Cambridge Series in Statistical and Probabilistic Mathematics

# Probability

## Theory and Examples

Fourth Edition

Rick Durrett

CAMBRIDGE

CAMBRIDGE

www.cambridge.org/9780521765398

This page intentionally left blank

# Probability

## Theory and Examples

### Fourth Edition

This book is an introduction to probability theory covering laws of large numbers, central limit theorems, random walks, martingales, Markov chains, ergodic theorems, and Brownian motion. It is a comprehensive treatment concentrating on the results that are the most useful for applications. Its philosophy is that the best way to learn probability is to see it in action, so there are 200 examples and 450 problems.

Rick Durrett received his Ph.D. in operations research from Stanford University in 1976. After nine years at UCLA and twenty-five at Cornell University, he moved to Duke University in 2010, where he is a professor of mathematics. He is the author of 8 books and more than 170 journal articles on a wide variety of topics, and he has supervised more than 40 Ph.D. students. He is a member of the National Academy of Science and the American Academy of Arts and Sciences and a Fellow of the Institute of Mathematical Statistics.

CAMBRIDGE SERIES IN STATISTICAL AND PROBABILISTIC MATHEMATICS



This series of high-quality upper-division textbooks and expository monographs covers all aspects of stochastic applicable mathematics. The topics range from pure and applied statistics to probability theory, operations research, optimization, and mathematical programming. The books contain clear presentations of new developments in the field and also of the state of the art in classical methods. While emphasizing rigorous treatment of theoretical methods, the books also contain applications and discussions of new techniques made possible by advances in computational practice.

*Already Published*

1. *Bootstrap Methods and Their Application*, by A. C. Davison and D. V. Hinkley
2. *Markov Chains*, by J. Norris
3. *Asymptotic Statistics*, by A. W. van der Vaart
4. *Wavelet Methods for Time Series Analysis*, by Donald B. Percival and Andrew T. Walden
5. *Bayesian Methods*, by Thomas Leonard and John S. J. Hsu
6. *Empirical Processes in M-Estimation*, by Sara van de Geer
7. *Numerical Methods of Statistics*, by John F. Monahan
8. *A User's Guide to Measure Theoretic Probability*, by David Pollard
9. *The Estimation and Tracking of Frequency*, by B. G. Quinn and E. J. Hannan
10. *Data Analysis and Graphics Using R*, by John Maindonald and John Braun
11. *Statistical Models*, by A. C. Davison
12. *Semiparametric Regression*, by D. Ruppert, M. P. Wand, and R. J. Carroll
13. *Exercise in Probability*, by Loic Chaumont and Marc Yor
14. *Statistical Analysis of Stochastic Processes in Time*, by J. K. Lindsey
15. *Measure Theory and Filtering*, by Lakhdar Aggoun and Robert Elliott
16. *Essentials of Statistical Inference*, by G. A. Young and R. L. Smith
17. *Elements of Distribution Theory*, by Thomas A. Severini
18. *Statistical Mechanics of Disordered Systems*, by Anton Bovier
19. *The Coordinate-Free Approach to Linear Models*, by Michael J. Wichura
20. *Random Graph Dynamics*, by Rick Durrett
21. *Networks*, by Peter Whittle
22. *Saddlepoint Approximations with Applications*, by Ronald W. Butler
23. *Applied Asymptotics*, by A. R. Brazzale, A. C. Davison, and N. Reid
24. *Random Networks for Communication*, by Massimo Franceschetti and Ronald Meester
25. *Design of Comparative Experiments*, by R. A. Bailey
26. *Symmetry Studies*, by Marlos A. G. Viana
27. *Model Selection and Model Averaging*, by Gerda Claeskens and Nils Lid Hjort
28. *Bayesian Nonparametrics*, by Nils Lid Hjort, Peter Müller, and Stephen G. Walker
29. *From Finite Sample to Asymptotic Methods in Statistics*, by Pranab K. Sen, Julio M. Singer, and Antonio C. Pedroso de Lima
30. *Brownian Motion*, by Peter Mörters and Yuval Peres

# Probability

## Theory and Examples

Fourth Edition

RICK DURRETT
*Department of Mathematics, Duke University*

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
32 Avenue of the Americas, New York, NY 10013-2473, USA

www.cambridge.org
Information on this title: www.cambridge.org/9780521765398



First edition published 1991 by Wadsworth Publishing
Second edition published 1995 by Duxbury Press
Third edition published 2004 by Duxbury Press
Fourth edition published 2010 by Cambridge University Press

Printed in the United States of America

*A catalog record for this publication is available from the British Library.*

*Library of Congress Cataloging in Publication data*

Durrett, Richard, 1951–
Probability : theory and examples / Rick Durrett. – 4th ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-521-76539-8 (hardback)
1. Probabilities. I. Title.
QA273.D865 2010
519.2–dc22 2010013387

ISBN 978-0-521-76539-8 Hardback

# Contents

# Preface

In 1989 when the first edition of this book was completed, my sons David and Greg were 3 and 1, and the cover picture showed the Dow Jones at 2650. The past 20 years have brought many changes, but the song remains the same. The title of the book indicates that as we develop the theory, we will focus our attention on examples. Hoping that the book would be a useful reference for people who apply probability in their work, we have tried to emphasize the results that are important for applications, and have illustrated their use with roughly 200 examples. Probability is not a spectator sport, so the book contains almost 450 exercises to challenge readers and to deepen their understanding.

This fourth edition has two major changes (in addition to a new publisher):

(i) The book has been converted from TeX to LaTeX. The systematic use of labels should eventually eliminate problems with references to other points in the text. In addition, the picture environment and graphicx package has allowed the figures lost from the third edition to be reintroduced and a number of new ones to be added.

(ii) Four sections of the old appendix have been combined with the first three sections of Chapter 1 to make a new first chapter on measure theory, which should allow the book to be used by people who do not have this background without making the text tedious for those who have.

**Acknowledgments.** I am always grateful to the many people who sent me comments and typos. Helping to correct the first edition were David Aldous, Ken Alexander, Daren Cline, Ted Cox, Robert Dalang, Joe Glover, David Griffeath, Phil Griffin, Joe Horowitz, Olav Kallenberg, Jim Kuelbs, Robin Pemantle, Yuval Peres, Ken Ross, Steve Samuels, Byron Schmuland, Jon Wellner, and Ruth Williams.

The third edition benefited from input from Manel Baucells, Eric Blair, Zhen-Qing Chen, Finn Christensen, Ted Cox, Bradford Crain, Winston Crandall, Amir Dembo, Neil Falkner, Changyong Feng, Brighten Godfrey, Boris Granovsky, Jan Hannig, Andrew Hayen, Martin Hildebrand, Kyoungmun Jang, Anatole Joffe, Daniel Kifer, Steve Krone, Greg Lawler, T. Y. Lee, Shlomo Levental, Torgny Lindvall, Arif Mardin, Carl Mueller, Robin Pemantle, Yuval Peres, Mark Pinsky, Ross Pinsky, Boris Pittel, David Pokorny, Vinayak Prabhu, Brett Presnell, Jim Propp,

Yossi Schwarzfuchs, Rami Shakarchi, Lian Shen, Marc Shivers, Rich Sowers, Bob Strain, Tsachy Weissman, and Hao Zhang.

New helpers for the fourth edition include John Angus, Phillipe Charmony, Adam Cruz, Ricky Der, Justin Dyer, Piet Groeneboom, Vlad Island, Elena Kosygina, Richard Laugesen, Sungchul Lee, Shlomo Levental, Ping Li, Fredddy López, Lutz Mattner, Piotr Milos, Davey Owen, Brett Presnell, Igal Sason, Alex Smith, Laurent Tournier, Harsha Wabgaonkar, John Walsh, Tsachy Weissman, Neil Wu, Ofer Zeitouni, Martin Zerner, and Andrei Zherebtsov. I apologize to those whose names have been omitted or are new typos.

**Family update.** David graduated from Ithaca College in May 2009 with a degree in print journalism, and like many of his peers is struggling to find work. Greg has one semester to go at MIT and is applying to graduate schools in computer science. He says he wants to do research in "machine learning," so perhaps he can write a program to find and correct the typos in my books.

After 25 years in Ithaca, we moved to Durham in June 2010 and I have taken a position in the math department at Duke. Everyone seems to focus on the fact that we are trading very cold winters for hotter summers and a much longer growing season, but the real attraction is the excellent opportunities for interdisciplinary research in the Research Triangle.

The more things change, the more they stay the same: inevitably there will be typos in the new version. You can email me at rtd@math.duke.edu

*Rick Durrett, July 2010*

# 1

# Measure Theory

In this chapter, we recall some definitions and results from measure theory. Our purpose here is to provide an introduction for readers who have not seen these concepts before and to review that material for those who have. Harder proofs, especially those that do not contribute much to one's intuition, are hidden away in the Appendix. Readers with a solid background in measure theory can skip Sections 1.4, 1.5, and 1.7, which were previously part of the Appendix.

## 1.1 Probability Spaces

Here and throughout the book, terms being defined are set in **boldface**. We begin with the most basic quantity. A **probability space** is a triple $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set of "outcomes," $\mathcal{F}$ is a set of "events," and $P : \mathcal{F} \to [0, 1]$ is a function that assigns probabilities to events. We assume that $\mathcal{F}$ is a $\sigma$**-field** (or $\sigma$**-algebra**), that is, a (nonempty) collection of subsets of $\Omega$ that satisfy

(i) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, and
(ii) if $A_i \in \mathcal{F}$ is a countable sequence of sets then $\cup_i A_i \in \mathcal{F}$.

Here and in what follows, **countable** means finite or countably infinite. Since $\cap_i A_i = (\cup_i A_i^c)^c$, it follows that a $\sigma$-field is closed under countable intersections. We omit the last property from the definition to make it easier to check.

Without $P$, $(\Omega, \mathcal{F})$ is called a **measurable space**, that is, it is a space on which we can put a measure. A **measure** is a nonnegative countably additive set function; that is, a function $\mu : \mathcal{F} \to \mathbf{R}$ with

(i) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$, and
(ii) if $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets, then

$$\mu(\cup_i A_i) = \sum_i \mu(A_i)$$

If $\mu(\Omega) = 1$, we call $\mu$ a **probability measure**. In this book, probability measures are usually denoted by $P$.

The next result gives some consequences of the definition of a measure that we will need later. In all cases, we assume that the sets we mention are in $\mathcal{F}$.

**Theorem 1.1.1.** *Let $\mu$ be a measure on $(\Omega, \mathcal{F})$*

*(i)* **Monotonicity.** *If $A \subset B$ then $\mu(A) \le \mu(B)$.*

*(ii)* **Subadditivity.** *If $A \subset \cup_{m=1}^\infty A_m$ then $\mu(A) \le \sum_{m=1}^\infty \mu(A_m)$.*

*(iii)* **Continuity from below.** *If $A_i \uparrow A$ (i.e., $A_1 \subset A_2 \subset \dots$ and $\cup_i A_i = A$) then $\mu(A_i) \uparrow \mu(A)$.*

*(iv)* **Continuity from above.** *If $A_i \downarrow A$ (i.e., $A_1 \supset A_2 \supset \dots$ and $\cap_i A_i = A$), with $\mu(A_1) < \infty$ then $\mu(A_i) \downarrow \mu(A)$.*

*Proof.*

(i) Let $B - A = B \cap A^c$ be the **difference** of the two sets. Using $+$ to denote disjoint union, $B = A + (B - A)$ so

$$\mu(B) = \mu(A) + \mu(B - A) \ge \mu(A).$$

(ii) Let $A'_n = A_n \cap A$, $B_1 = A'_1$ and for $n > 1$, $B_n = A'_n - \cup_{m=1}^{n-1}(A'_m)^c$. Since the $B_n$ are disjoint and have union $A$, we have, using (i) of the definition of measure, $B_m \subset A_m$, and (i) of this theorem,

$$\mu(A) = \sum_{m=1}^\infty \mu(B_m) \le \sum_{m=1}^\infty \mu(A_m)$$

(iii) Let $B_n = A_n - A_{n-1}$. Then the $B_n$ are disjoint and have $\cup_{m=1}^\infty B_m = A$, $\cup_{m=1}^n B_m = A_n$ so

$$\mu(A) = \sum_{m=1}^\infty \mu(B_m) = \lim_{n\to\infty} \sum_{m=1}^n \mu(B_m) = \lim_{n\to\infty} \mu(A_n)$$

(iv) $A_1 - A_n \uparrow A_1 - A$ so (iii) implies $\mu(A_1 - A_n) \uparrow \mu(A_1 - A)$. Since $A_1 \supset B$, we have $\mu(A_1 - B) = \mu(A_1) - \mu(B)$ and it follows that $\mu(A_n) \downarrow \mu(A)$. ■

The simplest setting, which should be familiar from undergraduate probability, is:

**Example 1.1.1. Discrete probability spaces.** Let $\Omega =$ a countable set, that is, finite or countably infinite. Let $\mathcal{F} =$ the set of all subsets of $\Omega$. Let

$$P(A) = \sum_{\omega\in A} p(\omega) \text{ where } p(\omega) \ge 0 \text{ and } \sum_{\omega\in\Omega} p(\omega) = 1$$

A little thought reveals that this is the most general probability measure on this space. In many cases when $\Omega$ is a finite set, we have $p(\omega) = 1/|\Omega|$ where $|\Omega| =$ the number of points in $\Omega$.

For a simple concrete example that requires this level of generality, consider the astragali, dice used in ancient Egypt made from the ankle bones of sheep. This die could come to rest on the top side of the bone for four points or on the bottom for three points. The side of the bone was slightly rounded. The die could come to rest on a flat and narrow piece for six points or somewhere on the rest of the side for one point. There is no reason to think that all four outcomes are equally likely, so we need probabilities $p_1$, $p_3$, $p_4$, and $p_6$ to describe $P$.

To prepare for our next definition, we need:

**Exercise 1.1.1.** (i) If $\mathcal{F}_i$, $i \in I$ are $\sigma$-fields, then $\cap_{i\in I}\mathcal{F}_i$ is. Here $I \neq \emptyset$ is an arbitrary index set (i.e., possibly uncountable). (ii) Use the result in (i) to show that if we are given a set $\Omega$ and a collection $\mathcal{A}$ of subsets of $\Omega$, then there is a smallest $\sigma$-field containing $\mathcal{A}$. We will call this the **$\sigma$-field generated by** $\mathcal{A}$ and denote it by $\sigma(\mathcal{A})$.

Let $\mathbf{R}^d$ be the set of vectors $(x_1, \ldots x_d)$ of real numbers and $\mathcal{R}^d$ be the **Borel sets,** the smallest $\sigma$-field containing the open sets. When $d = 1$, we drop the superscript.

**Example 1.1.2. Measures on the real line.** Measures on $(\mathbf{R}, \mathcal{R})$ are defined by giving probability a **Stieltjes measure function** with the following properties:
(i) $F$ is nondecreasing.
(ii) $F$ is right continuous, that is, $\lim_{y\downarrow x} F(y) = F(x)$.

**Theorem 1.1.2.** *Associated with each Stieltjes measure function $F$ there is a unique measure $\mu$ on $(\mathbf{R}, \mathcal{R})$ with $\mu(a, b]) = F(b) - F(a)$*

$$\mu((a, b]) = F(b) - F(a) \tag{1.1.1}$$

When $F(x) = x$ the resulting measure is called **Lebesgue measure**.

The proof of Theorem 1.1.2 is a long and winding road, so we will content ourselves with describing the main ideas involved in this section and hide the remaining details in the Appendix in Section A.1. The choice of "closed on the right" in $(a, b]$ is dictated by the fact that if $b_n \downarrow b$ then we have

$$\cap_n (a, b_n] = (a, b]$$

The next definition will explain the choice of "open on the left."

A collection $\mathcal{S}$ of sets is said to be a **semialgebra** if (i) it is closed under intersection, that is, $S, T \in \mathcal{S}$ implies $S \cap T \in \mathcal{S}$, and (ii) if $S \in \mathcal{S}$ then $S^c$ is a finite disjoint union of sets in $\mathcal{S}$. An important example of a semialgebra is:

**Example 1.1.3.** $\mathcal{S}_d =$ the empty set plus all sets of the form

$$(a_1, b_1] \times \cdots \times (a_d, b_d] \subset \mathbf{R}^d \quad \text{where } -\infty \le a_i < b_i \le \infty$$

The definition in (1.1.1) gives the values of $\mu$ on the semialgebra $\mathcal{S}_1$. To go from semialgebra to $\sigma$-algebra we use an intermediate step. A collection $\mathcal{A}$ of subsets of $\Omega$ is called an **algebra** (or **field**) if $A, B \in \mathcal{A}$ implies $A^c$ and $A \cup B$ are in $\mathcal{A}$. Since $A \cap B = (A^c \cup B^c)^c$, it follows that $A \cap B \in \mathcal{A}$. Obviously a $\sigma$-algebra is an algebra. An example in which the converse is false is:

**Example 1.1.4.** Let $\Omega = \mathbf{Z} =$ the integers. $\mathcal{A} =$ the collection of $A \subset \mathbf{Z}$ so that $A$ or $A^c$ is finite is an algebra.

**Lemma 1.1.3.** *If $\mathcal{S}$ is a semialgebra, then $\bar{\mathcal{S}} = \{$finite disjoint unions of sets in $\mathcal{S}\}$ is an algebra, called the* **algebra generated by** $\mathcal{S}$.

*Proof.* Suppose $A = +_i S_i$ and $B = +_j T_j$, where $+$ denotes disjoint union and we assume the index sets are finite. Then $A \cap B = +_{i,j}\, S_i \cap T_j \in \bar{\mathcal{S}}$. As for complements, if $A = +_i S_i$ then $A^c = \cap_i S_i^c$. The definition of $\mathcal{S}$ implies $S_i^c \in \bar{\mathcal{S}}$. We have shown that $\bar{\mathcal{S}}$ is closed under intersection, so it follows by induction that $A^c \in \bar{\mathcal{S}}$. ∎

**Example 1.1.5.** Let $\Omega = \mathbf{R}$ and $\mathcal{S} = \mathcal{S}_1$. Then $\bar{\mathcal{S}}_1 =$ the empty set plus all sets of the form

$$\cup_{i=1}^k (a_i, b_i] \quad \text{where } -\infty \le a_i < b_i \le \infty$$

Given a set function $\mu$ on $\mathcal{S}$, we can extend it to $\bar{\mathcal{S}}$ by

$$\mu\left(+_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i)$$

By a **measure on an algebra** $\mathcal{A}$, we mean a set function $\mu$ with

(i) $\mu(A) \ge \mu(\emptyset) = 0$ for all $A \in \mathcal{A}$, and
(ii) if $A_i \in \mathcal{A}$ are disjoint *and their union is in* $\mathcal{A}$, then

$$\mu\left(\cup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$$

$\mu$ is said to be $\sigma$**-finite** if there is a sequence of sets $A_n \in \mathcal{A}$ so that $\mu(A_n) < \infty$ and $\cup_n A_n = \Omega$. Letting $A_1' = A_1$ and for $n \ge 2$,

$$A_n' = \cup_{m=1}^n A_m \quad \text{or} \quad A_n' = A_n \cap \left(\cap_{m=1}^{n-1} A_m^c\right) \in \mathcal{A}$$

we can without loss of generality assume that $A_n \uparrow \Omega$ or the $A_n$ are disjoint.

