bullet}(s) = \pi^{\frac{\rho+2}{2}}\, \frac{\Gamma(\frac{1+\rho}{2})\Gamma(\frac{1-\rho}{2})}{\Gamma(\frac{\rho+2}{2})}\, s^{\frac{1-\rho}{2}} \left[I_{\frac{\rho-1}{2}}(2\pi s) - I_{\frac{\rho+1}{2}}(2\pi s) \right] \qquad s > 0, \tag{2.79}$$

we find (*loc. cit.*)

$$\begin{aligned} w(\sigma) &= -\frac{2i\,\pi^{\frac{\rho+4}{2}}}{\Gamma(\frac{\rho+2}{2})} \int_0^{\infty} s^{\frac{\rho+1}{2}}\, e^{-2\pi s\sigma} \left[J_{\frac{\rho-1}{2}}(2\pi s) + i\, J_{\frac{\rho+1}{2}}(2\pi s) \right] ds \\ &= (1+\sigma^2)^{-\frac{\rho+2}{2}} (1 - i\sigma). \end{aligned} \tag{2.80}$$

Thus, in the first case, w coincides with the function introduced in (2.20) as $w^0_{\rho,0}$; in the second case, it agrees with the function $w^0_{\rho,1}$: of course, this is only for the purpose of showing the practical value of the "inversion formula" provided by Proposition 2.5, since the function $v^{\bullet}$ we started from is none other, in the first case, than the function $(v^0_{\rho,0})^{\rm ram}$, as can be verified from (2.22); in the second case, $v_1 = (v^0_{\rho,1})^{\rm ram}$, a consequence of (2.23).

The following is a converse to Theorem 1.8, leading to a characterization of the space $\mathfrak{A}$ which will make the higher-dimensional generalization possible.

Theorem 2.6. *Let u be an entire function of one variable, such that $|u(z)| \leq C\, e^{\pi R\,|z|^2}$ for some pair (C, R) of positive constants. Recalling* (1.26)*, set, for σ real and large enough,*

$$\begin{aligned} w_0(\sigma) &= \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, u(x\, e^{-\frac{i\pi}{4}})\, dx, \\ w_1(\sigma) &= \frac{1-i}{2} \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, x\, u(x\, e^{-\frac{i\pi}{4}})\, dx, \end{aligned} \tag{2.81}$$

and assume that the functions w_0 and w_1 extend as analytic functions on the real line and admit when $|\sigma|$ is large the convergent expansions (1.27)*. Then $u \in \mathfrak{A}$.*

Proof. In view of Definition 2.2, the validity of the series expansions under consideration show that the function w_0 lies in the space $\widehat{C}^{\omega}_{-\frac{1}{2},0}$ and that w_1 lies in $\widehat{C}^{\omega}_{\frac{1}{2},1}$. Set $v_0 = \mathcal{F}^{-1}\, w_0 \in C^{\omega}_{-\frac{1}{2},0}$ and $v_1 = \mathcal{F}^{-1}\, w_1 \in C^{\omega}_{\frac{1}{2},1}$. Using the decomposition of the functions v_0 and v_1 into their entire and ramified parts as provided by Proposition 2.3, we set

$$\begin{aligned} f_0(x) &= 2^{-\frac{1}{2}}\, x\, v_0^{\rm ent}\left(\frac{x^2}{2}\right) + v_0^{\rm ram}\left(\frac{x^2}{2}\right), \\ f_1(x) &= v_1^{\rm ent}\left(\frac{x^2}{2}\right) + 2^{-\frac{1}{2}}\, x\, v_1^{\rm ram}\left(\frac{x^2}{2}\right), \end{aligned} \tag{2.82}$$

so that the functions f_0 and f_1 extend as entire functions. Using (2.33), in the present case

$$\begin{aligned} v_0(s) &= v_0^{\rm ent}(s) + |s|^{-\frac{1}{2}}\, v_0^{\rm ram}(s), \\ v_1(s) &= v_1^{\rm ent}(s) + \langle s\rangle^{\frac{1}{2}}\, v_1^{\rm ram}(s), \end{aligned} \tag{2.83}$$

a pair of equations valid for every real number s, we find that, *on the positive half-line*, one has

$$f_0(x) = 2^{-\frac{1}{2}}\, x\, v_0\left(\frac{x^2}{2}\right), \qquad f_1(x) = v_1\left(\frac{x^2}{2}\right): \tag{2.84}$$

by the same considerations, we find that the functions $f_{i,0}$ and $f_{i,1}$, defined according to (1.8), are given as

$$f_{i,0}(x) = 2^{-\frac{1}{2}}\, x\, v_0\left(-\frac{x^2}{2}\right), \qquad f_{i,1}(x) = v_1\left(-\frac{x^2}{2}\right), \qquad x > 0. \tag{2.85}$$

Applying the results of Proposition 2.3 again, it follows that the four functions f_0, f_1, $f_{i,0}$, $f_{i,1}$ are nice in the sense of Definition 1.1. Next, we use (2.70), in this case

$$\begin{aligned} w_0(\sigma) &= 2^{\frac{1}{2}} \int_0^\infty s^{-\frac{1}{2}}\, e^{-2\pi s\sigma}\, v_0^{\mathrm{ram}}(-is)\, ds &&= 2\int_0^\infty e^{-\pi\sigma x^2}\, v_0^{\mathrm{ram}}\left(-\frac{ix^2}{2}\right) dx, \\ w_1(\sigma) &= -2^{\frac{1}{2}}\, i \int_0^\infty s^{\frac{1}{2}}\, e^{-2\pi s\sigma}\, v_1^{\mathrm{ram}}(-is)\, ds &&= -i\int_0^\infty e^{-\pi\sigma x^2}\, x^2\, v_1^{\mathrm{ram}}\left(-\frac{ix^2}{2}\right) dx. \end{aligned} \tag{2.86}$$

Since, in view of (2.82), one has

$$(f_0)_{\mathrm{even}}(x\, e^{-\frac{i\pi}{4}}) = v_0^{\mathrm{ram}}\left(-\frac{i\,x^2}{2}\right), \qquad (f_1)_{\mathrm{odd}}(x\, e^{-\frac{i\pi}{4}}) = \frac{1-i}{2}\, x\, v_1^{\mathrm{ram}}\left(-\frac{i\,x^2}{2}\right), \tag{2.87}$$

the equations (2.81) will remain valid if, in the first (*resp.* the second) one, one substitutes for the function u the function f_0 (*resp.* f_1). By an elementary property of the Laplace transformation, it follows that the even part of u coincides with that of f_0 and that the odd part of u coincides with that of f_1. □

It will be handy in the sequel to have an explicit expression of

$$\pi_{\rho,\varepsilon}(g)v = \mathcal{F}^{-1}\, \hat{\pi}_{\rho,\varepsilon}(g)\, \mathcal{F}\, v. \tag{2.88}$$

Proposition 2.7. *Assume that* $\rho \in \mathbb{C}$, $\rho \neq 0$, *and that* $-1 < \mathrm{Re}\ \rho < 1$. *Let* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, *and let* $v \in C^\infty_{\rho,\varepsilon}$. *If* $b = 0$, *one has*

$$(\pi_{\rho,\varepsilon}(g)v)(s) = |a|_\varepsilon^{-1+\rho}\, e^{2i\pi\frac{c}{a}s}\, v(a^{-2}s). \tag{2.89}$$

If $b \neq 0$, *one has*

$$(\pi_{\rho,\varepsilon}(g)v)(s) = \int_{-\infty}^{\infty} e^{\frac{2i\pi}{b}(ds+at)}\, k_{\rho,\varepsilon}(b;s,t)\, v(t)\, dt, \tag{2.90}$$

with

$$\begin{aligned} k_{\rho,0}(b;s,t) = |b|^{-1} \Bigg[& 4\cos\frac{\pi\rho}{2}\, \mathrm{char}(st<0)\, \left(-\frac{s}{t}\right)^{\frac{\rho}{2}}\, K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right) \\ & + \frac{\pi}{\sin\frac{\pi\rho}{2}}\, \mathrm{char}(st>0)\, (\frac{s}{t})^{\frac{\rho}{2}}\, \left(J_{-\rho}\left(\frac{4\pi}{|b|}\sqrt{st}\right) - J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\right)\Bigg] \end{aligned} \tag{2.91}$$

and

$$k_{\rho,1}(b;s,t) = b^{-1}\left[4i\,\sin\frac{\pi\rho}{2}\,\text{char}(st<0)\;\langle s\rangle^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\;K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right)\right.$$
$$\left. -\frac{i\pi}{\cos\frac{\pi\rho}{2}}\,\text{char}(st>0)\;\langle s\rangle^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\;\left(J_{-\rho}\left(\frac{4\pi}{|b|}\sqrt{st}\right)+J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\right)\right]. \tag{2.92}$$

Proof. In the case when $b=0$, the result immediately follows from the equation

$$(\mathcal{F}\,\pi_{\rho,\varepsilon}(g)v)(\sigma) = |a|_\varepsilon^{1+\rho}\,(\mathcal{F}\,v)\left(a^2\left(\sigma-\frac{c}{a}\right)\right), \tag{2.93}$$

itself a consequence of (2.3). Assuming $b\neq 0$, and starting from (2.3) again, we find

$$(\pi_{\rho,\varepsilon}(g)v)(s) = \int_{-\infty}^{\infty} v(t)\,dt\int_{-\infty}^{\infty} |-b\sigma+d|_\varepsilon^{-1-\rho}\,\exp 2i\pi\left(s\sigma - t\frac{a\sigma-c}{-b\sigma+d}\right)d\sigma$$
$$= |b|_\varepsilon^{-1-\rho}\int_{-\infty}^{\infty} v(t)\,A_{\rho,\varepsilon}(b;s,t)\,e^{\frac{2i\pi}{b}(ds+at)}\,dt \tag{2.94}$$

with

$$A_{\rho,\varepsilon}(b;s,t) = \int_{-\infty}^{\infty} |-\sigma|_\varepsilon^{-1-\rho}\,e^{2i\pi(s\sigma+\frac{t}{b^2\sigma})}\,d\sigma, \tag{2.95}$$

a semi-convergent integral in view of the assumptions made about $\text{Re}\,\rho$. We thus need a lemma:

Lemma 2.8. *If* $st<0$,

$$A_{\rho,0}(b;s,t) = 4\,\cos\frac{\pi\rho}{2}\,|b|^\rho\,|s|^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\,K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right)$$

and

$$A_{\rho,1}(b;s,t) = 4i\,\sin\frac{\pi\rho}{2}\,|b|^\rho\,\langle s\rangle^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\,K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right). \tag{2.96}$$

If $st>0$,

$$A_{\rho,0}(b;s,t) = \frac{\pi}{\sin\frac{\pi\rho}{2}}\,|b|^\rho\,|s|^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\,\left(J_{-\rho}\left(\frac{4\pi}{|b|}\sqrt{st}\right)-J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\right)$$

and

$$A_{\rho,1}(b;s,t) = -\frac{i\pi}{\cos\frac{\pi\rho}{2}}\,|b|^\rho\,\langle s\rangle^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\,\left(J_{-\rho}\left(\frac{4\pi}{|b|}\sqrt{st}\right)+J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\right). \tag{2.97}$$

Proof. One must evaluate, for $s\in\mathbb{R}$ and $\beta\in\mathbb{R}$, the integral

$$I=\int_0^\infty \sigma^{-1-\rho}e^{2i\pi(s\sigma-\frac{\beta}{\sigma})}\,d\sigma\,: \tag{2.98}$$

for complex s and β with $\mathrm{Im}\,s>0$, $\mathrm{Im}\,\beta<0$, one has [17, p. 85]

$$I=2\,(-is)^{\frac{\rho}{2}}\,(i\beta)^{-\frac{\rho}{2}}\,K_\rho(2\,(-2i\pi s)^{\frac12}(2i\pi\beta)^{\frac12}). \tag{2.99}$$

Following with care the complex determinations, one then finds

$$I=2\,e^{-\frac{i\pi\rho}{2}}\,s^{\frac{\rho}{2}}\,\beta^{-\frac{\rho}{2}}\,K_\rho(4\pi\sqrt{s\beta})\qquad\text{if}\quad s>0,\,\beta>0 \tag{2.100}$$

and (using also [17, p. 66–67])

$$\begin{aligned}I&=2\,s^{\frac{\rho}{2}}\,|\beta|^{-\frac{\rho}{2}}\,K_\rho(-4i\pi\sqrt{-s\beta})\\&=\frac{\pi}{\sin\pi\rho}\,s^{\frac{\rho}{2}}\,|\beta|^{-\frac{\rho}{2}}\left(e^{\frac{i\pi\rho}{2}}\,J_{-\rho}(4\pi\sqrt{-s\beta})-e^{-\frac{i\pi\rho}{2}}\,J_\rho(4\pi\sqrt{-s\beta})\right)\end{aligned} \tag{2.101}$$

if $s>0$, $\beta<0$; the two other cases can be obtained by complex conjugation. □

End of Proof of Proposition 2.7. All that needs being done is plugging the results of Lemma 2.8 in (2.94). □

The following theorem proves the existence of the anaplectic representation of the group $SL(2,\mathbb{R})$ in the space $\mathfrak{A}$ as asserted in Theorem 1.20, at the same time showing the equivalence between this representation and the direct sum of a pair of classical irreducible representations.

Theorem 2.9. *There is a unique representation* Ana *of the group* $SL(2,\mathbb{R})$ *in the space* $\mathfrak{A}$ *satisfying the conditions* (i) *and* (ii) *of Theorem* 1.20. *It is equivalent to the direct sum of irreducible representations* $\pi_{-\frac12,0}\oplus\pi_{\frac12,1}$, *restricted to the space* $C^\omega_{-\frac12,0}\oplus C^\omega_{\frac12,1}$. *The intertwining operator is given by the map* $\Theta\colon(v_0,v_1)\mapsto u$, *as defined in* (2.82), *to be completed by the equations* $u_{\text{even}}=(f_0)_{\text{even}}$, $u_{\text{odd}}=(f_1)_{\text{odd}}$.

Proof. The net result of Theorems 1.8 and 2.6, together with Definition 2.2, is that the operator $(v_0,\,v_1)\mapsto(f_0,\,f_1)$ defined in (2.82) is a linear isomorphism from $C^\omega_{-\frac12,0}\oplus C^\omega_{\frac12,1}$ onto the space $\mathfrak{A}$, when functions lying in the latter space are expressed in their $\mathbb{C}^2$-realization. Next, we show that this operator Θ intertwines the representation $\pi_{-\frac12,0}\oplus\pi_{\frac12,1}$ with a representation of $SL(2,\mathbb{R})$ on $\mathfrak{A}$ satisfying the properties (i) and (ii) of the statement of Theorem 1.20, hence taken as a definition of the anaplectic representation. Let us start with the more difficult one: it follows from Proposition 2.7 that, for $s>0$, one has

$$\begin{aligned}(\pi_{-\frac12,0}(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right))\,v_0)(s)=\int_{-\infty}^{\infty}|s|^{-\frac12}\Big[&\mathrm{char}\,(t<0)\,e^{-4\pi\sqrt{s|t|}}\\&+\mathrm{char}\,(t>0)\,(\cos 4\pi\sqrt{st}-\sin 4\pi\sqrt{st})\Big]\,v(t)\,dt\end{aligned} \tag{2.102}$$

and

$$(\pi_{\frac{1}{2},1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_1)(s) = i \int_{-\infty}^{\infty} |t|^{-\frac{1}{2}} \Big[\text{char}\,(t<0)\, e^{-4\pi\sqrt{s|t|}} \\ -\ \text{char}\,(t>0)\,(\cos\, 4\pi\sqrt{st} + \sin\, 4\pi\sqrt{st})\Big]\, v(t)\, dt. \tag{2.103}$$

It is obvious how to extract the ramified parts of the two functions of s just made explicit: setting $s = \frac{x^2}{2}$, where we assume that $x > 0$ as well, and making the change of variable $t = \pm\frac{y^2}{2}$ on each of the two half-lines, one obtains

$$(\pi_{-\frac{1}{2},0}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))v_0)^{\text{ram}}\left(\frac{x^2}{2}\right) = \int_0^\infty y\left[v_0\left(\frac{y^2}{2}\right)\cos 2\pi xy + v_0\left(-\frac{y^2}{2}\right)\cosh 2\pi xy\right] dy \tag{2.104}$$

and

$$2^{-\frac{1}{2}}\, x\,(\pi_{\frac{1}{2},1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_1)^{\text{ram}}\left(\frac{x^2}{2}\right) \\ = -2^{\frac{1}{2}}\, i \int_0^\infty \left[v_1\left(\frac{y^2}{2}\right)\, \sin\, 2\pi xy + v_1\left(-\frac{y^2}{2}\right)\, \sinh\, 2\pi xy\right]\, dy \tag{2.105}$$

or, using (2.84) and (2.85),

$$(\pi_{-\frac{1}{2},0}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_0)^{\text{ram}}\left(\frac{x^2}{2}\right) = 2^{\frac{1}{2}} \int_0^\infty [f_0(y)\, \cos\, 2\pi xy + f_{i,0}(y)\, \cosh\, 2\pi xy]\, dy \tag{2.106}$$

and

$$2^{-\frac{1}{2}}\, x\,(\pi_{\frac{1}{2},1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_1)^{\text{ram}}\left(\frac{x^2}{2}\right) \\ = -2^{\frac{1}{2}}\, i \int_0^\infty [f_1(y)\, \sin\, 2\pi xy + f_{i,1}(y)\, \sinh\, 2\pi xy]\, dy. \tag{2.107}$$

According to (2.82), what remains to be done is checking that the last two functions of $x > 0$ just computed agree respectively with the even and odd parts of the function $\mathcal{F}_{\text{ana}}\, u$: this follows from (1.81).

To check that the transformation $(v_0, v_1) \mapsto u$ also intertwines the restrictions of the anaplectic representation and of the representation $\pi_{-\frac{1}{2},0} \oplus \pi_{\frac{1}{2},1}$ to the subgroup of $SL(2,\mathbb{R})$ consisting of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $b = 0$ is an easy matter, in view of the formulas, displayed in Proposition 2.7,

$$\left(\pi_{-\frac{1}{2},0}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right) v_0\right)(s) = |a|^{-\frac{3}{2}}\, v_0(a^{-2}s), \quad \left(\pi_{\frac{1}{2},1}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right) v_1\right)(s) = \langle a\rangle^{-\frac{1}{2}}\, v_1(a^{-2}s), \\ \left(\pi_{-\frac{1}{2},0}\left(\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)\right) v_0\right)(s) = e^{2i\pi cs} v_0(s), \qquad \left(\pi_{\frac{1}{2},1}\left(\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)\right) v_1\right)(s) = e^{2i\pi cs} v_1(s): \tag{2.108}$$

using (2.82) to extract the even and odd parts of the function corresponding to the pair $(\pi_{-\frac{1}{2},0}\,((\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}))\,v_0,\,\pi_{\frac{1}{2},1}\,((\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}))\,v_1)$, we only have to check that the ramified part (in $C^\omega_{-\frac{1}{2},0}$) of the function $s \mapsto v_0(a^{-2}s)$ is the function $s \mapsto |a|\,v_0^{\mathrm{ram}}(a^{-2}s)$ and that the ramified part (in $C^\omega_{\frac{1}{2},1}$) of the function $s \mapsto v_1(a^{-2}s)$ is the function $s \mapsto |a|^{-1}\,v_1^{\mathrm{ram}}(a^{-2}s)$: this takes care of the first of the two matrices considered in (2.108), while what concerns the second one is even simpler. □

Remark 2.2. The use of the map $u \mapsto u_i$ has been found to be essential in the whole development of anaplectic analysis, as it made the consideration of the four-vector $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ possible: then, allowing real translations in the picture also demands that translations by pure imaginary numbers should be considered as well. This is why we have been led to the use of entire functions or, equivalently, to the consideration of analytic vectors only of the pair of representations $(\pi_{-\frac{1}{2},0},\,\pi_{\frac{1}{2},1})$. Of course, some enlargement (for instance, using C^∞ vectors rather than analytic vectors) would be possible, but then the functions u would cease to live on the complex plane, only on the union $\mathbb{R}\cup i\,\mathbb{R}$. One should also remark that complex rotations $x \mapsto \lambda\, x$, $|\lambda| = 1$, do not preserve the space $\mathfrak{A}$ unless $\lambda^4 = 1$.

For a later use, we compute the image under Θ of the pair $(v^0_{-\frac{1}{2},0},\,0)$, where the first function was defined in (2.22):

$$v^0_{-\frac{1}{2},0}(s) = \frac{2\,\pi^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}\,|s|^{-\frac{1}{4}}\,K_{\frac{1}{4}}(2\pi\,|s|) = \frac{2^{\frac{1}{2}}\,\pi^{\frac{5}{4}}}{\Gamma(\frac{1}{4})}\,|s|^{-\frac{1}{4}}\,(I_{-\frac{1}{4}}(2\pi\,|s|) - I_{\frac{1}{4}}(2\pi\,|s|)) : \tag{2.109}$$

then, (2.33) yields

$$\begin{aligned} (v^0_{-\frac{1}{2},0})^{\mathrm{ent}}(s) &= -\frac{2^{\frac{1}{2}}\,\pi^{\frac{5}{4}}}{\Gamma(\frac{1}{4})}\,|s|^{-\frac{1}{4}}\,I_{\frac{1}{4}}(2\pi\,|s|), \\ (v^0_{-\frac{1}{2},0})^{\mathrm{ram}}(s) &= \frac{2^{\frac{1}{2}}\,\pi^{\frac{5}{4}}}{\Gamma(\frac{1}{4})}\,|s|^{\frac{1}{4}}\,I_{-\frac{1}{4}}(2\pi\,|s|), \end{aligned} \tag{2.110}$$

so that, for $x > 0$, (2.82) gives

$$f_0(x) = \frac{2^{\frac{1}{4}}\,\pi^{\frac{5}{4}}}{\Gamma(\frac{1}{4})}\,x^{\frac{1}{2}}\,(-I_{\frac{1}{4}}(\pi\,x^2) + I_{-\frac{1}{4}}(\pi\,x^2)) \tag{2.111}$$

and, finally,

$$\Theta\begin{pmatrix} v^0_{-\frac{1}{2},0} \\ 0 \end{pmatrix} = (f_0)_{\mathrm{even}} = \frac{2^{\frac{1}{4}}\,\pi^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}\,\phi, \tag{2.112}$$

with ϕ as introduced in Proposition 1.2.

We may now complete the proof of Theorem 1.20.

Theorem 2.10. *The anaplectic representation of $SL(2,\mathbb{R})$ in $\mathfrak{A}$ is pseudo-unitary with respect to the (pseudo-)scalar product introduced in Proposition* 1.14*, and so is the Heisenberg representation in the same space. Defining $\mathcal{R}$ as in Proposition* 1.13*, there is a unique extension of the anaplectic representation as a representation of the group $SL_i(2,\mathbb{R})$ such that the condition* (iii) *of Theorem* 1.20 *should be satisfied. The equation* (1.91) *defining the adjoint action of operators from the anaplectic representation on those from the Heisenberg representation is valid.*

Proof. The representation $\pi_{-\frac{1}{2},0} \oplus \pi_{\frac{1}{2},1}$ preserves the pseudo-scalar product on the space $C^{\omega}_{-\frac{1}{2},0} \oplus C^{\omega}_{\frac{1}{2},1}$ defined, if $\boldsymbol{u} = \left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right)$ and $\boldsymbol{v} = \left(\begin{smallmatrix} v_0 \\ v_1 \end{smallmatrix}\right)$, by the equation

$$\begin{aligned}(\boldsymbol{u}|\boldsymbol{v}) &= (u_0|v_0)_{-\frac{1}{2},0} + (u_1|v_1)_{\frac{1}{2},1} \\ &= \int_{-\infty}^{\infty} [|s|_0^{\frac{1}{2}}\, \bar{u}_0(s)\, v_0(s) + |s|_1^{-\frac{1}{2}}\, \bar{u}_1(s)\, v_1(s)]\, ds. \end{aligned} \tag{2.113}$$

Thus,

$$\begin{aligned}(\boldsymbol{v}|\boldsymbol{v}) &= \int_0^{\infty} [s^{\frac{1}{2}}\, |v_0(s)|^2 + s^{\frac{1}{2}}\, |v_0(-s)|^2 + s^{-\frac{1}{2}}\, |v_0(s)|^2 - s^{-\frac{1}{2}}\, |v_0(s)|^2] ds \\ &= 2^{\frac{1}{2}} \int_0^{\infty} \left[\frac{x^2}{2} \left| v_0\left(\frac{x^2}{2}\right)\right|^2 + \frac{x^2}{2} \left| v_0\left(-\frac{x^2}{2}\right)\right|^2 + \left| v_1\left(\frac{x^2}{2}\right)\right|^2 - \left| v_1\left(-\frac{x^2}{2}\right)\right|^2 \right] dx. \end{aligned} \tag{2.114}$$

Using (2.84) and (2.85), one sees that this reduces to

$$(\boldsymbol{v}|\boldsymbol{v}) = 2^{\frac{1}{2}} \int_0^{\infty} [|f_0(x)|^2 + |f_{i,0}(x)|^2 + |f_1(x)|^2 - |f_{i,1}(x)|^2]\, dx, \tag{2.115}$$

which is just the definition (1.68) of $(\boldsymbol{f}|\boldsymbol{f})$, where $\boldsymbol{f}$ is the $\mathbb{C}^4$-realization of the image of v under the intertwining operator Θ.

Next, we verify (1.91) in the case when $g \in SL(2,\mathbb{R})$, which can be done by looking only at the action of generators of this group. The only non-trivial equations to be proven are

$$\mathcal{F}_{\text{ana}}\, e^{2i\pi\, \eta Q} = e^{-2i\pi\, \eta P}\, \mathcal{F}_{\text{ana}}, \qquad \mathcal{F}_{\text{ana}}\, e^{2i\pi\, yP} = e^{2i\pi\, yQ}\, \mathcal{F}_{\text{ana}}, \tag{2.116}$$

which can be done in the usual way, starting from Definition 1.17 of the anaplectic Fourier transformation and using the invariance of the linear form Int under translations.

To show that the Heisenberg representation is pseudo-unitary reduces, thanks to (1.91), to the fact that the operators $e^{2i\pi\, \eta Q}$ with $\eta \in \mathbb{R}$ preserve the scalar product. Now, this is a consequence of Proposition 1.14 defining the scalar product together with the fact (a straightforward if tedious verification) that, with

$(g_0,\, g_1,\, g_{i,0},\, g_{i,1})$ as defined in (1.60) and (1.61), one has $|g_{i,0}|^2 - |g_{i,1}|^2 = |f_{i,0}|^2 - |f_{i,1}|^2$.

Some remaining details concerning the operator $\mathcal{R}$ must still be checked. From (2.89), the image, under the representation $\pi_{-\frac{1}{2},0} \oplus \pi_{\frac{1}{2},1}$, of the matrix $-I$, is the operator $(v_0,\, v_1) \mapsto (v_0,\, -v_1)$: under the intertwining operator (2.82), it transfers to the operator $(f_0,\, f_1) \mapsto (f_0,\, -f_1)$, which is the expression, in the $\mathbb{C}^2$-realization, of the symmetry operator $u \mapsto \check{u}$. This shows that $\mathcal{R}^2 = \mathrm{Ana}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ and, to complete the proof that Ana extends as a representation of the group generated by $SL(2,\mathbb{R})$ together with the matrix $\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$, we must still show that

$$\mathcal{R}\,\mathrm{Ana}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\mathcal{R}^{-1} = \mathrm{Ana}\left(\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\right) \tag{2.117}$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. The case when $b = 0$ is immediate in view of the condition (i) taken from the statement of Theorem 1.20 (and used towards the definition of the anaplectic representation). What remains to be done is checking the equation

$$(\mathcal{R}\,\mathcal{F}_{\mathrm{ana}}\,\check{u})(x) = (\mathcal{F}_{\mathrm{ana}}\,\mathcal{R}\,u)(x). \tag{2.118}$$

Now, (1.67) gives the $\mathbb{C}^4$-realization of $\mathcal{R}\,u$ in terms of that of u: then, the expanded version (1.81) of the anaplectic Fourier transformation yields

$$\begin{aligned}(\mathcal{F}_{\mathrm{ana}}\,\mathcal{R}\,u)(x) = 2^{\frac{1}{2}} \int_0^\infty f_{i,0}(y)\, \cos 2\pi xy\, dy - 2^{\frac{1}{2}} \int_0^\infty f_{i,1}(y)\, \sin 2\pi xy\, dy \\ + 2^{\frac{1}{2}} \int_0^\infty f_0(y)\, \cosh 2\pi xy\, dy - 2^{\frac{1}{2}} \int_0^\infty f_1(y)\, \sinh 2\pi xy\, dy,\end{aligned} \tag{2.119}$$

and a similar calculation gives the left-hand side of (2.118). Finally, extending (1.91) to the case when $g \in SL_i(2,\mathbb{R})$ is immediate, and we have already signalled that the pseudo-unitarity of the anaplectic representation ceases to hold when extending it from $SL(2,\mathbb{R})$ to $SL_i(2,\mathbb{R})$.

This concludes the proof of Theorem 2.10, accordingly that of Theorem 1.20 as well. □

The infinitesimal version

$$\mathrm{Ana}(g)\,Q\,\mathrm{Ana}(g^{-1}) = d\,Q - b\,P, \qquad \mathrm{Ana}(g)\,P\,\mathrm{Ana}(g^{-1}) = -c\,Q + a\,P, \tag{2.120}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, of the relation (1.91), is of course valid too, and can be proved (though other ways of doing it are possible) by differentiation of (1.91) along the generator of a one-parameter subgroup of $SL(2,\mathbb{R})$: going the other way around (from the infinitesimal relation to its exponentiated version) would of course be prevented by the lack of unitarity. In particular, one obtains the familiar-looking relations

$$\mathcal{F}_{\mathrm{ana}}\,Q\,\mathcal{F}_{\mathrm{ana}}^{-1} = -P, \qquad \mathcal{F}_{\mathrm{ana}}\,P\,\mathcal{F}_{\mathrm{ana}}^{-1} = Q. \tag{2.121}$$

Starting from the function ϕ as defined in Proposition 1.2, we now study the harmonic oscillator L, introduced in (1.84), in the anaplectic setting.

Theorem 2.11. *The spectrum of the harmonic oscillator in the space* $\mathfrak{A}$ *is* $\mathbb{Z}$, *and for every* $j \in \mathbb{Z}$ *the eigenspace corresponding to the eigenvalue* j *is generated by the function* ϕ^j, *with*

$$\phi^j = A^{*j}\,\phi \quad \text{if } j \geq 0, \qquad \phi^j = A^{|j|}\,\phi \quad \text{if } j \leq 0, \tag{2.122}$$

where the operators A^* *and* A *are defined by the usual equations*

$$A^* = \pi^{\frac{1}{2}}\left(x - \frac{1}{2\pi}\frac{d}{dx}\right), \qquad A = \pi^{\frac{1}{2}}\left(x + \frac{1}{2\pi}\frac{d}{dx}\right). \tag{2.123}$$

The functions ϕ^j, $j \in \mathbb{Z}$ *are pairwise orthogonal with respect to the* (*pseudo*)*-scalar product* (1.68). *The function* ϕ *is normalized and one has* $(\phi^{k+1}\,|\,\phi^{k+1}) = (k+\frac{1}{2})\,(\phi^k\,|\,\phi^k)$ *and* $(\phi^{-k}\,|\,\phi^{-k}) = (-1)^k\,(\phi^k\,|\,\phi^k)$ *for* $k \geq 0$.

Proof. One has $\phi^0 = \phi$, a function in the kernel of L as mentioned in the proof of Proposition 1.18, and the usual formal argument then shows that $L\,\phi^j = j\,\phi^j$ for every $j \in \mathbb{Z}$. With the help of the matrix decomposition, valid for $0 < t < \pi$,

$$g_t := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \operatorname{cotan} t & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} (\sin t)^{-1} & 0 \\ 0 & \sin t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \operatorname{cotan} t & 1 \end{pmatrix}, \tag{2.124}$$

one finds, for every $u \in \mathfrak{A}$, and $t \in]0, \pi[$,

$$(\mathrm{Ana}(g_t)\,u)(x) = e^{i\pi\,x^2 \operatorname{cotan} t}\,\mathcal{F}_{\mathrm{ana}}\,(y \mapsto (\sin t)^{\frac{1}{2}}\,u(y\,\sin t)\,e^{i\pi\,y^2 \operatorname{cotan} t})(x) \tag{2.125}$$

so that

$$-\frac{1}{i}\,\frac{d}{dt}\bigg|_{t=\frac{\pi}{2}} (\mathrm{Ana}(g_t)\,u)(x) = \left[\pi\,x^2 - \frac{1}{4\pi}\,\frac{d^2}{dx^2}\right](\mathcal{F}_{\mathrm{ana}}\,u)(x) \tag{2.126}$$

or, using the group property,

$$L = -\frac{1}{i}\,\frac{d}{dt}\bigg|_{t=0} \mathrm{Ana}\left(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right). \tag{2.127}$$

If one compares this equation to (2.10) (noting that here, t substitutes for $\frac{t}{2}$) and one makes use of the equivalence between the representations Ana and $\pi_{-\frac{1}{2},0} \oplus \pi_{\frac{1}{2},1}$ provided by Theorem 2.9, one sees, from the argument developed between (2.20) and (2.21), that the operator L has no eigenfunctions besides those already found: the ones corresponding to even (*resp.* odd) eigenvalues are the images, under the appropriate intertwining operator, of eigenvalues of the operator $L_{-\frac{1}{2}}$ (*resp.* $L_{\frac{1}{2}}$) in the space $C^{\omega}_{-\frac{1}{2},0}$ (*resp.* $C^{\omega}_{\frac{1}{2},1}$). In each case, the eigenvalue of the operator $L_{\pm\frac{1}{2}}$ to consider is half the corresponding eigenvalue of L.

That ϕ is normalized was proved in Proposition 1.18. That the ϕ^j's are pairwise orthogonal can most quickly be seen from the fact that the ones with an even (*resp.* odd) j arise from eigenfunctions of the operator $L_{-\frac{1}{2},0}$ (*cf.* (2.13)) or $L_{\frac{1}{2},1}$ under the intertwining operator Θ, and the fact that this latter operator lets the pseudo-scalar products on $C^\omega_{-\frac{1}{2},0} \oplus C^\omega_{\frac{1}{2},1}$ and on $\mathfrak{A}$ correspond to each other (2.115). The computation of $(\phi^k \,|\, \phi^k)$, based on the relations $A^*A = L - \frac{1}{2}$ and $AA^* = L + \frac{1}{2}$, is done in the usual way: that the result is different is of course a consequence of the shift in the eigenvalues. □

The following pair of technical propositions will be useful towards the end of Section 6.

Proposition 2.12. *Given any complex number* ρ *such that* $\rho \notin \mathbb{Z}$ *and* $\mathrm{Re}\,\rho > -1$, *and* $\varepsilon = 0$ *or* 1, *each of the two operators*

$$R = 1 - \frac{1}{2\pi}\frac{d}{ds} \qquad \text{and} \qquad T = 1 + \frac{1}{2\pi}\frac{d}{ds} \tag{2.128}$$

operates from $C^\infty_{\rho,\varepsilon}$ *to* $C^\infty_{\rho-1,1-\varepsilon}$ *and from* $C^\omega_{\rho,\varepsilon}$ *to* $C^\omega_{\rho-1,1-\varepsilon}$. *Each of the two operators*

$$R^\dagger_\rho = \frac{1}{2\pi}\left(s\frac{d}{ds} + 2\pi s - \rho\right) \qquad \text{and} \qquad T^\dagger_\rho = \frac{1}{2\pi}\left(-s\frac{d}{ds} + 2\pi s + \rho\right) \tag{2.129}$$

acts from $C^\infty_{\rho,\varepsilon}$ *to* $C^\infty_{\rho+1,1-\varepsilon}$ *and from* $C^\omega_{\rho,\varepsilon}$ *to* $C^\omega_{\rho+1,1-\varepsilon}$. *If* ρ *is real and* $-1 < \rho < 0$, *one has*

$$(v_1 \,|\, R\, v_2)_{\rho,1-\varepsilon} = (R^\dagger_\rho v_1 \,|\, v_2)_{\rho+1,\varepsilon} \tag{2.130}$$

and

$$(v_1 \,|\, T\, v_2)_{\rho,1-\varepsilon} = (T^\dagger_\rho v_1 \,|\, v_2)_{\rho+1,\varepsilon} \tag{2.131}$$

whenever $v_1 \in C^\infty_{\rho,1-\varepsilon}$ *and* $v_2 \in C^\infty_{\rho+1,\varepsilon}$.

Proof. With some restrictions on $\mathrm{Re}\,\rho$, one could use Proposition 2.1 for the first part, only noting that

$$\begin{aligned} |s|^\rho_\varepsilon &= |s|^{\rho-1}_{1-\varepsilon}\, s, \\ \frac{d}{ds}\,|s|^\rho_\varepsilon &= \rho\,|s|^{\rho-1}_{1-\varepsilon}, \\ \left(s\frac{d}{ds} - \rho\right)\,|s|^\rho_\varepsilon &= 0. \end{aligned} \tag{2.132}$$

However, for general values of ρ, one must go back to the definition (2.17) of the space $\widehat{C}^\infty_{\rho,\varepsilon}$ and, with $\widehat{R} = \mathcal{F}\,R\,\mathcal{F}^{-1}$, $\widehat{T} = \mathcal{F}\,T\,\mathcal{F}^{-1}$, $\widehat{R}^\dagger_\rho = \mathcal{F}\,R^\dagger_\rho\,\mathcal{F}^{-1}$, $\widehat{T}^\dagger_\rho = \mathcal{F}\,T^\dagger_\rho\,\mathcal{F}^{-1}$, one must compute

$$\begin{aligned} \widehat{R} &= 1 - i\sigma, & \widehat{T} &= 1 + i\sigma, \\ \widehat{R}^\dagger_\rho &= -\frac{1}{2\pi}\left(\sigma\frac{d}{d\sigma} - i\frac{d}{d\sigma} + \rho + 1\right), & \widehat{T}^\dagger_\rho &= \frac{1}{2\pi}\left(\sigma\frac{d}{d\sigma} + i\frac{d}{d\sigma} + \rho + 1\right). \end{aligned} \tag{2.133}$$

Obviously, the four operators just made explicit preserve the space of C^∞ (or analytic) functions on the real line; also (going back to (2.17)), one may check the identities

$$|\sigma|_{1-\varepsilon}^{-\rho}\,(\widehat{R}w)\left(-\frac{1}{\sigma}\right) = i\,\widehat{R}\left(|\sigma|_\varepsilon^{-1-\rho} w\left(-\frac{1}{\sigma}\right)\right),$$
$$|\sigma|_{1-\varepsilon}^{-2-\rho}\,(\widehat{R}_\rho^\dagger w)\left(-\frac{1}{\sigma}\right) = -i\,\widehat{R}_\rho^\dagger\left(|\sigma|_\varepsilon^{-1-\rho} w\left(-\frac{1}{\sigma}\right)\right), \tag{2.134}$$

and the same goes with the pair $(\widehat{T}, \widehat{T}_\rho^\dagger)$.

The second part of Proposition 2.12 is obtained from (2.30), with the help of an integration by parts, also using (2.132) again. □

The importance of the four operators discussed (or their images under the commutation by the Fourier transformation) stems in particular from the following formulas, connecting the eigenfunctions of various operators $L_{\rho,\varepsilon}$ as introduced in (2.20) and (2.21).

Proposition 2.13. *One has*

$$R\,v_{\rho,0}^k = v_{\rho-1,1}^k, \qquad R\,v_{\rho,1}^k = v_{\rho-1,0}^{k+1},$$
$$T\,v_{\rho,0}^k = v_{\rho-1,1}^{k-1}, \qquad T\,v_{\rho,1}^k = v_{\rho-1,0}^k \tag{2.135}$$

and

$$R_\rho^\dagger\,v_{\rho,0}^k = -\frac{1}{2\pi}\,(\rho-2k+1)\,v_{\rho+1,1}^{k-1}, \qquad R_\rho^\dagger\,v_{\rho,1}^k = -\frac{1}{2\pi}\,(\rho-2k)\,v_{\rho+1,0}^k,$$
$$T_\rho^\dagger\,v_{\rho,0}^k = \frac{1}{2\pi}\,(\rho+2k+1)\,v_{\rho+1,1}^k, \qquad T_\rho^\dagger\,v_{\rho,1}^k = \frac{1}{2\pi}\,(\rho+2k+2)\,v_{\rho+1,0}^{k+1}. \tag{2.136}$$

Finally, one has the identities

$$T_{\rho-1}^\dagger\,R = R\,T_\rho^\dagger, \qquad R_{\rho-1}^\dagger\,T = T\,R_\rho^\dagger. \tag{2.137}$$

Proof. Using (2.21), one is reduced, in order to prove (2.136), to a corresponding set of equations in which all operators should be replaced by their hat-covered versions, and the v's by the corresponding w's: since the latter ones were made explicit in (2.20), the verification is straightforward, and so is that of the last pair of equations. □

Chapter 2

The n-dimensional Anaplectic Analysis

A definition of the space $\mathfrak{A}$ which generalizes to the n-dimensional setting is obtained with the help of Theorem 1.8. There, the fact that an entire function u lies in $\mathfrak{A}$ is expressed in terms of the behavior of its pair (w_0, w_1) of quadratic transforms, and of the analytic continuation thereof. Something similar will be taken as a definition of the space $\mathfrak{A}^{(n)}$ in Section 4, but it is essential to realize that this is far from a straightforward generalization. Indeed, starting with functions on $\mathbb{R}^n$, one ends up, under the quadratic transformation, with pairs of functions (the second one vector-valued) on some space Sym_n of dimension $\frac{n(n+1)}{2}$: thus, except in the one-dimensional case, the quadratic transformation can only identify the space of functions on $\mathbb{R}^n$ under consideration with a very small space (*cf.* Remark 5.2) of functions on Sym_n. Also, the (non-unitary) representation theory of the symplectic group is not as helpful in the n-dimensional case as in the one-dimensional one.

The first section of this chapter is the easy one. We introduce the function Φ which is the rotation-invariant function in the null space of the n-dimensional harmonic oscillator. In just the same way that, in the usual analysis, all Hermite functions can be obtained by applying differential operators with polynomial coefficients to the rotation-invariant Gaussian function, we here obtain the "anaplectic Hermite functions". There is, however, a major novelty (and difficulty) in the fact that the null space of the harmonic oscillator is now infinite-dimensional, whereas its classical counterpart is generated by a single ground state.

Serious work starts in Section 4. Representing functions on $\mathbb{R}^n$ by their quadratic transforms, which live on Sym_n, we are faced with the problem of letting fractional-linear transformations associated with symplectic matrices act on this space. As will be recalled in Section 6, there is a well-defined non-singular action of the symplectic group by holomorphic transformations of the Siegel domain which is the complex tube $\mathrm{Sym}_n + i\,\mathbb{R}^{\frac{n(n+1)}{2}}$, and this action can be used to provide a definition of the metaplectic representation. Here, the action is more troublesome since it is quite singular. We must then first add to the space Sym_n its "points at infinity", *i.e.,* compactify it by means of a Cayley transform, next move to some finite cover of the result. This leads to the correct definition of the space $\mathfrak{A}^{(n)}$: it is, however, not trivial to prove (Theorem 4.18) that the function Φ lies in that space.

Section 5 is concerned with the definition and study of the anaplectic representation. Section 6 points to both resemblances and differences between the metaplectic and anaplectic representations. A Hecke style theorem exists in connection with the spherical decomposition of the anaplectic Fourier transformation. On the other hand, far from being invariant by the anaplectic action of the maximal compact subgroup of the symplectic group, the "median state" Φ of the harmonic oscillator transforms in a complicated way: we make some calculations explicit in the two-dimensional case.

There is no question that the anaplectic representation is a more complicated object than the metaplectic representation. This can be seen from the fact that, in the usual analysis, there is a class of very simple functions, namely the Gaussian functions, that is stable under all metaplectic transformations. Nothing of the sort exists in the anaplectic analysis. Nevertheless, the anaplectic analysis has a coherence of its own, which will show again, in a striking way as we hope, in Section 10, devoted for the most part to the one-dimensional anaplectic Weyl calculus (which also makes use of the space $\mathfrak{A}^{(2)}$): some of this coherence subsists in the higher-dimensional case.

3 The anaplectic harmonic oscillator in dimension ≥ 2

First, we need to generalize Proposition 2.3 by the consideration of integral values of the parameter ρ: we also need to consider the case when ρ is half an integer $\geq \frac{1}{2}$, but this case has already been taken care of by Proposition 2.3. There is no change in the definition of the space $\widehat{C}^{\omega}_{\rho,\varepsilon}$ of analytic functions w on the real line which satisfy the property that the function

$$\sigma \mapsto |\sigma|_{\varepsilon}^{-1-\rho}\, w\left(-\frac{1}{\sigma}\right) \qquad \text{is analytic near } 0 \tag{3.1}$$

as well. We are really interested only in the pairs $(\rho, 0)$ with $\rho = 0, 1, \ldots$ and $(\rho, 1)$ with $\rho = 1, 2, \ldots$. The result depends on the parity of $\rho + \varepsilon$.

Proposition 3.1. *Let the pair (ρ, ε) with $\rho = 0, 1, \ldots$ and $\varepsilon = 0$ or 1 be given. In the case when $\rho + \varepsilon$ is even, the image of the space $\widehat{C}^{\omega}_{\rho,\varepsilon}$ under the inverse Fourier transformation $\mathcal{F}^{-1}$ is the space $C^{\omega}_{\rho,\varepsilon}$ of functions v on $\mathbb{R}\backslash\{0\}$ with the following properties:*

(i) *there exist two entire functions* $v^{\rm ent}$ *and* $v^{\rm ram}$ (*corresponding to the entire part and the ramified part of* v) *such that*

$$v(s) = v^{\rm ent}(s) + s^\rho \,(\log{(2\pi|s|)}\, v^{\rm ram}(s), \quad s \in \mathbb{R}\,; \tag{3.2}$$

(ii) *there exists* $R > 0$ *and* $C > 0$ *such that*

$$|v^{\rm ent}(s)| + |v^{\rm ram}(s)| \leq C\, e^{4\pi R|s|}, \qquad s \in \mathbb{C}\,; \tag{3.3}$$

(iii) *there exist* $\delta > 0$ *and* $C > 0$ *such that*

$$|v(s)| \leq C\, e^{-2\pi\delta|s|}, \qquad s \text{ real},\ |s| > 1. \tag{3.4}$$

In the case when $\rho + \varepsilon$ *is odd, the condition* (i) *must be replaced by the condition*

(i)′ *there exist two entire functions* $v^{\rm ent}$ *and* $v^{\rm ram}$ *such that*

$$v(s) = v^{\rm ent}(s) + |s|_\varepsilon^\rho\, v^{\rm ram}(s), \quad s \in \mathbb{R} \tag{3.5}$$

(*so that there is in the case when* $\rho + \varepsilon$ *is odd no change from the statement of Proposition* 2.3, *originally valid only for* Re $\rho > -1$, ρ *not an integer*).

Proof. Before we give it, let us note that a modification, or suppression, of the factor 2π within the logarithm would not change the ramified part of u (it is the one we are mostly interested in), only its entire part. There are few changes to be made in order to prove Proposition 3.1, along the lines of that of Proposition 2.3. In the proof of the direct part (properties of $v = \mathcal{F}^{-1} w$ under the assumption that $w \in \widehat{C}^\omega_{\rho,\varepsilon}$), we may still, without loss of generality, consider only the case when $\varepsilon = 0$, and the only change occurs in Lemma 2.4. Indeed, we must now set

$$\begin{aligned} B_0^\infty(s) &= \mathrm{Pf} \int_0^\infty e^{2i\pi s\sigma}\, \sigma^{-1-\rho}\, d\sigma \\ &= \lim{}_{\delta\to 0} \left[\frac{(2i\pi s)^\rho}{\rho\,!} \log\delta + \sum_{k=0}^{\rho-1} \frac{(2i\pi s)^k}{k\,!\,(k-\rho)}\, \delta^{k-\rho} + \int_\delta^\infty e^{2i\pi s\sigma}\, \sigma^{-1-\rho}\, d\sigma \right] \end{aligned} \tag{3.6}$$

where the symbol Pf stands for "finite part". From [10, p. 335],

$$\int_\delta^\infty e^{2i\pi s\sigma}\, \sigma^{-1-\rho}\, d\sigma = \frac{(2i\pi s)^\rho}{\rho\,!} \left[\Gamma(0, -2i\pi\delta s) - e^{2i\pi\delta s} \sum_{m=0}^{\rho-1} \frac{(-1)^m\, m\,!}{(-2i\pi\delta s)^{m+1}} \right] \tag{3.7}$$

with

$$\Gamma(0, -2i\pi\delta s) = -\gamma - \log{(2\pi|s|)} - \log\delta + \frac{i\pi}{2}\,\mathrm{sign}\, s + \mathrm{o}(\delta) \tag{3.8}$$

so that, up to some error term which goes to 0 as $\delta \to 0$, one has

$$\int_\delta^\infty e^{2i\pi s\sigma}\,\sigma^{-1-\rho}\,d\sigma \sim \frac{(2i\pi s)^\rho}{\rho\,!}\left(1+\frac{1}{2}+\cdots+\frac{1}{\rho}\right) - \frac{(2i\pi s)^\rho}{\rho\,!}\left(\gamma+\log(2\pi|s|)+\log\delta-\frac{i\pi}{2}\,\mathrm{sign}\,s\right) + \frac{\delta^{-\rho}}{\rho\,!}\sum_{m=0}^{\rho-1}\sum_{j=0}^{m}\frac{m\,!}{j\,!}\,\delta^{j-m-1}\,(2i\pi s)^{j-m-1+\rho}\ : \tag{3.9}$$

setting $k=j-m-1-\rho$, one can write the last sum as

$$\frac{1}{\rho\,!}\sum_{k=0}^{\rho-1}(2i\pi s)^k\,\delta^{k-\rho}\sum_{m=\rho-k-1}^{\rho-1}\frac{m\,!}{[m-(\rho-k-1)]\,!} = \sum_{k=0}^{\rho-1}(2i\pi s)^k\,\delta^{k-\rho}\,\frac{\rho\,!}{(\rho-k)\,k\,!}. \tag{3.10}$$

This finally yields

$$B_0^\infty(s) = \frac{(2i\pi s)^\rho}{\rho\,!}\left(1+\frac{1}{2}+\cdots+\frac{1}{\rho}-\gamma-\log(2\pi|s|)+\frac{i\pi}{2}\,\mathrm{sign}\,s\right). \tag{3.11}$$

Since, if ρ is even,

$$\frac{1}{2}\left(B_0^\infty(s)+B_0^\infty(-s)\right) = \frac{(2i\pi s)^\rho}{\rho\,!}\left(1+\frac{1}{2}+\cdots+\frac{1}{\rho}-\gamma-\log(2\pi|s|)\right) \tag{3.12}$$

whereas, if ρ is odd,

$$\frac{1}{2}\left(B_0^\infty(s)+B_0^\infty(-s)\right) = \frac{i\pi}{2}\,\frac{(2i\pi|s|)^\rho}{\rho\,!}, \tag{3.13}$$

a look at the part of the proof of Proposition 2.3 that immediately follows the proof of Lemma 2.4 yields the desired result.

The following special cases are illuminative: set, as in Section 2, $w_{0,0}(\sigma) = (1+\sigma^2)^{-\frac{1}{2}}$ and $w_{1,0}(\sigma) = (1+\sigma^2)^{-1}$ so that $w_{0,0} \in \widehat{C}^\omega_{0,0}$ and $w_{1,0} \in \widehat{C}^\omega_{1,0}$. Then

$$\begin{aligned}\frac{1}{2}\,(\mathcal{F}^{-1}\,w_{0,0})(s) &= K_0(2\pi|s|)\\ &= -\gamma\,I_0(2\pi|s|)+2\sum_{n\geq 1}\frac{1}{n}\,I_{2n}(2\pi|s|)-\log(\pi|s|)\,I_0(2\pi|s|)\end{aligned} \tag{3.14}$$

and

$$(\mathcal{F}^{-1}\,w_{1,0})(s) = \pi\,e^{-2\pi|s|}\ : \tag{3.15}$$

the condition (i) or (i)$'$ is indeed satisfied; also, $(\mathcal{F}^{-1}\,w_{0,0})^{\mathrm{ram}}(s) = -2\,I_0(2\pi|s|)$ and $(\mathcal{F}^{-1}\,w_{1,0})^{\mathrm{ram}}(s) = \pi\,e^{-2\pi s}$.

In the reverse direction, the proof of Proposition 2.3 adapts without modification in the case when $\rho+\varepsilon$ is odd, so we assume that it is even. Again, we may assume that v, or $w = \mathcal{F}\,v$, has the parity associated with $\eta = 0$ or 1: then, $v^{\rm ent}$ (*resp.* $v^{\rm ram}$) still has the parity associated with η (*resp.* $\varepsilon+\eta$). Following the part of the proof of Proposition 2.3 that starts at (2.62), the first modification occurs at (2.69), which must be replaced by

$$\begin{aligned}&(-1)^\eta \tilde{v}(s\,e^{\frac{i\pi}{2}}) - \tilde{v}(s\,e^{-\frac{i\pi}{2}})\\ &= (-1)^\eta\, s^\rho\, v^{\rm ram}(s) \left[e^{\frac{i\pi\rho}{2}} \left(\log{(2\pi s)} + \frac{i\pi}{2}\right) - (-1)^\varepsilon\, e^{-\frac{i\pi\rho}{2}} \left(\log{(2\pi s)} + \frac{i\pi}{2}\right)\right] :\end{aligned} \tag{3.16}$$

since $\rho+\varepsilon$ is even, the bracket on the right-hand side reduces if $\varepsilon = 0$ to $-i\pi\,\cos\frac{\pi\rho}{2}$, and if $\varepsilon = 1$ to $-i\pi\,\sin\frac{\pi\rho}{2}$. Thus, in all cases,

$$w(\sigma) = -\pi\,\left(\cos\frac{\pi\rho}{2} + \sin\frac{\pi\rho}{2}\right) \int_0^\infty s^\rho\, e^{-2\pi s\sigma}\, v^{\rm ram}(-is)\, ds, \tag{3.17}$$

which leads after the same changes of contour as in Section 4 to the equation, valid for large σ,

$$\tilde{w}(\sigma\, e^{i\pi}) = (-1)^{\eta+1}\, w(\sigma), \tag{3.18}$$

in our present case just the same as (2.66). □

Next, we note that Proposition 2.7, devoted to the explicit calculation of the transformations $\pi_{\rho,\varepsilon}(g)$, $g \in SL(2,\mathbb{R})$, still applies without modification in the case when ρ is half an integer. Assuming thus that $\rho = 0, 1, \ldots$, one notes that there is no change from (2.89) in the case when $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with $b = 0$. On the contrary, the integral kernel $k_{\rho,\varepsilon}(b; s, t)$ that occurs in (2.90) is now given by the equations

$$\begin{aligned} k_{\rho,0} &= \frac{2\,(-1)^{\frac{\rho}{2}}}{|b|}\,|\frac{s}{t}|^{\frac{\rho}{2}}\\ &\quad\times\left[2\,K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right)\,\mathrm{char}\,(st<0) - \pi\,Y_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\,\mathrm{char}\,(st>0)\right],\\ k_{\rho,1} &= \frac{2i\pi}{b}\,(-1)^{\frac{\rho+2}{2}}\,\langle s\rangle^{\frac{\rho}{2}}\,|t|^{-\frac{\rho}{2}}\,J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\,\mathrm{char}\,(st>0)\end{aligned} \tag{3.19}$$

if ρ is even, and

$$\begin{aligned} k_{\rho,0} &= \frac{2\pi}{|b|}\,(-1)^{\frac{\rho+1}{2}}\,\left(\frac{s}{t}\right)^{\frac{\rho}{2}}\,J_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\,\mathrm{char}\,(st>0),\\ k_{\rho,1} &= \frac{2i\,(-1)^{\frac{\rho-1}{2}}}{b}\,\left(\frac{\langle s\rangle}{|t|}\right)^{\frac{\rho}{2}}\\ &\quad\times\left[2\,K_\rho\left(\frac{4\pi}{|b|}\sqrt{-st}\right)\,\mathrm{char}\,(st<0) - \pi\,Y_\rho\left(\frac{4\pi}{|b|}\sqrt{st}\right)\,\mathrm{char}\,(st>0)\right]\end{aligned} \tag{3.20}$$

if ρ is odd: here Y_ρ denotes, in the usual way [17, p. 66], the second solution of Bessel's differential equation, the use of which cannot be avoided in the case when ρ is an integer so that J_ρ and $J_{-\rho}$ are proportional. To prove the equations that precede, the easiest way is to take the limit as ρ tends to some integer of the integral kernels as made explicit in Proposition 2.7.

We now proceed towards a description of the eigenfunctions of the anaplectic harmonic oscillator

$$L^{(n)} = \pi\,|x|^2 - \frac{1}{4\pi}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}, \tag{3.21}$$

which is formally the same as the usual harmonic oscillator in dimension n: however, just as in the one-dimensional case, we are interested in its eigenfunctions of a quite different nature. Recall that the eigenfunctions of the usual harmonic oscillator are the so-called Hermite functions: suffice it to say that the linear space they generate coincides with the set of products of the function $x \mapsto \exp(-\pi\,|x|^2)$ by arbitrary polynomials in the variables x_j. We first generalize and analyze this notion.

Definition 3.2. Define on $\mathbb{R}^n$ the analytic function

$$\Phi(x) = |x|^{\frac{2-n}{2}}\, I_{\frac{n-2}{4}}(\pi\,|x|^2). \tag{3.22}$$

The anaplectic Hermite functions are the images of Φ under arbitrary differential operators in the algebra $\mathbb{C}[x, \frac{\partial}{\partial x}]$ generated by the operators of multiplication by x_j and the operators $\frac{\partial}{\partial x_j}$, $1 \leq j \leq n$.

In Section 2, we described the anaplectic Hermite functions in the one-dimensional case, and we shall tacitly assume that $n \geq 2$ in what follows. In the n-dimensional case, the anaplectic Hermite functions are not quite as easy to visualize as the usual Hermite functions: for, on one hand, they are not the products of some fixed function by polynomials; on the other hand, the tensor product of n eigenfunctions of the one-dimensional anaplectic harmonic oscillator is not an n-dimensional anaplectic Hermite function. For a good understanding of the anaplectic Hermite functions, we need to reduce the operator $L^{(n)}$ by means of the action of the group $O(n)$, which commutes with it. Recall that, if one identifies $\mathbb{R}^n\backslash\{0\}$ with the product $(0,\infty) \times S^{n-1}$ by the use of "polar coordinates" (r, ξ), one can write the usual Laplacian $\sum \frac{\partial^2}{\partial x_j^2}$ as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_{S^{n-1}}, \tag{3.23}$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on the unit sphere of $\mathbb{R}^n$. The spectral decomposition of $L^2(S^{n-1})$ calls for the very classical use of spherical harmonics: let us recall (*cf.* for instance [27, p.324]) that, given $\ell = 0, 1, \ldots$ a spherical harmonic of degree ℓ is the restriction to the sphere of any harmonic polynomial homogeneous of degree ℓ. Denote as $\mathrm{Pol}(n,\ell)$ the space of all polynomials on $\mathbb{R}^n$ homogeneous of degree ℓ, and denote as $\mathrm{Harm}(n,\ell)$ its subspace consisting of harmonic polynomials: one has the direct space decomposition

$$\mathrm{Pol}(n,\ell) = \oplus_{p=0}^{[\frac{\ell}{2}]} |x|^{2p}\, \mathrm{Harm}(n,\ell-2p), \tag{3.24}$$

where $|x|^{2p}$ stands for the operator of multiplication by that function. From the equation (3.23), it follows that if $\mathcal{Y}^\ell \in \mathrm{Harm}(n,\ell)$, its restriction Y^ℓ to the unit sphere is an eigenfunction of $\Delta_{S^{n-1}}$ for the eigenvalue $-\ell(\ell+n-2)$: also, all eigenfunctions can be obtained in this way. A function u on $\mathbb{R}^n\backslash\{0\}$ written in polar coordinates as

$$u(r,\xi) = f(r)\, Y^\ell(\xi), \tag{3.25}$$

where Y^ℓ is a spherical harmonic of degree ℓ, satisfies in $\mathbb{R}^n\backslash\{0\}$ the equation $L^{(n)}u = \kappa\, u$ for some $\kappa \in \mathbb{R}$ if and only if the function f satisfies the differential equation

$$f''(r) + \frac{n-1}{r}\, f'(r) - \left[\frac{\ell(\ell+n-2)}{r^2} + 4\pi^2 r^2 - 4\pi\kappa\right] f(r) = 0. \tag{3.26}$$

Any function $u \in C^\infty(\mathbb{R}^n\backslash\{0\})$ which is a simultaneous eigenfunction of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ (this makes sense since the use of polar coordinates makes it possible to consider that $\Delta_{S^{n-1}}$ acts on functions defined on $\mathbb{R}^n\backslash\{0\}$) for the pair of eigenvalues $(\kappa,\, -\ell(\ell+n-2))$, with $\ell = 0,1,\ldots$, is a function of the kind (3.25). Indeed, for a given ℓ, let $(Y^{\ell,m})_{1\leq m\leq d_\ell}$ be an orthonormal basis of the space of spherical harmonics of degree ℓ and, for $r > 0$ and $\xi \in S^{n-1}$, write $u(r\xi) = \sum_m f_m(r)\, Y^{\ell,m}(\xi)$ with

$$f_m(r) = \int_{S^{n-1}} u(r\eta)\, \overline{Y}^{\ell,m}(\eta)\, d\sigma(\eta). \tag{3.27}$$

Computing the left-hand side of (3.26) with f replaced by f_m and using (3.23) together with the equation $\Delta_{S^{n-1}} Y^{\ell,m} = -\ell(\ell+n-2)\, Y^{\ell,m}$, one finds, as a consequence of the equations $L^{(n)}\, u = \kappa\, u$ and $\Delta_{S^{n-1}}\, u = -\ell(\ell+n-2)\, u$, that f_m satisfies the equation (3.26). Now, this equation is of Fuchs type, and the roots of its indicial equation are ℓ and $2-n-\ell$: when $n \geq 2$, only the root ℓ is non-negative for $\ell \geq 0$. Solving the equation by means of indeterminate coefficients, one finds the following:

Lemma 3.3. *Let $\kappa \in \mathbb{Z}$ and let $\ell = 0,1,\ldots$. The linear space $E_{\kappa,\ell}$ of analytic functions u on $\mathbb{R}^n$ which are joint eigenfunctions of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ for the pair of eigenvalues $(\kappa,\, -\ell(\ell+n-2))$ consists of the functions $u(r,\xi) = f(r)\, Y^\ell(\xi)$, in which Y^ℓ is an arbitrary spherical harmonic of degree ℓ and f is a solution of (3.26) of the form $f(r) = r^\ell\, h(\frac{r^2}{2})$, where h is analytic in a neighborhood of $[0,\infty[$. Given ℓ, the space of such functions f is one-dimensional.*

The anaplectic Hermite functions introduced in Definition 3.2 can be characterized as follows.

Theorem 3.4. *The linear space of anaplectic Hermite functions is generated by the union of the spaces* $E_{\kappa,\ell}$ *with* $\kappa \in \mathbb{Z}$, $\ell = 0, 1, \dots,$ *and* $\kappa + \ell$ *even.*

Remarks 3.1. (i) An anaplectic Hermite function may coincide with an ordinary Hermite function, but only under the assumption that $n \equiv 0 \mod 4$: in this case, for every $\ell = 0, 1, \dots$ and $j = 0, 1, \dots,$ the functions in the space $E_{\kappa,\ell}$ are Hermite functions in both the ordinary and the anaplectic sense whenever $\kappa = \frac{n}{2} + \ell + 2j$. Since every ordinary Hermite function can be obtained from the ground state of the usual harmonic oscillator by the application of some operator in the algebra $\mathbb{C}[x, \frac{\partial}{\partial x}]$, it follows that, in the case when the dimension is divisible by 4, all usual Hermite functions are also anaplectic Hermite functions. For instance, if $n = 4$, an application of (3.45) below shows that $(\sum {A_j^*}^2)\, \Phi = -2^{\frac{3}{2}}\, e^{-\pi|x|^2}$ and $\sum A_j^2\, \Phi = -2^{\frac{3}{2}}\, e^{\pi|x|^2}$: these two anaplectic Hermite functions correspond respectively to $\kappa = 2$ and -2, and the first one is also, of course, an ordinary Hermite function.

(ii) Theorem 3.4 is also valid in the one-dimensional case. Note that, in this case, the notion of spherical harmonic still makes sense as the restriction of a harmonic polynomial: the condition that $\kappa + \ell$ should be even excludes, as it should, the function $\phi^{\natural}$ introduced at the end of Section 1 and, more generally, all functions in the space $\mathfrak{A}^{\natural}$. The function $\phi^{\natural}$ will reappear at the end of Section 10.

In order to prove Theorem 3.4, we first substitute for the operators x_j, $\frac{\partial}{\partial x_j}$ the operators

$$A_j^* = \pi^{\frac{1}{2}} \left(x_j - \frac{1}{2\pi} \frac{\partial}{\partial x_j} \right), \qquad A_j = \pi^{\frac{1}{2}} \left(x_j + \frac{1}{2\pi} \frac{\partial}{\partial x_j} \right), \tag{3.28}$$

which generate the same algebra.

Next, if $u(r, \xi) = f(r)\, Y^\ell(\xi)$, where Y^ℓ is a spherical harmonic of degree ℓ, if $\mathcal{Y}^\ell$ is the homogeneous extension of Y^ℓ to $\mathbb{R}^n$ of degree ℓ, which is a harmonic polynomial, and if $f(r) = r^\ell\, h(\frac{r^2}{2})$, one has, reverting to the coordinates x_j,

$$u(x) = h\left(\frac{|x|^2}{2}\right) \mathcal{Y}^\ell(x) : \tag{3.29}$$

also, the equation (3.26) can be rewritten, in terms of h, as

$$\left[s\, \frac{d^2}{ds^2} + (\ell + \frac{n}{2})\, \frac{d}{ds} - 4\pi^2\, s + 2\pi\kappa \right] h(s) = 0 : \tag{3.30}$$

let us denote the differential operator on the left-hand side as $M_{n,\kappa,\ell}$.

The proof of Theorem 3.4 requires several lemmas which will also be useful later (Section 8), which explains why these are stated in a more precise way than what would really be needed for our present purpose. In the first one, we link solutions of the equation (3.30) relative to the pair (κ, ℓ) or to the pair $(\kappa{\pm}1, \ell{+}1)$.

Lemma 3.5. *Given an analytic function* h *on the real line, and* $\ell \geq 0$*, set*

$$\begin{aligned}
(\partial h)(s) &= \pi^{\frac{1}{2}} \left(h(s) + \frac{1}{2\pi}\, h'(s) \right), \\
(\delta_\ell\, h)(s) &= \pi^{\frac{1}{2}} \left[s \left(h(s) + \frac{1}{2\pi}\, h'(s) \right) + \frac{\frac{n}{2} + \ell - 1}{2\pi}\, h(s) \right], \\
(\partial^\sharp h)(s) &= \pi^{\frac{1}{2}} \left(h(s) - \frac{1}{2\pi}\, h'(s) \right), \\
(\delta_\ell^\sharp\, h)(s) &= \pi^{\frac{1}{2}} \left[s \left(h(s) - \frac{1}{2\pi}\, h'(s) \right) - \frac{\frac{n}{2} + \ell - 1}{2\pi}\, h(s) \right].
\end{aligned} \tag{3.31}$$

If the equation $M_{n,\kappa,\ell}\, h = 0$ *is satisfied, then both equations* $M_{n,\kappa+1,\ell+1}\, \partial^\sharp h = 0$ *and* $M_{n,\kappa-1,\ell+1}\, \partial h = 0$ *are; in the other direction, if any of these two latter equations is satisfied and if, moreover, the equation* $M_{n,\kappa,\ell}\, h = 0$ *is satisfied at the origin, it is satisfied at every point of* $[0, \infty[$*. If the equation* $M_{n,\kappa,\ell}\, h = 0$ *is satisfied, both equations* $M_{n,\kappa+1,\ell-1}\, \delta_\ell^\sharp\, h = 0$ *and* $M_{n,\kappa-1,\ell-1}\, \delta_\ell\, h = 0$ *are.*

Proof. A straightforward computation yields the equations

$$\begin{aligned}
\left(\frac{d}{ds} + 2\pi \right) M_{n,\kappa,\ell} &= M_{n,\kappa-1,\ell+1} \left(\frac{d}{ds} + 2\pi \right), \\
\left(\frac{d}{ds} - 2\pi \right) M_{n,\kappa,\ell} &= M_{n,\kappa+1,\ell+1} \left(\frac{d}{ds} - 2\pi \right),
\end{aligned} \tag{3.32}$$

as well as the equations

$$\begin{aligned}
\left(s\, \frac{d}{ds} - 2\pi s + \frac{n}{2} + \ell - 1 \right) M_{n,\kappa,\ell} &= M_{n,\kappa+1,\ell-1} \left(s\, \frac{d}{ds} - 2\pi s + \frac{n}{2} + \ell - 1 \right), \\
\left(s\, \frac{d}{ds} + 2\pi s + \frac{n}{2} + \ell - 1 \right) M_{n,\kappa,\ell} &= M_{n,\kappa-1,\ell-1} \left(s\, \frac{d}{ds} + 2\pi s + \frac{n}{2} + \ell - 1 \right).
\end{aligned} \tag{3.33}$$

The lemma immediately follows. □

Lemma 3.6. *Given* $\mathcal{Y}^\ell \in \mathrm{Harm}(n, \ell)$ *with* $\ell \geq 0$*, and* $\alpha \in \mathbb{C}^n$*, set*

$$\begin{aligned}
S_\alpha^-\, \mathcal{Y}^\ell &= \left(\frac{n}{2} + \ell - 1 \right)^{-1} \langle\, \alpha,\, \nabla \mathcal{Y}^\ell\, \rangle, \\
S_\alpha^+\, \mathcal{Y}^\ell &= \langle \alpha,\, x \rangle\, \mathcal{Y}^\ell - \frac{|x|^2}{2}\, S_\alpha^-\, \mathcal{Y}^\ell,
\end{aligned} \tag{3.34}$$

where $\langle\, \alpha,\, \nabla \mathcal{Y}^\ell\, \rangle = \sum_j \alpha_j\, \frac{\partial \mathcal{Y}^\ell}{\partial x_j}$*, and the function* $S_\alpha^-\, \mathcal{Y}^\ell$ *is to be interpreted as zero in the case when* $\ell = 0$*, even if* $n = 2$*. Then,*

$$S_\alpha^-\, \mathcal{Y}^\ell \in \mathrm{Harm}(n, \ell - 1), \qquad S_\alpha^+\, \mathcal{Y}^\ell \in \mathrm{Harm}(n, \ell + 1). \tag{3.35}$$

Proof. Since $\Delta\,\mathcal{Y}^\ell = 0$, one has

$$\Delta\,(S_\alpha^+\,\mathcal{Y}^\ell) = 2\,\langle\,\alpha,\,\nabla\,\mathcal{Y}^\ell\,\rangle - n\,S_\alpha^-\,\mathcal{Y}^\ell - 2\,\langle\,x,\,\nabla\,(S_\alpha^-\,\mathcal{Y}^\ell)\,\rangle, \tag{3.36}$$

which reduces to zero as a consequence of Euler's identity relative to the function $S_\alpha^-\,\mathcal{Y}^\ell$. □

It will be convenient to denote as $h \boxtimes \mathcal{Y}^\ell$ the function on the right-hand side of (3.29):

$$(h \boxtimes \mathcal{Y}^\ell)(x) = h\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^\ell(x). \tag{3.37}$$

Lemma 3.7. *Given $\alpha \in \mathbb{C}^n$, set $A_\alpha = \sum_j \alpha_j\,A_j$, $A_\alpha^\sharp = \sum_j \alpha_j\,A_j^*$. Then, for any function $u \in E_{\kappa,\ell}$, one has*

$$A_\alpha\,u \in E_{\kappa-1,\ell-1} \oplus E_{\kappa-1,\ell+1}, \qquad A_\alpha^\sharp\,u \in E_{\kappa+1,\ell-1} \oplus E_{\kappa+1,\ell+1}: \tag{3.38}$$

more precisely, in the case when $u = h \boxtimes \mathcal{Y}^\ell$, the decompositions just quoted can be made explicit as

$$A_\alpha\,u = T_\alpha^-\,u + T_\alpha^+ u, \qquad A_\alpha^\sharp\,u = T_\alpha^{\sharp\,-}\,u + T_\alpha^{\sharp\,+}\,u, \tag{3.39}$$

with

$$\begin{aligned} T_\alpha^-\,(h \boxtimes \mathcal{Y}^\ell) &= (\delta_\ell\,h) \boxtimes S_\alpha^-\,\mathcal{Y}^\ell, &\qquad T_\alpha^+\,(h \boxtimes \mathcal{Y}^\ell) &= (\partial h) \boxtimes S_\alpha^+\,\mathcal{Y}^\ell,\\ T_\alpha^{\sharp\,-}\,(h \boxtimes \mathcal{Y}^\ell) &= (\delta_\ell^\sharp\,h) \boxtimes S_\alpha^-\,\mathcal{Y}^\ell, &\qquad T_\alpha^{\sharp\,+}\,(h \boxtimes \mathcal{Y}^\ell) &= (\partial^\sharp h) \boxtimes S_\alpha^+\,\mathcal{Y}^\ell. \end{aligned} \tag{3.40}$$

Proof. Since $[L^{(n)}, A_\alpha] = -A_\alpha$ and $[L^{(n)}, A_\alpha^\sharp] = A_\alpha^\sharp$, one has

$$A_\alpha\,(h \boxtimes \mathcal{Y}^\ell) \in \operatorname{Ker}\,(L^{(n)} - \kappa + 1) \qquad \text{and} \qquad A_\alpha^\sharp\,(h \boxtimes \mathcal{Y}^\ell) \in \operatorname{Ker}\,(L^{(n)} - \kappa - 1). \tag{3.41}$$

Writing for short

$$\mathcal{Z}^{\ell-1} = S_\alpha^-\,\mathcal{Y}^\ell, \qquad \mathcal{Z}^{\ell+1} = S_\alpha^+\,\mathcal{Y}^\ell, \tag{3.42}$$

so that

$$\langle \alpha, x\rangle\,\mathcal{Y}^\ell = \frac{|x|^2}{2}\,\mathcal{Z}^{\ell-1} + \mathcal{Z}^{\ell+1}, \tag{3.43}$$

one has

$$\begin{aligned} A_\alpha\,(h \boxtimes \mathcal{Y}^\ell) &= \pi^{\frac12}\left[\langle \alpha, x\rangle + \frac{1}{2\pi}\left\langle \alpha, \frac{\partial}{\partial x}\right\rangle\right]\left(h\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^\ell\right)\\ &= \pi^{\frac12}\,h\left(\frac{|x|^2}{2}\right)\,\langle \alpha, x\rangle\,\mathcal{Y}^\ell + \frac{1}{2\pi^{\frac12}}\left[\langle \alpha, x\rangle\,h'\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^\ell + h\left(\frac{|x|^2}{2}\right)\,\langle \alpha,\,\nabla\,\mathcal{Y}^\ell\right]\\ &= \left[\pi^{\frac12}\,h\left(\frac{|x|^2}{2}\right) + \frac{1}{2\pi^{\frac12}}\,h'\left(\frac{|x|^2}{2}\right)\right]\\ &\quad\times\left(\frac{|x|^2}{2}\,\mathcal{Z}^{\ell-1} + \mathcal{Z}^{\ell+1}\right) + \frac{1}{2\pi^{\frac12}}\left(\frac{n}{2} + \ell - 1\right)\,h\left(\frac{|x|^2}{2}\right)\,\mathcal{Z}^{\ell+1}, \end{aligned} \tag{3.44}$$

which leads to the expression (3.39) of $A_\alpha\,(h \boxtimes \mathcal{Y}^\ell)$, once the notation of the present lemma and that of Lemma 3.5 have been sorted-out. The same goes for the function $A^\sharp_\alpha\,(h \boxtimes \mathcal{Y}^\ell)$.

That $T^+_\alpha\,(h \boxtimes \mathcal{Y}^\ell)$ lies in $E_{\kappa-1,\ell+1}$ and the three analogous statements is a consequence of Lemma 3.5. □

Proof of Theorem 3.4. First, the function Φ itself lies in $E_{0,0}$, *i.e.,* is invariant under rotations and an eigenfunction of $L^{(n)}$ for the eigenvalue 0: this follows most easily from its definition (3.22) together with the differential equation (3.30). Then, that all anaplectic Hermite functions lie in the space linearly generated by all spaces $E_{\kappa,\ell}$ with $\kappa+\ell$ even follows from (3.38).

In the reverse direction, we use induction with respect to ℓ. First, we show that every function in some space $E_{\kappa,0}$ with κ even is an anaplectic Hermite function. The operator $\sum A_j^{*2}$ (*resp.* $\sum A_j^2$) commutes with rotations and sends the space $E_{\kappa,0}$ into $E_{\kappa+2,0}$ (*resp.* $E_{\kappa-2,0}$), and all spaces under consideration are one-dimensional: we only need to show that if $u(x) = h(\frac{|x|^2}{2})$ is analytic, non-zero and satisfies the equation $L^{(n)}u = \kappa\, u$ for some $\kappa = 0, 2, \dots$, it cannot be annihilated, say, by the operator

$$\sum A_j^{*2} = \pi\,|x|^2 + \frac{1}{4\pi}\,\Delta - \sum x_j\,\frac{\partial}{\partial x_j} - \frac{n}{2}. \tag{3.45}$$

In terms of the function $h = h(s)$ such that $u(x) = h(\frac{|x|^2}{2})$, the pair of operators $\sum A_j^{*2}$ and $L^{(n)} - \kappa$ can be written as

$$\begin{aligned} \sum A_j^{*2} &= \frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + \left(\frac{n}{2} - 4\pi s\right)\frac{d}{ds} + 4\pi^2 s - \pi n\right], \\ L^{(n)} - \kappa &= -\frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + \frac{n}{2}\,\frac{d}{ds} - 4\pi^2 s + 2\pi\kappa\right]: \end{aligned} \tag{3.46}$$

using the pair of differential equations $\sum A_j^{*2}\,u = 0$, $(L^{(n)} - \kappa)\,u = 0$ in the same way as the one we would use to compute the resultant of two polynomials, lowering the order of one equation at each step, we find that a non-zero solution can exist only if $2\kappa = n-4$ and $h(s) = C\,s^{\frac{2-n}{2}}\,e^{2\pi s}$, or $2\kappa = -n$ and $h(s) = e^{2\pi s}$: the second case can be discarded since we are only interested in the case when $\kappa \geq 0$, and the first one, when not reducing to the second one (when $n=2$) can be discarded because h would be non-analytic near 0. When $\kappa = 0, -2, \dots$, it works in the same way, only replacing the operator $\sum A_j^{*2}$ by $\sum A_j^2$.

We are now in a position to start the induction with respect to ℓ. Let $v \in E_{\kappa',\ell+1}$ for some $\ell = 0, 1, \dots$ with $\ell + 1 + \kappa'$ even. If $\kappa' \geq 0$, we set $\kappa' = \kappa + 1$ and will show that v is a linear combination $v = \sum A_j^*\,w_j$, each w_j lying in the space $E_{\kappa,\ell}$: similarly, if $\kappa' < 0$, we would set $\kappa' = \kappa - 1$ and use the operators A_j instead of the A_j^*'s; let us consider the first case only. By Lemma

3.3, one has $v(x) = g(\frac{|x|^2}{2})\,\mathcal{X}^{\ell+1}(x)$ for some polynomial $\mathcal{X}^{\ell+1} \in \mathrm{Harm}(n, \ell+1)$. Define h as the solution of the differential equation $\pi^{\frac{1}{2}}\,(h - \frac{1}{2\pi}\,h') = g$, *i.e.*, $\partial^\sharp\, h = g$, such that $(M_{n,\kappa,\ell}\, h)(0) = 0$, *i.e.*, $(\ell + \frac{n}{2})\, h'(0) + 2\pi\kappa\, h(0) = 0$: this is indeed possible for some appropriate choice of the constant of integration since $2\pi\,(\ell + \frac{n}{2} + \kappa) \neq 0$ (as $\ell \geq 0$, $\kappa \geq -1$ and $\ell + \kappa$ is even). Since g is a solution of the equation $M_{n,\kappa+1,\ell+1}\, g = 0$, it follows from Lemma 3.5 that h satisfies the equation $M_{n,\kappa,\ell}\, h = 0$.

For every $j = 1, \dots, n$, apply Lemma 3.6 to the harmonic polynomial $\frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j}$, of degree ℓ. Denoting as (j) the jth vector of the canonical basis of $\mathbb{R}^n$, one has (since $\mathcal{X}^{\ell+1}$ is harmonic)

$$\sum_j S^-_{(j)}\, \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j} = \Big(\frac{n}{2} + \ell - 1\Big)^{-1} \sum_j \frac{\partial}{\partial x_j}\Big(\frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j}\Big) = 0 \tag{3.47}$$

so that

$$\sum_j S^+_{(j)}\, \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j} = \sum_j x_j\, \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j} = (\ell+1)\,\mathcal{X}^{\ell+1} : \tag{3.48}$$

consequently, Lemma 3.7 implies that

$$\begin{aligned}\sum_j A_j^*\,(h \boxtimes \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j}) &= (\partial^\sharp\, h) \boxtimes \sum_j S^+_{(j)}\, \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j} \\ &= (\ell+1)\, g \boxtimes \mathcal{X}^{\ell+1} = (\ell+1)\, v :\end{aligned} \tag{3.49}$$

this concludes the proof since, for every j, the function $h \boxtimes \frac{\partial\,\mathcal{X}^{\ell+1}}{\partial x_j}$ lies in $E_{\kappa,\ell}$. □

We now connect the radial part of an anaplectic Hermite function, also a joint eigenfunction of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ of the kind introduced in Lemma 3.3, to the spaces of analytic vectors of the representations $\pi_{\rho,\varepsilon}$.

Theorem 3.8. *Assume $n \geq 2$, and let $\ell = 0, 1, \dots$. If $\mathcal{Y}^\ell$ is a harmonic polynomial homogeneous of degree ℓ and $u(x) = h(\frac{|x|^2}{2})\,\mathcal{Y}^\ell(x)$ is a joint eigenfunction of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ for the pair of eigenvalues $(\kappa,\, -\ell(\ell+n-2))$ with $\kappa+\ell$ an even integer, so that u is an anaplectic Hermite function according to Theorem* 3.4*, the function h is the ramified part of some function in the space $C^\omega_{\frac{n-2}{2}+\ell,\,\ell \bmod 2}$.*

Proof. Whatever the value of (ρ, ε), one may define, as in (2.20),

$$w^k_{\rho,\varepsilon}(\sigma) = (1+\sigma^2)^{-\frac{\rho+1+\varepsilon}{2} - k}\,(1 - i\sigma)^{2k+\varepsilon} \tag{3.50}$$

and the function $v^k_{\rho,\varepsilon} = \mathcal{F}^{-1}\, w^k_{\rho,\varepsilon}$, which lies in $C^\omega_{\rho,\varepsilon}$. Set $\rho = \frac{n-2}{2} + \ell$, and $h^k_{\rho,\varepsilon} = (v^k_{\rho,\varepsilon})^{\mathrm{ram}}$. It suffices to prove, for every $k \in \mathbb{Z}$, the equations

$$M_{n,2k,\ell}\; h^k_{\rho,0} = 0 \qquad \text{and} \qquad M_{n,2k+1,\ell}\; h^k_{\rho,1} = 0 \tag{3.51}$$

since, up to the multiplication by some constant, each of these two equations has only one solution analytic near zero. It is immediate that the function $w = w^k_{\rho,\varepsilon}$ satisfies the differential equation

$$\left[(1+\sigma^2)\,\frac{d}{d\sigma} + (\rho+1)\,\sigma + 2k + \varepsilon\right]\, w = 0, \tag{3.52}$$

so that its inverse Fourier transform $v = v^k_{\rho,\varepsilon}$ satisfies the equation

$$\left[s\frac{d^2}{ds^2} + (1-\rho)\,\frac{d}{ds} - 4\pi^2 s + 2\pi\,(2k+\varepsilon)\right]\, v = 0. \tag{3.53}$$

According to Proposition 2.3 and Proposition 3.1, in the case when $\rho + \varepsilon \notin 2\mathbb{Z}$, the function $s \mapsto |s|^{\rho}_{\varepsilon}\, h(s)$ must satisfy the same equation; in the case when $\rho + \varepsilon \in 2\mathbb{Z}$, it is the part involving the factor $\log|s|$ of the image of the function $s \mapsto s^{\rho}\,(\log|s|)\, h(s)$ under the same differential operator that must vanish. In both cases, we find the equation

$$\left[s\,\frac{d^2}{ds^2} + (1+\rho)\,\frac{d}{ds} - 4\pi^2 s + 2\pi\,(2k+\varepsilon)\right]\, h = 0, \tag{3.54}$$

which is exactly the sought-after equation. □

4 Analysis on the space of Lagrangian subspaces of $\mathbb{R}^{2n}$

The first thing to do, so as to generalize the anaplectic analysis to the n-dimensional setting, is to define an appropriate space $\mathfrak{A}^{(n)}$, generalizing the space $\mathfrak{A}$ of analytic functions introduced, in dimension 1, in Definition 1.1; also, we need to define the proper notion of anaplectic integral, generalizing the linear form Int defined in Proposition 1.16. Though the space $\mathfrak{A}$ has, according to Section 1, several possible definitions, the proper definition of the n-dimensional generalization $\mathfrak{A}^{(n)}$ must be based on Theorem 1.8.

