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Complex, Contact and Symmetric Manifolds

In Honor of L. Vanhecke

Oldřich Kowalski
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Editors

Birkhäuser
Boston • Basel • Berlin

Oldřich Kowalski
Charles University
Faculty of Mathematics and Physics
186 75 Praha
Czech Republic

Emilio Musso
Università di L'Aquila
Dipartimento di Matematica Pura
ed Applicata
67100 L'Aquila
Italy

Domenico Perrone
Università degli Studi di Lecce
Dipartimento di Matematica "E. De Giorgi"
73100 Lecce
Italy

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Preface

This volume contains the extended versions of almost all lectures delivered during the International Conference “Curvature in Geometry” held in Lecce (Italy), 11–14 June 2003, in honour of Professor Lieven Vanhecke.

Prof. Lieven Vanhecke began his professional career at the Catholic University of Leuven (Belgium) where he obtained his PhD in 1966. He has been teaching at that University since the academic year 1965–1966 and was appointed full professor in 1972. Since 1972, he has been the head the Section of Geometry of the Mathematics Department of the Catholic University of Leuven. From 1972 until 1985 he also taught at the University of Antwerp as a part-time professor and became an Honorary Professor there in 1985.

Prof. Lieven Vanhecke has done research mainly in the field of differential geometry and, more particularly, in Riemannian and pseudo-Riemannian geometry. Throughout his scientific work, the study of curvature and of its properties has always played a central role. He started with classical topics on line congruences and minimal varieties. Later, he investigated Lorentzian, Hermitian and Kaehlerian manifolds, almost complex and almost contact manifolds, volumes of geodesic spheres and tubes, homogeneous structures on Riemannian manifolds, harmonic spaces, generalized Heisenberg groups and Damek-Ricci spaces, geodesic symmetries and reflections on Riemannian manifolds, Sasakian manifolds, various generalizations of symmetric spaces (e.g., naturally reductive, weakly symmetric and D’Atri spaces), curvature homogeneous spaces, foliations, the geometry of the tangent bundle and of the unit tangent bundle, geodesic transformations, special vector fields on Riemannian manifolds (minimal, harmonic), etc.

He has given more than one hundred lectures in almost as many universities and research centers around the world, and visited many of these universities as a researcher.

The almost 80 mathematicians from many different countries with whom Prof. Lieven Vanhecke has collaborated testify both to the wide range of interesting problems covered by his research and, above all, to his uncommon personal qualities. This has made him one of the world’s leading researchers in the field of Riemannian geometry. Most of the papers published in this volume are written by mathematicians who have been at some point either his students or collaborators.

We dedicate this volume to Professor Lieven Vanhecke with great affection and deep respect.

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Jerusalem *Yaakov Friedman*
October, 2004

Curvature of Contact Metric Manifolds*

David E. Blair

Department of Mathematics,
Michigan State University, East Lansing, MI 48824
blair@math.msu.edu

Dedicated to Professor Lieven Vanhecke

Summary. This essay surveys a number of results and open questions concerning the curvature of Riemannian metrics associated to a contact form.

In 1975, when the author was on sabbatical in Strasbourg, it was an open question whether or not the 5-torus carried a contact structure. The author, being interested in the Riemannian geometry of contact manifolds, proved at that time ([4]) that on a contact manifold of dimension ≥ 5, there are no flat associated metrics. Shortly thereafter, R. Lutz [31] proved that the 5-torus does indeed admit a contact structure and hence the natural flat metric on the 5-torus is not an associated metric. The non-flatness result of 1975 was generalized by Z. Olszak [35], who proved in 1978 that a contact metric manifold of constant curvature c and dimension ≥ 5 is Sasakian and of constant curvature $+1$. In dimension 3, the only constant curvature cases are of curvature 0 and 1 as we will note below. Sometimes one has an intuitive sense that the existence of a contact form tends to tighten up the manifold. The non-existence of flat associated metrics does raise the question as to whether, aside from the flat 3-dimensional case, any contact metric manifold must have some positive sectional curvature. If the manifold is compact, it is known ([7] p. 99) that there is no associated metric of strictly negative curvature. This follows from a deep result of A. Zeghib [48] on geodesic plane fields. Recall that a plane field on a Riemannian manifold is said to be *geodesic* if any geodesic tangent to the plane field at some point is everywhere tangent to it. Zeghib proves that a compact negatively curved Riemannian manifold has no C^1 geodesic plane field (of non-trivial dimension). Since for any associated metric the integral curves of the characteristic vector field, or Reeb vector field, are geodesics, the characteristic vector field determines a geodesic line field to which we can apply the theorem of Zeghib to obtain the following result.

Theorem. *On a compact contact manifold, there is no associated metric of strictly negative curvature.*

* This essay is an expanded version of the author's lecture given at the conference "Curvature in Geometry" in honor of Professor Lieven Vanhecke in Lecce, Italy, 11–14 June 2003.

The author conjectures that this and a bit more is true locally, namely, that except for the flat 3-dimensional case, any contact metric manifold has some positive sectional curvature.

Before giving other curvature results, we must review the structure tensors of a contact metric manifold. By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; ξ is called the *characteristic vector field* or *Reeb vector field* of the contact form η. A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^n y^i dx^i$. We denote the *contact subbundle* or *contact distribution* defined by the subspaces $\{X \in T_m M : \eta(X) = 0\}$ by $\mathcal{D}$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle is as far from being integrable as possible. In fact, for a contact manifold the maximum dimension of an integral submanifold of $\mathcal{D}$ is only n, whereas a subbundle defined by a 1-form η is integrable if and only if $\eta \wedge d\eta \equiv 0$. A Riemannian metric g is an *associated metric* for a contact form η if, first of all,

$$\eta(X) = g(X, \xi), \text{ i.e. the characteristic vector field is orthogonal to } \mathcal{D}$$

and secondly, there exists a field of endomorphisms ϕ such that

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad d\eta(X, Y) = g(X, \phi Y).$$

We refer to (ϕ, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold*. The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

and the underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be expressed as $N = 0$ where N is defined by

$$N(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi,$$

$[\phi, \phi]$ being the Nijenhuis tensor of ϕ. A *Sasakian manifold* is a normal contact metric manifold. In terms of the curvature tensor a contact metric structure is Sasakian if and only if

$$R_{XY}\xi = \eta(Y)X - \eta(X)Y.$$

In terms of the covariant derivative of ϕ the Sasakian condition is

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

A contact metric structure for which ξ is Killing is said to be *K-contact* and it is easy to see that a Sasakian manifold is K-contact. In dimension 3, a K-contact manifold is necessarily Sasakian but this is not true in higher dimensions. In the theory of contact metric manifolds there is another tensor field that plays a fundamental role, viz. $h = \frac{1}{2}\pounds_\xi \phi$ where $\pounds$ denotes Lie differentiation. The operator h is symmetric, it anti-commutes with ϕ, $h\xi = 0$ and h vanishes if and only if the contact metric structure is K-contact. The complexification of the tangent bundle of a contact metric manifold admits a holomorphic subbundle $\mathcal{H} = \{X - i\phi|_{\mathcal{D}}X : X \in \mathcal{D}\}$ and its Levi form is given by $-d\eta(X, \phi|_{\mathcal{D}}Y)$, $X, Y \in \mathcal{D}$. In this way a contact metric manifold becomes a (non-integrable) strongly pseudo-convex CR-manifold. The CR-structure is integrable if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. Tanno [46] showed that the integrability is equivalent to $(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. For later use, we mention briefly the idea of a $\mathcal{D}$-*homothetic deformation* of a contact metric structure. Let a be a positive constant and define a new structure by,

$$\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a-1)\eta \otimes \eta.$$

The new structure is again a contact metric structure and if the original structure is a Sasakian, a K-contact, or a strongly pseudo-convex integrable CR-structure, so is the new structure. For details and additional information on the above development, see the author's book [7].

Returning to the positivity of curvature question, we briefly mention the following. A celebrated theorem of Myers [33] states that a complete Riemannian manifold whose Ricci curvature satisfies $Ric \geq \delta > 0$ is compact. In [27] I. Hasegawa and M. Seino generalized Myers' theorem for a K-contact manifold by proving that a complete K-contact manifold for which $Ric \geq -\delta > -2$ is compact. Note however that in the K-contact case, all sectional curvatures of plane sections containing ξ are equal to 1 and hence there is a certain amount of positive curvature from the outset. In an attempt to weaken the K-contact requirement in this result, R. Sharma and the author [11] considered a contact metric manifold M^{2n+1} for which ξ is an eigenvector field of the Ricci operator. In this case if $Ric \geq -\delta > -2$ and the sectional curvatures of plane sections containing ξ are $\geq \epsilon > \delta' \geq 0$ where

$$\delta' = 2\sqrt{n(\delta - 2\sqrt{2\delta} + n + 2)} - (\delta - 2\sqrt{2\delta} + 1 + 2n),$$

then M^{2n+1} is compact. The condition that ξ be an eigenvector field of the Ricci operator is not only a natural generalization of the K-contact condition, but an important condition in its own right. D. Perrone [40] recently showed that ξ is an eigenvector field of the Ricci operator if and only if ξ is a harmonic vector field. Moreover, all complete 3-dimensional contact metric manifolds for which ξ is an eigenvector of the Ricci operator and for which the Ricci operator has constant eigenvalue are known (Koufogiorgos [29]); these are either Sasakian or particular Lie groups.

The next curvature result to discuss is the following [5].

Theorem. *A contact metric manifold M^{2n+1} satisfies $R_{XY}\xi = 0$ if and only if it is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

This structure is the standard contact metric structure on the tangent sphere bundle of Euclidean space; the standard normalizations give the curvature of the sphere factor as +4. For brevity we will not discuss the contact metric structure on the tangent sphere bundle T_1M of a Riemannian manifold M; suffice it to note that the characteristic vector field is (to within a factor of 2) the geodesic flow (again see [7], Section 9.2 for details). Now $E^{n+1} \times S^n(4)$ is a symmetric space and one can ask first when the tangent sphere bundle is locally symmetric and, more generally, whether one can classify all locally symmetric contact metric manifolds. For the first question the author proved the following result in [6].

Theorem. *The standard contact metric structure on T_1M is locally symmetric if and only if either the base manifold M is flat or 2-dimensional and of constant curvature* $+1$.

For the more general question we have the following results of Blair-Sharma [12] and A. M. Pastore [37] respectively.

Theorem. *A 3-dimensional contact metric manifold is locally symmetric if and only if it is of constant curvature* 0 *or* $+1$.

Theorem. *A 5-dimensional contact metric manifold is locally symmetric if and only if it is locally isometric to $S^5(1)$ or $E^3 \times S^2(4)$.*

Very early in the development of the Riemannian geometry of contact manifolds the following had been shown.

Theorem. *A locally symmetric K-contact manifold is of constant curvature* $+1$ *and Sasakian.*

This result was due to Tanno in 1967 [43] and in the Sasakian with dimension ≥ 5 case to Okumura in 1962 [34].

We now consider briefly the analog of holomorphic sectional curvature, namely ϕ-sectional curvature. A plane section in T_mM^{2n+1} is called a *ϕ-section* if there exists a vector $X \in T_mM^{2n+1}$ orthogonal to ξ such that $\{X, \phi X\}$ span the section and the sectional curvature is called *ϕ-sectional curvature*. Just as the sectional curvatures of a Riemannian manifold and the holomorphic sectional curvatures of a Kähler manifold determine the curvature completely, on a Sasakian manifold the ϕ-sectional curvatures determine the curvature completely. Moreover, on a Sasakian manifold of dimension ≥ 5, if at each point the ϕ-sectional curvature is independent of the choice of ϕ-section at the point, it is constant on the manifold and the curvature tensor is given by,

$$\begin{aligned} R_{XY}Z = {} & \frac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\big). \end{aligned}$$

A Sasakian manifold of constant ϕ-sectional curvature is called a *Sasakian space form*. A well-known result of Tanno [44] is that a complete simply connected Sasakian manifold of constant ϕ-sectional curvature c is isometric to one of certain model spaces depending on whether $c > -3$, $c = -3$ or $c < -3$. The model space for $c > -3$ is a sphere with a $\mathcal{D}$-homothetic deformation of the standard structure. For $c = -3$ the model space is $\mathbb{R}^{2n+1}$ with the contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$, the factor of $\frac{1}{2}$ being convenient for normalization reasons, together with the metric $ds^2 = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}((dx^i)^2 + (dy^i)^2)$. For the $c < -3$ case one has a canonically defined contact metric structure on the product $B^n \times \mathbb{R}$ where B^n is a simply connected bounded domain in $\mathbb{C}^n$ with a Kähler structure of constant negative holomorphic curvature. In particular, Sasakian space forms exist for all values of c. In the general context of contact metric manifolds, J. T. Cho [23] recently introduced the notion of a *contact Riemannian space form*. We get at this notion in the following way. In [47] Tanno showed that the CR-structure of the tangent sphere bundle with its standard contact metric structure is integrable if and only if the base manifold is of constant curvature. Cho first computes the covariant derivative of h in this case obtaining

$$(\nabla_X h)Y = g((h - h^2)\phi X, Y)\xi + \eta(Y)(h - h^2)\phi X - \mu\eta(X)h\phi Y,$$

where μ is a constant. He then abstracts this idea and defines the class $\mathfrak{Q}$ of contact Riemannian manifolds with integrable CR-structure for which the covariant derivative of h satisfies the above condition. We remark that in the study of contact manifolds in general, lack of control of the covariant derivative of h is often an obstacle to further results. Now for a contact metric manifold M^{2n+1} of class $\mathfrak{Q}$ for which the ϕ-sectional curvature is independent of the choice of ϕ-section, Cho shows that the ϕ-sectional curvature is constant on M^{2n+1} and computes the curvature tensor explicitly. He then defines a *contact Riemannian space form* to be a complete, simply connected contact metric manifold of class $\mathfrak{Q}$ of constant ϕ-sectional curvature. Cho also gives a number of non-Sasakian examples and shows that a contact Riemannian space form is locally homogeneous and is strongly locally ϕ-symmetric (see below).

We noted above that a locally symmetric K-contact manifold is of constant curvature $+1$ and Sasakian. For K-contact geometry this can be regarded as saying that the idea of being locally symmetric is too strong. For this reason Takahashi [41] introduced the following notion: A Sasakian manifold is said to be a *Sasakian locally ϕ-symmetric space* if

$$\phi^2(\nabla_V R)_{XY}Z = 0,$$

for all vector fields V, X, Y, Z orthogonal to ξ. It is easy to check that Sasakian space forms are locally ϕ-symmetric spaces. Note that on a K-contact manifold, a geodesic that is initially orthogonal to ξ remains orthogonal to ξ. We call such a geodesic a *ϕ-geodesic*. A local diffeomorphism s_m of M^{2n+1}, $m \in M^{2n+1}$, is a *ϕ-geodesic symmetry* if its domain contains a (possibly) smaller domain $\mathcal{U}$ such that for every ϕ-geodesic $\gamma(s)$ parametrized by arc length with $\gamma(0) \in \mathcal{U}$ and on the integral curve of ξ through m,

$$(s_m \circ \gamma)(s) = \gamma(-s),$$

for all s with $\gamma(\pm s) \in \mathcal{U}$. Takahashi defines a Sasakian manifold to be a *Sasakian globally ϕ-symmetric space* by requiring that any ϕ-geodesic symmetry can be extended to a global automorphism of the structure and that the Killing vector field ξ generates a 1-parameter group of global transformations. Among the main results of Takahashi are the following three theorems.

Theorem. *A Sasakian locally ϕ-symmetric space is locally isometric to a Sasakian globally ϕ-symmetric space and a complete, connected, simply-connected Sasakian locally ϕ-symmetric space is a globally ϕ-symmetric space.*

Theorem. *A Sasakian manifold is locally ϕ-symmetric if and only if it admits a ϕ-geodesic symmetry at every point which is a local automorphism of the structure.*

Now suppose that $\mathcal{U}$ is a neighborhood on M^{2n+1} on which ξ is a regular vector field, then since M^{2n+1} is Sasakian, the projection $\pi : \mathcal{U} \longrightarrow \mathcal{V} = \mathcal{U}/\xi$ gives a Kähler structure on $\mathcal{V}$. Furthermore if $\underline{s}_{\pi(m)}$ denotes the geodesic symmetry on $\mathcal{V}$ at $\pi(m)$, then $\underline{s}_{\pi(m)} \circ \pi = \pi \circ s_m$.

Theorem. *A Sasakian manifold is locally ϕ-symmetric if and only if each Kähler manifold which is the base of a local fibering is a Hermitian locally symmetric space.*

Recall that a Riemannian manifold is locally symmetric if and only if the local geodesic symmetries are isometries. From the Takahashi theorems we note that on a Sasakian locally ϕ-symmetric space, local ϕ-geodesic symmetries are isometries. Conversely in [13], L. Vanhecke and the author proved that if on a Sasakian manifold the local ϕ-geodesic symmetries are isometries, the manifold is a Sasakian locally ϕ-symmetric space. This was extended to the K-contact case by Bueken and Vanhecke [19] and we have the following Theorem.

Theorem. *If on a K-contact manifold the local ϕ-geodesic symmetries are isometries, the manifold is a Sasakian locally ϕ-symmetric space.*

Finally J. A. Jiménez and O. Kowalski [28] classified complete simply-connected globally ϕ-symmetric spaces.

We now ask what is the best notion of a locally ϕ-symmetric space for a general contact metric manifold? One could use the same definition, namely,

$$\phi^2(\nabla_V R)_{XY}Z = 0,$$

for all vector fields V, X, Y, Z orthogonal to ξ and this condition gives what is known as a *weakly locally ϕ-symmetric space*. Now without the K-contact property one loses the fact that a geodesic, initially orthogonal to ξ, remains orthogonal to ξ. However we have just seen that in the Sasakian case local ϕ-symmetry is equivalent to reflections in the integral curves of the characteristic vector field being isometries. E. Boeckx and L. Vanhecke [17] proposed this property as the definition for local ϕ-symmetry in the contact metric case and call a contact metric manifold with this property a *strongly locally ϕ-symmetric space*. From the work of B.-Y. Chen and L. Vanhecke [22] one can see that on a strongly locally ϕ-symmetric space,

$$g((\nabla^{2k}_{X\cdots X}R)_{XY}X, \xi) = 0,$$

$$g((\nabla^{2k+1}_{X\cdots X}R)_{XY}X, Z) = 0,$$

$$g((\nabla^{2k+1}_{X\cdots X}R)_{X\xi}X, \xi) = 0,$$

for all X, Y, Z orthogonal to ξ and all $k \in \mathbb{N}$. Conversely, in the analytic case these conditions are sufficient for the contact metric manifold to be a strongly locally ϕ-symmetric space. In particular, taking $k = 0$ in the second condition, we note that a strongly locally ϕ-symmetric space is weakly locally ϕ-symmetric. In [21], G. Calvaruso, D. Perrone and L. Vanhecke determined all 3-dimensional strongly locally

ϕ-symmetric spaces. In [18] E. Boeckx, P. Bueken and L. Vanhecke gave an example of a non-unimodular Lie group with a weakly locally ϕ-symmetric contact metric structure which is not strongly locally ϕ-symmetric.

As a generalization of both $R_{XY}\xi = 0$ and the Sasakian case, $R_{XY}\xi = \eta(Y)X - \eta(X)Y$, Th. Koufogiorgos, B. Papantoniou and the author [10] considered the (κ, μ)-*nullity condition*,

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where κ and μ are constants and gave several reasons for studying it. We refer to a contact metric manifold satisfying this condition as a (κ, μ)-*manifold*. On a (κ, μ)-manifold, $\kappa \leq 1$. If $\kappa = 1$, the structure is Sasakian and if $\kappa < 1$, the (κ, μ)-nullity condition determines the curvature of M^{2n+1} completely. When $\kappa < 1$, the non-zero eigenvalues of h are $\pm\sqrt{1-\kappa}$ each with multiplicity n. Th. Koufogiorgos and C. Tsichlias [30] considered this condition where κ and μ are functions; they showed that in dimensions ≥ 5, κ and μ must be constants but that in dimension 3 these "generalized (κ, μ)-manifolds" exist. The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (κ, μ)-nullity condition if and only if the base manifold M is of constant curvature. In particular if M has constant curvature c, then $\kappa = c(2-c)$ and $\mu = -2c$. A $\mathcal{D}$-homothetic deformation destroys a condition like $R_{XY}\xi = 0$ or

$$R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y).$$

However, the form of the (κ, μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \tilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (κ, μ)-manifold M, E. Boeckx [15] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}},$$

and showed that for two non-Sasakian (κ, μ)-manifolds $(M_i, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two spaces are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ, μ)-manifolds locally as soon as we have, for every odd dimension $2n + 1$ and for every possible value of the invariant I, one (κ, μ)-manifold (M, ϕ, ξ, η, g) with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = (1 + c)/|1 - c|$. Boeckx also gives a Lie algebra construction for any odd dimension and any value of $I \leq -1$.

Returning to the strongly locally ϕ-symmetric spaces, we note that the non-Sasakian (κ, μ)-spaces are strongly locally ϕ-symmetric as was shown by E. Boeckx [14]. Special cases of these are the non-Abelian 3-dimensional unimodular Lie groups with left-invariant contact metric structures. To see these examples, we first note the classification of simply connected homogeneous 3-dimensional contact metric manifolds as given by D. Perrone in [39]. Let τ denote the scalar curvature and

$$w = \frac{1}{8}(\tau - Ric(\xi) + 4),$$

the Webster scalar curvature. The classification of 3-dimensional Lie groups and their left invariant metrics was given by Milnor [32].

Theorem. *Let (M^3, η, g) be a simply connected homogeneous contact metric manifold. Then M^3 is a Lie group G and both g and η are left-invariant. More precisely we have the following classification:* (1) *If G is unimodular, then it is one of the following Lie groups:*

1. *The Heisenberg group when $w = |\pounds_\xi g| = 0$;*
2. *$SU(2)$ when $4\sqrt{2}w > |\pounds_\xi g|$;*
3. *the universal covering of the group of rigid motions of the Euclidean plane when $4\sqrt{2}w = |\pounds_\xi g| > 0$;*
4. *the universal covering of $SL(2, \mathbb{R})$ when $-|\pounds_\xi g| \neq 4\sqrt{2}w < |\pounds_\xi g|$;*
5. *the group of rigid motions of the Minkowski plane when $4\sqrt{2}w = -|\pounds \xi g| < 0$.*

(2) *If G is non-unimodular, its Lie algebra is given by*

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = \gamma e_2, \quad [e_2, \xi] = 0,$$

where $\alpha \neq 0$, $e_1, e_2 = \phi e_1 \in \mathcal{D}$ and $4\sqrt{2}w < |\pounds_\xi g|$. Moreover, if $\gamma = 0$, the structure is Sasakian and $w = -\alpha^2/4$.

The structures on the unimodular Lie groups in this theorem satisfy the (κ, μ)-nullity condition and hence they are strongly locally ϕ-symmetric. The weak locally ϕ-symmetric contact metric structure which is not the strong locally ϕ-symmetric given by Boeckx, Bueken and Vanhecke [18] is the non-unimodular case with $\gamma = 2$. Notice also, in the unimodular case, the role played by the invariant $p = (4\sqrt{2}w)/|\pounds_\xi g|$. Moreover $w = (2 - \mu)/4$ and $|\pounds_\xi g| = 2\sqrt{2}\sqrt{1-\kappa}$; thus $p = (2-\mu)/(2\sqrt{1-\kappa})$ which is the above invariant I_M of Boeckx.

A special case of the (κ, μ)-spaces is, of course, the case where ξ belongs to the κ-nullity distribution, i.e. $\mu = 0$ and we call such a contact metric manifold an *$N(\kappa)$-contact metric manifold*. Using the Boeckx invariant we construct an example of a $(2n+1)$-dimensional $N(1 - (\frac{1}{n}))$-manifold, $n > 1$.

Example. Since the Boeckx invariant for a $(1 - (\frac{1}{n}), 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $(1 - (1/n), 0)$-manifold. That is, for $\kappa = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{\kappa + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a},$$

for a and c. The result is

$$c = \frac{(\sqrt{n} \pm 1)^2}{n-1}, \quad a = 1 + c,$$

and taking c and a to be these values we obtain a $N(1 - (\frac{1}{n}))$-manifold.

Now as a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold is said to be semi-symmetric if its curvature tensor satisfies $R_{XY} \cdot R = 0$, where R_{XY} acts on R as a derivation. In [45] Tanno showed that a semi-symmetric K-contact manifold is locally isometric to $S^{2n+1}(1)$. In [38] D. Perrone began the study of semi-symmetric contact metric manifolds and in [36] B. Papantoniou showed that a semi-symmetric (κ, μ)-space of dimension ≥ 5 is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. Similarly Ch. Baikoussis and Th. Koufogiorgos [1] showed that an $N(\kappa)$-contact metric manifold satisfying $R_{\xi X} \cdot W = 0$, W being the Weyl conformal curvature tensor, is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. In [16] E. Boeckx and G. Calvaruso showed that the tangent sphere bundle is semi-symmetric if and only if it is locally symmetric and therefore the base manifold is either flat or 2-dimensional and of constant curvature $+1$. With this in mind it is surprising that the concircular curvature tensor,

$$Z_{XY}V = R_{XY}V - \frac{\tau}{2n(2n+1)}(g(Y, V)X - g(X, V)Y),$$

leads to other cases. Recently J.-S. Kim, M. Tripathi and the author [8] proved the following.

Theorem. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies*

$$Z_{\xi X} \cdot Z = 0,$$

if and only if M is 3-dimensional and flat, or locally isometric to the sphere $S^{2n+1}(1)$, or M is locally isometric to the above example of an $N(1 - \frac{1}{n})$-manifold.

We close this essay with the question of conformally flat contact metric manifolds, a question in which there has recently been renewed interest. Early on, Okumura [34] had shown that a conformally flat Sasakian manifold of dimension ≥ 5 is of constant curvature $+1$ and in [42] and [43] Tanno extended this result to the K-contact case and for dimensions ≥ 3. Thus a conformally flat K-contact manifold is of constant curvature $+1$ and Sasakian. Recently Ghosh, Koufogiorgos and Sharma [24] have shown that a conformally flat contact metric manifold of dimension ≥ 5 with a strongly pseudo-convex integrable CR-structure is of constant curvature $+1$. As we have seen, in dimension ≥ 5, a contact metric structure of constant curvature must be of constant curvature $+1$ and is Sasakian; and in dimension 3, a contact metric structure of constant curvature must be of constant curvature 0 or $+1$, the latter case being Sasakian. For simplicity set $lX = R_{X\xi}\xi$, then l is a symmetric operator. K. Bang [2] showed that in dimension ≥ 5 there are no conformally flat contact metric structures with $l = 0$, even though there are many contact metric manifolds satisfying $l = 0$, ([2] or see

[7] p. 153). Bang's result was extended to dimension 3 and generalized by F. Gouli-Andreou and Ph. Xenos [26] who showed that in dimension 3 the only conformally flat contact metric structures satisfying $\nabla_\xi l = 0$ (equivalently $\nabla_\xi h = 0$, Perrone [38]) are those of constant curvature 0 or 1. In [25] F. Gouli-Andreou and N. Tsolakidou showed that a conformally flat contact metric manifold M^{2n+1} with $l = -\kappa\phi^2$ for some function κ is of constant curvature. In the case of the standard contact metric structure on the tangent sphere bundle, Th. Koufogiorgos and the author [9] showed that the metric is conformally flat, if and only if the base manifold is a surface of constant Gaussian curvature 0 or 1. The (κ, μ)-spaces are conformally flat only in the constant curvature cases. In dimension 3, this was shown by F. Gouli-Andreou and Ph. Xenos [26], even when κ and μ are functions. In higher dimensions the proof is straightforward: Let W denote the Weyl conformal curvature tensor. $W_{X\xi}\xi = 0$ with $X \perp \xi$ yields $[2(n-1)(\mu-1)/2n-1]hX = 0$; if $n = 1$ we have the case studied by Gouli-Andreou and Xenos and if $h = 0$ we have the K-contact case. If $\mu = 1$, $h \neq 0$ and $n > 1$, we can choose two orthogonal unit eigenvectors X and Y of h with eigenvalue $\lambda > 0$ and set $Z = \phi Y$. Then using Theorem 1 of [10], $W_{XY}Z = 0$ yields $\kappa = 1$ $(\lambda = 0)$, contradicting $\lambda > 0$. In [9] Th. Koufogiorgos and the author showed that a conformally flat contact metric manifold on which the Ricci operator commutes with ϕ is of constant curvature. Then in [21] G. Calvaruso, D. Perrone and L. Vanhecke showed that in dimension 3 the only conformally flat contact metric structures, for which ξ is an eigenvector of the Ricci operator, are those of constant curvature 0 or 1. An attempt was made in [24] to generalize this to higher dimensions by assuming another condition in addition to ξ being an eigenvector of the Ricci operator. However ξ being an eigenvector of the Ricci operator is the essential condition and we now have a recent result of K. Bang and the author [3] generalizing the Calvaruso, Perrone and Vanhecke result to higher dimensions.

Theorem. *A conformally flat contact metric manifold for which the characteristic vector field is an eigenvector of the Ricci operator is of constant curvature.*

In view of these strong curvature results, one may ask if there are any conformally flat contact metric structures which are not of constant curvature. In [7] (pp. 108–110), the author shows that 3-dimensional conformally flat contact metric manifolds of non-constant curvature do exist. These examples were studied further by Calvaruso [20]; he showed that these examples satisfy $\nabla_\xi h = ah\phi$, where a is a non-constant function. He also showed that if a is a constant $\neq 2$, then a 3-dimensional conformally flat contact metric manifold satisfying $\nabla_\xi h = ah\phi$ has constant curvature. It is not known if there exist conformally flat contact metric manifolds of dimension ≥ 5 which are not locally isometric to the standard Sasakian structure on the unit sphere.

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A Case for Curvature: the Unit Tangent Bundle

Eric Boeckx

Department of Mathematics, Katholieke Universiteit Leuven,
Celestijnenlaan 200B, 3001 Leuven, Belgium
eric.boeckx@wis.kuleuven.ac.be

Dedicated to Professor Lieven Vanhecke

1 Curvature theory

In the scientific work of L. Vanhecke, the notion of curvature is never more than a step away, if not studied explicitly. This is only right, since, in the words of R. Osserman, "curvature is *the* central concept (in differential geometry and, more in particular, in Riemannian geometry), distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic or topological". The reason for this can be seen as follows:

— if we equip a differentiable manifold M with a metric g, then its curvature is completely determined. If the metric g has nice properties (e.g., a large group of isometries), then this is reflected in a 'nice' curvature;
— conversely, we can often deduce information about the metric from special properties of the curvature. In some cases, knowledge about the curvature even suffices to completely determine the metric (at least locally). Locally symmetric spaces are the prime example here: they are distinguished from non-symmetric spaces by their parallel curvature and, starting from the curvature, one can reconstruct the manifold and its metric (locally).

The curvature information is contained in the Riemannian curvature tensor R. This is an analytic object, a (0, 4)-tensor which is not easy to handle, in general, despite its many symmetries. It is often very difficult to extract the geometrical information which is, as it were, encoded within. For this reason, the famous geometer M. Gromov calls the curvature tensor "a little monster of multilinear algebra whose full geometric meaning

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remains obscure". One therefore works not only with the curvature tensor R itself, but with other forms of curvature or related operators as well, which have a more direct geometric interpretation or which are easier to deal with. We mention the sectional curvature, the Ricci curvature, the scalar curvature and the Jacobi operators. However, not all of these contain the same amount of curvature information. Curvature theory has as its explicit aim the shedding of light on the interplay between the curvature of a Riemannian manifold and its geometric properties, in spite of the difficulties mentioned before.

The study of manifolds from the point of view of curvature has two complementary aspects, roughly corresponding to the two passages: from the metric (and all the geometry that it entails) to the curvature and from the curvature to the metric.

1. *Direct theory.* First, one looks at 'simple' manifolds. By this we mean Riemannian manifolds with a high degree of symmetry and hence with a relatively easy curvature tensor. In some cases, one can even write it down explicitly. As examples of such spaces, we mention locally symmetric spaces, homogeneous spaces and two-point homogeneous spaces. One studies their geometric properties, which are often generalizations of properties from classical Euclidean geometry. In particular, one also studies associated objects like small geodesic spheres, tubes about curves and submanifolds, tangent and unit tangent bundles, special transformations, ...

2. *Inverse theory.* Next, one compares more general manifolds with one of these 'simple' spaces: one takes the latter as a model and investigates which of its properties (or those of its associated objects) are *characteristic* for the model space. In other words: can one recognize the model space based on those specific properties? If not, one tries to find a complete classification of Riemannian manifolds with those properties. The technical details at this stage differ considerably from those in the direct theory. Indeed, for general manifolds, no explicit description of the curvature is available. Further, quantities such as, e.g., the volume of small geodesic spheres and balls can no longer be written down in closed form. Instead, one often uses series expansions for these quantities, the coefficients of which depend on the curvature. Geometric information concerning, e.g., the volumes of the small geodesic spheres then lead to restrictions on the curvature via the series expansions. In other situations, the geometric properties considered have natural consequences for the Jacobi operators or other forms of curvature. In this way, one collects curvature information and hopes to be able to draw conclusions from this concerning the metric. Curvature acts here as the bridge between the geometric properties and the metric itself.

The contributions of L. Vanhecke to the field of curvature theory in the above spirit are too numerous to specify and his influence on geometry and on geometers worldwide, the present one included, can readily be discerned. In this note, I only intend to illustrate the above program using the geometry of the unit tangent bundle as a showcase. On this topic, I have worked for some years now, often in collaboration with L. Vanhecke and other colleagues. For a survey of earlier results, see [5]. Here, I will concentrate on two aspects of the unit tangent bundle: its geodesics and the question of reducibility. The presentation will be rather brief. Full statements and proofs can be found in the articles [1] and [3].

2 The unit tangent bundle

We first recall a few of the basic facts and formulas about the unit tangent bundle of a Riemannian manifold. A more elaborate exposition and further references can be found in [4].

The tangent bundle TM of a Riemannian manifold (M, g) consists of pairs (x, u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi : TM \to M : (x, u) \mapsto x$ is the natural projection from TM onto M. It is well-known that the tangent space to TM at a point (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the Levi Civita connection ∇ of (M, g): $T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}$.

For $w \in T_xM$, there exists a unique horizontal vector $w^h \in HTM_{(x,u)}$ for which $\pi_*(w^h) = w$. It is called the *horizontal lift* of w to (x, u). There is also a unique vertical vector $w^v \in VTM_{(x,u)}$ for which $w^v(df) = w(f)$ for all functions f on M. It is called the *vertical lift* of w to (x, u). These lifts define isomorphisms between T_xM and $HTM_{(x,u)}$ and $VTM_{(x,u)}$ respectively. Hence, every tangent vector to TM at (x, u) can be written as the sum of a horizontal and a vertical lift of uniquely defined tangent vectors to M at x. The *horizontal* (respectively *vertical*) *lift of a vector field* X on M to TM is defined in the same way by lifting X pointwise. Further, if T is a tensor field of type $(1, s)$ on M and $X_1, \dots, X_{s-1}$ are vector fields on M, then we denote by $T(X_1, \dots, u, \dots, X_{s-1})^v$ the vertical vector field on TM which at (x, w) takes the value $T(X_{1x}, \dots, w, \dots, X_{s-1\,x})^v$, and similarly for the horizontal lift. In general, these are *not* the vertical or horizontal lifts of a vector field on M.

The *Sasaki metric* g_S on TM is completely determined by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X, Y) \circ \pi, \quad g_S(X^h, Y^v) = 0,$$

for vector fields X and Y on M.

Our interest lies in the unit tangent bundle T_1M, which is the hypersurface of TM consisting of all tangent vectors to (M, g) of length 1. It is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector field N to T_1M is given by the vertical vector field u^v. We see that horizontal lifts to $(x, u) \in T_1M$ are tangents to T_1M, but vertical lifts in general are not. For that reason, we define the *tangential lift* w^t of $w \in T_xM$ to $(x, u) \in T_1M$ by $w^t = w^v - g(w, u)N$. Clearly, the tangent space to T_1M at (x, u) is spanned by horizontal and tangential lifts of tangent vectors to M at x. One defines the *tangential lift of a vector field* X on M in the obvious way. For the sake of notational clarity, we will use $\bar{X}$ as a shorthand for $X - g(X, u)u$. Then $X^t = \bar{X}^v$. Further, we denote by VT_1M the $(n-1)$-dimensional distribution of vertical tangent vectors to T_1M.

If we consider T_1M with the metric induced from the Sasaki metric g_S of TM, also denoted by g_S, we turn T_1M into a Riemannian manifold. Its Levi Civita connection $\bar{\nabla}$ is described completely by

$$\begin{aligned} \bar{\nabla}_{X^t} Y^t &= -g(Y, u)X^t, \\ \bar{\nabla}_{X^t} Y^h &= \frac{1}{2}\,(R(u, X)Y)^h, \end{aligned} \tag{1}$$

$$\bar{\nabla}_{X^h} Y^t = (\nabla_X Y)^t + \frac{1}{2}\,(R(u,Y)X)^h,$$
$$\bar{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}\,(R(X,Y)u)^t,$$

for vector fields X and Y on M. Its Riemann curvature tensor $\bar{R}$ is given by

$$\begin{aligned}
\bar{R}(X^t, Y^t)Z^t &= g(\bar{Y}, \bar{Z})\, X^t - g(\bar{Z}, \bar{X})\, Y^t,\\
\bar{R}(X^t, Y^t)Z^h &= (R(\bar{X}, \bar{Y})Z)^h + \frac{1}{4}([R(u,X), R(u,Y)]Z)^h,\\
\bar{R}(X^h, Y^t)Z^t &= -\frac{1}{2}(R(\bar{Y}, \bar{Z})X)^h - \frac{1}{4}(R(u,Y)R(u,Z)X)^h,\\
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}(R(X,Z)\bar{Y})^t - \frac{1}{4}(R(X, R(u,Y)Z)u)^t\\
&\quad + \frac{1}{2}((\nabla_X R)(u,Y)Z)^h, \qquad (2)\\
\bar{R}(X^h, Y^h)Z^t &= (R(X,Y)\bar{Z})^t\\
&\quad + \frac{1}{4}(R(Y, R(u,Z)X)u - R(X, R(u,Z)Y)u)^t\\
&\quad + \frac{1}{2}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^h,\\
\bar{R}(X^h, Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}(R(u, R(X,Y)u)Z)^h\\
&\quad - \frac{1}{4}(R(u, R(Y,Z)u)X - R(u, R(X,Z)u)Y)^h\\
&\quad + \frac{1}{2}((\nabla_Z R)(X,Y)u)^t,
\end{aligned}$$

for vector fields X, Y and Z on M.

From these formulas, it is clear how the curvature of the base manifold interferes in the geometry and the curvature of the unit tangent bundle. Conversely, we will be able to 'translate' information on the unit tangent bundle to the base manifold using these formulas. This should not surprise us, as the metric structure on the base manifold completely determines that of the bundle.

3 Geodesics on the unit tangent bundle

As a first illustration of the role of curvature in geometric problems, we are interested in geodesics of the unit tangent bundle. Any curve $\gamma(t) = (x(t), V(t))$ in the unit tangent bundle can be considered as a curve $x(t)$ in the base manifold M, together with a unit vector field $V(t)$ along it. The geodesic equation in (T_1M, g_S) can be readily deduced from the formulas (1) for the Levi Civita connection. We find that $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), if and only if

$$\nabla_{\dot{x}}\dot{x} = -R(V, \nabla_{\dot{x}} V)\dot{x}, \tag{3}$$
$$\nabla_{\dot{x}}\nabla_{\dot{x}} V = -c^2 V,$$

where $c^2 = g(\nabla_{\dot{x}} V, \nabla_{\dot{x}} V)$ is a constant along $x(t)$. (See, e.g., [9].)

For general Riemannian manifolds, it is hopeless to try and solve the system of differential equations (3). For 'simple' base spaces, however, some results can be obtained. For two-dimensional base spaces, a full solution was given in [7]. When the base manifold is a space of constant curvature c, the curvature can be written as $R(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)$ and the equation (3) becomes much simpler. S. Sasaki ([10]) has explicitly determined all geodesics in this setting. As a side-result of his description, we state

Proposition 1. *If (M^n, g) is a space of constant curvature and $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), then the projected curve $x(t) = \pi(\gamma(t))$ in M^n has constant curvatures κ_1 and κ_2 and vanishing third curvature κ_3.*

For a locally symmetric base manifold, P. Nagy showed a result in the same spirit in [8].

Proposition 2. *If (M^n, g) is a locally symmetric space and $\gamma(t) = (x(t), V(t))$ is a geodesic of (T_1M, g_S), then the curve $x(t)$ in M has constant curvatures κ_i, $i = 1, \ldots, n-1$.*

The proofs for both propositions are based on the same idea. Since both $|\dot{\gamma}|^2 = |\dot{x}|^2 + |\nabla_{\dot{x}} V|^2$ and $|\nabla_{\dot{x}} V|^2 = c^2$ are constant, we can reparametrize $\gamma(t)$ (and $x(t)$) so that $|\dot{x}| = 1$. Hence we can take $T = \dot{x}$ as the first vector in the Frenet frame $\{T, N_1, \ldots, N_{n-1}\}$ along x and we have for the first three covariant derivatives of $\dot{x}$:

$$\begin{aligned}
\dot{x}^{(1)} &= \nabla_{\dot{x}}\dot{x} = \kappa_1 N_1,\\
\dot{x}^{(2)} &= \nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} = -{\kappa_1}^2\, T + \kappa_1' N_1 + \kappa_1\kappa_2\, N_2,\\
\dot{x}^{(3)} &= \nabla_{\dot{x}}\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} = -3\kappa_1\kappa_1'\, T + (\kappa_1'' - \kappa_1({\kappa_1}^2 + {\kappa_2}^2))\, N_1\\
&\qquad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')\, N_2 + \kappa_1\kappa_2\kappa_3\, N_3,
\end{aligned} \tag{4}$$

and similar expressions for the higher order derivatives of x. On the other hand, using (3), we can calculate

$$\begin{aligned}
\dot{x}^{(1)} &= -R(V, \dot{V})\dot{x},\\
\dot{x}^{(2)} &= -(\nabla_{\dot{x}} R)(V, \dot{V})\dot{x} + R(V, \dot{V})^2\dot{x},\\
\dot{x}^{(3)} &= -(\nabla^{(2)}_{\dot{x}\dot{x}} R)(V, \dot{V})\dot{x} + (\nabla_{R(V,\dot{V})\dot{x}} R)(V, \dot{V})\dot{x}\\
&\quad + 2(\nabla_{\dot{x}} R)(V, \dot{V})R(V, \dot{V})\dot{x} + R(V, \dot{V})(\nabla_{\dot{x}} R)(V, \dot{V})\dot{x}\\
&\quad - R(V, \dot{V})^3\dot{x},
\end{aligned} \tag{5}$$

where we have put $\dot{V} = \nabla_{\dot{x}} V$ for simplicity. Again, similar expressions can be derived for higher order derivatives of x. In particular, for a locally symmetric base space, this leads to the simple formula

$$\dot{x}^{(k)} = (-1)^k R(V, \dot{V})^k \dot{x}.$$

It is easy to see from this formula that $\dot{x}^{(k)}$ has constant length for all k. Combining this with the corresponding formulas (4) for arbitrary $\dot{x}^{(k)}$, $k = 1, \ldots, n-1$, one proves by induction that all curvatures κ_i are constant. The vanishing of κ_3 for base spaces of constant curvature is a consequence of the special form of the curvature tensor.

Both propositions above are examples of *direct* results. In [1], we have looked at possible converses, at *indirect* results. We comment on the role of curvature in this context.

As concerns the converse of Proposition 2, we note that explicit expressions can be given for the curvatures κ_i in terms of the curvature tensor R and its covariant derivatives via (4) and (5). However, these expressions quickly become rather complicated and of little practical use. For this reason, we only consider the case where the first curvature κ_1 is constant. For this function, we find the expression

$$\kappa_1{}^2 = g(R(V, \dot{V})\dot{x}, R(V, \dot{V})\dot{x}). \tag{6}$$

Taking the covariant derivative along $x(t)$, we find

Proposition 3. *Let (M, g) be a Riemannian manifold. Then for any geodesic γ of (T_1M, g_S), the projected curve $x = \pi \circ \gamma$ has constant first curvature κ_1 if and only if the curvature condition*

$$g((\nabla_Y R)(V, W)Y, R(V, W)Y) = 0, \tag{7}$$

is satisfied for all vector fields Y, V and W on M.

The curvature condition (7) is the starting point for our search for a possible converse to Proposition 2. It implies several conditions on the Jacobi operators $R_\sigma = R(\cdot\,, \dot{\sigma})\dot{\sigma}$ along geodesics σ on (M, g):

1. the eigenvalues of R_σ are constant along σ for each geodesic σ of (M, g), i.e., the manifold (M, g) is a $\mathfrak{C}$-*space*;
2. the operator $R_\sigma{}^2$ is parallel along each geodesic σ of (M, g).

In the literature, a lot of results on the Jacobi operator can be found. Using those, we can obtain converse statements to Proposition 2 for several classes of Riemannian manifolds, but so far not for the general case. For the precise statements, we refer to [1].

Next, we consider a converse of Proposition 1. We will look more generally at spaces (M, g) for which projections of geodesics on (T_1M, g_S) have vanishing curvature κ_1, κ_2 or κ_3.

The case $\kappa_1 \equiv 0$ is easily dealt with. From (4) and (5) we see that the base manifold must necessarily be flat.

Next, suppose that $\kappa_2 \equiv 0$ for every projected geodesic. Comparing the two different descriptions of $\dot{x}^{(2)}$, we find

Proposition 4. *Let (M, g) be a Riemannian space. Then any geodesic γ of (T_1M, g_S) projects to a curve x of M for which $\kappa_2 \equiv 0$ if and only if*

$$R(V,W)^2Y = -|R(V,W)Y|^2\, Y, \tag{8}$$

$$|R(V,W)Y|^2\, (\nabla_Y R)(V,W)Y = g((\nabla_Y R)(V,W)Y, R(V,W)Y)\, R(V,W)Y, \tag{9}$$

for all vector fields V, W *and* Y *on* M *with* $|Y| = 1$.

In this way, we have again translated the original geometric data about geodesics of (T_1M, g_S) into a curvature condition on (M, g). In particular, it follows from (9) that every Jacobi operator R_σ on (M, g) has parallel eigenspaces along the geodesic σ, i.e., (M, g) is a $\mathfrak{P}$-*space.* Since the only Riemannian manifolds which are both $\mathfrak{C}$- and $\mathfrak{P}$-spaces are the locally symmetric ones (see [2]), we find

Proposition 5. *Let* (M, g) *be a Riemannian space and suppose that any geodesic* γ *of* (T_1M, g_S) *projects to a curve* x *of* M *with constant* κ_1 *and vanishing* κ_2. *Then* (M, g) *is locally symmetric.*

Restricting now to locally symmetric base spaces, we can prove

Theorem 6. *Let* (M, g) *be a non-flat locally symmetric space and suppose that any geodesic* γ *of* (T_1M, g_S) *projects to a curve* x *in* M *with vanishing second curvature* κ_2. *Then* (M, g) *is two-dimensional.*

The proof of this result uses different techniques. First, one shows that the rank of the universal covering $(\tilde{M}, \tilde{g})$ of (M, g) must be one. For this, we use the root space decomposition of the Lie algebra corresponding to a representation G/H of $\tilde{M}$ as a homogeneous space. The condition (8) is fundamental here. In a second step, we show easily that no four-dimensional locally irreducible symmetric spaces exist which satisfy (8). Finally, we use the classification by B.-Y. Chen and T. Nagano of maximal totally geodesic submanifolds of rank-one symmetric spaces ([6]) to finish the proof.

To treat the case of vanishing third curvature $\kappa_3 \equiv 0$, we restrict at once to locally symmetric spaces.

Proposition 7. *Let* (M, g) *be a locally symmetric space. Then any geodesic* γ *of* (T_1M, g_S) *projects to a curve* x *in* M *for which* $\kappa_3 \equiv 0$, *if and only if*

$$R(V,W)^3Y + ({\kappa_1}^2 + {\kappa_2}^2)\, R(V,W)Y = 0, \tag{10}$$

for all vector fields V, W *and* Y *on* M. *The coefficient* ${\kappa_1}^2 + {\kappa_2}^2$ *only depends on* V *and* W, *not on* Y. *Its value is given by*

$${\kappa_1}^2 + {\kappa_2}^2 = |R(V,W)^2Y|^2/|R(V,W)Y|^2,$$

for any Y *such that* $R(V,W)Y \neq 0$.

Again using a mixture of Lie group theory and results on totally geodesic submanifolds in symmetric spaces, we are able to prove from this curvature condition the following converse to Proposition 1.

Theorem 8. *Let* (M^n, g), $n \geq 3$, *be a locally symmetric space such that the projection* $x = \pi \circ \gamma$ *of any geodesic* γ *of* (T_1M, g_S) *has vanishing third curvature* κ_3. *Then,* (M^n, g) *is either a space of constant curvature or a local product of a flat space and a space of constant curvature.*

4 Reducibility of the unit tangent bundle

As a second illustration of the programme set out in the first section, we consider the question: when is the unit tangent bundle of a Riemannian manifold reducible? In answering this question, the curvature tensor will again be the main actor, even if completely different techniques are needed compared to the ones used in the preceding section. On the whole, the answer to the above question requires a lot of calculations, but the underlying ideas are very simple. We will outline the argument and refer to [3] for the technical details.

The existence of a local decomposition implies some special behavior of the Riemann curvature tensor. Indeed, any curvature operator $\bar{R}(U, V)$ acting on a vector tangent to one of the components gives again a vector tangent to the same component. In particular, if in the expression $\bar{R}(U, V)W$, one of the vectors U, V, W is tangent to one component and another vector to the other component, then $\bar{R}(U, V)W$ will necessarily be zero. This is a very simple consequence of reducibility which is by no means equivalent to the existence of a local product decomposition. Still, it will bring us very far, as we will see. An additional advantage is that the curvature condition is a pointwise condition and no knowledge about covariant derivatives is needed.

Suppose first that, at a point (x, u) of T_1M, the tangent space to one of the factors, say to M_1, contains a non-zero vertical vector X^t, $X \in T_xM$ and X orthogonal to u. Then it holds

$$\bar{R}(Y^t, X^t)X^t = g(X, X)\, Y^t - g(X, Y)\, X^t \in T_{(x,u)}M_1$$

for all vectors $Y \in T_xM$. As a consequence, $VT_1M_{(x,u)} \subset T_{(x,u)}M_1$, and M_1 is at least $(n-1)$-dimensional. Hence, *if at a point of T_1M one of the factors contains a non-zero vertical vector, it contains the complete vertical distribution at that point.* We call the decomposition *vertical at* (x, u) in such a situation. Note that this is the case as soon as $\max\{\dim M_1, \dim M_2\} > n$. So, the only possibility for the decomposition not to be vertical at (x, u) is that $\dim M_1 = n$, $\dim M_2 = n - 1$ (or conversely) and neither factor is tangent to a vertical vector. We call this a *diagonal decomposition at* (x, u).

4.1 Diagonal decomposition

Suppose for now that we have a diagonal decomposition $T_1M \simeq M_1 \times M_2$ at (x, u) with $\dim M_1 = n$ and $\dim M_2 = n - 1$. The following technical result allows us to work with suitable bases for $T_{(x,u)}M_1$ and $T_{(x,u)}M_2$. Its proof uses the symmetries of the curvature tensor.

Lemma 9. *If $T_1M \simeq M_1 \times M_2$ is a diagonal decomposition at (x, u) with $\dim M_1 = n$ and $\dim M_2 = n - 1$, then there exist two orthonormal bases $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_{n-1}, u\}$ of T_xM and $\lambda > 0$, such that an orthogonal basis for $T_{(x,u)}M_1$ is given by*

$$X_1{}^h + \lambda Y_1{}^t, \ldots, X_{n-1}{}^h + \lambda Y_{n-1}{}^t, X_n{}^h$$

and an orthogonal basis for $T_{(x,u)}M_2$ is given by

$$\lambda X_1{}^h - Y_1{}^t, \ldots, \lambda X_{n-1}{}^h - Y_{n-1}^t.$$

Note that the decomposition at (x, u) gives rise to *two* special orthonormal bases of T_xM.

Next, we express that $\bar{R}(U, V)W = 0$, if U is one of the vectors in the above basis for $T_{(x,u)}M_1$ and W one of the vectors in the basis for $T_{(x,u)}M_2$. This gives a list of curvature conditions on (M, g). The ones we will need further on are given by

$$R(u, Y_j)R(u, Y_l)X_i + R(u, Y_l)R(u, Y_j)X_i = 4(\delta_{il}X_j - 2\delta_{jl}X_i + \delta_{ij}X_l), \tag{11}$$

$$\begin{aligned} &R(u, Y_j)R(u, Y_l)X_n + R(u, Y_l)R(u, Y_j)X_n \\ &\qquad = -2g(R(u, Y_j)X_n, R(u, Y_l)X_n)\, X_n, \end{aligned} \tag{12}$$

$$\begin{aligned} 4R(Y_l, Y_j)X_i &= R(u, Y_j)R(u, Y_l)X_i - R(u, Y_l)R(u, Y_j)X_i \\ &\quad - 4(\delta_{il}X_j - \delta_{ij}X_l), \end{aligned} \tag{13}$$

$$4R(Y_l, Y_j)X_n = R(u, Y_j)R(u, Y_l)X_n - R(u, Y_l)R(u, Y_j)X_n, \tag{14}$$

$$\begin{aligned} 4R(X_i, X_j)X_l &= \frac{4(\lambda^4 - \lambda^2 + 1)}{\lambda^2}\,(\delta_{jl}X_i - \delta_{il}X_j) \\ &\quad + R(u, R(X_j, X_l)u)X_i - R(u, R(X_i, X_l)u)X_j \\ &\quad - 2R(u, R(X_i, X_j)u)X_l, \end{aligned} \tag{15}$$

$$\begin{aligned} 4R(X_n, X_j)X_l &= \frac{1}{\lambda^2}\, g(R(u, Y_j)X_n, R(u, Y_l)X_n)\, X_n \\ &\quad + R(u, R(X_j, X_l)u)X_n - R(u, R(X_n, X_l)u)X_j \\ &\quad - 2R(u, R(X_n, X_j)u)X_l, \end{aligned} \tag{16}$$

where $i, j, k \in \{1, \dots, n-1\}$.

Two remarks are important here. First, if we can determine the operators $R(u, Y_l)$, $l = 1, \dots, n-1$, satisfying (11) and (12), then we can compute consecutively the operators $R(Y_l, Y_j)$, $l, j = 1, \dots, n-1$, from (13) and (14) and $R(X_i, X_j)$, $i, j = 1, \dots, n-1$, from (15) and (16). The operators $R(u, Y_l)$ are therefore the most fundamental. (Note also that this gives *two* descriptions for the curvature operators $R(V, W)$: one in the basis $\{Y_1, \dots, Y_{n-1}, u\}$ and another in the basis $\{X_1, \dots, X_n\}$.) Second, the conditions (11) and (12) remind one of the Clifford relations $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}$, though they are not quite right. Both remarks inspire us to study the operators $R(u, Y_l)$ in some more detail.

From conditions (11) and (12), it follows readily that

$$\begin{aligned} &R(u, Y_l)^2 X_l = 0, \\ &R(u, Y_l)^2 X_i = -4X_i, \quad i \neq l, \\ &R(u, Y_l)^2 X_n = -|R(u, Y_l)X_n|^2\, X_n. \end{aligned}$$

Since $R(u, Y_j)$ is skew-symmetric, the non-zero eigenvalues of $R(u, Y_j)^2$ must have even multiplicity. Hence,

— if n is even, the eigenvalue -4 has even multiplicity $n-2$ on $\{X_j, X_n\}^\perp$. Hence, the eigenvalue corresponding to X_n must be zero. This implies $R(u, Y_j)X_n = 0$

for $j = 1, \ldots, n-1$. By (14), also $R(Y_j, Y_k)X_n = 0$ for $j, k = 1, \ldots, n-1$. We conclude that X_n belongs to the nullity distribution of the curvature tensor R_x. In this case, the conditions (12), (14) and (16) are trivially satisfied;

— if n is odd, the eigenvalue -4 has odd multiplicity $n-2$ on $\{X_j, X_n\}^\perp$. The eigenvalue corresponding to X_n must then be -4 as well. So, it holds, $|R(u, Y_j)X_n|^2 = 4$, for $j = 1, \ldots, n-1$. It even holds, $|R(u, Y)X_n|^2 = 4$, for every unit vector Y orthogonal to u and $g(R(u, Y)X_n, R(u, Z)X_n) = 4g(Y, Z)$, for all vectors Y and Z orthogonal to u. In particular, the right-hand side of (12) equals $-8\delta_{jl}X_n$. In this case, conditions (12) and (14) are included in (11) and (13) if we allow the index i to be n.

This indicates that the cases where n is even and those where n is odd will have to be treated separately.

When n is even, consider the operators $\mathcal{R}_i$, $i = 1, \ldots, n-1$, acting on $V^n = T_xM$ by

$$\mathcal{R}_i = \frac{1}{2} R(u, Y_i) - \langle X_n, \cdot \rangle X_i + \langle X_i, \cdot \rangle X_n,$$

where $\langle \cdot, \cdot \rangle = g_x$. One can show that they satisfy the Clifford relations

$$\mathcal{R}_i \circ \mathcal{R}_j + \mathcal{R}_j \circ \mathcal{R}_i = -2\delta_{ij} \,\mathrm{id}. \tag{17}$$

Hence, they correspond to a Clifford representation of an $(n-1)$-dimensional Clifford algebra on an n-dimensional vector space.

When n is odd, define the operators $\mathcal{R}_i$, $i = 1, \ldots, n-1$, acting on $V^{n+1} = T_xM \oplus \mathbb{R}X_0$ by

$$\mathcal{R}_i = \frac{1}{2} R(u, Y_i) - \langle X_0, \cdot \rangle X_i + \langle X_i, \cdot \rangle X_0,$$

where $\langle \cdot, \cdot \rangle = g_x \oplus g_0$ with $g_0(aX_0, bX_0) = ab$. Again these satisfy the relations (17) and we obtain a Clifford representation of an $(n-1)$-dimensional Clifford algebra on an $(n+1)$-dimensional vector space.

It is well-known, however, that the dimension of a Clifford algebra and that of a module over it are closely related. (See, e.g., the table in [3].) In particular, it follows that Clifford representations as above can only exist for dimensions $n = 1, 2, 3, 4, 7$ and 8. So, only for those dimensions for the base manifold (M, g) can a diagonal decomposition exist for the unit tangent bundle. Moreover, the case $n = 1$ is irrelevant, since a one-dimensional manifold is never reducible.

Finally, treating these remaining cases separately, one can show that the two descriptions for the curvature tensor mentioned higher, one in the basis $\{Y_1, \ldots, Y_{n-1}, u\}$ and the other in the basis $\{X_1, \ldots, X_n\}$, are incompatible, except when $n = 2$. Then, the base manifold is necessarily flat. We conclude that diagonal decompositions for the unit tangent bundle exist only for a flat surface as base space.

4.2 Vertical decomposition

Suppose now that we have a vertical decomposition $T_1M \simeq M_1 \times M_2$ such that $VT_1M_{(x,u)} \subset T_{(x,u)}M_1$ everywhere. In this situation, if $(x, u) \in M_1 \times \{q\}$, for some

$q \in M_2$, it holds that $\pi^{-1}(x) \subset M_1 \times \{q\}$. Consequently, we have $M_1 \times \{q\} = \pi^{-1}(\pi(M_1 \times \{q\}))$. So, the leaves $M_1 \times \{q\}$, corresponding to the product, project under π to a foliation $\mathcal{L}_1$ on (M, g) and $\pi^{-1}(\mathcal{L}_1) = \{M_1 \times \{q\}, q \in M_2\}$. Let L_1 be the distribution on M tangent to $\mathcal{L}_1$. Define the distribution L_2 to be the orthogonal distribution to L_1 on M. Then,

$$T_{(x,u)}(M_1 \times \{q\}) = VT_rM_{(x,u)} \oplus h(L_{1x}), \quad T_{(x,u)}(\{p\} \times M_2) = h(L_{2x}),$$

where h denotes the horizontal lift. In particular, we can describe the tangent spaces to the factors of the (local) product using horizontal and vertical lifts. From the expressions (1) for the Levi Civita connection, it is easy to deduce that also L_2 is integrable, with associated foliation $\mathcal{L}_2$ with flat leaves, and that (M, g) is locally isometric to a Riemannian product $M \simeq M' \times \mathbb{R}^k$ where $k = \dim L_2 \leq n$. Conversely, it is almost immediate that a (local) decomposition $M \simeq M' \times \mathbb{R}^k$ with $k > 0$ gives rise to a (local) decomposition of (T_1M, g_S). This proves

Theorem 10 (Local version). *The unit tangent bundle* (T_1M, g_S) *of a Riemannian manifold* (M^n, g)*,* $n \geq 2$*, is locally reducible if and only if* (M, g) *has a flat factor, i.e.,* (M, g) *is locally isometric to a product* $(M', g') \times (\mathbb{R}^k, g_0)$ *where* $1 \leq k \leq n$ *and* g_0 *denotes the standard Euclidean metric on* $\mathbb{R}^k$*.*

With a little more effort (not involving curvature), we can even show the corresponding global result.

Theorem 11 (Global version). *Let* (M^n, g)*,* $n \geq 3$*, be a Riemannian manifold and suppose that* (T_1M, g_S) *is a global Riemannian product. Then,* (M, g) *is either flat or it is also a global Riemannian product, with a flat factor.*

Conversely, if (M, g) *is a global product space* $(M', g') \times (F^k, g_0)$ *where* $1 \leq k \leq n$ *and* F *is a connected and simply connected flat space, then* (T_1M, g_S) *is a global Riemannian product, also with* (F, g_0) *as a flat factor.*

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Convex Hypersurfaces in Hadamard Manifolds ⋆

A. A. Borisenko

Geometry Department, Mathematical Faculty,
Kharkov National University, Pl.Svobodi, 4, 61077, Kharkov, Ukraine
borisenk@univer.kharkov.ua

Dedicated to Professor Lieven Vanhecke

Summary. We prove a theorem about an extremal property of Lobachevsky space among simply connected Riemannian manifolds of nonpositive curvature.

Hadamard proved the following theorem. Let φ be an immersion of a compact oriented n-dimensional manifold M in Euclidean space E^{n+1} ($n \geq 2$) with positive Gaussian curvature everywhere. Then $\varphi(M)$ is a convex hypersurface [1].

Chern and Lashof [2] generalized this theorem. Let φ be an immersion of a compact oriented n-dimensional manifold M in E^{n+1}. Then the following two assertions are equivalent:

(i) The degree of the spherical mapping equals ± 1, and the Gaussian curvature does not change sign (i.e., it is either nonnegative or nonpositive everywhere);
(ii) $\varphi(M)$ is a convex hypersurface.

By Gaussian curvature, we mean the product of the principal curvatures.

S. Alexander generalized the Hadamard theorem for compact hypersurfaces in any complete, simply connected Riemannian manifold of nonpositive sectional curvature [3].

A topological immersion $f : N^n \to M$ of a manifold N^n into a Riemannian manifold M is called *locally convex* at a point $x \in N^n$ if x has a neighborhood U such that $f(U)$ is a part of the boundary of a convex set in M.

Heijenoort proved the following theorem. Let $f : N^n \to E^{n+1}$, where $n \geq 2$, be a topological immersion of a connected manifold N^n. If f is locally convex at all points and has at least one point of local strict support and N^n is complete in the metric induced by the immersion, then f is an embedding and $F = f(N^n)$ is the boundary of a convex body [4].

In [5] this theorem was generalized to h-locally convex (i.e., such that each point has a neighborhood lying on one side from a horosphere) regular hypersurfaces in Lobachevsky space and in [6], to nonregular hypersurfaces.

⋆ I thank the referee for some helpful comments and for correcting my English style.

In this section we shall recall some definitions and we shall state the notation.

A *Hadamard manifold* is a complete simply connected Riemannian manifold with sectional curvature $K \leq 0$.

Like in the hyperbolic space, a *horoball in a Hadamard manifold M* is the domain obtained as the limit of the balls with their centers in a geodesic ray going to infinity, and their corresponding geodesic spheres containing a fixed point. The boundary of a horoball is a *horosphere*. In general, a horosphere is a C^2 hypersurface. An *h-convex set* in a Hadamard manifold M of dimension $n+1$ is a subset $\Omega \subset M$ with boundary $\partial\Omega$ satisfying that, for every $P \in \partial\Omega$ there is a horosphere H of M through P such that Ω is locally contained in the horoball of M bounded by H. This H is called a *supporting horosphere* of Ω (and $\partial\Omega$).

For Hadamard manifolds M satisfying $-k_1^2 \geq K \geq -k_2^2$, $k_1, k_2 > 0$, if H is a horosphere, at each point of H where the normal curvature k_n is well defined, it satisfies $k_1 \leq k_n \leq k_2$.

For geodesic spheres of radius r their normal curvatures k_n satisfy the inequalities

$$k_1 \coth k_1 r \leq k_n \leq k_2 \coth k_2 r.$$

Note that the value $k \coth kr$ is the geodesic curvature of a circumference of radius r in the Lobachevsky plane of curvature $-k^2$.

An orientable regular (C^2 or more) hypersurface F of a Hadamard manifold M is λ-convex if, for a selection of its unit normal vector, the normal curvature k_n of F satisfies $k_n \geq \lambda$. A domain $\Omega \subset M$ is λ-convex if for every point $P \in \partial\Omega$ there is a regular λ-convex hypersurface F through P leaving a neighborhood of P in the convex side (the side where the unit normal vectors points) of F. If $\partial\Omega$ is regular, then it is a regular λ-convex hypersurface.

Given any set $\Omega \subset M$, an *inscribed ball (inball* for short) is a ball in M contained in Ω with maximum radius. Its radius is called the *inradius* of Ω, and it will be always denoted by r. Moreover, we shall denote by O the (not necessarily unique) center of an inball of Ω, and by d the distance in M to O.

A circumscribed ball (or circumball) is a ball in M^{n+1} containing Ω with minimum radius. Its radius is called circumradius in Ω and is denoted by R.

Now we shall prove the following theorems.

Theorem 1. *Let M^{n+1} be a simply connected complete Riemannian manifold with sectional curvature K satisfying*

$$-k_1^2 \geq K \geq -k_2^2, \quad k_2 \geq k_1 > 0.$$

Suppose that $F \subset M^{n+1}$ be a complete immersed hypersurface with normal curvatures

$$k_n \geq k_2.$$

Then either

I) *F^n is a compact convex hypersurface diffeomorphic to the sphere S^n and*

$$R - r < k_2 \ln 2$$

or

II) *F^n is a horosphere in M^{n+1} and, when the norm of covariant derivative of the curvature tensor of M^{n+1} is bounded, the ambient space M^{n+1} is the hyperbolic space of constant curvature $-k_2^2$.*

Remark. The condition $|\nabla\mathcal{R}| \leq C$ added in the part II of Theorem 1 in order we can conclude that M^{n+1} has constant curvature is used only to assure the regularity of horospheres of class C^3. We think that the result is still true without this hypothesis.

For more strong conditions on the normal curvatures of F^n it is true.

Theorem 2. *Let M^{n+1} be a Hadamard manifold with sectional curvature K satisfying*

$$-k_1^2 \geq K \geq -k_2^2, \quad k_2 \geq k_1 \geq 0.$$

Let F^n be a complete immersed hypersurface with normal curvatures bigger or equal $k_2 \coth k_2 r_0$ at any point F^n. Then either

I) *F^n is a compact convex hypersurface diffeomorphic sphere S^n and radius of circumscribed ball*

$$R < r_0$$

or

II) *F^n is a sphere of radius r_0 which is the boundary of the ball Ω. The ball Ω is isometric to the ball of radius r_0 of the hyperbolic space with constant curvature $-k_2^2$.*

The ambient space M^{n+1} is a C^3 regular Riemannian manifold. At any point of a smooth hypersurface F^n in a Hadamard manifold there are two tangent horospheres. If the normal curvatures of F^n at some point $P \in F^n$ with respect some normal are greater than zero, than the horosphere tangent to F^n at P with positive normal curvature with respect the same normal is called *tangent horosphere.*

Proof of Theorem 1. From the conditions of the theorem it follows that the normal curvatures k_n/H^n of any horosphere H^n in M^{n+1} satisfy

$$k_n/_{H^n} \leq k_2.$$

And for every point $P \in F^n$, the normal curvatures of F^n and the tangent horosphere H^n in the corresponding directions a satisfy the inequality

$$k_n(a)/_{F^n} \geq k_n(a)/_{H^n}.$$

I) Suppose that at one point P_0 the following strong inequality is true

$$k_n(a)/_{F^n} > k_n(a)/_{H^n}. \tag{1}$$

Let n_0 be the unit normal at the point P_0, such that the normal curvatures of F^n at the point $P_0 \in F^n$ with respect to normal n_0 are positive, and H^n be the tangent horosphere at the point P_0 with the normal n_0. From the inequality (1) it follows that there exists some neighborhood of the point P_0 on F^n such that it lies inside the horoball bounded by the horosphere H^n. Let us take a horospherical system of the coordinates in M^{n+1} with the base H^n, t is a length parameter along geodesic line orthogonal to H^n whose positive direction coincides with the normal n_0 at P_0. From another side t is the distance in M^{n+1} to the horosphere H^n. Let the function $f = t$ be the restriction of t to the hypersurface F^n. At the point P_0, the function f has a strong minimum. Let φ be the angle between the direction $\partial/\partial t$ and the unit normal N of the hypersurface F^n. Along the integral curves of the vector field $X = \text{grad } f/_{F^n}$ on the hypersurface F^n, the angle φ satisfies the equation [8],

$$k_n = \mu \cos\varphi + \sin\varphi \frac{d\varphi}{dt}, \tag{2}$$

where k_n is the normal curvature of F^n in the direction X at the point $P \in F^n$, μ is the normal curvature of the coordinate horosphere at the point $P \in F^n$ in the direction Y, which is the orthogonal projection of the vector X on the tangent space of the coordinate horosphere at the point P.

As $k_n \geq k_2$ and the normal curvatures of the horosphere $\mu \leq k_2$, from (2), for $\varphi \leq \pi/2$, it follows

$$k_2(1 - \cos\varphi) \leq \sin\varphi \frac{d\varphi}{dt};$$

$$\frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}} \geq e^{\frac{k_2}{2}(t-t_0)};$$

$$\sin\frac{\varphi}{2} \geq \sin\frac{\varphi_0}{2} e^{\frac{k_2}{2}(t-t_0)},$$

where φ_0 is the angle between $\partial/\partial t$ and the normal N for small t_0. It follows from inequality (1) at the point P_0 that $\varphi_0 > 0$. The angle φ monotonically increases along the integral curve and, for

$$t \leq \frac{2}{k_2} \ln \frac{e^{\frac{k_2 t_0}{2}}}{\sqrt{2}(\sin\frac{\varphi hi_0}{2})},$$

it reaches the value $\pi/2$. For $\varphi \geq \pi/2$ we have

$$k_2 \leq \sin\varphi \frac{d\varphi}{dt};$$

$$\cos\varphi \leq 1 - k_2(t - t_1),$$

where $\varphi(t_1) = \pi/2$ and, for $t_2 \leq t_1 + (2/k_2)$, the angle φ reaches the value π and function $f = t/_{F^n}$ at this point achieves a strong maximum.

The length of the integral curve on the hypersurface F^n of the vector field $X = \text{grad } f/_{F^n}$ satisfies the inequalities

$$k_2(1-\cos\varphi) \le \frac{d\varphi}{ds};$$

$$s \le s_0 + \frac{\cot\frac{\varphi_0}{2}}{k_2}.$$

It follows that point Q_0, where $\varphi = \pi$, does not go to infinity. Let t_2 be the infimum of the value t on the integral curves of the vector field $X = \text{grad } f/_{F^n}$ such that $\varphi(t_2) = \pi$. The level hypersurfaces of the function $f = t$ for $0 < t < t_2$ are spheres S^{n-1}, points P_0 and Q_0 are the strong minimum and the strong maximum respectively and function f is a Morse function on F^n with two critical points. Therefore the hypersurface F^n is homeomorphic to the sphere S^n. From the condition $k_n \ge k_2$ we obtain that the second quadratic form of F^n is positive definite at any point. From a theorem of S. Alexander [4] it follows that F^n is an embedded compact convex hypersurface diffeomorphic to S^n and bounds a convex domain Ω. The domain Ω is k_2-convex and satisfies the condition of theorem 3.1 [9] and

$$\max d(O, \partial\Omega) < k_2 \ln 2,$$

where O is the center of the inscribed ball.

II) Suppose that at any point $P \in F^n$ there exists a direction $a \in T_pF^n$ such that

$$k_n(a)/_{F^n} = k_n(a)/_{H^n}.$$

1) Let us show that some neighborhood $U \subset F^n$ of a point $P_0 \in F^n$ lies in the horoball bounded by tangent horosphere H^n. Let us take a horospherical system of coordinates with base H^n.

The metric M^{n+1} has the form

$$ds^2 = dt^2 + g_{ij}(t,\theta)d\theta^i d\theta^j. \tag{3}$$

The equation of the hypersurface F^n in the neighborhood $P_0 \in F^n$ is

$$t = \rho(\theta).$$

The unit normal vector N to F^n has coordinates

$$\xi^k = -\frac{\rho^k}{\sqrt{1+\langle \text{grad } \rho, \text{grad } \rho\rangle}}, \quad k = 1, \dots, n \tag{4}$$

$$\xi^{n+1} = \frac{1}{\sqrt{1+\langle \text{grad } \rho, \text{grad } \rho\rangle}},$$

where

$$\rho^k = g^{ks}\rho_s, \quad \langle \text{grad } \rho, \text{grad } \rho\rangle = g_{ij}\rho^i\rho^j, \tag{5}$$

$$\rho_i = \frac{\partial \rho}{\partial \theta^i}.$$

The coefficients of the second fundamental form of F^n are [10]

$$\Omega_{ij} = \cos\varphi \left[\rho_{i,j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial t} - \frac{1}{2}\frac{\partial g_{jk}}{\partial t}\rho_i\rho^k - \frac{1}{2}\frac{\partial g_{ik}}{\partial t}\rho_j\rho^k\right], \tag{6}$$

where φ is the angle between $\partial/\partial t$ and the normal N,

$$\cos\varphi = \frac{1}{\sqrt{1 + \langle \text{grad } \rho, \text{grad } \rho\rangle}}, \quad \rho_{i,j} = \rho_{ij} - \Gamma^k_{ij/g}\rho_k, \tag{7}$$

where $\Gamma^k_{ij/g}$ are the Kristoffel symbols of the metric,

$$d\sigma^2 = g_{ij} d\theta^i \, d\theta^j.$$

The coefficients of the metric tensor F^n have the form

$$a_{ij} = g_{ij} + \rho_i\rho_j. \tag{8}$$

From the conditions of the theorem, the normal curvatures

$$k_n/_{F^n} \geq k_2,$$

and, from (6), it follows that, for any tangent vector $b \in F^n$, $b = (b^1, \dots, b^n)$,

$$\cos\varphi \left[\rho_{i,j} b^i b^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial t} b^i b^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial t}\rho_i b^i \rho^k b^j - \frac{1}{2}\frac{\partial g_{ik}}{\partial t}\rho_j b^j \rho^k b^i\right] \geq k_2(g_{ij} b^i b^j + (\rho_i b^i)^2). \tag{9}$$

Let us introduce the function $h = e^{k_2\rho(\theta)}$.

$$h_i = k_2 e^{k_2\rho}\rho_i;$$

$$h_{ij} = k_2^2 e^{k_2\rho}\rho_i\rho_j + k_2 e^{k_2\rho}\rho_{ij}.$$

Hence

$$\rho_i = \frac{h_i}{h}\frac{1}{k_2};$$

$$\rho_{ij} = \frac{1}{k_2}\frac{h_{ij}}{h} - \frac{1}{k_2}\frac{h_i}{h}\frac{h_j}{h}; \tag{10}$$

$$\rho_{i,j} = \frac{1}{k_2}\frac{h h_{i,j} - h_i h_j}{h^2}.$$

We rewrite the inequality (9) in the following way:

$$\cos\varphi\left[\frac{1}{k_2}hh_{i,j}b^ib^j-\frac{1}{k_2}(h_ib^i)^2-\frac{1}{2}\frac{\partial g_{jk}}{\partial t}b^ib^jh^2-\frac{1}{2k_2^2}\frac{\partial g_{ik}}{\partial t}h_ib^ih^kb^j\right.$$
$$\left.-\frac{1}{2k_2^2}\frac{\partial g_{ik}}{\partial t}h_jb^jh^kb^i\right]\geq k_2\left[h^2g_{ij}b^ib^j+\frac{1}{k_2^2}(h_ib^i)^2\right]. \quad (11)$$

Since the normal curvature of any horosphere in M^{n+1} is lower or equal than k_2, then

$$-\frac{1}{2}\frac{\partial g_{ij}}{\partial t}b^ib^j\leq k_2g_{ij}b^ib^j, \quad (12)$$

where $A_{ij}=-\frac{1}{2}\frac{\partial g_{ij}}{\partial t}$ are coefficients of the second fundamental form of the horosphere $t=\text{const}$.

$$\left|-\frac{1}{2}\frac{\partial g_{jk}}{\partial t}h_ib^ih^kb^j\right|=(A_{jk}h^kb^j)|h_ib^i|$$
$$\leq\sqrt{(A_{jk}h^kh^j)A_{jk}b^kb^j}\,|h_ib^i|\leq k_2|\operatorname{grad}h||b||h_ib^i|, \quad (13)$$

where $|b|^2=g_{ij}b^ib^j,\quad |\operatorname{grad}h|^2=g_{ij}h^ih^j$.

Let us we substitute (12), (13) in (11) and obtain

$$\cos\varphi\frac{1}{k_2}hh_{i,j}b^ib^j\geq k_2h^2(1-\cos\varphi)|b|^2+\frac{1}{k_2}(1+\cos\varphi)(h_ib^i)^2$$
$$-2\frac{1}{k_2}|\operatorname{grad}h||b||(h_ib^i)|. \quad (14)$$

The expression on the right side is a quadratic expression with respect to $|(h_ib^i)|$. The discriminant of this polynomial is

$$\frac{1}{k_2^2}|\operatorname{grad}h|^2|b|^2-h^2\sin^2\varphi|b|^2. \quad (15)$$

But

$$\cos^2\varphi=\frac{1}{1+|\operatorname{grad}\rho|^2}=\frac{k_2^2h^2}{k_2^2h^2+|\operatorname{grad}h|^2},$$

$$\sin^2\varphi=\frac{|\operatorname{grad}h|^2}{k_2^2h^2+|\operatorname{grad}h|^2}.$$

We rewrite (15) in the form

$$\frac{|b|^2}{k_2^2}\left(\frac{|\operatorname{grad} h|^4}{k_2^2h^2+|\operatorname{grad} h|^2}\right)\geq 0. \tag{16}$$

From (14) it follows

$$h_{i,j}b^ib^j \geq 0. \tag{17}$$

Let L be the lines on F^n which satisfy the system of equations

$$\frac{\partial^2\theta^k}{\partial s^2}+\Gamma^k_{ij/g}(\theta,\rho(\theta))\frac{\partial\theta^i}{\partial s}\frac{\partial\theta^s}{\partial s}=0. \tag{18}$$

Through any point and in any direction there is only one line in this family. We call these lines the g-geodesics. We take the restriction of the function h on this line

$$\begin{aligned}\theta^i &= \theta^i(s);\\ h_s &= h_i\frac{d\theta^i}{ds};\\ h_{ss} &= h_{ij}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds}+h_k\frac{d^2\theta^k}{d^2s}.\end{aligned} \tag{19}$$

If we substitute (18) in (19), then

$$h_{ss}=h_{i,j}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds}\geq 0. \tag{20}$$

At the point P_0, $h=1, h_s=0$ and, from (20), it follows that along the g-geodesic lines which go through the point P_0, $h\geq 1$. Then

$$t=\rho(\theta)\geq 0,$$

and the hypersurface F^n lies on one side of the tangent horosphere H^n.

2) Let P_0 be an arbitrary fixed point in F^n, and let $H^n(P_0)$ be the tangent horosphere of F^n at P_0. From 1) it follows that some neighborhood of the point $P_0\in F^n$ is contained in the horoball bounded by the horosphere $H^n(P_0)$. Let us take the dual tangent horosphere $\tilde{H}^n(P_0)$. This horosphere is defined by the opposite point at infinity on the geodesic line going in the direction of the normal n_0 at the point $P_0\in F^n$. Let $\tilde{H}^n(\tau)$ be the parallel horospheres $\tilde{H}(0)=\tilde{H}^n(P_0)$, $M_\tau=F^n\bigcap\tilde{H}^n(\tau)$, τ is the distance from the horosphere $\tilde{H}^n(P_0)$. Let $f=\tau/_{F^n}$ be the restriction of the function τ to the hypersurface F^n. For the function f the point P_0 is a strong local minimum then, for small τ, the set $M_\tau=F^n\bigcap\tilde{H}^n(\tau)$ is diffeomorphic to the sphere S^{n-1} and bounds on F^n a domain D_τ homeomorphic to a ball and contains a unique critical point P_0 of the function $f=\tau/_{F^n}$. On the horosphere $\tilde{H}^n(\tau)$, the set M_τ bounds a convex domain homeomorphic to a ball. Actually, the normal ν to M_τ in $\tilde{H}^n(\tau)$ has the form

$$\nu = \lambda_1 n_1 + \lambda_2 N,$$

where n_1 is unit normal to $\tilde{H}^n(\tau)$, N is a normal to F^n, $\langle \nu, n_1 \rangle = 0$. Therefore

$$\nu = \langle n_1, N \rangle n_1 + N.$$

Let X be a unit vector field tangent to M_τ. Then

$$\langle \nu, \nabla_X X \rangle = \langle n_1, N \rangle \mu + k_n/_{F^n},$$

where μ is the normal curvature of the horosphere $\tilde{H}(\tau)$. Since $k_n/_{F^n} \geq k_2$ and $\mu \leq k_2$, then $\langle \nu, \nabla_X X \rangle > 0$, that is, the second quadratic form of M_τ in $\tilde{H}^n(\tau)$ is positive definite and the domain in $\tilde{H}^n(\tau)$ bounded by M_τ is a convex domain homeomorphic to a ball.

Let's consider the body $Q(\tau)$, bounded $\mathcal{D}_\tau$ and $\tilde{H}^n(\tau)$ for small τ. At any boundary point there exists a local supporting horosphere. It is a global supporting horosphere too, and the body $Q(\tau)$ is situated in the horoball bounded by supporting horospheres. In other words, the body $Q(\tau)$ is h-convex. Let τ^* be a supremum value of τ, for which the body $Q(\tau)$ is h-convex, $\mathcal{D}^* = \bigcup \mathcal{D}_\tau$. Let's show $\tau^* = \infty$. Let us assume the contrary. There are three possible cases:

a) $\mathcal{D}^* = F^n$;

b) $\mathcal{D}^* \neq F^n$ and, on the boundary S^* of the domain $\mathcal{D}^*$, there are critical points of the function $f = \tau/F^n$.

c) $\mathcal{D}^* \neq F^n$ and S^* doesn't contain any critical point of the function f.

The case *c*) is impossible. Really for $\tau > \tau^*$ the set M_τ is homeomorphic to the sphere also. It bounds a convex domain in $\tilde{H}^n(\tau)$ and at any boundary point $Q(\tau)$ there exists a local supporting horosphere. It follows that $Q(\tau)$ is a h-convex set for $\tau > \tau^*$ and τ^* is not the supremum.

For the case *b*) the set S^* contains a critical point P of the function f. At the point $P \in S^*$, the horosphere $\tilde{H}^n(\tau^*)$ is the tangent supporting horosphere to F^n, $S^* \subset \tilde{H}^n(\tau^*) \bigcap F^n$ is the boundary of the convex domain homeomorphic to a ball in $\tilde{H}^n(\tau^*)$. Let show that $\tilde{H}^n(\tau^*)$ is the tangent horosphere at all points S^*. Actually, some neighborhood U of the point $P \in F^n$ lies to one side with respect to $\tilde{H}^n(\tau^*)$, $U \bigcap S^*$ belongs to $\tilde{H}^n(\tau^*)$. If the horosphere $\tilde{H}^n(\tau^*)$ isn't tangent in some point $Q \in U \bigcap S^*$ then U doesn't lie to one side of $\tilde{H}^n(\tau^*)$. The set S^* is homeomorphic to the sphere S^{n-1} and the set of the points of S^*, such that the horosphere $\tilde{H}^n(\tau^*)$ is tangent, and open and closed at the same time. This set isn't empty and coincides with S^*. Let $Q(\tau^*)$ be the body bounded by $\mathcal{D}^*$ and the domain with boundary S^* on $\tilde{H}^n(\tau^*)$. It is a compact h-convex body with smooth boundary. Let $S(r)$ be a circumscribed sphere of $Q(\tau^*)$.

Suppose that a tangent point $P \in S(r)$ to the boundary $Q(\tau^*)$ belongs to $\tilde{H}^n(\tau^*)$. In this case the sphere $S(r)$ supports the horosphere $\tilde{H}^n(\tau^*)$ at the point P. The sphere $S(r)$ and $\tilde{H}^n(\tau^*)$ are tangent at the point P and the convex sides have the same direction. This is impossible.

Let $Q_0 \in \mathcal{D}^*$ be a tangent point of the sphere $S(r)$. For Hadamard manifolds the following is true.

Lemma 1. *Let $S(r)$ and $S(R)$ $(r < R)$ be tangent spheres at a point Q in a Hadamard manifold of sectional curvature $K \leq 0$.*

Suppose that at the point Q the convex sides of the spheres are the same. Then at the point Q the normal curvatures of the sphere $S(R)$ are lower than the normal curvatures of the sphere $S(r)$ in corresponding directions.

Proof. Let us take in M^{n+1} a spherical system coordinate with pole O, where O is the center of the sphere $S(R)$. In the neighborhood of the point Q the sphere $S(r)$ has the following parametrization

$$t = h(\theta^1, \dots, \theta^n),$$

where $t, \theta^1, \dots, \theta^n$ are spherical coordinates in M^{n+1} with metric

$$ds^2 = dt^2 + g_{ij}(t, \theta) d\theta^i d\theta^j.$$

The normal curvature of $S(r)$ at the tangent point Q of the spheres in the direction $b = (b^1, \dots, b^n)$ is equal to:

$$k_n = \frac{\left(\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} - \dfrac{1}{2} \dfrac{\partial g_{ij}}{\partial t} \right) b^i b^j}{g_{ij} b^i b^j}$$

$$= k_n(b)/_{S(R)} + \frac{\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} b^i b^j}{g_{ij} b^i b^j}. \tag{21}$$

Let us take the map:

$$\exp_{\mathrm{O}}^{-1} : M^{n+1} \to T_{\mathrm{O}} M^{n+1} = E^{n+1}.$$

The image of the sphere $S(R)$ is the sphere $\bar{S}(R)$ with the center $\bar{\mathrm{O}} = \exp^{-1}(\mathrm{O})$ and radius R. The image of the sphere $S(r)$ lies in a closed ball in E^{n+1} of radius r with the center $\bar{P} = \exp_{\mathrm{O}}^{-1}(P)$, where P is the center of the sphere $S(r)$.