The next result helps us to extend a measure defined on a semialgebra $\mathcal{S}$ to the $\sigma$-algebra it generates, $\sigma(\mathcal{S})$

**Theorem 1.1.4.** *Let $\mathcal{S}$ be a semialgebra and let $\mu$ defined on $\mathcal{S}$ have $\mu(\emptyset) = 0$. Suppose (i) if $S \in \mathcal{S}$ is a finite disjoint union of sets $S_i \in \mathcal{S}$ then $\mu(S) = \sum_i \mu(S_i)$, and (ii) if $S_i, S \in \mathcal{S}$ with $S = +_{i \ge 1} S_i$ then $\mu(S) \le \sum_{i \ge 1} \mu(S_i)$. Then $\mu$ has a unique* *extension $\bar{\mu}$ that is a measure on $\bar{\mathcal{S}}$ the algebra generated by $\mathcal{S}$. If $\bar{\mu}$ is sigma-finite then there is a unique extension $\nu$ that is a measure on $\sigma(\mathcal{S})$.*

In (ii) above, and in what follows, $i \geq 1$ indicates a countable union, while a plain subscript $i$ or $j$ indicates a finite union. The proof of Theorems 1.1.4 is rather involved, so it is given in Section A.1. To check condition (ii) in the theorem, the following is useful.

**Lemma 1.1.5.** *Suppose only that (i) holds.*
*(a) If $A, B_i \in \bar{\mathcal{S}}$ with $A = +_{i=1}^n B_i$ then $\bar{\mu}(A) = \sum_i \bar{\mu}(B_i)$.*
*(b) If $A, B_i \in \bar{\mathcal{S}}$ with $A \subset \cup_{i=1}^n B_i$ then $\bar{\mu}(A) \leq \sum_i \bar{\mu}(B_i)$.*

*Proof.* Observe that it follows from the definition that if $A = +_i B_i$ is a finite disjoint union of sets in $\bar{\mathcal{S}}$ and $B_i = +_j S_{i,j}$, then

$$\bar{\mu}(A) = \sum_{i,j} \mu(S_{i,j}) = \sum_i \bar{\mu}(B_i)$$

To prove (b), we begin with the case $n = 1$, $B_1 = B$. $B = A + (B \cap A^c)$ and $B \cap A^c \in \bar{\mathcal{S}}$, so

$$\bar{\mu}(A) \leq \bar{\mu}(A) + \bar{\mu}(B \cap A^c) = \bar{\mu}(B)$$

To handle $n > 1$ now, let $F_k = B_1^c \cap \cdots \cap B_{k-1}^c \cap B_k$ and note

$$\cup_i B_i = F_1 + \cdots + F_n$$
$$A = A \cap (\cup_i B_i) = (A \cap F_1) + \cdots + (A \cap F_n)$$

so using (a), (b) with $n = 1$, and (a) again

$$\bar{\mu}(A) = \sum_{k=1}^n \bar{\mu}(A \cap F_k) \leq \sum_{k=1}^n \bar{\mu}(F_k) = \bar{\mu}(\cup_i B_i) \qquad \blacksquare$$

*Proof of Theorem 1.1.2.* Let $\mathcal{S}$ be the semialgebra of half-open intervals $(a, b]$ with $-\infty \leq a < b \leq \infty$. To define $\mu$ on $\mathcal{S}$, we begin by observing that

$$F(\infty) = \lim_{x \uparrow \infty} F(x) \quad \text{and} \quad F(-\infty) = \lim_{x \downarrow -\infty} F(x) \quad \text{exist}$$

and $\mu((a, b]) = F(b) - F(a)$ makes sense for all $-\infty \leq a < b \leq \infty$ since $F(\infty) > -\infty$ and $F(-\infty) < \infty$.

If $(a, b] = +_{i=1}^n (a_i, b_i]$ then after relabeling the intervals we must have $a_1 = a$, $b_n = b$, and $a_i = b_{i-1}$ for $2 \leq i \leq n$, so condition (i) in Theorem 1.1.4 holds. To check (ii), suppose first that $-\infty < a < b < \infty$, and $(a, b] \subset \cup_{i \geq 1}(a_i, b_i]$ where (without loss of generality) $-\infty < a_i < b_i < \infty$. Pick $\delta > 0$ so that $F(a + \delta) < F(a) + \epsilon$ and pick $\eta_i$ so that

$$F(b_i + \eta_i) < F(b_i) + \epsilon 2^{-i}$$

The open intervals $(a_i, b_i + \eta_i)$ cover $[a+\delta, b]$, so there is a finite subcover $(\alpha_j, \beta_j)$, $1 \le j \le J$. Since $(a+\delta, b] \subset \cup_{j=1}^J (\alpha_j, \beta_j]$, (b) in Lemma 1.1.5 implies

$$F(b) - F(a+\delta) \le \sum_{j=1}^J F(\beta_j) - F(\alpha_j) \le \sum_{i=1}^\infty (F(b_i + \eta_i) - F(a_i))$$

So, by the choice of $\delta$ and $\eta_i$,

$$F(b) - F(a) \le 2\epsilon + \sum_{i=1}^\infty (F(b_i) - F(a_i))$$

and since $\epsilon$ is arbitrary, we have proved the result in the case $-\infty < a < b < \infty$. To remove the last restriction, observe that if $(a, b] \subset \cup_i (a_i, b_i]$ and $(A, B] \subset (a, b]$ has $-\infty < A < B < \infty$, then we have

$$F(B) - F(A) \le \sum_{i=1}^\infty (F(b_i) - F(a_i))$$

Since the last result holds for any finite $(A, B] \subset (a, b]$, the desired result follows. ■

## *Measures on* $\mathbf{R^d}$

Our next goal is to prove a version of Theorem 1.1.2 for $\mathbf{R}^d$. The first step is to introduce the assumptions on the defining function $F$. By analogy with the case $d = 1$ it is natural to assume:

(i) It is nondecreasing, that is, if $x \le y$ (meaning $x_i \le y_i$ for all $i$), then $F(x) \le F(y)$.
(ii) $F$ is right continuous, that is, $\lim_{y \downarrow x} F(y) = F(x)$ (here $y \downarrow x$ means each $y_i \downarrow x_i$).

However this time it is not enough. Consider the following $F$:

$$F(x_1, x_2) = \begin{cases} 1 & \text{if } x_1, x_2 \ge 1 \\ 2/3 & \text{if } x_1 \ge 1 \text{ and } 0 \le x_2 < 1 \\ 2/3 & \text{if } x_2 \ge 1 \text{ and } 0 \le x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

See Figure 1.1 for a picture. A little thought shows that

$$\begin{aligned} \mu((a_1, b_1] \times (a_2, b_2]) = \mu((-\infty, b_1] \times (-\infty, b_2]) - \mu((-\infty, a_1] \times (-\infty, b_2]) \\ - \mu((-\infty, b_1] \times (-\infty, a_2]) + \mu((-\infty, a_1] \times (-\infty, a_2]) \\ = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \end{aligned}$$

Using this with $a_1 = a_2 = 1 - \epsilon$ and $b_1 = b_2 = 1$ and letting $\epsilon \to 0$, we see that

$$\mu(\{1, 1\}) = 1 - 2/3 - 2/3 + 0 = -1/3$$

| | | |
|---|---|---|
| 0 | 2/3 | 1 |
| 0 | 0 | 2/3 |
| 0 | 0 | 0 |

Figure 1.1. Picture of the counterexample.

Similar reasoning shows that $\mu(\{1, 0\}) = \mu(\{0, 1\} = 2/3$.

To formulate the third and final condition for $F$ to define a measure, let

$$A = (a_1, b_1] \times \cdots \times (a_d, b_d]$$
$$V = \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}$$

where $-\infty < a_i < b_i < \infty$. To emphasize that $\infty$'s are not allowed, we will call $A$ a finite rectangle. Then $V =$ the vertices of the rectangle $A$. If $v \in V$, let

$$\text{sgn}\,(v) = (-1)^{\#\text{ of } a\text{'s in } v}$$
$$\Delta_A F = \sum_{v \in V} \text{sgn}\,(v) F(v)$$

We will let $\mu(A) = \Delta_A F$, so we must assume

(iii) $\Delta_A F \geq 0$ for all rectangles $A$.

**Theorem 1.1.6.** *Suppose $F : \mathbf{R}^d \to [0, 1]$ satisfies (i)–(iii) given above. Then there is a unique probability measure $\mu$ on $(\mathbf{R}^d, \mathcal{R}^d)$ so that $\mu(A) = \Delta_A F$ for all finite rectangles.*

**Example 1.1.6.** Suppose $F(x) = \prod_{i=1}^d F_i(x)$, where the $F_i$ satisfy (i) and (ii) of Theorem 1.1.2. In this case,

$$\Delta_A F = \prod_{i=1}^{d} (F_i(b_i) - F_i(a_i))$$

When $F_i(x) = x$ for all $i$, the resulting measure is Lebesgue measure on $\mathbf{R}^d$.

*Proof.* We let $\mu(A) = \Delta_A F$ for all finite rectangles and then use monotonicity to extend the definition to $\mathcal{S}_d$. To check (i) of Theorem 1.1.4, call $A = +_k B_k$ a **regular subdivision** of $A$ if there are sequences $a_i = \alpha_{i,0} < \alpha_{i,1} \ldots < \alpha_{i,n_i} = b_i$ so that each rectangle $B_k$ has the form

$$(\alpha_{1,j_1-1}, \alpha_{1,j_1}] \times \cdots \times (\alpha_{d,j_d-1}, \alpha_{d,j_d}] \quad \text{where } 1 \leq j_i \leq n_i$$

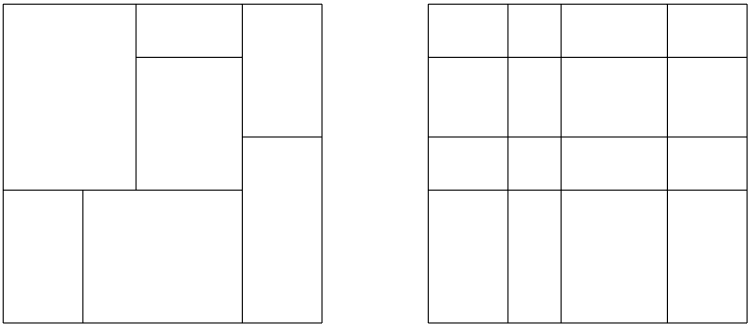


Figure 1.2. Conversion of a subdivision to a regular one.

It is easy to see that for regular subdivisions $\lambda(A) = \sum_k \lambda(B_k)$. (First consider the case in which all the endpoints are finite, and then take limits to get the general case.) To extend this result to a general finite subdivision $A = +_j A_j$, subdivide further to get a regular one see Figure 1.2.

The proof of (ii) is almost identical to that in Theorem 1.1.2. To make things easier to write and to bring out the analogies with Theorem 1.1.2, we let

$$(x, y) = (x_1, y_1) \times \cdots \times (x_d, y_d)$$
$$(x, y] = (x_1, y_1] \times \cdots \times (x_d, y_d]$$
$$[x, y] = [x_1, y_1] \times \cdots \times [x_d, y_d]$$

for $x, y \in \mathbf{R}^d$. Suppose first that $-\infty < a < b < \infty$, where the inequalities mean that each component is finite, and suppose $(a, b] \subset \cup_{i\ge 1}(a^i, b^i]$, where (without loss of generality) $-\infty < a^i < b^i < \infty$. Let $\bar{1} = (1, \dots, 1)$, pick $\delta > 0$ so that

$$\mu((a + \delta\bar{1}, b]) < \mu((a, b]) + \epsilon$$

and pick $\eta_i$ so that

$$\mu((a, b^i + \eta_i\bar{1}]) < \mu((a^i, b^i]) + \epsilon 2^{-i}$$

The open rectangles $(a^i, b^i + \eta_i\bar{1})$ cover $[a + \delta\bar{1}, b]$, so there is a finite subcover $(\alpha^j, \beta^j)$, $1 \le j \le J$. Since $(a + \delta\bar{1}, b] \subset \cup_{j=1}^J (\alpha^j, \beta^j]$, (b) in Lemma 1.1.5 implies

$$\mu([a + \delta\bar{1}, b]) \le \sum_{j=1}^{J} \mu((\alpha^j, \beta^j]) \le \sum_{i=1}^{\infty} \mu((a^i, b^i + \eta_i\bar{1}])$$

So, by the choice of $\delta$ and $\eta_i$,

$$\mu((a, b]) \le 2\epsilon + \sum_{i=1}^{\infty} \mu((a^i, b^i])$$

and since $\epsilon$ is arbitrary, we have proved the result in the case $-\infty < a < b < \infty$. The proof can now be completed exactly as before. ■

***Exercises***

**1.1.2.** Let $\Omega = \mathbf{R}$, $\mathcal{F} =$ all subsets so that $A$ or $A^c$ is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

**1.1.3.** Recall the definition of $\mathcal{S}_d$ from Example 1.1.3. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$, the Borel subsets of $\mathbf{R}^d$.

**1.1.4.** A $\sigma$-field $\mathcal{F}$ is said to be **countably generated** if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is countably generated.

**1.1.5.** (i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ are $\sigma$-algebras, then $\cup_i \mathcal{F}_i$ is an algebra. (ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a $\sigma$-algebra.

**1.1.6.** A set $A \subset \{1, 2, \ldots\}$ is said to have **asymptotic density** $\theta$ if

$$\lim_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a $\sigma$-algebra? an algebra?

## 1.2 Distributions

Probability spaces become a little more interesting when we define random variables on them. A real-valued function $X$ defined on $\Omega$ is said to be a **random variable** if for every Borel set $B \subset \mathbf{R}$ we have $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$. When we need to emphasize the $\sigma$-field, we will say that $X$ is $\mathcal{F}$-**measurable** or write $X \in \mathcal{F}$. If $\Omega$ is a discrete probability space (see Example 1.1.1), then any function $X : \Omega \to \mathbf{R}$ is a random variable. A second trivial, but useful, type of example of a random variable is the **indicator function** of a set $A \in \mathcal{F}$:

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

The notation is supposed to remind you that this function is 1 on $A$. Analysts call this object the characteristic function of $A$. In probability, that term is used for something quite different. (See Section 3.3.)

If $X$ is a random variable, then $X$ induces a probability measure on $\mathbf{R}$ called its **distribution** by setting $\mu(A) = P(X \in A)$ for Borel sets $A$. Using the notation introduced above, the right-hand side can be written as $P(X^{-1}(A))$. In words, we pull $A \in \mathcal{R}$ back to $X^{-1}(A) \in \mathcal{F}$ and then take $P$ of that set.

To check that $\mu$ is a probability measure we observe that if the $A_i$ are disjoint, then using the definition of $\mu$; the fact that $X$ lands in the union if and only if it lands in one of the $A_i$; the fact that if the sets $A_i \in \mathcal{R}$ are disjoint then the events $\{X \in A_i\}$ are disjoint; and the definition of $\mu$ again, we have:

$$\mu(\cup_i A_i) = P(X \in \cup_i A_i) = P(\cup_i \{X \in A_i\}) = \sum_i P(X \in A_i) = \sum_i \mu(A_i)$$

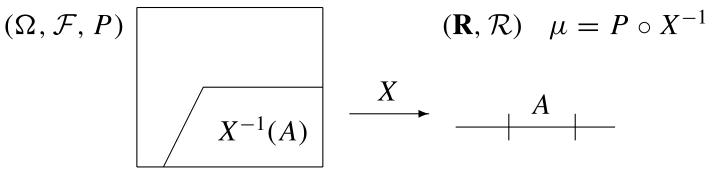



Figure 1.3. Definition of the distribution of $X$.

The distribution of a random variable $X$ is usually described by giving its **distribution function**, $F(x) = P(X \le x)$.

**Theorem 1.2.1.** *Any distribution function $F$ has the following properties:*
*(i)* *$F$ is nondecreasing.*
*(ii)* $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$.
*(iii)* *$F$ is right continuous, that is,* $\lim_{y\downarrow x} F(y) = F(x)$.
*(iv)* *If* $F(x-) = \lim_{y\uparrow x} F(y)$ *then* $F(x-) = P(X < x)$.
*(v)* $P(X = x) = F(x) - F(x-)$.

*Proof.* To prove (i), note that if $x \le y$ then $\{X \le x\} \subset \{X \le y\}$, and then use (i) in Theorem 1.1.1 to conclude that $P(X \le x) \le P(X \le y)$.

To prove (ii), we observe that if $x \uparrow \infty$, then $\{X \le x\} \uparrow \Omega$, while if $x \downarrow -\infty$, then $\{X \le x\} \downarrow \emptyset$, and then use (iii) and (iv) of Theorem 1.1.1.

To prove (iii), we observe that if $y \downarrow x$, then $\{X \le y\} \downarrow \{X \le x\}$.

To prove (iv), we observe that if $y \uparrow x$, then $\{X \le y\} \uparrow \{X < x\}$.

For (v), note $P(X = x) = P(X \le x) - P(X < x)$ and use (iii) and (iv). ■

The next result shows that we have found more than enough properties to characterize distribution functions.

**Theorem 1.2.2.** *If $F$ satisfies (i), (ii), and (iii) in Theroem 1.2.1, then it is the distribution function of some random variable.*

*Proof.* Let $\Omega = (0, 1)$, $\mathcal{F}$ = the Borel sets, and $P$ = Lebesgue measure. If $\omega \in (0, 1)$, let

$$X(\omega) = \sup\{y : F(y) < \omega\}$$

Once we show that

$$(*) \qquad \{\omega : X(\omega) \le x\} = \{\omega : \omega \le F(x)\}$$

the desired result follows immediately since $P(\omega : \omega \le F(x)) = F(x)$. (Recall $P$ is Lebesgue measure.) To check ($\star$), we observe that if $\omega \le F(x)$ then $X(\omega) \le x$, since $x \notin \{y : F(y) < \omega\}$. On the other hand if $\omega > F(x)$, then since $F$ is right continuous, there is an $\epsilon > 0$ so that $F(x + \epsilon) < \omega$ and $X(\omega) \ge x + \epsilon > x$. ■

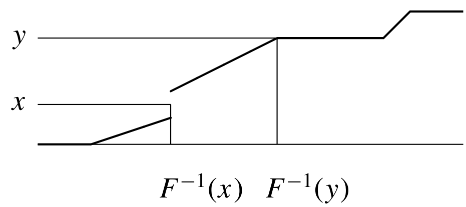



Figure 1.4. Picture of the inverse defined in the proof of Theorem 1.2.2.

Even though $F$ may not be 1-1 and onto, we will call $X$ the inverse of $F$ and denote it by $F^{-1}$. The scheme in the proof of Theorem 1.2.2 is useful in generating random variables on a computer. Standard algorithms generate random variables $U$ with a uniform distribution; then one applies the inverse of the distribution function defined in Theorem 1.2.2 to get a random variable $F^{-1}(U)$ with distribution function $F$.

If $X$ and $Y$ induce the same distribution $\mu$ on $(\mathbf{R}, \mathcal{R})$, we say $X$ and $Y$ are **equal in distribution**. In view of Theorem 1.1.2, this holds if and only if $X$ and $Y$ have the same distribution function, that is, $P(X \le x) = P(Y \le x)$ for all $x$. When $X$ and $Y$ have the same distribution, we like to write

$$X \stackrel{d}{=} Y$$

but this is too tall to use in text, so for typographical reasons we will also use $X =_d Y$.

When the distribution function $F(x) = P(X \le x)$ has the form

$$F(x) = \int_{-\infty}^{x} f(y)\, dy \tag{1.2.1}$$

we say that $X$ has **density function** $f$. In remembering formulas, it is often useful to think of $f(x)$ as being $P(X = x)$ although

$$P(X = x) = \lim_{\epsilon \to 0} \int_{x-\epsilon}^{x+\epsilon} f(y)\, dy = 0$$

By popular demand we have ceased our previous practice of writing $P(X = x)$ for the density function. Instead we will use things like the lovely and informative $f_X(x)$.