Most of the difficulties of the n-dimensional case are linked to the analysis of fractional-linear transformations on the space of symmetric matrices. We denote as Γ_n the cone of positive-definite matrices in the linear space Sym_n of all real symmetric matrices of size $n \times n$, a subspace of the space $\mathrm{Sym}_n^{\mathbb{C}}$, consisting of symmetric matrices with complex entries. If $\sigma \in \mathrm{Sym}_n$, we shall also write $\sigma \succ 0$ to mean that σ lies in Γ_n, and we shall denote as I the identity matrix of the appropriate size. On the space $\mathrm{Sym}_n^{\mathbb{C}}$, we shall use the norm $\sigma \mapsto \|\sigma\|$ which is the operator norm associated to the canonical norm $|\ |$ on $\mathbb{C}^n$.

Definition 4.1. Given any entire function u on $\mathbb{C}^n$ with the property that for some $C > 0$, one has $|u(z)| \leq C\, e^{\pi R |z|^2}$ for all $z \in \mathbb{C}^n$, we define the quadratic transform, or $\mathcal{Q}$-transform of u, as the pair of functions defined on the part of $\Gamma_n + i\,\mathrm{Sym}_n$ characterized by the condition $\mathrm{Re}\ (\sigma - R\,I) \succ 0$ as follows: $(\mathcal{Q}\,u)_0$ is the scalar function defined as

$$(\mathcal{Q}\,u)_0(\sigma) = \int_{\mathbb{R}^n} e^{-\pi\,\langle \sigma x, x\rangle}\, u(x\, e^{-\frac{i\pi}{4}})\, dx, \tag{4.1}$$

and $(\mathcal{Q}\,u)_1$ is the $\mathbb{C}^n$-valued function defined as

$$(\mathcal{Q}\,u)_1(\sigma) = \int_{\mathbb{R}^n} (I + i\,\sigma)\, x.\, e^{-\pi\,\langle \sigma x, x\rangle}\, u(x\, e^{-\frac{i\pi}{4}})\, dx, \tag{4.2}$$

in other words the function the jth component of which is

$$(\mathcal{Q}\,u)_1^{(j)}(\sigma) = \int_{\mathbb{R}^n} \left(x_j + i \sum_k \sigma_{jk}\, x_k \right) e^{-\pi\,\langle \sigma x, x\rangle}\, u(x\, e^{-\frac{i\pi}{4}})\, dx, \qquad j = 1, \dots, n. \tag{4.3}$$

As a preparation for the study of the action of fractional-linear transformations on the quadratic transform of u, we need a combinatorial lemma, best proved with the help of Wick's theorem, a tool more familiar in connection with Feynman diagrams [9].

Lemma 4.2. *For every multi-index $\alpha \in \mathbb{N}^n$, the function of $\tau \in \Gamma_n$ defined as*

$$I_\alpha(\tau) = (\det \tau)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\pi \langle \tau^{-1} x, x\rangle}\, x^\alpha\, dx \tag{4.4}$$

is a polynomial in the entries of the matrix τ. *Also, for every* $\tau \in \mathrm{Sym}_n^{\mathbb{C}}$,

$$(2\pi)^{|\alpha|}\, |I_\alpha(\tau)| \leq 2^{1-\frac{|\alpha|}{2}}\, \frac{\Gamma(|\alpha|)}{\Gamma(\frac{|\alpha|}{2})}\, (\max |\tau_{jk}|)^{\frac{|\alpha|}{2}}, \tag{4.5}$$

where the Gamma ratio is to be understood as $\frac{1}{2}$ *in the case when* $\alpha = 0$. *Given a multi-index* β *and an index* $j = 1, \dots, n$, *set*

$$J_{\beta,\,j}(\tau) = (\det \tau)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\pi \langle \tau^{-1} x, x\rangle}\, x^\beta\, (\tau^{-1}x)_j\; dx : \tag{4.6}$$

the first part of the statement concerning the function I_α *holds just as well for* $J_{\beta,\,j}$, *and the inequality* (4.5) *is still valid for this function after one has replaced* $|\alpha|$ *by* $|\beta| - 1$.

Proof. Given $N \geq 1$ and $2N$ distinct letters $1, \dots, 2N$, a pairing of the set $\{1, \dots, 2N\}$ is any unordered partition ϖ of this set into blocks with two elements each: thus, the number of possible pairings of such a set is $1.3.\dots.(2N-1)$, a number denoted as $(2N-1)!!$ by physicists. If a value k_i in the set $\{1, 2, \dots, n\}$ is ascribed to each letter i, a pairing of the set $\{1, \dots, 2N\}$ is compatible with this assignment of values if the two letters in any given block are assigned the same value. Clearly, there is no pairing compatible with the given assignment unless, for each $j = 1, \dots, n$, the number of letters assigned the value j is an even number $2N_j$: if such is the case (then, of course, $\sum N_j = N$), the number of pairings compatible with the assignment is $\prod_j (2N_j - 1)!!$. Wick's theorem is the fact that the value of the integral

$$\int_{\mathbb{R}^n} e^{-\pi\, |x|^2}\, x_{k_1} \cdots x_{k_{2N}}\, dx \tag{4.7}$$

coincides with $(2\pi)^{-N}$ times the number of pairings of the set $\{1, \dots, 2N\}$ compatible with the assignment of values $i \mapsto k_i$. Of course, this is immediate to see since

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\pi\, |x|^2}\, x_{k_1} \dots x_{k_{2N}}\, dx &= \prod_j \left(-\frac{1}{\pi}\,\frac{d}{d\lambda_j}\right)^{N_j}\Bigg|_{\lambda_j = 1} \int_{-\infty}^{\infty} e^{-\pi \lambda_j\, x_j^2}\, dx_j \\ &= (2\pi)^{-N} \prod_j (2N_j - 1)!!,\end{aligned} \tag{4.8}$$

but it is very useful when Feynman diagrams are considered.

Given $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2N$, fix any sequence $j_1, \dots, j_{2N}$ of integers in $[1, n]$ such that the number of such integers equal to 1 is $\alpha_1, \dots$ the number of such integers equal to n is α_n, so that, after a change of variables,

$$I_\alpha(\tau) = \int_{\mathbb{R}^n} e^{-\pi |x|^2}\, (\tau^{\frac{1}{2}} x)_{j_1} \cdots (\tau^{\frac{1}{2}} x)_{j_{2N}}\, dx. \tag{4.9}$$

Let $\omega \in \mathrm{O}(n)$ be such that $\tau^{\frac{1}{2}} = \omega\,\mu\,\omega^{-1}$, where μ is diagonal with entries $\mu_1, \dots, \mu_n$: after an orthogonal change of variables, one finds

$$I_\alpha(\tau) = \sum_{k_1,\dots,k_{2N}} (\omega_{j_1,k_1}\,\mu_{k_1}) \cdots (\omega_{j_{2N},k_{2N}}\,\mu_{k_{2N}}) \int_{\mathbb{R}^n} e^{-\pi|x|^2} x_{k_1} \cdots x_{k_{2N}}\, dx. \tag{4.10}$$

Using Wick's theorem, one can write

$$(2\pi)^N\, I_\alpha(\tau) = \sum_{k_1,\dots,k_{2N}} \sum_{\varpi} (\omega_{j_1,k_1}\,\mu_{k_1}) \cdots (\omega_{j_{2N},k_{2N}}\,\mu_{k_{2N}}), \tag{4.11}$$

where ϖ describes the set of all pairings of $\{1, \dots, 2N\}$ compatible with the assignment $i \mapsto k_i$: the effect of the sum over ϖ is simply to put the right coefficient ($(2\pi)^N$ times the value of the integral immediately above) in front of the sum that follows. One can commute the two summations, writing instead

$$(2\pi)^N\, I_\alpha(\tau) = \sum_{\varpi} \sum_{\substack{\text{assignments } i\mapsto k_i \\ \text{compatible with } \varpi}} (\omega_{j_1,k_1}\,\mu_{k_1}) \cdots (\omega_{j_{2N},k_{2N}}\,\mu_{k_{2N}}), \tag{4.12}$$

where, this time, ϖ runs through all pairings of $\{1, \dots, 2N\}$. Describe the generic pairing ϖ as the collection of sets $\{a_\ell, b_\ell\}$, with $1 \leq \ell \leq N$. The term associated with ϖ in the sum (4.12) is thus obtained when choosing all assignments $i \mapsto k_i$ such that $k_{a_\ell} = k_{b_\ell}$ for all ℓ: we denote as r_ℓ the common value of these two numbers. Finally, the term associated with ϖ in the sum (4.12) can be written as

$$\sum_{r_1,\dots,r_N \in [1,n]} \prod_{\ell=1}^{N} \omega_{j_{a_\ell},r_\ell}\, \mu_{r_\ell}^2\, \omega_{j_{b_\ell},r_\ell} : \tag{4.13}$$

since $\tau = \omega\,\mu^2\,\omega^{\text{transpose}}$, one has

$$\sum_{r_\ell=1}^{N} \omega_{j_{a_\ell},r_\ell}\, \mu_{r_\ell}^2\, \omega_{j_{b_\ell},r_\ell} = \tau_{j_{a_\ell},j_{b_\ell}}, \tag{4.14}$$

so that the term we are interested in from the sum (4.12) reduces to $\prod_\ell \tau_{j_{a_\ell},j_{b_\ell}}$.

Also, if $\alpha \neq 0$ and $\tau \in \mathrm{Sym}_n^{\mathbb{C}}$,

$$\begin{aligned}(2\pi)^{|\alpha|}\, |I_\alpha(\tau)| &\leq (|\alpha - 1|)\,!!\, (\max|\tau_{jk}|)^{\frac{|\alpha|}{2}} \\ &= 2^{1-\frac{|\alpha|}{2}}\, \frac{\Gamma(|\alpha|)}{\Gamma(\frac{|\alpha|}{2})}\, (\max|\tau_{jk}|)^{\frac{|\alpha|}{2}},\end{aligned} \tag{4.15}$$

which completes the proof of the part of the lemma concerning I_α.

The integral defining $J_{\beta,\,j}(\tau)$ can be non-zero only if $|\beta|$ is odd, say $2N-1$. Since

$$(\tau^{-\frac{1}{2}}x)_j = \sum_k (\tau^{-1})_{jk}\, (\tau^{\frac{1}{2}}x)_k, \tag{4.16}$$

it is immediate that

$$J_{\beta,\,j}(\tau) = \sum_k (\tau^{-1})_{jk}\, I_{\beta+(k)}(\tau), \tag{4.17}$$

denoting as (k) the multi-index of length 1 with a 1 at the kth place. Thus, going back to (4.9), we must associate with β a sequence $j_1, \dots, j_{2N-1}$ as explained there, substituting for the last integer j_{2N} an index k free to run from 1 to n. Using the result of the preceding computation, as reported right after (4.14), we must extract from each term $\prod_\ell \tau_{j_{a_\ell},\, j_{b_\ell}}$, in which it is no loss of generality to assume that $j_{b_\ell} > j_{a_\ell}$ for every ℓ, the only factor $\tau_{j_{a_\ell},\, j_{b_\ell}}$ for which $b_\ell = k$, and sum it against $(\tau^{-1})_{jk}$, ending up with $\delta_{a_\ell,\, j}$: this completes the proof of the lemma. □

As an example of this recipe, in any dimension,

$$(\det \tau)^{-\frac{1}{2}} \int e^{-\pi \langle \tau^{-1} x, x\rangle} x_{r_1}\, x_{r_2}\, x_{r_3}\, x_{r_4}\, dx$$
$$= \frac{1}{4\pi^2}\left[\tau_{r_1,r_2}\,\tau_{r_3,r_4} + \tau_{r_1,r_3}\,\tau_{r_2,r_4} + \tau_{r_1,r_4}\,\tau_{r_2,r_3}\right]. \tag{4.18}$$

Corollary 4.3. *Let u be an entire function on $\mathbb{C}^n$ satisfying for some pair of constants the estimate $|u(z)| \le C\, e^{\pi R\,|z|^2}$. The functions defined, for $\tau \in \mathrm{Sym}_n$ such that $\det \tau \neq 0$ and $\tau^{-1} - R\,I \succ 0$, as*

$$\tau \mapsto (\det \tau)^{-\frac{1}{2}}\, (\mathcal{Q}\, u)_0(\tau^{-1})$$

and the vector-valued one

$$\tau \mapsto (\det \tau)^{-\frac{1}{2}}\, (\mathcal{Q}\, u)_1(\tau^{-1}) \tag{4.19}$$

extend as analytic functions, denoted as $(\mathcal{Q}^{\mathrm{inv}}\, u)_0$ and $(\mathcal{Q}^{\mathrm{inv}}\, u)_1$ respectively, to the neighborhood of 0 in $\mathrm{Sym}_n^{\mathbb{C}}$ defined by the inequalities $\max |\tau_{jk}| < \frac{2\pi}{nR}$.

Proof. If $u(x) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha\, x^\alpha$, one has if $\tau^{-1} - R\,I \succ 0$ the series expansion

$$(\mathcal{Q}^{\mathrm{inv}}\, u)_0(\tau) = (\det \tau)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\pi\langle \tau^{-1} x, x\rangle}\, u(x\, e^{-\frac{i\pi}{4}})\, dx$$
$$= \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} c_\alpha\, e^{-\frac{i\pi\,|\alpha|}{4}}\, I_\alpha(\tau). \tag{4.20}$$

On the other hand, one has $u(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha\, z^\alpha$, where the assumption relative to u and Cauchy's integral formula on the polydisk with radii $(\rho, \dots, \rho)$ make it possible to write $|c_\alpha| \le C\, \rho^{-|\alpha|}\, e^{\pi n R \rho^2}$: by Stirling's formula,

$$|c_\alpha| \le C\, \frac{(\pi n R)^{\frac{|\alpha|}{2}}}{\Gamma(\frac{|\alpha|}{2})}, \qquad \alpha \neq 0. \tag{4.21}$$

The convergence of the series (4.20) in the neighborhood of 0 indicated follows from these inequalities together with the inequalities (4.5) and a new application of Stirling's formula. The proof is similar for what concerns the function $(\mathcal{Q}^{\mathrm{inv}}\, u)_1$: only, it is the terms with $|\alpha|$ odd from the series expansion of u which will play a role now since, with the notation used in (4.17),

$$(\mathcal{Q}^{\mathrm{inv}}\, u)_1^{(j)}(\tau) \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\ \mathrm{odd}}} c_\alpha\, e^{-\frac{i\pi\,|\alpha|}{4}} \left[I_{\alpha+(j)}(\tau) + i\, J_{\alpha,\,j}(\tau)\right]. \tag{4.22}$$

□

We now need to recall the following basic facts (*cf.* for instance [16]) regarding the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, the subgroup of $SL(2n,\mathbb{R})$ whose elements g are characterized as follows. Consider the symplectic form $[\ ,\ ]$ which is the bilinear form on $\mathbb{R}^{2n}\times\mathbb{R}^{2n} = (\mathbb{R}^n\times\mathbb{R}^n)\times(\mathbb{R}^n\times\mathbb{R}^n)$ such that $[\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right), \left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right)] = -\langle x,\eta\rangle + \langle y,\xi\rangle$: a linear transformation g of $\mathbb{R}^{2n}$ lies in $\mathrm{Sp}(n,\mathbb{R})$ if and only if one has $[gX, gY] = [X,Y]$ for every pair (X,Y) of vectors of $\mathbb{R}^{2n}$. Writing matrices of size $(2n)\times(2n)$ in block-form, with all blocks of size $n\times n$, one can characterize the fact that a matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ lies in $\mathrm{Sp}(n,\mathbb{R})$ by the condition that $g^{-1} = \left(\begin{smallmatrix} D' & -B' \\ -C' & A' \end{smallmatrix}\right)$, denoting as $A\mapsto A'$ the transposition map. This can be made explicit as

$$AD' - BC' = I, \qquad AB' = BA', \qquad CD' = DC' \tag{4.23}$$

or, equivalently,

$$D'A - B'C = I, \qquad A'C = C'A, \qquad B'D = D'B. \tag{4.24}$$

It is then an elementary thing to verify that if such is the case, and if a matrix $\sigma\in\mathrm{Sym}_n$ is such that the matrix $-B'\sigma + D'$ is invertible, then the matrix $(A'\sigma - C')(-B'\sigma + D')^{-1}$ is also symmetric.

This fractional-linear transformation on the argument of functions on Sym_n will enter our definition of the anaplectic representation.

Lemma 4.4. *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\in\mathrm{Sp}(n,\mathbb{R})$. *Assuming* $B\neq 0$, *let* $\varepsilon>0$ *be the smallest non-zero eigenvalue of the matrix* BB'. *For* $\sigma\in\Gamma_n$ *such that* $\sigma - \varepsilon^{-\frac{1}{2}}\,\|\,D\,\|\,I\succ 0$, *the matrix* $\sigma B - D$ *is invertible. If* $A\neq 0$, *the matrix* $\sigma A - C$ *is invertible if* $\sigma - \varepsilon'^{-\frac{1}{2}}\,\|\,C\,\|\,I\succ 0$, *denoting as* ε' *the smallest non-zero eigenvalue of* AA'. *If* $B=0$ (resp. $A=0$), *the matrix* D (resp. C) *is invertible.*

Proof. Regarding BB' as a positive-definite endomorphism of its image $\mathrm{Im}\,(BB')$, one may define $(BB')^{-1}$ as well as $(BB')^{-\frac{1}{2}}$: for any $z\in\mathrm{Im}\ B'$, one has $z = B'\,(BB')^{-1}\,Bz$ since the difference between the two sides lies in $\mathrm{Im}\ B'\cap\mathrm{Ker}\,B = \{0\}$, which implies that

$$|z| = |B'\,(BB')^{-1}\,Bz| = |(BB')^{-\frac{1}{2}}\,Bz| \leq \varepsilon^{-\frac{1}{2}}\,|Bz|. \tag{4.25}$$

Next, we observe that $\operatorname{Ker} B \cap \operatorname{Ker} D = \{0\}$ since $A'D - C'B = I$. Given $x \in \mathbb{R}^n$, $x \neq 0$, set $x = y + z$ with $y \in \operatorname{Ker} B$ and $z \in \operatorname{Im}\, B'$: then

$$\begin{aligned}\langle\, (\sigma B - D)x,\, Bz\rangle &= \langle \sigma Bz,\, Bz\rangle - \langle Dx,\, Bz\rangle \\ &= \langle \sigma Bz,\, Bz\rangle - \langle B'Dx,\, z\rangle \\ &= \langle \sigma Bz,\, Bz\rangle - \langle D'Bx,\, z\rangle \\ &= \langle \sigma Bz,\, Bz\rangle - \langle Bz,\, Dz\rangle. \end{aligned} \tag{4.26}$$

Thus

$$\begin{aligned}|(\sigma B - D)\, x|\; |Bz| &\geq \|\, \sigma^{-1}\, \|^{-1}\, |Bz|^2 - |Bz|\, |Dz| \\ &\geq \|\, \sigma^{-1}\, \|^{-1}\, |Bz|^2 - |Bz|\, \|\, D\, \|\, |z| \\ &\geq \Big[\|\, \sigma^{-1}\, \|^{-1} - \varepsilon^{-\frac{1}{2}}\, \|\, D\, \| \Big]\, |Bz|^2, \end{aligned} \tag{4.27}$$

where we have used (4.25). Since $Bz = Bx$, one sees that, under the condition $\sigma - \varepsilon^{-\frac{1}{2}}\, \|\, D\, \|\, I \succ 0$, one has $(\sigma B - D)\, x \neq 0$ if $Bx \neq 0$: but, since $\operatorname{Ker} B \cap \operatorname{Ker} D = \{0\}$, $(\sigma B - D)\, x$ cannot vanish either if $Bx = 0$ since this would imply $Dx = 0$ as well.

The part concerning the matrix $\sigma A - C$ follows from the fact that the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -B & A \\ -D & C \end{smallmatrix}\right)$ lies in $\operatorname{Sp}(n, \mathbb{R})$ just as well. □

Corollary 4.5. *Given a symplectic matrix* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$*, the open subset of* Sym_n *consisting of all real symmetric matrices* σ *such that* $-B'\,\sigma + D'$ *is invertible is dense in* Sym_n*.*

Proof. Let $\sigma_0 \in \operatorname{Sym}_n$ be arbitrary. The matrix

$$\begin{pmatrix} \sigma_0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & \sigma_0 \end{pmatrix} = \begin{pmatrix} -\sigma_0\, B + D & \sigma_0\, A - C + \sigma_0\, B\, \sigma_0 - D\, \sigma_0 \\ -B & A + B\, \sigma_0 \end{pmatrix} \tag{4.28}$$

lies in $\operatorname{Sp}(n, \mathbb{R})$ too, so that, applying Lemma 4.4, the matrix $\tau\, (-\sigma_0\, B + D) + B$ is invertible whenever $\tau \in \Gamma_n$ is such that $\tau - R\, I \succ 0$ with R large enough. Taking $\tau = (\sigma_0 - \sigma)^{-1}$ with $\sigma_0 - \sigma$ in Γ_n with a norm $< R^{-1}$, we conclude that $(\sigma_0 - \sigma)^{-1}\, (-\sigma_0\, B + D) + B$ is invertible, and so is, consequently, $-\sigma\, B + D$. □

A full description of the singularities of the fractional-linear transformation $\sigma \mapsto (A'\,\sigma - C')\,(-B'\,\sigma + D')^{-1}$, which will enter the definition of the anaplectic representation, is now required. Note that this is the transformation associated to the inverse of g if one defines $(g, \sigma) \mapsto g.\sigma = (C + D\,\sigma)\,(A + B\,\sigma)^{-1}$ when $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. This would be a group action of $\operatorname{Sp}(n, \mathbb{R})$ in Sym_n if it were not for the fact that it is not everywhere defined. It is so, however, in the case when $B = 0$, in which it reduces to $\sigma \mapsto D^{-1}\,(\sigma - DC')\,D'^{-1}$. Writing a matrix $g \in \operatorname{Sp}(n, \mathbb{R})$ as $g = g_1\, g_2$, where the upper-right block of g_1 is zero, reduces the analysis of the fractional-linear transformation associated to g^{-1} to that associated to g_2^{-1}. Thus, the following lemma makes a first reduction of the problem possible.

Lemma 4.6. *Given* $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$*, there exists a matrix* Ω *in the compact group* $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$ (*isomorphic to* $U(n)$ [11, p. 453] *under the map* $R+iS \mapsto \left(\begin{smallmatrix} R & S \\ -S & R \end{smallmatrix}\right)$*, where* R *and* S *are the real and imaginary parts of the unitary matrix* $R+iS$)*, such that the upper right block of the matrix* $g\,\Omega$ *is zero.*

Proof. Set $R = A'\,(AA'+BB')^{-\frac{1}{2}}$, $S = -B'\,(AA'+BB')^{-\frac{1}{2}}$: this makes sense since, applying an observation made in the beginning of the proof of Lemma 4.4 to the matrix $g^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}$, we note that $\mathrm{Ker}\,B' \cap \mathrm{Ker}\,A' = \{0\}$. Using (4.23), one checks that $R'R + S'S = I$, $R'S = S'R$, which is one of the (two) ways to check that the matrix $\Omega = \left(\begin{smallmatrix} R & S \\ -S & R \end{smallmatrix}\right)$ lies in the group $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$. Applying again the equation $AB' = BA'$, one sees that, indeed, the upper-left block $AS+BR$ of the product $g\,\Omega$ is zero. □

Next, we desingularize, by means of the Cayley transform [16, p. 35] and compactification, the action of the compact group $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$: the verification of the following proposition is straightforward.

Proposition 4.7. *Let* $\Sigma\colon = U(n) \cap \mathrm{Sym}_n^{\mathbb{C}}$*, and let* Σ^{reg} *be the dense subset consisting of all matrices in* Σ *whose set of eigenvalues does not include* 1*. The Cayley map* $\sigma \mapsto Z = (\sigma - i\,I)\,(\sigma + i\,I)^{-1}$ *is a one-to-one map from* Sym_n *into* Σ*, the image of which is exactly* Σ^{reg}*; the inverse map is* $Z \mapsto i\,(I+Z)\,(I-Z)^{-1}$*. The* (*almost always defined,* cf. *Corollary* 4.5) *action of the compact subgroup* $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$ *of* $\mathrm{Sp}(n,\mathbb{R})$ *on* Sym_n *extends as an action on the compactification of this space, as follows. The generic element of this group is the matrix* $\Omega = \left(\begin{smallmatrix} R & S \\ -S & R \end{smallmatrix}\right)$*, where* R *and* S *are the real and imaginary parts of a generic matrix* $R+i\,S$ *in the group* $U(n)$ *and, under conjugation under the Cayley transform, the action of such a matrix transfers as the map* $Z \mapsto (R - i\,S)\,Z\,(R+i\,S)^{-1}$*, an action without singularities on the whole of* Σ*.*

As a consequence of the group action, the compactification Σ can be identified with the homogeneous space $U(n)/O(n)$, and it has a natural base point. It is also immediate that this space can be identified with the set of Lagrangian subspaces (of dimension n) of $\mathbb{R}^{2n}$: it suffices to associate with the class of the matrix $R+i\,S \in U(n)$ the linear subspace of $\mathbb{R}^n \oplus \mathbb{R}^n \sim \mathbb{C}^n$ which is the image of $\mathbb{R}^n$ under $R+i\,S$, since the canonical symplectic form on $\mathbb{R}^n \oplus \mathbb{R}^n$ vanishes when evaluated on pairs $(Rx \oplus Sx,\ Ry \oplus Sy)$ in view of the relation $-S'\,R + R'\,S = 0$. The map $Z \mapsto \det Z$ from Σ onto S^1 is a fibration [20, p. 31], the fiber of which can be identified with the homogeneous space $SU^{\pm}(n)/O(n) \sim SU(n)/SO(n)$, a simply connected space, so that (*loc. cit.*, p. 91) the map $\det$ induces an isomorphism from the fundamental group $\pi_1(\Sigma)$ onto $\pi_1(S^1) \sim \mathbb{Z}$.

The space $U(n)/SO(n)$, a twofold covering of Σ, can be identified with the space of oriented Lagrangian subspaces of $\mathbb{R}^{2n}$. Giving a point in this space above some point $Z \in \Sigma$ is tantamount to choosing one of the two square roots of $\det Z$. We shall need, mostly, the n-fold covering $\Sigma^{(n)}$ of Σ, which is the space suitable for a definition of $(\det Z)^{\frac{1}{n}}$.

Definition 4.8. Noting that Σ, as a smooth submanifold of $U(n)$, is an analytic space, we shall say that a dense open subset of Σ, or of any of the finite covering spaces of Σ, is connected in the strong sense if its complementary is contained in some analytic subset of codimension ≥ 2. An *admissible* function on $\Sigma^{(n)}$ shall be any partly defined analytic function on $\Sigma^{(n)}$, with a domain of definition connected in the strong sense.

Remark 4.1. Our demands regarding the domain of definition of F may look a little technical: they are made so as to ensure that any finite set of admissible functions should have a common *connected* dense domain.

The group $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n) \approx U(n)$ acts on $\Sigma^{(n)}$ if $n=1$ or 2, but for no other value: for, under the action made explicit in Proposition 4.7, $\det Z$ transforms to $(\det(R+i\,S))^{-2}\det Z$, which, given $R+i\,S \in U(n)$, only makes it possible to define, in a canonical way, the square root of the determinant of $(R-i\,S)\,Z\,(R+i\,S)^{-1}$ in terms of that of $\det Z$. On the other hand, when $n=1$, one might use instead of $U(n)$ the quotient of this group by $\{\pm I\}$. To get at the nth root of the determinant, we must substitute for $U(n)$, minimally, the n-fold covering of the group $U(n)/\{\pm I\}$: however, representation-theoretic reasons will impose later that we use instead the n-fold covering $U^{(n)}(n)$ of $U(n)$. A point in $U^{(n)}(n)$ is characterized by a point $R+i\,S$ in $U(n)$ together with an nth root of $\det(R+i\,S)$. The map $R+i\,S \mapsto \left(\begin{smallmatrix} R & S \\ -S & R\end{smallmatrix}\right)$ extends as an isomorphism from $U^{(n)}(n)$ onto the n-fold covering $\mathrm{Sp}^{(n)}_{\mathrm{comp}}(n,\mathbb{R})$ of $\mathrm{Sp}_{\mathrm{comp}}(n,\mathbb{R}) = \mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$. Finally, this covering is a maximal compact subgroup of the n-fold covering $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ of $\mathrm{Sp}(n,\mathbb{R})$ (recall that the twofold covering of $\mathrm{Sp}(n,\mathbb{R})$ is the *metaplectic group*). To make matters perfectly clear, as will be needed in the next section, if a point of $\mathrm{Sp}^{(n)}_{\mathrm{comp}}(n,\mathbb{R})$ is characterized by a member of $\mathrm{Sp}_{\mathrm{comp}}(n,\mathbb{R})$ together with the value $(\det(R-i\,S))^{\frac{1}{n}}$ of some nth root of $\det(R-i\,S)$), it should act on $\Sigma^{(n)}$ as the map $Z^{\#} \mapsto Z_1^{\#}$ inducing on Σ the map $Z \mapsto Z_1 = (R-i\,S)\,Z\,(R+i\,S)^{-1}$ and completely characterized by the equation $(\det Z_1^{\#})^{\frac{1}{n}} = (\det(R-i\,S))^{\frac{2}{n}}\,(\det Z^{\#})^{\frac{1}{n}}$.

As a homogeneous space of $U(n)$, Σ has an invariant measure $d\mu(Z)$, unique up to normalization: it will be convenient to characterize its restriction to the dense subset Σ^{reg} in terms of the inverse Cayley transform $\sigma = i\,(I+Z)\,(I-Z)^{-1}$ of Z. As can be found in [16, p. 33], the jacobian of the transformation $\sigma \mapsto \tau = (C+D\,\sigma)\,(A+B\,\sigma)^{-1}$, where defined, is $(\det(A+B\,\sigma))^{-n-1}$: thus, specializing to the case when the symplectic matrix under consideration is the matrix Ω that occurs in Proposition 4.7 and noting that, in that case,

$$\det(\tau \pm i\,I) = (\det(R+S\,\sigma))^{-1}\det(R \pm i\,S)\det(\sigma \pm i\,I), \tag{4.29}$$

one sees that one can take, as a $U(n)$-invariant measure on Σ,

$$d\mu(Z) = (\det(I+\sigma^2))^{-\frac{n+1}{2}}\,dm(\sigma), \tag{4.30}$$

where dm is a Lebesgue measure on the linear space Sym_n.

The action of the full symplectic group on Σ is not as simple as that of the maximal compact subgroup considered in Proposition 4.7: however, it is still non-singular, as shown by the following lemma together with Lemma 4.6 and the observation that if $g = \left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$, $g.\sigma = D\,\sigma\, D' + C\, D'$.

Lemma 4.9. *Given $A \in GL(n, \mathbb{R})$, the function f on Σ which transfers and extends the transformation $\sigma \mapsto A'\,\sigma\, A$ of Sym_n is analytic. Given $C \in \mathrm{Sym}_n$, the function h on Σ which transfers and extends the transformation $\sigma \mapsto \sigma - C$ is analytic as well. Both analytic homeomorphisms keep the matrix $Z = I$ fixed. The direct images of the invariant measure $d\mu$ on Σ under the diffeomorphisms f and h are given by the equations*

$$\frac{f_*\, d\mu}{d\mu} = |\det A|^{n+1} \left(\frac{|\det (I - f(Z))|}{|\det (I - Z)|} \right)^{n+1},$$

$$\frac{h_*\, d\mu}{d\mu} = \left(\frac{|\det (I - h(Z))|}{|\det (I - Z)|} \right)^{n+1}. \tag{4.31}$$

Proof. Set $P = (AA')^{-1}$, a positive-definite symmetric matrix, which shows (use the spectral decomposition of P) that the matrix $I + \frac{I-P}{I+P}\, Z$ is invertible if Z is unitary: note that we allow ourselves the fractional notation when dealing with commuting matrices. One has

$$\left(\frac{I-P}{I+P} + Z \right) \left(I + \frac{I-P}{I+P}\, Z \right)^{-1}$$
$$= (I+P)^{-1}\, [I - P + (I+P)\, Z]\, [I + P + (I-P)\, Z]^{-1}\, (I+P), \tag{4.32}$$

hence, since $I + P = A'^{-1}\, (A' + A^{-1})$,

$$(A' + A^{-1}) \left(\frac{I-P}{I+P} + Z \right) \left(I + \frac{I-P}{I+P}\, Z \right)^{-1} (A' + A^{-1})^{-1}$$
$$= A'\, [I + Z - P\,(I - Z)]\, [I + Z + P\,(I-Z)]^{-1}\, A'^{-1}$$
$$= [A'\,(I+Z) - A^{-1}\,(I-Z)]\, [A'\,(I+Z) + A^{-1}\,(I-Z)]^{-1} : \tag{4.33}$$

when $\det (I - Z) \neq 0$, *i.e.*, when Z lies in the image Σ^{reg} of the Cayley map, this can also be written as

$$[A'\,(I+Z)(I-Z)^{-1}\, A - I]\, [A'\,(I+Z)(I-Z)^{-1}\, A + I]^{-1}, \tag{4.34}$$

which is just the definition, according to Proposition 4.7, of the matrix $f(Z)$ in the case : however, the new formula

$$f(Z) = (A' + A^{-1}) \left(\frac{I-P}{I+P} + Z \right) \left(I + \frac{I-P}{I+P}\, Z \right)^{-1} (A' + A^{-1})^{-1} \tag{4.35}$$

defines $f(Z)$ as a non-singular function of Z on the whole of Σ. Still, we must show that the right-hand side of (4.35) lies in Σ for every $Z \in \Sigma$: however, the tiresome verification can be dispensed with, remembering that Σ^{reg} is a dense subset of Σ.

In just the same way, given $C \in \mathrm{Sym}_n$, the matrix $I + \frac{C}{2i\,I-C}\,Z$, is invertible since $\|\frac{C}{2i\,I-C}\| < 1$ (again, use the spectral decomposition of C), and we consider the matrix

$$(2i\,I + C)\left[-\frac{C}{2i\,I+C} + Z\right]\left[I + \frac{C}{2i\,I-C}\,Z\right]^{-1}(2i\,I - C)^{-1}, \tag{4.36}$$

which depends analytically on $Z \in \Sigma$. In the case when $Z \in \Sigma^{\mathrm{reg}}$, it can also be written as

$$\begin{aligned}
&[-C + (2i\,I + C)\,Z\,]\;[2i\,I - C + C\,Z\,]^{-1} \\
&\quad = [i\,(I+Z) - (C + i\,I)\,(I-Z)\,]\;[i\,(I+Z) + (-C + i\,I)\,(I-Z)\,]^{-1} \\
&\quad = \left(i\,\frac{I+Z}{I-Z} - C - i\,I\right)\left(i\,\frac{I+Z}{I-Z} - C + i\,I\right)^{-1},
\end{aligned} \tag{4.37}$$

which is just the definition of $h(Z)$ in this case.

That f and h keep the unit matrix in Σ fixed is seen by direct inspection of the equations (4.35) and (4.36).

The computation of the Radon–Nikodym derivatives of the measures $f_*\,d\mu$ and $h_*\,d\mu$ with respect to $d\mu$ is an immediate consequence of (4.30) together with the equation $dm\,(A'\,\sigma\,A) = |\det A|^{n+1}\,dm(\sigma)$ and the observation that, if Z is the Cayley transform of σ, one has $\det(I+\sigma^2) = |\det \frac{1}{2}\,(I-Z)\,|^{-2}$. □

Corollary 4.10. *There is a unique homomorphism χ from $\mathrm{Sp}\,(n,\mathbb{R})$ into the group of analytic automorphisms of Σ which extends the homomorphism from $\mathrm{Sp}\,(n,\mathbb{R}) \cap O(2n)$ into the latter group provided by Proposition 4.7, and reduces to the transformation f or h when dealing with one of the two special elements of $\mathrm{Sp}\,(n,\mathbb{R})$ considered in Lemma 4.9: this action of $\mathrm{Sp}\,(n,\mathbb{R})$ on Σ extends the transfer, under the Cayley map, of the partially defined action $(g,\sigma) \mapsto g.\sigma = (C + D\,\sigma)\,(A + B\,\sigma)^{-1}$ of the symplectic group on Sym_n. There is a unique continuous homomorphism $\tilde\chi$ from the group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ into the group of analytic automorphisms of $\Sigma^{(n)}$ with the following property: given $\tilde g \in \mathrm{Sp}^{(n)}(n,\mathbb{R})$ above some point $g \in \mathrm{Sp}\,(n,\mathbb{R})$, the analytic automorphism $\tilde\chi(\tilde g)$ of $\Sigma^{(n)}$ lies above the analytic automorphism $\chi(g)$ of Σ.*

Proof. The part concerning χ follows from Corollary 4.5, Lemma 4.6, Proposition 4.7 and Lemma 4.9. We have already shown, in the remark that followed Definition 4.8, how to let the canonical maximal compact subgroup of the metaplectic group operate on $\Sigma^{(n)}$. Extending this action to an action of the full group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ only requires that one should lift to an action on $\Sigma^{(n)}$ the action on Σ of symplectic matrices of the kind $\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right)$ with $A = A' > 0$, or of the kind $\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$ with

$C = C'$: this is possible in a unique way since each of these two sets of matrices is homeomorphic to a linear space, namely Sym_n. □

We now rephrase Corollary 4.3 in a way suitable to our needs, combining it with the use of the Cayley map. First, we note that if $Z \in \Sigma^{\mathrm{reg}}$ is such that $\det(I+Z) \neq 0$, the eigenvalues μ_j of the unitary matrix Z can be uniquely written as $\mu_j = e^{-i\theta_j}$ with $-\pi < \theta_j < \pi$: since this leads to a determination of the argument of $\det Z$, the set of such matrices Z naturally imbeds into $\Sigma^{(n)}$. Setting $\tau = -i\,(I-Z)(I+Z)^{-1}$, so that Z should be the image under the Cayley map of the matrix τ^{-1}, one sees that the eigenvalues of the real symmetric matrix τ^{-1} are the numbers

$$i\,\frac{1+\mu_j}{1-\mu_j} = \operatorname{cotan}\frac{\theta_j}{2}, \tag{4.38}$$

so that the condition $\tau^{-1} - R\,I \succ 0$ which occurs in Corollary 4.3 can be written as $0 < \tan\frac{\theta_j}{2} < \frac{1}{R}$.

Theorem 4.11. *Let u be an entire function on $\mathbb{C}^n$ satisfying for some pair of constants (C,R) the estimate $|u(z)| \leq C\,e^{\pi R|z|^2}$. Let V^+ be the set of matrices $Z \in \Sigma^{\mathrm{reg}}$ the eigenvalues of which can be written as $\mu_j = e^{-i\theta_j}$ with $0 < \theta_j < \pi$ and $0 < \tan\frac{\theta_j}{2} < \frac{1}{R}$: when $Z \in V^+$, the matrix $\tau \in \mathrm{Sym}_n$ defined as $\tau = -i\,(I-Z)(I+Z)^{-1}$ is invertible and satisfies the condition $\tau^{-1} - R\,I \succ 0$, which makes it possible to define*

$$(\mathcal{K}\,u)_0(Z) = |\det(I-Z)|^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_0(i\,(I+Z)(I-Z)^{-1})$$

and the vector-valued function

$$(\mathcal{K}\,u)_1(Z) = |\det(I-Z)|^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_1(i\,(I+Z)(I-Z)^{-1}). \tag{4.39}$$

The functions $(\mathcal{K}\,u)_0$ and $(\mathcal{K}\,u)_1$, hereafter referred to as the $\mathcal{K}$-transforms of u, extend as analytic functions to the neighborhood V of I in Σ consisting of all matrices Z whose eigenvalues can be written as $e^{-i\theta_j}$ with $-\pi < \theta_j < \pi$ and $|\tan\frac{\theta_j}{2}| < \frac{2\pi}{nR}$.

Proof. In terms of the eigenvalues of Z as denoted above, those of the real symmetric matrix $\tau = -i\,(I-Z)(I+Z)^{-1}$ are the numbers $\tan\frac{\theta_j}{2}$: thus, in the case when $\tau^{-1} - R\,I \succ 0$, one has

$$\begin{aligned}(\mathcal{K}\,u)_0(Z) &= \left|\det\frac{I+i\,\tau}{2}\right|^{\frac{1}{2}}(\mathcal{Q}^{\mathrm{inv}}\,u)_0(\tau)\\ &= 2^{-\frac{n}{2}}\,(\det(I+\tau^2))^{\frac{1}{4}}\,(\mathcal{Q}^{\mathrm{inv}}\,u)_0(\tau),\end{aligned} \tag{4.40}$$

so that it follows from Corollary 4.3 and the fact that the map $Z \mapsto \tau$ is analytic in V that $(\mathcal{K}\,u)_0(Z)$ extends as an analytic function of Z in the mentioned neighborhood of the identity matrix: note here that all entries of a positive real symmetric matrix are majorized by its greatest eigenvalue. The same works of course with the other $\mathcal{Q}$-transform of u. We shall also use the inversion formula

$$(\mathcal{Q}\,u)_0(\sigma) = 2^{\frac{n}{2}}\,(\det{(I+\sigma^2)})^{-\frac{1}{4}}\,(\mathcal{K}\,u)_0((\sigma - i\,I)(\sigma + i\,I)^{-1}), \tag{4.41}$$

valid when $Z = (\sigma - i\,I)(\sigma + i\,I)^{-1}$ lies in V^+. □

Remark 4.2. It is important to realize that, under the Cayley map, the image Z of a matrix σ will go to the identity matrix in Σ if and only if each eigenvalue of σ goes to $\pm\infty$: the theorem that precedes thus goes much further than the Definition 4.1 of the $\mathcal{Q}$-transforms of u, originally given under the assumption that all eigenvalues of σ (here, $\sigma = \tau^{-1}$) are close to $+\infty$. Note that the function $(\mathcal{Q}\,u)_0$ (*resp.* $(\mathcal{K}\,u)_0$) lives on a part of Sym_n (*resp.* Σ): the domain of the second one is small enough, so that one may regard it as well, as explained just before the statement of Theorem 4.11, as defined in some neighborhood of the base point of $\Sigma^{(n)}$.

Definition 4.12. Let u be an entire function on $\mathbb{C}^n$, satisfying for some pair (C, R_1) of positive numbers the estimate $|u(z)| \leq C\,e^{\pi R_1\,|z|^2}$ for every $z \in \mathbb{C}^n$. Let $(\mathcal{Q}\,u)_0(\sigma)$ and $(\mathcal{Q}\,u)_1(\sigma)$ be its $\mathcal{Q}$-transforms, as defined in (4.1) and (4.2) under the assumption that $\sigma \in \mathrm{Sym}_n$ is such that $\sigma - R_1\,I \succ 0$. Let $(\mathcal{K}\,u)_0(Z)$ and $(\mathcal{K}\,u)_1(Z)$ be the functions of $Z \in \Sigma$ defined, under the assumption that the matrix Z is close enough to the identity matrix, in Theorem 4.11.

We shall say that the function u lies in the space $\mathfrak{A}^{(n)}$ if the following condition is realized: the functions $(\mathcal{K}\,u)_0$ and $(\mathcal{K}\,u)_1$ initially defined in some neighborhood of the base point of Σ, also regarded, as explained before Theorem 4.11, as defined in a neighborhood of the base point of $\Sigma^{(n)}$, extend as admissible functions on the space $\Sigma^{(n)}$: recall from Definition 4.8 that this means that they can be given a dense and open domain of definition, connected in the strong sense, in which they are to be analytic functions.

If $u \in \mathfrak{A}^{(n)}$ and if the domain of definition of $(\mathcal{K}\,u)_0$ contains the point $e^{i\pi}\,I = e^{-i\pi}\,I$ (*cf.* Definition 4.8), we set

$$\mathrm{Int}\,[u] = 2^{\frac{n}{2}}\,(\mathcal{K}\,u)_0(e^{-i\pi}\,I). \tag{4.42}$$

Remark 4.3. The image, under the Cayley map, of the origin of Sym_n, is the point $-I = e^{i\pi}\,I \in \Sigma$: the equation (4.42) may thus be considered as a version of what might seem to be the more natural equation $\mathrm{Int}\,[u] = (\mathcal{Q}\,u)_0(0)$; however, the Cayley map sends Sym_n into Σ, not $\Sigma^{(n)}$ (unless $n = 1$) and it may not be possible in general to use (4.41) to find a natural analytic extension of $(\mathcal{Q}\,u)_0$ to some open dense subset of Sym_n. For instance, in the two-dimensional case, Theorem 4.19 will give the example of an important function u whose $\mathcal{K}$-transform (defined on the twofold cover $\Sigma^{(2)}$ of Σ) is regular at the point $\begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix}$ but singular at the point $\begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix}$.

Theorem 4.13. *Let* $u \in \mathfrak{A}^{(n)}$. *For every* $A \in GL(n, \mathbb{R})$, *the function* v *on* $\mathbb{C}^n$ *defined as* $v(x) = |\det A|^{-\frac{1}{2}} u(A^{-1}x)$ *lies in* $\mathfrak{A}^{(n)}$ *too. For every* $C \in \mathrm{Sym}_n$, *the function* w *on* $\mathbb{C}^n$ *defined as* $w(x) = u(x)\, e^{i\pi \langle Cx, x\rangle}$ *lies in* $\mathfrak{A}^{(n)}$ *as well.*

Proof. It is immediate, from Definition 4.1, that if $\sigma - RI \succ 0$, one has

$$(\mathcal{Q}\, v)_0(\sigma) = |\det A|^{\frac{1}{2}}\, (\mathcal{Q}\, u)_0(A'\, \sigma\, A),$$
$$(\mathcal{Q}\, v)_1(\sigma) = |\det A|^{\frac{1}{2}}\, (I + i\, \sigma)\, (A^{-1} + i\, A'\, \sigma)^{-1}\, (\mathcal{Q}\, u)_1(A'\, \sigma\, A). \tag{4.43}$$

Let f be the analytic homeomorphism of Σ defined in Lemma 4.9, which is the transfer, under the Cayley map, of the automorphism $\sigma \mapsto A'\, \sigma\, A$ of Sym_n. Using Theorem 4.11, one sees that, whenever Z is the image, under the Cayley map, of some $\sigma \in \mathrm{Sym}_n$ with $\sigma - RI \succ 0$ and R large enough, so that one should have both $\sigma - RI \succ 0$ and $A'\, \sigma\, A - RI \succ 0$, one has

$$(\mathcal{K}\, v)_0(Z) = |\det A|^{\frac{1}{2}} \left| \frac{\det\,(I - f(Z))}{\det\,(I - Z)} \right|^{\frac{1}{2}} (\mathcal{K}\, u)_0(f(Z)). \tag{4.44}$$

Note, using (4.31), that this can also be written as

$$(\mathcal{K}\, v)_0(Z) = \left(\frac{f_*\, d\mu}{d\mu}(Z) \right)^{\frac{1}{2(n+1)}} (\mathcal{K}\, u)_0(f(Z)) : \tag{4.45}$$

in particular, the extra factor extends as an analytic non-zero function on Σ. For safety, one may also check this last fact from the equation (a consequence of (4.35) with the notation of the proof of Lemma 4.9)

$$I - f(Z) = (A' + A^{-1}) \left[I - (\frac{I-P}{I+P} + Z)\, (I + \frac{I-P}{I+P}\, Z)^{-1} \right] (A' + A^{-1})^{-1}, \tag{4.46}$$

which yields, after a short calculation,

$$|\det A|^{\frac{1}{2}} \left| \frac{\det\,(I - f(Z))}{\det\,(I - Z)} \right|^{\frac{1}{2}} = \left| \det \frac{A' + A^{-1}}{2} \right|^{-\frac{1}{2}} \times \left| \det\,(I + \frac{I-P}{I+P}\, Z) \right|^{-\frac{1}{2}}. \tag{4.47}$$

How to make f a bijection of $\Sigma^{(n)}$ (rather than Σ) onto itself was indicated at the end of the proof of Corollary 4.10. The same kind of computation works with the function $(\mathcal{K}\, v)_1$, only putting in front of the right-hand side of the equation playing the same role as (4.44) the extra linear operator $(I + i\, \sigma)\, (A^{-1} + i\, A'\, \sigma)^{-1} = 2\, (I + \frac{I-P}{I+P}\, Z^{-1})^{-1}\, (A^{-1} + A')^{-1}$ corresponding to the operator in front of the right-hand side of the second equation (4.43): after a straightforward computation, this can also be written as

$$\frac{1}{2} \left[A + A'^{-1} - (A - A'^{-1})\, (f(Z))^{-1} \right], \tag{4.48}$$

an expression which will be useful later.

Concerning the function w, we find that

$$(\mathcal{Q}\,w)_0(\sigma) = (\mathcal{Q}\,u)_0(\sigma - C),$$
$$(\mathcal{Q}\,w)_1(\sigma) = (I + i\,\sigma)\,(I + i\,(\sigma - C))^{-1}\,(\mathcal{Q}\,u)_1(\sigma - C), \tag{4.49}$$

so that, with h as defined in Lemma 4.9, and appealing to (4.31) again,

$$\begin{aligned}(\mathcal{K}\,w)_0(Z) &= \left| \frac{\det(I - h(Z))}{\det(I - Z)} \right|^{\frac{1}{2}} (\mathcal{K}\,u)_0(h(Z)) \\ &= \left(\frac{h_*\,d\mu}{d\mu}(Z) \right)^{\frac{1}{2(n+1)}} (\mathcal{K}\,u)_0(h(Z)).\end{aligned} \tag{4.50}$$

One may also check that

$$(I - Z)^{-1}\,(I - h(Z)) = \left(I + \frac{i\,C}{2}\,(I - Z) \right)^{-1}\;; \tag{4.51}$$

in the equation for $(\mathcal{K}\,w)_1(Z)$, we need to put in the extra linear operator

$$\left(I - \frac{i\,C}{2} + \frac{i\,C}{2}\,Z^{-1} \right)^{-1} = I + \frac{i\,C}{2} - \frac{i\,C}{2}\,(h(Z))^{-1} \tag{4.52}$$

in front of the right-hand side, as it follows from the second equation (4.49).

We note for future reference the following equation: if the domain of the $\mathcal{K}$-transforms of $u \in \mathfrak{A}^{(n)}$ contains the point $e^{i\pi}\,I \in \Sigma^{(n)}$, one has

$$\mathrm{Int}\,[x \mapsto u(x)\,e^{i\pi\,\langle Cx,x\rangle}] = 2^{\frac{n}{2}}\,(\det(I + C^2))^{-\frac{1}{4}}\,(\mathcal{K}\,u)_0\left(e^{i\pi}\,\frac{I - i\,C}{I + i\,C} \right) : \tag{4.53}$$

before proving it, we note that the eigenvalues of the matrix $\frac{I-i\,C}{I+i\,C}$ cannot be real numbers ≤ 0, so that they have arguments in $]-\pi,\pi[$; as explained just before Theorem 4.11, this matrix can thus be viewed as an element of $\Sigma^{(n)}$ rather than Σ, which gives the matrix $e^{i\pi}\,\frac{I-i\,C}{I+i\,C}$ a meaning as an element of the same covering space. In order to prove (4.53), we write in the case when Z is the image under the Cayley map of some matrix $\sigma \in \mathrm{Sym}_n$ such that $\sigma \succ R\,I$ with R large, using (4.50), (4.51) and (4.37),

$$\begin{aligned}&(\mathcal{K}\,(x \mapsto u(x)\,e^{i\pi\,\langle Cx,x\rangle}))_0(Z) \\ &\quad = \left[\det\left(I + \frac{i\,C}{2}\,(I - Z) \right) \right]^{-\frac{1}{4}} \left[\det\left(I - \frac{i\,C}{2}\,(I - \bar{Z}) \right) \right]^{-\frac{1}{4}} \\ &\qquad \times (\mathcal{K}\,u)_0\,((-C + (2i\,I + C)\,Z)\,(2i\,I - C + C\,Z)^{-1}) :\end{aligned} \tag{4.54}$$

this leads to (4.53) after one has noted that the question of determination of the fourth roots that occurs in the last expression can be dealt with by means of a homotopy argument, since C connects to the zero matrix in Sym_n. □

Theorem 4.14. *In the one-dimensional case, the space* $\mathfrak{A}^{(1)}$ *coincides with the space* $\mathfrak{A}$ *introduced in Section* 2.

Proof. Assume that $u \in \mathfrak{A}$: in terms of the pair of functions (w_0, w_1) used in Theorem 1.8, one has

$$(\mathcal{Q}\, u)_0(\sigma) = w_0(\sigma), \qquad (\mathcal{Q}\, u)_1(\sigma) = (1+i)\,(1+i\,\sigma)\,w_1(\sigma). \tag{4.55}$$

Under the relation $z = \frac{\sigma - i}{\sigma + i}$ the points of $\mathbb{R}$ with σ large correspond to the points $z = e^{-i\theta}$ of $\Sigma = S^1$ with $\theta > 0$ small: then, $\sigma = \text{cotan}\,\frac{\theta}{2}$. Under this condition, the expansion at infinity of w_0 provided by the above-mentioned theorem yields

$$(\mathcal{K}\, u)_0(e^{-i\theta}) = \left(2\,\cos\,\frac{\theta}{2}\right)^{-\frac{1}{2}} \sum_{n \geq 0} a_n \left(\tan\,\frac{\theta}{2}\right)^n. \tag{4.56}$$

Next, under the same conditions, (4.41) gives

$$(\mathcal{Q}\, u)_0(\sigma) = \left(2\,\sin\,\frac{\theta}{2}\right)^{\frac{1}{2}} (\mathcal{K}\, u)_0(e^{-i\theta}). \tag{4.57}$$

Using (4.56) and (4.57), one first sees that since the function $(\mathcal{Q}\, u)_0$ is analytic on the real line, the function $(\mathcal{K}\, u)_0$ extends as an analytic function to the neighborhood of the base point of the universal cover of Σ characterized by the inequalities $-\varepsilon < \theta < 2\pi$ for some $\varepsilon > 0$ (use the first of the two equations mentioned when θ is close to 0, the second one in $]0, 2\pi[$).

Assuming again that σ is large, we observe that when the real number r goes from 1 to -1, the point $r\sigma$ moves from σ to $-\sigma$ in Sym_n. In the process, the Cayley transform z of $r\sigma$ moves along the path $r \mapsto \frac{r\,(1+e^{-i\theta})-(1-e^{-i\theta})}{r\,(1+e^{-i\theta})+(1-e^{-i\theta})} = \frac{r - i\,\tan\,\frac{\theta}{2}}{r + i\,\tan\,\frac{\theta}{2}}$, so that the argument of z undergoes a change from the value $-\theta$ to the value $-(2\pi - \theta)$. Thus, one has, starting from (4.41),

$$(\mathcal{Q}\, u)_0(-\sigma) = \left(2\,\sin\,\frac{\theta}{2}\right)^{\frac{1}{2}} (\mathcal{K}\, u)_0(e^{i\,(\theta - 2\pi)}), \tag{4.58}$$

it being understood that, in the right-hand side of this equation, it is now the above-mentioned analytic extension of the function $(\mathcal{K}\, u)_0$ that is now being used. The same formula links the functions $(\mathcal{Q}\, u)_1$ and $(\mathcal{K}\, u)_1$.

On the other hand, still for large σ, (1.27) yields

$$\begin{aligned} \tilde{w}_0(\sigma\, e^{i\pi}) = -i\,\sigma^{-\frac{1}{2}} \sum_{n \geq 0} (-1)^n\, a_n\, \sigma^{-n} &= -i \left(\tan\,\frac{\theta}{2}\right)^{\frac{1}{2}} \sum_{n \geq 0} (-1)^n\, a_n \left(\tan\,\frac{\theta}{2}\right)^n \\ &= -i \left(2\,\sin\,\frac{\theta}{2}\right)^{\frac{1}{2}} (\mathcal{K}\, u)_0(e^{i\theta}) \end{aligned} \tag{4.59}$$

so that the condition (1.28) from Theorem 1.8 is equivalent to

$$(\mathcal{K}\, u)_0(e^{i\,(\theta - 2\pi)}) = (\mathcal{K}\, u)_0(e^{i\theta}). \tag{4.60}$$

It shows that the function $(\mathcal{K}\, u)_0$ satisfies the demands made in Definition 4.12, as it extends in this case as an analytic function on $\Sigma = S^1$. The same computations are valid if interested instead in the function $(\mathcal{K}\, u)_1$.

The argument that precedes can be reversed, proving the identity of the spaces $\mathfrak{A}^{(1)}$ and $\mathfrak{A}$.

In the particular case when $w_0(\sigma) = (1+\sigma^2)^{-\frac{1}{4}}$, which corresponds, up to normalization, to the case when u is the median state of the anaplectic harmonic oscillator, one may note that the function $2^{\frac{1}{2}}\,(\mathcal{K}\, u)_0$ reduces to the constant 1: nothing so simple will, or can occur in the higher-dimensional case. Finally, our present definition of the anaplectic "integral" coincides, in the one-dimensional case, with that of Section 1. The argument goes as follows. From (4.42), one has $\mathrm{Int}\,[u] = 2^{\frac{1}{2}}\,(\mathcal{K}\, u)_0(e^{\pm i\pi})$: from (4.57) and (4.55), this is the same as $w_0(0)$ the value at $\sigma = 0$ of the analytic extension of w_0 on the real line. To find this value, we appeal to the equation (1.33), which provides an integral representation on the whole real line of w_0: setting $\sigma = 0$ in the right-hand side of this equation, we find $\mathrm{Int}\,[u]$ as defined in (1.74). □

Theorem 4.15. *For general n, any Gaussian function $x \mapsto e^{-\pi\, Q(x)}$, where $Q(x)$ is a quadratic function of x with a positive-definite real part, lies in $\mathfrak{A}^{(n)}$ if and only if the dimension is divisible by 4.*

Proof. Take a Gaussian function of the species indicated in the statement of the theorem: in order to prove that it lies in $\mathfrak{A}^{(n)}$ there is no loss of generality, as a consequence of Theorem 4.13, in assuming that $u(x) = e^{-\pi\,|x|^2}$. One has, if $\sigma \succ 0$, $(\mathcal{Q}\, u)_0(\sigma) = (\det(\sigma - i\,I))^{-\frac{1}{2}}$, and the other quadratic transform of u is zero. Here,

$$(\det(\sigma - i\,I))^{-\frac{1}{2}} = \prod_j (\sigma_j - i)^{-\frac{1}{2}} = e^{\frac{i\pi n}{4}} \prod_j (1 + i\,\sigma_j)^{-\frac{1}{2}}, \tag{4.61}$$

where, as a constant policy, square roots of complex numbers in the plane cut along the negative half-line always have positive real parts: the function on the right-hand side is analytic in the whole linear space Sym_n. Applying the definition of $(\mathcal{Q}^{\mathrm{inv}}\, u)_0$ in Corollary 4.3, we find

$$(\mathcal{Q}^{\mathrm{inv}}\, u)_0(\tau) = (\det \tau)^{-\frac{1}{2}}\, (\mathcal{Q}\, u)_0(\tau^{-1}) = \prod_j (1 - i\,\tau_j)^{-\frac{1}{2}}. \tag{4.62}$$

If the eigenvalues $\mu_j = e^{-i\theta_j}$ of some matrix $Z \in \Sigma$ satisfy $0 < \theta_j < \pi$ one has, using (4.38) and (4.41),

$$(\mathcal{K}\, u)_0(Z) = \prod_j (2\,\cos\frac{\theta_j}{2})^{-\frac{1}{2}} \prod_j \left(1 - i\,\tan\frac{\theta_j}{2}\right)^{-\frac{1}{2}} = 2^{-\frac{n}{2}}\,(\det Z)^{-\frac{1}{4}}, \tag{4.63}$$

a function which extends analytically to the n-tuple cover $\Sigma^{(n)}$ of Σ if and only if n is divisible by 4. One should note that, if such is the case, whether one is dealing with the standard n-dimensional Gaussian function or a more general one, the function $(\mathcal{K}\,u)_0$ is always globally defined on the whole of $\Sigma^{(n)}$, as a consequence of Lemma 4.9. □

Remark 4.4. It follows from Remark 3.1 that if $n \equiv 0 \mod 4$, $e^{-\pi|x|^2}$ is even an anaplectic Hermite function; since $\frac{n-2}{4} \in -\frac{1}{2} + \mathbb{Z}$, the function Φ recalled in Theorem 4.18 to follow is in this case an elementary (not only a Bessel) function [17, p. 72]. Instead of a positive-definite quadratic form Q in the exponent, we might as well have taken a negative-definite one: in the next section, it will be shown that, quite generally, the space $\mathfrak{A}^{(n)}$ is invariant under the change $x \mapsto ix$.

In order to study our next, and main, example, we need a pair of lemmas.

Lemma 4.16. *Assume that the dimension n is ≥ 2. The subset Δ of Σ consisting of matrices Z at least one eigenvalue of which is not simple is an analytic subset of codimension 2.*

Proof. We first make a few observations concerning the structure of eigenspaces of matrices $Z \in \Sigma$, *i.e.,* matrices which are both unitary and symmetric. The complex conjugate of such a matrix is also its inverse, so that if $\xi \in \mathbb{C}^n$ is an eigenvector of Z corresponding to the eigenvalue λ, one has $\bar{\lambda} = \lambda^{-1}$ and $Z^{-1}\,\bar{\xi} = \lambda^{-1}\,\bar{\xi}$ as well as $Z\,\xi = \lambda\,\xi$: it follows that one has the two equations $Z\,(\mathrm{Re}\;\xi) = \lambda\,\mathrm{Re}\;\xi$ and $Z\,(\mathrm{Im}\;\xi) = \lambda\,\mathrm{Im}\;\xi$. Next, assuming that the eigenvalue λ of Z has multiplicity at least 2, one can find some complex two-dimensional linear subspace E of the eigenspace of Z corresponding to the given eigenvalue with a basis consisting of two vectors in $\mathbb{R}^n$. Indeed, if the vectors $\mathrm{Re}\;\xi$ and $\mathrm{Im}\;\xi$ considered so far are not proportional, there is nothing more to be done; if this is not the case, doing the same as before with a new eigenvector η of Z corresponding to the same eigenvalue and $\mathbb{C}$-independent from ξ, we consider either the pair $(\mathrm{Re}\;\eta, \mathrm{Im}\;\eta)$ if these two vectors are independent, or a pair consisting of non-zero vectors, one taken from the pair $(\mathrm{Re}\;\xi, \mathrm{Im}\;\xi)$ and one from the pair $(\mathrm{Re}\;\eta, \mathrm{Im}\;\eta)$. Next, we observe that both E and the orthogonal subspace $E^{\perp}$ of $\mathbb{C}^n$ are stable under the action of Z and that each of them has a linear basis consisting of real vectors: in particular, considering the restrictions of Z to each of these two subspaces, one gets a unitary transformation (of E or $E^{\perp}$) which is again symmetric in the sense made possible by such a basis.

The set Δ is of course an algebraic subset of Σ (it can be defined by the vanishing of the discriminant of the characteristic polynomial of Z), and its dimension is one more than that of the set consisting, for any given $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, of all pairs (E, Y) with the following properties: E should be a two-dimensional subspace of $\mathbb{C}^n$ generated by a pair of vectors in $\mathbb{R}^n$, and Y should be a unitary and symmetric linear automorphism of $E^{\perp}$. Now, the space of two-planes E has dimension $2n-4$ and, for any given E, the set of unitary symmetric transformations of $E^{\perp}$ has the dimension of Sym_{n-2}, *i.e.,* $\frac{(n-2)(n-1)}{2}$, which gives Δ the dimension $\frac{n(n+1)}{2} - 2$.

One may note that, when $n = 2$, the set Δ reduces to the circle $\{Z = e^{i\phi} I,\ \phi \in \mathbb{R}\}$, while Sym_2 is three-dimensional. $\square$

Contrary to Σ, $\Sigma\backslash\Delta$ has a complicated fundamental group, the more so in higher dimensions. However, we shall take advantage of the following:

Lemma 4.17. *Let $Z^0 \in \Sigma^{\mathrm{reg}}$ be a matrix with only simple eigenvalues. The sequence*

$$\pi_1\,(\Sigma^{\mathrm{reg}} \cap (\Sigma\backslash\Delta),\, Z^0) \longrightarrow \pi_1(\Sigma\backslash\Delta,\, Z^0) \longrightarrow \pi_1(\Sigma,\, Z^0) \longrightarrow 0 \tag{4.64}$$

induced by the embeddings $\Sigma^{\mathrm{reg}} \cap (\Sigma\backslash\Delta) \to \Sigma\backslash\Delta \to \Sigma$ is exact.

Proof. The composition map : $\Sigma^{\mathrm{reg}} \cap (\Sigma\backslash\Delta) \to \Sigma$ can be factored through the space Σ^{reg}, homeomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}}$, so that the corresponding composition of homomorphisms is zero. Next, that the homomorphism in the middle of the sequence above is onto is a consequence of Lemma 4.16, since Δ cannot disconnect Σ either globally or locally. We shall prove the following, which is slightly more than what remains to be done.

Let $\beta_0 \colon t \mapsto Z(t)$, $0 \leq t \leq 1$, be a continuous loop in $\Sigma\backslash\Delta$, starting from a point $Z(0) \in \Sigma^{\mathrm{reg}}$: assume that the loop β_0 is homotopically trivial as a loop in Σ. Then, there exists a homotopy $(\beta_s)_{0\leq s\leq 1}$ of loops at $Z(0)$ in the space $\Sigma\backslash\Delta$, with the following two properties: for any pair $(s,\, t)$, the matrices $\beta_s(t)$ and $\beta_0(t)$ are proportional (by a scalar factor of modulus 1), and the image of β_1 is entirely contained in Σ^{reg}.

To start with, there exist uniquely defined numbers θ_j^0, $1 \leq j \leq n$, with $0 < \theta_1^0 < \cdots < \theta_n^0 < 2\pi$, such that the eigenvalues of $Z(0)$ are the numbers $\mu_j^0 = e^{-i\,\theta_j^0}$. Since the eigenvalues of $Z(t)$ are always distinct, they can be followed up as continuous functions $t \mapsto \mu_j(t)$, $0 \leq t \leq 1$ for $j = 1, 2, \ldots, n$; also, one can uniquely set $\mu_j(t) = e^{-i\,\theta_j(t)}$, θ_j being a continuous function : $[0,1] \to \mathbb{R}$ with $\theta_j(0) = \theta_j^0$. Of necessity, one has $\theta_1(t) < \cdots < \theta_n(t)$ for all values of t. Since $|\theta_j^0 - \theta_k^0| < 2\pi$ for every pair j, k and $\theta_j(t) - \theta_k(t) \notin 2\pi\,\mathbb{Z}$ for $j \neq k$, one has $|\theta_j(t) - \theta_k(t)| < 2\pi$ for every pair j, k and every $t \in [0, 1]$.

As $\theta_n(t) - 2\pi < \theta_1(t)$ for all t and $\theta_n^0 - 2\pi < 0 < \theta_1^0$, one can construct a continuous function $\omega \colon [0,1] \to \mathbb{R}$ with $\omega(0) = 0$ and $\theta_n(t) - 2\pi < \omega(t) < \theta_1(t)$ for all t. We now show that one may also demand that $\omega(1) = 0$, for which it suffices to prove that $\theta_n(1) = \theta_n^0$ and $\theta_1(1) = \theta_1^0$. Indeed, since β_0 is a loop, there is for every j an integer $m_j \in \mathbb{Z}$ such that $\theta_j(1) - \theta_j(0) = 2\pi\, m_j$. For any pair $(j,\, k)$, one has

$$\begin{aligned} 2\pi\,(m_j - m_k) &= (\theta_j(1) - \theta_j(0)) - (\theta_k(1) - \theta_k(0)) \\ &= (\theta_j(1) - \theta_k(1)) - (\theta_j(0) - \theta_k(0)), \end{aligned} \tag{4.65}$$

a number $< 2\pi$ in absolute value since both terms are and they have the same sign. Thus $m_j = m$, a constant. Now, $\det Z(t) = e^{-i\,\sum \theta_j(t)}$, a continuous determination of the logarithm of which is the function $t \mapsto -i \sum \theta_j(t)$. As the loop

$t \mapsto Z(t)$ is homotopically trivial in Σ, one has

$$0 = \sum \theta_j(1) - \sum \theta_j(0) = \sum (\theta_j(1) - \theta_j(0)) = 2\pi\, nm, \tag{4.66}$$

so that $m = 0$.

Then, for every j,

$$0 < \theta_1(t) - \omega(t) \leq \theta_j(t) - \omega(t) \leq \theta_n(t) - \omega(t) < 2\pi, \tag{4.67}$$

so that, setting $\beta_s(t) = e^{i\, s\omega(t)}\, Z(t)$, we are done. □

Theorem 4.18. *In any dimension, the function* $\Phi(x) = |x|^{\frac{2-n}{2}}\, I_{\frac{n-2}{4}}(\pi\,|x|^2)$ *introduced in* (3.22) *lies in* $\mathfrak{A}^{(n)}$. *The* $\mathcal{K}$*-transform of* Φ *is analytic on the part of* $\Sigma^{(n)}$ *lying above* $\Sigma \backslash \Delta$.

Proof. We shall assume $n \geq 2$, since the one-dimensional case has already been treated. From [17, p. 84], setting $C_n = \frac{2^{\frac{2-n}{4}}\, \pi^{\frac{n}{4}-1}}{\Gamma(\frac{n}{4})}$, one has

$$\Phi(x) = C_n \int_{-1}^{1} e^{-\pi |x|^2 t}\, (1 - t^2)^{\frac{n}{4}-1}\, dt. \tag{4.68}$$

Definition 4.1 implies that, for $\sigma \succ 0$, $(\mathcal{Q}\,\Phi)_0(\sigma)$ is well defined (while the odd $\mathcal{Q}$-transform of Φ is zero) and that

$$\begin{aligned}
(\mathcal{Q}\,\Phi)_0(\sigma) &= C_n \int_{-1}^{1} (1-t^2)^{\frac{n}{4}-1}\, dt \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\, e^{i\pi t\,|x|^2}\, dx \\
&= C_n \int_{-1}^{1} (1-t^2)^{\frac{n}{4}-1}\, [\det(\sigma - it)]^{-\frac{1}{2}}\, dt \\
&= C_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\frac{n-2}{2}} \omega\, \prod_j (\sigma_j - i\, \sin\omega)^{-\frac{1}{2}}\, d\omega
\end{aligned} \tag{4.69}$$

if $\{\sigma_1, \dots, \sigma_n\}$ is the set of (positive) eigenvalues of σ. Moving to the half-circle γ from $-i$ to i on the right of 0, one can write

$$(\mathcal{Q}\,\Phi)_0(\sigma) = \frac{2\, C_n}{i} \int_\gamma (1+z^2)^{\frac{n-2}{2}} \prod_j (1 + 2\sigma_j\, z - z^2)^{-\frac{1}{2}}\, dz : \tag{4.70}$$

it is understood (recalling that, for the time being, $\sigma_j > 0$ for every j) that, when $z \in \gamma$, the arguments of $1+z^2$ and $1+2\sigma_j\, z - z^2$ are to be taken in $]-\frac{\pi}{2}, \frac{\pi}{2}[$. After a change of contour, made possible since, under the present assumptions regarding σ, the zeros of $1 + 2\sigma_j\, z - z^2$ can only lie outside the strip $0 \leq \mathrm{Re}\, z \leq 1$, one may transform the hyperelliptic integral above to

$$(\mathcal{Q}\,\Phi)_0(\sigma) = 2\, C_n \int_{-1}^{1} (1-t^2)^{\frac{n-2}{2}} \prod_j (1 + 2i\,\sigma_j\, t + t^2)^{-\frac{1}{2}}\, dt. \tag{4.71}$$

This equation shows that the function $\sigma \mapsto (\mathcal{Q}\,\Phi)_0(\sigma)$, initially defined for $\sigma \in \Gamma_n$, extends as an analytic function on Sym_n, even as a holomorphic function on the open subset of $\mathrm{Sym}_n^{\mathbb{C}}$ consisting of matrices such that no eigenvalue σ_j of σ should lie in the union of the closed half-lines from i to $i\infty$ and from $-i$ to $-i\infty$. At the same time, the last equation shows that $(\mathcal{Q}\,\Phi)_0(-\sigma) = (\mathcal{Q}\,\Phi)_0(\sigma)$ for all σ.

If $Z = (\sigma - i\,I)(\sigma + i\,I)^{-1}$ is the image of $\sigma \in \mathrm{Sym}_n$ under the Cayley map, the definition of the function $(\mathcal{K}\,\Phi)_0$ given in Theorem 4.11 yields

$$\begin{aligned}(\mathcal{K}\,\Phi)_0(Z) &= |\det(I - Z)|^{-\frac{1}{2}}\,(\mathcal{Q}\,\Phi)_0(\sigma)\\ &= 2^{-\frac{n}{2}}\,(\det(I + \sigma^2))^{\frac{1}{4}}\,(\mathcal{Q}\,\Phi)_0(\sigma),\end{aligned} \tag{4.72}$$

and the equation (4.71) provides a definition of the function $(\mathcal{K}\,\Phi)_0$ as an analytic function on the image Σ^{reg} of the Cayley map.

Next, we show that the function $(\mathcal{K}\,\Phi)_0$ takes the same value at any pair of matrices Z and $e^{i\phi}\,Z$, assuming that both matrices lie in the open set Σ^{reg}. To do this, we go back to the expression in the middle of (4.69) of $(\mathcal{Q}\,\Phi)_0(\sigma)$ and transform it further, under the renewed assumption that $\sigma \in \Gamma_n$, setting $\sin\omega = \tanh\xi$ so that

$$(\mathcal{Q}\,\Phi)_0(\sigma) = C_n \int_{-\infty}^{\infty} \prod_j (\sigma_j\,\cosh\xi - i\,\sinh\xi)^{-\frac{1}{2}}\,d\xi\ : \tag{4.73}$$

since $\sigma_j = \mathrm{cotan}\,\frac{\theta_j}{2}$ if $\{e^{-i\theta_1}, \ldots, e^{-i\theta_n}\}$ is the set of eigenvalues of the Cayley image Z of σ, one can write, provided that $0 < \theta_j < \pi$ for all j,

$$\begin{aligned}(\mathcal{K}\,\Phi)_0(Z) &= 2^{-\frac{n}{2}} \prod_j (\sin\frac{\theta_j}{2})^{-\frac{1}{2}}\,C_n \int_{-\infty}^{\infty} \prod_j \left(\cosh\xi\,\mathrm{cotan}\,\frac{\theta_j}{2} - i\,\sinh\xi\right)^{-\frac{1}{2}} d\xi\\ &= 2^{-\frac{n}{2}}\,C_n \int_{-\infty}^{\infty} \prod_j \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} d\xi.\end{aligned} \tag{4.74}$$

Obviously, the last formula remains valid whenever $-\pi < \theta_j < \pi$ for every j. On the other hand, a change of contour of integration $\xi \mapsto \xi - \frac{i\,\phi}{2}$ is possible provided that all numbers $\theta_j + \phi$ remain in the interval $]-\pi, \pi[$ as well, leading to the required invariance. In particular, in the case when $\theta_j \in]0, \pi[$ for every j, one may also write

$$(\mathcal{K}\,\Phi)_0(Z) = 2^{-\frac{n}{2}}\,C_n \int_{-\infty}^{\infty} \prod_j \left(\cosh\left(\xi - \frac{i\,(\theta_j - \pi)}{2}\right)\right)^{-\frac{1}{2}} d\xi\ : \tag{4.75}$$

this last equation is actually valid under the sole assumption that $\theta_j \in]0, 2\pi[$ for every j, thus provides an explicit integral expression of $(\mathcal{K}\,\Phi)_0$ to the whole of Σ^{reg}.

We now need to continue analytically the function $(\mathcal{K}\,\Phi)_0$ from this open dense subset of Σ to the part of the universal cover $\Sigma^{(\infty)}$ of Σ lying above $\Sigma\backslash\Delta$, and to show that the resulting function is invariant under the covering map of the identity of degree n. For the first part, we have to show that if two continuous loops γ_1 and γ_2 within $\Sigma\backslash\Delta$ are homotopic as loops within Σ, the analytic continuation of $(\mathcal{K}\,\Phi)_0$ along either loop ends up with the same final value. It is no loss of generality to assume that the initial point Z^0 of the loops under consideration lies in $\Sigma^{\rm reg}$. These two loops do not, generally, define the same element of $\pi_1\,(\Sigma\backslash\Delta,\, Z^0)$: however, there exists a loop β_0 at Z^0, within $\Sigma\backslash\Delta$, such that $\gamma_2 \sim \beta_0\,\gamma_1$, where the right-hand side denotes the composition of loops [20, p. 60]: of course, since γ_1 and γ_2 are homotopic as loops in Σ, the first factor β_0 is homotopically trivial as a loop in Σ. What has to be shown is that the continuation of the function $(\mathcal{K}\,\Phi)_0$ along such a loop yields the same value at the initial and final points. Taking advantage of Lemma 4.17, we may substitute for the loop β_0 the loop β_1 as introduced there: since the equation (4.75) provides a global definition of $(\mathcal{K}\,\Phi)_0$ in $\Sigma^{\rm reg}$, it is clear that, continuing this function along β_1, one will reach the same value at the two endpoints.

Finally, the element of $\pi_1\,(\Sigma,\, Z^0)$ associated with the loop $t \mapsto e^{2i\pi\, t}\, Z^0$, $0 \leq t \leq 1$, is n times the canonical generator of this fundamental group: again, $(\mathcal{K}\,\Phi)_0$ remains constant along any such loop.

The proof of Theorem 4.18 is over. Let us emphasize again, however, that when restricted to $\Sigma^{\rm reg}$, the function under study is analytic everywhere, without our having to exclude the points of Δ: this is what made it possible to avoid the very complicated discussion of the fundamental group of $\Sigma\backslash\Delta$. Unless $n=1$, the continuation of $(\mathcal{K}\,\Phi)_0$ to $\Sigma^{(n)}$ will have genuine singularities at some points of this space lying above Δ. The following theorem will make matters especially clear in this respect. □

Theorem 4.19. *Assume that* $n=2$. *When* $Z \in \Sigma^{\rm reg}$, *denoting the eigenvalues of* Z *as* $e^{-i\theta_1}$ *and* $e^{-i\theta_2}$ *with* θ_1 *and* θ_2 *in* $]0,\, 2\pi[$, *one has*

$$(\mathcal{K}\,\Phi)_0(Z) = \frac{1}{2\,\cos\,\frac{\theta_1-\theta_2}{4}}\ {}_2F_1\left(\frac{1}{2},\,\frac{1}{2}\,;\,1\,;\,-\tan^2\,\frac{\theta_1-\theta_2}{4}\right). \qquad (4.76)$$

If $Z^\# \in \Sigma^{(2)}$ *lies above the matrix* Z *just introduced but* $Z^\# \neq Z$ (i.e., $Z^\#$ *lies in the other sheet of the twofold cover of* Σ), *and under the additional assumption that* $Z \notin \Delta$, *one has*

$$(\mathcal{K}\,\Phi)_0(Z^\#) = \frac{1}{2\,|\sin\,\frac{\theta_1-\theta_2}{4}|}\ {}_2F_1\left(\frac{1}{2},\,\frac{1}{2}\,;\,1\,;\,-\mathrm{cotan}^{\,2}\,\frac{\theta_1-\theta_2}{4}\right). \qquad (4.77)$$

Proof. The beginning of the computation is valid in any dimension. We temporarily assume again that $Z \in \Sigma$ has the eigenvalues $e^{-i\theta_j}$ with $0<\theta_j<\pi$. Starting from (4.73), one may write

$$(\mathcal{Q}\,\Phi)_0(\sigma) = C_n \int_{-\infty}^{\infty} d\xi\, \prod_j \int_0^\infty \exp\left(-\pi t_j\,(\sigma_j\,\cosh\xi - i\,\sinh\xi)\right) t_j^{-\frac{1}{2}}\, dt_j \,: \quad (4.78)$$

using [17, p. 86], this can be written as

$$(\mathcal{Q}\,\Phi)_0(\sigma) = 2C_n \int_{\mathbb{R}_+^n} \left(\prod t_j\right)^{-\frac{1}{2}}\; K_0\left(\pi\,\sqrt{(\sum t_j)^2 + (\sum t_j\,\sigma_j)^2}\right) dt. \quad (4.79)$$

As seen after changing t_j to $\tau_j\,t_j$ in the integral that precedes, one has

$$\begin{aligned}(\mathcal{Q}^{\mathrm{inv}}\,\Phi)_0(\tau)\colon &= (\det\tau)^{-\frac{1}{2}}\,(\mathcal{Q}\,\Phi)_0(\tau^{-1})\\ &= (\mathcal{Q}\,\Phi)_0(\tau), \end{aligned} \quad (4.80)$$

an equation to be used later.

Using (4.72) and performing in the integral (4.79) the change of variables $t_j \mapsto t_j\,\sin\,\frac{\theta_j}{2}$, one gets the final expression

$$(\mathcal{K}\,\Phi)_0(Z) = 2^{\frac{2-n}{2}}\,C_n \int_{\mathbb{R}_+^n} \left(\prod t_j\right)^{-\frac{1}{2}}\; K_0\left(\pi\left|\,\sum t_j\,e^{\frac{i\theta_j}{2}}\right|\right) dt, \quad (4.81)$$

an expression obviously valid whenever $\theta_j \in]-\pi,\,\pi[$ for every j.

We now specialize to the two-dimensional case, in which one can compute $(\mathcal{K}\,\Phi)_0(Z)$ explicitly. Setting $\phi = \frac{\theta_1-\theta_2}{2}$ and assuming $\max\left(|\theta_1|,|\theta_2|\right) < \pi$, one has

$$(\mathcal{K}\,\Phi)_0(Z) = \frac{1}{\pi}\int_0^\infty\int_0^\infty K_0\left(\pi\,\sqrt{t_1^2 + 2\,t_1t_2\,\cos\,\phi + t_2^2}\right)\frac{dt_1\,dt_2}{\sqrt{t_1t_2}}. \quad (4.82)$$

Using the new variables $t = \frac{t_1-t_2}{2} \in \mathbb{R}$, $r = \frac{4\,t_1t_2}{(t_1-t_2)^2} > 0$, so that $\frac{dt_1\,dt_2}{\sqrt{t_1t_2}} = \frac{dt\,dr}{\sqrt{r(1+r)}}$, one finds

$$\begin{aligned}(\mathcal{K}\,\Phi)_0(Z) &= \frac{1}{\pi}\int_0^\infty \frac{dr}{\sqrt{r(1+r)}}\int_{-\infty}^\infty K_0\left(2\pi\,|t|\,\sqrt{1+r\,\cos^2\,\frac{\phi}{2}}\right) dt\\ &= \frac{1}{2\pi}\int_0^\infty \frac{dr}{\sqrt{r(1+r)(1+r\,\cos^2\,\frac{\phi}{2})}} \end{aligned} \quad (4.83)$$

or, performing the new change of variable $x = \frac{1}{1+r}$ and using [17, p. 54] the integral representation of the hypergeometric function,

$$\begin{aligned}(\mathcal{K}\,\Phi)_0(Z) &= \frac{1}{2\pi\,\cos\,\frac{\phi}{2}}\int_0^1 x^{-\frac{1}{2}}\,(1-x)^{-\frac{1}{2}}\left(1 + x\,\tan^2\,\frac{\phi}{2}\right)^{-\frac{1}{2}} dx\\ &= \frac{1}{2\,\cos\,\frac{\theta_1-\theta_2}{4}}\;{}_2F_1\left(\frac{1}{2},\,\frac{1}{2}\,;\,1\,;\,-\tan^2\,\frac{\theta_1-\theta_2}{4}\right) : \end{aligned} \quad (4.84)$$

this proves the equation (4.76).