In fact, let us consider triangles OPX, $\bar{O}\bar{P}\bar{X}$, where $X \in S(r)$, $\bar{X} = \exp_{\mathrm{O}}^{-1}(X)$; $OP = \bar{O}\bar{P} = R - r$, $OX = \bar{O}\bar{X} = h$ and $\angle POX = \angle \bar{P}\bar{O}\bar{X}$. From nonpositivity of the sectional curvature M^{n+1} and comparison theorem for triangles it follows that $\bar{P}\bar{X} \leq PX$. In a spherical system of coordinates with pole $\bar{O}$, the metric E^{n+1} has the form

$$ds^2 = dt^2 + G_{ij}(t, \theta) d\theta^i d\theta^j.$$

The normal curvature of the image of the sphere $S(r)$ at the point $\bar{Q}$ is equal

$$\bar{k}_n = \frac{\left(\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} - \dfrac{1}{2} \dfrac{\partial G_{ij}}{\partial t} \right) b^i b^j}{G_{ij} b^i b^j} \tag{22}$$

$$= \frac{\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} b^i b^j}{G_{ij} b^i b^j} + \frac{1}{R^2}.$$

As the image $S(r)$ lies in a closed ball of radius r with center $\bar{P}$, then

$$\bar{k}_n \geq \frac{1}{r}.$$

From (22) it follows at the point Q

$$\frac{\partial^2 h(\theta)}{\partial\theta^i \partial\theta^j} b^i b^j > 0 \tag{23}$$

From (21) and (23) we obtain the statement of Lemma 1.

It follows from the lemma that normal curvatures of the horosphere are lower than normal curvatures of the tangent sphere which lies inside the horoball, bounded by the horosphere. Therefore at the point $Q_0 \in F^n$, normal curvatures F^n satisfy the inequality:

$$k_n/F^n \geq k_n/S(r) > k_n/H^n,$$

where H^n is the supporting tangent horosphere. But this contradicts the assumption that at any point F^n there exists direction a, such that

$$k_n(a)/_{F^n} = k_n(a)/_{H^n}.$$

The case b) is impossible. The case a) is possible only for $\tau^* = \infty$, otherwise, the arguments of the case b) work here to give a contradiction again. We have proved that any tangent horosphere is globally supporting.

Let P_1, P_2 be different arbitrary points F^n and tangent supporting horospheres H_1, H_2 also be different. Then F^n belongs to the intersection of horoballs bounded by horospheres H_1, H_2. Intersection of horoballs is a compact bounded set if the sectional curvature of Hadamard manifold

$$K_\sigma \leq -k_1^2 < 0.$$

Therefore $\tau^* < \infty$, but this is impossible. Hence horospheres H_1 and H_2 coincide and F^n is a horosphere in Hadamard manifold M^{n+1}.

3) Let us introduce the horospherical system of coordinates with base F^n in the manifold M^{n+1}. The metric of the ambient space has the form (3). For $t = 0$, we obtain the hypersurface F^n.The principal curvatures of the horosphere $t = $ const satisfy the inequalities $k_2 \geq \lambda_i \geq k_1$, and the sectional curvature of M^{n+1} satisfies inequality

$$-k_1^2 \geq K \geq -k_2^2.$$

By the conditions of the theorem the principal curvatures of F^n satisfy inequality $\lambda_i \geq k_2$ and we obtain that $\lambda_i = k_2$ and the horosphere F^n is an umbilical hypersurface. Principal curvature of equidistant horospheres $t = $ const satisfy the Riccati equation,

$$\frac{d\lambda}{dt} = \lambda^2 + K_\sigma,$$

where K_σ is the sectional curvature in the direction of the two-dimensional plane spanned by the normal to the horosphere and the corresponding principal direction. Ricatti equation has sense because, since $|\nabla\mathcal{R}| \leq C$, the horospheres are of class C^3 [7]. Since $K_\sigma \geq -k_2^2$,

$$\frac{d\lambda}{dt} \geq \lambda^2 - k_2^2, \qquad \lambda(0) = \lambda_0.$$

Solving this inequality we obtain,

$$\lambda \geq k_2 \frac{(k_2 + \lambda_0)e^{-2k_2 t} - (k_2 - \lambda_0)}{(k_2 + \lambda_0)e^{-2k_2 t} + (k_2 - \lambda_0)},$$

for $\lambda_0 = k_2$, $\lambda \geq k_2$, from another side $\lambda \leq k_2$, and we get $\lambda = k_2$ for all values t. Therefore the coefficients of the metric tensor g_{ij} of the ambient space M^{n+1} satisfy the equations:

$$-\frac{1}{2}\frac{\partial g_{ij}}{\partial t} = k_2 g_{ij},$$

and $g_{ij}(\theta, t) = g_{ij}(\theta, 0)e^{-2k_2 t}$. The metric M^{n+1} has the form

$$ds^2 = dt^2 + e^{-2k_2 t} d\sigma^2,$$

where $d\sigma^2$ is the metric of the base horosphere F^n. Let us show that the metric of F^n is flat. Suppose that at some point of F^n in some two-dimensional plane the sectional curvature $\gamma_2 \neq 0$. Then the sectional curvatures of the coordinates horosphere $t = \text{const}$ in the corresponding point and direction is equal to $\gamma_2 e^{2k_2 t}$. From Gauss' formula we get that the sectional curvature of the ambient space M^{n+1} at the same direction is equal to

$$\gamma_2 e^{2k_2 t} - k_2^2, \quad -\infty \leq t < +\infty.$$

As the sectional curvature M^{n+1} satisfies the inequality

$$-k_1^2 \geq K \geq -k_2^2,$$

it follows that $\gamma_2 = 0$ and the manifold M^{n+1} is a space of constant curvature $-k_2^2$.

Proof of Theorem 2. From the part I) of theorem 1 it follows that F^n is a compact convex hypersurface diffeomorphic to S^n. Similarly, from the proof of the theorem 3.1 [9] we obtain that every tangent sphere of radius r_0 is globally supporting and F^n belongs to closed balls bounded by these spheres. Two cases are possible:

I) There exist two different points $P_1, P_2 \in F^n$ such that tangent spheres $S_1(r_0)$, $S_2(r_0)$ at these points of radius r_0 don't coincide. Than F^n lies at the intersection of the balls bounded of these spheres. In the Hadamard manifold the intersection of different balls of radius r_0 belongs to the ball of lower radius r_0.

II) At all points F^n the tangent sphere of radius r_0 is the same and F^n coincides with the sphere of radius r_0. Analogous to the proof of part II). 3) of theorem 1 we obtain that the ball bounded of this sphere isometric to a ball of radius r_0 in Lobachevsky space of curvature $-k_2^2$.

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Contact Metric Geometry of the Unit Tangent Sphere Bundle

G. Calvaruso

Dipartimento di Matematica "E. De Giorgi",
Universitá degli Studi di Lecce, Lecce, Italy
giovanni.calvaruso@unile.it

Dedicated to Professor Lieven Vanhecke

Summary. We review some of the most interesting results concerning the interactions between the geometry of a Riemannian manifold and the one of its unit tangent sphere bundle, equipped with its natural contact metric structure.

1 Introduction

The study of the relationships between the geometric properties of a Riemannian manifold M and those of its unit tangent sphere bundle T_1M, represents a well-known and interesting research field in Riemannian geometry.

On equipping T_1M with its "natural" metric g_S (the one induced by the Sasaki metric of the tangent bundle), as well as with the contact metric $\bar{g}$ of the standard contact metric structure $(\xi, \eta, \varphi, \bar{g})$ of T_1M, the curvature properties of T_1M influence those of the base manifold M itself, and vice versa. In many aspects, the properties of $(T_1M, \bar{g})$ are faithfully reflected by the ones of (T_1M, g_S), since these two metrics are homothetic. Nevertheless, some results are more strictly related to the special features of the contact metric manifolds. A survey concerning the geometry of (T_1M, g_S) can be found in [BV4]. The aim of this paper is to review the main known results which involve the contact metric structure of T_1M.

The paper is organized in the following way. Section 2 will be devoted to recall some basic facts and results about contact metric manifolds and, in particular, unit tangent sphere bundles. In Section 3, we shall review the historical development of the study of the geometry of a Riemannian manifold via the contact metric structure of its unit tangent sphere bundle. In Section 4, we shall describe the characterization of

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semi-symmetric unit tangent sphere bundles, recently obtained by the author in joint works with E. Boeckx [BC] and D. Perrone [CP2].

2 Preliminaries

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (η, φ, ξ) where ξ is a global vector field (called the *characteristic vector field*) and φ a global tensor of type (1,1) such that

$$\eta(\xi) = 1\,, \quad \varphi\xi = 0\,, \quad \eta\varphi = 0\,, \quad \varphi^2 = -I + \eta \otimes \xi\,.$$

A Riemannian metric g can be found such that

$$\eta = g(\xi, \cdot)\,, \quad d\eta = g(\cdot, \varphi\cdot)\,, \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)\,.$$

We refer to (M, η, g) or to $(M, \eta, g, \xi, \varphi)$ as a contact metric (or Riemannian) manifold. If L denotes the Lie differentiation, we denote by h and ℓ the operators defined by

$$h = \frac{1}{2} L_\xi \varphi, \quad \ell X = R(X, \xi)\xi.$$

The tensor h is symmetric and satisfies

$$\nabla\xi = -\varphi - \varphi h, \quad \nabla_\xi \varphi = 0, \quad h\varphi = -\varphi h, \quad h\xi = 0. \tag{1}$$

A *K-contact* manifold is a contact metric manifold such that ξ is a Killing vector field with respect to g. Clearly, M is K-contact if and only if $h = 0$. If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, M is said to be *Sasakian*. Any Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. It is easy to prove that if M is a contact metric three-manifold of constant sectional curvature 1, then M is necessarily Sasakian. We refer to [Bl2] for more information about contact metric manifolds.

Next, let $\bar{\pi} : TM \to M$ be the tangent bundle of a Riemannian manifold (M, g). The tangent space to TM at a point (x, u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = ker\pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the Levi-Civita connection ∇ of M. If X is a vector field on M, X^h and X^v denote respectively the horizontal and the vertical lift of X on TM. The map $X \mapsto X^h$ (respectively, $X \mapsto X^v$) is an isomorphism between $T_x M$ and $HTM_{(x,u)}$ (respectively, $T_x M$ and $VTM_{(x,u)}$). The Sasaki metric g_S on TM is defined by

$$g_S(A, B) = g(\bar{\pi}_* A, \bar{\pi}_* B) + g(KA, KB),$$

where A, B are the vector field on TM and K is the connection map corresponding to the Levi-Civita connection of M. TM admits an almost complex structure J defined by $JX^h = X^v$ and $JX^v = -X^h$.

The *tangent sphere bundle* $\pi\colon T_1M \to M$ is the hypersurface of TM defined by $T_1M = \{(x,u) \in TM : g_x(u,u) = 1\}$. We shall denote again by g_S the metric induced on T_1M by the Sasaki metric of TM.

The *geodesic flow* of (M, g) is the horizontal vector field of TM defined by

$$\xi'_u = -JN = u^i \left(\frac{\partial}{\partial x^i}\right)^h,$$

where $(x,u) \in TM$ and $u = u^i \frac{\partial}{\partial x^i}$ in local coordinates. If $(x,z) \in T_1M$, then ${\xi'}_z$ is tangent to T_1M. Hence, ξ' can be considered as a vector field on T_1M. Let η' be the 1-form on T_1M dual to ξ' with respect to g_S, and φ' the (1,1) tensor given by $\varphi' X = JX - \eta'(X)N$. Then,

$$(\xi, \eta, \varphi, \bar{g}) = \left(\frac{1}{2}\eta', 2\xi', \varphi', \frac{1}{4}g_S\right)$$

is the standard contact metric structure on T_1M. Note that the contact metric of T_1M is given by $\bar{g} = \frac{1}{4}g_S$. So, since $\bar{g}$ is homothetic to g_S, $(T_1M, \bar{g})$ and (T_1M, g_S) share many geometric properties. For example, the former is reducible if and only if the latter is, and $(T_1M, \bar{g})$ is locally symmetric, respectively semi-symmetric, if and only if (T_1M, g_S) has the same property.

The following characterization of locally reducible unit tangent sphere bundles was proved recently by E. Boeckx:

Theorem 1 ([B4]). *The unit tangent sphere bundle (T_1M, g_S) of a Riemannian manifold M, of dimension greater than two, is locally reducible if and only if the base manifold has a flat factor.*

We now describe the curvature tensor of $(T_1M, \bar{g})$. In general, the vertical lift of a vector (field) is not tangent to T_1M. For this reason, we define the *tangential lift* of $X \in T_xM$ by

$$X^t_{(x,u)} = (X - g(X,u)u)^v = \bar{X}^v,$$

where we put $\bar{X} = X - g(X,u)$. The metric $\bar{g}$ is then described explicitly by

$$4\bar{g}(X^t, Y^t) = g(\bar{X}, \bar{Y}) = g(X,Y) - g(X,u)g(Y,u),$$
$$4\bar{g}(X^t, Y^h) = 0,$$
$$4\bar{g}(X^h, Y^h) = g(X,Y),$$

at any point $(x,u) \in T_1M$. The Levi-Civita connection $\bar{\nabla}$ associated to $\bar{g}$ is given at any point (x,u) by

$$
\begin{aligned}
\bar{\nabla}_{X^t} Y^t &= -g(Y,u)X^t, \\
\bar{\nabla}_{X^t} Y^h &= \frac{1}{2}(R(u,X)Y)^h, \\
\bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(u,Y)X)^h, \\
\bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t,
\end{aligned} \tag{2}
$$

where $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature tensor of M.

The covariant derivatives of $\xi = 2u^i \left(\frac{\partial}{\partial x^i}\right)^h$ can be easily derived from the formulas above. Explicitly, we have

$$
\bar{\nabla}_{X^h}\xi = -(R(X,u)u)^t, \quad \bar{\nabla}_{X^t}\xi = -2\varphi X^t - (R(X,u)u)^h. \tag{3}
$$

Taking into account (1), $h = \varphi\nabla\xi + \varphi^2$. So, we have

$$
hX^h = -\bar{X}^h + (R(X,u)u)^h, \quad hX^t = X^t - (R(X,u)u)^t. \tag{4}
$$

The curvature tensor $\bar{R}$ of (T_1M, g_S) is given by

$$
\begin{aligned}
\bar{R}(X^t,Y^t)Z^t &= -g(\bar{X},\bar{Z})Y^t + g(\bar{Y},\bar{Z})X^t, \\
\bar{R}(X^t,Y^t)Z^h &= (R(\bar{X},\bar{Y})Z)^h + \frac{1}{4}([R(u,X),R(u,Y)]Z)^h, \\
\bar{R}(X^h,Y^t)Z^t &= -\frac{1}{2}(R(\bar{Y},\bar{Z})X)^h - \frac{1}{4}(R(u,Y)R(u,Z)X)^h, \\
\bar{R}(X^h,Y^t)Z^h &= \frac{1}{2}(R(X,Z)\bar{Y})^t - \frac{1}{4}(R(X,R(u,Y)Z)u)^t \\
&\quad + \frac{1}{2}((\nabla_X R)(u,Y)Z)^h, \\
\bar{R}(X^h,Y^h)Z^t &= (R(X,Y)\bar{Z})^t \\
&\quad + \frac{1}{4}(R(Y,R(u,Z)X)u - R(X,R(u,Z)Y)u)^t \\
&\quad + \frac{1}{2}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^h, \\
\bar{R}(X^h,Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}(R(u,R(X,Y)u)Z)^h \\
&\quad - \frac{1}{4}(R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y)^h \\
&\quad + \frac{1}{2}((\nabla_Z R)(X,Y)u)^t.
\end{aligned} \tag{5}
$$

From formulas (5) we can also obtain the following formulas for ℓ:

$$\begin{aligned} \ell X^h &= 4(R(X,u)u)^h - 3(R(R(X,u)u,u)u)^h + 2(R'(X,u)u)^t, \\ \ell X^t &= (R^2(X,u)u)^t + 2(R'(X,u)u)^h, \end{aligned} \tag{6}$$

where $R'(X,u)u = (\nabla_u R)(\cdot,u)u$.

Note that, since $g_S = 4\bar{g}$, the Riemannian connection, the curvature tensor of type (1,3) and the Ricci tensor of (T_1M, g_S) coincide with the corresponding ones of $(T_1M, \bar{g})$, while the scalar curvature and the sectional curvature of (T_1M, g_S) are obtained by the ones of $(T_1M, \bar{g})$ divided by 4.

3 A historical survey

We now review some old and recent results concerning the geometry of the standard contact metric structure of the unit tangent sphere bundle T_1M and its influences on the base manifold (M, g). In some cases, we give short sketches of the proofs of the results. We refer to the original papers for the details.

K-contact unit tangent sphere bundles

As we already mentioned in the previous Section, a K-contact manifold is a contact metric manifold $(\bar{M}, \eta, \bar{g})$ such that ξ is a Killing vector field with respect to $\bar{g}$, which is equivalent to the condition $h = 0$. A Sasakian manifold is always K-contact and the converse also holds in dimension three. K-contact spaces are a field of special interest among contact metric manifolds. As concerns the unit tangent sphere bundle, Y. Tashiro proved the following

Theorem 2 ([Ts]). *$(T_1M, \eta, \bar{g})$ is K-contact if and only if (M, g) has constant sectional curvature* 1. *In this case, T_1M is Sasakian.*

Proof. (For all the details, see [Ts] or Chapter 9 of [Bl2].) Suppose first T_1M is K-contact. Then, by (1), $\nabla_X \xi = \varphi X$ for all X. Comparing with (3), one can show that on the base manifold (M, g) we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{7}$$

that is, (M, g) has constant curvature 1.

Conversely, if (7) holds, one can compute the covariant derivative of φ. It satisfies

$$(\nabla_X \varphi)Y = \bar{g}(X,Y)\xi - \eta(Y)X,$$

for all vector fields X, Y, and this characterizes Sasakian manifolds [Bl2, Chapter 6]. □

Locally symmetric unit tangent sphere bundles

It is well-known that the study of locally symmetric spaces is one of the main topics in Riemannian geometry. In the framework of contact metric geometry, local symmetry has been extensively investigated, obtaining many rigidity results. For example, K-contact locally symmetric spaces have constant sectional curvature 1 [O], [Tn], while three-dimensional locally symmetric contact metric manifolds must have constant sectional curvature 0 or 1 [BlS]. For the unit tangent sphere bundle, Blair proved the following.

Theorem 3 ([Bl1]). $(T_1M, \eta, \bar{g})$ *is locally symmetric if and only if either* (M, g) *is flat or it is a surface of constant sectional curvature* 1.

An alternative proof, which uses only curvature information, was obtained by Boeckx and Vanhecke [BV1].

In Section 4 we describe how Theorem 3 has recently been extended by replacing local symmetry by semi-symmetry ([CP2], [BC]). An extension of Theorem 3 to locally φ-symmetric spaces, obtained in [BV1], is described further in this Section.

Also local symmetry on the base manifold can be characterized by some geometric properties of its unit tangent sphere bundle. For example, taking into account (6), it is not difficult to show that (M, g) is locally symmetric if and only if ℓ maps horizontal (or equivalently, vertical) vectors into horizontal (vertical) vectors. A more complete characterization result is the following.

Theorem 4 ([BPV]). (M, g) *is locally symmetric if and only if one of the following statements holds:*

(a) ℓ *maps horizontal (equivalently, vertical) vectors into horizontal (vertical) vectors;*
(b) the horizontal (equivalently, vertical) lift of an eigenvector of $R_u = R(\cdot, u)u$ *is an eigenvector of* ℓ*, at any point* $(p, u) \in T_1M$*;*
(c) the horizontal (equivalently, vertical) distribution of T_1M *is anti-invariant with respect to* $\nabla_\xi h$*.*

Unit tangent sphere bundles satisfying $\nabla_\xi h = 0$

Let $(\bar{M}, \eta, \bar{g})$ be a contact metric manifold. The condition

$$\nabla_\xi h = 0 \tag{8}$$

appeared for the first time in a paper by Chern and Hamilton [ChH], concerning the study of compact contact three-manifolds. They conjectured that for a fixed contact form η, whose characteristic vector field ξ induces a Seifert foliation, there exists a CR-structure, and hence, a contact metric structure, satisfying (8). It can be noted that (8) is equivalent to the condition that the sectional curvature of all planes, at a given point, perpendicular to the contact subbundle, are equal. When M is compact, (8) is the critical point condition for the functional $I(g) =$"integral of the scalar curvature", defined in the set $A(\eta)$ of all metrics associated to η [P1]. In dimension three, (8) is related to the existence of a *taut contact circle* on a compact contact metric manifold

[CP1]. Note that K-contact spaces obviously satisfy (8). On the other hand, there are contact metric manifolds satisfying $\nabla_\xi h = 0$ which are not K-contact. This is also the case for the unit tangent sphere bundle on a flat manifold, as follows by comparing Theorem 2 with the following.

Theorem 5 ([P2]). $(T_1M, \eta, \bar{g})$ *satisfies* $\nabla_\xi h = 0$ *if and only if* (M, g) *has constant sectional curvature* 0 *or* 1.

Theorem 5 also extends Theorem 3, since locally symmetric contact metric manifolds satisfy (8) [P2]. A further generalization of Theorem 5 is given by the following result:

Theorem 6 ([BPV]). (M, g) *is locally isometric to a two-point homogeneous space if and only if* $(T_1M, \eta, \bar{g})$ *satisfies*

$$\nabla_\xi h = 2ah\varphi + 2b\varphi S,$$

where a and b are functions only depending on $p \in M$ *and S is the* $(1, 1)$*-tensor defined by* $S(X^h) = X^h$ *and* $S(X^t) = -X^t$ *for* $X \in TM$.

Conformally flat unit tangent sphere bundles

Conformally flat manifolds are a classical field of investigation in Riemannian geometry. As is well-known, a Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *conformally flat* if its Riemannian metric $\bar{g}$ is locally conformal to a flat metric, and this gives strong information on the curvature of the manifold. Making such information interact with the curvature properties of the unit tangent sphere bundle, the following result has been obtained:

Theorem 7 ([BlK]). $(T_1M, \eta, \bar{g})$ *is conformally flat if and only if* (M, g) *is a surface of constant Gaussian curvature* 0 *or* 1.

Unit tangent sphere bundles which are (k, μ)-spaces

A contact metric manifold $(\bar{M}, \eta, \bar{g})$ is said to be a (k, μ)*-space* if its characteristic vector field ξ belongs to the so-called (k, μ)*-distribution*, that is, if

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for all vector fields X, Y, where k and μ are some constants.

A Sasakian manifold is a (k, μ)-space with $k = 1$ (and μ arbitrary). Non-Sasakian (k, μ)-spaces have been completely classified by Boeckx in [B3]. In [BlKP], where (k, μ)-spaces have been introduced, the authors proved the following.

Theorem 8 ([BlKP]). $(T_1M, \eta, \bar{g})$ *is a* (k, μ)*-space if and only if* (M, g) *has constant sectional curvature.*

Note that, according to Theorems 2 and 8, the unit tangent sphere bundle T_1M of a Riemannian manifold of constant sectional curvature $\neq 1$, gives an example of a (k, μ)-space which is not K-contact.

Homogeneous unit tangent sphere bundles

A contact metric manifold $(\bar{M}, \eta, \bar{g})$ is said to be *homogeneous* if there exists a connected Lie group of isometries acting transitively on M and leaving η invariant. It is said to be *locally homogeneous* if the pseudogroup of local isometries acts transitively on M and leaves η invariant. For the unit tangent sphere bundle, we have:

Theorem 9 ([BV4]). *If $(M.g)$ is a two-point homogeneous space, then its unit tangent sphere bundle $(T_1M, \eta, \bar{g})$ is a homogeneous contact metric manifold.*

Theorem 9 is the contact metric version of the following classic result by Wolf [W] (see also [MTr]):

Theorem 10 ([W]). *If (M, g) is a two-point homogeneous space, then its unit tangent sphere bundle (T_1M, g_S) is homogeneous.*

The converse of Theorems 9 and 10 has been proved to hold in several special cases ([BV2], [BV4]), but the general case remains an interesting open problem.

A way to characterize two-point homogeneous spaces using the properties of their unit tangent sphere bundles is the following

Theorem 11 ([BPV]). *(M, g) is locally isometric to a two-point homogeneous space if and only if on $(T_1M, \eta, \bar{g})$ we have both*

(a) *the eigenvalues of h are constant along the fibers, and*
(b) *ℓ maps vertical vectors into vertical vectors.*

Note that, according to Theorem 4, condition (b) above means that (M, g) is locally symmetric.

Locally φ-symmetric unit tangent sphere bundles

As we already remarked, local symmetry represents a very rigid condition in the framework of contact metric geometry. In order to weaken such conditions, *locally φ-symmetric spaces* were first introduced in [Tk], as Sasakian manifolds satisfying,

$$\bar{g}((\nabla_X R)(Y, Z)V, W) = 0, \tag{9}$$

for all X, Y, Z, V, W orthogonal to ξ. For a Sasakian manifold, (9) is equivalent to requiring that *the reflections with respect to the integral curves of ξ are local isometries*, but for a general contact metric manifold $(\bar{M}, \eta, \bar{g})$ this geometric property is strictly stronger than (9) (see [BBV]). Using this stronger geometric property as definition, locally φ-symmetric spaces have also been introduced for contact metric spaces in [BV1]. Note that a non-Sasakian (k, μ)-space is locally φ-symmetric [B2]. The following result was proved in [BV1]:

Theorem 12 ([BV1]). *$(T_1M, \eta, \bar{g})$ is locally φ-symmetric if and only if (M, g) has constant sectional curvature.*

H-contact unit tangent sphere bundles

A unit vector field V on a Riemannian manifold (M, g) determines a map between (M, g) itself and its unit tangent sphere bundle (T_1M, g_S). If M^n is compact and orientable, the *energy* of V is defined as the energy of the corresponding map:

$$E(V) = \frac{1}{2}\int_M \|dV\|^2\,dv = \frac{n}{2}\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2\,dv.$$

V is called *harmonic* if it is a critical point for E in the set of all unit vector fields of M [Wo]. This has been proved to be equivalent to requiring that

$$\nu_V(X) = \mathrm{tr}(\nabla.(A_V^t)X) = 0 \text{ on } V^\perp, \tag{10}$$

where $A_V X = -\nabla_X V$ [Wi].

The corresponding map between (M, g) and (T_1M, g_S) is harmonic if and only if V is harmonic and moreover

$$\mathrm{tr}R(\nabla.V, V)\cdot = 0. \tag{11}$$

Note that (10) and (11) also make sense when M is non-orientable or non-compact. For this reason, (10) has been assumed as the definition of a harmonic vector field on an arbitrary Riemannian manifold [G-M]. For further details and references about harmonic vector fields, we refer to [BV3], where the harmonicity of the geodesic flow was investigated (see also Theorem 14 further on).

Among the unit vector fields of a contact metric manifold $(\bar{M}, \eta, \bar{g})$, the most important for determining the geometry of the manifold is by far its characteristic vector field ξ. A *H-contact space* is a contact metric manifold whose characteristic vector field is harmonic [P5]. The following characterization was proved in [P4]:

Theorem 13 ([P4]). *$(\bar{M}, \eta, \bar{g})$ is H-contact if and only if ξ is an eigenvector for the Ricci operator.*

K-contact spaces (in particular, Sasakian manifolds), (k, μ)-spaces, locally φ-symmetric spaces are all examples of H-contact manifolds. We can refer to [P5] for more details on H-contact spaces.

As concerns the unit tangent sphere bundle, Theorem 1 of [BV3] can be reformulated in the following way:

Theorem 14 ([BV3]). *If (M, g) is two-point homogeneous, then $(T_1M, \eta, \bar{g})$ is H-contact.*

Note that, according to Theorems 2, 12 and 14, if (M, g) is a two-point homogeneous space of non-constant sectional curvature, then its unit tangent sphere bundle $(T_1M, \eta, \bar{g})$ is an H-contact space which is neither locally φ-symmetric (in particular, it is not a (k, μ)-space), nor K-contact.

For the unit tangent sphere bundle, Proposition 2 of [BV3] can be rewritten in the following way, which extends Theorems 2, 8 and 12 in low dimension:

Theorem 15 ([BV3]). *If* $\dim M = 2$ *or* 3*, then* $(T_1M, \eta, \bar{g})$ *is* H*-contact if and only if* M *has constant curvature.*

To our knowledge, the general problem of characterizing H-contact unit tangent sphere bundles is an interesting open problem. More precisely, the following was stated in [BV3].

Question. *Is a Riemannian manifold, whose unit tangent sphere bundle is* H*-contact, a two-point homogeneous space?*

4 Semi-symmetric unit tangent sphere bundles

A Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *semi-symmetric* if its curvature tensor $\bar{R}$ satisfies:

$$\bar{R}(X, Y) \cdot \bar{R} = 0\,, \tag{12}$$

for all vector fields X, Y, where $\bar{R}(X, Y)$ acts as a derivation on $\bar{R}$. This is equivalent to saying that $\bar{R}_p$ is, for each $p \in \bar{M}$, the same as the curvature tensor of a symmetric space. This last space may vary with p. So, locally symmetric spaces are obviously semi-symmetric, but the converse is not true. The first example of a semi-symmetric space which is not locally symmetric was found by H. Takagi [T]. In all dimensions greater than one, there exist semi-symmetric spaces which are not locally symmetric (we can refer to [BKV] for a survey). Nevertheless, semi-symmetry implies local symmetry in several cases and it is an interesting problem, given a class of Riemannian manifolds, to decide whether inside that class semi-symmetry implies local symmetry or not (see for example [B1], [CV]). The author, in joint works with E. Boeckx and D. Perrone, obtained the following generalization of Theorem 3 by Blair:

Theorem 16 ([BC], [CP2]). *If the unit tangent sphere bundle* (T_1M, g_S) (equivalently, $(T_1M, \eta, \bar{g})$) *of a Riemannian manifold* (M, g) *is semi-symmetric, then it is locally symmetric. Therefore,* (T_1M, g_S) *is semi-symmetric if and only if either* (M, g) *is flat or it is locally isometric to* $S^2(1)$.

In order to prove this result, after recalling the local structure of a semi-symmetric space, we deal separately with the cases when T_1M is three-dimensional, where we make use of the special features of a three-dimensional contact metric manifold, locally irreducible and locally reducible.

Definition 1. The *nullity vector space* of the curvature tensor at a point p of a Riemannian manifold $(\bar{M}, \bar{g})$ is given by

$$E_{0p} = \{X \in T_p\bar{M} \ / \ \bar{R}(X, Y)Z = 0 \text{ for all } Y, Z \in T_p\bar{M}\}.$$

The *index of nullity at* p is the number $v(p) = \dim E_{0p}$. The *index of non-nullity* (or *co-nullity*) *at* p is the number $u(p) = \dim \bar{M} - v(p)$.

The index of nullity and the index of co-nullity permit us to distinguish the different irreducible factors in the local structure of a semi-symmetric space $(\bar{M}, \bar{g})$, which was described by Szabó [Sz] in the following way.

Theorem 17 ([Sz]). *There exists an open dense subset U of $\bar{M}$ such that around every point of U the manifold is locally isometric to the direct product of symmetric spaces, two-dimensional manifolds, spaces foliated by Euclidean leaves of codimension two, real cones and Kählerian cones.*

For more details and references about the irreducible factors of the local decomposition of a semi-symmetric space, we refer to [BKV].

The three-dimensional case

Let (M, η, g) be a three-dimensional contact metric manifold and m a point of M. Let U be the open subset of M, where $h \neq 0$, and V the open subset of points $m \in M$ such that $h = 0$ in a neighborhood of m. Then, $U \cup V$ is an open dense subset of M. For any point $m \in U \cup V$ there exists a local orthonormal basis (called a *φ-basis*) $\{\xi, e, \varphi e\}$ of smooth eigenvectors of h in a neighborhood of m. On U we put $he = \lambda e$, where λ is a non-vanishing smooth function which we suppose to be positive. From (1), we have $h\varphi e = -\lambda\varphi e$. We can refer to [CPV] for more details.

The components of the Ricci operator Q, with respect to $\{\xi, e, \varphi e\}$, are given by (see [P3] or [CPV])

$$\begin{cases} Q\xi = 2(1-\lambda^2)\xi + Ae + B\varphi e, \\ Qe = A\xi + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right) e + \xi(\lambda)\varphi e, \\ Q\varphi e = B\xi + \xi(\lambda)e + \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right)\varphi e. \end{cases} \tag{13}$$

Starting from (12) and using (13), one can characterize three-dimensional semi-symmetric contact metric manifolds by a list of algebraic formulas involving λ, a, A, B:

Lemma 1. [CP2] *Let (M, η, g) be a three-dimensional non-Sasakian contact metric manifold. Then M is semi-symmetric if and only if*

$$B(\lambda^2 - 1 + 2a\lambda) = A\xi(\lambda), \tag{14}$$

$$A(\lambda^2 - 1 - 2a\lambda) = B\xi(\lambda), \tag{15}$$

$$AB + \xi(\lambda)\left(\frac{r}{2} + 2\lambda^2 - 2\right) = 0, \tag{16}$$

$$B^2 - [\xi(\lambda)^2] + (\lambda^2 - 1 - 2a\lambda)\left(\frac{r}{2} + 3\lambda^2 - 3 + 2a\lambda\right) = 0, \tag{17}$$

$$A^2 - [\xi(\lambda)^2] + (\lambda^2 - 1 + 2a\lambda)\left(\frac{r}{2} + 3\lambda^2 - 3 - 2a\lambda\right) = 0, \tag{18}$$

where r denotes the scalar curvature of M.

Making use of formulas (14)–(18), it is possible to prove the following

Proposition 1 ([CP2]). *A three-dimensional semi-symmetric contact metric manifold* (M^3, η, g) *satisfying* $A = 0$ *or* $B = 0$*, either is flat or has constant curvature* 1.

We are now ready to prove the three-dimensional version of Theorem 16:

Theorem 18. *Let* (M^2, g) *be a Riemannian surface.* $(T_1M^2, \bar{g})$ (equivalently, (T_1M^2, g_S)) *is semi-symmetric if and only if* (M^2, g) *is either flat or locally isometric to* $S^2(1)$.

Proof. If (M^2, g) has constant Gaussian curvature 0 or 1, from Theorem 3 it follows that T_1M^2 is locally symmetric. In particular, it is semi-symmetric.

In order to prove the converse, we need the description of the contact metric structure of T_1M^2. Using isothermal local coordinates (x^1, x^2) on M^2, its Riemannian metric g is given by

$$g = e^{2f}((dx^1)^2 + (dx^2)^2),$$

where f is a C^∞ function on M^2. Set $u = -v^2\frac{\partial}{\partial x^1} + v^1\frac{\partial}{\partial x^2}$ and $v = v^1\frac{\partial}{\partial x^1} + v^2\frac{\partial}{\partial x^2}$. Then, $\{\xi = 2v^h, e = 2u^h, \varphi e = 2u^v\}$ is a φ-basis for $(T_1M^2, \eta, \bar{g})$, of eigenvectors of h. More precisely, $he = \lambda e$ with $\lambda = 1 - k$, where k denotes the Gaussian curvature of (M^2, g), considered the function on T_1M^2 defined by $k(p, v) = k(p)$. We can refer to [CP2] for all the details.

Next, we can compute the Ricci tensor of T_1M^2 starting from formulas (5) (see also [BV2]). We obtain

$$\bar{\varrho}_{(p,v)}(X^h, Y^h) = \varrho_p(X, Y) - \frac{1}{2}\sum_{i=1,2} g_p(R(v, E_i)X, R(v, E_i)Y),$$

where ϱ is the Ricci tensor of M^2 and $\{E_i\}$ is an orthonormal basis of T_pM^2. In particular, taking $\{u, v\}$ as orthonormal basis of T_pM^2, we get

$$\begin{aligned} A &= \bar{\varrho}_{(p,v)}(\xi, e) = \varrho_{(p,v)}(2v^h, 2u^h) \\ &= 4\varrho_p(v, u) - 2g_p(R(v, u)v, R(v, u)u) = -k(p)^2 g_p(v, u) = 0. \end{aligned}$$

So, if $(T_1M^2, \eta, \bar{g})$ is semi-symmetric, since $A = 0$, Proposition 1 implies that T_1M is flat or it has constant curvature 1. In both cases, T_1M is locally symmetric and the conclusion follows from Theorem 3. □

Irreducible semi-symmetric unit tangent sphere bundles

From now on, we shall always assume that $\dim M \geq 3$. We now prove the following.

Theorem 19. *Let* (M, g) *be a Riemannian manifold of dimension* $n \geq 3$*. If* (T_1M, g_S) *is semi-symmetric, then* T_1M *must be locally reducible.*

Proof. We assume that T_1M is semi-symmetric and locally irreducible and we show that M is flat. By Theorem 1, this gives a contradiction.

According to Szabó's classification (Theorem 17), T_1M is locally isometric to one of the Riemannian manifolds listed in Theorem 17. If T_1M is locally isometric to a symmetric space, then it is locally symmetric and so, M must be flat (Theorem 3). We shall now exclude all the other possibilities. Since $\dim T_1M = 2n - 1$, where $n = \dim M$, clearly T_1M cannot be locally isometric either to a Riemannian surface, or to a Kählerian cone, since manifolds of both type have even dimension (see [BKV] for the description of Kählerian cones).

Next, suppose now that T_1M is locally isometric to F^{r+2}, a Riemannian manifold foliated by two-dimensional Euclidean leaves. This means that the index of nullity is constant along F^{r+2} and equal to r [BKV]. Hence, at any point $p \in F^{r+2}$, we have

$$T_pF^{r+2} = E_{0p} \oplus V_p,$$

with $\dim V_p = 2$. Note that $r + 2 = 2n - 1 \geq 5$ and so, $r \geq 3$. After some easy calculations on the curvature tensor of T_1M, starting from formulas (5), we can show that when $r > 3$, there exist three linearly independent vectors in V_p, while, if $r = 3$, then (M, g) is flat. Since both cases give a contradiction, we can exclude that T_1M is locally isometric to F^{r+2}.

Also the case when T_1M is locally isometric to a real cone can be excluded using curvature information. In fact, comparing the formulas for the curvature of a semi-symmetric real cone with the information coming from formulas (5), we eventually obtain a contradiction. We refer to [BC] for the full detailed proof. □

Reducible semi-symmetric unit tangent sphere bundles

We now complete the proof of Theorem 16, by proving the following.

Theorem 20. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Then, (T_1M, g_S) is semi-symmetric if and only if M is flat.*

Proof. The "if" part is trivial. As concerns the "only if" part, by Theorem 19 it follows that T_1M is locally reducible. Thus, according to Theorem 1, there exist $k \geq 1$ and, unless M is flat, a Riemannian manifold M', without the flat factor, such that

$$M = M' \times \mathbb{R}^k.$$

In order to complete the proof, we want to prove that M is flat.

Fix a point (x, u) in T_1M. Then, $x = (x', v_0) \in M' \times \mathbb{R}^k$ and the tangent space T_xM splits into the direct sum of $T_{x'}M'$ and $T_{v_0}\mathbb{R}^k$. Consider in the tangent space $T_{(x,u)}T_1M$ the following distributions:

$$L_1 = V_{(x,u)}(T_xM) \oplus H_{(x,u)}(T_{x'}M'),$$
$$L_2 = H_{(x,u)}(T_{v_0}\mathbb{R}^k).$$

Then, $T_{(x,u)}T_1M = L_1 \oplus L_2$.

We denote by X, Y, respectively U, V, some vector fields tangent to M', respectively to $\mathbb{R}^k$, while A, B denote generic vector fields tangent to M. Computing the Levi-Civita connection of (T_1M, g_S), we proved that L_1 and L_2 are two complementary, mutually orthogonal, totally geodesic and totally parallel distributions (L_2 is also flat) [BC]. Therefore, $\mathcal{L}_1$ and $\mathcal{L}_2$, the foliations determined respectively by L_1 and L_2, are leaves of a Riemannian product $M_1 \times M_2 = T_1M$. In particular, let M_1 be the integral manifold of L_1 through a point $((x', 0), u)$. Then, $T_1M' \subset M_1$.

Note that, since $T_1M = M_1 \times M_2$ is semi-symmetric and M_2 is flat, M_1 itself is semi-symmetric. We proved also that T_1M' is semi-symmetric (see [BC]). Now, if T_1M' is locally reducible, from Theorem 1 it follows that M' must have a flat factor, which gives a contradiction. Hence, T_1M' is locally irreducible and so, according to Theorems 18 and 19, M' must be isometric to $S^2(1)$. So, we end the proof of Theorem 20 by proving the following

Proposition 2. *The unit tangent sphere bundle of the Riemannian manifold* $M = S^2(1) \times \mathbb{R}^k$*, with* $k \geq 1$*, is not semi-symmetric.*

Proof. Suppose T_1M is semi-symmetric. Consider $x = (x', v_0) \in M$ and a unit vector $u \in T_xM$. Then, there exist unit vectors u_1, u_2, tangent respectively to $S^2(1)$ at x' and to $\mathbb{R}^k$ at v_0, such that $u = \cos\theta u_1 + \sin\theta u_2$. We choose u in such a way that $\cos\theta \neq 0 \neq \sin\theta$.

Next, we consider the following vectors tangent to M:

$$X = u_1 + X_2, \quad Y = u_1 + Y_2, \quad U = u_1 + U_2,$$
$$V = v_1 + V_2, \quad W = v_1 + W_2,$$

where v_1 is a unit tangent vector to $S^2(1)$ at x', orthogonal to u_1.

Applying the semi-symmetry condition (12), we have

$$(\bar{R}(X^h, Y^t) \cdot \bar{R})(U^h, V^t)W^t = 0. \tag{19}$$

We use formulas for the curvature tensor of T_1M and, after some standard but quite long calculations, from (19) we get

$$-\frac{3}{4}(1 - g(Y, u)\cos\theta)(u_1^t + g(W, u)\cos\theta u_2^t) = 0. \tag{20}$$

Note that $g(Y, u) = \cos\theta + \sin\theta g(Y_2, u_2)$ and $g(W, u) = \sin\theta\, g(W_2, u_2)$. If we take $Y_2 = W_2 = 0$, then (20) gives $u_1^t = 0$, that is,

$$(u_1 - g(u_1, u)u)^v = 0$$

and so,

$$0 = u_1 - g(u_1, u)u = \sin^2\theta u_1 - \sin\theta\cos\theta u_2. \tag{21}$$

Therefore, taking into account $\cos\theta \neq 0 \neq \sin\theta$, (21) gives $u_1 = u_2 = 0$, which clearly cannot happen. So, T_1M is not semi-symmetric.

Remark 1. The *tangent sphere bundle* $T_r M$ of arbitrary radius r is the submanifold of the tangent bundle TM defined by $T_r M = \{(x, u) \in TM : g_x(u, u) = r^2\}$ and equipped with the metric g_S induced by the Sasaki metric on TM (see for example [KS] and [KSVl]). The geometric properties of $(T_r M, g_S)$ may change with the radius. Its Levi-Civita connection and Riemann curvature tensor have been calculated in [KS]. One obtains similar expressions to the ones above for $T_1 M$, up to an occasional factor $1/r^2$.

As proved in [B4], Theorem 1 holds for tangent sphere bundles $(T_r M, g_S)$ of any radius r. Further, Theorem 3 by D. Blair can be generalized, obtaining that *the tangent sphere bundle* $(T_r M, g_S)$*,* $r > 0$*, of a Riemannian manifold* (M, g) *is locally symmetric if and only if either* (M, g) *is flat or it is locally isometric to the two-dimensional sphere* $S^2(r)$ *of radius* r*.*

With these ingredients, one can proceed as in the case of the unit tangent sphere bundle to prove the following

Theorem 21 ([BC]). *If the tangent sphere bundle* $(T_r M, g_S)$*,* $r > 0$*, of a Riemannian manifold* (M, g) *is semi-symmetric, then it is locally symmetric. Therefore,* $(T_r M, g_S)$ *is semi-symmetric if and only if either* (M, g) *is flat or it is locally isometric to* $S^2(r)$*.*

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Topological–antitopological Fusion Equations, Pluriharmonic Maps and Special Kähler Manifolds*

Vicente Cortés[1] and Lars Schäfer[1,2]

[1]Institut Élie Cartan de Mathématiques, Université Henri Poincaré - Nancy 1, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France
cortes@iecn.u-nancy.fr
[2]Mathematisches Institut der Universität Bonn, Beringstraße 1, D-53115 Bonn, Germany
schaefer@math.uni-bonn.de
schafer@iecn.u-nancy.fr

Dedicated to Professor Lieven Vanhecke

Summary. We introduce the notion of a tt*-bundle. It provides a simple definition, purely in terms of real differential geometry, for geometric structures which are solutions of a general version of the equations of topological–anti a topological fusion considered by Cecotti-Vafa, Dubrovin and Hertling. Then we give a simple characterization of the tangent bundles of special complex and special Kähler manifolds as particular types of tt*-bundles. We illustrate the relation between metric tt*-bundles of rank r and pluriharmonic maps into the pseudo-Riemannian symmetric space $\mathrm{GL}(r)/\mathrm{O}(p,q)$ in the case of a special Kähler manifold of signature $(p,q) = (2k, 2l)$. It is shown that the pluriharmonic map coincides with the dual Gauss map, which is a holomorphic map into the pseudo-Hermitian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l) \subset \mathrm{SL}(2n)/\mathrm{SO}(p,q) \subset \mathrm{GL}(2n)/\mathrm{O}(p,q)$, where $n = k + l$.

1 tt*-equations and pluriharmonic maps

Definition 1. A *tt*-bundle* (E, D, S) over a complex manifold (M, J) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ which satisfy the *tt*-equation*

$$R^\theta = 0, \quad \text{for all} \quad \theta \in \mathbb{R}, \tag{1.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + (\cos\theta) S_X + (\sin\theta) S_{JX}, \quad \text{for all} \quad X \in TM. \tag{1.2}$$

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A *metric tt*-bundle* (E, D, S, g) is a tt*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g such that for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z), \quad \text{for all} \quad X, Y, Z \in T_pM. \tag{1.3}$$

A *unimodular metric tt*-bundle* (E, D, S, g) is a metric tt*-bundle (E, D, S, g) such that $\operatorname{tr} S_X = 0$ for all $X \in TM$. An *oriented unimodular metric tt*-bundle* (E, D, S, g, or) is a unimodular metric tt*-bundle endowed with an orientation *or* of the bundle E.

Remarks. 1) In special cases, particularly emphasized in the literature, such as the moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [H], the complexified tt*-bundle $E^{\mathbb{C}}$ is identified with $T^{1,0}M$ and the metric g is positive definite. Here we will consider the case $E = TM$, and hence $E^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$. This includes special complex and special Kähler manifolds, as we shall see.

2) If (E, D, S) is a tt*-bundle then (E, D, S^θ) is a tt*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = (\cos\theta)S + (\sin\theta)S_J. \tag{1.4}$$

The same remark applies to metric tt*-bundles.

3) Notice that an oriented unimodular metric tt*-bundle (E, D, S, g, or) carries a canonical metric volume element $\nu \in \Gamma(\wedge^r E^*)$, $r = \operatorname{rk} E$, determined by g and *or*, which is D^θ parallel for all $\theta \in \mathbb{R}$.

Proposition 1. *Let $E \to M$ be a real vector bundle over a complex manifold (M, J) such that E is endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$. Then (E, D, S) is a tt*-bundle if and only if the following equations are satisfied.*

(i) $d^D S = d^D S_J = 0$, *where S and S_J are considered as one-forms with values in* End E *and d^D is the exterior covariant derivative defined by D,*
(ii) $[S_X, S_Y] = [S_{JX}, S_{JY}]$ *for all X and Y,*
(iii) $R^D(X, Y) + [S_X, S_Y] = 0$ *for all X and Y.*

Proof. Using the relations $2\cos\theta\sin\theta = \sin 2\theta$, $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin^2\theta = 1 - \cos 2\theta$, we obtain a (finite) Fourier decomposition of R^{D^θ} in the variable θ. The tt*-equation $R^{D^\theta} = 0$ shows that all Fourier components are zero. This yields (i-iii).□

Definition 2. Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold. A map $f : M \to N$ is called *pluriharmonic* if $f|_C$ is harmonic for all complex curves $C \subset M$.

Notice that the harmonicity of $f|_C$ is independent of the choice of a Riemannian metric in the conformal class of C, by conformal invariance of the harmonic map equation for (real) surfaces.

Proposition 2. *Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D a connection on M which satisfies*

$$D_{JY}X = JD_YX, \tag{1.5}$$

for all vector fields which satisfy $\mathcal{L}_X J = 0$ (i.e. for which $X - iJX$ is holomorphic), and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.

(i) *A map $f : M \to N$ is pluriharmonic if and only if it satisfies the following equation*

$$\nabla'' \partial f = 0, \tag{1.6}$$

where $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}} (TN)^{\mathbb{C}})$ is the $(1,0)$-component of df and ∇'' is the $(0,1)$-component of $\nabla = \nabla' + \nabla''$.

(ii) *Any complex manifold (M, J) admits a torsion-free complex connection, i.e. a torsion-free connection D which satisfies $DJ = 0$.*

(iii) *Any torsion-free complex connection D satisfies (1.5).*

Proof. The condition (1.5) means that $D''Z = 0$ for all local holomorphic vector fields Z, i.e. $\Gamma^{\gamma}_{\bar{\alpha}\beta} = \Gamma^{\bar{\gamma}}_{\bar{\alpha}\beta} = 0$ in terms of the Christoffel symbols of D with respect to holomorphic coordinates z^{α}. This implies that the Christoffel symbols of D do not contribute to equation (1.6). The equation is therefore independent of the choice of connection D. In fact, it is straightforward to check that the restriction of (1.6) to every complex curve C reduces to the harmonic map equation for $f|_C : C \to N$.
(ii) is well known, see [KN].
(iii) The conditions $T^D = 0$ and $DJ = 0$ imply that

$$\begin{aligned} D_{JY}X - JD_YX &= [JY, X] + D_X(JY) - JD_YX \\ &= [JY, X] + J[X, Y] = -(\mathcal{L}_X J)Y. \end{aligned} \tag{1.7}$$

The right-hand side vanishes if $\mathcal{L}_X J = 0$. □

Given a Hermitian metric γ on $T^{1,0}M$, or, more generally, a pseudo-Hermitian metric, the *Chern connection* of γ is the unique Hermitian connection $\mathcal{D}$ in the holomorphic bundle $T^{1,0}M$ which satisfies $\mathcal{D}''Z = 0$ for all holomorphic local sections Z of $T^{1,0}M$. The last property is usually written as $\mathcal{D}'' = \bar{\partial}$.

Proposition 3. *Let (M, J) be a complex manifold and $\mathcal{D}$ the Chern connection of a pseudo-Hermitian metric γ on $T^{1,0}M$. Then there is a unique connection D in the real tangent bundle TM such that $DZ = \mathcal{D}Z$ for all local sections Z of $T^{1,0}M$, where D has been complex bilinearly extended to a connection on the complexified tangent bundle. The connection D satisfies (1.5), $DJ = 0$ and $Dg = 0$, where g is the J-invariant pseudo-Riemannian metric defined by*

$$g(X, X) = 2\gamma(X^{1,0}, X^{1,0}), \quad X^{1,0} := \frac{1}{2}(X - iJX), \tag{1.8}$$

for all $X \in TM$.

Conversely, let g be a J-invariant pseudo-Riemannian metric on a complex manifold (M, J). Then there exists a unique connection D in TM, which satisfies the conditions (1.5), $DJ = 0$ and $Dg = 0$. Moreover, D induces a connection in $T^{1,0}M$, which is the Chern connection of the pseudo-Hermitian metric γ on $T^{1,0}M$ defined by (1.8).

The factor 2 is chosen such that γ coincides with the restriction to $T^{1,0}M$ of the sesquilinear extension of g to the complexified tangent bundle.

Proof. We define a connection D in the complexified tangent bundle $(TM)^{\mathbb{C}}$ by

$$D_X Z := \mathcal{D}_X Z \quad \text{and} \quad D_X \bar{Z} := \overline{\mathcal{D}_{\bar{X}} Z}, \tag{1.9}$$

for all local sections X of $(TM)^{\mathbb{C}}$ and Z of $T^{1,0}M$. By construction D is real, i.e. is the complex bilinear extension of a connection in TM, which we denote by the same symbol D. Obviously, it is the only real connection such that $DZ = \mathcal{D}Z$ for all local sections Z of $T^{1,0}M$. The equation (1.5) follows from $\mathcal{D}'' = \bar{\partial}$. By construction, D preserves the decomposition $(TM)^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$. Therefore, $DJ = 0$. Finally, $Dg = 0$ follows from the fact that $\mathcal{D}$ is Hermitian.

Conversely, let (M, J, g) be a pseudo-Hermitian manifold. Then we can define a pseudo-Hermitian metric γ in $T^{1,0}M$ by (1.8) and consider its Chern connection $\mathcal{D}$. As we know, it induces a connection D in TM which satisfies (1.5), $DJ = 0$ and $Dg = 0$. To prove the uniqueness, let $\tilde{D}$ be an other connection satisfying (1.5), $\tilde{D}J = 0$ and $\tilde{D}g = 0$. $\tilde{D}$ induces a connection $\tilde{\mathcal{D}}$ in $T^{1,0}M$, which satisfies $\tilde{\mathcal{D}}'' = \bar{\partial}$, due to (1.5), and which is Hermitian with respect to γ. Therefore, $\tilde{\mathcal{D}}$ is the Chern connection of γ, i.e. $\tilde{\mathcal{D}} = \mathcal{D}$. This implies $D = \tilde{D}$, by the first part of the proof. □

Given a metric tt*-bundle (E, D, S, g), we consider the flat connection D^θ for $\theta = 0$: $\nabla := D^0$. Any parallel frame $s = (s_1, \ldots, s_r)$ of E with respect to ∇ defines a map

$$G = G^{(s)} : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{GL}(r) | A^t = A \quad \text{has signature } (p, q)\}$$
$$x \mapsto G(x) := (g_x(s_i(x), s_j(x))), \tag{1.10}$$

where (p, q) is the signature of the metric g.

Similarly, for an oriented unimodular metric tt*-bundle (E, D, S, g, ν) with canonical volume element ν and a ∇-parallel frame $s = (s_1, \ldots, s_r)$ such that $\nu(s_1, s_2, \ldots, s_r) = 1$ we have a map

$$G = G^{(s)} : M \to \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{Sym}_{p,q}(\mathbb{R}^r) | \det A = (-1)^q\}. \tag{1.11}$$

By Sylvester's Theorem, the general linear group $\mathrm{GL}(r)$ acts transitively on the manifold $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$, which we can identify with the pseudo-Riemannian symmetric space

$$S(p, q) := \mathrm{GL}(r)/\mathrm{O}(p, q). \tag{1.12}$$

The subgroup $\mathrm{O}(p, q) \subset \mathrm{GL}(r)$ is the stabilizer of the matrix

$$I_{p,q} = \mathrm{diag}(\mathbf{1}_p, -\mathbf{1}_q).$$

We shall identify the tangent space of the coset space $S(p, q)$ at the canonical base point $o = e\mathrm{O}(p, q)$ with the vector space

$$\mathrm{sym}(p, q) := \{A \in \mathfrak{gl}(r) | \eta(A\cdot, \cdot) = \eta(\cdot, A\cdot)\} \tag{1.13}$$

of symmetric endomorphisms of $\mathbb{R}^r$ with respect to the standard scalar product $\eta = \eta_{p,q}$ of signature (p, q), which is represented by the matrix $I_{p,q}$. The structure of a symmetric space is defined by the symmetric decomposition

$$\mathfrak{gl}(r) = \mathfrak{o}(p, q) + \mathrm{sym}(p, q). \tag{1.14}$$

The pseudo-Riemannian metric is defined by an $O(p, q)$-invariant pseudo-Euclidean scalar product on $\mathrm{sym}(p, q)$. For instance, we may choose the metric induced by the trace form:

$$\mathfrak{gl}(r) \ni (X, Y) \mapsto \mathrm{tr}\, XY. \tag{1.15}$$

Similarly, $SL(r)$ acts transitively on the manifold $\mathrm{Sym}^1_{p,q}(\mathbb{R}^r)$, which we can identify with the pseudo-Riemannian symmetric space

$$S_1(p, q) := SL(r)/SO(p, q). \tag{1.16}$$

We have the de Rham decomposition

$$S(p, q) = \mathbb{R} \times S_1(p, q), \tag{1.17}$$

where the flat factor corresponds to the connected central subgroup

$$\mathbb{R}^{>0} = \{\lambda \mathrm{Id} | \lambda > 0\} \subset GL(r), \tag{1.18}$$

and the other factor is always indecomposable and even irreducible if $(p, q) \neq (1, 1)$. The tangent space of $SL(r)/SO(p, q)$ at the canonical base point $o = eSO(p, q)$ is identified with the trace-free η-symmetric matrices:

$$\mathrm{sym}_0(p, q) := \{A \in \mathrm{sym}(p, q) | \mathrm{tr}\, A = 0\}. \tag{1.19}$$

Under a change of parallel (respectively, parallel unimodular) frame $s \to su, u \in GL(r)$ (respectively, $u \in SL(r)$), the map $G = G^{(s)}$ transforms as

$$G^{(su)} = u^{-1} \cdot G^{(s)} = u^t G^{(s)} u, \tag{1.20}$$

where the dot stands for the action of $GL(r)$ on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$.

The following theorem is proven in [S2], cf. [S1]. In the case where $E^{\mathbb{C}} = T^{1,0}M$ and the metric g is positive definite it is due to Dubrovin [D].

Theorem 1. *Let (E, D, S, g) be a metric tt*-bundle over a simply connected complex manifold M. Then the map,*

$$G^{(s)} = (g(s_i, s_j)) : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) \cong GL(r)/O(p, q) = S(p, q), \tag{1.21}$$

associated to a parallel frame $s = (s_1, \dots, s_r)$ of E with respect to the flat connection $\nabla = D^0$ is pluriharmonic. Moreover, for all $x \in M$, the image of $T^{1,0}_x M \subset (T_x M) \otimes \mathbb{C}$ under the complex linear extension of $dL_u^{-1} dG_x : T_x M \to T_o S(p, q) = \mathrm{sym}(p, q)$ consists of commuting matrices, where $u \in GL(r)$ is any element such that $G(x) = u \cdot o$ and $L_u : S(p, q) \to S(p, q)$ is the isometry of $S(p, q)$ induced by the left-multiplication by u in $GL(r)$.

Conversely, let M be a simply connected complex manifold and $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) \cong S(p, q)$ a pluriharmonic map such that, for all $x \in M$, the image *of* $T_x^{1,0}M$ *under the complex linear extension of* $dL_u^{-1}df_x : T_xM \to T_oS(p,q) = \mathrm{sym}(p,q)$ *consists of commuting matrices, where* $u \in \mathrm{GL}(r)$ *is any element such that* $f(x) = u \cdot o$. *Then there exists a metric tt*-bundle* (E, D, S, g) *over* M *and a parallel frame* s *such that* $f = G^{(s)}$. *The condition on the image of* $T_x^{1,0}M$ *is automatically satisfied if* $pq = 0$, *which corresponds to a positive or negative definite metric* g.

The same correspondence holds for oriented unimodular tt-bundles and pluriharmonic maps into* $\mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \cong \mathrm{SL}(r)/\mathrm{SO}(p,q) = S_1(p,q)$.

Now we shall explain in more detail the condition on the image of $T^{1,0}M$ under the differential of f in the theorem. Above we have always identified $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ with $S(p,q)$. Let us denote by

$$\varphi : \mathrm{Sym}_{p,q}(\mathbb{R}^r) \to S(p,q)\,, \quad S \mapsto \tilde{S} = \varphi(S), \tag{1.22}$$

that identification, which is $\mathrm{GL}(r)$-equivariant and maps $I = I_{p,q}$ to the canonical base point o. We can identify the tangent space $T_S\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ at $S \in \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric matrices:

$$T_S\mathrm{Sym}_{p,q}(\mathbb{R}^r) = \mathrm{Sym}(\mathbb{R}^r) := \{A \in \mathrm{Mat}(r,\mathbb{R}) | A^t = A\}. \tag{1.23}$$

As above for $S = I$, the tangent space $T_{\tilde{S}}S(p,q)$ is canonically identified with the vector space of S-symmetric matrices:

$$T_{\tilde{S}}S(p,q) = \mathrm{sym}(S) := \{A \in \mathfrak{gl}\,(r) | A^tS = SA\}. \tag{1.24}$$

Note that $\mathrm{sym}(I_{p,q}) = \mathrm{sym}(p,q)$.

Proposition 4. *The differential of* φ *at* $S \in \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ *is given by*

$$\mathrm{Sym}(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2}S^{-1}X \in S^{-1}\mathrm{Sym}(\mathbb{R}^r) = \mathrm{sym}(S). \tag{1.25}$$

Let us now consider the differential

$$df_x : T_xM \to \mathrm{Sym}(\mathbb{R}^r) \tag{1.26}$$

of $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ at $x \in M$ and the differential

$$d\tilde{f}_x : T_xM \to \mathrm{sym}(f(x)) \tag{1.27}$$

of $\tilde{f} = \varphi \circ f : M \to S(p,q)$. Then the condition on the image of the differential of f in the theorem is that

$$dL_u^{-1}d\tilde{f}(T_x^{1,0}M) \subset \mathrm{sym}(p,q) \otimes \mathbb{C} \quad \text{consists of commuting matrices,} \tag{1.28}$$

where $\tilde{f}(x) = uo$. This is equivalent to the condition that $d\tilde{f}(T_x^{1,0}M) \subset \mathrm{sym}(\tilde{f}(x)) \otimes \mathbb{C}$ consists of commuting matrices. This follows from the fact that

$$dL_u : T_oS(p,q) \to T_{uo}S(p,q) = T_{\tilde{f}(x)}S(p,q), \tag{1.29}$$

corresponds to

$$Ad_u : \mathrm{sym}(p, q) = \mathrm{sym}(I) \to \mathrm{sym}(u \cdot I) = \mathrm{sym}(\tilde{f}(x)), \tag{1.30}$$

and that the adjoint representation preserves the Lie bracket.

Finally, $d\tilde{f}_x = d\varphi\, df_x = -\frac{1}{2} f(x)^{-1} df_x$ and, therefore,

$$d\tilde{f}(T_x^{1,0}M) = f(x)^{-1} df_x(T_x^{1,0}M). \tag{1.31}$$

This shows that f satisfies the condition (1.28) if and only if the matrices $f(x)^{-1}df_x(Z)$ and $f(x)^{-1}df_x(W)$ commute for all $Z, W \in T_x^{1,0}M$. This is equivalent to

$$[f(x)^{-1}df_x(JX), f(x)^{-1}df_x(JY)] = [f(x)^{-1}df_x(X), f(x)^{-1}df_x(Y)], \tag{1.32}$$

for all $X, Y \in T_xM$.

2 Special complex and special Kähler manifolds

In this section we recall some basic results on special complex manifolds and special Kähler manifolds. For more detailed information the reader is referred to [ACD], see also [F].

Definition 3. A *special complex manifold* (M, J, ∇) is a complex manifold (M, J) endowed with a flat torsion-free connection ∇ (on the real tangent-bundle) such that ∇J is symmetric.

A *special Kähler manifold* (M, J, ∇, ω) is a special complex manifold (M, J, ∇), for which ω is J-invariant and ∇-parallel. The (pseudo)-Kähler-metric $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ is called the *special Kähler metric* of the special Kähler manifold (M, J, ∇, ω).

Given a complex manifold (M, J) with a flat connection ∇, we define its *conjugate connection* by

$$\nabla^J_X = \nabla_X - J\nabla_X J \text{ with } X \in TM. \tag{2.1}$$

On a special complex manifold (M, J, ∇) the connection ∇^J is torsion-free. In addition, one can introduce a torsion-free connection

$$D := \frac{1}{2}(\nabla + \nabla^J) = \nabla - S, \text{ where } S := \frac{1}{2}J\nabla J, \tag{2.2}$$

which satisfies $DJ = 0$, as follows from a short calculation.

In the case of a special Kähler manifold (M, J, ∇, ω) the connection D is the Levi-Civita connection of the special Kähler metric g and the endomorphism-field S anticommutes with the complex structure J, i.e.:

$$JS_X = -S_X J, \quad \text{for all} \quad X \in TM. \tag{2.3}$$

Now we explain part of the extrinsic construction of special Kähler-manifolds given in [ACD]. In order to do this, we consider the complex vector space $V = T^*\mathbb{C}^n =$

$\mathbb{C}^{2n}$ with canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ endowed with the standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$ and the standard real structure $\tau : V \to V$ with fixed points $V^\tau = T^*\mathbb{R}^n$. These define a Hermitian form $\gamma := \Omega(\cdot, \tau\cdot)$.

Let (M, J) be a complex manifold (M, J) of complex dimension n. We call a holomorphic immersion $\phi : M \to V$ nondegenerate (respectively Lagrangian) if $\phi^*\gamma$ is nondegenerate (respectively, if $\phi^*\Omega = 0$). If ϕ is nondegenerate it defines a, possibly indefinite, Kähler metric $g = Re\,\phi^*\gamma$ on the complex manifold (M, J) and the corresponding Kähler form $g(\cdot, J\cdot)$ is a $J-$invariant symplectic form.

The following theorem gives a description of simply connected special Kähler-manifolds in terms of the above data:

Theorem 2 ([ACD]). *Let (M, J, ∇, ω) be a simply connected special Kähler manifold of complex dimension n, then there exists a holomorphic nondegenerate Lagrangian immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the Kähler metric g, the connection ∇ and the symplectic form $\omega = 2\phi^*\left(\sum_{i=1}^n dx^i \wedge dy_i\right) = g(\cdot, J\cdot)$ on M. Moreover, ϕ is unique up to an affine tranformation of V preserving the complex symplectic form Ω and the real structure τ. The flat connection ∇ is completely determined by the condition $\nabla\phi^*dx^i = \nabla\phi^*dy_i = 0$, $i = 1, \ldots, n$, where $x^i = Re\,z^i$ and $y_i = Re\,w_i$.*

3 Special complex and special Kähler manifolds as solutions of the tt*-equations

Let (E, D, S) be a tt*-bundle over a complex manifold (M, J). We are now interested in the case $E = TM$. In that case it is natural to consider tt*-bundles for which the connection $D^\theta = D + (\cos\theta)S + (\sin\theta)S_J$ is torsion-free.

Definition 4. A tt*-bundle (TM, D, S) over a complex manifold (M, J) is called *special* if D^θ is torsion-free and special, i.e. $D^\theta J$ is symmetric for all θ.

Proposition 5. *A tt*-bundle (TM, D, S) is special if and only if D is torsion-free and DJ, S and S_J are symmetric.*

Proof. The torsion T^θ of D^θ is given by

$$T^\theta(X, Y) = T(X, Y) + \cos\theta(S_X Y - S_Y X) + \sin\theta(S_{JX} Y - S_{JY} X)\,, \qquad (3.1)$$

where T is the torsion of D. This shows that $T^\theta = 0$ for all θ if and only if $T = 0$ and S and S_J are symmetric. The equation,

$$(D^\theta_X J)Y = (D_X J)Y + \cos\theta[S_X, J]Y + \sin\theta[S_{JX}, J]Y, \qquad (3.2)$$

shows that $D^\theta J$ is symmetric if DJ, S and S_J are symmetric. Conversely, if $T^\theta = 0$ and $D^\theta J$ is symmetric, then, by the first part of the proof, S and S_J are symmetric and equation (3.2) shows that DJ is symmetric. □

Theorem 3. (i) *Let* (M, J, ∇) *be a special complex manifold. Put* $S := \frac{1}{2} J\nabla J$ *and* $D := \nabla - S$. *Then* (TM, D, S) *is a special tt*-bundle, which satisfies the additional conditions:*
a) $S_X J = -J S_X$ *for all* $X \in TM$ *and*
b) $DJ = 0$.
This defines a map Φ *from special complex manifolds to special tt*-bundles.*

(ii) *Let* (TM, D, S) *be a special tt*-bundle over a complex manifold* (M, J). *Then* $(M, J, \nabla := D + S)$ *is a special complex manifold. This defines a map* Ψ *from special tt*-bundles to special complex manifolds such that* $\Psi \circ \Phi = \mathrm{Id}$. *If* (TM, D, S) *is a special tt*-bundle satisfying the conditions a) and b) in (i), then* $\Phi(\Psi(TM, D, S)) = (TM, D, S)$.

(iii) *Let* (M, J, g, ∇) *be a special Kähler manifold with S and D defined as in (i). Then* (TM, D, S, g) *is a special metric tt*-bundle. This defines a map* Φ *from special Kähler manifolds to special metric tt*-bundles.*

(iv) *Let* (TM, D, S, g) *be a special metric tt*-bundle over a pseudo-Hermitian manifold* (M, J, g). *Then* $(M, J, \nabla := D + S, g)$ *is a special Kähler manifold. In particular, we have a map* Ψ *from special metric tt*-bundles over pseudo-Hermitian manifolds to special Kähler manifolds such that* $\Psi \circ \Phi = \mathrm{Id}$. *If* (TM, D, S, g) *is a special metric tt*-bundle satisfying the conditions a) and b) in (i), then* $\Phi(\Psi(TM, D, S, g)) = (TM, D, S, g)$.

(v) *Let* (TM, D, S, g) *be a metric tt*-bundle over a pseudo-Hermitian manifold* (M, J, g) *such that D is torsion-free. Then it is special if and only if* $(M, J, \nabla := D + S, g)$ *is a special Kähler manifold.*

Proof. (i) Let (M, J, ∇) be a special complex manifold with S and D defined as above. Then,

$$\nabla^\theta = e^{\theta J} \circ \nabla \circ e^{-\theta J}, \tag{3.3}$$

is a family of flat torsion-free special connections. Using $\nabla = D + S$ and (2.3) we can write

$$\nabla^\theta_X = D_X + e^{2\theta J} S_X. \tag{3.4}$$

The following calculation shows that $\nabla^\theta = D^{-2\theta}$, where D^θ is defined in (1.2):

$$\begin{aligned}\nabla^\theta_X - D_X &= e^{2\theta J} S_X = \cos(2\theta) S_X + \sin(2\theta) J S_X \\ &\overset{(*)}{=} \cos(-2\theta) S_X + \sin(-2\theta) S_{JX} = D^{-2\theta}_X - D_X, \quad X \in TM.\end{aligned}$$

At $(*)$ we have used that $JS_X = -S_{JX}$, which follows from

$$JS_X Y = JS_Y X = -S_Y JX = -S_{JX} Y, \quad X, Y \in T_p M. \tag{3.5}$$

Here we used the symmetry of S and (2.3). This shows that (TM, D, S) is a special tt*-bundle.

(ii) Let (TM, D, S) be a special tt*-bundle. This means that D^θ is flat, torsion-free and special. In particular, $\nabla = D + S = D^0$ is flat, torsion-free and special and (M, J, ∇) is a special complex manifold. It is clear that $\Psi \circ \Phi = \mathrm{Id}$.

Conversely, let (TM, D, S) be a special tt*-bundle such that $DJ = 0$ and $S_X J = -JS_X$ for all X. Then we can recover D and S from $\nabla = D + S$ by the formulas $S = \frac{1}{2}J\nabla J$ and $D = \nabla - S$. In fact, Let (TM, D', S') be an other special tt*-bundle over (M, J) such that $D'J = 0$ and $S'_X J = -JS'_X$ for all $X \in TM$ and $\nabla = D + S = D' + S'$. Then,

$$0 = D'_X J = \nabla_X J - [S'_X, J] = \nabla_X J + 2JS'_X, \tag{3.6}$$

for all $X \in TM$. This shows that $S'_X = \frac{1}{2}J\nabla_X J = S_X$ and $D' = \nabla - S' = \nabla - S = D$.

(iii) Let (M, J, g, ∇) be a special Kähler manifold with S and D defined as in (i). Then, by (i), (TM, D, S) is a special tt*-bundle and satifies a) and b). To prove that it is a *metric* tt*-bundle we have to check that $Dg = 0$ and that (1.3) is satisfied. Since $DJ = 0$, by b), the equation $Dg = 0$ is equivalent to the following claim:

Claim: The Kähler form ω is D-parallel; $D\omega = 0$.

In fact $\nabla\omega = 0$ and $S_X = \frac{1}{2}J\nabla_X J$, $X \in TM$, is the product of two anticommuting ω-skew-symmetric endomorphisms $A = \frac{1}{2}J$ and $B = \nabla_X J$. This implies that S_X is ω-skew-symmetric and, thus, $D\omega = 0$.

The endomorphism S_X is ω-skew-symmetric and anticommutes with J, by a). Therefore S_X is symmetric with respect to $g = \omega(J\cdot, \cdot)$.

(iv) Let (TM, D, S, g) be a special metric tt*-bundle over a pseudo-Hermitian manifold (M, J, g). Thanks to (i), we know already that $(M, J, \nabla := D + S)$ is a special complex manifold. Therefore it suffices to prove that $\nabla\omega = 0$. The assumption $Dg = 0$ and property b), which follows from (i), imply that $D\omega = 0$. Now it is sufficient to observe that the endomorphisms S_X, $X \in TM$, are ω-skew-symmetric. In fact, S_X is g-symmetric in virtue of (1.3) and anticommutes with J, by a). This shows that (M, J, ∇, g) is a special Kähler manifold. The remaining statements follow from (ii).

(v) Let (TM, D, S, g) be a metric tt*-bundle over a pseudo-Hermitian manifold (M, J, g) such that $(M, J, \nabla := D + S, g)$ is a special Kähler manifold. If D is torsion-free, then it is the Levi-Civita connection of g and, thus, $D = \nabla - \frac{1}{2}J\nabla J$, see section 2. Now we can conclude that $\Phi(M, J, \nabla, g) = (TM, D, S, g)$. This shows that (TM, D, S, g) is a special metric tt*-bundle. □

Corollary 1. Any special metric tt*-bundle (TM, D, S, g) over a pseudo-Hermitian manifold (M, J, g) is oriented and unimodular.

Proof. TM is canonically oriented by the complex structure J. By Theorem 3, $(M, J, g, \nabla = D + S)$ is a special Kähler manifold. Its Kähler form is parallel with respect to D and ∇ and hence invariant under $S_X = \nabla_X - D_X$ for all $X \in TM$. This shows that $\operatorname{tr} S_X = 0$. □

In [H] special complex and special Kähler geometry is interpreted in terms of variations of Hodge structure of weight 1 on the complexified tangent bundle. From this interpretation and his discussion of tt*-geometry, it follows that any special complex (respectively, special Kähler) manifold defines a tt*-bundle (respectively, a metric tt*-bundle) in the sense of our definition.

4 The pluriharmonic map in the case of a special Kähler manifold

4.1 The Gauss maps of a special Kähler manifold

Let (M, J, g, ∇) be a special Kähler manifold of complex dimension $n = k + l$ and of Hermitian signature (k, l), i.e. g has signature $(2k, 2l)$. Let $(\widetilde{M}, J, g, \nabla)$ be its universal covering with the pullback special Kähler structure, which is again denoted by (J, g, ∇). According to Theorem 2, there exists a (holomorphic) Kählerian–Lagrangian immersion $\phi : \widetilde{M} \to V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$, which is unique up to a complex affine transformation of V with linear part in $\mathrm{Sp}(\mathbb{R}^{2n})$. We consider the *dual Gauss map* of ϕ

$$L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\widetilde{M} := d\phi_p T_p \widetilde{M} \subset V \tag{4.1}$$

into the Grassmannian of complex Lagrangian subspaces $W \subset V$ of *signature (k,l)*, i.e. such that the restriction of γ to W is a Hermitian form of signature (k, l). The map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ is in fact the dual of the *Gauss map*

$$L^\perp : \widetilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n}), \quad p \mapsto L(p)^\perp = \overline{L(p)} \cong L(p)^*. \tag{4.2}$$

Here $L(p)^\perp$ stands for the γ-orthogonal complement of $L(p)$ and the isomorphism $\overline{L(p)} \cong L(p)^*$ is induced by the symplectic form Ω on $V = L(p) \oplus \overline{L(p)}$.

The Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open subset of the complex Grassmannian $Gr_0(\mathbb{C}^{2n})$ of complex Lagrangian subspaces $W \subset V$ and hence a complex submanifold.

Proposition 6. i) *The dual Gauss map* $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ *is holomorphic*

ii) *The Gauss map* $L^\perp : \widetilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n})$ *is antiholomorphic.*

Proof. The holomorphicity of L follows from that of ϕ. Part (ii) follows from (i), since $L^\perp = \overline{L} : p \mapsto \overline{L(p)}$. □

The real symplectic group $\mathrm{Sp}(\mathbb{R}^{2n})$ acts transitively on $Gr_0^{k,l}(\mathbb{C}^{2n})$ and we have the following identification:

$$Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k, l). \tag{4.3}$$

Here $\mathrm{U}(k, l) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ is defined as the stabilizer of

$$W_o = \mathrm{span}\left\{\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial z^k} + i\frac{\partial}{\partial w_k}, \frac{\partial}{\partial z^{k+1}} - i\frac{\partial}{\partial w_{k+1}}, \cdots, \frac{\partial}{\partial z^n} - i\frac{\partial}{\partial w_n}\right\}. \tag{4.4}$$

The Gauss maps L and $L^\perp$ induce Gauss maps,

$$L_M : M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n}), \tag{4.5}$$

$$L_M^\perp : M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n}), \tag{4.6}$$

into the quotient of the Grassmannian by the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ of the flat symplectic connection ∇.

Corollary 2. (i) The dual Gauss map $L_M : M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ of M is holomorphic.
(ii) The Gauss map $L_M^{\perp} : M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n})$ is antiholomorphic.

The Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ is a pseudo-Hermitian symmetric space and, in particular, a homogeneous pseudo-Kähler manifold. If $\Gamma \subset \mathrm{Sp}(\mathbb{R}^{2n})$ acts properly discontinuously on $Gr_0^{k,l}(\mathbb{C}^{2n})$ then $\Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ is a locally symmetric space of pseudo-Hermitian type.

4.2 Holomorphic coordinates on the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ of complex Lagrangian subspaces of signature (k, l)

In this section we shall introduce a local model for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ and determine the corresponding local expression for the dual Gauss map. This model is a pseudo-Riemannian analogue of the Siegel upper half-space

$$\mathrm{Sym}^{+}(\mathbb{C}^n) := \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A \text{ and } \mathrm{Im}\, A \text{ is positive definite}\}. \tag{4.7}$$

Our aim is to construct holomorphic coordinates for the complex manifold $Gr_0^{k,l}(\mathbb{C}^{2n})$ in a Zariski-open neighborhood of a point W_0 of the Grassmannian represented by a Lagrangian subspace $W_0 \subset V$ of signature (k, l). Using a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$ we can assume that $W_0 = W_o$, see (4.4). Let $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ be the open subset consisting of $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ such that the projection,

$$\pi_{(z)} : V = T^*\mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^*\mathbb{C}^n \tag{4.8}$$

onto the first summand (z-space) induces an isomorphism,

$$\pi_{(z)}|_W : W \xrightarrow{\sim} \mathbb{C}^n. \tag{4.9}$$

Notice that $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open neighborhood of the base point W_0. For elements $W \in U_0$ we can express w_i as a function of $z = (z^1, \dots, z^n)$. In fact,

$$w_i = \sum C_{ij} z^j, \tag{4.10}$$

where

$$\begin{aligned} (C_{ij}) \in \mathrm{Sym}_{k,l}(\mathbb{C}^n) = \\ \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A \quad \text{and } \mathrm{Im}\, A \text{ has signature } (k, l)\}. \end{aligned} \tag{4.11}$$

Proposition 7. *The map*

$$C : U_0 \to \mathrm{Sym}_{k,l}(\mathbb{C}^n)\,,\; W \mapsto C(W) := (C_{ij}) \tag{4.12}$$

is a local holomorphic chart for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$.

Remark. The open subset $\mathrm{Sym}_{k,l}(\mathbb{C}^n) \subset \mathrm{Sym}(\mathbb{C}^n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) | A^t = A\}$ is a generalization of the famous Siegel upper half-space $\mathrm{Sym}_{n,0}(\mathbb{C}^n) = \mathrm{Sym}^+(\mathbb{C}^n)$, which is a Siegel domain of type I. In the latter case, we have $U_0 = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n)$ and a global coordinate chart

$$C : Gr_0^{n,0}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n) \xrightarrow{\sim} \mathrm{Sym}_{n,0}(\mathbb{C}^n). \tag{4.13}$$

We shall now describe the dual Gauss map L in local holomorphic coordinates in neighborhoods of $p_0 \in \widetilde{M}$ and $L(p_0) \in Gr_0^{k,l}(\mathbb{C}^{2n})$. Applying a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$, if necessary, we can assume that $L(p_0) \in U_0$. We put $U := L^{-1}(U_0)$. The open subset $U \subset \widetilde{M}$ is a neighborhood of p_0.

Let $\phi : \widetilde{M} \to T^*\mathbb{C}^n$ be the Kählerian–Lagrangian immersion. It defines a system of local (special) holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi|_U : U \xrightarrow{\sim} U' \subset \mathbb{C}^n, \quad p \mapsto (z^1(\phi(p)), \cdots, z^n(\phi(p))), \tag{4.14}$$

and we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \mathrm{Sym}_{k,l}(\mathbb{C}^n)\,, \end{array} \tag{4.15}$$

where the vertical arrows are holomorphic diffeomorphisms and L_U at $z = (z^1, \dots z^n)$ is given by

$$L_U(z) = (F_{ij}(z)) := \left(\frac{\partial^2 F(z)}{\partial z^i \partial z^j} \right). \tag{4.16}$$

Here $F = F(z)$ is a holomorphic function on $U' \subset \mathbb{C}^n$ determined, up to a constant, by the equations

$$w_j(\phi(p)) = \left. \frac{\partial F}{\partial z^j} \right|_{z(\phi(p))}. \tag{4.17}$$

Summarizing, we obtain the following proposition.

Proposition 8. *The dual Gauss map L has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij}), \tag{4.18}$$

where $\varphi : U \to \mathbb{C}^n$ is the (special) holomorphic chart of $\widetilde{M}$ associated to the Kählerian–Lagrangian immersion ϕ, see (4.14), and $C : U_0 \to \mathrm{Sym}(\mathbb{C}^n)$ is the holomorphic chart of $Gr_0^{k,l}(\mathbb{C}^{2n})$ constructed in (4.12).

4.3 The special Kähler metric in affine coordinates

As before, let (M, J, g, ∇) be a special Kähler manifold of Hermitian signature (k, l), $k + l = n = \dim_{\mathbb{C}} M$, and $(\widetilde{M}, J, g, \nabla)$ its universal covering. As in section 1, we shall now consider the metric g in a ∇-parallel frame. Such a frame is provided by the Kählerian–Lagrangian immersion $\phi : \widetilde{M} \to V$. In fact, any point $p \in \widetilde{M}$ has a neighborhood in which the functions $\widetilde{x}^i := \mathrm{Re}\, z^i \circ \phi$, $\widetilde{y}_i := \mathrm{Re}\, w_i \circ \phi$, $i = 1, \ldots, n$, form a system of local ∇-affine coordinates. We recall that the ∇-parallel Kähler form is given by $\omega = 2 \sum d\widetilde{x}^i \wedge d\widetilde{y}_i$. This implies that the globally defined one-forms $\sqrt{2} d\widetilde{x}^i$, $\sqrt{2} d\widetilde{y}_i$ constitute a ∇-parallel unimodular frame,

$$(e^a)_{a=1,\ldots,2n} = (e^1, \ldots, e^{2n}) := (\sqrt{2} d\widetilde{x}^1, \ldots, \sqrt{2} d\widetilde{x}^n, \sqrt{2} d\widetilde{y}_1, \ldots, \sqrt{2} d\widetilde{y}_n), \tag{4.19}$$

of $T^*\widetilde{M}$ with respect to the metric volume form $\nu = (-1)^{n+1} \frac{\omega^n}{n!} = 2^n d\widetilde{x}^1 \wedge \ldots \wedge d\widetilde{y}_n$. The dual frame (e_a) of $T\widetilde{M}$ is also ∇-parallel and unimodular. The metric defines a smooth map,

$$G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \{A \in \mathrm{Mat}(2n, \mathbb{R}) | A^t = A\,,\ \det A = 1 \text{ has signature } (2k, 2l)\}, \tag{4.20}$$

by

$$p \mapsto G(p) := (g_{ab}(p)) := (g_p(e_a, e_b)). \tag{4.21}$$

We call $G = (g_{ab})$ *the fundamental matrix* of ϕ. As before, we identify

$$\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l). \tag{4.22}$$

This is a pseudo-Riemannian symmetric space. For conventional reasons, in this section, $\mathrm{SO}(2k, 2l) \subset \mathrm{SL}(2n, \mathbb{R})$ is defined as the stabilizer of the symmetric matrix

$$E_o := \mathrm{diag}(\mathbf{1}_k, -\mathbf{1}_l, \mathbf{1}_k, -\mathbf{1}_l). \tag{4.23}$$

The fundamental matrix induces a map,

$$G_M : M \to \Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}), \tag{4.24}$$

into the quotient of $\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ by the action of the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \subset \mathrm{SL}(2n, \mathbb{R})$. The target $\Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ is a pseudo-Riemannian locally symmetric space, provided that Γ acts properly discontinuously.

Theorem 4. *The fundamental matrix,*

$$G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l),$$

takes values in the totally geodesic submanifold,

$$\iota : Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k, l) \hookrightarrow \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l), \tag{4.25}$$

and coincides with the dual Gauss map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$*:* $G = \iota \circ L$.

Proof. The proof follows from a geometric description of the inclusion ι. With any Lagrangian subspace $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ we can associate the scalar product $g^W :=$ $\mathrm{Re}\,\gamma|_W$ of signature $(2k, 2l)$ on $W \subset V$. The projection onto the real points,

$$\mathrm{Re} : V = T^*\mathbb{C}^n \to T^*\mathbb{R}^n = \mathbb{R}^{2n}\,, \quad v \mapsto \mathrm{Re}\, v = \frac{1}{2}(v + \overline{v}), \tag{4.26}$$

induces an isomorphism of real vector spaces $W \xrightarrow{\sim} \mathbb{R}^{2n}$ the inverse of which we denote by $\psi = \psi_W$. We claim that

$$\iota(W) = \psi^* g^W =: (g^W_{ab}) =: G^W. \tag{4.27}$$

To check this, it is sufficient to prove that the map

$$Gr_0^{k,l}(\mathbb{C}^{2n}) \ni W \mapsto G^W \in \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{4.28}$$

is $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariant and maps the base point W_o with stabilizer $\mathrm{U}(k,l)$, see (4.4), to the base point E_o with stabilizer $\mathrm{SO}(2k, 2l)$, see (4.23). Let us verify that indeed $G^{W_o} = E_o$.

Using the definition of γ, one finds for the basis

$$(e_j^{\pm}) := \left(\frac{\partial}{\partial z^j} \pm i \frac{\partial}{\partial w_j} \right) \tag{4.29}$$

of V that the only non-vanishing components of γ are $\gamma(e_j^{\pm}, e_j^{\pm}) = \pm 2$. This shows that $g^{W_o} = \mathrm{Re}\,\gamma|_{W_o}$ is represented by the matrix $2E_o$ with respect to the basis,

$$(e_1^+, \ldots, e_k^+, e_1^-, \ldots, e_l^-, ie_1^+, \ldots, ie_k^+, ie_1^-, \ldots, ie_l^-). \tag{4.30}$$

In order to calculate $G^{W_o} = (g^{W_o}_{ab}) = (g(\psi e_a, \psi e_b))$, we need to pass from the real basis (4.30) of W_o to the real basis (ψe_a).

Recall that the real structure τ is complex conjugate with respect to the coordinates (z^i, w_i). This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(ie_j^+) = -\frac{\partial}{\partial y_j} = -\sqrt{2}e_{n+j}\,, \;\; j = 1, \ldots, k,$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j\,, \quad \psi^{-1}(ie_j^-) = \frac{\partial}{\partial y_j} = \sqrt{2}e_{n+j}\,, \quad j = 1, \ldots, l.$$

This shows that $G^{W_o} = E_o$.