We can start with $f$ and use (1.2.1) to define a distribution function $F$. In order to end up with a distribution function it is necessary and sufficient that $f(x) \ge 0$ and $\int f(x)\, dx = 1$. Three examples that will be important in what follows are:

**Example 1.2.1. Uniform distribution on (0,1).** $f(x) = 1$ for $x \in (0, 1)$ and 0 otherwise. Distribution function:

$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

**Example 1.2.2. Exponential distribution with rate $\lambda$.** $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and 0 otherwise. Distribution function:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

**Example 1.2.3. Standard normal distribution.**

$$f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$$

In this case, there is no closed-form expression for $F(x)$, but we have the following bounds that are useful for large $x$:

**Theorem 1.2.3.** *For $x > 0$,*

$$(x^{-1} - x^{-3}) \exp(-x^2/2) \leq \int_x^\infty \exp(-y^2/2) dy \leq x^{-1} \exp(-x^2/2)$$

*Proof.* Changing variables $y = x + z$ and using $\exp(-z^2/2) \leq 1$ gives

$$\int_x^\infty \exp(-y^2/2)\, dy \leq \exp(-x^2/2) \int_0^\infty \exp(-xz)\, dz = x^{-1} \exp(-x^2/2)$$

For the other direction, we observe

$$\int_x^\infty (1 - 3y^{-4}) \exp(-y^2/2)\, dy = (x^{-1} - x^{-3}) \exp(-x^2/2) \qquad \blacksquare$$

A distribution function on **R** is said to be **absolutely continuous** if it has a density and **singular** if the corresponding measure is singular w.r.t. Lebesgue measure. See Section A.4 for more on these notions. An example of a singular distribution is:

**Example 1.2.4. Uniform distribution on the Cantor set.** The Cantor set $C$ is defined by removing $(1/3, 2/3)$ from [0,1] and then removing the middle third of each interval that remains. We define an associated distribution function by setting $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq 1$, $F(x) = 1/2$ for $x \in [1/3, 2/3]$, $F(x) = 1/4$ for $x \in [1/9, 2/9]$, $F(x) = 3/4$ for $x \in [7/9, 8/9]$, ... There is no $f$ for which (1.2.1) holds because such an $f$ would be equal to 0 on a set of measure 1. From the definition, it is immediate that the corresponding measure has $\mu(C^c) = 0$.

A probability measure $P$ (or its associated distribution function) is said to be **discrete** if there is a countable set $S$ with $P(S^c) = 0$. The simplest example of a discrete distribution is

**Example 1.2.5. Point mass at 0.** $F(x) = 1$ for $x \geq 0$, $F(x) = 0$ for $x < 0$.

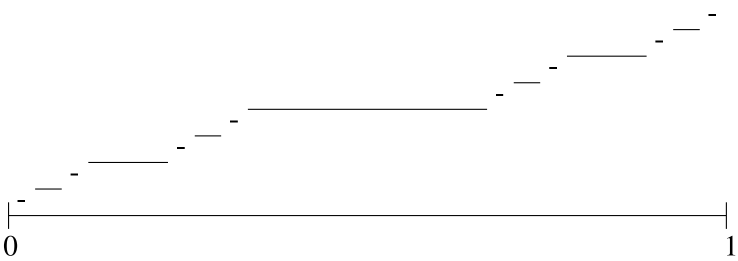


Figure 1.5. Cantor distribution function.

In Section 1.6, we will see the Bernoulli, Poisson, and geometric distributions. The next example shows that the distribution function associated with a discrete probability measure can be quite wild.

**Example 1.2.6. Dense discontinuities.** Let $q_1, q_2, \ldots$ be an enumeration of the rationals. Let $\alpha_i > 0$ have $\sum_{i=1}^{\infty} \alpha_1 = 1$ and let

$$F(x) = \sum_{i=1}^{\infty} \alpha_i 1_{[q_i,\infty)}$$

where $1_{[\theta,\infty)}(x) = 1$ if $x \in [\theta, \infty) = 0$ otherwise.

### *Exercises*

**1.2.1.** Suppose $X$ and $Y$ are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then $Z$ is a random variable.

**1.2.2.** Let $\chi$ have the standard normal distribution. Use Theorem 1.2.3 to get upper and lower bounds on $P(\chi \geq 4)$.

**1.2.3.** Show that a distribution function has at most countably many discontinuities.

**1.2.4.** Show that if $F(x) = P(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on (0,1), that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

**1.2.5.** Suppose $X$ has continuous density $f$, $P(\alpha \leq X \leq \beta) = 1$ and $g$ is a function that is strictly increasing and differentiable on $(\alpha, \beta)$. Then $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = (y - b)/a$, so the answer is $(1/a) f((y - b)/a)$.

**1.2.6.** Suppose $X$ has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. (The answer is called the **lognormal distribution**.)

**1.2.7.** (i) Suppose $X$ has density function $f$. Compute the distribution function of $X^2$ and then differentiate to find its density function. (ii) Work out the answer when $X$ has a standard normal distribution to find the density of the **chi-square distribution**.

## 1.3 Random Variables

In this section, we will develop some results that will help us later to prove that quantities we define are random variables, that is, they are measurable. Since most of what we have to say is true for random elements of an arbitrary measurable space $(S, \mathcal{S})$ and the proofs are the same (sometimes easier), we will develop our results in that generality. First we need a definition. A function $X : \Omega \to S$ is said to be a **measurable map** from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$ if

$$X^{-1}(B) \equiv \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \text{for all } B \in \mathcal{S}$$

If $(S, \mathcal{S}) = (\mathbf{R}^d, \mathcal{R}^d)$ and $d > 1$, then $X$ is called a **random vector**. Of course, if $d = 1$, $X$ is called a **random variable**, or r.v. for short.

The next result is useful for proving that maps are measurable.

**Theorem 1.3.1.** *If $\{\omega : X(\omega) \in A\} \in \mathcal{F}$ for all $A \in \mathcal{A}$ and $\mathcal{A}$* **generates** *$\mathcal{S}$ (i.e., $\mathcal{S}$ is the smallest $\sigma$-field that contains $\mathcal{A}$), then $X$ is measurable.*

*Proof.* Writing $\{X \in B\}$ as shorthand for $\{\omega : X(\omega) \in B\}$, we have

$$\{X \in \cup_i B_i\} = \cup_i \{X \in B_i\}$$
$$\{X \in B^c\} = \{X \in B\}^c$$

So the class of sets $\mathcal{B} = \{B : \{X \in B\} \in \mathcal{F}\}$ is a $\sigma$-field. Since $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{A}$ generates $\mathcal{S}$, $\mathcal{B} \supset \mathcal{S}$. ■

It follows from the two equations displayed in the previous proof that if $\mathcal{S}$ is a $\sigma$-field, then $\{\{X \in B\} : B \in \mathcal{S}\}$ is a $\sigma$-field. It is the smallest $\sigma$-field on $\Omega$ that makes $X$ a measurable map. It is called the **$\sigma$-field generated by** $X$ and denoted $\sigma(X)$. For future reference we note that

$$\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\} \tag{1.3.1}$$

**Example 1.3.1.** If $(S, \mathcal{S}) = (\mathbf{R}, \mathcal{R})$, then possible choices of $\mathcal{A}$ in Theorem 1.3.1 are $\{(-\infty, x] : x \in \mathbf{R}\}$ or $\{(-\infty, x) : x \in \mathbf{Q}\}$ where $\mathbf{Q} =$ the rationals.

**Example 1.3.2.** If $(S, \mathcal{S}) = (\mathbf{R}^d, \mathcal{R}^d)$, a useful choice of $\mathcal{A}$ is

$$\{(a_1, b_1) \times \cdots \times (a_d, b_d) : -\infty < a_i < b_i < \infty\}$$

or occasionally the larger collection of open sets.

**Theorem 1.3.2.** *If $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $f : (S, \mathcal{S}) \to (T, \mathcal{T})$ are measurable maps, then $f(X)$ is a measurable map from $(\Omega, \mathcal{F})$ to $(T, \mathcal{T})$*

*Proof.* Let $B \in \mathcal{T}$. $\{\omega : f(X(\omega)) \in B\} = \{\omega : X(\omega) \in f^{-1}(B)\} \in \mathcal{F}$, since by assumption $f^{-1}(B) \in \mathcal{S}$. ■

From Theorem 1.3.2, it follows immediately that if $X$ is a random variable then so is $cX$ for all $c \in \mathbf{R}$, $X^2$, $\sin(X)$, and so on. The next result shows why we wanted to prove Theorem 1.3.2 for measurable maps.

**Theorem 1.3.3.** *If $X_1, \dots X_n$ are random variables and $f : (\mathbf{R}^n, \mathcal{R}^n) \to (\mathbf{R}, \mathcal{R})$ is measurable, then $f(X_1, \dots, X_n)$ is a random variable.*

*Proof.* In view of Theorem 1.3.2, it suffices to show that $(X_1, \dots, X_n)$ is a random vector. To do this, we observe that if $A_1, \dots, A_n$ are Borel sets then

$$\{(X_1, \dots, X_n) \in A_1 \times \cdots \times A_n\} = \cap_i \{X_i \in A_i\} \in \mathcal{F}$$

Since sets of the form $A_1 \times \cdots \times A_n$ generate $\mathcal{R}^n$, the desired result follows from Theorem 1.3.1. ∎

**Theorem 1.3.4.** *If $X_1, \dots, X_n$ are random variables then $X_1 + \cdots + X_n$ is a random variable.*

*Proof.* In view of Theorem 1.3.3, it suffices to show that $f(x_1, \dots, x_n) = x_1 + \cdots + x_n$ is measurable. To do this, we use Example 1.3.1 and note that $\{x : x_1 + \cdots + x_n < a\}$ is an open set and hence is in $\mathcal{R}^n$. ∎

**Theorem 1.3.5.** *If $X_1, X_2, \dots$ are random variables then so are*

$$\inf_n X_n \qquad \sup_n X_n \qquad \limsup_n X_n \qquad \liminf_n X_n$$

*Proof.* Since the infimum of a sequence is $< a$ if and only if some term is $< a$ (if all terms are $\geq a$, then so is the infimum), we have

$$\{\inf_n X_n < a\} = \cup_n \{X_n < a\} \in \mathcal{F}$$

A similar argument shows $\{\sup_n X_n > a\} = \cup_n \{X_n > a\} \in \mathcal{F}$. For the last two, we observe

$$\liminf_{n\to\infty} X_n = \sup_n \left( \inf_{m \geq n} X_m \right)$$

$$\limsup_{n\to\infty} X_n = \inf_n \left( \sup_{m \geq n} X_m \right)$$

To complete the proof in the first case, note that $Y_n = \inf_{m \geq n} X_m$ is a random variable for each $n$, so $\sup_n Y_n$ is as well. ∎

From Theorem 1.3.5, we see that

$$\Omega_o \equiv \{\omega : \lim_{n\to\infty} X_n \text{ exists }\} = \{\omega : \limsup_{n\to\infty} X_n - \liminf_{n\to\infty} X_n = 0\}$$

is a measurable set. (Here $\equiv$ indicates that the first equality is a definition.) If $P(\Omega_o) = 1$, we say that $X_n$ **converges almost surely**, or a.s. for short. This type of convergence is called almost everywhere in measure theory. To have a limit defined on the whole space, it is convenient to let

$$X_\infty = \limsup_{n\to\infty} X_n$$

but this random variable may take the value $+\infty$ or $-\infty$. To accommodate this and some other headaches, we will generalize the definition of random variable.

A function whose domain is a set $D \in \mathcal{F}$ and whose range is $\mathbf{R}^* \equiv [-\infty, \infty]$ is said to be a **random variable** if for all $B \in \mathcal{R}^*$ we have $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$. Here $\mathcal{R}^* =$ the Borel subsets of $\mathbf{R}^*$ with $\mathbf{R}^*$ given the usual topology, that is, the one generated by intervals of the form $[-\infty, a)$, $(a, b)$ and $(b, \infty]$ where $a, b \in \mathbf{R}$. The reader should note that the **extended real line** $(\mathbf{R}^*, \mathcal{R}^*)$ is a measurable space, so all the results above generalize immediately.

### *Exercises*

**1.3.1.** Show that if $\mathcal{A}$ generates $\mathcal{S}$, then $X^{-1}(\mathcal{A}) \equiv \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$.

**1.3.2.** Prove Theorem 1.3.4 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.

**1.3.3.** Show that if $f$ is continuous and $X_n \to X$ almost surely, then $f(X_n) \to f(X)$ almost surely.

**1.3.4.** (i) Show that a continuous function from $\mathbf{R}^d \to \mathbf{R}$ is a measurable map from $(\mathbf{R}^d, \mathcal{R}^d)$ to $(\mathbf{R}, \mathcal{R})$. (ii) Show that $\mathcal{R}^d$ is the smallest $\sigma$-field that makes all the continuous functions measurable.

**1.3.5.** A function $f$ is said to be **lower semicontinuous** or l.s.c. if

$$\liminf_{y\to x} f(y) \ge f(x)$$

and **upper semicontinuous** (u.s.c.) if $-f$ is l.s.c. Show that $f$ is l.s.c. if and only if $\{x : f(x) \le a\}$ is closed for each $a \in \mathbf{R}$ and conclude that semicontinuous functions are measurable.

**1.3.6.** Let $f : \mathbf{R}^d \to \mathbf{R}$ be an arbitrary function and let $f^\delta(x) = \sup\{f(y) : |y - x| < \delta\}$ and $f_\delta(x) = \inf\{f(y) : |y - x| < \delta\}$ where $|z| = (z_1^2 + \cdots + z_d^2)^{1/2}$. Show that $f^\delta$ is l.s.c. and $f_\delta$ is u.s.c. Let $f^0 = \lim_{\delta\downarrow 0} f^\delta$, $f_0 = \lim_{\delta\downarrow 0} f_\delta$, and conclude that the set of points at which $f$ is discontinuous $= \{f^0 \ne f_0\}$ is measurable.

**1.3.7.** A function $\varphi : \Omega \to \mathbf{R}$ is said to be **simple** if

$$\varphi(\omega) = \sum_{m=1}^{n} c_m 1_{A_m}(\omega)$$

where the $c_m$ are real numbers and $A_m \in \mathcal{F}$. Show that the class of $\mathcal{F}$ measurable functions is the smallest class containing the simple functions and closed under pointwise limits.

**1.3.8.** Use the previous exercise to conclude that $Y$ is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$ where $f : \mathbf{R} \to \mathbf{R}$ is measurable.

**1.3.9.** To get a constructive proof of the last result, note that $\{\omega : m2^{-n} \le Y < (m+1)2^{-n}\} = \{X \in B_{m,n}\}$ for some $B_{m,n} \in \mathcal{R}$ and set $f_n(x) = m2^{-n}$ for $x \in B_{m,n}$ and show that as $n \to \infty$ $f_n(x) \to f(x)$ and $Y = f(X)$.

## 1.4 Integration

Let $\mu$ be a $\sigma$-finite measure on $(\Omega, \mathcal{F})$. We will be primarily interested in the special case $\mu$ is a probability measure, but we will sometimes need to integrate with respect to infinite measure, and and it is no harder to develop the results in general.

In this section we will define $\int f \, d\mu$ for a class of measurable functions. This is a four-step procedure:

1. Simple functions
2. Bounded functions
3. Nonnegative functions
4. General functions

This sequence of four steps is also useful in proving integration formulas. See, for example, the proofs of Theorems 1.6.9 and 1.7.2.

*Step 1.* $\varphi$ is said to be a **simple function** if $\varphi(\omega) = \sum_{i=1}^n a_i 1_{A_i}$ and $A_i$ are disjoint sets with $\mu(A_i) < \infty$. If $\varphi$ is a simple function, we let

$$\int \varphi \, d\mu = \sum_{i=1}^n a_i \mu(A_i)$$

The representation of $\varphi$ is not unique since we have not supposed that the $a_i$ are distinct. However, it is easy to see that the last definition does not contradict itself.

We will prove the next three conclusions four times, but before we can state them for the first time, we need a definition. $\varphi \ge \psi$ $\mu$-**almost everywhere** (or $\varphi \ge \psi$ $\mu$-a.e.) means $\mu(\{\omega : \varphi(\omega) < \psi(\omega)\}) = 0$. When there is no doubt about what measure we are referring to, we drop the $\mu$.

**Lemma 1.4.1.** *Let $\varphi$ and $\psi$ be simple functions.*
*(i) If $\varphi \ge 0$ a.e. then $\int \varphi \, d\mu \ge 0$.*
*(ii) For any $a \in \mathbf{R}$, $\int a\varphi \, d\mu = a \int \varphi \, d\mu$.*
*(iii) $\int \varphi + \psi \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu$.*

*Proof.* (i) and (ii) are immediate consequences of the definition. To prove (iii), suppose

$$\varphi = \sum_{i=1}^{m} a_i 1_{A_i} \quad \text{and} \quad \psi = \sum_{j=1}^{n} b_j 1_{B_j}$$

To make the supports of the two functions the same, we let $A_0 = \cup_i B_i - \cup_i A_i$, let $B_0 = \cup_i A_i - \cup_i B_i$, and let $a_0 = b_0 = 0$. Now

$$\varphi + \psi = \sum_{i=0}^{m} \sum_{j=0}^{n} (a_i + b_j) 1_{(A_i \cap B_j)}$$

and the $A_i \cap B_j$ are pairwise disjoint, so

$$\begin{aligned}
\int (\varphi + \psi)\, d\mu &= \sum_{i=0}^{m} \sum_{j=0}^{n} (a_i + b_j) \mu(A_i \cap B_j) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i \mu(A_i \cap B_j) + \sum_{j=0}^{n} \sum_{i=0}^{m} b_j \mu(A_i \cap B_j) \\
&= \sum_{i=0}^{m} a_i\, \mu(A_i) + \sum_{j=0}^{n} b_j\, \mu(B_j) = \int \varphi\, d\mu + \int \psi\, d\mu
\end{aligned}$$

In the next-to-last step, we used $A_i = +_j (A_i \cap B_j)$ and $B_j = +_i (A_i \cap B_j)$, where $+$ denotes a disjoint union. ■

We will prove (i)–(iii) three more times as we generalize our integral. As a consequence of (i)–(iii), we get three more useful properties. To keep from repeating their proofs, which do not change, we will prove:

**Lemma 1.4.2.** *If (i) and (iii) hold then we have:*
*(iv) If $\varphi \le \psi$ a.e. then $\int \varphi\, d\mu \le \int \psi\, d\mu$.*
*(v) If $\varphi = \psi$ a.e. then $\int \varphi\, d\mu = \int \psi\, d\mu$.*
*If, in addition, (ii) holds when $a = -1$ we have*
*(vi) $|\int \phi\, d\mu| \le \int |\phi|\, d\mu$*

*Proof.* By (iii), $\int \psi\, d\mu = \int \phi\, d\mu + \int (\psi - \phi)\, d\mu$ and the second integral is $\ge 0$ by (i), so (iv) holds. $\varphi = \psi$ a.e. implies $\varphi \le \psi$ a.e. and $\psi \le \varphi$ a.e, so (v) follows from two applications of (iv). To prove (vi) now, notice that $\phi \le |\phi|$, so (iv) implies $\int \phi\, d\mu \le \int |\phi|\, d\mu$. $-\phi \le |\phi|$, so (iv) and (ii) imply $-\int \phi\, d\mu \le \int |\phi|\, d\mu$. Since $|y| = \max(y, -y)$, the result follows. ■

*Step 2.* Let $E$ be a set with $\mu(E) < \infty$ and let $f$ be a bounded function that vanishes on $E^c$. To define the integral of $f$, we observe that if $\varphi, \psi$ are simple functions that have $\varphi \le f \le \psi$, then we want to have

$$\int \varphi \, d\mu \le \int f \, d\mu \le \int \psi \, d\mu$$

so we let

$$\int f \, d\mu = \sup_{\phi \le f} \int \varphi \, d\mu = \inf_{\psi \ge f} \int \psi \, d\mu \tag{1.4.1}$$

Here and for the rest of Step 2, we assume that $\varphi$ and $\psi$ vanish on $E^c$. To justify the definition, we have to prove that the sup and inf are equal. It follows from (iv) in Lemma 1.4.2 that

$$\sup_{\phi \le f} \int \varphi \, d\mu \le \inf_{\psi \ge f} \int \psi \, d\mu$$

To prove the other inequality, suppose $|f| \le M$ and let

$$E_k = \left\{ x \in E : \frac{kM}{n} \ge f(x) > \frac{(k-1)M}{n} \right\} \quad \text{for } -n \le k \le n$$

$$\psi_n(x) = \sum_{k=-n}^{n} \frac{kM}{n} 1_{E_k} \qquad \varphi_n(x) = \sum_{k=-n}^{n} \frac{(k-1)M}{n} 1_{E_k}$$