To compute $(\mathcal{K}\,\Phi)_0(Z^\#)$ in the case when $Z^\#$ is a point above $\Sigma\backslash\Delta$ in the other sheet of $\Sigma^{(2)}$, we must use analytic continuation. With θ_1 and θ_2 distinct and both in $]0, 2\pi[$, we may assume that $\theta_2 < \theta_1 < \theta_2 + \pi$, since this can be relaxed later, using analytic continuation; it is no loss of generality to assume that, simply, $Z = \begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{-i\theta_2}\end{pmatrix}$. In the computation that follows, it is handy, taking advantage of the invariance of the function $(\mathcal{K}\,\Phi)_0$ under rotations $Z \mapsto e^{i\,\phi}\,Z$, to set $\theta_2 = 0$: of course, one then does not have $Z \in \Sigma^{\rm reg}$ any longer, since one of the two eigenvalues of Z becomes 1, and the matrix $\begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & 1\end{pmatrix}$ must be interpreted as the point of $\Sigma^{(2)}$ which is the limit, as $\varepsilon > 0$ goes to 0, of the matrix $\begin{pmatrix} e^{-i\,(\theta_1+\varepsilon)} & 0 \\ 0 & e^{-i\,(\theta_2+\varepsilon)}\end{pmatrix}$. One can now make explicit a loop in Σ at Z, with an image contained in $\Sigma\backslash\Delta$, the class of which will be a generator of the fundamental group of Σ at Z. This is the loop $t \mapsto Z(t)$ defined as

$$Z(t) = \begin{cases} \begin{pmatrix} \frac{(1-4t)\,\cos\frac{\theta_1}{2} - i\,\sin\frac{\theta_1}{2}}{(1-4t)\,\cos\frac{\theta_1}{2} + i\,\sin\frac{\theta_1}{2}} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t \in [0,\, \tfrac{1}{2}], \\[2ex] \exp\, i\,\theta_1 \begin{pmatrix} 3-4t & \sqrt{(2t-1)(2-2t)} \\ \sqrt{(2t-1)(2-2t)} & 0 \end{pmatrix} & \text{if } t \in [\tfrac{1}{2}\, 1]. \end{cases} \tag{4.85}$$

Note that $Z(\frac{1}{2}) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & 1\end{pmatrix}$. The first half of the path is the limit, as $\varepsilon \to 0$, of a path that lies entirely in $\Sigma^{\rm reg}$ (it is given explicitly, with a small abuse of language, as the Cayley map of the path $t \mapsto \begin{pmatrix} (1-4t)\,\text{cotan}\,\frac{\theta_1}{2} & 0 \\ 0 & +\infty\end{pmatrix}$, where it is essential to distinguish the entries $+\infty$ and $-\infty$ in the lower-right corner, which would correspond to the same point of Σ but not to the same point of $\Sigma^{(2)}$). Along this path, the pair of arguments of the eigenvalues of $Z(t)$ moves from $(-\theta_1,\, 0)$ to $(-(2\pi - \theta_1),\, 0)$: consequently, the logarithm of $\det Z(t)$ moves from $-i\,\theta_1$ to $-i\,(2\pi - \theta_1)$. On the other hand, the trace $i\,\theta_1\,(3-4t)$ of the matrix in the exponent of the definition of the second part of the path $t \mapsto Z(t)$ decreases by $2i\,\theta$ as t goes from $\frac{1}{2}$ to 1 so that, at $t = 1$, $\log\det Z(t)$ reaches the value $-i\,(\theta_1 + 2\pi)$. Thus, the loop under examination indeed defines a generator of the fundamental group $\pi_1(\Sigma,\, Z(0))$.

Following up the function $(\mathcal{K}\,\Phi)_0$ along the first half of the loop is easy, as this path *almost* lies in $\Sigma^{\rm reg}$: one simply has to perform the change $\theta_1 \mapsto 2\pi - \theta_1$ in (4.75), replacing also the ambiguous factor $(\cosh(\xi + \frac{i\,\pi}{2}))^{-\frac{1}{2}}$ by the value $|\sinh\xi|^{-\frac{1}{2}}\, e^{-\frac{i\pi}{4}\,\text{sign}\,\xi}$ obtained by the limiting process as $\varepsilon > 0$ goes to 0. On the other hand, the eigenvalues of the matrix $\begin{pmatrix} 3-4t & \sqrt{(2t-1)(2-2t)} \\ \sqrt{(2t-1)(2-2t)} & 0\end{pmatrix}$ are the numbers $2-2t$ and $1-2t$: consequently, for $\frac{1}{2} \le t \le 1$, the eigenvalues of the matrix $Z(t)$ are the numbers $e^{i\theta_1\,(2-2t)}$ and $e^{i\theta_1\,(1-2t)}$. From what has been said above concerning the first part of the loop, the pair of arguments of the eigenvalues of $Z(\frac{1}{2})$ must be set at the values $\theta_1 - 2\pi$ and 0; for $\frac{1}{2} \le t \le 1$, the arguments relative to $Z(t)$ must thus be set at the pair of values $\theta_1\,(2-2t) - 2\pi$

and $\theta_1\,(1-2t)$, ending up at $t=1$ with the pair of arguments $(-2\pi,\,-\theta_1)$. Moving along the loop $t\mapsto Z(t)$ thus calls for substituting the pair $(2\pi,\,\theta_1)$ for the pair $(\theta_1,\,0)$. This finally leads to the equation (4.77) if one uses again the invariance of the function $(\mathcal{K}\,\Phi)_0$ under the map $Z\mapsto e^{-i\,\theta_2}\,Z$, which concludes the proof of Theorem 4.19. □

One should note that, after it has been extended to the part of $\Sigma^{(2)}$ above $\Sigma\backslash\Delta$, the function $(\mathcal{K}\,\Phi)_0$ is singular exactly on the closure of the set of points above $\Sigma^{\rm reg}\cap\Delta$ not in $\Sigma^{\rm reg}$. Thus, the pullback in $\Sigma^{(2)}$ of $\Delta\subset\Sigma$ is the disjoint union of two circles, one of which is the exact singular set of this function.

To have some idea of the size and nature of the singularity, let us revert, in the case when $Z^\sharp$ lies in the second sheet of $\Sigma^{(2)}$, to coordinates relative to the point $\sigma\in{\rm Sym}_2$ of which Z is the Cayley transform. We may set $\sigma=\left(\begin{smallmatrix}p+q & r\\ r & p-q\end{smallmatrix}\right)$: then the determinant and trace of σ are $p^2-q^2-r^2$ and $2p$, and the discriminant of the characteristic polynomial of σ is

$$\mathrm{disc}\,(\sigma)=(\sigma_1-\sigma_2)^2=4\,(q^2+r^2). \tag{4.86}$$

Still denoting as $e^{-i\theta_1}$ and $e^{-i\theta_2}$ the eigenvalues of Z one has

$$e^{-i\,(\theta_1-\theta_2)}=\frac{(\sigma_1-i)\,(\sigma_2+i)}{(\sigma_1+i)\,(\sigma_2-i)}=\frac{[1+\sigma_1\,\sigma_2+i\,(\sigma_1-\sigma_2)]^2}{(1+\sigma_1^2)\,(1+\sigma_2^2)}, \tag{4.87}$$

so that

$$\sin^2\frac{\theta_1-\theta_2}{2}=\frac{\mathrm{disc}\,(\sigma)}{(1+\sigma_1^2)\,(1+\sigma_2^2)}=\frac{4\,(q^2+r^2)}{4\,p^2+(1-p^2+q^2+r^2)^2}, \tag{4.88}$$

a quantity equivalent to $\frac{\mathrm{disc}\,(\sigma)}{(1+p^2)^2}$ as $\mathrm{disc}\,(\sigma)\to 0$: note that this condition just means of course that σ approaches Δ. If such is the case, we set $x=\mathrm{cotan}^2\,\frac{\theta_1-\theta_2}{4}$, an expression which goes to infinity just like

$$\begin{aligned}1+x&=\frac{1}{\sin^2\frac{\theta_1-\theta_2}{4}}\sim\frac{4}{\sin^2\frac{\theta_1-\theta_2}{2}}\\ &\sim\frac{[4+(\mathrm{tr}\,\sigma)^2]^2}{\mathrm{disc}\,(\sigma)}.\end{aligned} \tag{4.89}$$

Finally, in view of the expansion [17, p. 48] of the hypergeometric function at the argument $-x$ with $x>1$, one finds

$$\begin{aligned}(\mathcal{K}\,\Phi)_0(Z^\sharp)=&\frac{1}{2\pi^2}\,(1+x^{-1})^{\frac12}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+\frac12)}{m\,!}\right)^2\\ &\times\left[\log x-\psi\left(m+\frac12\right)-\psi\left(\frac12-m\right)+2\,\psi(m+1)\right](-x)^{-m},\end{aligned} \tag{4.90}$$

where ψ is the logarithmic derivative of the Gamma function. Thus, the size of the singularity is that of the logarithm of $\mathrm{disc}\,(\sigma)$.

5 The n-dimensional anaplectic representation

We here define and study the anaplectic representation of the group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ in the space $\mathfrak{A}^{(n)}$. Given $u \in \mathfrak{A}^{(n)}$, it will not be possible to define $\mathrm{Ana}(g)\,u$ for every $g \in \mathrm{Sp}^{(n)}(n,\mathbb{R})$, only for all g in some open dense subset of that group, depending on u.

Let us comment briefly on this point, lest it should unduly worry the reader and lead him to conclude that this disqualifies the object under study as a genuine representation. Indeed, one may also regard the anaplectic representation as a homomorphism into a group of only partially defined operators. In this way, this is not too different from the case with more usual representations, for instance the metaplectic representation. Consider the one-dimensional function $x \mapsto e^{2i\pi x}$ and its transform under the metaplectic transformation associated with one of the two points above some matrix $g \in SL(2,\mathbb{R})$; let $g = nak$ be the decomposition of g associated with the usual Iwasawa decomposition $G = NAK$ of the group $G = SL(2,\mathbb{R})$. Then, the transform under study will be an (analytic) function on the real line for every g except for those such that $k = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, since the corresponding metaplectic transformation is either $e^{-\frac{i\pi}{4}}\,\mathcal{F}$ or $e^{\frac{i\pi}{4}}\,\mathcal{F}^{-1}$. Of course, the answer to the difficulty (in the metaplectic case) lies in the fact that the space of all analytic functions on the line is not a well-chosen one. The genuine difficulty with the anaplectic representation is a "topological" one, and is linked to the fact that it is not unitarizable: *cf.* Section 8, however, for steps towards the construction of some appropriate pseudoscalar product. The constructions in the present section rely on elementary algebraic geometry, not on Hilbert space methods. It is clear, however, that at some point, it will be necessary to give a characterization of appropriate spaces of analytic functions, or analytic functionals, on $\mathbb{R}^n$, with a more controlled behavior, by means of corresponding properties of their $\mathcal{K}$-transforms: Remark 5.2 below at least gives some indication in this direction.

We must also emphasize the fact that our aim is not to construct the anaplectic representation as an abstract one, *i.e.,* up to equivalence. The space on which it operates has to be realized as a space of functions on $\mathbb{R}^n$, in a way which makes the anaplectic representation and the Heisenberg representation act in a coherent way.

That the anaplectic representation should be more singular than the metaplectic representation can be traced to several reasons, one of which has to do with the fact that, corresponding to any "energy level" of the anaplectic harmonic oscillator, there is (unless $n = 1$) an infinite-dimensional eigenspace: contrary to the ground state of the usual harmonic oscillator, which is invariant under all transformations in the image, under the metaplectic representation, of the maximal compact subgroup (isomorphic to $U(n)$) of the metaplectic group $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$, the *median* state Φ of the anaplectic harmonic oscillator is only invariant under the action of the subgroup of $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ above the much smaller group $O(n)$. This is the crux of the matter since, as will be seen, Φ does not admit *partial* anaplectic Fourier transforms in any obvious sense, only a global one: in contrast, the standard Gaussian function is invariant under all partial Fourier transformations.

We first recall from Corollary 4.10 that the partially defined action of the symplectic group $\mathrm{Sp}\,(n,\mathbb{R})$ on Sym_n, as given just before Lemma 4.6, extends as an analytic, fully regular action on Σ: one may also lift this as an action of the group $\mathrm{Sp}^{(n)}\,(n,\mathbb{R})$ on $\Sigma^{(n)}$. Set

$$\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} = 2^{-\frac12} \begin{pmatrix} I & i\,I \\ i\,I & I \end{pmatrix} . \begin{pmatrix} A & B \\ C & D \end{pmatrix} . \, 2^{-\frac12} \begin{pmatrix} I & -i\,I \\ -i\,I & I \end{pmatrix}, \tag{5.1}$$

i.e.,

$$\alpha = \frac12\,(A - i\,B + i\,C + D), \qquad \beta = \frac12\,(-i\,A + B + C + i\,D): \tag{5.2}$$

since $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} D' & -B' \\ -C' & A' \end{smallmatrix}\right)$, one has $\left(\begin{smallmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \bar\alpha' & -\beta' \\ -\bar\beta' & \alpha' \end{smallmatrix}\right)$, so that

$$\alpha\,\bar\alpha' - \beta\,\bar\beta' = I, \qquad -\alpha\,\beta' + \beta\,\alpha' = 0 \tag{5.3}$$

or, equivalently,

$$\bar\alpha'\,\alpha - \beta'\,\bar\beta = I, \qquad \bar\alpha'\,\beta - \beta'\,\bar\alpha = 0. \tag{5.4}$$

With the help of Proposition 4.7 and Lemma 4.9, one may check that the action $(g, Z) \mapsto [g]\,(Z)$ of $\mathrm{Sp}\,(n,\mathbb{R})$ on Σ is given by the equation

$$[g^{-1}]\,(Z) = (i\,\beta' + \bar\alpha'\,Z)\,(\alpha' - i\,\bar\beta'\,Z)^{-1}. \tag{5.5}$$

Exercising care, one may also denote as $[g^{-1}]$ the analytic homeomorphism of $\Sigma^{(n)}$ associated with some element g^{-1} of $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ in conformity with Corollary 4.10.

The aim of the present section is to prove the existence of a representation $g \mapsto \mathrm{Ana}\,(g)$ of the group $\mathrm{Sp}^{(n)}\,(n,\mathbb{R})$ in the linear space $\mathfrak{A}^{(n)}$, characterized by the equations, where Z stands for a point of $\Sigma^{(n)}$ not in the image under $[g]$ of the singular set of the $\mathcal{K}$-transforms of $u \in \mathfrak{A}^{(n)}$,

$$\begin{aligned} (\mathcal{K}\,\mathrm{Ana}\,(g)\,u)_0(Z) &= \left(\frac{[g^{-1}]_*\,d\mu}{d\mu}\,(Z)\right)^{\frac{1}{2(n+1)}} (\mathcal{K}\,u)_0([g^{-1}]\,(Z)), \\ (\mathcal{K}\,\mathrm{Ana}\,(g)\,u)_1(Z) &= \left(\frac{[g^{-1}]_*\,d\mu}{d\mu}\,(Z)\right)^{\frac{1}{2(n+1)}} \\ &\qquad \times \left[\alpha - i\,\beta\,([g^{-1}]\,(Z))^{-1}\right] . \,(\mathcal{K}\,u)_1([g^{-1}]\,(Z)): \end{aligned} \tag{5.6}$$

note that, in the matrix $\alpha - i\,\beta\,([g^{-1}]\,(Z))^{-1}$ on the second line, and contrary to its occurrence elsewhere in these two equations, $[g^{-1}]\,(Z)$ must be interpreted as a point of Σ rather than $\Sigma^{(n)}$: in this section, we decided against the too heavy notation $Z^{\#}$ used elsewhere to distinguish such a point of $\Sigma^{(n)}$ from its projection

Z in Σ. In the case when g reduces to the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$ with $\det A > 0$ (*resp.* $\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$), we shall be led to defining $\mathrm{Ana}\,(g)$ as one of the two transformations $u \mapsto v$ or $u \mapsto w$ that occurred in Theorem 4.13: that the formulas above are correct in this case will be checked in the proof of Theorem 5.10. In general, for a given g, $\mathrm{Ana}\,(g)\, u$ will be defined only under some assumption relative to the singular set of $\mathcal{K}\, u$. Let us refer the reader to the beginning of this section for a discussion of this point.

We now start the work necessary towards the completion of the announced result: essentially, we need to construct the anaplectic Fourier transformation and see its connection with the Heisenberg representation. First, we consider the operator Θ connecting functions on some part of Sym_n to functions on some part of Σ, defined as in Theorem 4.11 by the equation

$$(\Theta\, w)(Z) = |\det\,(I - Z)|^{-\frac{1}{2}}\, w\left(i\,\frac{I+Z}{I-Z}\right): \tag{5.7}$$

when using this operator, it will always be tacitly assumed that Z lies in the image, under the Cayley transform, of the set of matrices $\sigma \in \mathrm{Sym}_n$ with $\sigma - R\,I \succ 0$ for some large R.

But first, we must fix the notation concerning a non-redundant set of coordinates on the linear space Sym_n. If $\sigma = (\sigma_{jk})_{1 \leq j,k \leq n}$, we set $\sigma_{(jj)} = \sigma_{jj}$ and, if $j \neq k$, $\sigma_{(jk)} = 2\,\sigma_{jk}$: thus, whether $j \neq k$ or not, $\sigma_{(jk)}$ is thought of as depending on the *unordered* pair (j,k). In this way, we get coordinates on Sym_n, and we note the equations

$$\frac{\partial}{\partial \sigma_{(jk)}}\, \langle\, \sigma x, x\, \rangle = x_j\, x_k, \qquad \frac{\partial}{\partial x_k}\, \langle\, \sigma x, x\, \rangle = 2 \sum_j \sigma_{jk}\, x_j. \tag{5.8}$$

Lemma 5.1. *Given j, ℓ with $1 \leq j, \ell \leq n$, let $\tau^{j\ell}$ be the symmetric matrix such that $\langle \tau^{j\ell}\, x, x \rangle = x_j\, x_\ell$ for all $x \in \mathbb{R}^n$. One has for every C^1 function w on Sym_n the equation*

$$\Theta\left(\frac{\partial w}{\partial \sigma_{(j\ell)}}\right) = \mathcal{D}^{(j\ell)}\, \Theta\, w, \tag{5.9}$$

where, denoting as $V_Z^{j\ell}$ the matrix

$$V_Z^{j\ell} = \frac{1}{2i}\,(I - Z)\,\tau^{j\ell}\,(I - Z), \tag{5.10}$$

one has

$$\begin{aligned} \mathcal{D}^{(j\ell)} &= \nabla_{V_Z^{j\ell}} - \frac{1}{2}\,\mathrm{Re}\ \mathrm{Tr}\,((I-Z)^{-1}\, V_Z^{j\ell}) \\ &= \nabla_{V_Z^{j\ell}} - \frac{1}{4}\,\mathrm{Im}\ (I - Z)_{j\ell}\, : \end{aligned} \tag{5.11}$$

the operator $\nabla_{V_Z^{j\ell}}$ denotes the first-order derivative along the vector $V_Z^{j\ell}$, tangent to $\Sigma^{(n)}$ at Z.

Proof. Let us abbreviate as τ the matrix $\tau^{j\ell}$. Using (4.50) and (4.36), we obtain

$$\Theta\,(\frac{\partial w}{\partial \sigma_{(j\ell)}}) = \frac{d}{dt}\Big|_{t=0} \left[\,\left|\frac{\det\,(I-h_t(Z))}{\det\,(I-Z)}\right|^{\frac{1}{2}} w(h_t(Z))\right] \tag{5.12}$$

with

$$h_t(Z) = (2i\,I - t\,\tau)\,\left[\frac{t\,\tau}{2i\,I-t\,\tau}+Z\right]\,\left[I-\frac{t\,\tau}{2i\,I+t\,\tau}\,Z\right]^{-1}(2i\,I+t\,\tau)^{-1}. \tag{5.13}$$

Here, the analytic transformation h_t can be regarded just as well as an analytic automorphism of Σ or $\Sigma^{(n)}$, as explained in Corollary 4.10. Of course, $h_0(Z)=Z$, and we note that

$$V_Z^{j\ell}:\;=\frac{d}{dt}\Big|_{t=0} h_t(Z) = \frac{1}{2i}\,(I-Z)\,\tau^{j\ell}\,(I-Z): \tag{5.14}$$

for safety, one may check that this is, as it should be, a vector tangent to Σ or $\Sigma^{(n)}$ at Z since it is a symmetric matrix and its image under the multiplication on the left by Z^{-1} is indeed antihermitian. On the other hand,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\left|\frac{\det\,(I-h_t(Z))}{\det\,(I-Z)}\right|^{\frac{1}{2}} &= \frac{d}{dt}\Big|_{t=0}\left|\det\,(I-t\,(I-Z)^{-1}\,V_Z^{j\ell})\right|^{\frac{1}{2}}\\ &= -\frac{1}{2}\,\mathrm{Re}\;\mathrm{Tr}\,((I-Z)^{-1}\,V_Z^{j\ell}).\end{aligned} \tag{5.15}$$

We also note, using the equation

$$(\tau^{j\ell})_{rs} = \frac{1}{2}\,(\delta_{jr}\delta_{ls}+\delta_{js}\delta_{lr}) \tag{5.16}$$

for the entries of the matrix $\tau^{j\ell}$, that the vector $(I-Z)^{-1}\,V_Z^{j\ell} = \frac{1}{2i}\,\tau^{j\ell}\,(I-Z)$ satisfies

$$\mathrm{Tr}\,((I-Z)^{-1}\,V_Z^{j\ell}) = -\frac{i}{2}\,(I-Z)_{j\ell}. \tag{5.17}$$

□

Remark 5.1. With the differential operator $\mathcal{D}^{(j\ell)}$, we have the first instance of a situation that will (fortunately) recur in the present section: the coefficients of this operator are analytic functions defined on the whole of Σ. Therefore, there will be no difficulty when the need arises to let such an operator act on functions defined on some open subset of Σ or of any covering space of Σ.

Proposition 5.2. *If u lies in $\mathfrak{A}^{(n)}$, so does $Q_\ell\,u = x_\ell\,u$ for every $\ell = 1,\dots,n$.*

Proof. As an immediate consequence of Definition 4.1, one has if $\sigma - R\,I \succ 0$ with R large enough

$$(\mathcal{Q}\,(x_\ell\, u))_0(\sigma) = e^{-\frac{i\pi}{4}}\,((I + i\,\sigma)^{-1}\,(\mathcal{Q}\,u)_1(\sigma))^{(\ell)} \tag{5.18}$$

(the ℓth component of the vector-valued function on the right-hand side): in other words

$$(\mathcal{Q}\,(x_\ell\, u))_0(\sigma) == e^{-\frac{i\pi}{4}}\,\sum_k (I + i\,\sigma)^{-1}_{\ell k}\,(\mathcal{Q}\,u)_1^{(k)}(\sigma)), \tag{5.19}$$

denoting as $(I + i\,\sigma)^{-1}_{\ell k}$ the (ℓk)th entry of the matrix $(I + i\,\sigma)^{-1}$. Also, using the first of the two equations (5.8), one finds

$$(\mathcal{Q}\,(x_\ell\, u))_1(\sigma) = -\frac{e^{-\frac{i\pi}{4}}}{\pi}\,(I + i\,\sigma)\;(\mathcal{R}\,u)(\sigma), \tag{5.20}$$

where $\mathcal{R}u$ is the vector-valued function such that

$$(\mathcal{R}\,u)^{(j)}(\sigma) = \frac{\partial}{\partial\sigma_{(j\ell)}}\,(\mathcal{Q}\,u)_0(\sigma), \tag{5.21}$$

in other words

$$(\mathcal{Q}\,(x_\ell\, u))^{(j)}(\sigma) = -\frac{e^{-\frac{i\pi}{4}}}{\pi}\,\sum_k (I + i\,\sigma)_{jk}\,\frac{\partial}{\partial\sigma_{(k\ell)}}\,(\mathcal{Q}\,u)_0(\sigma). \tag{5.22}$$

According to Lemma 5.1, one has

$$(\mathcal{K}\,(x_\ell\, u))^{(j)}(Z) = -\frac{e^{-\frac{i\pi}{4}}}{\pi}\,\sum_k \left(-\frac{2Z}{I - Z}\right)_{jk}\,\mathcal{D}^{(k\ell)}\,(\mathcal{K}\,u)_0(Z), \tag{5.23}$$

and what remains to be done is showing that, for every r, the coefficients of the operator $\sum_k (I - Z)^{-1}_{rk}\,\mathcal{D}^{(k\ell)}$ extend as analytic functions on Σ: indeed, when this has been done, the equation just found, linking two functions defined in some part of Σ, will automatically provide an extension of the left-hand side to the connected dense open subset of $\Sigma^{(n)}$ where the right-hand side is defined and analytic.

In the operator under study, there is a scalar term $\sum_k (I - Z)^{-1}_{rk}\,(I - Z)_{k\ell}$ which has obviously no singularity on Σ, and a first-order operator which is the derivative along the vector $\sum_k (I - Z)^{-1}_{rk}\,V_Z^{k\ell}$: this latter operator is obtained just like the first-order part of the operator in (5.11), except for the fact that the matrix $\tau^{j\ell} \in \mathrm{Sym}_n$ must be replaced by the matrix

$$\sum_k (I - Z)^{-1}_{rk}\,\tau^{k\ell} = \frac{1}{2}\,\left[E^{\ell r}\,(I - Z)^{-1} + (I - Z)^{-1}\,E^{r\ell}\right], \tag{5.24}$$

where $E^{\ell r}$ is the matrix with a 1 at the place indicated and zeros elsewhere: the extra matrix $I - Z$ which occurs on both sides in the right-hand side of (5.10) is exactly what is needed so that this vector, when viewed in the Z-coordinates, should become extendable as an analytic function on the whole of Σ, which finally proves that $Q_k\, u$ lies in $\mathfrak{A}^{(n)}$. □

Proposition 5.3. *If u lies in $\mathfrak{A}^{(n)}$, so does $P_\ell\, u = \frac{1}{2i\pi}\,\frac{\partial}{\partial x_\ell}\, u$ for every $\ell = 1, \dots, n$.*

Proof. Using (4.1), an integration by parts and (5.8), we find

$$\begin{aligned}(\mathcal{Q}\,(P_\ell\, u))_0(\sigma) &= e^{-\frac{i\pi}{4}} \sum_k \sigma_{\ell k} \int_{\mathbb{R}^n} x_k\, e^{-\pi\,\langle \sigma\, x, x\rangle}\, u(x\, e^{-\frac{i\pi}{4}})\, dx \\ &= e^{-\frac{i\pi}{4}} \left(\frac{\sigma}{I + i\,\sigma}\,(\mathcal{Q}\, u)_1 \right)^{(\ell)} (\sigma),\end{aligned} \tag{5.25}$$

the ℓth component of the vector-valued function $e^{-\frac{i\pi}{4}}\,\sigma\,(I + i\,\sigma)^{-1}\,(\mathcal{Q}\,u)_1(\sigma)$. On the other hand,

$$\begin{aligned}(\mathcal{Q}\,(P_\ell\, u))_1^{(j)}(\sigma) &= \frac{e^{-\frac{i\pi}{4}}}{2\pi} \int_{\mathbb{R}^n} (x_j + i \sum_k \sigma_{jk}\, x_k)\, e^{-\pi\,\langle \sigma\, x, x\rangle}\, \frac{\partial}{\partial x_\ell}\,(u(x\, e^{-\frac{i\pi}{4}}))\, dx \\ &= \frac{e^{-\frac{i\pi}{4}}}{2\pi} \int_{\mathbb{R}^n} e^{-\pi\,\langle \sigma\, x, x\rangle}\,(u(x\, e^{-\frac{i\pi}{4}})) \\ &\quad \times \left[2\pi\,(x_j + i \sum_k \sigma_{jk}\, x_k) \sum_r \sigma_{\ell r}\, x_r - \delta_{j\ell} - i\,\sigma_{j\ell} \right] dx,\end{aligned} \tag{5.26}$$

which may also be written, using (5.8) again, as

$$(\mathcal{Q}(P_\ell u))_1^{(j)}(\sigma) = -\frac{e^{-\frac{i\pi}{4}}}{2\pi} \left[2 \sum_r \sigma_{\ell r} \Big(\frac{\partial}{\partial \sigma_{(jr)}} + i \sum_k \sigma_{jk} \frac{\partial}{\partial \sigma_{(kr)}} \Big) + \delta_{j\ell} + i \sigma_{j\ell} \right] (\mathcal{Q} u)_0(\sigma). \tag{5.27}$$

We must study the transfer, under the operator Θ defined in (5.7), of the operators

$$\begin{aligned} L_{j\ell} &= 2 \sum_r \sigma_{\ell r}\, \frac{\partial}{\partial \sigma_{(jr)}} + \delta_{j\ell}, \\ M_{j\ell} &= 2 \sum_{kr} \sigma_{\ell r}\, \sigma_{jk}\, \frac{\partial}{\partial \sigma_{(kr)}} + (\sigma_{j\ell}),\end{aligned} \tag{5.28}$$

where $(\sigma_{j\ell})$ stands for the corresponding operator of multiplication. Since the first one is much easier to study than the second, we concentrate on the latter one. The operator $M^{(j\ell)} = \Theta\, M_{j\ell}\, \Theta^{-1}$, expresses itself, as a consequence of Lemma 5.1, as

$$M^{(j\ell)} = 2\, \nabla_{X_Z^{j\ell}} - \mathrm{Im}\ \beta(Z) + \left(i\, \frac{I + Z}{I - Z} \right)_{j\ell} \tag{5.29}$$

with

$$\begin{aligned} X_Z^{j\ell} &= \sum_{kr} \left(i\, \frac{I + Z}{I - Z} \right)_{\ell r} \left(i\, \frac{I + Z}{I - Z} \right)_{jk} V_Z^{kr}, \\ \beta(Z) &= \frac{1}{2} \sum_{kr} \left(i\, \frac{I + Z}{I - Z} \right)_{\ell r} \left(i\, \frac{I + Z}{I - Z} \right)_{jk} (I - Z)_{kr},\end{aligned} \tag{5.30}$$

and we must show that the coefficients of the operator $M^{(j\ell)}$ extend as analytic functions on Σ: again, the whole difficulty lies in the presence of coefficients of the matrix $(I-Z)^{-1}$.

First, using (5.16), we find that the linear combination

$$A^{j\ell}:\ = \sum_{kr} \left(i\,\frac{I+Z}{I-Z}\right)_{jk} \tau^{kr} \left(i\,\frac{I+Z}{I-Z}\right)_{\ell r} \tag{5.31}$$

has its entries given by the formula

$$(A^{j\ell})_{pq} = \frac{1}{2}\left[\left(i\,\frac{I+Z}{I-Z}\right)_{jp} \left(i\,\frac{I+Z}{I-Z}\right)_{q\ell} \left(i\,\frac{I+Z}{I-Z}\right)_{jq} \left(i\,\frac{I+Z}{I-Z}\right)_{p\ell}\right]: \tag{5.32}$$

it follows that the entries of the matrix

$$X_Z^{j\ell} = \frac{1}{2i}\,(I-Z)\,A^{j\ell}\,(I-Z) \tag{5.33}$$

are given as

$$(X_Z^{j\ell})_{rs} = \frac{i}{4}\,\left[(I+Z)_{jr}\,(I+Z)_{\ell s} + (I+Z)_{js}\,(I+Z)_{\ell r}\,\right] \tag{5.34}$$

and, consequently, extend as analytic functions on Σ. Next, we write

$$\begin{aligned} \beta(Z) &= \frac{1}{2}\sum_r \left(i\,\frac{I+Z}{I-Z}\right)_{\ell r} (i\,(I+Z))_{jr}, \\ &\frac{i}{2}\left(i\,\frac{I+Z}{I-Z}.\,(2\,I-(I-Z))\right)_{\ell j} \\ &= i\left(i\,\frac{I+Z}{I-Z}\right)_{\ell j} + \frac{1}{2}\,(I+Z)_{\ell j}: \end{aligned} \tag{5.35}$$

remembering that the matrix $i\,\frac{I+Z}{I-Z}$ is real – it is the inverse Cayley transform of Z – we finally get

$$\text{Im}\ \beta(Z) = \left(i\,\frac{I+Z}{I-Z}\right)_{\ell j} + \frac{1}{2}\,\text{Im}\ (I+Z)_{\ell j}: \tag{5.36}$$

therefore, the first term on the right-hand side will just annihilate, in (5.29), the last singular term in the same equation; the remaining term $\frac{1}{2}\,\text{Im}\ (I+Z)_{\ell j}$ does extend analytically to the whole of Σ. □

Corollary 5.4. *All anaplectic Hermite functions lie in* $\mathfrak{A}^{(n)}$.

Proof. This follows from Definition 3.2 and the last two propositions. □

Theorem 5.5. *Let* $u \in \mathfrak{A}^{(n)}$*. For every* $\eta \in \mathbb{C}^n$*, the function* u_η *defined as* $u_\eta(x) = u(x)\, e^{-2i\pi\,\langle \eta, x\rangle}$ *lies in* $\mathfrak{A}^{(n)}$*.*

Proof. For $\sigma \in \mathrm{Sym}_n$ with $\sigma - RI \succ 0$ with R large enough, one has

$$\begin{aligned}(\mathcal{Q}\, u_\eta)_0(\sigma) &= \int_{\mathbb{R}^n} e^{-\pi\,\langle \sigma x, x\rangle}\ \exp\left(-\,2\pi\, e^{\frac{i\pi}{4}} \langle \eta, x\rangle\right)\ u(x\, e^{-\frac{i\pi}{4}})\, dx,\\ &= \int_{\mathbb{R}^n} e^{-\pi\,\langle \sigma x, x\rangle} \sum_{\alpha \in \mathbb{N}^n} \frac{(-2\pi\, e^{\frac{i\pi}{4}}\, \eta)^\alpha}{\alpha\,!}\ x^\alpha\ u(x\, e^{-\frac{i\pi}{4}})\, dx. \end{aligned} \tag{5.37}$$

Given any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with an even length $|\alpha|$, denote as $[\frac{\alpha}{2}]$ the multi-index $\beta \in \mathbb{N}^n$ such that $\beta_j = [\frac{\alpha_j}{2}]$ for all j: then $\alpha = 2\,[\frac{\alpha}{2}] + \gamma$ for some multi-index γ all components of which are 0 or 1, which we split as a sum of multi-indices of length 2 in an arbitrary way. The first of the two equations (5.8) then makes it possible to associate to each multi-index $\alpha \in N^n$ with an even length a multi-index $\tilde{\alpha} \in \mathbb{N}^{\frac{n(n+1)}{2}}$ with the following properties:

$$x^\alpha\, e^{-\pi\langle \sigma x, x\rangle} = \left(-\frac{1}{\pi}\,\frac{\partial}{\partial \sigma}\right)^{\tilde{\alpha}}\ e^{-\pi\langle \sigma x, x\rangle}, \qquad |\tilde{\alpha}| = \frac{1}{2}\,|\alpha| \quad \text{and} \quad \tilde{\alpha}\,! = \left[\frac{\alpha}{2}\right]!\ . \tag{5.38}$$

We must now check that the $\mathcal{K}$-transforms of u_η satisfy the hypotheses of Definition 4.12. It is convenient, to that effect, to split u into its even and odd parts and examine its $\mathcal{K}$-transforms one at a time. So far as estimates are concerned, we shall concentrate on the study of $(\mathcal{K}\, u)_0$ under the assumption that u is even: the complications related to the study of the three other types of $\mathcal{K}$-transforms are of an inessential nature, only calling for the consideration of multi-indices of possibly odd length rather than even, in which case another appeal to Proposition 5.2 may be needed. From Lemma 5.1, we get the identity

$$(\mathcal{K}\,(x_j\, x_\ell\, u))_0(Z) = \frac{i}{\pi}\, \mathcal{D}^{(j\ell)}\, (\mathcal{K}\, u)_0(Z) \tag{5.39}$$

with $\mathcal{D}^{(j\ell)}$ as defined there. Starting from (5.37), and noting that, from their definition in Lemma 5.1, the operators $\mathcal{D}^{(j\ell)}$ commute pairwise, we find, with $\tilde{\alpha}$ as introduced in (5.38),

$$(\mathcal{K}\, u_\eta)_0(Z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|\ \text{even}}} \frac{(2\pi^{\frac{1}{2}}\, e^{-\frac{i\pi}{4}}\, \eta)^\alpha}{\alpha\,!}\, \mathcal{D}^{\tilde{\alpha}}\, (\mathcal{K}\, u)_0(Z) : \tag{5.40}$$

finally, given any compact subset of the open, dense and connected subset Ω of $\Sigma^{(n)}$ to which the $\mathcal{K}$-transforms of u extend as analytic functions, there is some constant $C > 0$ such that the estimate

$$|\, \mathcal{D}^{\tilde{\alpha}}\, (\mathcal{K}\, u)_0(Z)\, | \leq \left[\frac{\alpha}{2}\right]!\ C^{\frac{|\alpha|}{2}}, \tag{5.41}$$

holds there, a consequence of the analyticity of the coefficients of the operator $\mathcal{D}^{(j\ell)}$ on the compact space $\Sigma^{(n)}$: the analyticity on Ω of the function $(\mathcal{K}\,u_\eta)_0$ follows. □

Under the assumption that the point $e^{\pm i\pi}\,I \in \Sigma^{(n)}$ lies in the domain of analyticity of the $\mathcal{K}$-transforms of u, we shall exploit later the identity

$$\mathrm{Int}\,[u_\eta] = \sum_{\alpha\in\mathbb{N}^n} \frac{(-2i\pi\,\eta)^\alpha}{\alpha\,!}\,\mathrm{Int}\,[x \mapsto x^\alpha\,u(x)], \tag{5.42}$$

a consequence of (5.40) together with (5.39) and Definition 4.12: it is valid without any assumption of parity regarding u.

Theorem 5.6. *Let $u \in \mathfrak{A}^{(n)}$, satisfying the property that the point $e^{i\pi}\,I \in \Sigma^{(n)}$ lies in the domain of analyticity of its $\mathcal{K}$-transforms, and let $x \in \mathbb{R}^n$. Setting $u_x(y) = u(y)\,e^{-2i\pi\,\langle x,y\rangle}$, so that the function u_x lies in $\mathfrak{A}^{(n)}$ according to Theorem* 5.5, *we set*

$$(\mathcal{F}_{\mathrm{ana}}u)(x) = \mathrm{Int}\,[u_x] \tag{5.43}$$

and call the transform $u \mapsto \mathcal{F}_{\mathrm{ana}}u$ the anaplectic Fourier transformation. The function $\mathcal{F}_{\mathrm{ana}}\,u$ is well defined as an element of the space $\mathfrak{A}^{(n)}$. One has, for every $u \in \mathfrak{A}^{(n)}$,

$$\mathcal{F}_{\mathrm{ana}}\left(\frac{1}{2i\pi}\,\frac{\partial u}{\partial x_j}\right) = x_j\,(\mathcal{F}_{\mathrm{ana}}\,u) \qquad \text{and} \qquad \mathcal{F}_{\mathrm{ana}}\,(x_j\,u) = -\frac{1}{2i\pi}\,\frac{\partial}{\partial x_j}\,(\mathcal{F}_{\mathrm{ana}}\,u). \tag{5.44}$$

Proof. That, for every $x \in \mathbb{C}^n$, the function $y \mapsto u_x(y)$ lies in $\mathfrak{A}^{(n)}$ was proved in Theorem 5.5. Then, by definition, $\mathrm{Int}\,[u_x] = 2^{\frac{n}{4}}\,(\mathcal{K}\,u_x)_0(e^{i\pi}\,I)$, and we first prove that, as a function of x, this is an entire function majorized by $C\,e^{\pi R|x|^2}$ for some pair (C, R). Indeed, assuming, say, that u is even, we find, starting from (5.40) and using (5.41) in the middle of the sequence that follows,

$$\begin{aligned} |\,\mathrm{Int}\,[u_x]| \;&=\; 2^{\frac{n}{4}}\left|\;\sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\ \mathrm{even}}} \frac{(2\pi^{\frac12}\,e^{-\frac{i\pi}{4}}\,x)^\alpha}{\alpha\,!}\,\mathcal{D}^{\tilde\alpha}\,(\mathcal{K}\,u)_0(e^{i\pi}\,I)\right| \\ &\le\; 2^{\frac{n}{4}}\sum_{\beta\in\mathbb{N}^n}\frac{\beta\,!}{(2\beta)\,!}\,(4\pi\,C)^{|\beta|}\,x^{2\beta} \\ &\le\; 2^{\frac{n}{4}}\,\exp\,(4\pi C\,|x|^2)\,: \end{aligned} \tag{5.45}$$

a similar estimate holds in the case when u is odd, using the analyticity property of the vector-valued function $(\mathcal{F}_{\mathrm{ana}}\,u)_1$.

Next, we show that one has

$$\begin{aligned} &(\mathcal{K}\,(\mathcal{F}_{\mathrm{ana}}\,u))_0(Z) = (\mathcal{K}\,u)_0(e^{i\pi}\,Z), \\ &(\mathcal{K}\,(\mathcal{F}_{\mathrm{ana}}\,u))_1(Z) = -i\,(\mathcal{K}\,u)_1(e^{i\pi}\,Z)\,: \end{aligned} \tag{5.46}$$

one may observe at once that this amounts to a verification of (5.6) if one decides that the anaplectic representation should associate the anaplectic Fourier transformation to the matrix $g = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Starting from (5.42), one finds

$$(\mathcal{F}_{\text{ana}}\, u)(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-2i\pi)^{|\alpha|}}{\alpha\,!}\, x^\alpha \operatorname{Int}[y \mapsto y^\alpha\, u(y)]. \tag{5.47}$$

We now prove the second equation (5.44). Let (j) be the multi-index of length 1 with 1 at the jth place: from (5.47),

$$-\frac{1}{2i\pi}\frac{\partial}{\partial x_j}(\mathcal{F}_{\text{ana}}\, u)(x) = -\frac{1}{2i\pi} \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta_j \geq 1}} \frac{(-2i\pi)^{|\beta|}}{\beta\,!}\, \beta_j\, x^{\beta-(j)} \operatorname{Int}[y \mapsto y^\beta\, u(y)]\,: \tag{5.48}$$

it suffices to set $\beta = \alpha + (j)$ to find the expression, again obtained from (5.47), of $(\mathcal{F}_{\text{ana}}\,(y_j\, u))(x)$. The same trick works towards the proof of the first equation (5.44).

Next, assuming $\sigma \succ RI$ with R large, one can write (using (5.47))

$$\begin{aligned}(\mathcal{Q}\,\mathcal{F}_{\text{ana}}\, u)_0(\sigma) &= \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\,(\mathcal{F}_{\text{ana}}\, u)(x\, e^{-\frac{i\pi}{4}})\, dx \\ &= \sum_\alpha \frac{(-2i\pi)^{|\alpha|}}{\alpha\,!} \operatorname{Int}[y \mapsto y^\alpha\, u(y)] \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\,(x\, e^{-\frac{i\pi}{4}})^\alpha\, dx \\ &= \sum_\alpha \frac{(-2\pi\, e^{\frac{i\pi}{4}})^{|\alpha|}}{\alpha\,!} \operatorname{Int}[y \mapsto y^\alpha\, u(y)] \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\, x^\alpha\, dx, \end{aligned} \tag{5.49}$$

where the summability is ensured by the estimate

$$|\operatorname{Int}[y \mapsto y^\alpha\, u(y)]| \leq \left[\frac{\alpha}{2}\right]!\, C^{\frac{|\alpha|}{2}}, \tag{5.50}$$

a consequence of the definition (4.42) of the linear form Int together with (5.39) and (5.41), and the easily proven inequality, valid when $\sigma - RI \succ 0$,

$$\left|\int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\, x^\alpha\, dx\right| \leq (\pi\, R)^{-\frac{n+|\alpha|}{2}} \left[\frac{\alpha}{2}\right]!. \tag{5.51}$$

The same estimates make it possible to write, under the same assumptions regarding σ,

$$\begin{aligned}(\mathcal{Q}\,\mathcal{F}_{\text{ana}}\, u)_0(\sigma) &= \operatorname{Int}\left[y \mapsto u(y) \sum_\alpha \frac{(-2\pi\, e^{\frac{i\pi}{4}}\, y)^\alpha}{\alpha\,!} \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle}\, x^\alpha\, dx\right] \\ &= \operatorname{Int}\left[y \mapsto u(y) \int_{\mathbb{R}^n} e^{-\pi\langle \sigma x, x\rangle} \exp\left(-2\pi\, e^{\frac{i\pi}{4}}\, \langle x, y\rangle\right) dx\right] \\ &= \operatorname{Int}\left[y \mapsto u(y)\,(\det \sigma)^{-\frac{1}{2}}\, e^{i\pi\langle \sigma^{-1} y, y\rangle}\right]. \end{aligned} \tag{5.52}$$

Thus, applying the definition contained in Theorem 4.11, we find, under the assumption that Z lies in the domain denoted as V^+ there,

$$(\mathcal{K}\,\mathcal{F}_{\mathrm{ana}}\,u)_0(Z) = |\det(I+Z)|^{-\frac{1}{2}}\,\mathrm{Int}\left[y \mapsto u(y)\,\exp\left(\pi\,\langle(\frac{I-Z}{I+Z})\,y,y\rangle\right)\right]: \quad (5.53)$$

we may then apply (4.53), finding as a result

$$(\mathcal{K}\,\mathcal{F}_{\mathrm{ana}}\,u)_0(Z) = |\det(I+Z)|^{-\frac{1}{2}}\,2^{\frac{n}{2}}\,(\det(I+C^2))^{-\frac{1}{4}}\,(\mathcal{K}\,u)_0\left(e^{i\pi}\,\frac{I-i\,C}{I+i\,C}\right) \quad (5.54)$$

with $C = \frac{1}{i}\,\frac{I-Z}{I+Z} \in \mathrm{Sym}_n$, which reduces, since then $\frac{I-i\,C}{I+i\,C} = Z$ and $\det(I+C^2) = 2^{2n}\,|\det(I+Z)|^{-2}$, to the first of the two equations (5.46).

In order to prove the second equation, we start with a convenient expression of $(\mathcal{K}\,u)_1(Z)$ for $u \in \mathfrak{A}^{(n)}$ and $Z \in V^+$: recalling (4.2), we find from (4.39) the equation

$$(\mathcal{K}\,u)_1(Z) = |\det(I-Z)|^{-\frac{1}{2}}\int_{\mathbb{R}^n}\left(-\frac{2\,Z}{I-Z}\right)x.\,\exp\left(-i\pi\left\langle\left(\frac{I+Z}{I-Z}\right)x,x\right\rangle\right)$$
$$u(x\,e^{-\frac{i\pi}{4}})\,dx \quad (5.55)$$

or, extending, componentwise, the definition of the first $\mathcal{Q}$-transform to the case of vector-valued functions,

$$(\mathcal{K}\,u)_1(Z) = e^{\frac{i\pi}{4}}\left(\mathcal{K}\left(x \mapsto (-\frac{2\,Z}{I-Z})\,x.\,u(x)\right)\right)_0(Z). \quad (5.56)$$

To compute $(\mathcal{Q}\,\mathcal{F}_{\mathrm{ana}}\,u)_1^{(j)}(\sigma)$ according to (4.3), we have to insert, on the second line of (5.52), the extra factor $x_j + i\sum_k \sigma_{jk}\,x_k$, which would also arise from the application to the integral under the integral sign of the operator $-\frac{e^{-\frac{i\pi}{4}}}{2\pi}\left[\frac{\partial}{\partial y_j} + i\sum_k \sigma_{jk}\,\frac{\partial}{\partial y_k}\right]$: with the help of (5.8), this leads to the equation

$$(\mathcal{Q}\,\mathcal{F}_{\mathrm{ana}}\,u)_1^{(j)}(\sigma) = -e^{\frac{i\pi}{4}}\,\mathrm{Int}\left[y \mapsto u(y)\,(\det\sigma)^{-\frac{1}{2}}\,e^{i\pi\,\langle\sigma^{-1}y,y\rangle}\right.$$
$$\left.\sum_\ell\left((\sigma^{-1})_{j\ell} + i\sum_k \sigma_{jk}\,(\sigma^{-1})_{k\ell}\right)y_\ell\right] \quad (5.57)$$

or

$$(\mathcal{Q}\,\mathcal{F}_{\mathrm{ana}}\,u)_1^{(j)}(\sigma) = e^{-\frac{i\pi}{4}}\,\mathrm{Int}\left[y \mapsto u(y)\,(\det\sigma)^{-\frac{1}{2}}\,e^{i\pi\,\langle\sigma^{-1}y,y\rangle}\right.$$
$$\left.\left(y_j - i\sum_\ell(\sigma^{-1})_{j\ell}\,y_\ell\right)\right] \quad (5.58)$$

or finally, in vector form,

$$(\mathcal{Q}\mathcal{F}_{\rm ana}u)_1(\sigma)=e^{-\frac{i\pi}{4}}\,(\det\sigma)^{-\frac{1}{2}}\,{\rm Int}\left[y\mapsto(I-i\sigma^{-1})y.u(y)\,e^{i\pi\,\langle\sigma^{-1}y,y\rangle}\right]. \tag{5.59}$$

From the last equation, applying again (4.39) and (4.53), we find for $Z\in V^+$ the equation

$$\begin{aligned}(\mathcal{K}\,\mathcal{F}_{\rm ana}\,u)_1(Z) &= |\det\,(I-Z)|^{-\frac{1}{2}}\,(\mathcal{Q}\,\mathcal{F}_{\rm ana}\,u)_1\left(i\,\frac{I+Z}{I-Z}\right)\\ &= |\det\,(I-Z)|^{-\frac{1}{2}}\,e^{-\frac{i\pi}{4}}\left(\det i\,\frac{I+Z}{I-Z}\right)^{-\frac{1}{2}}\,2^{\frac{n}{4}}\,(\det\,(I+C^2))^{-\frac{1}{4}}\\ &\qquad\left(\mathcal{K}\left(y\mapsto\,\frac{2\,Z}{I+Z}\,y.\,u(y)\right)\right)_0\left(e^{i\pi}\,\frac{I-i\,C}{I+i\,C}\right),\end{aligned} \tag{5.60}$$

with $C=\frac{1}{i}\,\frac{I-Z}{I+Z}\in{\rm Sym}_n$: finishing the computation as was done right after (5.54) and comparing the equations (5.60) and (5.56), one sees that the second equation (5.46) is satisfied.

The proof of Theorem 5.6 is over. It is essential, at the same time, to remark that the identities (5.18) and (5.20) used in the proof of Proposition 5.2, together with (5.47), show that the knowledge of the $\mathcal{Q}$, or $\mathcal{K}$-transforms of $u\in\mathfrak{A}^{(n)}$ entails that of the function u itself: note that we have not characterized, as this was not needed, the image under the pair $u\mapsto((\mathcal{Q}\,u)_0,\,(\mathcal{Q}\,u)_1)$ of the space $\mathfrak{A}^{(n)}$. □

The space $\mathfrak{A}^{(n)}$ is invariant under the Heisenberg representation:

Theorem 5.7. *Let $u\in\mathfrak{A}^{(n)}$. For every $(y,\eta)\in\mathbb{C}^n\times\mathbb{C}^n$, the functions $u_\eta\colon x\mapsto u(x)\,e^{2i\pi\,\langle\eta,x\rangle}$ and $v_y\colon x\mapsto u(x-y)$ lie in $\mathfrak{A}^{(n)}$. The singular set of the $\mathcal{K}$-transform of v_y* (i.e., *the complement, in $\Sigma^{(n)}$, of the open set where both $\mathcal{K}$-transforms of v_y are analytic*) *is independent of y, and the linear form* Int*, when defined, is invariant under translations.*

Proof. The fact that the first function lies in $\mathfrak{A}^{(n)}$ was the object of Theorem 5.5: the proof also showed that the singular set of u_η is independent of η. It relied, essentially, on the fact that the coefficients of the operators $\mathcal{D}^{(j\ell)}$, involved in the proof of Proposition 5.2, are analytic on the whole of Σ: the same proof will do for the function v_y, only replacing the family of operators $\{\mathcal{D}^{(j\ell)}\}$ by the family of operators $\{L^{(j\ell)},\,M^{(j\ell)}\}$, the conjugates under Θ of the operators introduced in (5.28). The equation (5.37) must be replaced by

$$\begin{aligned}(\mathcal{Q}\,v_y)_0(\sigma) &= e^{-i\pi\,\langle\sigma a,a\rangle}\int_{\mathbb{R}^n} e^{-\pi\,\langle\sigma x,x\rangle}\;\exp\left(-2\pi\,e^{\frac{i\pi}{4}}\,\langle\sigma y,x\rangle\right)\,u(x\,e^{-\frac{i\pi}{4}})\,dx,\\ &= e^{-i\pi\,\langle\sigma a,a\rangle}\int_{\mathbb{R}^n} e^{-\pi\,\langle\sigma x,x\rangle}\sum_{\alpha\in\mathbb{N}^n}\frac{(-2\pi\,e^{\frac{i\pi}{4}}\,\sigma y)^\alpha}{\alpha\,!}\,x^\alpha\,u(x\,e^{-\frac{i\pi}{4}})\,dx:\end{aligned} \tag{5.61}$$

then, for $Z \in V^+$, the definition given in Theorem 4.11 yields

$$(\mathcal{K}\, v_y)_0(Z) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} \frac{(2\pi^{\frac{1}{2}}\, e^{-\frac{i\pi}{4}})^{|\alpha|}}{\alpha\,!} \left(i\, \frac{I+Z}{I-Z}\cdot y\right)^{\alpha} \mathcal{D}^{\tilde{\alpha}}\, (\mathcal{K}\, u)_0(Z), \tag{5.62}$$

a sum all terms of which, except the first one, vanish at $Z = e^{i\pi}\, I$ so that, applying the definition (4.42) of the linear form Int, one sees that $\mathrm{Int}\,[v_y] = \mathrm{Int}\,[u]$. To answer a possible question, let us note that it would not have been possible to reduce the study of the operator $u \mapsto v_y$ to that of the operator $u \mapsto u_\eta$ by an application of the obvious exponentiated analogue of (5.44) because, contrary to the definition of v_y, that of $\mathcal{F}_{\text{ana}}\, u$ demands that the point $e^{i\pi}\, I$ should lie outside the singular set of the $\mathcal{K}$-transform of u. □

As a generalization of Definition 1.19, we now introduce the general anaplectic group.

Definition 5.8. The anaplectic group $\mathrm{Sp}_i(n,\mathbb{R})$ is the matrix group generated by the symplectic group together with the element (in block-form) $\begin{pmatrix} -i\, I & 0 \\ 0 & i\, I \end{pmatrix}$.

We now complete our list of generators of the anaplectic representation by the definition of the transformation $\mathcal{R}$ of the space $\mathfrak{A}^{(n)}$ that shall eventually be associated with the above matrix.

Theorem 5.9. *Set* $u_i(z) = u(i\, z)$, $z \in \mathbb{C}^n$. *The linear map* $\mathcal{R}\colon u \mapsto u_i$ *is an automorphism of the space* $\mathfrak{A}^{(n)}$. *The* $\mathcal{K}$*-transforms of the function* u_i *are given by the equations*

$$\begin{aligned} (\mathcal{K}\, u_i)_0(Z) &= (\mathcal{K}\, u)_0(Z^{-1}), \\ (\mathcal{K}\, u_i)_1(Z) &= i\, Z\, (\mathcal{K}\, u)_1(Z^{-1}). \end{aligned} \tag{5.63}$$

Here, in its occurrence as the argument of the $\mathcal{K}$*-transform of* u, Z^{-1} *is defined by means of the global analytic automorphism of* $\Sigma^{(n)}$ *that extends the map* $Z \mapsto Z^{-1}$ *from* V^+ *to* V *as introduced in Theorem* 4.11.

Proof. Setting $u(x) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha\, x^\alpha$, one finds as a consequence of Corollary 4.3 and with the notation of the proof there of the equation, valid if $\sigma - R\, I \succ 0$ for some large R,

$$(\mathcal{Q}\, u)_0(\sigma) = (\det\sigma)^{-\frac{1}{2}}\, (\mathcal{Q}^{\text{inv}}\, u)_0(\sigma^{-1}) = (\det\sigma)^{-\frac{1}{2}} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha| \text{ even}}} c_\alpha\, e^{-\frac{i\pi\,|\alpha|}{4}}\, I_\alpha(\sigma^{-1}). \tag{5.64}$$

To compute $(\mathcal{Q}\, u_i)_0(\sigma)$, we must substitute for the argument $x\, e^{-\frac{i\pi}{4}}$ of the function u under the integral in (4.20) the new value $x\, e^{\frac{i\pi}{4}}$ so that we must multiply

$I_\alpha(\sigma^{-1})$ by $e^{\frac{i\pi\,|\alpha|}{2}}$, getting

$$(\mathcal{Q}\,u_i)_0(\sigma) = (\det\sigma)^{-\frac{1}{2}} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\text{ even}}} c_\alpha\, e^{\frac{i\pi\,|\alpha|}{4}}\, I_\alpha(\sigma^{-1}) \tag{5.65}$$

as a result. Consequently, for $Z \in V^+$, the subset of Σ or $\Sigma^{(n)}$ introduced in Theorem 4.11,

$$(\mathcal{K}\,u)_0(Z) = |\det(I-Z)|^{-\frac{1}{2}} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\text{ even}}} c_\alpha\, e^{-\frac{i\pi\,|\alpha|}{4}}\, I_\alpha\left(-i\,\frac{I-Z}{I+Z}\right),$$

$$(\mathcal{K}\,u_i)_0(Z) = |\det(I-Z)|^{-\frac{1}{2}} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\text{ even}}} c_\alpha\, e^{\frac{i\pi\,|\alpha|}{4}}\, I_\alpha\left(-i\,\frac{I-Z}{I+Z}\right). \tag{5.66}$$

Now, changing Z to Z^{-1} changes $-i\,\frac{I-Z}{I+Z}$ to its negative: since, as it follows from the proof of Lemma 4.2, the polynomial I_α is homogeneous of degree $\frac{|\alpha|}{2}$ in the entries of its matrix argument, one finds that the first of the two equations (5.63) is indeed satisfied. There are two new changes to consider if interested in the vector-valued $\mathcal{K}$-transform of u_i in terms of that of u. First, from (4.2), we have to change the extra matrix $I+i\,\sigma$ to $I-i\,\sigma$, which amounts, in the transformation from $(\mathcal{K}\,u)_1(Z)$ to $(\mathcal{K}\,u_i)_1(Z)$, to inserting the extra matrix $\frac{I-i\,\tau}{I+i\,\tau}$ computed at $\tau = i\,\frac{I+Z^{-1}}{I-Z^{-1}}$, *i.e.,* the matrix $-Z$: next, with obvious notations, one has $x\,u(x\,e^{-\frac{i\pi}{4}}) = e^{\frac{i\pi}{4}}\,(x\,u)(x\,e^{-\frac{i\pi}{4}})$ but $x\,u(x\,e^{\frac{i\pi}{4}}) = e^{-\frac{i\pi}{4}}\,(x\,u)(x\,e^{\frac{i\pi}{4}})$ so that we must also insert an extra factor $-i$ in order to get a correct formula, finally ending up with the second equation (5.63). Considering the analytic continuation of $(\mathcal{K}\,u_i)_0$ or $(\mathcal{K}\,u_i)_1$ as characterized by this pair of equations, one sees that u_i lies in the space $\mathfrak{A}^{(n)}$. □

Now, we note that the symplectic group is generated by the set of matrices of the species $\begin{pmatrix} A & 0 \\ 0 & A'^{-1}\end{pmatrix}$, $\begin{pmatrix} I & 0 \\ C & 0\end{pmatrix}$ together with the matrix $\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$. Indeed, from Lemma 4.6, any matrix g in the symplectic group decomposes there as

$$g = \begin{pmatrix} A_1 & 0 \\ C_1 & {A'_1}^{-1}\end{pmatrix}\begin{pmatrix} R & S \\ -S & R\end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & {A'_1}^{-1}\end{pmatrix}\begin{pmatrix} I & 0 \\ A'_1\,C_1 & I\end{pmatrix}\begin{pmatrix} R & S \\ -S & R\end{pmatrix} \tag{5.67}$$

with a matrix $\begin{pmatrix} R & S \\ -S & R\end{pmatrix}$ associated to some unitary matrix $R+i\,S$.

Next, if S is invertible,

$$\begin{pmatrix} R & S \\ -S & R\end{pmatrix} = \begin{pmatrix} S & 0 \\ R & S+R\,S^{-1}\,R\end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\begin{pmatrix} I & 0 \\ S^{-1}\,R & I\end{pmatrix}. \tag{5.68}$$

Finally, even if S is not invertible, the matrix $-R\,\sin\varepsilon + S\,\cos\varepsilon$ which occurs in the product $\begin{pmatrix} R & S \\ -S & R\end{pmatrix}\begin{pmatrix} \cos\varepsilon\, I & -\sin\varepsilon\, I \\ \sin\varepsilon\, I & \cos\varepsilon\, I\end{pmatrix}$ is for generic values of ε. Indeed, since

$S\,R' = R\,S'$, the matrix $R'\,S$ preserves $\mathrm{Im}\,(R'\,R)$, and $R'\,(-R\,\sin\varepsilon + S\,\cos\varepsilon)$ is generically invertible as an endomorphism of that space; similarly, $S'\,(-R\,\sin\varepsilon + S\,\cos\varepsilon)$ is an invertible endomorphism of $\mathrm{Im}\,(S'\,S)$ except for a finite set of values of ε: finally, $R'\,R + S'\,S = I$.

Remark 5.2. We have not characterized the image of $\mathfrak{A}^{(n)}$ under the $\mathcal{Q}$ or $\mathcal{K}$-transforms: still, one may note that the scalar $\mathcal{Q}$-transform of any $u \in \mathfrak{A}^{(n)}$ must lie in the kernel of every differential operator of the kind $\frac{\partial}{\partial\sigma_{(jk)}}\,\frac{\partial}{\partial\sigma_{(\ell m)}} - \frac{\partial}{\partial\sigma_{(j\ell)}}\,\frac{\partial}{\partial\sigma_{(km)}}$, as a consequence of the first equation (5.8). What we need to know, however, is that a function u is characterized by its pair of $\mathcal{Q}$ or $\mathcal{K}$-transforms: obviously, it amounts to the same, in this context, to use the transforms of one or the other species, and we shall use the first ones. Denoting, for $r > 0$ as $u_{\mathrm{rad}}(r)$ the average of the function u on the sphere of $\mathbb{R}^n$ centered at the origin with radius r, it follows from (4.1) and the elementary properties of the Laplace transformation that if $u \in \mathfrak{A}^{(n)}$ is such that $(\mathcal{Q}\,u)_0(\sigma) = 0$ for all $\sigma \succ R\,I$ with R large enough, then u_{rad} is identically zero. Using Lemma 5.1 and (5.18), one sees that, assuming also that the vector-valued $\mathcal{Q}$-transform of u vanishes under the same assumption regarding σ, the radial part of the product of u by an arbitrary polynomial must be zero too, thus u itself must be zero.

We are now in a position to state the main result of the present section. Before doing so, let us refer the reader again to the slight abuse of language explained right after (5.6): in the two equations below, $[g^{-1}](Z)$, as an argument of the function $(\mathcal{K}\,u)_0$ or $(\mathcal{K}\,u)_1$, denotes an element of $\Sigma^{(n)}$; but in its last occurrence in the second equation, it denotes the corresponding element of $\Sigma = U(n) \cap \mathrm{Sym}_n^{\mathbb{C}}$. The same could be said about the occurrence of Z as an argument of the Radon–Nikodym derivative in both formulas, though in this case it does not really matter: for all transformations $[g^{-1}]$ of $\Sigma^{(n)}$ associated with elements of $\mathrm{Sp}^{(n)}_{\mathrm{comp}}(n,\mathbb{R})$ (*cf.* remark following Definition 4.8) are measure-preserving.

Theorem 5.10. *Let $g \in \mathrm{Sp}^{(n)}(n,\mathbb{R})$ and $u \in \mathfrak{A}^{(n)}$, and assume that the point $[g^{-1}](I)$, as defined as an element of $\Sigma^{(n)}$ by* (5.5) *together with Corollary* 4.10*, does not lie in the singular set of the pair $\mathcal{K}\,u\colon = ((\mathcal{K}\,u)_0,\, (\mathcal{K}\,u)_1)$. Then, there is a unique function $v \in \mathfrak{A}^{(n)}$ such that the pair of equations introduced in* (5.6)

$$(\mathcal{K}\,v)_0(Z) = \left(\frac{[g^{-1}]_*\,d\mu}{d\mu}\,(Z)\right)^{\frac{1}{2(n+1)}}\ (\mathcal{K}\,u)_0([g^{-1}]\,(Z)), \tag{5.69}$$

$$(\mathcal{K}\,v)_1(Z) = \left(\frac{[g^{-1}]_*\,d\mu}{d\mu}\,(Z)\right)^{\frac{1}{2(n+1)}}\ \left[\alpha - i\,\beta\,([g^{-1}]\,(Z))^{-1}\right].\,(\mathcal{K}\,u)_1([g^{-1}]\,(Z))$$

holds. Setting $v = \mathrm{Ana}(g)\,u$, one gets a representation by means of partially defined operators in the following sense: if g and g_1 lie in $\mathrm{Sp}^{(n)}(n,\mathbb{R})$, if I does not lie in the image under $[g]$ or under $[g_1\,g]$ of the singular set of $\mathcal{K}\,u$, one has

$$\mathrm{Ana}(g_1)\,\mathrm{Ana}(g)\,u = \mathrm{Ana}(g_1\,g)\,u. \tag{5.70}$$

This representation extends as a representation of the group $\mathrm{Sp}^{(n)}_{i}(n,\mathbb{R})$*, the covering of the anaplectic group generated by* $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ *together with the matrix* $\begin{pmatrix} -e^{\frac{i\pi}{2}}I & 0 \\ 0 & e^{\frac{i\pi}{2}}I \end{pmatrix}$*, in which the transformation associated with this matrix is the transformation* $\mathcal{R}$ *introduced in Theorem* 5.9.

Finally, the anaplectic representation combines with the Heisenberg representation in a way which can be characterized by the following formulas, involving the (vector-valued) infinitesimal operators $Q=(x_j)_{1\leq j\leq n}$ *and* $P=(\frac{1}{2i\pi}\frac{\partial}{\partial x_j})_{1\leq j\leq n}$ *of the latter one:*

$$\mathrm{Ana}\begin{pmatrix} A & B \\ C & D \end{pmatrix} Q\,\mathrm{Ana}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\right) = D'\,Q - B'\,P,$$

$$\mathrm{Ana}\begin{pmatrix} A & B \\ C & D \end{pmatrix} P\,\mathrm{Ana}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\right) = -C'\,Q + A'\,P. \tag{5.71}$$

Proof. What is left to do is only combining results already obtained here and there, though there is quite a number of details to check. First, if $\mathrm{Ana}(g)\,u$ is well defined, the singular set of its $\mathcal{K}$-transform is included in the image under $[g]$ of the singular set of the $\mathcal{K}$-transform of u: the two conditions before (5.70), which mean that $\mathrm{Ana}(g)\,u$ and $\mathrm{Ana}(g_1\,g)\,u$ are well defined as elements of $\mathfrak{A}^{(n)}$, thus imply that the left-hand side of this equation is well defined too.

We start with the verification of (5.69) for g in some set of generators of the group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$. If $g=\left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$ with $\det A>0$ or $g=\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$, we are dealing with a matrix, in the symplectic group, that connects to the identity matrix by means of a path within a set of matrices of the same species, so that it can also, in a canonical way, be interpreted as an element of the covering group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$, and there are no difficulties involved in the distinction between this space and the symplectic group itself. The image, under the anaplectic representation, of these two matrices, is defined as

$$\left(\mathrm{Ana}\left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)u\right)(x) = (\det A)^{-\frac{1}{2}}\,u(A^{-1}x),$$

$$\left(\mathrm{Ana}\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)u\right)(x) = u(x)\,e^{i\pi\,\langle Cx,x\rangle}\;: \tag{5.72}$$

we now check that the equations (5.69) are correct in this case. In the first (*resp.* the second) case, we have, with the recipe provided in (5.2), $(\alpha=\frac{1}{2}(A+A'^{-1}),\ \beta=\frac{i}{2}(A'^{-1}-A))$ (*resp.* $(\alpha=I+\frac{iC}{2},\ \beta=\frac{C}{2})$), which makes it immediate to check that $[g^{-1}]$ coincides in the two cases under study with the transformation f (*resp.* h) introduced in the third line of (4.33) or the second one of (4.37): then, the equations concerned with the scalar $\mathcal{K}$-transforms were given in (4.45) and (4.50); finally, the extra matrix $\alpha-i\beta([g^{-1}](Z))^{-1}$ is the same as that which occurred in the proof of Theorem 4.13, in (4.48) or (4.52).

Let us consider now the matrix $g = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$: it can also be considered as an element of $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ if one writes it, in the usual way, as $g = \left(\begin{smallmatrix} (\cos\frac{\pi}{2})\,I & (\sin\frac{\pi}{2})\,I \\ -(\sin\frac{\pi}{2})\,I & (\cos\frac{\pi}{2})\,I \end{smallmatrix}\right)$. We associate with it the anaplectic Fourier transformation $\mathcal{F}_{\text{ana}}$: in this case, $\alpha = -i\,I$, $\beta = 0$, which would lead us to setting $[g^{-1}](Z) = -Z$. However, from the remark following Definition 4.8, since $(\det((\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})\,I))^{\frac{1}{2}} = e^{\frac{i\pi n}{4}}$, $[g^{-1}]$ must be regarded more properly as the automorphism of $\Sigma^{(n)}$ such that $[g^{-1}](Z) = e^{i\pi}\,Z$: that the equations (5.69) are correct with this meaning are then a consequence of (5.46).

The elements of group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ considered so far constitute a set of generators in the case when the dimension is odd. What remains to be considered, as a consequence of (5.67) and (5.68), but only in the case when n is even, is the case of a matrix $g = \left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$ with $\det A < 0$, and orthogonal if so wished. One can extend the definition given in (5.72), but only after one has specified a determination of a square root of $\det A$, which requires of course that one should substitute for the matrix $g \in \mathrm{Sp}(n,\mathbb{R})$ one of the two elements of the metaplectic group above it: since $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ is a covering of the metaplectic group when n is even, this choice can be made within the former group. There is then a unique choice that makes the equations (5.69) valid.

In order to prove that our present definition of $\mathrm{Ana}(g)$ for g in a set of generators of the group $\mathrm{Sp}^{(n)}(n,\mathbb{R})$ extends as a representation, we can then consider instead the same problem dealing, in place of functions $u \in \mathfrak{A}^{(n)}$, with pairs (ψ_0, ψ_1) of functions on $\Sigma^{(n)}$ (the second one vector-valued), subject to the transformations characterized by the pair of equations (5.69). In view of Corollary 4.10, the first of these two equations certainly defines a group action: what remains to be done is tracing the effect of the extra linear factor in the second equation, and checking that it leads to the equation (5.70) too. Setting $\begin{pmatrix} \alpha_2 & \beta_2 \\ \bar\beta_2 & \bar\alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \bar\beta_1 & \bar\alpha_1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$, one has to check the equation

$$\begin{aligned} &\alpha_2 - i\,\beta_2\,([(g_1\,g)^{-1}](Z))^{-1} \\ &\quad = \left(\alpha_1 - i\,\beta_1\,([g_1^{-1}](Z))^{-1}\right)\,\left(\alpha - i\,\beta\,([(g_1\,g)^{-1}](Z))^{-1}\right) \end{aligned} \tag{5.73}$$

or, setting $W = [g_1^{-1}](Z)$, the equation

$$\begin{aligned} &\alpha_2 - i\,\beta_2\,([g^{-1}](W))^{-1} \\ &\quad = \left(\alpha_1 - i\,\beta_1\,W^{-1}\right)\,\left(\alpha - i\,\beta\,([g^{-1}](W))^{-1}\right) \end{aligned} \tag{5.74}$$

or, finally, the equation

$$\begin{aligned} &\alpha_2\,(i\,\beta' + \bar\alpha'\,W) - i\,\beta_2\,(\alpha' - i\,\bar\beta'\,W) \\ &\quad = (\alpha_1 - i\,\beta_1\,W^{-1})\,[\alpha\,(i\,\beta' + \bar\alpha'\,W) - i\,\beta\,(\alpha' - i\,\bar\beta'\,W)], \end{aligned} \tag{5.75}$$

where the difference of the coefficients of the matrices I, W and W^{-1} in the two sides are given respectively as

$$\begin{aligned}&i\,(\alpha_2\,\beta'-\beta_2\,\alpha')-i\,\alpha_1\,(\alpha\,\beta'-\beta\,\alpha')+i\,\beta_1\,(\alpha\,\bar{\alpha}'-\beta\,\bar{\beta}'),\\&\alpha_2\,\bar{\alpha}'-\beta_2\,\bar{\beta}'-\alpha_1\,(\alpha\,\bar{\alpha}'-\beta\,\bar{\beta}')\end{aligned}\tag{5.76}$$

and

$$-\beta(\alpha\,\beta'-\beta\,\alpha').\tag{5.77}$$

Using the equations $\alpha_2=\alpha_1\,\alpha+\beta_1\,\bar{\beta}$, $\beta_2=\alpha_1\,\beta+\beta_1\,\bar{\alpha}$, together with (5.3), this is the result of a straightforward computation.

In order to extend the representation Ana as a representation of $\mathrm{Sp}^{(n)}_{i}(n,\mathbb{R})$, we have to check that $\mathcal{R}^2=\mathrm{Ana}\left(\left(\begin{smallmatrix}e^{i\pi}I & 0\\ 0 & e^{i\pi}I\end{smallmatrix}\right)\right)$, which is obvious by the very definition of the transform on the right-hand side, and that

$$\mathcal{R}\ \mathrm{Ana}\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right)=\mathrm{Ana}\left(\begin{pmatrix}A & -B\\ -C & D\end{pmatrix}\right)\mathcal{R},\tag{5.78}$$

say for every $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\mathrm{Sp}(n,\mathbb{R})$ close to the identity so as not to have to distinguish between such a matrix and its interpretation as an element of $\mathrm{Sp}^{(n)}(n,\mathbb{R})$. Recalling (5.63), noting that the diffeomorphism $Z\mapsto Z^{-1}$ of Σ or $\Sigma^{(n)}$ is measure-preserving and that, under the map $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}A & -B\\ -C & D\end{smallmatrix}\right)$, the pair $(\alpha,\,\beta)$ changes to $(\bar{\alpha},\,-\bar{\beta})$, we only have to verify the equation concerned with the vector-valued $\mathcal{K}$-transforms of the two sides of (5.78), which demands checking that

$$\begin{aligned}&i\,Z\,[\alpha-i\,\beta\,(\alpha'-i\,\beta'\,Z^{-1})\,(i\,\beta'+\bar{\alpha}'\,Z^{-1})^{-1}]\\&\quad=[\bar{\alpha}+i\,\bar{\beta}\,(\bar{\alpha}'+i\,\beta'\,Z)\,(-i\,\bar{\beta}'+\alpha'\,Z)^{-1}]\,\times\,i\,(-i\,\bar{\beta}'+\alpha'\,Z)\,(\bar{\alpha}'+i\,\beta'\,Z)^{-1}:\end{aligned}\tag{5.79}$$

applying (5.3) again, this is a straightforward task.

Finally, checking (5.71) can be done by a case-by-case verification, assuming that $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ reduces to $\left(\begin{smallmatrix}A & 0\\ 0 & A'^{-1}\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}I & 0\\ C & I\end{smallmatrix}\right)$ or the element above $\left(\begin{smallmatrix}0 & I\\ -I & 0\end{smallmatrix}\right)$ that is associated with the anaplectic Fourier transformation: in the last case, this is just (5.44), while in the first two cases, it follows from (5.72).

Let us note that the version of (5.71) in which the Heisenberg representation itself, rather than its infinitesimal version, is involved, is not more difficult to ascertain. $\square$

6 Comparing the anaplectic and metaplectic representations

In the last section, we introduced the anaplectic representation and anaplectic Fourier transformation: to familiarize ourselves with it, we shall discuss here some matters pointing both to resemblances and differences with the usual analysis. One question that certainly calls for an answer is the reason why partial anaplectic Fourier transforms, or whichever transformations take their place, are so much harder to discuss than their classical counterparts; also, combining the new analysis with the theory of spherical harmonics, we shall bring to light the analogue of Hecke's theorem.

So as to really understand the profound similarity between the anaplectic and metaplectic representations, we shall start with a realization of the latter one in terms fully comparable to our construction of the former one: though the fact that the metaplectic representation is conveniently described through its action on Gaussian functions could not escape anyone interested in it [22, 12], the following does not seem (to our knowledge) to have been written in explicit terms.

Theorem 6.1. *Given* $u \in L^2(\mathbb{R}^n)$ *set, for* $\sigma \in \mathrm{Sym}_n^{\mathbb{C}}$ *with* $\mathrm{Re}\ \sigma \succ 0$,

$$(\mathcal{M}\,u)_0(\sigma) = \int_{\mathbb{R}^n} e^{-\pi\,\langle \sigma x, x\rangle}\, u(x)\, dx,$$
$$(\mathcal{M}\,u)_1(\sigma) = \int_{\mathbb{R}^n} (I + i\,\sigma)\, x.\, e^{-\pi\,\langle \sigma x, x\rangle}\, u(x)\, dx, \tag{6.1}$$

a definition which should of course be compared to (4.1), (4.2). *The metaplectic representation* Met *of* $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ *in* $L^2(\mathbb{R}^n)$ *can be traced on the* $\mathcal{M}$*-transforms as follows. For every element* g *of the metaplectic group* $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ *above some element* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *of the symplectic group, there is a continuous choice of a determination of the square root of* $\det(i\,B'\,\sigma + D')$ *for* $\sigma \in \mathrm{Sym}_n^{\mathbb{C}}$ *with* $\mathrm{Re}\ \sigma \succ 0$, *such that, for every* $u \in L^2(\mathbb{R}^n)$, *the following pair of equations, to be compared to Theorem* 5.10, *should hold:*

$$\begin{aligned}
&(\mathcal{M}\,\mathrm{Met}\,(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right))\ u)_0(\sigma)\\
&\qquad = [\det(i\,B'\,\sigma + D')]^{-\frac{1}{2}}\,(\mathcal{M}\,u)_0((A'\,\sigma - i\,C')\,(i\,B'\,\sigma + D')^{-1}),\\
&(\mathcal{M}\,\mathrm{Met}\,(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right))\ u)_1(\sigma)\\
&\qquad = [\det(i\,B'\,\sigma + D')]^{-\frac{1}{2}}\,(I + i\,\sigma)\,[i\,B'\,\sigma + D' + i\,(A'\,\sigma - i\,C')]^{-1}\\
&\qquad\qquad (\mathcal{M}\,u)_1((A'\,\sigma - i\,C')\,(i\,B'\,\sigma + D')^{-1}).
\end{aligned} \tag{6.2}$$

Proof. This is much easier than the corresponding theorem concerning the anaplectic representation, partly (but not only) due to the fact that we may rely on the already known existence of the metaplectic representation. Let us recall that it is characterized by the fact that it associates with the matrices $g = \left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$ with

$\det A > 0$ or $g = \left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$ the same transformations (acting this time on $L^2(\mathbb{R}^n)$) as those defined in (5.72), and that it associates with the matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, regarded as the value at the time $t = \frac{\pi}{2}$ of the block-matrix $\left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$ in $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$, the transformation $e^{-\frac{i\pi n}{4}} \mathcal{F}$. It then suffices to verify the following two facts: first, by a case-by-case study, that the equations (6.2) are correct whenever $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ has one of the special three forms indicated; next, that the set of equations (6.2) is compatible with the group structure.

For what concerns the first part, only the case of the matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is really different from the corresponding analysis in the proof of Theorem 5.10. One has the equation

$$(\mathcal{M}\,(e^{-\frac{i\pi n}{4}}\,\mathcal{F} u))_0(\sigma) = e^{-\frac{i\pi n}{4}}\,(\det\sigma)^{-\frac{1}{2}}\,(\mathcal{M}\,u)_0(\sigma) \tag{6.3}$$

and, a consequence of

$$\mathcal{F}\,((I + i\,\sigma)\,x.\,e^{-\pi\langle\sigma x, x\rangle}) = (\det\sigma)^{-\frac{1}{2}}\,(I - i\,\sigma^{-1})\,x.\,e^{-\pi\langle\sigma^{-1}x, x\rangle}, \tag{6.4}$$

the equation

$$(\mathcal{M}\,(e^{-\frac{i\pi n}{4}}\,\mathcal{F} u))_1(\sigma) = e^{-\frac{i\pi n}{4}}\,(\det\sigma)^{-\frac{1}{2}}\,\frac{I - i\,\sigma^{-1}}{I + i\,\sigma^{-1}}\,(\mathcal{M}\,u)_1(\sigma), \tag{6.5}$$

from which checking the pair of equations (6.2) is immediate.

For the second part, the first thing to do is of course checking that the map $(g, \sigma) \mapsto (D\,\sigma + i\,C)\,(-i\,B\,\sigma + A)^{-1}$, with $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, is a group action. Now this is the conjugate by the matrix $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ of the map $(g, \sigma) \mapsto (A\,\sigma - i\,B)\,(i\,C\,\sigma + D)^{-1}$: that this is an action follows from the fact [16, p. 32] that it corresponds to the usual action $Z \mapsto (A\,Z + B)\,(C\,Z + D)^{-1}$ on the generalized upper half-plane $\mathrm{Sym}_n^{\mathbb{C}} + i\,\Gamma_n$. The rest can be proved in just the same way as was developed, in the anaplectic case, between (5.73) and (5.76). All that has to be done concerns the extra linear factor that occurs in the vector-valued $\mathcal{M}$-transforms, and requires that we should check the identity

$$\begin{aligned} &[i\,B_1'\,\sigma + D_1' + i\,(A_1'\,\sigma - i\,C_1')]^{-1}\,[I + i\,(A_1'\,\sigma - i\,C_1')\,(i\,B_1'\,\sigma + D_1')^{-1}] \\ &\times [i\,B'\,(A_1'\,\sigma - i\,C_1')\,(i\,B_1'\,\sigma + D_1')^{-1} + D' + i\,A'\,(A_1'\,\sigma - i\,C_1')\,(i\,B_1'\,\sigma + D_1')^{-1} + C']^{-1} \\ &= [i\,(A_2' + B_2')\,\sigma + C_2' + D_2']^{-1} \end{aligned} \tag{6.6}$$

if $\left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, a straightforward task. □

Remark 6.1. The comparison between this presentation of the metaplectic representation and the definition, in Theorem 5.10, of the anaplectic representation, puts forward the role of polarizations of the phase space $\mathbb{R}^n \times \mathbb{R}^n$, considered only as a linear space with a symplectic structure (an alternate non-degenerate bilinear form denoted as $(X_1, X_2) \mapsto [X_1, X_2]$), in both constructions. In the first case (*cf.*

[4, 13] in connection with theta functions, [23, p. 218] in connection with pseudodifferential analysis), a complex polarization means a (linear) complex structure characterized by an automorphism J of the phase space such that $J^2 = -I$ satisfying the extra conditions that J should preserve the symplectic form and that the symmetric bilinear form $(X_1, X_2) \mapsto [J X_1, X_2]$ should be positive-definite. A real polarization is just a Lagrangian subspace of the phase space, *i.e.,* a maximal linear subspace on which the symplectic form vanishes identically. Now, the set of complex polarizations is just (*loc. cit.*) the complex tube consisting of all $\sigma \in \mathrm{Sym}_n^{\mathbb{C}}$ with $\mathrm{Re}\ \sigma \succ 0$, and the set of real polarizations is the space Σ of Proposition 4.7. Thus, Theorem 6.1 (*resp.* 5.10) is based on the realization of a certain space of functions on $\mathbb{R}^n$ as a space of vector-valued functions on the set of complex polarizations (*resp.* on some finite covering of the set of real polarizations) of the phase space.

One of the most striking differences between the metaplectic representation and the anaplectic one (when $n \geq 2$) lies in their action on the *ground* state (metaplectic case) or *median* state (anaplectic case) of the harmonic oscillator. For the ground state is invariant, up to phase factors of absolute value 1, under the image by the metaplectic representation of all elements above matrices in the compact subgroup $\mathrm{Sp}_{\mathrm{comp}}(n, \mathbb{R})$: it will be seen that, on the contrary, some one-parameter subgroups of this group lead, under the anaplectic representation, to a quite non-trivial action on the median state: this is so even though the groups under consideration also yield operators which commute with the harmonic oscillator. The same phenomenon holds at all energy levels of the harmonic oscillator, and the reason for this lies in the fact that eigenspaces of this operator are finite-dimensional in the usual case, and always infinite-dimensional in the anaplectic analysis. For simplicity, we shall consider here only the case when the dimension is two.