It remains to check the equivariance of $W \mapsto G^W = \psi^* g$. Using the definition of the map $\psi = \psi_W : \mathbb{R}^{2n} \to W$, one easily checks that, under the action of $\Lambda \in \mathrm{Sp}(\mathbb{R}^{2n})$, ψ transforms as

$$\psi_{\Lambda W} = \Lambda \circ \psi_W \circ \Lambda^{-1}|_{\mathbb{R}^{2n}}. \tag{4.31}$$

From this we deduce the transformation law of G^W:

$$G^{\Lambda W} = \psi^*_{\Lambda W} g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W.$$

The above claim (4.27), together with the fact that

$$g^{L(p)} = g_p \quad \text{and} \quad G^{L(p)} = G(p), \tag{4.32}$$

for all $p \in \widetilde{M}$, implies that

$$\iota(L(p)) = G^{L(p)} = G(p). \tag{4.33}$$

□

Corollary 3. The fundamental matrix $G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ is pluriharmonic.

Proof. $G = \iota \circ L$ is the composition of the holomorphic map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ with the totally geodesic inclusion $Gr_0^{k,l}(\mathbb{C}^{2n}) \subset \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$. The composition of a holomorphic map with a totally geodesic map is pluriharmonic. □

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$\mathbb{Z}_2$ and $\mathbb{Z}$-Deformation Theory for Holomorphic and Symplectic Manifolds*

Paolo de Bartolomeis

Dipartimento di Matematica Applicata G. Sansone, Università di Firenze
Via S. Marta 3, 50139 Firenze, Italy
deba@poincare.dma.unifi.it

Dedicated to Professor Lieven Vanhecke

Summary. We present and investigate, within the general frame of deformation theory, new $\mathbb{Z}_2$-constructions for generalized moduli spaces of holomorphic and symplectic structures.

1 Introduction

Deformation theories are one of the keystone settings of contemporary geometry, appearing in very different areas and providing, through moduli space constructions, a highly powerful tool to produce new invariants (cf. [6] and [7] for recent accounts).

This paper, in the first part, describes a tuning up of a general machine for deformation theory, enhancing the relationships between $\mathbb{Z}$ and $\mathbb{Z}_2$-theories. Then, after presenting equivalence classes of A^∞-algebras as an example of deformation space, to show how vast the range covered by deformation theories is, it deals with complex/holomorphic deformations and symplectic deformation. In the latter case, a totally new non-naïf theory is constructed.

By means of the results established in the first part, both in the complex/holomorphic case and the symplectic case, we define and discuss the corresponding $\mathbb{Z}_2$-theories (complex/holomorphic and supersymplectic structures).

2 $\mathbb{Z}_2$-theory and $\mathbb{Z}$-theory of deformations of DLA

2.1 $\mathbb{Z}_2$-theory: superstructures

We start with a quick overview of superstructures (or $\mathbb{Z}_2$-structures).

Definition 1. 1. A *super vector space* is a vector space V together with a decomposition

$$V = V^{(0)} \oplus V^{(1)}.$$

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Vectors in $V^{(0)}$ and in $V^{(1)}$ are called *homogeneous* of *degree* (or *parity*) 0 and 1, respectively. The degree of the homogeneous vector v is denoted by $|v|$.
A vector supersubspace of V is a subspace $W \subset V$ of the form

$$W = W^{(0)} \oplus W^{(1)},$$

where $W^{(j)}$ is a vector subspace of $V^{(j)}$, $j = 0, 1$.

2. A *super algebra* is an algebra A together with a vector space decomposition

$$A = A^{(0)} \oplus A^{(1)},$$

in such a way that

$$A^{(j)} A^{(k)} \subset A^{(j+k)}, \quad j, k \in \mathbb{Z}_2.$$

The bracket [,], defined on homogeneous elements as

$$[a, b] := ab - (-1)^{|a||b|} ba,$$

is called the *super commutator* of A and A is said to be *supercommutative* if its super commutator vanishes identically.

3. A *super Lie algebra* is a super vector space $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ together with a bilinear map,

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

such that
 a) $[\mathfrak{g}^{(j)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(j+k)}$, $j, k \in \mathbb{Z}_2$
 b) for homogeneous elements a, b, c, we have:
  i. $[a, b] = -(-1)^{|a||b|}[b, a]$,
  ii. $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

Note that:

- given a vector space V, the exterior algebra $\wedge^* V$ has a natural structure of super-commutative super algebra, just setting

$$V^{(0)} = \wedge^{\text{even}} V \ , \ V^{(1)} = \wedge^{\text{odd}} V;$$

- given a super vector space $V = V^{(0)} \oplus V^{(1)}$ with projections p_1 and p_2, then $End(V)$ has a natural structure of super algebra, just setting

$$End(V)^{(0)} := \{f \in End(V) \mid f(V^{(j)}) \subset V^{(j)} \ , \ j \in \mathbb{Z}_2\},$$

$$End(V)^{(1)} := \{f \in End(V) \mid f(V^{(j)}) \subset V^{(j+1)} \ , \ j \in \mathbb{Z}_2\},$$

the relation,

$$f = (p_1 \circ f \circ p_1 + p_2 \circ f \circ p_2) + (p_1 \circ f \circ p_2 + p_2 \circ f \circ p_1),$$

proves that

$$End(V) = End(V)^{(0)} \oplus End(V)^{(1)};$$

- given a super algebra A, the super commutator [,] defines on A the structure of super Lie algebra.

Definition 2. Let

$$A = A^{(0)} \oplus A^{(1)}$$

be a super algebra.

1. A *super derivation* D of degree $|D|$ on A is an element of $End(A)^{(|D|)}$ such that

   $$D(ab) = (Da)b + (-1)^{|D||a|}a(Db).$$

   This amounts to say that, for every $a \in A$, we have

   $$[D, L_a] - L_{Da} = 0,$$

   L_a being the left multiplication by a.
2. A *differential* d on A is a super derivation of degree 1 such that $d^2 = 0$. The couple (A, d) is called a *differential super algebra* (DSA).

Definition 3. Let

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$$

be a super Lie algebra.

1. A *super derivation* D of degree $|D|$ on $\mathfrak{g}$ is an element of $End(\mathfrak{g})^{(|D|)}$ such that:

   $$D[a, b] = [Da, b] + (-1)^{|D||a|}[a, Db].$$

2. A *differential* d on $\mathfrak{g}$ is a super derivation of degree 1 such that $d^2 = 0$. The couple $(\mathfrak{g}, d)$ is called a *differential super Lie algebra* (DSLA).

Note that, in both cases, the super commutator of the super derivations D and F is a super derivation of degree $|D| + |F|$.

Let $(\mathfrak{g}, [\,,\,], d)$ be a DSLA. We set

$$Z^{(p)} = Z^{(p)}(\mathfrak{g}, d) := \{a \in \mathfrak{g}^{(p)} \,|\, da = 0\}\ ,\ \ Z = Z^{(0)} \oplus Z^{(1)},$$

$$B^{(p)} = B^{(p)}(\mathfrak{g}, d) := d(\mathfrak{g}^{(p-1)})\ ,\ \ B = B^{(0)} \oplus B^{(1)},$$

$$H^{(p)} = H^{(p)}(\mathfrak{g}, d) = Z^{(p)}(\mathfrak{g}, d)/B^{(p)}(\mathfrak{g}, d)\ ,\ \ H = H^{(0)} \oplus H^{(1)}.$$

Let $(\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, [\,,\,], d)$ be a DSLA. For $\gamma \in \mathfrak{g}^{(1)}$, set

$$d_\gamma a \ := \ da + [\gamma, a].$$

Clearly,

$$d_\gamma[a, b] \ = \ [d_\gamma a, b] + (-1)^{|a|}[a, d_\gamma b]$$

and

$$d\gamma \ + \ \frac{1}{2}[\gamma, \gamma] \ = \ 0 \Longrightarrow d_\gamma^2 \ = \ 0,$$

i.e.,

$$\gamma \quad \text{satisfies the Maurer-Cartan (MC) equation} \Longrightarrow d_\gamma^2 \ = \ 0.$$

Set

$$\mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g}) = \{\gamma \in \mathfrak{g}^{(1)} \mid \gamma \text{ satisfies } MC\}.$$

Given $\alpha \in \mathfrak{g}$, set

$$ad(\exp(\alpha)) := \sum_{h=0}^{\infty} \frac{1}{h!}(\alpha\delta(\alpha))^h$$

(where, of course, $\alpha\delta(\alpha)(\epsilon) := [\alpha, \epsilon]$). Therefore $\exp(\mathfrak{g})$ acts on the left on $\mathfrak{g}$. $\exp(\mathfrak{g}^{(0)})$ acts on the d_γ's on the left as

$$d_\gamma \mapsto ad(\exp(\alpha))d_\gamma ad(\exp(-\alpha))$$

and this induces a left action of $\exp(\mathfrak{g}^{(0)})$ on $\mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})$ given by

$$(\alpha, \gamma) \mapsto \chi(\alpha)\gamma := \gamma - \sum_{h=0}^{\infty} \frac{1}{(h+1)!}(\alpha\delta(\alpha))^h (d_\gamma \alpha).$$

As usual, the results of the present section hold in the framework of formal power series, i.e., modulo convergence. Convergence can be rigorously established in the class of Artin rings and their projective limits.

Set

$$\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) := \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})/\exp(\mathfrak{g}^{(0)}).$$

Definition 4. $\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g})$ is called the $\mathbb{Z}_2$-deformation space of the DSLA $\mathfrak{g}$.

Note that:

- if $t \mapsto \gamma(t)$ is a smooth curve in $\mathfrak{MC}(\mathfrak{g})$, with $\gamma(0) = \gamma$, then,

  $$d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$$

  and so

  $$0 = d\gamma\prime(0) + [\gamma, \gamma\prime(0)] = d_\gamma \gamma\prime(0).$$

  Consequently,

  $$T_\gamma \mathfrak{MC}(\mathfrak{g}) \subset Z^{(1)}(\mathfrak{g}, d_\gamma).$$

- $$\frac{d}{dt}\chi(t\alpha)\gamma_{|t=0} = -d_\gamma \alpha$$

  and so,

  $$\gamma \mapsto -d_\gamma \alpha,$$

  represents the fundamental vector field of α associated to the given action.

- Setting

$$\hat{\gamma}(t) := \chi(t\alpha)\gamma(t),$$

we get

$$\hat{\gamma}'(0) = \gamma'(0) - d_\gamma \alpha.$$

Consequently, if $\epsilon \in Z^{(1)}(\mathfrak{g}, d_\gamma)$ is tangent to $\mathfrak{MC}(\mathfrak{g})$ at γ, then any element of $[\epsilon] \in H^{(1)}(\mathfrak{g}, d_\gamma)$ is tangent to $\mathfrak{MC}(\mathfrak{g})$ at γ and is related to ϵ by the action induced by χ. Therefore, if $\langle\gamma\rangle \in \mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g})$ then

$$T_{\langle\gamma\rangle}\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) \subset H^{(1)}(\mathfrak{g}, d_\gamma).$$

We set the following

Definition 5. If

$$T_{\langle\gamma\rangle}\mathrm{Def}_{\mathbb{Z}_2}(\mathfrak{g}) = H^{(1)}(\mathfrak{g}, d_\gamma),$$

we say that the deformation theory of the DSLA $\mathfrak{g}$ is totally unobstructed at $\langle\gamma\rangle$.

We are mainly interested, as we shall see, in infinitesimal deformations at 0 or, more precisely, formal developments of deformations.

2.2 $\mathbb{Z}$-theory

As a special case of superstructures we have $\mathbb{Z}$-graded structures.

Definition 6. A *graded vector space* is a vector space V together with a decomposition,

$$V = \bigoplus_{p\in\mathbb{Z}} V_p,$$

with the agreement that $V_p = \{0\}$, if $p < 0$; again vectors in the V_p's are called *homogeneous* and they are assigned to have *degree* p. In the same way, we can consider graded algebras, graded Lie algebras, differential graded algebras (DGA), differential graded Lie algebras (DGLA) etc., with the same definitions as before (indices in $\mathbb{Z}$).

In particular, if

$$\left(\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p,\ [\ ,\ ], d\right)$$

is a DGLA, we set

$$\mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g}) = \{\gamma \in \mathfrak{g}_1 \mid \gamma \text{ satisfies } MC\}.$$

Then, exactly as the $\mathbb{Z}_2$-case, we have a left action of $\exp(\mathfrak{g}_0)$ on $\mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$ and we set

$$\mathrm{Def}_{\mathbb{Z}}(\mathfrak{g}) := \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})/\exp(\mathfrak{g}_0).$$

Note that any graded structure has a natural underlying superstructure.

2.3 Formal deformations

We want to describe the basic setting of formal deformations. Let $(\mathfrak{g}, [\ ,\ ], d)$ be a DSLA and let H be its cohomology. Let H^* be the super vector space dual of H and let $K := \Bbbk[[H^*]]$ be the completed supersymmetric algebra of H^*.

In particular, if $\{v_1, \ldots, v_N\}$ is a super basis of H, the dual superbasis $\{x_1, \ldots, x_N\}$ satisfies $|x_j| = |v_j| - 1\ ,\ 1 \leq j \leq N$.

Set

$$\mathfrak{g}_K := \mathfrak{g} \otimes K\ ,\ d_K := d \otimes 1 \text{ etc.},$$

extend the structure of DSLA to $\mathfrak{g}_K$ in the standard way, i.e.,

- $[a \otimes \alpha,\ b \otimes \beta] = (-1)^{|\alpha||b|}[a,\ b] \otimes \alpha\beta$,
- $|a \otimes \alpha| = |a||\alpha|$.

Finally, let $\mathfrak{m}_K$ be the maximal ideal of K.
Note that

- $\mathfrak{g} \otimes \mathfrak{m}_K$ an ideal (and hence a subalgebra) of $\mathfrak{g}_K$;
- $\omega \in \mathfrak{g}_K$ can be written as

$$\omega = \sum_{j=0}^{\infty} \omega_j,$$

  where the ω_j's are homogeneous polynomials of degree j in the H^*-variables;
- $v_h \mapsto v_h x_h\ ,\ 1 \leq h \leq N\ ,$ identifies H with a degree-one homogeneous polynomial in $(H \otimes \mathfrak{m}_K)^{(1)}$.

Set

$$\mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] := \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g} \otimes \mathfrak{m}_K) = \left\{\gamma \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(1)}\ |\ d_K\gamma + \frac{1}{2}[\gamma,\ \gamma] = 0\right\},$$

$$\mathrm{Def}_{\mathbb{Z}_2}[[\mathfrak{g}]] := \mathrm{Def}(\mathfrak{g} \otimes \mathfrak{m}_K).$$

Definition 7. We say that the deformation theory of the DSLA $\mathfrak{g}$ is formally totally unobstructed (at $\langle\gamma\rangle$), if the deformation theory of $\mathfrak{g} \otimes \mathfrak{m}_K$ is totally unobstructed at $\langle\gamma\rangle$.

2.4 $\mathbb{Z}$-theory versus $\mathbb{Z}_2$-theory

It is a very interesting fact that $\mathbb{Z}_2$-deformation theory fibers in a natural manner over $\mathbb{Z}$-deformation theory.

In fact, let $(\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p,\ [\ ,\ ], d)$ be a DGLA, let $\pi_j\ :\ \mathfrak{g} \longrightarrow \mathfrak{g}_j\ ,\ j \in \mathbb{Z}$ be the natural projections, and let $\tilde{\mathfrak{g}} := \bigoplus_{j>1} \mathfrak{g}_j$; consider on $\mathfrak{g}$ the underlying structure of DSLA. Then we have:

Lemma 1. $\pi_1\ :\ \mathfrak{g} \longrightarrow \mathfrak{g}_1$ *induces a surjective map*

$$\pi\ :\ \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g}) \longrightarrow \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$$

such that:

1. *for every* $\gamma_1 \in \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g})$,

$$\pi^{-1}(\gamma_1) = \gamma_1 + \mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}}, d_{\gamma_1});$$

2. *for every* $\alpha \in \mathfrak{g}^{(0)}$,

$$\pi_1 \circ \chi(\alpha) = \chi(\pi_0(\alpha)) \circ \pi_1,$$

and thus we obtain a surjective map

$$\pi : Def_{\mathbb{Z}_2}(\mathfrak{g}) \longrightarrow Def_{\mathbb{Z}}(\mathfrak{g})$$

and

$$\pi^{-1}(\langle\gamma_1\rangle) \approx Def_{\mathbb{Z}_2}(\tilde{\mathfrak{g}}, d_{\gamma_1}).$$

Proof. Let $\gamma \in \mathfrak{MC}_{\mathbb{Z}_2}(\mathfrak{g})$. Write $\gamma = \gamma_1 + \sigma$ with $\gamma_1 = \pi_1(\gamma)$ and $\sigma \in \tilde{\mathfrak{g}} \cap \mathfrak{g}^{(1)}$. Then,

$$\begin{aligned} d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \\ &= d\gamma_1 + d\sigma + \frac{1}{2}[\gamma_1, \gamma_1] + \frac{1}{2}[\sigma, , \sigma] + [\gamma_1 , \sigma] \\ &= d\gamma_1 + \frac{1}{2}[\gamma_1, \gamma_1] + d_{\gamma_1}\sigma + \frac{1}{2}[\sigma, , \sigma], \end{aligned}$$

and thus,

$$\pi_1(\gamma) \in \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g}), \qquad \sigma \in \mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}}).$$

This gives the surjectivity and 1. at once. 2. is now obvious.

At formal level, we have:

$$\mathfrak{g} \otimes \mathfrak{m}_K = \bigoplus_{p \in \{\mathbb{Z}} (\mathfrak{g} \otimes \mathfrak{m}_K)_p,$$

where

$$(\mathfrak{g} \otimes (\mathfrak{m}_K))_p = \bigoplus_{r+s=p} (\mathfrak{g})_r \otimes (\mathfrak{m}_K)_s.$$

In particular,

$$(\mathfrak{g} \otimes (\mathfrak{m}_K)_1 = \mathfrak{g}_0 \otimes (\mathfrak{m}_K)_1 \mathfrak{g}_1 \otimes \mathfrak{m}_K)_0,$$

and

$$(\mathfrak{m}_K)_0 = \mathfrak{m}_{\hat{K}},$$

where

$$\hat{K} = \mathbb{k}[[x_1, \ldots, x_n]] \qquad \text{with} \qquad n = dim_{\mathbb{k}} H^1.$$

Therefore, we have a further reduction; the results are summarized in the following lemma, which can be proved exactly as the previous one.

Lemma 2. *Let*

$$p \,:\, (\mathfrak{g} \otimes \mathfrak{m}_K)_1 \longrightarrow \mathfrak{g}_1(\otimes \mathfrak{m}_K)_0$$

be the natural projection. Then

1. p *induces a surjective map:*

$$p \,:\, \mathfrak{MC}_{\mathbb{Z}}(\mathfrak{g} \otimes \mathfrak{m}_K) \longrightarrow \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]] := \left\{ \alpha \in \mathfrak{g}_1 \otimes \mathfrak{m}_K)_0 \mid d\alpha + \frac{1}{2}[\alpha,\, \alpha] = 0 \right\};$$

2. *for every* $\gamma_1 \in \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]$,

$$p^{-1}(\gamma_1) = \gamma_1 + \mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g}, d_{\gamma_1}]] := \left\{ \alpha \in \mathfrak{g}_0 \otimes \mathfrak{m}_K)_1 \mid d_{\gamma_1}\alpha + \frac{1}{2}[\alpha,\, \alpha] = 0 \right\};$$

3. p *is* $\exp(\mathfrak{g}_0 \otimes (\mathfrak{m}_K)_0$*-invariant;*
4. *setting* $\tilde{\pi} := p \circ \pi_1$ *we have that, for every* $\gamma_1 \in \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]$,

$$\tilde{\pi}_1(\gamma_1) = \gamma_1 + \mathfrak{F}(\gamma_1),$$

where

$$\mathfrak{F}(\gamma_1) = \{(\beta,\, \sigma) \mid \beta \in \mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g}, d_{\gamma_1}]],\ \sigma \in \mathfrak{MC}_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\, d_{\gamma_1+\beta})\};$$

5. *we obtain a surjective map,*

$$\tilde{\pi} \,:\, Def_{\mathbb{Z}_2}[[\mathfrak{g}]] \longrightarrow Def_{\mathbb{Z}}[[\mathfrak{g}]] := \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]]/\exp(\mathfrak{g}_0 \otimes (\mathfrak{m}_K)_0),$$

and $\tilde{\pi}^{-1}(\langle\gamma_1\rangle) \approx \{(\langle\beta\rangle,\, \langle\sigma\rangle)\}$, *where*

$$\langle\beta\rangle \in \mathfrak{E}_{\mathbb{Z}}[[\mathfrak{g}, d_{\gamma_1}]]/\exp(\mathfrak{g}_0 \otimes (\mathfrak{m}_K)_0), \quad \langle\sigma\rangle \in Def_{\mathbb{Z}_2}(\tilde{\mathfrak{g}},\, d_{\gamma_1+\beta}),$$

2.5 A special case

Let us begin with some general facts.

Definition 8. A differential $\Bbbk$-vector space $(V,\, d)$ is a $\Bbbk$-vector space V equipped with $d \in Hom_{\Bbbk}(V,\, V)$ satisfying $d^2 = 0$; set $Z := \mathrm{Ker}\, d,\ B := Im\, d,\ H := Z/B$.

Lemma 3. *Let* $(V,\, d)$ *be a differential* $\Bbbk$*-vector space; then there exist vector subspaces* $\mathcal{H}$ *and* S *with*

1 $\mathcal{H} \oplus B = Z$ *(and so* $\mathcal{H} \approx H$ *),*
2 $S \cap Z = \{0\}$,

in such a way that

$$V = \mathcal{H} \oplus dS \oplus S. \tag{1}$$

(1) *is called a* Hodge decomposition *for* $(V,\, d)$.

Moreover, given (1), $Q \in Hom_{\mathbb{k}}(V, V)$ *is defined in such a way that*

$$\alpha = \pi_{\mathcal{H}}(\alpha) + dQ(\alpha) + Q(d\alpha),$$

i.e., Q is a cohomological homotopy between I and $\pi_{\mathcal{H}}$ and $\alpha \in \mathfrak{g}$ is d-exact if and only if $d\alpha = 0$ and $\pi_{\mathcal{H}}(\alpha) = 0$ and in this case $\alpha = dQ(\alpha)$. Finally, if V is a super vector space (resp. a graded vector space) and d is compatible with the grading, then it is possible to choose $\mathcal{H}$ and S to be supersubspaces (resp. graded subspaces) obtaining a super (resp., graded) Hodge decomposition.

Proof. Let $\mathcal{H} \subset Z$ be a vector subspace such that

$$\mathcal{H} \oplus B = Z.$$

Let $R \subset \mathfrak{g}$ be a vector subspace such that:

- $\mathfrak{g} = \mathcal{H} \oplus R$,
- $B \subset R$.

Clearly, $R \cap Z = B$.

Let $S \subset R$ be a supersubspace such that $R = B \oplus S$. Then

$$S \cap Z = 0 \quad \text{and} \quad B = dS.$$

Finally, if

$$\alpha = \pi_{\mathcal{H}}(\alpha) + d\beta + \gamma,$$

just set $Q(\alpha) = \beta$. Then $dQ(d\alpha) = d\alpha = d\gamma$ and thus $\gamma = Q(d\alpha)$; note also that $Q^2 = 0$. Concerning the last statement, just observe that we can perform the whole construction preserving the grading.

We have now the following

Lemma 4. *Let* $(\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}, d)$ *be a DSLA. Then the following facts are equivalent:*

1. *there exists a quasi isomorphism,*

$$\phi : (\mathfrak{g}, [\ ,\ ], d) \longrightarrow (H, 0, 0);$$

2. *we have:*

$$[\mathfrak{g}, \mathfrak{g}] \cap Z \subset B; \tag{2}$$

3. *there exists super Hodge decomposition* $\mathfrak{g} = \mathcal{H} \oplus dS \oplus S$, *such that*

$$[\mathfrak{g}, \mathfrak{g}] \subset dS \oplus S. \tag{3}$$

Proof. 1. $\Longrightarrow$ 2. Since Φ is a quasi-isomorphism, we have, in particular

$$[\mathfrak{g}, \mathfrak{g}] \cap Z \subset \operatorname{Ker} \Phi \cap Z = B.$$

2. $\Longrightarrow$ 3. Let $\mathcal{H} \subset Z$ be a supersubspace such that

$$\mathcal{H} \oplus B = Z.$$

Clearly $\mathcal{H} \cap [\mathfrak{g}, \ \mathfrak{g}] = \{0\}$. Then as in the general construction of Hodge decomposition, just choose $R \subset \mathfrak{g}$ to be a supersubspace such that:
- $\mathfrak{g} = \mathcal{H} \oplus R$,
- $[\mathfrak{g}, \ \mathfrak{g}] + B \subset R$.

3. $\Longrightarrow$ 1. Just set $\Phi(\alpha) := [\pi_{\mathcal{H}}(\alpha)]$.

We recall that a dGBV algebra $(A, \ \Delta, \ d)$ satisfying the Δd-lemma is an example of DSLA meeting the condition of lemma (4) (cf.[3] and [5] [1]).

We have the following

Lemma 5. *Assume the DSLA* $(\mathfrak{g}, \ [\ ,\ ], \ d)$ *satisfies the conditions of lemma* (4)*; fix* $\mathcal{H}$, S *and hence* Φ *and* Q*. Let*

$$a \ : \ \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] \ \longrightarrow \ (Z \otimes \mathfrak{m}_K)^{(1)},$$

be defined by

$$a(\gamma) := \gamma + \frac{1}{2} Q_K([\gamma, \gamma]).$$

Then:

1. a *is one-to-one with inverse map,*

$$b := \alpha = \sum_{j=1}^{\infty} \alpha_j \ \mapsto \gamma = \sum_{j=1}^{\infty} \gamma_j,$$

where:

$$\gamma_1 = \alpha_1$$
$$\vdots$$
$$\gamma_j = -\frac{1}{2} \sum_{r+s=j} Q_K([\gamma_r, \ \gamma_s]) + \alpha_j.$$

2.

$$a(\chi(\beta)\gamma) = a(\gamma) \, mod((B \otimes \mathfrak{m}_K)_1)$$
$$a^{-1}(\alpha + d\epsilon) = a^{-1}(\epsilon) \, \mathrm{mod}(\exp((\mathfrak{g} \otimes \mathfrak{m}_K)_0))$$

and so

$$a^* \ : \ \langle \gamma \rangle \ \mapsto \ [a(\gamma)]$$

establishes a bijection

$$Def_{\mathbb{Z}_2}[[\mathfrak{g}]] \ \longrightarrow \ (H \otimes \mathfrak{m}_K)^{(1)}.$$

3. *Let*

$$\widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]] := \{\gamma \in (\mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] \mid \gamma_j \in \operatorname{Ker}\Phi \otimes \mathfrak{m}_K \ , \ j \geq 2\}.$$

Then

a) $\widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]]$ *is* $\exp((\mathfrak{g}\otimes\mathfrak{m}_K)^{(0)}) - invariant$

b) $a^* \ : \ \widetilde{Def}_{\mathbb{Z}_2}[[\mathfrak{g}]] \ := \ \widetilde{\mathfrak{MC}_{\mathbb{Z}_2}}[[\mathfrak{g}]]/\exp((\mathfrak{g}\otimes\mathfrak{m}_K)^{(0)}) \longrightarrow \ H$

Proof. 1. First note that, given $\gamma \in \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]]$, we have that $[\gamma,\ \gamma]$ is d_K-exact and, because of (4),

$$[\gamma,\ \gamma] = d_K Q_K([\gamma,\ \gamma]).$$

Therefore

$$da(\gamma) = d_K\gamma + \frac{1}{2} d_K Q_K([\gamma,\ \gamma]) = 0.$$

Now we can first check that, given $\alpha \in (Z\otimes\mathfrak{m}_K)^{(1)}$, we have

$$db(\alpha) + \frac{1}{2}[b(\alpha),\ b(\alpha)] = 0. \tag{4}$$

Now (4) amounts to

$$d\gamma_j = -\frac{1}{2}\sum_{r+s=j}[\gamma_r,\ \gamma_s],$$

and this can be shown recursively. It is certainly true for $j = 1$. Assume it is true for $l < j$, then:

$$\begin{aligned} d\sum_{r+s=j}[\gamma_r,\ \gamma_s] &= \sum_{r+s=j}([d\gamma_r,\ \gamma_s] - [\gamma_r,\ d\gamma_s]) \\ &= -\frac{1}{2}\sum_{r+s=j}\sum_{p+q=r}[[\gamma_p,\ \gamma_q],\ \gamma_s] + \frac{1}{2}\sum_{r+s=j}\sum_{t+u=s}[\gamma_r,\ [\gamma_t],\ \gamma_u] \\ &= -\sum_{r+s+t=j}[[\gamma_r,\ \gamma_s],\ \gamma_t] \\ &= 0, \quad \text{by Jacobi identity.} \end{aligned}$$

Therefore,

$$b \ : \ (Z\otimes\mathfrak{m}_K)^{(1)} \ \longrightarrow \ \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]].$$

Then,

$$ab(\alpha) = \sum_{j=1}^{\infty}\beta_j = b(\alpha) + \frac{1}{2}Q([\underline{(}\alpha),\ b(\alpha)]) = \alpha.$$

In fact,

$$\beta_j = \gamma_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r,\ \gamma_s])$$
$$= -\frac{1}{2}\sum_{r+s=j} Q([\gamma_r,\ \gamma_s]) + \alpha_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r,\ \gamma_s]) = \alpha_j,$$

$$ba(\gamma) = \sum_{j=0}^{\infty} \epsilon_j = \gamma,$$

can be shown recursively. Definitely true for $j = 1$, assume it holds true for $l < j$. Then,

$$\epsilon_j = -\frac{1}{2}\sum_{r+s=j} Q([\epsilon_r,\ \epsilon_s]) + \gamma_j + \frac{1}{2}\sum_{r+s=j} Q([\gamma_r,\ \gamma_s]) = \gamma_j.$$

2. We can easily show by direct computation that:

$$a(\chi(\eta)\gamma) = a(\gamma) + dQ(\chi(\eta)\gamma - \gamma).$$

Vice versa, given $\epsilon \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}$, we can construct recursively $\eta \in (\mathfrak{g} \otimes \mathfrak{m}_K)^{(0)}$ such that,

$$a^{-1}(\alpha + d\epsilon) = \chi(\eta)a^{-1}(\alpha),$$

i.e.,

$$\alpha + d\epsilon = \alpha + dQ(\chi(\eta)a^{-1}(\alpha) - a^{-1}(\alpha)).$$

Set $\eta_1 = \epsilon_1$ and assume η_l has been constructed for $l < j$. Note that, in general,

$$(\chi(\eta)\gamma - \gamma)_j = A_j - d\eta_j,$$

where A_j depends on $\gamma_r,\ \eta_s$ for $0 < r,\ s < j$.
Therefore:

$$(dQ(\chi(\eta)\gamma - \gamma))_j = dQ(A_j) - d\eta_j.$$

Thus choose $\eta_j = Q(A_j) - \epsilon_j$.

3. is clear.

Finally, if $(\mathfrak{g},\ [\ ,\ ],\ d)$ is a DGLA, we have the following, easy to prove lemma:

Lemma 6. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{MC}_{\mathbb{Z}_2}[[\mathfrak{g}]] & \xrightarrow{a} & (Z \otimes \mathfrak{m}_K)^{(1)} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathfrak{MC}_{\mathbb{Z}}[[\mathfrak{g}]] & \xrightarrow{a} & (Z \otimes \mathfrak{m}_K)^{(1)} \end{array}$$

Moreover,

$$a^* \circ \tilde{\pi}^* = \tilde{\pi}^* \circ a^*$$

and analogous results hold true for $\widetilde{\mathfrak{MC}}$, *provided* $|\Phi| = 0$ *in* $\mathbb{Z}$.

Finally, note that, if we want to efficiently define a at the $\mathbb{Z}$-level only, we just need to replace (2) with

$$[\mathfrak{g}_1,\ \mathfrak{g}_1] \cap Z_2 \subset B_2.$$

3 An example: A^∞-algebras and deformation theory

As a first example of deformation space, we consider the following. Let $(V = V^{(0)} \oplus V^{(1)},\ d)$ be a differentiable $\Bbbk$-super vector space. We can extend d and the superstructure to the tensor algebra $\mathbb{T}(V)$. In particular,

- $d(R \otimes S) = dR \otimes S + (-1)^{|R|} R \otimes dS$;
- if $L \in \mathrm{Hom}_{\Bbbk}(V^{\otimes r},\ V^{\otimes s})$, then

  $$dL = d \circ L - (-1)^{|L|} L \circ d$$

  and

  $$d(L \circ M) = dL \circ M + (-1)^{|L|} L \circ dM.$$

Set

$$C^p(V) := \mathrm{Hom}_{\Bbbk}(V^{\otimes(p+1)},\ V),$$

and given $R \in C^p(V)$, set

$$\|R\| = (|R| + p) \mod 2.$$

Given $R \in C^p(V),\ S \in C^q(V)$, let $[R,\ S] \in C^{p+q}(V)$ be defined as

$$[R,\ S] := \sum_{k=1}^{p+1} (-1)^{p(k-1)} R \circ (I^{\otimes(k-1)} \otimes S \otimes I^{\otimes((p+1-k)}) +$$

$$- (-1)^{\|R\|\|S\|} \sum_{k=1}^{q+1} (-1)^{q(k-1)} S \circ (I^{\otimes(k-1)} \otimes R \otimes I^{\otimes((q+1-k)}).$$

Then,

$$d[R,\ S] = [dR,\ S] + (-1)^{\|R\|}[R,\ dS]$$

and

$$\left(C(V) := \bigoplus_{p \in \mathbb{Z}} C^p(V),\ [\ ,\ ],\ d \right),$$

is a DSLA.

Let $\mathcal{A}(V)$ be the completion of $C(V)$ and extend in an obvious way the DSLA structure to $\mathcal{A}(V)$. Let $\mathcal{A}^*(V)$ be the sub DSLA of $\mathcal{A}(V)$ of elements with no components in $C^0(V)$. Then,

a structure of A^∞-algebra on V is a solution of the MC equation in $\mathcal{A}^*(V)$.

See [11] and [12] for examples of A^∞-algebras related to complex and symplectic geometry.

4 Complex and holomorphic deformation theory

4.1 Preliminaries

Let

$$J_n = \begin{pmatrix} O & -I_n \\ I_n & 0 \end{pmatrix}.$$

We consider the faithful representation,

$$\rho \,:\, \mathfrak{gl}(n,\,\mathbb{C}) \longrightarrow \mathfrak{gl}(2n,\,\mathbb{R}),$$
$$\rho \,:\, A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

In the sequel, we shall identify

$$\mathfrak{gl}(n,\,\mathbb{C}) \quad \text{with} \quad \rho(\mathfrak{gl}(n,\,\mathbb{C})) = \{X \in \mathfrak{gl}(2n,\,\mathbb{R}) \mid XJ_n - J_nX = 0\}.$$

Moreover,

$$\mathfrak{gl}(2n,\,\mathbb{R}) = \mathfrak{gl}(n,\,\mathbb{C}) \oplus \mathfrak{s}(n),$$

where

$$s(n) := \{X \in \mathfrak{gl}(2n,\,\mathbb{R}) \mid XJ_n + J_nX = 0\},$$

with projections

$$R \,:\, \mathfrak{gl}(2n,\,\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\,\mathbb{C}), \qquad X \mapsto \frac{1}{2}(X - J_nXJ_n),$$
$$S \,:\, \mathfrak{gl}(2n,\,\mathbb{R}) \longrightarrow \mathfrak{s}(n), \quad X \mapsto \frac{1}{2}(X + J_nXJ_n).$$

Let

$$\mathfrak{W}(n) := \{P \in GL(2n,\,\mathbb{R}) \mid P^2 = -I\}.$$

Clearly,

- $P \in \mathfrak{W}(n) \iff P = AJ_nA^{-1}$,
- $P = AJ_nA^{-1} = BJ_nB^{-1} \iff B^{-1}A \in GL(n,\,\mathbb{C})$.

Consequently,

$$\mathfrak{W}(n) = GL(2n,\,\mathbb{R})/GL(n,\,\mathbb{C})$$

and

$$GL(2n,\,\mathbb{R}) \mapsto \mathfrak{W}(n)$$

is a $GL(n,\,\mathbb{C})$-principal bundle with projection $\pi(A) = AJ_nA^{-1}$. In particular, there exists a neighborhood U of J_n and a section σ over U, i.e., a map $\sigma \,:\, U \longrightarrow GL(2n, \mathbb{R})$ such that:

a. $\sigma(J_n) = I$,
b. for every $P \in U$, $\sigma(P)J_n\sigma(P)^{-1} = P$.

Moreover, since $R(\sigma(J_n)) = I$, if U is sufficiently small, then, for every $P \in U$, $R(\sigma(P)) \in GL(n, \mathbb{C})$ and so $\tilde{\sigma}(P) := \sigma(P)(R(\sigma(P)))^{-1}$ is a section over U with $R(\tilde{\sigma}(P)) \equiv I$. It is obvious that $\tilde{\sigma}$ is uniquely characterized by these conditions, namely,

- $\tilde{\sigma}(J_n) = I$,
- $R(\tilde{\sigma})(P) \equiv I$.

In other words, every $P \in U$ can be expressed in a unique way as

$$P = (I + L)J_n(I + L)^{-1} \quad \text{with} \quad LJ_n = -J_nL. \tag{5}$$

We can give a complete description of those elements in $\mathfrak{W}(n)$ which are expressible as (5). Let

$$\mathcal{A}(n) := \{X \in \mathfrak{s}(n) \mid det(I + X) \neq 0\},$$

$$\mathfrak{P}(n) := \{P \in \mathfrak{W}(n) \mid det(I - J_nP) \neq 0\}.$$

Then, we have the following:

Lemma 7. *Set*

$$r(P) := (I - J_nP)^{-1}(I + J_nP).$$

Then r diffeomorphically sends $\mathfrak{P}(n)$ into $\mathcal{A}(n)$

Proof. Just note that

$$r(P) = 2(I - J_nP)^{-1} - I = -(I - PJ_n)^{-1}(I + PJ_n),$$

and that, clearly,

$$r^{-1}(L) = (I + L)J_n(I + L)^{-1}.$$

Note also that the elements $P \in \mathfrak{W}(n)$ are in one-to-one correspondence with complex subspaces W of $\mathbb{C}^{2n} = (\mathbb{R}^{2n})^{\mathbb{C}}$, satisfying

$$\mathbb{C}^{2n} = W \oplus \bar{W}. \tag{6}$$

In fact, given $P \in \mathfrak{W}(n)$, just set $W = V_P^{0,1}$; vice versa, given W satisfying (6), set $P = -it_2 \circ t_1^{-1}$, where

$$t_1 := p_{1|\bar{W}} \, : \, \bar{W} \longrightarrow \mathbb{R}^{2n},$$

$$t_2 := p_{2|\bar{W}} \, : \, \bar{W} \longrightarrow i\mathbb{R}^{2n}.$$

Given $\bar{W}$ sufficiently close to $V_{J_n}^{0,1}$, $\bar{W}$ can be described as the graph of a $\mathbb{C}$-linear map $L \, : \, V_{J_n}^{0,1} \longrightarrow V_{J_n}^{1,0}$ (and so $LJ_n = -J_nL$). Consequently,

$$\bar{W} = \{(I + L)X + i(I + L)JX \mid X \in \mathbb{R}^{2n}\},$$

and the corresponding element of $\mathfrak{W}(n)$ is $P = (I + L)J_n(I + L)^{-1}$.

4.2 Starting deformation theory

Let $(M,\ J)$ be a complex manifold and let $\mathfrak{H}(M)$ be the Lie algebra of smooth vector fields on M. Given $X,\ Y\in\mathfrak{H}(M)$, set

$$\begin{aligned}(\bar\partial_J X)(Y) &:= \frac{1}{4}([Y,\ X]+J[JY,\ X]+[JY, JX]-J[Y, JX])\\ &= \frac{1}{2}([Y,\ X]+J[JY,\ X])-\frac{1}{4}N_J(X,\ Y),\end{aligned} \tag{7}$$

where, as usual,

$$N_J\in\wedge_J^{0,2}(M)\otimes TM,$$

defined as

$$N_J(X,\ Y):=[JX,\ JY]-[X,\ Y]-J[JX,\ Y]-J[X,\ JY],$$

is the Nijenhuis tensor of J and

$$N_J=0 \quad\Longleftrightarrow\quad J \text{ is integrable.}$$

Then we have:

- $\bar\partial_J X\in\wedge_J^{0,1}\otimes TM$,
- $\bar\partial_J JX=J\bar\partial_J X$, i.e., $\bar\partial_J J=0$.

Note also that, given $f\in C^\infty(M,\ \mathbb{C})$, then

$$(\bar\partial_J)^2 f(X,\ Y)=-\frac{1}{8}(N_J(X,\ Y)-iJN_J(X,\ Y))f.$$

Let $(M,\ J)$ be a holomorphic manifold and set

- $\mathfrak{g}=\mathcal{A}:=\wedge_J^{0,*}(M)\otimes TM$,
- $[X,\ Y]=[X*Y]:=\frac{1}{2}([X,\ Y]-[JX,\ JY])$, for $X,\ Y\in\mathfrak{H}(M)$.
  A straightforward computation shows that $[\ *\ ]$ is a Lie algebra bracket (note that for a general complex structure J, we have:

$$\mathfrak{S}[X*[Y*Z]]=\frac{1}{4}\mathfrak{S}[JN,\ N_J(JY,\ Z)]),$$

- $d=\bar\partial_J$ where, now, for $X,\ Y\in\mathfrak{H}$,

$$(\bar\partial_J X)(Y):=\frac{1}{2}([Y,\ X]+J[JY,\ X]).$$

Then:

1. Define $|\alpha\otimes X|:=|\alpha|$ and so,

$$\mathcal{A}=\bigoplus_{p\in\mathbb{Z}}\mathcal{A}_p,$$

where

$$\mathcal{A}_p = \begin{cases} \wedge_J^{0,p}(M) \otimes TM, & \text{if } 0 \le p \le n, \\ 0\ , & \text{otherwise.} \end{cases}$$

2. Extend $[\,*\,]$ to $\mathcal{A}$ in the following way:
   a) if $L \in \wedge_J^{0,1}(M) \otimes TM$, define $[L * L]$ by means of the formula

$$[L * L](X,\ Y) = [L(X) * L(Y)] - L([L(X) * Y] + [X * L(Y)] - L([X * Y]));$$

   b) given $R,\ S \in \wedge_J^{0,1}(M) \otimes TM$, define $[R * S]$ by polarization, i.e.,

$$[R * S] := \frac{1}{2}([R + S * R + S] - [R * R] - [S * S]);$$

   c) given $\alpha \in \wedge_J^p(M),\ \beta \in \wedge_J^q(M)$, define

$$[\alpha \wedge R * \beta \wedge S] := (-1)^q \alpha \wedge \beta \wedge [R * S];$$

   d) extend to the general case by bilinearity.

   Note that, in terms of local complex coordinates $z_1, \ldots, z_n$, under the identification

$$TM \longleftrightarrow T^{1,0}M, \quad X \longleftrightarrow \frac{1}{2}(X - iJX),$$

   we have that, given $R \in \mathcal{A}_p\ ,\ S \in \mathcal{A}_q$,

$$R = \sum_{j=1}^{n} \sum_{|I|=p} r_{jI} d\bar{z}_I \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} r_j \otimes \frac{\partial}{\partial z_j},$$

$$S = \sum_{j=1}^{n} \sum_{|K|=q} s_{jK} d\bar{z}_I \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} s_j \otimes \frac{\partial}{\partial z_j}.$$

   Then,

$$[R * S] = \sum_{j,k=1}^{n} \left(r_j \wedge \frac{\partial}{\partial z_j} s_k - (-1)^{pq} s_j \wedge \frac{\partial}{\partial z_j} r_k \right) \otimes \frac{\partial}{\partial z_k},$$

   where, of course,

$$\frac{\partial}{\partial z_j} s_k = \sum_{|K|=q} \frac{\partial}{\partial z_j} s_{kK} d\bar{z}_K,$$

   (see e.g. [8]).
3. Extend $\bar{\partial}$ to $\mathcal{A}$ by setting

$$\bar{\partial}_J(\alpha \otimes X) = \bar{\partial}_J \alpha \otimes X + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_J X.$$

Then $(\mathcal{A},\ [\,*\,],\ \bar\partial_J)$ is a DGLA; note that $\mathcal{A}_0 = \mathfrak{H}(M)$ and, consequently, $\exp(\mathcal{A}_0)$, is the connected component of id_M in $\mathcal{D}\mathrm{iff}(M)$. Let $\tilde J$ be another complex structure on M with $det(I - J\tilde J) \neq 0$. Then we can write in a unique way,

$$\tilde J = AJA^{-1},$$

with

$$A = I + L \quad \text{and} \quad LJ + JL = 0,$$

i.e.,

$$L \in \wedge_J^{0,1}(M) \otimes TM.$$

A tedious but straightforward computation yields the following

Lemma 8. *Let $L,\ A,\ \tilde J$ be as before and let*

$$\rho(A) \ := \ (A^*)^{-1} \otimes A \in Aut(T^*M \otimes TM).$$

Then:

- $\rho^{-1}(A)N_{\tilde J} = -4(\bar\partial_J L + \frac{1}{2}[L * L])$;
- $\rho^{-1}(A) \circ \bar\partial_{\tilde J} \circ \rho(A) = \bar\partial_J + [L * \cdot]$,

i.e., on TM :

- $A^{-1}N_{\tilde J}(AX,\ AY) = -4(\bar\partial_J L + \frac{1}{2}[L * L](X,\ Y))$,
- $A^{-1}(\bar\partial_{\tilde J} AX)(AY) = (\bar\partial_J X)(Y) + [L * X](Y)$.

Proof. It is enough to consider the case

$$J = J_n, \quad A(0) = I \ \text{ (i.e., } \mathrm{L}(0) = 0),$$

and perform the computations at 0.

Consequently,

- $(\bar\partial_J)_L = \bar\partial_J + [L * \cdot]$ corresponds to $\bar\partial_{\tilde J}$;
- $L \in \mathfrak{MC}_{\mathbb{Z}}(\mathcal{A})$, $det(I + L) \neq 0$, $\Longleftrightarrow$, $\tilde J = (I + L)J(I + L)^{-1}$ is a holomorphic structure and so $L \ \mapsto \ (I + L)J(I + L)^{-1}$ establishes a bijection:

$$\mathfrak{MC}^*_{\mathbb{Z}}(\mathcal{A}) := \{L \in \mathfrak{MC}_{\mathbb{Z}}(\mathcal{A}) \,|\, det(I + L) \neq 0\}$$
$$\downarrow$$
$$\{\text{holomorphic strucures } \tilde J \text{ s.t. } det(I - \tilde J J) \neq 0\};$$

- two $\exp(\mathcal{A}_0)$-equivalent elements of $\mathfrak{MC}_{\mathbb{Z}}(\mathcal{A})$ correspond to diffeomorphic holomorphic structures.

We have also the following, easy to prove

Lemma 9. *Let L, A, $\tilde{J}$ be as before. Then, on $\wedge_J^{0,*}(M)$ we have*

$$\rho^{-1}(A) \circ \bar{\partial}_{\tilde{J}} \circ \rho(A) \;=\; \bar{\partial}_J + L \wedge \partial_J,$$

where, more generally,

$$\wedge \;:\; (\wedge_J^{0,p}(M) \otimes T_J^{1,0}M) \times (\wedge_J^{0,q}(M) \otimes \wedge_J^{1,0}(M)) \;\longrightarrow\; \wedge_J^{0,p+q}(M)$$

is defined by means of the duality pairing.

Lemma 9 suggests the possibility of considering operators on $\wedge_J^{0,*}(M)$ of the form:

$$\alpha \mapsto \bar{\partial}_J\alpha + L \wedge \partial_J\alpha,$$

with

$$L \in \wedge_J^{\mathrm{odd}}(M) \otimes TM\,, \quad L = \sum_{p=1}^{[\frac{n+1}{2}]} L_p \quad \text{with} \quad L_p \in \wedge_J^{0,2p-1}M,$$

possibly with $L_1 = 0$, i.e., including L_1 into a new $\tilde{J}$ on the basis of Lemma 8. Therefore, we can set the following:

Definition 9. A supercomplex (resp. superholomorphic) structure on M is the datum $\mathcal{J} = (J,\, L)$ of a complex (resp. holomorphic) structure J on M and $L \in \mathcal{A}^{(1)} = \wedge_J^{\mathrm{odd}}(M) \otimes TM$.

Given a superholomorphic structure $\mathcal{J} = (J,\, L)$, set, on $\wedge_J^{0,*}(M)$:

$$\bar{\daleth} = \bar{\daleth}_{\mathcal{J}} = \bar{\partial}_J + L \wedge \partial_J.$$

Clearly $\bar{\daleth}$ is a parity one derivation and

$$\bar{\daleth}^2 = (\bar{\partial}_J L + \frac{1}{2}[L * L]) \wedge (\partial + L \wedge \bar{\partial}_J).$$

Moreover, $\bar{\daleth}$ extends to $\mathcal{A}$ as

$$\bar{\daleth} = \bar{\partial}_J + [L * \cdot\,]\,,$$

and it satisfies

$$\bar{\daleth}(\alpha \otimes X) = \bar{\daleth}\alpha \otimes X + (-1)^{|\alpha|}\alpha \wedge \bar{\daleth}X,$$

for $X \in \mathcal{A}_0$, $\alpha \in \wedge_J^{0,*}(M)$. Clearly $\bar{\daleth}$ reflects the $\mathbb{Z}_2$-deformation theory of $\mathcal{A}$. Thus, in particular, we have

$$\bar{\daleth}^2 = 0 \quad \Longleftrightarrow \quad \bar{\partial}_J L + \frac{1}{2}[L * L] = 0,$$

which gives by lemma 1,

$$\bar{\partial}_J L_1 + \frac{1}{2}[L_1 * L_1] = 0,$$

i.e.,

$$\tilde{J} := (I + L)J(I + L)^{-1}$$

is holomorphic. This leads to a superholomorphic structure $\tilde{\mathcal{J}} = (\tilde{J}, \tilde{L})$ with $\tilde{L}_1 = 0$. Note that:

- if $n = 2$, then superholomorphic structures coincide with complex structures (because $L = L_1$!),
- if $n = 3$, then $L = L_1 + L_2$ and

$$\bar{\partial}_J L + \frac{1}{2}[L * L] = 0 \quad \Longleftrightarrow \quad \bar{\partial}_J L_1 + \frac{1}{2}[L_1 * L_1] = 0,$$

  and so, assuming $L_1 = 0$, we obtain for

$$\alpha \in \wedge_J^{0,*}(M), \quad \alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, \quad \text{with } \alpha_p \in \wedge_J^{0,p}(M) \;\; 0 \le p \le 3 :$$

$$\bar{\daleth}\alpha = 0 \quad \Longleftrightarrow \quad \begin{cases} \bar{\partial}_J \alpha_0 = 0, \\ \bar{\partial}_J \alpha_1 = 0, \\ \bar{\partial}_J \alpha_2 + L_2 \wedge \partial_J \alpha_0 = 0. \end{cases}$$

4.3 A very simple example

Let $M = \mathbb{T}^{2n} = \mathbb{C}^n / \mathbb{Z}^{2n}$ and let $\mathcal{J} = (J, L)$, where

- $J = J_{sdt}$,
- $L = \sum_{p=2}^{n} L_p \;,\; L_p = \sum_{j=1}^{n} \sum_{|I|=2p-1} a_{\bar{I}j} dz_{\bar{I}} \otimes \frac{\partial}{\partial z_j} \;,\; a_{\bar{I}j} \in \mathbb{C}.$

Clearly,

$$\bar{\daleth}^2_{\mathcal{J}} = 0.$$

5 Symplectic deformation theory

5.1 Preliminaries

Let (V, κ) be a $2n$-dimensional symplectic vector space. Define the *symplectic Hodge operator*

$$\bigstar : \wedge^r V^* \longrightarrow \wedge^{2n-r} V^*,$$

by means of the relation,

$$\alpha \wedge \bigstar \beta = \kappa(\alpha, \beta)\frac{\kappa^n}{n!},$$

$\alpha, \beta \in \wedge^r V^*$. It is easy to check that $\bigstar^2 = I$.

Consider the following endomorphisms of $\wedge^* V^*$:

- $L : \alpha \mapsto \kappa \wedge \alpha$,
- $\Lambda := -\bigstar L \bigstar$,
- $H = \sum_{r=0}^{2n} (n-r) p_r$, where

$$p_r : \wedge^* V^* \longrightarrow \wedge^r V^*,$$

  is the natural projection.

It is easy to check that

$$[L, \Lambda] = H \ , \ [L, H] = -2L \ , \ [\Lambda, H] = 2\Lambda,$$

and so $\wedge^* V^*$ has the natural structure of the $\mathfrak{sl}(2, \mathbb{C})$-module.

We have

Lemma 10. *For* $0 \leq p \leq n$,

$$L^p : \wedge^{n-p} V^* \longrightarrow \wedge^{n+p} V^*,$$

is an isomorphism and so, in particular, for $0 \leq p < n$,

$$L : \wedge^p V^* \longrightarrow \wedge^{p+2} V^*,$$

is injective.

We have also

Lemma 11. *Let* $0 \leq p \leq n$.

- *If* $\alpha \in \wedge^p V^*$, *then*

$$\bigstar(\alpha \wedge \kappa^{n-p}) = (-1)^{\frac{1}{2} p(p-1)} (n-p)! (\alpha + \Lambda\alpha \wedge \kappa). \tag{8}$$

-

$$\bigstar \kappa^p = \frac{p!}{(n-p)!} \kappa^{n-p}. \tag{9}$$

For every $A \in End(V)$, we define ${}^T A \in End(V)$ by means of the relation

$$\kappa(Av, w) = \kappa(v, {}^T A w).$$

Let,

$$\mathcal{S}_\kappa(V) := \{A \in End(V) \mid A = {}^T A\},$$
$$\mathcal{S}_\kappa^*(V) = \mathcal{S}_\kappa(V) \cap Aut(V).$$

We can immediately check that

$$A \in \mathcal{S}_\kappa^*(V) \quad \Longleftrightarrow \quad A^{-1} \in \mathcal{S}_\kappa^*(V).$$

Clearly, given $A \in \mathcal{S}_\kappa(V)$,

$$\kappa_A(v, w) := \kappa(Av, w)$$

defines an element of $\wedge^2 V^*$ and

$$\natural_\kappa : \mathcal{S}_\kappa(V) \longrightarrow \wedge^2 V^* , \; A \mapsto \kappa_A,$$

is a bijection sending $\mathcal{S}^*_\kappa(V)$ into symplectic forms.

Let $\tilde{\kappa}$ now be a symplectic form on V. Then there exists a uniquely defined $A \in \mathcal{S}_\kappa(V)$ §[a] such that:

$$\tilde{\kappa} = \kappa_A.$$

Consequently, if $\alpha, \beta \in \wedge^r V^*$, then,

$$\tilde{\kappa}(\alpha, \beta) = \kappa(\rho(A)\alpha, \beta) = \kappa(\alpha, \rho(A)\beta),$$

where, as before,

$$\rho(A)(\zeta_1 \wedge \cdots \wedge \zeta_r) = (A^*)^{-1}\zeta_1 \wedge \cdots \wedge (A^*)^{-1}\zeta_r.$$

Moreover,

$$\tilde{\kappa}^n = e^{n\lambda}\kappa^n,$$

where $\lambda = \lambda(A) = \frac{1}{2n} \log |det\, A|$.

Therefore, if $\tilde{\bigstar}$ is the symplectic Hodge operator with respect to $\tilde{\kappa}$, we have

$$\alpha \wedge \tilde{\bigstar}\beta = \tilde{\kappa}(\alpha, \beta)\frac{\tilde{\kappa}^n}{n!} = \kappa(\alpha, e^{n\lambda}\rho(A)\beta)\frac{\kappa^n}{n!} = \alpha \wedge \bigstar e^{n\lambda}\rho(A)\beta,$$

and so, setting $C = C(A) := e^{n\lambda}\rho(A)$, we have

$$\tilde{\bigstar} = \bigstar C = C^{-1}\bigstar.$$

Let (M, κ) be an almost symplectic manifold. Set

$$d^\bigstar := (-1)^{r+1}\bigstar d\bigstar,$$

on r-forms. Clearly, $(d^\bigstar)^2 = 0$ and if $\tilde{\kappa}$ is another almost symplectic structure, then

$$d^{\tilde{\bigstar}} = C^{-1}d^\bigstar C.$$

We have the following

Lemma 12. *Let (M, κ) be an almost symplectic manifold. Set*

$$\eth_\kappa := [L, d^\bigstar].$$

Then the following facts are equivalent:

1. $d\kappa = 0$*, i.e.,* κ *defines a symplectic structure on* M*;*
2. $\eth_\kappa = d$*;*
3. $Q := [d,\ \Lambda] - d^\bigstar = 0$*;*
4. $[d,\ d^\bigstar] = 0$*;*
5. $\eth_\kappa$ *is a differential, i.e., it is a derivation of parity* 1 *and* $\eth_\kappa^2 = 0$.

Proof. Note first that 2. and 3. are obviously equivalent and that Q is $C^\infty(M)$-linear (cf. [4]);

1. $\Longrightarrow$ 3. It is a basic symplectic identity (cf. [4]).

3. $\Longrightarrow$ 1. From $Q = 0$, it follows

a. $0 = Q\kappa^n = [d,\ \Lambda]\kappa^n = d\Lambda\kappa^n$. Now,

$$\Lambda\kappa^n = -\bigstar L \bigstar \kappa^n = -n!\bigstar\kappa = -n\kappa^{n-1},$$

and so,

$$Q\kappa^n = 0 \Longrightarrow d\kappa^{n-1} = 0, \quad \text{i.e., } d^\bigstar\kappa = 0.$$

If $n = 2$, there is nothing else to prove, otherwise,

b. $0 = Q\kappa = [d,\ \Lambda]\kappa = -\Lambda d\kappa$.

c. From $[a.]$ we obtain,

$$Q\kappa^{n-1} = [d,\ \Lambda]\kappa^{n-1} - d^\bigstar\kappa^{n-1} = d\Lambda\kappa^{n-1} - d^\bigstar\kappa^{n-1}.$$

Now,

$$\Lambda\kappa^{n-1} = -\bigstar L \bigstar \kappa^{n-1} = -(n-1)!\bigstar\kappa^2 = -2(n-1)\kappa^{n-2}.$$

From (8), it follows

$$d^\bigstar\kappa^{n-1} = -\bigstar d \bigstar \kappa^{n-1} = -(n-1)!\bigstar d\kappa = (n-1)(n-2)d\kappa \wedge \kappa^{n-3}.$$

Finally,

$$Q\kappa^{n-1} = -2(n-1)d\kappa^{n-2} - (n-1)(n-2)d\kappa \wedge \kappa^{n-3}$$
$$= -3(n-1)(n-2)d\kappa \wedge \kappa^{n-3},$$

and thus, by Lemma 10, $Q\kappa^{n-1} = 0$ gives $d\kappa = 0$.

1. $\Longrightarrow$ 4.

$$[d,\ d^\bigstar] = [d,\ [d,\ \Lambda]] = [[d,\ d],\Lambda] - [d,\ [d,\ \Lambda]] = 0.$$

4. $\Longrightarrow$ 1. Let $f \in C^\infty(M)$. Then

$$Qdf = -d^\bigstar df$$

and so

$$[d,\ d^\bigstar] = 0 \quad \Longrightarrow \quad Q = 0 \text{ on } \wedge^1(M).$$

Let $\alpha \in \wedge^1(M)$ s.t. $d^\bigstar\alpha = 0$ (and so $\Lambda d\alpha = 0$) Thus, again using (8), we obtain:

$$d^\bigstar d\alpha = 0 = -\bigstar d \bigstar(d\alpha) = (n-2)!\bigstar(d\alpha \wedge d\kappa^{n-2}),$$

which gives $d\kappa^{n-2} = 0$ and so $d\kappa = 0$.

1. $\Longrightarrow$ 5. It is now obvious.
5. $\Longrightarrow$ 2. If $f \in C^{\infty}(M)$, then $\eth_\kappa f = df - f d^{\bigstar}\kappa$ and so

$$\eth_\kappa \quad \text{is a derivation} \quad \Longrightarrow \quad \eth_\kappa 1 = 0 \quad \Longrightarrow \quad d^{\bigstar}\kappa = 0.$$

Thus if $\eth_\kappa$ is a derivation, it coincides with d on functions and, since it satisfies $\eth_\kappa^2 = 0$ *it is* d.

5.2 Starting deformation theory once more

Let (M, κ) be a compact symplectic manifold. Therefore,

$$Sym(M) := \{\text{simplectic forms on } M\},$$

is not empty. Set,

$$Sym_0^{(\kappa)}(M) := \{\tilde{\kappa} \in Sym(M) \,|\, \tilde{\kappa}^n = const.\kappa^n\}.$$

By Moser's lemma,

$$Sym(M) = \mathcal{D}\text{iff}(M) Sym_0^{(\kappa)}(M).$$

It is well known that $(\wedge^*(M),\ d^{\bigstar}, d)$ is a dGBV algebra, and so, in particular, for every $\alpha \in \wedge^*(M)$ defining,

$$\urcorner_\alpha : \wedge^*(M) \longrightarrow \wedge^*(M),$$

as

$$\urcorner_\alpha \beta := (-1)^{|\alpha|} d^{\bigstar}(\alpha \wedge \beta) - (-1)^{|\alpha|} d^{\bigstar}\alpha \wedge \beta - \alpha \wedge d^{\bigstar}\beta,$$

we obtain

1. $\urcorner_\alpha$ is a derivation,
2. setting $[\alpha \bullet \beta] := \urcorner_\alpha \beta$, we obtain that $(\wedge^*(M),\ [\bullet],\ d)$ is an odd dGLA.

Let $\tilde{\kappa}$ now be another almost symplectic structure on M. Write $\tilde{\kappa}(X, Y) = \kappa(AX, Y)$ and $\tilde{\kappa}^n = e^{n\lambda}\kappa^n$. Then,

$$C \eth_\kappa C^{-1} = C[\tilde{L}, \tilde{d^{\bigstar}}]C^{-1} = [C\tilde{L}C^{-1}, d^{\bigstar}].$$

Now

$$C\tilde{L}C^{-1} = \rho(A)\tilde{L}\rho(A)^{-1} = e(\rho(A)\tilde{\kappa}),$$

where, for any $\gamma \in \wedge^*(M)$, we denote by $e(\gamma)$ the left multiplication by γ, i.e., $e(\gamma)(\alpha) = \gamma \wedge \alpha$. Note also that $\rho(A)\tilde{\kappa}(X, Y) = \kappa_{A^{-1}}(X, Y) = \kappa(A^{-1}X, Y)$.

Write $\rho(A)\tilde{\kappa} = \kappa - \epsilon$ and assume $d^{\bigstar}\rho(A)\tilde{\kappa} = 0$, i.e., $d^{\bigstar}\epsilon = 0$. Thus

$$C\tilde{L}C^{-1} = [L, d^{\bigstar}] - [e(\epsilon), d^{\bigstar}] = d + \urcorner_\epsilon.$$

Consequently, defining $\mathcal{MC} \,:\, \wedge^*(M) \longrightarrow \wedge^*(M)$ as

$$\mathcal{MC}(\alpha) := d\alpha + \frac{1}{2}[\alpha \bullet \alpha],$$

we obtain

$$\mathfrak{d}_{\tilde\kappa}^2 = 0 \iff \mathcal{MC}(\epsilon) = 0 \iff d\tilde\kappa = 0.$$

Note also that

$$\begin{cases} d^{\bigstar}\rho(A)\tilde\kappa = 0 \\ d^{\tilde\bigstar}\tilde\kappa = 0 \end{cases} \qquad \Longrightarrow \qquad \lambda(A) = const.$$

Vice versa, given $\epsilon \in \operatorname{Ker} d^{\bigstar} \cap \wedge^2(M)$, with $det(\natural_\kappa^{-1}(\kappa - \epsilon)) \neq 0$, let $\tilde\kappa$ be defined by the equation,

$$\rho(A)\tilde\kappa = \kappa - \epsilon.$$

If $\mathcal{MC}(\epsilon) = 0$, then, again, $\mathfrak{d}_{\tilde\kappa}^2 = 0$ and so $\tilde\kappa \in Sym_0^{(\kappa)}(M)$.

Note once more that, given $\tilde\kappa$ almost symplectic, by Moser's lemma, there exists $\phi \in \mathcal{D}\mathrm{iff}(M)$ such that $\hat\kappa := \phi^*(\tilde\kappa)$ satisfies $\hat\kappa^n = e^{n\lambda}\kappa^n$ with $\lambda = const.$

Summarizing, let $(M,\ \kappa)$ be a symplectic manifold and let

$$\mathcal{A} = \left(\operatorname{Ker} d^{\bigstar} \cap \bigoplus_{p>0} \wedge^p(M) \right) [1],$$

where, as usual, [1] is the degree -1 shift. Consequently,

$$\mathcal{A}_p = \begin{cases} \wedge^{p+1}(M) \cap \operatorname{Ker} d^{\bigstar}, & \text{if } 0 \le p \le 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathcal{A},\ [\,\bullet\,],\ d)$ is the dGLA that governs the deformation theory of the symplectic structure κ. In particular, if

$$\mathfrak{MC}(\mathcal{A}) := \{\epsilon \in \mathcal{A}_1 \mid \mathcal{MC}(\epsilon) = 0\},$$

and

$$\mathfrak{MC}^*(\mathcal{A}) = \{\epsilon \in \mathfrak{MC}(\mathcal{A}) \mid det(\natural_\kappa^{-1}(\kappa - \epsilon)) \neq 0\},$$

then,

$$A \mapsto I - A^{-1},$$

induces a bijection,

$$Sym_0^{(\kappa)}(M) \longleftrightarrow \mathfrak{MC}^*(\mathcal{A}).$$

Note that, if

$$\mathcal{D}\mathrm{iff}_0^\kappa(M) = \{\phi \in \mathcal{D}\mathrm{iff}(M) \mid \phi^*(\kappa^n) = \kappa^n\ ,\ \phi \text{ is isotopic to the identity}\},$$

then the action of $\mathcal{D}\mathrm{iff}_0^\kappa(M)$ on $\wedge^2(M)$ corresponds to the action of $\exp(\mathcal{A}^{(0)})$ on $\mathfrak{MC}^*(\mathcal{A})$. In fact, given $X \in \mathfrak{H}(M)$, then

1. $$d^{\bigstar}\#_{\kappa}(X) = \operatorname{div} X;$$

2. on $\wedge^r(M)$, we have

$$\iota_X = (-1)^r \bigstar e(\#_{\kappa}(X))\bigstar$$

and, consequently, on $\mathcal{A}$

$$\bigstar L_X \bigstar = \urcorner_{\#_{\kappa}(X)};$$

3. $\exp(X) \in \mathcal{D}\mathrm{iff}_0^{\kappa}(M)$ sends d to $ad(\bigstar\rho((\exp(X)_*)\bigstar)d$ and so the infinitesimal action is

$$\alpha \mapsto \bigstar L_X \bigstar \alpha = \urcorner_{\#_{\kappa}(X)}\alpha = [\#_{\kappa}(X) \bullet \alpha].$$

Consequently,

$$\mathfrak{MC}^*(\mathcal{A})/\exp(\mathcal{A}_0)$$

is the moduli space of (infinitesimal) constant volume deformations of the symplectic structure κ.

We want to show now that the theory is totally unobstructed.

Let (M, κ) be a compact symplectic manifold, and assume

$$\int_M \kappa^n = 1.$$

Let $\tilde{\kappa}$ be an almost symplectic form, and let

$$e^{nc} := \int_M \tilde{\kappa}^n > 0.$$

Then,

$$\int_M (e^c \kappa)^n = \int \tilde{\kappa}^n,$$

and so, by Moser's lemma, there exists $\phi \in \mathcal{D}\mathrm{iff}(M)$ s.t.

$$[\phi^*(\tilde{\kappa})]^n = e^{nc}\kappa^n.$$

Let now $\alpha \in \wedge^2(M)$, $d\alpha = 0$. Set $\kappa_t := \kappa + t\alpha$. Let $t \mapsto \phi_t$ be a smooth curve in $\mathcal{D}\mathrm{iff}(M)$ s.t.

1. $\phi_0 = id_M$,
2. $\phi_t*(\kappa_t)$ has constant volume density, i.e.,

$$\phi_t^*(\kappa_t^n) = e^{nc(t)}\kappa^n.$$

Now,

$$e^{nc(t)} = \int_M \phi_t^*(\kappa_t^n) = \int_M \kappa_t^n = 1 + nt \int \alpha \wedge \kappa^{n-1} \; + o(t).$$

Write

$$\alpha = -\frac{1}{n}\Lambda\alpha\kappa + \beta, \qquad \text{with } \Lambda\beta = 0 \quad \text{i.e., } \beta \wedge \kappa^{n-1} = 0.$$

Therefore,

$$e^{nc(t)} = 1 - t \int_M \Lambda\alpha\kappa^n \; + o(t).$$

Now let $X \in \mathfrak{H}(M)$ s.t. its associated flow $\{\psi_t^X\}$ satisfies

$$\frac{d}{dt}(\psi_t^X)^*(\kappa_t)_{|t=0} = \frac{d}{dt}\phi_t^*(\kappa_t)_{|t=0}.$$

Consequently,

$$\frac{d}{dt}\phi_t^*(\kappa_t^n)_{|t=0} = L_X\kappa^n + n\alpha \wedge \kappa^{n-1} = -q\kappa^n,$$

where

$$q = \int_M \Lambda\alpha\kappa^n,$$

and thus, if $\gamma = \#_\kappa(X)$,

$$n\left(\alpha + d\gamma + \frac{1}{n}q\kappa\right) \wedge \kappa^{n-1} = 0,$$

i.e.,

$$\Lambda\left(\alpha + \frac{1}{n}q\kappa + d\gamma\right) = 0,$$

and so

$$d^\bigstar(\alpha + d\gamma) = 0.$$

Note that, if $\Lambda\alpha = const$ (i.e., $d^\bigstar\alpha = 0$), then

$$\Lambda\left(\alpha + \frac{1}{n}q\kappa\right) = \Lambda\alpha - \int_M \Lambda\alpha\kappa^n = 0,$$

and so $\Lambda d\gamma = 0$ and $d^\bigstar\gamma = 0$. Finally,

$$\frac{d}{dt}(\psi_t^X)^*(\kappa_t)_{|t=0} = \alpha + d\gamma,$$

and so $t \mapsto \kappa_t$ corresponds to a curve in $\mathfrak{MC}(\mathcal{A})$, with tangent $\alpha + d\gamma$ at 0.

It is clear that, if we consider the underlying $\mathbb{Z}_2$-deformation theory, we are led to the notion of *supersymplectic structure*.

Definition 10. A *supersymplectic structure* on the $2n$-dimensional differentiable manifold M is the datum of

$$\kappa \in \wedge^{\text{even}}(M)\ ,\ \kappa = \sum_{p=1}^{n} \kappa_p\ ,\ \kappa_p \in \wedge^{2p}(M)\ ,\ 1 \leq p \leq n$$

such that:

1. $\kappa^n \neq 0$, i.e., $\kappa_1^n \neq 0$,
2. $d\kappa = 0$, i.e., $d\kappa_p = 0$, $1 \leq p \leq n$,
3. $d^\bigstar \kappa = 0$, i.e., $d^\bigstar \kappa_p = 0$, $1 \leq p \leq n$, where $\bigstar$ is computed with respect to κ_1.

Therefore, if κ is a supersymmetric structure on M, then κ_1 is a simplectic structure. Vice versa, from a symplectic structure κ_1, we can always construct a supersymplectic structure, just setting

$$\kappa := \sum_{p=1}^{n} \kappa_1^p.$$

Note that, in general, the DSLA $(\mathcal{A},\ d)$ does not satisfy the condition of lemma 4 (because, in general, the dGBV algebra $(\mathcal{A},\ d^\bigstar,\ d)$ does not satisfy the $dd^\bigstar$-lemma). Therefore, in contrast with the $\mathbb{Z}$-case, we cannot conclude that the theory is totally (formally) unobstructed; this is true whenever the symplectic manifold $(M,\ \kappa_1)$ satisfies the Hard Lefschetz Condition (cf. [2], [9], [10], [4], [13]).

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Commutative Condition on the Second Fundamental Form of CR-submanifolds of Maximal CR-dimension of a Kähler Manifold

Mirjana Djorić

Faculty of Mathematics, University of Belgrade,
Studentski trg 16, pb. 550, 11000 Belgrade, Serbia and Montenegro
mdjoric@matf.bg.ac.yu

Dedicated to Professor Lieven Vanhecke

Summary. We study m-dimensional real submanifolds of codimension p with $(m-1)$-dimensional maximal holomorphic tangent subspace in a Kähler manifold. Consequently, on these manifolds there exists an almost contact structure (F, u, U, g) naturally induced from the ambient space. In this paper, we study a certain commutative condition on the almost contact structure and on the second fundamental form of these submanifolds.

1 Introduction

Let N be a real hypersurface of an almost Hermitian manifold $\overline{N}$. In [23] Y. Tashiro showed that in this case N is equipped with an almost contact metric structure naturally induced by the almost Hermitian structure on $\overline{N}$. This has been a fertile field for many authors, in particular when $\overline{N}$ is a complex space form. See [3], [11] and [20] for more details and further references. Above all, M. Okumura, S. Montiel and A. Romero gave a geometric meaning of the commutativity of the second fundamental tensor of the real hypersurface of a complex space form, and its almost induced contact structure ([16], [20]).

In [8] and [9] M. Okumura and the author of this paper considered similar problems by studying CR-submanifolds of maximal CR-dimension in complex space forms. Namely, let M^m be a real submanifold of the complex manifold $(\overline{M}^{m+p}, \overline{g})$ with complex structure J. If, for any $x \in M$, the tangent space $T_x(M)$ of M at x satisfies $\dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = m - 1$, then M is called a CR-submanifold of maximal CR-dimension. It follows that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \mathrm{span}\{\xi_x\}$, for any $x \in M$. A real hypersurface is a typical example of a CR-submanifold of maximal CR-dimension and the generalization of some results which are valid for real hypersurfaces to CR-submanifolds of maximal CR-dimension may be expected, see for example [5]. In the real hypersurface case and in particular when $\overline{M}$ is a Kähler manifold, many results have been obtained. See, for example,

[17] for the fundamental definitions and results and for further references. On the other hand, for arbitrary codimension, less detailed results are known but may be expected. For example, we refer to [21], [5] and [6].

In the present article, we continue the above-mentioned study and we discuss a certain commutative condition on the almost contact structure and on the second fundamental form of these submanifolds. Namely, in [9] M. Okumura and the author of this paper investigated CR-submanifolds of maximal CR-dimension in complex space forms which satisfy the condition $h(FX, Y) + h(X, FY) = 0$, under the assumption that the distinguished vector field ξ is parallel in the normal bundle, where F is the induced almost contact structure and h is the second fundamental form of M. We proved that under this condition there exists a totally geodesic complex space form M' of $\overline{M}$, such that M is a real hypersurface of M'. Therefore, it was possible to apply the results of real hypersurface theory ([16], [20]) and prove some classification theorems for CR-submanifolds of maximal CR-dimension in complex projective space, complex hyperbolic space and complex Euclidean space.

Moreover, it was proved in [8] that if a CR-submanifold of maximal CR-dimension in a complex space form satisfies the commutative condition $h(FX, Y) - h(X, FY) = 0$, under the assumption that the distinguished vector field ξ is parallel in the normal bundle, then the holomorphic sectional curvature of the ambient manifold is non-positive.

The main purpose of this paper is to continue the study of the commutative condition $h(FX, Y) - h(X, FY) = 0$, but after leaving out the assumption that the distinguished normal vector field ξ is parallel with respect to the normal connection and in the case when the ambient manifold is not necessarily a complex space form, but is a Kähler manifold. In section 2, we recall some general preliminary facts concerning submanifolds (see [4] and [14] for more details and further references) and especially CR-submanifolds (see [26], [25] and [1]) and derive useful formulas for later use. Section 3 is devoted to the study of CR-submanifolds of maximal CR-dimension in Kähler manifolds which satisfy the condition $h(FX, Y) - h(X, FY) = 0$.

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2 CR-submanifolds of maximal CR-dimension of a Kähler manifold

Let $\overline{M}$ be an $(m + p)$-dimensional Kähler manifold with Kähler structure $(J, \bar{g})$ and M an m-dimensional real submanifold of $\overline{M}$ with immersion $\imath$ of M into $\overline{M}$. Also, we denote by $\imath$ the differential of the immersion, or we omit to mention $\imath$, for brevity of notation. The Riemannian metric g of M is induced from the Riemannian metric $\bar{g}$ of $\overline{M}$ in such a way that $g(X, Y) = \bar{g}(\imath X, \imath Y)$, where $X, Y \in T(M)$. We denote by $T(M)$ and $T^{\perp}(M)$ the tangent bundle and the normal bundle of M, respectively.

Next, it is known that, for any $x \in M$, the subspace $H_x(M) = JT_x(M) \cap T_x(M)$ is the maximal J-invariant subspace of the tangent space $T_x(M)$ at x, and it is called the holomorphic tangent space to M at x. In general, the dimension of $H_x(M)$ varies with x (see [6], for example), but if the subspace $H_x(M)$ has constant dimension for any $x \in M$, a submanifold M is called the Cauchy–Riemann submanifold or briefly CR-submanifold and the constant complex dimension of $H_x(M)$ is called the CR-dimension of M ([18], [25]). It is well-known that a real hypersurface is one of the typical examples of CR-submanifolds whose CR-dimension is $(m-1)/2$, where m is the dimension of a hypersurface. It is easily seen that if M is a CR-submanifold in the sense of Bejancu's definition given in [1], M is also a CR-submanifold in the sense of the above-given definition. In the case when M is a CR-submanifold of CR-dimension $(m-1)/2$, these definitions coincide. On the other hand, when the CR-dimension is less than $(m-1)/2$, the converse is not true. We refer to [6] for more details and examples of CR-submanifolds of maximal CR-dimension.

In the sequel we consider CR-submanifolds of maximal CR-dimension, that is, $\dim_{\mathbb{R}} H_x(M) = \dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = m-1$. Then it follows that M is odd-dimensional and that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \operatorname{span}\{\xi_x\}$, for any $x \in M$. Hence, for any $X \in T(M)$, choosing a local orthonormal basis $\xi, \xi_1, \ldots, \xi_{p-1}$ of vectors normal to M, we have the following decomposition into tangential and normal components:

$$J\imath X = \imath FX + u(X)\xi, \tag{1}$$

$$J\xi = -\imath U + P\xi, \tag{2}$$

$$J\xi_a = -\imath U_a + P\xi_a \qquad (a = 1, \ldots, p-1). \tag{3}$$

Here F and P are skew-symmetric endomorphisms acting on $T(M)$ and $T^{\perp}(M)$ respectively, $U, U_a, a = 1, \ldots, p-1$ are tangent vector fields and u is one form on M. Furthermore, using (1), (2) and (3), the Hermitian property of J implies,

$$g(U, X) = u(X), \quad U_a = 0 \quad (a = 1, \ldots, p-1), \tag{4}$$

$$F^2 X = -X + u(X)U, \tag{5}$$

$$u(FX) = 0, \quad FU = 0, \quad P\xi = 0. \tag{6}$$

Hence, relations (2) and (3) may be written in the form

$$J\xi = -\imath U, \quad J\xi_a = P\xi_a \quad (a = 1, \ldots, p-1). \tag{7}$$

Moreover, these relations imply that (F, u, U, g) defines an almost contact metric structure on M (see [24], [2]).

Since $\{\eta \in T^{\perp}(M), \eta \perp \xi\}$ is J-invariant, from now on let us denote the orthonormal basis of $T^{\perp}(M)$ by $\xi, \xi_1, \ldots, \xi_q, \xi_{1^*}, \ldots, \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = (p-1)/2$. Next, let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{M}$ and M respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. They are related by the following well-known Gauss equation

$$\overline{\nabla}_{\imath X}\imath Y = \imath \nabla_X Y + h(X, Y), \tag{8}$$

where h denotes the second fundamental form, and by Weingarten equations,

$$\overline{\nabla}_{\imath X}\xi = -\imath AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}, \tag{9}$$

$$\overline{\nabla}_{\imath X}\xi_a = -\imath A_a X - s_a(X)\xi + \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\}, \tag{10}$$

$$\overline{\nabla}_{\imath X}\xi_{a^*} = -\imath A_{a^*} X - s_{a^*}(X)\xi + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\}, \tag{11}$$

where the s's are the coefficients of the normal connection D and A, A_a, A_{a^*}, $a = 1, \dots, q$, are the shape operators corresponding to the normals ξ, ξ_a, ξ_{a^*}, respectively. They are related to the second fundamental form by

$$h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^{q}\{g(A_a X, Y)\xi_a + g(A_{a^*}X, Y)\xi_{a^*}\}. \tag{12}$$

Since the ambient manifold is a Kähler manifold, using (1), (7), (10) and (11), it follows that

$$A_{a^*}X = FA_a X - s_a(X)U, \tag{13}$$

$$A_a X = -FA_{a^*}X + s_{a^*}(X)U, \tag{14}$$

$$s_{a^*}(X) = u(A_a X) = g(A_a X, U) = g(A_a U, X), \tag{15}$$

$$s_a(X) = -u(A_{a^*}X) = -g(A_{a^*}X, U) = -g(A_{a^*}U, X), \tag{16}$$

$$s_{a^*b^*} = s_{ab}, \quad s_{a^*b} = -s_{ab^*}, \tag{17}$$

for all X, Y tangent to M and $a, b = 1, \dots, q$. Therefore,

$$\mathrm{trace} A_a = s_{a^*}(U), \quad \mathrm{trace} A_{a^*} = -s_a(U), \quad (a = 1, \dots, q). \tag{18}$$

Moreover, since F is skew-symmetric, and A_a and A_{a^*}, $(a = 1, \dots, q)$ are symmetric, (14) and (16) imply

$$g((A_a F + FA_a)X, Y) = u(Y)s_a(X) - u(X)s_a(Y), \tag{19}$$

$$g((A_{a^*}F + FA_{a^*})X, Y) = u(Y)s_{a^*}(X) - u(X)s_{a^*}(Y), \tag{20}$$

for all $a = 1, \dots, q$.

Next, differentiating relation (1), using (7), (8), (9) and (12), and comparing the tangential and normal parts, we get,

$$(\nabla_Y F)X = u(X)AY - g(AY, X)U, \tag{21}$$

$$(\nabla_Y u)(X) = g(FAY, X), \tag{22}$$

$$\nabla_X U = FAX. \tag{23}$$

Furthermore, the Gauss equation and the Codazzi equation (for the distinguished vector ξ) become

$$\overline{g}(\overline{R}_{iX\,iY}iZ, iW) = g(R_{XY}Z, W) - g(AY, Z)g(AX, W) + g(AX, Z)g(AY, W) \tag{24}$$

$$-\sum_{b=1}^{q}\{g(A_bY,Z)g(A_bX,W)-g(A_bX,Z)g(A_bY,W)\}$$
$$-\sum_{b=1}^{q}\{g(A_{b^*}Y,Z)g(A_{b^*}X,W)-g(A_{b^*}X,Z)g(A_{b^*}Y,W)\},$$

$$\overline{g}(\overline{R}_{iX\,iY}iZ,\xi)=g((\nabla_XA)Y-(\nabla_YA)X,Z)$$
$$+\sum_{b=1}^{q}\{-g(A_bY,Z)s_b(X)+g(A_bX,Z)s_b(Y)\}$$
$$+\sum_{b=1}^{q}\{-g(A_{b^*}Y,Z)s_{b^*}(X)+g(A_{b^*}X,Z)s_{b^*}(Y)\}, \qquad (25)$$

for all X, Y, Z tangent to M. Here, $\overline{R}$ and R denote the Riemannian curvature tensors of $\overline{M}$ and M respectively. Moreover, the Codazzi equations for normal vectors ξ_a, $a=1,\dots,p-1$ are

$$\overline{g}(\overline{R}_{iX\,iY}iZ,\xi_a)=g((\nabla_XA_a)Y-(\nabla_YA_a)X,Z)$$
$$-g(AY,Z)s_a(X)+g(AX,Z)s_a(Y)$$
$$+\sum_{b=1}^{q}\{g(A_bY,Z)s_{ba}(X)-g(A_bX,Z)s_{ba}(Y)\}$$
$$+\sum_{b=1}^{q}\{g(A_{b^*}Y,Z)s_{b^*a}(X)-g(A_{b^*}X,Z)s_{b^*a}(Y)\}. \qquad (26)$$

Finally, if the ambient manifold $\overline{M}$ is a complex space form, i.e. a Kähler manifold of constant holomorphic sectional curvature $4k$, then

$$\overline{R}_{\overline{X}\,\overline{Y}}\overline{Z}=k\left\{\bar{g}(\overline{Y},\overline{Z})\overline{X}-\bar{g}(\overline{X},\overline{Z})\overline{Y}+\bar{g}(J\overline{Y},\overline{Z})J\overline{X}\right.$$
$$\left.-\bar{g}(J\overline{X},\overline{Z})J\overline{Y}-2\bar{g}(J\overline{X},\overline{Y})J\overline{Z}\right\}, \qquad (27)$$

for $\overline{X}$, $\overline{Y}$, $\overline{Z}$ tangent to $\overline{M}$.

3 CR-submanifolds satisfying $h(FX,Y)-h(X,FY)=0$

CR-submanifolds M^m of maximal CR-dimension of complex space forms whose distinguished normal vector field ξ is parallel with respect to the normal connection and which satisfy the condition

$$h(FX,Y)-h(X,FY)=0, \qquad (28)$$

for all $X, Y \in T(M)$, were studied in [8] and the following theorem was proved:

Theorem 1 ([8]). *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a complex space form. If the distinguished normal vector field ξ is parallel with respect to the normal connection, and if the condition (28) is satisfied, then the holomorphic sectional curvature of the ambient manifold is non-positive.*

Now, we continue this study, dropping the condition of the parallelism (with respect to the normal connection) of the distinguished normal vector field ξ.

Using relation (12), we obtain,

$$h(FX,Y)-h(X,FY)=\{g(AFX,Y)-g(AX,FY)\}\xi + \sum_{a=1}^{q}\{[g(A_aFX,Y)-g(A_aX,FY)]\xi_a + [g(A_{a^*}FX,Y)-g(A_{a^*}X,FY)]\xi_{a^*}\}. \tag{29}$$

Therefore, since F is skew-symmetric, it follows that the relation (28) is equivalent to

$$AF+FA=0, \tag{30}$$

$$A_aF+FA_a=0, \tag{31}$$

$$A_{a^*}F+FA_{a^*}=0. \tag{32}$$

We continue this section by recalling the definition of the Levi form and by deducing one more sufficient condition for a submanifold of a Kähler manifold to be Levi-flat. For more details, we refer to [6], [7], [10], [13].

For this purpose, let us assume that M is a CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. Further, let $H_x^C(M)$ be the complexification of $H_x(M)$ and

$$H_x^{(0,1)}(M)=\left\{\imath X+\sqrt{-1}J\imath X | X\in H_x(M)\right\},$$

$$H_x^{(1,0)}(M)=\left\{\imath X-\sqrt{-1}J\imath X | X\in H_x(M)\right\}.$$

Then $H_x^C(M)=H_x^{(0,1)}(M)\oplus H_x^{(1,0)}(M)$ and we can define the following sub-bundles of the complexification of the tangent bundle $T^C(M)$: $H^C(M)=\bigcup_{x\in M}H_x^C(M)$, $H^{(0,1)}(M)=\bigcup_{x\in M}H_x^{(0,1)}(M)$, $H^{(1,0)}(M)=\bigcup_{x\in M}H_x^{(1,0)}(M)$.