By definition, $\psi_n(x) - \varphi_n(x) = (M/n)1_E$, so

$$\int \psi_n(x) - \varphi_n(x) \, d\mu = \frac{M}{n} \mu(E)$$

Since $\varphi_n(x) \le f(x) \le \psi_n(x)$, it follows from (iii) in Lemma 1.4.1 that

$$\begin{aligned} \sup_{\phi \le f} \int \varphi \, d\mu \ge \int \varphi_n \, d\mu &= -\frac{M}{n}\mu(E) + \int \psi_n \, d\mu \\ &\ge -\frac{M}{n}\mu(E) + \inf_{\psi \ge f} \int \psi \, d\mu \end{aligned}$$

The last inequality holds for all $n$, so the proof is complete. ■

**Lemma 1.4.3.** *Let $E$ be a set with $\mu(E) < \infty$. If $f$ and $g$ are bounded functions that vanish on $E^c$ then:*

*(i) If $f \ge 0$ a.e. then $\int f \, d\mu \ge 0$.*
*(ii) For any $a \in \mathbf{R}$, $\int af \, d\mu = a \int f \, d\mu$.*
*(iii) $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.*
*(iv) If $g \le f$ a.e. then $\int g \, d\mu \le \int f \, d\mu$.*
*(v) If $g = f$ a.e. then $\int g \, d\mu = \int f \, d\mu$.*
*(vi) $|\int f \, d\mu| \le \int |f| \, d\mu$.*

*Proof.* Since we can take $\phi \equiv 0$, (i) is clear from the definition. To prove (ii), we observe that if $a > 0$, then $a\varphi \le af$ if and only if $\varphi \le f$, so

$$\int af\,d\mu = \sup_{\phi \le f} \int a\varphi\,d\mu = \sup_{\phi \le f} a \int \varphi\,d\mu = a \sup_{\phi \le f} \int \varphi\,d\mu = a \int f\,d\mu$$

For $a < 0$, we observe that $a\varphi \le af$ if and only if $\varphi \ge f$, so

$$\int af\,d\mu = \sup_{\phi \ge f} \int a\varphi\,d\mu = \sup_{\phi \ge f} a \int \varphi\,d\mu = a \inf_{\phi \ge f} \int \varphi\,d\mu = a \int f\,d\mu$$

To prove (iii), we observe that if $\psi_1 \ge f$ and $\psi_2 \ge g$, then $\psi_1 + \psi_2 \ge f + g$, so

$$\inf_{\psi \ge f+g} \int \psi\,d\mu \le \inf_{\psi_1 \ge f, \psi_2 \ge g} \int \psi_1 + \psi_2\,d\mu$$

Using linearity for simple functions, it follows that

$$\begin{aligned}\int f + g\,d\mu &= \inf_{\psi \ge f+g} \int \psi\,d\mu \\ &\le \inf_{\psi_1 \ge f, \psi_2 \ge g} \int \psi_1\,d\mu + \int \psi_2\,d\mu = \int f\,d\mu + \int g\,d\mu\end{aligned}$$

To prove the other inequality, observe that the last conclusion applied to $-f$ and $-g$ and (ii) imply

$$-\int f + g\,d\mu \le -\int f\,d\mu - \int g\,d\mu$$

(iv)–(vi) follow from (i)–(iii) by Lemma 1.4.2. ∎

**Notation.** We define the integral of $f$ over the set $E$:

$$\int_E f\,d\mu \equiv \int f \cdot 1_E\,d\mu$$

*Step 3.* If $f \ge 0$, then we let

$$\int f\,d\mu = \sup\left\{\int h\,d\mu : 0 \le h \le f, h \text{ is bounded and } \mu(\{x : h(x) > 0\}) < \infty\right\}$$

The last definition is nice since it is clear that this is well defined. The next result will help us compute the value of the integral.

**Lemma 1.4.4.** *Let $E_n \uparrow \Omega$ have $\mu(E_n) < \infty$ and let $a \wedge b = \min(a, b)$. Then*

$$\int_{E_n} f \wedge n\,d\mu \uparrow \int f\,d\mu \quad \text{as } n \uparrow \infty$$

*Proof.* It is clear that from (iv) in Lemma 1.4.3 that the left-hand side increases as $n$ does. Since $h = (f \wedge n)1_{E_n}$ is a possibility in the sup, each term is smaller than the integral on the right. To prove that the limit is $\int f\,d\mu$, observe that if $0 \le h \le f$, $h \le M$, and $\mu(\{x : h(x) > 0\}) < \infty$, then for $n \ge M$ using $h \le M$, (iv), and (iii),

$$\int_{E_n} f \wedge n\,d\mu \ge \int_{E_n} h\,d\mu = \int h\,d\mu - \int_{E_n^c} h\,d\mu$$

Now $0 \le \int_{E_n^c} h\,d\mu \le M\mu(E_n^c \cap \{x : h(x) > 0\}) \to 0$ as $n \to \infty$, so

$$\liminf_{n\to\infty} \int_{E_n} f \wedge n\,d\mu \ge \int h\,d\mu$$

which proves the desired result since $h$ is an arbitrary member of the class that defines the integral of $f$. ■

**Lemma 1.4.5.** *Suppose $f, g \ge 0$.*
*(i) $\int f\,d\mu \ge 0$*
*(ii) If $a > 0$ then $\int af\,d\mu = a\int f\,d\mu$.*
*(iii) $\int f + g\,d\mu = \int f\,d\mu + \int g\,d\mu$*
*(iv) If $0 \le g \le f$ a.e. then $\int g\,d\mu \le \int f\,d\mu$.*
*(v) If $0 \le g = f$ a.e. then $\int g\,d\mu = \int f\,d\mu$.*

Here we have dropped (vi) because it is trivial for $f \ge 0$.

*Proof.* (i) is trivial from the definition. (ii) is clear, since when $a > 0$, $ah \le af$ if and only if $h \le f$ and we have $\int ah\,d\mu = a\int h\,du$ for $h$ in the defining class. For (iii), we observe that if $f \ge h$ and $g \ge k$, then $f + g \ge h + k$ so taking the sup over $h$ and $k$ in the defining classes for $f$ and $g$ gives

$$\int f + g\,d\mu \ge \int f\,d\mu + \int g\,d\mu$$

To prove the other direction, we observe $(a + b) \wedge n \le (a \wedge n) + (b \wedge n)$, so (iv) from Lemma 1.4.3 and (iii) from Lemma 1.4.4 imply

$$\int_{E_n} (f + g) \wedge n\,d\mu \le \int_{E_n} f \wedge n\,d\mu + \int_{E_n} g \wedge n\,d\mu$$

Letting $n \to \infty$ and using Lemma 1.4.4 gives (iii). As before, (iv) and (v) follow from (i), (iii), and Lemma 1.4.2. ■

*Step 4.* We say $f$ is integrable if $\int |f|\,d\mu < \infty$. Let

$$f^+(x) = f(x) \vee 0 \quad \text{and} \quad f^-(x) = (-f(x)) \vee 0$$

where $a \vee b = \max(a, b)$. Clearly,

$$f(x) = f^+(x) - f^-(x) \quad \text{and} \quad |f(x)| = f^+(x) + f^-(x)$$

We define the integral of $f$ by

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu$$

The right-hand side is well defined since $f^+, f^- \le |f|$ and we have (iv) in Lemma 1.4.5. For the final time, we will prove our six properties. To do this, it is useful to know:

**Lemma 1.4.6.** *If $f = f_1 - f_2$ where $f_1, f_2 \ge 0$ and $\int f_i\, d\mu < \infty$, then*

$$\int f\, d\mu = \int f_1\, d\mu - \int f_2\, d\mu$$

*Proof.* $f_1 + f^- = f_2 + f^+$ and all four functions are $\ge 0$, so by (iii) of Lemma 1.4.5,

$$\int f_1\, d\mu + \int f^-\, d\mu = \int f_1 + f^-\, d\mu = \int f_2 + f^+\, d\mu = \int f_2\, d\mu + \int f^+\, d\mu$$

Rearranging gives the desired conclusion. ■

**Theorem 1.4.7.** *Suppose $f$ and $g$ are integrable.*

*(i) If $f \ge 0$ a.e. then $\int f\, d\mu \ge 0$.*
*(ii) For all $a \in \mathbf{R}$, $\int af\, d\mu = a\int f\, d\mu$.*
*(iii) $\int f + g\, d\mu = \int f\, d\mu + \int g\, d\mu$.*
*(iv) If $g \le f$ a.e. then $\int g\, d\mu \le \int f\, d\mu$.*
*(v) If $g = f$ a.e. then $\int g\, d\mu = \int f\, d\mu$.*
*(vi) $|\int f\, d\mu| \le \int |f|\, d\mu$.*

*Proof.* (i) is trivial. (ii) is clear since if $a > 0$, then $(af)^+ = a(f^+)$, and so on. To prove (iii), observe that $f + g = (f^+ + g^+) - (f^- + g^-)$, so using Lemma 1.4.6 and Lemma 1.4.5,

$$\begin{aligned}\int f + g\, d\mu &= \int f^+ + g^+\, d\mu - \int f^- + g^-\, d\mu \\ &= \int f^+\, d\mu + \int g^+\, d\mu - \int f^-\, d\mu - \int g^-\, d\mu\end{aligned}$$

As usual, (iv)–(vi) follow from (i)–(iii) and Lemma 1.4.2. ■

**Notation for special cases**

(a) When $(\Omega, \mathcal{F}, \mu) = (\mathbf{R}^d, \mathcal{R}^d, \lambda)$, we write $\int f(x)\, dx$ for $\int f\, d\lambda$.
(b) When $(\Omega, \mathcal{F}, \mu) = (\mathbf{R}, \mathcal{R}, \lambda)$ and $E = [a, b]$, we write $\int_a^b f(x)\, dx$ for $\int_E f\, d\lambda$.
(c) When $(\Omega, \mathcal{F}, \mu) = (\mathbf{R}, \mathcal{R}, \mu)$ with $\mu((a, b]) = G(b) - G(a)$ for $a < b$, we write $\int f(x)\, dG(x)$ for $\int f\, d\mu$.
(d) When $\Omega$ is a countable set, $\mathcal{F} =$ all subsets of $\Omega$, and $\mu$ is counting measure, we write $\sum_{i\in\Omega} f(i)$ for $\int f\, d\mu$.

We mention example (d) primarily to indicate that results for sums follow from those for integrals. The notation for the special case in which $\mu$ is a probability measure will be taken up in Section 1.6.

***Exercises***

**1.4.1.** Show that if $f \geq 0$ and $\int f \, d\mu = 0$, then $f = 0$ a.e.

**1.4.2.** Let $f \geq 0$ and $E_{n,m} = \{x : m/2^n \leq f(x) < (m+1)/2^n\}$. As $n \uparrow \infty$,

$$\sum_{m=1}^{\infty} \frac{m}{2^n} \mu(E_{n,m}) \uparrow \int f \, d\mu$$

**1.4.3.** Let $g$ be an integrable function on $\mathbf{R}$ and $\epsilon > 0$. (i) Use the definition of the integral to conclude there is a simple function $\varphi = \sum_k b_k 1_{A_k}$ with $\int |g - \varphi| \, dx < \epsilon$. (ii) Use Exercise A.2.1 to approximate the $A_k$ by finite unions of intervals to get a **step function**

$$q = \sum_{j=1}^{k} c_j 1_{(a_{j-1}, a_j)}$$

with $a_0 < a_1 < \cdots < a_k$, so that $\int |\varphi - q| < \epsilon$. (iii) Round the corners of $q$ to get a continuous function $r$ so that $\int |q - r| \, dx < \epsilon$.

**1.4.4.** Prove the **Riemann-Lebesgue lemma.** If $g$ is integrable then

$$\lim_{n \to \infty} \int g(x) \cos nx \, dx = 0$$

Hint: If $g$ is a step function, this is easy. Now use the previous exercise.

## 1.5 Properties of the Integral

In this section, we will develop properties of the integral defined in the last section. Our first result generalizes (vi) from Theorem 1.4.7.

**Theorem 1.5.1. Jensen's inequality.** *Suppose $\varphi$ is convex, that is,*

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \geq \varphi(\lambda x + (1-\lambda)y)$$

*for all $\lambda \in (0,1)$ and $x, y \in \mathbf{R}$. If $\mu$ is a probability measure, and $f$ and $\varphi(f)$ are integrable then*

$$\varphi\left(\int f \, d\mu\right) \leq \int \varphi(f) \, d\mu$$

*Proof.* Let $c = \int f \, d\mu$ and let $\ell(x) = ax + b$ be a linear function that has $\ell(c) = \varphi(c)$ and $\varphi(x) \geq \ell(x)$. To see that such a function exists, recall that convexity implies

$$\lim_{h \downarrow 0} \frac{\varphi(c) - \varphi(c-h)}{h} \leq \lim_{h \downarrow 0} \frac{\varphi(c+h) - \varphi(c)}{h}$$

(The limits exist since the sequences are monotone.) If we let $a$ be any number between the two limits and let $\ell(x) = a(x-c) + \varphi(c)$, then $\ell$ has the desired properties. With the existence of $\ell$ established, the rest is easy. (iv) in Theorem 1.4.7 implies

$$\int \varphi(f)\,d\mu \geq \int (af+b)\,d\mu = a\int f\,d\mu + b = \ell\left(\int f\,d\mu\right) = \varphi\left(\int f\,d\mu\right)$$

since $c = \int f\,d\mu$ and $\ell(c) = \phi(c)$. ∎

Let $\|f\|_p = (\int |f|^p\,d\mu)^{1/p}$ for $1 \leq p < \infty$, and notice $\|cf\|_p = |c| \cdot \|f\|_p$ for any real number $c$.

**Theorem 1.5.2. Hölder's inequality.** *If $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, then*

$$\int |fg|\,d\mu \leq \|f\|_p \|g\|_q$$

*Proof.* If $\|f\|_p$ or $\|g\|_q = 0$, then $|fg| = 0$ a.e., so it suffices to prove the result when $\|f\|_p$ and $\|g\|_q > 0$ or by dividing both sides by $\|f\|_p\|g\|_q$, when $\|f\|_p = \|g\|_q = 1$. Fix $y \geq 0$ and let

$$\varphi(x) = x^p/p + y^q/q - xy \quad \text{for } x \geq 0$$
$$\varphi'(x) = x^{p-1} - y \quad \text{and} \quad \varphi''(x) = (p-1)x^{p-2}$$

so $\varphi$ has a minimum at $x_o = y^{1/(p-1)}$. $q = p/(p-1)$ and $x_o^p = y^{p/(p-1)} = y^q$, so

$$\varphi(x_o) = y^q(1/p + 1/q) - y^{1/(p-1)}y = 0$$

Since $x_o$ is the minimum, it follows that $xy \leq x^p/p + y^q/q$. Letting $x = |f|$, $y = |g|$, and integrating

$$\int |fg|\,d\mu \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p\|g\|_q$$ ∎

**Remark.** The special case $p = q = 2$ is called the **Cauchy-Schwarz inequality**. One can give a direct proof of the result in this case by observing that for any $\theta$,

$$0 \leq \int (f+\theta g)^2\,d\mu = \int f^2\,d\mu + \theta\left(2\int fg\,d\mu\right) + \theta^2\left(\int g^2\,d\mu\right)$$

so the quadratic $a\theta^2 + b\theta + c$ on the right-hand side has at most one real root. Recalling the formula for the roots of a quadratic

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we see $b^2 - 4ac \leq 0$, which is the desired result.

Our next goal is to give conditions that guarantee

$$\lim_{n\to\infty} \int f_n\,d\mu = \int \left(\lim_{n\to\infty} f_n\right)\,d\mu$$

First, we need a definition. We say that $f_n \to f$ **in measure**, that is, for any $\epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$ as $n \to \infty$. On a space of finite measure, this is a weaker assumption than $f_n \to f$ a.e., but the next result is easier to prove in the greater generality.

**Theorem 1.5.3. Bounded convergence theorem.** *Let $E$ be a set with $\mu(E) < \infty$. Suppose $f_n$ vanishes on $E^c$, $|f_n(x)| \le M$, and $f_n \to f$ in measure. Then*

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

**Example 1.5.1.** Consider the real line **R** equipped with the Borel sets $\mathcal{R}$ and Lebesgue measure $\lambda$. The functions $f_n(x) = 1/n$ on $[0, n]$ and 0 otherwise on show that the conclusion of Theorem 1.5.3 does not hold when $\mu(E) = \infty$.

*Proof.* Let $\epsilon > 0$, $G_n = \{x : |f_n(x) - f(x)| < \epsilon\}$ and $B_n = E - G_n$. Using (iii) and (vi) from Theorem 1.4.7,

$$\begin{aligned}\left|\int f\,d\mu - \int f_n\,d\mu\right| &= \left|\int (f - f_n)\,d\mu\right| \le \int |f - f_n|\,d\mu \\ &= \int_{G_n} |f - f_n|\,d\mu + \int_{B_n} |f - f_n|\,d\mu \\ &\le \epsilon\mu(E) + 2M\mu(B_n)\end{aligned}$$

$f_n \to f$ in measure implies $\mu(B_n) \to 0$. $\epsilon > 0$ is arbitrary and $\mu(E) < \infty$, so the proof is complete. ■

**Theorem 1.5.4. Fatou's lemma.** *If $f_n \ge 0$ then*

$$\liminf_{n\to\infty} \int f_n\,d\mu \ge \int \left(\liminf_{n\to\infty} f_n\right)\,d\mu$$

**Example 1.5.2.** Example 1.5.1 shows that we may have strict inequality in Theorem 1.5.4. The functions $f_n(x) = n1_{(0,1/n]}(x)$ on (0,1) equipped with the Borel sets and Lebesgue measure show that this can happen on a space of finite measure.

*Proof.* Let $g_n(x) = \inf_{m\ge n} f_m(x)$. $f_n(x) \ge g_n(x)$ and as $n \uparrow \infty$,

$$g_n(x) \uparrow g(x) = \liminf_{n\to\infty} f_n(x)$$

Since $\int f_n\,d\mu \ge \int g_n\,d\mu$, it suffices then to show that

$$\liminf_{n\to\infty} \int g_n\,d\mu \ge \int g\,d\mu$$

Let $E_m \uparrow \Omega$ be sets of finite measure. Since $g_n \ge 0$ and for fixed $m$

$$(g_n \wedge m)\cdot 1_{E_m} \to (g \wedge m)\cdot 1_{E_m} \quad \text{a.e.}$$

the bounded convergence theorem, 1.5.3, implies

$$\liminf_{n\to\infty} \int g_n \, d\mu \geq \int_{E_m} g_n \wedge m \, d\mu \to \int_{E_m} g \wedge m \, d\mu$$

Taking the sup over $m$ and using Theorem 1.4.4 gives the desired result. ■

**Theorem 1.5.5. Monotone convergence theorem.** *If $f_n \geq 0$ and $f_n \uparrow f$ then*

$$\int f_n \, d\mu \uparrow \int f \, d\mu$$

*Proof.* Fatou's lemma, Theorem 1.5.4, implies $\liminf \int f_n \, d\mu \geq \int f \, d\mu$. On the other hand, $f_n \leq f$ implies $\limsup \int f_n \, d\mu \leq \int f \, d\mu$. ■

**Theorem 1.5.6. Dominated convergence theorem.** *If $f_n \to f$ a.e., $|f_n| \leq g$ for all $n$, and $g$ is integrable, then $\int f_n \, d\mu \to \int f \, d\mu$.*

*Proof.* $f_n + g \geq 0$ so Fatou's lemma implies

$$\liminf_{n\to\infty} \int f_n + g \, d\mu \geq \int f + g \, d\mu$$

Subtracting $\int g \, d\mu$ from both sides gives

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int f \, d\mu$$

Applying the last result to $-f_n$, we get

$$\limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$$

and the proof is complete. ■

## *Exercises*

**1.5.1.** Let $\|f\|_\infty = \inf\{M : \mu(\{x : |f(x)| > M\}) = 0\}$. Prove that

$$\int |fg| d\mu \leq \|f\|_1 \|g\|_\infty$$

**1.5.2.** Show that if $\mu$ is a probability measure then

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$$

**1.5.3. Minkowski's inequality.** (i) Suppose $p \in (1, \infty)$. The inequality $|f + g|^p \leq 2^p(|f|^p + |g|^p)$ shows that if $\|f\|_p$ and $\|g\|_p$ are $< \infty$ then $\|f + g\|_p < \infty$. Apply Hölder's inequality to $|f||f + g|^{p-1}$ and $|g||f + g|^{p-1}$ to show $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. (ii) Show that the last result remains true when $p = 1$ or $p = \infty$.