Lemma 6.2. *Assume that the dimension is* 2 *and set*

$$L_{jk} = \pi\, x_j\, x_k - \frac{1}{4\pi}\, \frac{\partial^2}{\partial x_j\, \partial x_k}, \qquad \Omega = \frac{1}{i} \left(x_1\, \frac{\partial}{\partial x_2} - x_2\, \frac{\partial}{\partial x_1} \right) \tag{6.7}$$

and recall that the (formal) harmonic oscillator is $L = L_{11} + L_{22}$, *an operator which commutes with* Ω, *and that*

$$A_j^* = \pi^{\frac{1}{2}} \left(x_j - \frac{1}{2\pi}\, \frac{\partial}{\partial x_j} \right), \qquad A_j = \pi^{\frac{1}{2}} \left(x_j + \frac{1}{2\pi}\, \frac{\partial}{\partial x_j} \right). \tag{6.8}$$

The only non-trivial commutation relations among the four operators $A_1 - i\, A_2$, $A_1 + i\, A_2$, $A_1^* - i\, A_2^*$, $A_1^* + i\, A_2^*$, *are*

$$[A_1 - i\, A_2,\ A_1^* + i\, A_2^*] = [A_1 + i\, A_2,\ A_1^* - i\, A_2^*] = 2. \tag{6.9}$$

One has

$$2\, L_{11} = A_1\, A_1^* + A_1^*\, A_1, \qquad 2\, L_{22} = A_2\, A_2^* + A_2^*\, A_2 \tag{6.10}$$

and

$$(A_1+i\,A_2)\,(A_1^*-i\,A_2^*) = L+1-\Omega, \qquad (A_1-i\,A_2)\,(A_1^*+i\,A_2^*) = L+1+\Omega: \tag{6.11}$$

in particular, setting

$$B^* = \frac{1}{2}\,(A_1^*-i\,A_2^*)\,(A_1-i\,A_2), \qquad B = \frac{1}{2}\,(A_1^*+i\,A_2^*)\,(A_1+i\,A_2), \tag{6.12}$$

one can see that the linear space generated by the (formal) operators B, B^* *and* Ω *has the structure of the Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$ *since*

$$[\Omega,\,B] = 2\,B, \quad [\Omega,\,B^*] = -2\,B^*, \quad [B,\,B^*] = \Omega. \tag{6.13}$$

Proof. The straightforward computations are made somewhat easier by the use of the complex coordinate $z = x_1 + i\,x_2$, so that $\Omega = z\,\frac{\partial}{\partial z} - \bar{z}\,\frac{\partial}{\partial \bar{z}}$ $L = \pi\,|z|^2 - \frac{1}{\pi}\,\frac{\partial^2}{\partial z\,\partial \bar{z}}$ and

$$A_1 - i\,A_2 = \pi^{\frac{1}{2}}\left(\bar{z} + \frac{1}{\pi}\,\frac{\partial}{\partial z}\right), \qquad A_1 + i\,A_2 = \pi^{\frac{1}{2}}\left(z + \frac{1}{\pi}\,\frac{\partial}{\partial \bar{z}}\right),$$
$$A_1^* + i\,A_2^* = \pi^{\frac{1}{2}}\left(z - \frac{1}{\pi}\,\frac{\partial}{\partial \bar{z}}\right), \qquad A_1^* - i\,A_2^* = \pi^{\frac{1}{2}}\left(\bar{z} - \frac{1}{\pi}\,\frac{\partial}{\partial z}\right). \tag{6.14}$$

Using (6.9) and (6.11), one finds

$$B^*\,B = \frac{1}{4}\,(L-1-\Omega)\,(L+1+\Omega), \quad B\,B^* = \frac{1}{4}\,(L-1+\Omega)\,(L+1-\Omega). \tag{6.15}$$

□

We here interest ourselves in the restriction of the anaplectic representation of $\mathrm{Sp}_{\mathrm{comp}}(2,\mathbb{R})$ to the subspace $\mathrm{Ker}\,L$ of $\mathfrak{A}^{(2)}$ consisting of all functions in the kernel of the anaplectic harmonic oscillator. This question has no analogue in classical analysis, for all eigenspaces of the harmonic oscillator are finite-dimensional: in particular that corresponding to the lowest eigenvalue is one-dimensional. Here, all such eigenspaces are infinite-dimensional, and within the space $\mathrm{Ker}\,L$, the action of the afore-mentioned compact group remains to be analyzed: of course, the decomposition of this space under the action of rotations is a special case of the study made in Section 3 – though we shall make it even more explicit in this case – and what is left is analyzing the group generated by an operator such as L_{11}, *i.e.,* in some sense, the group $t \mapsto \exp{(it\,L_{11})}$. Of course, no Stone's theorem is available in the present context, since there is no Hilbert space structure in the anaplectic analysis, but, at least on the germ level (*cf.* Theorem 5.10), this can be defined as the map $t \mapsto \mathrm{Ana}(g_t)$, with g_t defined as below, in Proposition 6.4. In particular, for $t = -\frac{\pi}{2}$, we should get some anaplectic analogue of the partial Fourier transformation with respect to the first variable.

We should warn against the emotional appeal, in the anaplectic environment, of such concepts as partial Fourier transformations. There is no natural embedding of the group of one-dimensional anaplectic operators into the group of two-dimensional anaplectic operators that would "freeze" the remaining variable in the way familiar in the usual analysis.

Lemma 6.3. *Assume* $n = 2$. *Within the space* $\mathrm{Ker}\, L$, *the operators* L_{11} *and* $\frac{1}{2}(B+B^*)$ *coincide. A linear basis of the subspace consisting of anaplectic Hermite functions* (cf. *Section* 3) *is given as the family* $(\Phi^j)_{j\in\mathbb{Z}}$ *with* $\Phi_0 = \Phi$ *defined by the equation* $\Phi(x) = I_0(\pi\,|x|^2)$ *and*

$$\Phi^j = B^{*j}\,\Phi \quad \text{if } j \geq 0, \qquad \Phi^j = B^{|j|}\,\Phi \quad \text{if } j \leq 0. \tag{6.16}$$

Explicitly, one has, with $x_1 + i\,x_2 = |x|\,e^{i\theta}$, *if* $j \geq 0$,

$$\Phi^j(x) = \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})}\, I_j(\pi\,|x|^2)\; e^{-2ij\,\theta}, \tag{6.17}$$

and $\Phi^{-j} = \overline{\Phi^j}$.

Proof. Using the relations (a consequence of (6.14) and of the expression of L in terms of the complex coordinate z)

$$[L,\, A_1 \pm i\,A_2] = -(A_1 \pm i\,A_2), \qquad [L,\, A_1^* \pm i\,A_2^*] = A_1^* \pm i\,A_2^*, \tag{6.18}$$

it is immediate that $[L,\, B] = [L,\, B^*] = 0$ so that B and B^* act as endomorphisms of $\mathrm{Ker}\, L$: when acting on an eigenfunction of Ω, B (*resp.* B^*) increases (*resp.* decreases) the eigenvalue by 2 (a consequence of (6.13)) so that one has, reverting to the complex coordinate z,

$$\Phi^j(z) = g_j(\pi\, z\bar{z})\, z^{-j}\, \bar{z}^{\,j} \tag{6.19}$$

for some holomorphic function g_j to be determined, with $g_0(\rho) = I_0(\rho)$: it is no loss of generality to assume that $j \geq 0$. One immediately finds

$$\begin{aligned}(A_1 - i\,A_2)\,\Phi^j &= \pi^{\frac{1}{2}}\, g_j(\pi\, z\bar{z})\, z^{-j}\, \bar{z}^{\,j+1} \\ &\quad + \pi^{\frac{1}{2}}\, g_j'(\pi\, z\bar{z})\, z^{-j}\, \bar{z}^{\,j+1} - \pi^{-\frac{1}{2}}\, j\, g_j(\pi\, z\bar{z})\, z^{-j-1}\, \bar{z}^{\,j}\end{aligned} \tag{6.20}$$

and, applying $\frac{1}{2}(A_1^* - i\,A_2^*)$ to the preceding result, one sees that the equation (6.19) relative to Φ^{j+1} will still be verified provided that

$$g_{j+1}(\rho) = \frac{1}{2}\left[-\rho\,\frac{d^2}{d\rho^2} + 2j\,\frac{d}{d\rho} + \rho - \frac{j(j+1)}{\rho}\right]\, g_j(\rho): \tag{6.21}$$

using the equation $I_j' - \frac{j}{\rho}\, I_j = I_{j+1}$ as well as the second-order differential equation of modified Bessel functions (to be found in all books on special functions), we end up with (6.17). □

Using (6.15) and remembering that the functions Φ^j lie in $\operatorname{Ker} L$ and satisfy $\Omega\,\Phi^j = -2j\,\Phi^j$, one sees that

$$B\,\Phi^j = -\left(j-\frac{1}{2}\right)^2 \Phi^{j-1} \quad\text{and}\quad B^*\,\Phi^{-j} = -\left(j-\frac{1}{2}\right)^2 \Phi^{-(j-1)} \quad \text{if } j \geq 1. \tag{6.22}$$

Of course, $B^*\,\Phi^j = \Phi^{j+1}$ and $B\,\Phi^{-j} = \Phi^{-(j+1)}$ under the same assumption.

If Ψ lies in $\operatorname{Ker} L$, one has $(L_{11}+L_{22})\,\Psi = 0$ and (as a consequence of (6.10))

$$\begin{aligned} L_{11}\,\Psi &= \frac{1}{2}\,(L_{11}-L_{22})\,\Psi = \frac{1}{4}\,(A_1\,A_1^* + A_1^*\,A_1 - A_2\,A_2^* - A_2^*\,A_2)\,\Psi \\ &= \frac{1}{2}\,(A_1^*\,A_1 - A_2^*\,A_2)\,\Psi, \end{aligned} \tag{6.23}$$

which is just the same as $\frac{1}{2}\,(B+B^*)\,\Psi$ as it follows from (6.12). This should make the operator $\exp\,(it\,L_{11})$, considered within $\operatorname{Ker} L$, computable in principle, in view of (6.13).

To do this, we shall make use of the generalized (projective) discrete series of representations of $SL(2,\mathbb{R})$, of which we hardly need to know more than the definition. Recall that a projective representation π is just like a representation, except for the fact that instead of an equation like $\pi(g_1)\,\pi(g) = \pi(g_1 g)$, one only demands that one should have $\pi(g_1)\,\pi(g) = \theta(g_1,g)\,\pi(g_1 g)$, where $\theta(g_1,g)$ is some complex number of modulus 1. Under the assumption that $\tau > 0$, consider the Hilbert space of all holomorphic functions f in Π with

$$\|f\|^2 := \int_\Pi |f(z)|^2 (\operatorname{Im} z)^{\tau+1}\,d\mu(z) < \infty. \tag{6.24}$$

There exists a unitary projective representation π of $SL(2,\mathbb{R})$ in this Hilbert space, characterized up to scalar factors in the group $\exp(2i\pi\tau\mathbb{Z})$ by the fact that

$$(\pi(g)f)(z) = (-cz+a)^{-\tau-1} f\left(\frac{dz-b}{-cz+a}\right) \tag{6.25}$$

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $c < 0$. In all this the fractional powers which occur are those associated with the principal determination of the logarithm in the upper half-plane.

Consider now the linear basis $\{H,\,X,\,Y\}$ of $\mathfrak{sl}(2,\mathbb{R})$ defined as

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6.26}$$

so that the triple above satisfies the same commutation relations as the triple $\{\Omega,\,B,\,B^*\}$ in (6.13), and set

$$\mathcal{L}_H = d\pi(H) = \left.\frac{d}{dt}\right|_{t=0} \pi(\exp\,t\,H) \tag{6.27}$$

and, in a similar way, $\mathcal{L}_X = d\pi(X)$, $\mathcal{L}_Y = d\pi(Y)$: explicitly, one has

$$\mathcal{L}_H = -2\,z\,\frac{d}{dz} - (\tau+1), \qquad \mathcal{L}_X = -\frac{d}{dz}, \qquad \mathcal{L}_Y = z^2\,\frac{d}{dz} + (\tau+1)\,z: \tag{6.28}$$

by construction, the three operators just defined, taken in this order, satisfy the same commutation relations as the three operators Ω, B, B^*. On the other hand, an easy computation yields the corresponding Casimir operators (when the operators in the latter triple are considered as acting only on $\mathrm{Ker}\,L$) as

$$-\frac{1}{2}\,(\mathcal{L}_X\,\mathcal{L}_Y + \mathcal{L}_Y\,\mathcal{L}_X) - \frac{1}{4}\,\mathcal{L}_H^2 = \frac{1-\tau^2}{4}, \qquad -\frac{1}{2}\,(B\,B^* + B^*\,B) - \frac{1}{4}\,\Omega^2 = \frac{1}{4}, \tag{6.29}$$

and it is immediate to check that in the case when $\tau = 0$, which we assume from now on, the algebras of formal differential operators generated by the two triples $\{\Omega,\, B,\, B^*\}$ and $\{\mathcal{L}_H,\, \mathcal{L}_X,\, \mathcal{L}_Y\}$ are isomorphic. We now let the function Φ^0, a solution of the equation $\Omega\,\Phi^0 = 0$, correspond to the function $\phi^0(z) = z^{-\frac{1}{2}}$ in the upper half-plane (where the argument of z is chosen in $(0,\pi)$), since this is a solution of the corresponding equation $\mathcal{L}_H\,\phi^0 = 0$: then, for $j \geq 1$, we set

$$\begin{aligned}\phi^j(z) &= \left(z^2\,\frac{d}{dz} + z\right)^j (z^{-\frac{1}{2}}) = \frac{1}{2}.\frac{3}{2}\ldots\left(j-\frac{1}{2}\right)\,z^{j-\frac{1}{2}},\\ \phi^{-j}(z) &= \left(-\frac{d}{dz}\right)^j (z^{-\frac{1}{2}}) = \frac{1}{2}.\frac{3}{2}\ldots\left(j-\frac{1}{2}\right)\,z^{-j-\frac{1}{2}},\end{aligned} \tag{6.30}$$

and one verifies the equations

$$\mathcal{L}_X\,\phi^j = -\left(j-\frac{1}{2}\right)^2 \phi^{j-1}, \qquad \mathcal{L}_Y\,\phi^{-j} = -\left(j-\frac{1}{2}\right)^2 \phi^{-j+1}, \quad j \geq 1, \tag{6.31}$$

which are the analogue of (6.22).

With $\mathcal{L}_X + \mathcal{L}_Y = (z^2-1)\,\frac{d}{dz} + z$, the operator that corresponds to $B + B^*$, one can compute the exponential $\exp\left(\frac{it}{2}\,(\mathcal{L}_X + \mathcal{L}_Y)\right)\colon f \mapsto f_t$ in an explicit way, say for t in the range $0 \leq t < \frac{\pi}{2}$. There is no need to state any definite Cauchy problem to that effect, since we only want to use the result to guess a formula regarding the exponential of the operator $\frac{1}{2}\,(B + B^*)$. Still, we start with the remark that if $0 < t_0 < \frac{\pi}{2}$ so that $0 \leq \tan\,\frac{t}{2} \leq \lambda_0 < 1$ when $0 \leq t \leq t_0$, the open subset ω of the upper half-plane defined by the conditions

$$|z|^2 - 1 + (\lambda_0^{-1} - \lambda_0)\,\mathrm{Im}\,z > 0, \qquad \lambda_0 < |z| < \frac{1}{\lambda_0}, \quad z \in \omega \tag{6.32}$$

is non-void. Given a holomorphic function f in the upper half-plane, the differential equation $f_t = \exp\left(\frac{it}{2}\,(\mathcal{L}_X + \mathcal{L}_Y)\right) f$ can then be solved at least for $z \in \omega$ and $0 \leq t < t_0$ by the formula

$$f_t(z) = \frac{1}{\cos\,\frac{t}{2}}\,(1 - i\,z\,\tan\,\frac{t}{2})^{-1}\,f\left(\frac{z - i\,\tan\,\frac{t}{2}}{1 - i\,z\,\tan\,\frac{t}{2}}\right) \tag{6.33}$$

since the first condition (6.32) ensures that the argument of f on the right-hand side of this equation lies in the upper half-plane: indeed, the imaginary part of the argument has the sign of $(\mathrm{Im}\, z)\,(1-\tan^2\frac{t}{2})+(|z|^2-1)\,\tan\frac{t}{2}\geq(\tan\frac{t}{2})\,[|z|^2-1+(\lambda_0^{-1}-\lambda_0)\,\mathrm{Im}\, z]$. In particular, denoting as $\zeta_{\mathrm{right}}^{-\frac{1}{2}}$ the square-root of ζ^{-1} computed, under the assumption that $\mathrm{Re}\,\zeta>0$, with the usual rule in this half-plane, one finds, recalling that $\phi^0(z)=z^{-\frac{1}{2}}$,

$$(\phi^0)_t(z)=\frac{1}{\cos\frac{t}{2}}\,z^{-\frac{1}{2}}\left(1-i\,z\,\tan\frac{t}{2}\right)_{\mathrm{right}}^{-\frac{1}{2}}\left(1-i\,z^{-1}\,\tan\frac{t}{2}\right)_{\mathrm{right}}^{-\frac{1}{2}}: \tag{6.34}$$

the second condition (6.32) makes it possible to expand each of the two factors as a series, getting as a result

$$(\phi^0)_t(z)=\frac{1}{\pi\,\cos\frac{t}{2}}\sum_{m,n\geq 0}\frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2}+n)}{m\,!\,n\,!}\left(i\,\tan\frac{t}{2}\right)^{m+n}\,z^{m-n-\frac{1}{2}} \tag{6.35}$$

or, using the functions ϕ^j defined in (6.30),

$$(\phi^0)_t(z)=\sum_{j\in\mathbb{Z}}c_j(t)\,\phi^j(z) \tag{6.36}$$

with

$$\begin{aligned}c_{\pm j}(t)&=\frac{1}{\pi\,\cos\frac{t}{2}}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+j)}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2}+j+m)}{\Gamma(1+j+m)}\,\frac{(i\,\tan\frac{t}{2})^{2m+j}}{m\,!}\\&=\frac{(i\,\tan\frac{t}{2})^j}{j\,!\,\cos\frac{t}{2}}\,{}_2F_1\left(j+\frac{1}{2},\frac{1}{2};j+1;-\tan^2\frac{t}{2}\right)\end{aligned} \tag{6.37}$$

for every $j\geq 0$.

Proposition 6.4. *Set* $g_t=\begin{pmatrix}\cos t&0&-\sin t&0\\0&1&0&0\\\sin t&0&\cos t&0\\0&0&0&1\end{pmatrix}$. *For* $|t|<\frac{\pi}{2}$, *one has the convergent expansion*

$$\mathrm{Ana}(g_t)\,\Phi=\sum_{j\in\mathbb{Z}}\frac{(i\,\tan\frac{t}{2})^{|j|}}{|j|\,!\,\cos\frac{t}{2}}\,{}_2F_1\left(|j|+\frac{1}{2},\frac{1}{2};|j|+1;-\tan^2\frac{t}{2}\right)\,\Phi^j. \tag{6.38}$$

Proof. We first prove that the sum of the series on the right-hand side, evaluated at $z\in\mathbb{C}^2$ is absolutely convergent and (for $|t|<\frac{\pi}{2}$) majorized by $C\,e^{\pi R\,|z|^2}$ for any $R>\frac{3}{2}$. Starting from the expression (6.17) of $\Phi^j(z)$, and using [17, p. 84], we find

$$\Phi^j(z)=\frac{1}{\pi}\,(z_1-i\,z_2)^{2j}\,(\frac{\pi}{2})^j\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\pi\,(z_1^2+z_2^2)\,\sin\phi}\,\cos^{2j}\phi\,d\phi: \tag{6.39}$$

using then the first expression (6.37) of the coefficients $c_j(t)$, we obtain

$$\begin{aligned}\sum_{j\geq 0}|c_j(t)|\,|\Phi^j(z)|&\\ &\leq \frac{\pi^{-\frac{1}{2}}}{\cos\frac{t}{2}}\sum_{\substack{j\geq 0\\ m\geq 0}}\left(\frac{\pi\,|z|^2}{2}\right)^j e^{\pi\,|z|^2}\,\frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2}+j+m)}{\Gamma(\frac{1}{2}+j)\,\Gamma(1+j+m)}\,\frac{|\tan\frac{t}{2}|^{2m+j}}{m\,!}\\ &\leq \pi^{-\frac{1}{2}}\,e^{\pi\,|z|^2}\sum_{j\geq 0}\frac{\left(\frac{\pi\,|z|^2}{2}\,|\tan\frac{t}{2}|\right)^j}{\Gamma(\frac{1}{2}+j)}\sum_{m\geq 0}(\tan\frac{t}{2})^{2m},\end{aligned} \tag{6.40}$$

which leads to the absolute convergence of the series on the left-hand side and the claimed estimate.

Next, we show that, denoting as Ψ_t the function on the right-hand side of (6.38), so that in particular $\Psi_0=\Phi$, one has

$$\frac{d\,\Psi_t}{dt}=i\,L_{11}\,\Psi_t. \tag{6.41}$$

By the very definition of the function $(\phi^0)_t$ above as a solution of the differential equation $\frac{d(\phi^0)_t}{dt}=\frac{i}{2}\,(\mathcal{L}_X+\mathcal{L}_Y)\,(\phi^0)_t$ and the expansion (6.36), one has

$$\sum_{j\in\mathbb{Z}}c_j'(t)\,\phi^j=\frac{i}{2}\sum_{j\in\mathbb{Z}}c_j(t)\,(\mathcal{L}_X+\mathcal{L}_Y)\,\phi^j, \tag{6.42}$$

which leads, in view of (6.31), to

$$\begin{aligned}\sum_{j\in\mathbb{Z}}c_j'(t)\,\phi^j=\frac{i}{2}&\left[c_0(t)\,(\phi^1+\phi^{-1})+\sum_{j\geq 1}c_j(t)\left(\phi^{j+1}-\left(j-\frac{1}{2}\right)^2\phi^{j-1}\right)\right.\\ &\left.+\sum_{j\leq -1}c_j(t)\left(\phi^{j-1}-\left(|j|-\frac{1}{2}\right)^2\phi^{j+1}\right)\right]:\end{aligned} \tag{6.43}$$

this is nothing but the shortest way to check that the complicated (hypergeometric) coefficients $c_j(t)$ verify a certain collection of differential equations, one of which is, for instance,

$$c_j'(t)=\frac{i}{2}\left(c_{j-1}(t)-\left(j+\frac{1}{2}\right)^2c_{j+1}(t)\right)\qquad j\geq 2. \tag{6.44}$$

In view of the analogous equations (6.22), relative to the sequence $(\Phi^j)_{j\in\mathbb{Z}}$, and of the fact that $L_{11}\,\Phi^j=\frac{1}{2}\,(B+B^*)\,\Phi^j$ for every $j\in\mathbb{Z}$, one finds the differential equation (6.41) as a consequence of the definition of Ψ_t as the right-hand side of the expansion (6.38), the necessary estimates involved having been treated, essentially, in the beginning of the present proof.

Under the assumption that $|t| < \frac{\pi}{2}$, we are now in a position to define the quadratic transforms $(\mathcal{Q}\,\Psi_t)_p(\sigma)$ and $(\mathcal{K}\,\Psi_t)_p(Z)$ (where $p = 0$ or 1), the first one for $\sigma \in \mathrm{Sym}_2$ with $\sigma - RI \succ 0$ for some $R > \frac{3}{2}$, the second one for $Z \in \Sigma^{(2)}$ close enough from the identity matrix. The operator L_{11} transfers as an operator on the scalar part of the $\mathcal{Q}$-transform as the operator

$$i\,\frac{\partial}{\partial\sigma_{(11)}} + i\sum_{k\ell}\sigma_{1k}\,\sigma_{1\ell}\,\frac{\partial}{\partial\sigma_{(k\ell)}} + \frac{i}{2}\,\sigma_{11} = i\,\frac{\partial}{\partial\sigma_{11}} + \frac{i}{2}\,M_{11} \tag{6.45}$$

in terms of the set of operators introduced in (5.28): this is proved by an application of (5.8) to the definition (4.1) of the scalar $\mathcal{Q}$-transform. Next, Lemma 5.1 and the proof of Proposition 5.3, between (5.29) and (5.36), make it possible to obtain the transfer of the operator L_{11} as the operator

$$i\,\mathcal{D}^{(11)} + \frac{i}{2}\,M^{(11)} = i\left[\nabla_{V_Z^{11}} - \frac{1}{4}\,\mathrm{Im}\ \ (I-Z)_{11} + \nabla_{X_Z^{11}} - \frac{1}{4}\,\mathrm{Im}\ \ (I+Z)_{11}\right] \tag{6.46}$$

acting on the scalar part of the $\mathcal{K}$-transform. Recall from (5.10) that $V_Z^{11} = \frac{1}{2i}\,(I-Z)\,\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\,(I-Z)$ and from (5.34) that $X^{11} = \frac{i}{2}\,(I+Z)\,\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\,(I+Z)$. Consequently, the operator L_{11} transfers to the operator

$$\nabla_\Xi \qquad \text{with} \qquad \Xi = -\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)Z + Z\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\right). \tag{6.47}$$

On the other hand, note that $g_t^{-1} = \left(\begin{smallmatrix}R&S\\-S&R\end{smallmatrix}\right)$ with $R = \left(\begin{smallmatrix}\cos t&0\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}\sin t&0\\0&0\end{smallmatrix}\right)$, so that (5.69), which we here recall, using Proposition 4.7, as

$$(\mathcal{K}\,\mathrm{Ana}\,(g_t)\,u)_0(Z) = (\mathcal{K}\,u)_0\,\left(\left(\begin{smallmatrix}e^{-it}&0\\0&1\end{smallmatrix}\right)\,Z\,\left(\begin{smallmatrix}e^{-it}&0\\0&1\end{smallmatrix}\right)\right), \tag{6.48}$$

yields

$$\frac{d}{dt}\,[(\mathcal{K}\,\mathrm{Ana}\,(g_t)\,u)_0(Z)] = -i\,\nabla_{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)Z+Z\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)}\,[(\mathcal{K}\,\mathrm{Ana}\,(g_t)\,u)_0(Z)]: \tag{6.49}$$

comparing this differential equation to (6.41) with the help of (6.47), we obtain the claim of Proposition 6.4 since we do not have to worry about the vector-valued parts of the $\mathcal{K}$-transforms as we are dealing only with even functions here. □

Remarks 6.2. (i) From the result of Theorem 5.10 and the analysis of singularities of the function $(\mathcal{K}\,\Phi)_0$ in (4.84), it follows that the function $\mathrm{Ana}(g_t)\,\Phi$ is well defined as an entire function satisfying the usual estimate not only for $|t| < \frac{\pi}{2}$, but in fact for $|t| < \pi$. On the other hand, for t in this range of values, and $z \in \mathbb{C}$, the series on the right-hand side of (6.38), evaluated at z, is absolutely convergent and its sum is majorized by $\frac{4}{\pi}\,(\cos\frac{t}{2})^{-1}\,e^{\pi(1+|\tan\frac{t}{2}|)\,|z|^2}$: this is a consequence of the estimates

$$|\Phi^j(z)| \leq \frac{2}{\pi}\left(\frac{\pi\,|z|^2}{2}\right)^{|j|} e^{\pi\,|z|^2}, \qquad j \in \mathbb{Z},\ z \in \mathbb{C} \tag{6.50}$$

and $|{}_2F_1\,(|j|+\frac{1}{2},\frac{1}{2};|j|+1;-\tan^2\frac{t}{2})|\leq 1$, where the last estimate follows from the integral representation [17, p. 54] of the hypergeometric function, and the first one is based on the expression (6.17) of Φ^j together with the classical integral representation [17, p. 84] of the modified Bessel function I_j. As a consequence of Proposition 6.4 and of [17, p. 40], one thus has in particular

$$\mathrm{Ana}(g_{\pm\frac{\pi}{2}})\,\Phi=\pi^{\frac{1}{2}}\sum_{j\in\mathbb{Z}}\frac{(\mp\frac{i}{2})^{|j|}}{(\Gamma(\frac{|j|}{2}+\frac{3}{4}))^2}\,\Phi^j\;: \tag{6.51}$$

again, this is quite different from the usual analysis, in which the ground state of the harmonic oscillator is invariant under partial Fourier transformations as well as under the global Fourier transformation.

A closed formula for $\mathrm{Ana}(g_{\frac{\pi}{2}})\,\Phi$ will be given in the course of the proof of Theorem 10.13.

(ii) The same works with the other energy levels of the (anaplectic) harmonic oscillator: assuming that $\kappa\in\mathbb{Z}$ is even, so that (*cf.* Section 3) the space $\mathrm{Ker}\,(L-\kappa)$ should contain a radial function, we start with such a function $\Phi^{\kappa,0}$. How to construct such a function was indicated in the proof of Lemma 3.6: if $\kappa=0,2,\dots,$ and $h_1(t)=[\frac{d^2}{dt^2}+(1-2t)\,\frac{d}{dt}+t-1]\,h(t)$, then the function $h_1(\pi\,|x|^2)$ will lie in $\mathrm{Ker}\,(L-\kappa-2)$ if the function $h(\pi\,|x|^2)$ lies in $\mathrm{Ker}\,(L-\kappa)$; something analogous works on the other side of the spectrum of L, only replacing the operator $\sum(A_j^*)^2$ used in the proof of Lemma 3.6 by $\sum A_j^2$. The operators B and B^* also operate within the space $\mathrm{Ker}\,(L-\kappa)$, and one defines, just as in Lemma 6.2,

$$\Phi^{\kappa,j}=B^{*j}\,\Phi^{\kappa,0}\;\text{ if } j\geq 0,\qquad \Phi^{\kappa,j}=B^{|j|}\,\Phi^{\kappa,0}\;\text{ if } j\leq 0. \tag{6.52}$$

Using (6.15), one finds, for $j\geq 1$,

$$B\,\Phi^{\kappa,j}=-{\scriptstyle(j-\frac{\kappa+1}{2})\,(j+\frac{\kappa-1}{2})}\,\Phi^{\kappa,j-1},\qquad B^*\,\Phi^{\kappa,-j}=-{\scriptstyle(j-\frac{\kappa+1}{2})\,(j+\frac{\kappa-1}{2})}\,\Phi^{\kappa,-j+1}. \tag{6.53}$$

Since the second equation in (6.29) must be replaced by

$$-\frac{1}{2}\,(B\,B^*+B^*\,B)-\frac{1}{4}\,\Omega^2=\frac{1-\kappa^2}{4}, \tag{6.54}$$

we must take this time $\tau=|\kappa|$ which leads, since the first operator in (6.28) is $-2\,(z\,\frac{d}{dz}+\frac{\tau+1}{2})=-2\,(z^{-\frac{\tau}{2}})\,(z\,\frac{d}{dz}+\frac{1}{2})\,(z^{\frac{\tau}{2}})$, to substituting for (6.30) the defining set of equations $\phi^j(z)=\frac{\Gamma(\frac{1}{2}+|j|)}{\Gamma(\frac{1}{2})}\,z^{j-\frac{\tau+1}{2}}$. Since, now, $\mathcal{L}_X+\mathcal{L}_Y=(z^2-1)\,\frac{d}{dz}+(\tau+1)\,z$, the equations (6.33) and (6.34) must be replaced by

$$f_t(z)=\left(\cos\,\frac{t}{2}\right)^{-\tau-1}\left(1-i\,z\,\tan\,\frac{t}{2}\right)^{-\tau-1}\,f\left(\frac{z-i\,\tan\,\frac{t}{2}}{1-i\,z\,\tan\,\frac{t}{2}}\right) \tag{6.55}$$

and

$$(\phi^0)_t(z) = \left(\cos\,\frac{t}{2}\right)^{-\tau-1}\, z^{-\frac{\tau+1}{2}} \left(1 - i\,z\,\tan\,\frac{t}{2}\right)_{\text{right}}^{-\frac{\tau+1}{2}} \left(1 - i\,z^{-1}\,\tan\,\frac{t}{2}\right)_{\text{right}}^{-\frac{\tau+1}{2}} : \tag{6.56}$$

this finally leads, under the same conditions as before, to the expansion

$$\mathrm{Ana}(g_t)\,\Phi^{\kappa,0} = \sum_{j\in\mathbb{Z}} c_j(t)\,\Phi^{\kappa,j} \tag{6.57}$$

with

$$c_{\pm j}(t) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{\tau+1}{2}+j)}{\Gamma(\frac{1}{2}+j)\,\Gamma(\frac{\tau+1}{2})}\,\frac{(i\,\tan\,\frac{t}{2})^j}{j\,!\,(\cos\,\frac{t}{2})^{\tau+1}}\;{}_2F_1\left(j+\frac{\tau+1}{2}, \frac{\tau+1}{2}; j+1; -\tan^2\,\frac{t}{2}\right) \tag{6.58}$$

for every $j \geq 0$.

Proposition 6.4 has brought to light a major difference between the anaplectic analysis and the usual one: on the contrary, the following shows a strong analogy, the passage from the usual analysis to the anaplectic one only calling for the replacement of the discrete series of $SL(2,\mathbb{R})$ by the (full, non-unitary) principal series.

Hecke's theorem, in classical Fourier analysis on $\mathbb{R}^n$, is the equation [7, 21]

$$\mathcal{F}\left(h\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^\ell\right) = h_1\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^\ell, \tag{6.59}$$

where $\mathcal{Y}^\ell$ denotes any harmonic polynomial homogeneous of degree ℓ, and the map $h \mapsto h_1$ is given by the Hankel transformation

$$h_1(s) = 2\pi\,i^{-\ell}\int_0^\infty h(t)\left(\frac{t}{s}\right)^{\frac{n-2}{4}+\frac{\ell}{2}}\; J_{\frac{n-2}{2}+\ell}(4\pi\sqrt{st})\,dt. \tag{6.60}$$

In other words, setting $k(t) = t^{\frac{n-2}{2}+\ell}\,h(t)$ and $k_1(t) = t^{\frac{n-2}{2}+\ell}\,h_1(t)$, one has

$$k_1(s) = 2\pi\,i^{-\ell}\int_0^\infty k(t)\left(\frac{s}{t}\right)^{\frac{n-2}{4}+\frac{\ell}{2}}\; J_{\frac{n-2}{2}+\ell}(4\pi\sqrt{st})\,dt : \tag{6.61}$$

in terms of the discrete series $(\mathcal{D}_{\frac{m}{2}+1})_{m=-1,0,\dots}$ of the *metaplectic* group $\widetilde{SL}(2,\mathbb{R})$ as normalized in [25, p. 61] (we considered another realization of the same series of representations in (6.25)), one has

$$k_1 = i^n\,\mathcal{D}_{\frac{n-2}{2}+\ell+1}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\,k, \tag{6.62}$$

where the matrix is to be interpreted as $\left(\begin{smallmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{smallmatrix}\right)$.

We shall generalize Hecke's formula to the anaplectic case: but first, we need to make an observation concerning the expansion of Fourier transforms of functions $v \in C^{\omega}_{\rho,\varepsilon}$. Under the Cayley map $\sigma \mapsto e^{-i\,\theta} = \frac{\sigma-i}{\sigma+i}$, $0 < \theta < 2\pi$, the condition for $w = \mathcal{F}\,v$ to belong to the space $\widehat{C}^{\omega}_{\rho,\varepsilon}$ given in Definition 2.2 transfers to the fact that the function $e^{-\frac{i\theta}{2}} \mapsto |\sin\frac{\theta}{2}|_{\varepsilon}^{-1-\rho}\, w(\text{cotan}\,\frac{\theta}{2})$ extends as an analytic function on the whole circle, invariant (*resp.* changing to its negative) under the map $\theta \mapsto \theta + 2\pi \mod 4\pi$ in the case when $\varepsilon = 0$ (*resp.* 1). Consequently, there exists a unique sequence $(c_k)_{k\in\mathbb{Z}}$ of complex numbers, satisfying the estimate $|c_k| \leq C\,(1+\epsilon)^{-|k|}$ for some $\epsilon > 0$ and $C > 0$, such that, for every $\sigma \in \mathbb{R}$,

$$w(\sigma) = (1+\sigma^2)^{-\frac{\rho+1}{2}} \sum_{k\in\mathbb{Z}} c_k \left(\frac{1-i\,\sigma}{1+i\,\sigma}\right)^k \qquad \text{if } \varepsilon = 0,$$

$$w(\sigma) = (1+\sigma^2)^{-\frac{\rho+2}{2}} \sum_{k\in\mathbb{Z}} c_k \left(\frac{1-i\,\sigma}{1+i\,\sigma}\right)^k (1-i\,\sigma) \qquad \text{if } \varepsilon = 1. \tag{6.63}$$

In the theorem that follows, we shall have to assume a little more, namely that the function on the circle considered above extends as a holomorphic function (of $e^{\frac{i\theta}{2}}$ in general, or even $e^{i\theta}$ in the case when $\varepsilon = 0$) on $\mathbb{C}\backslash\{0\}$. This is tantamount to saying that the sequence $(c_k)_{k\in\mathbb{Z}}$ satisfies the following condition: given any number $M \geq 1$ there is some $C > 0$ such that $|c_k| \leq C\,M^{-|k|}$ for all $k \in \mathbb{Z}$.

Theorem 6.5. *Let $\mathcal{Y}^{\ell}$ be a homogeneous harmonic polynomial of degree ℓ; let $\varepsilon = 0$ or 1 according to whether ℓ is even or odd. Let $v \in C^{\omega}_{\frac{n-2}{2}+\ell,\varepsilon}$ and, setting $\rho = \frac{n-2}{2} + \ell$ and $w = \mathcal{F}\,v$, assume that the function $e^{-\frac{i\theta}{2}} \mapsto |\sin\frac{\theta}{2}|_{\varepsilon}^{-1-\rho}\, w(\text{cotan}\,\frac{\theta}{2})$ extends as a holomorphic function on the punctured complex plane. Then the function of $x \in \mathbb{R}^n$ defined as $u(x) = v^{\text{ram}}(\frac{|x|^2}{2})\,\mathcal{Y}^{\ell}(x)$ lies in the space $\mathfrak{A}^{(n)}$. Its anaplectic Fourier transform can be obtained by the formula*

$$\left(\mathcal{F}_{\text{ana}}\left(y \mapsto v^{\text{ram}}\left(\frac{|y|^2}{2}\right)\,\mathcal{Y}^{\ell}(y)\right)\right)(x) = v_1^{\text{ram}}\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^{\ell}(x), \tag{6.64}$$

where the function v_1 is linked to v by the equation (to be compared to (6.62))

$$v_1 = \pi_{\frac{n-2}{2}+\ell,\varepsilon}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\, v. \tag{6.65}$$

Proof. Using the notation in (2.20), (2.21), which extends without modification to the case when ρ is an integer, one derives from (6.63) the series expansion $v = \sum_{k\in\mathbb{Z}} c_k\, v^k_{\rho,\varepsilon}$, so that, in a sense which will be analyzed at the end of the proof,

$$u = c_0\, u^0_{\ell} + \sum_{k\geq 1} [c_k\, u^k_{\ell} + c_{-k}\, u^{-k}_{\ell}] \tag{6.66}$$

with

$$u^k_{\ell}(x) = (v^k_{\rho,\varepsilon})^{\text{ram}}\left(\frac{|x|^2}{2}\right)\,\mathcal{Y}^{\ell}(x): \tag{6.67}$$

recall that $\ell+\varepsilon$ is even. From now on, we shall assume that

$$\mathcal{Y}^\ell(x) = (x_1 - i\,x_2)^\ell : \tag{6.68}$$

this does not lead to any loss of generality since we can use the invariance of the anaplectic Fourier transformation under rotations together with the fact that the group $O(n)$ acts irreducibly on the linear space of harmonic polynomials of a given degree ℓ. From the results of Section 3, in particular Theorem 3.8 and the proof thereof, it follows that all the functions u_ℓ^k are anaplectic Hermite functions in the sense of Definition 3.2: indeed, from (3.51), one sees that such a function is an eigenvalue of the harmonic oscillator for the eigenvalue $\kappa = 2k+\varepsilon$ so that, from one assumption of Theorem 6.5, $\kappa+\ell$ is even.

Lemma 6.6. *For every $\ell \geq 0$, setting $\rho = \frac{n-2}{2} + \ell$ with $\ell+\varepsilon$ even, one has*

$$u_{\ell+1}^0(x) = (\rho+1-\varepsilon)^{-1} \left[2\pi\,(-1)^\varepsilon\,(x_1 - i\,x_2) - \frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2} \right] u_\ell^0(x). \tag{6.69}$$

For every $k = 0, 1, \dots,$ one then has

$$\begin{aligned} u_\ell^k &= \left(\frac{1}{2}\right)^k \frac{\Gamma(\frac{n}{4} + \frac{\ell+\varepsilon}{2})}{\Gamma(\frac{n}{4} + \frac{\ell+\varepsilon}{2} + k)} \left(\sum A_j^{*2}\right)^k u_\ell^0, \\ u_\ell^{-k} &= \left(-\frac{1}{2}\right)^k \frac{\Gamma(\frac{n}{4} + \frac{\ell-\varepsilon}{2})}{\Gamma(\frac{n}{4} + \frac{\ell-\varepsilon}{2} + k)} \left(\sum A_j^{2}\right)^k u_\ell^0. \end{aligned} \tag{6.70}$$

One has

$$\mathcal{F}_{\text{ana}}\, u_\ell^k = \begin{cases} (-1)^k\; u_\ell^k & \text{if } \ell \text{ is even,} \\ (-1)^k\,(-i)\; u_\ell^k & \text{if } \ell \text{ is odd.} \end{cases} \tag{6.71}$$

Proof. We first perform a certain number of calculations under the additional assumption that ρ is an arbitrary complex number such that $\rho+\varepsilon \notin 2\mathbb{Z}$ and $\operatorname{Re}\rho > -1$. A consequence of Proposition 2.12 is the following set of equations, in which we assume that v is some element of $\mathfrak{C}_{\rho,\varepsilon}^\omega$:

$$\begin{aligned} (T_\rho^\dagger\, v)^{\text{ram}} &= R\, v^{\text{ram}}, &\qquad (R\, v)^{\text{ram}} &= T_{-\rho}^\dagger\, v^{\text{ram}}, \\ (R_\rho^\dagger\, v)^{\text{ram}} &= T\, v^{\text{ram}}, &\qquad (T\, v)^{\text{ram}} &= R_{-\rho}^\dagger\, v^{\text{ram}}; \end{aligned} \tag{6.72}$$

it is understood that, in what precedes, the ramified part of v makes reference to the decomposition, provided by Proposition 2.3 or Proposition 3.1, of elements of $\mathfrak{C}_{\rho,\varepsilon}^\omega$, while the ramified part of $T_\rho^\dagger\, v$ or $R_\rho^\dagger\, v$ is taken, according to Proposition 2.12, in the space $\mathfrak{C}_{\rho+1,1-\varepsilon}^\omega$, and that of $T\,v$ or $R\,v$ is taken in $\mathfrak{C}_{\rho-1,1-\varepsilon}^\omega$. For instance, to prove the first of these four equations, it suffices to remark that $(|s|_{1-\varepsilon}^{-\rho-1})\, T_\rho^\dagger\, (|s|_\varepsilon^\rho) = R$.

We first use this set of equations to prove (6.69). Proposition 2.13 gives

$$T_\rho^\dagger\, v_{\rho,0}^0 = \frac{\rho+1}{2\pi}\, v_{\rho+1,1}^0, \qquad R_\rho^\dagger\, v_{\rho,1}^0 = -\frac{\rho}{2\pi}\, v_{\rho+1,0}^0, \tag{6.73}$$

so that

$$(v^0_{\rho+1,1})^{\mathrm{ram}}(s) = \frac{2\pi}{\rho+1}\, R\, (v^0_{\rho,0})^{\mathrm{ram}}(s), \qquad (v^0_{\rho+1,0})^{\mathrm{ram}}(s) = -\frac{2\pi}{\rho}\, T\, (v^0_{\rho,1})^{\mathrm{ram}}(s), \tag{6.74}$$

a pair of equations that can be summed up as

$$(v^0_{\rho+1,1-\varepsilon})^{\mathrm{ram}}(s) = (\rho+1-\varepsilon)^{-1}\left[2\pi\,(-1)^{\varepsilon} - \frac{d}{ds}\right](v^0_{\rho,\varepsilon})^{\mathrm{ram}}(s). \tag{6.75}$$

This equation is also valid at any integral point $\rho_0 \geq 0$ with $\rho_0+\varepsilon$ even, in view of the fact that, if such is the case, one has

$$(v^k_{\rho_0,\varepsilon})^{\mathrm{ram}}(s) = \mathrm{Res}_{\rho=\rho_0}\left((v^k_{\rho,\varepsilon})^{\mathrm{ram}}(s)\right): \tag{6.76}$$

to prove this relation, we use the defining relation

$$v^k_{\rho,\varepsilon}(s) = (v^k_{\rho,\varepsilon})^{\mathrm{int}}(s) + |s|^{\rho}_{\varepsilon}\,(v^k_{\rho,\varepsilon})^{\mathrm{ram}}(s) \tag{6.77}$$

together with the Taylor expansion

$$|s|^{\rho}_{\varepsilon} = |s|^{\rho_0}_{\varepsilon}\,|s|^{\rho-\rho_0} = s^{\rho_0}\,[1+(\rho-\rho_0)\log|s|+\cdots]. \tag{6.78}$$

Noting that, for any smooth function v, one has

$$\left(\frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\right)\left[(x_1 - i\,x_2)^{\ell}\, v\left(\frac{|x|^2}{2}\right)\right] = (x_1 - i\,x_2)^{\ell+1}\, v'\left(\frac{|x|^2}{2}\right), \tag{6.79}$$

one finds (6.69) as a consequence of (6.75).

Next, Proposition 2.13 yields for every $k = 0, 1, \dots$ the pair of relations

$$R\,T^{\dagger}_{\rho}\, v^k_{\rho,\varepsilon} = \frac{\rho+2k+1+\varepsilon}{2\pi}\, v^{k+1}_{\rho,\varepsilon}, \qquad T\,R^{\dagger}_{\rho}\, v^{-k}_{\rho,\varepsilon} = -\frac{\rho+2k+1-\varepsilon}{2\pi}\, v^{-k-1}_{\rho,\varepsilon}. \tag{6.80}$$

Using (under the renewed assumption that $\mathrm{Re}\,\rho > -1$ and $\rho+\varepsilon \notin 2\mathbb{Z}$) the equations (a consequence of (6.72), in which all ramified parts are taken with respect to the space $C^{\omega}_{\rho,\varepsilon}$)

$$(R\,T^{\dagger}_{\rho}\, v)^{\mathrm{ram}} = T^{\dagger}_{-\rho-1}\, R\, v^{\mathrm{ram}}, \qquad (T\,R^{\dagger}_{\rho}\, v)^{\mathrm{ram}} = R^{\dagger}_{-\rho-1}\, T\, v^{\mathrm{ram}} \tag{6.81}$$

and making the compositions of two differential operators involved on the right-hand side explicit, one thus obtains the equations, valid for $k = 0, 1, \dots,$

$$\begin{aligned}&(\rho+1+2k+\varepsilon)\,\left(v^{k+1}_{\rho,\varepsilon}\right)^{\mathrm{ram}}\\ &\quad = \frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + (\rho+1-4\pi\,s)\,\frac{d}{ds} + 4\pi^2\,s - 2\pi\,(\rho+1)\right]\left(v^k_{\rho,\varepsilon}\right)^{\mathrm{ram}}, \end{aligned} \tag{6.82}$$

$$\begin{aligned}&-(\rho+1+2k-\varepsilon)\,\left(v^{-k-1}_{\rho,\varepsilon}\right)^{\mathrm{ram}}\\ &\quad = \frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + (\rho+1+4\pi\,s)\,\frac{d}{ds} + 4\pi^2\,s + 2\pi\,(\rho+1)\right]\left(v^{-k}_{\rho,\varepsilon}\right)^{\mathrm{ram}}. \end{aligned} \tag{6.83}$$

Again, these last two equations are still valid if $\rho = 0, 1, \dots$, and $\rho + \varepsilon$ is even, by the argument using (6.76).

On the other hand, for any smooth function v, one has the relations

$$\begin{aligned}
&\Delta\left((x_1 - i\,x_2)^\ell\, v\left(\frac{|x|^2}{2}\right)\right)\\
&\qquad = (x_1 - i\,x_2)^\ell\, \Delta\left(v\left(\frac{|x|^2}{2}\right)\right) + 2\ell\,(x_1 - i\,x_2)^\ell\, v'\left(\frac{|x|^2}{2}\right),\\
&\left(\sum x_j\,\frac{\partial}{\partial x_j}\right)\left((x_1 - i\,x_2)^\ell\, v\left(\frac{|x|^2}{2}\right)\right)\\
&\qquad = (x_1 - i\,x_2)^\ell \left(\sum x_j\,\frac{\partial}{\partial x_j} + \ell\right)\left(v\left(\frac{|x|^2}{2}\right)\right):
\end{aligned} \tag{6.84}$$

using also the expression (3.46) of the radial part of the operator $\sum A_j^{*\,2}$ and a similar one concerning the operator $\sum A_j^2$, one finds

$$\begin{aligned}
\left(\sum A_j^{*\,2}\right)\left((x_1 - i\,x_2)^\ell\, v\left(\frac{|x|^2}{2}\right)\right) &= (x_1 - i\,x_2)^\ell\,(D^- v)\left(\frac{|x|^2}{2}\right),\\
\left(\sum A_j^2\right)\left((x_1 - i\,x_2)^\ell\, v\left(\frac{|x|^2}{2}\right)\right) &= (x_1 - i\,x_2)^\ell\,(D^+ v)\left(\frac{|x|^2}{2}\right),
\end{aligned} \tag{6.85}$$

where

$$\begin{aligned}
D^- &= \frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + \left(\frac{n}{2} + \ell - 4\pi s\right)\frac{d}{ds} + 4\pi^2\,s - 2\pi\left(\frac{n}{2} + \ell\right)\right],\\
D^+ &= \frac{1}{2\pi}\left[s\,\frac{d^2}{ds^2} + \left(\frac{n}{2} + \ell + 4\pi s\right)\frac{d}{ds} + 4\pi^2\,s + 2\pi\left(\frac{n}{2} + \ell\right)\right]:
\end{aligned} \tag{6.86}$$

comparing this pair of operators to the ones occurring in (6.83), we obtain (6.70).

We first prove (6.71) in the case when $\ell = 0$: to start with, it follows from (2.22) that the ramified part of the function $v^0_{\frac{n-2}{2},0}(s)$ is a constant times $|s|^{-\frac{n-2}{4}}\, I_{\frac{n-2}{4}}(2\pi\,|s|)$, so that, according to the definition of the function Φ in Definition 3.2, the function u_0^0 agrees with Φ up to the multiplication by some constant. Also, u_0^0 is invariant under the anaplectic Fourier transformation, a consequence of (4.81) together with (5.46). The equation (6.69), together with the pair of equations (5.44), shows that the equation (6.71) is indeed the correct one when $\ell = 0$: to derive the general case, involving possibly non-zero values of k, one may rely on the pair of equations (6.70) together with

$$\mathcal{F}_{\rm ana}\, A_j^* = -i\,A_j^*\,\mathcal{F}_{\rm ana}, \qquad \mathcal{F}_{\rm ana}\, A_j = i\,A_j\,\mathcal{F}_{\rm ana}, \tag{6.87}$$

again a consequence of (5.44). □

End of Proof of Theorem 6.5. We now proceed to show, with the help of (6.70) again, that u itself lies in $\mathfrak{A}^{(n)}$. In view of Definition 4.12, the main problem is to transfer the operator $\sum A_j^{*2}$ or $\sum A_j^2$ to an operator acting on the $\mathcal{K}$-realization, in the sense of Theorem 4.11, of functions in $\mathfrak{A}^{(n)}$.

Starting from the expression (3.45) of $\sum A_j^{*2}$, we use Definition 4.1 to observe that the operator $(\pi\,|x|^2)$ of multiplication by the function indicated transfers to the $\mathcal{Q}$-realization (only scalar such transforms have to be considered here since we are dealing with even functions only) as the operator $i\sum_j \frac{\partial}{\partial\sigma_{(jj)}}$; using Lemma 5.1, we then obtain that the same operator transfers to the $\mathcal{K}$-realization as the operator $i\left[\nabla_{\frac{(I-Z)^2}{2i}} + \frac{1}{4}\,\mathrm{Im}\ \mathrm{Tr}\,Z\right]$. Next, $-\frac{1}{4\pi}\,\Delta = \mathcal{F}_{\mathrm{ana}}\,(\pi\,|x|^2)\,\mathcal{F}_{\mathrm{ana}}^{-1}$ transfers to the latter realization, as a consequence of the first equation (5.46), as the operator $i\left[-\nabla_{\frac{(I+Z)^2}{2i}} - \frac{1}{4}\,\mathrm{Im}\ \mathrm{Tr}\,Z\right]$. Finally, the operator $-\sum x_j\,\frac{\partial}{\partial x_j} - \frac{n}{2}$ transfers to the first realization as the operator $2\sum_{j,k=1}^n \sigma_{jk}\,\frac{\partial}{\partial\sigma_{(jk)}} + \frac{n}{2}$, to the second one as the operator $\nabla_{I-Z^2} - \frac{1}{2}\,\mathrm{Re}\ \mathrm{Tr}\,Z$, as seen after a short calculation using Lemma 5.1 again, also (5.16). Adding the results of the three computations that precede, we see that the operator $\sum A_j^{*2}$ transfers to the $\mathcal{K}$-realization as the operator

$$
\begin{aligned}
i\left[\nabla_{\frac{(I-Z)^2}{2i}} + \nabla_{\frac{(I+Z)^2}{2i}}\right] &+ \frac{i}{2}\,\mathrm{Im}\ \mathrm{Tr}\,Z + \nabla_{I-Z^2} - \frac{1}{2}\,\mathrm{Re}\ \mathrm{Tr}\,Z \\
&= 2\,\nabla_I + \frac{1}{2}\,\mathrm{Tr}\,Z = 2\,\mathrm{Tr}\,\frac{\partial}{\partial Z} - \frac{1}{2}\,\mathrm{Tr}\,(Z^{-1}) : \qquad (6.88)
\end{aligned}
$$

note that the matrix I is a linear combination with *complex* coefficients of the two matrices $\frac{(I-Z)^2}{2i}$ and $\frac{(I+Z)^2}{2i}$, both in the tangent space to $\Sigma^{(n)}$ at Z: alternatively, one may regard the derivative along the vector I as the operator $\sum \frac{\partial}{\partial Z_{(jj)}}$, since we are dealing with functions analytic on open subsets of Σ^2, the complexification of which agrees locally with $\mathrm{Sym}_n^{\mathbb{C}}$. Only the last addition has to be performed again to find that $\sum A_j^2$ transfers to $2\,\nabla_{Z^2} + \frac{1}{2}\,\mathrm{Tr}\,Z = 2\,\mathrm{Tr}\,(Z\,\frac{\partial}{\partial Z}\,Z) - \frac{1}{2}\,\mathrm{Tr}\,Z$. From the estimate regarding the sequence $(c_k)_{k\in\mathbb{Z}}$ made explicit just before the statement of the present proposition, it follows that the series (6.66) defining u converges in the space $\mathfrak{A}^{(n)}$, since the corresponding series of $\mathcal{K}$-transforms converges in the space of functions analytic in the open subset of $\Sigma^{n)}$ where the function $(\mathcal{K}\,u_\ell^0)_0$ is analytic as a consequence of Theorem 4.18 and (6.69).

The Hecke-type formula announced in (6.64) finally follows from the equation

$$
\pi_{\frac{n-2}{2}+\ell,\varepsilon}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) v^k_{\frac{n-2}{2}+\ell,\varepsilon} = \begin{cases} (-1)^k\ v^k_{\frac{n-2}{2}+\ell,\varepsilon} & \text{if } \varepsilon = 0, \\ (-1)^k\,(-i)\ v^k_{\frac{n-2}{2}+\ell,\varepsilon} & \text{if } \varepsilon = 1, \end{cases} \qquad (6.89)
$$

obtained as a consequence of (2.20). □

7 The dual anaplectic analysis

Eigenfunctions of the usual type of the harmonic oscillator $L^{(n)}$ with the same parity correspond to eigenvalues of $L^{(n)}$ that differ by some even integer, since even (*resp.* odd) such eigenfunctions can only correspond to eigenvalues κ with $\kappa - \frac{n}{2}$ even (*resp.* odd). The same is the case with anaplectic Hermite functions, except for the fact that the shift by $\frac{n}{2}$ is not present any more: this has been shown in Theorem 3.4. In the usual analysis, there is nothing one can do about this. However, as we shall see in the present short section, there exists a *dual anaplectic* analysis, the development of which starts with the consideration of the spaces $E_{\kappa,\ell}$ of eigenfunctions of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ with $\kappa + \ell$ odd. Our interest in this matter does not reduce to the need for completeness. Indeed, such a kind of eigenfunctions of the harmonic oscillator may enter the picture whether we want it or not: an example will arise in Theorem 10.14, to be preceded by (10.73).

The first example of a function in $E_{0,1}$ is the function $x \mapsto x_1\,|x|^{-\frac{n}{2}}\, I_{\frac{n}{4}}(\pi\,|x|^2)$. Note that, in the one-dimensional case, it is a multiple of the function $\phi^\natural$ introduced in (1.94).

Theorem 7.1. *Let* $x \mapsto \langle \xi,\, x\rangle$ *be an arbitrary non-zero real linear form on* $\mathbb{R}^n$. *The linear space consisting of all images of the odd function*

$$\Psi^\natural_\xi(x) = \langle \xi,\, x\rangle\,|x|^{-\frac{n}{2}}\, I_{\frac{n}{4}}(\pi\,|x|^2) \tag{7.1}$$

under arbitrary operators in the algebra $\mathbb{C}\,[x,\, \frac{\partial}{\partial x}]$ *is generated by the union of the spaces* $E_{\kappa,\ell}$ *with* $\kappa \in \mathbb{Z}$, $\ell = 0, 1, \dots$ *and* $\kappa + \ell$ *odd. It contains the even function*

$$\Psi^\natural(x) = |x|^{\frac{4-n}{2}}\,[I_{\frac{n}{4}}(\pi\,|x|^2) - I_{\frac{n-4}{4}}(\pi\,|x|^2)] \tag{7.2}$$

and, unless $n = 2$, *it could be generated in the same way after* $\Psi^\natural$ *has been substituted for* $\Psi^\natural_\xi$.

Proof. Using the rotation operators $x_j\,\frac{\partial}{\partial x_k} - x_k\,\frac{\partial}{\partial x_j}$, one sees that it is no loss of generality to assume that $\langle \xi,\, x\rangle = x_j$ for some j, in which case we denote $\Psi^\natural_\xi$ as $\Psi^\natural_j$. That $\Psi^\natural_j$ lies in the null space of the formal harmonic oscillator is a consequence of the equation (3.26) with $f(r) = r^{\frac{2-n}{2}}\, I_{\frac{n}{4}}(\pi\, r^2)$, $\ell = 1$ and $\kappa = 0$. The proof of Theorem 3.4 adapts with trivial changes. Concerning the function $\Psi^\natural$, we only observe that one has

$$\Psi^\natural = \pi^{-\frac{1}{2}} \sum_{j=1}^{n} A_j^*\, \Psi^\natural_j, \qquad \pi^{\frac{1}{2}}\, A_k\, \Psi^\natural = \frac{2-n}{2}\, \Psi^\natural_k, \tag{7.3}$$

as seen by a straightforward computation. □

Remark 7.1. When $n = 2$, $\Psi^{\natural}(x) = -2^{\frac{1}{2}}\,\pi^{-1}\,e^{-\pi|x|^2}$, which explains why this function is annihilated by the lowering operator A_k. More generally, the Gaussian function $x \mapsto e^{-\pi|x|^2}$, which lies in $E_{\kappa,\ell}$ with $\kappa = \frac{n}{2}$ and $\ell = 0$ when n is even, is a *dual anaplectic* Hermite function if and only if $n \equiv 2 \mod 4$: compare Remark 3.1. When this is the case, all Hermite functions of the usual type are also dual anaplectic Hermite functions.

To prepare for the definition of the dual anaplectic space $(\mathfrak{A}^{(n)})^{\natural}$, let us introduce the $(2n)$-fold covering $\Sigma^{(2n)}$ of Σ, *i.e.*, the space suitable for a definition of $(\det Z)^{\frac{1}{2n}}$, also a twofold covering of $\Sigma^{(n)}$. A loop $\phi \mapsto e^{i\phi}\,Z$, $0 \le \phi \le 2\pi$ in Σ, lifts as a loop in $\Sigma^{(n)}$ but not in $\Sigma^{(2n)}$: the corresponding map: $\Sigma^{(2n)} \to \Sigma^{(2n)}$ from the initial point of the path to its end point defines the unique non-trivial covering automorphism $\omega_{(n)}^{(2n)}$ of $\Sigma^{(2n)}$ above the identity of $\Sigma^{(n)}$. The following generalizes Definition 4.12.

Definition 7.2. Under the same initial assumptions regarding the analytic function u on $\mathbb{C}^n$ as in Definition 4.12, we shall say that u lies in $(\mathfrak{A}^{(n)})^{\natural}$ if the following holds: the functions $(\mathcal{K}\,u)_0$ and $(\mathcal{K}\,u)_1$ extend as analytic functions in some open subset of $\Sigma^{(2n)}$ connected in the strong sense and invariant under the automorphism $\omega_{(n)}^{(2n)}$; moreover, each of these two functions changes to its negative under $\omega_{(n)}^{(2n)}$.

Remark 7.2. Recall that the points $e^{i\pi}\,I$ and $e^{-i\pi}\,I$ of $\Sigma^{(2n)}$ are distinct. It is necessary – for the sake of coherence with the parameter-dependent situation of Section 11 – to generalize the definition (4.42) of the linear form Int, in the present context denoted as $\mathrm{Int}^{\natural}$, according to the following normalization:

$$\mathrm{Int}^{\natural}\,[u] = 2^{\frac{n}{2}}\,(\mathcal{K}\,u)_0(e^{-i\pi}\,I). \tag{7.4}$$

The following generalizes Theorem 4.14.

Theorem 7.3. *The space* $(\mathfrak{A}^{(1)})^{\natural}$ *consists of all entire functions* u *of one variable which satisfy the following condition: there is a* $\mathbb{C}^4$*-valued function* $\boldsymbol{f} = (f_0, f_1, f_{i,0}, f_{i,1})$*, the components of which are nice functions in the sense of Definition* 1.1*, related by the equations* (*to be compared to* (1.8)*, noting the sign changes*)*,*

$$f_{i,0}(z) = \frac{1+i}{2}\,(f_0(iz) - i\,f_0(-iz)),$$
$$f_{i,1}(z) = \frac{1-i}{2}\,(f_1(iz) + i\,f_1(-iz)), \tag{7.5}$$

such that the even part of u *coincides with that of* f_0 *and the odd part of* u *coincides with that of* f_1*. If such is the case, the* 4*-tuple* $\boldsymbol{f}$ *is unique. One has*

$$\mathrm{Int}^{\natural}\,[u] = 2^{\frac{1}{2}}\,i\int_0^{\infty}(f_0(x) - f_{i,0}(x))\,dx. \tag{7.6}$$

Proof. We shall not rewrite the variant of the part of Section 2 needed to that effect though, as pointed out in Remark 2.1, this is a straightforward if time-consuming task. Since only a change of sign is required here and there, we shall be satisfied with the verification that all that precedes works in the case when $u = \phi^\natural$, the basic function of the dual anaplectic analysis introduced in (1.94). Consider the function introduced in (1.11) and defined, for $x > 0$, by the equation $\psi(x) = 2^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}\,x^{\frac{1}{2}}\,K_{\frac{1}{4}}(\pi x^2)$. We now consider the 4-tuple $\boldsymbol{f} = (0,\,\psi,\,0,\,\psi)$ instead of the vector $(\psi,\,0,\,\psi,0)$ considered in the proof of Proposition 1.2. The odd part of ψ is just the function $\phi^\natural$, so that this function lies in the space $\mathfrak{A}^\natural$ introduced in Remark 1.3. We have to show that it lies in the space $(\mathfrak{A}^{(1)})^\natural$ according to Definition 7.2.

Starting from Definition 4.1, we obtain, for $\sigma > 0$,

$$\begin{aligned}(\mathcal{Q}\,\phi^\natural)_1(\sigma) &= (1+i\,\sigma)\int_{-\infty}^{\infty} x\;\phi^\natural(x\,e^{-\frac{i\pi}{4}})\,e^{-\pi\sigma x^2}\,dx\\ &= -2^{\frac{1}{2}}\,\pi\,(1-i)\,(1+i\,\sigma)\int_0^{\infty} x^{\frac{3}{2}}\,J_{\frac{1}{4}}(\pi x^2)\,e^{-\pi\sigma x^2}\,dx\\ &= -2^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}\,\Gamma\left(\frac{3}{4}\right)\,(1-i)\,(1+i\,\sigma)\,(1+\sigma^2)^{-\frac{3}{4}}. \end{aligned} \tag{7.7}$$

Then, from (4.39), if $0 < \theta < \pi$, recalling that $\sigma = \text{cotan}\,\frac{\theta}{2}$ is the point of $\mathbb{R}$ corresponding under the Cayley map to the point $e^{-i\theta} \in S^1$,

$$(\mathcal{K}\,\phi^\natural)_1(e^{-i\,\theta}) = -\pi^{-\frac{1}{2}}\,\Gamma\left(\frac{3}{4}\right)\,(1+i)\,e^{-\frac{i\,\theta}{2}}, \tag{7.8}$$

indeed a function of $Z = e^{-i\,\theta}$ that extends as a function on the twofold covering of S^1 and changes to its negative under the map $\theta \mapsto \theta + 2\pi$.

We finally check the normalization constant that enters the equation (7.6), to be compared to (1.74). To do this, of course, we must use this time an even function in $\mathfrak{A}^\natural = (\mathfrak{A}^{(1)})^\natural$, for instance the function $x \mapsto x\,\phi^\natural(x)$ which is associated to the 4-tuple $\boldsymbol{h} = (h_0,\,0,\,h_{i,0},\,0)$ with $h_0(x) = x\,\psi(x)$, *i.e.*, $\boldsymbol{h} = (x\,\psi,\,0,\,-x\,\psi,\,0)$. One has

$$\begin{aligned}(\mathcal{Q}\,(x\,\phi^\natural))_0(\sigma) &= \int_{-\infty}^{\infty} e^{-\frac{i\pi}{4}}\,x\;\phi^\natural(x\,e^{-\frac{i\pi}{4}})\,e^{-\pi\sigma x^2}\,dx\\ &= 2i\,\pi^{\frac{1}{2}}\int_0^{\infty} x^{\frac{3}{2}}\,J_{\frac{1}{4}}(\pi x^2)\,e^{-\pi\sigma x^2}\,dx\\ &= 2^{\frac{1}{4}}\,i\,\pi^{-\frac{5}{4}}\,\Gamma\left(\frac{3}{4}\right)\,(1+\sigma^2)^{-\frac{3}{4}}\;: \end{aligned} \tag{7.9}$$

hence, for $0 < \theta < \pi$,

$$(\mathcal{K}\,(x\,\phi^\natural))_0(e^{-i\,\theta}) = 2^{-\frac{1}{4}}\,i\,\pi^{-\frac{5}{4}}\,\Gamma\left(\frac{3}{4}\right)\,\sin\frac{\theta}{2} \tag{7.10}$$

and, in particular,

$$\mathrm{Int}^\natural\,[x\,\phi^\natural] = 2^{\frac{1}{2}}\,(\mathcal{K}\,(x\,\phi^\natural))_0(e^{-i\,\pi}) = 2^{\frac{1}{4}}\,i\,\pi^{-\frac{5}{4}}\,\Gamma\left(\frac{3}{4}\right). \tag{7.11}$$

On the other hand, the integral

$$2^{\frac{1}{2}}\,i\int_0^\infty (f_0(x) - f_{i,0}(x))\,dx = 4\,\pi^{-\frac{1}{2}}\,i\int_0^\infty x^{\frac{3}{2}}\,K_{\frac{1}{4}}(\pi x^2)\,dx \tag{7.12}$$

has the same value, according to [17, p. 91] again. □

The following analogue of Theorem 4.15 is immediate.

Theorem 7.4. *The Gaussian function* $x \mapsto e^{-\pi|x|^2}$ (*or any function* $x \mapsto e^{-\pi\,Q(x)}$, *where* $Q(x)$ *is a quadratic function of* x *with a positive-definite real part*) *lies in the space* $(\mathfrak{A}^{(n)})^\natural$ *if and only if* $n \equiv 2 \mod 4$.

Proof. It suffices to use the equation (4.63), here recalled,

$$(\mathcal{K}\,u)_0(Z) = 2^{-\frac{n}{2}}\,(\det Z)^{-\frac{1}{4}}, \tag{7.13}$$

observing that the function on the right-hand side extends as an analytic function on $\Sigma^{(2n)}$ if and only n is even: if $n \equiv 0$ (*resp.* $n \equiv 2$) mod 4, this function is invariant (*resp.* changes to its negative) under the automorphism $\omega^{(2n)}_{(n)}$ of $\Sigma^{(2n)}$ defined just before Definition 7.2. □

One can develop the dual anaplectic analysis in the same way as that used, in Sections 4 and 5, for the anaplectic analysis: there are no changes worth mentioning – apart from a sign change in the analogue of (1.72), which will be mentioned later – except in the proof of the following analogue of Theorem 4.18.

Theorem 7.5. *In any dimension* n, *the dual anaplectic Hermite functions introduced in Theorem* 7.1 *lie in the space* $(\mathfrak{A}^{(n)})^\natural$. *The singular set of the* $\mathcal{K}$*-transforms of any such function is contained in the pullback of the set* Δ *introduced in Lemma* 4.16.

Proof. The analogue of Proposition 5.2, to the effect that the space $\mathfrak{A}^{(n)}$ is invariant under the algebra $\mathbb{C}\,[x, \frac{\partial}{\partial x}]$, works just as well for the space $(\mathfrak{A}^{(n)})^\natural$. In particular, the one-dimensional case of Theorem 7.5 is thus a consequence of Theorem 7.3. We first consider the case when $n \geq 3$, since the case when $n = 2$ is somewhat special. Indeed, from Theorem 7.1, we may in the first case reduce the problem to proving that the function $\Psi^\natural$ lies in $(\mathfrak{A}^{(n)})^\natural$.

Using [17, p. 84], one can write (7.2) as the integral

$$\begin{aligned}\Psi^\natural(x) &= 2^{1-\frac{n}{4}}\,\pi^{\frac{n}{4}-\frac{3}{2}} \int_{-1}^{1} e^{-\pi|x|^2 t} \left[\frac{2\pi\,|x|^2}{n-2}\,(1-t^2)^{\frac{n}{4}-\frac{1}{2}} - (1-t^2)^{\frac{n}{4}-\frac{3}{2}}\right] dt \\ &= -2^{1-\frac{n}{4}}\,\pi^{\frac{n}{4}-\frac{3}{2}} \int_{-1}^{1} (1+t)\,(1-t^2)^{\frac{n}{4}-\frac{3}{2}}\, e^{-\pi|x|^2 t}\, dt. \qquad (7.14)\end{aligned}$$

The same computation as in (4.69) leads, if $\sigma \succ 0$, to

$$(\mathcal{Q}\,\Psi^\natural)_0(\sigma) = D_n \int_{-1}^{1} (1+t)\,(1-t^2)^{\frac{n}{4}-\frac{3}{2}}\, [\det(\sigma - it)]^{-\frac{1}{2}}\, dt \qquad (7.15)$$

with $D_n = -2^{1-\frac{n}{4}}\,\pi^{\frac{n}{4}-\frac{3}{2}}$. Then, just as in (4.74), but with the extra factor $(1+t)\,(1-t^2)^{-\frac{1}{2}}$ which becomes e^ξ when $t = \tanh\xi$, one finds, with the notation there,

$$(\mathcal{K}\,\Psi^\natural)_0(Z) = 2^{-\frac{n}{2}}\, D_n \int_{-\infty}^{\infty} \prod_j \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} e^\xi\, d\xi \qquad (7.16)$$

if $-\pi < \theta_j < \pi$ for every j, or

$$(\mathcal{K}\,\Psi^\natural)_0(Z) = 2^{-\frac{n}{2}}\, D_n\, i \int_{-\infty}^{\infty} \prod_j \left(\cosh\left(\xi - \frac{i\,(\theta_j - \pi)}{2}\right)\right)^{-\frac{1}{2}} e^\xi\, d\xi \qquad (7.17)$$

if $\theta_j \in]0, 2\pi[$ for every j, an assumption that covers the case when $Z \in \Sigma^{\text{reg}} \cap (\Sigma\backslash\Delta)$.

With the help of Lemma 4.17 and of the proof of this lemma, the problem of continuing analytically this integral reduces to the same problem along the special paths $\phi \mapsto e^{i\phi}\, Z$. The change of contour of integration $\xi \mapsto \xi - \frac{i\phi}{2}$ shows that along such a path, $(\mathcal{K}\,\Psi^\natural)_0(Z)$ undergoes the multiplication by $e^{-\frac{i\phi}{2}}$: this concludes the proof of Theorem 7.5 in the case when $n \neq 2$.

The case when $n = 2$ is more complicated. We must consider instead of $\Psi^\natural$ the odd function

$$\begin{aligned}\Psi_1^\natural(x) &= \frac{x_1}{|x|}\, I_{\frac{1}{2}}(\pi\,|x|^2) = 2^{\frac{1}{2}}\,\pi^{-1}\, x_1\, \frac{\sinh \pi|x|^2}{|x|^2} \\ &= 2^{-\frac{1}{2}}\, x_1 \int_{-1}^{1} e^{-\pi|x|^2 t}\, dt. \qquad (7.18)\end{aligned}$$

Using Definition 4.1, one obtains the vector-valued part of the $\mathcal{Q}$-transform of $\Psi_1^\natural$ as

$$(\mathcal{Q}\,\Psi_1^\natural)_1(\sigma) = 2^{-\frac{1}{2}}\, e^{-\frac{i\pi}{4}} \int_{\mathbb{R}^2} (I + i\,\sigma) \begin{pmatrix} x_1^2 \\ x_1\, x_2 \end{pmatrix} e^{-\pi\,\langle \sigma x,\, x\rangle}\, dx \int_{-1}^{1} e^{i\pi\,|x|^2 t}\, dt. \quad (7.19)$$

Setting $\sigma = \left(\begin{smallmatrix} p & r \\ r & q \end{smallmatrix}\right)$, one can write this as

$$(\mathcal{Q}\,\Psi_1^\natural)_1(\sigma) = -2^{-\frac{1}{2}}\,\pi^{-1}\,e^{-\frac{i\pi}{4}} \begin{pmatrix} (1+i\,p)\,\frac{\partial}{\partial p} + \frac{ir}{2}\,\frac{\partial}{\partial r} \\ ir\,\frac{\partial}{\partial p} + \frac{1+i\,q}{2}\,\frac{\partial}{\partial r} \end{pmatrix} V(\sigma) \tag{7.20}$$

with

$$V(\sigma) = \int_{\mathbb{R}^2} e^{-\pi\,\langle \sigma x,\, x\rangle}\,dx \int_{-1}^{1} e^{i\pi\,|x|^2 t}\,dt = \int_{-1}^{1} [\det(\sigma - it)]^{-\frac{1}{2}}\,dt, \tag{7.21}$$

an equation valid when $\sigma \succ 0$. Hence

$$(\mathcal{Q}\,\Psi_1^\natural)_1(\sigma) = 2^{-\frac{3}{2}}\,\pi^{-1}\,e^{-\frac{i\pi}{4}} \int_{-1}^{1} \begin{pmatrix} (1+i\,p)\,(q-i\,t) - i\,r^2 \\ r\,(t-1) \end{pmatrix} [\det(\sigma - it)]^{-\frac{3}{2}}\,dt. \tag{7.22}$$

The vector in the integrand can be decomposed as

$$\begin{pmatrix} (1+i\,p)\,(q-i\,t) - i\,r^2 \\ r\,(t-1) \end{pmatrix} = i \begin{pmatrix} \det(\sigma - i\,t) - (1+i\,q)\,(1-t) + (1-t)^2 \\ i\,r\,(t-1) \end{pmatrix}, \tag{7.23}$$

so that

$$(\mathcal{Q}\,\Psi_1^\natural)_1(\sigma) = 2^{-\frac{1}{2}}\,\pi^{-1}\,e^{\frac{i\pi}{4}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} (W_0(\sigma) + W_2(\sigma)) + \begin{pmatrix} -1-i\,q \\ i\,r \end{pmatrix} W_1(\sigma) \right] \tag{7.24}$$

with

$$\begin{aligned} W_0(\sigma) &= \int_{-1}^{1} [\det(\sigma - i\,t)]^{-\frac{1}{2}}\,dt, \\ W_1(\sigma) &= \int_{-1}^{1} (1-t)\,[\det(\sigma - i\,t)]^{-\frac{3}{2}}\,dt, \\ W_2(\sigma) &= \int_{-1}^{1} (1-t)^2\,[\det(\sigma - i\,t)]^{-\frac{3}{2}}\,dt. \end{aligned} \tag{7.25}$$

Our problem is to first extend the three terms as analytic functions on the whole space Sym_2, next, after having expressed the results in terms of the Cayley transform Z of σ and multiplied each term by $|\det(I-Z)|^{-\frac{1}{2}}$ – recall that this operation from functions of σ to functions of Z changes a $\mathcal{Q}$-transform into the corresponding $\mathcal{K}$-transform – to analyse the analytic continuation of the linear combination in (7.24) to $\Sigma^{(4)}$. The proof follows closely that of Theorem 4.18. Denoting as σ_1 and σ_2 the eigenvalues of σ initially assumed to be positive-definite, and setting $t = \tanh \xi$ in the integrals that precede, we find

$$W_0(\sigma) = \int_{-\infty}^{\infty} \prod_{j=1,2} (\sigma_j \cosh \xi - i \sinh \xi)^{-\frac{1}{2}} \frac{d\xi}{\cosh \xi},$$

$$W_1(\sigma) = \int_{-\infty}^{\infty} \prod_{j=1,2} (\sigma_j \cosh \xi - i \sinh \xi)^{-\frac{3}{2}} e^{\xi} d\xi,$$

$$W_2(\sigma) = \int_{-\infty}^{\infty} \prod_{j=1,2} (\sigma_j \cosh \xi - i \sinh \xi)^{-\frac{3}{2}} \frac{e^{2\xi}}{\cosh \xi} d\xi. \tag{7.26}$$

If the eigenvalues of the Cayley transform Z of σ are $e^{-i\theta_j}$, *i.e.*, if $\sigma_j = \text{cotan}\, \frac{\theta_j}{2}$, this can be rewritten if $0 < \theta_j < \pi$ as

$$W_0(\sigma) = \left(\sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \prod_{j=1,2} \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} \frac{d\xi}{\cosh \xi},$$

$$W_1(\sigma) = \left(\sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \prod_{j=1,2} \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{3}{2}} e^{\xi} d\xi,$$

$$W_2(\sigma) = \left(\sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \prod_{j=1,2} \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{3}{2}} \frac{e^{2\xi}}{\cosh \xi} d\xi. \tag{7.27}$$

Notwithstanding the presence of $\cosh \xi$ in the denominator of two of the integrands, the change of contour $\xi \mapsto \xi + \frac{i\,\omega}{2}$ is still possible when $|\omega| < \pi$, leading to the expression

$$W_0(\sigma) = \left(\sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\right)^{\frac{1}{2}} \int_{\text{Im } \xi = \frac{\omega}{2}} \prod_{j=1,2} \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} \frac{d\xi}{\cosh \xi}, \tag{7.28}$$

which makes it possible, if one chooses ω close to π, to cover the case when θ_1 and θ_2 both lie in $]0, 2\pi[$, *i.e,* the case when $Z \in \Sigma^{\text{reg}}$. One has in this case $|\det (I - Z)|^{-\frac{1}{2}} = \frac{1}{2} (\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2})^{-\frac{1}{2}}$, a scalar factor the multiplication by which transforms each of the factors in front of the right-hand sides of (7.27) into a function with an obvious analytic continuation to Σ. Our sole remaining problem lies with the three integrals in (7.27), the second one accompanied, as indicated by (7.24), by the vector $\begin{pmatrix} -1-i\,q \\ i\,r \end{pmatrix}$. This goes essentially like the end of the proof of Theorem 4.18, starting with the use of Lemma 4.17, which reduces the problem of analytic continuation to that along paths $\phi \mapsto e^{-i\phi} Z$, originating at points $Z \in \Sigma^{\text{reg}} \cap (\Sigma \backslash \Delta)$. It is no loss of generality to assume that Z is diagonal, say $Z = \begin{pmatrix} e^{-i\,\theta_1} & \\ & e^{-i\,\theta_2} \end{pmatrix}$, in which case the extra vector in front of $W_1(\sigma)$ in (7.24) reduces, on the path, to $\begin{pmatrix} -1-i\,e^{-i\,(\theta_1+\phi)} \\ 0 \end{pmatrix}$, a vector which comes back to its initial value when $\phi = 2\pi$.

There is, however, a novelty, in that in the continuation along such a path, we shall have to cross poles of the function $\xi \mapsto (\cosh \xi)^{-1}$ in the integrands of $W_0(\sigma)$ and $W_2(\sigma)$. Fortunately, the residues at $\xi = \frac{i\pi}{2}$ (and, eventually, $\xi = \frac{i\pi}{2} + i\,k\pi$), to wit $\prod (\sin \frac{\theta_j}{2})^{-\frac{1}{2}}$ and $-\prod (\sin \frac{\theta_j}{2})^{-\frac{3}{2}}$, cancel off in the sum $W_0(\sigma) + W_2(\sigma)$ in view of the extra factors in (7.27). To obtain the analytic continuation of an integral such as (7.28) along the path indicated, we may accompany the change $\theta_j \mapsto \theta_j + i\,\phi$ (which corresponds to the map $Z \mapsto e^{-i\,\phi}\, Z$) by the change of contour $\xi \mapsto \xi + \frac{i\,\phi}{2}$, only jumping over the values $\phi = \pi + 2k\pi$, as made possible by the analysis above of residues. When $\phi = 2\pi$, we then obtain instead of the right-hand side of (7.28) the expression

$$\left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}\right)^{\frac{1}{2}} \int_{\mathrm{Im}\ \xi = \frac{\omega}{2}} \prod_{j=1,2} \left(\cosh \left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} \frac{d\xi}{\cosh(\xi + i\,\pi)}, \qquad (7.29)$$

the negative of the preceding one: under the map $\xi \mapsto \xi + i\,\pi$, the same sign change occurs in the functions e^{ξ} and $\frac{e^{2\xi}}{\cosh \xi}$ from the integrals (7.27) defining $W_1(\sigma)$ and $W_2(\sigma)$. This concludes the proof of Theorem 7.5. $\square$

Remark 7.3. In the one-dimensional case, let $\natural \colon u \mapsto u^{\natural}$ be the linear operator: $\mathfrak{A} \to \mathfrak{A}^{\natural}$ defined in the $\mathbb{C}^4$-realization as $(f_0, f_1, f_{i,0}, f_{i,1}) \mapsto (f_1, f_0, f_{i,1}, f_{i,0})$. It intertwines the two versions (on $\mathfrak{A}$ and $\mathfrak{A}^{\natural}$) of the Heisenberg representation: indeed, the equations (1.58) and (1.60) which express the operators from the Heisenberg representation in terms of the $\mathbb{C}^4$-realization lead to the formulas

$$\tau_y\, u^{\natural} = (\tau_y\, u)^{\natural}, \qquad \tau^{\eta}\, u^{\natural} = (\tau^{\eta}\, u)^{\natural}. \qquad (7.30)$$

However, the anaplectic and dual anaplectic representations are not equivalent, as seen from the equation $(\mathcal{F}_{\mathrm{ana}}\, u)^{\natural} = -\mathcal{F}^{\natural}_{\mathrm{ana}}\, u^{\natural}$: it is understood that the dual anaplectic Fourier transformation that occurs on the right-hand side should again be defined by the equation (1.80), after the linear form Int has been replaced by $\mathrm{Int}^{\natural}$. We now proceed towards the necessary verifications: uninteresting as they are, these depend on rather extensive calculations which we now sum up.