Then, it is well-known that both distributions $H^{(0,1)}(M)$ and $H^{(1,0)}(M)$ are involutive. However, this does not imply that $H^C(M)=H^{(0,1)}(M)\oplus H^{(1,0)}(M)$ is involutive and the Levi form is defined in such a way that it measures the degree to which $H^C(M)$ fails to be involutive ([10], [13]). Hence,

Definition 1. The Levi form L is the projection of $J[J\imath X,\imath Y]$ to $T(M)^\perp$ for $X,Y\in H(M)$.

Therefore, $H^C(M)$ is involutive if and only if the Levi form vanishes identically. Further, we need the following well-known

Theorem 2 ([10]). *Let $\overline{M}$ be a Kähler manifold and M be a real submanifold of $\overline{M}$. Then,*

$$L(X,Y)=h(X,Y)+h(FX,FY), \tag{33}$$

for $X,Y\in H(M)$, where h denotes the second fundamental form with respect to the Riemannian connection $\overline{\nabla}$ of $\overline{M}$ and F is the skew-symmetric endomorphism acting on $T(M)$.

Finally, when in M the Levi form vanishes identically, M is called a Levi-flat submanifold. In his recent work Siu ([22]) proved that there does not exist a smooth Levi-flat hypersurface of complex projective space of dimension ≥ 3. However, if we do not assume that M is complete, the example of a real hypersurface, which is given by Kimura ([12]), is Levi-flat.

As an immediate consequence, using (5), we have from (30) and (33):

Proposition 1. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then the manifold M is Levi-flat.*

Further, if relation (30) holds at a point of the submanifold M, using (6), we get

$$AU = \alpha U, \tag{34}$$

where we have put $\alpha = u(AU)$. Thus, the following lemma holds:

Lemma 1. *Let M be an m-dimensional CR-submanifold of maximal CR-dimension of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then U is an eigenvector of the shape operator A with respect to distinguished normal vector field ξ, at any point of M.*

Further, let M be an m-dimensional CR-submanifold of maximal CR-dimension of a Kähler manifold $\overline{M}$ such that the condition (30) is satisfied. Denote by $\mathcal{D}$ the distribution spanned by all vectors orthogonal to U. Then, we can easily see that $\mathcal{D}$ is an involutive distribution. Namely, let $X, Y \in \mathcal{D}$, whereas using relation (23), it follows that $g([X,Y],U) = -g((FA+AF)X,Y)$. Therefore, relation (30) implies $g([X,Y],U) = 0$, i.e. that $[X,Y] \in \mathcal{D}$, which shows that $\mathcal{D}$ is involutive. So we have

Lemma 2. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then the distribution $\mathcal{D}$, which is spanned by all vectors orthogonal to U, is involutive.*

Further, we have

Theorem 3. *Let M be an m-dimensional CR-submanifold of maximal CR-dimension of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then the integral submanifold $M_{\mathcal{D}}$ of the distribution $\mathcal{D}$, which is spanned by all vectors orthogonal to U, is a Kähler manifold.*

Proof. Let us denote by j the immersion of $M_{\mathcal{D}}$ in M. Since U is a unit normal to $M_{\mathcal{D}}$ in M, we note that $g(U, jX') = 0$ for X' tangent to $M_{\mathcal{D}}$. Then, we have the following decomposition into tangential and normal components:

$$FjX' = jF'X' + u'(X')U. \tag{35}$$

Thus, from (6) it follows that

$$F^2 jX' = jF'^2 X' + u'(F'X')U. \tag{36}$$

Further, since $u(jX') = 0$, using (5), we have

$$F^2 jX' = -jX'. \tag{37}$$

Now, (36) and (37) yield $u'(F'X') = 0$ and thus, we have

$$F'^2 X' = -X', \tag{38}$$

i.e. F' is an almost complex structure on $M_{\mathcal{D}}$.

Further, relation (35) implies

$$FjX' = jF'X'. \tag{39}$$

Finally, using the Gauss equation (8) and relation (21), and comparing tangential and normal components, it follows that

$$(\nabla'_{X'} F')Y' = 0,$$

and hence, $M_{\mathcal{D}}$ is a Kähler manifold.

In what follows, we shall consider cosymplectic manifolds. For this purpose, we first recall that an odd-dimensional differentiable manifold M^{2k-1} is an almost cosymplectic manifold, if there exist a 1-form φ and a 2-form π such that

$$\varphi \wedge \pi^{k-1} \neq 0, \tag{40}$$

at each point of M^{2k-1} (see, for example [15]). The pair (φ, π) is called an almost cosymplectic structure on M^{2k-1}. If, on an almost cosymplectic manifold, the 1-form φ and the 2-form π are both closed, the manifold is called a cosymplectic manifold. Further, M. Okumura in [19] studied the cosymplectic hypersurfaces of complex space forms. Having found a certain condition for the structure to be cosymplectic, he studied non-existence of the cosymplectic hypersurfaces under certain conditions for the scalar curvature of a complex space form and proved that the scalar curvature of a cosymplectic hypersurface of a complex space form is a non-positive constant.

Now, let M be an $m = 2l + 1$-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$, with an almost contact metric structure (F, u, U, g). Further, let ω be a two-form defined by $\omega(X, Y) = g(FX, Y)$. Then, since F has rank $2l$, it follows that

$$u \wedge \omega^l \neq 0,$$

which shows that M is an almost cosymplectic manifold. Moreover, since F is a skew-symmetric endomorphism, using (22), it follows that

$$du(X, Y) = g(FAX + AFX, Y)$$

and therefore,

$$du = 0 \Leftrightarrow FA + AF = 0. \tag{41}$$

It is easily seen, using relation (1), that $\omega = \iota^*\Omega$, where Ω is a Kähler form of $\overline{M}$, and therefore, $d\omega = 0$. Combining this with relation (41), we obtain:

Theorem 4. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then M is a cosymplectic manifold.*

Further, denoting the second eigenvalue of the shape operator A with respect to distinguished normal vector field ξ by λ, and the corresponding eigenvector by X, that is $AX = \lambda X$, relation (30) implies that FX is an eigenvector of A corresponding to the eigenvalue $-\lambda$. Therefore, we proved

Lemma 3. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. If the condition (30) is satisfied, then if $X \in T(M)$ and if $X \perp U$ is an eigenvector with eigenvalue λ of the shape operator A with respect to distinguished normal vector field ξ, then FX is an eigenvector of A with corresponding eigenvalue $-\lambda$, and, therefore, with respect to a suitable orthonormal frame, A has the following form:*

$$A = \begin{pmatrix} \alpha & & & & & & \\ & \lambda^1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda^q & & & \\ & & & & -\lambda^1 & & \\ & & & & & \ddots & \\ & & & & & & -\lambda^q \end{pmatrix}.$$

Consequently, $\operatorname{trace} A = \alpha$.

In the above consideration we used only the condition (30). Now, we continue our study by exploring the conditions (31) and (32). Proceeding in a similar way as in the proof of Lemma 3, and using (31) and (32), we easily prove:

Lemma 4. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$. If the condition (28) is satisfied, then $A_a U = \alpha_a U$ and $A_{a^*} U = \alpha_{a^*} U$ for $a = 1, \ldots, q$, where A_a, A_{a^*}, $a = 1, \ldots, q$, are the shape operators corresponding to the normals ξ_a, ξ_{a^*} respectively. Moreover, with respect to a suitable orthonormal frame,*

$$A_a = \begin{pmatrix} \alpha_a & & & & & & \\ & \lambda_a^1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_a^q & & & \\ & & & & -\lambda_a^1 & & \\ & & & & & \ddots & \\ & & & & & & -\lambda_a^q \end{pmatrix}$$

and, similarly,

$$A_{a^*} = \begin{pmatrix} \alpha_{a^*} & & & & & & \\ & \lambda^1_{a^*} & & & & & \\ & & \ddots & & & & \\ & & & \lambda^q_{a^*} & & & \\ & & & & -\lambda^1_{a^*} & & \\ & & & & & \ddots & \\ & & & & & & -\lambda^q_{a^*} \end{pmatrix}.$$

Further, it follows from (19), (20), (31) and (32) that

$$u(Y)s_a(X) = u(X)s_a(Y), \quad u(Y)s_{a^*}(X) = u(X)s_{a^*}(Y),\ a = 1, \dots, q, \tag{42}$$

and therefore,

$$s_a(X) = u(X)s_a(U), \quad s_{a^*}(X) = u(X)s_{a^*}(U),\ a = 1, \dots, q. \tag{43}$$

Lemma 1 implies that U is an eigenvector of the shape operator A with respect to distinguished normal vector field ξ, at any point of M, that is $AU = \alpha U$. Hence, using (23), it follows that

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, U) + g(AFAX, Y) - g(AFAY, X) \\ &\quad = (X\alpha)g(U, Y) - (Y\alpha)g(U, X) + \alpha g(FAX, Y) - \alpha g(FAY, X). \end{aligned} \tag{44}$$

From now on, we suppose that the ambient manifold $\overline{M}$ is a Kähler manifold of constant holomorphic sectional curvature $4k$. Then we have, by straightforward computation and using (1) and (27)

$$\overline{g}(\overline{R}(\imath X, \imath Y)\imath Z, \xi) = k\{g(FY, Z)u(X) - g(FX, Z)u(Y) - 2g(FX, Y)u(Z)\}. \tag{45}$$

Hence, using relations (25) and (45), it follows that

$$\begin{aligned} &k\{g(FY, Z)u(X) - g(FX, Z)u(Y) - 2g(FX, Y)u(Z)\} = g((\nabla_X A)Y - (\nabla_Y A)X, Z) \\ &\quad + \sum_{b=1}^{q}\{-g(A_b Y, Z)s_b(X) + g(A_b X, Z)s_b(Y)\} \\ &\quad + \sum_{b=1}^{q}\{-g(A_{b^*} Y, Z)s_{b^*}(X) + g(A_{b^*} X, Z)s_{b^*}(Y)\}. \end{aligned} \tag{46}$$

Replacing Z by U and using relation (43), the last equation reduces to

$$g((\nabla_X A)Y - (\nabla_Y A)X, U) = -2k\, g(FX, Y). \tag{47}$$

Further, since F is skew-symmetric, condition (30) and relations (44) and (47) imply

$$-2k\,g(FX,Y)-2\,g(FA^2X,Y)=(X\alpha)u(Y)-(Y\alpha)u(X). \tag{48}$$

Replacing Y by U in the last equation, direct computation yields

$$k\,g(FX,Y)+g(FA^2X,Y)=0, \tag{49}$$

and thus, using (5), it follows that

$$A^2X+k\,X+\beta U=0. \tag{50}$$

Therefore, if $X\in T(M)$, $X\perp U$ is an eigenvector of the shape operator A with respect to distinguished normal vector field ξ, namely $AX=\lambda\,X$, then

$$\lambda^2+k=0. \tag{51}$$

Hence we have:

Lemma 5. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$ with constant holomorphic sectional curvature $4k$. If the condition (28) is satisfied, then the shape operator A with respect to distinguished normal vector field ξ admits at most three distinct eigenvalues: α (corresponding to U), $\sqrt{-k}$ and $-\sqrt{-k}$.*

As an immediate consequence we have from (28) and (51):

Corollary 1. There exists no CR-submanifold of maximal CR-dimension of a Kähler manifold with constant positive holomorphic sectional curvature satisfying the condition (28).

Therefore, assuming that the ambient manifold $\overline{M}$ is of constant non-positive holomorphic sectional curvature, and using the above consideration, we conclude that the shape operator A with respect to the distinguished normal vector field ξ can be diagonalized as follows that

$$A=\begin{pmatrix}\alpha & & & & & & \\ & \sqrt{-k} & & & & & \\ & & \ddots & & & & \\ & & & \sqrt{-k} & & & \\ & & & & -\sqrt{-k} & & \\ & & & & & \ddots & \\ & & & & & & -\sqrt{-k}\end{pmatrix}.$$

Further, we consider the case when the shape operator A with respect to the distinguished normal vector field ξ has three distinct eigenvalues: α (of multiplicity one), $\sqrt{-k}$ and $-\sqrt{-k}$. For that purpose, let

$$\mathcal{D}_+ = \{X | AX = \sqrt{-k}X, g(X, U) = 0\}, \tag{52}$$

$$\mathcal{D}_- = \{X | AX = -\sqrt{-k}X, g(X, U) = 0\}. \tag{53}$$

Then, it follows that

$$\mathcal{D}(p) = D_+(p) \oplus D_-(p)$$

at each point of the integral submanifold $M_{\mathcal{D}}$ of $\mathcal{D}$. Moreover,

Proposition 2. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$ with constant holomorphic sectional curvature $4k$. If the condition (28) is satisfied, then the distributions $\mathcal{D}_+$ and $\mathcal{D}_-$, defined by (52) and (53), are both integrable.*

Proof. Let us consider the case $X, Y \in \mathcal{D}_+$, having in mind that the proof of the latter case is analogous.

Differentiating $g(X, U) = 0$ by $Y \in \mathcal{D}_+$ and using (23), it follows that $g(\nabla_Y X, U) = -g(X, FAY)$. Therefore, since F is skew-symmetric and using (30), it follows that $g([X, Y], U) = 0$.

Further, in order to prove $[X, Y] \in \mathcal{D}_+$, for $X, Y \in \mathcal{D}_+$, by differentiating $AX = \sqrt{-k}$ and $AY = \sqrt{-k}$, we obtain

$$(\nabla_X A)Y - (\nabla_Y A)X + A[X, Y] = \sqrt{-k}[X, Y]. \tag{54}$$

According to the assumption, it follows that $FX \in \mathcal{D}_-$ (see Lemma 3). Moreover, using (43) and (46) it follows that $A[X, Y] = \sqrt{-k}[X, Y]$ and thus $[X, Y] \in \mathcal{D}_+$.

Lemma 6. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a complex space form $\overline{M}$. If the condition (28) is satisfied, then it follows that on $\mathcal{D}$*

$$AA_a + A_a A = 0, \quad AA_{a^*} + A_{a^*} A = 0, \quad a = 1, \ldots, q. \tag{55}$$

Proof. According to Lemma 4, U is an eigenvector of the shape operator A_a, that is $A_a U = \alpha_a U$. Differentiating this equation covariantly and using relations (23) and (30), we obtain

$$\begin{aligned} &g((\nabla_X A_a)Y - (\nabla_Y A_a)X, U) + g((A_a FA + AFA_a)X, Y) \\ &\quad = (X\alpha_a)u(Y) - (Y\alpha_a)u(X). \end{aligned} \tag{56}$$

Next, since $\overline{M}$ is a complex space form, using relation (27) and Codazzi equation (26), we get,

$$\begin{aligned} g((\nabla_X A_a)Y - (\nabla_Y A_a)X, Z) &= g(AY, Z)s_a(X) - g(AX, Z)s_a(Y) \\ &\quad + \sum_{b=1}^{q} \{g(A_b X, Z)s_{ba}(Y) - g(A_b Y, Z)s_{ba}(X)\} \\ &\quad + \sum_{b=1}^{q} \{g(A_{b^*} X, Z)s_{b^*a}(Y) - g(A_{b^*} Y, Z)s_{b^*a}(X)\}. \end{aligned} \tag{57}$$

Replacing Z by U in the last relation, and using relations (15) and (16), it follows that

$$g((\nabla_X A_a)Y - (\nabla_Y A_a)X, U) = g(AY, U)s_a(X) - g(AX, U)s_a(Y) + \sum_{b=1}^{q}\{s_{b^*}(X)s_{ba}(Y) - s_{b^*}(Y)s_{ba}(X) - s_b(X)s_{b^*a}(Y) + s_b(Y)s_{b^*a}(X)\}. \tag{58}$$

Further, using (34) and (43), the last relation reduces to

$$g((\nabla_X A_a)Y - (\nabla_Y A_a)X, U) = u(X)\sum_{b=1}^{q}\{s_{b^*}(U)s_{ba}(Y) - s_b(U)s_{b^*a}(Y)\} + u(Y)\sum_{b=1}^{q}\{s_b(U)s_{b^*a}(X) - s_{b^*}(U)s_{ba}(X)\}. \tag{59}$$

Now, relations (56) and (59) yield

$$u(X)\sum_{b=1}^{q}\{s_{b^*}(U)s_{ba}(Y) - s_b(U)s_{b^*a}(Y)\} + u(Y)\sum_{b=1}^{q}\{s_b(U)s_{b^*a}(X) - s_{b^*}(U)s_{ba}(X)\} + g((A_aFA + AFA_a)X, Y) = (X\alpha_a)u(Y) - (Y\alpha_a)u(X). \tag{60}$$

Replacing Y by U in the last relation, since $A_aU = \alpha_aU$, we get

$$\sum_{b=1}^{q}\{s_b(U)s_{b^*a}(X) - s_{b^*}(U)s_{ba}(X)\} = X\alpha_a - (U\alpha_a)u(X) - u(X)\sum_{b=1}^{q}\{s_{b^*}(U)s_{ba}(U) - s_b(U)s_{b^*a}(U)\}. \tag{61}$$

Now, relations (60) and (61) imply

$$g((A_aFA + AFA_a)X, Y) = 0. \tag{62}$$

Finally, relations (62), (30), (31) and (5) prove the required result.

The proof of the latter case is analogous.

Corollary 2. Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a complex space form $\overline{M}$. If the condition (28) is satisfied then if $X \in T(M)$ and if $X \perp U$ is an eigenvector with eigenvalue λ of the shape operator A with respect to the distinguished normal vector field ξ, then A_aX is an eigenvector of A with corresponding eigenvalue $-\lambda$, and, analogously for A_{a^*}.

In Theorem 3 we proved that the integral submanifold $M_{\mathcal{D}}$ is a Kähler manifold if the condition (30) is satisfied and if $\overline{M}$ is a Kähler manifold. Now, we prove

Theorem 5. *Let M be an m-dimensional CR-submanifold of CR-dimension $(m-1)/2$ of a Kähler manifold $\overline{M}$ of constant holomorphic sectional curvature $4k$. If the condition (28) is satisfied, then the holomorphic sectional curvature of the integral submanifold $M_{\mathcal{D}}$ of the distribution $\mathcal{D}$, which is spanned by all vectors orthogonal to U, is non-positive.*

Proof. First we note that the holomorphic sectional curvature $H'(X')$ of the integral submanifold $M_{\mathcal{D}}$ of the distribution $\mathcal{D}$ is given by

$$H'(X') = \frac{g'(R'_{X'\,F'X'}F'X', X')}{g'(X', X')^2}, \tag{63}$$

for X' tangent to $M_{\mathcal{D}}$, where R' denotes the Riemannian curvature tensor of $M_{\mathcal{D}}$ and j is the immersion of $M_{\mathcal{D}}$ in M, as defined in Theorem 3. Moreover, since U is a unit normal to $M_{\mathcal{D}}$ in M, we note that $g(U, jX') = 0$ for X' tangent to $M_{\mathcal{D}}$ and we recall relations (38) and (39), which are important for later considerations.

Since another case can be proved quite analogously, we will suppose that $X \in \mathcal{D}_+$ and we will identify X' and X from now on. This means that $X \perp U$, $AX = \lambda X$, $\lambda = \sqrt{-k}$, and therefore, using Lemma 3 it follows that $FX \in \mathcal{D}_-$, i.e., $AFX = -\lambda FX$. Moreover, Corollary 2 yields $AA_aX = -\lambda A_aX$ and $g(A_aX, X) = 0$ and analogously $g(A_{a^*}X, X) = 0$.

Therefore, using the Gauss equation for $M_{\mathcal{D}}$ in M, we have

$$\begin{aligned} g(R_{X\,FX}FX, X) = {} & g'(R'_{X\,F'X}F'X, X) - g'(A'F'X, F'X)g'(A'X, X) \\ & + g'(A'X, F'X)g'(A'F'X, X), \end{aligned} \tag{64}$$

where A' is the shape operator of $M_{\mathcal{D}}$ in M. Now, using (23), we obtain

$$g'(A'X, Y) = -g(FAX, Y). \tag{65}$$

Thus, (64) and (65) yield

$$\begin{aligned} g'(R'_{X\,F'X}F'X, X) = {} & g(R_{X\,FX}FX, X) + g(FAFX, FX)g(FAX, X) \\ & - g(FAX, FX)g(FAFX, X). \end{aligned}$$

Moreover, since $X \in \mathcal{D}_+$, it follows that

$$g'(R'_{X\,F'X}F'X, X) = g(R_{X\,FX}FX, X) - \lambda^2 g(X, X)^2. \tag{66}$$

Now, using the Gauss equation (24), we obtain

$$\begin{aligned} \overline{g}(\overline{R}_{iX\,iFX}iFX, iX) = {} & g(R_{X\,FX}FX, X) \\ & + \lambda^2\, g(X, X)^2 + \sum_{b=1}^{q} g(A_bFX, X)^2 + \sum_{b=1}^{q} g(A_{b^*}FX, X)^2. \end{aligned} \tag{67}$$

On the other hand, since $\overline{M}$ is a complex space form of constant holomorphic sectional curvature $4k$, $k = -\lambda^2$, using (27) and (5), it follows that

$$\overline{g}(\overline{R}_{iX\,iFX}iFX, iX) = 4\,k\,g(X, X)^2. \tag{68}$$

Finally, using (67), (68) and (66), direct computation yields,

$$g'(R'_{X\,F'X}F'X, X) = 6k\,g(X, X)^2 - \sum_{a=1}^{q}\{g(A_aFX, X)^2 + g(A_{a^*}FX, X)^2\}. \tag{69}$$

The result follows now at once.

Corollary 3. Let M be a real hypersurface of a Kähler manifold $\overline{M}(k)$ of constant holomorphic sectional curvature $4k$. If the condition (28) is satisfied, then the holomorphic sectional curvature of the integral submanifold $M_{\mathcal{D}}$ of the distribution $\mathcal{D}$, which is spanned by all vectors orthogonal to U, is equal to $6k$, $k \leq 0$.

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The Geography of Non-Formal Manifolds

Marisa Fernández[1] and Vicente Muñoz[2]

[1]Departamento de Matemáticas, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
mtpferol@lg.ehu.es
[2]Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain
vicente.munoz@uam.es

Dedicated to Professor Lieven Vanhecke

Summary. We show that there exist non-formal compact oriented manifolds of dimension n and with first Betti number $b_1 = b \geq 0$ if and only if $n \geq 3$ and $b \geq 2$, or $n \geq (7 - 2b)$ and $0 \leq b \leq 2$. Moreover, we present explicit examples for each one of these cases.

1 Introduction

Simply connected compact manifolds of dimension less than or equal to 6 are formal [11, 10, 5]. A method to construct non-formal simply connected compact manifolds of any dimension $n \geq 7$ was given by the authors in [6]. An alternative method is given in [3] (see also [12] for an example in dimension 7). A natural question is whether there are examples of non-formal compact manifolds of any dimension whose first Betti number $b_1 = b \geq 0$ is arbitrary. We consider the following problem on the *geography* of manifolds:

For which pairs (n, b) with $n \geq 1$ and $b \geq 0$ are there compact oriented manifolds of dimension n and with $b_1 = b$ which are non-formal? Note that we can restrict to just considering connected manifolds. In this paper, we solve this problem completely by proving the following theorem.

Theorem 1. *There are compact oriented n-dimensional manifolds with $b_1 = b$ which are non-formal if and only if $n \geq 3$ and $b \geq 2$, or $n \geq (7 - 2b)$ and $0 \leq b \leq 2$.*

In the case of a simply connected manifold M, formality for M is equivalent to saying that its real homotopy type is determined by its real cohomology algebra. In the non-simply connected case, things are a little bit more complicated. If M is nilpotent,

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i.e., $\pi_1(M)$ is nilpotent and it acts nilpotently on $\pi_i(M)$ for $i \geq 2$, then formality means again that the real homotopy type is determined by the real cohomology algebra. In general, we shall say that M is formal, if the minimal model of the manifold (which is, by definition, the minimal model of the algebra of differential forms $\Omega^*(M)$) is determined by the real cohomology algebra (see Section 2 for precise definitions). Note that there are alternative (and non-equivalent) definitions of formality in the non-nilpotent situation (see [8]). This punctualization is important because the non-formal manifolds that we construct in Section 3 are necessarily not nilpotent (see Section 5). In the following table, the big dots mark the pairs (n, b_1) for which all manifolds of dimension n and first Betti number b_1 are formal. For any of the small dots, there are examples of non-formal manifolds. To prove Theorem 1 we need to do two things. On one hand, we need to verify that manifolds of dimension $n \leq 6$ with $b_1 = 0$ and manifolds of dimension $n \leq 4$ with $b_1 = 1$ are *always* formal. For this we use the results of [5]. On the other hand, we need to present examples of *non-formal* manifolds of dimension $n \geq 7$ with $b_1 = 0$, of dimension $n \geq 5$ with $b_1 = 1$ and of dimension $n \geq 3$ for any other $b_1 \geq 2$. For this we use a similar method to that of [6]. Note that both questions for the case $b_1 = 0$ are already solved, so here we have to focus on the case $b_1 = 1$.

Table 1. Geography of non-formal manifolds

$n \geq 7$	·	·	·	...
$n = 6$	•	·	·	...
$n = 5$	•	·	·	...
$n = 4$	•	•	·	...
$n = 3$	•	•	·	...
$n = 2$	•	•	•	•
	$b_1 = 0$	$b_1 = 1$	$b_1 = 2$	$b_1 \geq 3$

2 Minimal models and formality

We recall some definitions and results about minimal models [2, 7, 13]. Let (A, d) be a *differential algebra*, that is, A is a graded commutative algebra over the real numbers, with a differential d which is a derivation, i.e., $\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{\deg(a)} a \cdot (\mathrm{d}b)$, where $\deg(a)$ is the degree of a. Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. A differential algebra (A, d) is said to be *minimal* if:

1. A is free as an algebra, that is, A is the free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and
2. there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well-ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each $\mathrm{d}a_\tau$ is expressed in terms of preceding a_μ $(\mu < \tau)$. This implies that $\mathrm{d}a_\tau$ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

We shall say that a minimal differential algebra $(\bigwedge V, \mathrm{d})$ is a *minimal model* for a connected differentiable manifold M, if there exists a morphism of differential graded algebras $\rho\colon (\bigwedge V, \mathrm{d}) \longrightarrow (\Omega M, \mathrm{d})$, where ΩM is the de Rham complex of differential forms on M, inducing an isomorphism

$$\rho^*\colon H^*(\bigwedge V) \longrightarrow H^*(\Omega M, \mathrm{d}) = H^*(M)$$

on cohomology. If M is a simply connected manifold (or, more generally, a nilpotent space), the dual of the real homotopy vector space $\pi_i(M)\otimes\mathbf{R}$ is isomorphic to V^i for any i. Halperin in [7] proved that any connected manifold M has a minimal model unique up to isomorphism, regardless of its fundamental group. A minimal model $(\bigwedge V, \mathrm{d})$ of a manifold M is said to be *formal*, and M is said to be *formal*, if there is a morphism of differential algebras $\psi\colon (\bigwedge V, \mathrm{d}) \longrightarrow (H^*(M), \mathrm{d} = 0)$ that induces the identity on cohomology. Alternatively, the above property means that $(\bigwedge V, \mathrm{d})$ is a minimal model of the differential algebra $(H^*(M), 0)$. Therefore, $(\Omega M, \mathrm{d})$ and $(H^*(M), 0)$ share their minimal model, i.e., one can obtain the minimal model of M out of its real cohomology algebra. When M is nilpotent, the minimal model encodes its real homotopy type. In order to detect non-formality, we have Massey products. Let us recall its definition. Let M be a (not necessarily simply connected) manifold and let $a_i \in H^{p_i}(M)$, $1 \le i \le 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in M with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = \mathrm{d}\xi$, $\alpha_2 \wedge \alpha_3 = \mathrm{d}\eta$. The Massey product of the classes a_i is defined as

$$\langle a_1, a_2, a_3\rangle = [\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \alpha_3] \in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3}.$$

We have the following result, for whose proof we refer to [2, 13, 14].

Theorem 2. *If M has a non-trivial Massey product, then M is non-formal.*

Therefore, the existence of a non-zero Massey product is an obstruction to the formality.

In order to prove formality, we extract the following notion from [5].

Definition 1. Let $(\bigwedge V, \mathrm{d})$ be a minimal model of a differentiable manifold M. We say that $(\bigwedge V, \mathrm{d})$ is s-*formal*, or M is a s-*formal manifold* ($s \ge 0$) if for each $i \le s$ one can get a space of generators V^i of elements of degree i that decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

1. $\mathrm{d}(C^i) = 0$,
2. the differential map $\mathrm{d}\colon N^i \longrightarrow \bigwedge V$ is injective,
3. any closed element in the ideal $I_s = I(\bigoplus_{i\le s} N^i)$, generated by $\bigoplus_{i\le s} N^i$ in $\bigwedge(\bigoplus_{i\le s} V^i)$, is exact in $\bigwedge V$.

The condition of s-formality is weaker than that of formality. However, we have the following positive result proved in [5].

Theorem 3. *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if is $(n-1)$-formal (that is, if and only if M is s-formal, for $s = n-1$, according to the previous definition).*

This result is very useful because it allows us to check that a manifold M is formal by looking at its s-stage minimal model, that is, $\bigwedge(\bigoplus_{i\leq s} V^i)$. In general, when computing the minimal model of M, after we pass the middle dimension, the number of generators starts to grow quite dramatically. This is due to the fact that Poincaré duality imposes that the Betti numbers do not grow and therefore there are a large number of cup products in cohomology vanishing, which must be killed in the minimal model by introducing elements in N^i, for i above the middle dimension. This makes Theorem 3 a very useful tool for checking formality in practice.

3 Non-formal manifolds with $b_1 = 1$ and dimensions 5 and 6

The 5-dimensional example

Let H be the Heisenberg group, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbf{R}$. Then a global system of coordinates x, y, z for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that a basis for the left invariant 1-forms on H consists of $\{dx, dy, dz - x\, dy\}$. Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. So the quotient space $N = \Gamma\backslash H$ is a compact 3-dimensional nilmanifold. Hence the forms dx, dy, $dz - x\, dy$ descend to 1-forms α, β, γ on N and

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta.$$

The non-formality of N is detected by a non-zero triple Massey product

$$\langle[\beta], [\alpha], [\alpha]\rangle = [-\alpha \wedge \gamma] \in \frac{H^2(N)}{[\beta] \cup H^1(N) + H^1(N) \cup [\alpha]} = H^2(N).$$

Now, let us consider the 5-dimensional manifold $X = N \times \mathbf{T}^2$, where $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. The coordinates of $\mathbf{R}^2$ will be denoted x_1, x_2. So $\{dx_1, dx_2\}$ defines a basis $\{\delta_1, \delta_2\}$ for the 1-forms on $\mathbf{T}^2$. We get a non-zero triple Massey product as follows:

$$\langle[\beta \wedge \delta_1], [\alpha], [\alpha]\rangle = [-\gamma \wedge \alpha \wedge \delta_1]. \tag{1}$$

Our aim now is to kill the fundamental group of X by performing a suitable surgery construction, in order to obtain a manifold with $b_1 = 1$. The projection $p(x, y, z) = (x, y)$ describes N as a fiber bundle $p : N \to \mathbf{T}^2$ with fiber $\mathbf{S}^1$. Actually, N is the total space of the unit circle bundle of the line bundle of degree 1 over the 2-torus. The fundamental group of N is therefore

$$\pi_1(N) \cong \Gamma = \langle \lambda_1, \lambda_2, \lambda_3 \mid [\lambda_1, \lambda_2] = \lambda_3, \lambda_3 \text{ central}\rangle, \tag{2}$$

where λ_3 corresponds to the fiber. The fundamental group of $X = N \times \mathbf{T}^2$ is

$$\pi_1(X) = \pi_1(N) \oplus \mathbf{Z}^2. \tag{3}$$

Consider the following submanifolds embedded in X:

$$T_1 = p^{-1}(\{0\} \times \mathbf{S}^1) \times \{0\} \times \{0\},$$
$$T_2 = \{\xi\} \times \mathbf{S}^1 \times \mathbf{S}^1,$$

with ξ a point in N. These are 2-dimensional tori with trivial normal bundle. Consider now another 5-manifold Y with an embedded 2-dimensional torus T with trivial normal bundle. Then, we may perform the *fiber connected sum* of X and Y identifying T_1 and T, denoted $X\#_{T_1=T}Y$, in the following way: take (open) tubular neighborhoods $\nu_1 \subset X$ and $\nu \subset Y$ of T_1 and T respectively; then $\partial\nu_1 \cong \mathbf{T}^2 \times \mathbf{S}^2$ and $\partial\nu \cong \mathbf{T}^2 \times \mathbf{S}^2$; take an orientation reversing diffeomorphism $\phi : \partial\nu_1 \xrightarrow{\cong} \partial\nu$; the fiber connected sum is defined to be the (oriented) manifold obtained by gluing $X - \nu_1$ and $Y - \nu$ along their boundaries by the diffeomorphism ϕ. In general, the resulting manifold depends on the identification ϕ, but this will not be relevant for our purposes.

Lemma 1. *Suppose Y is simply connected, then the fundamental group of $X\#_{T_1=T}Y$ is the quotient of $\pi_1(X)$ by the image of $\pi_1(T_1)$.*

Proof. Since the codimension of T_1 is bigger than or equal to 3, we have that $\pi_1(X - \nu_1) = \pi_1(X - T_1)$ is isomorphic to $\pi_1(X)$. The Seifert–Van Kampen theorem establishes that $\pi_1(X\#_{T_1=T}Y)$ is the amalgamated sum of $\pi_1(X - \nu_1) = \pi_1(X)$ and $\pi_1(Y - \nu) = \pi_1(Y) = 1$ over the image of $\pi_1(\partial\nu_1) = \pi_1(T_1 \times \mathbf{S}^2) = \pi_1(T_1)$, as required. □

We shall take for Y the 5-sphere $\mathbf{S}^5$. We embed a 2-dimensional torus $\mathbf{T}^2$ in $\mathbf{R}^5$. This torus has a trivial normal bundle since its tangent bundle is trivial (being parallelizable) and the tangent bundle of $\mathbf{R}^5$ is also trivial. After compactifying $\mathbf{R}^5$ by one point, we get a 2-dimensional torus $T \subset \mathbf{S}^5$ with trivial normal bundle. In the same way, we may consider another copy of the 2-dimensional torus $T \subset \mathbf{S}^5$ and perform the fiber connected sum of X and $\mathbf{S}^5$ identifying T_2 and T. We may do both fiber connected sums along T_1 and T_2 simultaneously, since T_1 and T_2 are disjoint. Call

$$M = X\#_{T_1=T}\mathbf{S}^5\#_{T_2=T}\mathbf{S}^5 \tag{4}$$

the resulting manifold. By Lemma 1, $\pi_1(M)$ is the quotient of $\pi_1(X)$ by the images of $\pi_1(T_1)$ and $\pi_1(T_2)$. This kills the $\mathbf{Z}^2$ summand in (3) and it also kills λ_2 and λ_3 in (2). Therefore $\pi_1(M) = \langle\lambda_1\rangle \cong \mathbf{Z}$, i.e., $b_1(M) = 1$.

Our goal now is to prove that M is non-formal. We shall do this by proving the non-vanishing of a suitable triple Massey product. More specifically, let us prove that the Massey product (1) survives to M. For this, let us describe geometrically the cohomology classes $[\alpha \wedge \delta_1]$ and $[\beta]$. Consider the following submanifolds of X:

$$B_1 = p^{-1}(\mathbf{S}^1 \times \{a_2\}) \times \{b_1\} \times \mathbf{S}^1,$$
$$B_2 = p^{-1}(\{a_1\} \times \mathbf{S}^1) \times \mathbf{S}^1 \times \mathbf{S}^1,$$

where the a_i and b_i are generic points of $\mathbf{S}^1$. It is easy to check that $B_i \cap T_j = \emptyset$ for all i and j. So B_i may be also considered as submanifolds of M. Let η_i be the 2-forms representing the Poincaré dual to B_i in X. By [1], η_i can be taken supported in a small tubular neighborhood of B_i. Therefore the support of η_i lies inside $X - T_1 - T_2$, so we also have naturally $\eta_i \in \Omega^2(M)$. Note that in X we have clearly that $[\eta_1] = [\beta \wedge e_1]$ and $[\eta_2] = [\alpha]$, where e_1 is (the pull-back of) a differential 1-form on $\mathbf{S}^1$ (considered as the first of the two circle factors in $X = N \times \mathbf{S}^1 \times \mathbf{S}^1$) cohomologous to δ_1 and supported in a neighborhood of $b_1 \in \mathbf{S}^1$. Thus $[\eta_1] = [\beta \wedge \delta_1]$ in X.

Lemma 2. *The triple Massey product* $\langle [\eta_1], [\eta_2], [\eta_2] \rangle$ *is well-defined on M and equals to* $[-\gamma \wedge \alpha \wedge e_1]$.

Proof. Let α' be the pull-back to N of the 1-form supported in a neighborhood of a_1 in the first factor of $\mathbf{S}^1 \times \mathbf{S}^1$ under the projection $p : N \to \mathbf{T}^2$. Analogously, let β' be the pull-back to N of the 1-form supported in a neighborhood of a_2 in the second factor of $\mathbf{S}^1 \times \mathbf{S}^1$. Therefore $[\alpha'] = [\alpha]$ and $[\beta'] = [\beta]$. Clearly,

$$(\alpha' \wedge e_1) \wedge \beta' = \mathrm{d}\gamma' \wedge e_1,$$

where $\mathrm{d}\gamma' = \alpha' \wedge \beta'$. It can be supposed easily that γ' is zero in a neighborhood of $\xi \in N$. Therefore the support of $\gamma' \wedge e_1$ is disjoint from T_1 and T_2. Hence $\gamma' \wedge e_1$ is well-defined as a form in M. So the triple Massey product

$$\langle [\eta_1], [\eta_2], [\eta_2] \rangle = [-\gamma' \wedge \alpha \wedge e_1]$$

is well-defined in M. □

Finally, let us see that this Massey product is non-zero in

$$\frac{H^3(M)}{[\beta' \wedge e_1] \cup H^1(M) + H^2(M) \cup [\alpha']}.$$

Consider $B_3 = p^{-1}(\mathbf{S}^1 \times \{a_3\}) \times \mathbf{S}^1 \times \{b_2\}$, for generic points a_3, b_2 of $\mathbf{S}^1$. Then the Poincaré dual of B_3 is defined by a 2-form $\beta'' \wedge e_2$ supported near B_3, where β'' is Poincaré dual to $p^{-1}(\mathbf{S}^1 \times \{a_3\})$, $[\beta''] = [\beta]$, and e_2 is (the pull-back of) a differential 1-form on $\mathbf{S}^1$ (considered as the second of the two circle factors in $X = N \times \mathbf{S}^1 \times \mathbf{S}^1$) cohomologous to δ_2 and supported in a neighborhood of $b_2 \in \mathbf{S}^1$. Again this 2-form can be considered as a form in M. Now, for any $[\varphi] \in H^1(M)$, $[\varphi'] \in H^2(M)$ we have,

$$([\gamma' \wedge \alpha \wedge e_1] + [\beta' \wedge e_1 \wedge \varphi] + [\alpha' \wedge \varphi']) \cdot [\beta'' \wedge e_2] = 1,$$

since the first product gives 1, the second is zero and the third is zero because $\alpha' \wedge \beta''$ is exact in N and hence in M. This result and Theorem 2 prove the following:

Theorem 4. *The manifold M, defined by (4), is a compact oriented non-formal* 5-*manifold with* $b_1 = 1$.

The 6-dimensional example

A compact oriented non-simply connected and non-formal manifold M' of dimension 6 is obtained in an analogous fashion to the construction of the 5-dimensional manifold M. We start with $X' = N \times \mathbf{T}^3$ and consider the 3-dimensional tori with trivial normal bundle

$$T_1' = p^{-1}(\{0\} \times \mathbf{S}^1) \times \{0\} \times \{0\} \times \mathbf{S}^1,$$

$$T_2' = \{\xi\} \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1.$$

Define

$$M' = X' \#_{T_1'=T'} \mathbf{S}^6 \#_{T_2'=T'} \mathbf{S}^6, \tag{5}$$

where T' is an embedded 3-torus in $\mathbf{S}^6$ with trivial normal bundle. Then M' is a non-formal 6-manifold with $b_1 = 1$, which can be proved in a similar way to Theorem 4.

4 Proof of theorem 1

Let us first prove the affirmative results in Theorem 1.

Proposition 1. *Let M be a connected, compact and orientable manifold of dimension n and first Betti number $b_1 = b$.*

- *If $n \leq 2$, then M is formal.*
- *If $n \leq 6$ and $b = 0$, then M is formal.*
- *If $n \leq 4$ and $b = 1$, then M is formal.*

Proof. The first item is well-known: The circle and any oriented surface are formal. However, it follows from Theorem 3 very easily. Since M is connected, M is 0-formal. Hence M is formal as $n \leq 2$. Second item follows from [5, 10, 11]. Let us recall briefly the proof. Since M has $b_1 = 0$, it follows that in the minimal model $V^1 = 0$. This implies that $N^2 = 0$ since there are no decomposable elements of degree 3 and hence no element of V^2 can kill any element of degree 3 in the minimal model. Thus M is 2-formal and hence formal, by Theorem 3, since $n \leq 6$. The third item is proved similarly. Since M has $b_1 = 1$, in the minimal model $(\bigwedge V, d)$ we have that $V^1 = C^1$ is generated by one element ξ. There cannot be any element in N^1 since there are no decomposable elements of degree 2 (the only such element is $\xi \cdot \xi = 0$). Thus M is 1-formal and hence formal, by Theorem 3, since $n \leq 4$. □

With this result, we only need to find non-formal (connected, compact, orientable) manifolds under the conditions $n \geq \max\{3, 7 - 2b_1\}$ to complete the proof of Theorem 1.

- Non-formal manifolds with $n \geq 7$ and $b_1 = 0$ are constructed by the authors in [6]. Actually, those examples are simply connected. An alternative method is given in [3]. Oprea [12] also constructed examples of dimension 7 for other purposes.
- Non-formal manifolds of dimensions $n = 5$ or 6 and first Betti number $b_1 = 1$. These are the manifolds M and M' given by (4) and (5) in Section 3.

- Non-formal manifolds of dimension $n \geq 7$ and $b_1 = 1$. Take the non-formal 5-dimensional manifold M of Section 3 and consider $M \times \mathbf{S}^{n-5}$. This is again non-formal (by [5, Lemma 2.11]) and has $b_1(M \times \mathbf{S}^{n-5}) = b_1(M) = 1$.
- Case $n = 3$ and $b_1 = 2$. The manifold N considered as the beginning of Section 3 is non-formal.
- Case $n = 3$ and $b_1 \geq 3$. Consider $N\#(b_1-2)(\mathbf{S}^1 \times \mathbf{S}^2)$, which is non-formal because the Massey product $\langle[\beta], [\alpha], [\alpha]\rangle = [\alpha \wedge \gamma]$ is again defined and non-zero (as it happened for N).
- Case $n = 4$ and $b_1 \geq 3$. Consider $\left(N\#(b_1 - 3)(\mathbf{S}^1 \times \mathbf{S}^2)\right) \times \mathbf{S}^1$, which is non-formal being a product of a non-formal manifold with other manifold.
- Case $n \geq 5$ and $b_1 \geq 2$. We just consider $\left(N\#(b_1 - 2)(\mathbf{S}^1 \times \mathbf{S}^2)\right) \times \mathbf{S}^{n-3}$.
- Case $n = 4$ and $b_1 = 2$. A non-formal example can be constructed by a nilmanifold which is non-formal. For example (see [4]), let E be the total space of the $\mathbf{S}^1$-bundle over N with Chern class $c_1 = [\beta \wedge \gamma] \in H^2(N)$. The nilmanifold E is defined by the equations,

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta, \quad d\eta = \beta \wedge \gamma,$$

  where $\{\alpha, \beta, \gamma, \eta\}$ is a basis for the differential 1-forms on E. Then $[\beta] \cup [\alpha] = [\alpha] \cup [\alpha] = 0$, so that the Massey product $\langle[\beta], [\alpha], [\alpha]\rangle$ is well-defined, and it is non-zero because it is represented by the cohomology class of $\gamma \wedge \alpha$ which is non-zero in cohomology.

5 Final remarks

Note that the examples of non-formal manifolds with $b_1 = 1$ that we have constructed have Abelian fundamental group, since it is isomorphic to $\mathbf{Z}$. However, these manifolds are not nilpotent. Actually, if a manifold M with $b_1 = 1$ is nilpotent, then M is 2-formal. Furthermore, if the dimension is $n \leq 6$ and M is compact oriented, then it is formal. To prove that for a nilpotent manifold M with $b_1 = 1$ we have that M is 2-formal, it is enough to check that $N^2 = 0$. This would follow from the fact that no decomposable element of degree 3 (i.e., elements in $V^1 \cdot V^2$) is exact. Let ξ be the generator of V^1 and let $a \in V^2$ be a non-zero closed element. Suppose that $[\xi] \cup [a] = 0$ and let us reach to a contradiction. We use the following lemma of Lalonde–McDuff–Polterovich [9], which has been communicated to us by J. Oprea.

Lemma 3. *Suppose that $\gamma \in \pi_1(M)$, $A \in \pi_2(M)$, $h \in H^1(M; \mathbf{Z})$ and $\alpha \in H^2(M; \mathbf{Z})$, satisfy that $h(\gamma) \neq 0$ and $\alpha(A) \neq 0$. Then if $\alpha \cup h = 0$, the action of γ on A is non-trivial.*

In our case, take $h = [\xi] \in H^1(M)$ (after suitable rescaling if necessary to make it an integral class). Let $\gamma \in \pi_1(M)$ be any element with $h(\gamma) \neq 0$. Then, $h(\gamma^n) \neq 0$ for any $n > 0$. Now take $\alpha = [a]$ and consider any element $A \in \pi_2(M)$ with $\alpha(A) \neq 0$ (this exists since we are assuming that M is nilpotent and in this case $V^2 = (\pi_2(M) \otimes \mathbf{R})^*$). Then Lemma 3 implies that γ^n acts on A non-trivially. Hence γ acts non-nilpotently on $\pi_2(M)$, which is a contradiction.

We end up with some questions that arise naturally once Theorem 1 is answered.

1. Are there any restrictions on the Betti numbers for the existence of non-formal manifolds? Alternatively, solve the following *geography problem*:
   For which tuples $(n, b_1, \dots, b_s)$ with $n \geq 1$, $s = [n/2]$ and $b_i \geq 0$ is there a compact oriented manifold M of dimension n, with Betti numbers $b_i(M) = b_i$, $i = 1, \dots, s$, and which is *non-formal*?
2. Another alternative question is the following: Given a finitely presented group Γ and an integer n with $n \geq \max\{3, 2b_1(\Gamma) - 7\}$, are there always non-formal n-manifolds M with fundamental group $\pi_1(M) \cong \Gamma$?

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Total Scalar Curvatures of Geodesic Spheres and of Boundaries of Geodesic Disks*

J. C. Díaz-Ramos, E. García-Río, and L. Hervella

Department of Geometry and Topology, Faculty of Mathematics,
University of Santiago de Compostela, 15782 Santiago de Compostela, Spain
xtjosec@usc.es, xtedugr@usc.es, hervella@zmat.usc.es

Dedicated to Professor Lieven Vanhecke

Summary. Total curvatures of boundaries of geodesic disks in Riemannian manifolds are investigated. The first terms in the corresponding power series expansions are obtained for the total scalar curvature and the L^2-norms of the scalar curvature, the Ricci tensor and the curvature tensor. As an application, it is shown that these functions characterize the local geometry of most of the two-point homogeneous spaces.

1 Introduction

In the study of geometric properties of a Riemannian manifold (M, g), it is often useful to consider geometric objects naturally associated to (M, g). These can be special hypersurfaces like small geodesic spheres and tubes around geodesics, bundles with (M, g) as base manifold, or families of transformations reflecting symmetry properties of (M, g) [V88]. The existence of a relationship between the curvature of a Riemannian manifold and the volume of its geodesic spheres and tubes led some authors to state the following question:

> To what extent is the curvature (or the geometry) of a given Riemannian manifold (M, g) influenced, or even determined, by the volume properties of certain naturally defined families of geometric objects (for example geodesic spheres and tubes) in M?

This problem seems very difficult to handle in such a generality. However, when one looks at manifolds with a high degree of symmetry (e.g., two-point homogeneous spaces), these geometric objects have nice properties and one may expect to obtain characterizations of those spaces by means of such properties. For instance, the two-point homogeneous spaces may be characterized by using the spectrum of their geodesic

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spheres [CV81] or in most cases by the L^2-norm of the curvature tensor of geodesic spheres [DGH]. (See also [DGV] for more information on total curvatures of geodesic spheres.)

This work fits into the general program above. The family of geometric objects to be considered are the geodesic disks, which were previously investigated by O. Kowalski and L. Vanhecke with special attention to their volume properties [KV82], [KV83], [KV85]. Here, we are interested in the intrinsic geometry of the boundaries of these disks and we devote our attention to the study of their total scalar curvatures obtained by integrating the scalar curvature and the quadratic curvature invariants on these boundaries. In doing that, we compute the first terms in their power series expansions. Several conclusions are obtained from those coefficients. In particular, we note that

two-point homogeneous spaces are characterized by some of the total curvatures of the boundaries of geodesic disks among Riemannian manifolds with adapted holonomy.

The paper is organized as follows. In Section 2, we recall some notation and basic notions on scalar curvature invariants. The first terms in the power series expansions of the corresponding total invariants are derived in §2.2. These are used in Section 3 to obtain the first terms in the power series expansions of the total curvatures of the boundaries. Finally, Section 4 is devoted to point out some applications of those expressions.

2 Preliminaries

Let (M^n, g) be an n-dimensional smooth Riemannian manifold of class C^∞. We denote by ∇ the Levi–Civita connection and put $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for the curvature tensor, where X, Y are vector fields on M. Also, $R_{XYZW} = g(R_{XY}Z, W)$ and the Ricci tensor and the scalar curvature are given by $\rho_{XY} = \sum_{i=1}^n R_{Xe_iYe_i}$ and $\tau = \sum_{i=1}^n \rho_{e_ie_i}$ respectively, and with respect to an orthonormal basis $\{e_1, \dots, e_n\}$. For simplicity, here and in what follows, we use the notation $\rho_{ij} = \rho_{e_ie_j}$, $R_{ijkl} = R_{e_ie_je_ke_l}$, $\nabla_{ijk\dots} = \nabla_{e_ie_je_k\dots}$ and so on.

Finally, note that to avoid problems with the domains of exponential maps, the geodesic spheres and disks considered here are sufficiently small, i.e., their radius is always smaller than the injectivity radius at their center.

2.1 Scalar curvature invariants

A *scalar curvature invariant* is a polynomial in the components of the curvature tensor that does not depend on the choice of the orthonormal basis used to build it. The *order* of a scalar curvature invariant is, by definition, the number of derivatives of the metric tensor involved in it. Let $I(k, n)$ be the vector space of curvature invariants of order $2k$, $m \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space at m, T_mM.

For $n \geq 2$, $I(1,n)$ has dimension 1 and is generated by τ. For $n \geq 4$, $I(2,n)$ has dimension 4 and a basis is given by,

$$\tau^2, \qquad \|\rho\|^2 = \sum \rho_{ij}^2, \qquad \|R\|^2 = \sum R_{ijkl}^2, \qquad \Delta\tau = \sum \nabla_{ii}^2 \tau. \quad (1)$$

A basis for $I(3,n)$ is given in [GV79]. For our purposes here, only invariants of order two and four are needed. Indeed, those allow to characterize important classes of Riemannian manifolds. We have for $n > 2$ [CV81]:

For any n-dimensional Riemannian manifold,

$$\|\rho\|^2 \geq \frac{1}{n}\tau^2, \tag{2}$$

with equality if and only if the manifold is an Einstein space.

For any n-dimensional Riemannian manifold,

$$\|R\|^2 \geq \frac{2}{n-1}\|\rho\|^2, \tag{3}$$

with equality if and only if the manifold has constant sectional curvature.

For a $2n$-dimensional Kähler manifold,

$$\|R\|^2 \geq \frac{4}{n+1}\|\rho\|^2, \tag{4}$$

with the equality holding if and only if M has constant holomorphic sectional curvature.

For a $4n$-dimensional quaternionic Kähler manifold,

$$\|R\|^2 \geq \frac{5n+1}{(n+2)^2}\|\rho\|^2, \tag{5}$$

with the equality holding precisely for the quaternionic space forms.

2.2 Total scalar curvatures of geodesic spheres

Our purpose here is to obtain the first two terms in the power series expansions of the integrals of the curvature invariants of order two and four on geodesic spheres. We denote by $G_m(r)$ the geodesic sphere with center $m \in M$ and radius r, that is, $G_m(r) = \{m' \in M/d(m,m') = r\}$. Since $r > 0$ is supposed to be smaller than the injectivity radius at m, the geodesic sphere $G_m(r)$ is a hypersurface of M and $G_m(r)$ $= \exp_m(S^{n-1}(r))$, where $S^{n-1}(r) = \{y \in T_mM/\|x\| = r\}$ is the sphere of radius r in the tangent space to M at the basepoint m. Moreover as a matter of notation, let $\widetilde{\tau}$, $\|\widetilde{\rho}\|^2, \ldots$ denote the scalar curvature, the square norm of the Ricci tensor, ... of the geodesic sphere $G_m(r)$, and set τ, $\|\rho\|^2, \ldots$ for the corresponding objects for the ambient manifold (M,g).

First of all, note that we will not consider the Laplacian of the scalar curvature since $\int_{G_m(r)} \widetilde{\Delta}\widetilde{\tau}du = 0$. Also, in what follows, $c_{n-1} = \frac{n\pi^{n/2}}{(n/2)!}$ where $(n/2)! = \Gamma((n/2)+1)$ stands for the volume of the unit sphere in the Euclidean n-space (cf. [G90]). In the lemma below, the first terms in the power series expansions of the total scalar curvature [CV81] and the L^2-norms of the scalar curvature, the Ricci tensor and the curvature tensor [DGH] of sufficiently small geodesic spheres are given.

Lemma 1 ([CV81], [DGH]). *Let (M, g) be an n-dimensional Riemannian manifold and $m \in M$. Then, we have:*

$$\int_{G_m(r)} \widetilde{\tau} = c_{n-1} r^{n-1} \left\{ \frac{(n-2)(n-1)}{r^2} - \frac{(n-3)(n-2)}{6n} \tau(m) \right.$$
$$+ \frac{1}{n(n+2)} \left[-\frac{(n+2)(n+3)}{120} \|R\|^2 + \frac{n^2+5n+21}{45} \|\rho\|^2 \right.$$
$$\left. \left. + \frac{n^2-7n-6}{72} \tau^2 - \frac{(n-3)(n-2)}{20} \Delta\tau \right](m) r^2 + O(r^3) \right\},$$

$$\int_{G_m(r)} \widetilde{\tau}^2 = c_{n-1} r^{n-1} \left\{ (n-2)^2 (n-1)^2 r^{-4} - \frac{(n-5)(n-2)^2(n-1)}{6n} \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n(n+2)} \left[-\frac{(n-2)(n-1)(n^2+13n+10)}{120} \|R\|^2 \right.$$
$$+ \frac{n^4+10n^3+43n^2-14n+120}{45} \|\rho\|^2$$
$$+ \frac{n^4-14n^3+29n^2-60n-188}{72} \tau^2$$
$$\left. \left. - \frac{(n-5)(n-2)^2(n-1)}{20} \Delta\tau \right](m) + O\left(r^2\right) \right\},$$

$$\int_{G_m(r)} \|\widetilde{\rho}\|^2 = c_{n-1} r^{n-1} \left\{ (n-2)^2 (n-1) r^{-4} - \frac{(n-5)(n-2)^2}{6n} \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n(n+2)} \left[-\frac{n^3-9n^2-16n-20}{120} \|R\|^2 \right.$$
$$+ \frac{n^3+31n^2-16n-120}{45} \|\rho\|^2 + \frac{n^3-13n^2-16n+44}{72} \tau^2$$
$$\left. \left. - \frac{(n-5)(n-2)^2}{20} \Delta\tau \right](m) + O\left(r^2\right) \right\},$$

$$\int_{G_m(r)} \|\widetilde{R}\|^2 = c_{n-1} r^{n-1} \left\{ 2(n-2)(n-1) r^{-4} - \frac{(n-5)(n-2)}{3n} \tau(m) r^{-2} \right.$$
$$+ \frac{1}{n(n+2)} \left[\frac{59n^2-93n-10}{60} \|R\|^2 + \frac{2\left(n^2-37n+60\right)}{45} \|\rho\|^2 \right.$$
$$\left. \left. + \frac{n^2-11n+2}{36} \tau^2 - \frac{(n-5)(n-2)}{10} \Delta\tau \right](m) + O\left(r^2\right) \right\}.$$

3 Total scalar curvatures of boundaries of geodesic disks

Geodesic disks were introduced by O. Kowalski and L. Vanhecke as a generalization of the notion of a two-dimensional disk in the Euclidean space $\mathbb{E}^3$. In a series of papers ([KV82], [KV83], [KV85]) they investigated their volume properties in relation to local homogeneity and obtained a characterization of the two-point homogeneous spaces by means of the volumes of their small geodesic disks. Since the boundaries of geodesic disks are compact submanifolds, we are interested in their total scalar curvatures obtained by integrating the corresponding scalar curvature invariants of order two and four.

Recall that the *geodesic disk* $\overline{D}_m^{\xi}(r)$ of radius r, centered at $m \in M$ and orthogonal to $\xi \in T_mM$, is defined by

$$\begin{aligned}\overline{D}_m^{\xi}(r) &= \{\exp_m(su)/u \in T_mM,\ \|u\| = 1,\ g(u,\xi) = 0,\ 0 \le s \le r\} \\ &= \{m' \in M/d(m,m') \le r\} \cap \exp_m(\{\xi\}^{\perp})\end{aligned}$$

where $\exp_m : T_mM \to M$ is the exponential map at m. For the purpose of this paper and the investigation of total scalar curvatures, we consider the boundaries

$$D_m^{\xi}(r) = \{m' \in M/d(m,m') = r\} \cap \exp_m(\{\xi\}^{\perp}).$$

In order to obtain the first terms in the power series expansions of the total curvatures of these boundaries, the following result will be extensively used. It relates scalar curvature invariants of order two and four of $\exp_m(\{\xi\}^{\perp})$ with the corresponding objects in the ambient space.

Lemma 2. *Let* (M, g) *be an n-dimensional Riemannian manifold and* $\xi \in T_mM$ *a unit vector. If* $\widetilde{R}$, $\widetilde{\rho}$, $\widetilde{\tau}$, ... *denote the objects in* $exp_m(\{\xi\}^{\perp})$ *and* R, ρ, τ, ... *denote the corresponding objects on* (M, g)*, then the following hold at* m*:*

$$\begin{aligned}\|\widetilde{R}\|^2 &= \|R\|^2 + 4\sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - 4\sum_{i,j,k=1}^{n} R_{\xi ijk}^2, \\ \|\widetilde{\rho}\|^2 &= \|\rho\|^2 + \rho_{\xi\xi}^2 - 2\sum_{i=1}^{n}\rho_{\xi i}^2 + \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - 2\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i \xi j}, \\ \widetilde{\tau} &= \tau - 2\rho_{\xi\xi}, \\ \widetilde{\Delta}\widetilde{\tau} &= \Delta\tau - 2\Delta\rho_{\xi\xi} + 2\nabla_{\xi\xi}^2\rho_{\xi\xi} - \nabla_{\xi\xi}^2\tau + \frac{4}{9}\rho_{\xi\xi}^2 \\ &\quad - \frac{4}{9}\sum_{i=1}^{n}\rho_{\xi i}^2 + \frac{4}{3}\sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - \frac{2}{3}\sum_{i,j,k=1}^{n} R_{\xi ijk}^2.\end{aligned}$$

Proof. It follows from the work in [KV82], after some calculations. □

Now, the first terms in the power series expansions of the total curvatures of the boundaries of geodesic disks are obtained from the corresponding ones for the geodesic spheres in Lemma 1 after using the identities in Lemma 2. As a matter of notation $\hat{\tau}$, $\|\hat{\rho}\|^2, \dots$ denote the curvature objects on the boundaries $D_m^\xi(r)$, while τ, $\|\rho\|^2, \dots$ stand for the corresponding objects on (M, g). We omit the calculations which are straightforward and immediately state the different expansions separately in Theorem 1–Theorem 4.

Theorem 1. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the total scalar curvature of the boundaries $D_m^\xi(r)$:*

$$\int_{D_m^\xi(r)} \hat{\tau} = c_{n-2} r^{n-2} \left\{ (n-2)(n-3)r^{-2} + A_{(0)}(m) + A_{(2)}(m) r^2 + O(r) \right\}$$

where

$$A_{(0)} = -\frac{(n-3)(n-4)}{6(n-1)}[\tau - 2\rho_{\xi\xi}],$$

$$\begin{aligned} A_{(2)} = \frac{1}{(n-1)(n+1)} \Bigg\{ & \frac{n^2-9n+2}{72}\tau^2 - \frac{(n+2)(n+1)}{120}\|R\|^2 + \frac{n^2+3n+17}{45}\|\rho\|^2 \\ & - \frac{(n-3)(n-4)}{20}\Delta\tau + \frac{(n-3)(n-4)}{20}[\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] \\ & - \frac{(n+2)(n+11)}{45}\sum_{i=1}^n \rho_{\xi i}^2 - \frac{(n-4)(7n-11)}{90} \\ & \times \sum_{i,j=1}^n R_{\xi i \xi j}^2 - \frac{2(n^2+3n+17)}{45}\sum_{i,j=1}^n R_{\xi i \xi j}\rho_{ij} + \frac{n^2-2n+7}{15} \\ & \times \sum_{i,j,k=1}^n R_{\xi ijk}^2 - \frac{n^2-9n+2}{18}\tau\rho_{\xi\xi} + \frac{(n-1)(n-4)}{18}\rho_{\xi\xi}^2 \Bigg\}. \end{aligned}$$

Theorem 2. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the scalar curvature of the boundaries $D_m^\xi(r)$:*

$$\int_{D_m^\xi(r)} \hat{\tau}^2 = c_{n-2} r^{n-2} \left\{ (n-2)^2(n-3)^2 r^{-4} + B_{(-2)}(m) r^{-2} + B_{(0)}(m) + O(r) \right\}$$

where

$$B_{(-2)} = -\frac{(n-3)^2(n-6)(n-2)}{6(n-1)}[\tau - 2\rho_{\xi\xi}],$$

$$
\begin{aligned}
B_{(0)} = \frac{1}{(n-1)(n+1)} \Bigg\{ & \frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{72}\tau^2 \\
& - \frac{(n-2)(n-3)(n^2+11n-2)}{120}\|R\|^2 + \frac{n^4+6n^3+19n^2-74n+168}{45}\|\rho\|^2 \\
& - \frac{(n-3)^2(n-6)(n-2)}{20}\Delta\tau + \frac{(n-3)^2(n-6)(n-2)}{20} \\
& \times [\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] - \frac{n^4 + 26n^3 - 31n^2 - 4n + 228}{45} \\
& \times \sum_{i=1}^{n} \rho^2_{\xi i} - \frac{7n^4 - 78n^3 + 223n^2 - 488n + 276}{90} \\
& \times \sum_{i,j=1}^{n} R^2_{\xi i \xi j} - \frac{2(n^4 + 6n^3 + 19n^2 - 74n + 168)}{45} \\
& \times \sum_{i,j=1}^{n} R_{\xi i \xi j}\rho_{ij} + \frac{(n-2)(n-3)(n^2+n+8)}{15} \\
& \times \sum_{i,j,k=1}^{n} R^2_{\xi ijk} - \frac{n^4 - 18n^3 + 77n^2 - 164n - 84}{18}\tau\rho_{\xi\xi} \\
& + \frac{n^4 - 10n^3 + 57n^2 - 136n - 60}{18}\rho^2_{\xi\xi} \Bigg\}.
\end{aligned}
$$

Theorem 3. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the Ricci tensor of the boundaries $D^{\xi}_m(r)$:*

$$
\int_{D^{\xi}_m(r)} \|\hat{\rho}\|^2 = c_{n-2} r^{n-2} \left\{ (n-2)(n-3)^2 r^{-4} + C_{(-2)}(m) r^{-2} + C_{(0)}(m) + O(r) \right\}
$$

where

$$
C_{(-2)} = -\frac{(n-3)^2(n-6)}{6(n-1)}[\tau - 2\rho_{\xi\xi}],
$$

$$
\begin{aligned}
C_{(0)} = \frac{1}{(n-1)(n+1)} \Bigg\{ & \frac{n^3 - 16n^2 + 13n + 46}{72}\tau^2 - \frac{n^3 - 12n^2 + 5n - 14}{120}\|R\|^2 \\
& + \frac{n^3 + 28n^2 - 75n - 74}{45}\|\rho\|^2 - \frac{(n-3)^2(n-6)}{20}\Delta\tau + \frac{(n-3)^2(n-6)}{20} \\
& \times [\nabla^2_{\xi\xi}\tau - 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}] - \frac{n^3 + 68n^2 - 195n - 94}{45}
\end{aligned}
$$

$$\times \sum_{i=1}^{n} \rho_{\xi i}^2 - \frac{7n^3 - 164n^2 + 435n - 218}{90}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - \frac{2(n^3 + 28n^2 - 75n - 74)}{45}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j} \rho_{ij} + \frac{n^3 - 12n^2 + 25n - 34}{15}$$

$$\times \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 - \frac{n^3 - 16n^2 + 13n + 46}{18} \tau \rho_{\xi\xi} + \frac{n^3 - 35n + 38}{18} \rho_{\xi\xi}^2 \Big\} .$$

Theorem 4. *Let (M, g) be an n-dimensional Riemannian manifold, $m \in M$ and $\xi \in T_m M$ a unit vector. Then, for sufficiently small radius r, one has the following expansion for the L^2-norm of the curvature tensor of the boundaries $D_m^{\xi}(r)$:*

$$\int_{D_m^{\xi}(r)} \|\hat{R}\|^2 = c_{n-2} r^{n-2} \left\{ 2(n-2)(n-3) r^{-4} + D_{(-2)}(m) r^{-2} + D_{(0)}(m) + O(r) \right\}$$

where

$$D_{(-2)} = -\frac{(n-3)(n-6)}{3(n-1)} [\tau - 2\rho_{\xi\xi}],$$

$$D_{(0)} = \frac{1}{(n-1)(n+1)} \Big\{ \frac{n^2 - 13n + 14}{36} \tau^2 + \frac{59n^2 - 211n + 142}{60} \|R\|^2$$

$$+ \frac{2(n^2 - 39n + 98)}{45} \|\rho\|^2 - \frac{(n-3)(n-6)}{10} \Delta\tau$$

$$+ \frac{(n-3)(n-6)}{10} [\nabla_{\xi\xi}^2 \tau - 2\nabla_{\xi\xi}^2 \rho_{\xi\xi} + 2\Delta\rho_{\xi\xi}]$$

$$- \frac{2(n^2 - 69n + 178)}{45} \sum_{i=1}^{n} \rho_{\xi i}^2 + \frac{173n^2 - 657n + 514}{45}$$

$$\times \sum_{i,j=1}^{n} R_{\xi i \xi j}^2 - \frac{4(n^2 - 39n + 98)}{45} \sum_{i,j=1}^{n} R_{\xi i \xi j} \rho_{ij} - \frac{2(29n^2 - 101n + 62)}{15}$$

$$\times \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 - \frac{n^2 - 13n + 14}{9} \tau \rho_{\xi\xi} + \frac{(n-2)(n-23)}{9} \rho_{\xi\xi}^2 \Big\} .$$

4 Characterizations of the model spaces

The purpose of this section is to obtain characterizations of the two-point homogeneous spaces by means of the total curvatures of the boundaries of geodesic disks as an application of the expansions in Theorems 1–4. First of all, we recall that by a two-point homogeneous space we mean one of the following spaces: Euclidean n-space, the n-dimensional spheres and the hyperbolic spaces, the projective and hyperbolic n-spaces over the complex numbers or over the quaternions, and the Cayley projective or hyperbolic plane. Furthermore, we say that the holonomy of a Riemannian manifold (M, g) is *adapted* to one of these models, if the holonomy group of (M, g) is a subgroup of the holonomy group of the given model space, that is, the holonomy of (M, g) is contained in $O(n)$, $U(n)$, $Sp(1) \cdot Sp(n)$ or Spin(9) respectively. Moreover, note that in what follows, we will omit the Cayley plane since its holonomy group completely characterizes its local geometry. In fact, if a manifold has holonomy group contained in Spin(9), then it is flat or locally isometric to the Cayley plane or its non-compact dual [A67].

We begin with the following:

Lemma 3. *Let (M, g) be an n-dimensional Riemannian manifold. Suppose that one of the following holds:*

(i) $4 < n$ *and the total scalar curvature of the boundaries of geodesic disks coincides with the corresponding one in an Einstein manifold;*
(ii) $3 < n \neq 6$ *and any of the L^2-norms of the scalar curvature, the Ricci tensor or the curvature tensor of the boundaries of geodesic disks coincides with the corresponding one in an Einstein manifold.*

Then, (M, g) is an Einstein manifold with the same scalar curvature as the model space.

Proof. (i) is obtained from the coefficient $A_{(0)}$ in Theorem 1 and (ii) follows immediately from the corresponding coefficients of r^{-2} in the expansions in Theorems 2–4. □

Recall that a Riemannian manifold is said to be 2-*stein* if (M, g) is Einsteinian and satisfies

$$\sum_{i,j=1}^{n} R^2_{xixj} = \lambda g(x, x)^2$$

for all x. Also, (M, g) is said to be *super-Einstein* if it is Einstein and

$$\sum_{i,j,k=1}^{n} R^2_{xijk} = \mu g(x, x)$$

for all x. It was shown in [CV81] that 2-stein manifolds are super-Einstein, but the converse is not true. (For instance, irreducible symmetric spaces are super-Einstein, but they are not necessarily 2-stein.)

Lemma 4. *Let (M, g) be an n-dimensional Einstein manifold. If*

$$a\|R\|^2 + b \sum_{i,j,k=1}^{n} R^2_{\xi ijk} + c \sum_{i,j=1}^{n} R^2_{\xi i\xi j} = k \tag{6}$$

for some real constants a, b, c, k with $(n + 4)b + 3c \neq 0$, $c \neq 0$ and for all unit vectors ξ, then (M, g) is 2-stein.

Proof. Put

$$\omega_{xyvw} = \sum_{i,j=1}^{n} R_{xiyj} R_{viwj}, \qquad \eta_{xy} = \sum_{i,j,k=1}^{n} R_{xijk} R_{yijk}.$$

Then, for all vectors $x, y \in T_m M$ and all $\alpha, \beta \in \mathbb{R}$, it follows from (6) that

$$a\|R\|^2 g(\alpha x + \beta y, \alpha x + \beta y)^2 + b\eta_{\alpha x+\beta y,\alpha x+\beta y} g(\alpha x + \beta y, \alpha x + \beta y) \\ + c\omega_{\alpha x+\beta y,\dots,\alpha x+\beta y} = kg(\alpha x + \beta y, \alpha x + \beta y)^2.$$

Expand the previous expression and take the coefficients of $\alpha^2\beta^2$. Then, put $y = e_i$ and take the trace to obtain

$$2a\|R\|^2(n+2)g(x,x) + b(\|R\|^2 g(x,x) + (n+4)\eta_{xx}) \\ + 2c\left(\sum_{i,j=1}^{n} \rho_{ij} R_{xixj} + \frac{3}{2}\eta_{xx}\right) = 2(n+2)kg(x,x). \quad (7)$$

Since (M, g) is assumed to be Einsteinian, (7) becomes

$$[b(n+4) + 3c]\eta_{xx} = -\left[2(n+2)a\|R\|^2 + b\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\right] g_{xx}, \quad (8)$$

and contracting this gives

$$[b(n+4)+3c]\|R\|^2 = -n\left[2(n+2)a\|R\|^2 + b\|R\|^2 + \frac{2c\tau^2}{n^2} - 2(n+2)k\right]. \quad (9)$$

Now, from (8) and (9), one has

$$[b(n+4) + 3c]\eta_{xx} = \frac{b(n+4) + 3c}{n}\|R\|^2 g_{xx},$$

and thus $\eta = \frac{\|R\|^2}{n} g$. Hence, it follows from (6) that

$$\omega_{xxxx} = -\frac{1}{c}\left(\frac{na+b}{n}\|R\|^2 - k\right) g_{xx}^2$$

which shows that (M, g) is 2-stein. □

Lemma 5. *Let (M, g) be an n-dimensional Riemannian manifold. Suppose that one of the following holds:*

(i) $4 < n$ *and the total scalar curvature of the boundaries $D_m^\xi(r)$ does not depend on the normal direction ξ, or*

(ii) $3 < n \neq 6$ *and any of the L^2-norms of the scalar curvature, the Ricci tensor or the curvature tensor of the boundaries $D_m^\xi(r)$ does not depend on the normal direction ξ.*

Then, (M, g) is 2-stein.

Proof. We first show (i). Since the total scalar curvature of the boundaries of geodesic disks does not depend on the normal direction, the coefficients $A_{(0)}$ and $A_{(2)}$ are independent of the unit ξ. Therefore, it follows from $A_{(0)} = -[(n-3)(n-4)/6(n-1)][\tau - 2\rho_{\xi\xi}]$ that (M, g) is an Einstein space. Moreover, for an Einstein manifold, $\rho = (\tau/n)g$ holds and so, the coefficient $A_{(2)}$ becomes

$$A_{(2)} = \frac{1}{(n-1)(n+1)}\left\{\frac{(n-4)(5n^3-37n^2+62n+92)}{360n^2}\tau^2 - \frac{(n+2)(n+1)}{120}\|R\|^2 - \frac{(n-4)(7n-11)}{90}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{n^2-2n+7}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}.$$

So, (M, g) is 2-stein as an application of Lemma 4. The case (ii) is obtained in an analogous way. Indeed, if one assumes either $B_{(-2)}$ or $C_{(-2)}$ or $D_{(-2)}$ to be independent of ξ and $\dim M \neq 3, 6$, then (M, g) is an Einstein space. The fact that it is also 2-stein follows from Lemma 4 after consideration of the coefficients $B_{(0)}$, $C_{(0)}$ and $D_{(0)}$, which now become

$$B_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^6-102n^5+789n^4-2712n^3+3352n^2+1520n-5712}{360n^2}\tau^2 - \frac{(n-2)(n-3)(n^2+11n-2)}{120}\|R\|^2 - \frac{7n^4-78n^3+223n^2-488n+276}{90} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{(n-2)(n-3)(n^2+n+8)}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}, \tag{10}$$

$$C_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^5-92n^4+605n^3-1622n^2+548n+2696}{360n^2}\tau^2 - \frac{n^3-12n^2+5n-14}{120}\|R\|^2 - \frac{7n^3-164n^2+435n-218}{90} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \frac{n^3-12n^2+25n-34}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}, \tag{11}$$

$$D_{(0)} = \frac{1}{(n-1)(n+1)}\left\{\frac{5n^4-77n^3+14n^2+1180n-2072}{180n^2}\tau^2 + \frac{59n^2-211n+142}{60}\|R\|^2 + \frac{173n^2-657n+514}{45} \times \sum_{i,j=1}^{n} R^2_{\xi i\xi j} - \frac{2(29n^2-101n+62)}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}\right\}. \tag{12}$$

□

Now we are ready to derive the desired characterizations of the two-point homogeneous spaces for $n > 4$.

Theorem 5. *Let (M, g) be an n-dimensional Riemannian manifold with holonomy adapted to a two-point homogeneous space. If $4 < n$ and the total scalar curvature of sufficiently small boundaries $D_m^\xi(r)$ coincides with that of a two-point homogeneous space, then (M, g) is locally isometric to that model space.*

Proof. It follows from Lemma 5-(i) that (M, g) is 2-stein and thus super-Einstein [CV81], from where it follows that

$$\sum_{i,j=1}^{n} R_{\xi i \xi j}^2 = \frac{1}{n(n+2)}\left(\frac{3}{2}\|R\|^2 + \frac{1}{n}\tau^2\right), \qquad \sum_{i,j,k=1}^{n} R_{\xi ijk}^2 = \frac{1}{n}\|R\|^2. \tag{13}$$

Then, the coefficient $A_{(0)}$ in the power series expansion of the total scalar curvature of the boundaries of geodesic disks becomes

$$A_{(2)} = \frac{1}{n(n^2-1)(n+2)}\left\{\frac{(n-4)(5n^4-27n^3-12n^2+188n+228)}{360n}\tau^2 - \frac{(n-4)(n^3+n^2+26n+6)}{120}\|R\|^2\right\}.$$

Now the result is obtained by just comparing this with the corresponding coefficient $A_{(2)}$ in the model spaces and using the equations (2)–(5). □

Here it is worthwhile to emphasize that dimension four is excluded in previous theorem. Since the boundaries of the geodesic disks in a 4-dimensional manifold are compact surfaces, the total curvature $\int_{D_m^\xi(r)} \hat{\tau}$ is the Gauss Bonnet integral, and thus a topological invariant.

Theorem 6. *Let (M, g) be an n-dimensional Riemannian manifold with holonomy adapted to a two-point homogeneous space. If $3 < n \neq 6$ and the L^2-norms of the scalar curvature or the Ricci tensor or the curvature tensor of sufficiently small boundaries of geodesic disks coincides with that of a two-point homogeneous space, then (M, g) is locally isometric to that model space.*

Proof. Proceeding as in the previous theorem and using (13), the equations (10), (11) and (12) of the corresponding coefficients become

$$B_{(0)} = \frac{1}{n(n^2-1)(n+2)} \times \left\{\frac{5n^7-92n^6+585n^5-1162n^4-1760n^3+7332n^2-720n-12528}{360n}\tau^2 - \frac{n^6-9n^4-190n^3+714n^2-840n-216}{120}\|R\|^2\right\},$$

$$C_{(0)} = \frac{n-3}{n(n^2-1)(n+2)} \left\{ \frac{5n^5 - 67n^4 + 220n^3 + 220n^2 - 1380n - 2088}{360n} \tau^2 \right.$$
$$\left. - \frac{(n^2 - 14n - 2)(n^2 - n + 18)}{120} \|R\|^2 \right\},$$

$$D_{(0)} = \frac{1}{n(n^2-1)(n+2)} \left\{ \frac{(n-3)(5n^4 - 52n^3 - 296n^2 + 1012n + 696}{180n} \tau^2 \right.$$
$$\left. + \frac{(n-3)(59n^3 - 148n^2 - 34n - 12)}{60} \|R\|^2 \right\}.$$

Now the result follows by comparing these with the corresponding coefficients in the model spaces and using the characterizations (2)–(5). □

Explicit formulas for the total scalar curvatures of the boundaries of geodesic disks in the two-point homogeneous spaces are not yet available. However, by making use of the expansions in Theorems 1–4, the first terms in their power series expansions can be explicitly computed.

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Curvature Homogeneous Pseudo-Riemannian Manifolds which are not Locally Homogeneous

Corey Dunn and Peter B. Gilkey

Mathematics Department, University of Oregon, Eugene OR 97403 USA
cdunn@darkwing.uoregon.edu
gilkey@darkwing.uoregon.edu

Dedicated to Professor Lieven Vanhecke

Summary. We construct a family of balanced signature pseudo-Riemannian manifolds, which arise as hypersurfaces in flat space, that are curvature homogeneous, that are modeled on a symmetric space, and that are not locally homogeneous.

1 Introduction

Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g) of signature (p, q). Following Kowalski, Tricerri, and Vanhecke [16, 17], we say that (M, g) is *curvature homogeneous* if given any two points $P, Q \in M$. There is a linear isomorphism $\Psi : T_P M \to T_Q M$ such that $\Psi^* g_Q = g_P$, and such that $\Psi^* R_Q = R_P$; this notion has also been called 0 curvature homogeneous when considering a similar condition for the higher covariant derivatives of the curvature tensor.

Similarly, (M, g) is said to be *locally homogeneous* if given any two points P and Q, there are neighborhoods U_P and U_Q of P and Q respectively, and an isometry $\psi : U_P \to U_Q$ such that $\psi P = Q$. Taking $\Psi := \psi_*$ shows that locally homogeneous manifolds are curvature homogeneous. The somewhat surprising fact is that the converse fails – there are curvature homogeneous manifolds which are **not** locally homogeneous.

There is by now an extensive literature on the subject in the Riemannian setting, see, for example, the discussion in [1, 2, 14, 23–25]. There are also a number of papers in the Lorentzian setting [5–7] and also in the affine setting [15, 18]. There are, however, almost no papers in the higher dimensional setting – and those that exist appear in the study of 4-dimensional neutral signature Osserman manifolds, see, for example, [3, 8].

Key words: curvature homogeneous, balanced signature, hypersurfaces.
Subject Classifications: 53C50

In this brief note, we exhibit a family of examples in signature (p, p) for any $p \geq 3$ which are curvature homogeneous, but not locally homogeneous; this family first arose in the study of Szabó Osserman IP Pseudo-Riemannian manifolds [10, 11].

Let $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_p)$ be the usual coordinates on $\mathbb{R}^{2p}$. Let $f(x)$ be a smooth function on an open subset $\mathcal{O} \subset \mathbb{R}^p$. We define a non-degenerate pseudo-Riemannian metric g_f of balanced signature (p, p) on $M := \mathcal{O} \times \mathbb{R}^p$:

$$g_f(\partial_i^x, \partial_j^x) = \partial_i^x f \cdot \partial_j^x f, \quad g_f(\partial_i^x, \partial_j^y) = \delta_{ij}, \quad \text{and} \quad g_f(\partial_i^y, \partial_j^y) = 0\,. \tag{1}$$

This is closely related to the so called 'deformed complete lift' of a metric on $\mathcal{O}$ to $T\mathcal{O}$, see, for example, the discussion in [4, 13, 20].

The pseudo-Riemannian manifold (M, g_f) arises as a hypersurface in a flat space. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w}_1\}$ be a basis for a vector space W. Define an inner product $\langle \cdot, \cdot \rangle$ of signature $(p, p+1)$ on W by setting

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0, \quad \langle \mathbf{u}_i, \mathbf{v}_j \rangle = \delta_{ij}, \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0,$$

$$\langle \mathbf{u}_i, \mathbf{w}_1 \rangle = 0, \quad \langle \mathbf{v}_i, \mathbf{w}_1 \rangle = 0, \quad \langle \mathbf{w}_1, \mathbf{w}_1 \rangle = 1\,.$$

Let $F(x, y) = x_1\mathbf{u}_1 + \dots + x_p\mathbf{u}_p + y_1\mathbf{v}_1 + \dots + y_p\mathbf{v}_p + f(x)\mathbf{w}_1$ define an embedding of M in W. Then, g_f is the induced metric on the embedded hypersurface. The normal ν to the hypersurface is given $\nu := \mathbf{w}_1 - \partial_1^x f\ \mathbf{v}_1 - \dots - \partial_p^x f\ \mathbf{v}_p$. Thus, the second fundamental form L_f of the embedding is given by the Hessian

$$L_f(\partial_i^x, \partial_j^x) = \partial_i^x \partial_j^x f, \quad L_f(\partial_i^x, \partial_j^y) = 0, \quad \text{and} \quad L_f(\partial_i^y, \partial_j^y) = 0\,.$$

We define distributions

$$\mathcal{X} := \text{Span}\{\partial_1^x, \dots, \partial_p^x\} \quad \text{and} \quad \mathcal{Y} := \text{Span}\{\partial_1^y, \dots, \partial_p^y\}.$$

We then have $L(Z_1, Z_2) = 0$ if $Z_1 \in \mathcal{Y}$ or $Z_2 \in \mathcal{Y}$, so the restriction $L_f^{\mathcal{X}}$ of L to the distribution $\mathcal{X}$ carries the essential information. The following is the main result of this paper:

Theorem 1. *If the quadratic form $L_f^{\mathcal{X}}$ is positive definite, then (M, g_f) is curvature homogeneous. Furthermore, if $p \geq 3$, then (M, g_f) is not locally homogeneous for generic f.*

As noted above, these manifolds first arose in an entirely different setting. Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g). Let ∇R be the covariant derivative of R. Let J, S and $\mathcal{R}$ be the associated Jacobi operator, Szabó operator, and skew-symmetric curvature operator respectively. Let $X \in TM$ and let $\{Y, Z\}$ be an oriented orthonormal basis for an oriented space-like or time-like 2-plane π. These operators are defined by the identities:

$$g(J(X)U, V) = R(U, X, X, V),$$

$$g(S(X)U, V) = \nabla R(U, X, X, V; X),$$

$$g(\mathcal{R}(\pi)U, V) = R(Y, Z, U, V).$$

Stanilov and Videv [21] have defined a *higher-order Jacobi operator* by setting

$$J(\pi) := g(X_1, X_1)J(X_1) + \ldots + g(X_\ell, X_\ell)J(X_\ell),$$

where $\{X_1, \ldots, X_\ell\}$ is any orthonormal basis for a non-degenerate subspace $\pi \subset TM$.

Definition 1. Let (N, g) be a pseudo-Riemannian manifold. Then, (N, g) is

1. *space-like Jordan Osserman* (resp. *time-like Jordan Osserman*), if the Jordan normal form of $J(X)$ is constant on the bundle of unit space-like (resp. unit time-like) vectors.
2. *space-like Szabó* (resp. *time-like Szabó*), if the eigenvalues of $S(X)$ are constant on the bundle of unit space-like (resp. unit time-like) vectors.
3. *space-like Jordan IP* (resp. *time-like Jordan IP*), if the Jordan normal form of $\mathcal{R}(\pi)$ is constant on the Grassmannian of oriented space-like (resp. time-like) 2-planes in TM.
4. *Jordan Osserman* of type (r, s), if the Jordan normal form of $J(\pi)$ is constant on the Grassmannian of non-degenerate subspaces of type (r, s) in TM.

The spectral geometry of the Jacobi operator, of the skew-symmetric curvature operator, and of the Szabó operator were first considered in the Riemannian setting by Osserman [19], by Ivanova and Stanilov [12], and by Szabó [22] respectively. We refer to [9] for further details. The manifolds (M, g_f) provide examples of these manifolds. We refer to [10, 11] for the proof of:

Theorem 2. *If the quadratic form $L_f^{\mathcal{X}}$ is positive definite, then (M, g_f) is space-like Jordan Osserman, time-like Jordan Osserman, space-like Szabó, time-like Szabó, space-like Jordan IP, and time-like Jordan IP. Furthermore, (M, g_f) is Jordan Osserman of types $(r, 0)$, $(0, r)$, $(p - r, p)$ and $(p, p - r)$ and is not Jordan Osserman of type (r, s) otherwise.*

Note that there are no known Jordan Szabó manifolds which are not symmetric.

Here is a brief guide to the paper. In Section 2, we determine the tensors R_f and ∇R_f which are defined by the metric g_f and show (M, g_f) is curvature homogeneous. In Section 3, we complete the proof of Theorem 1 by showing that (M, g_f) is not locally homogeneous for generic f. We conclude in Remark 1 by showing the 'model space' for the curvature tensor for (M, g_f) is that of a symmetric space.

2 The tensors R_f and ∇R_f

We begin the proof of Theorem 1 by determining R_f and ∇R_f.