**1.5.4.** If $f$ is integrable and $E_m$ are disjoint sets with union $E$ then

$$\sum_{m=0}^{\infty} \int_{E_m} f \, d\mu = \int_E f \, d\mu$$

So if $f \geq 0$, then $\nu(E) = \int_E f \, d\mu$ defines a measure.

**1.5.5.** If $g_n \uparrow g$ and $\int g_1^- \, d\mu < \infty$ then $\int g_n \, d\mu \uparrow \int g \, d\mu$.

**1.5.6.** If $g_m \geq 0$ then $\int \sum_{m=0}^{\infty} g_m \, d\mu = \sum_{m=0}^{\infty} \int g_m \, d\mu$.

**1.5.7.** Let $f \geq 0$. (i) Show that $\int f \wedge n \, d\mu \uparrow \int f \, d\mu$ as $n \to \infty$. (ii) Use (i) to conclude that if $g$ is integrable and $\epsilon > 0$, then we can pick $\delta > 0$ so that $\mu(A) < \delta$ implies $\int_A |g| d\mu < \epsilon$.

**1.5.8.** Show that if $f$ is integrable on $[a, b]$, $g(x) = \int_{[a,x]} f(y) \, dy$ is continuous on $(a, b)$.

**1.5.9.** Show that if $f$ has $\|f\|_p = (\int |f|^p d\mu)^{1/p} < \infty$, then there are simple functions $\phi_n$ so that $\|\phi_n - f\|_p \to 0$.

**1.5.10.** Show that if $\sum_n \int |f_n| d\mu < \infty$ then $\sum_n \int f_n \, d\mu = \int \sum_n f_n \, d\mu$.

## 1.6 Expected Value

We now specialize to integration with respect to a probability measure $P$. If $X \geq 0$ is a random variable on $(\Omega, \mathcal{F}, P)$ then we define its **expected value** to be $EX = \int X \, dP$, which always makes sense, but may be $\infty$. To reduce the general case to the nonnegative case, let $x^+ = \max\{x, 0\}$ be the **positive part** and let $x^- = \max\{-x, 0\}$ be the **negative part** of $x$. We declare that $EX$ **exists** and set $EX = EX^+ - EX^-$ whenever the subtraction makes sense, that is, $EX^+ < \infty$ or $EX^- < \infty$.

$EX$ is often called the **mean** of $X$ and denoted by $\mu$. $EX$ is defined by integrating $X$, so it has all the properties that integrals do. From Theorems 1.4.5 and 1.4.7 and the trivial observation that $E(b) = b$ for any real number $b$, we get the following:

**Theorem 1.6.1.** *Suppose $X, Y \geq 0$ or $E|X|, E|Y| < \infty$.*
*(a) $E(X + Y) = EX + EY$.*
*(b) $E(aX + b) = aE(X) + b$ for any real numbers $a, b$.*
*(c) If $X \geq Y$ then $EX \geq EY$.*

In this section, we will restate some properties of the integral derived in the last section in terms of expected value and prove some new ones. To organize things, we will divide the developments into three subsections.

### 1.6.1 *Inequalities*

For probability measures, Theorem 1.5.1 becomes:

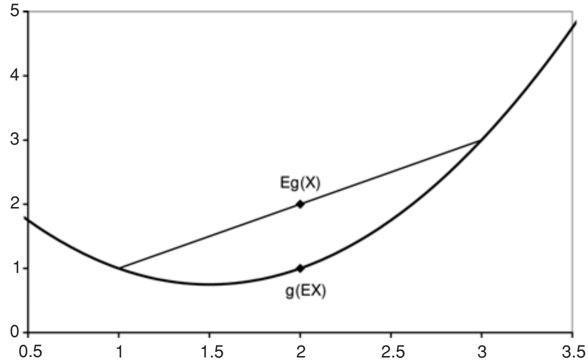



Figure 1.6. Jensen's inequality for $g(x) = x^2 - 3x + 3$, $P(X = 1) = P(X = 3) = 1/2$.

**Theorem 1.6.2. Jensen's inequality.** *Suppose $\varphi$ is* **convex***, that is,*

$$\lambda\varphi(x) + (1 - \lambda)\varphi(y) \geq \varphi(\lambda x + (1 - \lambda)y)$$

*for all $\lambda \in (0, 1)$ and $x, y \in \mathbf{R}$. Then*

$$E(\varphi(X)) \geq \varphi(EX)$$

*provided both expectations exist, that is, $E|X|$ and $E|\varphi(X)| < \infty$.*

To recall the direction in which the inequality goes, note that if $P(X = x) = \lambda$ and $P(X = y) = 1 - \lambda$, then (see Figure 1.6)

$$E\phi(X) = \lambda\varphi(x) + (1 - \lambda)\varphi(y) \geq \varphi(\lambda x + (1 - \lambda)y) = \phi(EX)$$

Two useful special cases are $|EX| \leq E|X|$ and $(EX)^2 \leq E(X^2)$.

**Theorem 1.6.3. Hölder's inequality.** *If $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, then*

$$E|XY| \leq \|X\|_p \|Y\|_q$$

*Here $\|X\|_r = (E|X|^r)^{1/r}$ for $r \in [1, \infty)$; $\|X\|_\infty = \inf\{M : P(|X| > M) = 0\}$.*

To state our next result, we need some notation. If we only integrate over $A \subset \Omega$, we write

$$E(X; A) = \int_A X\, dP$$

**Theorem 1.6.4. Chebyshev's inequality.** *Suppose $\varphi : \mathbf{R} \to \mathbf{R}$ has $\varphi \geq 0$, let $A \in \mathcal{R}$ and let $i_A = \inf\{\varphi(y) : y \in A\}$.*

$$i_A P(X \in A) \leq E(\varphi(X); X \in A) \leq E\varphi(X)$$

*Proof.* The definition of $i_A$ and the fact that $\phi \geq 0$ imply that

$$i_A 1_{(X \in A)} \leq \varphi(X) 1_{(X \in A)} \leq \varphi(X)$$

So taking expected values and using part (c) of Theorem 1.6.1 gives the desired result. ■

**Remark.** Some authors call this result **Markov's inequality** and use the name Chebyshev's inequality for the special case in which $\varphi(x) = x^2$ and $A = \{x : |x| \geq a\}$:

$$a^2 P(|X| \geq a) \leq EX^2 \tag{1.6.1}$$

### 1.6.2 *Integration to the Limit*

Our next step is to restate the three classic results from the previous section about what happens when we interchange limits and integrals.

**Theorem 1.6.5. Fatou's lemma.** *If $X_n \geq 0$ then*

$$\liminf_{n\to\infty} EX_n \geq E(\liminf_{n\to\infty} X_n)$$

**Theorem 1.6.6. Monotone convergence theorem.** *If $0 \leq X_n \uparrow X$ then $EX_n \uparrow EX$.*

**Theorem 1.6.7. Dominated convergence theorem.** *If $X_n \to X$ a.s., $|X_n| \leq Y$ for all $n$, and $EY < \infty$, then $EX_n \to EX$.*

The special case of Theorem 1.6.7 in which $Y$ is constant is called the **bounded convergence theorem**.

In the developments below, we will need another result on integration to the limit. Perhaps the most important special case of this result occurs when $g(x) = |x|^p$ with $p > 1$ and $h(x) = x$.

**Theorem 1.6.8.** *Suppose $X_n \to X$ a.s. Let $g, h$ be continuous functions with*
*(i) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$,*
*(ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$, and*
*(iii) $Eg(X_n) \leq K < \infty$ for all $n$.*
*Then $Eh(X_n) \to Eh(X)$.*

*Proof.* By subtracting a constant from $h$, we can suppose without loss of generality that $h(0) = 0$. Pick $M$ large so that $P(|X| = M) = 0$ and $g(x) > 0$ when $|x| \geq M$. Given a random variable $Y$, let $\bar{Y} = Y1_{(|Y| \leq M)}$. Since $P(|X| = M) = 0$, $\bar{X}_n \to \bar{X}$ a.s. Since $h(\bar{X}_n)$ is bounded and $h$ is continuous, it follows from the bounded convergence theorem that

(a) $$Eh(\bar{X}_n) \to Eh(\bar{X})$$

To control the effect of the truncation, we use the following:

(b) $|Eh(\bar{Y}) - Eh(Y)| \le E|h(\bar{Y}) - h(Y)| \le E(|h(Y)|; |Y| > M) \le \epsilon_M Eg(Y)$

where $\epsilon_M = \sup\{|h(x)|/g(x) : |x| \ge M\}$. To check the second inequality, note that when $|Y| \le M$, $\bar{Y} = Y$, and we have supposed $h(0) = 0$. The third inequality follows from the definition of $\epsilon_M$.

Taking $Y = X_n$ in (b) and using (iii), it follows that

(c) $$|Eh(\bar{X}_n) - Eh(X_n)| \le K\epsilon_M$$

To estimate $|Eh(\bar{X}) - Eh(X)|$, we observe that $g \ge 0$ and $g$ is continuous, so Fatou's lemma implies

$$Eg(X) \le \liminf_{n\to\infty} Eg(X_n) \le K$$

Taking $Y = X$ in (b) gives

(d) $$|Eh(\bar{X}) - Eh(X)| \le K\epsilon_M$$

The triangle inequality implies

$$\begin{aligned}|Eh(X_n) - Eh(X)| \le{}& |Eh(X_n) - Eh(\bar{X}_n)| \\ &+ |Eh(\bar{X}_n) - Eh(\bar{X})| + |Eh(\bar{X}) - Eh(X)|\end{aligned}$$

Taking limits and using (a), (c), (d), we have

$$\limsup_{n\to\infty} |Eh(X_n) - Eh(X)| \le 2K\epsilon_M$$

which proves the desired result since $K < \infty$ and $\epsilon_M \to 0$ as $M \to \infty$. ■

### 1.6.3 *Computing Expected Values*

Integrating over $(\Omega, \mathcal{F}, P)$ is nice in theory, but to do computations we have to shift to a space on which we can do calculus. In most cases, we will apply the next result with $S = \mathbf{R}^d$.

**Theorem 1.6.9. Change of variables formula.** *Let $X$ be a random element of $(S, \mathcal{S})$ with distribution $\mu$, that is, $\mu(A) = P(X \in A)$. If $f$ is a measurable function from $(S, \mathcal{S})$ to $(\mathbf{R}, \mathcal{R})$ so that $f \ge 0$ or $E|f(X)| < \infty$, then*

$$Ef(X) = \int_S f(y)\,\mu(dy)$$

**Remark.** To explain the name, write $h$ for $X$ and $P \circ h^{-1}$ for $\mu$ to get

$$\int_\Omega f(h(\omega))\,dP = \int_S f(y)\,d(P \circ h^{-1})$$

*Proof.* We will prove this result by verifying it in four increasingly general special cases that parallel the way that the integral was defined in Section 1.4. The reader should note the method employed, since it will be used several times below.

CASE 1: INDICATOR FUNCTIONS. If $B \in \mathcal{S}$ and $f = 1_B$, then recalling the relevant definitions shows

$$E1_B(X) = P(X \in B) = \mu(B) = \int_S 1_B(y)\,\mu(dy)$$

CASE 2: SIMPLE FUNCTIONS. Let $f(x) = \sum_{m=1}^n c_m 1_{B_m}$ where $c_m \in \mathbf{R}$, $B_m \in \mathcal{S}$. The linearity of expected value, the result of Case 1, and the linearity of integration imply

$$\begin{aligned} Ef(X) &= \sum_{m=1}^n c_m E1_{B_m}(X) \\ &= \sum_{m=1}^n c_m \int_S 1_{B_m}(y)\,\mu(dy) = \int_S f(y)\,\mu(dy) \end{aligned}$$

CASE 3: NONEGATIVE FUNCTIONS. Now if $f \geq 0$ and we let

$$f_n(x) = ([2^n f(x)]/2^n) \wedge n$$

where $[x] =$ the largest integer $\leq x$ and $a \wedge b = \min\{a, b\}$, then the $f_n$ are simple and $f_n \uparrow f$, so using the result for simple functions and the monotone convergence theorem:

$$Ef(X) = \lim_n Ef_n(X) = \lim_n \int_S f_n(y)\,\mu(dy) = \int_S f(y)\,\mu(dy)$$

CASE 4: INTEGRABLE FUNCTIONS. The general case now follows by writing $f(x) = f(x)^+ - f(x)^-$. The condition $E|f(X)| < \infty$ guarantees that $Ef(X)^+$ and $Ef(X)^-$ are finite. So using the result for nonnegative functions and linearity of expected value and integration:

$$\begin{aligned} Ef(X) = Ef(X)^+ - Ef(X)^- &= \int_S f(y)^+\,\mu(dy) - \int_S f(y)^-\,\mu(dy) \\ &= \int_S f(y)\,\mu(dy) \end{aligned}$$

which completes the proof. ■

A consequence of Theorem 1.6.9 is that we can compute expected values of functions of random variables by performing integrals on the real line. Before we can treat some examples, we need to introduce the terminology for what we are about to compute. If $k$ is a positive integer, then $EX^k$ is called the **kth moment** of $X$. The first moment $EX$ is usually called the **mean** and denoted by $\mu$. If $EX^2 < \infty$, then the **variance** of $X$ is defined to be $\text{var}(X) = E(X-\mu)^2$. To compute the variance the following formula is useful:

$$\begin{aligned} \text{var}(X) &= E(X-\mu)^2 \\ &= EX^2 - 2\mu EX + \mu^2 = EX^2 - \mu^2 \end{aligned} \tag{1.6.2}$$

From this it is immediate that

$$\text{var}(X) \le EX^2 \tag{1.6.3}$$

Here $EX^2$ is the expected value of $X^2$. When we want the square of $EX$, we will write $(EX)^2$. Since $E(aX+b) = aEX + b$ by (b) of Theorem 1.6.1, it follows easily from the definition that

$$\begin{aligned} \text{var}(aX+b) &= E(aX+b-E(aX+b))^2 \\ &= a^2E(X-EX)^2 = a^2\,\text{var}(X) \end{aligned} \tag{1.6.4}$$

We turn now to concrete examples and leave the calculus in the first two examples to the reader. (Integrate by parts.)

**Example 1.6.1.** If $X$ has an **exponential distribution** with rate 1, then

$$EX^k = \int_0^\infty x^k e^{-x}\,dx = k!$$

So the mean of $X$ is 1 and variance is $EX^2 - (EX)^2 = 2 - 1^2 = 1$. If we let $Y = X/\lambda$, then by Exercise 1.2.5, $Y$ has density $\lambda e^{-\lambda y}$ for $y \ge 0$, the **exponential density** with parameter $\lambda$. From (b) of Theorem 1.6.1 and (1.6.4), it follows that $Y$ has mean $1/\lambda$ and variance $1/\lambda^2$.

**Example 1.6.2.** If $X$ has a **standard normal distribution**,

$$EX = \int x(2\pi)^{-1/2}\exp(-x^2/2)\,dx = 0 \quad \text{(by symmetry)}$$

$$\text{var}(X) = EX^2 = \int x^2(2\pi)^{-1/2}\exp(-x^2/2)\,dx = 1$$

If we let $\sigma > 0$, $\mu \in \mathbf{R}$, and $Y = \sigma X + \mu$, then (b) of Theorem 1.6.1 and (1.6.4) imply $EY = \mu$ and $\text{var}(Y) = \sigma^2$. By Exercise 1.2.5, $Y$ has density

$$(2\pi\sigma^2)^{-1/2}\exp(-(y-\mu)^2/2\sigma^2)$$

the **normal distribution** with mean $\mu$ and variance $\sigma^2$.

We will next consider some discrete distributions. The first is very simple, but will be useful several times below, so we record it here.

**Example 1.6.3.** We say that $X$ has a **Bernoulli distribution** with parameter $p$ if $P(X = 1) = p$ and $P(X = 0) = 1 - p$. Clearly,

$$EX = p \cdot 1 + (1 - p) \cdot 0 = p$$

Since $X^2 = X$, we have $EX^2 = EX = p$ and

$$\text{var}\,(X) = EX^2 - (EX)^2 = p - p^2 = p(1 - p)$$

**Example 1.6.4.** We say that $X$ has a **Poisson distribution** with parameter $\lambda$ if

$$P(X = k) = e^{-\lambda}\lambda^k/k! \text{ for } k = 0, 1, 2, \ldots$$

To evaluate the moments of the Poisson random variable, we use a little inspiration to observe that for $k \geq 1$

$$\begin{aligned} E(X(X-1)\cdots(X-k+1)) &= \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)e^{-\lambda}\frac{\lambda^j}{j!} \\ &= \lambda^k \sum_{j=k}^{\infty} e^{-\lambda}\frac{\lambda^{j-k}}{(j-k)!} = \lambda^k \end{aligned}$$

where the equalities follow from (i) the fact that $j(j-1)\cdots(j-k+1) = 0$ when $j < k$, (ii) canceling part of the factorial, and (iii) the fact that Poisson distribution has total mass 1. Using the last formula, it follows that $EX = \lambda$ while

$$\text{var}\,(X) = EX^2 - (EX)^2 = E(X(X-1)) + EX - \lambda^2 = \lambda$$

**Example 1.6.5.** $N$ is said to have a **geometric distribution** with success probability $p \in (0, 1)$ if

$$P(N = k) = p(1-p)^{k-1} \quad \text{for } k = 1, 2, \ldots$$

$N$ is the number of independent trials needed to observe an event with probability $p$. Differentiating the identity

$$\sum_{k=0}^{\infty}(1-p)^k = 1/p$$

and referring to Example A.5.3 for the justification gives

$$-\sum_{k=1}^{\infty} k(1-p)^{k-1} = -1/p^2$$

$$\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} = 2/p^3$$

From this it follows that

$$EN = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = 1/p$$

$$EN(N-1) = \sum_{k=1}^{\infty} k(k-1)p(1-p)^{k-1} = 2(1-p)/p^2$$

$$\begin{aligned} \text{var}\,(N) &= EN^2 - (EN)^2 = EN(N-1) + EN - (EN)^2 \\ &= \frac{2(1-p)}{p^2} + \frac{p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \end{aligned}$$

## *Exercises*

**1.6.1.** Suppose $\varphi$ is strictly convex, that is, $>$ holds for $\lambda \in (0, 1)$. Show that, under the assumptions of Theorem 1.6.2, $\varphi(EX) = E\varphi(X)$ implies $X = EX$ a.s.

**1.6.2.** Suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ is convex. Imitate the proof of Theorem 1.5.1 to show

$$E\phi(X_1, \dots, X_n) \geq \phi(EX_1, \dots, EX_n)$$

provided $E|\phi(X_1, \dots, X_n)| < \infty$ and $E|X_i| < \infty$ for all $i$.