The first point is to note that the $\mathbb{C}^4$-realizations

$$\boldsymbol{f} = (f_0, f_1, f_{i,0}, f_{i,1}) \qquad \text{and} \qquad \tilde{\boldsymbol{f}} = (\tilde{f}_0, \tilde{f}_1, \tilde{f}_{i,0}, \tilde{f}_{i,1})$$

of $u \in \mathfrak{A}$ and $\mathcal{F}_{\mathrm{ana}}\, u$ are linked by the relations

$$\begin{aligned}
\tilde{f}_0(x) &= 2^{\frac{1}{2}} \int_0^{\infty} [(\cos 2\pi xy - \sin 2\pi xy)\, f_0(y) + e^{-2\pi xy}\, f_{i,0}(y)]\, dy,\\
\tilde{f}_1(x) &= 2^{\frac{1}{2}}\, i \int_0^{\infty} [-(\cos 2\pi xy + \sin 2\pi xy)\, f_1(y) + e^{-2\pi xy}\, f_{i,1}(y)]\, dy,\\
\tilde{f}_{i,0}(x) &= 2^{\frac{1}{2}} \int_0^{\infty} [e^{-2\pi xy}\, f_0(y) + (\cos 2\pi xy - \sin 2\pi xy)\, f_{i,0}(y)]\, dy,\\
\tilde{f}_{i,1}(x) &= 2^{\frac{1}{2}}\, i \int_0^{\infty} [-e^{-2\pi xy}\, f_1(y) + (\cos 2\pi xy + \sin 2\pi xy)\, f_{i,1}(y)]\, dy. \qquad (7.31)
\end{aligned}$$

This can be verified as a consequence of the results of Section 2. Express $\boldsymbol{f}$ in terms of $(v_0,\, v_1) \in C^{\omega}_{-\frac{1}{2},0} \oplus C^{\omega}_{\frac{1}{2},1}$ by means of (2.84) and (2.85), and do the same for $\tilde{\boldsymbol{f}}$ in terms of $(\tilde{v}_0,\, \tilde{v}_1)$. According to Theorem 2.9, one has $\tilde{v}_0 = \pi_{-\frac{1}{2},0}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_0$ and $\tilde{v}_1 = \pi_{\frac{1}{2},1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\, v_1$, and the operators $v_0 \mapsto \tilde{v}_0$ and $v_1 \mapsto \tilde{v}_1$ involved here were made explicit in (2.102) and (2.103). This, and some patience, leads to the four equations (7.31). Now, set $\boldsymbol{h} = (h_0,\, h_1,\, h_{i,0},\, h_{i,1}) = (f_1,\, f_0,\, f_{i,1},\, f_{i,0})$, the $\mathbb{C}^4$-realization of $u^{\natural}$. On one hand, the transform under $\mathcal{F}^{\natural}_{\text{ana}}$ of this function is given (compare (1.81)) as

$$(\mathcal{F}^{\natural}_{\text{ana}}\, u^{\natural})(x) = 2^{\frac{1}{2}}\, i \int_0^{\infty} \big[h_0(y)\, \cos 2\pi xy\; dy - i\, h_1(y)\, \sin 2\pi xy \\ - h_{i,0}(y)\, \cosh 2\pi xy - i\, h_{i,1}(y)\, \sinh 2\pi xy\big]\; dy, \quad (7.32)$$

where we must explain the sign changes from the former reference. First, comparing the definitions of Int and $\text{Int}^{\natural}$, one sees the reason for the presence of the coefficient $2^{\frac{1}{2}}\, i$ in front of the new integral, as well as the need for a global sign change of the sum of the last two terms. However, there is still another sign change, in the term concerning $h_{i,1}$, due to the following reason: our derivation of (1.81) was based on the use of the linear form Int and of the first equation of each of the two pairs (1.71) and (1.72): now, in the *dual* anaplectic analysis, it is immediate that, so that the formulas remain true, one should change the coefficient i that appears on the right-hand side of the second formula to $-i$. On the other hand, the function $(\mathcal{F}_{\text{ana}}\, u)^{\natural}$ can be obtained as the sum of the even part of $\tilde{f}_1$ and the odd part of $\tilde{f}_0$, which leads to the negative of the preceding result.

8 The pseudoscalar product in n-dimensional anaplectic analysis

The anaplectic analysis does not reduce to a study of the anaplectic representation: it is also meant for providing an analysis on appropriate spaces of functions containing the anaplectic Hermite functions, as introduced in Definition 3.2. It is in this latter sense only that the present section provides an answer to the construction of a "natural" pseudoscalar product, actually a unique one up to normalization.

We start with the construction of a non-degenerate pseudoscalar product on the linear subspace of $\mathfrak{A}^{(n)}$ generated by the anaplectic Hermite functions such that, for every j, the operators A_j and A_j^* should be formally adjoint to each other. We shall show that such a pseudoscalar product exists if and only if $n \not\equiv 0$ mod 4: similarly, such a pseudoscalar product exists on the comparable subspace of $(\mathfrak{A}^{(n)})^{\natural}$ if and only if $n \not\equiv 2$ mod 4. We tacitly assume that $n \geq 2$ in this section.

In the usual analysis, it is easy to define spaces of functions, such as $\mathcal{S}(\mathbb{R}^n)$, containing all Hermite functions and invariant under the Heisenberg representation: the situation is more complicated in the anaplectic analysis. Though the definition of a Heisenberg-invariant linear space, on which the pseudoscalar product would be meaningful, might still be possible, we have chosen to make a simpler construction, based on the consideration of a pair of spaces: then, the Heisenberg representation, restricted to the smaller of the two spaces involved and valued into the larger one, will preserve the pseudoscalar product.

Let us start with the necessity of the condition $n \not\equiv 0$ mod 4 (*resp.* $n \not\equiv 2$ mod 4) in the anaplectic (*resp.* dual-anaplectic) analysis. Set

$$B^* = \sum A_j^{*\,2}, \qquad B = \sum A_j^2, \tag{8.1}$$

and recall from the beginning of the proof of Theorem 3.4 that

$$B^* \colon E_{\kappa,\,0} \to E_{\kappa+2,\,0}, \qquad B \colon E_{\kappa,\,0} \to E_{\kappa-2,\,0}. \tag{8.2}$$

If $u \in E_{\kappa,0}$, $u(x) = h(\frac{|x|^2}{2})$, one has $(B^*u)(x) = h_1(\frac{|x|^2}{2})$, where the function h_1 is given in terms of u by the equation (3.46): moreover, if $h(0) = 1$, it follows from the equation (3.30) (with $\ell = 0$) that $h'(0) = -\frac{4\pi\kappa}{n}$, so that $h_1(0) = -\kappa - \frac{n}{2}$. Consequently, if $f_{\kappa,0}$ denotes the (unique) function in $E_{\kappa,0}$ such that $f_{\kappa,0}(0) = 1$, one has

$$B^* f_{\kappa,0} = -(\kappa + \frac{n}{2})\, f_{\kappa+2,0}. \tag{8.3}$$

Similarly,

$$B\, f_{\kappa,0} = (\frac{n}{2} - \kappa)\, f_{\kappa-2,0}. \tag{8.4}$$

In particular, $B^* f_{\kappa,0} \neq 0$ when $\kappa \geq 0$. On the other hand, assuming that we are dealing with a pseudoscalar product for which the operators A_j and A_j^* are formally adjoint to each other for every j, we may compute $(B^* f_{\kappa,0} \,|\, B^* f_{\kappa,0})$ as $(BB^* f_{\kappa,0} \,|\, f_{\kappa,0})$: it follows from the preceding two equations that

$$BB^* f_{\kappa,0} = \left(\kappa - \frac{n}{2} + 2\right)\left(\kappa + \frac{n}{2}\right) f_{\kappa,0}. \tag{8.5}$$

As a consequence, $(B^* f_{\kappa,0} \,|\, B^* f_{\kappa,0}) = 0$ in the case when $\kappa = \frac{n}{2} - 2$ (this is an admissible value of κ in the case when $n \equiv 0 \mod 4$, as indicated in Theorem 3.4), so that the pseudoscalar product is zero when restricted to the one-dimensional joint eigenspace of the pair $(L^{(n)}, \Delta_{S^{n-1}})$ generated by the function $f_{\frac{n}{2},0}$: hence no non-degenerate pseudoscalar product satisfying our demands can exist on $\mathfrak{A}^{(n)}$. The case of the dual anaplectic analysis (on $(\mathfrak{A}^{(n)})^\natural$) follows just as well.

Denote as $d\sigma$ the usual rotation-invariant measure on the sphere S^{n-1} with total mass $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. For any pair (κ, ℓ), recall that the equation (3.30) has only a one-dimensional space of solutions analytic at 0, and that such a solution does not vanish at 0 if not identically zero. We may thus use on the finite-dimensional vector space $E_{\kappa,\ell}$ the (Hilbert) norm defined as

$$\| u \|_\ell = |h(0)| \left[\int_{S^{n-1}} |\mathcal{Y}^\ell(x)|^2 \, d\sigma(x) \right]^{\frac{1}{2}} \tag{8.6}$$

if $u = h \boxtimes \mathcal{Y}^\ell$.

Lemma 8.1. *If $\mathcal{Y}^\ell$ is a harmonic polynomial, homogeneous of degree ℓ, one has*

$$\int_{S^{n-1}} |\nabla \mathcal{Y}^\ell|^2 \, d\sigma(x) = \ell\,(n + 2\ell - 2) \int_{S^{n-1}} |\mathcal{Y}^\ell|^2 \, d\sigma(x). \tag{8.7}$$

Proof. Using the homogeneity, one may transform the left-hand side of the identity to be proven into an integral on the unit ball, finding

$$\int_{S^{n-1}} |\nabla \mathcal{Y}^\ell|^2 \, d\sigma(x) = (n + 2\ell - 2) \int_{B^n} \sum_j \left| \frac{\partial \mathcal{Y}^\ell}{\partial x_j} \right|^2 dx. \tag{8.8}$$

Since $\Delta \mathcal{Y}^\ell = 0$, this reduces, by Stokes's formula, to

$$\begin{aligned} \int_{S^{n-1}} |\nabla \mathcal{Y}^\ell|^2 \, d\sigma(x) &= (n + 2\ell - 2) \sum_j \int_{S^{n-1}} \mathcal{Y}^\ell \, x_j \, \frac{\partial \overline{\mathcal{Y}^\ell}}{\partial x_j} \, d\sigma(x) \\ &= \ell\,(n + 2\ell - 2) \int_{S^{n-1}} |\mathcal{Y}^\ell|^2 \, d\sigma(x). \end{aligned} \tag{8.9}$$

□

Theorem 8.2. *Assume that $n \not\equiv 0 \mod 4$, and let $E^{(n)} = \oplus_{\kappa \in \mathbb{Z},\, \ell \in \mathbb{N},\ \kappa+\ell \text{ even}} E_{\kappa,\ell}$ be the linear space generated by all anaplectic Hermite functions on $\mathbb{R}^n$. Consider on*

$E^{(n)}$ the pseudoscalar product characterized by the fact that the subspaces $E_{\kappa,\ell}$ corresponding to different pairs (κ,ℓ) are pairwise orthogonal, together with the equation, valid for $u \in E_{\kappa,\ell}$,

$$(u\,|\,u) = \gamma_{\kappa,\ell}\,\|\,u\,\|_\ell^2, \tag{8.10}$$

where

$$\gamma_{\kappa,\ell} = 2^{-\frac{n}{2}}\,(2\pi)^{-\frac{n}{2}-\ell}\left(\Gamma\left(\frac{n}{2}+\ell\right)\right)^2\,\frac{\Gamma(1-\frac{n}{4}+\frac{\kappa-\ell}{2})}{\Gamma(\frac{n}{4}+\frac{\kappa+\ell}{2})}\;: \tag{8.11}$$

recall that the norm $\|\;\|_\ell$ on $E_{\kappa,\ell}$ has been defined in (8.6). The pseudoscalar product so defined is non-degenerate on $E^{(n)}$ and, for every j, the operators A_j and A_j^* are formally adjoint to each other with respect to it. The pseudoscalar product is characterized by the last property up to the multiplication by an arbitrary non-zero constant.

All that precedes goes as well with the linear space similarly defined in relation to the dual anaplectic analysis, provided that $n \not\equiv 2 \mod 4$.

Proof. Assume that $u \in E_{\kappa,\ell}$ is of the kind $u = h \boxtimes \mathcal{Y}^\ell$, with the notation (3.37). We have to show that, whenever $v \in E_{\kappa-1,\ell+1} \oplus E_{\kappa-1,\ell-1}$, one has $(v\,|\,A_j u) = (A_j^* v\,|\,u)$: indeed, unless v lies in the direct sum indicated, both sides of the equation to be verified are zero, a consequence of Lemma 3.7. Thus, set

$$v = g_1 \boxtimes \mathcal{X}^{\ell+1} + g_2 \boxtimes \mathcal{X}^{\ell-1}. \tag{8.12}$$

In accordance with Lemma 3.6, set

$$x_j\,\mathcal{Y}^\ell = \mathcal{Z}^{\ell+1} + \frac{|x|^2}{2}\,\mathcal{Z}^{\ell-1},$$

$$x_j\,\mathcal{X}^{\ell+1} = \mathcal{T}^{\ell+2} + \frac{|x|^2}{2}\,\mathcal{T}^{\ell}, \qquad x_j\,\mathcal{Y}^{\ell-1} = \mathcal{S}^{\ell} + \frac{|x|^2}{2}\,\mathcal{S}^{\ell-2}, \tag{8.13}$$

where each of the polynomials $\mathcal{Z}^{\ell+1}$, $\mathcal{Z}^{\ell-1}$, $\mathcal{T}^{\ell+2}$, $\mathcal{T}^{\ell}$, $\mathcal{S}^{\ell}$, $\mathcal{S}^{\ell-2}$ is harmonic and homogeneous of the degree indicated in the exponent.

From the equation (3.30), note that

$$h'(0) = -\frac{2\pi\kappa}{\ell+\frac{n}{2}}\,h(0), \qquad g_2'(0) = -\frac{2\pi(\kappa-1)}{\ell+\frac{n}{2}-1}\,g_2(0). \tag{8.14}$$

Applying the polarized version of the definition of the pseudoscalar product given in the statement of the present theorem together with Lemma 3.7, one gets on one hand

$$\begin{aligned}\pi^{-\frac{1}{2}}\,(v\,|\,A_j u) = \gamma_{\kappa-1,\ell+1}\;\bar{g}_1(0)\,(h(0)+\frac{1}{2\pi}\,h'(0))\int_{S^{n-1}}\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Z}^{\ell+1}(x)\,d\sigma(x)\\ +\,\gamma_{\kappa-1,\ell-1}\;\bar{g}_2(0)\,\frac{\frac{n}{2}+\ell-1}{2\pi}\,h(0)\int_{S^{n-1}}\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Z}^{\ell-1}(x)\,d\sigma(x).\end{aligned} \tag{8.15}$$

Now, $h(0)+\frac{1}{2\pi}\,h'(0)=\frac{\ell+\frac{n}{2}-\kappa}{\ell+\frac{n}{2}}\,h(0)$. Also, since two spherical harmonics of different degrees are always orthogonal, it follows from (8.13) that

$$\begin{aligned}\int_{S^{n-1}}\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Z}^{\ell+1}(x)\,d\sigma(x)&=\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x),\\ \int_{S^{n-1}}\overline{\mathcal{X}}^{\ell-1}(x)\,\mathcal{Z}^{\ell-1}(x)\,d\sigma(x)&=2\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell-1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x),\end{aligned}\tag{8.16}$$

so that

$$\begin{aligned}\pi^{-\frac{1}{2}}\,(v\,|\,A_ju)=\gamma_{\kappa-1,\ell+1}\,\frac{\ell+\frac{n}{2}-\kappa}{\ell+\frac{n}{2}}\,\bar{g}_1(0)\,h(0)\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x)\\ +\gamma_{\kappa-1,\ell-1}\,\frac{\ell+\frac{n}{2}-1}{\pi}\,\bar{g}_2(0)\,h(0)\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell-1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x).\end{aligned}\tag{8.17}$$

On the other hand, in the same way of proof,

$$\begin{aligned}\pi^{-\frac{1}{2}}\,(A_j^*v\,|\,u)&=\gamma_{\kappa,\ell}\left(\frac{-(\frac{n}{2}+\ell)}{2\pi}\right)\bar{g}_1(0)\,h(0)\int_{S^{n-1}}\overline{\mathcal{T}}^{\ell}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x)\\ &\quad+\gamma_{\kappa,\ell}\,(\bar{g}_2(0)-\frac{1}{2\pi}\,\bar{g}_2'(0))\,h(0)\int_{S^{n-1}}\overline{\mathcal{S}}^{\ell}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x)\\ &=-\,\gamma_{\kappa,\ell}\,\frac{\ell+\frac{n}{2}}{\pi}\,\bar{g}_1(0)\,h(0)\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell+1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x)\\ &\quad+\gamma_{\kappa,\ell}\,\frac{\kappa+\ell+\frac{n}{2}-2}{\ell+\frac{n}{2}-1}\,\bar{g}_2(0)\,h(0)\int_{S^{n-1}}x_j\,\overline{\mathcal{X}}^{\ell-1}(x)\,\mathcal{Y}^{\ell}(x)\,d\sigma(x).\end{aligned}\tag{8.18}$$

In order to show that the operators A_j and A_j^* are formally adjoint to each other with respect to the given pseudoscalar product, it thus simply remains to check the two equations

$$\begin{aligned}\gamma_{\kappa-1,\ell+1}\,\frac{\ell+\frac{n}{2}-\kappa}{\ell+\frac{n}{2}}&=-\gamma_{\kappa,\ell}\,\frac{\ell+\frac{n}{2}}{\pi},\\ \gamma_{\kappa-1,\ell-1}\,\frac{\ell+\frac{n}{2}-1}{\pi}&=\gamma_{\kappa,\ell}\,\frac{\kappa+\ell+\frac{n}{2}-2}{\frac{n}{2}+\ell-1},\end{aligned}\tag{8.19}$$

an elementary task.

Since $\kappa\pm\ell$ is even (*resp.* odd) when considering the anaplectic (*resp.* dual anaplectic) analysis, it is clear from a look at the coefficients $\gamma_{\kappa,\ell}$ that the pseudoscalar product under consideration is always non-degenerate. More precisely, in the case of the anaplectic analysis, set

$$k_1=\frac{\kappa+\ell}{2}+\left[\frac{n}{4}\right],\qquad k_2=\frac{\kappa-\ell}{2}-\left[\frac{n}{4}\right]:\tag{8.20}$$

it is then immediate that the sign of $\gamma_{\kappa,\ell}$ is that of $\varepsilon_1\,\varepsilon_2$, where $\varepsilon_j = 1$ if k_j is non-negative or even, and $\varepsilon_j = -1$ if k_j is negative and odd. The same goes in the dual anaplectic analysis, setting this time

$$k_1 = \frac{\kappa+\ell+1}{2} + \left[\frac{n}{4} - \frac{1}{2}\right], \qquad k_2 = \frac{\kappa-\ell-1}{2} - \left[\frac{n}{4} - \frac{1}{2}\right]. \tag{8.21}$$

It is only in the one-dimensional case that the condition κ even (*resp.* κ odd) is sufficient, in the anaplectic (*resp.* dual anaplectic) analysis, to ensure that $\gamma_{\kappa,\ell} > 0$: this is the reason why, in this case, the restriction of the anaplectic representation to the even (*resp.* odd) functions is unitarizable.

We now turn to the question of uniqueness, starting with an arbitrary non-zero pseudoscalar product for which the operators A_j and A_j^* are adjoint to each other for every j. In the anaplectic analysis – from now on in this section, we shall satisfy ourselves with this case only – we may assume that the equation (8.10) is valid for u in the one-dimensional space $E_{0,0}$. Next, we show that if this equation is valid for $u \in E_{\kappa,0}$ for a certain number κ, it is also valid for the pair $(\kappa+2,0)$: indeed, with the notation from the beginning of this section, and recalling (8.3) and (8.5), one finds

$$(f_{\kappa+2,0}\,|\,f_{\kappa+2,0}) = \frac{\kappa - \frac{n}{2} + 2}{\kappa + \frac{n}{2}}\,(f_{\kappa,0}\,|\,f_{\kappa,0}), \tag{8.22}$$

and it suffices to check the equation $\gamma_{\kappa+2,0} = \frac{\kappa-\frac{n}{2}+2}{\kappa+\frac{n}{2}}\,\gamma_{\kappa,0}$. Substituting B for B^*, one may instead move from a pair $(\kappa,0)$ to the pair $(\kappa-2,0)$ so that the equation (8.10) is now valid whenever $\ell = 0$ (in which case, of necessity, κ is even).

To finish the proof, it remains to be shown that if (8.10) is valid for some pair (κ,ℓ), it is also valid for the pair $(\kappa+1,\ell+1)$. Indeed, since, as a consequence of our assumptions, the operators $L^{(n)}$ and $\Delta_{S^{n-1}}$ are formally self-adjoint on $E^{(n)}$ there is no need to consider a pair of functions lying in two different spaces of the decomposition $E^{(n)} = \oplus\, E_{\kappa,\ell}$. Thus, let $v_1 = g_1 \boxtimes \mathcal{X}_1^{\ell+1}$ and $v_2 = g_2 \boxtimes \mathcal{X}_2^{\ell+1}$ be two elements of $E_{\kappa+1,\ell+1}$. Going back to the end of the proof of Theorem 3.4, we solve the equation $\partial^\sharp\, h_2 = g_2$ with h_2 in the null space of the operator $M_{n,\kappa,\ell}$: this can be done in the case when $\kappa \geq -1$, and we have indicated in the proof of Theorem 3.4 the modification to be done in the other case. Then $h_2 \boxtimes \frac{\partial \mathcal{X}_2^{\ell+1}}{\partial x_j}$ lies in $E_{\kappa,\ell}$ for every j, and one has (3.49)

$$\sum_j A_j^* \left(h_2 \boxtimes \frac{\partial\, \mathcal{X}_2^{\ell+1}}{\partial x_j}\right) = (\ell+1)\, v_2. \tag{8.23}$$

Thus

$$(v_2\,|\,v_1) = (\ell+1)^{-1} \sum_j \left(\left(h_2 \boxtimes \frac{\partial\, \mathcal{X}_2^{\ell+1}}{\partial x_j}\right)\,|\,A_j\, v_1\right), \tag{8.24}$$

a number which we can compute by assumption after we have substituted for $A_j\, v_1$ its projection (*cf.* Lemma 3.7)

$$T^-_{(j)}\, v_1 = (\delta_{\ell+1}\, g_1) \boxtimes S^-_{(j)}\, \mathcal{X}_1^{\ell+1} = \left(\frac{n}{2}+\ell\right)^{-1} (\delta_{\ell+1}\, g_1) \boxtimes \frac{\partial \mathcal{X}_1^{\ell+1}}{\partial x_j} \tag{8.25}$$

on the space $E_{\kappa,\ell}$. Recalling from Lemma 3.5 that $(\delta_{\ell+1}\, g_1)(0) = \frac{\frac{n}{2}+\ell}{2\pi^{\frac{1}{2}}}\, g_1(0)$, we obtain

$$(v_2\,|\,v_1) = \gamma_{\kappa,\ell}\ (\ell+1)^{-1}\, \bar{h}_2(0)\, \frac{1}{2\pi^{\frac{1}{2}}}\, g_1(0) \sum_j \int_{S^{n-1}} \frac{\partial \bar{\mathcal{X}}_2^{\ell+1}}{\partial x_j}\, \frac{\partial \mathcal{X}_1^{\ell+1}}{\partial x_j}\, d\sigma(x). \tag{8.26}$$

Using the equation

$$g_2(0) = \pi^{\frac{1}{2}}\, \frac{\frac{n}{2}+\ell+\kappa}{\frac{n}{2}+\ell}\, h_2(0), \tag{8.27}$$

a consequence of the definition of the operator $\partial^\sharp$ in Lemma 3.5 and of the equation (3.30) relative to h_2, and the equation

$$\sum_j \int_{S^{n-1}} \frac{\partial \bar{\mathcal{X}}_2^{\ell+1}}{\partial x_j}\, \frac{\partial \mathcal{X}_1^{\ell+1}}{\partial x_j}\, d\sigma(x) = (\ell+1)(n+2\ell) \int_{S^{n-1}} \bar{\mathcal{X}}_2^{\ell+1}\, \mathcal{X}_1^{\ell+1}\, d\sigma(x), \tag{8.28}$$

a polarized version of Lemma 8.1, we obtain

$$(v_2\,|\,v_1) = \gamma_{\kappa,\ell} \times \frac{1}{\pi}\, \frac{(\frac{n}{2}+\ell)^2}{\frac{n}{2}+\ell+\kappa}\, \bar{g}_2(0)\, g_1(0) \int_{S^{n-1}} \bar{\mathcal{X}}_2^{\ell+1}\, \mathcal{X}_1^{\ell+1}\, d\sigma(x) : \tag{8.29}$$

this is the desired result since the coefficient in front of the right-hand side agrees with $\gamma_{\kappa+1,\ell+1}$. □

Remarks 8.1. (i) Note that $\gamma_{-\kappa,\ell} = (-1)^\kappa\, \gamma_{\kappa,\ell}$.

(ii) In the case when $n \equiv 0 \mod 4$, the coefficient $\gamma_{\kappa,\ell}$ is still defined, in the anaplectic analysis (*i.e.,* when $\kappa+\ell$ is even), if and only if $\kappa-\ell = \frac{n}{2}+2j$ for $j = 0, 1, \dots$: then, this coefficient is also non-zero. But the direct sum of the corresponding spaces $E_{\kappa,\ell}$ is nothing else, as observed in Remark 3.1, than the space of usual Hermite functions. A variant, to wit that obtained by substituting $(-1)^\kappa\, \frac{\Gamma(1-\frac{n}{4}-\frac{\kappa+\ell}{2})}{\Gamma(\frac{n}{4}+\frac{\ell-\kappa}{2})}$ for the – *formally* identical – fraction $\frac{\Gamma(1-\frac{n}{4}+\frac{\kappa-\ell}{2})}{\Gamma(\frac{n}{4}+\frac{\kappa+\ell}{2})}$ taken from the right-hand side of (8.11), would make it possible to consider instead the *mock*-Hermite functions, defined as the images of the usual Hermite functions under the change of coordinates $x \mapsto ix$: though absolutely devoid of applications, this observation is still necessary for a good comprehension.

While we are at it, let us briefly show that, in the case when $n \equiv 0 \mod 4$, the anaplectic and metaplectic representations coincide when both are regarded only on the space generated by usual Hermite functions: in other words, the anaplectic analysis does not bring anything new in this case. Indeed, if u is such a function, it follows from (4.42) together with some easy considerations starting from the definition of $(\mathcal{K}\,u)_0$ in Theorem 4.11 that

$$\text{Int}\,[u] = \lim\nolimits_{\varepsilon\to 0} \int_{\mathbb{R}^n} u(x\,e^{-i(\frac{\pi}{4}-\varepsilon)})\,dx = e^{\frac{i\pi n}{4}} \int_{\mathbb{R}^n} u(x)\,dx. \tag{8.30}$$

Then, the definition of $\mathcal{F}_{\text{ana}}$ in Theorem 5.6 shows that $\mathcal{F}_{\text{ana}}\,u = e^{\frac{i\pi n}{4}}\,\mathcal{F}\,u$, where the last transformation is also the image, under the metaplectic representation, of the matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. The rest of the assertion follows.

In Theorem 8.2, the pseudoscalar product $(u\,|\,u)$, with $u \in E^{(n)}$, has been defined in terms of the decomposition $u = \sum_{\kappa,\ell} u_{\kappa,\ell}$. This double series is very annoying so far as questions of convergence are concerned. We now show how to get rid of the index κ, at the benign price of having to substitute (ℓ, m), $1 \leq m \leq d_\ell$, the dimension of the space of spherical harmonics of degree ℓ, for ℓ. In the process, all the coefficients $\gamma_{\kappa,\ell}$ will disappear.

The first thing to do is to separate the even and odd values of $\kappa \in \mathbb{Z}$, setting $\kappa = 2k+\varepsilon$, $\varepsilon = 0$ or 1: since $E^{(n)}$ is the space of anaplectic Hermite functions, $\kappa+\ell$ must be even, which means that ε is characterized as having the same parity as ℓ. For every $\ell = 0, 1, \ldots$, let $(Y^{\ell,m})_{1\leq m\leq d_\ell}$ be an orthonormal basis of the subspace of $L^2(S^{n-1})$ consisting of spherical harmonics of degree ℓ, and extend $Y^{\ell,m}$ as a homogeneous polynomial $\mathcal{Y}^{\ell,m}$ of degree ℓ. According to Theorem 3.8, one has for every pair (k, ℓ) the equation $u_{2k+\varepsilon,\ell} = (v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\text{ram}} \boxtimes \mathcal{Y}^\ell$, where the function $v^k_{\rho,\varepsilon}$ was defined in (2.20), (2.21), and the harmonic polynomial $\mathcal{Y}^\ell$, dependent on (k,ℓ), can be decomposed as $\mathcal{Y}^\ell = \sum_m \alpha_{k,\ell,m}\,\mathcal{Y}^{\ell,m}$.

Instead of the decomposition of u considered so far, we now set

$$u = \sum_{\ell,m} P_{\ell,m}\,u, \tag{8.31}$$

where

$$P_{\ell,m}\,u = \sum_k \alpha_{k,\ell,m}\,((v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\text{ram}}) \boxtimes \mathcal{Y}^{\ell,m}. \tag{8.32}$$

Setting

$$g_{\ell,m} = \sum_k \alpha_{k,\ell,m}\,v^k_{\frac{n-2}{2}+\ell,\varepsilon}, \tag{8.33}$$

a function in the space $C^\omega_{\frac{n-2}{2}+\ell,\varepsilon}$ (do not forget that we are currently dealing with functions u in the algebraic sum of the spaces $E_{\kappa,\ell}$), we may finally write

$$u = \sum_{\ell,m} (g_{\ell,m})^{\text{ram}} \boxtimes \mathcal{Y}^{\ell,m}, \tag{8.34}$$

where the ramified part is taken with a reference to the space $C^\omega_{\frac{n-2}{2}+\ell,\varepsilon}$.

We shall presently show that $(u\,|\,u)$ then reduces to an expression

$$(u\,|\,u) = \sum_{\ell,m} c_\ell\,(g_{\ell,m}\,|\,g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}, \tag{8.35}$$

where the (pseudo-)scalar product on the right-hand side denotes that introduced in (2.30). There is, however, a difficulty, in that the number $\frac{n-2}{2}+\ell$ which occurs here at the place of ρ is generally far from being < 1: we thus first extend this notion, without any change to the notation.

Lemma 8.3. *Let a function $v_2 \in C^{\omega}_{\rho,\varepsilon}$ be given for all complex ρ with $\mathrm{Re}\ \rho > -1$, and assume that, as a function of ρ valued in $C^\infty(\mathbb{R}^\times)$, it is holomorphic in the given half-plane; make the same assumptions regarding another function v_1, only changing ρ to $\bar\rho$. Then, as a function of ρ, the expression $(v_1\,|\,v_2)_{\rho,\varepsilon}$, as defined in* (2.30) *in the case when ρ is real and $0 < |\rho| < 1$, extends as a holomorphic function of ρ in the half-plane from which all the points such that $\rho - \varepsilon = 1, 3, \dots$ have been deleted.*

Proof. Clearly, in view of the properties of v_1 and v_2 expressed in Proposition 2.1, and (in the case when $\rho = \varepsilon = 0$) in Proposition 3.1, the integral (2.30) continues to make sense for any complex ρ with $-1 < \mathrm{Re}\ \rho < 1$, thus providing the sought-after continuation in the given strip. For ρ real with $0 < \rho < 1$, we switch to the expression (2.31), with $w_1 = \mathcal{F}\,v_1$ and $w_2 = \mathcal{F}\,v_2$. Since, when ρ is complex with a positive real part, the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\sigma-\tau|^{\mathrm{Re}\ \rho-1}\,[(1+\sigma^2)\,(1+\tau^2)]^{-\frac{1}{2}\,(\mathrm{Re}\ \rho+1)}\,d\sigma\,d\tau \tag{8.36}$$

is convergent, it follows from the estimates $|w_j(\sigma)| \leq C\,(1+\sigma^2)^{-\frac{1}{2}\,(\mathrm{Re}\ \rho+1)}$ (a consequence of (2.17)) that, under the given assumptions, the integral (2.31) makes sense, and is a holomorphic function of ρ, in the half-plane $\mathrm{Re}\ \rho > 0$. Locating the poles of the coefficient in front of (2.31), one obtains the present lemma. □

Corollary 8.4. *Assume $n \not\equiv 0 \bmod 4$, and let $\ell = 0, 1, \dots$, $\varepsilon = 0$ or 1 with $\varepsilon + \ell$ even. Then, for every $k \in \mathbb{Z}$,*

$$(v^k_{\frac{n-2}{2}+\ell,\varepsilon}\,|\,v^k_{\frac{n-2}{2}+\ell,\varepsilon})_{\frac{n-2}{2}+\ell,\varepsilon} = \pi^{\frac{n}{2}+\ell}\,\frac{\Gamma(1-\frac{n}{4}+k+\frac{\varepsilon-\ell}{2})}{\Gamma(\frac{n}{4}+k+\frac{\varepsilon+\ell}{2})}. \tag{8.37}$$

Proof. Since, with $\rho = \frac{n-2}{2}+\ell$, one necessarily has $\rho - \varepsilon - 1 \notin 2\mathbb{Z}$, one can use analytic continuation, deriving this result from Lemma 8.3. □

Remark 8.2. Of course, even when $\mathrm{Re}\ \rho \geq 1$, the equation (2.30) remains valid in the case when each of the two functions v_1 and v_2 is square-integrable with respect to the measure $|s|^{-\rho}\,ds$ on the real line.

We now need to compute $(v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\mathrm{ram}}(0)$, under the assumptions of Corollary 8.4.

Lemma 8.5. *Assume* Re $\rho > -1$ *and* $\varepsilon = 0$ *or* 1*. One has*

$$(v^0_{\rho,\varepsilon})^{\mathrm{ram}}(0) = \begin{cases} \frac{\pi^{\rho+\frac{3}{2}}}{\sin\frac{\pi}{2}(\varepsilon-\rho)} & \times \frac{1}{\Gamma(\frac{\rho+2-\varepsilon}{2})\,\Gamma(\frac{\rho+1+\varepsilon}{2})} & \text{if } \rho-\varepsilon \neq 0, 2, \dots, \\ -\frac{2\,\pi^{\rho+\frac{1}{2}}}{\cos\frac{\pi}{2}(\varepsilon-\rho)} & \times \frac{1}{\Gamma(\frac{\rho+2-\varepsilon}{2})\,\Gamma(\frac{\rho+1+\varepsilon}{2})} & \text{if } \rho-\varepsilon = 0, 2, \dots. \end{cases} \tag{8.38}$$

Proof. From (2.22), (2.23) and (2.25), one finds that

$$(v^0_{\rho,\varepsilon})^{\mathrm{ram}}(0) = \pi^{\rho+\frac{1}{2}}\,\frac{\Gamma(\frac{-\rho+\varepsilon}{2})}{\Gamma(\frac{\rho+1+\varepsilon}{2})}, \qquad \rho-\varepsilon \neq 0, 2, \cdots : \tag{8.39}$$

this reduces to the first formula above after one has treated the Gamma function on the top by the formula of complements. The second case is then an immediate consequence of the residue formula (6.76). □

Corollary 8.6. *Assume that* $n \not\equiv 0$ *mod* 4*, and that* $\varepsilon + \ell$ *is even. For any* $k \in \mathbb{Z}$*, one has*

$$(v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\mathrm{ram}}(0) = (-1)^k\,\varepsilon_n\,\times\,(-1)^{\frac{\varepsilon-\ell}{2}}\,\frac{(2\pi)^{\frac{n}{2}+\ell}}{\Gamma(\frac{n}{2}+\ell)}, \tag{8.40}$$

with

$$\varepsilon_n = \begin{cases} \frac{1}{2\,\cos\frac{\pi n}{4}} & \text{if } n \text{ is odd,} \\ -\frac{1}{\pi\,\sin\frac{\pi n}{4}} & \text{if } n \equiv 2 \bmod 4. \end{cases} \tag{8.41}$$

Proof. Consider first the case when $k = 0$. With $\rho = \frac{n-2}{2} + \ell$, the condition that $\rho - \varepsilon \neq 0, 2, \dots$ just means that n is odd (use the assumptions relative to n and to $\varepsilon + \ell$). In this case, an application of the first equation (8.38) gives

$$(v^0_{\frac{n-2}{2}+\ell,\varepsilon})^{\mathrm{ram}}(0) = (-1)^{\frac{\varepsilon-\ell}{2}}\,\frac{\pi^{\frac{n+1}{2}+\ell}}{\cos\frac{\pi n}{4}} \times \frac{1}{\Gamma(\frac{n}{4}+\frac{\ell}{2}+\frac{1-\varepsilon}{2})\,\Gamma(\frac{n}{4}+\frac{\ell}{2}+\frac{\varepsilon}{2})}, \tag{8.42}$$

and one obtains the claimed formula as a consequence of the duplication formula of the Gamma function; the same goes in the other case.

Set, as in (3.51), $h^k_{\frac{n-2}{2}+\ell,\varepsilon} = (v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\mathrm{ram}}$. It has been found there that one has $M_{n,2k+\varepsilon,\ell}\;h^k_{\frac{n-2}{2}+\ell,\varepsilon} = 0$, where the differential operator involved was defined in (3.30). As a consequence,

$$(h^k_{\frac{n-2}{2}+\ell,\varepsilon})'(0) = -\frac{2\pi\,(2k+\varepsilon)}{\ell+\frac{n}{2}}\,h^k_{\frac{n-2}{2}+\ell,\varepsilon}(0). \tag{8.43}$$

Finally, using the equation (6.82), here giving

$$\left(\frac{n}{2}+\ell+2k+\varepsilon\right)\,h^{k+1}_{\frac{n-2}{2}+\ell,\varepsilon}(0) = \frac{\frac{n}{2}+\ell}{2\pi}\,(h^{k+1}_{\frac{n-2}{2}+\ell,\varepsilon})'(0) - \left(\frac{n}{2}+\ell\right)\,h^{k+1}_{\frac{n-2}{2}+\ell,\varepsilon}(0), \tag{8.44}$$

one finds that

$$h^{k+1}_{\frac{n-2}{2}+\ell,\varepsilon}(0) = (-1)^k\,h^k_{\frac{n-2}{2}+\ell,\varepsilon}(0). \tag{8.45}$$

□

Theorem 8.7. *Assume that* $n \not\equiv 0 \mod 4$. *Under the assumption that* $u \in E^{(n)}$ *is given as the linear combination* (8.34), *here recalled for convenience,*

$$u = \sum_{\ell,m} (g_{\ell,m})^{\mathrm{ram}} \boxtimes \mathcal{Y}^{\ell,m}, \tag{8.46}$$

one has

$$(u \,|\, u) = \varepsilon_n^2 \sum_{\ell,m} 2^\ell \, (g_{\ell,m} \,|\, g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}, \tag{8.47}$$

with $\varepsilon + \ell$ *even and* $\varepsilon_n^2 = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd,} \\ \pi^{-2} & \text{if } n \equiv 2 \bmod 4. \end{cases}$

Proof. According to Theorem 8.2, and with the notation in (8.31)–(8.34), one has

$$\begin{aligned}(u \,|\, u) &= \sum_{k,\ell} \gamma_{2k+\varepsilon,\ell} \, |(v^k_{\frac{n-2}{2}+\ell,\varepsilon})^{\mathrm{ram}}(0)|^2 \sum_m |\alpha_{k,\ell,m}|^2 \\ &= \varepsilon_n^2 \sum_{k,\ell,m} 2^\ell \, \pi^{\frac{n}{2}+\ell} \, \frac{\Gamma(1 - \frac{n}{4} + k + \frac{\varepsilon-\ell}{2})}{\Gamma(\frac{n}{4} + k + \frac{\varepsilon+\ell}{2})} \, |\alpha_{k,\ell,m}|^2, \end{aligned} \tag{8.48}$$

whereas

$$(g_{\ell,m} \,|\, g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon} = \sum_k |\alpha_{k,\ell,m}|^2 \, (v^k_{\frac{n-2}{2}+\ell,\varepsilon} \,|\, v^k_{\frac{n-2}{2}+\ell,\varepsilon})_{\frac{n-2}{2}+\ell,\varepsilon}. \tag{8.49}$$

It suffices to compare the last expression to (8.48) and to make use of Corollary 8.4. □

Remarks 8.3. (i) We have assumed $n \geq 2$ throughout. However, one may check with the help of (2.113), (2.82) and Theorem 2.9 that, in the one-dimensional case, the expression of $(u \,|\, u)$ above agrees with that introduced in Section 1. Indeed, there are only two normalized "spherical" harmonics to be considered, to wit $\mathcal{Y}^0(x) = 2^{-\frac{1}{2}}$ and $\mathcal{Y}^1(x) = 2^{-\frac{1}{2}} x$, in which case, comparing (8.46) to (2.82), one obtains the relations $g_0 = 2^{\frac{1}{2}} v_0$, $g_1 = v_1$.

(ii) If, instead of the decomposition (8.46), the function u expresses itself as a linear combination

$$u = \sum_{\ell,p} (g_{\ell,p})^{\mathrm{ram}} \boxtimes \mathcal{Y}^{\ell,p}, \tag{8.50}$$

in which, for each ℓ, the set of functions $\mathcal{Y}^{\ell,p}$ is an arbitrary finite collection of harmonic polynomials homogeneous of degree ℓ, the equation (8.47) transforms to

$$(u \,|\, u) = \varepsilon_n^2 \sum_\ell 2^\ell \sum_{p,q} (g_{\ell,p} \,|\, g_{\ell,p})_{\frac{n-2}{2}+\ell,\varepsilon} \int_{S^{n-1}} \bar{\mathcal{Y}}^{\ell,p}(x) \, \mathcal{Y}^{\ell,q}(x) \, d\sigma(x). \tag{8.51}$$

(iii) We shall also have to consider functions u given by the decomposition (8.46), in which, for each pair (ℓ, m), $g_{\ell,m}$ will be an arbitrary function in $C^{\omega}_{\frac{n-2}{2}+\ell,\varepsilon}$, not necessarily a finite linear combination of the functions $v^k_{\frac{n-2}{2}+\ell,\varepsilon}$ as in (8.33). Provided that the series on the right-hand side of (8.47) is convergent, it will then still serve as a definition of $(u\,|\,u)$: but one must first verify that the map $g \mapsto g^{\mathrm{ram}}$, $g \in C^{\omega}_{\frac{n-2}{2}+\ell,\varepsilon}$, is one-to-one; the argument given just before Proposition 2.5 applies just as well in the case when $\rho + \varepsilon \in 2\mathbb{Z}, \mathrm{Re}\,\rho > -1$.

Our aim is to define a pair of linear spaces of functions, containing $E^{(n)}$, such that the operators of the Heisenberg representation map the smaller one into the larger one, the definition of the pseudoscalar product on a pair of functions from the larger space being possible. To start with, with the help of results of Section 3, we shall study the operator of multiplication by $e^{2i\pi\langle\alpha,x\rangle}$, as expressed by means of the series of the exponential. First, some algebra:

Lemma 8.8. *Let $\mathcal{Y}^{\ell}$ be a harmonic polynomial, homogeneous of degree ℓ, and let $g \in C^{\omega}_{\frac{n-2}{2}+\ell,\varepsilon}$, with $\ell+\varepsilon$ even. Set $u = g^{\mathrm{ram}} \boxtimes \mathcal{Y}^{\ell}$ and $g = g(s)$. With the notation of Lemma* 3.6, *one has the identity*

$$\langle\alpha, x\rangle\, u = (sg)^{\mathrm{ram}} \boxtimes S^+_{\alpha}\, \mathcal{Y}^{\ell} + g^{\mathrm{ram}} \boxtimes S^-_{\alpha}\, \mathcal{Y}^{\ell} : \tag{8.52}$$

it is understood that the ramified part of a function h occurring in some expression explicitly written in the form $h^{\mathrm{ram}} \boxtimes \mathcal{Y}^{\ell'}$ is always taken with a reference to the $C^{\omega}_{\frac{n-2}{2}+\ell',\varepsilon'}$-theory, with $\ell'+\varepsilon'$ even. In a similar way, one has for every $\beta \in \mathbb{C}^n$ the equation

$$\langle\beta, \nabla\rangle\, u = \left[s\,g' - \left(\frac{n-2}{2}+\ell\right)\, g\right]^{\mathrm{ram}} \boxtimes S^+_{\beta}\, \mathcal{Y}^{\ell} + (g')^{\mathrm{ram}} \boxtimes S^-_{\beta}\, \mathcal{Y}^{\ell}. \tag{8.53}$$

Proof. Before we give it, let us remind the reader that the definition, in Proposition 2.3, of the ramified part of some function v in the space $C^{\omega}_{\rho,\varepsilon}$, was dependent on the choice of the pair (ρ, ε), even though we chose to dispense with making this dependence explicit in the notation. Clearly, from (2.33), if v lies in this space, it also lies in the space $C^{\omega}_{\rho-1,1-\varepsilon}$, but its ramified part in the latter space is the function $s \mapsto s\, v^{\mathrm{ram}}(s)$ if v^{ram} is its ramified part in the first space. On the other hand, v^{ram} is also the ramified part of the function $s \mapsto s\, v(s)$ in the $C^{\omega}_{\rho+1,1-\varepsilon}$-theory. The preceding justification works only if $\rho+\varepsilon$ is not an even integer, but the exceptional case is covered just as well with the help of Proposition 3.1.

Starting from Lemma 3.6, *i.e.,*

$$\langle\alpha, x\rangle\, \mathcal{Y}^{\ell} = S^+_{\alpha}\, \mathcal{Y}^{\ell} + \frac{|x|^2}{2}\, S^-_{\alpha}\, \mathcal{Y}^{\ell}, \tag{8.54}$$

we write

$$\langle\alpha, x\rangle\, u(x) = g^{\mathrm{ram}}\left(\frac{|x|^2}{2}\right)\, (S^+_{\alpha}\, \mathcal{Y}^{\ell})(x) + \frac{|x|^2}{2}\, g^{\mathrm{ram}}\left(\frac{|x|^2}{2}\right)\, (S^-_{\alpha}\, \mathcal{Y}^{\ell})(x), \tag{8.55}$$

where g^{ram} is the ramified part of g in the $C^{\omega}_{\frac{n-2}{2}+\ell,\varepsilon}$-theory. One obtains the first formula with the help of the argument in the beginning of the present proof. For the second one, starting from Lemma 3.6 again, we write

$$\begin{aligned}(\langle\beta,\nabla\rangle\, u)(x) = \left(\frac{n-2}{2}+\ell\right)\, g^{\mathrm{ram}}\left(\frac{|x|^2}{2}\right)\,(S_\beta^-\,\mathcal{Y}^\ell)(x)\\ +\,(g^{\mathrm{ram}})'\left(\frac{|x|^2}{2}\right)\left[(S_\beta^+\,\mathcal{Y}^\ell)(x)+\frac{|x|^2}{2}\,(S_\beta^-\,\mathcal{Y}^\ell)(x)\right],\end{aligned} \tag{8.56}$$

and we observe, setting $\rho=\frac{n-2}{2}+\ell$, that $\rho\, g^{\mathrm{ram}}+s\,(g^{\mathrm{ram}})'$ is the ramified part of g' in $C^{\omega}_{\rho-1,1-\varepsilon}$ and that $(g^{\mathrm{ram}})'$ is the ramified part of $sg'-\rho\, g$ in $C^{\omega}_{\rho+1,1-\varepsilon}$. $\square$

Under the same assumptions as in Lemma 8.8, one can give a formula for the function $\langle\alpha,x\rangle^j\, u$, $j=0,1,\dots$, to wit

$$\langle\alpha,x\rangle^j\, u=\sum_{\iota_1=\pm1,\dots,\iota_j=\pm1}(s^{\iota_+}\, g)^{\mathrm{ram}}\boxtimes S_\alpha^{\iota_1}\dots S_\alpha^{\iota_j}\,\mathcal{Y}: \tag{8.57}$$

in the preceding expression, ι_+ stands for the number of $+1$'s in the sequence $\iota_1,\dots,\iota_j$. Of course, the ramified part is taken within the space $C^{\omega}_{\frac{n-2}{2}+\ell+\bar\iota}$ where, by definition, $\bar\iota=\iota_1+\dots+\iota_j=2\iota_+-j$.

Consequently, if $u=\sum_{\ell,m}(g_{\ell,m})^{\mathrm{ram}}\boxtimes\mathcal{Y}^{\ell,m}$, our aim is to define $u_\alpha = e^{2i\pi\langle\alpha,\,Q\rangle}\, u$ by means of the series

$$e^{2i\pi\langle\alpha,\,x\rangle}\, u=\sum_\ell G_\ell \tag{8.58}$$

with

$$G_\ell=\sum_j\frac{(2\pi)^j}{j\,!}\sum_{\iota_1=\pm1,\dots,\iota_j=\pm1}\ \sum_{m'}(s^{\iota_+}\, g_{\ell-\bar\iota,m'})^{\mathrm{ram}}\boxtimes S_\alpha^{\iota_1}\dots S_\alpha^{\iota_j}\,\mathcal{Y}^{\ell-\bar\iota,m'}, \tag{8.59}$$

in which the index m' characterizes the choice of an arbitrary vector in an orthonormal basis of the subspace of $L^2(S^{n-1})$ consisting of spherical harmonics of degree $\ell-\bar\iota$: of course there is no such index unless $\ell-\bar\iota\geq0$.

The functions $(g_\alpha)^{\mathrm{ram}}_{\ell,m}$ which enter the decomposition $G_\ell=\sum_m(g_\alpha)^{\mathrm{ram}}_{\ell,m}\boxtimes\mathcal{Y}^{\ell,m}$ associated to the choice of an orthonormal basis of the space of spherical harmonics of degree ℓ can be obtained as linear combinations of the functions $(s^{\iota_+}\, g_{\ell-\bar\iota,m'})^{\mathrm{ram}}$, where, in view of the following lemma, the coefficients are all less than $\frac{(2\pi)^j}{j\,!}\times 2^{\frac{j-\iota_+}{2}}\, 3^{\frac{\iota_+}{2}}\,|\alpha|^j$.

Lemma 8.9. *With the notation of Lemma* 3.6, *one has*

$$\begin{aligned}\int_{S^{n-1}}|S_\alpha^-\,\mathcal{Y}^\ell|^2\, d\sigma(x)&\leq\frac{2\ell}{\frac{n}{2}+\ell-1}\,|\alpha|^2\int_{S^{n-1}}|\mathcal{Y}^\ell|^2\, d\sigma(x),\\ \int_{S^{n-1}}|S_\alpha^+\,\mathcal{Y}^\ell|^2\, d\sigma(x)&\leq\frac{n+3\ell-2}{\frac{n}{2}+\ell-1}\,|\alpha|^2\int_{S^{n-1}}|\mathcal{Y}^\ell|^2\, d\sigma(x).\end{aligned} \tag{8.60}$$

Proof. From the inequality $|\langle\,\alpha,\,\nabla\,\mathcal{Y}^\ell\,\rangle| \leq |\alpha|\,|\nabla\,\mathcal{Y}^\ell|$ and the definition of $S_\alpha^-\,\mathcal{Y}^\ell$ in Lemma 3.6, one finds, with the help of Lemma 8.1,

$$\int_{S^{n-1}} |S_\alpha^-\,\mathcal{Y}^\ell|^2\,d\sigma(x) \leq \left(\frac{n}{2}+\ell-1\right)^{-2} |\alpha|^2\,\ell\,(n+2\ell-2)\int_{S^{n-1}} |\mathcal{Y}^\ell|^2\,d\sigma(x), \tag{8.61}$$

which leads to the first inequality. From the second equation (3.34), one finds

$$\int_{S^{n-1}} |S_\alpha^+\,\mathcal{Y}^\ell|^2\,d\sigma(x) \leq 2\left[\int_{S^{n-1}} |\langle\alpha,\,x\rangle\,\mathcal{Y}^\ell|^2\,d\sigma(x) + \frac{1}{4}\int_{S^{n-1}} |S_\alpha^-\,\mathcal{Y}^\ell|^2\,d\sigma(x)\right], \tag{8.62}$$

which leads to the second inequality. □

Definition 8.10. We shall say that a real-analytic function u on $\mathbb{R}^n$ lies in the space $\mathcal{S}^\bullet(\mathbb{R}^n)$ if the following conditions hold:

(i) for every (ℓ, m), the function $u_{\ell,m}$ defined on $]0,\infty[$ by the equation

$$u_{\ell,m}(\frac{t^2}{2}) = \int_{S^{n-1}} u(t\xi)\,\bar{\mathcal{Y}}^{\ell,m}(t\xi)\,d\sigma(\xi) \tag{8.63}$$

can be written as $u_{\ell,m} = (g_{\ell,m})^{\mathrm{ram}}$ for a (necessarily unique: *cf.* Remark 8.2) function $g_{\ell,m} \in C^\omega_{\frac{n-2}{2}+\ell,\varepsilon}$;

(ii) one has for some constant $C>0$ depending only on u the estimates

$$\left|\frac{d^p}{d\sigma^p}\,(\mathcal{F}\,g_{\ell,m})(\sigma)\right| \leq C^{p+\ell+1}\,p!\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell+p)}. \tag{8.64}$$

We shall say that the function u lies in the space $\mathcal{S}_\bullet(\mathbb{R}^n)$ if the condition (ii) is replaced by the condition (ii)′:

$$\left|\left[\sigma^2\frac{d}{d\sigma} + \left(\frac{n}{2}+\ell\right)\,\sigma\right]^p (\mathcal{F}\,g_{\ell,m})(\sigma)\right| \leq C^{p+\ell+1}\,p!\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell)}\,(1+\sigma^{-2})^{-\frac{p}{2}}. \tag{8.65}$$

Remarks 8.4. (i) Since the dimension $d_\ell = \frac{(2\ell+n-2)(n+\ell-3)\,!}{(n-2)\,!\,\ell\,!}$ of the space of spherical harmonics of degree ℓ is a $\mathrm{O}(\ell^{n-2})$, the definition is independent of the choice, for every $\ell = 0, 1, \ldots$, of the orthonormal basis $(Y^{\ell,m})_m$.

(ii) As will be proved presently, the space $\mathcal{S}^\bullet(\mathbb{R}^n)$ is well adapted to the study of an operator such as $e^{2i\pi\langle\alpha,\,Q\rangle}$, and the space $\mathcal{S}_\bullet(\mathbb{R}^n)$ to that of the (translation) operator $e^{2i\pi\langle\beta,\,P\rangle}$. What can be seen without difficulty, from Lemma 8.8, is that the operator Q (*resp.* P) is an endomorphism of the first (*resp.* the second) of these two spaces. We now describe how these are related to each other.

The equation (8.63) leads to the series (8.46), and we define, as a generalization of Theorem 6.5,

$$\mathcal{F}_{\mathrm{ana}}\,u = \sum_{\ell,m}\left(\pi_{\frac{n-2}{2}+\ell,\varepsilon}(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right))\,g_{\ell,m}\right)^{\mathrm{ram}} \boxtimes \mathcal{Y}^{\ell,m}. \tag{8.66}$$

The space $\mathcal{S}^\bullet(\mathbb{R}^n)$ may not be contained in the space $\mathfrak{A}^{(n)}$, defined by completely different means – we have not found it necessary to examine this seemingly difficult question – so that we may not rely on the definition of the anaplectic Fourier transformation as given there. We must of course not rely, then, on the properties of the anaplectic Fourier transformation as given in Section 5: however, the only fact needed in what follows is the equation $\mathcal{F}_{\text{ana}}\,\langle\beta,\,P\rangle\,\mathcal{F}_{\text{ana}}^{-1} = \langle\beta,\,Q\rangle$, which can be seen from an application of Lemma 8.8 and (8.66), only paying attention to the fact that ramified parts have to be taken, in each instance, in the appropriate space. From (2.3), one has

$$(\mathcal{F}\,\pi_{\frac{n-2}{2}+\ell,\varepsilon}(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right))\,g_{\ell,m})(\sigma) = |-\sigma|_\varepsilon^{-\frac{n}{2}-\ell}\,(\mathcal{F}\,g_{\ell,m})\left(-\frac{1}{\sigma}\right) \tag{8.67}$$

and, under the transformation $w \mapsto w_1$, with $w_1(\sigma) = |-\sigma|_\varepsilon^{-\frac{n}{2}-\ell}\,w(-\frac{1}{\sigma})$, the operator $X = \frac{d}{d\sigma}$ transfers to $(-1)^\varepsilon\,Y_\ell$ with $Y_\ell = \sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell)\,\sigma$. This shows that the spaces $\mathcal{S}^\bullet(\mathbb{R}^n)$ and $\mathcal{S}_\bullet(\mathbb{R}^n)$ are the images of each other under the anaplectic Fourier transformation.

(iii) Our third remark concerns the estimate of $(g_{\ell,m}\,|\,g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}$, as rendered necessary towards the convergence of the series (8.47) defining $(u\,|\,u)$. On one hand, it follows from (2.31) and some elementary transformations of the coefficient in front of the integral there that, unless $n=2$ and $\ell=0$,

$$\begin{aligned}&(g_{\ell,m}\,|\,g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}\\ &= \frac{(-1)^{\frac{\ell-\varepsilon}{2}}\,(-i)^\varepsilon}{4\sin\frac{\pi n}{4}}\,\frac{(2\pi)^{\frac{n}{2}+\ell}}{\Gamma(\frac{n-2}{2}+\ell)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma-\tau|_\varepsilon^{\frac{n-4}{2}+\ell}\,\overline{(\mathcal{F}g_{\ell,m})(\sigma)}\,(\mathcal{F}g_{\ell,m})(\tau)\;d\sigma\,d\tau.\end{aligned} \tag{8.68}$$

On the other hand, one has

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\sigma-\tau|_\varepsilon^{\frac{n-4}{2}+\ell}\,(1+\sigma^2)^{-\frac{n}{4}-\frac{\ell}{2}}\,(1+\tau^2)^{-\frac{n}{4}-\frac{\ell}{2}}\,d\sigma\,d\tau\\ &\leq 2^{\frac{n-4}{2}+\ell}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max\left(|\sigma|,\,|\tau|\right)^{\frac{n-4}{2}+\ell}\,(1+\sigma^2)^{-\frac{n}{4}-\frac{\ell}{2}}\,(1+\tau^2)^{-\frac{n}{4}-\frac{\ell}{2}}\,d\sigma\,d\tau\\ &\leq 2^{\frac{n-2}{2}+\ell}\,\pi\left(\frac{\Gamma(\frac{n}{4}+\frac{\ell}{2}-\frac{1}{2})}{\Gamma(\frac{n}{4}+\frac{\ell}{2})}\right)^2 \leq 2^{\frac{n-2}{2}+\ell}\,\pi^2.\end{aligned} \tag{8.69}$$

From the $\frac{1}{\Gamma}$-factor in front of the right-hand side of (8.68), it is clear that the series defining $(u\,|\,u)$ converges in the case when $u \in \mathcal{S}^\bullet(\mathbb{R}^n)$.

Also, since the pseudoscalar product $(g_{\ell,m}\,|\,g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}$ is left unchanged when $g_{\ell,m}$ is changed to its image under $\pi_{\frac{n-2}{2}+\ell,\varepsilon}(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right))$, the same can be said in the case when $u \in \mathcal{S}_\bullet(\mathbb{R}^n)$, and the map $\mathcal{F}_{\text{ana}}$ from $\mathcal{S}_\bullet(\mathbb{R}^n)$ to $\mathcal{S}^\bullet(\mathbb{R}^n)$ defined by (8.66) preserves the pseudoscalar product.

(iv) As a last remark, all functions in the space $E^{(n)}$ lie in $\mathcal{S}^{\bullet}(\mathbb{R}^n)$ (then, a consequence of a preceding remark, also in $\mathcal{S}_{\bullet}(\mathbb{R}^n)$). Indeed, from the finite decomposition (8.33) and the expression (2.20) of $w^k_{\frac{n-2}{2}+\ell,\varepsilon} = \mathcal{F}\, v^k_{\frac{n-2}{2}+\ell,\varepsilon}$, one sees that it suffices to prove the following lemma.

Lemma 8.11. *For some absolute constant $C > 0$ one has*

$$\left| \frac{d^p}{d\sigma^p} \left((1+\sigma^2)^{-\frac{\rho+1}{2}-a}\,(1-i\,\sigma)^{2a}\right)\right| \leq C^{1+\rho+|a|+p}\, p\,!\; (1+\sigma^2)^{-\frac{\rho+1+p}{2}} \tag{8.70}$$

for every $\rho > -1$, $2a \in \mathbb{Z}$ and $p = 0, 1, \dots$.

Proof. Use Cauchy's formula

$$\frac{d^p}{d\sigma^p} \left((1+\sigma^2)^{-\frac{\rho+1}{2}-a}\,(1-i\,\sigma)^{2a}\right) = \frac{p\,!}{2i\pi} \int_{|z-\sigma|=\frac{1}{2}} \frac{(1+z^2)^{-\frac{\rho+1}{2}-a}\,(1-i\,z)^{2a}}{(z-\sigma)^{p+1}}\, dz. \tag{8.71}$$

□

Proposition 8.12. *Let $u \in \mathcal{S}^{\bullet}(\mathbb{R}^n)$. For every $\alpha \in \mathbb{C}^n$ the function $u_\alpha = e^{2i\pi\langle \alpha,\, Q\rangle}\, u$ lies in $\mathcal{S}^{\bullet}(\mathbb{R}^n)$ too. If $\alpha \in \mathbb{R}^n$, one has $(u_\alpha \,|\, u_\alpha) = (u \,|\, u)$.*

Proof. We first study the function u_α, for small $|\alpha|$, by means of the series (8.58). The problem lies in giving suitable estimates for the functions $\frac{d^p}{d\sigma^p}\,\mathcal{F}\,(g_\alpha)_{\ell,m}$, where the functions $(g_\alpha)^{\mathrm{ram}}_{\ell,m}$ are the ones that enter the decomposition of G_ℓ as defined right after (8.59). Since

$$\begin{aligned} \left| \frac{d^p}{d\sigma^p}\,(\mathcal{F}\,(s^{\iota_+}\, g_{\ell-\bar\iota,m})(\sigma))\right| &= (2\pi)^{-\iota_+} \left| \frac{d^{p+\iota_+}}{d\sigma^{p+\iota_+}}\,(\mathcal{F}\, g_{\ell-\bar\iota,m}(\sigma))\right| \\ &\leq (2\pi)^{-\iota_+}\, C^{p+\iota_++\ell-\bar\iota+1}\,(p+\iota_+)\,!\;(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell-\bar\iota+p+\iota_+)}, \end{aligned} \tag{8.72}$$

one finds, writing $\iota_+ - \bar\iota = j - \iota_+$, observing that $\iota_+ - \bar\iota \geq 0$ and using the estimates regarding some coefficients proved just after (8.59), that

$$\begin{aligned} \left| \frac{d^p}{d\sigma^p}\,(\mathcal{F}\,(g_\alpha)_{\ell,m})(\sigma)\right| \leq \sum_{j\geq 0}\; \sum_{\iota_1=\pm 1,\dots,\iota_j=\pm 1}\; \sum_{m'} \frac{(2\pi)^j}{j\,!} \times 2^{\frac{j-\iota_+}{2}}\, 3^{\frac{\iota_+}{2}}\, |\alpha|^j \\ (2\pi)^{-\iota_+}\, C^{(p+\ell+1)+(j-\iota_+)}\,(p+\iota_+)\,!\;(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell+p)}. \end{aligned} \tag{8.73}$$

Now, the dimension of the space of spherical harmonics of degree $\ell - \bar\iota$ is at most

$$\begin{aligned} C_1\,(1+\ell-\bar\iota)^{n-2} &= C_1\,(1+\ell+j-2\iota_+)^{n-2} \\ &\leq C_1\,(\ell+1)^{n-2}\,(1+j-\iota_+)^{n-2} \leq C_1\,(\ell+1)^{n-2}\, e^{(n-2)(j-\iota_+)}. \end{aligned} \tag{8.74}$$

On the other hand,

$$(C\, e^{n-2})^{j-\iota_+}\,(p+\iota_+)\,! \leq (p+j)\,! + C_2^{j+1} \tag{8.75}$$

for some constant C_2 depending only on (C, n). Thus,

$$\left| \frac{d^p}{d\sigma^p} (\mathcal{F}\,(g_\alpha)_{\ell,m})(\sigma) \right| \leq C_1\, C^{p+\ell+1}\,(\ell+1)^{n-2}\,(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell+p)}$$
$$\sum_{j\geq 0} \frac{(2^{\frac{5}{2}}\,\pi\,|\alpha|)^j}{j\,!}\,((p+j)\,!+C_2^{j+1}) : \quad (8.76)$$

this leads to the required estimates in the case when $|\alpha| \leq \frac{1}{2}\,(2^{\frac{5}{2}}\,\pi)^{-1}$, since the last series can then be made explicit as

$$p\,!\,(1-2^{\frac{5}{2}}\,\pi\,|\alpha|)^{-p-1} + C_2\,\exp\,(2^{\frac{5}{2}}\,\pi\,C_2\,|\alpha|). \quad (8.77)$$

Of course, the new constant C which makes the analogue of (8.64) has changed in the move $u \mapsto u_\alpha$, but the domain of convergence of the series does not depend on C, so that the operator $e^{2i\pi\langle\alpha,\,Q\rangle}$ can be iterated as many times as needed, and is thus seen to act within the space $\mathcal{S}^\bullet(\mathbb{R}^n)$ without any restriction on $|\alpha|$. □

The preceding proposition also shows that the space $\mathcal{S}_\bullet(\mathbb{R}^n)$ is invariant under the action of translations. However, some more effort is needed in order to combine the operators from the Heisenberg representation of the two different species.

Definition 8.13. We shall say that a real-analytic function u on $\mathbb{R}^n$ lies in the space $\mathcal{S}^\bullet_\bullet(\mathbb{R}^n)$ if, with the same notation as in Definition 8.10, the following conditions hold: for some constant $C > 0$ depending only on u and all integers p, q, ℓ, r such that $p \geq 0$, $q \geq 0$, $p \geq r$ and $\ell \geq 0$, $\ell + r \geq 0$, one has

$$\left| [\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell+r)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(\mathcal{F}\,g_{\ell,m})(\sigma) \right|$$
$$\leq C^{\ell+p+q+1}\,(p+q)\,!\,\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell+r)}\,(1+\sigma^{-2})^{-\frac{q}{2}}. \quad (8.78)$$

The first thing to note is that $\mathcal{S}^\bullet_\bullet(\mathbb{R}^n) \subset \mathcal{S}^\bullet(\mathbb{R}^n) \cap \mathcal{S}_\bullet(\mathbb{R}^n)$: for the first inclusion, take $q = 0$ and $p = r$; for the second one, take $p = r = 0$.

Next, we shall prove that anaplectic Hermite functions lie in the space $\mathcal{S}^\bullet_\bullet(\mathbb{R}^n)$: this requires a pair of lemmas.

Lemma 8.14. *Under the assumptions that* p, q, ℓ, r *are integers such that* $p \geq 0$, $q \geq 0$, $p \geq r$ *and* $\ell \geq 0$, $\ell + r \geq 0$, *one has for some absolute constant* $C > 0$,

$$\left| [\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell+r)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell)} \right|$$
$$\leq C^{\ell+p+q+1}\,(p+q)\,!\,\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell+r)}\,(1+\sigma^{-2})^{-\frac{q}{2}}. \quad (8.79)$$

Proof. Set $\mu = \frac{1}{2}\left(\frac{n}{2}+\ell\right)$. One sees by induction that

$$\frac{d^p}{d\sigma^p}(1+\sigma^2)^{-\mu} = \begin{cases} \sum_{m=\frac{p}{2}}^{p} a_m^p\,(1+\sigma^2)^{-\mu-m} & \text{if } p \text{ is even,} \\ \sum_{m=\frac{p+1}{2}}^{p} b_m^p\,\sigma\,(1+\sigma^2)^{-\mu-m} & \text{if } p \text{ is odd,} \end{cases} \tag{8.80}$$

with

$$\begin{aligned} b_m^{p+1} &= -2\,(\mu+m-1)\,a_{m-1}^p, \\ a_m^{p+1} &= 2\,(\mu+m-1)\,b_{m-1}^p - 2\left(\mu+m-\frac{1}{2}\right) b_m^p \;: \end{aligned} \tag{8.81}$$

now, $\mu \geq \frac{1}{2}$ so that $|\mu+m-1| \leq \mu+m$, which is $\leq \mu+p$ for $m \leq p$: from the last two equations it thus follows that

$$\max\left(\sum_{m=\frac{p}{2}}^{p} |a_m^p|\;,\;\sum_{m=\frac{p+1}{2}}^{p} |b_m^p|\right) \leq 4^{2\mu+p}\,\frac{\Gamma(2\mu+p)}{\Gamma(2\mu)}. \tag{8.82}$$

Next, we show by induction on q that

$$\begin{aligned} &[\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell+r)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(1+\sigma^2)^{-\mu} \\ &\quad = \begin{cases} \sum_{\frac{r}{2}\leq m\leq p+q} A_m^{q,p}\,(1+\sigma^2)^{-\mu-m} & \text{if } p+q \text{ is even,} \\ \sum_{\frac{r+1}{2}\leq m\leq p+q} B_m^{q,p}\,\sigma\,(1+\sigma^2)^{-\mu-m} & \text{if } p+q \text{ is odd :} \end{cases} \end{aligned} \tag{8.83}$$

here, the index m only runs through integers in both equations, and one has

$$\begin{aligned} B_m^{q+1,p} &= (r-2m)\,A_m^{q,p} + 2\,(\mu+m-1)\,A_{m-1}^{q,p}, \\ A_m^{q+1,p} &= (r-1-2m)\,B_{m+1}^{q,p} + (4m+2\mu-r-1)\,B_m^{q,p} - 2\,(\mu+m-1)\,B_{m-1}^{q,p}. \end{aligned} \tag{8.84}$$

Note that $|r-2m| + 2\,|\mu+m-1| \leq 2m-r+2\,(m+\mu) \leq 4m+2\mu+\ell \leq 4\,(m+\mu) \leq 4\,(\mu+p+q)$ and that $|r+1-2m|+|4m+2\mu-r-1|+2\,|\mu+m-1| \leq (2m-r+1)+(4m+2\mu-r-1)+2\,(\mu+m) = 8m+4\mu-2r \leq 8m+4\mu+2\ell \leq 8\,(m+\mu)$: it follows that

$$\max\left(\sum_m |A_m^{q,p}|\;,\;\sum_m |B_m^{q,p}|\right) \leq 8^{2\mu+p+q}\,\frac{\Gamma(2\mu+p+q)}{\Gamma(2\mu)}. \tag{8.85}$$

This implies the announced estimate in the case when $|\sigma| \geq 1$, in view of the inequality

$$\frac{\Gamma(\nu+p)}{\Gamma(\nu)} \leq 8^{\nu+p}\,p\,! \tag{8.86}$$

valid if $\nu \geq 0$ and $p = 0, 1, \ldots$. Indeed, if $\nu \leq p$, one has

$$\frac{\Gamma(\nu+p)}{\Gamma(\nu)} \leq \frac{(2p-1)\,!}{(p-1)\,!} \leq 4^p\,p\,! \tag{8.87}$$

by Stirling's formula, while by the same, if $\nu \geq p \geq 1$,

$$\frac{\Gamma(\nu+p)}{\Gamma(\nu)} \leq (2\nu)^p = \left(\frac{2\nu}{p}.\,p\right)^p \leq e^{2\nu}\,p^p \leq (e^2)^{\nu+\frac{p}{2}}\,p!\,. \tag{8.88}$$

To prove (8.79) for small values of $|\sigma|$, another expression of the left-hand side of (8.83) is needed: setting $\eta = 0$ if p is even, $\eta = 1$ if p is odd, one can see that

$$\left[\sigma^2\,\frac{d}{d\sigma} + \left(\frac{n}{2}+\ell\right)\,\sigma\right]^q\,\frac{d^p}{d\sigma^p}\,(1+\sigma^2)^{-\mu} = \sigma^{q+\eta}\sum_{m=\frac{p+\eta}{2}}^{p+q} C_m^{q,p}\,(1+\sigma^2)^{-\mu-m} \tag{8.89}$$

with

$$C_m^{q+1,p} = (r+q+\eta-2m)\,C_m^{q,p} + 2\,(\mu+m-1)\,C_{m-1}^{q,p}. \tag{8.90}$$

Again, $|\mu+m-1| \leq \mu+p+q$ when $m \leq p+q$. On the other hand, $|r+q+\eta-2m| \leq \max(\ell,p) + |q+\eta-2m| \leq 3\,(\mu+p+q)+1$. From all this one finds, by induction, that

$$\left|\,[\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell+r)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(1+\sigma^2)^{-\frac{1}{2}(\frac{n}{2}+\ell)}\,\right| \leq C^{2\mu+p+q}\,\frac{\Gamma(2\mu+p+q)}{\Gamma(2\mu)}\,\sigma^q\,(1+\sigma^2)^{-\mu-\frac{p}{2}}, \tag{8.91}$$

which implies (8.79) for $|\sigma| \leq 1$, if one remarks also that $\mu + \frac{p}{2} \geq \frac{1}{2}\,(\frac{n}{2}+\ell+p) \geq \frac{1}{2}\,(\frac{n}{2}+\ell+r)$. □

Lemma 8.15. *Given $\rho \geq 0$, let V_ρ be the linear space generated by all functions of the kind $(\frac{d}{d\sigma})^j\,(1+\sigma^2)^{-\frac{\rho+1}{2}-m}$ with $j = 0,1,\dots$ and $m = 0,1,\dots$. If a and b are two integers such that $0 \leq a \leq 2b$, the function $\psi_{a,b}$ defined as*

$$\psi_{a,b}(\sigma) = (1+\sigma^2)^{-\frac{\rho+1}{2}-b}\,(1-i\,\sigma)^a \tag{8.92}$$

lies in V_ρ.

Proof. If $a = 1$, one must have $b \geq 1$ so that $\psi_{0,b-1} \in V_\rho$. One can then write

$$\psi_{1,b} = \psi_{0,b} + \frac{i}{\rho+1+2b}\,\frac{d}{d\sigma}\,\psi_{0,b-1}. \tag{8.93}$$

Next, assume that for some $a \geq 1$, the function $\psi_{a,b}$ lies in V_ρ whenever $a \leq 2b$, and consider a function $\psi_{a+1,b}$ with $a+1 \leq 2b$: then, to start with, the functions $\psi_{a-1,b-1}$ and $\psi_{a-1,b}$ lie in V_ρ. Now, writing $(1-i\,\sigma)^2 = -(\sigma^2+1)+2-2i\sigma$, one obtains

$$\psi_{a+1,b} = -\psi_{a-1,b-1} + 2\,\psi_{a-1,b} - 2i\,(1+\sigma^2)^{-\frac{\rho+1}{2}-b}\,\sigma\,(1-i\,\sigma)^{a-1} \tag{8.94}$$

and it only remains to prove that the last function on the right-hand side lies in V_ρ too. This follows from the equation

$$\frac{d}{d\sigma}\,\psi_{a-1,b-1}+i\,(a-1)\,\psi_{a-2,b-1}=-\,(\rho+1+2b)\,(1+\sigma^2)^{-\frac{\rho+1}{2}-b}\,\sigma\,(1-i\,\sigma)^{a-1}, \quad (8.95)$$

in which the second term on the left-hand side is present only if $a\neq 1$. □

Proposition 8.16. *All anaplectic Hermite functions lie in the space* $\mathcal{S}_{\bullet}^{\bullet}(\mathbb{R}^n)$.

Proof. It suffices to show that this space contains every function of the kind $g^{\text{ram}}\boxtimes\mathcal{Y}^\ell$ with $w=\mathcal{F}g$ coinciding with the function $w^k_{\rho,0}$ as made explicit in (2.20), with $\rho=\frac{n-2}{2}+\ell$ and ℓ even, as well as every function of the kind $g^{\text{ram}}\boxtimes\mathcal{Y}^{\ell+1}$ with $w=\mathcal{F}g$ coinciding this time with the function $w^k_{\rho+1,1}$. We first show that all the functions $w^k_{\rho,0}$ and $w^k_{\rho+1,1}$ lie in the space V_ρ introduced in Lemma 8.15. If $k=0,1,\dots$, this follows from this lemma and from the equations (2.20), here rewritten for convenience:

$$\begin{aligned} w^k_{\frac{n-2}{2}+\ell,0}(\sigma)&=(1+\sigma^2)^{-\frac12(\frac n2+\ell)-k}\,(1-i\,\sigma)^{2k},\\ w^k_{\frac{n-2}{2}+\ell+1,1}(\sigma)&=(1+\sigma^2)^{-\frac12(\frac n2+\ell)-k-1}\,(1-i\,\sigma)^{2k+1}; \end{aligned} \quad (8.96)$$

when $k\geq 1$, one may also write

$$w^{-k}_{\frac{n-2}{2}+\ell,0}=\overline{w^k_{\frac{n-2}{2}+\ell,0}},\qquad w^{-k}_{\frac{n-2}{2}+\ell+1,1}=\overline{w^{k-1}_{\frac{n-2}{2}+\ell+1,1}}, \quad (8.97)$$

so that the general case when $k\in\mathbb{Z}$ is taken care of, as the space V_ρ is stable under complex conjugation.

To settle the even case of the proposition, we thus have to show that, with

$$w=\left(\frac{d}{d\sigma}\right)^j\,(1+\sigma^2)^{-\frac12(\frac n2+\ell)-m}, \quad (8.98)$$

where $j=0,1,\dots$ and $m=0,1,\dots$, and under the assumptions relative to the set $(p,\,q,\,\ell,\,r)$ taken from Definition 8.13, one has

$$\left|\,[\sigma^2\,\frac{d}{d\sigma}+(\frac n2+\ell+r)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,w(\sigma)\right| \leq C^{\ell+p+q+1}\,(p+q)\,!\;(1+\sigma^2)^{-\frac12(\frac n2+\ell+r)}\,(1+\sigma^{-2})^{-\frac q2}. \quad (8.99)$$

This is an immediate application of Lemma 8.14. Only, some shift in the parameters is needed, replacing the set $(p,\,q,\,\ell,\,r)$ there by the set $(p',\,q,\,\ell',\,r')$ with $p'=p+j$, $\ell'=\ell+2m$, $r'=r-2m$, so that the conditions $p'\geq 0$, $q\geq 0$, $p'\geq r'$, $\ell'\geq 0$, $\ell'+r'\geq 0$ are still satisfied. Also, on the right-hand side, the coefficient $C^{\ell+p+q+1}\,(p+q)\,!$ will have to be replaced by

$$C^{\ell+p+q+1}\,(p+q)\,!\times C^{2m+j}\,\frac{(p+q+j)\,!}{(p+q)\,!}\leq C^{\ell+p+q+1}\,(p+q)\,!\times C^{2m+j}\,8^{p+q+1+j}\,j\,!, \quad (8.100)$$

which is all right since, in the situation under discussion, the pair (m, j) is fixed. The odd case of the proposition is taken care of in the same way, only observing (because of the condition $p' \geq r+1$ to be checked now in the case when $m = 0$, so as to be able to substitute $\ell + 1$ for ℓ in the exponent on the right-hand side of (8.99)) that one now has $j \geq 1$. □

Theorem 8.17. *All the operators from the Heisenberg representation map the space* $\mathcal{S}^{\bullet}_{\bullet}(\mathbb{R}^n)$ *into* $\mathcal{S}_{\bullet}(\mathbb{R}^n)$. *They preserve the pseudoscalar product.*

Proof. We already know that the operators $e^{2i\pi\langle\alpha,\,Q\rangle}$ preserve the space $\mathcal{S}^{\bullet}(\mathbb{R}^n)$ and that the operators $e^{2i\pi\langle\beta,\,P\rangle}$ preserve the space $\mathcal{S}_{\bullet}(\mathbb{R}^n)$. It thus suffices to show that, for $|\alpha|$ small in some absolute way ($|\alpha| < 2^{-\frac{5}{2}}\,\pi^{-1}$ will do), the operator $e^{2i\pi\langle\alpha,\,Q\rangle}$ preserves the space $\mathcal{S}^{\bullet}_{\bullet}(\mathbb{R}^n)$. The proof of this follows that of Proposition 8.12 with almost no change. One may rewrite (8.78) as

$$\left|\,[\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(\mathcal{F}\,((s^{\iota_+}\,(g_{\ell-\bar{\iota},m}))(\sigma)\,\right|$$
$$\leq (2\pi)^{-\iota_+}\,C^{\ell-\bar{\iota}+p+\iota_++q+1}\,(p+\iota_++q)\,!\,(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell)}\,(1+\sigma^{-2})^{-\frac{q}{2}}. \tag{8.101}$$

The estimate (8.73) becomes

$$\left|\,[\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(\mathcal{F}\,(g_\alpha)_{\ell,m})(\sigma)\,\right| \leq \sum_{j\geq 0}\ \sum_{\iota_1=\pm1,\dots,\iota_j=\pm1}\ \sum_{m'} \frac{(2\pi)^{j-\iota_+}}{j\,!}$$
$$\times\, 2^{\frac{j-\iota_+}{2}}\,3^{\frac{\iota_+}{2}}\,|\alpha|^j\,C^{\ell-\bar{\iota}+p+\iota_++q+1}\,(p+\iota_++q)\,!\,(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell)}\,(1+\sigma^{-2})^{-\frac{q}{2}}, \tag{8.102}$$

and (8.76) is changed to

$$\left|\,[\sigma^2\,\frac{d}{d\sigma} + (\frac{n}{2}+\ell)\,\sigma]^q\,\frac{d^p}{d\sigma^p}\,(\mathcal{F}\,(g_\alpha)_{\ell,m})(\sigma)\,\right|$$
$$\leq C_1\,C^{p+q+\ell+1}\,(\ell+1)^{n-2}\,(1+\sigma^2)^{-\frac{1}{2}\,(\frac{n}{2}+\ell)}$$
$$\times\,(1+\sigma^{-2})^{-\frac{q}{2}}\sum_{j\geq 0}\frac{(2^{\frac{5}{2}}\,\pi\,|\alpha|)^j}{j\,!}\,((p+j)\,!+C_2^{j+1})\,: \tag{8.103}$$

the last series can also be made explicit, this time as

$$(p+q)\,!\,(1-2^{\frac{5}{2}}\,\pi\,|\alpha|)^{-p-q-1} + C_2\,\exp\,(2^{\frac{5}{2}}\,\pi\,C_2\,|\alpha|). \tag{8.104}$$

That, in the present context, an operator such as $e^{2i\pi\langle\beta,\,P\rangle}\,e^{2i\pi\langle\alpha,\,Q\rangle}$, with α and β in $\mathbb{R}^n$, preserves the pseudoscalar product is a consequence of Proposition 8.12 and Remark 8.4 (iii). □

As observed in Remark 8.4 (i), the definition of the space $\mathcal{S}^{\bullet}(\mathbb{R}^n)$, though associated to the choice of the canonical Euclidean structure on $\mathbb{R}^n$, does not depend on the choice, for every $\ell = 0, 1, \dots,$ of the orthonormal basis $(Y_{\ell,m})_m$ of spherical harmonics; the same is true of the spaces $\mathcal{S}_\bullet(\mathbb{R}^n)$ and $\mathcal{S}_\bullet^\bullet(\mathbb{R}^n)$. Moreover, the pseudoscalar product does not depend on such a choice.

Another transformation easy to study is the transformation $u \mapsto u_\lambda$ with $u_\lambda(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, where $\lambda > 0$. If u is given by the series (8.46), one has $u_\lambda = \sum_{\ell,m} f_{\lambda,m} \boxtimes \mathcal{Y}^{\ell,m}$ with

$$f_{\ell,m}(s) = \lambda^{\frac{n}{2}} g_{\ell,m}^{\mathrm{ram}}(\lambda^2 s) : \tag{8.105}$$

now, this function is just the ramified part, in the $C^\omega_{\frac{n-2}{2}+\ell,\varepsilon}$-theory, of the function $h_{\ell,m}$ defined as

$$h_{\ell,m}(s) = \lambda^{2-\frac{n}{2}-\ell} g_{\ell,m}(\lambda^2 s). \tag{8.106}$$

This provides a decomposition of the form (8.46) for the function u_λ and, since $(h_{\ell,m} \,|\, h_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon} = (g_{\ell,m} \,|\, g_{\ell,m})_{\frac{n-2}{2}+\ell,\varepsilon}$ as can be verified from the definition of these expressions in Lemma 8.3, the transformation $u \mapsto u_\lambda$ preserves the pseudoscalar product defined as an extension of (8.47). It is immediate, on the other hand, that this transformation preserves each of the spaces $\mathcal{S}^\bullet(\mathbb{R}^n)$, $\mathcal{S}_\bullet(\mathbb{R}^n)$ and $\mathcal{S}_\bullet^\bullet(\mathbb{R}^n)$.

Remark 8.5. One question which we shall leave unanswered at present is whether the pseudoscalar product is left unchanged, on some appropriate space, by more general operators from the anaplectic representation. It is true that, since the infinitesimal generators Q_j and P_k $(j, k = 1, \dots, n)$ of the Heisenberg representation are formally self-adjoint, on $E^{(n)}$ or on any of the spaces introduced in Definition 8.10 or 8.13, there may be some hope that their symmetrized quadratic combinations, still in some sense the infinitesimal operators of the anaplectic representation, might enjoy the same property. However, the problem has to do with the construction of some invariant space of functions, on which the pseudoscalar product would make sense. For all we know, it is not excluded that a space comparable to those already introduced might do, in the case of transformations associated to linear changes of coordinates. However, let us indicate some rough reason why it is very unlikely that a space such as $\mathcal{S}^\bullet(\mathbb{R}^n)$, $\mathcal{S}_\bullet(\mathbb{R}^n)$ or $\mathcal{S}_\bullet^\bullet(\mathbb{R}^n)$ could have such an invariance with respect to general transformations from the anaplectic representation. Going back to Sections 4 and 5, one sees that the $\mathcal{Q}$-transform of any anaplectic Hermite function is analytic on the whole of Sym_n: in other words, the $\mathcal{K}$-transform of such a function is analytic on the subset of Σ, or $\Sigma^{(n)}$, denoted as Σ^{reg}, in Proposition 4.7. This property is likely to hold also in the case of a function taken from a space consisting of series of anaplectic Hermite functions – though, in the case of the three spaces just referred to, we have not made such a verification. But, except for the case of linear changes of coordinates, the equations (5.69) from Theorem 5.10 show that, in general, the $\mathcal{K}$-transform of the image of the basic function Φ of Theorem 4.18 (the rotation-invariant median state of the anaplectic harmonic oscillator) under some anaplectic transformation ceases to be regular on Σ^{reg}, because it has singularities which are moved in the process.