Lemma 1. *Let $Z_1, \ldots$ be coordinate vector fields on $M := \mathcal{O} \times \mathbb{R}^p$. Let the metric g_f be given by equation* (1). *Then,*

1. $\nabla_{Z_1} Z_2 = 0$*, if* $Z_1 \in \mathcal{Y}$ *or if* $Z_2 \in \mathcal{Y}$*;*
2. $R(Z_1, Z_2, Z_3, Z_4) = L(Z_1, Z_4)L(Z_2, Z_3) - L(Z_1, Z_3)L(Z_2, Z_4)$. *This vanishes if one of the* $Z_i \in \mathcal{Y}$ *for* $1 \le i \le 4$*;*

3. $\nabla R(Z_1, Z_2, Z_3, Z_4; Z_5) = Z_5\{R(Z_1, Z_2, Z_3, Z_4)\}$. *This vanishes if one of the* $Z_i \in \mathcal{Y}$ *for* $1 \le i \le 5$.

Proof. We have

$$(\nabla_{Z_1} Z_2, Z_3) = \frac{1}{2}\{Z_2 g_f(Z_1, Z_3) + Z_1 g_f(Z_2, Z_3) - Z_3 g_f(Z_1, Z_2)\}.$$

This vanishes if any of the $Z_i \in \mathcal{Y}$. Assertion (1) now follows. We now define $g_{ij}^x := g(\partial_i^x, \partial_j^x)$ and let $\Gamma_{ijk}^x := (1/2)(\partial_i^x g_{jk}^x + \partial_j^x g_{ik}^x - \partial_k^x g_{ij}^x)$. We adopt the Einstein convention and sum over repeated indices to see

$$\nabla_{\partial_i^x} \partial_j^x = \Gamma_{ijk}^x \partial_k^y, \quad \nabla_{\partial_i^x} \partial_j^y = \nabla_{\partial_j^y} \partial_i^x = 0, \quad \text{and} \quad \nabla_{\partial_i^y} \partial_j^y = 0.$$

It now follows that $R(Z_1, Z_2, Z_3, Z_4) = 0$ if any of the $Z_i \in \mathcal{Y}$. Furthermore,

$$R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = \partial_i \Gamma_{jkl}^x - \partial_j \Gamma_{ikl}^x.$$

Assertion (2) now follows. This also, of course, follows from the classical formula which expresses the curvature tensor of a hypersurface in flat space in terms of the second fundamental form.

Since $\nabla_{Z_5} Z_i \in \mathcal{Y}$ and since $R(\cdot, \cdot, \cdot, \cdot)$ vanishes if any of the entries belong to $\mathcal{Y}$, Assertion (3) follows from Assertion (2).

We show that (M, g_f) is curvature homogeneous by showing:

Lemma 2. *Let* $P \in M$. *Assume* $L_f^{\mathcal{X}}$ *is positive definite. Then there exists a basis* $\{X_1, \ldots, X_p, Y_1, \ldots, Y_p\}$ *for* $T_P M$ *so that:*

1. $g_f(X_i, X_j) = 0$, $g_f(X_i, Y_j) = \delta_{ij}$, *and* $g_f(Y_i, Y_j) = 0$.
2. $R_f(X_i, X_j, X_k, X_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$.
3. $R_f(\cdot, \cdot, \cdot, \cdot) = 0$ *if any of the entries is one of the vector fields* $\{Y_1, \ldots, Y_p\}$.

Proof. Fix $P \in M$. We diagonalize the quadratic form $L_f^{\mathcal{X}}$ at P to choose tangent vectors $\bar{X}_i = a_{ij}\partial_j^x \in T_P M$, so that $L(\bar{X}_i, \bar{X}_j) = \delta_{ij}$. Let $\bar{Y}_i := a^{ji}\partial_j^y$ where a^{ij} is the inverse matrix. Then,

$$g_f(\bar{X}_i, \bar{Y}_j) = a_{ik} a^{\ell j} g_f(\partial_k^x, \partial_\ell^y) = a_{ik} a^{kj} = \delta_{ij},$$

$$g_f(\bar{Y}_i, \bar{Y}_j) = 0,$$

$$R_f(\bar{X}_i, \bar{X}_j, \bar{X}_k, \bar{X}_\ell) = \delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell},$$

and $R_f(\cdot, \cdot, \cdot, \cdot) = 0$ if any entry is $\bar{Y}_i$. We define,

$$X_i := \bar{X}_i - \frac{1}{2} g_f(\bar{X}_i, \bar{X}_j)\bar{Y}_j \quad \text{and} \quad Y_i := \bar{Y}_i,$$

to ensure $g_f(X_i, X_j) = 0$. It follows that the frame $\{X_1, \ldots, X_p, Y_1, \ldots, Y_p\}$ satisfies the normalizations of the Lemma.

3 Homogeneity

We begin our discussion with a technical observation. Let V be a finite-dimensional real vector space. A 4-tensor $R \in \otimes^4 V^*$ is said to be an *algebraic curvature tensor* if it satisfies the symmetries of the Riemann curvature tensor

$$R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = -R(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4) = R(\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2) \quad \text{and}$$
$$R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) + R(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_4) + R(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4) = 0.$$

If ϕ is a symmetric bilinear form on V, then we may define an algebraic curvature tensor R_ϕ on V by setting

$$R_\phi(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) := \phi(\mathbf{v}_1, \mathbf{v}_4)\phi(\mathbf{v}_2, \mathbf{v}_3) - \phi(\mathbf{v}_1, \mathbf{v}_3)\phi(\mathbf{v}_2, \mathbf{v}_4).$$

Lemma 3. *Let ϕ_1 and ϕ_2 be symmetric positive definite bilinear forms on a vector space V of dimension at least* 3. *If $R_{\phi_1} = R_{\phi_2}$, then $\phi_1 = \phi_2$.*

We note that Lemma 3 fails if $\dim V \le 2$.

Proof. Since ϕ_1 is positive definite, we can diagonalize ϕ_2 with respect to ϕ_1 and choose a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ for V so that $\phi_1(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ and so that $\phi_2(\mathbf{e}_i, \mathbf{e}_j) = \lambda_i \delta_{ij}$. If $i \neq j$, then

$$\begin{aligned} 1 &= \phi_1(\mathbf{e}_i, \mathbf{e}_i)\phi_1(\mathbf{e}_j, \mathbf{e}_j) - \phi_1(\mathbf{e}_i, \mathbf{e}_j)\phi_1(\mathbf{e}_i, \mathbf{e}_j) = R_{\phi_1}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_i) \\ &= R_{\phi_2}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j, \mathbf{e}_i) = \phi_2(\mathbf{e}_i, \mathbf{e}_i)\phi_2(\mathbf{e}_j, \mathbf{e}_j) - \phi_2(\mathbf{e}_i, \mathbf{e}_j)\phi_2(\mathbf{e}_i, \mathbf{e}_j) \\ &= \lambda_i \lambda_j. \end{aligned} \tag{1}$$

Since $r \ge 3$, we can choose k so $\{i, j, k\}$ are distinct indices. By equation (1), $1 = \lambda_i \lambda_k = \lambda_j \lambda_k$, so $\lambda_i = \lambda_j$ for all i, j. Since $1 = \lambda_i \lambda_j = \lambda_i^2$ and since ϕ_2 is positive definite, $\lambda_i = 1$ for all i and hence $\phi_1 = \phi_2$.

We say that $\mathcal{B} := (X_1, \dots, X_p, Y_1, \dots, Y_p)$ is an *admissible* basis for $T_P M$ if $\mathcal{B}$ satisfies the normalizations of Lemma 2. We can now define a useful invariant.

Lemma 4. *Suppose $L_f^{\mathcal{X}}$ is positive definite. Let $P \in M$. Let $\mathcal{B}$ be an admissible basis for $T_P M$. Let $\alpha_f(P, \mathcal{B}) := \sum_{i,j,k,l,n} \nabla R_f(X_i, X_j, X_k, X_l; X_n)(P)^2$.*

1. *$\alpha_f(P, \mathcal{B})$ is independent of the particular admissible basis $\mathcal{B}$ chosen.*
2. *If (M, g_f) is locally homogeneous, then α_f is the constant function.*

Proof. The distribution $\mathcal{Y}$ is invariantly defined being characterized by

$$\mathcal{Y}_P = \{Y \in T_P M : R(Z_1, Z_2, Z_3, Y) = 0 \quad \text{for all} \quad Z_i \in T_P M\}.$$

The subspace $\mathcal{X}$ on the other hand is not invariantly defined. Denote the standard projection by π from $T_P M$ to $T_P M/\mathcal{Y}_P$. As

$$L(\cdot, \cdot) = 0, \quad R_f(\cdot, \cdot, \cdot, \cdot) = 0 \quad \text{and} \quad \nabla R_f(\cdot, \cdot, \cdot, \cdot; \cdot) = 0,$$

if any entry belongs to Y, these tensors induce corresponding structures $\bar{L}_f$, $\bar{R}_f$ and $\mathfrak{R}_f$ on $T_PM/\mathcal{Y}_P$ so that,

$$L_f = \pi^*\bar{L}_f, \quad R_f = \pi^*\bar{R}_f, \quad \text{and} \quad \nabla R_f = \pi^*\mathfrak{R}_f.$$

If $\mathcal{B}$ is an admissible basis, then we may define a quadratic form $\phi_{\mathcal{B}}$ on $T_PM/\mathcal{Y}_P$ by requiring that $\{\pi X_1, \dots, \pi X_p\}$ is orthonormal with respect to this quadratic form. We then have $\bar{R}_f = R_{\phi_{\mathcal{B}}}$. By Lemma 3, $\phi = \phi_{\mathcal{B}}$ is independent of the particular basis chosen and is invariantly defined. This defines a positive definite inner product on $T_PM/\mathcal{Y}_P$ which we use to raise and lower indices and to contract tensors. The invariant α is then given by $||\mathfrak{R}_f||^2_\phi$ and is invariantly defined. Since the structures involved are preserved by isometries, the Lemma now follows. What we have done, of course, is to prove that the second fundamental form is preserved by a local isometry of (M, g_f) here.

Proof of Theorem 1. In light of Lemma 4, to complete the proof of Theorem 1, it suffices to construct f so that α_f is constant on no open subset of $\mathbb{R}^p$; the fact that such f are generic will then follow using standard arguments. Let $f_{;i} = \partial_i^x f$, $f_{;ij} := \partial_i^x \partial_j^x f$, and so forth. We use Lemma 1 to see

$$R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = f_{;il} f_{;jk} - f_{;ik} f_{;jl},$$
$$\nabla R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x; \partial_n^x) = \partial_n^x \{f_{;il} f_{;jk} - f_{;ik} f_{;jl}\}.$$

Let $\Theta = \Theta(x_1)$ be a smooth function on $\mathbb{R}$ so that $|\Theta_{;11}| < 1$. Set

$$f(x) := \frac{1}{2}\{x_1^2 + \dots + x_p^2\} + \Theta(x_1).$$

We may then compute, up to the usual $\mathbb{Z}_2$ symmetries, that the non-zero components of R_f and of ∇R_f are

$$R_f(\partial_1^x, \partial_i^x, \partial_i^x, \partial_1^x) = 1 + \Theta_{;11} \quad \text{for} \quad 2 \le i \le p,$$
$$R_f(\partial_i^x, \partial_j^x, \partial_j^x, \partial_i^x) = 1 \quad \text{for} \quad 2 \le i < j \le p,$$
$$\nabla R_f(\partial_1^x, \partial_i^x, \partial_i^x, \partial_1^x; \partial_1^x) = \Theta_{;111} \quad \text{for} \quad 2 \le i \le p\,.$$

Consequently, after taking into account to normalize the basis for the tangent bundle suitably, we have,

$$\alpha_f = \frac{4(p-1)\Theta_{;111}^2}{(1+\Theta_{;11})^3}.$$

It is now clear the metric g_f is not be locally homogeneous for generic Θ. □

Remark 1. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p, \mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for a vector space V of dimension $2p$. Define an innerproduct $(\cdot, \cdot)$ and an algebraic curvature tensor R on V whose non-zero entries are

$$(\mathbf{u}_i, \mathbf{v}_j) = \delta_{ij} \quad \text{and} \quad R(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k, \mathbf{u}_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}.$$

Then by Lemma 2, $(V, (\cdot, \cdot), R)$ is a model for the metric and curvature tensor of all the manifolds (M, g_f) considered above. If we set $\Theta = 0$, then,

$$f_0 = \frac{1}{2}\left\{x_1^2 + \ldots + x_p^2\right\}.$$

Since $\nabla R = 0$, (M, g_{f_0}) is a symmetric space and hence locally homogeneous. This shows that $(V, (\cdot, \cdot), R)$ is the model for a symmetric space. Thus there exist pseudo-Riemannian manifolds which are not locally homogeneous, whose metric and curvature tensor is modeled on those of a symmetric space.

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On Hermitian Geometry of Complex Surfaces

A. Fujiki[1] and M. Pontecorvo[2]

[1]Department of Mathematics, Osaka University, Japan
fujiki@math.sci.osaka-u.ac.jp
[2]Dipartimento di Matematica, Università di Roma Tre, Italy
max@mat.uniroma3.it

Dedicated to Professor Lieven Vanhecke

1 Introduction

The aim of this exposition is to place our recent joint work on anti-self-dual Hermitian surfaces in the more general context of *locally conformal Kähler metrics*—which literally means that the metric is conformal to a Kähler metric, locally. From now on we will adopt the standard notation $l.c.K.$ for these metrics which were introduced and studied by Vaisman in the 1970s.

We start by recalling some preliminaries. Throughout this work S will denote a *smooth* complex surface—a complex manifold of complex dimension 2—with complex structure $J \in \mathrm{A}ut(TM)$ with $J^2 = -id$. A Riemannian metric g on the underlining real four-manifold S is said to be *Hermitian*, if it is compatible with the complex structure in the sense that J acts as an isometry: for all tangent vectors X and Y in TM,

$$g(JX, JY) = g(X, Y).$$

In this situation, we can define a non-degenerate 2-form $\omega \in \Lambda^{1,1}(S)$ usually called the Kähler form of the Hermitian metric by prescribing

$$\omega(X, Y) = g(X, JY),$$

and consider the linear map from one-forms to three-forms defined by taking wedge product with ω

$$Ł : \Lambda^1(S) \longrightarrow \Lambda^3(S),$$
$$\eta \mapsto \omega \wedge \eta.$$

Using the fact that ω is non-degenerate, the linear map L is always injective and therefore is an isomorfism because S is of real dimension four. We conclude that in this dimension there always is a unique one-form $\theta \in \Lambda^1(S)$ such that,

$$d\omega = \omega \wedge \theta.$$

θ is usually called the *Lee form* of the metric and it is easily seen to satisfy the following properties:

1. $\theta = 0$ – the Lee form vanishes $\iff$ g is a Kähler metric – i.e., $d\omega = 0$.
2. $\theta = df$ – the Lee form is exact $\iff$ the metric $e^{-f} g$ is Kähler – i.e., g is globally conformal Kähler.
3. $d\theta = 0$ – the Lee form is closed (locally exact) $\iff$ g is $l.c.K.$
4. We will also consider the case of *parallel Lee form* $\nabla\theta = 0$ where ∇ is the Levi-Civita connection of g; this of course implies $d\theta = 0$ and therefore it is a special class of $l.c.K.$ metrics also called *generalized Hopf manifolds* by Vaisman [36]. Notice that such surfaces must have vanishing Euler characteristic: $\chi(S) = 0$, when S is compact.

The main purpose of this note is to address the following question of Vaisman.

Question 1.1 ([37, p.122]) *Which compact complex surfaces* (S, J) *can admit* $l.c.K.$ *metrics?*

We will take the natural approach of first reducing the problem to minimal surfaces and then look at the Enriques–Kodaira classification. The rest of the section is devoted to give a brief account of these notions.

We start by explaining the minimal model of a surface introduced by Kodaira [16]: If one applies the classical monoidal transformation of blowing up a point on S, the result is a new complex surface $\tilde{S}$ containing a smooth rational curve C of self-intersection $C^2 = -1$. The blown up surface $\tilde{S}$ is diffeomorphic to the connected sum $S\#\overline{\mathbb{CP}}_2$. Conversely, a smooth rational curve C of self-intersection $C^2 = -1$ on a complex surface $\tilde{S}$ can always be blown down to a smooth point and the resulting smooth surface S will have second Betti number $b_2(\tilde{S}) - 1$; therefore if $\tilde{S}$ is compact, after a finite number of blowing down we will obtain that S is *minimal* – i.e., without rational curves of self-intersection -1. Such an S is called a *minimal-model* for the compact complex surface $\tilde{S}$ and in general is not unique.

It is then enough to understand *minimal* complex surfaces and this is the general philosophy of the classification which however is also very suitable to address the geometrical problem of Question 1.1 because of the following result of Tricerri which generalizes the analogous result in the Kähler case:

Proposition 1.2 ([34]) *A complex manifold* M *is* $l.c.K.$ *if and only if the blow up of* M *at point is* $l.c.K.$

As noticed in [34, Remark 4.3], this reduces the above question of Vaisman to minimal surfaces, for this reason from now on we can assume that S is a *minimal compact* complex surface and heavily rely on the famous Enriques–Kodaira classification which is summarized in the following table taken from the book of Barth–Peters–Van de Ven [5, p.188]. The classification divides all minimal surfaces into ten classes belonging to four groups according to the possible values of the *Kodaira dimension*, $\mathrm{Kod}(S) = -\infty, 0, 1, 2$ which appears in the second column of the table, while in the other columns we have indicated the *algebraic dimension* $a(S)$, the *Euler characteristic* $\chi(S)$ and the *first Betti number* $b_1(S)$.

Table 1. Table of Enriques–Kodaira classification

Class of S	Kod(S)	$a(S)$	$\chi(S)$	$b_1(S)$
1) rational surfaces		2	3,4	0
2) class VII_0 surfaces	$-\infty$	0,1	≥ 0	1
3) ruled surfaces of genus g		2	$4(1-g)$	2g
4) Enriques surfaces		2	12	0
5) Hyperelliptic surfaces		2	0	2
6) Kodaira surfaces	0	1	0	1,3
7) $K3$-surfaces		0,1,2	24	0
8) tori		0,1,2	0	4
9) properly elliptic surfaces	1	2	≥ 0	even
		1	0	odd
10) surfaces of general type	2	2	> 0	even

2 The case $b_1(S)$ even

It is well-known from Hodge theory that any compact Kähler manifold M must have odd de Rham cohomology of even dimension. Vice-versa, in the special case of surfaces, due to the fact that $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$ whether b_1 is even or odd [5, p.117], we have the following result of Vaisman:

Proposition 2.1 ([35, Prop 2.3]) *Every l.c.K. metric on a compact surface with even first Betti number is actually globally conformal Kähler.*

Therefore, in the case b_1 even Vaisman's question reduces to the more classical one of finding Kähler metrics on surfaces. As conjectured by Kodaira and Morrow [17] the answer is the following:

Theorem 2.2 ([27, 31]) *A compact complex surface is Kähler if and only if $b_1(S)$ is even.*

The original proof of this result was done case by case using Enriques–Kodaira classification of minimal surfaces. We give a brief account of the proof following the table of the previous section.

Because every Moischezon surface S – i.e., of top algebraic dimension $a(S) = 2$ – is actually projective algebraic [5, p.127] it follows that surfaces in 1), 3), 4), 5), and 10) are certainly Kähler because they are submanifolds of $\mathbb{CP}_n$. Tori 8) admit flat Kähler metrics while elliptic surfaces 9) with b_1 even are Kähler by a result of Miyaoka [27]. The problem remained open for the only class left, namely for K3 surfaces, until it was solved by Siu [31] building on preliminary work of Todorov.

It is also interesting to notice that quite recently Buchdhal and Lamari found two unified proofs of this theorem—i.e., not using Kodaira's classification. Their works are independent—using different complex analytical methods – and appeared in the same issue of the same journal [6, 18].

3 The case $b_1(S)$ odd and $\chi(S) = 0$

From now on we can assume that S is a minimal compact complex surface with odd first Betti number and look for strictly $l.c.K.$ metrics on S—i.e., not globally conformal Kähler. We see from Kodaira's classification of these surfaces that the Euler characteristic $\chi(S)$ cannot be negative and in our treatment we distinguish two main cases: The first one $\chi(S) = 0$ is completely understood both from the point of view of the classification of the complex structure [2] and the existence of $l.c.K.$ metrics [1]; notice that $\chi(S) = 0$ is also a necessary condition for the metric to have parallel Lee form.

We start by presenting a brief description of the complex structure of these surfaces in order of decreasing Kodaira dimension.

Properly elliptic surfaces with b_1 odd

A surface S is said to be elliptic if it admits a holomorphic map to a curve B with generic fiber an elliptic curve. It was shown by Kodaira [5, 16, 26] that when S is minimal with $b_1(S)$ odd, the singular fibers can only be multiple fibers; in this situation S admits an unbranched covering $\tilde{S}$ which is a (topologically non-trivial) elliptic fiber bundle over a smooth complex curve B with $b_1(\tilde{S}) = b_1(B) + 1$ and $b_2(\tilde{S}) = 2b_1(B)$. In particular, we conclude that $\chi(S) = 0$ for any minimal elliptic surface with $b_1(S)$ odd.

Finally, an elliptic surface S is called *properly elliptic* if $Kod(S) = 1$; when $b_1(S)$ is odd this amounts to say that the base B has genus $g \geq 2$. Furthermore, every surface of algebraic dimension 1 turns out to be elliptic [5, p.194].

Kodaira surfaces

By definition they are surfaces with $b_1(S)$ odd and $Kod(S) = 0$. They are divided into primary and secondary Kodaira surfaces according to whether b_1 is equal to 3 or 1. Primary Kodaira surfaces are elliptic fiber bundles over an elliptic curve and they provide interesting examples in differential geometry and topology. In fact it is shown in [30] that the complex structure J of a primary Kodaira surface anti-commutes with a symplectic structure I—generating in that way an almost hypercomplex structure on S; (S, I) was cited by Thurston as the first example of a compact symplectic manifold, which is not Kähler because $b_1 = 3$ [33]; and S also represents an interesting example in rational homotopy theory. Finally, secondary Kodaira surfaces are finite quotients of primary ones [5, p.147].

It follows from the classification table that the remaining minimal surfaces S with $b_1(S)$ odd and $\chi(S) = 0$ belong to class VII_0 – i.e., satisfy $Kod(S) = -\infty$ and $b_1(S) = 1$. The classification of surfaces in class VII_0 is known only in the special case $\chi(S) = 0$ and a theorem of Bogomolov [2] also proved by Yau et al. [22] and by Teleman [32] states that a surface in this class is either a Hopf surface or a Inoue–Bombieri surface, which we now describe briefly.

Hopf surfaces

By the work of Kodaira, a Hopf surface is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by a discrete group of biholomorphisms which is a finite extension of the infinite cyclic group generated by the contraction:

$$(z, w) \mapsto (az, bw + \lambda z^n),$$

where $a, b, \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ satisfy $0 < |a| < |b| < 1$ and $\lambda(a - b^n) = 0$; we say that a Hopf surface is *diagonal* if $\lambda = 0$ (class 1 in the terminology used by Belgun). A Hopf surface is elliptic exactly when $\lambda = 0$ and $a^p = b^q$ for some $p, q \in \mathbb{N}$ while an elliptic surface with b_1 odd must be a Hopf surface when the base $B \cong \mathbb{CP}_1$.

Bombieri–Inoue surfaces

These surfaces were independently discovered at the same time [14] and [3], their universal cover is $\mathbb{C} \times \mathcal{H}$ where $\mathcal{H}$ denotes the upper-half plane and contrary to Hopf surfaces which always have at least one elliptic curve (namely the image of $z = 0$) Bombieri–Inoue surfaces have no complex curves at all. They come in three different families which for simplicity we denote by S_m, S_n^- and $S_{n,u}^-$ with $u \in \mathbb{C}$.

Now that we have an idea of the complex structure of minimal surfaces with odd first Betti number and zero Euler characteristic, we want to investigate which of them admit $l.c.K.$ metrics. This problem has been solved by Belgun [1] in his doctoral thesis completing the work of several authors as Vaisman, Tricerri, Gauduchon–Ornea. In fact Belgun even classified surfaces which admit metrics with parallel Lee form and his powerful results can be summarized as follows:

Theorem 3.1 ([1]) *The complete list of compact complex surfaces S with b_1 odd admitting $l.c.K.$ metrics with parallel Lee form is the following:*

1. *Properly elliptic surfaces – i.e., all surfaces with $Kod(S) = 1$.*
2. *Kodaira surfaces, primary or secondary – i.e., all surfaces with $Kod(S) = 0$.*
3. *Diagonal Hopf surfaces – i.e., Hopf surfaces with with $\lambda = 0$.*

Belgun was also able to construct $l.c.K.$ metrics on every non-diagonal Hopf surface improving therefore the previous work of Gauduchon–Ornea [10] to show that:

Theorem 3.2 ([1]) *Every Hopf surface admits a $l.c.K.$ metric.*

The only case left is that of Inoue–Bombieri surfaces whose geometry was first studied by Tricerri who constructed $l.c.K.$ metrics on all of them except for $S_{n,u}^-$ and $u \notin \mathbb{R}$ [34]. Then another remarkable theorem of Belgun is that Tricerri's result is in fact sharp.

Theorem 3.3 ([1]) *The Inoue–Bombieri surfaces $S_{n,u}^-$ with $u \notin \mathbb{R}$ do not admit $l.c.K.$ metrics at all.*

An interesting consequence is that, contrary to the Kähler case, $l.c.K.$ metrics are not stable under small deformations [1].

4 Anti-self-dual Hermitian metrics on surfaces of class VII_0 with $b_2 > 0$

As seen in the previous section, the work of Belgun completely answered the question of Vaisman in the case of zero Euler characteristic. It follows from the classification that the only other possible case is that of surfaces of class VII_0 with $0 < \chi = b_2$, because $b_1 = 1$. There is no classification of these surfaces but only several examples due to Inoue, Hirzebruch, Enoki, Kato, Nakamura, and Dloussky. These examples all turn out to have small deformations which are not minimal. They are blown-up Hopf surfaces.

On the *topology* of these surfaces we can therefore say that all known examples of S are diffeomorphic to $(S^1 \times S^3)\#m\overline{\mathbb{CP}}_2$ where $m = b_2(S) \geq 1$.

There are also some very basic open questions about the *complex structures*; for example it is not known whether every surface $S \in VII_0$ with $b_2(S) \geq 1$ admits a curve [26].

As far as the *Hermitian geometry* of these surfaces is concerned, very little is known. We only have examples by LeBrun [19] who constructed anti-self-dual Hermitian metrics with semi-free S^1-action on parabolic Inoue surfaces using his hyperbolic ansatz. The action must in fact be holomorphic by [29] and this fits well with a result of Hausen [11] asserting that the only surfaces in this class admitting a 1-dimensional group of biholomorphisms with fixed points are parabolic Inoue surfaces.

The crucial link here is that LeBrun's metrics are automatically $l.c.K.$ by the following result of Boyer; see also [28] for an alternative twistor proof.

Theorem 4.1 ([4]) *Let S be a compact surface with $b_1(S)$ odd admitting an anti-self-dual Hermitian metric g. Then g is $l.c.K.$ and S belongs to class VII.*

In what follows, we present a new twistor construction of anti-self-dual Hermitian metrics on class VII surfaces; by Boyer's result these metrics are automatically $l.c.K.$ and notice that all known examples of $l.c.K.$ metrics on surfaces of class VII_0 with $b_2 > 0$ are indeed anti-self-dual Hermitian. The details and the proofs of our construction will appear elsewhere [8].

4.1 Surfaces with positive b_2, according to Nakamura

Although it is still an open question whether all the class VII_0 surfaces with $b_2 > 0$ must have a curve, it is known for example that they can only have elliptic or rational curves; in fact at most one-elliptic curve and at most $b_2(S)$ rational curves some of them forming a cycle C, there can be at most two cycles of rational curves in S. More precisely, some of these surfaces can be characterized by the configuration of curves that they contain. This is the case for Inoue and Enoki surfaces which always have $b_2(S)$ rational curves and can be identified by the presence of an elliptic curve or by the number of cycles and their self-intersection numbers. Rather than giving the original definition of each specific class we will simply refer to the excellent exposition in [26] from which we extract the useful table 2.

Our construction is very much inspired by the work of Nakamura [23, 25] on rational degenerations of class VII surfaces. In what follows, we briefly explain how, Inoue and Enoki surfaces can be constructed starting from a completely different class of surfaces, namely toric surfaces which are blow-ups of $\mathbb{CP}_2$ over a fixed point of the action.

Let $p \in \mathbb{CP}_2$ be a fixed point of a standard $(\mathbb{C}^* \times \mathbb{C}^*)$-action and let $H \subset \mathbb{CP}_2$ denote the hyperplane class. We have $-K = 3H$ for the anti-canonical class which

Table 2. Table of Enoki and Inoue surfaces with $b_2 > 0$.

curves	surfaces
an elliptic curve on a cycle	parabolic Inoue surfaces
two cycles	hyperbolic Inoue surfaces
a cycle C with $C^2 < 0$ and $b_2(S) = b_2(C)$	half Inoue surfaces
a cycle C with $C^2 = 0$	Enoki surfaces

can therefore be represented by a cycle of three rational curves—each of them having self-intersection number $+1$—and let p be one of the three corners. Blowing up $\mathbb{CP}_2$ at the point p yields the Hirzebruch surface Σ_1 with anti-canonical divisor $-K$ which is a cycle of four rational curves with self-intersection numbers $-1, 0, +1, 0$.

One can go on like this by always blowing up *one of the two corners of the last exceptional divisor*. After m times the result is again a toric surface $\tilde{D}$ diffeomorphic to $\mathbb{CP}_2\#m\overline{\mathbb{CP}}_2$ with a unique $+1$-rational curve denoted by H which is disjoint from the exceptional divisor of the last blow-up, denoted by E. They are part of a cycle of $(m+2)$-rational curves which represents the anti-canonical class of the surface $\tilde{D}$,

$$-K = E + B_1 + \cdots + B_i + H + B_{i+1} + \cdots + B_m,$$

the important point here is that by always blowing up one of the two corners of the (-1)-curve E we produced an anti-canonical cycle $-K$, whose -1 components always intersect E—in other words $B_j^2 = -1$ implies $j = 1$ or $j = m$. This is the property that makes this construction produce *minimal* surfaces with $b_1 = 1$.

From this smooth toric surface $\tilde{D}$, we now construct a singular surface D': Take $\phi : H \to E$ to be a biholomorphism of the complex projective line sending the two corners of H to those of E and consider the rational surface with ordinary double curve given by the quotient

$$D' = \tilde{D}/\phi.$$

Notice that D' is a singular surface with normal crossings along the double curve $F = \phi(H) = \phi(E)$ satisfying the d-semistable condition $\nu_H \otimes \nu_E \cong \mathcal{O}(+1) \otimes \mathcal{O}(-1) = \mathcal{O}_{\mathbb{CP}_1}$.

In this setting we know from a more general result of Nakamura [23, 24] that the Kuranishi family of D' is unobstructed, the general element D_t is a smooth surface in class VII containing a global spherical shell and diffeomorphic to $(S^1 \times S^3)\#m\overline{\mathbb{CP}}_2$ with $m = b_2(\tilde{D}) - 2$. In fact he shows that every class VII surface with global spherical shell admits a rational degeneration (not necessarely toric).

Because we want to obtain VII_0 surfaces with a particular configuration of curves (as described in the table), we consider deformations of the singular pair (D', B') where $\tilde{D}$ is toric and $B' = \phi(B_1 + \cdots B_m) \subset D'$ is the normal crossing divisor given by the image of the divisor $-K - H - E$ in $\tilde{D}$. We then have the following result:

Theorem 4.2 *The Kuranishi family of the singular pair (D', B') is unobstructed, the general member D_t is either an Inoue or an Enoki surface of class VII_0 with $b_2 = m$.*

In fact more precisely, one obtains a half Inoue surface if B'/ϕ consists of just one cycle. In other cases, ϕ identifies the four end-points of B' in order to form two cycles of rational curves and we obtain a hyperbolic Inoue surface when $i \geq 2$, or a parabolic Inoue surface when $i = 1$, because in this case one of the cycles in B'/ϕ consists of just one rational curve with a double point which is deformed to a smooth elliptic curve.

Finally, in order to obtain Enoki surfaces we need $i = 1$ and to actually neglect B_1 so that the general member D_t has only a cycle of rational curves with zero self-intersection number and no elliptic curve.

4.2 Twistor construction

Now that we understand the complex structure of our surfaces as smooth deformations of the singular pair (D', B'), we are going to produce anti-self-dual Hermitian metrics by imbedding (D', B') into a singular twistor space Z'. The construction of Z' is suggested by the work of Donaldson–Friedman [7] which for our purposes fits very well with Nakamura's construction of surfaces in class VII.

The starting point is a result of Joyce [13] who constructed self-dual metrics on the connected sum of m copies of $\mathbb{CP}_2$ (denoted by $m\mathbb{CP}_2$ from now on) with isometry group $S^1 \times S^1$ and their twistor spaces were studied by Fujiki in [9]. Let $t : Z \to m\mathbb{CP}_2$ be the twistor fibration from a Joyce twistor space to a Joyce metric, as usual each fiber $t^{-1}(p) \cong \mathbb{CP}_1$ is a complex submanifold of Z with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ called *twistor line*; these fibers are invariant with respect to the *real* structure $\sigma : Z \to Z$ which is an anti-holomorphic involution which restricts to the antipodal map on each twistor line, and is therefore fixed-point free.

What is important for our purposes is that every Joyce twistor space contains a pair of *degree*-1 divisors D and $\bar{D}$ (in fact a generic Z contains exactly $(m + 3)$ such pairs) by which we mean the following: D is an effective divisor in Z with intersection number 1 with a twistor line and $\bar{D} = \sigma(D)$. The generic twistor line intersects D at one point and there is exactly one twistor line $L_1 \subset D$, by reality it is also contained in $\bar{D}$ so that $L_1 = D \cap \bar{D}$. The restriction of the twistor map $t : D \to M$ is orientation reversing and shows that D is diffeomorphic to a blow-up of $\mathbb{CP}_2$: $D \cong \mathbb{CP}_2 \# m\overline{\mathbb{CP}}_2$ [20, prop.6].

In fact it is shown in [9] that each of this degree-1 divisors are toric surfaces with respect to a holomorphic $\mathbb{C}^* \times \mathbb{C}^*$-action on Z which is a complexification of the isometric action on M, given by the twistor correspondence.

L_1 is the component of self-intersection $+1$ in the anti-canonical cycle $-K$ of the toric surface $D \subset Z$, and in order to apply the Donaldson–Friedman construction let L_2 be the twistor line passing through one of the two corners of a (-1)-component of anti-canonical cycle $-K$. We can then follow the prescription of [7] and blow up the twistor space Z at L_1 and L_2 to obtain a smooth 3-fold $\tilde{Z}$ containing two exceptional quadrics Q_1 and Q_2 each with normal bundle $\mathcal{O}(-1, 1)$ and finally produce a singular twistor space Z' by using a biholomorphism $\psi : Q_1 \to Q_2$ which extends ϕ switching the two $\mathbb{CP}_1$-factors of the quadrics and taking the quotient space

$$Z' = \tilde{Z}/\psi.$$

According to general theory [7], Z' is a complex 3-fold with only normal crossing singularities along the smooth quadric $\psi(Q_1) = \psi(Q_2)$ satisfying the d-semistable condition and we can prove that its deformation theory is unobstructed so that it always admits smooth deformations which are twistor spaces of anti-self-dual metrics on the self-connected sum of $m\mathbb{CP}_2$ (with reversed orientation) which is $(S^1 \times S^3)\#m\mathbb{CP}_2$ – i.e., exactly what we want, topologically.

However, our construction gives us for free a lot more geometrical structure: The proper transform $\tilde{D}$ of D in the blown up twistor space $\tilde{Z}$ is exactly one of the toric surfaces considered in the previous section and is now *disjoint* from the proper transform $\tilde{\bar{D}}$ of $\bar{D}$. The divisors $\tilde{D}$ and $\tilde{\bar{D}}$ are isomorphic as toric surfaces and intersect transversely the two exceptional quadrics Q_1 and Q_2. The biholomorphism $\psi : Q_1 \to Q_2$ extends the identification ϕ so that the singular surface D' of the previous section is contained inside the singular twistor space Z' together with $\bar{D}'$ because the construction is compatible with real structures. In fact D' and $\bar{D}'$ are disjoint Cartier divisors in Z' with chains of rational curves $B' \subset D'$ and $\bar{B}' \subset \bar{D}'$. We then set $S' = D' + \bar{D}'$ and $C' = B + \bar{B}'$ and consider the triple of singular complex spaces with real structure. The deformation theory of such triples was studied by Honda [12] and we are able to prove the following result.

Theorem 4.3 *The Kuranishi family of the singular triple (Z', S', C') is unobstructed, the general member Z_t is smooth and contains a class VII$_0$ surface D_t with curves B_t.*

Because the triple (Z', S', C') has a real structure we know from general theory [7, 12, 15] that for t generic and real, Z_t is a twistor space with a degree-1 divisor D_t which is disjoint from $\bar{D}_t$ and isomorphic to one of the surfaces of 4.2. This is the key to prove the following result.

Theorem 4.4 *Every minimal hyperbolic or half Inoue surface with $b_2 = m$ admits an m-dimensional family of anti-self-dual Hermitian metrics. The same result holds on some Enoki and some parabolic Inoue minimal surfaces with $b_2 = m$.*

Altough it is not yet clear which parabolic Inoue surfaces admit anti-self-dual Hermitian metrics, let us notice that our metrics on these surfaces admit an S^1-action and should therefore be conformally isometric to LeBrun's by the general result of [21].

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Unit Vector Fields that are Critical Points of the Volume and of the Energy: Characterization and Examples*

Olga Gil-Medrano

Departament de Geometria i Topologia, Universitat de València, 46100 Burjassot, Valencia, Spain
olga.gil@uv.es

Dedicated to Professor Lieven Vanhecke

Summary. In the last few years, many works have appeared containing examples and general results on harmonicity and minimality of vector fields in different geometrical situations. This survey will be devoted to describe many of the known examples, as well as the general results from where they are obtained.

1 Introduction

The tangent bundle of a Riemannian manifold (M, g) admits a natural metric g^S known as the Sasaki metric. Since a vector field $V : M \to (TM, g^S)$ is an immersion, it is natural to consider the problem of characterizing those vector fields for which $V(M)$ is a minimal submanifold, or those for which V is a harmonic map. We can also look for vector fields that are critical points of the volume, or of the energy, when restricted to variations among vector fields.

There are several reasons why it is interesting to obtain these characterizations. For example, we can use them to find new examples of harmonic maps and of minimal immersions into manifolds with an interesting and highly non-trivial geometry: the tangent bundle or the unit tangent bundle.

Besides, we can use these critical conditions to obtain, on a given compact manifold, some information concerning the infimum of the volume or the energy of unit vector fields and to detect the possible minimizers, if they exist.

Finally, there are many situations in which a distinguished vector field appears in a natural way, for example the characteristic vector field of a contact metric manifold, the radial vector field on a normal neighborhood of a point of a Riemannian manifold and the geodesic vector field on the unit tangent bundle. In this case, it is interesting to

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study how the criticality of the vector field is related to the geometry of the manifold. To do so, it is important to obtain a convenient expression for the conditions of criticality.

Many authors have been interested in the study of the volume and the energy of vector fields, and also of sections of more general fiber bundles. In the last five years, an important number of works have appeared containing examples and general results on harmonicity and minimality of vector fields in different geometrical situations. The aim of this paper is to survey the advances made on these aspects of the problem, this is a subject in which the important contribution due to Professor Lieven Vanhecke plays a central role.

Many advances have been done also concerning the problem of finding the minima of the volume or of the energy among unit vector fields, the majority of the results obtained are for the sphere and this was the subject of our survey [Gil02]; for more recent developments the reader can see [GV02], [BG**], [GH**], [BS**] and the references therein.

Apart from volume and energy, other related function on the space of vector fields have been studied with similar methods, as for example, the corrected energy defined in [Bri00], the generalized energy considered in [Gil01], [GL01] and [GH**] and the space-like energy of time-like vector fields on Lorentzian manifolds, defined in [GH04].

The paper is organized as follows: In Section 2, we give the definitions and the characterization of critical vector fields, as well as the first examples. The characterization of critical sections of more general bundles is very similar and some results have been obtained on this subject of which we give a brief account. Section 3 is devoted to describe many of the known examples, as well as the general results from where they are obtained.

2 Definitions and first results

The energy of a smooth map $\varphi : (M, g) \to (N, h)$ from a Riemannian manifold to another is defined as,

$$E(\varphi) = \frac{1}{2}\int_M \operatorname{tr}(L_\varphi)\mathrm{dv}_g,$$

where L_φ is the endomorphism field completely determined by $(\varphi^*h)(X, Y) = g(L_\varphi(X), Y)$. If $\{E_i\}$ is a g-orthonormal frame, then

$$\operatorname{tr}(L_\varphi) = \sum_{i=1}^{n}(h \circ \varphi)(\varphi_*(E_i), \varphi_*(E_i)).$$

The volume of an immersion $\varphi : M \to (N, h)$ is the volume of the Riemannian manifold (M, φ^*h), that is

$$\operatorname{Vol}(\varphi) = \int_M \mathrm{dv}_{\varphi^*h}.$$

If we choose a metric g on M, then

$$\mathrm{Vol}(\varphi) = \int_M \sqrt{\det(L_\varphi)}\mathrm{dv}_g.$$

It is well known that the Euler–Lagrange equations, of the corresponding variational problems, give rise to the definition of *tension* of a map and of *mean curvature* of an immersion. Both of them are vector fields along the map and their vanishing defines harmonic maps and minimal immersions respectively. In a g-orthonormal frame as above, the tension is expressed in terms of the Levi-Civita connections ∇^g and ∇^h as,

$$\tau_g(\varphi) = \sum_{i=1}^{n} \Big(\nabla^h_{E_i} \varphi_*(E_i) - \varphi_*(\nabla^g_{E_i} E_i) \Big).$$

The mean curvature vector field coincides with the tension of the map $\varphi : (M, \varphi^* h) \to (N, h)$.

2.1 Volume and energy of vector fields

If we consider the tangent bundle $\pi : TM \to M$ and a metric g_0 on M, we can construct a natural metric on TM as follows: at each point $v \in TM$, we consider on the vertical sub-space of $T_v(TM)$ the inner product g_0 (up to the usual identification with T_pM, where $p = \pi(v)$). We take the horizontal sub-space determined by the Levi-Civita connection as a supplementary of the vertical and we declare them to be orthogonal. Finally, we define the inner product of horizontal vectors as the product of their projections, with the metric g_0. The so constructed metric g_0^S is sometimes referred as *the Sasaki metric*.

The geometry of (TM, g_0^S) is well known and a good description can be found in [Bla02]. We have used it in [Gil01] to compute $\tau_g(V)$, the tension of a vector field V in M considered as a map $V : (M, g) \to (TM, g_0^S)$. We will represent by ∇ the Levi-Civita connection of g_0, and by R the $(1, 3)$ curvature tensor given by

$$R(X, Y, Z) := R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Proposition 1. *Let V be a vector field in M. Then*

$$\tau_g(V) = \Big(\sum_i R((\nabla V)(\tilde{E}_i), V, \tilde{E}_i) + \tau_g(\mathrm{Id}) \Big)^{hor} + \Big((\nabla V)(\tau_g(\mathrm{Id})) + \sum_i (\nabla_{\tilde{E}_i}(\nabla V))(\tilde{E}_i) \Big)^{ver},$$

where for a vector field X we have represented by X^{ver} its vertical lift and by X^{hor} its horizontal lift. $\{\tilde{E}_i\}$ is any g-orthonormal frame and $\tau_g(\mathrm{Id})$ is the tension of $\mathrm{Id} : (M, g) \to (M, g_0)$.

In the particular case $g = g_0$, the expression above simplifies due to the vanishing of $\tau_g(\mathrm{Id})$ and in fact, the vertical part of the tension is just the rough Laplacian of V. This result goes back to Ishihara [Ish79].

For a compact manifold M, the vector field

$$(\nabla V)(\tau_g(\mathrm{Id})) + \sum_i (\nabla_{\tilde{E}_i}(\nabla V))(\tilde{E}_i)$$

vanishes if and only if $\nabla V = 0$, as can be seen combining Corollary 15 and Proposition 17 of [Gil01]. Then it is more interesting to consider the problem of determining which unit vector fields, with respect to the metric g_0, give harmonic maps from (M, g) to the unit tangent bundle (T^1M, g_0^S). It is not difficult to see that

Theorem 1 ([Gil01]). *Given a unit vector field V in a Riemannian manifold (M, g_0) and a metric g on M, the map $V : (M, g) \to (T^1M, g_0^S)$ is harmonic if and only if*

(a) $\sum_i R((\nabla V)(\tilde{E}_i), V, \tilde{E}_i) + \tau_g(\mathrm{Id}) = 0$ *and*
(b) $(\nabla V)(\tau_g(\mathrm{Id})) + \sum_i (\nabla_{\tilde{E}_i}(\nabla V))(\tilde{E}_i) = \lambda V$ *for some* $\lambda \in C^\infty(M)$.

Apart from the particular case $g = g_0$ mentioned above, the other special case involving only one metric on M is $g = V^*g_0$. The tension computed in Proposition 1 is then the mean curvature vector field of the immersion $V : M \to (T^1M, g_0^S)$. In [Gil01], we have shown that for $g = V^*g_0^S$ the condition (b) implies the condition (a) and then

Proposition 2. *Given a unit vector field V in a Riemannian manifold (M, g_0), the immersion $V : M \to (T^1M, g_0^S)$ is minimal if and only if there is $\lambda \in C^\infty(M)$ such that*

$$(\nabla V)(\tau_{V^*g_0^S}(\mathrm{Id})) + \sum_i (\nabla_{\tilde{E}_i}(\nabla V))(\tilde{E}_i) = \lambda V.$$

A different question is: *When a unit vector field is a critical point of the energy restricted to unit vector fields*? Since we are now concerned with a variational problem with constraints, the condition will be the vanishing of the projection of the tension $\tau_g(V) \in T_V(C^\infty(M, T^1M))$ onto the sub-space $T_V(\Gamma^\infty(M, T^1M))$, consisting in those elements that are tangent to the submanifold of all smooth sections of the unit tangent bundle. It is easy to see that if η is a vector field on T^1M along V, then it is tangent to the manifold of unit sections if and only if the vector $\eta(x) \in T_{V(x)}(T^1M)$ is vertical, for all $x \in M$. As a consequence,

Corollary 1. A unit vector field is a critical point of the energy restricted to unit vector fields if and only if it satisfies condition (b). It is a critical point of the volume restricted to unit vector fields if and only if it is a minimal immersion.

This is not the usual approach to the problem. Instead, we can go back to the definition of energy and volume of a map and, since for a vector field V we have $V^*g_0^S(X, Y) = g_0(X, Y) + g_0(\nabla_X V, \nabla_Y V)$ and then $L_V = \mathrm{Id} + (\nabla V)^t(\nabla V)$, we can see that the energy of the map $V : (M, g_0) \to (T^1M, g_0^S)$, that is, by definition the energy of the vector field, is given by,

$$E(V) = \frac{n}{2} + \frac{1}{2}\int_M \|\nabla V\|^2 \mathrm{dv}_0$$

and the volume of the vector field is

$$\mathrm{Vol}(V) = \int_M \sqrt{\det(L_V)}\mathrm{dv}_0.$$

The relevant part of the energy, $B(V) = \int_M \|\nabla V\|^2 \mathrm{dv}_0$ is sometimes called the total bending of the vector field.

The condition for a unit vector field to be a critical point has been obtained by direct computation of the Euler–Lagrange equation of these variational problems. The second order differential operators involved are the rough Laplacian $\nabla^*\nabla$ and the operator $\nabla^* D$ where D is, as ∇, a first order differential operator from the space of vector fields to the space of (1, 1)-tensor fields. It is given by $DV = \sqrt{\det(L_V)}(\nabla V)L_V^{-1}$.

Let us recall that ∇^* represents the formal adjoint of ∇ that can be expressed in an orthonormal frame as $\nabla^*(K) = \sum(\nabla_{E_i} K)E_i$. Moreover, the 1-form associated by the metric to this vector field is $\sum(\nabla_{E_i} K^t)^i$.

Theorem 2. *Given a unit vector field V in a Riemannian manifold (M, g_0), then*

1. ([Wie95]) *V is a critical point of the energy if and only if $\nabla^*\nabla V = \lambda V$, for some smooth function λ.*
2. ([GL02]) *V is a critical point of the volume if and only if $\nabla^* DV = \lambda V$, for some smooth function λ. Moreover, it is critical if and only if it defines a minimal immersion.*

Remark. In [Gil01] we show, by a direct argument, that the condition in part 2 above and that of Proposition 2 are equivalent.

The covariant version of Theorem 2, as it appears in [GL02], has been very useful for the study of particular examples and also to compute the second variation of the volume in [GL01]. The second variation of the energy was previously computed by a different method in [Wie95].

Proposition 3. *Let V be a unit vector field in a Riemannian manifold (M, g_0). Let us denote by ω_V (resp. $\tilde{\omega}_V$) the 1-form associated by the metric to $\nabla^*\nabla V$ (resp. $\nabla^* DV$) and by ν_V the 1-form is given by*

$$\nu_V(X) = \sum_i R((\nabla V)(E_i), V, E_i, X).$$

V is a critical point of the energy if and only if $\omega_V(X) = 0$ for all $X \in V^\perp$. In that case V will be called a unit harmonic vector field. V defines a harmonic map into the unit tangent bundle if and only if it is harmonic and $\nu_V = 0$. Finally, V is a critical point of the volume if and only if $\tilde{\omega}_V(X) = 0$ for all $X \in V^\perp$; in that case V defines a minimal immersion.

If we represent by V^* the 1-form associated to a unit vector field V, the rough Laplacian is related with the Hodge Laplacian by the Weitzenböck formula

$$\Delta V^* = \nabla^*\nabla V^* + \rho_V,$$

where ρ_V is 1-form associated to the Ricci tensor, i.e., $\rho_V(X) = \rho(V, X)$. Then, when a vector field is harmonic, the associated 1-form verifies $\Delta V^* = \rho_V$ and, in general, it is not Hodge-harmonic.

2.2 First examples

If V is a unit Killing vector field, we can write the conditions in Proposition 3 in terms of the curvature.

Proposition 4. *Given a unit Killing vector field V in a Riemannian manifold (M, g_0) then*

1. ([Wie95]) *V is harmonic if and only if $\rho_V(X) = 0$ for all $X \in V^\perp$.*
2. ([GL01]) *V is minimal if and only if $\tilde{\rho}_V(X) = 0$ for all $X \in V^\perp$, where $\tilde{\rho}_V$ is defined, in terms of an orthonormal frame of $V^\perp$, as follows:*

$$\tilde{\rho}_V(X) = \sum_i \Big(R(L_V^{-1}(\nabla_X V), L_V^{-1}(\nabla_{E_i} V), V, E_i) + R(\nabla_X V, \nabla_{E_i} V, V, E_i) \Big).$$

It is well known that Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbb{C}P^m$ determines a foliation of S^{2m+1} by great circles and that a unit vector field can be chosen as a generator of this distribution. It is given by $\xi = JN$, where N represents the unit normal to the sphere and J the usual complex structure on $\mathbb{R}^{2m+2}$. ξ is the standard Hopf vector field, but we will call a *Hopf vector field* any vector field in S^{2m+1} obtained as JN for J a complex structure on $\mathbb{R}^{2m+2}$, that is $J \in$ End $(\mathbb{R}^{2m+2})$ such that $J^t \circ J = \mathrm{Id}$, $J^2 = -\mathrm{Id}$. On the other hand, Hopf vector fields are exactly unit Killing vector fields of S^{2m+1}.

An important result due to Gluck and Ziller [GZ86] is that if $m = 1$, the Hopf vector fields are exactly the unit vector fields with minimum volume. For $m > 1$ it is shown, by Johnson in [Joh88], that for any variation of a Hopf vector field by unit vector fields, the first derivative of the volume vanishes. In view of Theorem 2, this means that Hopf vector fields are minimal immersions. In fact we have,

Proposition 5. *Let V be a unit Killing vector field on a Riemannian manifold M. Then*

1. ([Wie95]) *If M is an Einstein manifold of constant Ricci curvature κ, V is harmonic with energy $E(V) = (n + \kappa/2)\mathrm{Vol}(M)$.*
2. ([GL01]) *If M is of constant curvature k, V defines a minimal immersions with volume $\mathrm{Vol}(V) = (1 + k)^{(n-1/2)}\mathrm{Vol}(M)$.*

Moreover, it is easy to see, using Proposition 3 that Hopf vector fields are harmonic maps. An interesting result of Han and Yim is the following:

Proposition 6 ([HY98]). *The only unit vector fields on S^3 that are harmonic maps are Hopf vector fields.*

It has been improved by Gluck and Gu ([GG01]) who have shown that the Hopf vector fields are the only unit vector fields on S^3, such that $\nu_V = 0$.

The proofs are rather technical and can not be extended to higher dimensional spheres. The particular behaviour of 3-dimensional manifolds, in what concerns volume and energy of vector fields, is very frequent. For instance, in [GL01] we have shown

Proposition 7. *A unit Killing vector field V on a 3-dimensional manifold is minimal if and only if $\rho_V(V^\perp) = 0$ (that in this case is equivalent to $R(X, Y, V) = 0$, for all $X, Y \in V^\perp$) but this is no longer true if the dimension of the manifold is greater than 3.*

Let us now consider the vector fields W defined on $S^n - \{-p\}$ by parallel transport of a given $w \in T_p^1 S^n$, along the great circles of S^n passing through p. We will call such a W, *a parallel translation unit vector field.*

In [Ped93], it is shown that the generalized Pontryagin cycle is minimal at each smooth point as a submanifold of the corresponding Stiefel manifold. Since $W(S^n - \{-p\})$ can be seen as the set of smooth points of one of these cycles, then W is a minimal unit vector field. In [GL02], we use direct computation of W and ∇W to show that $\tilde{\omega}_W(W^\perp) = 0$. The expression of ∇W can also used be to compute ω_W (the details can be seen in [Lli02]) and then

Proposition 8. *Parallel translation unit vector fields defined on $M = S^n - \{-p\}$ are minimal immersions into the unit tangent bundle but they are critical for the energy only if the dimension is* 2.

As this example shows, the problems concerning the volume and the energy of vector fields, presents certain peculiarities on 2-dimensional manifolds. If we assume M to be compact, the only possible manifolds admitting unit vector fields are tori. The critical vector fields for the volume and the energy have been studied in [Joh88] and [Wie96], respectively. In [GL01], we have shown that, on any 2-dimensional manifold, if a unit vector field is harmonic or minimal then it is stable.

2.3 Beyond the tangent bundle

Let $\pi : P \to (M, g)$ be a fiber bundle over a Riemannian manifold. Suppose that the fibers are endowed with Riemannian metrics (such that the structure group of the bundle acts on them by isometries), and that the bundle is equipped with a connection. With a construction completely analogous to that described for the Sasaki metric on the tangent bundle, we can define a metric on P that is also known as the Kaluza–Klein metric.

In the case of a vector bundle, Konderak has written, in [Kon99], the condition for a section $\sigma : (M, g) \to (P, g^S)$ to be a harmonic map. To do so, he has computed the tension of σ.

The weaker condition of σ to be a critical point of the energy restricted to sections, i.e., to be a harmonic section, has been considered by Wood (see [Woo90], [Woo00]) and studied for a number of different vector bundles (or sub-bundles of vector bundles) of geometrical interest. For example, for sections of the twistor bundle (i.e., almost Hermitian structures) in [Woo95] and for almost-product structures in [Woo94]. Salvai ([Sal02] and [Sal02b]) studies the harmonicity of sections of some unit sub-bundles of trivializable bundles over certain parallelizable manifolds.

In [Woo03], Wood has obtained the condition for a section of an homogeneous fiber bundle to be harmonic.

A unit vector field on a Riemannian manifold gives rise to a 1-dimensional distribution on it. If we consider a general k-dimensional distribution on (M, g), the problem of determining when it is a harmonic map into de Grassmann bundle has been solved by Choi and Yim in [CY03]. They have used this characterization to study the harmonicity of a G-invariant distribution of a reductive homogeneous spaces $M = G/K$. In particular, they show that *the Hopf* 3*-dimensional distribution is harmonic*. The same conclusion has been obtained in [Woo94], by the study of the corresponding almost product-structure, and in [CNW01] by considering the Grassmann bundle as a sub-bundle of the exterior bundle.

If $\pi : P \to M$ is a tensor bundle over M, (i.e., the fiber at each point P_x is a vector space consisting of tensors of the tangent space T_xM), the only data needed to construct the Sasaki metric in P is a Riemannian metric g_0 on M, since we take its extension to tensors as the inner product on the fiber and the extension of the Levi-Civita connection, as the fiber connection. We have obtained in [GGV**] with González–Dávila and Vanhecke *the condition for a section* $\sigma : (M, g) \to (P, g_0^S)$ *to be a harmonic map* and a harmonic section as far as the corresponding results for sections of the unit bundle. If we assume $g = g_0$, we recover the results mentioned above. As for the particular case where P is the tangent bundle, described in subsection 2.1, if we assume that $g = \sigma^* g_0^S$, we obtain *the condition for a section to define a minimal immersion.*

We have used these results to obtain the condition for a distribution to be a harmonic map from (M, g) to the Grassmann bundle of M endowed with the restriction of the Sasaki metric g_0^S of the conveniently chosen exterior bundle. We show that *the* 3*-dimensional distribution of* S^{4m+3} *tangent to the quaternionic Hopf fibration defines not only a harmonic map but also a minimal immersion* and we extend these results to more general situations coming from 3-Sasakian and quaternionic geometry.

3 A cascade of examples

To obtain examples of critical vector fields, and of critical sections in general, it is necessary to have a good knowledge of the covariant derivative and the curvature of the manifold, in order to compute the tension or the mean curvature of the vector field. So, it is natural to look first at manifolds whose geometry is well known. Also the properties of the section itself are important, and some general intuition is that criticality would imply some beauty: symmetry (or skew-symmetry) of the covariant derivative, naturallity of the definition, etc. In the last five years, many papers have been published on the subject of volume and energy of unit vector fields, where several authors have provided examples, and studied the properties, of minimal and harmonic unit vector fields in different Riemannian manifolds. Many of the examples are obtained as a consequence of general results just by exhibiting some manifolds and sections fulfilling the hypothesis; but others arise in a particular manifold and they appear as isolated, by the moment. Probably, how they fit in the general picture will be understood in the future. We have divided this section in several subsections although some of the examples are in the intersection. We do not pretend to give an exhaustive list, almost all the works mentioned here contain more examples than those we have reported.

3.1 Invariant vector fields on Lie groups and homogeneous spaces

Let (G, g) be a Lie group endowed with a left-invariant metric, and let $\mathcal{S}$ represent the unit sphere of the Lie algebra $\mathfrak{g}$. The condition for a left-invariant unit vector field V to be minimal or harmonic can be expressed in terms of the corresponding element of $\mathcal{S}$ and of the elements of $\mathfrak{g}^*$ determined by ω_V and $\tilde{\omega}_V$. We have a first existence result:

Proposition 9 ([TV00b],[GV02b]). *Any odd-dimensional Lie group with left-invariant metric, admits harmonic, and minimal, left-invariant unit vector fields.*

The maps $e(V) = \mathrm{tr}(L_V)$ and $f(V) = \sqrt{\det(L_V)}$ defined on the manifold of unit vector fields and with values in the space of smooth functions, give rise to real valued maps on $\mathcal{S}$ that will be represented by the same symbols. Tsukada and Vanhecke in [TV00b] for the volume and González–Dávila and Vanhecke in [GV02b] for the energy have shown

Proposition 10. *Let (G, g) be a Lie group endowed with a left-invariant metric and let $V \in \mathcal{S}$. The corresponding invariant vector field is harmonic if and only if for all $X \in V^{\perp}$,*

$$de_V(X) = \mathrm{tr}(\mathrm{ad}_{(\nabla V)^t(X)}).$$

It is minimal if and only if for all $X \in V^{\perp}$,

$$df_V(X) = \mathrm{tr}(\mathrm{ad}_{K_V(X)}).$$

Let us recall that a Lie group is said to be unimodular if $\mathrm{tr}(\mathrm{ad}_X) = 0$, for all $X \in \mathfrak{g}$ and that, for a non-unimodular Lie group, the unimodular kernel is the codimension one ideal of $\mathfrak{g}$ defined by,

$$\mathfrak{u} = \{X \in \mathfrak{g} \ ; \ \mathrm{tr}(\mathrm{ad}_X) = 0\}.$$

Considering a unit vector $H \in \mathfrak{u}^{\perp}$, we will say that ad_H is symmetric if its restriction to $\mathfrak{u}$ is symmetric with respect to the inner product on $\mathfrak{u}$ determined by the metric.

Theorem 3 ([TV00b],[GV02b]). *Let G be either a unimodular Lie group or a non-unimodular Lie group such that* ad_H *is symmetric. Then a left-invariant unit vector field on G is harmonic if and only if it is a critical point of e and it is minimal if and only if it is a critical point of f. In particular, on any such group there exists at least one harmonic and one minimal left-invariant vector field.*

As an application, for the unimodular case, one can find in [TV00b] (resp. in [GV02b]) a complete classification of left-invariant minimal (resp. harmonic) unit vector fields on the classical Heisenberg group of dimension $2k + 1$. For such a vector field, it is equivalent to be harmonic, to define a harmonic map and to be minimal.

Examples of non-unimodular Lie groups with the symmetry condition are Damek–Ricci spaces and those obtained from solvable metric Lie algebras of Iwasawa type.

A Lie group for which there is $l \in \mathfrak{g}^*$, $l \neq 0$, such that

$$[X, Y] = l(X)Y - l(Y)X$$

is also of this kind, and in [TV00] one can also find a complete classification of left-invariant minimal unit vector fields on them.

For a 3-dimensional Lie group, the complete classification of minimal left-invariant unit vector fields has been given in [TV00], and of those that are harmonic or defined harmonic maps in [GV02]. By comparing the results in both papers, one can see that in the unimodular case, minimality and harmonicity are equivalent but do not necessarily imply the harmonicity as a map.

It is worth noting that, although for a non-unimodular case it was not possible to use always Theorem 3, the computations needed to apply Proposition 10 are not very complicated due to the dimension restriction. For arbitrary dimension, González–Dávila and Vanhecke have shown, in [GV00],

Proposition 11. *Unit left-invariant vector fields orthogonal to the unimodular kernel are minimal in any dimension.*

The same authors in [GV00] and [GV02b] have found many examples, among others, on the different types of generalized Heisenberg groups and on Damek–Ricci spaces.

Another situation that is well-understood is that of Lie groups with a bi-invariant metric. In [GV02b], the authors have been able to determine the full set of left-invariant harmonic vector fields. Moreover, they show that in this case, to be harmonic is equivalent to define a harmonic map. In particular, they have shown

Proposition 12. *On a compact connected semi-simple Lie group with the usual metric, given by the negative of the Killing form, every left-invariant unit vector field determines a harmonic map into the unit tangent bundle.*

On the other hand, the study of the volume of unit vector fields on a compact semi-simple Lie group, has been done by Salvai in [Sal03], using the characterization of minimality of [TV00b] and the structure and properties of semi-simple Lie groups, in a development where the roots of the Lie algebra play a central role. To describe the results we need some definitions.

Let $\mathfrak{g}$ be the Lie algebra of a compact semi-simple Lie group, G. Let $\mathfrak{t}$ be a maximal abelian subalgebra, Δ the corresponding root system, C a Weyl chamber and Φ the associated basis of Δ. Given $\alpha \in \Phi$ there is a unique $v_\alpha \in \mathfrak{t}$ such that $v_\alpha \in \mathrm{Ker}\beta$ for all $\beta \neq \alpha$, $\alpha(v_\alpha) > 0$ and $|v_\alpha| = 1$. Each vector v_α is a vertex of the simplex

$$\bar{C}^1 = \bar{C} \cap \{v \in \mathfrak{g} \;\; ; \;\; |v| = 1\}.$$

This vector is maximal singular (i.e., its $Ad(G)$-orbit has dimension strictly less than the orbit of any other unit vector in a neighborhood). Each maximal singular unit vector belongs to the $Ad(G)$-orbit of exactly one of this vertex.

Theorem 4 ([Sal03]). *On a compact semi-simple Lie group, for any maximal singular unit vector, the corresponding left-invariant and right-invariant unit vector fields are minimal.*

Moreover, if $\alpha \neq \beta$ and the unit vector fields corresponding to v_α and v_β have the same volume, then there is v in the edge joining v_α with v_β (i.e., $\gamma(v) = 0$ for all $\gamma \neq \alpha, \gamma \neq \beta$) and different from v_α and v_β, such that the corresponding left-invariant and right-invariant unit vector fields are minimal.

This result provides a lower bound for the number of inequivalent (i.e., they are not related by any element of the identity component of the isometry group) minimal vector fields. Furthermore, the same author has computed the expression of the volume of the left-invariant (and the right-invariant) vector field determined by a unit $v \in \mathfrak{g}$, in terms of the root system, obtaining,

$$\mathrm{vol}(G) \prod_{\phi \in \Delta^+} \left(1 + \frac{1}{4}\phi(v)^2\right).$$

When applied to particular groups one can show, for example

Proposition 13 ([Sal03]).

a) The number of non-equivalent minimal unit vector fields on $SU(n+1)$ *is not smaller than* $2n + 2[n/2]$.
b) The number of non-equivalent minimal unit vector fields on $SO(5)$ *and in* G_2 *is not smaller than* 4.
c) The number of non-equivalent minimal unit vector fields on $SO(8)$ *is not smaller than* 14.

Let us finish this subsection by the extension of these results on Lie groups to the more general situation of homogeneous spaces. We have constructed in [GGV01], with González–Dávila and Vanhecke, many examples of harmonic and minimal unit vector fields among the invariant vector fields of homogeneous spaces. To do so, we have derived a criterion for the minimality and harmonicity of such a vector field using the framework of homogeneous structures [TrV83] and infinitesimal models [TrV89].

Let $(M = G/G_0, g)$ be a homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$ and let $\tilde{\nabla}$ be the adapted canonical connection. We will consider the associated homogeneous structure $S = \nabla - \tilde{\nabla}$ and its trace $\eta = \sum_i S_{E_i} E_i$ where $\{E_i\}$ is an orthonormal basis of $\mathfrak{m}$.

The associated infinitesimal model is $\mathfrak{M} = (\mathfrak{m}, \tilde{T}, \tilde{R}, \langle , \rangle)$, where $\tilde{T}$ and $\tilde{R}$ are the torsion and curvature of $\tilde{\nabla}$. The sub-space of invariant vectors of $\mathfrak{m}$ is defined as,

$$\mathrm{Inv}\,\mathfrak{M} = \bigcap_{X,Y \in \mathfrak{m}} \mathrm{Ker}\,\tilde{R}_{X,Y}.$$

Theorem 5 ([GGV01]). *Let* $(M = G/G_0, g)$ *be a homogeneous Riemannian manifold with reductive decomposition* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$*, a left-invariant unit vector field V is harmonic if and only if*

$$de_V(X) = -\langle \nabla_\eta V, X \rangle = -\langle \eta, (\nabla V)^t(X) \rangle,$$

and it is minimal if and only if

$$df_V(X) = -\langle \eta, K_V(X) \rangle,$$

for all $X \in \mathrm{Inv}\,\mathfrak{M} \cap V^\perp$.

Among the different classes of homogeneous structures given in [TrV83], we can find the hyperbolic spaces.

Proposition 14 ([GGV01]). *On the hyperbolic plane every left-invariant vector field is harmonic and minimal. For $n \geq 3$, on the hyperbolic space $H^n = \{y \in \mathbb{R}^n \ ; \ y_1 > 0\}$, the unit vector fields $\pm V$, tangent to parametric curves of parameter y_1, and those determined by unit vectors in $V^\perp$ are exactly the harmonic invariant unit vector fields. The only minimal unit invariant vector fields are $\pm V$. There are no harmonic maps among unit invariant vector fields.*

In contrast with this situation, in the class of naturally reductive spaces,

Proposition 15 ([GGV01]). *Every harmonic invariant unit vector field on a naturally reductive homogeneous manifold defines a harmonic map into its unit tangent bundle.*

In the proof of this result, we use that on naturally reductive homogeneous manifolds invariant vector fields should be Killing. This fact enables us to use frames adapted to the endomorphism ∇V, to write minimality and harmonicity conditions in terms of $\tilde{T}$ in order to show:

Proposition 16 ([GGV01]). *On a naturally reductive homogeneous manifold such that the space* $\operatorname{Inv}\mathfrak{M}$ *is* 1*-dimensional, the unit generators are minimal and harmonic.*

Proposition 17 ([GGV01]). *A unit invariant vector field on a naturally reductive homogeneous manifold of dimension ≤ 5 is minimal if and only if it is harmonic.*

The same conclusion is also true, independently of the dimension, for unit invariant V such that ∇V is of rank 2. With all these properties, we determine all invariant minimal and harmonic unit vector fields on naturally reductive homogeneous spaces of dimension is ≤ 5.

Another interesting class of homogeneous structures S is defined by the conditions $\eta = 0$ and $\langle S_X Y, Z\rangle + \langle S_Z X, Y\rangle + \langle S_Y Z, X\rangle = 0$. The full classification of connected, complete and simply connected Riemannian manifolds of dimension-3 and 4 which admit a non-trivial structure was obtained by Kowalski and Tricerri in [KTr87]. In [GGV01], we combine this classification with Theorem 5 to determine, all invariant minimal and harmonic unit vector fields on homogeneous spaces of dimension-3 and 4.

3.2 Examples coming from contact geometry

The first examples of critical vector fields, Hopf vector fields, are in fact the characteristic vector fields of the usual Sasakian structure of odd-dimensional spheres. We have obtained that they are critical since they are Killing, but the same can be concluded also as a particular case of the following general result.

Theorem 6 ([Wie95],[GL02]). *The characteristic vector field of any Sasakian manifold is harmonic and a minimal immersion.*

Let us recall some definitions. For more details on contact geometry, the reader is referred to the book by Blair [Bla02].

A contact structure on a $(2n+1)$-dimensional manifold M is a globally defined 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. There exists a unique vector field ξ such that $\eta(\xi) = 1$

and $\iota_\xi d\eta = 0$; it is called the characteristic vector field. A Riemannian metric g on M is said to be associated if $\eta(X) = g(\xi, X)$ and the $(1,1)$-tensor field ϕ defined by $g(X, \phi(Y)) = d\eta(X, Y)$ verifies $\phi^2(X) + X = \eta(X)\xi$. In that case, (η, g) is a contact metric structure and the usual notation for a contact metric manifold is (M, η, g, ξ, ϕ).

If the characteristic vector field is Killing, the manifold is said K-contact, for such a manifold

$$R(X, \xi, Y, \xi) = g(X, Y),$$

and then $\rho(\xi, X) = 2ng(\xi, X)$. Moreover,

$$R(X, Y)\xi = g(X, \xi)Y - g(Y, \xi)X,$$

the manifold is called Sasakian. Another class of contact metric manifolds is that of the (k, μ)-spaces, introduced by Blair, Koufogiorgos and Papantoniou in [BKP95] as those such that,

$$R(X, Y)\xi = k\big(g(X, \xi)Y - g(Y, \xi)X\big) + \frac{\mu}{2}\big(g(X, \xi)(\mathcal{L}_\xi\phi)(Y) - g(Y, \xi)(\mathcal{L}_\xi\phi)(X)\big),$$

where k and μ are constants. In the particular case of a three-dimensional manifold, one can consider the class of generalized (k, μ)-spaces where k and μ are functions (see [KT00]).

The (k, μ)-spaces have been completely classified by Boeckx in [Boe00]. In particular, the unit tangent manifold of a constant curvature space is of this type, so the next result provides examples of minimal and harmonic vector fields.

Proposition 18 ([GV00],[Per]).** *The characteristic vector fields of K-contact manifolds and of (k, μ)-spaces are minimal and harmonic.*

Since a contact metric manifold is Sasakian if and only if it is a K-contact (k, μ)-space, the proposition above generalizes Theorem 6 in two different directions.

The minimality of the characteristic vector field of a K-contact manifold was also obtained by Rukimbira in [Ruk02] by a different method, namely by showing that the characteristic vector field of a contact metric manifold defines a contact invariant submanifold of the unit tangent bundle, endowed with the natural contact structure defined by the metric.

In the particular case of three-dimensional contact manifolds, a convenient expression of the Levi-Civita connection, obtained by Calvaruso, Perrone and Vanhecke in [CPV99], was used by González–Dávila and Vanhecke in [GV01] to show

Proposition 19. *The characteristic vector field ξ of a three-dimensional contact metric manifold is harmonic if and only if $\rho_\xi(\xi^\perp) = 0$.*

They also obtained the conditions that ξ should verify in order to define a harmonic map and to be minimal, and used them to study the cases where the manifold has constant scalar curvature or it is locally homogeneous. With the same methods, these results have been recently improved by Perrone in [Per03].

Proposition 20 ([Per03]). *The characteristic vector field of a three-dimensional contact metric manifold defines a harmonic map if and only if the manifold is a generalized (k, μ)-space on an open and dense subset. It is harmonic and is minimal if and only if the manifold is either Sasakian or a unimodular Lie group equipped with a non Sasakian left-invariant contact metric structure.*

As a consequence, it is shown in [Per03] that a compact 3-manifold admits a contact metric structure whose characteristic vector field is harmonic and is minimal if and only if it is diffeomorphic to a left quotient of the Lie group G under a discrete subgroup, where G is one of $SU(2)$, the Heisenberg group, the group of motions of the Minkowski plane or the universal cover of $SL(2, \mathbb{R})$, or the universal cover of the group of motions of the Euclidean plane.

In [Per**], the same author has generalized Proposition 19 to any dimension as a consequence of the following:

Proposition 21 ([Per]).** *Let (M, ξ, g) be a contact metric manifold. Then $g(\nabla^*\nabla\xi, X) = \rho_\xi(X) - 2ng(\xi, X)$, for all vector field X.*

This result leads the author to define the class of *H-contact manifolds* as those with harmonic characteristic vector field, or equivalently those for which $\rho_\xi(\xi^\perp) = 0$. This condition on the Ricci tensor appeared in [GPT89], in relation with the vanishing of the first Betti number of a contact manifold with critical metric. Such a metric is defined as a critical point of the Chern–Hamilton functional that is given, up to constants, by the energy of ξ. Let us recall that, in fact, the energy functional on $\mathfrak{X}(M)$ is the restriction of a functional defined on $\mathfrak{X}(M) \times \mathcal{M}$, where $\mathcal{M}$ is the space of metrics on M (see [Gil01]). Then, critical metrics are critical points for variations by associated metrics and with fixed ξ, and harmonic unit vector fields are critical points for variations by unit vector fields, and with fixed g. In [Per**], H-contact manifolds with critical metrics are called *critical contact metric structures* and the author obtains a complete description—up to diffeomorphism—of all compact 3-dimensional manifolds admitting such a structure.

3.3 The hopf vector field of a hypersurface of a complex manifold

The Hopf unit vector fields on odd-dimensional spheres are a particular case of a general construction on any orientable real hypersurface (M, g) of a Kähler manifold $(\bar{M}, \bar{g}, J)$. We can define on M the unit vector field $\xi = JN$, where N represents the unit normal to the hypersurface. In fact, ξ is the characteristic vector field of an almost-contact metric structure on M. We can define an endomorphism field on the submanifold by the expression

$$\phi(X) = JX - \bar{g}(JX, N)N.$$

The submanifold M is said to be a *Hopf hypersurface* if ξ determines a principal direction, that is, if $S_\xi = \alpha\xi$, where S represents the shape operator. Tsukada and Vanhecke have shown

Theorem 7 ([TV00]). *Let M be an orientable real hypersurface of a Kähler manifold. Then the Hopf vector field ξ is harmonic if and only if*

$$X(h) = g(\phi S^2\xi, X) - \bar{\rho}(X, N),$$

for all $X \in \xi^{\perp}$, where h denotes the mean curvature of M and $\bar{\rho}$ is the Ricci tensor of $\bar{M}$.

The corresponding result for the particular case of a complex space form was previously obtained by the same authors in [TV01]. The principal curvature α of a Hopf hypersurface, in a complex space form of non-zero curvature, is constant and the other principal curvatures are constant along the integral curves of ξ. Therefore, $\xi(h) = 0$ and then

Corollary 2 ([TV01]). The Hopf vector field of a Hopf hypersurface in a complex space form, of non-zero curvature, is harmonic if and only if the mean curvature is constant.

Under the same hypothesis, the authors show that if the mean curvature is constant then the Hopf vector field defines a harmonic map. Moreover, among constant mean curvature orientable real hypersurfaces in a complex space form, of non-zero curvature, the Hopf hypersurfaces are characterized by the property of their Hopf vector fields defining harmonic maps.

For more general manifolds they show:

Corollary 3 ([TV00]). Let M be a Hopf hypersurface of an Einstein–Kähler manifold.

a) If M has constant mean curvature, then ξ is harmonic.
b) If ξ is harmonic and $\phi \circ S + S \circ \phi \neq 0$ on an open and dense subset, then the mean curvature of M is constant.

Using these results, they obtain new examples of harmonic vector fields. In particular, they describe in [TV00] examples of hypersurfaces of the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. This manifold carries a Kähler structure J and a quaternionic Kähler structure $\mathcal{J}$. For $m \geq 3$, they consider real hypersurfaces with normal bundle $M^{\perp}$ such that $J(M^{\perp})$ and $\mathcal{J}(M^{\perp})$ are invariant under the action of the shape operator. They proved that the corresponding unit Hopf vector fields of these hypersurfaces are harmonic. Moreover, using the results concerning this class of submanifolds by Berndt [Ber97] and by Berndt and Suh [BS99], they compute ν_ξ and $\tilde{\omega}_\xi$ and show that ξ also defines a harmonic map and a minimal immersion. Similar results hold for the non-compact dual space $G_2(\mathbb{C}^{m+2})^*$.

The condition characterizing the minimality of the Hopf vector field is more complicated. The function involved here is not the mean curvature but the following:

$$\tilde{h} = \sum_{i=1}^{2n} \operatorname{arc\,cot} \lambda_i,$$

where $\{\lambda_i \; ; \; i = 1, \ldots, 2n\}$ are the principal curvatures of $\xi^{\perp}$. This function is defined only in the open dense set where the multiplicities of the principal curvatures are locally constant.

Proposition 22 ([TV01]). *The Hopf vector field of a Hopf hypersurface in a complex space form is minimal if and only if $\tilde{h}$ is constant.*

It is a consequence that the Hopf vector field of Hopf hypersurface with constant principal curvatures in a complex space form, of non-zero curvature, provide simultaneously new examples of unit vector fields that are minimal and harmonic maps. Tubes about complex submanifolds in complex space forms are examples of Hopf hypersurfaces. Those with constant principal curvatures have been classified in [Ber89] and [Kim86].

A very surprising result is the following:

Proposition 23 ([TV01]). *The Hopf vector field of a Hopf hypersurface of a complex space form of constant holomorphic sectional curvature* 4 *is minimal.*

The minimal ruled real hypersurfaces form another class of particular examples. Let us recall a real hypersurface M is said to be ruled if $\xi^{\perp}$ defines a foliation with totally geodesic leaves.

Proposition 24 ([TV01]). *The Hopf vector field of a minimal real ruled hypersurface of a complex space form of non-zero curvature is harmonic, minimal but never defines a harmonic map.*

Since examples of this kind of hypersurfaces were known previously, the result gives more examples of critical vector fields.

3.4 Radial unit vector fields

For a given point $p_0 \in S^n$, we can define on $S^n - \{p_0, -p_0\}$ a unit radial vector field V_R, which is at each point the unit vector tangent to the unique geodesic passing through p_0 and the point. Radial vector fields play an important role in what concerns volume and energy; let us see why.

On one hand, since a radial vector field is geodesic and its orthogonal distribution defines a foliation with umbilical leaves, the endomorphism L_V can be expressed in terms of the principal curvature function α of the leaves, as a diagonal matrix with entries $(1+\alpha^2, \ldots, 1+\alpha^2, 1)$ and then, it is not difficult to show that

Proposition 25. *The unit radial vector fields on the sphere are critical points of the energy, minimal immersions but they are not harmonic maps.*

Using the expression of L_V, we can also compute the energy and the volume. In particular, the energy of the radial unit vector field on the 2-dimensional sphere is infinite. There is a general result of Boeckx, González–Dávila and Vanhecke in [BGV03] showing that, in fact,

Proposition 26. *The tori are the only compact oriented surfaces admitting unit vector fields with finite energy and with at most a finite number of singular points.*

In [BGV03], they have computed the explicit value of the energy of radial vector fields around points, and about special classes of totally geodesic submanifolds, on the other compact rank-one symmetric spaces, using the fact that all these unit vector fields are geodesic, that their orthogonal distribution coincides at each point with the tangent space of the geodesic sphere, or the tube, and that the principal curvatures of these hypersurfaces are well known.

In all the cases considered, the set of singularities of the vector field consists on a finite number of pair-wise disjoint closed submanifolds.

On the other hand, it has been shown by Brito and Walczak in [BW00] that $E(V) \geq E(V_R)$ for all unit vector fields on the sphere with isolated singularities with equality only when $V = V_R$; and by Brito, Chacón and Naveira in [BCN] that $\mathrm{Vol}(V) \geq \mathrm{Vol}(V_R)$ for all smooth unit vector fields V. In fact, $E(V_R)$ is not only a lower bound, but it is the infimum of the energy of smooth unit vector fields on S^n if n is odd and $n > 3$ as we have seen in [BBG03]. The analogous question concerning the volume is more delicate, as we have shown in [BG**], and to find the infimum of the volume of unit vector fields on odd-dimensional spheres is still an open problem.

In [BV01], Boeckx and Vanhecke made a local study of radial vector fields about points and about totally geodesic submanifolds of rank one symmetric spaces. They are defined in general only on tubular neighborhoods of these points and submanifolds, excluding the points and the submanifolds themselves. Their results can be summarized as follows:

Theorem 8. *On a two-point homogeneous space, any radial unit vector field defined in a (pointed) normal neighborhood of a point is minimal and harmonic but it determines a harmonic map into the unit tangent bundle if and only if the manifold is flat.*

Furthermore, if a two-dimensional manifold has the property that every radial unit vector field is either harmonic or minimal, then it has constant curvature.

Theorem 9. *Let (M, g) be a Riemannian manifold and let P be a totally geodesic submanifold. The radial unit vector field defined in any tubular neighborhood of P is minimal and harmonic in the following cases:*

a) (M, g) has constant curvature.
b) (M, g) is one of the two-point homogeneous spaces with complex structure J and P is J-invariant.
c) (M, g) is a $2m$-dimensional Kähler manifold of constant holomorphic sectional curvature and P is a m-dimensional anti-invariant submanifold.
In all cases, the vector field determines a harmonic map into the unit tangent bundle only if the manifold is flat.

Let us recall that a Sasakian manifold has constant ϕ-sectional curvature c if and only if every plane admitting an orthonormal base of the form $\{X, \phi X\}$ has curvature c. For this kind of manifolds, the same authors have shown the following:

Proposition 27 ([BV01]). *On a Sasakian space form of constant ϕ-sectional curvature, the radial vector field on a tubular neighborhood of any characteristic line is both minimal and harmonic, but it does not determine a harmonic map.*

Since a radial vector field is, up to normalization, the gradient of the distance function (to the submanifold or to the point), it is a particular case of a more general situation that was studied by the same authors in [BV01b], wherein, they considered vector fields defined outside the set of critical points of isoparametric functions, as the gradient divided by its norm. A real valued function f on a Riemannian manifold is said to be isoparametric if

$$\|df\|^2 = b(f) \quad \text{and} \quad \Delta f = a(f),$$

where a is a continuous function and b is smooth.They have shown

Proposition 28. *The unit vector fields obtained from the gradient of isoparametric functions, up to normalization, are harmonic on Einstein manifolds, minimal on spaces of constant curvature and they determine harmonic maps only on flat manifolds.*

With this approach, they also obtain many examples of minimal and harmonic unit vector fields defined by the flow normal to foliations provided by certain homogeneous hypersurfaces of complex and quaternionic spaces forms. These examples never determine harmonic maps into the unit tangent bundle.

A connected closed submanifold F of a complete Riemannian manifold M is said to be reflective if the geodesic reflection of M in F is a well-defined global isometry. If M is a symmetric space, there exists another reflective submanifold associated to F, denoted $F^{\perp}$. Both are totally geodesic and then symmetric spaces. For this kind of submanifolds, Berndt, Vanhecke and Verhóczki [BVV03] have shown the following

Theorem 10 ([BVV03]). *Let M be a Riemannian symmetric space of compact or of non-compact type, and let P be a reflective submanifold of M such that its codimension is greater than one and the rank of the complementary reflective submanifold $P^{\perp}$ is equal to one. Then the radial unit vector field, tangent to the geodesics emanating perpendicularly from P, is harmonic and minimal.*

Under the hypotheses, P is the singular orbit of a cohomogeneity one action on M, the radial vector field is then defined in the open and dense set formed by the union of principal orbits. The authors provide a list of pairs (M, P) with the required conditions, finding new explicit examples of unit vector fields that are harmonic and minimal.

Apart from being an important source of natural examples, the harmonicity of radial vector fields give rise to a very nice new characterization of harmonic manifolds. Let us recall that this name is used for Riemannian spaces such that every small geodesic sphere has constant mean curvature. Boeckx and Vanhecke [BV00b] have shown

Theorem 11 ([BV00b]). *A Riemannian manifold is harmonic if and only if all radial vector fields on pointed normal neighborhoods of arbitrary points are harmonic vector fields. They determine harmonic maps only if the manifold is flat.*

3.5 Unit vector fields on tangent and unit tangent bundles

It is well known that given a Riemannian manifold (M, g) a natural vector field on T^1M, called the geodesic vector field and denoted by G, can be defined as follows: for $u \in T^1_pM$ let γ_u be the geodesic passing through p with tangent vector u and let $c(t)$ be the curve on T^1M given by $c(t) = \gamma_u'(t) \in T^1_{\gamma_u(t)}$ which is a horizontal lift of γ_u. We define $G(u) = c'(0)$. It is easy to see that G is a unit vector field of (T^1M, g^S).

The geometry of the tangent or unit tangent bundle has been deeply studied and involves the geometry of the base manifold in a way that, although very natural, it produces complicated expressions if the curvature of the manifold has no particular properties. Let us recall that the curvature tensor appears in the covariant derivative of the Sasaki metric. So, one can not expect to obtain results valid for a general manifold, without curvature assumptions.

The better results by the moment are due to Boeckx and Vanhecke [BV00] that have shown

Theorem 12 ([BV00]). *Let (M, g) be a two-point homogeneous space. Then the geodesic flow $G : T^1M \to (T^1(T^1M), (g^S)^S)$ is a minimal immersion and a harmonic map from (T^1M, g^S). Moreover, if the dimension of M is either 2 or 3 and G is either minimal or harmonic, then M has constant curvature.*

Although in the unit tangent bundle of a Riemannian manifold the only distinguished unit vector field is the geodesic field, as far as the manifold is endowed with an almost hermitian structure (M, g, J), it is possible to define a one parameter family of special horizontal unit vector fields on T^1M by $G_\alpha = \cos\alpha\, G + \sin\alpha\, (G \circ J)$. Furthermore, a vertical unit vector field on the unit tangent bundle is well defined by $\tilde{G}(u) = (Ju)^v$. In [BV00], it has been shown

Theorem 13. *If (M, g, J) is a complex space form, $\tilde{G}$ and G_α, for all α, define harmonic maps and minimal immersions.*

The above results continue to hold for the corresponding scaled vector fields defined on the sphere bundles of any radius (T^rM, g^S) and also for their natural extensions defined on TM outside the zero section.

The same authors with González–Dávila have studied, in [BGV02], the Hessian of the energy functional at G and at $\tilde{G}$ for certain variations. They are able to show that if the curvature of M fulfills certain inequalities, that depend on the dimension, then they are unstable not only as harmonic maps, but also as harmonic vector fields.