**1.6.3. Chebyshev's inequality is and is not sharp.** (i) Show that Theorem 1.6.4 is sharp by showing that if $0 < b \leq a$ are fixed, there is an $X$ with $EX^2 = b^2$ for which $P(|X| \geq a) = b^2/a^2$. (ii) Show that Theorem 1.6.4 is not sharp by showing that if $X$ has $0 < EX^2 < \infty$, then

$$\lim_{a\to\infty} a^2 P(|X| \geq a)/EX^2 = 0$$

**1.6.4. One-sided Chebyshev bound.** (i) Let $a > b > 0, 0 < p < 1$, and let $X$ have $P(X = a) = p$ and $P(X = -b) = 1 - p$. Apply Theorem 1.6.4 to $\phi(x) = (x + b)^2$ and conclude that if $Y$ is any random variable with $EY = EX$ and $\text{var}\,(Y) = \text{var}\,(X)$, then $P(Y \geq a) \leq p$ and equality holds when $Y = X$.
(ii) Suppose $EY = 0$, $\text{var}\,(Y) = \sigma^2$, and $a > 0$. Show that $P(Y \geq a) \leq \sigma^2/(a^2 + \sigma^2)$, and there is a $Y$ for which equality holds.

**1.6.5. Two nonexistent lower bounds.**
Show that: (i) if $\epsilon > 0$, $\inf\{P(|X| > \epsilon) : EX = 0,\ \text{var}\,(X) = 1\} = 0$.
(ii) if $y \geq 1$, $\sigma^2 \in (0, \infty)$, $\inf\{P(|X| > y) : EX = 1,\ \text{var}\,(X) = \sigma^2\} = 0$.

**1.6.6. A useful lower bound.** Let $Y \geq 0$ with $EY^2 < \infty$. Apply the Cauchy-Schwarz inequality to $Y 1_{(Y>0)}$ and conclude

$$P(Y > 0) \geq (EY)^2/EY^2$$

**1.6.7.** Let $\Omega = (0, 1)$ equipped with the Borel sets and Lebesgue measure. Let $\alpha \in (1, 2)$ and $X_n = n^\alpha 1_{(1/(n+1),1/n)} \to 0$ a.s. Show that Theorem 1.6.8 can be applied with $h(x) = x$ and $g(x) = |x|^{2/\alpha}$, but the $X_n$ are not dominated by an integrable function.

**1.6.8.** Suppose that the probability measure $\mu$ has $\mu(A) = \int_A f(x)\,dx$ for all $A \in \mathcal{R}$. Use the proof technique of Theorem 1.6.9 to show that for any $g$ with $g \geq 0$ or $\int |g(x)|\,\mu(dx) < \infty$, we have

$$\int g(x)\,\mu(dx) = \int g(x) f(x)\,dx$$

**1.6.9. Inclusion-exclusion formula**. Let $A_1, A_2, \dots A_n$ be events and $A = \cup_{i=1}^n A_i$. Prove that $1_A = 1 - \prod_{i=1}^n (1 - 1_{A_i})$. Expand out the right-hand side, then take expected value to conclude

$$P\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(\cap_{i=1}^n A_i)$$

**1.6.10. Bonferroni inequalities.** Let $A_1, A_2, \dots A_n$ be events and $A = \cup_{i=1}^n A_i$. Show that $1_A \leq \sum_{i=1}^n 1_{A_i}$, and so forth, and then take expected values to conclude

$$P\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i)$$

$$P\left(\cup_{i=1}^n A_i\right) \geq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$P\left(\cup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

In general, if we stop the inclusion-exclusion formula after an even (odd) number of sums, we get an lower (upper) bound.

**1.6.11.** If $E|X|^k < \infty$ then for $0 < j < k$, $E|X|^j < \infty$, and furthermore

$$E|X|^j \leq (E|X|^k)^{j/k}$$

**1.6.12.** Apply Jensen's inequality with $\varphi(x) = e^x$ and $P(X = \log\ y_m) = p(m)$ to conclude that if $\sum_{m=1}^n p(m) = 1$ and $p(m), y_m > 0$, then

$$\sum_{m=1}^n p(m) y_m \geq \prod_{m=1}^n y_m^{p(m)}$$

When $p(m) = 1/n$, this says the arithmetic mean exceeds the geometric mean.

**1.6.13.** If $EX_1^- < \infty$ and $X_n \uparrow X$, then $EX_n \uparrow EX$.

**1.6.14.** Let $X \geq 0$ but do NOT assume $E(1/X) < \infty$. Show

$$\lim_{y \to \infty} yE(1/X; X > y) = 0, \qquad \lim_{y \downarrow 0} yE(1/X; X > y) = 0.$$

**1.6.15.** If $X_n \geq 0$, then $E(\sum_{n=0}^{\infty} X_n) = \sum_{n=0}^{\infty} EX_n$.

**1.6.16.** If $X$ is integrable and $A_n$ are disjoint sets with union $A$, then

$$\sum_{n=0}^{\infty} E(X; A_n) = E(X; A)$$

that is, the sum converges absolutely and has the value on the right.

## 1.7 Product Measures, Fubini's Theorem

Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \mu_2)$ be two $\sigma$-finite measure spaces. Let

$$\Omega = X \times Y = \{(x, y) : x \in X, y \in Y\}$$
$$\mathcal{S} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

Sets in $\mathcal{S}$ are called **rectangles**. It is easy to see that $\mathcal{S}$ is a semialgebra:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$
$$(A \times B)^c = (A^c \times B) \cup (A \times B^c) \cup (A^c \times B^c)$$

Let $\mathcal{F} = \mathcal{A} \times \mathcal{B}$ be the $\sigma$-algebra generated by $\mathcal{S}$.

**Theorem 1.7.1.** *There is a unique measure $\mu$ on $\mathcal{F}$ with*

$$\mu(A \times B) = \mu_1(A)\mu_2(B)$$

**Notation.** $\mu$ is often denoted by $\mu_1 \times \mu_2$.

*Proof.* By Theorem 1.1.4 it is enough to show that if $A \times B = +_i(A_i \times B_i)$ is a finite or countable disjoint union, then

$$\mu(A \times B) = \sum_i \mu(A_i \times B_i)$$

For each $x \in A$, let $I(x) = \{i : x \in A_i\}$. $B = +_{i \in I(x)} B_i$, so

$$1_A(x)\mu_2(B) = \sum_i 1_{A_i}(x)\mu_2(B_i)$$

Integrating with respect to $\mu_1$ and using Exercise 1.5.6 gives

$$\mu_1(A)\mu_2(B) = \sum_i \mu_1(A_i)\mu_2(B_i)$$

which proves the result. ∎

Using Theorem 1.7.1 and induction, it follows that if $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, \ldots, n$, are $\sigma$-finite measure spaces and $\Omega = \Omega_1 \times \cdots \times \Omega_n$, there is a unique measure $\mu$ on the $\sigma$-algebra $\mathcal{F}$ generated by sets of the form $A_1 \times \cdots \times A_n$, $A_i \in \mathcal{F}_i$, that has

$$\mu(A_1 \times \cdots \times A_n) = \prod_{m=1}^{n} \mu_m(A_m)$$

When $(\Omega_i, \mathcal{F}_i, \mu_i) = (\mathbf{R}, \mathcal{R}, \lambda)$ for all $i$, the result is Lebesgue measure on the Borel subsets of $n$ dimensional Euclidean space $\mathbf{R}^n$.

Returning to the case in which $(\Omega, \mathcal{F}, \mu)$ is the product of two measure spaces, $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, our next goal is to prove:

**Theorem 1.7.2. Fubini's theorem.** *If $f \geq 0$ or $\int |f| \, d\mu < \infty$, then*

$$(*) \quad \int_X \int_Y f(x, y) \, \mu_2(dy) \, \mu_1(dx) = \int_{X \times Y} f \, d\mu = \int_Y \int_X f(x, y) \, \mu_1(dx) \, \mu_2(dy)$$

*Proof.* We will prove only the first equality, since the second follows by symmetry. Two technical things that need to be proved before we can assert that the first integral makes sense are:

When $x$ is fixed, $y \to f(x, y)$ is $\mathcal{B}$ measurable.

$x \to \int_Y f(x, y)\mu_2(dy)$ is $\mathcal{A}$ measurable.

We begin with the case $f = 1_E$. Let $E_x = \{y : (x, y) \in E\}$ be the **cross-section** at $x$.

**Lemma 1.7.3.** *If $E \in \mathcal{F}$ then $E_x \in \mathcal{B}$.*

*Proof.* $(E^c)_x = (E_x)^c$ and $(\cup_i E_i)_x = \cup_i (E_i)_x$, so if $\mathcal{E}$ is the collection of sets $E$ for which $E_x \in \mathcal{B}$, then $\mathcal{E}$ is a $\sigma$-algebra. Since $\mathcal{E}$ contains the rectangles, the result follows. ■

**Lemma 1.7.4.** *If $E \in \mathcal{F}$, then $g(x) \equiv \mu_2(E_x)$ is $\mathcal{A}$ measurable and*

$$\int_X g \, d\mu_1 = \mu(E)$$

Notice that it is not obvious that the collection of sets for which the conclusion is true is a $\sigma$-algebra since $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2)$. Dynkin's $\pi - \lambda$ Theorem (A.1.4) was tailor-made for situations like this.

*Proof.* If conclusions hold for $E_n$ and $E_n \uparrow E$, then Theorem 1.3.5 and the monotone convergence theorem imply that they hold for $E$. Since $\mu_1$ and $\mu_2$ are $\sigma$-finite, it is enough then to prove the result for $E \subset F \times G$ with $\mu_1(F) < \infty$ and $\mu_2(G) < \infty$, or taking $\Omega = F \times G$ we can suppose without loss of generality that $\mu(\Omega) < \infty$. Let $\mathcal{L}$ be the collection of sets $E$ for which the conclusions hold.

We will now check that $\mathcal{L}$ is a $\lambda$-system. Property (i) of a $\lambda$-system is trivial. (iii) follows from the first sentence in the proof. To check (ii) we observe that

$$\mu_2((A - B)_x) = \mu_2(A_x - B_x) = \mu_2(A_x) - \mu_2(B_x)$$

and integrating over $x$ gives the second conclusion. Since $\mathcal{L}$ contains the rectangles, a $\pi$-system that generates $\mathcal{F}$, the desired result follows from the $\pi - \lambda$ theorem. ∎

We are now ready to prove Theorem 1.7.2 by verifying it in four increasingly general special cases.

Case 1. If $E \in \mathcal{F}$ and $f = 1_E$, then $(*)$ follows from Lemma 1.7.4

Case 2. Since each integral is linear in $f$, it follows that $(*)$ holds for simple functions.

Case 3. Now if $f \geq 0$ and we let $f_n(x) = ([2^n f(x)]/2^n) \wedge n$, where $[x] =$ the largest integer $\leq x$, then the $f_n$ are simple and $f_n \uparrow f$, so it follows from the monotone convergence theorem that $(*)$ holds for all $f \geq 0$.

Case 4. The general case now follows by writing $f(x) = f(x)^+ - f(x)^-$ and applying Case 3 to $f^+$, $f^-$, and $|f|$. ∎

To illustrate why the various hypotheses of Theorem 1.7.2 are needed, we will now give some examples where the conclusion fails.

**Example 1.7.1.** Let $X = Y = \{1, 2, \dots\}$ with $\mathcal{A} = \mathcal{B} =$ all subsets and $\mu_1 = \mu_2 =$ counting measure. For $m \geq 1$, let $f(m, m) = 1$ and $f(m + 1, m) = -1$, and let $f(m, n) = 0$ otherwise. We claim that

$$\sum_m \sum_n f(m, n) = 1 \quad \text{but} \quad \sum_n \sum_m f(m, n) = 0$$

A picture is worth several dozen words:

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 1 & \dots \\
\uparrow & 0 & 0 & 1 & -1 & \dots \\
n & 0 & 1 & -1 & 0 & \dots \\
 & 1 & -1 & 0 & 0 & \dots \\
 & & m & \rightarrow & & 
\end{array}$$

In words, if we sum the columns first, the first one gives us a 1 and the others 0, while if we sum the rows each one gives us a 0.

**Example 1.7.2.** Let $X = (0, 1)$, $Y = (1, \infty)$, both equipped with the Borel sets and Lebesgue measure. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$.

$$\int_0^1 \int_1^\infty f(x, y)\, dy\, dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})\, dx > 0$$

$$\int_1^\infty \int_0^1 f(x, y)\, dx\, dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})\, dy < 0$$

The next example indicates why $\mu_1$ and $\mu_2$ must be $\sigma$-finite.

**Example 1.7.3.** Let $X = (0, 1)$ with $\mathcal{A} =$ the Borel sets and $\mu_1 =$ Lebesgue measure. Let $Y = (0, 1)$ with $\mathcal{B} =$ all subsets and $\mu_2 =$ counting measure. Let $f(x, y) = 1$ if $x = y$ and 0 otherwise

$$\int_Y f(x, y)\, \mu_2(dy) = 1 \quad \text{for all } x \text{ so} \qquad \int_X \int_Y f(x, y)\, \mu_2(dy)\, \mu_1(dx) = 1$$

$$\int_X f(x, y)\, \mu_1(dx) = 0 \quad \text{for all } y \text{ so} \qquad \int_Y \int_X f(x, y)\, \mu_1(dy)\, \mu_2(dx) = 0$$

Our last example shows that measurability is important or maybe that some of the axioms of set theory are not as innocent as they seem.

**Example 1.7.4.** By the axiom of choice and the continuum hypothesis one can define an order relation $<'$ on (0,1) so that $\{x : x <' y\}$ is countable for each $y$. Let $X = Y = (0, 1)$, let $\mathcal{A} = \mathcal{B} =$ the Borel sets and $\mu_1 = \mu_2 =$ Lebesgue measure. Let $f(x, y) = 1$ if $x <' y$, $= 0$ otherwise. Since $\{x : x <' y\}$ and $\{y : x <' y\}^c$ are countable,

$$\int_X f(x, y)\, \mu_1(dx) = 0 \quad \text{for all } y$$

$$\int_Y f(x, y)\, \mu_2(dy) = 1 \quad \text{for all } x$$

## *Exercises*

**1.7.1.** If $\int_X \int_Y |f(x, y)| \mu_2(dy)\mu_1(dx) < \infty$, then

$$\int_X \int_Y f(x, y)\mu_2(dy)\mu_1(dx) = \int_{X\times Y} f\, d(\mu_1 \times \mu_2) = \int_Y \int_X f(x, y)\mu_1(dx)\mu_2(dy)$$

**Corollary.** Let $X = \{1, 2, \ldots\}$, $\mathcal{A} =$ all subsets of $X$, and $\mu_1 =$ counting measure. If $\sum_n \int |f_n| d\mu < \infty$, then $\sum_n \int f_n\, d\mu = \int \sum_n f_n\, d\mu$.

**1.7.2.** Let $g \geq 0$ be a measurable function on $(X, \mathcal{A}, \mu)$. Use Theorem 1.7.2 to conclude that

$$\int_X g\,d\mu = (\mu \times \lambda)(\{(x, y) : 0 \leq y < g(x)\}) = \int_0^\infty \mu(\{x : g(x) > y\})\,dy$$

**1.7.3.** Let $F$, $G$ be Stieltjes measure functions, and let $\mu$, $\nu$ be the corresponding measures on $(\mathbf{R}, \mathcal{R})$. Show that

(i) $\int_{(a,b]} \{F(y) - F(a)\} dG(y) = (\mu \times \nu)(\{(x, y) : a < x \leq y \leq b\})$
(ii) $\int_{(a,b]} F(y)\,dG(y) + \int_{(a,b]} G(y)\,dF(y)$

$$= F(b)G(b) - F(a)G(a) + \sum_{x \in (a,b]} \mu(\{x\})\nu(\{x\})$$

(iii) If $F = G$ is continuous, then $\int_{(a,b]} 2F(y) dF(y) = F^2(b) - F^2(a)$.

To see that the second term in (ii) is needed, let $F(x) = G(x) = 1_{[0,\infty)}(x)$ and $a < 0 < b$.

**1.7.4.** Let $\mu$ be a finite measure on $\mathbf{R}$ and $F(x) = \mu((-\infty, x])$. Show that

$$\int (F(x + c) - F(x))\,dx = c\mu(\mathbf{R})$$

**1.7.5.** Show that $e^{-xy} \sin x$ is integrable in the strip $0 < x < a$, $0 < y$. Perform the double integral in the two orders to get

$$\int_0^a \frac{\sin x}{x}\,dx = (\arctan a) - (\cos a)\int_0^\infty \frac{e^{-ay}}{1+y^2}\,dy - (\sin a)\int_0^\infty \frac{ye^{-ay}}{1+y^2}\,dy$$

and replace $1 + y^2$ by 1 to conclude $\left|\int_0^a (\sin x)/x\,dx - (\arctan a)\right| \leq 2/a$ for $a \geq 1$.

# 2

# Laws of Large Numbers

## 2.1 Independence

Measure theory ends and probability begins with the definition of independence. We begin with what we hope is a familiar definition and then work our way up to a definition that is appropriate for our current setting.

Two events $A$ and $B$ are **independent** if $P(A \cap B) = P(A)P(B)$.

Two random variables $X$ and $Y$ are **independent** if for all $C, D \in \mathcal{R}$,

$$P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$$

that is, the events $A = \{X \in C\}$ and $B = \{Y \in D\}$ are independent.

Two $\sigma$-fields $\mathcal{F}$ and $\mathcal{G}$ are **independent** if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$ the events $A$ and $B$ are independent.

As the next exercise shows, the second definition is a special case of the third.

**Exercise 2.1.1.** (i) Show that if $X$ and $Y$ are independent then $\sigma(X)$ and $\sigma(Y)$ are. (ii) Conversely, if $\mathcal{F}$ and $\mathcal{G}$ are independent, $X \in \mathcal{F}$, and $Y \in \mathcal{G}$, then $X$ and $Y$ are independent.

The first definition is, in turn, a special case of the second.

**Exercise 2.1.2.** (i) Show that if $A$ and $B$ are independent, then so are $A^c$ and $B$, $A$ and $B^c$, and $A^c$ and $B^c$. (ii) Conclude that events $A$ and $B$ are independent if and only if their indicator random variables $1_A$ and $1_B$ are independent.

In view of the fact that the first definition is a special case of the second, which is a special case of the third, we take things in the opposite order when we say what it means for several things to be independent. We begin by reducing to the case of finitely many objects. An infinite collection of objects ($\sigma$-fields, random variables, or sets) is said to be independent if every finite subcollection is.

$\sigma$-fields $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ are **independent** if whenever $A_i \in \mathcal{F}_i$ for $i = 1, \dots, n$, we have

$$P\left(\cap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i)$$

Random variables $X_1, \dots, X_n$ are **independent** if whenever $B_i \in \mathcal{R}$ for $i = 1, \dots, n$ we have

$$P\left(\cap_{i=1}^n \{X_i \in B_i\}\right) = \prod_{i=1}^n P(X_i \in B_i)$$

Sets $A_1, \dots, A_n$ are **independent** if whenever $I \subset \{1, \dots n\}$ we have

$$P\left(\cap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

At first glance, it might seem that the last definition does not match the other two. However, if you think about it for a minute, you will see that if the indicator variables $1_{A_i}$, $1 \le i \le n$ are independent and we take $B_i = \{1\}$ for $i \in I$, and $B_i = \mathbf{R}$ for $i \notin I$ then the condition in the definition results. Conversely,

**Exercise 2.1.3.** Let $A_1, A_2, \dots, A_n$ be independent. Show (i) $A_1^c, A_2, \dots, A_n$ are independent; (ii) $1_{A_1}, \dots, 1_{A_n}$ are independent.

One of the first things to understand about the definition of independent events is that it is not enough to assume $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \ne j$. A sequence of events $A_1, \dots, A_n$ with the last property is called **pairwise independent**. It is clear that independent events are pairwise independent. The next example shows that the converse is not true.