Chapter 3

Towards the Anaplectic Symbolic Calculi

In the first section of this chapter, we study the anaplectic analogue of the Bargmann–Fock realization of functions on the real line. It is, again, a realization by means of functions of two variables: however, the differential equation that characterizes the image of the transformation is associated to a second-order operator rather than a first-order one (the gauge-transformed $\frac{\partial}{\partial z}$-operator of the classical theory). Just as in the usual analysis, the Bargmann–Fock transformation depends on some additional structure: a harmonic oscillator or, equivalently, a complex structure on the phase space (here, $\mathbb{R}^2$, as we consider only the one-dimensional analysis) compatible in some sense with the symplectic structure. The same is needed again, both in the usual analysis and here, when introducing the Wick symbolic "calculus" of operators. As will be observed, the fact that there is, in the anaplectic analysis, no *fundamental* difference between the raising and lowering operators (in contradiction to the creation and annihilation operators of the classical analysis), since they are conjugate to each other under the complex rotation by ninety degrees, has an algebraically interesting consequence (*cf.* Proposition 9.11) in the Wick calculus, and in the related notion of "Wick ordering" of operators.

Section 9 should not be regarded as an important one: only it answers some very natural questions, besides stating and proving a few lemmas for a later use. All practitioners of pseudodifferential analysis are familiar with the fact that the Wick symbol of an operator is only a very smoothed up version of another species of symbol, to wit the Weyl symbol, thus leading to a considerable loss of information (which does not mean that the consideration of such a kind of Gaussian-regularized symbol in pseudodifferential analysis is not helpful [22], but it should not be used *in place of* the Weyl symbol in a systematic way).

The second section of this chapter should be regarded as an introduction to the anaplectic analogue of the Weyl calculus: recall that, at least on $\mathbb{R}^n$, the

Weyl calculus is often considered as synonymous with pseudodifferential analysis. A concept dual to that of Weyl symbolic calculus is that of Wigner function: in the usual analysis, the Wigner function associated with a pair of functions on the real line is a function of two variables with two possible interpretations, one of which is as the Weyl symbol of the rank-one operator associated with the given pair of functions. In anaplectic analysis, the corresponding concept is also important in another respect. It provides the simplest bilinear operation from pairs of functions in some appropriate subspace of $\mathfrak{A}$ to functions in the space $\mathfrak{A}^{(2)}$: even the tensor product is more difficult to deal with. In particular – though this is harder to prove than the corresponding fact from classical analysis – the Wigner function of two anaplectic Hermite functions of one variable is an anaplectic Hermite function of two variables.

In higher dimension, however, the Weyl calculus has been hardly touched upon, in Remark 10.5. It is of course only when the formal structure of the anaplectic analysis has been completely elucidated that one can seriously hope to find for this new pseudodifferential analysis a proper domain of application: one should still consider all this, at present, as an exercise in harmonic analysis rather than a new tool in partial differential equations. The main difficulty with anaplectic analysis, more heavily felt in the higher-dimensional case, has already been experienced in Section 6, and is related to the fact that there is no class of very simple functions, stable under the anaplectic representation, that would play the role usually played by Gaussian functions: but this is also, after all, what makes it the source of developments of possible independent interest, such as Corollary 10.12 or Theorem 10.14. Though it may be too early to tell, we also feel that the approach to a possible extension of the Lax–Phillips scattering theory briefly reported in Remark 10.4 may be promising.

9 The Bargmann–Fock transformation in the anaplectic setting

The usual Bargmann–Fock transformation is an isometric linear transformation from the space $L^2(\mathbb{R})$ to a subspace of $L^2(\mathbb{R}^2)$, to wit that consisting of functions V on $\mathbb{R}^2$ which, in terms of the complex variable $z = x + iy$, become antiholomorphic after they have been multiplied by the function $z \mapsto \exp\left(\frac{\pi}{2}\,|z|^2\right)$. The Bargmann–Fock transformation $u \mapsto V$ is defined by the equation

$$V(x+iy) = (e^{2i\pi\,(yQ-xP)}\chi \,|\, u) \tag{9.1}$$

involving the scalar product in $L^2(\mathbb{R})$ (linear with respect to the second entry), the Heisenberg transformation $\pi(x,y) = e^{2i\pi\,(yQ-xP)}$ introduced in (1.1), and the normalized fundamental state $\chi(t) = 2^{\frac{1}{4}}\, e^{-\pi t^2}$ of the harmonic oscillator.

The condition characterizing the functions V in the image of the Bargmann–Fock transformation can be stated as the differential equation

$$\left(\frac{\partial}{\partial z}+\frac{\pi\bar{z}}{2}\right)V=0. \tag{9.2}$$

Set

$$\begin{aligned}\Lambda&=-\frac{1}{2\pi}\left[\left(\frac{\partial}{\partial x}-i\pi y\right)^2+\left(\frac{\partial}{\partial y}+i\pi x\right)^2\right]\\&=-\frac{1}{2\pi}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-2i\pi\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)-\pi^2\left(x^2+y^2\right)\right],\end{aligned} \tag{9.3}$$

so that

$$-\frac{2}{\pi}\left(\frac{\partial}{\partial\bar{z}}-\frac{\pi\,z}{2}\right)\left(\frac{\partial}{\partial z}+\frac{\pi\,\bar{z}}{2}\right)=\Lambda-1. \tag{9.4}$$

Consequently, for V in the image of the Bargmann–Fock transformation, one has $(\Lambda-1)\,V=0$ but, of course, this second-order equation carries less information than the equation (9.2).

Remark 9.1. The Bargmann–Fock transformation depends on the choice of the canonical complex structure on $\mathbb{R}^2$ (any other *specified* complex structure compatible with the symplectic structure of $\mathbb{R}^2$, *i.e.,* for which the imaginary part of a product $z\,\bar{z}'$ of complex coordinates should coincide with the value $-xy'+yx'$ of the canonical two-form of $\mathbb{R}^2$ on the corresponding pair of points, would do just as well, only changing χ to the ground state of some transformed version of the harmonic oscillator).

We now study the analogue of the Bargmann–Fock transformation in the anaplectic setting, substituting for $L^2(\mathbb{R})$ the space $\mathfrak{A}$ together with its indefinite scalar product, and for χ the function ϕ introduced in Proposition 1.2.

Proposition 9.1. *Let ϕ be the median state of the harmonic oscillator introduced in Proposition* 1.2. *Given $u\in\mathfrak{A}$, define the function U on $\mathbb{R}^2$ through the equation*

$$U(x,y)=(e^{2i\pi\,(yQ-xP)}\phi\,|\,u) \tag{9.5}$$

involving the Heisenberg representation and the indefinite scalar product in $\mathfrak{A}$. The function U extends as an antiholomorphic function of (x,y) in $\mathbb{C}^2$, after x and y are given independent *complex values. For some constant $C>0$, the estimate $|U(x,y)|\leq C\,\exp\left(\pi(x^2+y^2)\right)$ holds for every $(x,y)\in\mathbb{R}^2$. Moreover, U satisfies the differential equation*

$$\Lambda\,U=0, \tag{9.6}$$

with Λ as introduced in (9.3).

Proof. Recall from the proof of Proposition 1.2 that the $\mathbb{C}^4$-realization of ϕ is $(\psi,\, 0,\, \psi,\, 0)$, with ψ as defined in (1.11). Combining (1.58) and (1.71) with the definition (1.1) of $\pi(x,y)$, we find that the $\mathbb{C}^4$-realization of $e^{i\pi xy}\, e^{2i\pi\,(yQ-xP)}\phi$ is the vector $(h_0,\, h_1,\, h_{i,0},\, h_{i,1})$ with

$$\begin{aligned} h_0(t) &= \frac{1}{2}\,[\psi(t-x)\, e^{2i\pi yt} + \psi(t+x)\, e^{-2i\pi yt}],\\ h_1(t) &= \frac{1}{2}\,[\psi(t-x)\, e^{2i\pi yt} - \psi(t+x)\, e^{-2i\pi yt}],\\ h_{i,0}(t) &= \frac{1}{2}\,[\psi(t-ix)\, e^{2\pi yt} + \psi(t+ix)\, e^{-2\pi yt}],\\ h_{i,1}(t) &= \frac{i}{2}\,[-\psi(t-ix)\, e^{2\pi yt} + \psi(t+ix)\, e^{-2\pi yt}]. \end{aligned} \tag{9.7}$$

Since these four functions extend as entire functions of the complex variables x and y, it follows from the definition, in Proposition 1.14, of the scalar product in $\mathfrak{A}$, that the function U extends as an antiholomorphic function of (x,y) in $\mathbb{C}^2$. The estimate concerning $U(x,y)$ is obvious, in the case when $y=0$, from the relations (9.7) and the estimate of the function $I_{\pm\frac{1}{4}}(\pi x^2)$ at infinity. In general, if $x+iy = r\,e^{i\theta}$, set $g = \left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$: one has $\mathrm{Ana}(g)\phi = \phi$ since, under the intertwining operator Θ introduced in Theorem 2.9, this equation is equivalent to the equation $\pi_{-\frac{1}{2},0}(g)\, v^0_{-\frac{1}{2},0} = v^0_{-\frac{1}{2},0}$ proved in (2.15). Using (1.91), one can then write

$$U(x,y) = (\mathrm{Ana}(g)\, e^{-2i\pi\, rP}\,\phi\,|\,u) = (e^{-2i\pi\, rP}\,\phi\,|\,\mathrm{Ana}(g^{-1})\,u), \tag{9.8}$$

which reduces the estimate of $U(x,y)$ to that of $U'(r,0)$, where U' is defined by the same equation as U, only in association with the function $\mathrm{Ana}(g^{-1})\,u$ instead of u. Set

$$F(x,y;t) = e^{-i\pi xy}\, e^{2i\pi yt}\,\psi(t-x) \tag{9.9}$$

and observe after a straightforward calculation using the equation

$$\psi''(t) - 4\pi^2 t^2\,\psi(t), \tag{9.10}$$

which expresses that ψ, just like ϕ, lies in the kernel of the harmonic oscillator, that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2i\pi\left(y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\right) - \pi^2\,(x^2+y^2)\right] F(x,y;t) = 0: \tag{9.11}$$

the same equation holds if $F(-x,-y;t)$, $F(ix,-iy;t)$ or $F(-ix,iy;t)$ is substituted for $F(x,y;t)$. Thus, after having been multiplied by $e^{-i\pi xy}$, the four functions in (9.7) are annihilated by the second-order differential operator just introduced, which, not forgetting the complex conjugation since the function $e^{2i\pi\,(yQ-xP)}\phi$ occurs on the left of the scalar product that defines $U(x,y)$, finishes the proof of Proposition 9.1. $\square$

Remark 9.2. Let us emphasize that it is the equation $\Lambda U = 0$, not the fact that U is an antiholomorphic function of *two* variables, that plays here the same role as the antiholomorphy condition (9.2) of the usual Bargmann-Fock transformation.

To characterize the image of $\mathfrak{A}$ under the map $u \mapsto U$ defined in (9.5), we need a number of lemmas. This is a little more cumbersome than one would like since, in view of the absence of a genuine (positive) scalar product on $\mathfrak{A}$, one has to use instead only that on $C^{\omega}_{-\frac{1}{2},0}$, depending on Propositions 2.12 and 2.13 to cover the other part $C^{\omega}_{\frac{1}{2},1}$ as well, finally transferring everything on $\mathfrak{A}$ with the help of Θ.

Lemma 9.2. *Let $\rho \in \mathbb{C}$, $\rho \neq 0$, $-1 < \mathrm{Re}\ \rho < 1$. For every $k \in \mathbb{Z}$, one has*

$$|s|_{\varepsilon}^{-\rho}\, v_{\rho,\varepsilon}^{k} = \pi^{\rho}\, \frac{\Gamma(\frac{1-\rho+\varepsilon+2k}{2})}{\Gamma(\frac{1+\rho+\varepsilon+2k}{2})}\, v_{-\rho,\varepsilon}^{k}. \tag{9.12}$$

Proof. From (2.22) and (2.23), one sees that the equation to be proved indeed holds in the case when $k = 0$. Next, Proposition 2.12 implies that

$$\begin{aligned} T_{\rho-1}^{\dagger}\, R\, v_{\rho,\varepsilon}^{k} &= \frac{1}{2\pi}\,(\rho + 2k + 1 + \varepsilon)\, v_{\rho,\varepsilon}^{k+1}, \\ R\, T_{-\rho}^{\dagger}\, v_{-\rho,\varepsilon}^{k} &= \frac{1}{2\pi}\,(-\rho + 2k + 1 + \varepsilon)\, v_{-\rho,\varepsilon}^{k+1}\ : \end{aligned} \tag{9.13}$$

note that the two equations are related since, as observed in Proposition 2.12, $T_{\rho-1}^{\dagger}\, R = R\, T_{\rho}^{\dagger}$. Starting from the product-of-operators formula

$$|s|_{\varepsilon}^{-\rho}\, \frac{d}{ds}\, |s|_{\varepsilon}^{\rho} = \frac{d}{ds} + \frac{\rho}{s}, \tag{9.14}$$

we also note that

$$|s|_{\varepsilon}^{-\rho}\, T_{\rho-1}^{\dagger}\, |s|_{\varepsilon}^{\rho} = R\,(s), \qquad |s|_{\varepsilon}^{-\rho}\, R\, |s|_{\varepsilon}^{\rho} = (s^{-1})\, T_{-\rho}^{\dagger}, \tag{9.15}$$

where $(s^{\pm 1})$ is the operator of multiplication by $s^{\pm 1}$: consequently,

$$|s|_{\varepsilon}^{-\rho}\ (T_{\rho-1}^{\dagger}\ R)\ |s|_{\varepsilon}^{\rho} = R\, T_{-\rho}^{\dagger}. \tag{9.16}$$

From (9.13), (9.16) and the case $k = 0$ of (9.12), one can derive (9.12), for every $k \geq 0$, by induction. The case when $k < 0$ is handled in a similar way, using instead of the operators which occur in (9.13) the operators $R_{\rho-1}^{\dagger}\, T$ and $T\, R_{-\rho}^{\dagger}$, which lower the energy level. □

Lemma 9.3. *Let $\rho \in \mathbb{R}$, $0 < |\rho| < 1$. One has*

$$(v_{\rho,\varepsilon}^{k} \mid v_{\rho,\varepsilon}^{k})_{\rho,\varepsilon} = \pi^{\rho+1}\, \frac{\Gamma(\frac{1-\rho+\varepsilon+2k}{2})}{\Gamma(\frac{1+\rho+\varepsilon+2k}{2})}. \tag{9.17}$$

Proof. From the definition (2.30) of the scalar product, together with the preceding lemma, one finds, using Plancherel's formula in the middle,

$$\begin{aligned}(v^k_{\rho,\varepsilon} \,|\, v^k_{\rho,\varepsilon})_{\rho,\varepsilon} &= \pi^\rho \,\frac{\Gamma(\frac{1-\rho+\varepsilon+2k}{2})}{\Gamma(\frac{1+\rho+\varepsilon+2k}{2})} \int_{-\infty}^{\infty} \bar{v}^k_{\rho,\varepsilon}(s)\, v^k_{-\rho,\varepsilon}(s)\, ds \\ &= \pi^\rho \,\frac{\Gamma(\frac{1-\rho+\varepsilon+2k}{2})}{\Gamma(\frac{1+\rho+\varepsilon+2k}{2})} \int_{-\infty}^{\infty} \bar{w}^k_{\rho,\varepsilon}(\sigma)\, w^k_{-\rho,\varepsilon}(\sigma)\, d\sigma, \end{aligned} \tag{9.18}$$

which leads to the formula indicated after one has made use of (2.20). □

Remark 9.3. As a consequence, $(v^k_{\rho,0} \,|\, v^k_{\rho,0})_{\rho,0} > 0$ for all $k \in \mathbb{Z}$ but $(v^k_{\rho,1} \,|\, v^k_{\rho,1})_{\rho,1} > 0$ if and only if $k \geq 0$: for, otherwise, $\frac{2-\rho}{2} + k$ and $\frac{2+\rho}{2} + k$ are separated by the integer $k + 1 \leq 0$. Of course, this fits with the fact that $\pi_{\rho,\varepsilon}$ is associated with a definite scalar product if $\varepsilon = 0$, not if $\varepsilon = 1$.

Lemma 9.4. *Let $\rho \in \mathbb{R}$, $0 < |\rho| < 1$. Given any v in the space $C^\omega_{\rho,\varepsilon}$, there exist $C > 0$ and $\delta \in]0,1[$ such that*

$$|(v^k_{\rho,\varepsilon} \,|\, v)_{\rho,\varepsilon}| \leq C\, \delta^{|k|}, \qquad k \in \mathbb{Z}. \tag{9.19}$$

Conversely, given any sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers satisfying for some pair $C > 0$, $\delta < 1$ the estimate $|a_k| \leq C\,\delta^{|k|}$ for every $k \in \mathbb{Z}$, there is a unique $v \in C^\omega_{\rho,\varepsilon}$ such that $(v^k_{\rho,\varepsilon} \,|\, v)_{\rho,\varepsilon} = a_k$ for every k.

Proof. We first treat the case when $\varepsilon = 0$ to be in a position to use Hilbert space methods. Let $v \in C^\omega_{\rho,0}$, $v = \mathcal{F}^{-1} w$. In view of (2.30), one has

$$\begin{aligned}(v^k_{\rho,0} \,|\, v)_{\rho,0} &= \int_{-\infty}^{\infty} \bar{v}^k_{\rho,0}(s)\; |s|^{-\rho} v(s)\, ds \\ &= \int_{-\infty}^{\infty} \bar{w}^k_{\rho,0}(\sigma)\; w_1(\sigma)\, d\sigma \end{aligned} \tag{9.20}$$

with $w_1 = |D|^{-\rho} w \in \widehat{C}^\omega_{-\rho,0}$ (a consequence of Proposition 2.3), or

$$\begin{aligned}(v^k_{\rho,0} \,|\, v)_{\rho,0} &= \int_{-\infty}^{\infty} (1+\sigma^2)^{-\frac{\rho+1}{2}-k}\, (1+i\sigma)^{2k}\, w_1(\sigma)\, d\sigma \\ &= \int_{-\infty}^{\infty} (1+\sigma^2)^{-\frac{\rho+1}{2}} \left(\frac{1+i\sigma}{1-i\sigma}\right)^k w_1(\sigma)\, d\sigma : \end{aligned} \tag{9.21}$$

recall (2.32) that the functions $\sigma \mapsto w_1(\sigma)$ and $\sigma \mapsto |\sigma|^{\rho-1}\, w_1(-\frac{1}{\sigma})$ are analytic on the real line. In particular, as $\sigma \in \mathbb{C}$, $|\sigma| \to \infty$, one has for some constant $C > 0$ the estimate $|w_1(\sigma)| \leq C\,(1 + |\sigma|^2)^{\frac{\rho-1}{2}}$.

Assume that $k \geq 0$: for $0 < \epsilon < 1$, the integrand extends as a holomorphic function in a neighborhood of the closed domain limited by the real axis and the parabola $\{\sigma = x + i\epsilon\,(1+x^2):\ x \in \mathbb{R}\}$, and we first check that one can move the contour of integration from the real axis to the parabola by showing that, for fixed k, the integral of the same function as above, taken on the vertical segment $\{\sigma = A + it\sqrt{1+A^2}:\ 0 \leq t \leq \epsilon\sqrt{1+A^2}\}$, goes to zero as $A \to \infty$. Since, on this segment, $|d\sigma| = \sqrt{1+A^2}\,dt$ and the integrand (for fixed k) is at most

$$\begin{aligned} C\,|1+\sigma^2|^{-\frac{\rho+1}{2}}\,(1+|\sigma|^2)^{\frac{\rho-1}{2}} &\leq C\,|1+\sigma^2|^{-1} \\ &= C\,|1+A^2 - t^2(1+A^2) + 2iA\sqrt{1+A^2}\,t|^{-1} \\ &= C\,(1+A^2)^{-1}\,|1-t^2+\frac{2iA}{\sqrt{1+A^2}}\,t|^{-1}, \end{aligned} \tag{9.22}$$

where the second factor is integrable on $(0,\infty)$, we are done. We thus come back to (9.21), in which we interpret the integral, with the same integrand, as taking place on the above-specified parabola, and estimate the result as $k \to \infty$. From the second expression (9.21), all that remains to be done, so as to complete the proof of the first part of Lemma 9.4, in the case when $k > 0$, is to remark that the supremum of $|\frac{1+i\sigma}{1-i\sigma}|$ as σ lies on the parabola, is < 1, an elementary task. In the case when $k \to -\infty$, the same proof works, only substituting $-\epsilon$ for ϵ.

In the other direction, we rely on the fact, mentioned just after (2.15), that the sequence $(w_{\rho,0}^k)_{k\in\mathbb{Z}}$ is a complete orthogonal set in the Hilbert space consisting of all w with $|||\,w\,|||_\rho < \infty$, *i.e.,* with $|D|^{-\frac{\rho}{2}}w \in L^2(\mathbb{R})$. Thus v can be found as $v = \mathcal{F}^{-1}w$ provided that, defining w as

$$w = \sum_{k\in\mathbb{Z}} a_k\ |||\,w_{\rho,0}^k\,|||_\rho^{-2}\ w_{\rho,0}^k, \tag{9.23}$$

we are able to show that $w \in \widehat{C}_{\rho,0}^{\omega}$. We abbreviate the expression above as

$$w = \sum_{k\in\mathbb{Z}} b_k\, w_{\rho,0}^k, \tag{9.24}$$

where, in view of Lemma 9.3 and of Stirling's formula (which proves that $|||\,w_{\rho,0}^k\,|||_\rho^{-1} \leq C(1+|k|)^{|\rho|}$), the sequence $(b_k)_{k\in\mathbb{Z}}$ satisfies the same hypothesis as the one we have made about the sequence $(a_k)_{k\in\mathbb{Z}}$. Then,

$$w(\sigma) = \sum_{k\in\mathbb{Z}} b_k\,(1+\sigma^2)^{-\frac{\rho+1}{2}}\left(\frac{1+i\sigma}{1-i\sigma}\right)^k \tag{9.25}$$

and

$$|\sigma|^{-1-\rho}\,w(-\frac{1}{\sigma}) = \sum_{k\in\mathbb{Z}}(-1)^k\,b_k\,(1+\sigma^2)^{-\frac{\rho+1}{2}}\left(\frac{1+i\sigma}{1-i\sigma}\right)^k, \tag{9.26}$$

and all that remains to be proven is that both functions extend as holomorphic functions of σ to some neighborhood of the real line. Now, this comes from the fact that, given any positive number $\delta < 1$, and $A > 0$, one has

$$\sup_{-A\leq x\leq A}\left|\frac{1-i(x+i\epsilon)}{1+i(x+i\epsilon)}\right|^{\pm 1} = \left|\frac{1+\epsilon-ix}{1-\epsilon+ix}\right|^{\pm 1} \leq \delta^{-1} \tag{9.27}$$

if $\epsilon > 0$ is small enough. Alternatively, the last few lines can be replaced by the argument immediately preceding the statement of Theorem 6.5. This concludes the proof of Lemma 9.4 in the case when $\varepsilon = 0$, and we now treat the other case.

If $-1 < \rho < 0$, and $v \in C^{\omega}_{\rho,1}$, we use Proposition 2.13, then Proposition 2.12, writing

$$(v^k_{\rho,1}\,|\,v)_{\rho,1} = (R\,v^k_{\rho+1,0}\,|\,v)_{\rho,1} = (v^k_{\rho+1,0}\,|\,R^\dagger_\rho v)_{\rho+1,0} \tag{9.28}$$

to find the required estimate, since $R^\dagger_\rho v \in C^{\omega}_{\rho+1,0}$ according to Proposition 2.12. In the case when $0 < \rho < 1$, we use instead the equations

$$\begin{aligned}(v^k_{\rho,1}\,|\,v)_{\rho,1} &= -2\pi\,(\rho-2k-2)^{-1}\,(R^\dagger_{\rho-1}\,v^{k+1}_{\rho-1,0}\,|\,v)_{\rho,1}\\ &= -2\pi\,(\rho-2k-2)^{-1}\,(v^{k+1}_{\rho-1,0}\,|\,Rv)_{\rho-1,0}.\end{aligned} \tag{9.29}$$

Finally, the same lines work in the reverse direction since, in view of (2.133) and (2.134), the condition $R^\dagger_\rho v \in C^{\omega}_{\rho+1,0}$ or $Rv \in C^{\omega}_{\rho-1,0}$ implies that $v \in C^{\omega}_{\rho,1}$: in the first case, use Proposition 2.3 and observe that the operator $\widehat{R}^\dagger_\rho$ is elliptic on the real line. □

Proposition 9.5. *Let $(\phi_j)_{j\in\mathbb{Z}}$ be the sequence of eigenfunctions of the anaplectic harmonic oscillator introduced in Theorem* 2.11. *Given any function $u \in \mathfrak{A}$, the set of scalar products of u against the functions ϕ_j satisfies for some constants $C > 0$ and $\delta \in]0,1[$ the estimate*

$$|(\phi^j\,|\,u)| \leq C\left[\frac{|j|}{2}\right]!\;(2\delta)^{\frac{|j|}{2}}, \qquad j\in\mathbb{Z}. \tag{9.30}$$

Conversely, given any sequence $(a_j)_{j\in\mathbb{Z}}$ of complex numbers satisfying for some $C > 0$ and $\delta \in]0,1[$ the inequality

$$|a_j| \leq C\left[\frac{|j|}{2}\right]!\;(2\delta)^{\frac{|j|}{2}}, \qquad j\in\mathbb{Z}, \tag{9.31}$$

there exists a unique function $u \in \mathfrak{A}$ such that $a_j = (\phi^j\,|\,u)$ for all j.

Proof. Under the intertwining operator $\Theta\colon C^{\omega}_{-\frac12,0}\oplus C^{\omega}_{\frac12,1} \to \mathfrak{A}$ introduced in Theorem 2.9, this is essentially a rephrasing of Lemma 9.4: only, we have to take care of the normalization of the eigenvectors involved. Recall from Theorem 2.11 that, for every $j \geq 0$, one has $\phi^j = A^{*j}\phi$ and $\phi^{-j} = A^j\phi$ with $A^* = \pi^{\frac12}(Q-iP)$ and $A = \pi^{\frac12}(Q+iP)$. Under the isomorphism Θ the operators Q and P transfer to

$$\Theta^{-1}\,Q\,\Theta = 2^{\frac12}\begin{pmatrix}0 & 1\\ s & 0\end{pmatrix} \qquad \text{and} \qquad \Theta^{-1}\,P\,\Theta = \frac{2^{\frac12}}{2i\pi}\begin{pmatrix}0 & \frac{d}{ds}\\ s\,\frac{d}{ds}+\frac12 & 0\end{pmatrix} \tag{9.32}$$

so that, with the notation of Proposition 2.12, the operators A^* and A transfer respectively to

$$\Theta^{-1} A^* \Theta = (2\pi)^{\frac{1}{2}} \begin{pmatrix} 0 & R \\ T_\rho^\dagger & 0 \end{pmatrix} \qquad \text{and} \qquad \Theta^{-1} A \Theta = (2\pi)^{\frac{1}{2}} \begin{pmatrix} 0 & T \\ R_\rho^\dagger & 0 \end{pmatrix}. \tag{9.33}$$

It has been shown in (2.112) that, with $\alpha = \frac{2^{\frac{1}{4}}\pi^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}$, one has

$$\alpha\, \Theta^{-1} \phi = \begin{pmatrix} v^0_{-\frac{1}{2},0} \\ 0 \end{pmatrix} : \tag{9.34}$$

one can then see, using the set of formulas from Proposition 2.13, that, for $k \geq 0$, one has

$$\alpha\, \Theta^{-1} \phi^{2k} = 2^k\, \frac{\Gamma(\frac{1}{4}+k)}{\Gamma(\frac{1}{4})} \begin{pmatrix} v^k_{-\frac{1}{2},0} \\ 0 \end{pmatrix},$$

$$\alpha\, \Theta^{-1} \phi^{-2k} = (-1)^k\, 2^k\, \frac{\Gamma(\frac{1}{4}+k)}{\Gamma(\frac{1}{4})} \begin{pmatrix} v^{-k}_{-\frac{1}{2},0} \\ 0 \end{pmatrix} \tag{9.35}$$

and, for $k \geq 1$,

$$\alpha\, \Theta^{-1} \phi^{2k-1} = (2\pi)^{-\frac{1}{2}}\, 2^k\, \frac{\Gamma(\frac{1}{4}+k)}{\Gamma(\frac{1}{4})} \begin{pmatrix} 0 \\ v^{k-1}_{\frac{1}{2},1} \end{pmatrix},$$

$$\alpha\, \Theta^{-1} \phi^{-2k+1} = (2\pi)^{-\frac{1}{2}}\, (-1)^k\, 2^k\, \frac{\Gamma(\frac{1}{4}+k)}{\Gamma(\frac{1}{4})} \begin{pmatrix} 0 \\ v^{-k}_{\frac{1}{2},1} \end{pmatrix}. \tag{9.36}$$

The proposition is then a consequence of the characterization of the space $C^\omega_{\rho,\varepsilon}$ given in Lemma 9.4, and of the fact that the map Θ transforms the scalar product on $C^\omega_{-\frac{1}{2},0} \oplus C^\omega_{\frac{1}{2},1}$ into that on $\mathfrak{A}$. □

Theorem 9.6. *Let $u \in \mathfrak{A}$ and let $U(x,y)$ be the function defined on $\mathbb{R}^2$ in* (9.5) *or its extension as an antiholomorphic function in $\mathbb{C}^2$. Recall that it satisfies for $(x,y) \in \mathbb{R}^2$ the equation $\Lambda U = 0$, with Λ as introduced in* (9.3)*. For some $C > 0$ and some $\delta \in]0,1[$, the estimate*

$$|U(z,iz)| + |U(z,-iz)| \leq C \exp\left(\frac{\pi\delta}{2}\, |z|^2\right) \tag{9.37}$$

holds for every $z \in \mathbb{C}$. Conversely, let U be any antiholomorphic function in $\mathbb{C}^2$, satisfying for $(x,y) \in \mathbb{R}^2$ the equation $\Lambda U = 0$. Assume moreover that, for some $C > 0$ and some $\delta \in]0,1[$, the inequality (9.37) *holds. Then, there exists a unique function $u \in \mathfrak{A}$ such that U is associated to u under* (9.5)*.*

Proof. Let $(x, y) \in \mathbb{R}^2$, and let $z = x + iy$. It is convenient, here, to set $\tau_z = e^{2i\pi\,(yQ-xP)}$, *i.e.*, for any $u \in \mathfrak{A}$, $(\tau_z u)(t) = u(t-x)e^{2i\pi(t-\frac{x}{2})y}$. Recalling that

$$A^* = \pi^{\frac{1}{2}}\left(t - \frac{1}{2\pi}\,\frac{d}{dt}\right) \qquad \text{and} \qquad A = \pi^{\frac{1}{2}}\left(t + \frac{1}{2\pi}\,\frac{d}{dt}\right), \tag{9.38}$$

we note that, for every $u \in \mathfrak{A}$, one has the pair of equations

$$\begin{aligned} \left(\frac{\partial}{\partial \bar z} + \frac{\pi\, z}{2}\right)\tau_z\, u &= -\pi^{\frac{1}{2}}\,\tau_z(Au), \\ \left(\frac{\partial}{\partial z} - \frac{\pi\, \bar z}{2}\right)\tau_z\, u &= \pi^{\frac{1}{2}}\,\tau_z(A^* u). \end{aligned} \tag{9.39}$$

Recalling Theorem 2.11, we thus find for every $j \geq 0$ the equations

$$\begin{aligned} \left(\frac{\partial}{\partial z} + \frac{\pi\, \bar z}{2}\right)^j U(x,y) &= (-1)^j\,\pi^{\frac{j}{2}}\,(\tau_z\,\phi^{-j}\,|\,u), \\ \left(\frac{\partial}{\partial \bar z} - \frac{\pi\, z}{2}\right)^j U(x,y) &= \pi^{\frac{j}{2}}\,(\tau_z\,\phi^{j}\,|\,u). \end{aligned} \tag{9.40}$$

In particular, this gives *some* of the Taylor coefficients of U at $0 \in \mathbb{R}^2$ in terms of the scalar products $(\phi^j\,|\,u)$, to wit

$$\left(\left(\frac{\partial}{\partial z}\right)^j U\right)(0) = (-1)^j\,\pi^{\frac{j}{2}}\,(\phi^{-j}\,|\,u), \qquad \left(\left(\frac{\partial}{\partial \bar z}\right)^j U\right)(0) = \pi^{\frac{j}{2}}\,(\phi^{j}\,|\,u). \tag{9.41}$$

As a consequence of Lemma 9.2, one thus finds for some pair C, δ with $\delta < 1$, and every $j = 0, 1, \dots$, the estimate

$$\left|\left(\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)^j U\right)(0)\right| + \left|\left(\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)^j U\right)(0)\right| \leq C\,\left[\frac{j}{2}\right]!\,(2\pi\,\delta)^{\frac{j}{2}}. \tag{9.42}$$

Since U extends as an antiholomorphic function of $(x, y) \in \mathbb{C}^2$, the two terms on the left-hand side express the derivatives of all orders, evaluated at 0, of the holomorphic functions of one variable $z \mapsto \overline{U}(z, iz)$ and $z \mapsto \overline{U}(z, -iz)$. Now, it is an elementary fact, based on the use of Cauchy's inequality

$$\frac{1}{j\,!}\,|f^{(j)}(0)| \leq \inf_{R>0}\,\left(R^{-j}\,\sup_{|z|=R}|f(z)|\right), \tag{9.43}$$

valid for any entire function of one variable, and on Stirling's estimate $j^{-\frac{j}{2}}\,e^{\frac{j}{2}} \sim \frac{1}{j\,!}\,[\frac{j}{2}]\,!\,2^{\frac{j}{2}}$, that the validity of the inequalities (9.42) is equivalent to that of (9.37).

All the arguments can be reversed, after we have proved that any antiholomorphic function U on $\mathbb{C}^2$, the restriction of which to $\mathbb{R}^2$ satisfies the differential equation $\Lambda U = 0$, is characterized by the pair of functions $z \mapsto U(z, iz)$ and $z \mapsto U(z, -iz)$. Now, if one knows the Taylor expansion at 0 of these two functions, one also knows, reverting to the notation $z = x + iy$ for x, y real, all derivatives $(\frac{\partial}{\partial z})^j\, U(0)$ or $(\frac{\partial}{\partial \bar{z}})^j\, U(0)$ relative to the restriction of U to $\mathbb{R}^2$. Finally, writing the differential equation as

$$\frac{2}{\pi}\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial \bar{z}}\,U = \left[z\,\frac{\partial}{\partial z} - \bar{z}\,\frac{\partial}{\partial \bar{z}} + \frac{\pi\,|z|^2}{2} \right] U, \tag{9.44}$$

one sees that, if $j \geq 0$, $k \geq 0$, one has

$$\frac{2}{\pi}\left(\left(\frac{\partial}{\partial z}\right)^{j+1}\left(\frac{\partial}{\partial \bar{z}}\right)^{k+1} U\right)(0) = \left(j - k + \frac{\pi}{2}\,jk\right)\left(\left(\frac{\partial}{\partial z}\right)^{j}\left(\frac{\partial}{\partial \bar{z}}\right)^{k} U\right)(0), \tag{9.45}$$

which makes it possible, by a descent process, to compute all mixed derivatives of U at 0 in terms of the marginal ones above. □

The following proposition will play an essential role at the end of the next section too. We wish to call the attention of the reader, especially the one interested in special function theory, to the pleasant phenomenon that occurs on the first line of (9.51) below: the integral to be computed appears as a sum of two integrals, neither of which could be expressed in such elementary terms (they would require the use of Struve's functions [17, p. 113]).

Proposition 9.7. *Let ϕ be the median state of the harmonic oscillator introduced in* (1.9). *For every $(y, \eta) \in \mathbb{R}^2$, one has*

$$(e^{2i\pi(\eta Q - yP)}\,\phi\,|\,\phi) = I_0\left(\frac{\pi}{2}(y^2 + \eta^2)\right) \tag{9.46}$$

where the pseudo-scalar product on the left-hand side is the one in $\mathfrak{A}$, introduced in Proposition 1.14.

Proof. We first recall that $\mathrm{Ana}(k_\theta)\phi = \phi$ for every $\theta \in \mathbb{R}$, and $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$: this was proved in the proof of Proposition 9.1, just before (9.8). Then, it suffices, as a consequence of (1.91), to prove (9.46) in the case when $y = 0$. Recall from (2.111) and (2.112) that the $\mathbb{C}^4$-realization of ϕ is the vector $\boldsymbol{f} = \begin{pmatrix} f_0 \\ 0 \\ f_0 \\ 0 \end{pmatrix}$ with

$$f_0(x) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\,x^{\frac{1}{2}}\,K_{\frac{1}{4}}(\pi\,x^2), \qquad x > 0. \tag{9.47}$$

We must apply (1.68), denoting as $\boldsymbol{h}$ the $\mathbb{C}^4$-realization of the function $e^{2i\pi\eta\,Q}\,\phi$: in our present case, not forgetting that ϕ is even,

$$\begin{aligned} h_0(x) &= f_0(x)\,\cos 2\pi\eta\,x, \\ h_{i,0}(x) &= f_0(x)\,\cosh 2\pi\eta\,x, \end{aligned} \tag{9.48}$$

and

$$\left(e^{2i\pi\eta Q}\,\phi\,|\,\phi\right) = 2^{\frac{1}{2}} \int_0^\infty (\cos 2\pi\eta x + \cosh 2\pi\eta x)\,(f_0(x))^2\,dx. \tag{9.49}$$

From the expression of f_0 recalled in (9.47) and [17, p. 98], one has

$$\begin{aligned}(f_0(x))^2 &= \frac{2^{\frac{3}{2}}}{\pi}\,x \int_0^\infty K_{\frac{1}{2}}(2\pi x^2 \sinh t)\, e^{-2\pi x^2 \cosh t}\,dt \\ &= \frac{2^{\frac{1}{2}}}{\pi} \int_0^\infty (\sinh t)^{-\frac{1}{2}}\, e^{-2\pi x^2 e^t}\,dt \end{aligned} \tag{9.50}$$

and

$$\begin{aligned}\left(e^{2i\pi\eta Q}\,\phi\,|\,\phi\right) &= \pi^{-1} \int_0^\infty (\sinh t)^{-\frac{1}{2}}\,dt \int_{-\infty}^\infty (e^{2i\pi\eta x} + e^{2\pi\eta x})\, e^{-2\pi x^2 e^t}\,dx \\ &= \frac{2^{-\frac{1}{2}}}{\pi} \int_0^\infty e^{-\frac{t}{2}} (\sinh t)^{-\frac{1}{2}} \left(e^{\frac{\pi}{2e^t}\eta^2} + e^{-\frac{\pi}{2e^t}\eta^2}\right) dt \\ &= \frac{2}{\pi} \int_0^1 \left(\cosh \frac{\pi\eta^2 s}{2}\right) \frac{ds}{\sqrt{1-s^2}} \\ &= I_0(\frac{\pi\eta^2}{2}) \end{aligned} \tag{9.51}$$

according to [17, p. 84]. □

Proposition 9.8. *With $x,\, y \in \mathbb{R}^2$, and $z = x + iy$, the anaplectic Bargmann–Fock transform $\Phi^j(x,y)$ of the j^{th}-eigenstate ϕ^j (in $\mathfrak{A}$) of the harmonic oscillator is given by the equations, valid for $j \geq 0$,*

$$\begin{aligned}\Phi^j(x,y) &= (-1)^j\, \pi^{\frac{j}{2}} \left(\frac{\bar{z}}{2}\right)^j \;\; I_{0,j}\left(\frac{\pi|z|^2}{2}\right), \\ \Phi^{-j}(x,y) &= \pi^{\frac{j}{2}} \left(\frac{z}{2}\right)^j \;\; I_{0,-j}\left(\frac{\pi|z|^2}{2}\right) \end{aligned} \tag{9.52}$$

with

$$I_{0,j}(t) = \left(\frac{d}{dt} - 1\right)^j I_0(t) \qquad \text{and} \qquad I_{0,-j}(t) = \left(\frac{d}{dt} + 1\right)^j I_0(t). \tag{9.53}$$

Proof. Proposition 9.7, together with (9.5), shows that the anaplectic Bargmann–Fock transform of the function ϕ itself is the function

$$\Phi(x,y) = I_0\left(\frac{\pi}{2}\,(x^2+y^2)\right). \tag{9.54}$$

Next, applying (9.40) with ϕ in place of u, one finds, for $j \geq 0$,

$$\begin{aligned}\Phi^j(x,y) &= (\tau_z\phi \,|\, \phi^j) = \overline{(\phi^j \,|\, \tau_z\phi)} = \overline{(\tau_{-z}\phi^j \,|\, \phi)} \\ &= \pi^{-\frac{j}{2}} \overline{\left(-\frac{\partial}{\partial \bar{z}} + \frac{\pi z}{2}\right)^j \Phi(x,y)} \\ &= (-1)^j\, \pi^{-\frac{j}{2}} \left(\frac{\partial}{\partial z} - \frac{\pi \bar{z}}{2}\right)^j \Phi(x,y) \end{aligned} \tag{9.55}$$

and, similarly,

$$\Phi^{-j}(x,y) = \pi^{-\frac{j}{2}} \left(\frac{\partial}{\partial \bar{z}} + \frac{\pi z}{2}\right)^j \Phi(x,y). \tag{9.56}$$

□

Remark 9.4. In relation to the characterization, given in Theorem 9.6, of the image of the anaplectic Bargmann–Fock transformation, note that $\Phi^j(z, iz)$ is identically zero if $j < 0$, and $\Phi^j(z, -iz)$ is zero if $j > 0$.

We now come to the question of defining the anaplectic analogue of the Wick symbol, or Wick "symbolic calculus" of operators. Our definition – which, as explained in the introduction of this chapter, can have at most limited value – is modelled after one way of introducing the notion of Wick symbol in the usual analysis, but such a comparison would be misleading if carried too far: for, in the usual case, there is coincidence between the notion of Wick symbol and, in the other direction, that of Wick ordering of an operator. It will turn out that this coincidence ceases to hold in the anaplectic case: this is why we shall call the species of symbol to be introduced now the *coherent state symbol* or CS-symbol.

Definition 9.9. Let T be a linear endomorphism of $\mathfrak{A}$. Recall that $\tau_z = e^{2i\pi(yQ-xP)}$ if $z = x+iy$. We define the CS-symbol of T as the function $\mathfrak{T}$ on $\mathbb{R}^2$ such that (abusing the notation $(z,\bar{z})$, in the physicists' way, to really mean (x,y))

$$\mathfrak{T}(z,\bar{z}) = (\tau_z\phi \,|\, T(\tau_z\phi)). \tag{9.57}$$

Proposition 9.10. *The CS-symbol is covariant under the Heisenberg transform,* i.e., *if* $\mathfrak{S}$ *is the CS-symbol of the operator* $S = \tau_{z'}\, T\, \tau_{z'^{-1}} = \tau_{z'}\, T\, \tau_{-z'}$, *one has*

$$\mathfrak{S}(z,\bar{z}) = \mathfrak{T}(z - z', \bar{z} - \bar{z}'). \tag{9.58}$$

Also, the CS-symbol is covariant under the restriction of the anaplectic representation to the subgroup $SO(2)$ *of* $SL_i(2,\mathbb{R})$, i.e., *setting* $\left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right).z = e^{i\theta}z$, *the CS-symbol of the operator* $\mathrm{Ana}(g)\, T\, \mathrm{Ana}(g^{-1})$ *is the function* $z \mapsto \mathfrak{T}(g^{-1}.z, \overline{g^{-1}.z})$ *whenever* $g \in SO(2)$.

Proof. Recalling (1.2), one has $\tau_{z'}\,\tau_z = \exp\left(i\pi\, \mathrm{Im}\ (z'\bar{z})\right)\tau_{z+z'}$ so that

$$\begin{aligned}\mathfrak{S}(z,\bar{z}) &= (\tau_z\phi\,|\,(\tau_{z'}\,T\,\tau_{-z'})(\tau_z\phi))\\ &= (\tau_{-z'}\,\tau_z\phi\,|\,T\,(\tau_{-z'}\,\tau_z\,\phi))\\ &= (\tau_{z-z'}\phi\,|\,T\,(\tau_{z-z'}\phi))\end{aligned} \tag{9.59}$$

since the two extra scalar factors of absolute value 1 on the two sides cancel off.

For the second part, we use the fact, shown on the occasion of the proof of Proposition 9.1, that ϕ is invariant under the transformations $\mathrm{Ana}(g)$, $g \in SO(2)$. Also, we use the equation (1.91) linking the Heisenberg and anaplectic representations, noting that when $g \in SO(2)$, the linear action of g on $\mathbb{R}^2$ that occurs in (1.91) and the action $z \mapsto g.z$ as defined in the proposition indeed correspond under the map $(x,y) \mapsto x+iy$. Thus,

$$\begin{aligned}(\tau_z\phi\,|\,\mathrm{Ana}(g)\,T\,\mathrm{Ana}(g^{-1})\,\tau_z\phi) &= (\mathrm{Ana}(g^{-1})\,\tau_z\phi\,|\,T\,\mathrm{Ana}(g^{-1})\,\tau_z\phi) \qquad (9.60)\\ &= (\tau_{g^{-1}.z}\,\mathrm{Ana}(g^{-1})\,\phi\,|\,T\,\tau_{g^{-1}.z}\,\mathrm{Ana}(g^{-1})\,\phi)\\ &= (\tau_{g^{-1}.z}\,\phi\,|\,T\,\tau_{g^{-1}.z}\,\phi)\\ &= \mathfrak{S}(g^{-1}.z,\,\overline{g^{-1}.z}).\end{aligned}$$

□

We now compute the CS-symbol of any operator in the algebra generated by the operators Q and P or, which amounts to the same, A and A^*. Given any polynomial in the formal indeterminates A and A^*, the associated Wick-ordered operator is that obtained by letting the *operator* A act before A^*, and the associated anti-Wick-ordered operator is that obtained by letting A^* act before A. For instance, given the polynomial $E(A^*,A) = A^{*j}\,A^k$, (where A^*, A are considered as formal indeterminates), its associated Wick-ordered operator is the operator $A^{*j}\,A^k$, and its associated anti-Wick-ordered operator is $A^k\,A^{*j}$.

Proposition 9.11. *The CS-symbol of the operator* $A^{*j}\,A^k$ *is*

$$(\tau_z\phi\,|\,A^{*j}\,A^k\,\tau_z\phi) = \sum_{\ell\geq 0}\binom{j}{l}\binom{k}{l}(-1)^\ell\,\frac{\Gamma(\frac{1}{2}+\ell)}{\Gamma(\frac{1}{2})}\,(\pi^{\frac{1}{2}}\bar{z})^{j-\ell}\,(\pi^{\frac{1}{2}}z)^{k-\ell} \tag{9.61}$$

and the CS-symbol of the operator $A^k\,A^{*j}$ *is*

$$(\tau_z\phi\,|\,A^k\,A^{*j}\,\tau_z\phi) = \sum_{\ell\geq 0}\binom{j}{l}\binom{k}{l}\,\frac{\Gamma(\frac{1}{2}+\ell)}{\Gamma(\frac{1}{2})}\,(\pi^{\frac{1}{2}}z)^{k-\ell}\,(\pi^{\frac{1}{2}}\bar{z})^{j-\ell}. \tag{9.62}$$

Proof. It is a straightforward consequence of the definition (2.123) of the operators A^* and A, and of the definition of τ_z that

$$A\,\tau_z = \tau_z\,A + \pi^{\frac{1}{2}}\,z\,\tau_z, \qquad A^*\,\tau_z = \tau_z\,A^* + \pi^{\frac{1}{2}}\,\bar{z}\,\tau_z. \tag{9.63}$$

By induction,

$$A^k\,\tau_z = \sum_{\ell=0}^{k} \binom{k}{\ell}\,(\pi^{\frac12} z)^{k-\ell}\,\tau_z\,A^\ell \qquad \text{and} \qquad A^{*j}\,\tau_z = \sum_{\ell=0}^{j} \binom{j}{\ell}\,(\pi^{\frac12}\bar z)^{j-\ell}\,\tau_z\,A^{*\ell}. \tag{9.64}$$

To conclude, it suffices, so as to prove (9.61), to write $(\tau_z\phi\,|\,A^{*j}\,A^k\,\tau_z\phi) = (A^j\,\tau_z\phi\,|\,A^k\,\tau_z\phi)$ and to use the first of the preceding equations, together with the fact that τ_z is a pseudo-unitary operator on $\mathfrak{A}$, finally to use the result of Proposition 2.11. The second equation is proved in the same way. □

Remarks 9.5: (i) It is essential to note that, despite the fact that the very definition (9.57) of the CS-symbol *looks like* the definition of the Wick calculus in the usual analysis, the CS-symbol treats the Wick and anti-Wick orderings on an absolutely equal footing, as can be seen from Proposition 9.11. An a priori explanation of this phenomenon lies in the existence of the basic symmetry $\mathcal{R}$ of $\mathfrak{A}$: as it follows from the pair of equations (2.123), the conjugation by $\mathcal{R}$ changes A into $i\,A^*$ and A^* into $i\,A$. This is in striking contrast with the usual analysis, in which the creation and annihilation operators bear no such relation.

For the sake of comparison, let us consider the case of the usual Wick calculus, in which the scalar product is that of $L^2(\mathbb{R})$ and the basic function χ is the normalized Gaussian function $\chi(t) = 2^{\frac14}\,e^{-\pi t^2}$. Exactly the same proof as above, together with the relation $\|A^{*j}\chi\|^2 = j\,!$ and the annihilation relation $A\,\chi = 0$, shows on one hand that

$$(\tau_z\chi\,|\,A^{*j}\,A^k\,\tau_z\chi)_{L^2(\mathbb{R})} = (\pi^{\frac12}\bar z)^j\,(\pi^{\frac12} z)^k, \tag{9.65}$$

on the other hand that

$$(\tau_z\chi\,|\,A^k\,A^{*j}\,\tau_z\chi)_{L^2(\mathbb{R})} = \sum_{\ell\geq 0} \binom{j}{l}\binom{k}{l}\,\ell\,!\,(\pi^{\frac12} z)^{k-\ell}\,(\pi^{\frac12}\bar z)^{j-\ell}\;: \tag{9.66}$$

of course there is in this setting a definite privilege for the Wick ordering of operators.

(ii) In view of the commutation relations

$$[A,\,A^{*j}] = j\,A^{*j-1} \qquad \text{and} \qquad [A^*,\,A^k] = -k\,A^{k-1}, \tag{9.67}$$

immediate by induction, one sees that if $\mathfrak{S}$ is the CS-symbol of $[A,\,T]$, where $\mathfrak{T}$ is the CS-symbol of T, one has $\mathfrak{S} = \pi^{-\frac12}\,\frac{\partial}{\partial\bar z}\,\mathfrak{T}$; similarly, the CS-symbol of $[A^*,\,T]$ is $\mathfrak{S} = -\pi^{-\frac12}\,\frac{\partial}{\partial z}\,\mathfrak{T}$.

(iii) If $S = A^{*j}A^k$, one has $\mathcal{R}\,S\,\mathcal{R}^{-1} = i^{j+k}\,A^j\,A^{*k}$ in view of the relations just recalled, so that, for every operator S with symbol $\mathfrak{S}$ in the algebra generated by A and A^*, the CS-symbol of $\mathcal{R}\,S\,\mathcal{R}^{-1}$ is the function $z \mapsto \mathfrak{S}(i\bar z, iz)$: observe that iz is not the conjugate of $i\bar z$ and that, in this expression, $\mathfrak{S}$ is implicitly extended as a holomorphic function of two *independent* variables. This relation is not a relation of covariance and, recalling that $\mathcal{R} = \mathrm{Ana}(\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right))$, the reason why the proof of Proposition 9.10 does not apply to this case is that $\mathcal{R}$ does not preserve the scalar product on $\mathfrak{A}$: instead, one always has $(\mathcal{R}v \,|\, \mathcal{R}u) = (\mathcal{R}^2 v \,|\, u)$, where $\mathcal{R}^2$ is the parity transform.

The properties (ii) and (iii) can be generalized to the case when T is an arbitrary linear endomorphism of $\mathfrak{A}$:

Proposition 9.12. *Let $\mathfrak{T}$ be the CS-symbol of an endomorphism T of $\mathfrak{A}$. The CS-symbol of the commutator $[A, T]$ is $\pi^{-\frac{1}{2}} \frac{\partial}{\partial \bar{z}} \mathfrak{T}$, and the CS-symbol of the commutator $[A^*, T]$ is $-\pi^{-\frac{1}{2}} \frac{\partial}{\partial z} \mathfrak{T}$. The CS-symbol of the operator $\mathcal{R}\, T\, \mathcal{R}^{-1}$ is the function $z \mapsto \mathfrak{T}(i\bar{z}, iz)$, after $\mathfrak{T}$ has been extended as a holomorphic function of two variables.*

Proof. Starting again from the relation $\tau_z = e^{2i\pi\,(yQ - xP)}$, one gets

$$[Q, T] = \frac{1}{2i\pi} \left.\frac{d}{dy'}\right|_{y'=0} \left(e^{2i\pi y' Q}\, T\, e^{-2i\pi y' Q}\right) \tag{9.68}$$

so that the CS-symbol of $[Q, T]$ is the function

$$\frac{1}{2i\pi} \left.\frac{d}{dy'}\right|_{y'=0} \mathfrak{T}(z - iy', \bar{z} + iy') = \frac{1}{2\pi}\left(-\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) \mathfrak{T}(z, \bar{z}); \tag{9.69}$$

in a similar way, the CS-symbol of $[P, T]$ is the function $\frac{1}{2i\pi}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) \mathfrak{T}$. This proves the first part.

For the second one, we are not entitled to the notation τ_z any longer, since x and y will take complex values, and we revert to the notation $\pi(x, y)$. It is a straightforward task to check the relations

$$\mathcal{R}\, \pi(x, \xi)\, \mathcal{R}^{-1} = \pi(-ix, iy), \qquad \mathcal{R}^{-1}\, \pi(x, \xi)\, \mathcal{R} = \pi(ix, -iy), \tag{9.70}$$

so that (remembering that $\mathcal{R}$ is not unitary, but self-adjoint) the symbol $\mathfrak{S}$ of $\mathcal{R}\, T\, \mathcal{R}^{-1}$ is given, for $z = x + iy$ with x, y real, as

$$\begin{aligned} \mathfrak{S}(x, y) &= (\pi(x, y)\, \phi \,|\, \mathcal{R}\, T\, \mathcal{R}^{-1}\, \pi(x, y)\, \phi) \\ &= (\mathcal{R}\, \pi(x, y)\, \phi \,|\, T\, \mathcal{R}^{-1}\, \pi(x, y)\, \phi) \\ &= (\pi(-ix, iy)\, \mathcal{R}\, \phi \,|\, T\, \pi(ix, -iy)\, \mathcal{R}^{-1}\, \phi) \\ &= (\pi(-ix, iy)\, \phi \,|\, T\, \pi(ix, -iy)\, \phi) \end{aligned} \tag{9.71}$$

since ϕ is invariant under $\mathcal{R}$. Such a function is the value on the pair $(ix, -iy)$ of a holomorphic function of two variables: it amounts to the same to say that it is the value on the pair $(x - iy, x + iy)$ of a holomorphic function of two variables. □

10 Towards the one-dimensional anaplectic Weyl calculus

The anaplectic analysis (not only representation) is quite different from the usual analysis in many aspects. The most immediate deep difference shows itself in the existence of invertible operators in the Lie algebra of the complexified Heisenberg representation. Besides being of possible independent interest, the following fact will be technically useful in the proof of Theorem 10.8.

Theorem 10.1. *For every* $\alpha \in \mathbb{C}$ *such that* $\mathrm{Re}\ \alpha \neq 0$*, the operator* $D_\alpha = P - \alpha\, Q$ *is an automorphism of the space* $\mathfrak{A}$*.*

Proof. Under the automorphism $\mathcal{R}$ of $\mathfrak{A}$ introduced in Proposition 1.13, the operator D_α transfers to $i\, D_{-\alpha}$: as a consequence, we may assume that $\mathrm{Im}\ \alpha > 0$. Next, with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\alpha = \frac{di+c}{bi+a}$, it follows from (2.120) that

$$P - \alpha Q = (a + bi)^{-1}\, \mathrm{Ana}(g)\, (P - iQ)\, \mathrm{Ana}(g^{-1}) : \tag{10.1}$$

it is thus no loss of generality to assume that $\alpha = i$, which we do from now on.

Writing the equation $D_i\, v = u$ to be solved as $Av = \pi^{\frac{1}{2}}\, u$, we note, as it follows from Theorem 2.11, that this implies $(\phi^k \,|\, v) = \pi^{\frac{1}{2}}\, (\phi^{k-1} \,|\, u)$ if $k \geq 1$, and (using also $A^*A = L + \frac{1}{2}$) $(k + \frac{1}{2})\, (\phi^k \,|\, v) = \pi^{\frac{1}{2}}\, (\phi^{k-1} \,|\, u)$ if $k \leq 0$. The estimates which, according to Proposition 9.5, permit us to recognize a sequence of numbers as being the sequence of scalar products of a function in $\mathfrak{A}$ against the ϕ_j 's make it possible to conclude. □

In the present section, we concern ourselves with the problem of building a good (Weyl style) symbolic calculus of operators in the anaplectic environment. It is a good idea to start with one version of the defining formula of the ordinary Weyl calculus:

$$\mathrm{Op}(\mathfrak{S}) = \int_{\mathbb{R}^2} (\mathcal{F}\, \mathfrak{S})(y, \eta)\, \exp\left(2i\pi\, (\eta Q - yP)\right) dy\, d\eta \tag{10.2}$$

and to analyse its ingredients. In the usual case, this formula expresses the operator $\mathrm{Op}(\mathfrak{S})$ – say, acting on Schwartz's space $\mathcal{S}(\mathbb{R})$) – with *symbol* $\mathfrak{S}$ as an integral superposition of operators taken from the Heisenberg representation. This rule makes the correspondence Op a *covariant* one under two distinct representations. First, under the Heisenberg representation: the symbol of the operator $e^{2i\pi\, (\eta Q - yP)}\, \mathrm{Op}(\mathfrak{S})\, e^{-2i\pi\, (\eta Q - yP)}$ is the function (or distribution) $(x, \xi) \mapsto \mathfrak{S}(x - y,\, \xi - \eta)$. Next, recalling that Met is the metaplectic representation (revisited in the beginning of Section 6), one has, for every $g \in \widetilde{SL}(2, \mathbb{R})$, the relation

$$\mathrm{Met}(g)\, \mathrm{Op}(\mathfrak{S})\, \mathrm{Met}(g^{-1}) = \mathrm{Op}(\mathfrak{S} \circ g^{-1}). \tag{10.3}$$

The second property holds provided one defines $\mathcal{F}$ as the *symplectic* Fourier transformation (the one that commutes with all linear transformations of $\mathbb{R}^2$ in $SL(2,\mathbb{R})$, not only the orthogonal ones):

$$(\mathcal{F}\,\mathfrak{S})(x,\xi) = \int_{\mathbb{R}^2} \mathfrak{S}(y,\eta)\, e^{2i\pi\,(x\eta - y\xi)}\, dy\, d\eta. \tag{10.4}$$

In the anaplectic case, it is important to realize that this definition can only work and be of interest if one goes anaplectic all the way: let us analyse the meaning of the ingredients of the formula (10.2) in the anaplectic analysis. We are certainly familiar with the Heisenberg representation, which is formally the same as in the usual analysis: it satisfies the Heisenberg relation (1.2) as well as the covariance relation (1.91), which is a good starting point. Now, the symbol $\mathfrak{S}$ should lie in $\mathfrak{A}^{(2)}$ or, to have a very general class of operators, in the dual of this latter space, and the Fourier transformation should of course be the anaplectic one. But this does not exhaust the list of ingredients of the formula (10.2). The most hidden one, in some sense the one most difficult to deal with in a satisfactory way, is concerned with the pointwise product of functions, which seems to occur on the right-hand side when you apply the operator under consideration to a function χ and evaluate the result at a given point. Now, the pointwise product is not a good operation in the anaplectic analysis, and one should interpret instead the formula above in a weak sense, *i.e.,* one should aim at defining the scalar product $(\psi\,|\,\mathrm{Op}(\mathfrak{S})\,\chi)_{\mathfrak{A}}$: recall from Proposition 1.14 that the scalar product, even though not a pre-Hilbert one, is still non-degenerate. The answer, again taken from the usual Weyl calculus, consists in writing this scalar product as the result $\langle\,\mathfrak{S},\, W(\psi,\,\chi)\,\rangle$ of testing the linear form $\mathfrak{S}$ against some function $W(\psi,\,\chi)$ of two variables, called the *Wigner function* of ψ and χ.

In the case of the usual analysis, the Wigner function $W(\psi,\chi)$ of a pair (χ,ψ) of functions on the line, say in Schwartz's space $\mathcal{S}(\mathbb{R})$, is the function on $\mathbb{R}^2$ defined by

$$W(\psi,\chi)(x,\xi) = 2\int_{-\infty}^{\infty} \bar{\psi}(x+t)\,\chi(x-t)\,e^{4i\pi t\xi}\,dt. \tag{10.5}$$

It enjoys a double status: first, it is the Weyl symbol of the rank-one operator $u \mapsto (\psi|u)\,\chi$; next, for every symbol $\mathfrak{S}\in\mathcal{S}'(\mathbb{R}^2)$, one has the identity

$$(\psi\,|\,\mathrm{Op}(\mathfrak{S})\chi) = \int_{\mathbb{R}^2} \mathfrak{S}(x,\xi)\, W(\psi,\chi)(x,\xi)\, dx\, d\xi, \tag{10.6}$$

which is exactly what we had in mind. By transposition, it is immediate that the two covariance formulas mentioned above have versions which express instead the covariance of the Wigner sesquilinear machine itself: we shall not display these here, but their anaplectic analogues, to be displayed and proved in Proposition 10.4, are fully similar.

From a certain point on, it will be necessary to consider functions lying in a specific linear subspace of $\mathfrak{A}$: note that we have already come across this type of function in Theorem 6.5.

Definition 10.2. A function $u \in \mathfrak{A}$ will be said to lie in $\mathfrak{A}_0$ if it satisfies the following condition, to be compared to the one in Proposition 9.5: for any $M > 0$, there exists some constant $C > 0$ such that

$$|(\phi^j \,|\, u)| \leq C \left[\frac{|j|}{2}\right]! \, M^{-|j|}, \qquad j \in \mathbb{Z}: \tag{10.7}$$

it amounts to the same to say that u admits a series expansion $u = \sum_j c_j \, \phi^j$, where the coefficients satisfy, for M arbitrarily large, some estimate $|c_j| \leq C\,([\frac{|j|}{2}]\,!)^{-1} \, M^{-|j|}$. In an equivalent way, if $\Theta^{-1} u = \left(\begin{smallmatrix} v_0 \\ v_1 \end{smallmatrix}\right)$ and if $w_0 = \mathcal{F} v_0$, $w_1 = \mathcal{F} v_1$, the functions $\theta \mapsto |\sin \frac{\theta}{2}|^{-\frac{1}{2}} \, w_0(\text{cotan}\, \frac{\theta}{2})$ and $\theta \mapsto \langle \sin \frac{\theta}{2} \rangle^{-\frac{3}{2}} \, w_1(\text{cotan}\, \frac{\theta}{2})$ extend as holomorphic functions on the punctured complex plane.

We generalize an alternative expression of (10.5) to the anaplectic setting, and shall show that the Wigner function of any two functions in $\mathfrak{A}_0$ is a function (of two variables) in the space $\mathfrak{A}^{(2)}$. When this is done, we can claim – we shall not return to this point – that we have defined, in a weak sense, the operator $\text{Op}(\mathfrak{S})$ with symbol $\mathfrak{S}$, as a linear operator from $\mathfrak{A}_0$ to the complex anti-dual of that space, whenever $\mathfrak{S}$ is linear form on $\mathfrak{A}^{(2)}$. The covariance of the calculus will follow provided that we define the linear action of $SL(2, \mathbb{R})$ on the dual space of $\mathfrak{A}^{(2)}$ by duality.

Definition 10.3. Given χ and $\psi \in \mathfrak{A}$, the anaplectic Wigner function $W(\psi, \chi)$ is the function on $\mathbb{R}^2$ defined as

$$W(\psi, \chi)(x, \xi) = 2\,(e^{4i\pi(-x\,P + \xi\,Q)} \, \check{\psi} \,|\, \chi)_{\mathfrak{A}}, \tag{10.8}$$

where the pseudo-scalar product on the right-hand side is the one introduced in Proposition 1.14.

The first thing to note is that, in the usual analysis, this definition – with the usual scalar product in place of the one in $\mathfrak{A}$ – would agree with the above-given ones: the verification is an easy matter, in view of the expression (1.1) of the operator $e^{2i\pi(\eta\,Q - y\,P)}$.

From now on, we drop the subscript $\mathfrak{A}$ to denote the pseudo-scalar product in that space. The covariance properties of the anaplectic Wigner function are expressed as follows:

Proposition 10.4. *Let $(y, \eta) \in \mathbb{R}^2$ and $g \in SL(2, \mathbb{R})$ be given. For any pair (ψ, χ) of functions in $\mathfrak{A}$, one has*

$$\begin{aligned} W(e^{2i\pi\,(\eta Q - yP)}\,\psi,\, e^{2i\pi\,(\eta Q - yP)}\,\chi)(x, \xi) &= W(\psi, \chi)(x - y, \xi - \eta), \\ W(\text{Ana}(g)\,\psi,\, \text{Ana}(g)\,\chi) &= W(\psi, \chi) \circ g^{-1}. \end{aligned} \tag{10.9}$$

Proof. From Definition 10.3, one has

$$W(e^{2i\pi\,(\eta Q-yP)}\,\psi,\,e^{2i\pi\,(\eta Q-yP)}\,\chi)(x,\xi)\\ = 2\,(e^{4i\pi(-x\,P+\xi\,Q)}\,e^{2i\pi(y\,P-\eta\,Q)}\,\check{\psi}\,|\,e^{2i\pi(y\,P-\eta\,Q)}\,\chi), \quad (10.10)$$

so that the first equation appears as a consequence of (1.2). On the other hand,

$$W(\mathrm{Ana}(g)\,\psi,\,\mathrm{Ana}(g)\,\chi)(x,\xi) = 2\,(e^{4i\pi(-x\,P+\xi\,Q)}\,\mathrm{Ana}(g)\,\check{\psi}\,|\,\mathrm{Ana}(g)\,\chi)\\ = 2\,(\mathrm{Ana}(g^{-1})\,e^{4i\pi(-x\,P+\xi\,Q)}\,\mathrm{Ana}(g)\,\check{\psi}\,|\,\chi): \quad (10.11)$$

according to Theorem 1.20, this is the same as $2\,(e^{4i\pi(-x'\,P+\xi'\,Q)}\,\check{\psi}\,|\,\chi)$ provided that $\binom{x'}{\xi'} = g^{-1}\,\binom{x}{\xi}$. □

We shall now show that the Wigner function of any two functions in $\mathfrak{A}_0$ lies in $\mathfrak{A}^{(2)}$. This requires a number of lemmas.

Lemma 10.5. *With $Q = (x)$ and $P = \frac{1}{2i\pi}\frac{d}{dx}$ one has the identities, valid for arbitrary pairs $(\psi,\chi) \in \mathfrak{A}$:*

$$W(\psi, Q\chi) = \left(x - \frac{1}{4i\pi}\frac{\partial}{\partial\xi}\right) W(\psi,\chi), \qquad W(\psi, P\chi) = \left(\xi + \frac{1}{4i\pi}\frac{\partial}{\partial x}\right) W(\psi,\chi),$$
$$W(Q\psi, \chi) = \left(x + \frac{1}{4i\pi}\frac{\partial}{\partial\xi}\right) W(\psi,\chi), \qquad W(P\psi, \chi) = \left(\xi - \frac{1}{4i\pi}\frac{\partial}{\partial x}\right) W(\psi,\chi). \quad (10.12)$$

Proof. One has the identities

$$W(\psi,\,e^{2i\pi\,(-yP+\eta Q)}\,\chi)(x,\xi) = e^{2i\pi\,(x\eta-y\xi)}\,W(\psi,\chi)\left(x-\frac{y}{2},\xi-\frac{\eta}{2}\right), \quad (10.13)$$

a consequence of Definition 10.3 and (1.2). Then,

$$W(\psi,Q\chi)(x,\xi) = \frac{1}{2i\pi}\,\frac{\partial}{\partial\eta}\Big|_{\eta=0}\,W(\psi, e^{2i\pi\eta Q}\,\chi)(x,\xi)\\ = \frac{1}{2i\pi}\,\frac{\partial}{\partial\eta}\Big|_{\eta=0}\left(e^{2i\pi x\eta}\,W(\psi,\chi)\left(x,\xi-\frac{\eta}{2}\right)\right), \quad (10.14)$$

which yields the first desired result, and the second identity on the first line is obtained in exactly the same way. The expression of the difference $W((\eta\,Q - y\,P)\psi,\,\chi) - W(\psi,\,(\eta\,Q - y\,P)\chi)$ can be thought of as an infinitesimal version of the first covariance relation stated in Proposition 10.4. □

Lemma 10.5, together with Theorem 10.1 (which shows in particular that any odd function in $\mathfrak{A}$ is the image of some even function in that space under an operator such as A) makes it possible to reduce the proof that the Wigner function $W(\psi,\,\chi)$ of any two functions in $\mathfrak{A}_0$ lies in $\mathfrak{A}^{(2)}$ to the case when χ and ψ are even. Then, the Wigner function is even too: when applying Definition 4.12 of $\mathfrak{A}^{(2)}$ towards proving that it lies in that space, we may then dispense with the study of the vector-valued $\mathcal{K}$-transform of $W(\psi,\,\chi)$ and concentrate on the study of $(\mathcal{K}\,W(\psi,\,\chi))_0$.

Theorem 10.6. *Let $\phi(x)=(\pi|x|)^{\frac{1}{2}}\,I_{-\frac{1}{4}}(\pi\,x^2)$ be the median state of the harmonic oscillator, as introduced in Proposition* 1.2. *One has*

$$W(\phi,\phi)(x,\xi)=2\,I_0(2\pi\,(x^2+\xi^2)). \tag{10.15}$$

Proof. This follows from Proposition 9.7. It should be compared to the equation $W(\chi,\chi)(x,\xi)=2\,\exp\left(-2\pi\,(x^2+\xi^2)\right)$ which, in the usual analysis, gives the Weyl symbol of the operator of projection, in $L^2(\mathbb{R})$, on the space generated by the normalized ground state $\chi(x)=2^{\frac{1}{4}}\,e^{-\pi x^2}$ of the usual harmonic oscillator. □

Remark 10.1. Just like its analogue in the classical theory (to wit, the function $2\,e^{-2\pi(x^2+\xi^2)}$), the function $W(\phi,\,\phi)$ is normalized in the sense corresponding to the pseudoscalar product introduced, in higher dimension – here 2 – in Section 8. Indeed, from Proposition 9.7, one has $(W(\phi,\,\phi)\,|\,W(\phi,\,\phi))\;=\;2\,(\Phi\,|\,\Phi)$ with $\Phi(x,\xi)\;=\;I_0(\pi(x^2\;+\;\xi^2))$. Now, going back to Theorem 8.7, one writes $\Phi\;=\;g_{0,0}^{\mathrm{ram}}\boxtimes\mathcal{Y}^{0,0}$ with $\mathcal{Y}^{0,0}=(2\pi)^{-\frac{1}{2}}$ (the normalized constant on the unit circle), so that $g_{0,0}^{\mathrm{ram}}(s)=(2\pi)^{\frac{1}{2}}\,I_0(2\pi s)$. From (3.14), one then finds that $g_{0,0}=-(\frac{\pi}{2})^{\frac{1}{2}}\,v_{0,0}^0$ so that, as shown in Corollary 8.4, $(g_{0,0}\,|\,g_{0,0})\;=\;\frac{\pi^2}{2}$. One concludes with the help of (8.47). This remark may have possible significance in the future, when the anaplectic pseudodifferential analysis is more fully developed: indeed, it would be nice if, just as in the usual case, the pseudoscalar product on the space of symbols would just correspond to the polarized form of some indefinite version of a Hilbert–Schmidt squared norm. At present, this is true if one stays entirely within the realm of anaplectic Hermite functions.

Theorem 10.7. *Define, on $\mathbb{R}^2$, the pair of differential operators*

$$\Lambda=2\pi\,(x^2+\xi^2)-\frac{1}{8\pi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\xi^2}\right),\qquad \Omega=\frac{1}{i}\left(x\,\frac{\partial}{\partial\xi}-\xi\,\frac{\partial}{\partial x}\right), \tag{10.16}$$

the first one of which is simply the conjugate, under the change of variables $(x,\xi)\mapsto(2^{\frac{1}{2}}x,2^{\frac{1}{2}}\xi)$, of the harmonic oscillator in two variables $L^{(2)}$ introduced in (3.21). *For every $j\in\mathbb{Z}$, let ϕ^j be the eigenfunction in $\mathfrak{A}$, as introduced in Proposition* 2.11, *of the one-dimensional harmonic oscillator corresponding to the eigenvalue j. Then, given any pair of integers j,k, the Wigner function $W(\phi^j,\phi^k)$ is the analytic function on $\mathbb{R}^2$ characterized, up to normalization, by the eigenvalue equations*

$$\Lambda\,W(\phi^j,\phi^k)=(j+k)\,W(\phi^j,\phi^k),\qquad \Omega\,W(\phi^j,\phi^k)=(j-k)\,W(\phi^j,\phi^k). \tag{10.17}$$

Proof. Introduce on $\mathbb{R}^2$ the complex structure for which $\zeta = x + i\xi$ is a complex coordinate and set, in the usual way,

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial \xi}\right), \qquad \frac{\partial}{\partial \bar\zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial \xi}\right). \tag{10.18}$$

Introducing the operators

$$\begin{aligned} \mathrm{Right}^\dagger &= \pi^{\frac12}\left(\bar\zeta - \frac{1}{2\pi}\,\frac{\partial}{\partial\zeta}\right), \qquad & \mathrm{Right} &= \pi^{\frac12}\left(\zeta + \frac{1}{2\pi}\,\frac{\partial}{\partial\bar\zeta}\right), \\ \mathrm{Left}^\dagger &= \pi^{\frac12}\left(\zeta - \frac{1}{2\pi}\,\frac{\partial}{\partial\bar\zeta}\right), \qquad & \mathrm{Left} &= \pi^{\frac12}\left(\bar\zeta + \frac{1}{2\pi}\,\frac{\partial}{\partial\zeta}\right), \end{aligned} \tag{10.19}$$

and recalling that

$$A^* = \pi^{\frac12}\,(Q - i\,P), \qquad A = \pi^{\frac12}\,(Q + i\,P), \tag{10.20}$$

one derives from Lemma 10.5 the equations

$$\begin{aligned} W(\psi, A^*\chi) &= \mathrm{Right}^\dagger\, W(\psi,\chi), \qquad & W(\psi, A\chi) &= \mathrm{Right}\, W(\psi,\chi), \\ W(A^*\psi, \chi) &= \mathrm{Left}^\dagger\, W(\psi,\chi), \qquad & W(A\psi, \chi) &= \mathrm{Left}\, W(\psi,\chi). \end{aligned} \tag{10.21}$$

On the other hand, noting that

$$\Lambda = 2\pi\,\zeta\,\bar\zeta - \frac{1}{2\pi}\,\frac{\partial^2}{\partial\zeta\,\partial\bar\zeta}, \tag{10.22}$$

one has the commutation relations

$$\begin{aligned} [\Lambda, \mathrm{Right}^\dagger] &= \mathrm{Right}^\dagger, \qquad & [\Lambda, \mathrm{Left}^\dagger] &= \mathrm{Left}^\dagger, \\ [\Lambda, \mathrm{Right}] &= -\mathrm{Right}, \qquad & [\Lambda, \mathrm{Left}] &= -\mathrm{Left}\, : \end{aligned} \tag{10.23}$$

the first of the two equations (10.17) follows after one has verified it directly, in the case when $j = k = 0$, from the result of Proposition 9.7. The second one is obtained from the expression $\Omega = \zeta\,\frac{\partial}{\partial\zeta} - \bar\zeta\,\frac{\partial}{\partial\bar\zeta}$, from which the commutation relations

$$\begin{aligned} [\Omega, \mathrm{Right}^\dagger] &= -\mathrm{Right}^\dagger, \qquad & [\Omega, \mathrm{Left}^\dagger] &= \mathrm{Left}^\dagger, \\ [\Omega, \mathrm{Right}] &= \mathrm{Right}, \qquad & [\Omega, \mathrm{Left}] &= -\mathrm{Left} \end{aligned} \tag{10.24}$$

follow. That the pair of eigenvalue equations as given has only a one-dimensional space of analytic solutions is an easy matter too. □

Theorem 10.8. *Let χ and $\psi \in \mathfrak{A}_0$, and let $W(\psi,\,\chi)$ be the Wigner function of the pair $\psi,\,\chi$ as introduced in Definition* 10.3*. The function $W(\psi,\,\chi)$ lies in the space $\mathfrak{A}^{(2)}$. More precisely, the $\mathcal{K}$-transform $((\mathcal{K}\,W(\psi,\,\chi))_0,\,(\mathcal{K}\,W(\psi,\,\chi))_1)$ of $W(\psi,\,\chi)$, as introduced in Theorem* 4.11*, extends as an analytic function on the open set of matrices $Z^\# \in \Sigma^{(2)}$ lying above $\Sigma\backslash\Delta$, where we recall that Δ is the set of matrices in $\Sigma^{(2)}$ with a double eigenvalue* (*these are scalar matrices since we are concerned here with the two-dimensional case*)*.*

Proof. First recall from the observation following the proof of Lemma 10.5 that we may assume that ψ and χ are even functions, and we have to study the analytic continuation of the function $(\mathcal{K}\, W(\psi,\, \chi))_0$. Using Definition 10.2, we write

$$\psi = \sum_{j\in\mathbb{Z}} b_j\, \phi^{2j}, \qquad \chi = \sum_{k\in\mathbb{Z}} c_k\, \phi^{2k}, \tag{10.25}$$

with the ϕ^j 's as introduced in Theorem 2.11, and the sequences (b_k) and (c_k) satisfying for every $M > 0$ the estimate $|b_k| + |c_k| \leq C\,(|k|\,!)^{-1}\, M^{-|k|}$. The problem is to prove the convergence towards an analytic function, in the part of $\Sigma^{(2)}$ above $\Sigma\backslash\Delta$, of four series, one of which (the other three ones are quite similar) is $\sum_{j\geq 0,\, k\geq 0} b_j\, c_k\ \mathrm{Left}^{2j}\ \mathrm{Right}^{2k}\ W(\psi,\chi)$. The rest of the proof is similar to that of Theorem 5.5: we show that an operator such as

$$\mathrm{Left}^2 = \pi\, \left[\bar\zeta^2 + \frac{1}{\pi}\,\bar\zeta\,\frac{\partial}{\partial\zeta} + \frac{1}{4\pi^2}\,\frac{\partial^2}{\partial\zeta^2}\right] \tag{10.26}$$

transfers on the $\mathcal{K}$-transform of a function on $\mathbb{R}^2$ as a first-order differential operator with coefficients analytic throughout $\Sigma^{(2)}$. Concerning the first term on the right-hand side of (10.26), this has already been proved in (5.39): the same goes for the last term $\frac{\partial^2}{\partial\zeta^2}$, which is the conjugate of a term such as the first one under a (two-dimensional) anaplectic Fourier transformation, this latter operation corresponding (*cf.* (5.46)) to a global analytic diffeomorphism of $\Sigma^{(2)}$. Next,

$$2\,\bar\zeta\,\frac{\partial}{\partial\zeta} = x\,\frac{\partial}{\partial x} + \xi\,\frac{\partial}{\partial\xi} + i\left(x\,\frac{\partial}{\partial\xi} - \xi\,\frac{\partial}{\partial x}\right): \tag{10.27}$$

the sum of the last two terms is an infinitesimal generator of the (analytic) action of the maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$. Finally, $x\,\frac{\partial}{\partial x} + \xi\,\frac{\partial}{\partial\xi}$ transfers on the $\mathcal{Q}$-transform level to the operator $-2\sum_{j,k}\sigma_{jk}\,\frac{\partial}{\partial\sigma_{(jk)}} - 2$, an operator already considered in the proof of Proposition 5.3, which would have been written there as $-\sum L_{jj}$ (*cf.* (5.28). This concludes the proof of Theorem 10.8. □

There is another approach to Theorem 10.8, based on a computation of the $\mathcal{Q}$-transform of the Wigner function of a general pair $(\psi,\, \chi)$, and on an attempt (which proved too complicated) at finding the analytic continuation of the integral obtained as a result. We reproduce part of the argument, since it gives new significance to a known identity involving hypergeometric functions, or Legendre functions.

Lemma 10.9. *Let $v \in \mathfrak{A}$, and assume that v satisfies, for some pair $(C,\, \varepsilon)$ of positive constants, the estimate*

$$|v(x\, e^{-\frac{i\pi}{4}})| \leq C\, e^{-\pi\varepsilon\, x^2}, x\in\mathbb{R}. \tag{10.28}$$

Then one has the identity

$$\mathrm{Int}\,[v] = \int_{-\infty}^{\infty} v(x\,e^{-\frac{i\pi}{4}})\,dx. \tag{10.29}$$

Proof. Let $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ be the $\mathbb{C}^4$-realization of v: as shown in (1.33), one has for large σ the equation

$$\int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, v(x\,e^{-\frac{i\pi}{4}})\,dx = 2^{\frac{1}{2}} \int_0^{\infty} [e^{-i\pi\sigma x^2}\, f_0(x) + e^{i\pi\sigma x^2}\, f_{i,0}(x)]\,dx : \tag{10.30}$$

now both sides of the equation extend as holomorphic functions of σ in some half-strip $\{\sigma \in \mathbb{C}\colon \mathrm{Re}\;\sigma > -\varepsilon,\ |\mathrm{Im}\;\sigma| < \varepsilon\}$. In particular, under the current assumptions, the identity just recalled is still valid for $\sigma = 0$: this proves the lemma, in view of the definition (1.74) of the linear form Int. □

The following lemma expresses a new connection between the anaplectic representation and the Heisenberg representation.