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On 3D-Riemannian Manifolds with Prescribed Ricci Eigenvalues

Oldřich Kowalski and Zdeněk Vlášek

Charles University Prague, Czech Republic
kowalski@karlin.mff.cuni.cz**
vlasek@karlin.mff.cuni.cz

Dedicated to Professor Lieven Vanhecke

Summary. In this paper, we deal with 3-dimensional Riemannian manifolds where some conditions are put on their principal Ricci curvatures. In Section 2 we classify locally all Riemannian 3-manifolds with prescribed distinct Ricci eigenvalues, which can be given as arbitrary real analytic functions. In Section 3 we recall, for the *constant* distinct Ricci eigenvalues, an explicit solution of the problem, but in a more compact form than it was presented in [17]. Finally, in Section 4 we give a survey of related results, mostly published earlier in various journals. Last but not least, we compare various PDE methods used for solving problems of this kind.

1 Introduction

The problem of how many Riemannian metrics exist on the open domains of R^3 with prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ was completely solved in [15] and [19]. The main existence theorem says that the local isometry classes of these metrics are always parametrized by *two arbitrary functions of one variable*. Some non-trivial explicit examples were presented in [15], as well. A more elegant but less rigorous proof of the main existence theorem was given in [5].

The case of distinct constant Ricci eigenvalues is more interesting. Here, the first examples were presented by K.Yamato in [33], namely a complete (but not locally homogeneous) metric defined on R^3 for each prescribed triplet $(\varrho_1, \varrho_2, \varrho_3)$ of constant distinct Ricci eigenvalues satisfying certain algebraic inequalities. Thus, these triplets

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form an open set in $\mathsf{R}^3[\varrho_1, \varrho_2, \varrho_3]$. This open set was essentially extended by new examples in [13]. Finally, in [17], non-trivial explicit examples were constructed for *every choice* of Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. These examples are not locally homogeneous but mostly local and not complete. (There is still an open problem for which triplets $\varrho_1 > \varrho_2 > \varrho_3$ a complete metric exists with such Ricci eigenvalues.)

The problem of '*how many* local isometry classes (or, more exactly, how many *isometry classes of germs*) of Riemannian metrics exist for prescribed constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$' was solved first by A. Spiro and F. Tricerri in [30], using the theory of formally integrable analytic differential systems. They proved that this "local moduli space" depends on an infinite number of parameters. This solution was not satisfactory enough for us and we succeeded to show in [26] that this local moduli space is parametrized, in fact, by (the germs of) three arbitrary functions of two variables. Moreover, the method of solution was completely "classical", based on the Cauchy–Kowalewski Theorem. Yet, for many mathematicians, this solution may be not completely satisfactory for a different reason: The partial differential equations expressing the geometric conditions are rather cumbersome (see Section 3), and one of the main steps of the proof is not transparent enough, because it depends heavily on a hard computer work (using Maple V) for the huge amount of routine symbolic manipulations with the corresponding PDE system.

In this paper, we prove the same result by a different method. Here the computer assistence (using Maple V) is also used, but in a much more transparent way. Namely, when using the new method, some cumbersome formulas occur again. Yet, for the main argument, we need only their qualitative properties and not the explicit expressions.

Moreover, by the new method, we are able to *generalize* the original result to the situation when the prescribed distinct Ricci eigenvalues are not constants but *arbitrary functions*. This is the content of Section 2.

In Section 3, we come back to the old version from [17] and [26] (with constant ϱ_i and a complicated PDE system) to show that there is a general *explicit* formula involving three parameters $\varrho_1 > \varrho_2 > \varrho_3$ and producing a Riemannian metric with the Ricci principal curvatures ϱ_i. This result was essentially proved already in [17], but now it is presented in a particularly simple form, in the spirit of the pioneering work by K. Yamato [33]. It is obvious that the new method from Section 2 is not suitable to produce such explicit examples and so one can compare the advantages and disadvantages of both (very different) methods.

The last Section 4 is mainly a survey of related results which have been published earlier (except the last subsection inspired by the work by S. Ivanov and I. Petrova [11]). The main purpose of Section 4 is to show that there are more geometric problems concerning prescribed properties of the Ricci eigenvalues for which a completely satisfactory geometric solution was found, but where "the method of the Ricci characteristic polynomial", introduced in Section 2, obviously fails. Namely, from the optics of this method, one comes to an overdetermined system of PDE. Yet, we are still able to describe "the size" of the general solutions of such systems just coming back from a known geometric result to the corresponding PDE system. This might be a useful contribution to the "philosophy of PDE methods" in Riemannian geometry.

2 The case of arbitrary distinct Ricci eigenvalues

Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined on a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Let (M, g) be a Riemannian manifold and $\mathcal{U}' \subset M$ a coordinate neighborhood. We say that *the metric g restricted to $\mathcal{U}'$ has principal Ricci curvatures $\varrho_1, \varrho_2, \varrho_3$*, if this is valid with respect to a local chart $\varphi : \mathcal{U}' \to \mathcal{U}$, i.e., when expressing $g_{|\mathcal{U}'}$ in the local coordinates x, y, z.

The main theorem of this paper is the following:

Theorem 1. *Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined on a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Then, the local moduli space of (local) Riemannian metrics with the prescribed principal Ricci curvatures $\varrho_1, \varrho_2, \varrho_3$ can be parametrized by three arbitrary functions of two variables.*

We shall start with the hard part of the proof, which is based on the following:

Theorem 2. *Let $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ be three real analytic functions defined in a domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$. Then all (local) diagonal Riemannian metrics with the principal Ricci curvatures ϱ_1, ϱ_2 and ϱ_3 depend on six arbitrary functions of two variables.*

The following Theorem should be considered as a "folklore", see e.g. [9].

Theorem 3. *Let (M, g) be a real analytic 3-dimensional Riemannian manifold. Then, in a neighborhood of each point $p \in M$, there is a system (x, y, z) of local coordinates in which g adopts a diagonal form. All coordinate transformations for which the diagonality of a metric is preserved depend on 3 arbitrary functions of two variables.*

Thus, in the sequel, we can assume that each Riemannian metric g in consideration has the matrix (g_{ij}) of components written in the form

$$(g_{ij}) = \begin{pmatrix} K(x,y,z) & 0 & 0 \\ 0 & L(x,y,z) & 0 \\ 0 & 0 & M(x,y,z) \end{pmatrix}, \ (i, j = 1, 2, 3).$$

Here, of course, the functions K, L and M are positive and real analytic in the corresponding domain $\mathcal{U} \subset \mathsf{R}^3[x, y, z]$.

A routine calculation gives the following expression for the Ricci *operator* Ric in the given local coordinates x, y, z. Here, we introduce the following abbreviated notation: If G denotes K, L or M, evaluated at a general point (x, y, z), we write,

$$G_1 = \frac{\partial G}{\partial x},\ G_2 = \frac{\partial G}{\partial y},\ G_3 = \frac{\partial G}{\partial z},$$
$$G_{11} = \frac{\partial^2 G}{(\partial x)^2},\ G_{12} = \frac{\partial^2 G}{\partial x \partial y},\ \dots,\ G_{33} = \frac{\partial^2 G}{(\partial z)^2},$$

evaluated at (x, y, z), as well. Now, in the abbreviated notation, we have the following formulas:

$$\begin{aligned}
\mathrm{Ric}_1^1 = &- (MK_{22} + LK_{33} + ML_{11} + LM_{11}) / (2KLM) + [LM^2K_1L_1 \\
&+ L^2MK_1M_1 + LM^2K_2^2 + KM^2K_2L_2 - KLMK_2M_2 + L^2MK_3^2 \\
&- KLMK_3L_3 + KL^2K_3M_3 + KM^2L_1^2 + KL^2M_1^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_2^2 = &- (MK_{22} + ML_{11} + KL_{33} + KM_{22}) / (2KLM) + [LM^2K_1L_1 \\
&+ LM^2K_2^2 + KM^2K_2L_2 - KLMK_3L_3 + KM^2L_1^2 - KLML_1M_1 \\
&+ K^2ML_2M_2 + K^2ML_3^2 + K^2LL_3M_3 + K^2LM_2^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_3^3 = &- (LK_{33} + KL_{33} + LM_{11} + KM_{22}) / (2KLM) + [L^2MK_1M_1 \\
&- KLMK_2L_2 + L^2MK_3^2 + KL^2K_3M_3 - KLML_1M_1 + K^2ML_2M_2 \\
&+ K^2ML_3^2 + K^2LL_3M_3 + KL^2M_1^2 + K^2LM_2^2\,]/(4K^2L^2M^2), \\
\mathrm{Ric}_1^2 = &\ \mathrm{Ric}_2^1 \\
= &-(2KLMM_{12} - LMK_2M_1 - KML_1M_2 - KLM_1M_2)/(4KL^2M^2), \\
\mathrm{Ric}_1^3 = &\ \mathrm{Ric}_3^1 \\
= &-(2KLML_{13} - LMK_3L_1 - KML_1L_3 - KLL_3M_1)/(4KL^2M^2), \\
\mathrm{Ric}_2^3 = &\ \mathrm{Ric}_3^2 \\
= &-(2KLMK_{23} - LMK_2K_3 - KMK_2L_3 - KLK_3M_2)/(4K^2LM^2).
\end{aligned}$$

We now express the above formulas in a shorter way:

$$\begin{aligned}
\mathrm{Ric}_1^1 &= - (MK_{22} + LK_{33} + ML_{11} + LM_{11}) / (2KLM) + G_1^1, \\
\mathrm{Ric}_2^2 &= - (MK_{22} + ML_{11} + KL_{33} + KM_{22}) / (2KLM) + G_2^2, \\
\mathrm{Ric}_3^3 &= - (LK_{33} + KL_{33} + LM_{11} + KM_{22}) / (2KLM) + G_3^3, \\
\mathrm{Ric}_1^2 &= \mathrm{Ric}_2^1 = -M_{12}/(2LM) + G_1^2, \\
\mathrm{Ric}_1^3 &= \mathrm{Ric}_3^1 = -L_{13}/(2LM) + G_1^3, \\
\mathrm{Ric}_2^3 &= \mathrm{Ric}_3^2 = -K_{23}/(2KM) + G_2^3,
\end{aligned} \tag{1}$$

where G_i^j are rational functions of $K, L, M, K_1, K_2, \ldots, M_3$, i.e., they depend only on the functions K, L, M and their first derivatives.

Consider now the prescribed Ricci eigenvalues $\varrho_1(x, y, z),\ \varrho_2(x, y, z),\ \varrho_3(x, y, z)$ (which are real analytic functions defined in the same domain as K, L and M). The corresponding geometric conditions can be expressed, in the simplest way, through the characteristic polynomial $\det(\lambda I - \mathrm{Ric}) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$ of the Ricci operator Ric, in the form

$$c_2 = -\sum_{i=1}^{3} \varrho_i, \quad c_1 = \sum_{1 \le i < j \le 3} \varrho_i \varrho_j, \quad c_0 = -\varrho_1\varrho_1\varrho_3. \tag{2}$$

This is a system of nonlinear PDE's of second order because

$$c_2 = -\sum_{i=1}^{3} \mathrm{Ric}^i_i, \quad c_1 = \sum_{1\le i<j\le 3} (\mathrm{Ric}^i_i \mathrm{Ric}^j_j - (\mathrm{Ric}^i_j)^2), \quad c_0 = -\det[\mathrm{Ric}^i_j]. \tag{3}$$

We can see easily from (1) that the only "non-mixed" second partial derivatives involved in the functions Ric^i_j are K_{22}, K_{33}, L_{11}, L_{33}, M_{11} and M_{22}. Hence we cannot use the Cauchy–Kowalewski Theorem in the basic setting. We shall try to remove this defect by a linear transformation of independent variables (which is optimal in some sense), namely,

$$u = z, \ v = y, \ w = x + y + z. \tag{4}$$

The metric g, if expressed in the new variables u, v, w, is not anymore in the diagonal form. The new Ricci components $\mathrm{Ric}^\alpha_\beta$ will become linear combinations of the original components Ric^i_j. Nevertheless, because, with respect to the new variables we get

$$[\mathrm{Ric}^\alpha_\beta] \ = \ [S][\mathrm{Ric}^i_j][S^{-1}],$$

where S is a constant regular matrix, the characteristic polynomial of Ric will remain invariant and the expression (3) have the same form for the old components Ric^i_j as for the new components $\mathrm{Ric}^\alpha_\beta$. Thus, we can still use the old components Ric^i_j in our computations and all to be done is to transform all Ric^i_j to the new variables u, v, w. As the first step, the original functions K, L, M and their partial derivatives have to be transformed.

We now introduce new positive functions U, V, W of three variables u, v, w by

$$\begin{aligned} U(u, v, w) &= K(w - u - v, v, u), \\ V(u, v, w) &= L(w - u - v, v, u), \\ W(u, v, w) &= M(w - u - v, v, u). \end{aligned} \tag{5}$$

Rewriting the old coordinates, we get,

$$\begin{aligned} K(x, y, z) &= U(z, y, x + y + z), \\ L(x, y, z) &= V(z, y, x + y + z), \\ M(x, y, z) &= W(z, y, x + y + z). \end{aligned} \tag{6}$$

If F denotes U, V or W evaluated at $(u, v, w) = (z, y, x + y + z)$, we shall write

$$F_1 = \frac{\partial F}{\partial u}, \ F_2 = \frac{\partial F}{\partial v}, \ F_3 = \frac{\partial F}{\partial w},$$
$$F_{11} = \frac{\partial^2 F}{(\partial u)^2}, \ F_{12} = \frac{\partial^2 F}{\partial u \partial v}, \ \ldots, \ F_{33} = \frac{\partial^2 F}{(\partial w)^2},$$

evaluated at the point $(u, v, w) = (z, y, x + y + z)$, as well.

We get easily, in our abbreviated form,

$$\begin{aligned} K &= U,\ K_1 = U_3,\ K_2 = U_2 + U_3,\ K_3 = U_1 + U_3,\ K_{11} = U_{33},\\ K_{12} &= U_{23} + U_{33},\ K_{13} = U_{13} + U_{33},\ K_{22} = U_{22} + 2U_{23} + U_{33},\\ K_{23} &= U_{12} + U_{13} + U_{23} + U_{33},\ K_{33} = U_{11} + 2U_{13} + U_{33},\\ L &= V,\ L_1 = V_3,\ L_2 = V_2 + V_3,\ L_3 = V_1 + V_3,\ L_{11} = V_{33},\\ L_{12} &= V_{23} + V_{33},\ L_{13} = V_{13} + V_{33},\ L_{22} = V_{22} + 2V_{23} + V_{33},\\ L_{23} &= V_{12} + V_{13} + V_{23} + V_{33},\ L_{33} = V_{11} + 2V_{13} + V_{33},\\ M &= W,\ M_1 = W_3,\ M_2 = W_2 + W_3,\ M_3 = W_1 + W_3,\ M_{11} = W_{33},\\ M_{12} &= W_{23} + W_{33},\ M_{13} = W_{13} + W_{33},\ M_{22} = W_{22} + 2W_{23} + W_{33},\\ M_{23} &= W_{12} + W_{13} + W_{23} + W_{33},\ M_{33} = W_{11} + 2W_{13} + W_{33}. \end{aligned} \tag{7}$$

Hence, we obtain, for the old components Ric^i_j evaluated at $(u, v, w) = (z, y, x + y + z)$,

$$\begin{aligned} \mathrm{Ric}^1_1 &= -\,((V+W)\,U_{33} + W V_{33} + V W_{33})\,/\,(2UVW) + F^1_1,\\ \mathrm{Ric}^2_2 &= -\,(W U_{33} + (U+W)\,V_{33} + U W_{33})\,/\,(2UVW) + F^2_2,\\ \mathrm{Ric}^3_3 &= -\,(V U_{33} + U V_{33} + (U+V)\,W_{33})\,/\,(2UVW) + F^3_3,\\ \mathrm{Ric}^2_1 &= \mathrm{Ric}^1_2 = -W_{33}/(2VW) + F^2_1,\\ \mathrm{Ric}^3_1 &= \mathrm{Ric}^1_3 = -V_{33}/(2VW) + F^3_1,\\ \mathrm{Ric}^3_2 &= \mathrm{Ric}^2_3 = -U_{33}/(2UW) + F^3_2, \end{aligned} \tag{8}$$

where F^i_j are rational functions of U, V, W, their first partial derivatives with respect to u, v, w, and their second partial derivatives which are different from U_{33}, V_{33} and W_{33}.

Now, we are going to prove that, in the new variables, the standard Cauchy–Kowalewski Theorem can be used for the solution of the PDE system (2). We only have to keep in mind that the prescribed Ricci eigenvalues ϱ_i mean here the functions $\overline{\varrho}_i(u, v, w) = \varrho_i(w - u - v, v, u)$, $i = 1, 2, 3$, defined in the same domain as U, V and W. The system (2) can be expressed explicitly in the new variables u, v, w. We get the first PDE in the form

$$\begin{aligned} c_2 &= ((V+W)\,U_{33} + (U+W)\,V_{33} + (U+V)\,W_{33})\,/\,(UVW) + H_2\\ &= -\sum_{i=1}^{3} \overline{\varrho}_i, \end{aligned} \tag{9}$$

where H_2 is a rational function of U, V, W, their first derivatives and their second derivatives which are not of the form U_{33}, V_{33} or W_{33}. H_2 is defined as a function of the variables u, v, w in the whole definition domain of the functions U, V, W. From here, we express

$$W_{33} = -\left((V + W)\, U_{33} + (U + W)\, V_{33}\right) / (U + V) + P, \tag{10}$$

where P is a rational function of $\overline{\varrho}_1, \overline{\varrho}_2, \overline{\varrho}_3, U, V, W$, the first derivatives of U, V, W, and their second derivatives except U_{33}, V_{33} and W_{33}. Anyway P is a real analytic function of u, v, w, U, V, W and of the corresponding derivatives.

Next, we substitute the expression for W_{33} from (10) in the formulas (8) and we obtain the Ricci components in the reduced form:

$$\begin{aligned}
\mathrm{Ric}^1_1 &= -\left((V + W)\, U_{33} + (W - V)\, V_{33}\right) / (2(U + V)VW) + P^1_1, \\
\mathrm{Ric}^2_2 &= -\left((W - U)\, U_{33} + (U + W)\, V_{33}\right) / (2(U + V)UW) + P^2_2, \\
\mathrm{Ric}^3_3 &= (U_{33} + V_{33}) / (2UV) + P^3_3, \\
\mathrm{Ric}^2_1 &= \mathrm{Ric}^1_2 = \left((V + W)\, U_{33} + (U + W)\, V_{33}\right) / (2(U + V)VW) + P^2_1, \\
\mathrm{Ric}^3_1 &= \mathrm{Ric}^1_3 = -V_{33}/ (2VW) + P^3_1, \\
\mathrm{Ric}^3_2 &= \mathrm{Ric}^2_3 = -U_{33}/ (2UW) + P^3_2,
\end{aligned} \tag{11}$$

where P^i_j are functions of the same type as P introduced in (10).

So, assuming that (9) is satisfied identically, we must write down the remaining two PDE's where the Ricci operator is expressed in the form (11). The second equation of (2) can be written in the form

$$f_1(U_{33})^2 + f_2 U_{33} V_{33} + f_3(V_{33})^2 + g_1 U_{33} + g_2 V_{33} = Q, \tag{12}$$

where Q is of the same type as P and P^i_j. Moreover, we get explicitly

$$\begin{aligned}
f_1 &= \frac{U^2(3V^2 + 3VW + 2W^2) + UV(2V^2 - VW + W^2) + V^2(V^2 + W^2)}{4(U + V)^2U^2V^2W^2}, \\
f_2 &= \frac{U^3(V + W) + U^2(-V^2 + VW + 2W^2) + UVW^2 + V^2W^2}{2(U + V)^2U^2V^2W^2}, \\
f_3 &= \frac{2U^3(U + V + W) + U^2(2V^2 - VW + 2W^2) + UVW(V + W) + V^2W^2}{4(U + V)^2U^2V^2W^2},
\end{aligned} \tag{13}$$

and g_1, g_2 are (more complicated) rational functions of the same type as H_2 in (9).

The third equation of (2) can be written in the form

$$\begin{aligned}
f_{30}(U_{33})^3 &+ f_{21}(U_{33})^2 V_{33} + f_{12} U_{33}(V_{33})^2 + f_{03}(V_{33})^3 + f_{20}(U_{33})^2 \\
&+ f_{11} U_{33} V_{33} + f_{02}(V_{33})^2 + f_{10} U_{33} + f_{01} V_{33} = S,
\end{aligned} \tag{14}$$

where S is of the same type as P, Q. Moreover, we get explicitly

$$\begin{aligned}
f_{30} &= (V + W)\, [(V^2 - 2VW - W^2)U + V^3 + VW^2]/d, \\
f_{21} &= [2(V^2 - W^2)U^2 + (V^3 + 3V^2W - 5VW^2 - 3W^3)U - V^4 \\
&\quad + V^3W + V^2W^2 + 3VW^3\,]/d,
\end{aligned} \tag{15}$$

$$f_{12} = [(V - W)\,U^3 + (V^2 + VW - 4W^2)U^2 + \left(4V^2W - VW^2 - 3W^3\right)U - V^2W^2 + 3VW^3\,]/d,$$

$$f_{03} = (U + W)\,(V - W)\left[U^2 + (V + W)\,U - VW\right]/d,$$

where $d = 8\,(U + V)^2\,U^2V^3W^3$.

The other coefficients are functions of the same type as H_2 in (9) (but occupying many pages in the explicit form).

It remains to analyze the system (12) + (14) of PDE. If this system can be solved in an explicit form

$$U_{33} = T_1,\ \ V_{33} = T_2, \tag{16}$$

where T_1 and T_2 are algebraic functions of $\overline{\varrho}_1, \overline{\varrho}_2, \overline{\varrho}_3, U, V, W$ and of the "admisible" derivatives of U, V, W, then the full system (10) + (16) can be solved by the use of the Cauchy–Kowalewski Theorem, which will prove Theorem 2. Of course, the solvability and the correctness of all calculations will depend on the initial conditions of the corresponding Cauchy problem. (Notice that a solution in the form (16) may have more branches but this is not too relevant for the proof of our Theorem).

First, let (u_0, v_0, w_0) be a point from the definition domain of the functions U, V, W. We define six functions of two variables u, v (the Cauchy initial conditions) in a neighborhood of (u_0, v_0) by the formulas,

$$F_1(u, v) = U(u, v, w_0),\ \ F_2(u, v) = V(u, v, w_0),\ \ F_3(u, v) = W(u, v, w_0), \tag{17}$$

$$G_1(u, v) = \frac{\partial U}{\partial w}(u, v, w_0),\ G_2(u, v) = \frac{\partial V}{\partial w}(u, v, w_0),\ \ G_3(u, v) = \frac{\partial W}{\partial w}(u, v, w_0).$$

Further, denote for a moment u, v, w as u_1, u_2, u_3. We shall define constants

$$a_i = F_i(u_0, v_0) > 0,\ a_{i,j} = \frac{\partial F_i}{\partial u_j}(u_0, v_0),\ a_{i,jk} = \frac{\partial F_i}{\partial u_j\,\partial u_k}(u_0, v_0),$$

$$b_{i,j} = \frac{\partial G_i}{\partial u_j}(u_0, v_0) \quad \text{for } i = 1, 2, 3 \text{ and } j, k = 1, 2. \tag{18}$$

It is obvious that, for every choice of the constants in (18), we can still define functions $F_i(u, v)$ and $G_i(u, v)$ satisfying (18) as *arbitrary* real analytic functions in a neighborhood of (u_0, v_0). (In fact, we are fixing only a finite number of initial Taylor coefficients of such functions.)

Next, if f is any real analytic function of the variables u, u, w, U, V, W, of the first derivatives of U, V, W, and of those second derivatives which are not of the form U_{33}, V_{33} or W_{33}, we shall denote by $\tilde{f}$ the corresponding value at the point (u_0, v_0, w_0). Obviously, each constant $\tilde{f}$ depends (in a real analytic way) on the constants u_0, v_0, w_0, $a_i, a_{i,j}, a_{i,jk}$ and $b_{i,j}$. In particular, we can make the substitution $u = u_0,\ v = v_0,\ w = w_0$ in the coefficients f_i, g_i, f_{ij}, Q and S of the equations (12) and (14). Let us choose the initial constants $a_i > 0,\ a_{i,j}, a_{i,jk}$ and $b_{i,j}$ in such a way that the equation (12) at the origin defines a generic real quadratic curve and the equation (14) a generic cubic curve, both in the coordinate plane $\mathsf{R}[X = U_{33}, Y = V_{33}]$. Moreover, we can make our choice in such a way that these two curves meet transversaly at some point (X_0, Y_0).

Now, using the real analytic version of the "implicit function theorem" for more variables, we obtain easily the following

Lemma 1. *Let $P(X, Y)$ and $Q(X, Y)$ be two polynomials of two variables X, Y and with the coefficients which are arbitrary parameters. If, for a fixed choice of these parameters, the equations $P(X, Y) = 0$ and $Q(X, Y) = 0$ have a common solution (X_0, Y_0) such that the Jacobian $\det \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix}$ is nonzero at (X_0, Y_0) then, in a neighborhood of (X_0, Y_0), the variables X, Y can be expressed from the above equations in a unique way as a real analytic function of the corresponding coefficients.*

Now, consider for a moment, the coefficients f_i, g_i, f_{ij}, Q and S in the equations (12) and (14) as arbitrary parameters. Applying Lemma 1 to this situation, we see that, in a neighborhood of the point (X_0, Y_0), the quantities U_{33} and V_{33} are expressed in a unique way as real analytic functions of the above coefficients and, consequently, as real analytic functions of u, v, w, U, V, W and their admissible derivatives in the neighborhood of the set $(u_0, v_0, w_0, a_i, a_{i,j}, b_{i,j})$ of initial values.

Then the Cauchy–Kowalewski Theorem can be applied to the system $\{(10), (12), (14)\}$ of PDE and the proof of Theorem 2 is completed. □

The proof of Theorem 1 now follows at once from the second part of Theorem 3. □

Remark 1. The same arguments which we used in the proof of Theorem 1 work also for the proof of the first part of Theorem 3 ! In the latter case, we are looking for a coordinate transformation $x = x(u^1, u^2, u^3)$, $y = y(u^1, u^2, u^3)$, $z = z(u^1, u^2, u^3)$ taking a general metric $g = \sum_{i,j=1}^{3} g_{ij} du^i du^j$ into a diagonal form. Here, we obtain a nonlinear PDE system of *first* order for 3 unknown functions. We need all the basic steps here as well (first a linear transformation of coordinates to ensure the applicability of the Cauchy–Kowalewski Theorem in the standard form, and, at the very end, the elementary "geometric analysis"). Instead of ensuring intersection of one quadratic curve and one cubic curve, we need in the latter case only to ensure intersection of two quadratic curves. Moreover, all the computations are much more simple and a computer aid is not needed at all.

Remark 2. The situation changes dramatically if two of the prescribed Ricci eigenvalues are asked to be equal. Consider the characteristic matrix $[\lambda I - \mathrm{Ric}]$ and substitute for λ the prescribed double Ricci eigenvalue $\varrho_1 = \varrho_2$. Then, the specified matrix has rank one and hence all sub-determinants of degree two must vanish. Because the matrix is symmetric, these conditions are obviously reduced to three independent algebraic conditions for the Ricci components Ric^i_j. We obtain three new PDE, which are of order 2 and of degree 2. Obviously, at least one of these new PDE is independent of (2). Hence we obtain an overdetermined system of PDE and the Cauchy–Kowalewski Theorem cannot be applied. We shall give a short survey about such kind of geometric problems, earlier results and corresponding methods in the last section.

Recall that we are always looking for a *geometrical* solution, i.e., we want to "parametrize" the local moduli space of Riemannian metrics for the given problem.

From this point of view, we shall see that a notion “overdetermined” and “underdetermined” PDE system has only a relative meaning, depending on the approach and method used in the particular situation.

3 The case of constant distinct Ricci eigenvalues

In [17], the first author and F. Prüfer solved the following problem: For every prescribed numbers $\varrho_1 > \varrho_2 > \varrho_3$, write down an explicit Riemannian metric g such that its Ricci eigenvalues are constant and equal to ϱ_i. A broad family of examples (so-called “generalized Yamato spaces”) was constructed there. Moreover, in [18], a geometrical characterization of this family was given inside the set of all Riemannian metrics with prescribed Ricci eigenvalues as above.

In this paper, we present, for each prescribed $\varrho_1 > \varrho_2 > \varrho_3$, a particularly simple example.

Theorem 4. *Consider fixed constants $\varrho_1 > \varrho_2 > \varrho_3$ and define the new constants α, λ_i and b as follows:*

$$\begin{aligned} &\alpha = \frac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2} < 0, \\ &\lambda_i = (\varrho_1 + \varrho_2 + \varrho_3)/2 - \varrho_i, \ i = 1, 2, 3, \\ &b = \frac{1}{\alpha + 1}\left\{-\alpha\lambda_2 + \frac{\alpha + 2}{\alpha}((\alpha + 1)\lambda_3 + \lambda_2)\right\} = \frac{(\varrho_3 - \varrho_2)(\varrho_1 + \varrho_3)}{\varrho_1 - \varrho_3}. \end{aligned} \tag{19}$$

Further, define a function $a^1_{21}(w)$ as follows:

$$\begin{aligned} &\text{(i)} \quad a^1_{21}(w) = -\frac{1}{\alpha\, w} \qquad \text{for } b = 0, \\ &\text{(ii)} \quad a^1_{21}(w) = \sqrt{\frac{b}{\alpha}} \tan\left(\sqrt{\alpha b}\, w\right) \quad \text{for } b < 0, \\ &\text{(iii)} \quad a^1_{21}(w) = \sqrt{\frac{b}{|\alpha|}} \tanh\left(\sqrt{|\alpha| b}\, w\right) \quad \text{for } b > 0. \end{aligned} \tag{20}$$

Let $I(w)$ be a maximal open interval on which $\left(a^1_{21}(w)\right)^2 > -\lambda_2/|\alpha + 1|$. Define other functions $a^i_{jk}(w)$ for $i, j, k = 1, 2, 3$ on $I(w)$ in a unique way so that $a^i_{jk}(w) + a^j_{ik}(w) = 0$ and

$$\begin{aligned} (\alpha + 1)(a^1_{21})^2 + (a^1_{23})^2 = \lambda_2, \quad a^1_{23} > 0, \\ -\alpha\, a^1_{23} a^1_{32} = (\alpha + 1)\lambda_3 + \lambda_2, \\ a^1_{22} = 0, \quad a^1_{31} = 0, \quad a^2_{31} = -a^1_{23}, \\ a^1_{33} = 0, \quad a^2_{32} = 0, \quad a^2_{33} = (\alpha + 1)\, a^1_{21}. \end{aligned} \tag{21}$$

Then, the metric $g = \sum_{i=1}^{3} \left(\omega^i\right)^2$ *defined on the strip* $I(w) \times \mathsf{R}^2[x, y] \subset \mathsf{R}^3[w, x, y]$ *by the orthonormal coframe*

$$\begin{aligned} \omega^1 &= \left[\left(a^1_{32} - a^1_{23}\right) y - a^1_{21} x\right] dw + dx, \\ \omega^2 &= dw, \\ \omega^3 &= dy + \left[(\alpha + 1)\, a^1_{21} y - \left(a^1_{32} + a^1_{23}\right) x\right] dw, \end{aligned} \tag{22}$$

has the following properties:

1) *The Ricci eigenvalues of g are* $\varrho_1 > \varrho_2 > \varrho_3$.
2) *The corresponding Christoffel symbols* Γ^i_{jk} *of g are the functions* $a^i_{jk}(w)$.
3) *The metric g is not locally homogeneous.*

Remark 3. The definition of the constant b in (19) is correct and the first equation (21) is always solvable because $\alpha + 1 = (\varrho_1 - \varrho_2)/(\varrho_3 - \varrho_2) < 0$. If $b > 0$ and $\lambda_2 > 0$, then we can put $I(w) = (-\infty, +\infty)$ and the metric g is defined on R^3.

Outline of the proof of Theorem 4. Instead of a direct check (which is a rather non-trivial task) we shall prove our Theorem on a broader background of "generalized Yamato spaces" as presented in [17] and [18]. (See Theorem 5 and Proposition 1 below).

Let (M, g) be a Riemannian 3-manifold of class C^∞ with distinct constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. Choose an open domain $\mathcal{U} \subset M$ and a smooth orthonormal moving frame $\{E_1, E_2, E_3\}$ consisting of the corresponding Ricci eigenvectors at each point of $\mathcal{U}$. Denoting by R_{ijkl} and R_{ij} the corresponding covariant components of the curvature tensor and of the Ricci form respectively, we obtain,

$$\begin{aligned} &R_{ii} = \varrho_i \ (i = 1, 2, 3), \qquad R_{ij} = 0 \quad \text{for } i \neq j, \\ &R_{1212} = \lambda_3,\ R_{1313} = \lambda_2,\ R_{2323} = \lambda_1,\ \text{where } \lambda_i \text{ are constants,} \\ &R_{ijkl} = 0 \quad \text{if at least three indices are distinct.} \end{aligned} \tag{23}$$

Moreover, the numbers λ_i are related to the numbers ϱ_i by the middle formula of (19) and we obviously get

$$\lambda_i - \lambda_j = -(\varrho_i - \varrho_j),\ i, j = 1, 2, 3. \tag{24}$$

In a neighborhood $\mathcal{U}_m$ of any point $m \in \mathcal{U}$, one can construct a local coordinate system (w, x, y) such that

$$E_3 = \frac{\partial}{\partial y} \quad \text{on } \mathcal{U}_m. \tag{25}$$

Consider the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ which is dual to $\{E_1, E_2, E_3\}$. Then, the coordinate expression of the coframe $\{\omega^1, \omega^2, \omega^3\}$ in $\mathcal{U}_m$ must be of the form

$$\begin{aligned} \omega^1 &= A\, dw + B\, dx, \\ \omega^2 &= C\, dw + D\, dx, \\ \omega^3 &= dy + G\, dw + H\, dx, \end{aligned} \tag{26}$$

where A, B, C, D, G, H are unknown functions to be determined.

Now, using the calculus of exterior forms and the standard structural equations for the connection form and the curvature form (cf. [6, 12]) one can derive the expressions for the components a^i_{jk} of the Levi-Civita connection with respect to the given frame. First, we introduce new functions $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ (where $\mathcal{D} \neq 0$) by

$$\mathcal{D} = AD - BC,\ \mathcal{E} = AH - BG,\ \mathcal{F} = CH - DG. \tag{27}$$

We also define a bracket of two functions f, g by

$$[f, g] = f'_y g - f g'_y. \tag{28}$$

Then we obtain, by a routine calculation,

$$\begin{aligned}
a^1_{21} &= \frac{1}{\mathcal{D}}(GB'_y - HA'_y + A'_x - B'_w),\ a^1_{31} = \frac{1}{\mathcal{D}}(DA'_y - CB'_y),\\
a^1_{22} &= \frac{1}{\mathcal{D}}(GD'_y - HC'_y + C'_x - D'_w),\ a^2_{32} = \frac{1}{\mathcal{D}}(AD'_y - BC'_y),\\
a^1_{33} &= \frac{1}{\mathcal{D}}(DG'_y - CH'_y),\ a^2_{33} = \frac{1}{\mathcal{D}}(AH'_y - BG'_y),\\
a^1_{23} &= \frac{1}{2\mathcal{D}}\left\{[C, D] + [A, B] - [G, H] + (G'_x - H'_w)\right\},\\
a^2_{31} &= \frac{1}{2\mathcal{D}}\left\{[C, D] - [A, B] + [G, H] - (G'_x - H'_w)\right\},\\
a^1_{32} &= \frac{1}{2\mathcal{D}}\left\{[C, D] - [A, B] - [G, H] + (G'_x - H'_w)\right\}.
\end{aligned} \tag{29}$$

From the structural equations for the connection form (ω^i_j) and for the curvature form (Ω^i_j), using the curvature conditions (23) and the subsequent exterior differentiation, we obtain the following relations for the Christoffel symbols a^i_{jk}:

$$a^2_{32} = \alpha\, a^1_{31},\ a^2_{33} = (\alpha + 1)\, a^1_{21},\ a^1_{33} = -\left(\frac{\alpha + 1}{\alpha}\right) a^1_{22}, \tag{30}$$

where α is the constant introduced in (19).

Now, assuming (30), the formulas (29) are equivalent to the following system of nine PDE for six basic Christoffel symbols a^1_{21}, a^1_{22}, a^1_{31}, a^1_{23}, a^2_{31} and a^1_{32}:

$$\begin{aligned}
A'_y &= A\, a^1_{31} + C\,(a^1_{32} - a^1_{23}),\\
B'_y &= B\, a^1_{31} + D\,(a^1_{32} - a^1_{23}),\\
C'_y &= A\,(a^1_{23} + a^2_{31}) + \alpha\, C a^1_{31},\\
D'_y &= B\,(a^1_{23} + a^2_{31}) + \alpha\, D a^1_{31},\\
G'_y &= (\alpha + 1)\, C a^1_{21} - \frac{\alpha + 1}{\alpha} A\, a^1_{22},\\
H'_y &= (\alpha + 1)\, D a^1_{21} - \frac{\alpha + 1}{\alpha} B\, a^1_{22}.
\end{aligned} \tag{31}$$

$$
\begin{aligned}
A'_x - B'_w &= \mathcal{D}a^1_{21} + \mathcal{E}a^1_{31} + \mathcal{F}(a^1_{32} - a^1_{23}),\\
C'_x - D'_w &= \mathcal{D}a^1_{22} + \mathcal{E}(a^1_{23} + a^2_{31}) + \alpha\mathcal{F}\,a^1_{31},\\
G'_x - H'_w &= \mathcal{D}(a^1_{32} - a^2_{31}) - \frac{\alpha+1}{\alpha}\,\mathcal{E}\,a^1_{22} + (\alpha+1)\,\mathcal{F}\,a^1_{21}.
\end{aligned}
\tag{32}
$$

Next, we express explicitly the geometric curvature conditions (23). Using again the structural equations for the curvature form (Ω^i_j), we obtain after lengthy but routine calculations the following system of nine PDE's for all nine Christoffel symbols, still having in mind the relations (30):

$$
\begin{aligned}
&A(a^1_{21})'_x - B(a^1_{21})'_w + C(a^1_{22})'_x - D(a^1_{22})'_w + G(a^1_{23})'_x - H(a^1_{23})'_w\\
&\qquad\qquad - \mathcal{D}(U_3 - \lambda_3) - \mathcal{E}V_3 - \mathcal{F}W_3 = 0,\\
&A(a^1_{21})'_y + C(a^1_{22})'_y + G(a^1_{23})'_y - (a^1_{23})'_w - AV_3 - CW_3 = 0,\\
&B(a^1_{21})'_y + D(a^1_{22})'_y + H(a^1_{23})'_y - (a^1_{23})'_x - BV_3 - DW_3 = 0,\\
&A(a^1_{31})'_x - B(a^1_{31})'_w + C(a^1_{32})'_x - D(a^1_{32})'_w + G(a^1_{33})'_x - H(a^1_{33})'_w\\
&\qquad\qquad - \mathcal{D}U_2 - \mathcal{E}(V_2 - \lambda_2) - \mathcal{F}W_2 = 0,\\
&A(a^1_{31})'_y + C(a^1_{32})'_y + G(a^1_{33})'_y - (a^1_{33})'_w - A(V_2 - \lambda_2) - CW_2 = 0,\\
&B(a^1_{31})'_y + D(a^1_{32})'_y + H(a^1_{33})'_y - (a^1_{33})'_x - B(V_2 - \lambda_2) - DW_2 = 0,\\
&A(a^2_{31})'_x - B(a^2_{31})'_w + C(a^2_{32})'_x - D(a^2_{32})'_w + G(a^2_{33})'_x - H(a^2_{33})'_w\\
&\qquad\qquad - \mathcal{D}U_1 - \mathcal{E}V_1 - \mathcal{F}(W_1 - \lambda_1) = 0,\\
&A(a^2_{31})'_y + C(a^2_{32})'_y + G(a^2_{33})'_y - (a^2_{33})'_w - AV_1 - C(W_1 - \lambda_1) = 0,\\
&B(a^2_{31})'_y + D(a^2_{32})'_y + H(a^2_{33})'_y - (a^2_{33})'_x - BV_1 - D(W_1 - \lambda_1) = 0.
\end{aligned}
\tag{33}
$$

Here, we put (using again only the "basic" six Christoffel symbols)

$$
\begin{aligned}
U_1 &= \alpha a^1_{21}a^2_{31} - (\alpha-1)a^1_{22}a^1_{31} - (\alpha+2)a^1_{21}a^1_{32},\\
V_1 &= \frac{(\alpha+1)(\alpha+2)}{\alpha}a^1_{21}a^1_{22} - (\alpha+1)a^1_{31}a^2_{31} - (\alpha-1)a^1_{31}a^1_{23},\\
W_1 &= \frac{\alpha+1}{\alpha}(a^1_{22})^2 - (\alpha+1)^2(a^1_{21})^2 - \alpha^2(a^1_{31})^2 + a^1_{23}a^2_{31} - a^1_{32}a^2_{31} + a^1_{32}a^1_{23},\\
U_2 &= \frac{1}{\alpha}a^1_{22}a^1_{32} + (\alpha-1)a^1_{21}a^1_{31} - \frac{2\alpha+1}{\alpha}a^1_{22}a^2_{31},\\
V_2 &= (\alpha+1)(a^1_{21})^2 - (a^1_{31})^2 - \left(\frac{\alpha+1}{\alpha}\right)^2(a^1_{22})^2 - a^1_{32}a^1_{23} - a^1_{32}a^2_{31} - a^1_{23}a^2_{31},
\end{aligned}
\tag{34}
$$

$$W_2 = (1-\alpha)a^1_{23}a^1_{31} - (\alpha+1)a^1_{32}a^1_{31} + \frac{(2\alpha+1)(\alpha+1)}{\alpha}a^1_{22}a^1_{21},$$

$$U_3 = -(a^1_{21})^2 - (a^1_{22})^2 - \alpha(a^1_{31})^2 + a^1_{23}a^2_{31} - a^1_{23}a^1_{32} + a^1_{32}a^2_{31},$$

$$V_3 = \frac{1}{\alpha}a^1_{22}a^1_{23} - (\alpha+2)a^1_{21}a^1_{31} - \frac{2\alpha+1}{\alpha}a^1_{22}a^2_{31},$$

$$W_3 = -\alpha a^1_{21}a^1_{23} - (\alpha+2)a^1_{21}a^1_{32} - (2\alpha+1)a^1_{22}a^1_{31}.$$

By the detailed analysis of the system of 18 PDE, (31)–(33) for 12 unknown functions $A, B, \dots, H, a^1_{21}, a^1_{22}, \dots, a^1_{32}$, the following result was obtained in [17]. (Here we present the more convenient local version of the corresponding theorem.)

Theorem 5. *Let a triplet $\varrho_1 > \varrho_2 > \varrho_3$ of constant Ricci eigenvalues be prescribed. Let a^i_{jk} be functions on $\mathcal{U} \subset \mathsf{R}^2[w, x]$ satisfying the following conditions:*

(N1) $a^1_{31} = 0,\ a^1_{23} + a^2_{31} = 0,\ a^1_{22} = 0,$

(N2) a^1_{23} *is an arbitrary function of class C^∞ on $\mathcal{U} \subset \mathsf{R}^2[w, x]$ such that*

(a) $(a^1_{23})'_w \neq 0,\ a^1_{23} > 0,$

(b) $(a^1_{23})^2 > \max\left\{\lambda_2, (\alpha+2)\left[(\alpha+1)\lambda_3 + \lambda_2\right]/\alpha^2\right\},$

(N3) $(\alpha+1)(a^1_{21})^2 + (a^1_{23})^2 = \lambda_2,\ a^1_{21} > 0,$

(N4) $-\alpha\, a^1_{23}\, a^1_{32} = (\alpha+1)\lambda_3 + \lambda_2.$

Then, there exist smooth functions A, B, C, D, G, H on $\mathcal{U} \times \mathsf{R}[y] \subset \mathsf{R}^3[w, x, y]$ (depending on two arbitrary functions of two variables and two arbitrary functions of one variable) such that the basic system of partial differential equations (31)–(33) *is satisfied.*

We shall now specify these functions. First, look at the function W_3 defined in (34). One can calculate explicitly from (N3) and (N4) that,

$$W_3 = f(a^1_{23}) = \sqrt{\frac{(a^1_{23})^2 - \lambda_2}{|\alpha+1|}}\left(\alpha\, a^1_{23} + (\alpha+2)\frac{|\alpha+1|\lambda_3 - \lambda_2}{\alpha\, a^1_{23}}\right). \quad (35)$$

We see that the inequalities in (N2)(b) just ensure that $f(a^1_{23})$ is non-zero everywhere in our domain $\mathcal{U}$ (but this can be always assumed in our local case, because $(a^1_{23})'_w \neq 0$).

Define now C, D as functions on $\mathcal{U}$ by,

$$C = -\frac{(a^1_{23})'_w}{f(a^1_{23})}, \quad D = -\frac{(a^1_{23})'_x}{f(a^1_{23})}. \quad (36)$$

It is shown in [17], that, if the Christoffel symbols are defined by (N1)–(N4) and the functions C, D are defined by (36), then for arbitrary choice of the functions A, B, G, H all PDE's (33) are satisfied. Further, the following was proved.

Proposition 1. *To satisfy the remaining* PDE (31) *and* (32)*, it is sufficient to define the functions* A, B, G, H *by*

$$A = C(a^1_{32} - a^1_{23})y + A_0(w, x), \ B = D(a^1_{32} - a^1_{23})y + B_0(w, x),$$
$$G = (\alpha + 1)\, C\, a^1_{21} y + G_0(w, x), \ H = (\alpha + 1)\, D\, a^1_{21} y + H_0(w, x), \tag{37}$$

where A_0, B_0, G_0, H_0 *are functions of class* C^∞ *on* $\mathcal{U} \subset \mathsf{R}^2[w, x]$ *satisfying the equations*

$$(A_0)'_x - (B_0)'_w = (DA_0 - CB_0)\, a^1_{21} + (DG_0 - CH_0)\left(a^1_{23} - a^1_{32}\right), \tag{38}$$
$$(G_0)'_x - (H_0)'_w = (DA_0 - CB_0)\left(a^1_{32} + a^1_{23}\right) - (DG_0 - CH_0)\,(\alpha + 1)\, a^1_{21}.$$

To obtain the explicit examples announced in Theorem 4, let us suppose that a^1_{23} depends on the variable w only and put $B_0 = 1, H_0 = 0$. Then, $C = -(a^1_{23})'(w)/f(a^1_{23}) \neq 0$ depends on w only and $D = 0, B = 1, H = 0$. For A and G we get explicit solutions

$$A = C\,(a^1_{32} - a^1_{23})y - C\, a^1_{21} x, \tag{39}$$
$$G = (\alpha + 1)\, C\, a^1_{21} y - C\,(a^1_{32} + a^1_{23})x.$$

It remains to verify that the formulas (19)–(22) in Theorem 4 follow from the previous ones and to make the final conclusions. First we see that if we solve the differential equation $(a^1_{23})'(w) = -f(a^1_{23})$, then the function $a^1_{23}(w)$ will be specified so that $C = 1$. If we pass from $a^1_{23}(w)$ to $a^1_{21}(w)$, we obtain a much simpler equation

$$(a^1_{21})'(w) = \alpha(a^1_{21})^2 + b. \tag{40}$$

Hence, the formulas (20) follow at once (neglecting the integration constant here). Further, we recall that the PDE system (33) is equivalent to the statement that $\varrho_1 > \varrho_2 > \varrho_3$ are corresponding Ricci eigenvalues and the PDE system (31) + (32) together with (30) says that $a^i_{jk}(w)$ defined by (21) are the corresponding Christoffel symbols. Finally, because the Christoffel symbols a^i_{jk} are calculated with respect to a Ricci-adapted frame (which is determined uniquely up to reflections at each point), and because not all a^i_{jk} are constant, the space (M, g) cannot be locally homogeneous. □

Remark 4. For the prescribed constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$, (even if they are not all distinct) there is *not always* a locally homogeneous space with such Ricci eigenvalues. Some necessary conditions were given in [27] and the complete answer can be found in [16].

Remark 5. The 3-dimensional Riemannian manifolds with constant Ricci eigenvalues belong to the broader family of so-called *curvature homogeneous spaces*. See, e.g., [1, 3, 23–25, 28, 29] and, in particular, a survey in [4]. This topic was developed with strong participation of F. Tricerri and L. Vanhecke; it was originally motivated by a conjecture of M. Gromov.

We also proved the following in [26]:

Theorem 6. *The general solution of the PDE system (31)–(33) depends on six arbitrary functions of two variables and six arbitrary functions of one variable.*

The proof depends strongly on the computer aid because one has to show that all integrability conditions coming from this PDE system are consequences of the original PDE's. This is a hard computer work which is not very transparent and difficult to check by hand. After showing this, one can use the Cauchy–Kowalewski Theorem in two successive steps to obtain the result.

Now we have the following geometric existence theorem which we reproduce in full from [17], including its short proof.

Theorem 7. *The isometry classes of germs of three-dimensional (real analytic) Riemannian metrics with prescribed constant Ricci eigenvalues are parametrized by triplets of germs of arbitrary (real analytic) functions of two variables.*

Proof. Let (M, g), $(\overline{M}, \overline{g})$ be two real analytic Riemannian 3-manifolds with the same constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$. Let $F : \mathcal{U} \to \overline{\mathcal{U}}$ be an isometry between two open domains of M and $\overline{M}$ respectively. We construct the "Ricci adapted" orthonormal coframes $\{\omega^i\}$, $\{\overline{\omega}^i\}$ and the local coordinate systems (w, x, y), $(\overline{w}, \overline{x}, \overline{y})$ in the neighborhoods $\mathcal{U}_m \subset \mathcal{U}$ and $\overline{\mathcal{U}}_{F(m)} = F(\mathcal{U}_m) \subset \overline{\mathcal{U}}$ respectively, such that g and $\overline{g}$ are both of the form (26). We obviously have

$$F^*(\overline{\omega}^i) = \varepsilon_i \omega^i, \quad \varepsilon_i \in \{-1, 1\}, \quad i = 1, 2, 3. \tag{41}$$

Hence, we see easily that the corresponding parametrization of F in local coordinates must be of the form,

$$\overline{w} = \Phi_1(w, x), \quad \overline{x} = \Phi_2(w, x), \quad \overline{y} = \varepsilon y + \Phi_3(w, x), \tag{42}$$

where $\varepsilon = \pm 1$ and $\Phi_i(w, x)$ are arbitrary (real analytic) functions of two variables. Conversely, every local transformation F of the form (42) determines a local isometry which preserves the formulas (26) through (41). The result now follows from Theorem 6. □

Let us notice that we neglect here six arbitrary functions of one variable. This is fully justified because, for the geometric conclusions, these functions are not relevant.

Remark 6. Looking at the proof carefully, we see that the same argument also works when ϱ_i are not constants but arbitrary functions! Hence, we have an alternative way to derive Theorem 1 from Theorem 2 where we don't need the second part of the "diagonalization theorem" 3.

Open problem. It is not known to the authors if an explicit construction as in Theorem 4 can be extended to non-constant Ricci eigenvalues.

4 Related problems with curvature restrictions

4.1 The Schur's theorem

Consider prescribed Ricci eigenvalues $\varrho_1(x, y, z)$, $\varrho_2(x, y, z)$, $\varrho_3(x, y, z)$ on (M, g) which are *all equal* and consider the corresponding system of partial differential equations (2). In this case, we have to add three new independent PDE's, namely $\mathrm{Ric}^i_j = 0$ for $1 \leq i < j \leq 3$, and the system of equations becomes strongly overdetermined. According to the Schur's Theorem, (M, g) must be a space of constant curvature. Hence the local moduli space depends only on one parameter. As a consequence of Theorem 3, the general solution of the corresponding overdetermined system depends on three arbitrary functions of two variables (and possibly, on some functions of one variable and some parameters – we shall not repeat this stipulation in the sequel).

4.2 The pseudo-symmetric spaces of constant type

A 3-dimensional *pseudo-symmetric space of constant type* is characterized by the following properties (cf. [7], [8], [20]–[22] and [4], Chap. 11): One of the Ricci eigenvalues is prescribed as a constant and the other two Ricci eigenvalues are required to be equal but arbitrary. (If the constant eigenvalue is zero, the space is said to be *semi-symmetric* (see [2], [3], [14], [31]–[32], and, in particular, [4] for more information). Then, we have only two PDE for the coefficients c_i of the Ricci characteristic polynomial but there are additional three quadratic equations for the Ricci components Ric^i_j involving an *arbitrary* function. Eliminating this arbitrary function, we are left with two additional PDEs, which are biquadratic. This system is not easy to analyse. Yet, using a different approach, we come to some satisfactory and surprising results.

Let us start with a 3-dimensional Riemannian manifold (M, g) whose Ricci tensor has the eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ with constant ϱ_3. One can construct easily, in a neighborhood of any fixed point $m \in M$, a Ricci adapted orthonormal moving frame $\{E_1, E_2, E_3\}$ and a system (w, x, y) of local coordinates such that $E_3 = \partial/\partial y$. We shall also consider the dual coframe $\{\omega^1, \omega^2, \omega^3\}$.

The Ricci tensor $\widehat{R}$ expressed with respect to $\{E_1, E_2, E_3\}$ has the form $\widehat{R}_{ij} = \varrho_i \delta_{ij}$. Because each ϱ_i is expressed through the sectional curvature K_{ij} by the formula $\varrho_i = \widehat{R}_{ii} = \sum_{j \neq i} K_{ij}$, there exist a function $k = k(w, x, y)$ of the variables w, x and y, and a constant $\tilde{c}$ such that

$$K_{12} = k,\ K_{13} = K_{23} = \tilde{c}, \qquad (43)$$
$$\varrho_1 = \varrho_2 = k + \tilde{c},\ \varrho_3 = 2\,\tilde{c}.$$

From the structural equations for the connection form (ω^i_j) and for the curvature form (Ω^i_j), using the curvature conditions (43), we obtain after a simple manipulation with the corresponding exterior differential forms ω^i, ω^i_j the following results:

Proposition 2. *In a normal neighborhood of any point $m \in M$ there exist an orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ and a local coordinate system (w, x, y) such that*

$$\begin{aligned} \omega^1 &= f\,dw,\\ \omega^2 &= A\,dx + C\,dw, \\ \omega^3 &= dy + H\,dw. \end{aligned} \tag{44}$$

Here f, A and C are smooth functions of the variables w, x and y, $fA \neq 0$, and H is a smooth function of the variables w and x.

Moreover, $fA = \sigma/(k - \tilde{c})$ for some non-zero function $\sigma = \sigma(w, x)$.

Next, we obtain easily the following expression for the components of the connection form:

$$\begin{aligned} \omega_2^1 &= -A\alpha\,dx + R\,dw + \beta\,dy,\\ \omega_3^1 &= A\beta\,dx + S\,dw, \\ \omega_3^2 &= A'_y\,dx + T\,dw, \end{aligned} \tag{45}$$

where

$$\begin{aligned} \alpha &= \chi(A'_w - C'_x - HA'_y),\\ \beta &= \frac{\chi}{2}(H'_x + AC'_y - CA'_y), \end{aligned} \tag{46}$$

and

$$\begin{aligned} R &= \chi f f'_x - C\alpha + H\beta,\\ S &= f'_y + C\beta, \\ T &= C'_y - f\beta, \end{aligned} \tag{47}$$

putting here χ for $1/fA$. The curvature conditions (43) (when used in the structural equations for the curvature form) then give a system of nine PDE's for our problem:

$$\begin{aligned} (A\alpha)'_y + \beta'_x &= 0,\\ R'_y - \beta'_w &= 0,\\ (A\alpha)'_w + R'_x + SA'_y - A\beta T &= -fAk,\\ A''_{yy} - A\beta^2 &= -\tilde{c}A,\\ -A''_{yw} + T'_x + A(\beta R + \alpha S) &= \tilde{c}AH,\\ T'_y - S\beta &= -\tilde{c}C,\\ (A\beta)'_y + A'_y\beta &= 0,\\ S'_x - (A\beta)'_w - (A\alpha T + A'_y R) &= 0,\\ S'_y + T\beta &= -\tilde{c}f. \end{aligned} \tag{48}$$

This is a reasonable PDE system, because two of the equations are consequences of the others and for the remaining equations we obtain a number of nice "first integrals" (like formulas (49)–(51) below).

Now, an important tool how to simplify the system (48) is the notion of *asymptotic leaf*. It is defined as a surface $N \subset M$ generated by the principal ϱ_3-lines and such that its tangent distribution is parallel along each principle ϱ_3-line in (M, g). (Here, naturally, principal ϱ_3-lines are integral curves of the local vector field E_3. They are known to be geodesic lines in (M, g).)

Now, the following result can be proved with some effort:

Proposition 3. *For any point $p \in M$ there are just four possibilities:*

a) *There is no asymptotic leaf through p ("elliptic point").*
b) *There are just two asymptotic leafs through p ("hyperbolic point").*
c) *There is just one asymptotic leaf through p ("parabolic point").*
d) *There are infinitely many asymptotic leafs through p ("planar point").*

We call a (local) space (M, g) to be *of elliptic type* if it consists of elliptic points only. Similarly, we define spaces of *hyperbolic, parabolic* and *planar type*. Thus, on such kind of spaces we can consider *asymptotic foliations*. If the space is not elliptic, at least one asymptotic foliation exists and one can define a new local coordinate system (w, x, y) such that, in addition, the integral manifolds of the equation $dw = 0$ are asymptotic leafs. Then a dramatic simplification of the system (48) takes place, enabling to write down the general solution in the explicit form!

One has the following main results ([10], [14], [20]–[22] and [4]) proved by the first author and M. Sekizawa:

A) The local moduli space of all spaces of elliptic type (or of hyperbolic type, or of parabolic type, or of planar type respectively) is parametrized by 3 arbitrary functions of 2 variables (or by 3, or by 2, or by 1 arbitrary functions of 2 variables respectively). Hence the corresponding "overdetermined" system of PDE for the Ricci components Ric^i_j is not really overdetermined because it has a general solution depending on 6 arbitrary functions of 2 variables — the same result as for the system (2) with distinct prescribed Ricci eigenvalues.
B) The local moduli space of all spaces of non-elliptic types can be expressed by a finite number of explicit formulas involving only algebraic operations, elementary functions, differentiation, integration, and depending explicitly on the corresponding number of arbitrary functions of two variables.
This is a rare phenomenon in the theory of nonlinear PDE systems.
C) The double Ricci eigenvalue, which was supposed to be arbitrary, is in fact not arbitrary! It must be of the form

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 y^2 + k_2 y + k_3} \quad \text{for } \varrho_3 = 0, \tag{49}$$

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 \cos(\lambda y) + k_2 \sin(\lambda y) + k_3} + 2\lambda^2 \quad \text{for } \varrho_3 = 2\lambda^2 > 0, \tag{50}$$

$$\varrho_1 = \varrho_2 = \frac{1}{k_1 \cos(\lambda y) + k_2 \sin(\lambda y) + k_3} - 2\lambda^2 \quad \text{for } \varrho_3 = -2\lambda^2 > 0, \tag{51}$$

where k_1, k_2, k_3 are arbitrary functions of 2 variables w, x. We are somehow on the "halfway" to the Schur's Theorem.

4.3 The semi-symmetric spaces of elliptic type with the *prescribed* non-constant double Ricci eigenvalue

The prescribed eigenvalue must be of the form (49). The problem was investigated in [14], pp. 471–474. The local moduli space depends here on one arbitrary function of 2 variables.

The corresponding system of PDE's for the Ricci components is again overdetermined and its general solution depends on 4 arbitrary functions of two variables.

4.4 The case of constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$

Here we have a specialized PDE system (48) in which k is a constant. As we mentioned in the Introduction (see [15], [5], [19]), the local moduli space of all possible metrics depends on 2 arbitrary functions of 1 variable.

The PDE system for the Ricci components is again "strongly overdetermined" and the general solution depends only on 3 arbitrary functions of 2 variables.

Notice that the local moduli space here is "much smaller" than in the case of three distinct constant Ricci eigenvalues! This is obviously due to the fact that the corresponding PDE system (2) gets overdetermined by adding new equations.

4.5 The 3-dimensional Riemannian manifolds with two zero Ricci eigenvalues and one arbitrary Ricci eigenvalue

The corresponding PDE system for the Ricci characteristic polynomial is here *rather special*. In fact, we get the conditions $c_1 = 0$, $c_0 = 0$ and the additional equations saying that the 2-dimensional sub-determinants of the matrix $[\mathrm{Ric}^i_j]$ are zero. It might be an interesting problem to solve the corresponding PDE system in order to obtain the information about general solution.

One can also proceed like in the subsection 4.2, and to write down a system of 9 PDE's of second order. But this system is very hard to solve and the "parametrization problem" for the moduli space still remains open.

The problem was raised, in fact, for general dimension by S. Ivanov and I. Petrova in [11] when the authors studied "the spaces with pointwise constant curvature eigenvalues" (in fact, eigenvalues of the skew-symmetric curvature operator $R(X, Y)$. The classification problem was solved completely by the above authors in dimension 4 and later by P. Gilkey, J. Leahy and H. Sadofsky in the higher dimensions except $n = 7$ and $n = 8$. Yet, it still remains open in dimension 3 (which is just the case described in the title of this paragraph – see Remark 2 and Remark 3 in the Introduction of [11]).

The only known results are isolated examples of the above spaces:

A) The group $SU(3)$ with a special left-invariant metric (see [27] and Remark 2 in [11]).

B) The metrics of the form

$$g = \frac{1}{p^2}\ e^{-2\lambda y} dw^2 + [p\ e^{\lambda y} dx + (r\ e^{\lambda y} + s\ e^{-\lambda y}) dw]^2 + dy^2, \tag{52}$$

where $p = p(w)$, $s = s(w)$, $r = -\lambda^2 p^2(w) s(w) x^2 + p'(w) x + \psi(w)$, and $p(w), s(w), \psi(w)$ are arbitrary functions. Here, $\varrho_1 = \varrho_2 = 0$, $\varrho_3 = -2\lambda^2$. These metrics are not locally homogeneous. (See [15], Example 5.8.)

C) The example by Ivanov–Petrova: (M, g) is a warped product $M^3 = B^1 \times_f N^2$, where $B^1 = B^1(y)$ is a real line, N^2 is a space form of constant curvature K, and the warping function $f(y)$ is $\sqrt{Ky^2 + Cy + D}$ with constant C, D such that $C^2 - 4KD \neq 0$. The Ricci eigenvalues are $\left(0,\ 0,\ \frac{1}{2}(C^2 - 4KD)/(Ky^2 + Cy + D)^2\right)$.

D) The new example found by V. Hájková and O. Kowalski:

$$g = y^{2p} dw^2 + y^{2(1-p)} dx^2 + dy^2,\ \text{where } p \text{ is a parameter.} \tag{53}$$

Here, $\varrho_1 = \varrho_2 = 0$ and $\varrho_3 = 2p(1-p)/y^2$. Further, $p(1-p)$ is a Riemannian invariant and the case $p = 1/2$ corresponds to the example C) for the particular choice $K = 0,\ C = 1,\ D = 0$.

E) (Added in proof). See Y. Nikolayevsky, On Riemannian manifolds cohose skew-symmetric curvature operator has constant curvature, preprint, to appear in *Bull. Austral. Math. Soc.*, 2004.

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Two Problems in Real and Complex Integral Geometry*

A. M. Naveira

Departamento de Geometría y Topología, Facultad de Matematicas,
46100-Burjassot Valencia, Spain
naveira@uv.es

Dedicated to Professor Lieven Vanhecke

Summary. In this article I state two problems related to Integral Geometry. In the first, I try to obtain analytical inequalities which become equivalent to the inequalities among the integrals of the mean curvatures of a hypersurface in the Euclidean space. The second problem is related to the Complex Integral Geometry. I try to analyze the "complex cross-sectional measures" of a convex body contained in the complex Euclidean space.

1 Introduction

The concept of mixed volume or integral cross-sectional measure (quermass integrale) is well known in the specialized mathematical literature, [5, 10, 12]. It refers to a family of intrinsic measures, among which there are a number of inequalities; for instance, "The classical Isoperimetric Inequality". A beautiful proof of the equivalence of its geometric and analytic formulations can be found in [4]. In the first part of this note, we pose the problem of generalizing the equivalence to other isoperimetric-like geometric inequalities formulated in terms of the different integral cross-sectional measures. We can say that the history of Integral Geometry begins with Buffon's needle problem at the end of 18th century. It grew up in an incipient way throughout the 19th century, but most of its development took place in the 20th century, basically with the work of Blaschke and his school, and later with Santaló and Chern, among others. Santaló's book has been a basic tool for the study of Integral Geometry in the last decades of the 20th century. It is, and will continue to be a very important reference for all researchers in this field. For convex sets in the Euclidean space, Santaló defines the real integral cross-sectional measures as an average integral on the real Grassmann manifold of real subspaces through the origin. He also considers that a study in depth of the Complex Integral Geometry could be interesting [10], p. 338. In [7, 8], many of the properties of densities in real spaces has been extended to the complex case. Here, it is also possible to define, by means of an integration in the corresponding complex

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Grassmann manifold, the concept of complex integral cross-sectional measures. In the second part of this note we pose the following question: How is the representation of the complex cross-sectional measures in terms of the integrals of the generalized mean curvatures?.

2 Inequalities of isoperimetric type

2.1 The mixed volumes of convex compact sets in the Euclidean space

The mixed volumes

To every pair of non-empty sets A, $B \subset \mathbb{R}^n$ their (vector) Minkowski sum is defined by $A+B = a+b/a \in A, b \in B$, while $\lambda A = \lambda a/a \in A$ is the result of the homotethy of A with coefficient λ. For the moment, I consider only non-empty convex compact subsets of the space $\mathbb{R}^n$, without saying it explicitly.

Theorem 1 (Theorem of Minkowski, [5], p. 136). *The volume of the linear combination of non-empty compact sets K_1, ..., K_s, $s \neq n$, in general, with non-negative coefficients $\lambda_1, \ldots, \lambda_s$ is a homogeneous polynomial of degree n with respect to $\lambda_1, \ldots, \lambda_s$:*

$$V(\Sigma_{i=1,\ldots,s}\lambda_i K_i) = \Sigma_{i_1=1,\ldots,s} \ldots \Sigma_{i_n=1,\ldots,s} V(K_{i_1}, \ldots, K_{i_n})\lambda_{i_1} \ldots \lambda_{i_n}. \quad (1)$$

It follows from the theorem that these coefficients depend only on those $K_1, \ldots, K_s$. The classical definition of mixed volume of non-empty convex compact sets $K_1, \ldots, K_n$ in $\mathbb{R}^n$(they are not necessarily distinct and their order plays no role) according to Minkowski is the following: this volume is the coefficient $V(K_1, \ldots, K_n)$ in the decomposition of the Minkowski theorem involving these sets.

It is convenient to write (1) in the form

$$V(\Sigma_{i=1}^s \lambda_i K_i) = \frac{n!}{p_1! \ldots p_r!} \Sigma_{p_1+\cdots+p_r=n} \; \Sigma_{1 \le i_1 < \cdots < i_r \le s} V(K_{i_1,p_1}; \ldots; K_{i_r,p_r})\lambda_{i_1}^{p_1} \ldots \lambda_{i_r}^{p_r}, \quad (2)$$

where

$$V(K_{i_1,p_1}; \ldots; K_{i_r,p_r}) = V(K_{i_1}, \ldots, (p_1), \ldots, K_{i_1}; K_{i_r}, \ldots, (p_r), \ldots, K_{i_r}). \quad (3)$$

Characteristic properties of mixed volumes can be found in [5].

In geometry, one usually considers only those mixed volumes for which, only two (rarely three) of the K_i differ. Most often one considers the so-called mth integral cross-sectional measures

$$V_m(K) = V(K, \ldots (m) \ldots, K, D, \ldots (n-m) \ldots, D), \quad (4)$$

where K is a non-empty convex compact set, while D is the closed unit ball in $\mathbb{R}^n$.

Their particular cases (with the appropriate normalization) are the volume $V(K)$ and the $(n-1)$-dimensional boundary area $S(K) : V(K) = V_n(K)$ and $S(K) = nV_{n-1}(K)$.

The Alexandrov–Fenchel Inequality (AFI)

The best-known inequality among the general inequalities relating to mixed volumes is the Alexandrov–Fenchel Inequality.

Theorem 2 ([5]). *Alexandrov–Fenchel Inequality (AFI) says that*

$$V^2(K_1, \ldots, K_n) \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n). \tag{5}$$

One of the most important consequence of the AFI is the following [5, p. 143]:

$$\begin{aligned} &V^m(K, \ldots, (i), \ldots, K, L, \ldots, (m-i), \ldots, L, C_{m+1}, \ldots, C_n) \\ &\geq V^i(K, \ldots, (m), \ldots, K, C_{m+1}, \ldots, C_n)V^{m-i}(L, \ldots, (m), \ldots, L, C_{m+1}, \ldots, C_n). \end{aligned}$$

2.2 Inequalities of isoperimetric type

As particular cases of the Alexandrov–Fenchel inequality, we can consider a whole series of well-known geometric inequalities:

i) The Classical Isoperimetric Inequality

$$S^n(K) \geqslant n^n v_n V^{n-1}(K), \tag{6}$$

where K is a non-empty convex-compact set in $\mathbb{R}^n$, $S(K) = n\ V^{n-1}(K)$ is the $(n-1)$-dimensional area of ∂K, $V(K)$ is its volume and v_n the volume of the unit ball D in $\mathbb{R}^n$. This inequality is known as the first Minkowski Inequality.

ii) The second Minkowski Inequality

$$V_1^n(K) \geq v_n^{n-1} V(K). \tag{7}$$

iii) The Minkowski Quadratic Inequalities

$$\begin{aligned} V_{n-1}^2(K) &\geq V_{n-2}(K)V(K) \\ V_1^2(K) &\geq v_n V_2(K). \end{aligned} \tag{8}$$

iv) The Favard Inequalities, which generalizes the Minkowski quadratic inequalities

$$V_i^2(K) \geq V_{i-1}(K)V_{i+1}(K). \tag{9}$$

v) The generalized Minkowski Inequalities, which generalizes the first and the second Minkowski Inequalities

$$V_i^n(K) \geq v_n^{n-i} V^i(K). \tag{10}$$

vi) Even more general is the Alexandrov Inequality

$$V_j^i(K) \geq v_n^{i-j} V_i^j(K), \tag{11}$$

for $j \leq i$. Remark that the inequalities of Alexandrov contain the classical isoperimetric inequality.

It is well known that for all these inequalities the equality holds if and only if K is the Euclidean ball.

When $n = 3$, for convex bodies the AFI yields series of inequalities among the volume V, the area of the surface S and the total mean curvature M:

$$S^3 \geq 36\pi V^2 \tag{12}$$
$$M^3 \geq 48\pi^2 V$$
$$M^2 \geq 4\pi S$$
$$S^2 \geq 3VM.$$

I think that it is interesting to make some remarks about proofs of the AFI. Alexandrov published two proofs of his inequality in 1936. The first of them is combinatorial. The second proof of Alexandrov is more analytical. In 1975, Khovanskii obtained an algebraic proof for $n = 2$. In 1978, Fedotov published an erroneous algebraic proof of this inequality for any n. In 1978, and independently, Tessier in Parı́s and Khovanskii in Moscow obtained a correct algebraic proof using the Hodge Index Theorem.

It is well known in the literature the Steiner decomposition for the volume of a parallel body in function of the volume, surface area and the integrals of mean curvature of the primitive body [5, 10]. The corresponding formula is

$$V_n(K + \lambda D) = V_n(K) + nV_{n-1}(K)\lambda + C_{n,2}V_{n-2}(K)\lambda^2 + \cdots + C_{n,n-1}V_1(K)\lambda^{n-1} + v_n\lambda^n. \tag{13}$$

If in the above formula we substitute $\lambda = \lambda_1 + \lambda_2$, then carry out this decomposition consecutively for λ_1 and λ_2 and equate the coefficients, then we obtain the Steiner decomposition for mth integral cross-sectional measures

$$V_m(K + \lambda D) = V_m(K) + mV_{m-1}(K)\lambda + C_{m,2}V_{m-2}(K)\lambda^2 + \ldots + C_{m,m-1}V_1(K)\lambda^{m-1} + v_n\lambda^m. \tag{14}$$

2.3 The Brunn–Minkowski Inequalities

As a consequence of the AFI in the form (2), it is possible to prove the following:

Corollary 1 ([5]). From the above theorem, we have the Brunn–Minkowski Inequality for integral cross-sectional measures; that is,

$$V_m^{1/m}(K + L) \geq V_m^{1/m}(K) + V_m^{1/m}(L). \tag{15}$$

In [4] we can find a beautiful proof of the Isoperimetric Inequality, using the Brunn–Minkowski Inequality for V_n.

This proof can be extended to all cross-sectional measures. In fact, from (14) and using (13), we have

$$\begin{aligned}\{V_m(K+\lambda D)\}^{\frac{1}{m}} &= \{V_m(K)+mV_{m-1}(K)\lambda + C_{m,2}V_{m-2}(K)\lambda^2 \\ &\quad + \ldots + C_{m,m-1}V_1(K)\lambda^{m-1} + v_n\lambda^m\}^{\frac{1}{m}} \\ &= V_m(K)^{\frac{1}{m}} + \frac{V_{m-1}(K)}{V_m(K)^{1-\frac{1}{m}}}\lambda \geq V_m(K)^{\frac{1}{m}} + V_m(\lambda D)^{\frac{1}{m}} \\ &= V_m(K)^{\frac{1}{m}} + v_m^{\frac{1}{m}}\lambda,\end{aligned}$$

and when λ tends to 0, it follows that

$$\frac{V_{m-1}(K)}{V_m(K)^{1-\frac{1}{m}}} \geq v_m^{\frac{1}{m}}. \tag{16}$$

Remark. The inequalities of isoperimetric type of Alexandrov and Fenchel for cross-sectional measures of convex domains in the Euclidean space were established for non-convex domains, subject to natural curvature conditions. The technique used is of Monge–Ampère type equations, [13, 14].

2.4 The theorem of Blaschke–Chern–Santaló

We can say that the history of the Integral Geometry begins with Buffon's needle problem at the end of 18th century. This was growing up in an incipient way through 19th century, but its more important development was in the 20th century, fundamentally with Blaschke's work, and his school and later with Santaló and Chern, among others.

Santaló's book has been basic for all the studies about Integral geometry that have appeared in the last decades of the 20th century. This book is, and will be, a very important reference for all the researchers on the subject.

It is well known that on a Lie group **G** it is possible to define a left-invariant volume form d**G**. Let **G** be a Lie group of dimension n and let **H** be a closed subgroup of **G** of dimension n-m. The set of left cosets g**H**, $g \in$ **G** is the homogeneous space **G/H**, which, as is known [16], admits a differentiable manifold structure of dimension m. We want to find the conditions for the existence of a non-zero m-form on **G/H** invariant under **G**. Such an m-form is called a *density* on **G/H** and it gives rise, by integration, to an *invariant measure* on **G/H**. Since **G** acts transitively on **G/H**, the invariant density, if it exists, is unique up to a constant factor. Note that **H** and g**H** are differentiable submanifolds of **G** and it is foliated by leaves which are dipheomorphic to g**H**. Then we know that g**H** are the integral manifolds of a completely Pfaffian system

$$\omega_1 = 0, \omega_2 = 0, \ldots, \omega_m = 0,$$

where ω_i are 1-forms on **G**. It is easy to prove that

$$d(\mathbf{G}/\mathbf{H}) = \omega_1 \bigwedge \omega_2 \bigwedge \cdots \bigwedge \omega_m \tag{17}$$

is invariant under **G** and up to a constant factor, it is unique m-form with this property. However, d**(G/H)** is not always a density for **(G/H)** because its value can change when the points $g \in$ **G** displace on the submanifold g**H**.

Theorem 3 (Blaschke–Chern–Santaló, [10]). *A necessary and sufficient condition for the m-form $d(\mathbf{G/H})$ to be a density for $(\mathbf{G/H})$ is that its exterior differential vanish; that is,*

$$d(d(\mathbf{G}/\mathbf{H})) = 0. \tag{18}$$

Historical Remark. I asked Professor Santaló how he had come to formulate the above theorem precisely in that way. His answer was that Blaschke had already studied many properties of densities in that way, and that Chern had found conditions for the existence of an invariant density on homogeneous spaces [6]. This is why I have named this theorem as of *Blaschke–Chern–Santaló*, since Professor Santaló states it this way in his book [11].

2.5 The group of motions in the Euclidean space

Definition 1. If $\mathbf{x} = (x_1, \dots, x_n), \mathbf{x}' = (x'_1, \dots, x'_n)$, then a motion from x to x' is given by

$$\mathbf{x}' = a\mathbf{x} + b, \tag{19}$$

where $a \in O(n), b \in \mathbb{R}^n$.

It is well known that the study of the group of motions can be done by the method of moving frames of Cartan. Let $((\mathbf{p})_0, \mathbf{e}_i^0) = ((\mathbf{p})_0, e_1^0, \dots, e_n^0)$ be an orthonormal fixed frame. To each motion g corresponds a moving frame $((\mathbf{p}), \mathbf{e}_i) = g((\mathbf{p})_0, \mathbf{e}_i^0)$, the transform by g of $((\mathbf{p})_0, \mathbf{e}_i^0)$. We have

$$d\mathbf{p} = \sum_{i=1,\dots,n} \omega_i e_i \qquad de_i = \sum_{i=1,\dots,n} \omega_{ji} e_j, \tag{20}$$

and

$$\omega_i = d\mathbf{p} \cdot e_i, \qquad \omega_{ji} = de_i \cdot e_j, \qquad \omega_{ji} + \omega_{ij} = 0.$$

The structure equations are:

$$d\omega_{ij} = -\sum_{i=1,\dots,n} \omega_{ih} \bigwedge \omega_{hj} \qquad d\omega_i = -\sum_{i=1,\dots,n} \omega_h \bigwedge \omega_{ih}. \tag{21}$$

Now, using the theorem of Blaschke–Chern–Santaló, we can give the density for r-subspaces $L_r \subset R^n$.

Proposition 1 ([10]). *The density for the r-subspaces is given by*

$$dL_r = \bigwedge_{\alpha} \omega_\alpha \bigwedge_{i,\beta} \omega_{i,\beta}, \tag{22}$$

where $\alpha, \beta = r+1, \dots, n$ *and* $i = 1, \dots, r$.

We represent by $L_{r[0]}$ the r-subspaces contained in $\mathbb{R}^n$ through the origin 0. It follows that $dL_{r[0]} = dL_{(n-r)[0]} = \bigwedge \omega_{i,\beta}$ and it is a density for the real Grassmann manifold $\mathbf{G}^R_{r,n-r}$, which is a compact manifold.

If $L^{(r)}_{i[0]}$ represent a r-subspace contained in $\mathbb{R}^i$ and if we denote by $L_{r[j]}$ the r-subspace contained in $\mathbb{R}^n$ that contain a subspace L_j, then we have a remarkable relation among densities:

$$dL^{(r)}_{i[0]} \bigwedge dL_{r[0]} = dL_{r[i+1]} \bigwedge dL_{(i+1)[0]}. \tag{23}$$

2.6 Quermass integrales of convex sets

I consider a convex body $\mathbf{K} \subset \mathbb{R}^n$ and their boundary $\partial\mathbf{K}$ is called a *convex hypersurface*. Let $\mathbf{K}$ be a convex set, and let $0 \in R^n$ be a fixed point. Consider all the $(n-r)$-subspaces $L_{n-r[0]}$ and let $\mathbf{K}'_{n-r}$ be the orthogonal projection of $\mathbf{K}$ into $L_{n-r[0]}$.

Denoting by $\mathbf{O}_i$ the surface area of the i-dimensional unit sphere, we define the mean value of the volume $V(\mathbf{K}'_{n-r})$ of these projected sets as the *quermass integrale* of order r:

$$W_r(\mathbf{K}) = \frac{(n-r)\mathbf{O}_{n-1}}{n\mathbf{O}_{n-r-1}} \frac{1}{\mathrm{vol}\mathbf{G}^R_{r,n-r}} I_r(\mathbf{K}),$$

where

$$I_r(\mathbf{K}) = \int_{\mathbf{G}^{R_{r,n-r}}} \mathrm{vol}(\mathbf{K}'_{n-r}) dL_{n-r[0]} \tag{24}$$

which is an important characteristic of the convex set $\mathbf{K}$.

Using the above relation among densities, it is possible to prove that this formula is recurrent; that is,

$$I_r(\mathbf{K}) = \frac{2}{\mathbf{O}_{r-1}} \int_{\mathbf{G}^R_{1,n-1}} I'_{r-1}(\mathbf{K}'_{n-1}) dL_{n-1[0]}.$$

For completeness, we put

$$W_0(\mathbf{K}) = I_0(\mathbf{K}) = \mathrm{vol}(\mathbf{K}),\ W_n(\mathbf{K}) = \frac{\mathbf{O}_{n-1}}{n}.$$

Theorem 4 (Cauchy's Formula, [10]). *If F denotes the $(n-1)$-dimensional surface area of $\partial\mathbf{K}$, then*

$$F = \frac{2(n-1)}{\mathbf{O}_{n-2}} \int_{\mathbf{G}^R_{1,n-1}} \mathrm{vol}(\mathbf{K}'_{n-1}) dL_n - 1[0] \tag{25}$$

Remark. Evidently, this construction justifies the term *integral cross-sectional measure.*

2.7 Steiner's formula for parallel convex sets

Definition 2. The *parallel domain* $\mathbf{K}_\lambda$ in the distance λ of a convex set $\mathbf{K}$ is the union of the solid spheres of radius λ, the centers of which are points of $\mathbf{K}$; that is, $\mathbf{K}_\lambda = \mathbf{K} + \lambda\mathbf{D}$. The boundary $\partial\mathbf{K}_\lambda$ is called the *parallel surface* of $\partial\mathbf{K}$ at the distance λ.

Theorem 5 (Steiner's Formula). *For parallel convex sets, we have*

$$\mathrm{vol}(\mathbf{K}_\lambda) = \sum_{i=0,\dots,n} W_i(\mathbf{K})\lambda^i. \tag{26}$$

Corollary 2. $W_i = V_{n-i}$; that is, the quermass integrale and the cross-sectional measure of the same order coincide.

2.8 Integrals of mean curvature and cross-sectional measures of convex domains

If the boundary $\partial\mathbf{K}$ of a convex set is a differentiable hypersurface, the cross-sectional measure $W_r(K)$ can be expressed by means of the integrals of mean curvature of $\partial\mathbf{K}$. It is known that at each point of a hypersurface $\partial\mathbf{K}$ there are $n-1$ principal directions and $n-1$ principal curvatures $\kappa_1, \dots, \kappa_{n-1}$. If $d\sigma$ denotes the area element of $\partial\mathbf{K}$, then the rth integral of mean curvature $Q_r(\partial\mathbf{K})$ is defined by

$$Q_r(\partial\mathbf{K}) = (C_{n-1,r})^{-1} \int_{\partial\mathbf{K}} \{\kappa_{i_1}, \dots, \kappa_{i_r}\} d\sigma, \tag{27}$$

where $\{\kappa_{i_1}, \dots, \kappa_{i_r}\}$ denotes the rth elementary symmetric function of the principal curvatures. The product $\kappa_1 \cdot \dots \cdot \kappa_{n-1}$ is called the Gauss–Kronecker curvature and is related to the area element du_{n-1} of the spherical image of $\partial\mathbf{K}$ by the equation

$$\kappa_1 \cdot \dots \cdot \kappa_{n-1} d\sigma = du_{n-1}. \tag{28}$$

If $R_i = (1/\kappa_i)$ are the principal radii of curvature, then

$$Q_r(\partial\mathbf{K}) = (C_{n-1,r})^{-1} \int_{S^{n-1}} R_{i_1}, \dots, R_{i_{n-r-1}} d\sigma. \tag{29}$$

From the above assumptions, the principal radii of curvature of $\partial\mathbf{K}_\lambda$ are $R_i + \lambda$ and the element area $d\sigma_\lambda$ becomes

$$d\sigma_\lambda = \prod_{i=0,\dots,n-1} (R_i + \lambda), \tag{30}$$

hence, the area of ∂K_λ is

$$\partial K_\lambda = \Sigma_{i=0,\dots,n-1} Q_r(\partial\mathbf{K})\lambda^r. \tag{31}$$

For the volume of S(K_λ), we have

$$\mathrm{vol}(\mathbf{K}_\lambda) = \mathrm{vol}(\mathbf{K}) + \sum \frac{1}{r+1} Q_r(\partial\mathbf{K})\lambda^{r+1}. \tag{32}$$

Comparison of this expression with Steiner's Formula gives

$$Q_r(\partial\mathbf{K}) = nW_{r+1}(\mathbf{K}).$$

2.9 Inequalities of isoperimetric type for arbitrary domains

In [14], the author extend some inequalities of isoperimetric type to arbitrary domains in the Euclidean space verifying a natural curvature condition.

Definition 3. A domain $\mathbf{K}$ with boundary $\partial\mathbf{K}$ is said to be k-convex, $0 \leqq k \leqq n-1$, if $Q_j(\partial\mathbf{K}) \geqq 0$ for all $j = 0, 1, \ldots, k$.

An arbitrary domain is clearly 0-convex while a differentiable domain is convex if and only if it is $(n-1)$-convex.

I recall that the quermass integrale (or equivalently) the integrals of mean curvatures are well defined for any bounded domain $\mathbf{K}$ with boundary $\partial\mathbf{K}$.

Proposition 2 ([14]). *The inequalities of isoperimetric type*

$$Q^i_j(\partial\mathbf{K}) \geqq v_n^{i-j} Q^j_i(\partial\mathbf{K})$$

are valid for any bounded and differentiable domain $\mathbf{K}$ *which is* $(n-m-1)$*-convex.*

Sobolev spaces and the classical isoperimetric inequality

It is well known that there is a connection between the classical isoperimetric inequality and the Sobolev embedding theorem. In fact, it is well known that for a measurable function f the p-norm is defined as

$$\|\|f\|\|_p = \int \|f\| d\mu^{\frac{1}{p}}, \tag{33}$$

where μ is a positive Radon measure.

Definition 4. Let (M^n, g) be a Riemannian manifold of dimension n. For a real function $\varphi \in \mathcal{C}^k(M^n)$, we define

$$\|\nabla^k\varphi\| = \nabla^{i_1}\nabla^{i_2}\ldots\nabla^{i_k}\varphi\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_k}\varphi. \tag{34}$$

In particular, $\|\nabla^0\varphi\| = \|\varphi\|$, $\|\nabla^1\varphi\|^2 = \|\nabla\varphi\|^2 = \nabla^i\varphi\nabla_i\varphi$, $\nabla^k\varphi$ will mean any k-th covariant derivative of φ.

Let us consider the vector space $\Re^p_k$ of functions φ, such that $\|\nabla^l\varphi\| \in L_p(M^n)$, for all l with $0 \leqq l \leqq k$, where k and l are integers and $p \geqq 1$ is a real number.