**Example 2.1.1.** Let $X_1, X_2, X_3$ be independent random variables with

$$P(X_i = 0) = P(X_i = 1) = 1/2$$

Let $A_1 = \{X_2 = X_3\}$, $A_2 = \{X_3 = X_1\}$ and $A_3 = \{X_1 = X_2\}$. These events are pairwise independent since if $i \ne j$, then

$$P(A_i \cap A_j) = P(X_1 = X_2 = X_3) = 1/4 = P(A_i)P(A_j)$$

but they are not independent since

$$P(A_1 \cap A_2 \cap A_3) = 1/4 \ne 1/8 = P(A_1)P(A_2)P(A_3)$$

In order to show that random variables $X$ and $Y$ are independent, we have to check that $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for all Borel sets $A$ and $B$. Since there are a lot of Borel sets, our next topic is

### 2.1.1 *Sufficient Conditions for Independence*

Our main result is Theorem 2.1.3. To state that result, we need a definition that generalizes all our earlier definitions.

Collections of sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \subset \mathcal{F}$ are said to be **independent** if whenever $A_i \in \mathcal{A}_i$ and $I \subset \{1, \ldots, n\}$ we have $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

If each collection is a single set, that is, $\mathcal{A}_i = \{A_i\}$, this definition reduces to the one for sets.

**Lemma 2.1.1.** *Without loss of generality we can suppose each $\mathcal{A}_i$ contains $\Omega$. In this case the condition is equivalent to*

$$P\left(\cap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i) \quad \textit{whenever } A_i \in \mathcal{A}_i$$

*since we can set $A_i = \Omega$ for $i \notin I$.*

*Proof.* If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent and $\bar{\mathcal{A}}_i = \mathcal{A}_i \cup \{\Omega\}$ then $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2, \ldots, \bar{\mathcal{A}}_n$ are independent, since if $A_i \in \bar{\mathcal{A}}_i$ and $I = \{j : A_j = \Omega\}$ $\cap_i A_i = \cap_{i \in I} A_i$. ■

The proof of Theorem 2.1.3 is based on Dynkin's $\pi - \lambda$ theorem. To state this result, we need two definitions. We say that $\mathcal{A}$ is a **$\pi$-system** if it is closed under intersection, that is, if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$. We say that $\mathcal{L}$ is a **$\lambda$-system** if (i) $\Omega \in \mathcal{L}$. (ii) If $A, B \in \mathcal{L}$ and $A \subset B$, then $B - A \in \mathcal{L}$. (iii) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$, then $A \in \mathcal{L}$.

**Theorem 2.1.2.** $\pi - \lambda$ **Theorem**. *If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*

The proof is hidden away in Section A.1 of the Appendix.

**Theorem 2.1.3.** *Suppose $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent and each $\mathcal{A}_i$ is a $\pi$-system. Then $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \ldots, \sigma(\mathcal{A}_n)$ are independent.*

*Proof.* Let $A_2, \ldots, A_n$ be sets with $A_i \in \mathcal{A}_i$, let $F = A_2 \cap \cdots \cap A_n$ and let $\mathcal{L} = \{A : P(A \cap F) = P(A)P(F)\}$. Since $P(\Omega \cap F) = P(\Omega)P(F)$, $\Omega \in \mathcal{L}$. To check (ii) of the definition of a $\lambda$-system, we note that if $A, B \in \mathcal{L}$ with $A \subset B$, then $(B - A) \cap F = (B \cap F) - (A \cap F)$. So, using (i) in Theorem 1.1.1, the fact $A, B \in \mathcal{L}$ and then (i) in Theorem 1.1.1 again:

$$\begin{aligned} P((B - A) \cap F) &= P(B \cap F) - P(A \cap F) = P(B)P(F) - P(A)P(F) \\ &= \{P(B) - P(A)\}P(F) = P(B - A)P(F) \end{aligned}$$

and we have $B - A \in \mathcal{L}$. To check (iii), let $B_k \in \mathcal{L}$ with $B_k \uparrow B$ and note that $(B_k \cap F) \uparrow (B \cap F)$, so using (iii) in Theorem 1.1.1, the fact that $B_k \in \mathcal{L}$, and then

(iii) in Theorem 1.1.1 again:

$$P(B \cap F) = \lim_k P(B_k \cap F) = \lim_k P(B_k)P(F) = P(B)P(F)$$

Applying the $\pi - \lambda$ theorem now gives $\mathcal{L} \supset \sigma(\mathcal{A}_1)$. It follows that if $A_1 \in \sigma(\mathcal{A}_1)$ and $A_i \in \mathcal{A}_i$ for $2 \le i \le n$, then

$$P(\cap_{i=1}^n A_i) = P(A_1)P(\cap_{i=2}^n A_i) = \prod_{i=1}^n P(A_i)$$

Using Lemma 2.1.1 now, we have

($*$) If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent then $\sigma(\mathcal{A}_1), \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent.

Applying ($*$) to $\mathcal{A}_2, \ldots, \mathcal{A}_n, \sigma(\mathcal{A}_1)$ (which are independent since the definition is unchanged by permuting the order) shows that $\sigma(\mathcal{A}_2), \mathcal{A}_3, \ldots, \mathcal{A}_n, \sigma(\mathcal{A}_1)$ are independent, and after $n$ iterations we have the desired result. ∎

**Remark.** The reader should note that it is not easy to show that if $A, B \in \mathcal{L}$ then $A \cap B \in \mathcal{L}$, or $A \cup B \in \mathcal{L}$, but it is easy to check that if $A, B \in \mathcal{L}$ with $A \subset B$ then $B - A \in \mathcal{L}$.

Having worked to establish Theorem 2.1.3, we get several corollaries.

**Theorem 2.1.4.** *In order for $X_1, \ldots, X_n$ to be independent, it is sufficient that for all $x_1, \ldots, x_n \in (-\infty, \infty]$*

$$P(X_1 \le x_1, \ldots, X_n \le x_n) = \prod_{i=1}^n P(X_i \le x_i)$$

*Proof.* Let $\mathcal{A}_i =$ the sets of the form $\{X_i \le x_i\}$. Since

$$\{X_i \le x\} \cap \{X_i \le y\} = \{X_i \le x \wedge y\},$$

where $(x \wedge y)_i = x_i \wedge y_i = \min\{x_i, y_i\}$. $\mathcal{A}_i$ is a $\pi$-system. Since we have allowed $x_i = \infty$, $\Omega \in \mathcal{A}_i$. Exercise 1.3.1 implies $\sigma(\mathcal{A}_i) = \sigma(X_i)$, so the result follows from Theorem 2.1.3. ∎

The last result expresses independence of random variables in terms of their distribution functions. The next two exercises treat density functions and discrete random variables.

**Exercise 2.1.4.** Suppose $(X_1, \ldots, X_n)$ has density $f(x_1, x_2, \ldots, x_n)$, that is

$$P((X_1, X_2, \ldots, X_n) \in A) = \int_A f(x)\,dx \text{ for } A \in \mathcal{R}^n$$

If $f(x)$ can be written as $g_1(x_1)\cdots g_n(x_n)$ where the $g_m \ge 0$ are measurable, then $X_1, X_2, \dots, X_n$ are independent. Note that the $g_m$ are not assumed to be probability densities.

**Exercise 2.1.5.** Suppose $X_1, \dots, X_n$ are random variables that take values in countable sets $S_1, \dots, S_n$. Then in order for $X_1, \dots, X_n$ to be independent, it is sufficient that whenever $x_i \in S_i$,

$$P(X_1 = x_1, \dots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i)$$

Our next goal is to prove that functions of disjoint collections of independent random variables are independent. See Theorem 2.1.6 for the precise statement. First we will prove an analogous result for $\sigma$-fields.

**Theorem 2.1.5.** *Suppose $\mathcal{F}_{i,j}$, $1 \le i \le n$, $1 \le j \le m(i)$ are independent and let $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j})$. Then $\mathcal{G}_1, \dots, \mathcal{G}_n$ are independent.*

*Proof.* Let $\mathcal{A}_i$ be the collection of sets of the form $\cap_j A_{i,j}$ where $A_{i,j} \in \mathcal{F}_{i,j}$. $\mathcal{A}_i$ is a $\pi$-system that contains $\Omega$ and contains $\cup_j \mathcal{F}_{i,j}$, so Theorem 2.1.3 implies $\sigma(\mathcal{A}_i) = \mathcal{G}_i$ are independent. ■

**Theorem 2.1.6.** *If for $1 \le i \le n$, $1 \le j \le m(i)$, $X_{i,j}$ are independent and $f_i : \mathbf{R}^{m(i)} \to \mathbf{R}$ are measurable, then $f_i(X_{i,1}, \dots, X_{i,m(i)})$ are independent.*

*Proof.* Let $\mathcal{F}_{i,j} = \sigma(X_{i,j})$ and $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j})$. Since $f_i(X_{i,1}, \dots, X_{i,m(i)}) \in \mathcal{G}_i$, the desired result follows from Theorem 2.1.5 and Exercise 2.1.1. ■

A concrete special case of Theorem 2.1.6 that we will use in a minute is: if $X_1, \dots, X_n$ are independent, then $X = X_1$ and $Y = X_2 \cdots X_n$ are independent. Later, when we study sums $S_m = X_1 + \cdots + X_m$ of independent random variables $X_1, \dots, X_n$, we will use Theorem 2.1.6 to conclude that if $m < n$ then $S_n - S_m$ is independent of the indicator function of the event $\{\max_{1\le k\le m} S_k > x\}$.

### 2.1.2 *Independence, Distribution, and Expectation*

Our next goal is to obtain formulas for the distribution and expectation of independent random variables.

**Theorem 2.1.7.** *Suppose $X_1, \dots, X_n$ are independent random variables and $X_i$ has distribution $\mu_i$. Then $(X_1, \dots, X_n)$ has distribution $\mu_1 \times \cdots \times \mu_n$.*

*Proof.* Using the definitions of (i) $A_1 \times \cdots \times A_n$, (ii) independence, (iii) $\mu_i$, and (iv) $\mu_1 \times \cdots \times \mu_n$,

$$P((X_1, \ldots, X_n) \in A_1 \times \cdots \times A_n) = P(X_1 \in A_1, \ldots, X_n \in A_n)$$
$$= \prod_{i=1}^n P(X_i \in A_i) = \prod_{i=1}^n \mu_i(A_i) = \mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n)$$

The last formula shows that the distribution of $(X_1, \ldots, X_n)$ and the measure $\mu_1 \times \cdots \times \mu_n$ agree on sets of the form $A_1 \times \cdots \times A_n$, a $\pi$-system that generates $\mathcal{R}^n$. So Theorem 2.1.2 implies they must agree. ■

**Theorem 2.1.8.** *Suppose $X$ and $Y$ are independent and have distributions $\mu$ and $\nu$. If $h : \mathbf{R}^2 \to \mathbf{R}$ is a measurable function with $h \geq 0$ or $E|h(X,Y)| < \infty$, then*

$$Eh(X,Y) = \iint h(x,y)\, \mu(dx)\, \nu(dy)$$

*In particular, if $h(x,y) = f(x)g(y)$ where $f, g : \mathbf{R} \to \mathbf{R}$ are measurable functions with $f, g \geq 0$ or $E|f(X)|$ and $E|g(Y)| < \infty$, then*

$$Ef(X)g(Y) = Ef(X) \cdot Eg(Y)$$

*Proof.* Using Theorem 1.6.9 and then Fubini's theorem (Theorem 1.7.2), we have

$$Eh(X,Y) = \int_{\mathbf{R}^2} h\, d(\mu \times \nu) = \iint h(x,y)\, \mu(dx)\, \nu(dy)$$

To prove the second result, we start with the result when $f, g \geq 0$. In this case, using the first result, the fact that $g(y)$ does not depend on $x$, and then Theorem 1.6.9 twice, we get

$$Ef(X)g(Y) = \iint f(x)g(y)\, \mu(dx)\, \nu(dy) = \int g(y) \int f(x)\, \mu(dx)\, \nu(dy)$$
$$= \int Ef(X)g(y)\, \nu(dy) = Ef(X)Eg(Y)$$

Applying the result for nonnegative $f$ and $g$ to $|f|$ and $|g|$ shows $E|f(X)g(Y)| = E|f(X)|E|g(Y)| < \infty$, and we can repeat the last argument to prove the desired result. ■

From Theorem 2.1.8, it is only a small step to

**Theorem 2.1.9.** *If $X_1, \ldots, X_n$ are independent and have (a) $X_i \geq 0$ for all $i$, or (b) $E|X_i| < \infty$ for all $i$, then*

$$E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n EX_i$$

*that is, the expectation on the left exists and has the value given on the right.*

*Proof.* $X = X_1$ and $Y = X_2 \cdots X_n$ are independent by Theorem 2.1.6, so taking $f(x) = |x|$ and $g(y) = |y|$, we have $E|X_1 \cdots X_n| = E|X_1|E|X_2 \cdots X_n|$, and it follows by induction that if $1 \le m \le n$,

$$E|X_m \cdots X_n| = \prod_{i=m}^{n} E|X_k|$$

If the $X_i \ge 0$, then $|X_i| = X_i$ and the desired result follows from the special case $m = 1$. To prove the result in general, note that the special case $m = 2$ implies $E|Y| = E|X_2 \cdots X_n| < \infty$, so using Theorem 2.1.8 with $f(x) = x$ and $g(y) = y$ shows $E(X_1 \cdots X_n) = EX_1 \cdot E(X_2 \cdots X_n)$, and the desired result follows by induction. ■

**Example 2.1.2.** It can happen that $E(XY) = EX \cdot EY$ without the variables being independent. Suppose the joint distribution of $X$ and $Y$ is given by the following table:

|  |  | $Y$ |  |  |
|---|---|---|---|---|
|  |  | 1 | 0 | −1 |
|  | 1 | 0 | $a$ | 0 |
| $X$ | 0 | $b$ | $c$ | $b$ |
|  | −1 | 0 | $a$ | 0 |

where $a, b > 0$, $c \ge 0$, and $2a + 2b + c = 1$. Things are arranged so that $XY \equiv 0$. Symmetry implies $EX = 0$ and $EY = 0$, so $E(XY) = 0 = EXEY$. The random variables are not independent since

$$P(X = 1, Y = 1) = 0 < ab = P(X = 1)P(Y = 1)$$

Two random variables $X$ and $Y$ with $EX^2, EY^2 < \infty$ that have $EXY = EXEY$ are said to be **uncorrelated**. The finite second moments are needed so that we know $E|XY| < \infty$ by the Cauchy-Schwarz inequality.

### 2.1.3 *Sums of Independent Random Variables*

**Theorem 2.1.10.** *If $X$ and $Y$ are independent, $F(x) = P(X \le x)$, and $G(y) = P(Y \le y)$, then*

$$P(X + Y \le z) = \int F(z - y)\, dG(y)$$

The integral on the right-hand side is called the **convolution** of $F$ and $G$ and is denoted $F * G(z)$. The meaning of $dG(y)$ will be explained in the proof.

*Proof.* Let $h(x, y) = 1_{(x+y \le z)}$. Let $\mu$ and $\nu$ be the probability measures with distribution functions $F$ and $G$. Since for fixed $y$

$$\int h(x, y)\, \mu(dx) = \int 1_{(-\infty, z-y]}(x)\, \mu(dx) = F(z - y)$$

using Theorem 2.1.8 gives

$$P(X+Y\le z) = \iint 1_{(x+y\le z)}\,\mu(dx)\,\nu(dy)$$
$$= \int F(z-y)\,\nu(dy) = \int F(z-y)\,dG(y)$$

The last equality is just a change of notation. We regard $dG(y)$ as a shorthand for "integrate with respect to the measure $\nu$ with distribution function $G$." ■

To treat concrete examples, we need a special case of Theorem 2.1.10.

**Theorem 2.1.11.** *Suppose that $X$ with density $f$ and $Y$ with distribution function $G$ are independent. Then $X+Y$ has density*

$$h(x) = \int f(x-y)\,dG(y)$$

*When $Y$ has density $g$, the last formula can be written as*

$$h(x) = \int f(x-y)\,g(y)\,dy$$

*Proof.* From Theorem 2.1.10, the definition of density function, and Fubini's theorem (Theorem 1.7.2), which is justified since everything is nonnegative, we get

$$P(X+Y\le z) = \int F(z-y)\,dG(y) = \int\int_{-\infty}^{z} f(x-y)\,dx\,dG(y)$$
$$= \int_{-\infty}^{z}\int f(x-y)\,dG(y)\,dx$$

The last equation says that $X+Y$ has density $h(x) = \int f(x-y)dG(y)$. The second formula follows from the first when we recall the meaning of $dG(y)$ given in the proof of Theorem 2.1.10 and use Exercise 1.6.8. ■

Theorem 2.1.11 plus some ugly calculus allows us to treat two standard examples. These facts should be familiar from undergraduate probability.

**Example 2.1.3.** The **gamma density** with parameters $\alpha$ and $\lambda$ is given by

$$f(x) = \begin{cases} \lambda^\alpha x^{\alpha-1}e^{-\lambda x}/\,\Gamma(\alpha) & \text{for } x\ge 0 \\ 0 & \text{for } x<0 \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

**Theorem 2.1.12.** *If $X = \text{gamma}(\alpha,\lambda)$ and $Y = \text{gamma}(\beta,\lambda)$ are independent, then $X+Y$ is gamma$(\alpha+\beta,\lambda)$. Consequently if $X_1,\dots X_n$ are independent exponential$(\lambda)$ r.v.'s, then $X_1+\cdots+X_n$, has a gamma$(n,\lambda)$ distribution.*

*Proof.* Writing $f_{X+Y}(z)$ for the density function of $X+Y$ and using Theorem 2.1.11

$$\begin{aligned} f_{X+Y}(x) &= \int_0^x \frac{\lambda^\alpha (x-y)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(x-y)} \frac{\lambda^\beta y^{\beta-1}}{\Gamma(\beta)} e^{-\lambda y}\, dy \\ &= \frac{\lambda^{\alpha+\beta} e^{-\lambda x}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x (x-y)^{\alpha-1} y^{\beta-1}\, dy \end{aligned}$$

so it suffices to show the integral is $x^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. To do this, we begin by changing variables $y = xu$, $dy = x\,du$ to get

$$x^{\alpha+\beta-1} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1}\, du = \int_0^x (x-y)^{\alpha-1} y^{\beta-1}\, dy \tag{2.1.1}$$

There are two ways to complete the proof at this point. The soft solution is to note that we have shown that the density $f_{X+Y}(x) = c_{\alpha,\beta} e^{-\lambda} \lambda^{\alpha+\beta} x^{\alpha+\beta-1}$ where

$$c_{\alpha,\beta} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1}\, du$$

There is only one norming constant $c_{\alpha,\beta}$ that makes this a probability distribution, so recalling the definition of the beta distribution, we must have $c_{\alpha,\beta} = 1/\Gamma(\alpha+\beta)$.