Lemma 10.10. *Let $u \in \mathfrak{A}$, satisfying for some pair of constants C, R the estimate $|u(z)| \leq C\,e^{\pi R|z|^2}$. Let a, c be a pair of real numbers $\geq R+1$. Then the function u_1 defined as the (ordinary) integral superposition*

$$u_1 = \int_{\mathbb{R}^2} e^{-2\pi\,(a\,x^2 + c\,\xi^2)}\, e^{4\pi\,e^{\frac{i\pi}{4}}\,(-x\,P + \xi\,Q)}\, u\;\; dx\,d\xi \tag{10.31}$$

lies in $\mathfrak{A}$ and can be made explicit as

$$u_1 = \frac{1}{2}\,(ac+1)^{-\frac{1}{2}}\;\mathrm{Ana}(g_1)u \qquad \text{with} \qquad g_1 = \begin{pmatrix} \frac{ac-1}{ac+1} & -\frac{2c}{ac+1} \\ \frac{2a}{ac+1} & \frac{ac-1}{ac+1} \end{pmatrix}. \tag{10.32}$$

Proof. We abbreviate $e^{\frac{i\pi}{4}}$ as κ and make u_1 explicit as

$$u_1(t) = \int_{\mathbb{R}^2} e^{-2\pi\,(a\,x^2 + c\,\xi^2)}\, u(t - 2\,\kappa^{-1}\,x)\, e^{4\pi\,\kappa\,(t - \kappa^{-1}\,x)\,\xi}\; dx\,d\xi. \tag{10.33}$$

Now

$$\int_{-\infty}^{\infty} e^{-2\pi c\xi^2}\, e^{-4\pi\,x\xi}\, e^{4\pi\,\kappa\,t\xi}\, d\xi = (2c)^{-\frac{1}{2}}\, e^{\frac{2\pi}{c}\,(x - \kappa\,t)^2}, \tag{10.34}$$

so that

$$u_1(t) = (2c)^{-\frac{1}{2}}\, e^{\frac{2i\pi}{c}\,t^2} \int_{-\infty}^{\infty} e^{-2\pi a x^2}\, e^{\frac{2\pi\,x^2}{c}}\, e^{-\frac{4\pi\,\kappa\,xt}{c}}\, u(t - 2\,\kappa^{-1}\,x)\,dx, \tag{10.35}$$

a convergent integral since $a - c^{-1} > R$. The same inequality makes it possible to use the deformation of contour associated with the translation $x \mapsto x + \frac{\kappa t}{2}$, followed by the change of variable $y = -2x$, ending up with the equation

$$u_1(t) \;=\; \frac{1}{(8c)^{\frac{1}{2}}}\, e^{\frac{2i\pi\,t^2}{c}} \int_{-\infty}^{\infty} e^{-\frac{\pi}{2}\,(a - c^{-1})\,(\kappa t - y)^2}\, e^{-\frac{2\pi\kappa}{c}\,(\kappa t - y)t}\;\, u(\kappa^{-1}y)\,dy \tag{10.36}$$

or

$$u_1(t) = \frac{1}{(8c)^{\frac{1}{2}}}\, e^{-i\pi\,\frac{ac-1}{2c}\,t^2} \int_{-\infty}^{\infty} e^{\frac{\pi\kappa}{c}\,(ac+1)\,ty}\, e^{-\pi\,\frac{ac-1}{2c}\,y^2}\, u(\kappa^{-1}y)\,dy. \tag{10.37}$$

Setting

$$v(y) = u(y)\, e^{-i\pi\,\frac{ac-1}{2c}\,y^2}\, e^{i\pi\,\frac{ac+1}{c}\,ty}, \tag{10.38}$$

the function v lies in $\mathfrak{A}$ according to Proposition 1.15, and it also satisfies the condition that makes the application of Lemma 10.9 possible: hence

$$u_1(t) = \frac{1}{(8c)^{\frac{1}{2}}}\, e^{-i\pi\,\frac{ac-1}{2c}\,t^2}\, \mathrm{Int}\,[v]. \tag{10.39}$$

Using Definition 1.17 of the anaplectic Fourier transformation, this is the same as

$$u_1(t) = \frac{1}{(8c)^{\frac{1}{2}}}\, e^{-i\pi\,\frac{ac-1}{2c}\,t^2}\, \mathcal{F}_{\mathrm{ana}}\left(y \mapsto u(y)\, e^{-i\pi\,\frac{ac-1}{2c}\,y^2}\right)\left(\frac{-ac-1}{2c}\,t\right) \tag{10.40}$$

or, from the characterization of the anaplectic representation given in Theorem 1.20,

$$\begin{aligned} u_1 &= \frac{1}{2}\,(ac+1)^{-\frac{1}{2}} \\ &\quad \times \mathrm{Ana}\left(\begin{pmatrix} 1 & 0 \\ \frac{1-ac}{2c} & 1 \end{pmatrix}\begin{pmatrix} \frac{-2c}{ac+1} & 0 \\ 0 & \frac{-ac-1}{2c} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1-ac}{2c} & 1 \end{pmatrix}\right), \end{aligned} \tag{10.41}$$

finally leading to the matrix g_1 indicated in the statement of the lemma. □

We now need to transfer to the space $\widehat{C}^{\omega}_{-\frac{1}{2},0}$ the scalar product of two even functions in $\mathfrak{A}$. If u^1 and u^2 are two such functions, let $\begin{pmatrix} w_0^1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} w_0^2 \\ 0 \end{pmatrix}$ be associated to u^1 and u^2 respectively by the map (1.26), recalled in (2.81). By Theorem 2.10, one has $(u^1 \,|\, u^2)_{\mathfrak{A}} = (w_0^1 \,|\, w_0^2)_{\widehat{\mathcal{H}}_{-\frac{1}{2},0}}$ where, as defined in (2.4),

$$(w_0^1 \,|\, w_0^2)_{\widehat{\mathcal{H}}_{-\frac{1}{2},0}} = \int_{-\infty}^{\infty} \bar{w}_0^1(t)\,(|D|^{\frac{1}{2}}\, w_0^2)(t)\,dt : \tag{10.42}$$

the variable denoted as t here was formerly denoted as σ, a letter reserved for another use here.

Lemma 10.11. *With the notation just introduced, set, for any number $z \neq 1$ on the unit circle,*

$$\begin{aligned} f^1(z) &= |1-z|^{-\frac{1}{2}}\, w_0^1\left(i\,\frac{1+z}{1-z}\right), \\ f^2(z) &= |1-z|^{-\frac{3}{2}}\,(|D|^{\frac{1}{2}}\, w_0^2)\left(i\,\frac{1+z}{1-z}\right). \end{aligned} \tag{10.43}$$

The functions f^1 *and* f^2 *extend as real-analytic functions on the unit circle. One has*

$$(w_0^1 \,|\, w_0^2)_{\widehat{\mathcal{H}}_{-\frac{1}{2},0}} = \frac{2}{i} \int_{|z|=1} \bar{f}^1(z)\, f^2(z)\, \frac{dz}{z}. \tag{10.44}$$

If $g_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, *setting*

$$\gamma = \frac{1}{2i} \left(\begin{smallmatrix} i & 1 \\ -i & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -1 \\ i & i \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_1 & \bar{\beta}_1 \\ \beta_1 & \bar{\alpha}_1 \end{smallmatrix}\right) \in SU(1,1), \tag{10.45}$$

one has

$$(\hat{\pi}_{-\frac{1}{2},0}(g_1)\, w_0^1 \,|\, w_0^2)_{\widehat{\mathcal{H}}_{-\frac{1}{2},0}} = \frac{2}{i} \int_{|z|=1} |-\bar{\beta}_1\, z + \bar{\alpha}_1|^{-\frac{1}{2}}\, \bar{f}^1 \left(\frac{\alpha_1\, z - \beta_1}{-\bar{\beta}_1\, z + \bar{\alpha}_1} \right) f^2(z)\, \frac{dz}{z}. \tag{10.46}$$

Proof. That f^1 is analytic on the unit circle was shown in Theorem 4.14, as a consequence of the expansion (1.27) of the function w_0^1 at infinity. Now, the function $|D|^{\frac{1}{2}}\, w_0^2$ lies in $\widehat{C}^{\omega}_{\frac{1}{2},0}$, thus admits at infinity a convergent expansion of the kind $\sum_{n \geq 0} a_n\, t^{-n}\, |t|^{-\frac{3}{2}}$, from which the analyticity of the function f^2 follows as well. Under the (one-dimensional) Cayley transform $t = i\, \frac{1+z}{1-z}$, or $z = \frac{t-i}{t+i}$, one has $\frac{2}{i}\, |1-z|^{-2}\, \frac{dz}{z} = dt$, from which one finds (10.44) from (10.42).

From the definition of the representation $\hat{\pi}_{-\frac{1}{2},0}$ in (2.3), one has

$$(\hat{\pi}_{-\frac{1}{2},0}(g_1)\, w_0^1)(t) = |-b_1\, t + d_1|^{-\frac{1}{2}}\, w_0^1 \left(\frac{a_1\, t - c_1}{-b_1\, t + d_1} \right), \tag{10.47}$$

and it is a straightforward matter, using the Cayley map, to obtain (10.46) as a consequence. □

Corollary 10.12. *For every* $x > 0$ *one has*

$$\frac{1+x}{2}\, {}_2F_1 \left(\frac{1}{4}, \frac{1}{2}; 1; 1 - x^4 \right) = {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; -\left(\frac{x-1}{x+1} \right)^2 \right). \tag{10.48}$$

Proof. With $\phi(x) = (\pi\, |x|)^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi x^2)$, and

$$u(x,\xi) = W(\phi,\phi)(x,\xi) = 2\, I_0(2\pi\, (x^2 + \xi^2)), \tag{10.49}$$

in other words, with the notation used in Theorem 4.18 (2-dimensional case),

$$u(x,\xi) = 2\, \Phi(2^{\frac{1}{2}}\, x,\, 2^{\frac{1}{2}}\, \xi), \tag{10.50}$$

from which it is immediate, denoting as $\sigma \in \mathrm{Sym}_2$ a symmetric matrix such that $\sigma \succ R\, I$ for some large R, that

$$(\mathcal{Q}\, u)_0(2\sigma) = (\mathcal{Q}\, \Phi)_0(\sigma), \tag{10.51}$$

an identity connecting the quadratic transforms of u and Φ. Let $e^{-i\theta_1}$ and $e^{-i\theta_2}$ be the eigenvalues of $Z=(\sigma-i\,I)(\sigma+i\,I)^{-1}$, and assume to avoid any discussion of sign that θ_1 and θ_2 are positive and small, and $\theta_1>\theta_2$. Since $\det(I-Z)=4\,\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}$, it follows from (4.84) that

$$(\mathcal{Q}\,\Phi)_0(\sigma)=\frac{(\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2})^{\frac{1}{2}}}{\cos\frac{\theta_1-\theta_2}{4}}\;{}_2F_1\left(\frac{1}{2},\,\frac{1}{2};\,1;\,-\tan^2\frac{\theta_1-\theta_2}{4}\right). \tag{10.52}$$

On the other hand, assuming from now on that $\sigma=\left(\begin{smallmatrix}a&0\\0&c\end{smallmatrix}\right)$, the result of Lemma 10.10, together with (10.8), is that

$$(\mathcal{Q}\,u)_0(2\sigma)=(ac+1)^{-\frac{1}{2}}\,(\mathrm{Ana}(g_1)\,\phi\,|\,\phi)_{\mathfrak{A}}. \tag{10.53}$$

It has been verified in the proof of Proposition 9.5 that

$$\Theta^{-1}\phi=\frac{\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}\,\pi^{\frac{3}{4}}}\begin{pmatrix}v^0_{-\frac{1}{2},0}\\0\end{pmatrix} \tag{10.54}$$

with $v^0_{-\frac{1}{2},0}$ as given in (2.22), and (2.20) gives

$$w^0_{-\frac{1}{2},0}(t)=(\mathcal{F}\,v^0_{-\frac{1}{2},0})(t)=\frac{\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}\,\pi^{\frac{3}{4}}}\,(1+t^2)^{-\frac{1}{4}}. \tag{10.55}$$

Then [17, p. 412]

$$\begin{aligned}(|D|^{\frac{1}{2}}\,w^0_{-\frac{1}{2},0})(t)&=\mathcal{F}\,(2^{\frac{3}{4}}\,\pi^{-\frac{1}{2}}\,|s|^{\frac{1}{4}}\,K_{\frac{1}{4}}(2\pi\,|s|))\,(t)\\&=2^{-\frac{1}{4}}\,\pi^{-\frac{5}{4}}\,\Gamma\left(\frac{3}{4}\right)\,(1+t^2)^{-\frac{3}{4}}.\end{aligned} \tag{10.56}$$

With the notation introduced just before Lemma 10.11, we thus set $w_0^1=w_0^2=w^0_{-\frac{1}{2},0}$, which leads (with the notation of this lemma) to

$$f^1(z)=2^{-\frac{3}{4}}\,\pi^{-\frac{3}{4}}\,\Gamma\left(\frac{1}{4}\right),\qquad f^2(z)=2^{-\frac{7}{4}}\,\pi^{-\frac{5}{4}}\,\Gamma\left(\frac{3}{4}\right): \tag{10.57}$$

consequently, $\bar{f}^1(z)\,f^2(z)=\frac{1}{4\pi}$. Applying Lemma 10.9, and noting that

$$\alpha=ac-1-i\,(a+c),\qquad \beta=i\,(c-a), \tag{10.58}$$

we obtain

$$(\mathcal{Q}\,u)_0(2\sigma)=\frac{1}{2i\pi}\int_{|z|=1}|i\,(c-a)\,z+ac-1+i\,(a+c)|^{-\frac{1}{2}}\,\frac{dz}{z}. \tag{10.59}$$

In terms of θ_1 and θ_2, one has $a=i\,\frac{1+e^{-i\theta_1}}{1-e^{-i\theta_1}}=\frac{1}{\tan\frac{\theta_1}{2}}$, thus

$$ac-1=\frac{\cos\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}},\quad a+c=\frac{\sin\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}},\quad c-a=\frac{\sin\frac{\theta_1-\theta_2}{2}}{\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}}. \tag{10.60}$$

Setting $\mu = \tan \frac{\theta_1 - \theta_2}{4}$ and $z = e^{i\,(\omega + \frac{\theta_1+\theta_2}{2})}$, we obtain

$$\begin{aligned}
\frac{(\mathcal{Q}\,u)_0(2\sigma)}{(\sin \frac{\theta_1}{2}\, \sin \frac{\theta_2}{2})^{\frac{1}{2}}} &= \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + \frac{2i\mu}{1+\mu^2}\, e^{i\omega} \right|^{-\frac{1}{2}} d\omega \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 + \frac{4\mu^2}{(1+\mu^2)^2} - \frac{4\mu\, \sin\omega}{1+\mu^2} \right]^{-\frac{1}{4}} d\omega \\
&= \frac{1}{\pi} \int_{-1}^{1} \left[1 + \frac{4\mu^2}{(1+\mu^2)^2} - \frac{4\mu\,\xi}{1+\mu^2} \right]^{-\frac{1}{4}} \frac{d\xi}{\sqrt{1-\xi^2}} \\
&= \frac{1}{\pi} \int_0^1 \left[1 + \frac{4\mu^2}{(1+\mu^2)^2} - \frac{4\mu}{1+\mu^2} + \frac{9\mu\,\eta}{1+\mu^2} \right]^{-\frac{1}{4}} \eta^{-\frac{1}{2}}\,(1-\eta)^{-\frac{1}{2}}\, d\eta \\
&= \frac{1}{\pi} \int_0^1 \left[\frac{(1-\mu)^4}{(1+\mu^2)^2} + \frac{8\mu\,\eta}{1+\mu^2} \right]^{-\frac{1}{4}} \eta^{-\frac{1}{2}}\,(1-\eta)^{-\frac{1}{2}}\, d\eta \\
&= \frac{1}{\pi}\, \frac{(1+\mu^2)^{\frac{1}{2}}}{1-\mu} \int_0^1 \left[1 + \frac{8\mu(1+\mu^2)}{(1-\mu)^4}\, \eta \right]^{-\frac{1}{4}} \eta^{-\frac{1}{2}}\,(1-\eta)^{-\frac{1}{2}}\, d\eta \\
&= \frac{(1+\mu^2)^{\frac{1}{2}}}{1-\mu}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; 1; -\frac{8\mu(1+\mu^2)}{(1-\mu)^4} \right).
\end{aligned} \tag{10.61}$$

Comparing this result to (10.52), we obtain

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\mu^2 \right) = (1-\mu)^{-1}\, {}_2F_1\left(\frac{1}{4}, \frac{1}{2}; 1; -\frac{8\mu(1+\mu^2)}{(1-\mu)^4} \right) \tag{10.62}$$

for $0 < \mu < 1$, but one still has $-\frac{8\mu(1+\mu^2)}{(1-\mu)^4} < 1$ if $\mu > -1$, which makes it possible to use analytic continuation to claim the validity of (10.62) for $-1 < \mu < 1$. Note that this equation provides the analytic continuation to all real values of μ of the integral that led to the right-hand side of (10.62): it is in another attempt at a proof of Theorem 10.8 that we were led to the series of transformations which produced this identity. Setting $x = \frac{1+\mu}{1-\mu}$, one obtains the aesthetically more satisfying equation (10.48).

Using [17, p. 51] and [17, p. 52], one may write this equation as

$$\mathfrak{P}_{-\frac{1}{4}} \left(\frac{1+x^4}{2x^2} \right) = \left(\frac{2x}{1+x^2} \right)^{\frac{1}{2}} \mathfrak{P}_{-\frac{1}{2}} \left(\frac{2x}{1+x^2} \right). \tag{10.63}$$

This last identity can also be found as a consequence of the equation (11) in [17, p. 157], applied to the right-hand side, and of the first equation concerning $\mathfrak{P}_\nu^\mu$ in [17, p. 153], applied to the left-hand side. □

In the usual Weyl calculus, there is a simple link between the Weyl symbol f of some operator and its integral kernel k, which may be expressed as the formula

$$k(s,t) = (\mathcal{F}_2^{-1}\, f) \left(\frac{s+t}{2}, s-t \right), \tag{10.64}$$

where $\mathcal{F}_2^{-1}$ denotes the inverse partial Fourier transformation with respect to the second variable. In the case of a rank-one operator, built from a pair of L^2 functions, this can be written as

$$\chi(s)\,\bar{\psi}(t) = (\mathcal{F}_2^{-1}\,W(\psi,\,\chi))\left(\frac{s+t}{2},\,s-t\right), \tag{10.65}$$

an equation which can also be derived from (10.5).

As a preparation – among other reasons – towards the study of the tensor product of two functions in the space $\mathfrak{A}_0$, we now generalize this formula to the case of the anaplectic analysis. Two important differences will appear. First, as already mentioned, there is no genuine concept of partial Fourier transformation, since "freezing" one of a pair of variables is not possible. However, as done in Section 6, one may consider instead the anaplectic transformation $\mathrm{Ana}(g)$, with g as introduced in Theorem 10.13 below: recall that this would yield exactly the transformation $\mathcal{F}_2^{-1}$ if the metaplectic representation were used in place of the anaplectic one.

In the anaplectic analysis, it will be necessary to add two or four terms of the preceding kind, on the right-hand side, to obtain an analogue of (10.65). In the null space of the formal harmonic oscillator, there are two linearly independent functions ϕ and $\phi^{\natural}$ (the odd one, introduced at the very end of Section 1): ultimately, this will imply that the Wigner function contains more information than the tensor product.

Theorem 10.13. *Let ψ and χ lie in the space $\mathfrak{A}_0$: then the tensor product $\chi\otimes\bar{\psi}$ lies in $\mathfrak{A}^{(2)}$. Moreover, assuming that ψ* (resp. χ) *has the parity associated with $\delta = 0$ or 1 mod 2* (resp. *that associated with* $\delta' = 0$ *or* 1)*, and setting $g = \left(\begin{smallmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)$, one has the identity*

$$\chi(s)\,\bar{\psi}(t) = \frac{1}{4}\sum_{\substack{\varepsilon=\pm 1\\ \varepsilon'=\pm 1}} \varepsilon^{\delta}\,{\varepsilon'}^{\delta'}\ \left(\mathrm{Ana}(g)\ W(\psi,\,\chi)\,\right)\left(\frac{\varepsilon'\,s+\varepsilon\,t}{2},\,\varepsilon'\,s-\varepsilon\,t\right). \tag{10.66}$$

Proof. We first prove the identity above in the case when $\psi = \chi = \phi$, the basic function introduced in Proposition 1.2. Since $\mathrm{Ana}(g)$ is not really a partial Fourier transformation, we must prove directly, in view of their immediate use, the unsurprising formulas (in which x,ξ is taken as the pair of independent variables on $\mathbb{R}^2$ and (ξ) stands for the operator of multiplication by ξ)

$$\mathrm{Ana}(g)\,\frac{\partial}{\partial\xi} = -2i\pi\,(\xi)\ \mathrm{Ana}(g), \qquad \mathrm{Ana}(g)\ (\xi) = \frac{1}{2i\pi}\,\frac{\partial}{\partial\xi}\ \mathrm{Ana}(g): \tag{10.67}$$

these are an immediate consequence of (5.71). On one hand, $W(\phi,\,\phi)$ satisfies the equation $\frac{1}{2i\pi}\,(x\,\frac{\partial}{\partial\xi} - \xi\,\frac{\partial}{\partial x})\,W(\phi,\,\phi) = 0$ (which expresses the invariance of $W(\phi,\,\phi)$ under rotations); on the other hand, $W(\phi,\,\phi)(x,\xi) = 2\,\Phi(2^{\frac{1}{2}}x,\,2^{\frac{1}{2}}\xi)$, with Φ as defined in Theorem 4.18, so that $W(\phi,\,\phi)$ lies also in the null space of the operator $2\pi\,(x^2+\xi^2)-\frac{1}{8\,\pi}\,(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial \xi^2})$ (a transfer of the two-dimensional harmonic oscillator). In view of (10.67), the image of $W(\phi,\,\phi)$ under $\mathrm{Ana}(g)$, as a function of (x,ξ), vanishes under the action of either of the operators

$$-x\,\xi+\frac{1}{4\pi}\,\frac{\partial^2}{\partial x\,\partial\xi}\qquad\text{and}\qquad \pi\left(2\,x^2+\frac{\xi^2}{2}\right)-\frac{1}{4\pi}\left(\frac{1}{2}\,\frac{\partial^2}{\partial x^2}+2\,\frac{\partial^2}{\partial \xi^2}\right). \tag{10.68}$$

Finally, setting $s=x+\frac{\xi}{2}$, $t=x-\frac{\xi}{2}$, one sees that, under this change of variables, the operators in (10.68) transfer to

$$\frac{1}{4\pi}\left[s^2-t^2-\frac{1}{4\pi^2}\,\frac{\partial^2}{\partial s^2}+\frac{1}{4\pi^2}\,\frac{\partial^2}{\partial t^2}\right]\qquad\text{and}\qquad \pi\,(s^2+t^2)-\frac{1}{4\pi}\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right) \tag{10.69}$$

respectively.

Hence, the function $(s,t)\mapsto(\mathrm{Ana}(g)\;W(\phi,\,\phi))(\frac{s+t}{2},\,s-t)$ lies in the null space of the two *formal* standard harmonic oscillators L_s and L_t, with respect to the two variables. It is not even with respect to s and t separately, but it is globally even. Since the null space of the one-dimensional standard harmonic oscillator is generated by ϕ and $\phi^{\natural}$, the function under consideration is a linear combination of the functions $\phi\otimes\phi$ and $\phi^{\natural}\otimes\phi^{\natural}$. Finally, the even part (with respect to s, t or both) of the function $(\mathrm{Ana}(g)\;W(\phi,\,\phi))(\frac{s+t}{2},\,s-t)$ coincides with a multiple of the function $\phi\otimes\phi$. The normalization constant does not play any role at present, and we may thus assume that the identity (10.66) is valid in the case when $\psi=\chi=\phi$, granted that we shall prove a more precise result later.

Next, we show that the identity (10.66) is valid for every pair $(\psi,\,\chi)$ of anaplectic Hermite functions, by showing that if it is true for such a pair, it is also true for the pair obtained by applying the raising or lowering operator to either of the functions ψ, χ: one of the four verifications goes as follows. In view of (10.21), one has $W(A\,\psi,\,\chi)=\mathrm{Left}\,W(\psi,\,\chi)$. Then, in view of (10.67), one has

$$\begin{aligned}\mathrm{Ana}(g)\,W(A\,\psi,\,\chi)&=\pi^{\frac{1}{2}}\,\mathrm{Ana}(g)\left(x-i\,\xi+\frac{1}{4\pi}\,(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial\xi})\right)W(\psi,\,\chi)\\&=\pi^{\frac{1}{2}}\left(x-\frac{1}{2\pi}\,\frac{\partial}{\partial\xi}+\frac{1}{4\pi}\,\frac{\partial}{\partial x}-\frac{\xi}{2}\right)\mathrm{Ana}(g)\,W(\psi,\,\chi),\end{aligned} \tag{10.70}$$

and the operator in front of the right-hand side transfers, under the change of variables $(s,t)\mapsto(x,\xi)=(\frac{\varepsilon'\,s+\varepsilon\,t}{2},\,\varepsilon'\,s-\varepsilon\,t)$, to the operator $\varepsilon\,\pi^{-\frac{1}{2}}\,(t+\frac{1}{2\pi}\,\frac{\partial}{\partial t})$. On one hand, $A\,\psi$ has the parity associated with $\delta+1$ mod 2 if ψ has the parity associated with δ; on the other hand, this operator indeed transforms the function $(\chi\otimes\bar{\psi})(s,t)$ into the function $(\chi\otimes\overline{A\,\psi})(s,t)$.

Finally, if ψ and χ lie in $\mathfrak{A}_0$, and each of the two functions has some definite parity, one may use the expansions (10.25) to obtain a convergent expansion of the $\mathcal{Q}$-transform of $(\chi\otimes\bar{\psi})(s,t)$ in the domain $\{\sigma\in\mathrm{Sym}_2\colon\sigma\succ R\,I\}$ for some large R; at the same time, the equation (10.66), together with Theorem 10.8, takes care of the convergence, in the space of functions analytic in the part of $\Sigma^{(2)}$ above $\Sigma\backslash\Delta$, of the corresponding series of $\mathcal{K}$-transforms. This provides the analytic continuation of the $\mathcal{K}$-transform of $(\chi \otimes \bar{\psi})(s,t)$, thus completing the proof of Theorem 10.13. □

The link between $W(\phi,\, \phi)$ and Φ, already used, can be written as $W(\phi,\, \phi) = 2^{\frac{1}{2}}\,\text{Ana}\,\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1}\end{smallmatrix}\right)\,\Phi$ with $A = 2^{-\frac{1}{2}}\,\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$: since $g\,\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1}\end{smallmatrix}\right)\,g^{-1} = \left(\begin{smallmatrix} B & 0 \\ 0 & B^{-1}\end{smallmatrix}\right)$ with $B = \begin{pmatrix} 2^{-\frac{1}{2}} & 0 \\ 0 & 2^{\frac{1}{2}}\end{pmatrix}$, one has

$$2^{-\frac{1}{2}}\,(\text{Ana}(g)\,W(\phi,\, \phi))(x,\xi) = (\text{Ana}(g)\,\Phi)\,(2^{\frac{1}{2}}x,\, 2^{-\frac{1}{2}}\xi). \tag{10.71}$$

We have seen in the proof of the last theorem that there exist two constants C_0 and C_1 such that the identity

$$2^{-\frac{1}{2}}\,(\text{Ana}(g)\,W(\phi,\, \phi))\left(\frac{s+t}{2},\, s-t\right) = C_0\;\phi(s)\,\phi(t) + C_1\;\phi^{\natural}(s)\,\phi^{\natural}(t) \tag{10.72}$$

holds. Setting

$$\Psi(x_1,\, x_2) = C_0\;\phi\left(\frac{x_1+x_2}{2^{\frac{1}{2}}}\right)\,\phi\left(\frac{x_1-x_2}{2^{\frac{1}{2}}}\right) + C_1\;\phi^{\natural}\left(\frac{x_1+x_2}{2^{\frac{1}{2}}}\right)\,\phi^{\natural}\left(\frac{x_1-x_2}{2^{\frac{1}{2}}}\right), \tag{10.73}$$

this can be written as the simple equation $\Psi = \text{Ana}(g)\,\Phi$. We shall find the values of the two unknown coefficients by a comparison of the quadratic transforms of the two sides, which will provide us with an opportunity to apply the definition of the two-dimensional anaplectic representation from Theorem 5.10.

Theorem 10.14. *The wave equation*

$$\left(\frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial r^2}\right)\; f = 0 \tag{10.74}$$

admits a solution satisfying on the plane $q = 0$ *the initial conditions*

$$\begin{aligned} f(p,\, 0,\, r) &= \frac{\pi^{-\frac{3}{2}}}{2}\left(\Gamma\left(\frac{1}{4}\right)\right)^2\;\; [4\,p^2 + (p^2 - r^2 - 1)^2]^{-\frac{1}{4}}, \\ \frac{\partial f}{\partial q}\,(p,\, 0,\, r) &= 2\,\pi^{-\frac{3}{2}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2\;\; [4\,p^2 + (p^2 - r^2 - 1)^2]^{-\frac{3}{4}}, \end{aligned} \tag{10.75}$$

given by the equation

$$f(p,\, q,\, r) = (a-q)^{-\frac{1}{2}}\;{}_2F_1\left(\frac{1}{2},\, \frac{1}{2};\, 1;\, -\frac{a+q}{a-q}\right), \tag{10.76}$$

where

$$a = \frac{1}{2}\,\sqrt{4\,p^2 + (p^2 - q^2 - r^2 - 1)^2}. \tag{10.77}$$

This solution is defined and analytic in the complement, in $\mathbb{R}^3$*, of the branch of hyperbola* $\{(p,q,r)\colon r=0,\ q=\sqrt{p^2+1}\}$.

The coefficients that make the equation (10.73) *valid are* $C_0 = 2^{-\frac{1}{2}}$ *and* $C_1 = \frac{2^{-\frac{1}{2}}}{i\,\pi}$. *If they are so chosen, one then has*

$$f(p,q,r) = (\mathcal{Q}\,\Psi)_0\,\left(\left(\begin{smallmatrix} p+q & r \\ r & p-q \end{smallmatrix}\right)\right). \tag{10.78}$$

Remark 10.2. Despite the fact that the function f is locally summable on $\mathbb{R}^3$, it is not a solution, in the distribution sense, of the wave equation in the whole of $\mathbb{R}^3$, only in its domain of analyticity. On the other hand, since the wave equation is invariant under the change $(p,\,q,\,r) \mapsto (p,\,-q,\,r)$, one can also find an explicit solution of the Cauchy problem obtained after one has changed the coefficients in front of the right-hand sides of the two equations (10.75) in an arbitrary way.

Proof. With $f(p,q,r)$ as defined by (10.76), and (when $q=0$) $a = \frac{1}{2}\sqrt{4\,p^2+(p^2-r^2-1)^2}$, one has, with the help of several formulas from [17, p. 40], together with the formula of complements,

$$f(p,\,0,\,r) = {}_2F_1\left(\frac{1}{2},\,\frac{1}{2};\,1;\,-1\right)\,a^{-\frac{1}{2}} = (2\pi)^{-\frac{3}{2}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2\,a^{-\frac{1}{2}} \tag{10.79}$$

and

$$\begin{aligned}\frac{\partial f}{\partial q}\,(p,\,0,\,r) &= \frac{1}{2}\,\left[{}_2F_1\left(\frac{1}{2},\,\frac{1}{2};\,1;\,-1\right) - {}_2F_1\left(\frac{3}{2},\,\frac{3}{2};\,2;\,-1\right)\right]\,a^{-\frac{3}{2}}\\ &= 2^{-\frac{1}{2}}\,\pi^{-\frac{3}{2}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2\,a^{-\frac{3}{2}}.\end{aligned} \tag{10.80}$$

Setting

$$b = \frac{1}{2}\,\sqrt{4r^2+(p^2-q^2-r^2+1)^2}, \tag{10.81}$$

one has $a = \sqrt{b^2+q^2}$. It follows that $a-q>0$ (while the argument of the hypergeometric function is always <1) except when $b=0$ and $q>0$, *i.e.*, $r=0$ and $q=\sqrt{p^2+1}$: this makes the domain of analyticity of the function f explicit, as indicated.

Take the generic matrix in Sym_n to be

$$\sigma = \begin{pmatrix} p+q & r \\ r & p-q \end{pmatrix}; \tag{10.82}$$

also, with $x=(x_1,x_2)$, set

$$U(x) = \phi\left(\frac{x_1+x_2}{2^{\frac{1}{2}}}\right)\,\phi\left(\frac{x_1-x_2}{2^{\frac{1}{2}}}\right), \qquad U^\natural(x) = \phi^\natural\left(\frac{x_1+x_2}{2^{\frac{1}{2}}}\right)\,\phi^\natural\left(\frac{x_1-x_2}{2^{\frac{1}{2}}}\right). \tag{10.83}$$

With $s = \frac{x_1+x_2}{2^{\frac{1}{2}}}$, $t = \frac{x_1-x_2}{2^{\frac{1}{2}}}$, one has

$$\langle \sigma\, x,\, x\rangle = (p+r)\, s^2 + 2q\, st + (p-r)\, t^2. \tag{10.84}$$

Let us recall Definition 4.1 of the quadratic transform:

$$(\mathcal{Q}\, U)_0(\sigma) = \int_{\mathbb{R}^2} e^{-\pi\, [(p+r)\, s^2 + 2q\, st + (p-r)\, t^2]}\ \phi(e^{-\frac{i\pi}{4}}\, s)\, \phi(e^{-\frac{i\pi}{4}}\, t)\ ds\, dt; \tag{10.85}$$

a similar definition holds for the $\mathcal{Q}$-transform of $U^\natural$, also an even function of the *pair of variables* x_1, x_2, or s, t, even though, contrary to U, it is not even with respect to s and t separately. The equation (10.85) and some obvious parity considerations make it immediate that

$$\left.\frac{\partial}{\partial q}\right|_{q=0} (\mathcal{Q}\, U)_0(\sigma) = 0 \qquad \text{and} \qquad (\mathcal{Q}\, U^\natural)_0\ \left(\left(\begin{smallmatrix} p & r \\ r & p \end{smallmatrix}\right)\right) = 0. \tag{10.86}$$

On the other hand, in the case when $q = 0$, $\langle \sigma\, x,\, x\rangle$ reduces to $(p+r)\, s^2 + (p-r)\, t^2$. At such points σ the computation of the integral (10.85) reduces to computations from the one-dimensional case already made, and we get from (1.26) and (1.39) (the latter one with $n = 0$) that

$$\begin{aligned} (\mathcal{Q}\, \Psi)_0\ \left(\left(\begin{smallmatrix} p & r \\ r & p \end{smallmatrix}\right)\right) &= \frac{\pi^{-\frac{3}{2}}}{2} \left(\Gamma\left(\frac{1}{4}\right)\right)^2 (1 + (p+r)^2)^{-\frac{1}{4}}\, (1 + (p-r)^2)^{-\frac{1}{4}} \\ &= \frac{\pi^{-\frac{3}{2}}}{2} \left(\Gamma\left(\frac{1}{4}\right)\right)^2 [4\, p^2 + (p^2 - r^2 - 1)^2]^{-\frac{1}{4}}. \end{aligned} \tag{10.87}$$

Next,

$$\left.\frac{\partial}{\partial q}\right|_{q=0} (\mathcal{Q}\, U^\natural)_0(\sigma) = -2\pi \int_{\mathbb{R}^2} e^{-\pi\, [(p+r)\, s^2 + (p-r)\, t^2]}\ \ s\, t\ \ \phi^\natural(e^{-\frac{i\pi}{4}}\, s)\, \phi^\natural(e^{-\frac{i\pi}{4}}\, t)\ ds\, dt\ : \tag{10.88}$$

it is time to recall from (1.94) that

$$\phi^\natural(s) = -(\pi\, |s|)^{\frac{1}{2}}\, (\operatorname{sign} s)\, I_{\frac{1}{4}}(\pi\, s^2), \tag{10.89}$$

an odd function, so that $s \mapsto s\, \phi^\natural(e^{-\frac{i\pi}{4}}\, s)$ is an even function, coinciding for $s > 0$ with $-e^{-\frac{i\pi}{4}}\, \pi^{\frac{1}{2}}\, s^{\frac{3}{2}}\, J_{\frac{1}{4}}(\pi\, s^2)$. With the help of [17, p. 91], one finds

$$\left.\frac{\partial}{\partial q}\right|_{q=0} (\mathcal{Q}\, U^\natural)_0(\sigma) = i\, 2^{\frac{3}{2}}\, \pi^{-\frac{1}{2}} \left(\Gamma\left(\frac{3}{4}\right)\right)^2 [4\, p^2 + (p^2 - r^2 - 1)^2]^{-\frac{3}{4}}. \tag{10.90}$$

Comparing the Cauchy data, on the plane $q = 0$, of the function f, as given by (10.79) and (10.80), and those of $(\mathcal{Q}\, \Psi)_0$, with Ψ as defined in (10.73), we obtain that (10.78) can hold only if the coefficients C_0 and C_1 have the values indicated.

Taking Remark 5.2 into account, the only point that remains to be proved is that

$$f(p,q,r) = (\mathcal{Q}\,\mathrm{Ana}(g)\,\Phi)_0\,\left(\left(\begin{smallmatrix} p+q & r \\ r & p-q \end{smallmatrix}\right)\right). \tag{10.91}$$

Since we already know that both sides extend as analytic functions in the complement, in the space Sym_2, of some algebraic subset of dimension 1, it is sufficient to prove this identity for (p,q,r) in an arbitrary non-void open subset of $\mathbb{R}^3$: we shall assume that p is large and that $|q|$ and $|r|$ are small, so that the matrix σ should be close to a large multiple of the identity matrix.

Let $Z = \frac{\sigma - iI}{\sigma + iI}$ be the image of σ under the Cayley map. With our present $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ as defined in the proof of Theorem 10.13, the equations (5.2) and (5.5) give $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$, $\beta = 0$ and $[g^{-1}](Z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -i \end{smallmatrix}\right) Z \left(\begin{smallmatrix} 1 & 0 \\ 0 & -i \end{smallmatrix}\right)$. Here,

$$Z = \frac{1}{(p+i)^2 - q^2 - r^2} \begin{pmatrix} p^2-(q-i)^2-r^2 & 2i\,r \\ 2i\,r & p^2-(q+i)^2-r^2 \end{pmatrix},$$

so that

$$[g^{-1}](Z) = \frac{1}{(p+i)^2 - q^2 - r^2} \begin{pmatrix} p^2-(q-i)^2-r^2 & 2r \\ 2r & -p^2+(q+i)^2+r^2 \end{pmatrix}. \tag{10.92}$$

An application of Theorem 5.10 and Theorem 4.19 leads to an expression of $(\mathcal{K}\,\mathrm{Ana}(g)\,\Phi)(Z)$ as some function of the ratio of the eigenvalues $e^{-i\,\theta_1}$ and $e^{-i\,\theta_2}$ of the matrix $[g^{-1}](Z)$ that occurs in (10.92). One finds that, for some order of the two eigenvalues, one has

$$e^{i\,(\theta_1-\theta_2)} = \frac{i\,q - b}{i\,q + b}, \tag{10.93}$$

with b as defined in (10.81). However, the angle $\frac{\theta_1-\theta_2}{4}$ must be determined mod π, and we have to decide which of the two equations (4.76) and (4.77) must be taken. This can only be done by connecting the matrix g to the identity through the path $(g_t)_{0\leq t\leq \frac{\pi}{2}}$, where $g_t = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A = D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \cos t \end{smallmatrix}\right)$ and $C = -B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \sin t \end{smallmatrix}\right)$, which implies $\beta = 0$ again and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{it} \end{smallmatrix}\right)$, so that $[g_t^{-1}](Z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{-it} \end{smallmatrix}\right) Z \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{-it} \end{smallmatrix}\right)$.

In the case when $r = 0$, all these computations simplify, since

$$[g_t^{-1}](Z) = \begin{pmatrix} \frac{p+q-i}{p+q+i} & 0 \\ 0 & e^{-2it}\,\frac{p-q-i}{p-q+i} \end{pmatrix}, \tag{10.94}$$

the image under the Cayley map of the matrix $\begin{pmatrix} p+q & 0 \\ 0 & \frac{(p-q)\,\cos t - \sin t}{(p-q)\,\sin t + \cos t} \end{pmatrix}$, provided that $p > q$, which can be assumed. Theorem 4.19 thus implies that it is the equation (4.76) that has to be used under the assumptions above, in which $|r|$ and $p - |q|$ are small. In the case when $r = 0$, comparing the ratio of the eigenvalues of the matrix in (10.94) taken at $t = \frac{\pi}{2}$ to the expression (10.93), one sees that

$e^{-i\theta_1}$ (*resp.* $e^{-i\theta_2}$) actually denotes the upper-left (*resp.* lower right) entry of that matrix. The equation (10.94) again then shows that, as t moves from 0 to $\frac{\pi}{2}$, the first eigenvalue $e^{-i\theta_1}$ of $[g_t^{-1}](Z)$ is a constant, θ_1 being a small positive number, while the argument θ_2 of the inverse of the second eigenvalue moves from a small positive number to a number close to π. Thus, at the end of the path, $\sin\frac{\theta_1-\theta_2}{2} < 0$, and since, from (10.93), $e^{\frac{i(\theta_1-\theta_2)}{2}} = \pm\frac{q+ib}{a}$, one sees that $\cos\frac{\theta_1-\theta_2}{2} = -\frac{q}{a}$, from which $\cos^2\frac{\theta_1-\theta_2}{4} = \frac{a-q}{2a}$.

The equation (10.76) then follows from (4.76) together with the fact that if $\sigma\in\mathrm{Sym}_n$ and $Z\in\Sigma^{(2)}$ are linked by the Cayley relation, the $\mathcal{Q}$-transform of Ψ at σ and the $\mathcal{K}$-transform of this function at Z are linked by the relation (recalling (4.39))

$$(\mathcal{K}\,\Psi)_0(Z) = \frac{1}{2}\,(\det(I+\sigma^2))^{\frac{1}{4}}\,(\mathcal{Q}\,\Psi)_0(\sigma)\,: \tag{10.95}$$

with σ as in (10.82), $\det(I+\sigma^2) = 4\,a^2$. □

Remark 10.3. When $(p,q,r) = (\sinh\xi,\cosh\xi,0)$ describes the branch of hyperboloid which is the singular set of the function $(\mathcal{Q}\,\mathrm{Ana}(g)\,\Phi)_0$, the corresponding point $Z = \frac{e^\xi-i}{e^\xi+i}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ of the compactification Σ of Sym_2 moves from $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ to $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Thus, the closure of this branch in Σ, and *a fortiori* that in $\Sigma^{(2)}$, is not a closed curve: the second one is just one half of the singular support of the function $(\mathcal{K}\,\mathrm{Ana}(g)\,\Phi)_0$, which, according to what was said at the very end of Section 4, consists of one circle (in the appropriate parametrization: *cf.* end of proof of Lemma 4.16). The other circle of interest there, half of which lies in the Cayley image of Sym_2 (the *first* sheet of $\Sigma^{(2)}$), is defined here by the equations $r=0,\ q=-\sqrt{p^2+1}$: it does not intersect the singular support of $(\mathcal{K}\,\mathrm{Ana}(g)\,\Phi)_0$, but it constitutes half of the singular support of $(\mathcal{K}\,\mathrm{Ana}(g^{-1})\,\Phi)_0$.

Remark 10.4. Solutions of the wave equation (10.74) are of interest since [15] they provide solutions of the equation $(\frac{\partial^2}{\partial t^2} + (\Delta - \frac{1}{4}))\,h = 0$ involving the Laplace–Beltrami operator Δ on the upper half-plane Π, hence connect to the Lax–Phillips scattering theory for the automorphic wave equation. More precisely, consider the analytic diffeomorphism

$$(t,z)\mapsto\sigma = \begin{pmatrix}p+q & r\\ r & p-q\end{pmatrix} = e^{-t}\begin{pmatrix}\frac{1}{\mathrm{Im}\ z} & -\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z}\\ -\frac{\mathrm{Re}\ z}{\mathrm{Im}\ z} & \frac{|z|^2}{\mathrm{Im}\ z}\end{pmatrix} \tag{10.96}$$

from $\mathbb{R}\times\Pi$ to the cone C defined by the inequalities $(p^2-q^2-r^2>0,\ p>0)$. Under this change of variables, accompanied by the transformation $h\mapsto f = e^{\frac{t}{2}}\,h$, the wave equation (10.74) inside C relative to f is equivalent to the automorphic wave equation above concerning h. It is also an essential part of the Lax–Phillips theory that a solution h of the latter equation is characterized by its pair of Cauchy data $h_0(z) = h(0,z)$ and $h_1(z) = \frac{\partial h}{\partial t}(0,z)$.

Any quadratic transform, in the style of Definition 4.1, leads to a solution of (10.74): in particular, with any even-tempered distribution $\mathfrak{S}$ on $\mathbb{R}^2$, one may associate the function (compare (4.1))

$$f(p,\,q,\,r) = \int_{\mathbb{R}^2} \mathfrak{S}(x,\xi)\ e^{-\pi\,\langle\sigma\left(\begin{smallmatrix} x\\ \xi\end{smallmatrix}\right),\left(\begin{smallmatrix} x\\ \xi\end{smallmatrix}\right)\rangle}\ dx\, d\xi. \tag{10.97}$$

Noting that, under the change of coordinates (10.96), one has

$$\langle\sigma\left(\begin{smallmatrix} x\\ \xi\end{smallmatrix}\right),\left(\begin{smallmatrix} x\\ \xi\end{smallmatrix}\right)\rangle = e^{-t}\,\frac{|x - z\,\xi|^2}{\mathrm{Im}\ z}, \tag{10.98}$$

one sees that the Cauchy data, on the upper half-plane, of the function h associated to f in the manner indicated above, are

$$\begin{aligned} h_0(z) &= \int_{\mathbb{R}^2} \mathfrak{S}(x,\xi)\ \exp\left(-\pi\,\frac{|x-z\,\xi|^2}{\mathrm{Im}\ z}\right)\ dx\,d\xi,\\ h_1(z) &= \int_{\mathbb{R}^2} \mathfrak{S}(x,\xi)\ \exp\left(-\pi\,\frac{|x-z\,\xi|^2}{\mathrm{Im}\ z}\right)\ \left[\pi\,\frac{|x-z\,\xi|^2}{\mathrm{Im}\ z} - \frac{1}{2}\right]\ dx\,d\xi. \end{aligned} \tag{10.99}$$

Now, the pair of functions just made explicit has a nice interpretation in terms of the Weyl calculus on the real line. Indeed, consider the two functions $u_i(s) = 2^{\frac{1}{4}}\,e^{-\pi s^2}$ and $u_i^1(s) = 2^{\frac{5}{4}}\,\pi^{\frac{1}{2}}\,s\,e^{-\pi s^2}$, the normalized first two eigenstates of the (usual, or metaplectic) harmonic oscillator. For any $g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{R})$, with $z = \frac{ai+b}{ci+d}$, set $u_z = \mathrm{Met}(g)\,u_i$ and $u_z^1 = \mathrm{Met}(g)\,u_i$, where the indeterminacy, by the factor ± 1, inherent in the fact that we have not singled out either of the two points in the metaplectic group lying above g, is of no consequence in what follows. Renormalizing the Weyl calculus Op as $\mathrm{Op}_{\sqrt{2}}$ with $\mathrm{Op}_{\sqrt{2}}(\mathfrak{S}) = \mathrm{Op}((x,\xi) \mapsto \mathfrak{S}(2^{\frac{1}{2}}x,\, 2^{\frac{1}{2}}\xi))$, one has [25, p. 17]

$$h_0(z) = \left(u_z\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\,u_z\right), \qquad h_1(z) = \frac{1}{2}\,\left(u_z^1\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\,u_z^1\right). \tag{10.100}$$

This played a basic role in the development of automorphic pseudodifferential analysis (*loc. cit.*). It has also been a major incentive towards the development of anaplectic analysis and our interest in quadratic transforms in general. We have some reasons to believe that, along similar lines, some connection can be found between anaplectic pseudodifferential analysis and a variant of the Lax–Phillips theory, putting forward the one-sheeted hyperboloid in place of the two-sheeted one. Numerous unexpected facts, however, have occurred so far in the development of anaplectic analysis: as a consequence, we shall refrain, as yet, from making any conjecture as to whether this may be of any significance in connection with modular form theory.

Remark 10.5. Let us finally discuss whether anything can be done in the higher-dimensional case. Provided that $n \not\equiv 0 \mod 4$, a pseudoscalar product has been introduced in Section 8: it makes sense, in particular, on the space denoted as $\mathcal{S}_\bullet(\mathbb{R}^n)$ so that, making use of Theorem 8.17 and starting from two functions in the space $\mathcal{S}_\bullet^\bullet(\mathbb{R}^n)$, one can define their Wigner function by the equation

$$W(\psi,\chi)(x,\xi) = 2^n \,(e^{4i\pi(-x\,P+\xi\,Q)}\,\check{\psi}\,|\,\chi), \tag{10.101}$$

a generalization of (10.8): the new coefficient 2^n (instead of 2) is taken in analogy with the corresponding formula from the usual analysis.

As a consequence of Proposition 8.16, this definition makes sense, in particular, in the case when ϕ and ψ are anaplectic Hermite functions. Lemma 10.5 generalizes, leading to the equation

$$W(\psi, Q_j\,\chi) = \left(x_j - \frac{1}{4i\pi}\,\frac{\partial}{\partial\xi_j}\right) W(\psi,\chi), \tag{10.102}$$

in which the operator Q_j is the operator that multiplies functions of x by x_j, and to the three other analogous equations.

Denote as $\Phi^{(n)}$ the function denoted as Φ in Theorem 4.18, which is a rotation-invariant function in the null space of the harmonic oscillator $L^{(n)}$, and set $\Psi = W(\Phi^{(n)},\, \Phi^{(n)})$. One might expect that, as is the case in the one-dimensional analysis (Theorem 10.6), the function $(x,\,\xi) \mapsto \Psi(2^{-\frac{1}{2}}x,\, 2^{-\frac{1}{2}}\xi)$ should be a multiple of the function $\Phi^{(2n)}$: however, we shall explain the reason why this is not the case. The analogous fact, in the usual analysis, holds in any dimension (only Gaussian functions are involved then).

Let us use the equation (10.102) and the related ones. Since the function $\Phi^{(n)}$ is annihilated by the infinitesimal generators $M_{jk} = Q_j\,P_k - Q_k\,P_j$ of the rotation group, computing the Wigner function $W(\Phi^{(n)},\, M_{jk}\,\Phi^{(n)})$ as well as the one in which the operator M_{jk} acts on the left-hand side, one obtains that the function Ψ is annihilated by two operators: first, the first-order differential operator $x_j\,\frac{\partial}{\partial x_k} - x_k\,\frac{\partial}{\partial x_j} + \xi_j\,\frac{\partial}{\partial \xi_k} - \xi_k\,\frac{\partial}{\partial \xi_j}$, next a second-order differential operator N_{jk} which it is more difficult to take advantage of, and to which we shall come back later. Consequently, the function Ψ is invariant under the linear transformations associated with the matrices (in block-form) $\left(\begin{smallmatrix}\Omega & 0\\ 0 & \Omega\end{smallmatrix}\right)$ with $\Omega \in SO(n)$. Next, the function Φ lies in the null space of $L^{(n)}$: by the same trick, this leads again to a pair of equations, to wit

$$\left[\sum (x_j^2+\xi_j^2) - \frac{1}{16\pi^2}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}\right)\right]\Psi = 0, \qquad \sum \left(x_j\,\frac{\partial}{\partial \xi_j} - \xi_j\,\frac{\partial}{\partial x_j}\right)\Psi = 0. \tag{10.103}$$

The first of these equations means that the function $\Psi(2^{-\frac{1}{2}}x,\, 2^{-\frac{1}{2}}\xi)$ lies in the null space of the harmonic oscillator $L^{(2n)}$ in $2n$ variables; the second one means that Ψ is invariant under the one-parameter group of linear transformations, an infinitesimal generator of which is associated to the matrix $\left(\begin{smallmatrix}0 & I\\ -I & 0\end{smallmatrix}\right)$.

Now, except in dimension 1, a function Ψ of (x,ξ) invariant under the linear transformations $(x,\xi) \mapsto (\Omega x,\, \Omega\xi)$ with $\Omega \in SO(n)$ as well as under the transformation $(x,\xi) \mapsto (x\,\cos\theta - \xi\,\sin\theta,\, x\,\sin\theta + \xi\,\cos\theta)$ does not have to be rotation-invariant. In dimension 2 (*resp.* ≥ 3), it only has to be of the kind $F(\alpha,\,\beta)$ with $\alpha = \frac{1}{2}\,(|x|^2 + |\xi|^2)$ in both cases, and $\beta = x_1\xi_2 - x_2\xi_1$ (*resp.* $\frac{1}{2}\sum_{j,k}(x_j\,\xi_k - x_k\,\xi_j)^2$). The equation $N_{jk}\,\Psi = 0$ referred to above, in which, actually, $N_{jk} = x_j\,\xi_k - x_k\,\xi_j + \frac{1}{16\pi^2}\,(\frac{\partial^2}{\partial\xi_j\,\partial x_k} - \frac{\partial^2}{\partial\xi_k\,\partial x_j})$, becomes, in terms of $(\alpha,\,\beta)$, a second-order equation which reduces to the equation $\beta\,(\frac{\partial^2 F}{\partial\alpha^2} + \frac{\partial^2 F}{\partial\beta^2} - 16\,\pi^2\,F) + 2\alpha\,\frac{\partial^2 F}{\partial\alpha\,\partial\beta} = 0$ when $n = 2$, to a comparable one in higher dimension: together with the condition which expresses that the first equation (10.103) is satisfied, it should in principle make the function Ψ computable as a series, in a way resembling that used in the first part of Section 6; we have not completed the calculations.

Only observe that, as soon as $n \geq 2$, F cannot be a function of α alone. One may note that the method just developed gives a new proof of Theorem 10.6 (the one-dimensional case). Also, it may be useful to point again at the difference of structure – already felt in Section 6 – between the function $\Phi^{(n)}$ from the anaplectic analysis and the Gaussian function of the usual analysis. Contrary to the first one, the latter one is an eigenfunction of the *partial* harmonic oscillators as well (in particular, it is invariant under the partial Fourier transformations): this, ultimately, leads to the associated Wigner function being invariant under the group $\mathrm{Sp}(n,\mathbb{R}) \cap O(2n)$, which is sufficient to ensure that it depends only on $|x|^2 + |\xi|^2$.

Chapter 4

The One-dimensional Case Revisited

It has been our claim, based on the fact that the image of the ground state of the harmonic oscillator under the Heisenberg algebra is a dense subspace of $L^2(\mathbb{R}^n)$, that, in some sense, the harmonic oscillator leads in a natural way to a fairly wide section of classical analysis. In the first section of the book, it has been shown that, in the one-dimensional case, the same construction, starting with the even function in the null space of the harmonic oscillator in place of the Gaussian function, leads to a new species of analysis, to wit the anaplectic one: the spectrum of the harmonic oscillator is then $\mathbb{Z}$ instead of $\frac{1}{2} + \mathbb{N}$.

Our construction of the one-dimensional anaplectic analysis was based on the consideration of the complementary series of representations of $SL(2,\mathbb{R})$, as described in Section 2: it was only after some transformations that we arrived at the definition, given in Section 1, of the space $\mathfrak{A}$ in terms of the $\mathbb{C}^4$-realization of functions therein. As it turns out – but this is a fact that we discovered only long after we had embarked on the whole project – this presentation of the anaplectic analysis is just the one appropriate towards a more general theory, depending on one (complex) parameter.

We here show that, for any complex number ν mod 2, starting with the appropriate generalized eigenstate of the harmonic oscillator, one finds a new analysis, and a new space $\mathfrak{A}_\nu$. One has $\mathfrak{A}_{-\frac{1}{2}} = \mathfrak{A}$ but, for reasons to follow, the parameter ν must not be confused with the parameter ρ from the beginning of Section 2. We shall assume that $\nu \notin \mathbb{Z}$: for any value of $\nu \notin \mathbb{Z}$, the Heisenberg representation and some appropriate representation of some covering of $SL(2,\mathbb{R})$, to be called the ν-anaplectic representation, preserve the space $\mathfrak{A}_\nu$.

The ν-anaplectic representation coincides with the anaplectic representation from Section 1 in the case when $\nu = -\frac{1}{2}$ mod 2 and with the dual anaplectic representation as developed, on the space $\mathfrak{A}^\natural$, in Section 7, in the case when $\nu \in$

$\frac{1}{2}+2\,\mathbb{Z}$. If ν is a real number, half of it (its restriction to the subspace of even or odd functions according to whether $\nu \in]-1,0[+2\,\mathbb{Z}$ or $\nu \in]0,1[+2\,\mathbb{Z}$) is unitarizable and is equivalent to a certain representation taken from the series denoted as $C_q^{(\tau)}$ by Pukanszky in [18]. More precisely, the series of unitary representations from (*loc. cit.*) that fits within the present considerations is the one for which $q=\frac{3}{16}$: it does not participate in the Plancherel formula for the universal cover of $SL(2,\mathbb{R})$.

Our main interest in this development comes from the fact that it certainly changed our point of view about the role of the anaplectic representation and analysis in general, initially thought of as being *dual* to the usual analysis, but given now a more considerable range. To put things into a different perspective, in association with any complex number ν mod 2, there is a well-defined species of mathematical analysis (including a concept of integral and Fourier transformation and, in the case when $\nu \in \mathbb{R}$, a pseudo-scalar product) for which the spectrum of the even part of the harmonic oscillator is $\frac{1}{2}+\nu+2\,\mathbb{Z}$; that of the odd part is $-\frac{1}{2}+\nu+2\,\mathbb{Z}$. Only the case when $\nu \in \mathbb{Z}$ is excluded: it corresponds to the usual analysis.

11 The fourfold way and the ν-anaplectic representation

Let $L=\pi\,x^2-\frac{1}{4\pi}\frac{d^2}{dx^2}$ be the standard harmonic oscillator. Near each of the two endpoints $\pm\infty$ of the real line, the equation $Lf=(\nu+\frac{1}{2})\,f$ has two solutions, one that behaves like $|x|^{\nu}\,e^{-\pi\,x^2}$, the other like $|x|^{-\nu-1}\,e^{\pi\,x^2}$: this is an immediate consequence of the WKB method. It is only when $\nu=0,1,\ldots$ that one can find a solution of the equation on the whole real line (in this case a Hermite function) equivalent to a constant times $|x|^{\nu}\,e^{-\pi\,x^2}$ as well near $+\infty$ as near $-\infty$. If $\nu\notin\mathbb{N}$, one must satisfy oneself with a solution with a good behavior near $+\infty$ only, in which case it will, of course, be extremely far from lying in $L^2(\mathbb{R})$. In [17, chapter 8], such a generalized eigenfunction is called a parabolic cylinder function, and the solution of the equation $Lf=(\nu+\frac{1}{2})\,f$ normalized by the condition

$$f(x)\sim(2\,\pi^{\frac{1}{2}}\,x)^{\nu}\,e^{-\pi\,x^2},\qquad x\to+\infty, \tag{11.1}$$

is denoted as $f(x)=D_\nu(2\pi^{\frac{1}{2}}\,x)$.

Our first task in this section is to build an appropriate space $\mathfrak{A}_\nu$ of entire functions of one variable containing the function u that is the even part of the function f just defined, and stable under the Heisenberg algebra. We refer to Definition 1.1 for comparison: note that the definition that follows is exactly that of $\mathfrak{A}$ in the case when $\nu=-\frac{1}{2}$.

Definition 11.1. Let $\nu\notin\mathbb{Z}$, and consider the space of $\mathbb{C}^4$-valued functions $\boldsymbol{f}=(f_0,\,f_1,\,f_{i,0},\,f_{i,1})$ with the following properties: each component of $\boldsymbol{f}$ is a nice function in the sense of Definition 1.1, and the components are linked by the following equations:

$$f_{i,0}(x) = \frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{-\frac{i\pi}{2}(\nu+1)} f_0(ix) + e^{\frac{i\pi}{2}(\nu+1)} f_0(-ix) \right],$$
$$f_{i,1}(x) = \frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{-\frac{i\pi\nu}{2}} f_1(ix) + e^{\frac{i\pi\nu}{2}} f_1(-ix) \right]. \tag{11.2}$$

The space $\mathfrak{A}_\nu$ is the image of the space of functions so defined under the map $\boldsymbol{f} \mapsto u$, where the even (*resp.* odd) part of u is the even part of f_0 (*resp.* the odd part of f_1). We shall also refer to $\boldsymbol{f}$ as the $\mathbb{C}^4$-realization of u.

Remark 11.1: The space $\mathfrak{A}_\nu$ depends only on ν mod 2: for if $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ is a $\mathbb{C}^4$-realization of u when the parameter ν is considered, one may regard $(f_0,\, f_1,\, g_{i,0},\, g_{i,1})$, with

$$g_{i,0} = -((\nu+1)(\nu+2))^{-1} f_{i,0}, \qquad g_{i,1} = -((\nu+1)(\nu+2))^{-1} f_{i,1}, \tag{11.3}$$

as a $\mathbb{C}^4$-realization of u in the space $\mathfrak{A}_{\nu+2}$. We shall always assume, however, that a value of ν has been fixed: we do not make any limitation, to begin with, about the value of ν, save for the standing assumption that it should not be integral.

One should also note that, when $\nu \in \frac{1}{2} + 2\,\mathbb{Z}$, the space $\mathfrak{A}_\nu$ coincides with the space $\mathfrak{A}^\natural$ considered in Remark 1.3 and in Section 7: simply observe that if $(f_0,\, f_1,\, \tilde{f}_{i,0},\, \tilde{f}_{i,1})$ is the $\mathbb{C}^4$-realization of some function in $\mathfrak{A}^\natural$ in the sense of Theorem 7.3, then $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$, with $f_{i,0} = 2\,\tilde{f}_{i,0}$ and $f_{i,1} = -2\,\tilde{f}_{i,1}$, is a $\mathbb{C}^4$-realization of the same function in $\mathfrak{A}_{\frac{1}{2}}$, in the sense of Definition 11.1. The normalisation to follow of the integral – and, as a consequence, that of the Fourier transformation and of the whole analysis – is fully compatible, in the cases when $\nu = \mp\frac{1}{2}$, with that formerly introduced in $\mathfrak{A}$ and $\mathfrak{A}^\natural$ respectively.

Corollary 1.7, to the effect that the map $\boldsymbol{f} \mapsto u$ is one-to-one, immediately extends: that $\nu \notin 1 + 2\,\mathbb{Z}$ shows that f_0 is unique; that $\nu \notin 2\,\mathbb{Z}$ shows that f_1 is. Unless $\nu \in -\frac{1}{2} + \mathbb{Z}$, the space $\mathfrak{A}_\nu$ is not invariant under the complex rotation by ninety degrees. However, the following holds:

Proposition 11.2. *The map $u \mapsto u_i$, with $u_i(x) = u(ix)$, is a linear isomorphism from $\mathfrak{A}_\nu$ to $\mathfrak{A}_{-\nu-1}$. If $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ is the $\mathbb{C}^4$-realization of u in the $\mathfrak{A}_\nu$-analysis, that of u_i in the $\mathfrak{A}_{-\nu-1}$-analysis is*

$$(h_0,\, h_1,\, h_{i,0},\, h_{i,1}) = C_\nu\, (f_{i,0},\, -i\, f_{i,1},\, f_0,\, -i\, f_1,) \tag{11.4}$$

with

$$C_\nu = 2^{\nu+\frac{1}{2}}\, \frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})}. \tag{11.5}$$

Proof. The equations (11.2) can be inverted as

$$f_0(x) = \frac{\Gamma(1+\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{\frac{i\pi\nu}{2}} f_{i,0}(ix) + e^{-\frac{i\pi\nu}{2}} f_{i,0}(-ix) \right],$$
$$f_1(x) = \frac{\Gamma(1+\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{\frac{i\pi}{2}(\nu+1)} f_{i,1}(ix) + e^{-\frac{i\pi}{2}(\nu+1)} f_{i,1}(-ix) \right]. \tag{11.6}$$

It is then immediate that the $\mathbb{C}^4$-valued function $\boldsymbol{h}$ defined above qualifies as the $\mathbb{C}^4$-realization of some function in $\mathfrak{A}_{-\nu-1}$: that this function is just u_i is proved in a straightforward way, only relying on the duplication formula (1.89). □

Theorem 11.3. *For any $\nu \notin \mathbb{Z}$, the Heisenberg transformations $\pi(y,\eta)$, with $(y,\eta) \in \mathbb{C}^2$, preserve the space $\mathfrak{A}_\nu$. The $\mathbb{C}^4$-realizations $\boldsymbol{h}$ and $\boldsymbol{g}$ of $\pi(y,0)\,u$ and $\pi(0,\eta)\,u$ respectively are given in terms of the $\mathbb{C}^4$-realization of u by the same relations as in Proposition* 1.11.

Proof. For instance, we claim that

$$h_0(x) = \frac{1}{2}\,(f_0(x-y) + f_0(x+y) + f_1(x-y) - f_1(x+y)),$$
$$h_1(x) = \frac{1}{2}\,(f_0(x-y) - f_0(x+y) + f_1(x-y) + f_1(x+y)),$$
$$h_{i,0}(x) = \frac{1}{2}\,(f_{i,0}(x-iy) + f_{i,0}(x+iy) + i\,f_{i,1}(x-iy) - i\,f_{i,1}(x+iy)),$$
$$h_{i,1}(x) = \frac{1}{2}\,(-i\,f_{i,0}(x-iy) + i\,f_{i,0}(x+iy) + f_{i,1}(x-iy) + f_{i,1}(x+iy)): \tag{11.7}$$

the very tedious verification that the link between (h_0,h_1) on one hand, $(h_{i,0},h_{i,1})$ on the other hand, is still given by the equations (11.2), is of course straightforward; that $(\pi(y,0)\,u)_{\text{even}}$ (*resp.* $(\pi(y,0)\,u)_{\text{odd}}$) coincides with the even part of h_0 (*resp.* the odd part of h_1) does not require a new verification. □

We define the $\mathcal{Q}$-transform and the $\mathcal{K}$-transform of a function $u \in \mathfrak{A}_\nu$ by the same formulas as the ones from Section 4:

$$(\mathcal{Q}\,u)_0(\sigma) = \int_{-\infty}^{\infty} e^{-\pi\,\sigma x^2}\, u(x\,e^{-\frac{i\pi}{4}})\,dx, \qquad \sigma \text{ large} \tag{11.8}$$

and, for z on the unit circle, $z = e^{-i\theta}$ with $\theta > 0$ and small,

$$(\mathcal{K}\,u)_0(z) = |1-z|^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_0\left(i\,\frac{1+z}{1-z}\right). \tag{11.9}$$

Similarly, (*cf.* Definition 4.1),

$$(\mathcal{Q}\,u)_1(\sigma) = \int_{-\infty}^{\infty} (1+i\,\sigma)\,x\; e^{-\pi\,\sigma x^2}\, u(x\,e^{-\frac{i\pi}{4}})\,dx, \tag{11.10}$$

and the transform from $(\mathcal{Q}\,u)_1$ to $(\mathcal{K}\,u)_1$ is the same as that from $(\mathcal{Q}\,u)_0$ to $(\mathcal{K}\,u)_0$.

Proposition 11.4. *Let $u \in \mathfrak{A}_\nu$ be associated with the four-vector $\boldsymbol{f}$. The $\mathcal{Q}$-transform of u extends as an analytic function on the real line, given by the equations*

$$(\mathcal{Q}\,u)_0(\sigma) = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})} \times \left[2\,\cos\frac{\pi\nu}{2}\int_0^\infty e^{-i\pi\sigma\,x^2}\,f_0(x)\,dx + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty e^{i\pi\sigma\,x^2}\,f_{i,0}(x)\,dx\right] \tag{11.11}$$

and

$$(\mathcal{Q}\,u)_1(\sigma) = e^{\frac{i\pi\,\nu}{2}}\left[2i\,\cos\frac{\pi\nu}{2}\int_0^\infty (1+i\,\sigma)\,x\,e^{-i\pi\sigma\,x^2}\,f_1(x)\,dx + i\,\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty (1+i\,\sigma)\,x\,e^{i\pi\sigma\,x^2}\,f_{i,1}(x)\,dx\right]. \tag{11.12}$$

Proof. The proof is similar to the beginning of that of Theorem 1.8. We set $w_0 = (\mathcal{Q}\,u)_0$ (note that, for consistency with (1.26), we cannot use here the notation w_1 for $(\mathcal{Q}\,u)_1$) and

$$w_0^+(\sigma) = \int_0^\infty e^{-\pi\sigma\,x^2}\,f_0(x\,e^{-\frac{i\pi}{4}})\,dx = e^{\frac{i\pi}{4}}\int_0^\infty e^{-i\pi\sigma\,x^2}\,f_0(x)\,dx,$$
$$w_0^-(\sigma) = \int_0^\infty e^{-\pi\sigma\,x^2}\,f_0(x\,e^{\frac{3i\pi}{4}})\,dx. \tag{11.13}$$

With $\lambda = e^{\frac{i\pi}{4}}\,(1+e^{i\pi\nu})$, one has

$$w_0(\sigma) - \lambda\int_0^\infty e^{-i\pi\sigma\,x^2}\,f_0(x)\,dx = \int_0^\infty e^{-\pi\sigma\,x^2}\,[-e^{i\pi\nu}\,f_0(x\,e^{-\frac{i\pi}{4}}) + f_0(x\,e^{\frac{3i\pi}{4}})]\,dx$$
$$= \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\,e^{\frac{i\pi}{2}\,(\nu+1)}\int_0^\infty e^{-\pi\sigma\,x^2}\,f_{i,0}(x\,e^{\frac{i\pi}{4}})\,dx: \tag{11.14}$$

with a new change of contour of integration in the last integral written, this leads to the equation (11.11). The proof of (11.12) is the same, substituting $\mu = i\,(1+e^{i\pi\nu})$ for λ. □

We can now generalize Theorem 1.8, as well as a part of Theorem 4.14.

Theorem 11.5. *Let $\nu \in \mathbb{C}$, $\nu \notin \mathbb{Z}$, and let $u \in \mathfrak{A}_\nu$. For a certain sequence $(a_n)_{n\geq 0}$ of coefficients, one has for $\sigma \in \mathbb{R}$ with $|\sigma|$ large enough the convergent expansion*

$$(\mathcal{Q}\,u)_0(\sigma) = e^{-\frac{i\pi}{2}\,(\nu+\frac{1}{2})\,\mathrm{sign}\,\sigma}\sum_{n\geq 0} a_n\,\sigma^{-n}\,|\sigma|^{-\frac{1}{2}}, \tag{11.15}$$

and a similar one for $(\mathcal{Q}\,u)_1(\sigma)$.

The $\mathbb{C}^4$-realization of u can be recovered, in terms of u, by the following four formulas, valid for $x > 0$ and involving semi-convergent integrals:

$$\begin{aligned} f_0(x) &= \frac{e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})}}{2\cos\frac{\pi\nu}{2}}\, x \int_{-\infty}^{\infty} (\mathcal{Q}\,u)_0(\sigma)\, e^{i\pi\sigma\, x^2}\, d\sigma, \\ f_{i,0}(x) &= (2\pi)^{-\frac{1}{2}}\, e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})}\, \Gamma(-\nu)\, x \int_{-\infty}^{\infty} (\mathcal{Q}\,u)_0(\sigma)\, e^{-i\pi\sigma\, x^2}\, d\sigma, \\ f_1(x) &= \frac{e^{-\frac{i\pi(\nu+1)}{2}}}{2\cos\frac{\pi\nu}{2}} \int_{-\infty}^{\infty} \frac{(\mathcal{Q}\,u)_1(\sigma)}{1+i\,\sigma}\, e^{i\pi\sigma\, x^2}\, d\sigma, \\ f_{i,1}(x) &= (2\pi)^{-\frac{1}{2}}\, e^{-\frac{i\pi(\nu+1)}{2}}\, \Gamma(-\nu) \int_{-\infty}^{\infty} \frac{(\mathcal{Q}\,u)_1(\sigma)}{1+i\,\sigma}(\sigma)\, e^{-i\pi\sigma\, x^2}\, d\sigma. \end{aligned} \tag{11.16}$$

The $\mathcal{K}$-transform of u extends as a pair of analytic functions on the universal cover of the circle $\Sigma = \{z\colon |z| = 1\}$, such that

$$\begin{aligned} (\mathcal{K}\,u)_0(e^{i\theta}) &= e^{-i\pi(\nu+\frac{1}{2})}\, (\mathcal{K}\,u)_0(e^{i(\theta-2\pi)}), \\ (\mathcal{K}\,u)_1(e^{i\theta}) &= e^{-i\pi(\nu+\frac{1}{2})}\, (\mathcal{K}\,u)_1(e^{i(\theta-2\pi)}). \end{aligned} \tag{11.17}$$

Proof. The validity of the pair of expansions (11.15) for σ real and large is proved in just the same way as the corresponding one in Theorem 1.8: of course, for the time being, the extra phase factor in front of the right-hand sides of (11.15) should be regarded as a matter of convenience.

Again, we denote as $\tilde{w}_0 = (\widetilde{\mathcal{Q}\,u})_0$ and $(\widetilde{\mathcal{Q}\,u})_1$ the holomorphic extensions of $w_0 = (\mathcal{Q}\,u)_0$ and $(\mathcal{Q}\,u)_1$, initially considered for σ real and large, to the part of the Riemann surface of the square-root function lying above some set $\{z\colon |z| > R\}$ with a large R. The first equation (1.37) can be carried out without any change, and the second one becomes

$$(\widetilde{\mathcal{Q}\,u})_1(\sigma\, e^{i\pi}) = \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, (1 - i\,\sigma)\, x\, u_i(x\, e^{-\frac{i\pi}{4}})\, dx. \tag{11.18}$$

In other words, for large σ,

$$\tilde{w}_0(\sigma\, e^{i\pi}) = -i\, (\mathcal{Q}\,u_i)_0(\sigma), \qquad (\widetilde{\mathcal{Q}\,u})_1(\sigma\, e^{i\pi}) = \frac{1-i\,\sigma}{1+i\,\sigma}\, (\mathcal{Q}\,u_i)_1(\sigma). \tag{11.19}$$

Not forgetting that u_i lies in $\mathfrak{A}_{-\nu-1}$ rather than $\mathfrak{A}_\nu$, we may use the result of Proposition 11.2, coupled with that of Theorem 11.3, to find the following equation after some straightforward calculations with the Gamma function:

$$\begin{aligned} \tilde{w}_0(\sigma\, e^{i\pi}) = e^{-\frac{i\pi}{2}(\nu+\frac{3}{2})} \Bigg[& \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} \int_0^{\infty} e^{-i\pi\sigma\, x^2}\, f_{i,0}(x)\, dx \\ & + 2\cos\frac{\pi\nu}{2} \int_0^{\infty} e^{i\pi\sigma\, x^2}\, f_0(x)\, dx \Bigg], \end{aligned} \tag{11.20}$$

in other words

$$(\widetilde{\mathcal{Q}\,u})_0(\sigma\,e^{i\pi}) = e^{-i\pi\,(\nu+1)}\,(\mathcal{Q}\,u)_0(-\sigma):\tag{11.21}$$

in a similar way,

$$(\widetilde{\mathcal{Q}\,u})_1(\sigma\,e^{i\pi}) = e^{-i\pi\,(\nu+1)}\,(\mathcal{Q}\,u)_1(-\sigma).\tag{11.22}$$

The last two equations make it possible to derive the expansions (11.15) near $-\infty$ from those near $+\infty$.

The equations (11.16) can be obtained from (11.11) and (11.12) by inverting a Fourier transformation.

From (11.9) and (11.16) it follows (since $\sigma = \text{cotan}\,\frac{\theta}{2}$ corresponds to $z = e^{-i\theta}$ under the Cayley map, and $|1-z| = 2\,\sin\frac{\theta}{2}$ for $0<\theta<2\pi$) that, for $\theta>0$ and small, one has

$$(\mathcal{K}\,u)_0(e^{-i\theta}) = 2^{-\frac12}\,e^{-\frac{i\pi}{2}\,(\nu+\frac12)}\sum_{n\geq 0} a_n\left(\tan\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}\right)^{-\frac12}.\tag{11.23}$$

As explained in the proof of Theorem 4.14, if $0<\theta<2\pi$, the point $e^{-i(2\pi-\theta)}\in\Sigma$ can be reached as the image of $-\sigma$ under the Cayley map, so that, using (11.21),

$$(\mathcal{K}\,u)_0(e^{-i\,(2\pi-\theta)}) = 2^{-\frac12}\,e^{\frac{i\pi}{2}\,(\nu+\frac12)}\sum_{n\geq 0} a_n\left(-\tan\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}\right)^{-\frac12}.\tag{11.24}$$

On the other hand, the analytic extension of $(\mathcal{K}\,u)_0$ near $z=1$ (*cf.* Theorem 4.11) leads, for $\theta>0$ and small, starting from (11.23) again, to

$$(\mathcal{K}\,u)_0(e^{i\theta}) = 2^{-\frac12}\,e^{-\frac{i\pi}{2}\,(\nu+\frac12)}\sum_{n\geq 0} a_n\left(-\tan\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}\right)^{-\frac12}.\tag{11.25}$$

The first equation (11.17) follows; the second is proved in the same way. □

We now compute the $\mathcal{K}$-transform of some basic function $\phi^\nu\in\mathfrak{A}_\nu$. Since [17, p. 326] $D_{-\frac12}(2\pi^{\frac12}x) = \pi^{-\frac14}\,x^{\frac12}\,K_{\frac14}(\pi\,x^2)$, one sees, comparing $\phi^{-\frac12}$ to the function ϕ introduced in Proposition 1.2, that $\phi = 2^{\frac12}\,\pi^{-\frac14}\,\phi^{-\frac12}$: we have not deemed it necessary to have the present normalization of $\phi^{-\frac12}$ (made for simplicity in the definition of the $\mathbb{C}^4$-realization) agree, in the case when $\nu=-\frac12$, with that of ϕ from Section 1; anyway, it is only when ν is real and $\nu\in]-1,0[\cup 2\,\mathbb{Z}$ that the even part of the ν-anaplectic representation to be introduced below turns out to be unitarizable.

Theorem 11.6. *Let $\nu\notin\mathbb{Z}$, and let ϕ^ν be the function in $\mathfrak{A}_\nu$ the $\mathbb{C}^4$-realization of which is the function*

$$\boldsymbol{f}(x) = (D_\nu(2\pi^{\frac12}\,x),\,0,\,D_{-\nu-1}(2\pi^{\frac12}\,x),\,0).\tag{11.26}$$

One has, for $0 < \theta < 2\pi$,

$$(\mathcal{K}\,\phi^\nu)_0(e^{-i\theta}) = \frac{2^{\frac{\nu-1}{2}}\,\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})}\;e^{\frac{i}{2}\,(\nu+\frac{1}{2})\,\theta}. \tag{11.27}$$

Proof. That the functions f_0 and $f_{i,0}$ taken from the components of $\boldsymbol{f}$ are linked by the equation (11.2) is a consequence of [17, p. 325]. As given in [17, p. 330], one has if $\xi > 0$ the equation

$$\int_0^\infty e^{-\pi\xi\,x^2}\,D_\nu(2\pi^{\frac{1}{2}}\,x)\,dx = \frac{2^{\frac{\nu-2}{2}}}{\Gamma(\frac{2-\nu}{2})}\,(1+\xi)^{-\frac{1}{2}}\;{}_2F_1\left(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{2-\nu}{2};\,\frac{\xi-1}{\xi+1}\right): \tag{11.28}$$

thus, with the principal determination of the square root function in the right half-plane,

$$\begin{aligned}\int_0^\infty e^{-i\pi\sigma\,x^2}\,f_0(x)\,dx &= \frac{2^{\frac{\nu-2}{2}}}{\Gamma(\frac{2-\nu}{2})}\,(1+i\,\sigma)^{-\frac{1}{2}}\;{}_2F_1\left(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{2-\nu}{2};\,\frac{\sigma+i}{\sigma-i}\right),\\ \int_0^\infty e^{i\pi\sigma\,x^2}\,f_{i,0}(x)\,dx &= \frac{2^{\frac{-\nu-3}{2}}}{\Gamma(\frac{3+\nu}{2})}\,(1-i\,\sigma)^{-\frac{1}{2}}\;{}_2F_1\left(\frac{1+\nu}{2},\,\frac{1}{2};\,\frac{3+\nu}{2};\,\frac{\sigma-i}{\sigma+i}\right).\end{aligned} \tag{11.29}$$

Thus, after some calculations using the duplication formula of the Gamma function,

$$\begin{aligned}(\mathcal{Q}\,\phi^\nu)_0(\sigma) = A\,(1+i\,\sigma)^{-\frac{1}{2}}\;{}_2F_1\left(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{2-\nu}{2};\,\frac{\sigma+i}{\sigma-i}\right)&\\ +\,B\,(1-i\,\sigma)^{-\frac{1}{2}}\;{}_2F_1\left(\frac{1+\nu}{2},\,\frac{1}{2};\,\frac{3+\nu}{2};\,\frac{\sigma-i}{\sigma+i}\right)&\end{aligned} \tag{11.30}$$

with

$$A = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}\,\frac{2^{\frac{\nu}{2}}\,\pi}{\Gamma(\frac{1+\nu}{2})\,\Gamma(\frac{1-\nu}{2})\,\Gamma(\frac{2-\nu}{2})},\qquad B = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}\,\frac{2^{\frac{\nu}{2}}\,\pi}{\Gamma(\frac{1-\nu}{2})\,\Gamma(-\frac{\nu}{2})\,\Gamma(\frac{3+\nu}{2})}. \tag{11.31}$$

Now, if $z = e^{-i\theta}$, $0 < \theta < 2\pi$, one has for every $u \in \mathfrak{A}_\nu$,

$$(\mathcal{K}\,u)_0(z) = \left(2\,\sin\,\frac{\theta}{2}\right)^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_0\left(\mathrm{cotan}\,\frac{\theta}{2}\right), \tag{11.32}$$

which leads to

$$\begin{aligned}(\mathcal{K}\,\phi^\nu)_0(e^{-i\theta}) = 2^{-\frac{1}{2}}\Bigg[e^{-\frac{i\pi}{4}}\,e^{\frac{i\theta}{4}}\,A\;\,{}_2F_1\left(-\frac{\nu}{2},\,\frac{1}{2};\,\frac{2-\nu}{2};\,e^{i\theta}\right)&\\ +\,e^{\frac{i\pi}{4}}\,e^{-\frac{i\theta}{4}}\,B\;\,{}_2F_1\left(\frac{1+\nu}{2},\,\frac{1}{2};\,\frac{3+\nu}{2};\,e^{-i\theta}\right)\Bigg].&\end{aligned} \tag{11.33}$$

As unpalatable as they may seem, these computations offer two safeguards: first, the fact that we already know the result in the case when $\nu = -\frac{1}{2}$ – yes, (11.27) is compatible with the result in the last paragraph of the proof of Theorem 4.14, taking into consideration the coefficient α from (9.35) – next, that the function $\theta \mapsto (\mathcal{K}\,\phi^\nu)_0(e^{-i\theta})$ must extend as an analytic function near 0 in view of Theorem 4.11. Now, in view of the singularities of the two hypergeometric functions occurring in (11.33) near the point $z = 1$, this can only happen if the two singularities cancel off. They do, thanks to the equation [17, p. 48]

$$ {}_2F_1(\frac{1+\nu}{2}, \frac{1}{2}; \frac{3+\nu}{2}; e^{-i\theta}) = \frac{\Gamma(\frac{3+\nu}{2})\,\Gamma(-\frac{\nu}{2})}{\Gamma(\frac{1}{2})}\, e^{-\frac{i\pi\,(\nu+1)}{2}}\, e^{\frac{i(\nu+1)\,\theta}{2}} + \frac{\Gamma(\frac{3+\nu}{2})\,\Gamma(\frac{\nu}{2})}{\Gamma(\frac{1+\nu}{2})\,\Gamma(\frac{2+\nu}{2})}\,(-i\,e^{\frac{i\theta}{2}})\; {}_2F_1\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{2-\nu}{2}; e^{i\theta}\right): \tag{11.34} $$

the end of the computation is straightforward. □

Remark 11.2. As a consequence of [17, p. 330], the function ϕ^ν itself, *i.e.*, the even part of the first component of its C^4-realization (11.26), is characterized by the rather simple integral expression, valid if $-2 < \mathrm{Re}\,\nu < 0$,

$$ \phi^\nu(x\,e^{-\frac{i\pi}{4}}) = \frac{\pi^{-\frac{\nu}{2}}}{\Gamma(-\nu)}\, e^{\frac{i\pi\nu}{4}}\, e^{i\pi\,x^2} \int_0^\infty e^{\frac{i\pi\,t^2}{2}}\, t^{-\nu-1} \cos 2\pi x t\; dt. \tag{11.35} $$

Besides the function ϕ^ν introduced in Theorem 11.5, we now introduce, still in the $\mathfrak{A}_\nu$-analysis, the odd function $\psi^{\nu+1}$ with the $\mathbb{C}^4$-realization

$$ \boldsymbol{h}(x) = (0,\, D_{\nu+1}(2\pi^{\frac{1}{2}}\, x),\, 0,\, (\nu+1)\, D_{-\nu-2}(2\pi^{\frac{1}{2}}\, x)). \tag{11.36} $$

Again, the link (11.2) between the two non-zero components, which can be written, since $\frac{\Gamma(-\nu)}{\nu+1} = -\Gamma(-\nu-1)$, as

$$ \frac{\Gamma(-\nu-1)}{(2\pi)^{\frac{1}{2}}}\,[e^{-\frac{i\pi\nu}{2}}\, D_{\nu+1}(it) + e^{\frac{i\pi\nu}{2}}\, D_{\nu+1}(-it)] = -D_{-\nu-2}(t), \tag{11.37} $$

can be found in [17, p. 328].

In the next lemma and its proof, and in the proof of the proposition that follows, we (abusively) abbreviate as D_μ, for any value of μ, the function $x \mapsto D_\mu(2\pi^{\frac{1}{2}}\, x)$: no confusion can arise. Recalling that $A = \pi^{\frac{1}{2}}\,(x + \frac{1}{2\pi}\frac{d}{dx})$, $A^* = \pi^{\frac{1}{2}}\,(x - \frac{1}{2\pi}\frac{d}{dx})$, note the relations

$$ A\, D_\nu = \nu\, D_{\nu-1}, \qquad\qquad A^*\, D_\nu = D_{\nu+1} \tag{11.38} $$

which, made explicit as

$$ \begin{aligned} &\pi^{\frac{1}{2}} x\, D_\nu(2\pi^{\frac{1}{2}}\, x) + D'_\nu(2\pi^{\frac{1}{2}}\, x) = \nu\, D_{\nu-1}(2\pi^{\frac{1}{2}}\, x), \\ &\pi^{\frac{1}{2}} x\, D_\nu(2\pi^{\frac{1}{2}}\, x) - D'_\nu(2\pi^{\frac{1}{2}}\, x) = D_{\nu+1}(2\pi^{\frac{1}{2}}\, x), \end{aligned} \tag{11.39} $$

can again be found in (*loc. cit.*, p. 327).