Definition 5. The *Sobolev* space $H^p_k(M^n)$ is the completion of $\Re^p_k$ with respect to the norm

$$\|\|\nabla\varphi\|\|_{H^p_k} = \sum_{i=0}^{k} \|\|\nabla^\ell\varphi\|\|_p. \tag{35}$$

Theorem 6 ([1, 2, 15]). *If* $1 \leq n$*, all* $\varphi \in H^p_l(\mathbb{R}^\ltimes)$ *satisfy:*

$$\|\|\nabla\varphi\|\|_p \leqq K(n,q)\|\|\nabla\varphi\|\|_q \tag{36}$$

with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ *and*

$$K(n,q) = \frac{q-1}{n-q}\left\{\frac{n-q}{n(q-1)}\right\}^{\frac{1}{q}}\left[\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{q}\right)\Gamma\left(n+1-\frac{n}{q}\right)\omega_{n-1}}\right]^{\frac{1}{n}} \tag{37}$$

For $1 < q < n$, *and* $K(n,1) = (1/n)[n/(\omega_{n-1})]^{1/n}$. $K(n,q)$ *is the norm of the embedding* $H^q_\ell \subset L_p$, *and it is attained by the functions*

$$\varphi(x) = \left(\lambda + \| \|x\| \|^{\frac{q}{q-1}}\right)^{1-\frac{n}{q}},$$

where λ *is any positive real number.*

When $q = 1$, this gives the usual isoperimetric inequality, [5, 9]. The extremum functions are then the characteristic functions of the balls of $\mathbb{R}^n$.

As a particular case of the general situation, [3, p. 40], we have the following result of Federer and Fleming or Maz'ya and a proof of it is presented in [4], p. 300–302.

Theorem 7. *Let* $\mathbf{K}$ *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\mathbf{K}$. *The isoperimetric inequality*

$$V^n_{n-1}(\partial\mathbf{K}) \geqq v_n V^{n-1}_n(\mathbf{K})$$

for every such $\mathbf{K}$ *is equivalent to the inequality*

$$\left(\int_{\mathbf{K}} \|grad f\| d\mathbf{K}\right)^n \geqq v_n \left(\int_{\mathbf{K}} \|f\|^{\frac{n}{n-1}} d\mathbf{K}\right)^{n-1} \qquad for all\, f \in C_0^\infty(\mathbf{K}). \tag{38}$$

For the proof of the above result, it was the basic fact that the surface area of the level-hypersurfaces were positive decreasing functions, [4, p. 301].

Open Problem. In the direction of the (38), it seems natural to try obtain analytical inequalities which becomes equivalent to (6)–(11).

3 Complex cross-sectional measures

3.1 Some remarks about complex integral geometry

In the mathematical literature, in particular, in Santaló's book, the theory of the Real Integral Geometry is developed and supported in the Theorem of Blaschke–Chern–Santaló. Also, Santaló said in his book, [10], p. 338: "*Integral Geometry on complex spaces has not been sufficiently developed and probably deserves further study*". Some results are known now about the complex projective space and the unitary group. But, in general, this theory is far to be achieved.

Let $\mathbb{C}^n$ denote the space of n-tuples of complex numbers $(z_1, z_2, \dots, z_n)$. The r-dimensional complex linear subspaces of $\mathbb{C}^n$ will be called r-planes. We consider the group of complex motions

$$\mathbf{z}' = a\mathbf{z} + b,$$

where a is an element of the unitary group $\mathbb{U}^n$ and b is a complex number.

The study of this group can be easily done by the method of complex moving frames, using their real representation. Let (p, e_i, e_{i^*}) be a complex moving frame. So, applying the standard method, we have

$$\omega_i = d\mathbf{p}\cdot\mathbf{e}_i, \qquad \omega_{i^*} = d\mathbf{p}\cdot\mathbf{e}_{i^*},$$
$$\omega_{ji} = d\mathbf{e}_i\cdot\mathbf{e}_j, \qquad \omega_{ji^*} = d\mathbf{e}_{i^*}\cdot\mathbf{e}_j,$$
$$\omega_{ji} + \omega_{ij} = 0, \qquad \omega_{ji^*} + \omega_{i^*j} = 0,$$
$$\omega_{i^*j^*} + \omega_{j^*i^*} = 0$$

Now, the structure equations are

$$\begin{aligned} rcld\omega_{ij} &= -\omega_{ik}\bigwedge\omega_{kj} - \omega_{ik^*}\bigwedge\omega_{k^*j}\\ d\omega_{ij^*} &= -\omega_{ik}\bigwedge\omega_{kj^*} - \omega_{ik^*}\bigwedge\omega_{k^*j^*}\\ d\omega_i &= -\omega_{ik}\bigwedge\omega_k - \omega_{ik^*}\bigwedge\omega_{k^*}\\ d\omega_{i^*} &= \omega_{ik^*}\bigwedge\omega_k - \omega_{ik}\bigwedge\omega_{k^*}. \end{aligned} \tag{39}$$

The complex rotations about a point can be identified with the unitary group $\mathbb{U}(n)$ which is compact and has an invariant measure which was determined, among others, by Santaló [11]. Since the complex translations had also an invariant density, we have an invariant density for the group of the complex motions.

Then, we can generalize to the Complex Integral Geometry on $\mathbb{C}^n$ all results known pertaining to densities in the Real Integral Geometry and, mutatis mutandis, the formulae are the same.

The complex Grassmann manifold $\mathbf{G}^C_{r,n-r}$ of the complex r-planes through the origin is well-defined as $\mathbf{G}^C_{r,n-r} = [\mathbf{U}(n)/\mathbf{U}(r)x\mathbf{U}(n-r)]$ and we can determine its volume.

3.2 The complex quermass integrales

Also, we can consider a convex body $\mathbf{K} \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n$. If I assume that $\partial\mathbf{K}$ is sufficiently smooth, then the r-th complex quermass integrale $V^C_r(\mathbf{K})$, (complex cross-sectional measure), may be defined as a mean integral value of the r-dimensional volume of the projections of $\mathbf{K}$ on all r-dimensional complex subspaces. Explicitly, we can define

$$V^C(\mathbf{K}) = \frac{1}{\mathrm{vol}\mathbf{G}^C_{r,n-r}}\int H^r(P_\nu(\mathbf{K})d\mu(\nu), \tag{40}$$

where $\mathbf{G}^C_{r,n-r}$ is the complex Grassmann manifold of r-dimensional complex subspaces in $\mathbb{C}^n$, $P_\nu(\mathbf{K})$ is the orthogonal projection of $\mathbf{K}$ on the subspace $\nu \in \mathbf{G}^C_{r,n-r}$, H^r denotes the r-dimensional Hausdorff measure in $\mathbb{C}^n$ and μ denotes the normalized Haar measure on $\mathbf{G}^C_{r,n-r}$.

In the real case, it has been shown that the r-th quermass integrale could be represented by the integral of the symmetric functions of the principal curvatures.

Open Problem. How is the representation $V^C_m(\mathbf{K})$ in function of the integrals of the generalized mean curvatures?

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Notes on the Goldberg Conjecture in Dimension Four

Takashi Oguro[1] and Kouei Sekigawa[2]

[1]Department of Mathematical Sciences, School of Science and Engineering, Tokyo Denki University, Saitama, 350–0394, Japan
oguro@r.dendai.ac.jp
[2]Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-2181, Japan
sekigawa@sc.niigata-u.ac.jp

Dedicated to Professor Lieven Vanhecke

Summary. Concerning the integrability of almost Kähler manifolds, there is a longstanding conjecture by S.I. Goldberg, "*A compact almost Kähler Einstein manifold is Kähler*". The conjecture is true in the case where the scalar curvature is non-negative. However, the conjecture is still open in the remaining case. In this note, we shall give a brief survey on the recent progress concerning the conjecture in four-dimensional case.

1 Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold, if the Kähler form Ω is closed (or equivalently, $\underset{X,Y,Z}{\mathfrak{S}}\, g((\nabla_X J)Y, Z) = 0$, where $\underset{X,Y,Z}{\mathfrak{S}}$ denotes the cyclic sum with respect to any smooth vector fields X, Y, Z on M), namely, (M, Ω) is a symplectic manifold. By the definition, a Kähler manifold ($\nabla J = 0$) is necessarily an almost Kähler manifold. A non-Kähler, almost Kähler manifold is called a strictly almost Kähler manifold. The first example of compact strictly almost Kähler manifold was given by W.T. Thurston [25]. It is well-known that, if the almost complex structure J of almost Kähler manifold $M = (M, J, g)$ is integrable, then M is a Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S.I. Goldberg is known [10].

Conjecture. A compact almost Kähler Einstein manifold is integrable.

In [23], the second author showed that the above conjecture is true in the case where the scalar curvature is non-negative. However, the conjecture is still open in the remaining case. Other partial affirmative answers have been obtained by many authors under some additional curvature conditions ([1–6, 8–10, 14, 16, 19–24] and so on).

This article is a brief survey on the Goldberg conjecture in dimension four.

2 Preliminaries

In this section, we prepare some fundamental formulas on four-dimensional almost Kähler Einstein manifold.

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold and $\Omega(X, Y) = g(JX, Y)$, for $X, Y \in \mathfrak{X}(M)$ be the Kähler form of M, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. We assume that M is oriented by the volume form $dV = (1/2)\Omega^2$. Let ∇ be the Levi-Civita connection, and further, R, Ric, s the curvature tensor, the Ricci tensor, the scalar curvature of M defined respectively by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{2.1}$$

$$\mathrm{Ric}(X, Y) = \mathrm{Trace}\left\{Z \mapsto R(Z, X)Y\right\}, \tag{2.2}$$

$$s = \mathrm{Trace}_g \mathrm{Ric}, \tag{2.3}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Moreover, we define the Ricci $*$-tensor Ric^* by

$$\mathrm{Ric}^*(X, Y) = \mathrm{Trace}\left\{Z \mapsto -JR(Z, X)JY\right\}, \tag{2.4}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Further, we denote by s^* the $*$-scalar curvature of M which is the trace of the linear endomorphism Q^* defined by $g(Q^*X, Y) = \mathrm{Ric}^*(X, Y)$ for $X, Y \in \mathfrak{X}(M)$. By the definition, we see immediately

$$\mathrm{Ric}^*(X, Y) = \mathrm{Ric}^*(JY, JX), \tag{2.5}$$

and hence, Ric^* is symmetric if and only if Ric^* is J-invariant. We may also note that if M is Kähler, then $\mathrm{Ric}^* = \mathrm{Ric}$ holds on M. An almost Hermitian manifold M is called a weakly $*$-Einstein manifold if $\mathrm{Ric}^* = (s^*/4)g$ holds on M. Especially, if the $*$-scalar curvature of a weakly $*$-Einstein manifold M is constant, then M is called a $*$-Einstein manifold. It is known that the following identity holds for any four-dimensional almost Hermitian manifold [12], [24]:

$$\mathrm{Ric}^*(X, Y) + \mathrm{Ric}^*(Y, X) - \{\mathrm{Ric}(X, Y) + \mathrm{Ric}(JX, JY)\} = \frac{s^* - s}{2} g(X, Y), \tag{2.6}$$

for $X, Y \in \mathfrak{X}(M)$. We may regard the curvature tensor R as (0,4)-tensor and also an endomorphism of the vector bundle of 2-forms $\wedge^2 M = \wedge^2 T^*M$ as,

$$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

$$\langle R(\alpha_1 \wedge \beta_1), \alpha_2 \wedge \beta_2) \rangle = -R(\alpha_1^\sharp, \beta_1^\sharp, \alpha_2^\sharp, \beta_2^\sharp),$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ and $\alpha_i, \beta_i \in \Gamma(M, T^*M)$. Here, the symbol $\sharp$ is the natural isomorphism $T^*M \to TM$ by g and $\langle \cdot, \cdot \rangle$ is the inner product on $\wedge^2 M$ induced from g. The usual type decomposition $\wedge^2 M \otimes \mathbb{C} = \wedge^{2,0} M \oplus \wedge^{1,1} M \oplus \wedge^{0,2} M$ of complexified 2-forms induces the following decomposition of the vector bundle $\wedge^2 M$:

$$\wedge^2 M = \mathbb{R}\Omega \oplus \wedge_0^{1,1} M \oplus LM, \tag{2.7}$$

where $LM = (\wedge^{2,0}M \oplus \wedge^{0,2}M)_{\mathbb{R}}$ and $\wedge_0^{1,1}M$ is the orthogonal complement of $\mathbb{R}\Omega$ in $(\wedge^{1,1}M)_{\mathbb{R}}$. In the above decomposition, $\wedge_0^{1,1}M$ is the vector bundle of real primitive J-invariant 2-forms, LM the vector bundle of real primitive J-skew-invariant 2-forms over M respectively. The vector bundle $\wedge_0^{1,1}M$ is identified itself with the bundle $\wedge_-^2 M$ of anti-self-dual 2-forms, while the sum $\mathbb{R}\Omega \oplus LM$ is the bundle $\wedge_+^2 M$ of the self-dual 2-forms. Further, it is well-known that M is Einstein if and only if $\wedge_+^2 M$ and $\wedge_-^2 M$ are both preserved by the curvature operator [13]. We denote by π the orthogonal projection $\pi : \wedge^2 M \to LM$. The vector bundle LM is endowed with the natural complex structure (also denoted by J) which is defined by $J\Phi(X,Y) = -\Phi(JX,Y)$ $(X, Y \in \mathfrak{X}(M))$ for any local section Φ of LM.

Let $\{e_i\}$ be a local orthonormal frame field and set

$$R_{ijkl} = R(e_i, e_j, e_k, e_l), \qquad \mathrm{Ric}_{ij} = \mathrm{Ric}(e_i, e_j), \qquad \mathrm{Ric}^*_{ij} = \mathrm{Ric}^*(e_i, e_j),$$

where the Latin indices run over the range 1, 2, 3, 4. In the sequel, indices with bar are the ones with respect to $\{Je_i\}$, for example

$$R_{\bar{i}jkl} = R(Je_i, e_j, e_k, e_l).$$

Using these notational convention, we obtain

$$\mathrm{Ric}_{ij} = -\sum_a R_{iaja}, \qquad \mathrm{Ric}^*_{ij} = \sum_a R_{ia\bar{j}\bar{a}} = -\frac{1}{2}\sum_a R_{i\bar{j}a\bar{a}},$$
$$s = -\sum_{a,b} R_{abab}, \qquad s^* = -\frac{1}{2}\sum_{a,b} R_{a\bar{a}b\bar{b}}$$

and the Ricci $*$-tensor satisfies $\mathrm{Ric}^*_{ij} = \mathrm{Ric}^*_{\bar{j}\bar{i}}$ (2.5). We denote J_{ij} and $\nabla_i J_{jk}$ by

$$J_{ij} = g(Je_i, e_j), \qquad \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k).$$

$\{J_{ij}\}$ and $\{\nabla_i J_{jk}\}$ satisfy

$$J_{ij} = -J_{ji} = J_{\bar{i}\bar{j}}, \tag{2.8}$$
$$\nabla_i J_{jk} = -\nabla_i J_{\bar{j}\bar{k}}. \tag{2.9}$$

Now, let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any local unitary frame field and $\{e^i\}$ be its dual frame field. Then, the Kähler form Ω of M is represented by $\Omega = e^1 \wedge e^2 + e^3 \wedge e^4$. Further, we may easily observe that

$$\{\Phi, J\Phi\} = \left\{\frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\right\}, \tag{2.10}$$

and

$$\{\Psi_1, \Psi_2, \Psi_3\} = \left\{\frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^3 + e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 - e^2 \wedge e^3)\right\} \tag{2.11}$$

and local orthonormal frame fields of LM and $\wedge_0^{1,1}M$ respectively.

We denote by $\mathrm{inv}_J\mathrm{Ric}$, $\mathrm{anti}_J\mathrm{Ric}$, symRic^*, $\mathrm{skewRic}^*$ the following tensor fields of type (0, 2) defined respectively by

$$\begin{aligned} \mathrm{inv}_J\mathrm{Ric}(X,Y) &= \frac{1}{2}(\mathrm{Ric}(X,Y)+\mathrm{Ric}(JX,JY)), \\ \mathrm{anti}_J\mathrm{Ric}(X,Y) &= \frac{1}{2}(\mathrm{Ric}(X,Y)-\mathrm{Ric}(JX,JY)), \\ \mathrm{symRic}^*(X,Y) &= \frac{1}{2}(\mathrm{Ric}^*(X,Y)+\mathrm{Ric}^*(Y,X)), \\ \mathrm{skewRic}^*(X,Y) &= \frac{1}{2}(\mathrm{Ric}^*(X,Y)-\mathrm{Ric}^*(Y,X)), \end{aligned} \tag{2.12}$$

for $X, Y \in \mathfrak{X}(M)$. Then, the formula (2.6) can also be written as follows:

$$\mathrm{symRic}_0^* = \mathrm{invRic}_0,$$

where $\mathrm{Ric}_0 = \mathrm{Ric} - (s/4)g$ and $\mathrm{Ric}_0^* = \mathrm{Ric}^* - (s^*/4)g$.

The almost complex structure J induces an endomorphism on the vector bundle $\wedge^k M$ of k-forms (denoted also by J) defined by,

$$J\beta(X_1,\ldots,X_k) = -\beta(JX_1,X_2,\ldots,X_k),$$

for $\beta \in \Gamma(M;\wedge^k M)$ and $X_1,\ldots,X_k \in \mathfrak{X}(M)$. Then, the endomorphism J satisfies $J^2 = -\mathrm{Id}$, and defines a complex structure on LM.

We denote by W the Weyl (conformal) tensor of M. Then, we have

$$W = R - \frac{1}{2}\mathrm{Ric}_0 \owedge g - \frac{s}{24}\, g \owedge g, \tag{2.13}$$

where the symbol $\owedge$ is the Nomizu–Kulkarni product of symmetric tensors of type (0, 2) generating a curvature type tensor. The Weyl tensor W can also be regarded as a symmetric endomorphism of $\wedge^2 M$ which induces endomorphism W^+ of $\wedge^2_+ M$ (resp. W^- of $\wedge^2_- M$). By making use of the second Bianchi identity, the divergence δW of W is expressed as

$$\delta W = -\frac{1}{2} d_R\left(\mathrm{Ric} - \frac{s}{6}g\right), \tag{2.14}$$

where $d_R : \Gamma(M;\wedge^1 M \otimes \wedge^1 M) \to \Gamma(M;\wedge^1 M \otimes \wedge^2 M)$ is defined by,

$$(d_R T)(X,Y,Z) = (\nabla_Y T)(X,Z) - (\nabla_Z T)(X,Y),$$

for $T \in \Gamma(M;\wedge^1 M \otimes \wedge^1 M)$. From (2.12) and (2.13), by direct calculation, we have

$$W^+ = \frac{1}{8}\left(s^* - \frac{s}{3}\right)\Omega\otimes\Omega + W_{LM} + \frac{1}{2}(J\mathrm{skewRic}^*\otimes\Omega + \Omega\otimes J\mathrm{skewRic}^*), \tag{2.15}$$

where $W_{LM} = \pi \circ W^+|_{LM}$. According to the decomposition (2.7), we may decompose the operator W into several parts as follows:

$$W = W^+ + W^-, \tag{2.16}$$

$$W^+ = \left(\begin{array}{c|c} \dfrac{\kappa}{6} & W_2^+ \\ \hline (W_2^+)^* & W_3^+ - \dfrac{\kappa}{12}\mathrm{Id}^+ \end{array}\right),$$

where the function $\kappa = 3\langle W^+(\Omega), \Omega\rangle = (3s^* - s)/2$ is the conformal scalar curvature and Id^+ is the identity on $\wedge^2_+ M$. From (2.14), (2.15) and (2.16), we have

$$W_2^+ = \frac{1}{2}(\Omega \otimes J\mathrm{skewRic}^* + J\mathrm{skewRic}^* \otimes \Omega), \tag{2.17}$$

$$W_3^+ = W_{LM} + \frac{\kappa}{12}\mathrm{Id}^+. \tag{2.18}$$

There are three spacial type of four-dimensional almost Hermitian Einstein manifolds which are imposing one additional condition on the self-dual Weyl tensor W^+:

(W I) κ is constant (equivalently, the manifold has constant $*$-scalar curvature),

(W II) W_2^+ vanishes (equivalently, the manifold is weakly $*$-Einstein),

(W III) W_3^+ vanishes (equivalently, the Kähler form Ω is a root of the Weyl tensor W).

A four-dimensional almost Hermitian manifold satisfying $W_3^+ = 0$ is called a manifold with Hermitian Weyl tensor. Especially, we may see that a Kähler Einstein surface satisfies the above conditions (W I)–(W III).

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian Einstein manifold and $\{\Phi, J\Phi\}$ a local orthonormal frame of LM given by (2.10). Then, we have

$$W_3^+ = \begin{pmatrix} u + \dfrac{s^* - s}{8} & w \\ w & v + \dfrac{s^* - s}{8} \end{pmatrix},$$

where $u = \langle R(\Phi), \Phi\rangle$, $v = \langle R(J\Phi), J\Phi\rangle$, $w = \langle R(\Phi), J\Phi\rangle$. We denote by h the smooth function on M, defined by

$$h = \frac{(s^* - s)^2}{16} - 4 \det R_{LM},$$

where $R_{LM} = \pi \circ R|_{LM}$. Then, h can be expressed locally by

$$h = (u - v)^2 + 4w^2.$$

We may also observe that a four-dimensional almost Hermitian Einstein manifold is a manifold with Hermitian Weyl tensor if and only if h vanishes.

For a four-dimensional almost Hermitian manifold, the conditions (W I) κ is constant, (W II) $W_2^+ = 0$, (W III) $W_3^+ = 0$ are closely related to the following conditions (G_1)–(G_3) on the curvature tensor defined by Gray [11] (not necessarily in the four-dimensional context):

(G_1) $R(X, Y, Z, W) = R(X, Y, JZ, JW)$,

(G_2) $R(X, Y, Z, W) - R(JX, JY, Z, W) = R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$,

(G_3) $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$,

for $X, Y, Z \in \mathfrak{X}(M)$. Identity (G_i) will be called the ith Gray condition. A simple application of the first Bianchi identity yields the implications $(\mathrm{G}_1) \Rightarrow (\mathrm{G}_2) \Rightarrow (\mathrm{G}_3)$. We denote by $\mathcal{AK}$ and $\mathcal{AK}_i$ the class of almost Kähler manifolds and the class of manifolds in $\mathcal{AK}$, whose curvature tensors satisfy the condition (G_i). Then, we have the following inclusion relations:

$$\mathcal{AK} \supseteq \mathcal{AK}_3 \supseteq \mathcal{AK}_2 \supseteq \mathcal{AK}_1 \supseteq \mathcal{K},$$

where $\mathcal{K}$ denotes the class of Kähler manifolds.

In the remaining part of this section, we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then, in addition to (2.8) and (2.9), we get

$$\nabla_i J_{jk} = -\nabla_{\bar{i}} J_{\bar{j}k}, \qquad \sum_a \nabla_a J_{ai} = 0. \tag{2.19}$$

Taking account of (2.8)–(2.11) and (2.19), we may set

$$\nabla\Omega = \alpha \otimes \Phi - J\alpha \otimes J\Phi, \tag{2.20}$$

where α is a local 1-form and $\{\Phi, J\Phi\}$ is a local orthonormal frame of LM given by (2.10). Gray [11] has obtained the following curvature identity:

$$\begin{aligned} R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} \\ = 2\sum_a (\nabla_a J_{ij}) \nabla_a J_{kl} \end{aligned} \tag{2.21}$$

From (2.21), we get immediately

$$\mathrm{Ric}^*_{ij} + \mathrm{Ric}^*_{ji} - \mathrm{Ric}_{ij} - \mathrm{Ric}_{\bar{i}\bar{j}} = \sum_{a,k} (\nabla_a J_{ik}) \nabla_a J_{jk}, \tag{2.22}$$

and further

$$\|\nabla J\|^2 = 2(s^* - s). \tag{2.23}$$

We denote by φ the symmetric tensor field of type (0, 2) defined by,

$$\varphi(X, Y) = \sum_a g((\nabla_a J)X, (\nabla_a J)Y).$$

Then, we see that (2.21) and (2.22) imply

$$\langle R(\Phi), \Phi' \rangle + \langle R(J\Phi), J\Phi' \rangle = -\frac{1}{2} \sum_a \langle \nabla_a \Omega, \Phi \rangle \langle \nabla_a \Omega, \Phi' \rangle, \tag{2.24}$$

and

$$\varphi = 2(\mathrm{symRic}^* - \mathrm{inv}_J \mathrm{Ric}), \tag{2.25}$$

respectively. Further, from (2.6), (2.12) and (2.25), we have

$$\varphi = \frac{s^* - s}{2} g. \tag{2.26}$$

3 Some partial answers

In this section, we introduce several partial answers to the Goldberg conjecture and also some related results to the conjecture.

First, we recall the Chern–Weil formulae for four-dimensional compact oriented Riemannian manifolds and almost Hermitian manifolds. Suppose that $M = (M, g)$ is a four-dimensional compact oriented Riemannian manifold. Then, by the Chern–Weil theorem together with the Hirzebruch signature theorem, the signature $\tau(M)$ of M is given by

$$\tau(M) = \frac{1}{12\pi^2} \int_M \{\|W^+\|^2 - \|W^-\|^2\} dV. \tag{3.1}$$

Similarly, the generalized Gauss–Bonnet theorem, the Euler characteristic $\chi(M)$ of M is given by

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left\{ \|W^+\|^2 + \|W^-\|^2 + \frac{s^2}{24} - \frac{1}{2}\|\mathrm{Ric}_0\|^2 \right\} dV. \tag{3.2}$$

The following result is due to Wu [26].

Theorem 3.1. *Let $M = (M, J)$ be a four-dimensional compact almost complex manifold. Then we have*

$$c_1(M)^2 = 2\chi(M) + 3\tau(M),$$

where $c_1(M)$ is the first Chern class of M.

In [13], Hitchin proved that a four-dimensional compact oriented Einstein manifold $M = (M, g)$ must satisfy the following inequality (known as Hitchin–Thorpe inequality):

$$\chi(M) \geq \frac{3}{2}|\tau(M)|.$$

This result gives a topological obstruction for the existence of Einstein metrics on a four-dimensional compact oriented manifold.

Now, we shall discuss four-dimensional almost Kähler Einstein manifolds satisfying the conditions (W I)–(W III) interpreted in § 2 respectively.

First, consider the condition (W I). Armstrong [5] proved the following.

Theorem 3.2. *If $M = (M, J, g)$ is a four-dimensional compact almost Kähler Einstein manifold, then $s = s^*$ holds somewhere on M.*

From the above Theorem 3.2, we have readily the following.

Corollary 3.3. ([5]) *A four-dimensional compact almost Kähler Einstein manifold with constant $*$-scalar curvature is necessarily Kähler.*

The present authors obtained the local version of the above Corollary 3.3 [20].

Secondly, consider the condition (W II). In [19], the authors proved the following.

Theorem 3.4. *Let $M = (M, J, g)$ be a four-dimensional almost Kähler Einstein and $*$-Einstein manifold. Then M is Kähler Einstein.*

From the above Theorem 3.4 and (2.23), we have readily the following.

Corollary 3.5. *A four-dimensional almost Kähler manifold of constant sectional curvature is a flat Kähler surface.*

We remark that the assertion of the above Corollary 3.5 is also valid for higher dimensional case (cf. [6]). Further, in [21], the authors and Yamada proved the following.

Theorem 3.6. *Let $M = (M, J, g)$ be a four-dimensional strictly almost Kähler Einstein and weakly $*$-Einstein manifold. Then, M is a Ricci-flat space of pointwise constant holomorphic sectional curvature $s^*/8$ (and hence self-dual).*

Armstrong [6] obtained more detailed result than the above Theorem 3.6.

Corollary 3.7. *Every four-dimensional compact almost Kähler Einstein and weakly $*$-Einstein manifold is Kähler Einstein.*

Thirdly, we discuss the condition (W III). In [1], Apostolov and Armstrong proved the following.

Theorem 3.8. *Any four-dimensional compact almost Kähler manifold $M = (M, J, g)$ with J-invariant Ricci tensor and Hermitian Weyl tensor is Kähler provided that $5\chi(M) + 6\tau(M) \neq 0$.*

For a four-dimensional compact oriented (non-flat) Einstein manifold, the topological condition stated in the above Theorem 3.6 is verified as a consequence of the Hitchin–Thorpe inequality. Thus, we have

Corollary 3.9. ([1]) *Any four-dimensional compact almost Kähler Einstein manifold with Hermitian Weyl tensor is Kähler.*

Next, we discuss briefly four-dimensional almost Kähler manifolds satisfying the Gray conditions (G_1), (G_2), (G_3) interpreted in § 2.

First, it was proved that the equality $\mathcal{AK}_1 = \mathcal{K}$ holds locally. In [2], Apostrov, Armstrong and Draghici proved the following result.

Theorem 3.10. *The equality $\mathcal{AK}_2 = \mathcal{K}$ for every four-dimensional compact manifold.*

We remark that the example of Davidov and Muškarov [8], multiplied by compact Kähler manifolds, show that even in the case, the inclusion $\mathcal{AK}_2 \supset \mathcal{K}$ is strict in dimension $2n \geq 6$. We must also remark that there are known several examples of four-dimensional strictly non-compact almost Kähler manifolds belonging to the class $\mathcal{AK}_2$ including the example by Kowalski [15]. It was shown that the inclusion $\mathcal{AK} \supset \mathcal{AK}_3$ is strict. In four-dimensional compact case, Draghici [9] proved that the equality $\mathcal{AK}_3 = \mathcal{K}$ holds for four-dimensional compact manifold with second Betti number equal to 1.

As mentioned in § 1, the Goldberg Conjecture is still open in the case, where the scalar curvature is negative. In [14], Itoh gave a partial affirmative answer to the conjecture for the remaining case by making use of the Seiberg–Witten theory developed by Taubes and LeBrun.

Theorem 3.11. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold with negative scalar curvature s. If the equality*

$$\int_M s^2 dV = 32\pi^2\{2\chi(M) + 3\tau(M)\} \tag{3.3}$$

holds, then M is Kähler Einstein.

We note that the equality can be rewritten as

$$\int_M s^2 dV = 32\pi^2 c_1(M)^2$$

by virtue of Theorem 3.1. Quite recently, Sato [22] obtained an improvement of the above Theorem 3.11 without the Siberg–Witten theory, proved the following.

Theorem 3.12. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler manifold with $\pi(\delta W) = 0$. If the equality (3.3) in Theorem 3.11 holds, then M is Kähler Einstein.*

We shall introduce here the sketch of the proof. First, let $M = (M, J, g)$ be a four-dimensional almost Kähler manifold. By making use of the formulas (2.19)–(2.23), we can deduce the following formula

$$\begin{aligned}\Delta s^{*2} &= -8\|R_{LM}\|^2 - 2(\|\mathrm{Ric}^*\|^2 - \langle \mathrm{Ric}^*, \mathrm{Ric}\rangle) \\ &\quad - \sum \nabla_j((\delta R)_{ikl} J_{ij} J_{kl}) + \sum (\delta R)_{ikl} J_{ij} \nabla_j J_{kl} \\ &\quad - 4\sum \nabla_i((\nabla_i J_{ja})\mathrm{Ric}^*_{a\bar{j}}).\end{aligned} \tag{3.4}$$

We assume further that the manifold M under consideration is compact and satisfies the condition $\pi(\delta W) = 0$. Then, by (2.14), we get $\pi d_R(\mathrm{Ric} - (s/6)g) = 0$. By using this equality, we may get

$$\sum (\delta R)_{ikl} J_{ij} \nabla_j J_{kl} = 0. \tag{3.5}$$

Now, taking account of (2.26), we get

$$\begin{aligned}&\|\mathrm{Ric}^*\|^2 - \langle \mathrm{Ric}^*, \mathrm{Ric}\rangle = \|\mathrm{skewRic}^*\|^2 + \|\mathrm{symRic}^*\|^2 - \langle \mathrm{symRic}^*, \mathrm{inv}_J \mathrm{Ric}\rangle \\ &= \|\mathrm{skewRic}^*\|^2 + \frac{1}{2}\langle \mathrm{symRic}^*, \varphi\rangle \\ &= \|\mathrm{skewRic}^*\|^2 + \frac{s^*}{4}(s^* - s).\end{aligned} \tag{3.6}$$

From (2.15), we get

$$\|W^+\|^2 = \frac{1}{16}\left(s^* - \frac{s}{3}\right)^2 + \frac{1}{2}\|\mathrm{skewRic}^*\|^2 + \|W_{LM}\|^2. \tag{3.7}$$

Integrating (3.4) and taking account of (3.5), we have the following integral formula

$$4\int_M \|R_{LM}\|^2 dV + \int_M \|\mathrm{Ric}^*\|^2 dV - \int_M \langle \mathrm{Ric}^*, \mathrm{Ric}\rangle dV = 0. \tag{3.8}$$

Since R_{LM} does not have trace-free Ricci tensor as its part, by (2.13), we have

$$\|R_{LM}\|^2 = \left\| W_{LM} + \frac{s}{12}\mathrm{Id}_{LM} \right\|^2 = \|W_{LM}\|^2 + \frac{s}{6}\mathrm{Trace}(W_{LM}) + 2\left(\frac{s}{12}\right)^2.$$

Since $\mathrm{Trace}(W^+) = 0$, we have

$$\mathrm{Trace}(W_{LM}) = -\frac{1}{4}\left(s^* - \frac{s}{3}\right).$$

Hence, we obtain

$$\|R_{LM}\|^2 = \|W_{LM}\|^2 - \frac{ss^*}{24} + \frac{s^2}{36}. \tag{3.9}$$

Substituting (3.6) and (3.9) into (3.8), we have

$$\begin{aligned}
0 &= 4\int_M \left(\|W_{LM}\|^2 - \frac{ss^*}{24} + \frac{s^2}{36}\right) dV \\
&\quad + 2\int_M \left(\|W^+\|^2 - \|W_{LM}\|^2 - \frac{1}{16}\left(s^* - \frac{s}{3}\right)^2\right) dV + \frac{1}{4}\int_M s^*(s^* - s) dV \\
&= 2\left(\int_M \|W_{LM}\|^2 dV - \frac{1}{32}\int_M \left(s^* - \frac{s}{3}\right)^2 dV\right) \\
&\quad + \int_M \left(2\|W^+\|^2 - \frac{s^2}{12} - \frac{1}{2}\|\mathrm{Ric}_0\|^2\right) dV \\
&\quad + \frac{3}{16}\int_M (s^* - s)^2 dV + \frac{1}{2}\int_M \|\mathrm{Ric}_0\|^2 dV.
\end{aligned}$$

Thus, from the assumption (3.3) in Theorem 3.11 and the formulas (3.1) and (3.2), we have finally

$$\begin{aligned}
&2\left(\int_M \|W_{LM}\|^2 dV - \frac{1}{32}\int_M \left(s^* - \frac{s}{3}\right)^2 dV\right) \\
&\qquad + \frac{3}{16}\int_M (s^* - s)^2 dV + \frac{1}{2}\int_M \|\mathrm{Ric}_0\|^2 dV = 0.
\end{aligned} \tag{3.10}$$

In general, for $k \times k$ symmetric matrix A, we have $\mathrm{Trace}(A^2) \geq (\mathrm{Trace}(A))^2/k$, and the equality holds if and only if $A = c\,(\delta_{ij})$ for a constant c. Since we may regard W_{LM} as a 2×2 symmetric matrix pointwisely, we have

$$\|W_{LM}\|^2 = \mathrm{Trace}(W_{LM})^2 \geq \frac{1}{2}(\mathrm{Trace}(W_{LM}))^2 = \frac{1}{32}\left(s^* - \frac{s}{3}\right)^2.$$

Thus, we see that each term of the right hand side of (3.10) is non-negative, we must have $s^* - s = 0$ and $\mathrm{Ric}_0 = 0$ on M, and hence, M is Kähler Einstein.

As a corollary of Theorem 3.10, we may easily show the following.

Corollary 3.13. *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold with negative scalar curvature. If M satisfies*

$$\int_M s^2 dV \leq 24 \int_M \|W^+\|^2 dV,$$

or more strictly if $|s| \leq 2\sqrt{6}\,\|W^+\|$ at each point of M, then M must be Kähler Einstein. Here, W^+ is the self-dual Weyl curvature operator of the metric g.

As an application of the above Corollary 3.13, Lemence and the authors proved the following.

Theorem 3.14. ([16]) *Let $M = (M, J, g)$ be a four-dimensional compact almost Kähler Einstein manifold with negative scalar curvature. If M satisfies*

$$\int_M \{4\|\mathrm{skewRic}^*\|^2 + s(s^* - s)\} dV \geq 0,$$

then M is Kähler Einstein.

In [3], Apostolov, Draghici and Moroianu discussed the possibility of the existence of counter examples to the Conjecture. More precisely, they asserted that if there exists a compact irreducible Kähler surface with two distinct constant negative Ricci eigenvalues, then we can construct a compact strictly almost Kähler Einstein manifold with negative scalar curvature, namely, a counter example to the Goldberg conjecture. Further, they considered the structure of such Kähler surfaces. It must be also remarked that they obtained examples of non-compact homogeneous strictly almost Kähler Einstein manifolds of dimension $2n(\geq 6)$ with negative scalar curvature.

4 Examples

Until recently, it was not known whether or not there exist local examples of strictly almost Kähler Einstein manifold at all. The first such example was given by Nurowski and Przanowski [18], and then the example was generalized to give a family of examples by Tod. Tod's examples are all special cases of the Gibbons–Hawking ansatz and constructed from hyper-Kähler manifold by considering the opposite almost complex structure. Tod's examples are all Ricci-flat and weakly $*$-Einstein which are not $*$-Einstein. This fact supports the assertion of Theorems 3.4 and 3.6. In [4], Apostolov, Gauduchon and Calderbank gave a local example of 4-dimensional strictly almost Kähler Einstein (Ricci-flat) and not weakly $*$-Einstein manifold. We herewith explain their example. To do this, let S^2 be a unit sphere with the canonical metric g_{S^2} and Σ the corresponding Riemann sphere with the canonical symplectic structure ω_Σ. Let (x, y) be a local coordinate system around north pole N of S^2 defined by the stereographic projection form N to the plain containing the equator. Then we have $g_{S^2} = f^2(dx^2 + dy^2)$, $\omega_\Sigma = f^2 dx \wedge dy$, where $f = 2/(1 + x^2 + y^2)$. Further, let W be a positive harmonic function defined on a neighborhood D of the north pole N and chose a smooth function V on D in such a way that $H = V + iW$

is a non-constant holomorphic function on D. We define 1-forms e^1, e^2, e^3, e^4 on $M = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; (x, y) \in D,\ z > 0,\ t \in \mathbb{R} \right\}$ by

$$e^1 = \sqrt{Wz}\, f\, dx, \qquad e^2 = \sqrt{Wz}\, f\, dy, \qquad e^3 = \sqrt{\frac{W}{z}}\, dz, \qquad e^4 = \sqrt{\frac{z}{W}}\,(dt + \alpha), \tag{4.1}$$

where α is a 1-form on D satisfying $d\alpha = W\omega_\Sigma = f^2 W\, dx \wedge dy$. Here, we define almost Hermitian structure (J, g) on M as follows:

$$g = \sum_{i=1}^{4} e^i \otimes e^i, \quad \Omega = e^1 \wedge e^2 + e^3 \wedge e^4, \tag{4.2}$$

where Ω is the corresponding Kähler form of (J, g). Then, we may easily check that $d\Omega = 0$ (and hence, (M, J, g) is an almost Kähler manifold). Furthermore, we may also observe that (M, J, g) is a Ricci-flat strictly almost Kähler manifold, and the Ricci $*$-tensor Ric* is given by

$$(\mathrm{Ric}^*) = \begin{pmatrix} \frac{W_x^2 + W_y^2}{zW^3 f^2} & 0 & \frac{W_x}{zW^2 f} & \frac{W_y}{zW^2 f} \\ 0 & \frac{W_x^2 + W_y^2}{zW^3 f^2} & -\frac{W_y}{zW^2 f} & -\frac{W_x}{zW^2 f} \\ -\frac{W_x}{zW^2 f} & -\frac{W_y}{zW^2 f} & \frac{W_x^2 + W_y^2}{zW^3 f^2} & 0 \\ \frac{W_y}{zW^2 f} & \frac{W_x}{zW^2 f} & 0 & \frac{W_x^2 + W_y^2}{zW^3 f^2} \end{pmatrix}, \tag{4.3}$$

and hence (M, J, g) is not weakly $*$-Einstein.

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Curved Flats, Exterior Differential Systems, and Conservation Laws*

Chuu-Lian Terng[1] and Erxiao Wang[2]

[1]Northeastern University and UC Irvine
terng@neu.edu
[2]MSRI, Berkeley, CA 94720
ewang@msri.org

Dedicated to Professor Lieven Vanhecke

Summary. Let σ be an involution of a real semi-simple Lie group U, U_0 the subgroup fixed by σ, and U/U_0 the corresponding symmetric space. Ferus and Pedit called a submanifold M of a rank r-symmetric space U/U_0 a *curved flat*, if T_pM is tangent to an r-dimensional flat of U/U_0 at p for each $p \in M$. They noted that the equation for curved flats is an integrable system. Bryant used the involution σ to construct an involutive exterior differential system $\mathcal{I}_\sigma$ such that integral submanifolds of $\mathcal{I}_\sigma$ are curved flats. Terng used r first flows in the U/U_0-hierarchy of commuting Soliton equations to construct the U/U_0-system. She showed that the U/U_0-system and the curved flat system are gauge equivalent, used the inverse scattering theory to solve the Cauchy problem globally with smooth rapidly decaying initial data, used loop group factorization to construct infinitely many families of explicit solutions, and noted that many of these systems occur as the Gauss–Codazzi equations for submanifolds in space forms. The main goals of this paper are: (i) give a review of these known results, (ii) use techniques from Soliton theory to construct infinitely many integral submanifolds and conservation laws for the exterior differential system $\mathcal{I}_\sigma$.

1 Introduction

Let G be a complex semi-simple Lie group, τ an involution of G such that its differential at the identity e is complex conjugate linear, and σ an involution of G such that the differential is complex linear. Assume that

$$\tau\sigma = \sigma\tau. \tag{1}$$

Let U be the fixed point set of τ, i.e., a real form of G. We will still use τ, σ to denote $d\tau_e$ and $d\sigma_e$ respectively. Let $\mathcal{G}, \mathcal{U}$ denote the Lie algebras of G and U respectively.

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Since σ and τ commute, $\sigma(U) \subset U$. So $\sigma|U$ is an involution of U. Let $\mathcal{U}_0, \mathcal{U}_1$ denote the $+1, -1$ eigenspaces of σ on $\mathcal{U}$. Then,

$$[\mathcal{U}_0, \mathcal{U}_0] \subset \mathcal{U}_0, \quad [\mathcal{U}_0, \mathcal{U}_1] \subset \mathcal{U}_1, \quad [\mathcal{U}_1, \mathcal{U}_1] \subset \mathcal{U}_0.$$

The quotient space U/U_0 is a symmetric space, and the eigen-decomposition $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ is called a *Cartan decomposition*. Ferus and Pedit [8] called a submanifold M of a rank r symmetric space U/U_0 a *curved flat* if T_pM is tangent to an r-dimensional flat of U/U_0 at p for each $p \in M$. They noted that the equation for curved flats is an integrable system. Bryant [6] used the involution σ to construct a natural involutive exterior differential system $\mathcal{I}_\sigma$ such that integral submanifolds of $\mathcal{I}_\sigma$ in U project down to curved flats in U/U_0. Terng [12] used r first flows in the U/U_0-hierarchy of commuting Soliton equations to construct the U/U_0-system. She showed that the U/U_0-system and the curved flat system are gauge equivalent, used the inverse scattering theory to solve the Cauchy problem globally with smooth rapidly decaying initial data [12], used loop group factorization to construct infinitely many families of explicit solutions [14], and noted that many of these systems occur as the Gauss–Codazzi equations for submanifolds in space forms [12, 15]. The main goals of this paper are: (i) review some of these known results, (ii) use techniques from Soliton theory to construct infinitely many integral submanifolds and conservation laws for the exterior differential system $\mathcal{I}_\sigma$. We review the definitions of these systems next.

An element $a \in \mathcal{U}_1$ is called *regular* if

(i) $\mathcal{A} := \{y \in \mathcal{U}_1 \mid [a, y] = 0\}$ is a maximal Abelian subspace in $\mathcal{U}_1$,
(ii) $\mathrm{Ad}(U_0)(\mathcal{A})$ is open in $\mathcal{U}_1$.

Let $(,)$ be an ad-invariant, non-degenerate bilinear form on $\mathcal{U}$. Given a linear subspace V of $\mathcal{U}$ let

$$V^\perp = \{y \in \mathcal{U} \mid (y, V) = 0\}.$$

Assume that U/U_0 has rank r. Let $\mathcal{A}$ be a maximal Abelian subspace in $\mathcal{U}_1$, and let $a_1, \ldots, a_r \in \mathcal{A}$ be regular and form a basis of $\mathcal{A}$. The U/U_0-system (cf. [12]) is the following PDE for $v : \mathbb{R}^r \to \mathcal{U}_1 \cap \mathcal{A}^\perp$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \quad 1 \le i \ne j \le r. \tag{2}$$

These systems occur naturally in submanifold geometry. For example, the Gauss–Codazzi equations for isometric immersions of space forms in space forms [2, 9, 12], for isothermic surfaces in $\mathbb{R}^n$ [2, 7], and for flat Lagrangian submanifolds in $\mathbb{C}^n$ or in $\mathbb{C}P^n$ [15].

The U/U_0-system also arises naturally from Soliton theory (cf. [12]). In fact, given $1 \le i \le r$, $b \in \mathcal{A}$, and a positive integer j, the (b, j)-th Soliton flow in the U/U_0-hierarchy is a certain partial differential equation for $v : \mathbb{R}^2 \to \mathcal{U}_1 \cap \mathcal{A}^\perp$:

$$v_t = P_{b,j}(v).$$

For example, the second flow in the $SU(2)$-hierarchy is the NLS (non-linear Schrödinger equation), the third flow in the $SU(2)/SO(2)$-hierarchy is the modified KdV equation, and the first flow in the $SU(3)/SO(3)$-hierarchy is the 3-wave equation. The U/U_0-system (2) is given by the collection of the $(a_j, 1)$-flows with $1 \le j \le r$ in the U/U_0-hierarchy.

The *curved flat system associated to* U/U_0 (cf. [9]) is the following first-order system for $(A_1, \ldots, A_r) : \mathbb{R}^r \to \prod_{i=1}^r \mathcal{U}_1$:

$$\begin{cases} (A_i)_{x_j} = (A_j)_{x_i}, & i \neq j, \\ \left[A_i, A_j\right] = 0, & i \neq j. \end{cases} \tag{3}$$

It is known that solutions of the curved flat system give rise to curved flats in U/U_0 [8].

Let $\alpha = g^{-1}dg$ be the Maurer–Cartan form on U. Write $\alpha = \alpha_0 + \alpha_1$ with respect to the Cartan decomposition $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$. Let $\mathcal{I}_\sigma$ be the exterior differential ideal generated by α_0. It was observed by Bryant [6] that $(U, \mathcal{I}_\sigma)$ is involutive and the PDE for the exterior differential system $\mathcal{I}_\sigma$ is the curved flat system (3). If $f : \mathcal{O} \to U$ is a maximal integral submanifold of the exterior differential system (EDS) $(U, \mathcal{I}_\sigma)$, then $f^*(\alpha_0) = 0$. The $\mathcal{U}_0$ component of the Maurer–Cartan equation $d\alpha + (1/2)[\alpha, \alpha] = 0$ gives,

$$d\alpha_0 + \frac{1}{2}([\alpha_0, \alpha_0] + [\alpha_1, \alpha_1]) = 0.$$

So $f^*([\alpha_1, \alpha_1]) = 0$. This implies that $f^{-1}df$ is $\mathcal{U}_1$-valued and the subspace $f^{-1}\mathrm{Im}(df_p)$ is Abelian for all $p \in \mathcal{O}$. This means that $(f^{-1}f_{y_1}, \ldots, f^{-1}f_{y_r})$ is a solution of the curved flat system (3) with respect to any coordinate system y. Using the Cartan–Kähler Theorem we can see that the curved flat system should only depend on n functions of one variable, where $n = \dim(\mathcal{U}_1) - r$. But the curved flat system (3) is a system of $r(r-1)/2$ equations of nr functions. This indicates that the curved flat system has many redundant functions and we probably can use geometry to find a special coordinate system on integral submanifolds so that their PDE involves only n functions. This is indeed the case. We can find a special coordinate system x on an integral submanifold of $(U, \mathcal{I}_\sigma)$, so that the corresponding PDE written in x coordinate is gauge equivalent to the U/U_0-system.

Since the curved flat system is gauge equivalent to the U/U_0-system, we can use techniques from Soliton theory to construct infinitely many explicit integral submanifolds and conservation laws for the exterior differential system $\mathcal{I}_\sigma$.

This paper is organized as follows: We explain the gauge equivalence of the U/U_0-system and the curved flat system in Section 2, give a brief review of theory of exterior differential systems in Section 3, and give Bryant's proof that the exterior system $\mathcal{I}_\sigma$ on the Lie group U is involutive in Section 4. Finally, in Section 5, we explain how to use the Birkhoff loop group factorization to construct infinitely many families of explicit solutions and commuting flows for the U/U_0-system and conservation laws for $\mathcal{I}_\sigma$.

2 The U/U_0-system

Let G, τ, σ, U, U_0 be as in Section 1, U/U_0 the corresponding symmetric space, $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$ its Cartan decomposition, and (,) be an ad-invariant, non-degenerate bilinear form on $\mathcal{U}$.

Note that the U/U_0-system (2) can also be defined invariantly as a system for maps $v : \mathcal{A} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$, so that $[da, v]$ is flat, where da means the differential of the identity map $a(\xi) = \xi$ on $\mathcal{A}$. When we choose a basis $a_1, \ldots, a_r$ of $\mathcal{A}$, the system becomes (2). Changing basis of $\mathcal{A}$ amounts to a linear change of coordinates of $\mathbb{R}^r$.

The U/U_0-system (2) and the curved flat system (3) are gauge equivalent. To explain this, we first recall some known propositions, which can be proved by direct computations.

Proposition 2.1 *The following statements are equivalent for smooth maps $u_i : \mathbb{R}^n \to \mathcal{G}$, $1 \le i \le n$:*

1. $\sum_{i=1}^n u_i dx_i$ *is a flat $\mathcal{G}$-connection 1-form on $\mathbb{R}^n$,*
2. *the first-order system $E_{x_i} = Eu_i$, $\quad 1 \le i \le n$ is solvable,*
3. *there exists $g : \mathcal{O} \to G$ such that $g^{-1}dg = \sum_{i=1}^n u_i dx_i$ for some open subset $\mathcal{O}$ of the origin in $\mathbb{R}^n$.*

Proposition 2.2 ([12]) *The following statements are equivalent for $v : \mathbb{R}^n \to \mathcal{U}_1 \cap \mathcal{A}^\perp$:*

1. *v is a solution of the U/U_0-system* (2)*,*
2. $\sum_{i=1}^r [a_i, v]dx_i$ *is a $\mathcal{U}$-valued flat connection on $\mathbb{R}^r$,*
3. $$\theta_\lambda = \sum_{i=1}^r (a_i\lambda + [a_i, v])dx_i \tag{4}$$
   is a $\mathcal{G}$-valued flat connection on $\mathbb{R}^r$ for all parameter $\lambda \in \mathbb{C}$,
4. *there is an $s \in \mathbb{R}$, so that $\theta_s = \sum_{i=1}^r (a_i s + [a_i, v])dx_i$ is a $\mathcal{U}$-valued flat connection on $\mathbb{R}^r$.*

Proposition 2.3 *A smooth map $(A_1, \cdots, A_r) : \mathbb{R}^r \to \prod_{i=1}^r \mathcal{U}_1$ is a solution of the curved flat system* (3) *associated to U/U_0 if and only if*

$$\omega_\lambda = \sum_{i=1}^r \lambda A_i dx_i$$

is a flat $\mathcal{G}$-valued connection 1-form on $\mathbb{R}^r$ for all $\lambda \in \mathbb{C}$.

The flat connections θ_λ and ω_λ are called *Lax connections* of the U/U_0-system and the U/U_0-curved flat system.

A map $\xi : \mathbb{C} \to \mathcal{G}$ is said to satisfy the *U/U_0-reality condition* if

$$\tau(\xi(\bar{\lambda})) = \xi(\lambda), \quad \sigma(\xi(\lambda)) = \xi(-\lambda).$$

It follows from the definition that $\xi(\lambda) = \sum_j \xi_j \lambda^j$ satisfies the U/U_0-reality condition if and only if $\xi_j \in \mathcal{U}_0$, if j is even and $\xi_j \in \mathcal{U}_1$, if j is odd. Note that both Lax connections θ_λ and ω_λ satisfy the U/U_0-reality condition.

It follows from Proposition 2.1, that if v is a solution of the U/U_0-system then there exists a unique $E(x, \lambda)$ so that

$$\begin{cases} E^{-1}E_{x_i} = a_i\lambda + [a_i, v], \quad 1 \le i \le r, \\ E(0, \lambda) = e. \end{cases}$$

Since θ_λ satisfies the U/U_0-reality condition, E also satisfies the U/U_0-reality condition:

$$\tau((E(x,\bar{\lambda})) = E(x,\lambda), \quad \sigma(E(x,\lambda)) = E(x,-\lambda).$$

We call such E the *parallel frame* of the Lax connection θ_λ associated to v.

The following proposition says that solutions of the U/U_0-system give rise to solutions of the curved flat system.

Proposition 2.4 ([15]) *Let $v : \mathbb{R}^r \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ be a solution of the U/U_0-system* (2), *and $E(x,\lambda)$ the parallel frame of the corresponding Lax connection θ_λ defined by* (4). *Let $g(x) = E(x,0)$, and $A_i = ga_ig^{-1}$ for $1 \le i \le r$. Then,*

(i) *the gauge transformation of θ_λ by g is*

$$g * \theta_\lambda = \sum_{i=1}^{r} \lambda g a_i g^{-1} dx_i,$$

(ii) *$(A_1, \ldots, A_r)$ is a solution of the curved flat system* (3).

Theorem 2.5 ([9]) *If $(A_1, \ldots, A_r)$ is a solution of the curved flat system* (3) *associated to U/U_0, then there exists $f : \mathcal{O} \to U$ such that $f^{-1}f_{x_i} = A_i$ for all $1 \le i \le r$, and $\pi(f)$ is a curved flat in U/U_0, where $\pi : U \to U/U_0$ is the natural projection. Conversely, every curved flat in U/U_0 can be lifted to a map f to U so that $(f^{-1}f_{x_i}, \ldots, f^{-1}f_{x_r})$ is a solution of the curved flat system* (3).

A direct computation implies

Proposition 2.6 *If $(A_1, \cdots, A_r)$ is a solution of the curved flat system* (3) *associated to U/U_0, then there exists a smooth map $f : \mathbb{R}^r \to U$ such that f satisfies the following conditions:*

$$\begin{cases} f^{-1}f_{x_i} \in \mathcal{U}_1, \\ \left[f^{-1}f_{x_i}, f^{-1}f_{x_j}\right] = 0, \quad \text{for all } i \neq j. \end{cases} \tag{5}$$

Conversely, if $f : \mathbb{R}^r \to U$ is an immersion satisfying (5), *then*

$$(f^{-1}f_{x_i}, \ldots, f^{-1}f_{x_i})$$

is a solution of the curved flat system (3).

An immersed submanifold $f : \mathcal{O} \to \mathcal{U}_1$ is called *flat Abelian* [15], if

1. $[f_{y_i}, f_{y_j}] = 0$ for all $1 \le i \neq j \le n$,
2. the induced metric on $\mathcal{O}$ is flat.

The following theorems give explicit algorithms to construct flat Abelian submanifolds in $\mathcal{U}_1$ and curved flats in the symmetric space U/U_0 from solutions of the U/U_0-system. The proofs can be found in [15].

Theorem 2.7 ([15]) *Let v and E be as in Proposition 2.4. Set*

$$Y = \frac{\partial E}{\partial \lambda} E^{-1} |\_\lambda = 0.$$

Then Y is an immersed flat Abelian submanifold in $\mathcal{U}_1$. Conversely, locally all flat Abelian submanifolds in $\mathcal{U}_1$ can be constructed this way.

For $g \in U$ and $x \in U$,

$$g * x = gx\sigma(g)^{-1}$$

defines an action of U on U (it is called the σ-action). The orbit at e is

$$M = \{g\sigma(g)^{-1} \mid g \in U\} \subset U.$$

Since the isotropy subgroup at e is U_0, the orbit M is diffeomorphic to U/U_0. In fact, M is a totally geodesic submanifold of U and is isometric to the symmetric space U/U_0. This is the classical *Cartan embedding* of the symmetric space U/U_0 in U.

Theorem 2.8 ([15]) *With the same assumption as in Theorem 2.4, set*

$$\psi(x) = E(x, 1)E(x, -1)^{-1}.$$

Then ψ is a curved flat in the symmetric space U/U_0. Conversely, locally all curved flats in U/U_0 can be constructed this way.

Theorem 2.9 *Let $\mathcal{O}$ be an open neighborhood of* $0 \in \mathbb{R}^r$*. If $f : \mathcal{O} \to U$ is an immersion satisfying* (5)*, then there exists a local coordinate system x near* 0*, a regular basis $\{a_1, \ldots, a_r\}$ of the maximal Abelian subspace $\mathcal{A} = \mathrm{Im}(df_0)$, $g : \mathcal{O} \to U_0$, and a solution $v : \mathbb{R}^r \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ of the U/U_0-system* (2)*, so that*

$$\begin{cases} f^{-1} f_{x_i} = g a_i g^{-1}, \\ g^{-1} g_{x_i} = [a_i, v]. \end{cases} \tag{6}$$

Conversely, if v is a solution of the U/U_0-system, then $f(x) = E(x, 1)E(x, 0)^{-1}$ satisfies (5) *and* (6)*, where $E(x, \lambda)$ is the parallel frame for the Lax connection θ_λ corresponding to v.*

Proof. Since generically all maximal Abelian subalgebra are conjugate under elements of U_0, there exist $g : \mathcal{O} \to U_0$ and $b_1, \cdots, b_n : \mathcal{O} \to \mathcal{A}$ such that

$$f^{-1} f_{y_i} = g b_i g^{-1},$$

for $1 \leq i \leq n$. A direct computation implies

$$\begin{aligned} 0 = d(df) &= d\left(f \sum_{i=1}^{n} g b_i g^{-1} dy_i \right) \\ &= f \sum_{j \neq i} \left(g(b_i)_{y_j} g^{-1} + [g_{y_j} g^{-1}, g b_i g^{-1}] \right) dy_j \wedge dy_i \\ &= fg \left(\sum_{j \neq i} ((b_i)_{y_j} - [b_i, g^{-1} g_{y_j}]) dy_j \wedge dy_i \right) g^{-1}. \end{aligned}$$

This implies that

$$(b_i)_{y_j} - [b_i, g^{-1}g_{y_j}] = (b_j)_{y_i} - [b_j, g^{-1}g_{y_i}], \tag{7}$$

for all $i \neq j$. Let (,) denote a non-degenerate ad-invariant bilinear form on $\mathcal{U}$. Then, $(\mathcal{A}, [\mathcal{A}, \mathcal{U}]) = 0$ and $([\mathcal{U}_\mathcal{A}, [\mathcal{A}, \mathcal{U}]) = 0$. So $[\mathcal{A}, \mathcal{U}] \subset \mathcal{U}_\mathcal{A}^\perp$ and $[\mathcal{A}, \mathcal{U}] \subset \mathcal{A}^\perp$. Also we have

$$\mathcal{U}_1 = \mathcal{A} \oplus (\mathcal{A}^\perp \cap \mathcal{U}_1).$$

Note $(b_i)_{y_j} \in \mathcal{A}$ and $[b_i, g^{-1}g_{y_j}] \in [\mathcal{A}, \mathcal{U}_0]$ is contained in $\mathcal{A}^\perp$. By (7), we get

$$(b_i)_{y_j} = (b_j)_{y_i}, \tag{8a}$$

$$[b_i, g^{-1}g_{y_i}] = [b_j, g^{-1}g_{y_i}], \tag{8b}$$

for all $1 \leq i \neq j \leq n$. Equation (8a) implies that $\sum_{i=1}^n b_i dy_i$ is closed. So, there exist a local coordinate change $x = x(y)$ and constant $a_1, \ldots, a_n$ in $\mathcal{A}$, such that $\sum_{i=1}^n b_i dy_i = \sum_{i=1}^n a_i dx_i$.

Let $\beta = \sum_{i=1}^n b_i dy_i$. Equation (8b) can be rewritten as $[\beta, g^{-1}dg] = 0$. Write β and $g^{-1}dg$ in x coordinate to get $\beta = \sum_{i=1}^n a_i dx_i$ and $g^{-1}dg = \sum_{i=1}^n g^{-1}g_{x_i} dx_i$. Then,

$$0 = [\beta, g^{-1}dg] = \sum_{i \neq j} [a_i, g^{-1}g_{x_j}] dx_i \wedge dx_j.$$

So we have

$$\left[a_i, g^{-1}g_{x_j}\right] = [a_j, g^{-1}g_{x_i}], \quad \forall\, i \neq j.$$

Up to a linear change of coordinates of x, we may assume that a_i's are regular. Note the kernel of $\mathrm{ad}(a_i)$ on $\mathcal{U}_1$ is $\mathcal{A}$, and the tangent plane of the orbit $\mathrm{Ad}(U_0)(a_i)$ at a_i is $[a_i, \mathcal{U}_0]$. By assumption $\mathrm{Ad}(U_0)(\mathcal{A})$ is open in $\mathcal{U}_1$. So the dimension of the tangent plane of the principal $\mathrm{Ad}(U_0)$-orbit at a_i is equal to $\dim(\mathcal{U}_1) - \dim(\mathcal{A})$. Thus, $\mathrm{ad}(a_i)$ maps $\mathcal{A}^\perp \cap \mathcal{U}_1$ isomorphically onto $\mathcal{U}_0 \cap (\mathcal{U}_0)_\mathcal{A}^\perp$, where $(\mathcal{U}_0)_\mathcal{A} = \{\xi \in \mathcal{U}_0 \mid [\xi, \mathcal{A}] = 0\}$. Then, by (8b), there exists a $v : \mathcal{O} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$, so that

$$g^{-1}g_{x_i} = [a_i, v], \quad 1 \leq i \leq n.$$

But $g^{-1}dg = \sum_i [a_i, v] dx_i$ is a flat connection. By Proposition 2.2, v is a solution of the U/U_0-system (2).

To prove the converse, note that

$$E^{-1}dE = \theta_\lambda = \sum_{i=1}^r (a_i\lambda + [a_i, v]) dx_i.$$

Set $g(x) = E(x, 0)$ and $F(x, \lambda) = E(x, \lambda)E(x, 0)^{-1} = E(x, \lambda)g(x)^{-1}$. A direct computation implies that

$$F^{-1}dF = g\theta_\lambda g^{-1} - dgg^{-1} = \sum_{i=1}^{r} \lambda g a_i g^{-1} dx_i.$$

Note $f(x) = F(x, 1)$. So $f^{-1}df = \sum_{i=1}^{r} g a_i g^{-1} dx_i$.

Remark 2.10. The maps g and v in Theorem 2.9 are essentially unique. To see this, suppose we have $g, \tilde{g}$, so that

$$g a_i g^{-1} = \tilde{g} a_i \tilde{g}^{-1} = f^{-1} f_{x_i},$$

$g^{-1} g_{x_i} = [a_i, v]$, and $\tilde{g}^{-1}\tilde{g}_{x_i} = [a_i, \tilde{v}]$. Since $g^{-1}\tilde{g} a_i = a_i$, there exists $(\mathcal{U}_0)_{\mathcal{A}}$-valued map h such that $g^{-1}\tilde{g} = h$, i.e., $\tilde{g} = gh$. But,

$$\begin{aligned} \tilde{g}^{-1}\tilde{g}_{x_i} &= [a_i, \tilde{v}] = h^{-1}[a_i, v]h + h^{-1}h_{x_i} \\ &= [a_i, h^{-1}vh] + h^{-1}h_{x_i} \in \mathcal{U}_{\mathcal{A}}^{\perp} + \mathcal{U}_{\mathcal{A}}. \end{aligned}$$

Thus,

$$\begin{cases} h^{-1}h_{x_i} = 0, \\ \left[a_i, h^{-1}vh\right] = [a_i, \tilde{v}]. \end{cases}$$

The first equation implies h is a constant. Since $h^{-1}vh \in \mathcal{U}_1 \cap \mathcal{A}^{\perp}$ and $\mathrm{ad}(a_i)$ is injective on $\mathcal{U}_1 \cap \mathcal{A}^{\perp}$, the second equation implies that $h^{-1}vh = \tilde{v}$. This proves that $\tilde{g} = gh$ and $\tilde{v} = h^{-1}vh$ for some constant $h \in (\mathcal{U}_0)_{\mathcal{A}}$.

3 Basics of exterior differential systems

We give a brief account of Cartan–Kähler theory based on the lectures given by R. Bryant at MSRI in 1999 and 2003 (cf. [3] for details and references).

Let M be a smooth manifold, and $\Omega^*(M)$ the graded algebra of differential forms on M. An ideal $\mathcal{I}$ of $\Omega^*(M)$ is called a *differential ideal*, if $\mathcal{I}$ satisfies the following conditions:

1. $\mathcal{I} = \bigoplus_j \mathcal{I}^j$, where $\mathcal{I}^j = \Omega^j(M) \cap \mathcal{I}$;
2. $d\mathcal{I} \subset \mathcal{I}$.

An *exterior differential system* (EDS) is a pair $(M, \mathcal{I})$ consisting of a smooth manifold M and a differential ideal $\mathcal{I} \subset \Omega^*(M)$.

A submanifold $N \subset M$ is called an *integral submanifold* for the EDS $(M, \mathcal{I})$ if $i^*\mathcal{I} = 0$, where $i : N \hookrightarrow M$ is the inclusion. In local coordinates, this condition can be written as a system of PDE (or ODE).

A linear subspace $E \subset T_pM$ is said to be an *integral element* of $\mathcal{I}$ if $\varphi|_E = 0$ for all $\varphi \in \mathcal{I}$. The set of all integral elements of $\mathcal{I}$ of dimension n is denoted $v_n(\mathcal{I})$. A submanifold of M is an integral submanifold of $\mathcal{I}$ if and only if each of its tangent space is an integral element of $\mathcal{I}$.

Note that $v_n(\mathcal{I}) \cap Gr_n(T_pM)$ is a real algebraic sub-variety of $Gr_n(T_pM)$, which may be very complicated. The set of *ordinary integral elements* $v_n^o(\mathcal{I}) \subset v_n(\mathcal{I})$ consists of those which are locally cut out 'cleanly' by finite number of n-forms in $\mathcal{I}$, so that the connected components of $v_n^o(\mathcal{I})$ are smooth embedded submanifolds of $Gr_n(TM)$. The rigorous definition can be found in [3].

Let $\{e_1, \cdots, e_n\}$ be a basis of the linear subspace E of T_pM. The *polar space* of E is defined to be the vector space,

$$H(E) = \{v \in T_pM \mid \varphi(v, e_1, \cdots, e_n) = 0 \text{ for all } \varphi \in \mathcal{I}^{n+1}\}.$$

When $E \in v_n(\mathcal{I})$, a $(n+1)$-plane E^+ containing E is an integral element of $\mathcal{I}$ if and only if $E^+ \subset H(E)$. Define

$$r(E) = \dim H(E) - \dim E - 1.$$

This integer may jump up at certain points. An ordinary integral element E is called *regular* if r is locally constant in a neighborhood of E in $v_n^o(\mathcal{I})$. The set of regular integral elements is denoted $v_n^r(\mathcal{I})$ and is a dense open subset of $v_n^o(\mathcal{I})$. Thus, $v_n^r(\mathcal{I}) \subset v_n^o(\mathcal{I}) \subset v_n(\mathcal{I}) \subset Gr_n(TM)$. An integral submanifold is called *regular* if all of its tangent spaces are regular integral elements.

We state the following two theorems that are given in [5]:

Theorem 3.1 (Cartan–Kähler Theorem) *Suppose $(M, \mathcal{I})$ is a real analytic EDS and that $N \subset M$ is a connected real analytic regular n-dimensional integral submanifold of $\mathcal{I}$ with $r(N) \geq 0$. Let $R \subset M$ be a real analytic submanifold of codimension $r(N)$ containing N, such that*

$$\dim(T_pR \cap H(T_pN)\,) = n + 1, \text{ for all } p \in N.$$

Then, there exists a unique connected real analytic $(n+1)$-dimensional integral submanifold $\tilde{N}$ such that $N \subset \tilde{N} \subset R$.

A *regular flag* is a flag of integral elements

$$(0) = E_0 \subset E_1 \subset \cdots \subset E_n = E \subset T_pM,$$

where $E_j \in v_j^r(\mathcal{I})$ for $0 \leq j < n$ and $E_n \in v_n(\mathcal{I})$. Note that E_n may not be regular, but one can show that it must be ordinary. By applying Cartan–Kähler Theorem repeatedly to this flag, one can show that there is a real analytic n-dimensional integral manifold $N \subset M$ passing through p and satisfying $T_pN = E$. Set

$$c(E_j) = \dim(T_pM) - \dim H(E_j).$$

Theorem 3.2 (Cartan's Test) *Let $(M, \mathcal{I})$ be an EDS, and $F = (E_0, \cdots, E_n)$ an integral flag of $\mathcal{I}$. Then $v_n(\mathcal{I})$ has codimension at least*

$$c(F) = c(E_0) + \cdots + c(E_{n-1})$$

in $Gr_n(TM)$ at E_n. Moreover, F is a regular flag if and only if $v_n(\mathcal{I})$ is a smooth submanifold of $Gr_n(TM)$ in a neighborhood E_n and has codimension exactly $c(F)$.

The *Cartan characters* of the flag F are the numbers

$$s_j(F) = \dim H(E_{j-1}) - \dim H(E_j), \qquad 0 \le j \le n,$$

with the convention $c(E_{-1}) = 0$ or $H(E_{-1}) = T_pM$. These numbers exhibit the generality of integral submanifolds. Roughly speaking, the integral manifolds near N will depend on s_0 constants, s_1 functions of one variable, $\cdots$, s_n functions of n variables.

A connected open subset Z of $v_n^o(\mathcal{I})$ is called *involutive* if every $E \in Z$ is the terminus of a regular flag. When Z is clear from the context, we simply say that our EDS $(M, \mathcal{I})$ is involutive.

Suppose $(M, \mathcal{I})$ is an EDS with n-dimensional integral submanifold. A *conservation law* for $(M, \mathcal{I})$ is an $(n-1)$-form $\phi \in \Omega^{n-1}(M)$ such that $d(f^*\phi) = 0$ for every integral submanifold $f : N^n \hookrightarrow M$ of $\mathcal{I}$. Actually, one only considers as conservation laws those ϕ, such that $d\phi \in \mathcal{I}$. Two "trivial" type of conservation laws are $\phi \in \mathcal{I}^{n-1}$ or ϕ being exact on M. Factoring out these cases, *the space of conservation laws* is defined to be $\mathcal{C} = H^{n-1}(\Omega^*(M)/\mathcal{I})$. It also makes sense to factor out those conservation laws represented by closed $(n-1)$-forms on M (then the quotient space is called the space of *proper* conservation laws). One can study the symmetries of the EDS and then apply Noether's Theorem to compute the corresponding conservation laws (cf. [4] for details).

4 Involutivity of the EDS

Let G, τ, σ, U, $\mathcal{U}_0$ and $\mathcal{U}_1$ be as in Section 1. Let α be the canonical left-invariant 1-form $g^{-1}dg$ on U. Write

$$\alpha = \alpha_0 + \alpha_1,$$

with respect to the Cartan decomposition $\mathcal{U} = \mathcal{U}_0 + \mathcal{U}_1$. The $\mathcal{U}_j$-component of the Maurer–Cartan equation $d\alpha + (1/2)[\alpha, \alpha] = 0$ gives

$$\begin{cases} d\alpha_0 + \frac{1}{2}([\alpha_0, \alpha_0] + [\alpha_1, \alpha_1]) = 0, \\ d\alpha_1 + [\alpha_0, \alpha_1] = 0. \end{cases}$$

Let $\mathcal{I}_\sigma \subset \Omega^*(U)$ be the differential ideal generated by the components of α_0. It follows from the Maurer–Cartan equation that

$$\begin{aligned} \mathcal{I}_\sigma &= \langle \alpha_0, d\alpha_0 \rangle \\ &= \langle \alpha_0, [\alpha_1, \alpha_1] \rangle. \end{aligned}$$

Here $\langle\ ,\ \rangle$ denotes the algebraic ideal generated by the enclosed forms.

The following Proposition was proved by R. Bryant.

Proposition 4.1 ([6]) *The EDS $(U, \mathcal{I}_\sigma)$ is involutive.*

Proof. Since everything is homogeneous, we only need to look at the integral elements $E \subset T_eU = \mathcal{U}$. Note that $E \subset \mathcal{U}_1 = \cap_{j \neq 1} \ker(\alpha_j)$. For $E = (0) \in v_0(\mathcal{I}_\sigma)$, we have $H(E) = \mathcal{U}_1$ and $v_0(\mathcal{I}_\sigma) \cong U$. Thus, $v_0(\mathcal{I}_\sigma) = v_0^o(\mathcal{I}_\sigma) = v_0^r(\mathcal{I}_\sigma)$. Now consider $E = \mathbb{R}x \in v_1(\mathcal{I}_\sigma)$ for some $x \in \mathcal{U}_1 - \{0\}$. Its polar space is

$$H(E) = \{y \in \mathcal{U}_1 | [x, y] = 0\},$$

since $[\alpha_1, \alpha_1]_e(x, y) = [x, y]$. For generic such x, $H(\mathbb{R}x)$ will be a maximal Abelian subalgebra of $\mathcal{U}_1$, and set dim $H(\mathbb{R}x) = \dim \mathcal{A} = r$. Therefore,

$$v_1(\mathcal{I}_\sigma) = v_1^o(\mathcal{I}_\sigma) \supsetneqq v_1^r(\mathcal{I}_\sigma).$$

Furthermore, when $\mathbb{R}x \in v_1^r(\mathcal{I}_\sigma)$, every subspace E of $H(\mathbb{R}x)$ containing $\mathbb{R}x$ is also regular and has $H(E) = H(\mathbb{R}x)$. Thus, generic $E \in v_r^o(\mathcal{I}_\sigma)$ is the terminus of a regular flag, and our EDS is involutive. In fact, every regular integral curve of $\mathcal{I}_\sigma$ lies in a unique r-dimensional integral submanifold of $\mathcal{I}_\sigma$, or locally the integral submanifolds depend on $s_0 = \dim \mathcal{U} - \dim \mathcal{U}_1$ constants and $s_1 = \dim(\mathcal{U}_1) - r$ functions of one variable (since $s_2 = \cdots = s_r = 0$).

Corollary 4.2 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^r$, and $f : \mathcal{O} \to U$ an immersion. Then the following statements are equivalent:

1. f is a r-dimensional integral submanifold of $(U, \mathcal{I}_\sigma)$,
2. f satisfies (5),
3. $(f^{-1} f_{x_1}, \ldots , f^{-1} f_{x_r})$ is a solution of the curved flat system (3).

Hence the PDE for the EDS $(U, \mathcal{I}_\sigma)$ is the curved flat system (3) associated to U/U_0.

As a consequence of the Cartan–Kähler Theorem 3.1, Proposition 4.1 and Corollary 4.2, it follows that the real analytic curved flats in U/U_0 or the real analytic solutions of the curved flat systems (3) depend only on $\dim(\mathcal{U}_1 \cap \mathcal{A}^\perp)$ functions of one variable along a non-characteristic line.

By Theorem 2.9, there is a special coordinate system x on $\mathbb{R}^r$, so that the curved flat system (3) written in x coordinate system is gauge equivalent to the U/U_0-system (2). The Cartan–Kähler theory implies that the Cauchy problem of the U/U_0-system has a unique local solution for any given local real analytic initial data on the x_1-axis. But, it was also proved in [12], using the inverse scattering theory of Beals and Coifman [1], that given any smooth rapidly decaying function $v_0 : \mathbb{R} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$, there exists a unique smooth solution $v : \mathbb{R}^r \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ so that

$$v(x_1, \ldots , x_r) = v_0(x_1, 0, \ldots , 0).$$

Although the theory exterior differential system seems to give a weaker result concerning the Cauchy problem, it may prove to be a very good tool to detect "integrability".

Remark 4.3. Let G, τ, U be as above, and ρ an order k automorphism of G so that $d\rho_e$ is complex linear. Assume that

$$\tau\rho = \rho^{-1}\tau^{-1}.$$

Let $\mathcal{G}_j$ denote the eigenspace of $d\rho_e$ with eigenvalue $e^{(2\pi i j/k)}$, and $\mathcal{K}_j = \mathcal{U} \cap \mathcal{G}_j$. Then,

$$\mathcal{U} = \mathcal{K}_0 + \cdots + \mathcal{K}_{k-1}. \tag{9}$$

Let $\alpha = g^{-1}dg$, and

$$\alpha = \alpha_0 + \cdots + \alpha_{k-1},$$

the decomposition of α with respect to (9). Let $\mathcal{I}_\rho$ denote the differential ideal on U generated by $\alpha_0, \alpha_2, \ldots, \alpha_{k-1}$. Then,

$$\begin{aligned}\mathcal{I}_\rho &= \langle \alpha_0, \alpha_2, \ldots, \alpha_{k-1}, d\alpha_0, d\alpha_2, \ldots, d\alpha_{k-1}\rangle \\ &= \langle \alpha_0, \alpha_2, \ldots, \alpha_{k-1}, [\alpha_1, \alpha_1]\rangle.\end{aligned}$$

We define regular elements in $\mathcal{K}_1$ the same way as before, namely, $a \in \mathcal{K}_1$ is regular if it is contained in a maximal Abelian subspace $\mathcal{A}$ in $\mathcal{K}_1$ and $\mathrm{Ad}(U_0)(\mathcal{A})$ is open in $\mathcal{K}_1$. If $\mathcal{K}_1$ admits regular elements, then the proof of Proposition 4.1 works for $(U, \mathcal{I}_\rho)$. In fact, in this case, we have:

1. $(U, \mathcal{I}_\rho)$ is involutive.
2. If $\dim(\mathcal{A}) = r$, then any r-dimensional integral submanifold depend on $\dim(\mathcal{K}_1) - \dim(\mathcal{A})$ number of functions of one variable.
3. every regular integral curve is contained in a unique r-dimensional integral submanifold of $(U, \mathcal{I}_\rho)$.
4. The curved flat system associated to U/K is the system (3) for $(A_1, \ldots, A_r)$: $\mathbb{R}^r \to \mathcal{K}_1$, and the U/K-system is the system (2) for $v : \mathbb{R}^r \to \mathcal{K}_{k-1}$. Modulo a change of coordinate system of $\mathbb{R}^r$, these two system are gauge equivalent.
5. Given an immersion $f : \mathbb{R}^r \to U$, the following statements are equivalent:
   a) f is an integral submanifold of $(U, \mathcal{I}_\rho)$,
   b) $f^{-1}f_{x_i} \in \mathcal{K}_1$ and $[f^{-1}f_{x_i}, f^{-1}f_{x_j}] = 0$ for all i, j,
   c) $(f^{-1}f_{x_1}, \ldots, f^{-1}f_{x_r})$ is a solution of the curved flat system associated to U/K.
6. The PDE for the EDS $(U, \mathcal{I}_\rho)$ is the curved flat system associated to U/K.

5 Conservation laws and commuting flows

We construct infinitely many conservation laws and commuting flows for the U/U_0-system, and indicate how to construct infinitely many explicit solutions of the U/U_0-system.

First we review the Birkhoff Factorization Theorem (for details see [10]). Let $\epsilon > 0$ be a small number, and $\mathcal{O}_\epsilon = \{\lambda \in \mathbb{C} \mid (1/\epsilon) < |\lambda| \leq \infty\}$ the open neighborhood at ∞ in $S^2 = \mathbb{C} \cup \{\infty\}$. $L(G)$ denote the group of holomorphic maps $g : \mathcal{O}_\epsilon \setminus \{\infty\} \to G$, $L_+(G)$ the subgroup of $g \in L(G)$ such that g can be extended to a holomorphic map in $\mathbb{C}$, and $L_-(G)$ the subgroup of $g \in L(G)$ that can be extended to a holomorphic map in $\mathcal{O}_\epsilon$ and is equal to the identity e at ∞.

Theorem 5.1 (Birkhoff Factorization Theorem) *The multiplication map*

$$\mu : L + (G) \times L_-(G) \to L(G), \quad (g_+, g_-) \mapsto g_+g_-$$

is one to one, and the image is an open dense subset of $L(G)$.

In other words, for generic $g \in L(G)$, we can factor $g = g_+ g_-$ uniquely with $g_\pm \in L_\pm(G)$. Let $\hat{e}$ denote the constant map from $\mathcal{O}_e \setminus \{\infty\}$ to G with constant e. Since $\hat{e}$ lies in the image of the multiplication map μ, there is an open subset of $\hat{e}$, so that all elements in this open subset can be factored uniquely.

Let $\tilde{\tau}$ and $\hat{\sigma}$ denote the map on $L(G)$ defined by

$$(\tilde{\tau}(g))(\lambda) = \tau(g(\bar{\lambda})), \quad (\hat{\sigma}(g))(\lambda) = \sigma(g(-\lambda)).$$

It is easy to check that

1. $\tilde{\tau}$ and $\hat{\sigma}$ are conjugate linear and complex linear involutions of $L(G)$,
2. $g \in L(G)$ is a fixed point of both $\tilde{\tau}$ and $\hat{\sigma}$ if and only if g satisfies the U/U_0-reality condition: $\tau(g(\bar{\lambda})) = g(\lambda)$, $\sigma(g(\lambda)) = \sigma(-\lambda)$.
3. both $\tilde{\tau}$ and $\hat{\sigma}$ leave $L_\pm(G)$ invariant.

Let $L^{\tau,\sigma}(G)$ and $L_\pm^{\tau,\sigma}(G)$ denote the subgroup of fixed points of $\tilde{\tau}$ and $\hat{\sigma}$ of $L(G)$ and $L_\pm(G)$ respectively. Then we have:

Corollary 5.2 The multiplication map

$$L_+^{\tau,\sigma}(G) \times L_-^{\tau,\sigma}(G) \to L^{\tau,\sigma}(G)$$

is one to one and the image is open and dense in $L^{\tau,\sigma}(G)$.

We want to use this factorization to construct infinitely many solutions and commuting flows for the U/U_0-system. Given $b \in \mathcal{A}$ and $j > 1$ an odd integer, $x \in \mathbb{R}^r$, and $t \in \mathbb{R}$, let $e^A(x,t) \in L_+^{\tau,\sigma}(G)$ be defined by

$$e^A(x,t)(\lambda) = \exp((a_1 x_1 + \cdots + a_r x_r)\lambda + b\lambda^j t).$$

Given $f \in L_-^{\tau,\sigma}(G)$, since $e^A(0,0) = \hat{e}$ is the identity in $L^{\tau,\sigma}(G)$ and e^A is smooth from $\mathbb{R}^r \times \mathbb{R}$ to $L^{\tau,\sigma}(G)$, by Corollary 5.2 there is an open subset of $(0,0)$ in $\mathbb{R}^r \times \mathbb{R}$ so that we can factor $f^{-1}e^A(x,t)$ uniquely as

$$f^{-1}e^A(x,t) = E(x,t)m(x,t)^{-1}, \tag{10}$$

where $E(x,t) \in L_+^{\tau,\sigma}(G)$ and $m(x,t) \in L_-^{\tau,\sigma}(G)$.

Given $c \in \mathcal{A}$, let

$$m^{-1}cm = Q_{c,0} + Q_{c,1}\lambda^{-1} + Q_{c,2}\lambda^{-2} + \cdots \tag{11}$$

denote the Taylor series of $(m(x,t)^{-1}cm(x,t))(\lambda)$ at $\lambda = \infty$. Since $m(x,t)(\lambda) = e$ at $\lambda = \infty$,

$$Q_{c,0} = c. \tag{12}$$

We want to explain how to compute $Q_{c,n}$, $m^{-1}dm$ and $E^{-1}dE$ next. To do this, we take ∂_{x_i} of (10) to get

$$f^{-1}e^A(x,t)a_i\lambda = E_{x_i}m^{-1} - Em^{-1}m_{x_i}m^{-1}.$$

Multiply E^{-1} on the left and m on the right of the above equation and use (10) to get

$$m^{-1}a_i\lambda m = E^{-1}E_{x_i} - m^{-1}m_{x_i}. \tag{13}$$

Take ∂_t of (10) and use a similar calculation to get

$$m^{-1}b\lambda^j m = E^{-1}E_t - m^{-1}m_t. \tag{14}$$

Note that $E^{-1}E_{x_i}$ and $E^{-1}E_t$ lie in the Lie algebra $\mathcal{L}_+^{\tau,\sigma}(\mathcal{G})$ of $L_+^{\tau,\sigma}(G)$, and $m^{-1}m_{x_i}$ and $m^{-1}m_t$ lie in the Lie algebra $\mathcal{L}_-^{\tau,\sigma}(\mathcal{G})$ of $L_-^{\tau,\sigma}(G)$. But it follows from the factorization theorem that

$$\mathcal{L}(\mathcal{G}) = \mathcal{L}_+^{\tau,\sigma}(\mathcal{G}) \oplus \mathcal{L}_-^{\tau,\sigma}(\mathcal{G}),$$

as direct sum of vector spaces. Let $\xi_\pm$ denote the $\mathcal{L}_\pm^{\tau,\sigma}(\mathcal{G})$ component of $\xi \in \mathcal{L}^{\tau,\sigma}(\mathcal{G})$. Then, (13) and (14) imply that

$$E^{-1}E_{x_i} = (m^{-1}a_i m\lambda)_+, \tag{15a}$$
$$E^{-1}E_t = (m^{-1}bm\lambda^j)_+ \tag{15b}$$
$$m^{-1}m_{x_i} = -(m^{-1}a_i m\lambda)_-, \tag{15c}$$
$$m^{-1}m_t = -(m^{-1}bm\lambda^j)_-. \tag{15d}$$

Use (11) to see that

$$(m^{-1}a_i m\lambda)_+ = Q_{a_i,0}\lambda + Q_{a_i,1},$$
$$(m^{-1}bm\lambda^j)_+ = Q_{b,0}\lambda^j + Q_{b,1}\lambda^{j-1} + \cdots + Q_{b,j}.$$

So we get

$$\begin{cases} E^{-1}E_{x_i} = Q_{a_i,0}\lambda + Q_{a_i,1}, \\ E^{-1}E_t = Q_{b,0}\lambda^j + Q_{b,1}\lambda^{j-1} + \cdots + Q_{b,j}. \end{cases} \tag{16}$$

Lemma 5.3 *If $c_1, c_2 \in \mathcal{A}$, then*

$$[m^{-1}c_1 m,\ m^{-1}c_2 m] = 0, \tag{17a}$$
$$[m^{-1}c_1 m,\ -(m^{-1}c_2\lambda^n m)_-] = [m^{-1}c_1 m,\ (m^{-1}c_2\lambda^n m)_+]. \tag{17b}$$

Theorem 5.4 ([12]) *There exists $v : \mathbb{R}^r \times \mathbb{R} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ so that*

$$Q_{a_i,1} = [a_i, v].$$

Moreover, for each $t \in \mathbb{R}$, $v(\cdots, t)$ is a solution of the U/U_0-system.

Proof. By (17a),

$$\left[m^{-1}a_i m,\ m^{-1}a_j m\right] = 0, \quad 1 \le i \ne j \le r.$$

So, the coefficient of λ^{-1} of the left hand side has to be zero, i.e.,

$$\left[a_i, Q_{a_j,1}] + [Q_{a_i,1}, a_j\right] = 0, \quad 1 \leq i \neq j \leq r.$$

But this implies that there exists $v : \mathbb{R}^r \times \mathbb{R} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$, so that

$$Q_{a_i,1} = [a_i, v].$$

By (16) and Proposition 2.1, we see that $\sum_i (a_i\lambda + [a_i, v])dx_i$ is a flat $\mathcal{G}$-valued connection on $\mathbb{R}^r$ for all $\lambda \in \mathbb{C}$. Hence for each fixed t, $v(\cdots, t)$ is a solution of the U/U_0-system.

The following is well-known (cf. [11, 13]):

Theorem 5.5

1. *$Q_{b,j}(x,t)$ is a polynomial in $u, \partial_x v, \cdots, \partial_x^{j-1} v$,*
2. *$Q_{b,j}$ satisfies the following recursive formula*

$$(Q_{b,j})_{x_i} + [[a_i, v], Q_{b,j}] = [Q_{b,j+1}, a_i], \tag{18}$$

3. *$Q_{b,0} = b$, $Q_{b,1} = [b, v]$.*

Proof. A direct computation gives

$$\begin{aligned}
(m^{-1}bm)_{x_i} &= [m^{-1}bm, m^{-1}m_{x_i}], \quad \text{by (15a)} \\
&= [m^{-1}bm, -(m^{-1}a_i\lambda m)_-], \quad \text{by (17b)} \\
&= [m^{-1}bm, (m^{-1}a_i\lambda m)_+].
\end{aligned}$$

Substitute (11) to the above equation to get

$$(m^{-1}bm)_{x_i} = [m^{-1}bm, a_i\lambda + u]. \tag{19}$$

Compare coefficient of λ^{-j} of λ^{-j} of (19) to get (18).

It was proved in [11, 13] that $Q_{b,j}$ is a polynomial in $u_i, \partial_{x_i} u_i, \ldots, \partial_{x_i}^{j-1} u_i$, where $u_i = [a_i, v]$. Since $\mathrm{ad}(a_i)$ is a linear isomorphism between $\mathcal{U}_1 \cap \mathcal{A}^\perp$ and $\mathcal{U}_0 \cap \mathcal{U}_{\mathcal{A}}^\perp$, (1) follows. Since $m(\cdots, \infty) = \mathrm{I}$, $Q_{b,0} = b$. Use (18) to prove $Q_{b,1} = [b, v]$.

Use (16) and Proposition 2.1 to see that

$$\Theta_\lambda^{b,j} = \sum_{i=1}^r (a_i\lambda + [a_i, v])dx_i + (b\lambda^j + Q_{b,1}\lambda^{j-1} + \cdots + Q_{b,j})dt$$

is a flat connection on $\mathbb{R}^r \times \mathbb{R}$ for all $\lambda \in \mathbb{C}$. It follows from the recursive formula (18) and the flat equation

$$d\Theta_\lambda^{b,j} + \Theta_\lambda^{b,j} \wedge \Theta_\lambda^{b,j} = 0,$$

that we have

$$\begin{cases} [a_i, v_{x_j}] = [a_j, v_{x_i}] + [[a_i, v], [a_j, v]], & i \neq j, \\ [a_i, v_t] = (Q_{b,j})_x + [[a_i, v], Q_{b,j}], & 1 \leq i \leq r. \end{cases} \tag{20}$$

The first set of equations just means $v(\cdots, t)$ is a solution of the U/U_0-system for each t, and the second set of equations give the flow on the space of solutions of the U/U_0-system.

Let $A_{\mathbb{C}}$ denote the subgroup of G whose Lie subalgebra is $\mathcal{A} \otimes \mathbb{C}$, and $L_+^{\tau,\sigma}(A_{\mathbb{C}})$ is the subgroup of $f \in L_+^{\tau,\sigma}(G)$ such that $g(\lambda) \in A_{\mathbb{C}}$ for all $\lambda \in \mathbb{C}$. Given $b \in \mathcal{A}$ and j a positive integer, then $\xi_{b,j}$ lies in the Lie algebra $\mathcal{L}_+^{\tau,\sigma}(\mathcal{A} \otimes \mathbb{C})$, where $\xi_{b,j}(\lambda) = b\lambda^j$. Let $e_{b,j}(t)$ be the one-parameter subgroup in $L_+^{\tau,\sigma}(G)$ generated by $\xi_{b,j}$, i.e.,

$$e_{b,j}(t)(\lambda) = e^{b\lambda^j t}.$$

It was proved in [13] that if $v(x,t)$ is a solution of (20), then

$$e_{b,j}(t) \cdot v(\cdots, 0) := v(\cdots, t)$$

is the dressing action of $e_{b,j}(t) \in L_+^{\tau,\sigma}(A_{\mathbb{C}})$ on the space of solutions of the U/U_0-systems. The second set of equations of (20) is the vector field on the space of solutions of the U/U_0-system corresponding to the one-parameter subgroup generated by $\xi_{b,j}$. Since the group $L_+^{\tau,\sigma}(A_{\mathbb{C}})$ is Abelian, the flows generated by these $\xi_{b,j}$ are commuting. So, we have

Theorem 5.6 ([12, 13]) *Given $b \in \mathcal{A}$ and a positive integer j, the flow*

$$[a_i, v_t] = (Q_{b,j})_{x_i} + [[a_i, v], Q_{b,j}], \quad 1 \leq i \leq r, \tag{21}$$

leaves the space of solutions of the U/U_0-system (2) *invariant. Moreover, all these flows commute.*

We sketch the method of constructing solutions of the U/U_0-system below. Let $e^{A_0}(x) \in L_+^{\tau,\sigma}(G)$ be defined by,

$$e^{A_0(x)} = \exp\left(\sum_{i=1}^{r} a_i x_i \lambda\right).$$

Theorem 5.7 ([14]) *Given $f \in L_-^{\tau,\sigma}(G)$, factor*

$$f^{-1} e^{A_0(x)} = E(x) m^{-1}(x) \tag{22}$$

with $E(x) \in L_+^{\tau,\sigma}(G)$ and $m(x) \in L_-^{\tau,\sigma}(G)$. Expand $m(x)(\lambda)$ at $\lambda = \infty$:

$$m(x)(\lambda) = e + m_{-1}(x)\lambda^{-1} + m_2(x)\lambda^{-2} + \cdots.$$

Then,

1. $m_{-1}(x) \in \mathcal{U}_1$,

2. *$v = (m_{-1})^{\perp}$ is a solution of the U/U_0-system, where $(m_{-1})^{\perp}$ is the projection of m_{-1} onto $\mathcal{U}_1 \cap \mathcal{A}^{\perp}$ with respect to $\mathcal{U}_1 = \mathcal{A} \oplus (\mathcal{U}_1 \cap \mathcal{A}^{\perp})$.*

Proof. Use the same computation as for the proof of (15a) to conclude that

$$E^{-1}E_{x_i} = (m^{-1}a_i m)_+ = a_i\lambda + Q_{a_1,1}.$$

Expand $m(x)(\lambda)$ at $\lambda = \infty$:

$$m(x)(\lambda) = e + m_{-1}(x)\lambda^{-1} + m_{-2}(x)\lambda^{-2} + \cdots .$$

A direct computation implies that

$$m^{-1}a_i m = a_i + [a_i, m_{-1}]\lambda^{-1} + \cdots .$$

Therefore, $Q_{a_i,1} = [a_i, m_{-1}]$. Since $m \in L^{\tau,\sigma}(G)$, $m_{-1}(x) \in \mathcal{U}_1$. So,

$$[a_i, v] = [a_i, m_{-1}^{\perp}] = [a_i, m_{-1}] = Q_{a_i,1}.$$

Hence we have shown that

$$E^{-1}E_{x_i} = a_i\lambda + [a_i, v], \quad 1 \le i \le r.$$

By Proposition 2.2, v is a solution of the U/U_0-system.

Remark 5.8. It was proved in [14] that if each entry of $f \in L_{-}^{\tau,\sigma}(G)$ is a meromorphic function on $S^2 = \mathbb{C} \cup \{\infty\}$, then the factorization (22) can be carried out explicitly using residue calculus. In particular, $m(x)(\lambda)$ and $E(x,\lambda) = E(x)(\lambda)$ can be given by explicit formulas. Therefore, we get explicit solutions $v = (m_{-1})^{\perp}$ for the U/U_0-system. Since the parallel frame $E(x,\lambda)$ for the solution v is also given explicitly, it follows from Corollary 4.2 and Theorem 2.9 that $F(x) = E(x,1)E(x,0)^{-1}$ is an explicit integral submanifold of the EDS $(U, \mathcal{I}_\sigma)$.

Next we derive conservation laws of the flows for the U/U_0-system.

Theorem 5.9 *Let $c \in \mathcal{A}$, and n a positive integer. Then,*

$$(Q_{c,n}, a_i)_{x_j} = (Q_{c,n}, a_j)_{x_i}, \quad 1 \le i \ne j \le r. \tag{23}$$

In particular,

$$\phi_{c,n} := \sum_{i=1}^{r} (Q_{c,n}, a_i)\, dx_i \tag{24}$$

is a closed 1-form on $\mathbb{R}^r$.

Proof. Compute directly to get

$$\begin{aligned}(m^{-1}cm, a_i)_{x_j} &= ([m^{-1}cm, m^{-1}m_{x_j}], a_i), \text{ by (15c)},\\ &= ([m^{-1}cm, -(m^{-1}a_j\lambda m)_-], a_i), \text{ by (17b)},\\ &= ([m^{-1}cm, (m^{-1}a_j\lambda m)_+], a_i) = ([m^{-1}cm, a_j\lambda + Q_{a_j,1}], a_i).\end{aligned}$$

Use (11) and compare coefficient of λ^{-n} of the above equation to get

$$\begin{aligned}(Q_{c,n}, a_i)_{x_j} &= ([Q_{c,n}, Q_{a_j,1}], a_i) + ([Q_{c,n+1}, a_j], a_i)\\ &= (Q_{c,n}, [Q_{a_j,1}, a_i]) + (Q_{c,n+1}, [a_j, a_i]) = (Q_{c,n}, [[a_j, v], a_i]) + 0\\ &= (Q_{c,n}, [[a_i, v], a_j]) = ([Q_{c,n}, [a_i, v]], a_j), \quad \text{by (18)},\\ &= ((Q_{c,n})_{x_i} - [Q_{n+1,a_i}],\ a_j) = ((Q_{c,n})_{x_i},\ a_j).\end{aligned}$$

If $f : \mathcal{O} \to U$ is a r-dimensional integral submanifold of the EDS $(U, \mathcal{I}_\sigma)$, then by Theorem 2.9 and Corollary 4.2 there exist a special local coordinate system x of $\mathcal{O}$, $g : \mathcal{O} \to U_0$ and a solution v of the U/U_0-system (2) such that $f^{-1} f_{x_i} = g a_i g^{-1}$ and $g^{-1} g_{x_i} = [a_i, v]$ for all $1 \le i \le r$. Let $*$ denote the Hodge star operator for the Euclidean space $\mathbb{R}^r$. Given $1 \le i \ne j \le r$, let

$$\psi^{ij}_{c,n} = \phi_{c,n} \wedge (*(dx_i \wedge dx_j)) = \left(\sum_{\ell=1}^{r} (Q_{c,n}, a_\ell) dx_\ell\right) \wedge (*(dx_i \wedge dx_j)).$$

Then, $\psi^{ij}_{c,n}$ is a closed $(r-1)$-form on the integral submanifold. In other words, $\psi^{ij}_{c,n}$ is a conservation law for the EDS $(U, \mathcal{I}_\sigma)$ for all $1 \le i < j \le r$, $c \in \mathcal{A}$, and positive integer n.

Next we derive the conservation laws for the flow (21) on the space of solutions of the U/U_0-system (2). Given $a, c \in \mathcal{A}$, compute

$$\begin{aligned}(m^{-1}cm, a)_t &= ([m^{-1}cm, m^{-1}m_t], a), \quad \text{by (15d)}\\ &= ([m^{-1}cm, -(m^{-1}b\lambda^j m)_-], a), \quad \text{by (17b)}\\ &= ([m^{-1}cm, (m^{-1}b\lambda^j m)_+], a).\end{aligned}$$

Substitute (11) to the above equation and compare coefficient of λ^{-n} to get

$$((Q_{c,n})_t, a) = \sum_{i=0}^{j-1} ([Q_{c,n+i}, Q_{b,j-i}], a). \tag{25}$$

Here we have used

$$([Q_{c,n}, Q_{b,0}], a) = ([Q_{c,n}, b], a) = (Q_{c,n}, [b, a]) = 0.$$

We claim that

$$([Q_{c,n}, Q_{b,j}], a_i) = \sum_{i=1}^{j} (Q_{c,n+i-1}, Q_{b,j-i})_{x_i} \tag{26}$$

We prove this claim by induction on j. For $j = 1$, we have

$$\begin{aligned}(Q_{c,n}, Q_{b,1}], a_i) &= (Q_{c,n}, [Q_{b,1}, a_i]) = (Q_{c,n}, [[b, v], a_i]),\\ &= (Q_{c,n}, [[a_i, v], b]) = -([[a_i, v], Q_{c,n}], b)\\ &= -([Q_{c,n+1}, a_i] - (Q_{c,n})_{x_i}, b) = ((Q_{c,n})_{x_i}, b).\end{aligned}$$

(We used the Jacobi identity for the first line of the computation above). This proves (26) for $j = 1$. Now assume (26) is true for j and we want to prove the identity for $j + 1$. We compute,

$$\begin{aligned}([Q_{c,n}, Q_{b,j+1}], a_i) &= (Q_{c,n}, [Q_{b,j+1}, a_i]), \quad \text{by (18)}\\ &= (Q_{c,n}, (Q_{b,j})_{x_i} + [u_i, Q_{b,j}])\\ &= (Q_{c,n}, Q_{b,j})_{x_i} - ((Q_{c,n})_{x_i}, Q_{b,j}) + (Q_{c,n}, [u_i, Q_{b,j}])\\ &= (Q_{c,n}, Q_{b,j})_{x_i} - ((Q_{c,n})_{x_i}, Q_{b,j}) - ([u_i, Q_{c,n}], Q_{b,j})\\ &= (Q_{c,n}, Q_{b,j})_{x_i} + ([a, Q_{c,n+1}], Q_{b,j}), \quad \text{by (18)},\\ &= (Q_{c,n}, Q_{b,j})_{x_i} + (a, [Q_{c,n+1}, Q_{b,j}])\end{aligned}$$

Then the induction hypothesis implies (26) is true for $j + 1$.

It follows from (25) and (26) that we have:

Theorem 5.10 *Let $v : \mathbb{R}^r \times \mathbb{R} \to \mathcal{U}_1 \cap \mathcal{A}^\perp$ be a solution of* (20)*, $c \in \mathcal{A}$, and n a positive integer. Then,*

$$(Q_{c,n}, a_i)_t = \sum_{\ell=0}^{j-1} \sum_{s=1}^{j-\ell} (Q_{c,n+\ell+s-1}, Q_{b,j-\ell-s})_{x_i}. \tag{27}$$

As a consequence, we see that

$$\int_{\mathbb{R}^r} (Q_{c,n}, a_i) dx_1 \wedge \cdots \wedge dx_r$$

is a conserved quantity for the flow (21) on the space of rapidly decaying solutions of the U/U_0-system.

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Symmetric Submanifolds of Riemannian Symmetric Spaces and Symmetric R-spaces

Kazumi Tsukada

Ochanomizu University, Department of Mathematics, Tokyo 112-8610, Japan
tsukada@math.ocha.ac.jp

Dedicated to Professor Lieven Vanhecke

Summary. Symmetric submanifolds are defined analogously to Riemannian symmetric spaces in the theory of Riemannian submanifolds. This notion was introduced by D. Ferus ([2], 1980) firstly for a submanifold of a Euclidean space and can be easily extended to a submanifold of a general Riemannian manifold. One of the main problems is to classify symmetric submanifolds of Riemannian symmetric spaces. This problem has been studied by several mathematicians, and for Euclidean spaces and rank 1 symmetric spaces, complete and beautiful classifications of symmetric submanifolds have been given. In a recent joint work [1] J. Berndt et al. study symmetric submanifolds in irreducible Riemannian symmetric spaces of non-compact type and rank greater than one. This finishes the above classification problem completely. In this expository note, I would like to explain the similarity between the theories of Riemannian symmetric spaces and symmetric submanifolds, the ideas of classification in the framework of Grassmann geometry and our recent results.

1 Symmetric submanifolds and parallel submanifolds

Symmetric submanifolds are defined analogously to Riemannian symmetric spaces. Riemannian symmetric spaces admit an intrinsic symmetry at each point, whereas symmetric submanifolds admit an extrinsic symmetry at each point.

Definition 1.1. A (regular) connected submanifold M of a connected Riemannian manifold $\bar{M}$ is called *symmetric*, if at each point p in M there exists an involutive isometry t_p of $\bar{M}$ satisfying the following:

$$t_p(p) = p, t_p(M) = M,$$
$$(t_p)_*X = -X \quad \text{for all } X \in T_pM, \quad (t_p)_*\xi = \xi \quad \text{for all } \xi \in T_p^{\perp}M,$$

where T_pM and $T_p^{\perp}M$ denote the tangent space and the normal space of M at p respectively.

The isometry t_p is called the *extrinsic symmetry* of M at p. For the convenience of our theory, we also define the notion of symmetric immersions.

Definition 1.2. An isometric immersion $f : M \to \bar{M}$ of a connected Riemannian manifold M into a connected Riemannian manifold $\bar{M}$ is called *symmetric* if at each point $p \in M$, there exist an isometry s_p of M and an isometry t_p of $\bar{M}$ which satisfy the following:

$$s_p(p) = p, t_p \circ f = f \circ s_p, \text{(and then } t_p(f(p)) = f(p))$$
$$(t_p)_* f_* X = -f_* X \quad \text{for all } X \in T_pM, \quad (t_p)_*\xi = \xi \quad \text{for all } \xi \in T_p^{\perp}M.$$

A locally symmetric submanifold and a locally symmetric immersion are defined similarly as the local version. The inclusion map of a symmetric submanifold (resp. a locally symmetric submanifold) is a symmetric immersion (resp. a locally symmetric immersion). By the definition above, we see that a symmetric submanifold (or a Riemannian manifold which admits a symmetric immersion) is a Riemannian symmetric space and that a locally symmetric submanifold (or a Riemannian manifold which admits a locally symmetric immersion) is a locally Riemannian symmetric space.

The following two facts are fundamental in the theory of Riemannian symmetric spaces:

(A) A Riemannian manifold M is locally symmetric if and only if the covariant derivative of the curvature tensor R vanishes, namely, $\nabla R = 0$.
(B) A locally Riemannian symmetric space is determined by the curvature tensor R at one point.

We discuss similar properties to them.

Proposition 1.3. *Let $f : M \to \bar{M}$ be a locally symmetric immersion. Then, the covariant derivative of the second fundamental form α vanishes, i.e., $\bar{\nabla}\alpha = 0$. At each point p of M the subspaces f_*T_pM and $T_p^{\perp}M$ of $T_{f(p)}\bar{M}$ are curvature invariant. That is,*

$$\bar{R}(f_*T_pM, f_*T_pM)f_*T_pM \subset f_*T_pM \quad \text{and} \quad \bar{R}(T_p^{\perp}M, T_p^{\perp}M)T_p^{\perp}M \subset T_p^{\perp}M,$$

where $\bar{R}$ denotes the curvature tensor of $\bar{M}$.

The properties above easily follow from the fact, that for each point $p \in M$ $\bar{\nabla}\alpha$ and $\bar{R}$ are preserved by the differential t_{p*p} of the extrinsic symmetry t_p. A submanifold (resp. an isometric immersion) with parallel second fundamental form is called a *parallel submanifold* (resp. *parallel immersion*).

A parallel submanifold is determined by its second fundamental form at one point. This fact corresponds to the property (B) of Riemannian symmetric spaces.

Theorem 1.4. (cf. Naitoh [9]) *Let M_1 and M_2 be a simply connected complete Riemannian manifold and a complete Riemannian manifold respectively. Let $f_i : M_i \to \bar{M}$ $(i = 1, 2)$ be parallel immersions of M_i into $\bar{M}$ and α_i $(i = 1, 2)$ denote the second fundamental form of f_i. Assume that there exist points $p_1 \in M_1$, $p_2 \in M_2$ and a linear isometry $\phi : T_{p_1}M_1 \to T_{p_2}M_2$ which satisfy*

$$f_1(p_1) = f_2(p_2), \quad f_{2*p_2} \circ \phi = f_{1*p_1}, \quad \alpha_2(\phi X, \phi Y) = \alpha_1(X, Y),$$

$X, Y \in T_{p_1}M_1$. *Then there exists a Riemannian covering map* $\Phi : M_1 \to M_2$ *of* M_1 *onto* M_2, *which satisfies*

$$\Phi(p_1) = p_2, \quad \Phi_{*p_1} = \phi, \quad f_2 \circ \Phi = f_1.$$

If the ambient space $\bar{M}$ is locally symmetric, then the converse of Proposition 1.3 holds.

Theorem 1.5. (W. Strübing [16], Naitoh [9]) *Let* $\bar{M}$ *be a locally Riemannian symmetric space and* $f : M \to \bar{M}$ *an isometric immersion. Suppose that the second fundamental form* α *is parallel and that at each point* p *of* M $T_p^{\perp}M$ *is a curvature invariant subspace of* $T_{f(p)}\bar{M}$. *Then* $f : M \to \bar{M}$ *is a locally symmetric immersion.*

Remark 1. Since $\bar{\nabla}\alpha = 0$, by the equation of Codazzi f_*T_pM is curvature invariant. We assume that at a point p of a locally symmetric space $\bar{M}$, both a subspace V and its orthogonal complement $V^{\perp}$ in the tangent space $T_p\bar{M}$ are curvature invariant. We define a linear isometry λ of $T_p\bar{M}$ by $\lambda(X) = -X$ for $X \in V$ and $\lambda(\xi) = \xi$ for $\xi \in V^{\perp}$. Then the curvature tensor $\bar{R}$ is invariant by λ, namely $\lambda(\bar{R}(X, Y)Z) = \bar{R}(\lambda X, \lambda Y)\lambda Z$ for $X, Y, Z \in T_p\bar{M}$, and hence, there exists a local isometry t_p on a neighborhood of p which satisfies $t_p(p) = p$ and $t_{p*p} = \lambda$.

Remark 2. Proofs of Theorems 1.4 and 1.5 are essentially due to the property of geodesics of parallel submanifolds which was discovered by Strübing [16].

We recall another fact for Riemannian symmetric spaces.

(C) A Riemannian symmetric space is a homogeneous Riemannian manifold.

Compared with this property, we see that a symmetric submanifold M of a Riemannian manifold $\bar{M}$ is an equivariantly homogeneous submanifold in the following sense. We denote by $I(\bar{M})$ the isometry group of $\bar{M}$ and by $I^o(\bar{M})$ its identity component. Let G_M be the subgroup of $I(\bar{M})$ which is generated by all extrinsic symmetries t_p, $p \in M$. Then, $G_M^o = I^o(\bar{M}) \cap G_M$ acts transitively on M. For the detailed argument, we refer to [10].

2 Grassmann geometry and classification of symmetric submanifolds

At first, we recall the curvature property of the tangent spaces and the normal spaces of symmetric submanifolds. Let M be a symmetric submanifold of a Riemannian symmetric space $\bar{M}$. For each point $p \in M$, both tangent space T_pM and the normal space $T_p^{\perp}M$ are invariant under the curvature tensor $\bar{R}$ of $\bar{M}$. Therefore, there exist unique totally geodesic submanifolds N and N^* through p which are tangent to T_pM and $T_p^{\perp}M$ at p respectively. By Theorem 1.5, N is a symmetric submanifold of $\bar{M}$ (N^* is also a symmetric submanifold). We call $(\bar{M}, N)$ a totally geodesic symmetric submanifold associated with $(\bar{M}, M)$.

We recall Grassmann geometry introduced by R. Harvey and H.B. Lawson [3]. Let $\bar{M}$ be a Riemannian manifold and $Gr_m(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ of all m-dimensional linear subspaces of the tangent spaces of $\bar{M}$. We take a subset $\mathcal{S}$ of $Gr_m(T\bar{M})$. Then, an m-dimensional submanifold M of $\bar{M}$ is called an *$\mathcal{S}$-submanifold* if all the tangent spaces T_pM of M belong to the subset $\mathcal{S}$, and the collection of such $\mathcal{S}$-submanifolds is called the *$\mathcal{S}$-geometry*. Grassmann geometry is a general name for such $\mathcal{S}$-geometries. If a connected Lie group G acts isometrically on a Riemannian manifold $\bar{M}$, it also acts naturally on the Grassmann bundle $Gr_m(T\bar{M})$. Then, we may take a G-orbit $\mathcal{O}$ as a subset $\mathcal{S}$ in $Gr_m(T\bar{M})$ with respect to this action. Such a Grassmann geometry is called *an orbit type*. Now, we suppose that $\bar{M}$ is a simply connected, semi-simple Riemannian symmetric space and G is the identity component $I^o(\bar{M})$ of the isometry group of $\bar{M}$. For a totally geodesic symmetric submanifold N of $\bar{M}$, we consider the G-orbit $\mathcal{O}$ containing the tangent space T_pN ($\in Gr_m(T\bar{M})$). Then, any symmetric submanifold M of $\bar{M}$ tangent to N is an $\mathcal{O}$-submanifold, because of its equivariance and hence it belongs to the $\mathcal{O}$-geometry.

From the viewpoint of Grassmann geometry, the program of the classification of symmetric submanifolds in Riemannian symmetric spaces is the following:

(1) the decomposition theorem which reduces the problem to the irreducible ones;
(2) the classification of symmetric submanifolds of Euclidean spaces;
(3) the classification of totally geodesic, symmetric submanifolds $(\bar{M}, N)$ of simply connected semi-simple Riemannian symmetric spaces $\bar{M}$, in particular, the classification of irreducible ones;
(4) in the $\mathcal{O}$-geometries which are associated with $(\bar{M}, N)$ classified in (3), the classification of such $\mathcal{O}$-geometries which contain non-totally geodesic, symmetric submanifolds;
(4)′ in the $\mathcal{O}$-geometries which are associated with $(\bar{M}, N)$ classified in (3), the classification of such $\mathcal{O}$-geometries which contain non-totally geodesic $\mathcal{O}$-submanifolds;
(5) the classification of non-totally geodesic symmetric submanifolds belonging to $\mathcal{O}$-geometries which are classified in (4).

The case (2) is the classification due to Ferus [2]. The cases (1), (3) and (4)′ have been settled by Naitoh in a series of papers [11–14]. We comment on case (3). A totally geodesic symmetric submanifold M of a Riemannian symmetric space $\bar{M}$ is characterized as a connected submanifold of $\bar{M}$, such that the geodesic reflection of $\bar{M}$ in M is an isometry, in which case M is called a *reflective* submanifold. The reflective submanifolds of Riemannian symmetric spaces were classified by Leung [6, 7].

For the case (4)′, Naitoh obtained the following remarkable result.

Theorem 2.1. *Let $\mathcal{O}$ be the G-orbit defined from an irreducible, totally geodesic symmetric submanifold $(\bar{M}, N)$, where $\bar{M}$ is a simply connected, semi-simple Riemannian symmetric space. All $\mathcal{O}$-geometries except the following ones have only totally geodesic submanifolds:*

(1) *the geometry of k-dimensional $(0 < k < n)$ submanifolds of the sphere S^n resp. of the real hyperbolic space $\mathbb{R}H^n$ $(n \geq 2)$;*
(2) *the geometry of k-dimensional $(0 < k < n)$ complex submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*

(3) *the geometry of n-dimensional totally real submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*
(4) *the geometry of $2n$-dimensional totally complex submanifolds of the quaternionic projective space $\mathbb{H}P^n$ resp. of the quaternionic hyperbolic space $\mathbb{H}H^n$ $(n \geq 2)$;*
(5) *the geometries associated with irreducible symmetric R-spaces and their non-compact dual geometries.*

Remark. Naitoh proved Theorem 2.1 for simply connected irreducible Riemannian symmetric spaces $\bar{M}$ of compact type. However, it is easy to see that the proof also holds for the non-compact case.

The symmetric submanifolds belonging to the geometries of type (1)–(4) in Theorem 2.1 were classified by several authors, we refer to [15] and [18] for further details. The symmetric submanifolds belonging to the geometries of type (5) were classified by Naitoh in [10] for the compact case and recently by J. Berndt et al. [1] for the non-compact case. We explain our work in the next section.

3 Symmetric submanifolds associated with symmetric R-spaces

For details in this section, we refer to [1]. We will construct a one-parameter family of symmetric submanifolds in irreducible Riemannian symmetric spaces of non-compact type associated with symmetric R-spaces. First we show the typical examples – a one-parameter family of totally umbilical hypersurfaces of a real hyperboplic space. We start with the totally geodesic hypersurface of a real hyperbolic space. It bends slightly. Then, it is a totally umbilical hypersurface which is homothetic to the totally geodesic hypersurface. It bends more and more. Then it yields a so-called *horosphere*. It is a flat submanifold. After the horosphere, we have a totally umbilical sphere. It is a compact dual of a real hyperbolic space. Our construction can be viewed as a generalization of this family of totally umbilical hypersurfaces.

We recall the theory of symmetric R-spaces, for details we refer to Kobayashi and Nagano [5], Nagano [8] and Takeuchi [17]. Let $(\bar{\mathfrak{g}}, \sigma)$ be a positive definite symmetric graded Lie algebra, that is, $\bar{\mathfrak{g}}$ is a real semi-simple Lie algebra with a Cartan involution σ satisfying the following properties:

(1) $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1$ (vector space direct sum) and $[\bar{\mathfrak{g}}_p, \bar{\mathfrak{g}}_q] \subset \bar{\mathfrak{g}}_{p+q}$ $(p, q \in \{0, \pm 1\})$;
(2) $\sigma(\bar{\mathfrak{g}}_p) = \bar{\mathfrak{g}}_{-p}$ $(p \in \{0, \pm 1\})$;
(3) $\bar{\mathfrak{g}}_{-1} \neq \{0\}$, and the adjoint action of $\bar{\mathfrak{g}}_0$ on the vector space $\bar{\mathfrak{g}}_{-1}$ is effective.

For the classification of the positive definite symmetric graded Lie algebras, see [5, 17], and the table at the end of this paper. We define a linear isomorphism τ of $\bar{\mathfrak{g}}$ by $\tau(X) = (-1)^p X$ for $X \in \bar{\mathfrak{g}}_p$. Then, τ is an involutive automorphism of $\bar{\mathfrak{g}}$ with $\sigma\tau = \tau\sigma$. Let $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$ be the Cartan decomposition induced by σ. Then we have $\tau(\bar{\mathfrak{k}}) = \bar{\mathfrak{k}}$ and $\tau(\bar{\mathfrak{p}}) = \bar{\mathfrak{p}}$. Let $\bar{\mathfrak{k}} = \mathfrak{k}_+ + \mathfrak{k}_-$ and $\bar{\mathfrak{p}} = \mathfrak{p}_+ + \mathfrak{p}_-$ be the ± 1-eigenspace decompositions of $\bar{\mathfrak{k}}$ and $\bar{\mathfrak{p}}$ with respect to τ.

Since $\bar{\mathfrak{g}}$ is semi-simple, there exists a unique element $\nu \in \bar{\mathfrak{g}}_0$ such that

$$\bar{\mathfrak{g}}_p = \{X \in \bar{\mathfrak{g}} \mid \mathrm{ad}(\nu)X = pX\} \text{ for all } p \in \{0, \pm 1\}.$$

It is easy to see that $\nu \in \bar{\mathfrak{p}}$, and hence $\nu \in \mathfrak{p}_+$.

The restriction of the Killing form B of $\bar{\mathfrak{g}}$ to $\bar{\mathfrak{p}} \times \bar{\mathfrak{p}}$ is a positive definite inner product on $\bar{\mathfrak{p}}$, which will be denoted by $\langle \cdot, \cdot \rangle$. This inner product is invariant under the adjoint action of $\bar{\mathfrak{k}}$ on $\bar{\mathfrak{p}}$ and under the involution $\tau|\bar{\mathfrak{p}}$. In particular, $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are perpendicular to each other. Let $\bar{G}$ be the simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$ and $\bar{K}$ be the connected Lie subgroup of $\bar{G}$ with Lie algebra $\bar{\mathfrak{k}}$, and define the homogeneous space $\bar{M} = \bar{G}/\bar{K}$. Let $\bar{\pi} : \bar{G} \to \bar{M}$ be the natural projection and put $o = \bar{\pi}(e)$, where e is the identity of $\bar{G}$. The restriction to $\bar{\mathfrak{p}}$ of the differential $\bar{\pi}_{*e} : \bar{\mathfrak{g}} \to T_o\bar{M}$ of $\bar{\pi}$ at e yields a linear isomorphism from $\bar{\mathfrak{p}}$ onto $T_o\bar{M}$. In the following, we will always identify $\bar{\mathfrak{p}}$ and $T_o\bar{M}$ via this isomorphism. From the $\mathrm{Ad}(\bar{K})$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\bar{\mathfrak{p}} \cong T_o\bar{M}$, we get a $\bar{G}$-invariant Riemannian metric on $\bar{M}$. Then $\bar{M} = \bar{G}/\bar{K}$ is the simply connected Riemannian symmetric space of non-compact type associated with $(\bar{\mathfrak{g}}, \sigma, \langle \cdot, \cdot \rangle)$.

We put

$$K'_+ = \{k \in \bar{K} \mid \mathrm{Ad}(k)\nu = \nu\}.$$

Then K'_+ is a closed Lie subgroup whose Lie algebra is $\mathfrak{k}_+$. The homogeneous space $M' = \bar{K}/K'_+$ is diffeomorphic to the orbits $\mathrm{Ad}(\bar{K}) \cdot \nu \subset \bar{\mathfrak{p}}$ and $\bar{K} \cdot \bar{\pi}(\exp \nu) \subset \bar{M}$, where $\exp : \bar{\mathfrak{g}} \to \bar{G}$ denotes the Lie exponential map from $\bar{\mathfrak{g}}$ into $\bar{G}$. We equip M' with the induced Riemannian metric from $\bar{M}$. Then, M' is a compact Riemannian symmetric space associated with the orthogonal symmetric Lie algebra $(\bar{\mathfrak{k}}, \tau|\bar{\mathfrak{k}})$, where $\tau|\bar{\mathfrak{k}}$ is the restriction of τ to $\bar{\mathfrak{k}}$. The symmetric spaces M' arising in this manner are precisely the *symmetric R-spaces*. If $\bar{\mathfrak{g}}$ is simple, then M' is called an *irreducible symmetric R-space*. Symmetric R-spaces form a class of compact Riemannian symmetric spaces with remarkable properties.

The subspace $\mathfrak{p}_-$ is a Lie triple system in $\bar{\mathfrak{p}} = T_o\bar{M}$ and $[\mathfrak{p}_-, \mathfrak{p}_-] \subset \mathfrak{k}_+$. Thus, there exists a connected complete totally geodesic submanifold M of $\bar{M}$ with $o \in M$ and $T_oM = \mathfrak{p}_-$. Moreover, since $T_o^{\perp}M = \mathfrak{p}_+$ is also a Lie triple system, by Theorem 1.5 (global version) M is a symmetric submanifold. Since M is the image of $\mathfrak{p}_-$ under the exponential map of $\bar{M}$ at o, we see that M is simply connected. We define a subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ by $\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{p}_-$ and denote by G the connected Lie subgroup of $\bar{G}$ with Lie algebra $\mathfrak{g}$. Then, by construction, M is the G-orbit through o. The Lie algebra of the isotropy subgroup K_+ of this action at o is just $\mathfrak{k}_+$.The restriction $\tau|\mathfrak{g}$ of τ to $\mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$ and $(\mathfrak{g}, \tau|\mathfrak{g})$ is the orthogonal symmetric Lie algebra dual to $(\bar{\mathfrak{k}}, \tau|\bar{\mathfrak{k}})$. Moreover, M is the Riemannian symmetric space of non-compact type associated with $(\mathfrak{g}, \tau|\mathfrak{g})$.

We now introduce an $\mathcal{O}$-geometry on $\bar{M}$. We put $\dim \mathfrak{p}_- = m$ and denote by $\mathcal{O}$ the orbit through $\mathfrak{p}_-$ under the action of $\bar{G}$ on $Gr_m(T\bar{M})$. This $\mathcal{O}$-geometry is a geometry of type (5) in Theorem 2.1 for the non-compact case.

We will now construct a one-parameter family of symmetric submanifolds of $\bar{M}$ consisting of $\mathcal{O}$-submanifolds and containing the totally geodesic submanifold M and the symmetric R-space M'. For each $c \in \mathbb{R}$, we define a linear subspace $\mathfrak{p}_c$ of $\mathfrak{p}_- + \mathfrak{k}_- = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1$ by

$$\mathfrak{p}_c = \{X + c\,\mathrm{ad}(\nu)X \mid X \in \mathfrak{p}_-\}.$$

We note that the adjoint transformation $\mathrm{ad}(\nu)$ restricted to $\mathfrak{p}_-$ is an isomorphism from $\mathfrak{p}_-$ onto $\mathfrak{k}_-$. We put $\mathfrak{g}_c = \mathfrak{k}_+ + \mathfrak{p}_c$. Then, $\mathfrak{g}_c$ is a subalgebra of $\bar{\mathfrak{g}}$. $\mathfrak{g}_c$ is invariant under τ and $(\mathfrak{g}_c, \tau|\mathfrak{g}_c)$ is an orthogonal symmetric Lie algebra. We denote by G_c the connected Lie subgroup of $\bar{G}$ with Lie algebra $\mathfrak{g}_c$ and by M_c the G_c-orbit through o in $\bar{M}$.

Proposition 3.1. *For each $c \in \mathbb{R}$, M_c is a symmetric submanifold belonging to the $\mathcal{O}$-geometry of $\bar{M}$.*

The submanifolds M_c and M_{-c} are congruent via the geodesic symmetry s_o of $\bar{M}$ at o. Since $\mathfrak{g}_0 = \mathfrak{k}_+ + \mathfrak{p}_-$, M_0 coincides with the totally geodesic submanifold of $\bar{M}$. We explain the geometric properties of the submanifolds M_c $(c \geq 0)$ in more detail.

Theorem 3.2. *The submanifolds M_c, $0 \leq c < 1$, form a family of non-compact symmetric submanifolds which are homothetic to the totally geodesic submanifold M. The submanifolds M_c, $1 < c < \infty$, form a family of compact symmetric submanifolds which are homothetic to the symmetric R-space M'. The submanifold M_1 is a flat symmetric space which is isometric to a Euclidean space. The second fundamental form α_c of M_c is given by*

$$\alpha_c(X, Y) = c[\mathrm{ad}(\nu)X, Y] \in \mathfrak{p}_+ = T_o^{\perp}M_c\ ,\ \ X, Y \in \mathfrak{p}_- = T_oM_c.$$

In particular, all submanifolds M_c, $0 \leq c < \infty$, are pairwise non-congruent.

Remark. In the case of Table, No. 13, $i = 1$, $\bar{M}$ is a real hyperbolic space $\mathbb{R}H^n$, and the family of symmetric submanifolds M_c constructed as above is the family of complete totally umbilical hypersurfaces.

Next we explain the classification result. Let $\bar{M}$ be a simply connected Riemannian symmetric space of noncompact type introduced in this section. We consider the $\mathcal{O}$-geometry defined by the totally geodesic symmetric submanifold M of $\bar{M}$. Our main result is the following.

Theorem 3.3. *Let $\bar{M}$ be an irreducible Riemannian symmetric space as in the Table, except No. 13, for $i = 1$. Then, an $\mathcal{O}$-submanifold of $\bar{M}$ is locally congruent to some M_c constructed in this section.*

For the proof of this Theorem, see [1]. Applying this result, we obtain the classification of symmetric submanifolds.

Theorem 3.4. *Let $\bar{M}$ be an irreducible Riemannian symmetric space as in Theorem 3.3. Then, every symmetric submanifold M of $\bar{M}$ which belongs to the $\mathcal{O}$-geometry is congruent to some M_c as constructed in this section.*

This table is a modification of Table II in [10]. The notation for real semi-simple Lie algebras is as in [4].
$(\bar{\mathfrak{g}}, \sigma)$ $\bar{M}$: an irreducible Riemannian symmetric space $\bar{M}$ associated with a positive definite symmetric graded Lie algebra $(\bar{\mathfrak{g}}, \sigma)$,
$\mathfrak{p}_-$ (M): a totally geodesic submanifold M tangent to $\mathfrak{p}_-$,
$\mathfrak{k}_-$ (M'): a symmetric R-space.

Table 1.

No.	$(\bar{\mathfrak{g}}, \sigma)$ $\bar{M}$	$\mathfrak{p}_-$ (M)	$\mathfrak{k}_-$ (M')
1	$\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$\mathfrak{su}(i,n-i)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$	$\mathfrak{su}(n)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$
2	$\mathfrak{so}(2n;\mathbb{C})/\mathfrak{so}(2n)$	$\mathfrak{so}^*(2n)/\mathfrak{u}(n)$	$\mathfrak{so}(2n)/\mathfrak{u}(n)$
3	$\mathfrak{so}(n;\mathbb{C})/\mathfrak{so}(n)$	$\mathfrak{so}(n-2,2)/\mathfrak{so}(n-2)+\mathbb{T}$	$\mathfrak{so}(n)/\mathfrak{so}(n-2)+\mathbb{T}$
4	$\mathfrak{sp}(n;\mathbb{C})/\mathfrak{sp}(n)$	$\mathfrak{sp}(n;\mathbb{R})/\mathfrak{u}(n)$	$\mathfrak{sp}(n)/\mathfrak{u}(n)$
5	$E_6^{\mathbb{C}}/E_6$	$E_6^{-14}/\mathfrak{so}(10)+\mathbb{T}$	$E_6/\mathfrak{so}(10)+\mathbb{T}$
6	$E_7^{\mathbb{C}}/E_7$	$E_7^{-25}/E_6+\mathbb{T}$	$E_7/E_6+\mathbb{T}$
7	$\mathfrak{su}(n,n)/\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(n))$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$\mathbb{T}+\mathfrak{su}(n)+\mathfrak{su}(n)/\mathfrak{su}(n)$
8	$\mathfrak{so}^*(4n)/\mathfrak{u}(2n)$	$\mathbb{R}+\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$\mathbb{T}+\mathfrak{su}(2n)/\mathfrak{sp}(n)$
9	$\mathfrak{sp}(n;\mathbb{R})/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$\mathbb{T}+\mathfrak{su}(n)/\mathfrak{so}(n)$
10	$E_7^{-25}/E_6+\mathbb{T}$	$\mathbb{R}+E_6^{-26}/F_4$	$\mathbb{T}+E_6/F_4$
11	$\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$\mathfrak{so}(i,n-i)/\mathfrak{so}(i)+\mathfrak{so}(n-i)$	$\mathfrak{so}(n)/\mathfrak{so}(i)+\mathfrak{so}(n-i)$
12	$\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$\mathfrak{sp}(i,n-i)/\mathfrak{sp}(i)+\mathfrak{sp}(n-i)$	$\mathfrak{sp}(n)/\mathfrak{sp}(i)+\mathfrak{sp}(n-i)$
13	$\mathfrak{so}(i,n-i)/$ $\mathfrak{so}(i)+\mathfrak{so}(n-i)$	$\mathfrak{so}(i-1,1)/\mathfrak{so}(i-1)$ $+\mathfrak{so}(n-i-1,1)/\mathfrak{so}(n-i-1)$	$\mathfrak{so}(i)/\mathfrak{so}(i-1)$ $+\mathfrak{so}(n-i)/\mathfrak{so}(n-i-1)$
14	$\mathfrak{so}(n,n)/\mathfrak{so}(n)+\mathfrak{so}(n)$	$\mathfrak{so}(n;\mathbb{C})/\mathfrak{so}(n)$	$\mathfrak{so}(n)+\mathfrak{so}(n)/\mathfrak{so}(n)$
15	$\mathfrak{sp}(n,n)/\mathfrak{sp}(n)+\mathfrak{sp}(n)$	$\mathfrak{sp}(n;\mathbb{C})/\mathfrak{sp}(n)$	$\mathfrak{sp}(n)+\mathfrak{sp}(n)/\mathfrak{sp}(n)$
16	$E_6^6/\mathfrak{sp}(4)$	$\mathfrak{sp}(2,2)/\mathfrak{sp}(2)+\mathfrak{sp}(2)$	$\mathfrak{sp}(4)/\mathfrak{sp}(2)+\mathfrak{sp}(2)$
17	E_6^{-26}/F_4	$F_4^{-20}/\mathfrak{so}(9)$	$F_4/\mathfrak{so}(9)$
18	$E_7^7/\mathfrak{su}(8)$	$\mathfrak{su}^*(8)/\mathfrak{sp}(4)$	$\mathfrak{su}(8)/\mathfrak{sp}(4)$

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Complex Forms of Quaternionic Symmetric Spaces*

Joseph A. Wolf

Department of Mathematics, University of California, Berkeley CA 94720–3840
jawolf@math.berkeley.edu

Dedicated to Professor Lieven Vanhecke

Summary. We give a complete classification of the complex forms of quaternionic symmetric spaces.

1 Introduction

Some years ago, H. A. Jaffee found the real forms of Hermitian symmetric spaces ([J1], [J2]; or see [HÓ]). That classification turns out to be related to the classification of causal symmetric spaces. This was first observed by I. Satake ([S, Remark 2 on page 30] and [S, Remark on page 87]). Somewhat later, it was independently observed by J. Hilgert, G. Ólafsson and B. Ørsted; see [HÓ], especially Chapter 3 and the Notes at the end of that Chapter. I learned about that from Bent Ørsted. He and Gestur Ólafsson had informally discussed complex forms of quaternionic symmetric spaces and found examples for the classical groups, for G_2 , and perhaps for F_4. Ørsted told me about the classical ones, and we rediscovered examples for G_2 and F_4. I thank Bent Ørsted for agreeing to my incorporating those examples into this note. Later I used the computer program LiE [L] to find examples for E_6, E_7 and E_8.

In this note, I write down a complete classification for complex forms L/V of quaternionic symmetric spaces G/K. The definitions and some preliminary results are in Sections 2 and 3, the main results are stated in Section 4, and the proofs are in Sections 5, 6, 7 and 8. The case where G is a classical group and $\mathrm{rank}(L) = \mathrm{rank}(G)$ is handled, essentially by matrix considerations, in Section 5. That, of course, does not work comfortably for the exceptional groups, which must be approached by means of their root structure. The tool for this is a script for the use of the computer program LiE; it is described in Section 6 along with some examples of its application. Those examples have the interesting property that the complexifications $L_{\mathbb{C}}$ and $K_{\mathbb{C}}$ are conjugate in $G_{\mathbb{C}}$. They cover the delicate cases for G exceptional and $\mathrm{rank}(L) = \mathrm{rank}(G)$, and the

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remaining exceptional equal rank cases are settled in Section 7. Finally, the few cases of $\text{rank}(L) < \text{rank}(G)$ are worked out in Section 8.

A possible extension of the theory is mentioned in Section 9.

After this paper was written, I learned that quite a lot was published on totally complex submanifolds of quaternionic symmetric spaces from the viewpoint of differential geometry. See, for example, [ADM], [AM1], [AM2], [F], [JKS], [L1], [L2], [Ma], [Mo], [Ts] and [X], but especially the first three. I also learned that M. Takeuchi [Ta] had studied the maximal totally complex submanifolds of quaternionic symmetric spaces, reducing their classification to that of certain Satake diagrams and writing out the classification in the classical group cases. *A priori* that is not quite the same as the classification of complex forms of quaternionic symmetric spaces, but it is very close. On the other hand, it seems to me that the method given here is more efficient and more direct, and more explicit in the exceptional group cases. I thank Dmitry Alekseevsky for calling the above-cited papers to my attention.

2 Quaternionic symmetric spaces

We recall the structure of quaternionic symmetric spaces [W]. A quaternionic structure on a connected Riemannian manifold M is a parallel field A of quaternion algebras A_x on the real tangent spaces $T_x(M)$, such that every unimodular element of every A_x is an orthogonal linear transformation. Thus, A gives every tangent space the structure of quaternionic vector space, such that the Riemannian metric at x is Hermitian relative to the elements of A_x of square $-I$. If $n = \dim M$, then a quaternionic structure is the same as a reduction of the structure group of the tangent bundle from $O(n)$ to $Sp(n/4) \cdot Sp(1)$. Let K_x denote the holonomy group of M at x (we will see in a minute that this is appropriate notation for symmetric spaces with no Euclidean factor). Suppose that M is simply connected, so that the K_x are connected. Let $A = \{A_x\}$ be a quaternionic structure on M. Then A_x is stable under the action of K_x, so $K_x \cap A_x$ is a closed normal subgroup of K_x. Now, $K_x = K_x^{lin} \cdot K_x^{sca}$, where K_x^{lin} is the quaternion-linear part, centralizer of A_x in K_x, and $K_x^{sca} = K_x \cap A_x$ is the scalar part. We say that K_x has real scalar part if K_x^{sca} consists of real scalars, i.e., K_x^{sca} is $\{1\}$ or $\{\pm 1\}$. We say that K_x has complex scalar part if K_x^{sca} is contained in a complex subfield of A_x but not in the real subfield, and we say that K_x has quaternion scalar part if K_x^{sca} is not contained in a complex subfield of A_x. A Riemannian 4-manifold M with holonomy $U(2)$ has a dual role: it has a quaternionic structure A_1 generated by the $SU(2)$-factor in the holonomy; that has quaternionic scalar part, the same $SU(2)$, M; it has a second quaternionic structure A_2 where $A_{2,x}$ is the centralizer of $A_{1,x}$ in the algebra of $\mathbb{R}$-linear transformations of $T_x(M)$; it has complex scalar part, generated by the circle center of the holonomy $U(2)$. Thus, we have an interesting dual picture. The holonomy of M has quaternionic scalar part for A_1 and has complex scalar part for A_2.

Proposition 2.1. *The connected simply connected Riemannian symmetric spaces with quaternionic structure are the following.*

(i) *The Euclidean spaces of dimension divisible by* 4. *Here, the holonomy has real scalar part.*

(ii) *Products* $M = M_1 \times \cdots \times M_\ell$*, where each* M_i *is* (a) *the complex projective or hyperbolic plane with the quaternionic structure of complex scalar part, or* (b) *a product* $M_i' \times M_i''$ *where each factor is a complex projective line and a complex hyperbolic line. Here* $M = G/K$*,* K *is the holonomy, and the holonomy has complex scalar part.*

(iii) *Irreducible connected simply connected Riemannian symmetric spaces* $M = G/K$*, where* K *has an* $Sp(1)$ *factor that generates quaternion algebras on the tangent spaces of* M*. Here* K *is the holonomy, and the holonomy has quaternion scalar part.*

There is a structure theory for the spaces of Proposition 2.1(iii). There are two, a compact one and its non-compact dual, for each complex simple Lie algebra, and they are constructed from the highest root [W]. These spaces are listed in the Table 1 below. Here, we use the notation that G_2, F_4, E_6, E_7 and E_8 denote the compact connected simply connected groups of those Cartan classification types, and their non-compact forms listed in the Table are connected real forms contained as analytic subgroups in the corresponding complex simply connected groups. All known examples of compact connected simply connected quaternionic manifolds with holonomy of quaternionic scalar type are Riemannian symmetric spaces.

Table 1.

Irreducible Quaternionic Symmetric Spaces, Scalar Part of Holonomy Quaternionic			
compact $M = G/K$	non-compact $M' = G'/K$	Rank	Dimension/$\mathbb{H}$
$SU(r+2)/S(U(r) \times U(2))$	$SU(r,2)/S(U(r) \times U(2))$	$\min(r,2)$	r
$SO(r+4)/[SO(r) \times SO(4)]$	$SO(r,4)/[SO(r) \times SO(4)]$	$\min(r,4)$	r
$Sp(n+1)/[Sp(n) \times Sp(1)]$	$Sp(n,1)/[Sp(n) \times Sp(1)]$	1	n
$G_2/SO(4)$	$G_{2,A_1A_1}/SO(4)$	2	2
$F_4/[Sp(3) \cdot Sp(1)]$	$F_{4,C_3C_1}/[Sp(3) \cdot Sp(1)]$	4	7
$E_6/[SU(6) \cdot Sp(1)]$	$E_{6,A_5C_1}/[SU(6) \cdot Sp(1)]$	4	10
$E_7/[Spin(12) \cdot Sp(1)]$	$E_{7,D_6C_1}/[Spin(12) \cdot Sp(1)]$	4	16
$E_8/[E_7 \cdot Sp(1)]$	$E_{8,E_7C_1}/[E_7 \cdot Sp(1)]$	4	28

Thus, irreducible quaternionic symmetric spaces have rank 1, 2, 3 or 4. Curiously, quaternionic symmetric spaces for F_4, E_6, E_7, and E_8 all have restricted root systems of type F_4.

3 Complex forms of quaternionic manifolds

Let S be a smooth submanifold of a Riemannian manifold M. Let $A = \{A_x \mid x \in M\}$ denote a quaternionic structure on M. If $x \in S$, let A_x^S denote the subalgebra of all elements in A_x that preserve the real tangent space $T_x(S)$. We say that S is totally complex, if $A_x^S \cong \mathbb{C}$ and $T_x(S) \cap q(T_x(S)) = 0$ for all $q \in A_x \setminus A_x^s$, for all $x \in S$. If S is totally complex in M, then $A^S = \{A_x^S \mid x \in S\}$ restricts to a well-defined almost complex structure on S, parallel along S because A is parallel on M, so (see

[KN, Cor. 3.5, p. 145]) $(S, A^S|_S)$ is Kähler. If in addition, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, then we say that S is a maximal totally complex submanifold of M.

Let S be a maximal totally complex submanifold of M. Suppose that S is a topological component of the fixed point set of an involutive isometry σ of M. Then, we say that S is a complex form of M and that σ is the quaternion conjugation of M over S. The following is immediate.

Lemma 3.1. *Let (M, A) be a quaternionic symmetric space. If S is a complex form of M, then S is a totally geodesic submanifold. If S is a totally geodesic, totally complex submanifold of M, then S is an Hermitian symmetric space.*

Let $M = G/K$, irreducible quaternionic symmetric space, with base point $x_0 = 1K$, where $K = K' \cdot Sp(1)$ as in Proposition 2.1(iii) and Table 1. Let θ denote the involutive automorphism of G that is conjugation by the symmetry (say t) at x_0. Let $S \subset M$ be a totally geodesic submanifold through x_0. Then, S is a Riemannian symmetric space with symmetry $t|_S$ at x_0. Express $S = L(x_0) \cong L/V$, where L is the identity component of $\{g \in G \mid g(S) = S\}$ and $V = L \cap K$. Then $\theta(L) = L$.

The following three results are our basic tools for finding the complex forms $S = L/V$ of $M = G/K$, where $\operatorname{rank}(L) = \operatorname{rank}(G)$. Proposition 3.2 gives criteria for L/V to be an appropriate submanifold of G/K. Proposition 3.3 tells us that when L/V is identified abstractly, it in fact exists well positioned in G/K, and Proposition 3.4 is a uniqueness theorem showing when two complex forms are G-equivalent.

Proposition 3.2. *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let σ be an involutive inner automorphism of G that commutes with θ. Let L be the identity component of the fixed point set G^σ. Set $V = L \cap K$. Denote $S = L(x_0) \cong L/V$.*

1. *If $V \cap Sp(1)$ is a circle group, then S is a totally complex submanifold of M.*
2. *S is a complex form of M if and only if* (i) *$V \cap Sp(1)$ is a circle group, and* (ii) $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$.
3. *If S is a complex form of M, then $\sigma = Ad(s)$ where $s \in V$.*

Proposition 3.3. *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let L be a symmetric subgroup of equal rank in G that has an Hermitian symmetric quotient L/V, such that V is isomorphic to a symmetric subgroup $V' \subset K$. Then, L is conjugate to a θ–stable subgroup $L' \subset G$ such that $L' \cap K = V'$.*

Proposition 3.4. *Let $M = G/K$ be an irreducible quaternionic symmetric space, with base point $x_0 = 1K$, as above. Let $S_i = L_i(x_0) \cong K_i/V_i$ be two complex forms of M. If S_1 and S_2 are isometric, then some element of K carries S_1 onto S_2.*

Proof of Proposition 3.2. We can pass to the compact dual if necessary, so we may (and do) assume M compact. Decompose the Lie algebra $\mathfrak{g}$ of G under $d\theta$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ represents the real tangent space of M. Then $Sp(1)$ gives $\mathfrak{m}$ a quaternionic vector space structure, so any circle subgroup gives $\mathfrak{m}$ a complex vector space structure. If that circle is $V \cap Sp(1)$, it defines an L-invariant almost complex structure on S, and that is integrable because S is a Riemannian symmetric space. We have proved Statement 1.

For Statement 2, first suppose that S is a complex form of M. Since σ is inner by hypothesis, $\text{rank}(L) = \text{rank}(G)$. Since S is an Hermitian symmetric space, $\text{rank}(V) = \text{rank}(L)$. Now, V contains a Cartan subgroup T of G. Thus, $V \cap Sp(1)$ contains a circle group $T_1 := T \cap Sp(1)$. Now the only possibilities for $V \cap Sp(1)$ are (a) T_1, (b) the normalizer of T_1 in $Sp(1)$, and (c) all of $Sp(1)$. Here, (b) is excluded because it would prevent S from having an L-invariant almost complex structure, and (c) is excluded because it would prevent S from being totally complex, so $V \cap Sp(1)$ is a circle group. Finally, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$ because S is a maximal totally complex submanifold of M.

Conversely, suppose that $V \cap Sp(1)$ is a circle group and $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$. By Statement 1, S is a totally complex submanifold of M. By $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, it is a maximal totally complex submanifold. And we started with the symmetry σ, so S is a complex form of M.

For Statement 3 note, as above, that $s \in L$ because $\text{rank}(L) = \text{rank}(G)$, and now $s \in V$ because $\text{rank}(V) = \text{rank}(L)$. □

Proof of Proposition 3.3. All our groups have equal rank, so V' is the K-centralizer of some $v' \in V'$ with v'^2 central in K. Here, K contains the center of G, and those centers satisfy $Z_K/Z_G = \{1, z\}Z_G$ cyclic order 2. Let $\sigma' = \text{Ad}(v')$. If $v'^2 \in zZ_G$, then $\sigma'^2 = \theta$, so $d\sigma$ has eigenvalues $\pm\sqrt{-1}$ on $\mathfrak{m}$, and $L' = G^{\sigma'}$ has the property that $S' = L'(x_0) \cong L'/V'$ is Hermitian symmetric. Since $V \in L$ and $V' \in L'$ are symmetric subgroups of G, and their Hermitian symmetric subgroups are isomorphic, it follows from Table 1 and the classification of Riemannian symmetric spaces that $L \cong L'$. Now, L and L' are conjugate in G, so we may assume $L = L'$. Then, V and V' are isomorphic symmetric subgroups in L, so they are L-conjugate. This completes the proof. □

Proof of Proposition 3.4. Suppose that S_1 and S_2 are isometric, say $g : S_1 \cong S_2$ for some isometric map g. We can assume $g(x_0) = x_0$, so dg gives a Lie triple system isomorphism of $\mathfrak{l}_1 \cap \mathfrak{m}$ onto $\mathfrak{l}_2 \cap \mathfrak{m}$. Write $\mathfrak{l}_i = \mathfrak{l}_i' \oplus \mathfrak{z}_i$, where $\mathfrak{l}_i'$ is generated by $\mathfrak{l}_i \cap \mathfrak{m}$ and $\mathfrak{z}_i \subset \mathfrak{v}_i$ is a complementary ideal. Then dg gives a Lie algebra isomorphism of $\mathfrak{l}_1'$ onto $\mathfrak{l}_2'$. Let $\mathbf{j}_i \in \mathfrak{sp}(1)$ be orthogonal to the Lie algebra of the circle group $V_i \cap Sp(1)$. Then, $\mathbf{j}_i$ centralizes $\mathfrak{z}_i$ and $\mathfrak{m}$ is the real vector space direct sum of $\mathfrak{l}_i \cap \mathfrak{m}$ with $\text{ad}(\mathbf{j}_i)(\mathfrak{l}_i \cap \mathfrak{m})$. Now, $\text{ad}(\mathfrak{z}_i)|_{\mathfrak{m}} = 0$, so each $\mathfrak{z}_i = 0$, and $dg : \mathfrak{l}_1 \cong \mathfrak{l}_2$. Since $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are isomorphic symmetric subalgebras of $\mathfrak{g}$, they are $\text{Ad}(G)$–conjugate. Thus we may assume $g \in G$. As $g(x_0) = x_0$ now $g \in K$. Thus some $g \in K$ carries S_1 onto S_2. □

Propositions 3.2 and 3.4 will let us do the classification of complex forms $S = L/V$ of quaternionic symmetric spaces $M = G/K$ in case $\text{rank}(L) = \text{rank}(G)$. There are only a few cases where $\text{rank}(L) < \text{rank}(G)$, and we will handle them individually. That is not very elegant, but it is very efficient.

4 The classification of complex forms

In this section, we state the classification of complex forms $S = L/V$ of quaternionic symmetric spaces $M = G/K$ and $M' = G'/K$ whose holonomy has quaternion scalar part. The proofs are given in Sections 5, 7 and 8. We state the results separately for the compact and the non-compact cases.

Theorem 4.1. *Let $M = G/K$ be a compact simply connected irreducible quaternionic Riemannian symmetric space. Then the complex forms $S = L/V$ of M are exactly the following, and each is unique up to the action of G.*

1. $M = SU(r+2)/S(U(r) \times U(2))$. *Then* (1a) $S = \frac{SO(r+2)}{SO(r)\times SO(2)}$, *or*

   (1b) $S = P^u(\mathbb{C}) \times P^{r-u}(\mathbb{C}) = \frac{SU(u+1)}{S(U(u)\times U(1))} \times \frac{SU(r-u+1)}{S(U(r-u)\times U(1))}$, $0 \leqq u \leqq r$.
2. $M = SO(r+4)/[SO(r) \times SO(4)]$. *Then* (2a) $S = \frac{SU(r'+2)}{S(U(r')\times U(2))}$, $r = 2r'$ *even, or* (2b) $S = \frac{SO(u+2)}{[SO(u)\times SO(2)} \times \frac{SO(r-u+2)}{SO(r-u)\times SO(2)}$, $0 \leqq u \leqq r$.
3. $M = Sp(n+1)/[Sp(n) \times Sp(1)] = P^n(\mathbb{H})$. *Then* $S = P^n(\mathbb{C}) = \frac{U(n+1)}{[U(n)\times U(1)}$.
4. $M = G_2/SO(4)$. *Then* $S = P^1(\mathbb{C}) \times P^1(\mathbb{C}) = \frac{SO(4)}{SO(2)\times SO(2)}$.
5. $M = F_4/[Sp(3) \cdot Sp(1)]$. *Then* $S = \frac{Sp(3)}{U(3)} \times P^1(\mathbb{C})$.
6. $M = E_6/[SU(6) \cdot Sp(1)]$. *Then* (6a) $S = \frac{SU(6)}{S(U(3)\times U(3))} \times P^1(\mathbb{C})$, *or* (6b) $S = \frac{Sp(4)}{U(4)}$, *or* (6c) $S = \frac{SO(10)}{U(5)}$.
7. $M = E_7/[Spin(12) \cdot Sp(1)]$. *Then* (7a) $S = \frac{E_6}{Spin(10)\cdot U(1)}$, *or* (7b) $S = \frac{SU(8)}{S(U(4)\times U(4))}$, *or* (7c) $S = \frac{SO(12)}{U(6)} \times P^1(\mathbb{C})$.
8. $M = E_8/[E_7 \cdot Sp(1)]$. *Then* (8a) $S = \frac{E_7}{E_6 T_1} \times P^1(\mathbb{C})$ *or* (8b) $S = \frac{SO(16)}{U(8)}$.

Theorem 4.2. *Let $M = G/K$ be a non-compact irreducible quaternionic Riemannian symmetric space. Then, the complex forms $S = L/V$ of M are exactly the following, and each is unique up to the action of G.*

1. $M = SU(r,2)/S(U(r) \times U(2))$. *Then* (1a) $S = \frac{SO(r,2)}{SO(r)\times SO(2)}$, *or*

   (1b) $S = H^u(\mathbb{C}) \times H^{r-u}(\mathbb{C}) = \frac{SU(u,1)}{S(U(u)\times U(1))} \times \frac{SU(r-u,1)}{S(U(r-u)\times U(1))}$, $0 \leqq u \leqq r$.
2. $M = SO(r,4)/[SO(r) \times SO(4)]$. *Then* (2a) $S = \frac{SU(r',2)}{S(U(r')\times U(2))}$, $r = 2r'$ *even, or* (2b) $S = \frac{SO(u,2)}{[SO(u)\times SO(2)} \times \frac{SO(r-u,2)}{SO(r-u)\times SO(2)}$, $0 \leqq u \leqq r$.
3. $M = Sp(n,1)/[Sp(n) \times Sp(1)] = H^n(\mathbb{H})$. *Then* $S = H^n(\mathbb{C}) = \frac{U(n,1)}{[U(n)\times U(1)}$.
4. $M = G_{2,A_1A_1}/SO(4)$. *Then* $S = H^1(\mathbb{C}) \times H^1(\mathbb{C}) = \frac{SO(2,2)}{SO(2)\times SO(2)}$.
5. $M = F_{4,C_3C_1}/[Sp(3) \cdot Sp(1)]$. *Then* $S = \frac{Sp(3;\mathbb{R})}{U(3)} \times H^1(\mathbb{C})$.

6. $M = E_{6,A_5C_1}/[SU(6)\cdot Sp(1)]$. *Then* (6a) $S = \frac{SU(3,3)}{S(U(3)\times U(3))} \times H^1(\mathbb{C})$, *or* (6b) $S = \frac{Sp(4;\mathbb{R})}{U(4)}$, *or* (6c) $S = \frac{SO^*(10)}{U(5)}$.

7. $M = E_{7,D_6C_1}/[Spin(12)\cdot Sp(1)]$. *Then* (7a) $S = \frac{E_{6,D_5T_1}}{Spin(10)\cdot U(1)}$, *or* (7b) $S = \frac{SU(4,4)}{S(U(4)\times U(4))}$, *or* (7c) $S = \frac{SO^*(12)}{U(6)} \times P^1(\mathbb{C})$.

8. $M = E_{8,E_7C_1}/[E_7\cdot Sp(1)]$. *Then* (8a) $S = \frac{E_{7,E_6T_1}}{E_6T_1} \times P^1(\mathbb{C})$ *or* (8b) $S = \frac{SO^*(16)}{U(8)}$.

Of course, Theorem 4.2 is immediate from Theorem 4.1 by passage to the non-compact dual symmetric spaces. So, we need only prove Theorem 4.1. The proof of Theorem 4.1 consists of consolidating the results of Sections 5, 7 and 8.

5 The equal rank classification — classical cases

We run through the list of compact irreducible quaternionic symmetric spaces $M = G/K$ from Table 1, for the cases where G is a classical group. For each of them, we look at the possible symmetric subgroups L, that correspond to an Hermitian symmetric space $S = L/V$, such that $\text{rank}(L) = \text{rank}(G)$, $\dim_{\mathbb{C}} S = \dim_{\mathbb{H}} M$, $\text{rank}(S) \leqq \text{rank}(M)$, and V is isomorphic to a symmetric subgroup of K properly placed as in Proposition 3.2. The equal rank classification will follow using Proposition 3.4. We retain the notation used in Propositions 3.2 and 3.4. Fix $s \in K$, such that L is the identity component of $\sigma = \text{Ad}(s)$.

Case $M = SU(r+2)/S(U(r)\times U(2))$. First, suppose $r \geqq 2$. We may take s to be diagonal. It has only two distinct eigenvalues, and its component in the $U(2)$-factor of K must have both eigenvalues. Now $L \cong S(U(u+1)\times U(v+1))$, $V \cong S([U(u)\times U(1)]\times[U(v)\times U(1)])$, and S is the product $P^u(\mathbb{C})\times P^v(C)$ of complex projective spaces. Here, $\dim_{\mathbb{H}} M = r = u+v = \dim_{\mathbb{C}} S$. If $u, v \geqq 1$, then $\text{rank}(M) = 2 = \text{rank}(S)$. If $u = 0$, then the factor $P^u(\mathbb{C})$ is reduced to a point, $S \cong P^v(C)$, and $\text{rank}(S) = 1$. The analog holds, of course, if $v = 0$.

Now, consider the degenerate case $r = 1$. Then $M = P^2(\mathbb{C})$ and fits the dual pattern described in the paragraph just before the statement of Proposition 2.1. Relative to the quaternionic structure denoted A_1 there, the one with with quaternion scalar part, the matrix considerations above show that M has a complex form $S = P^1(\mathbb{C})$.

Case $M = SO(r+4)/[SO(r)\times SO(4)]$. As before, the matrix s has just two distinct eigenvalues, and each one must appear with multiplicity 2 in the $SO(4)$-factor of K. If $s^2 = I$, then $L = SO(u+2)\times SO(v+2)$ with $u+v = r$, where $V = L\cap K = [SO(u)\times SO(2)]\times[SO(v)\times SO(2)]$. Here the $SO(2)$-factors in V are the intersection with the $SO(4)$-factor of K. That gives us the forms $S = (SO(u+2)/[SO(u)\times SO(2)])\times(SO(v+2)/[SO(v)\times SO(2)])$ of M.

Now, suppose $s^2 = -I$. Then, $r = 2r'$ even, $L \cong U(r'+2)$, $V \cong U(r')\times U(2)$, and we have the complex form $S \cong SU(r'+2)/S(U(r')\times U(2))$ of M.

Case $M = Sp(n+1)/[Sp(n)\times Sp(1)] = P^n(\mathbb{H})$. The symmetric subgroups of $Sp(n+1)$ are the $Sp(u)\times Sp(v)$, $u+v = n+1$, and $U(n+1)$. The first case,

$L = Sp(u) \times Sp(v)$, would give $V = Sp(u) \times Sp(v-1) \times Sp(1)$, so $S = Sp(v)/[Sp(v-1) \times Sp(1)]$, which is not Hermitian symmetric. That leaves the case $L = U(n+1)$ and $V = U(n) \times U(1)$, where $S = P^n(\mathbb{C})$. It satisfies the conditions of Proposition 3.2 and thus is a complex form of M.

6 The LiE program

While the matrix computation methods of Section 5 work well for the classical group cases, it is more convenient to make use of the root structure in the exceptional group cases. In this section, we indicate just how we used the LiE program [L] to do that. We illustrate it for E_8, but it is the same for any simple group structure. Here, node refers to the simple root at which the negative of the maximal root is attached in the extended Dynkin diagram.

STEP 0: INITIALIZE.

> setdefault(E8) > rank = 8
> diagram ; *prints out the Dynkin diagram and numbers the simple roots.*
> node = 8 ; *the number of the simple root that defines K.*

STEP 1: POSITIVE ROOTS OF $\mathfrak{g}$.

> pos = pos_roots
> max_root = pos[n_rows(pos)]

STEP 2: POSITIVE ROOTS OF $\mathfrak{k}$.

> kkk = pos > for i = 1 to n_rows(kkk) do
if kkk[i,node] == 1 then kkk[i] = null(rank) fi od ; *zeroes rows $\mathfrak{m}$-roots*
> kk = unique(kkk) ; *eliminates duplicate rows*
> k = null(n_rows(kk)-1,rank) > for i = 1 to n_rows(k) do k[i] = kk[i+1] od
; *eliminates last zero row*
> Cartan_type(k) ; *verifies correct Cartan type for $\mathfrak{k}$, in this case E_7A_1*

STEP 3: POSITIVE ROOTS OF $\mathfrak{m}$.

> mmm = pos > for i = 1 to n_rows(mmm) do
if mmm[i,node] != 1 then mmm[i] = null(rank) fi od ; *zeroes rows for $\mathfrak{k}$–roots*
> mm = unique(mmm) ; *eliminates duplicate rows*
> m = null(n_rows(mm)-1,rank)
> for i = 1 to n_rows(m) do m[i] = mm[i+1] od ; *eliminates last zero row*

STEP 4: CHOICE OF sym WHERE $\sigma = \mathrm{Ad}(sym)$; DEFINITION OF $\mathfrak{l} = \mathfrak{g}^\sigma$.

> sym = null(rank + 1) ; *initializes sym as row vector*
> sym[node] = 1 ; *one possibility for nonzero element of sym*
> sym[rank+1] = 2 ; *normalizes 1–parameter group containing symm*
> l = cent_roots(sym) ; *defines $\mathfrak{l}$ as centralizer of sym*
> Cartan_type(l) ; *Cartan type of $\mathfrak{l}$, in this case E_7A_1*

STEP 5: POSITIVE ROOTS OF $\mathfrak{s} := \mathfrak{l} \cap \mathfrak{m}$ AND OF $\mathfrak{v} := \mathfrak{l} \cap \mathfrak{k}$.

> sss = l > for i = 1 to n_rows(sss) do
  if sss[i,node] != 1 then sss[i] = null(rank) fi od
> ss = unique(sss)
> s = null(n_rows(ss)-1,rank)
> for i = 1 to n_rows(s) do s[i] = ss[i+1] od
> vvv = l
> for i = 1 to n_rows(vvv) do if vvv[i,2] == 1 then vvv[i] = null(rank) fi od
> vv = unique(vvv)
> v = null(n_rows(vv)-1,rank)
> for i = 1 to n_rows(v) do v[i] = vv[i+1] od
> Cartan_type(v) ; *Cartan type of* $\mathfrak{v}$, *in this case* $E_6T_1T_1$
; *At this point we know that* $S = L/V \cong (E_7/[E_6 \times T_1]) \times (T_1/T_1)$,
; *so it is an hermitian symmetric subspace of* G/K.

STEP 6: VERIFY THAT S IS A MAXIMAL TOTALLY COMPLEX IN M.

> t = null(n_rows(s)-1,rank)
> for i=1 to n_rows(t) do t[i] = max_root - s[i] od
> u = null(n_rows(s) + n_rows(t) + n_rows(m), rank)
> for i = 1 to n_rows(s) do u[i] = s[i] od
> for i = 1 to n_rows(t) do u[n_rows(s) + i] = t[i] od
> for i = 1 to n_rows(m) do u[n_rows(s) + n_rows(t) + i] = m[i] od

; *now the rows of u are: positive roots of* $\mathfrak{s}$,
; *maximal root minus positive roots of* $\mathfrak{s}$,
; *positive roots of* $\mathfrak{m}$
> w = unique(u) ; *the rows of w are the positive roots of* $\mathfrak{m}$ *and non–root*
; *linear functionals* (*max root minus positive root of* $\mathfrak{s}$)
> n_rows(w) - n_rows(m) ; *number of non–root linear functionals in w*,
; *measures failure of S to be maximal totally complex*;
; *OK here because it returns* 0

We carry out the routine in some key cases. These are cases where K and L are conjugate in G.

CASE $G = B_7$. Here, node $= 2$, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = B_5A_1A_1$ and $V = B_4T_1T_1T_1$, thus to the complex form $S = SO(11)/[SO(9) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(15)/[SO(11) \times SO(4)$. More generally, for B_n with $n \geqq 3$, node $= 2$, and sym = $[0, 1, 0, \ldots, 0, 2]$ gives the complex form $S = SO(2n-3)/[SO(2n-5) \times SO(4)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(2n+1)/[SO(2n-3) \times SO(4)]$. This is the case $v = 2$, $u = r-2$ considered for $G = SO(r+4)$, r odd, in Section 5.

CASE $G = D_7$. Here, node $= 2$, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = D_5A_1A_1$ and $V = D_4T_1T_1T_1$, thus to the complex form $S = SO(10)/[SO(8) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(14)/[SO(10) \times SO(4)]$. More generally, for D_n with $n \geqq 3$, node $= 2$, and sym = $[0, 1, 0, \ldots, 0, 2]$ gives the complex form $S = SO(2n-4)/[SO(2n-6) \times SO(2)] \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = SO(2n)/[SO(2n-4) \times SO(4)]$. This is the case $v = 2$, $u = r-2$ considered for $G = SO(r+4)$, r even, in Section 5.

CASE $G = G_2$. Here, node = 2, and sym = [0, 1, 2] leads to $L = A_1A_1$ and $V = T_1T_1$, thus to the complex form $S = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ of $G/K = G_2/SO(4)$.

CASE $G = F_4$. Here, node = 1, and sym = [1, 0, 0, 0, 2] leads to $L = C_3C_1$ and $V = A_2T_1T_1$, thus to the complex form $S = [Sp(3)/U(3)] \times P^1(\mathbb{C})$ of $G/K = F_4/C_3C_1$.

CASE $G = E_6$. Here, node = 2, and sym = [0, 1, 0, 0, 0, 0, 2] leads to $L = A_5A_1$ and $V = A_2T_1A_2T_1$, thus to the complex form $S = [SU(6)/S(U(3) \times U(3))] \times P^1(\mathbb{C})$ of $G/K = E_6/A_5A_1$.

CASE $G = E_7$. Here, node = 2, and sym = [0, 1, 0, 0, 0, 0, 0, 2] leads to $L = D_6A_1$ and $V = A_5T_1T_1$, and thus to the complex form $S = [SO(12)/U(6)] \times P^1(\mathbb{C})$ of $G/K = E_7/D_6A_1$.

CASE $G = E_8$. As we saw, sym = [0, 0, 0, 0, 0, 0, 0, 1, 2] leads to $L = E_7A_1$ and $V = E_6T_1T_1$, and thus to the complex form $S = (E_7/[E_6 \times T_1]) \times P^1(\mathbb{C})$ of $G/K = E_8/E_7A_1$.

CASES A_7 and C_7. Here, the computation using LiE has not yet produced complex forms S of M. In other words, I have not yet guessed the appropriate vectors sym to define toral elements of G whose centralizers are appropriate subgroups $L \subset G$.

7 The equal rank classification – exceptional cases

In this section, we complete the classification for the equal rank exceptional group cases.

CASE $G = G_2$. The only symmetric subgroup of G_2 is $SO(4)$, so here the only complex form of $M = G_2/SO(4)$ is $S = P^1(\mathbb{C}) \times P^1(\mathbb{C})$ as described in Section 6.

CASE $G = F_4$. The only symmetric subgroups of F_4 are $Sp(3) \cdot Sp(1)$ and $Spin(9)$. If $L = Spin(9)$, then the Hermitian symmetric space $L/V = Spin(9)/[Spin(7) \times Spin(2)]$. That would place the $Spin(7)$-factor of V in the $Sp(3)$-factor of K; but $Sp(3) \subset SU(6)$ while $Spin(7)$ has no non-trivial representation of degree < 7. Thus, $L \neq Spin(9)$, so, here the only complex form of $M = F_4/C_3C_1$ is $S = [Sp(3)/U(3)] \times P^1(\mathbb{C})$ as described in Section 6.

CASE $G = E_6$. The only symmetric subgroups of maximal rank in E_6 are A_5A_1 and D_5T_1.

If $L = D_5T_1$, then the Hermitian symmetric space $S = L/V$ must be either $SO(10)/[SO(8) \times SO(2)]$ with $V = [SO(8) \times SO(2)] \cdot SO(2)$, or $[SO(10)/U(5)]$ with $V = U(5) \cdot SO(2)$. The first is excluded because $\dim_{\mathbb{C}} SO(10)/[SO(8) \times SO(2)] = 8 < 10 = \dim_{\mathbb{H}} M$. The second of these is a complex form of $M = E_6/A_5A_1$ by Propositions 3.2 and 3.3.

$L = A_5A_1$ gives another complex form $S = [SU(6)/S(U(3) \times U(3))] \times P^1(\mathbb{C})$ of $M = E_6/A_5A_1$ as described in Section 6.

CASE $G = E_7$. The only symmetric subgroups of E_7 are D_6A_1, A_7 and E_6T_1.

If $L \cong E_6T_1$, then the Hermitian symmetric space $S = L/V$ must be E_6/D_5T_1 with $V = D_5T_1T_1$. It is a complex form of $M = E_7/D_6A_1$ by Propositions 3.2 and 3.3.

If $L = A_7$, then the Hermitian symmetric space $S = L/V$ must be $SU(8)/S(U(u)\times U(v))$ with $u + v = 8$. Here $\dim_{\mathbb{C}} L/V = UV$ while $\dim_{\mathbb{H}} M = 16$, so $u = v = 4$. That would place the $[SU(4)\times SU(4)]$-factor of V in the $Spin(12)$-factor of K. It could only sit there as $Spin(6) \times Spin(6)$, which is the identity component of its $Spin(12)$-normalizer because it is a symmetric subgroup of $Spin(12)$, so the circle center of V is contained in the $Sp(1)$-factor of K. Thus, S is a complex form of $M = E_7/D_6A_1$ by Propositions 3.2 and 3.3.

$L = D_6A_1$ gives another complex form $S = [SO(12)/U(6)] \times P^1(\mathbb{C})$ of $M = E_7/D_6A_1$ as as described in Section 6.

Case $G = E_8$. The only symmetric subgroups of E_8 are E_7A_1 and D_8.

If $L = D_8$, then the Hermitian symmetric space $S = L/V$ either must be $SO(16)/[SO(14) \times SO(2)]$ with $V = [SO(14) \times SO(2)]$, or $SO(16)/U(8)$ with $V = U(8)$. The first of these is excluded because $\dim_{\mathbb{C}} SO(16)/[SO(14)\times SO(2)] = 14 < 28 = \dim_{\mathbb{H}} M$. The second of these is a complex form of $M = E_8/E_7A_1$ by Propositions 3.2 and 3.3.

$L \cong E_7A_1$ gives another complex form $S = (E_7/[E_6 \times T_1]) \times P^1(\mathbb{C})$ of $M = E_8/E_7A_1$ as described in Section 6.

8 The unequal rank classification

In this section, we deal with the cases $\text{rank}(L) < \text{rank}(G)$. Here, G is of type A_n, D_n or E_6.

Case $M = SU(r+2)/S(U(r) \times U(2))$. The only symmetric subgroups of lower rank in $SU(r+2)$ are $SO(r+2)$ and, for $r = 2r'$ even, $Sp(r'+1)$.

If $L = Sp(r'+1)$, $r = 2r'$ even, then $S = Sp(r'+1)/U(r'+1)$ with $V = U(r'+1)$. Here, $\dim_{\mathbb{H}} M = 2r'$ and $\dim_{\mathbb{C}} S = \frac{1}{2}(r'+2)(r'+1)$, so those dimensions are equal just when $r'^2 - r' + 2 = 0$. That equation has no integral solution. Thus, $L \neq Sp(r'+1)$.

If $L = SO(r+2)$, then $S = SO(r+2)/[SO(r) \times SO(2)]$ with $V = [SO(r) \times SO(2)]$. The $SO(2)$–factor of V is contained in the derived group $SU(2)$ of the $U(2)$-factor of K, and $\dim_{\mathbb{C}} S = r = \dim_{\mathbb{H}} M$, so Proposition 3.2 shows that S is a complex form of M.

Case $M = SO(2n+4)/[SO(2n) \times SO(4)]$. The only symmetric subgroups of lower rank in $SO(2n+4)$ are $SO(2u+1) \times SO(2v+1)$, where $u + v = n + 1$. If $L = SO(2u+1)\times SO(2v+1)$ then $V = SO(2u-1)\times SO(2)\times SO(2v-1)\times SO(2)$ and $S = \{SO(2u+1)/[SO(2u-1) \times SO(2)]\} \times \{SO(2v+1)/[SO(2v-1) \times SO(2)]\}$, where the product of the two $SO(2)$-factors is contained in the $SO(4)$-factor of K. Since $\dim_{\mathbb{C}} S = (2u-1)+(2v-1) = n = \dim_{\mathbb{H}} M$, the argument of Proposition 3.2 shows that S is a complex form of M.

Case $M = E_6/A_5A_1$. The only symmetric subgroups of lower rank in E_6 are F_4 and C_4, and $L \neq F_4$ because F_4 has no Hermitian symmetric quotient space. If $L = Sp(4)$, then $S = Sp(4)/U(4)$ with $V = U(4)$. Here, V sits in K as follows. The semisimple part $[V, V] = U(4)/\{\pm I\} = SO(6) \subset SU(6) = A_5$. $[V, V]$ is a connected symmetric subgroup of the connected simple group A_5, so it is equal to the identity component of its normalizer in A_5. Thus, the projection $K = A_5 A_1 \to A_5$ annihilates the circle center of V. In other words, $V \cap Sp(1)$ is a circle group central in V. It follows as in Proposition 3.2(1) that S is a totally complex submanifold of M. Since $\dim_{\mathbb{C}} S = 10 = \dim_{\mathbb{H}} M$, it is a maximal totally complex submanifold, and being a symmetric submanifold it is a complex form.

This completes the proof of Theorems 4.1 and 4.2, the main results of this note.

9 Quaternionic forms

In this section, we look at the idea of quaternionic forms of symmetric spaces as suggested by the examples of projective planes $P^2(\mathbb{H}) \subset P^2(\mathbb{O})$ and hyperbolic planes $H^2(\mathbb{H}) \subset H^2(\mathbb{O})$. The meaning of Cayley structure is not entirely clear because of non-associativity, so we do not have a good definition for Cayley symmetric space. Here, we offer a tentative definition of quaternionic form and a number of examples, some interesting and some too artificial to be interesting.

Let M be a Riemannian symmetric, let σ be an involutive isometry of M, let S be a totally geodesic submanifold of M, and suppose that (i) S is a topological component of the fixed point set of σ, (ii) $\dim_{\mathbb{R}} S = \frac{1}{2} \dim_{\mathbb{R}} M$, and (iii) S has quaternionic structure for which its holonomy has quaternion scalar part. Then we will say that S is a quaternionic form of M.

Suppose that $M = G/K$ with base point $x_0 = 1K$ and $S = L(x_0) = L/V$, where L is the identity component of the group generated by transvections of S. Following Proposition 2.1, $S = L/V$ is one of the spaces listed in Table 1. That gives us interesting examples

$\frac{SU(r+2)}{S(U(r)\times U(2))} = \frac{U(r+2)}{U(r)\times U(2)}$ in $Sp(r+2)/[Sp(r) \times Sp(2)]$;

$\frac{SO(r+4)}{SO(r)\times SO(4)}$ in $U(r+4)/[U(r) \times U(4)] = SU(r+4)/S(U(r) \times SU(4))$;

$\frac{Sp(r+1)}{Sp(r)\times Sp(1)}$ in $U(2r+2)/[U(2r) \times U(2)] = SU(2r+2)/S(U(2r) \times U(2))$;

$\frac{SU(r+2)}{S(U(r)\times U(2))} = \frac{U(r+2)}{U(r)\times U(2)}$ in $SO(2r+4)/[SO(2r) \times SO(4)]$; $\frac{Sp(3)}{Sp(2)\times Sp(1)} = P^2(\mathbb{H})$ in $P^2(\mathbb{O}) = F_4/Spin(9)$ (computing with LiE);

$\frac{E_7}{Spin(12)\cdot Sp(1)}$ in $E_8/SO(16)$ (using Proposition 3.3 with $L = E_7 A_1$, as in §7).

It also gives us some other examples S in $S \times S$ as a factor or as the diagonal;

$\frac{SU(r+2)}{S(U(r)\times U(2))}$ in $SU(r+4)/S(U(r) \times U(4))$ or $SU(2r+2)/S(U(2r) \times U(2))$;

$\frac{SO(r+4)}{SO(r)\times SO(4)}$ in $SO(r+8)/[SO(r) \times SO(8)]$ or $SO(2r+4)/[SO(2r) \times SO(4)]$;

$\frac{Sp(r+1)}{Sp(r)\times Sp(1)}$ in $Sp(r+2)/[Sp(r) \times Sp(2)]$ or $Sp(2r+1)/[Sp(2r) \times Sp(1)]$.

Those other examples somehow seem too formal to be interesting. Of course with any of these compact examples $S \subset M$, we also have the non-compact duals $S' \subset M'$.

These examples indicate that a reasonable theory for quaternionic forms S of symmetric spaces M will require some additional structure on the normal bundle of S in M.

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