Lemma 11.7. *In the $\mathfrak{A}_\nu$-analysis, for every $j \in \mathbb{Z}$, the $\mathbb{C}^4$-realization of $\phi^{\nu+2j}$ is*

$$\left(D_{\nu+2j},\, 0,\, (-1)^j\, \frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}\, D_{-\nu-2j-1},\, 0\right)$$

and that of $\psi^{\nu+2j+1}$ is

$$\left(0,\, D_{\nu+2j+1},\, 0,\, (-1)^j\, \frac{\Gamma(\nu+2j+2)}{\Gamma(\nu+1)}\, D_{-\nu-2j-2}\right).$$

Proof. In the $\mathfrak{A}_{\nu+2j}$-analysis, the $\mathbb{C}^4$-realization of $\phi^{\nu+2j}$ is

$$(D_{\nu+2j},\, 0,\, D_{-\nu-2j-1},\, 0)$$

and that of $\psi^{\nu+2j+1}$ is

$$(0,\, D_{\nu+2j+1},\, 0,\, (\nu+2j+1)\, D_{-\nu-2j-2}).$$

On the other hand, Remark 11.1 indicates how to link the $\mathbb{C}^4$-realizations of the same function in any pair of spaces $\mathfrak{A}_\mu$ and $\mathfrak{A}_{\mu+2}$. Let us give one example of such a calculation, say that corresponding to an odd function $\psi^{\nu+2j+1}$, under the assumption that $j \geq 0$. Only the second and fourth components of the $\mathbb{C}^4$-realizations under consideration are non-zero: the second one is the same in both cases while, in order to transform the $f_{i,1}$-component with respect to the realization in the $\mathfrak{A}_{\nu+2j}$-analysis into the corresponding one in $\mathfrak{A}_\nu$-analysis, we must multiply it by $(-1)^j\, \frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}$. □

Proposition 11.8. *Let $\nu \in \mathbb{C}\backslash\mathbb{Z}$. The eigenfunctions of the* (*formal*) *harmonic oscillator L lying in the space $\mathfrak{A}_\nu$ are exactly, up to the multiplication by arbitrary constants, the functions $\phi^{\nu+2j}$ and $\psi^{\nu+2j+1}$ with $j \in \mathbb{Z}$: the first* (resp. *the second*) *one corresponds to the eigenvalue $\nu+2j+\frac{1}{2}$* (resp. $\nu+2j+\frac{3}{2}$). *One has the relations*

$$A\,\phi^{\nu+2j} = (\nu+2j)\,\psi^{\nu+2j-1}, \qquad A^*\,\phi^{\nu+2j} = \psi^{\nu+2j+1},$$
$$A\,\psi^{\nu+2j+1} = (\nu+2j+1)\,\phi^{\nu+2j}, \qquad A^*\,\psi^{\nu+2j+1} = \phi^{\nu+2j+2}. \tag{11.40}$$

Proof. Since it is just as easy here as in Section 1 to show that the space $\mathfrak{A}_\nu$ is invariant under the symmetry $u \mapsto \check{u}$, one may assume that one is dealing with an eigenfunction of L with a definite parity. Now, a function ϕ^μ or $\psi^{\mu+1}$ can only lie in $\mathfrak{A}_\nu$ if $\mu-\nu \in 2\,\mathbb{Z}$, as shown by the equations (11.17). In view of the expression of L as a product of annihilation and creation operators, all that remains to be done is checking the equations (11.40). In order to do this, we first observe that if $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ is the $\mathbb{C}^4$-realization of some function $u \in \mathfrak{A}_\nu$, that of $A\,u$ is $(A\,f_1,\, A\,f_0,\, A^*\,f_{i,1},\, -A^*\,f_{i,0})$ and that of $A^*\,u$ is $(A^*\,f_1,\, A^*\,f_0,\, A\,f_{i,1},\, -A\,f_{i,0})$. To see this, we just combine the two equations (1.62) and (1.63), still valid in our case (they are also the infinitesimal version of (11.7) and the analogous one, though a direct proof is even easier). Let us then check, for instance, the equation for $A\,\psi^{\nu+2j+1}$: the $\mathbb{C}^4$-realization of this function in $\mathfrak{A}_\nu$ is

$$\begin{aligned}(A\,D_{\nu+2j+1},\,0,\,A^*\,(-1)^j\,&\frac{\Gamma(\nu+2j+2)}{\Gamma(\nu+1)}\,D_{\nu-2j-2},\,0)\\ &=((\nu+2j+1)\,D_{\nu+2j},\,0,\,(-1)^j\,\frac{\Gamma(\nu+2j+2)}{\Gamma(\nu+1)}\,D_{\nu-2j-1},\,0)\\ &=(\nu+2j+1)\,(D_{\nu+2j},\,0,\,(-1)^j\,\frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}\,D_{\nu-2j-1},\,0),\end{aligned}\tag{11.41}$$

i.e., the $\mathbb{C}^4$-realization of the function $(\nu+2j+1)\,\phi^{\nu+2j}$. □

Here is a converse to Theorem 11.5, which extends Theorem 4.14.

Theorem 11.9. *Let u be an entire function of one variable, satisfying for some pair C, R of positive constants the estimate $|u(z)|\leq C\,e^{\pi R\,|z|^2}$: recall – as the one-dimensional case of Theorem* 4.11 *– that, for $\theta>0$ and small, the functions $(\mathcal{K}\,u)_0(e^{-i\theta})$ and $(\mathcal{K}\,u)_1(e^{-i\theta})$ are analytic functions of the variable $z=e^{-i\theta}$, and assume that they extend as analytic functions on the universal cover of $\Sigma=S^1$ satisfying the quasi-periodicity conditions* (11.17)*. Then u lies in the space $\mathfrak{A}_\nu$.*

Proof. It is no loss of generality to assume, in the present proof, that $-\frac{3}{2}\leq$ Re $\nu\leq\frac{1}{2}$ (*cf.* Remark 11.1), a condition invariant under the change $\nu\mapsto-1-\nu$. The proof splits into two quite similar parts according to the parity of u: we shall assume first that it is an even function. Denote as $z\mapsto z^{-(\frac{\nu}{2}+\frac{1}{4})}$ the analytic function on the real line (viewed as the universal cover of the unit circle) that coincides with $e^{\frac{i\theta}{2}\,(\nu+\frac{1}{2})}$ on the point $z=e^{-i\theta}$ with $0<\theta<2\pi$: the condition (11.17) means that $(\mathcal{K}\,u)_0(z)$ is the product of $z^{-(\frac{\nu}{2}+\frac{1}{4})}$ by a function analytic on the unit circle. We define the coefficients c_n, $n\in\mathbb{Z}$, by the equation

$$(\mathcal{K}\,u)_0(z)=z^{-(\frac{\nu}{2}+\frac{1}{4})}\sum_{n\in\mathbb{Z}}c_n\,z^n\,:\tag{11.42}$$

note that $|c_n|\leq C\,(1+\varepsilon)^{-|n|}$ for some pair of positive numbers C, ε. We shall take advantage of Theorem 11.6 and rebuild the $\mathbb{C}^4$-realization $(f_0,\,0,\,f_{i,0},\,0)$ of u by *defining* f_0 as the sum of the series

$$f_0(z)=\sum_{n\in\mathbb{Z}}c_n\,2^{n+\frac{1-\nu}{2}}\,\pi^{-\frac{1}{2}}\,\Gamma\left(\frac{1-\nu}{2}+n\right)\;D_{\nu-2n}(2\pi^{\frac{1}{2}}\,z).\tag{11.43}$$

We use when $n>\frac{\text{Re }\nu}{2}$ the integral formula [17, p. 328]

$$D_{\nu-2n}(2\pi^{\frac{1}{2}}\,z)=\frac{e^{-\pi z^2}}{\Gamma(-\nu+2n)}\int_0^\infty t^{-\nu+2n-1}\,e^{-(\frac{t^2}{2}+2\pi^{\frac{1}{2}}\,zt)}\,dt\tag{11.44}$$

and, when $n < \frac{\mathrm{Re}\ \nu+1}{2}$, the formula (*loc. cit.*)

$$D_{\nu-2n}(2\pi^{\frac{1}{2}}\, z) = (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{\pi z^2} \int_0^\infty e^{-\frac{t^2}{2}} \cos\left(\frac{\pi\nu}{2} - 2\pi^{\frac{1}{2}}\, zt\right) t^{\nu-2n}\, dt. \tag{11.45}$$

We note, for large n, the Stirling estimate

$$\left|\frac{\Gamma(\frac{1-\nu}{2}+n)}{\Gamma(-\nu+2n)}\right| = \pi^{\frac{1}{2}} \left|\frac{2^{1+\nu-2n}}{\Gamma(-\frac{\nu}{2}+n)}\right| \leq C\ 2^{-2n}\, n^{-n+\frac{1}{2}+\frac{\mathrm{Re}\ \nu}{2}}\, e^n \leq C\ 2^{-2n}\, n^{-n+\frac{3}{4}}\, e^n \tag{11.46}$$

and, when $-n$ is large, the estimate

$$\left|\Gamma(\frac{1-\nu}{2}+n)\right| \leq C\ |n|^{-|n|-\frac{\mathrm{Re}\ \nu}{2}}\, e^{|n|} \leq C\ |n|^{-|n|+\frac{3}{4}}\, e^{|n|}. \tag{11.47}$$

From what precedes, there exist two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\leq -1}$ such that

$$\sum_{n\geq 1} c_n\, 2^{n+\frac{1-\nu}{2}}\, \pi^{-\frac{1}{2}}\, \Gamma\left(\frac{1-\nu}{2}+n\right)\ D_{\nu-2n}(2\pi^{\frac{1}{2}}\, z)$$
$$= e^{-\pi z^2} \sum_{n\geq 1} a_n \int_0^\infty t^{-\nu+2n-1}\, e^{-(\frac{t^2}{2}+2\pi^{\frac{1}{2}}\, zt)}\, dt \tag{11.48}$$

and

$$\sum_{n\leq -1} \cdots = e^{\pi z^2} \sum_{n\leq -1} b_n \int_0^\infty t^{\nu-2n}\, e^{-\frac{t^2}{2}} \cos\left(\frac{\pi\nu}{2} - 2\pi^{\frac{1}{2}}\, zt\right) dt : \tag{11.49}$$

moreover, one has the estimates

$$\begin{aligned} |a_n| &\leq C\,(1+\varepsilon)^{-n}\, 2^{-n}\, n^{-n+\frac{3}{4}}\, e^n, \\ |b_n| &\leq C\,(1+\varepsilon)^{-|n|}\, 2^{-|n|}\, |n|^{-|n|+\frac{3}{4}}\, e^{|n|}, \end{aligned} \tag{11.50}$$

so that

$$\sum_{n\geq 1} |a_n|\, t^{-\mathrm{Re}\ \nu+2n-1} \leq C\, n^{\frac{5}{4}}\, (n\,!)^{-1}\, (2\,(1+\varepsilon))^{-n}\, t^{-\mathrm{Re}\ \nu+2n-1}$$
$$\leq C\left(1 + t^{\frac{5}{2}}\, \exp\, \frac{t^2}{2\,(1+\varepsilon)}\right) : \tag{11.51}$$

the same estimate holds for the sum $\sum_{n\leq -1} |b_n|\, t^{\mathrm{Re}\ \nu-2n}$. This shows the convergence of the series (11.43) defining $f_0(z)$, at the same time proving that the entire function $f_0(z)$ is bounded by some exponential of the kind $C\, e^{\pi R\, |z|^2}$.

Starting from f_0 as defined in (11.43) and using Lemma 11.7, one finds (changing n to $-n$ as an index in the series)

$$f_{i,0}(z) = \Gamma(-\nu) \sum_{n\in\mathbb{Z}} c_{-n}\, 2^{-n+\frac{1-\nu}{2}}\, \pi^{-\frac{1}{2}}\, \frac{\Gamma(\frac{1-\nu}{2}-n)}{\Gamma(-\nu-2n)}\, D_{-\nu-1-2n}(2\pi^{\frac{1}{2}}\, z), \qquad (11.52)$$

a series the coefficients of which, when estimated by means of Stirling's formula, behave in a way quite similar to those of the series for $f_0(z)$.

To analyze the behavior of $f_0(x)$ as $x \to \infty$, we use for $x > 0$ the equation (11.16):

$$D_\nu(2\pi^{\frac{1}{2}} x) = \frac{e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})}}{2\,\cos\frac{\pi\nu}{2}}\; x \int_{-\infty}^{\infty} (\mathcal{Q}\,\phi^\nu)_0(\sigma)\, e^{i\pi\sigma\, x^2}\, d\sigma, \qquad (11.53)$$

in which (a consequence of (11.33) together with (11.32))

$$(\mathcal{Q}\,\phi^\nu)_0(\sigma) = \frac{2^{\frac{\nu}{2}}\,\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})}\, (1+\sigma^2)^{-\frac{1}{4}} \left(\frac{\sigma+i}{\sigma-i}\right)^{\frac{1}{2}(\nu+\frac{1}{2})} : \qquad (11.54)$$

it is understood that the fractional power on the right-hand side is that which takes the value 1 at $\sigma = +\infty$.

Transforming the semi-convergent integral (11.53) into a convergent one by means of an integration by parts and using (11.43), we find for $x > 0$ the equation

$$-i\pi\, x\, f_0(x) = \frac{e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})}}{2^{\frac{1}{2}}\,\cos\frac{\pi\nu}{2}} \sum_{n\in\mathbb{Z}} c_n \times \int_{-\infty}^{\infty} e^{i\pi\sigma\, x^2}\, \frac{d}{d\sigma} \left[(1+\sigma^2)^{-\frac{1}{4}} \left(\frac{\sigma+i}{\sigma-i}\right)^{\frac{1}{2}(\nu+\frac{1}{2})-n} \right] d\sigma. \qquad (11.55)$$

For $0 < \delta_1 < 1$ one has $\frac{1-\delta_1}{1+\delta_1} \leq |\frac{\sigma+i\delta_1+i}{\sigma+i\delta_1-i}| \leq \frac{1+\delta_1}{1-\delta_1}$. With the help of the estimate $|c_n| \leq C\,(1+\varepsilon)^{-|n|}$, the contour deformation $\sigma \mapsto \sigma+i\delta_1$ makes it possible to verify that, as $x \to +\infty$, one has $|f_0(x)| \leq C\, e^{-\pi\delta x^2}$ for some pair of positive constants C, δ. The same estimate goes for the function $f_{i,0}(x)$. Finally, the function in $\mathfrak{A}_\nu$, a $\mathbb{C}^4$-realization of which is $(f_0,\, 0,\, f_{i,0},\, 0)$, certainly coincides with the even function u we started with, since the two functions have the same $\mathcal{Q}$-transform: now, if an entire function $v(z) = \sum_{k\geq 0} a_k\, z^k$ is bounded by some exponential $C\, e^{\pi R|z|^2}$, knowing the expansion, as $\sigma \to +\infty$, of $\sigma^{\frac{1}{2}}\,(\mathcal{Q}\,v)(\sigma)$ (*cf.* Theorem 4.11 and Corollary 4.3, in the present easy one-dimensional case) as a power series in σ^{-1}, makes it possible to recover the coefficients a_k.

In the odd case, nothing much is changed: only, we need to compute (just as we did in Theorem 11.6 in the even case) the $\mathcal{K}$-transform of a function such as $\psi^{\nu+1}$. The proof of the following theorem will thus make the proof of the present one complete. □

Theorem 11.10. *One has*

$$(\mathcal{K}\,\psi^{\nu+1})_1(e^{-i\theta}) = -e^{\frac{i\pi}{4}}\,\frac{2^{\frac{\nu-1}{2}}\,\nu}{\Gamma(\frac{1-\nu}{2})}\;e^{\frac{i}{2}\,(\nu-\frac{3}{2})\,\theta}. \tag{11.56}$$

Proof. From (11.8) and (11.10), one has for every $u \in \mathfrak{A}_\nu$ the identity

$$(\mathcal{Q}\,A^*\,u)_1(\sigma) = e^{\frac{i\pi}{4}}\,(1+i\,\sigma)\,\left(\mathcal{Q}\,\left(\left(\pi^{\frac{1}{2}}\,x^2 - \frac{1}{2\pi^{\frac{1}{2}}}\,x\,\frac{d}{dx}\right)\,u\right)\right)_0(\sigma): \tag{11.57}$$

now the operator on the right-hand side can also be written as

$$\pi^{\frac{1}{2}}\,x^2 - \frac{1}{2\pi^{\frac{1}{2}}}\,x\,\frac{d}{dx} = \frac{\pi^{-\frac{1}{2}}}{2}\left(L + A^2 - \frac{1}{2}\right): \tag{11.58}$$

it thus transforms the function ϕ^ν into the function

$$v = \frac{\pi^{-\frac{1}{2}}}{2}\,(\nu\,\phi^\nu + \nu(\nu-1)\,\phi^{\nu-2}), \tag{11.59}$$

a consequence of Proposition 11.8. This makes it possible to compute $(\mathcal{K}\,\psi^{\nu+1})_1 = (\mathcal{K}\,A^*\,\phi^\nu)_1$ since, using Theorem 11.6, one finds

$$\begin{aligned}(\mathcal{K}\,v)_0(e^{-i\theta}) &= \frac{\pi^{-\frac{1}{2}}}{2}\left[\nu\,\frac{2^{\frac{\nu-1}{2}}\,\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})}\,e^{\frac{i}{2}\,(\nu+\frac{1}{2})\,\theta} + \nu\,(\nu-1)\,\frac{2^{\frac{\nu-3}{2}}\,\pi^{\frac{1}{2}}}{\Gamma(\frac{3-\nu}{2})}\,e^{\frac{i}{2}\,(\nu-\frac{3}{2})\,\theta}\right]\\ &= \frac{2^{\frac{\nu-3}{2}}\,\nu}{\Gamma(\frac{1-\nu}{2})}\,e^{\frac{i}{2}\,(\nu+\frac{1}{2})\,\theta}\,(1-e^{-i\theta}).\end{aligned} \tag{11.60}$$

To obtain $(\mathcal{K}\,\psi^{\nu+1})_1(e^{-i\theta})$, we still have to multiply by $e^{\frac{i\pi}{4}}\,(1+i\,\mathrm{cotan}\,\frac{\theta}{2}) = -2\,e^{\frac{i\pi}{4}}\,\frac{e^{-i\theta}}{1-e^{-i\theta}}$, getting the result announced. □

Definition 11.11. Given $u \in \mathfrak{A}_\nu$, we set

$$\mathrm{Int}\,[u] = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}\left[2\,\cos\,\frac{\pi\nu}{2}\int_0^\infty f_0(x)\,dx + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty f_{i,0}(x)\,dx\right]. \tag{11.61}$$

Remark 11.3. In view of (11.3), the definition of the linear form Int only depends on ν mod 2: this is one of the reasons for the presence of the phase factor in front of the right-hand side.

Theorem 11.12. *The linear form* Int *is invariant under the* (*real or complex*) *Heisenberg translations* $\pi(y,0)$. *In terms of the* $\mathcal{K}$*-transform of* u, *one has*

$$\mathrm{Int}\,[u] = 2^{\frac{1}{2}}\,(\mathcal{K}\,u)_0(e^{-i\pi}). \tag{11.62}$$

Proof. Writing

$$\mathrm{Int}\,[u] = \alpha \int_0^\infty f_0(x)\,dx + \beta \int_0^\infty f_{i,0}(x)\,dx, \tag{11.63}$$

one notes the relation $\alpha = 2\beta\,\frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}}\,\cos\frac{\pi\nu}{2}$. One may then follow the proof of Proposition 1.16, using the first and third relations (11.7) so that, with $v = \pi(y,0)\,u$,

$$\begin{aligned}\mathrm{Int}\,[v] = \frac{1}{2}\int_0^\infty &[\alpha\,(f_0(x-y)+f_0(x+y)) + \beta\,(f_{i,0}(x-iy)+f_{i,0}(x+iy))]\,dx\\ +\frac{1}{2}\int_0^\infty &[\alpha\,(f_1(x-y)-f_1(x+y)) + i\,\beta\,(f_{i,1}(x-iy)-f_{i,1}(x+iy))]\,dx :\end{aligned} \tag{11.64}$$

just as in the proof of Proposition 1.16, one writes the second line as

$$\begin{aligned}&\frac{\alpha}{2}\int_{-y}^{y} f_1(z)\,dz - \frac{\beta}{2}\int_{-y}^{y} f_{i,1}(iz)\,dz\\ &\quad= \frac{\alpha}{2}\int_{-y}^{y} f_1(z)\,dz - \frac{\beta}{2}\,\frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}}\int_{-y}^{y}[e^{-\frac{i\pi\nu}{2}}\,f_1(-z) + e^{\frac{i\pi\nu}{2}}\,f_1(z)]\;dz,\end{aligned} \tag{11.65}$$

an expression that reduces to zero in view of the relation between α and β. Then

$$\begin{aligned}\mathrm{Int}\,[v-u] &= \frac{\alpha}{2}\left(\int_{-y}^{0} f_0(z)\,dz - \int_0^y f_0(z)\,dz\right)\\ &\qquad + \frac{i\,\beta}{2}\left(\int_{-y}^{0} f_{i,0}(iz)\,dz - \int_0^y f_{i,0}(iz)\,dz\right)\\ &= \frac{1}{2}\int_0^y [\alpha\,(f_0(-z)-f_0(z)) + i\,\beta\,(f_{i,0}(-iz)-f_{i,0}(iz))]\,dz,\end{aligned} \tag{11.66}$$

an expression which reduces to zero again in view of the same relation, together with (11.2).

Using the expression (11.11) of the analytic extension of the function $(\mathcal{Q}\,u)_0$ to the whole real line, one sees that

$$\mathrm{Int}\,[u] = (\mathcal{Q}\,u)_0(0) : \tag{11.67}$$

that this agrees with $2^{\frac{1}{2}}\,(\mathcal{K}\,u)_0(e^{-i\pi})$ is immediate in view of (11.9) since, as the points of $\Sigma = S^1$ in the image of $\mathrm{Sym}_1 = \mathbb{R}$ under the Cayley map are the points $e^{-i\theta}$ with $0 < \theta < 2\pi$, the point $0 \in \mathbb{R}$ is obtained as the inverse Cayley image of the point $z = e^{-i\pi}$. □

Definition 11.13. The ν-anaplectic Fourier transformation $\mathcal{F}^\nu_{\mathrm{ana}}$ is defined by the equation

$$(\mathcal{F}^\nu_{\mathrm{ana}}\,u)(x) = \mathrm{Int}\,[y \mapsto e^{-2i\pi\,xy}\,u(y)], \qquad u \in \mathfrak{A}_\nu. \tag{11.68}$$

It is of course the integral form Int from the ν-theory that has to be used here: again, the transformation $\mathcal{F}^{\nu}_{\text{ana}}$ only depends on ν mod 2.

Theorem 11.14. *The ν-anaplectic Fourier transformation is a linear automorphism of the space $\mathfrak{A}_\nu$. One has the equations*

$$(\mathcal{K}\,(\mathcal{F}^{\nu}_{\text{ana}}\,u))_0(z) = e^{i\pi\,(\nu+\frac{1}{2})}\,(\mathcal{K}\,u)_0(e^{i\pi}\,z),$$
$$(\mathcal{K}\,(\mathcal{F}^{\nu}_{\text{ana}}\,u))_1(z) = e^{i\pi\nu}\,(\mathcal{K}\,u)_1(e^{i\pi}\,z). \tag{11.69}$$

Proof. Set $v = \mathcal{F}^{\nu}_{\text{ana}}\,u$. From Theorem 11.3 (which sends us back to the equations (1.60) and (1.61) from Proposition 1.11) and Definition 11.11, one finds

$$v(x) = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}\left[2\,\cos\frac{\pi\nu}{2}\int_0^\infty (f_0(y)\,\cos 2\pi xy - i\,f_1(y)\,\sin 2\pi xy)\,dy\right.$$
$$\left.+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty (f_{i,0}(y)\,\cosh 2\pi xy - i\,f_{i,1}(y)\,\sinh 2\pi xy)\,dy\right]. \tag{11.70}$$

It is immediate that v extends as an entire function: using estimates of the kind

$$|e^{2\pi y\,(x+i\xi)}| \leq e^{\pi\,(\varepsilon\,y^2+\varepsilon^{-1}\,x^2)}, \tag{11.71}$$

one sees also that, for complex z, $v(z)$ is bounded by some function $C\,e^{\pi R|z|^2}$. Applying Theorem 11.9, the only remaining problem is computing the $\mathcal{K}$-transform of v. For $\sigma > 0$, one has

$$\int_{-\infty}^{\infty} e^{-\pi\sigma\,x^2}\,\cos\,(2\pi\,x\,e^{-\frac{i\pi}{4}}\,y)\,dx = \sigma^{-\frac{1}{2}}\,e^{\frac{i\pi y^2}{\sigma}},$$
$$\int_{-\infty}^{\infty} e^{-\pi\sigma\,x^2}\,\cosh\,(2\pi\,x\,e^{-\frac{i\pi}{4}}\,y)\,dx = \sigma^{-\frac{1}{2}}\,e^{-\frac{i\pi y^2}{\sigma}}, \tag{11.72}$$

so that (11.8) yields

$$(\mathcal{Q}\,v)_0(\sigma) = e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}\,\sigma^{-\frac{1}{2}}\left[2\,\cos\frac{\pi\nu}{2}\int_0^\infty f_0(y)\,e^{\frac{i\pi y^2}{\sigma}}\,dy\right.$$
$$\left.+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty f_{i,0}(y)\,e^{-\frac{i\pi y^2}{\sigma}}\,dy\right]; \tag{11.73}$$

similarly,

$$\int_{-\infty}^{\infty} e^{-\pi\sigma\,x^2}\,x\,\sin\,(2\pi\,x\,e^{-\frac{i\pi}{4}}\,y)\,dx = e^{\frac{i\pi}{4}}\,\sigma^{-\frac{3}{2}}\,y\,e^{\frac{i\pi y^2}{\sigma}},$$
$$\int_{-\infty}^{\infty} e^{-\pi\sigma\,x^2}\,x\,\sinh\,(2\pi\,x\,e^{-\frac{i\pi}{4}}\,y)\,dx = e^{\frac{i\pi}{4}}\,\sigma^{-\frac{3}{2}}\,y\,e^{-\frac{i\pi y^2}{\sigma}}, \tag{11.74}$$

and

$$(\mathcal{Q}\,v)_1(\sigma) = e^{\frac{i\pi\nu}{2}}\,\sigma^{-\frac{3}{2}}\,(1+i\,\sigma)\left[2\,\cos\frac{\pi\nu}{2}\int_0^\infty y\,f_1(y)\,e^{\frac{i\pi y^2}{\sigma}}\,dy \right. \\ \left. + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)}\int_0^\infty y\,f_{i,1}(y)\,e^{-\frac{i\pi y^2}{\sigma}}\,dy\right]. \quad (11.75)$$

Comparing (11.73) and (11.75) respectively to (11.11) and (11.11), we obtain for large σ the pair of relations

$$(\mathcal{Q}\,v)_0(\sigma) = \sigma^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_0(-\sigma^{-1}),$$
$$(\mathcal{Q}\,v)_1(\sigma) = -i\,\sigma^{-\frac{1}{2}}\,(\mathcal{Q}\,u)_1(-\sigma^{-1}), \quad (11.76)$$

from which the equations (11.69) follow, as a consequence of (11.15). □

Proposition 11.15. *Let* $\nu \in \mathbb{C}\backslash\mathbb{Z}$. *The maps* $u \mapsto v$, $v(x) = a^{-\frac{1}{2}}\,u(a^{-1}x)$, $a > 0$ *and* $u \mapsto w$, $w(x) = e^{i\pi c x^2}\,u(x)$, $c \in \mathbb{R}$, *are linear automorphisms of* $\mathfrak{A}_\nu$. *With* $z = e^{-i\theta}$, *let* $d\theta = \frac{dz}{iz}$ *be the rotation-invariant measure on* $\Sigma = S^1$. *Define the analytic automorphisms* F_a *and* F^c *of the circle by the equations*

$$F_a(z) = \frac{(a^2+1)\,z + a^2 - 1}{(a^2-1)\,z + a^2 + 1}, \qquad F^c(z) = \frac{(2-ic)\,z + ic}{-ic\,z + 2 + ic}, \quad (11.77)$$

and note that, given any covering $\widetilde{\Sigma}$ *of* Σ, *finite or infinite,* F_a *and* F^c *uniquely lift up as analytic automorphisms of* $\widetilde{\Sigma}$ *preserving the base point* $z = 1$. *The transfer of the two maps* $u \mapsto v$ *and* $u \mapsto w$ *just introduced to the* $\mathcal{K}$*-realization is given by the equations* (*involving the Radon–Nikodym derivatives of* F_a *and* F^c)

$$(\mathcal{K}\,v)_0(z) = \left(\frac{F_a^*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\,(\mathcal{K}\,u)_0(F_a(z)),$$
$$(\mathcal{K}\,v)_1(z) = \frac{1}{2}\left(a + a^{-1} - \frac{a - a^{-1}}{F_a(z)}\right)\left(\frac{F_a^*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\,(\mathcal{K}\,u)_1(F_a(z)), \quad (11.78)$$

and

$$(\mathcal{K}\,w)_0(z) = \left(\frac{(F^c)^*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\,(\mathcal{K}\,u)_0(F^c(z)),$$
$$(\mathcal{K}\,w)_1(z) = \frac{1}{2}\left(1 + \frac{ic}{2} - \frac{ic}{2\,F^c(z)}\right)\left(\frac{(F^c)^*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\,(\mathcal{K}\,u)_1(F_a(z)). \quad (11.79)$$

Proof. It is immediate that if $\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ is the $\mathbb{C}^4$-realization of u, the function v (*resp.* w) admits the $\mathbb{C}^4$-realization $x \mapsto a^{-\frac{1}{2}}\,\boldsymbol{f}(a^{-1}x)$ (*resp.* $(e^{i\pi cx^2}\,f_0,\, e^{i\pi cx^2}\,f_1,\, e^{-i\pi cx^2}\,f_{i,0},\, e^{-i\pi cx^2}\,f_{i,1})$).

On the other hand, for large σ,

$$(\mathcal{Q}\,v)_0(\sigma) = a^{\frac{1}{2}}\,(\mathcal{Q}\,u)_0(a^2\sigma), \quad (\mathcal{Q}\,v)_1(\sigma) = a^{\frac{3}{2}}\,\frac{1+i\,\sigma}{1+i\,a^2\sigma}\,(\mathcal{Q}\,u)_0(a^2\sigma),$$
$$(\mathcal{Q}\,w)_0(\sigma) = (\mathcal{Q}\,u)_0(\sigma-c), \quad (\mathcal{Q}\,w)_1(\sigma) = \frac{1+i\,\sigma}{1+i\,(\sigma-c)}\,(\mathcal{Q}\,u)_1(\sigma-c). \tag{11.80}$$

Under the Cayley transformation $\sigma \mapsto z$, the maps $\sigma \mapsto a^2\sigma$ and $\sigma \mapsto \sigma - c$ transfer to the maps given in (11.77): one may recopy (4.35) and (4.36), or redo the calculation in this trivial case. A straightforward calculation shows that the Radon–Nikodym derivatives of the transformations F_a and F^c of the circle are given as

$$\frac{F_a^*\,d\theta}{d\theta} = \left(a^2\,\cos^2\frac{\theta}{2} + a^{-2}\,\sin^2\frac{\theta}{2}\right)^{-1}, \quad \frac{(F^c)^*\,d\theta}{d\theta} = \left(1 - c\,\sin\theta + c^2\,\sin^2\frac{\theta}{2}\right)^{-1}, \tag{11.81}$$

which makes it immediate to verify the equations (4.31) in this case:

$$\frac{F_a^*\,d\theta}{d\theta} = a^2\left|\frac{1-F_a(z)}{1-z}\right|^2, \qquad \frac{(F^c)^*\,d\theta}{d\theta} = \left|\frac{1-F^c(z)}{1-z}\right|^2. \tag{11.82}$$

The proof of (11.78) and (11.79) then follows that of (4.44) and (4.48) on one hand, of (4.50) and (4.52) on the other hand. □

Consider the group G_2 consisting of matrices $\left(\begin{smallmatrix}\lambda & \mu\\ \nu & \rho\end{smallmatrix}\right)$ such that the fractional-linear transformation $z \mapsto \frac{\lambda\,z+\mu}{\nu\,z+\rho}$ preserves the unit circle, in other words the group of matrices characterized by the conditions

$$|\lambda|^2 + |\mu|^2 = |\nu|^2 + |\rho|^2, \qquad \lambda\,\bar\mu = \nu\,\bar\rho. \tag{11.83}$$

Since

$$\begin{aligned}1 = |\lambda\,\rho - \mu\,\nu|^2 &= |\lambda|^2\,|\rho|^2 - 2\,\mathrm{Re}\,\left(\lambda\,\bar\mu\,\nu\,\bar\rho\right) + |\mu|^2\,|\nu|^2\\ &= |\lambda|^2\,|\rho|^2 - 2\,|\lambda|^2\,|\mu|^2 + |\mu|^2\,|\nu|^2 = |\lambda|^2\,(|\rho|^2-|\mu|^2) - |\mu|^2\,(|\lambda|^2-|\nu|^2)\\ &= (|\lambda|^2-|\mu|^2)\,(|\lambda|^2-|\nu|^2),\end{aligned} \tag{11.84}$$

one has $|\lambda| \neq |\mu|$: let G_1 be the subset of G_2 characterized by the condition $|\lambda| > |\mu|$, which implies $|\lambda| > |\nu|$ too. Writing the first of the two conditions (11.83) as $|\lambda|^2 + \frac{|\nu|^2\,|\rho|^2}{|\lambda|^2} = |\nu|^2 + |\rho|^2$, or $(|\lambda|^2 - |\nu|^2)\,(|\lambda|^2 - |\rho|^2) = 0$, one sees that $|\lambda = |\rho|$, from which $|\mu| = |\nu|$ too. The equation $1 = \lambda\,\rho - \mu\,\nu$ then yields $\bar\lambda\bar\rho = |\lambda|^2\,|\rho|^2 - \bar\lambda\,\mu\,\bar\rho\,\nu = |\lambda|^2\,(|\rho|^2 - |\mu|^2) > 0$, and $\bar\mu\bar\nu = -|\mu|^2\,|\nu|^2 + \lambda\,\bar\mu\,\rho\,\bar\nu = |\nu|^2\,(|\rho|^2 - |\mu|^2) > 0$ as well, which (together with what precedes) implies that

$\rho = \bar\lambda$ and $\nu = \bar\mu$. Finally, G_1 consists of all matrices in $SL(2,\mathbb{C})$ of the kind $\left(\begin{smallmatrix} \lambda & \mu \\ \bar\mu & \bar\lambda \end{smallmatrix}\right)$. It is a group, isomorphic to $SL(2,\mathbb{R})$ under the map

$$g \mapsto T\,g\,T^{-1}, \qquad \text{where } T = 2^{-\frac{1}{2}} \left(\begin{smallmatrix} 1 & i \\ -1 & i \end{smallmatrix}\right). \tag{11.85}$$

The group G_0, isomorphic to $G_1/\{\pm I\}$, of transformations $z \mapsto \frac{\lambda\, z+\mu}{\bar\mu\, z+\bar\lambda}$ of the unit circle, is exactly the group of birational (in the z-"coordinate") automorphisms of the unit circle of topological degree $+1$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we denote as $[g]$ the analytic automorphism just considered in association to the matrix $\left(\begin{smallmatrix} \lambda & \mu \\ \bar\mu & \bar\lambda \end{smallmatrix}\right) = T\,g\,T^{-1}$: it is readily checked that this is in complete agreement with the equations (5.1) to (5.5) of the n-dimensional case. In an explicit way, one has

$$\begin{pmatrix} \lambda & \mu \\ \bar\mu & \bar\lambda \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a - i\,b + i\,c + d & -a - i\,b - i\,c + d \\ -a + i\,b + i\,c + d & a + i\,b - i\,c + d \end{pmatrix}: \tag{11.86}$$

in particular, with the notation of Proposition 11.15, $[g]$ coincides with F_a in the case when $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{-1}$, with F^c in the case when $g = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)^{-1}$; when $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{-1}$, $[g]$ is the rotation $z \mapsto -z = e^{i\pi} z$.

Let $N = 1, 2, \dots$ or ∞, and consider the $(2N)$-fold cover of G_0 (if $N = \infty$, this means the universal cover), which can be identified to the N-fold cover $G^{(N)}$ of $G = SL(2,\mathbb{R})$. On the other hand, if $\Sigma^{(2N)}$ denotes the $(2N)$-fold cover of $\Sigma = S^1$, the preceding group can also be identified with the group of analytic automorphisms of $\Sigma^{(2N)}$ lying above birational automorphisms of Σ of degree $+1$. In particular (compare Corollary 4.10), the N-fold cover $G^{(N)}$ of G acts on $\Sigma^{(N)}$, and we still denote as $g \mapsto [g]$ the corresponding group homomorphism.

In Theorem 11.9, the $\mathcal{K}$-transform of a function in $\mathfrak{A}_\nu$ has been defined as a pair of (everywhere analytic, in contradiction to the higher-dimensional case) functions on $\Sigma^{(\infty)}$: however, in view of the quasi-periodicity condition (11.17), we may also, in the case when ν is rational, regard it as defined on $\Sigma^{(N)}$ provided that $N\,(\frac{\nu}{2} + \frac{1}{4}) \in \mathbb{Z}$. All that precedes gives the following analogue of Theorem 5.10 a meaning.

Theorem 11.16. *Let $N = \infty$ if $\nu \notin \mathbb{Q}$; if $\nu \in \mathbb{Q}$, let N be either ∞ or any integer ≥ 1 such that $N\,(\frac{\nu}{2}+\frac{1}{4}) \in \mathbb{Z}$. In accordance with* (5.1)*, for every $g \in G^{(N)}$ above some matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, set $\left(\begin{smallmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{smallmatrix}\right) = S\,\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\,S^{-1}$ with $S = 2^{-\frac{1}{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$. Given $u \in \mathfrak{A}_\nu$, there is a unique function $v \in \mathfrak{A}_\nu$ such that, for every $z \in \Sigma^{(N)}$,*

$$(\mathcal{K}\,v)_0(z) = \left(\frac{[g^{-1}]_*\, d\theta}{d\theta}\,(z) \right)^{\frac{1}{4}} (\mathcal{K}\,u)_0([g^{-1}]\,(z)),$$

$$(\mathcal{K}\,v)_1(z) = \left(\frac{[g^{-1}]_*\, d\theta}{d\theta}\,(z) \right)^{\frac{1}{4}} \left[\alpha - i\,\beta\,([g^{-1}]\,(z))^{-1}\right] . (\mathcal{K}\,u)_1([g^{-1}]\,(z)). \tag{11.87}$$

Setting $v = \mathrm{Ana}_\nu(g)\,u$, one defines a representation Ana_ν of $G^{(N)}$ in $\mathfrak{A}_\nu$.

Let $P = \frac{1}{2i\pi}\frac{d}{dx}$ *and let* Q *be the operator of multiplication by* x*, the argument of all functions in* $\mathfrak{A}_\nu$ *under consideration. For any* $g \in G^{(N)}$ *above some matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$*, one has the relations*

$$\mathrm{Ana}_\nu(g)\, Q\, \mathrm{Ana}_\nu(g^{-1}) = d\, Q - b\, P, \qquad \mathrm{Ana}_\nu(g)\, P\, \mathrm{Ana}_\nu(g^{-1}) = -c\, Q + a\, P. \tag{11.88}$$

Proof. Based on Theorem 11.14 and Proposition 11.15, it follows the proof of Theorem 5.10, though it is of course simpler in two respects: this is the one-dimensional case, and the functions on $\Sigma^{(N)}$ under consideration have no singularities. On the right-hand side of the second equation (11.87) and (11.69), one finds

$$\mathrm{Ana}_\nu\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) = e^{-i\pi\,(\nu+\frac{1}{2})}\,\mathcal{F}^\nu_{\mathrm{ana}}, \tag{11.89}$$

where the matrix on the left-hand side is to be understood as the element

$$\exp\frac{\pi}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \quad \text{of} \quad G^{(N)}. \qquad \square$$

12 The pseudoscalar product on $\mathfrak{A}_\nu$

In this last section, we define an invariant pseudo-scalar product on $\mathfrak{A}_\nu$. Also, we shall identify the representation Ana_ν, acting on the even or odd (depending on ν) part of $\mathfrak{A}_\nu$, with a unitary representation from Pukanszky's list [18].

To harmonize the notation with those from (*loc. cit.*), we introduce the following linear basis of the Lie algebra of $SL(2,\mathbb{R})$ or of any of its covering groups:

$$\ell_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \ell_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \ell_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{12.1}$$

Next, we denote as $H_0,\, H_1,\, H_2$ the infinitesimal operators of the ν-anaplectic representation, normalized as

$$H_j = i\,\frac{d}{dt}\Big|_{t=0} \mathrm{Ana}_\nu(e^{t\,\ell_j}) : \tag{12.2}$$

our first task is to make these operators explicit. There is no question of domain here: all the operators to be considered preserve the space $\mathfrak{A}_\nu$.

Proposition 12.1. *One has*

$$H_0 = -\frac{1}{2}\,L, \qquad H_1 = -\frac{i}{2}\left(x\,\frac{d}{dx} + \frac{1}{2}\right), \qquad H_2 = -\frac{1}{2}\left(\pi\,x^2 + \frac{1}{4\pi}\,\frac{d^2}{dx^2}\right). \tag{12.3}$$

Proof. One has

$$e^{t\,\ell_1} = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \qquad e^{t\,(\ell_0+\ell_2)} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : \tag{12.4}$$

now, by the very construction of the ν-anaplectic representation, one has

$$(\mathrm{Ana}_\nu(\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix})\,u)(x) = e^{-\frac{t}{4}}\,u(e^{-\frac{t}{2}}\,x), \qquad (\mathrm{Ana}_\nu((\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}))\,u)(x) = e^{i\pi t\,x^2}\;u(x), \tag{12.5}$$

from which the computation of H_1 and $H_0 + H_2$, follows.

Of course, the computation of H_0 requires more care. Setting $g = e^{t\,\ell_0}$, one has $g = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}$, and the matrix $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ associated to g by the recipe of Theorem 11.16 is the matrix $\begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}$. Next, the matrix $\begin{pmatrix} \lambda & \mu \\ \bar\mu & \bar\lambda \end{pmatrix}$ associated to the matrix g^{-1} under the equation (11.86) is the matrix $\begin{pmatrix} e^{-\frac{it}{2}} & 0 \\ 0 & e^{\frac{it}{2}} \end{pmatrix}$, so that $[g^{-1}](z) = e^{-it}\,z$ for $z \in \Sigma$ or in any covering space of Σ. The formulas in Theorem 11.16 thus imply

$$\begin{aligned}
(\mathcal{K}\,\mathrm{Ana}_\nu(e^{t\,\ell_0})\,u)_0\,(z) &= (\mathcal{K}\,u)_0(e^{-it}\,z),\\
(\mathcal{K}\,\mathrm{Ana}_\nu(e^{t\,\ell_0})\,u)_1\,(z) &= e^{\frac{it}{2}}\,(\mathcal{K}\,u)_1(e^{-it}\,z) :
\end{aligned} \tag{12.6}$$

as a verification, one may note that $e^{-\pi \ell_0} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the matrix which gives $e^{-i\pi(\nu+\frac{1}{2})} \mathcal{F}^\nu_{\rm ana}$ under the ν-anaplectic representation, and that the equations (12.6) indeed reduce to (11.69) when $t = -\pi$. One thus has the identities

$$(\mathcal{K}\, H_0\, u)_0(e^{-i\theta}) = i\, \frac{d}{d\theta}\, (\mathcal{K}\, u)_0(e^{-i\theta})),$$
$$(\mathcal{K}\, H_0\, u)_1(e^{-i\theta}) = \left(-\frac{1}{2} + i\, \frac{d}{d\theta}\right)\, (\mathcal{K}\, u)_1(e^{-i\theta})). \tag{12.7}$$

We must now transfer the harmonic oscillator to the $\mathcal{K}$-realization. From (11.11), we find for large σ, after an integration by parts, the identity

$$(\mathcal{Q}\, L\, u)_0(\sigma) = \int_{-\infty}^{\infty} \left(-i\,\pi\, x^2 - \frac{i}{4\pi}\, \frac{d^2}{dx^2}\right)\, (e^{-\pi\sigma\, x^2})\ .\ u(x\, e^{-\frac{i\pi}{4}})\, dx$$
$$= i\, \left[(1+\sigma^2)\, \frac{d}{d\sigma} + \frac{1}{2}\,\sigma\right]\, (\mathcal{Q}\, u)_0(\sigma): \tag{12.8}$$

in view of the relation $(\mathcal{K}\, u)_0(e^{-i\theta}) = (2\, \sin \frac{\theta}{2})^{-\frac{1}{2}}\, (\mathcal{Q}\, u)_0(\text{cotan}\, \frac{\theta}{2})$, valid for $\theta > 0$ and small, and of the relation $(1+\sigma^2)\, \frac{d}{d\sigma} = -2\, \frac{d}{d\theta}$, one finds that $(\mathcal{K}\, L\, u)_0(e^{-i\theta}) = \mathcal{M}_0\, (\theta \mapsto (\mathcal{K}\, u)_0(e^{-i\theta}))$ where $\mathcal{M}_0$ expresses itself as the composition of operators

$$\mathcal{M}_0 = \left(\sin \frac{\theta}{2}\right)^{-\frac{1}{2}}\, i\left[-2\, \frac{d}{d\theta} + \frac{1}{2}\, \text{cotan}\, \frac{\theta}{2}\right] \left(\sin \frac{\theta}{2}\right)^{\frac{1}{2}} = -2i\, \frac{d}{d\theta}. \tag{12.9}$$

In a similar way, one finds that

$$(\mathcal{Q}\, L\, u)_1(\sigma) = \int_{-\infty}^{\infty} \left(-i\,\pi\, x^2 - \frac{i}{4\pi}\, \frac{d^2}{dx^2}\right)\, (e^{-\pi\sigma\, x^2}\, (1 + i\,\sigma)\, x)\ .\ u(x\, e^{-\frac{i\pi}{4}})\, dx$$
$$= i\left[(1+\sigma^2)\, \frac{d}{d\sigma} + \frac{1}{2}\,\sigma - i\right]\, (\mathcal{Q}\, u)_1(\sigma): \tag{12.10}$$

thus $(\mathcal{K}\, L\, u)_1(e^{-i\theta}) = \mathcal{M}_1\, (\theta \mapsto (\mathcal{K}\, u)_1(e^{-i\theta}))$ with $\mathcal{M}_1 = \mathcal{M}_0 + 1$. The first equation (12.3) follows. □

Remark 12.1. In some sense (no Stone's theorem is available here yet), one may write

$$e^{-i\pi\, (\nu+\frac{1}{2})}\, \mathcal{F}^\nu_{\rm ana} = \exp\left(-\frac{i\pi}{2}\, L\right), \tag{12.11}$$

a formula very close to the usual $e^{-\frac{i\pi}{4}}\, \mathcal{F} = \exp\,(-\frac{i\pi}{2}\, L)$: the classical case *roughly* corresponds to $\nu \in 2\,\mathbb{Z}$ – and, if one wishes to extend Proposition 11.2, the case when $\nu + 1 \in 2\,\mathbb{Z}$ corresponds to the conjugate of the classical analysis under the map $u \mapsto u_i$ – but there is a phase shift by the factor $e^{\frac{i\pi}{4}}$ inherent in the anaplectic analysis.

The following proposition is the result of an immediate calculation.

Proposition 12.2. *Set* $H_+ = H_1 + i\,H_2,\ H_- = H_1 - i\,H_2$. *One has*

$$A^*\,A = L - \frac{1}{2},\quad AA^* = L + \frac{1}{2},\quad A^2 = 2i\,H_+,\quad {A^*}^2 = -2i\,H_-. \tag{12.12}$$

On the other hand, $H_1^2 - H_2^2 - H_0^2 = \frac{3}{16}$.

Under the assumption that ν is real, $\nu \notin \mathbb{Z}$, we can now build a pseudo-scalar product on $\mathfrak{A}_\nu$, invariant under the ν-anaplectic representation.

We shall consider the case when $\nu \in]-1, 0[+2\,\mathbb{Z}$, in which, as will be seen, it is possible to build a pseudoscalar product on $\mathfrak{A}_\nu$ positive-definite on $(\mathfrak{A}_\nu)_{\rm even}$. In the case when $\nu \in]0, 1[+2\,\mathbb{Z}$, one can build a pseudoscalar product positive-definite on $(\mathfrak{A}_\nu)_{\rm odd}$. To avoid routine developments, we shall specialize in the first case.

In Theorem 12.3 below, we shall define the pseudoscalar product $(u|u)$ where $u \in (\mathfrak{A}_\nu)_{\rm even}$, in terms of the expansion (11.42) of the $\mathcal{K}$-transform of u. This would suffice to establish its existence and main properties. However, we deem it preferable to first give a more natural definition, based on the already known case when $\nu = -\frac{1}{2}$. If $u \in (\mathfrak{A})_{\rm even}$, the pseudoscalar product $(u\,|\,u)$ in $\mathfrak{A} = \mathfrak{A}_{-\frac{1}{2}}$ can be expressed as

$$(u\,|\,u) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} |\sigma - \tau|^{-\frac{3}{2}}\ \overline{(\mathcal{Q}\,u)_0}(\sigma)\,(\mathcal{Q}\,u)_0(\tau)\ d\sigma\,d\tau: \tag{12.13}$$

it is understood that the integral has to be interpreted in a regularized sense, as defined between (2.7) and (2.8), or with the help of an integration by parts or, as a third possibility, using analytic continuation with respect to the exponent. To check this equation, one goes back to the fact that if $u \in \mathfrak{A}$ is associated to the pair $\binom{v_0}{0}$ as in Theorem 2.9, one has, as given in (2.113),

$$(u\,|\,u) = \int_{-\infty}^{\infty} |s|^{\frac{1}{2}}\,|v_0(s)|^2\,ds \tag{12.14}$$

or, in terms of $w_0 = \mathcal{F}\,v_0$,

$$(u\,|\,u) = \int_{-\infty}^{\infty} \bar{w}_0(\sigma)\,(|D|^{\frac{1}{2}}\,w_0)(\sigma)\,d\sigma, \tag{12.15}$$

where the operator $|D|^{\frac{1}{2}}$ was defined right after (2.4). Since $(\mathcal{F}\,|s|^{\frac{1}{2}})\,(\sigma) = -\frac{1}{4\pi}\,|\sigma|^{-\frac{3}{2}}$, the equation (12.13) follows.

Transferring the result to the $\mathcal{K}$-realization of u in place of the $\mathcal{Q}$-realization, we obtain

$$(u\,|\,u) = -\frac{2^{-\frac{3}{2}}}{\pi}\int_{S^1\times S^1} |e^{-i\theta} - e^{-i\eta}|^{-\frac{3}{2}}\ \overline{(\mathcal{K}\,u)_0}(e^{-i\theta})\,(\mathcal{K}\,u)_0(e^{-i\eta})\ d\theta\,d\eta, \tag{12.16}$$

with the same precautions concerning the meaning of the integral.

We shall extend this expression to the case of the space $(\mathfrak{A}_\nu)_{\text{even}}$, but some modification has to be done first. If the matrix $\begin{pmatrix} \lambda & \mu \\ \bar\mu & \bar\lambda \end{pmatrix}$ is linked to $g \in SL(2,\mathbb{R})$ by the equation (11.86), and if one sets $e^{-i\theta} = [g](e^{-i\theta_1})$, it is immediate that $\frac{[g]_* \, d\theta_1}{d\theta_1} = |\bar\mu\, e^{-i\theta_1} + \bar\lambda|^{-2}$: setting also $e^{-i\eta} = [g](e^{-i\eta_1})$, and using (11.87) together with

$$[g](e^{-i\theta_1}) - [g](e^{-i\eta_1}) = \frac{e^{-i\theta_1} - e^{-i\eta_1}}{(\bar\mu\, e^{-i\theta_1} + \bar\lambda)\,(\bar\mu\, e^{-i\eta_1} + \bar\lambda)}, \tag{12.17}$$

one can see that the right-hand of (12.16), considered for $u \in \mathfrak{A}_\nu$, is invariant under the transformations $\text{Ana}(g)$ with g of the form $\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$ (there is no need for such matrices to replace $SL(2,\mathbb{R})$ by a covering group). However, it is not invariant under the transformations associated with elements of $G^{(N)}$ above matrices in $SO(2)$ because such transformations $[g]$ do not preserve the set $\{e^{-i\theta} \colon 0 < \theta < 2\pi\}$. We thus substitute for (12.16) the following definition, some ingredients of which will be defined immediately below:

$$(u\,|\,u) = \frac{e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}}{4\pi\,(\sin\pi\nu)\,(\cos\frac{\pi\nu}{2})} \\ \times \lim_{N_1\to\infty} \frac{1}{N_1} \int_{E_{N_1}} |e^{-i\theta} - e^{-i\eta}|^{-\frac{3}{2}}\, \overline{(\mathcal{K}\,u)_0}(e^{-i\theta})\,(\mathcal{K}\,u)_0(e^{-i\eta})\; d\theta\, d\eta. \tag{12.18}$$

Though it is not really needed, it is assumed, for simplicity, that N_1 goes to infinity through the integers. The set E_{N_1} is defined as the set of pairs $(\theta,\,\eta)$ such that $0 < \theta < 2\pi\, N_1,\ 0 < \eta < 2\pi\, N_1,\ 0 < \theta - \eta < 2\pi$. The divergent integral is given the meaning obtained by analytic continuation, replacing the exponent $-\frac{3}{2}$ by λ with $\text{Re}\,\lambda > -1$. It can be shown in a direct way – though there are a few lengthy details in the proof – that $(u\,|\,u)$, so defined, makes sense for every $u \in \mathfrak{A}_\nu$, and that one obtains in this way a Hermitian form invariant under the restriction of the ν-anaplectic representation to the subspace of even functions: there is no a priori explanation for the phase factor, needed for reality and positivity, assuming that $\nu \in]-1,\,0[$. We shall be satisfied, however, with using this definition in a heuristic way, computing with its help the scalar product of any two even eigenstates of the ν-anaplectic harmonic oscillator: it is understood that the polarized form of $(u\,|\,u)$ which we are using is antilinear with respect to the function on the left.

Recalling Theorem 11.6, one has $(\mathcal{K}\,\phi^{\nu+2j})(e^{-i\theta}) = C_{\nu+2j}\; e^{\frac{i}{2}\,(\nu+2j+\frac{1}{2})\,\theta}$, with $C_\mu = \frac{2^{\frac{\mu-1}{2}}\,\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\mu}{2})}$. Set, for $\text{Re}\,\lambda > -1$,

$$I_\lambda(\phi^{\nu+2j},\, \phi^{\nu+2k}) = C_{\nu+2j}\, C_{\nu+2k} \\ \times \lim_{N_1\to\infty} \frac{1}{N_1} \int_{E_{N_1}} |e^{-i\theta} - e^{-i\eta}|^{-\lambda}\, e^{-\frac{i}{2}\,(\nu+2j+\frac{1}{2})\,\theta}\, e^{\frac{i}{2}\,(\nu+2k+\frac{1}{2})\,\theta}\; d\theta\, d\eta. \tag{12.19}$$

Performing the change of variables $(\theta,\,\eta) \mapsto (\theta,\,\xi) = (\theta,\,\theta - \eta)$, the domain of the variable ξ is $(0,\,2\pi)$ and, for given ξ, that of the variable θ, to wit $(0,\,2\pi N_1) \cap$

$(\xi,\,\xi{+}2\pi N_1)$, differs very little from $(0,\,2\pi N_1)$ since the integral is to be multiplied by $\frac{1}{N_1}$: this shows that $I_\lambda(\phi^{\nu+2j},\,\phi^{\nu+2k})$ is zero if $j \neq k$, and that

$$I_\lambda(\phi^{\nu+2j},\,\phi^{\nu+2j}) = C^2_{\nu+2j}\;\times 2\pi \int_0^{2\pi} |e^{-i\xi}-1|^\lambda\, e^{-\frac{i}{2}\,(\nu+2j+\frac{1}{2})\,\xi}\,d\xi. \qquad (12.20)$$

The integral is to be found in [17, p. 8]:

$$2^{\lambda+1}\int_0^\pi (\sin\xi)^\lambda\, e^{-i\,(\nu+2j+\frac{1}{2})\,\xi}\,d\xi = 2\pi\,\frac{e^{-\frac{i\pi}{2}\,(\nu+\frac{1}{2}+2j)}\,\Gamma(1+\lambda)}{\Gamma(\frac{1}{2}(\lambda+\nu)+j+\frac{5}{4})\,\Gamma(\frac{1}{2}(\lambda-\nu)-j+\frac{3}{4})}. \qquad (12.21)$$

Using analytic continuation, we thus find

$$\begin{aligned}(\phi^{\nu+2j}\,|\,\phi^{\nu+2j}) &= \frac{e^{\frac{i\pi}{2}\,(\nu+\frac{1}{2})}}{4\pi\,(\sin\pi\nu)\,(\cos\frac{\pi\nu}{2})}\; I_{-\frac{3}{2}}(\phi^{\nu+2j},\,\phi^{\nu+2j})\\ &= \frac{1}{4\pi\,(\sin\pi\nu)\,(\cos\frac{\pi\nu}{2})}\;C^2_{\nu+2j}\,(2\pi)^2\,\frac{(-1)^j\,\Gamma(-\frac{1}{2})}{\Gamma(\frac{\nu+1}{2}+j)\,\Gamma(-\frac{\nu}{2}-j)}\\ &= \frac{1}{2}\,\Gamma(\nu+2j+1), \end{aligned} \qquad (12.22)$$

where the last expression occurs after some manipulations involving the formula of complements and the duplication formula of the Gamma function. As a verification, this gives $(\phi^{-\frac{1}{2}}\,|\,\phi^{-\frac{1}{2}}) = \frac{\pi^{\frac{1}{2}}}{2}$, an equation equivalent to the normalization condition for $\phi \in \mathfrak{A}$ since, as already observed just before Theorem 11.6, one has $\phi = 2^{\frac{1}{2}}\,\pi^{-\frac{1}{4}}\,\phi^{-\frac{1}{2}}$. One should note that, since $\nu \in]-1,\,0[$, this is a positive number, which explains our choice of the phase factor in the definition (12.18).

Since – for brevity – we have not given all details concerning the justification of the formula just quoted, we shall use the result of the preceding computation as a definition.

Theorem 12.3. *Let* $\nu \in]-1,\,0[$. *For every* $u \in (\mathfrak{A}_\nu)_{\rm even}$*, thus characterized, in the sense of* (11.42)*, by the expansion*

$$(\mathcal{K}\,u)_0(z) = z^{-(\frac{\nu}{2}+\frac{1}{4})}\sum_{j\in\mathbb{Z}} c_j\, z^{-j}, \qquad (12.23)$$

define $(u\,|\,u)$ *as the convergent series*

$$(u\,|\,u) = \frac{\pi^{\frac{1}{2}}}{\cos^2\frac{\pi\nu}{2}}\sum_{j\in\mathbb{Z}}\frac{\Gamma(\frac{\nu}{2}+j+1)}{\Gamma(\frac{1+\nu}{2}+j)}\;|c_j|^2. \qquad (12.24)$$

This is a positive-definite scalar product on $(\mathfrak{A}_\nu)_{\rm even}$*, which coincides when* $\nu = -\frac{1}{2}$ *with the one defined in Section* 1*. It is invariant under the restriction to this space of the* ν*-anaplectic representation. Finally, this latter representation, acting* *on the completion of* $(\mathfrak{A}_\nu)_{\text{even}}$ *with respect to the scalar product just introduced, is unitarily equivalent with the representation* $C_q^{(\tau)}$ *taken from* [18], *with* $q = \frac{3}{16}$ *and* $\tau \in [0,1[$ *characterized by the further condition* $\tau \equiv -\frac{1}{2}\,(\nu + \frac{1}{2})$ *mod* 1.

Proof. In view of Theorem 11.6, one may also write

$$u = \sum_{j \in \mathbb{Z}} 2^{\frac{1-\nu}{2} - j}\, \pi^{-\frac{1}{2}}\, \Gamma\left(\frac{1-\nu}{2} - j\right)\, c_j\, \phi^{\nu+2j}. \tag{12.25}$$

Since, for any $u \in \mathfrak{A}_\nu$, the $\mathcal{K}$-transform of u is analytic on $\Sigma^{(N)}$, one has $|c_j| \leq C\,(1+\varepsilon)^{-|j|}$ for some pair $C,\, \varepsilon$ of positive constants, a fact already used in the proof of Theorem 11.9: this proves the convergence of the series defining $(u\,|\,u)$. The verification that, when u reduces to some function $\phi^{\nu+2j}$, the expression of $(u\,|\,u)$ given by (12.24) coincides with that given by (12.22) is again a consequence of the duplication formula and of the formula of complements of the Gamma function. It follows in particular that, when $\nu = -\frac{1}{2}$, the new scalar product coincides with the one we already know.

The invariance of the scalar product under transformations $\mathrm{Ana}_\nu(g)$, $g \in G^{(N)}$ lying above some rotation matrix, is immediate. What remains to be checked is the effect of the transformations $\mathrm{Ana}_\nu(g)$ with $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, $a > 0$ or $g = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. Since Ana_ν has already been constructed as a representation within the space $(\mathfrak{A}_\nu)_{\text{even}}$, it suffices, in view of (12.2), to show that the operators H_1 and H_2 are formally self-adjoint within that space: using Proposition 12.2, one may prove instead that A^2 and $(A^*)^2$ are formally adjoint to each other or, which amounts to the same, that $(A^2\,\phi^{\nu+2j}\,|\,\phi^{\nu+2k}) = (\phi^{\nu+2j}\,|\,(A^*)^2\,\phi^{\nu+2k})$ for every pair (j,k). Now, with the help of Proposition 11.8, one has

$$\begin{aligned}(A^2\,\phi^{\nu+2j}\,|\,\phi^{\nu+2k}) &= \delta_{j-1,k}\ (\nu+2j)\,(\nu+2j-1)\,(\phi^{\nu+2j-2}\,|\,\phi^{\nu+2j-2}),\\ (\phi^{\nu+2j}\,|\,(A^*)^2\,\phi^{\nu+2k}) &= \delta_{j,k+1}\ (\phi^{\nu+2j}\,|\,\phi^{\nu+2j}),\end{aligned} \tag{12.26}$$

and the right-hand sides of these two equations coincide thanks to (12.22).

To compare the representation just discussed, after completion, with the ones from Pukanszky's list [18], we first recall, with the notation from the beginning of the present section borrowed from (*loc. cit.*) – of course the notation there does not refer to the ν-anaplectic representation, but to an arbitrary irreducible unitary representation of the universal cover of $SL(2,\mathbb{R})$ – that $\exp(-2i\pi\, H_0) = \mathrm{Ana}_\nu(e^{2\pi\ell_0})$ and, using the first equation (12.6), that one has the identity

$$(\mathcal{K}\,\mathrm{Ana}_\nu(e^{2\pi\ell_0})\,u)_0(z) = (\mathcal{K}\,u)_0(e^{-2i\pi}\,z): \tag{12.27}$$

this is the same as $e^{i\pi\,(\nu+\frac{1}{2})}\,(\mathcal{K}\,u)_0(z)$ as a consequence of (11.17). Now, in [18], a certain number $\tau \in [0,1[$ is characterized by the equation $\exp(-2i\pi\,H_0) = e^{-2i\pi\,\tau}\,I$: in our case, we must then take $\tau \equiv -\frac{1}{2}\,(\nu+\frac{1}{2})$ mod 1, as indicated. This implies that $\tau = -\frac{1}{2}\,(\nu+\frac{1}{2})$ if $\nu \in]-1, -\frac{1}{2}]$ and $\tau = -\frac{1}{2}\,(\nu - \frac{3}{2})$ if

$\nu \in]-\frac{1}{2}, 0[$: in both cases, $\tau\,(1-\tau) < q = \frac{3}{16}$, the number obtained in Proposition 12.2 from the consideration of the Casimir operator of the representation. This proves the sought-after unitary equivalence with the representation $C_q^{(\tau)}$; also recall from [18] that, since $q \leq \frac{1}{4}$, the representation under study does not occur in the Plancherel formula for the universal cover of $SL(2,\mathbb{R})$. It is elementary to verify that all values of $\tau \in [0,1[$ such that $\tau\,(1-\tau) < \frac{3}{16}$ are obtained in this way. Only, note that all that precedes would break down if we had $\nu \in]0,1[$ instead of $]-1,0[$: in this case, we would find a unitary representation from the consideration of the *odd* part of the ν-anaplectic representation, taking this time $\tau \equiv -\frac{1}{2}\,(\nu - \frac{1}{2})$ mod 1. □

It is now an easy matter to complete the invariant scalar product on $(\mathfrak{A}_\nu)_{\rm even}$ into an invariant *pseudo*-scalar product on $\mathfrak{A}_\nu$.

Theorem 12.4. *Let $\nu \in]-1,1[$. For every $u \in \mathfrak{A}_\nu$, characterized by the pair of expansions* (cf. (11.42))

$$(\mathcal{K}\,u)_0(z) = z^{-\frac{1}{2}\,(\nu+\frac{1}{2})} \sum_{j\in\mathbb{Z}} c_j\, z^{-j},$$
$$(\mathcal{K}\,u)_1(z) = z^{-\frac{1}{2}\,(\nu-\frac{3}{2})} \sum_{j\in\mathbb{Z}} c'_j\, z^{-j}, \tag{12.28}$$

define $(u\,|\,u)$ as the convergent series

$$(u\,|\,u) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1+\nu}{2}+j)\,\cos^2\frac{\pi\nu}{2}} \times \left[2\,\Gamma\left(\frac{\nu}{2}+j+1\right)\,|c_j|^2 + \pi\,\frac{\nu+1+2j}{\nu+2j}\,\Gamma\left(\frac{\nu}{2}+j\right)\,|c'_j|^2\right]. \tag{12.29}$$

The pseudoscalar product so defined on $\mathfrak{A}_\nu$ is invariant under the ν-anaplectic representation as well as under the Heisenberg representation, and coincides with the one already known when $\nu = -\frac{1}{2}$. It is positive-definite on the space generated by even eigenfunctions of the harmonic oscillator L, as well as on the space (*orthogonal to the precedent*) *generated by odd eigenfunctions of L with a positive eigenvalue; it is negative-definite on the orthogonal complement of the sum of these two spaces,* i.e., *on the space generated by odd eigenfunctions of L with a negative eigenvalue.*

Proof. Recall from (11.56) that

$$(\mathcal{K}\,\psi^{\nu+2j+1})_1(z) = -e^{\frac{i\pi}{4}}\,\frac{2^{\frac{\nu-1}{2}+j}\,(\nu+2j)}{\Gamma(\frac{1-\nu}{2}-j)}\,z^{-\frac{1}{2}\,(\nu-\frac{3}{2})}\,z^{-j}. \tag{12.30}$$

One now has

$$u = \sum_{j\in\mathbb{Z}} 2^{\frac{1-\nu}{2}-j}\,\Gamma(\frac{1-\nu}{2}-j)\,\left[\pi^{-\frac{1}{2}}\,c_j\,\phi^{\nu+2j} - e^{-\frac{i\pi}{4}}\,\frac{c'_j}{\nu+2j}\,\psi^{\nu+2j+1}\right]. \tag{12.31}$$

In particular, this leads to

$$
\begin{aligned}
(\psi^{\nu+2j+1} \,|\, \psi^{\nu+2j+1}) &= \frac{2^{\nu+2j-2}\,\pi^{\frac{3}{2}}}{\cos^2\frac{\pi\nu}{2}}\;\frac{(\nu+2j)\,(\nu+2j+1)\,\Gamma(\frac{\nu}{2}+j)}{\Gamma(\frac{1+\nu}{2}+j)\,(\Gamma(\frac{1-\nu}{2}-j))^2} \\
&= -\frac{2^{\nu+2j}\,\pi^{\frac{3}{2}}}{\sin\pi\nu}\,(\nu+2j+1)\,\left(\Gamma\left(-\frac{\nu}{2}-j\right)\Gamma\left(\frac{1-\nu}{2}-j\right)\right)^{-1} \\
&= \frac{1}{2}\,\Gamma(\nu+2j+2), \qquad (12.32)
\end{aligned}
$$

an expression which should be compared to (12.22): note that this scalar product is positive if $j = 0, 1, \ldots$ negative if $j = -1, -2, \ldots$.

What remains to be done is to show that A and A^* are adjoint to each other in the space $\mathfrak{A}_\nu$: now, using Proposition 11.8, one gets

$$
\begin{aligned}
(A\,\phi^{\nu+2j} \,|\, \psi^{\nu+2k-1}) &= \delta_{jk}\,(\nu+2j)\,(\psi^{\nu+2j-1} \,|\, \psi^{\nu+2j-1}), \\
(\phi^{\nu+2j} \,|\, A^*\,\psi^{\nu+2k-1}) &= \delta_{jk}\,(\phi^{\nu+2j} \,|\, \phi^{\nu+2j}), \qquad (12.33)
\end{aligned}
$$

and the two right-hand sides are identical in view of (12.22) and (12.32); in the same way,

$$
\begin{aligned}
(A^*\,\phi^{\nu+2j} \,|\, \psi^{\nu+2k+1}) &= \delta_{jk}\,(\psi^{\nu+2j+1} \,|\, \psi^{\nu+2j+1}), \\
(\phi^{\nu+2j} \,|\, A\,\psi^{\nu+2k+1}) &= \delta_{jk}\,(\nu+2j+1)\,(\phi^{\nu+2j} \,|\, \phi^{\nu+2j}) : \qquad (12.34)
\end{aligned}
$$

this concludes the proof of Theorem 12.4. □

In the next theorem, we generalize to the case of the ν-anaplectic analysis the definition of the pseudoscalar product given, when $\nu = -\frac{1}{2}$, in Proposition 1.14. The only proof we could find depends on computations, but the result certainly puts forward, again, the role played by the "fourfold" presentation of the analysis.

Theorem 12.5. *Let* $\nu \in]-1, 0[$. *For every* $u \in \mathfrak{A}_\nu$, *characterized by its* $\mathbb{C}^4$-*realization as introduced in Definition* 11.1, *one has*

$$
(u \,|\, u) = 2^{\frac{1}{2}} \int_0^\infty \left[|f_0|^2 + |f_1|^2 + \frac{\Gamma(\nu+1)}{\Gamma(-\nu)}\,(|f_{i,0}|^2 - |f_{i,1}|^2)\right]\,dx. \qquad (12.35)
$$

Proof. The proof consists in verifying that the polarized version, temporarily denoted as $((\ |\ ))$, of the right-hand side gives the correct result when applied to any pair of eigenfunctions of the ν-anaplectic harmonic oscillator.

Let us denote as $\vartriangle$ the logarithmic derivative (traditionally denoted as ψ) of the Gamma function, *i.e.*, $\vartriangle(z) = \frac{\Gamma'(z)}{\Gamma(z)}$: the formula of complements of the Gamma function can be written [17, p. 14] as

$$
\vartriangle(z) - \vartriangle(1-z) = -\pi\,\mathrm{cotan}\,\pi z. \qquad (12.36)
$$

Just as in Lemma 11.7 and its proof, we abbreviate as D_μ the function $x \mapsto D_\mu(2\pi^{\frac{1}{2}}\, x)$. The computation starts from the equation (*loc. cit.*, p. 330)

$$\int_0^\infty D_\mu^2\, dx = 2^{-\frac{5}{2}}\, \frac{\Delta(\frac{1-\mu}{2}) - \Delta(-\frac{\mu}{2})}{\Gamma(-\mu)}. \tag{12.37}$$

With $L = \pi\, x^2 - \frac{1}{4\pi}\,\frac{d^2}{dx^2}$, we use Green's formula

$$\int_0^\infty L\, h.\, f\, dx = \int_0^\infty h.\, L\, f\, dx + \frac{1}{4\pi}\,(h'(0)\, f(0) - f'(0)\, h(0)), \tag{12.38}$$

together with the following [17, p. 324]

$$D_\nu(0) = 2^{\frac{\nu}{2}}\, \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})}, \qquad D'_\nu(0) = -2^{\frac{\nu+1}{2}}\, \frac{\pi^{\frac{1}{2}}}{\Gamma(-\frac{\nu}{2})}. \tag{12.39}$$

Since $D_{\nu+2j}$ is an eigenfunction of L for the eigenvalue $\nu + \frac{1}{2} + 2j$, this yields after some trivial computations the result

$$(j-k)\int_0^\infty D_{\nu+2j}\, D_{\nu+2k}\, dx \tag{12.40}$$
$$= 2\pi^{\frac{1}{2}}.\, 2^{\nu+j+k-\frac{5}{2}}\left[\frac{1}{\Gamma(\frac{1-\nu}{2}-j)\,\Gamma(-\frac{\nu}{2}-k)} - \frac{1}{\Gamma(\frac{1-\nu}{2}-k)\,\Gamma(-\frac{\nu}{2}-j)}\right],$$

valid for any pair $(j,\, k) \in \mathbb{Z}^2$: similarly,

$$(k-j)\int_0^\infty D_{-\nu-2j-1}\, D_{-\nu-2k-1}\, dx \tag{12.41}$$
$$= 2\pi^{\frac{1}{2}}.\, 2^{-\nu-j-k-\frac{7}{2}}\left[\frac{1}{\Gamma(\frac{2+\nu}{2}-j)\,\Gamma(\frac{1+\nu}{2}+k)} - \frac{1}{\Gamma(\frac{2+\nu}{2}+k)\,\Gamma(\frac{1+\nu}{2}+j)}\right].$$

Since, as shown in Lemma 11.7, the $\mathbb{C}^4$-realization of the function $\phi^{\nu+2j}$ is $(D_{\nu+2j},\, 0,\, (-1)^j\, \frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}\, D_{-\nu-2j-1},\, 0)$, we easily verify, using also

$$\frac{\Gamma(\frac{1+\nu}{2}+j)\,\Gamma(\frac{2+\nu}{2}+k)}{\Gamma(-\nu)\,\Gamma(\nu+1)} = (-1)^{j+k}\, \frac{2\pi}{\Gamma(\frac{1-\nu}{2}-j)\,\Gamma(-\frac{\nu}{2}-k)}, \tag{12.42}$$

that, if $j \neq k$, one has $((\phi^{\nu+2j} \,|\, \phi^{\nu+2k})) = 0$.

The proof that $((\psi^{\nu+2j+1} \,|\, \psi^{\nu+2k+1})) = 0$ is entirely similar.

Now, let us compute the values of the form $((\ |\ ))$ on pairs of identical eigenfunctions of the ν-anaplectic harmonic oscillator, in which case we may start from a direct application of (12.37). One has

$$((\phi^{\nu+2j} \,|\, \phi^{\nu+2j})) = 2^{\frac{1}{2}}\left[\int_0^\infty D_{\nu+2j}^2\, dx + \frac{(\Gamma(\nu+2j+1))^2}{\Gamma(-\nu)\,\Gamma(\nu+1)}\int_0^\infty D_{-\nu-2j-1}^2\, dx\right]. \tag{12.43}$$

With the help of (12.37), this can be written as

$$\frac{1}{4\Gamma(-\nu-2j)}\left[\Delta\left(\frac{1-\nu}{2}-j\right)-\Delta\left(-\frac{\nu}{2}-j\right)+\Delta\left(\frac{2+\nu}{2}+j\right)-\Delta\left(\frac{1+\nu}{2}+j\right)\right], \tag{12.44}$$

an expression which, thanks to the functional equation (12.36), reduces to $\frac{1}{2}\Gamma(\nu+2j+1)$. Now, this agrees with the scalar product $(\phi^{\nu+2j}\,|\,\phi^{\nu+2j})$, as seen from (12.22). The same computation works when dealing with odd eigenstates of L in the space $\mathfrak{A}_\nu$. □

Finally, the ν-anaplectic analysis should lead, eventually, to a Weyl-like symbolic calculus of operators, which may start with the computation of a Wigner function, this being defined, as an extension of Definition 10.3, as

$$W^\nu(\psi,\chi)(x,\xi) = 2\,(e^{4i\pi(-x\,P+\xi\,Q)}\,\check{\psi}\,|\,\chi)_{\mathfrak{A}_\nu}. \tag{12.45}$$

It satisfies the same covariance properties as those expressed in Proposition 10.4 in the case when $\nu=-\frac{1}{2}$.

Proposition 12.6. *One has*

$$W^\nu(\phi^\nu,\phi^\nu)(x,\xi) = \Gamma(\nu+1)\,e^{-2\pi\,(x^2+\xi^2)}\;{}_1F_1\,(-\nu;\,1;\,4\pi\,(x^2+\xi^2)). \tag{12.46}$$

Proof. Our main interest in this proof is showing how easy it is to compute in the ν-anaplectic analysis, in view of the fact that the Heisenberg representation is still available.

First, we note that Lemma 10.5 extends with no change whatsoever to the ν-anaplectic analysis, and that most of Theorem 10.7 does, by way of consequence: the only argument from the proof of that theorem that does not is the computation (with the notation there) of $\Lambda\,W^\nu(\phi^\nu,\,\phi^\nu)$. However, one sees immediately that

$$\Lambda = \text{Right}^\dagger\,\text{Right} + \text{Left}^\dagger\,\text{Left} + 1\;: \tag{12.47}$$

consequently, using (10.21),

$$\begin{aligned}\Lambda\,W^\nu(\phi^\nu,\,\phi^\nu) &= W^\nu(\phi^\nu,\,A^*\,A\,\phi^\nu) + W^\nu(A^*\,A\,\phi^\nu,\,\phi^\nu) + W^\nu(\phi^\nu,\,\phi^\nu)\\ &= (2\nu+1)\,W^\nu(\phi^\nu,\,\phi^\nu)\end{aligned} \tag{12.48}$$

so that, again, $W^\nu(\phi^\nu,\,\phi^\nu)$ is an eigenfunction (with a new eigenvalue) of the (rescaled, in the same way as in Theorem 10.7) harmonic oscillator in two variables. Using the rotation invariance, one may write $W^\nu(\phi^\nu,\,\phi^\nu) = f(2\pi\,(x^2+\xi^2))$, where the function f is an analytic function on $[0,\,\infty[$ satisfying the differential equation

$$f''(t) + \frac{1}{t}\,f'(t) - \left(1-\frac{(2\nu+1)}{t}\right)\,f(t) = 0\;: \tag{12.49}$$

using this differential equation, and the already known value of $f(0) = 2(\phi^\nu \,|\, \phi^\nu)$, we arrive at the expression (12.46).

Needless to say, one could compute in just the same way, as in Theorem 10.7, the Wigner function of any two eigenfunctions of the ν-anaplectic harmonic oscillator on the line. $\square$

Remark 12.2. We shall stop short of developing a higher-dimensional ν-anaplectic analysis for general values of ν However, let us just indicate that, if one sets

$$W^\nu(\phi^\nu,\, \phi^\nu)(x,\xi) = \Gamma(\nu+1)\ \Phi_\nu\,(2^{\frac{1}{2}}x,\, 2^{\frac{1}{2}}\xi) \tag{12.50}$$

(this is the needed rescaling transformation: *cf.* Theorem 10.13; the coefficient is chosen so that $\Phi_\nu(0,0) = 1$), so that, a consequence of [17, p. 275]

$$\Phi_\nu(x,\xi) = -\frac{\sin \pi\nu}{\pi} \int_{-1}^{1} e^{-\pi\,(x^2+\xi^2)\,t}\,(1+t)^\nu\,(1-t)^{-1-\nu}\,dt. \tag{12.51}$$

It is an easy matter, generalizing what has been done in Section 4, to follow the continuation of the two-dimensional $\mathcal{K}$-transform of Φ_ν to the universal cover of $\Sigma = U(2) \cap \mathrm{Sym}_2^{\mathbb{C}}$. For, starting in the same way as in the proof of Theorem 4.18, we arrive at the following analogue of (4.74):

$$(\mathcal{K}\,\Phi_\nu)_0(Z) = \frac{2^{-\frac{3}{2}}}{\cos\frac{\pi\nu}{2}} \int_{-\infty}^{\infty} e^{(2\nu+1)\,\xi} \prod_{j=1,2} \left(\cosh\left(\xi - \frac{i\,\theta_j}{2}\right)\right)^{-\frac{1}{2}} d\xi\ : \tag{12.52}$$

here, $e^{-i\theta_1}$ and $e^{-i\theta_2}$ are the eigenvalues of $Z \in \Sigma$, and we start under the assumption that $0 < \theta_j < \pi$. With the method of Section 4, one then sees that the function $(\mathcal{K}\,\Phi_\nu)_0$ extends as an analytic function on the part of the universal cover of Σ lying above the subset of matrices with only simple eigenvalues, and that it is multiplied by the factor $e^{-(2\nu+1)\,i\pi}$ when Z is changed to $Z\,e^{2i\pi}$. In other words, the function $Z \mapsto (\det Z)^{\frac{1}{2}\,(\nu+\frac{1}{2})}\,(\mathcal{K}\,\Phi_\nu)_0(Z)$ extends as an analytic function on a connected dense open subset of $\Sigma^{(2)}$ (the twofold cover of Σ): the location and nature of the singularities of this function is similar to those of $(\mathcal{K}\,\Phi)_0$, as described at the very end of Section 4.

Bibliography

[1] C.M. Bender, M.V. Berry, A. Mandilara, *Generalized PT-symmetry and real spectra*, J. Phys. A: Math. Gen. **35** (2002), L467–L471.

[2] C.M. Bender, S. Boettcher, P.N. Meisinger, *PT-symmetric quantum mechanics*, J. of Math. Phys. **40** (1999), 2201–2229.

[3] J.D. Bjorken, S.D. Drell, *Relativistic quantum fields*, McGraw Hill, New York, 1965.

[4] P. Cartier, *Quantum mechanical commutation relations and theta functions*, Proc. Symp. Pure Math. IX, Amer. Math. Soc., (1966), 361–383.

[5] C. Cohen–Tannoudji, J.Dupont–Roc, G. Grynberg, *Photons et atomes: Introduction à l'électrodynamique quantique*, InterEditions/Editions du CNRS, Paris, 1987.

[6] A. Erdelyi, *Higher Transcendental Functions*, vol. 1, McGraw Hill, New York-Toronto-London, 1953.

[7] P. Eymard, *Analyse harmonique euclidienne*, *in* Les Cours du Cimpa, Nice (France), 1983.

[8] M. Gell–Mann, Y. Ne'eman, *The eightfold way*, W.A. Benjamin Inc., New York–Amsterdam, 1964.

[9] J. Glimm, A. Jaffe, *Quantum Physics: a functional integral point of view*, 2^{nd} edition, Springer-Verlag, Berlin–Heidelberg–New York, 1987.

[10] I.S. Gradstein, I.M. Ryshik, *Tables of Series, Products and Integrals*, vol. 2, Verlag Harri Deutsch, Thun–Frankfurt/M, 1981.

[11] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Acad. Press, New York–London, 1978.

[12] R. Howe, *The oscillator semigroup*, Proc.Symp.Pure Math. **48**, Amer. Math. Soc, Providence, 1988.

[13] J. Igusa, *Theta Functions*, Springer-Verlag, Berlin–Heidelberg–New York, 1972.

[14] A.W. Knapp, *Representation Theory of Semi-Simple Groups*, Princeton Univ. Press, Princeton, 1986.

[15] P.D. Lax, R.S. Phillips, *Scattering Theory for Automorphic Functions*, Ann. Math. Studies **87**, Princeton Univ. Press, 1976.

[16] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math. **216**, Springer-Verlag, Berlin–Heidelberg, 1971.

[17] W. Magnus, F.Oberhettinger, R.P. Soni, *Formulas and theorems for the special functions of mathematical physics*, $3^{\rm rd}$ edition, Springer-Verlag, Berlin, 1966.

[18] L. Pukanszky, *The Plancherel formula for the universal covering group of $SL(\mathbb{R},2)$*, Math. Annal. **156** (1964), 96–143.

[19] S.G. Pyatkov, *Operator theory. Nonclassical problems*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2002.

[20] N. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, 1951.

[21] E.M. Stein, G. Weiss, *Introduction to Fourier analysis in Euclidean spaces*, Princeton Univ. Press, Princeton, 1971.

[22] A. Unterberger, *Oscillateur harmonique et opérateurs pseudodifférentiels*, Ann. Inst. Fourier Grenoble **29**, 3 (1979), 201–221.

[23] A. Unterberger, *Les opérateurs métadifférentiels*, Lecture Notes in Physics **126** (1980), 205–241.

[24] A. Unterberger, *Quantization and non-holomorphic modular forms*, Lecture Notes in Math. **1742**, Springer-Verlag, Berlin–Heidelberg, 2000.

[25] A. Unterberger, *Automorphic pseudodifferential analysis and higher-level Weyl calculi*, Progress in Math., Birkhäuser, Basel-Boston, 2002.

[26] G. Valiron, *Théorie des Fonctions*, Masson, Paris, 1942.

[27] A. Wawrzyńczyk, *Group Representations and Special Functions*, Reidel Pub. Co., Dordrecht–Boston–Lancaster, 1984.

[28] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

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