The less elegant approach for those of us who cannot remember the definition of the beta is to prove the last equality by calculus. Rewriting (2.1.1) with the right-hand side on the left, multiplying each side of by $e^{-x}$, integrating from 0 to $\infty$, and then using Fubini's theorem on the right we have

$$\begin{aligned} &\Gamma(\alpha+\beta) \int_0^1 (1-u)^{\alpha-1} u^{\beta-1}\, du \\ &\quad = \int_0^\infty \int_0^x y^{\beta-1} e^{-y} (x-y)^{\alpha-1} e^{-(x-y)}\, dy\, dx \\ &\quad = \int_0^\infty y^{\beta-1} e^{-y} \int_x^\infty (x-y)^{\alpha-1} e^{-(x-y)}\, dx\, dy = \Gamma(\alpha)\Gamma(\beta) \end{aligned}$$

which gives the first result. The second follows from the fact that a gamma$(1,\lambda)$ is an exponential with parameter $\lambda$ and induction. ■

**Example 2.1.4. Normal distribution.** In Example 1.6.2, we introduced the normal density with mean $\mu$ and variance $a$,

$$(2\pi a)^{-1/2} \exp(-(x-\mu)^2/2a).$$

**Theorem 2.1.13.** *If $X = \text{normal}(\mu, a)$ and $Y = \text{normal}(\nu, b)$ are independent, then $X+Y = \text{normal}(\mu+\nu, a+b)$.*

*Proof.* It is enough to prove the result for $\mu = \nu = 0$. Suppose $Y_1 = \text{normal}(0, a)$ and $Y_2 = \text{normal}(0, b)$. Then Theorem 2.1.11 implies

$$f_{Y_1+Y_2}(z) = \frac{1}{2\pi\sqrt{ab}} \int e^{-x^2/2a} e^{-(z-x)^2/2b}\, dx$$

Dropping the constant in front, the integral can be rewritten as

$$\begin{aligned}
&\int \exp\left(-\frac{bx^2 + ax^2 - 2axz + az^2}{2ab}\right) dx \\
&\quad = \int \exp\left(-\frac{a+b}{2ab}\left\{x^2 - \frac{2a}{a+b}xz + \frac{a}{a+b}z^2\right\}\right) dx \\
&\quad = \int \exp\left(-\frac{a+b}{2ab}\left\{\left(x - \frac{a}{a+b}z\right)^2 + \frac{ab}{(a+b)^2}z^2\right\}\right) dx
\end{aligned}$$

since $-\{a/(a+b)\}^2 + \{a/(a+b)\} = ab/(a+b)^2$. Factoring out the term that does not depend on $x$, the last integral

$$\begin{aligned}
&= \exp\left(-\frac{z^2}{2(a+b)}\right) \int \exp\left(-\frac{a+b}{2ab}\left(x - \frac{a}{a+b}z\right)^2\right) dx \\
&= \exp\left(-\frac{z^2}{2(a+b)}\right) \sqrt{2\pi ab/(a+b)}
\end{aligned}$$

since the last integral is the normal density with parameters $\mu = az/(a+b)$ and $\sigma^2 = ab/(a+b)$ without its proper normalizing constant. Reintroducing the constant we dropped at the beginning,

$$f_{Y_1+Y_2}(z) = \frac{1}{2\pi\sqrt{ab}}\sqrt{2\pi ab/(a+b)}\exp\left(-\frac{z^2}{2(a+b)}\right) \qquad \blacksquare$$

### 2.1.4 *Constructing Independent Random Variables*

The last question that we have to address before we can study independent random variables is: do they exist? (If they don't exist, then there is no point in studying them!) If we are given a finite number of distribution functions $F_i$, $1 \le i \le n$, it is easy to construct independent random variables $X_1, \ldots, X_n$ with $P(X_i \le x) = F_i(x)$. Let $\Omega = \mathbf{R}^n$, $\mathcal{F} = \mathcal{R}^n$, $X_i(\omega_1, \ldots, \omega_n) = \omega_i$ (the $i$th coordinate of $\omega \in \mathbf{R}^n$), and let $P$ be the measure on $\mathcal{R}^n$ that has

$$P((a_1, b_1] \times \cdots \times (a_n, b_n]) = (F_1(b_1) - F_1(a_1)) \cdots (F_n(b_n) - F_n(a_n))$$

If $\mu_i$ is the measure with distribution function $F_i$ then $P = \mu_1 \times \cdots \times \mu_n$.

To construct an infinite sequence $X_1, X_2, \ldots$ of independent random variables with given distribution functions, we want to perform the last construction on the infinite product space

$$\mathbf{R}^{\mathbf{N}} = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \mathbf{R}\} = \{\text{functions } \omega : \mathbf{N} \to \mathbf{R}\}$$

where $\mathbf{N} = \{1, 2, \ldots\}$ and $\mathbf{N}$ stands for **natural numbers**. We define $X_i(\omega) = \omega_i$ and we equip $\mathbf{R}^{\mathbf{N}}$ with the product $\sigma$-field $\mathcal{R}^{\mathbf{N}}$, which is generated by the **finite dimensional sets** = sets of the form $\{\omega : \omega_i \in B_i, 1 \le i \le n\}$ where $B_i \in \mathcal{R}$. It is clear how we want to define $P$ for finite dimensional sets. To assert the existence of a unique extension to $\mathcal{R}^{\mathbf{N}}$, we use Theorem A.3.1:

**Theorem 2.1.14. Kolmogorov's extension theorem**. *Suppose we are given probability measures $\mu_n$ on $(\mathbf{R}^n, \mathcal{R}^n)$ that are consistent, that is,*

$$\mu_{n+1}((a_1, b_1] \times \cdots \times (a_n, b_n] \times \mathbf{R}) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

*Then there is a unique probability measure $P$ on $(\mathbf{R}^{\mathbf{N}}, \mathcal{R}^{\mathbf{N}})$ with*

$$P(\omega : \omega_i \in (a_i, b_i], 1 \le i \le n) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

In what follows we will need to construct sequences of random variables that take values in other measurable spaces $(S, \mathcal{S})$. Unfortunately, Theorem 2.1.14 is not valid for arbitrary measurable spaces. The first example (on an infinite product of different spaces $\Omega_1 \times \Omega_2 \times \ldots$) was constructed by Andersen and Jessen (1948). (See Halmos, 1950, p. 214, or Neveu, 1965, p. 84.) For an example in which all the spaces $\Omega_i$ are the same, see Wegner (1973). Fortunately, there is a class of spaces that is adequate for all of our results and for which the generalization of Kolmogorov's theorem is trivial.

$(S, \mathcal{S})$ is said to be **nice** if there is a 1-1 map $\varphi$ from $S$ into $\mathbf{R}$ so that $\varphi$ and $\varphi^{-1}$ are both measurable.

Such spaces are often called **standard Borel spaces**, but we already have too many things named after Borel. The next result shows that most spaces arising in applications are nice.

**Theorem 2.1.15.** *If $S$ is a Borel subset of a complete separable metric space $M$, and $\mathcal{S}$ is the collection of Borel subsets of $S$, then $(S, \mathcal{S})$ is nice.*

*Proof.* We begin with the special case $S = [0, 1)^{\mathbf{N}}$ with metric

$$\rho(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|/2^n$$

If $x = (x^1, x^2, x^3, \ldots)$, expand each component in binary $x^j = .x_1^j x_2^j x_3^j \ldots$ (taking the expansion with an infinite number of 0's). Let

$$\varphi_o(x) = .x_1^1 x_2^1 x_1^2 x_3^1 x_2^2 x_1^3 x_4^1 x_3^2 x_2^3 x_1^4 \ldots$$

To treat the general case, we observe that by letting

$$d(x, y) = \rho(x, y)/(1 + \rho(x, y))$$

(for more details, see Exercise 2.1.6), we can suppose that the metric has $d(x, y) < 1$ for all $x, y$. Let $q_1, q_2, \ldots$ be a countable dense set in S. Let

$$\psi(x) = (d(x, q_1), d(x, q_2), \ldots).$$

$\psi : S \to [0, 1)^N$ is continuous and 1-1. $\varphi_o \circ \psi$ gives the desired mapping. ■

**Exercise 2.1.6.** Let $\rho(x, y)$ be a metric. (i) Suppose $h$ is differentiable with $h(0) = 0$, $h'(x) > 0$ for $x > 0$, and $h'(x)$ decreasing on $[0, \infty)$. Then $h(\rho(x, y))$ is a metric. (ii) $h(x) = x/(x + 1)$ satisfies the hypotheses in (i).

**Caveat emptor.** The proof above is somewhat light when it comes to details. For a more comprehensive discussion, see Section 13.1 of Dudley (1989). An interesting consequence of the analysis there is that for Borel subsets of a complete separable metric space the continuum hypothesis is true: that is, all sets are either finite, countably infinite, or have the cardinality of the real numbers.

### *Exercises*

**2.1.7.** Let $\Omega = (0, 1)$, $\mathcal{F}$ = Borel sets, $P$ = Lebesgue measure. $X_n(\omega) = \sin(2\pi n\omega)$, $n = 1, 2, \ldots$ are uncorrelated but not independent.

**2.1.8.** (i) Show that if $X$ and $Y$ are independent with distributions $\mu$ and $\nu$, then

$$P(X + Y = 0) = \sum_y \mu(\{-y\})\nu(\{y\})$$

(ii) Conclude that if $X$ has continuous distribution, $P(X = Y) = 0$.

**2.1.9.** Prove directly from the definition that if $X$ and $Y$ are independent and $f$ and $g$ are measurable functions, then $f(X)$ and $g(Y)$ are independent.

**2.1.10.** Let $K \geq 3$ be a prime and let $X$ and $Y$ be independent random variables that are uniformly distributed on $\{0, 1, \ldots, K - 1\}$. For $0 \leq n < K$, let $Z_n = X + nY \bmod K$. Show that $Z_0, Z_1, \ldots, Z_{K-1}$ are **pairwise independent**, that is, each pair is independent. They are not independent because if we know the values of two of the variables, then we know the values of all the variables.

**2.1.11.** Find four random variables taking values in $\{-1, 1\}$ so that any three are independent but all four are not. Hint: Consider products of independent random variables.

**2.1.12.** Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F}$ = all subsets of $\Omega$, and $P(\{i\}) = 1/4$. Give an example of two collections of sets $\mathcal{A}_1$ and $\mathcal{A}_2$ that are independent but whose generated $\sigma$-fields are not.

**2.1.13.** Show that if $X$ and $Y$ are independent, integer-valued random variables, then

$$P(X+Y=n)=\sum_m P(X=m)P(Y=n-m)$$

**2.1.14.** In Example 1.6.4, we introduced the Poisson distribution with parameter $\lambda$, which is given by $P(Z=k)=e^{-\lambda}\lambda^k/k!$ for $k=0,1,2,\ldots$ Use the previous exercise to show that if $X=\text{Poisson}(\lambda)$ and $Y=\text{Poisson}(\mu)$ are independent, then $X+Y=\text{Poisson}(\lambda+\mu)$.

**2.1.15.** $X$ is said to have a Binomial$(n,p)$ distribution if

$$P(X=m)=\binom{n}{m}p^m(1-p)^{n-m}$$

(i) Show that if $X=\text{Binomial}(n,p)$ and $Y=\text{Binomial}(m,p)$ are independent, then $X+Y=\text{Binomial}(n+m,p)$. (ii) Look at Example 1.6.3 and use induction to conclude that the sum of $n$ independent Bernoulli$(p)$ random variables is Binomial$(n,p)$.

**2.1.16.** It should not be surprising that the distribution of $X+Y$ can be $F*G$ without the random variables being independent. Suppose $X,Y\in\{0,1,2\}$ and take each value with probability 1/3. (a) Find the distribution of $X+Y$ assuming $X$ and $Y$ are independent. (b) Find all the joint distributions $(X,Y)$ so that the distribution of $X+Y$ is the same as the answer to (a).

**2.1.17.** Let $X,Y\geq 0$ be independent with distribution functions $F$ and $G$. Find the distribution function of $XY$.

**2.1.18.** If we want an infinite sequence of coin tossings, we do not have to use Kolmogorov's theorem. Let $\Omega$ be the unit interval (0,1) equipped with the Borel sets $\mathcal{F}$ and Lebesgue measure P. Let $Y_n(\omega)=1$ if $[2^n\omega]$ is odd and 0 if $[2^n\omega]$ is even. Show that $Y_1,Y_2,\ldots$ are independent with $P(Y_k=0)=P(Y_k=1)=1/2$.

## 2.2 Weak Laws of Large Numbers

In this section, we will prove several "weak laws of large numbers." The first order of business is to define the mode of convergence that appears in the conclusions of the theorems. We say that $Y_n$ converges to $Y$ **in probability** if for all $\epsilon>0$, $P(|Y_n-Y|>\epsilon)\to 0$ as $n\to\infty$.

### 2.2.1 $L^2$ *Weak Laws*

Our first set of weak laws come from computing variances and using Chebyshev's inequality. Extending a definition given in Example 2.1.2 for two random variables, a family of random variables $X_i$, $i \in I$ with $EX_i^2 < \infty$ is said to be **uncorrelated** if we have

$$E(X_i X_j) = EX_i EX_j \quad \text{whenever } i \neq j$$

The key to our weak law for uncorrelated random variables, Theorem 2.2.3, is:

**Theorem 2.2.1.** *Let $X_1, \ldots, X_n$ have $E(X_i^2) < \infty$ and be uncorrelated. Then*

$$var(X_1 + \cdots + X_n) = var(X_1) + \cdots + var(X_n)$$

*where $var(Y)$ = the variance of $Y$.*

*Proof.* Let $\mu_i = EX_i$ and $S_n = \sum_{i=1}^n X_i$. Since $ES_n = \sum_{i=1}^n \mu_i$, using the definition of the variance, writing the square of the sum as the product of two copies of the sum, and then expanding, we have

$$\begin{aligned}
\text{var}(S_n) = E(S_n - ES_n)^2 &= E\left(\sum_{i=1}^n (X_i - \mu_i)\right)^2 \\
&= E\left(\sum_{i=1}^n \sum_{j=1}^n (X_i - \mu_i)(X_j - \mu_j)\right) \\
&= \sum_{i=1}^n E(X_i - \mu_i)^2 + 2\sum_{i=1}^n \sum_{j=1}^{i-1} E((X_i - \mu_i)(X_j - \mu_j))
\end{aligned}$$

where in the last equality we have separated out the diagonal terms $i = j$ and used the fact that the sum over $1 \leq i < j \leq n$ is the same as the sum over $1 \leq j < i \leq n$.

The first sum is $\text{var}(X_1) + \cdots + \text{var}(X_n)$, so we want to show that the second sum is zero. To do this, we observe

$$\begin{aligned}
E((X_i - \mu_i)(X_j - \mu_j)) &= EX_i X_j - \mu_i EX_j - \mu_j EX_i + \mu_i \mu_j \\
&= EX_i X_j - \mu_i \mu_j = 0
\end{aligned}$$

since $X_i$ and $X_j$ are uncorrelated. ■

In words, Theorem 2.2.1 says that for uncorrelated random variables, the variance of the sum is the sum of the variances. The second ingredient in our proof of Theorem 2.2.3 is the following consequence of (1.6.4):

$$\text{var}(cY) = c^2 \, \text{var}(Y)$$

The third and final ingredient is

**Lemma 2.2.2.** *If $p > 0$ and $E|Z_n|^p \to 0$ then $Z_n \to 0$ in probability.*

*Proof.* Chebyshev's inequality, Theorem 1.6.4, with $\varphi(x) = x^p$ and $X = |Z_n|$ implies that if $\epsilon > 0$ then $P(|Z_n| \geq \epsilon) \leq \epsilon^{-p} E|Z_n|^p \to 0$. ■

We can now easily prove

**Theorem 2.2.3.** $L^2$ **weak law.** *Let $X_1, X_2, \ldots$ be uncorrelated random variables with $EX_i = \mu$ and $\text{var}(X_i) \le C < \infty$. If $S_n = X_1 + \cdots + X_n$ then as $n \to \infty$, $S_n/n \to \mu$ in $L^2$ and in probability.*

*Proof.* To prove $L^2$ convergence, observe that $E(S_n/n) = \mu$, so

$$E(S_n/n - \mu)^2 = \text{var}(S_n/n) = \frac{1}{n^2}(\text{var}(X_1) + \cdots + \text{var}(X_n)) \le \frac{Cn}{n^2} \to 0$$

To conclude there is also convergence in probability, we apply the Lemma 2.2.2 to $Z_n = S_n/n - \mu$. ∎

The most important special case of Theorem 2.2.3 occurs when $X_1, X_2, \ldots$ are independent random variables that all have the same distribution. In the jargon, they are **independent and identically distributed**, or **i.i.d.** for short. Theorem 2.2.3 tells us in this case that if $EX_i^2 < \infty$, then $S_n/n$ converges to $\mu = EX_i$ in probability as $n \to \infty$. In Theorem 2.2.9 below, we will see that $E|X_i| < \infty$ is sufficient for the last conclusion, but for the moment we will concern ourselves with consequences of the weaker result.

Our first application is to a situation that on the surface has nothing to do with randomness.

**Example 2.2.1. Polynomial approximation**. Let $f$ be a continuous function on [0,1], and let

$$f_n(x) = \sum_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m} f(m/n) \quad \text{where} \quad \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

be the **Bernstein polynomial of degree** $n$ associated with $f$. Then as $n \to \infty$

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0$$

*Proof.* First observe that if $S_n$ is the sum of $n$ independent random variables with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$, then $EX_i = p$, $\text{var}(X_i) = p(1-p)$ and

$$P(S_n = m) = \binom{n}{m} p^m (1-p)^{n-m}$$

so $Ef(S_n/n) = f_n(p)$. Theorem 2.2.3 tells us that as $n \to \infty$, $S_n/n \to p$ in probability. The last two observations motivate the definition of $f_n(p)$, but to prove the desired conclusion we have to use the proof of Theorem 2.2.3 rather than the result itself.

Combining the proof of Theorem 2.2.3 with our formula for the variance of $X_i$ and the fact that $p(1-p) \le 1/4$ when $p \in [0,1]$, we have

$$P(|S_n/n - p| > \delta) \le \frac{\text{var}\,(S_n/n)}{\delta^2} = \frac{p(1-p)}{n\delta^2} \le \frac{1}{4n\delta^2}$$

To conclude now that $Ef(S_n/n) \to f(p)$, let $M = \sup_{x\in[0,1]} |f(x)|$, let $\epsilon > 0$, and pick $\delta > 0$ so that if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. (This is possible since a continuous function is uniformly continuous on each bounded interval.) Now, using Jensen's inequality, Theorem 1.6.2, gives

$$|Ef(S_n/n) - f(p)| \le E|f(S_n/n) - f(p)| \le \epsilon + 2MP(|S_n/n - p| > \delta)$$

Letting $n \to \infty$, we have $\limsup_{n\to\infty} |Ef(S_n/n) - f(p)| \le \epsilon$, but $\epsilon$ is arbitrary so this gives the desired result. ∎

Our next result is for comic relief.

**Example 2.2.2. A high-dimensional cube is almost the boundary of a ball.** Let $X_1, X_2, \ldots$ be independent and uniformly distributed on $(-1,1)$. Let $Y_i = X_i^2$, which are independent since they are functions of independent random variables. $EY_i = 1/3$ and $\text{var}\,(Y_i) \le EY_i^2 \le 1$, so Theorem 2.2.3 implies

$$(X_1^2 + \cdots + X_n^2)/n \to 1/3 \quad \text{in probability as } n \to \infty$$

Let $A_{n,\epsilon} = \{x \in \mathbf{R}^n : (1-\epsilon)\sqrt{n/3} < |x| < (1+\epsilon)\sqrt{n/3}\}$ where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. If we let $|S|$ denote the Lebesgue measure of $S$, then the last conclusion implies that for any $\epsilon > 0$, $|A_{n,\epsilon} \cap (-1,1)^n|/2^n \to 1$, or, in words, most of the volume of the cube $(-1,1)^n$ comes from $A_{n,\epsilon}$, which is almost the boundary of the ball of radius $\sqrt{n/3}$.

### 2.2.2 *Triangular Arrays*

Many classical limit theorems in probability concern arrays $X_{n,k}$, $1 \le k \le n$ of random variables and investigate the limiting behavior of their row sums $S_n = X_{n,1} + \cdots + X_{n,n}$. In most cases, we assume that the random variables on each row are independent, but for the next trivial (but useful) result, we do not need that assumption. Indeed, here $S_n$ can be any sequence of random variables.

**Theorem 2.2.4.** *Let $\mu_n = ES_n$, $\sigma_n^2 = var\,(S_n)$. If $\sigma_n^2/b_n^2 \to 0$ then*

$$\frac{S_n - \mu_n}{b_n} \to 0 \qquad \textit{in probability}$$

*Proof.* Our assumptions imply $E((S_n - \mu_n)/b_n)^2 = b_n^{-2}\,\text{var}\,(S_n) \to 0$, so the desired conclusion follows from Lemma 2.2.2. ∎

We will now give three applications of Theorem 2.2.4. For these three examples, the following calculation is useful:

$$\sum_{m=1}^{n} \frac{1}{m} \ge \int_1^n \frac{dx}{x} \ge \sum_{m=2}^{n} \frac{1}{m}$$

$$\log n \le \sum_{m=1}^{n} \frac{1}{m} \le 1 + \log n \tag{2.2.1}$$

**Example 2.2.3. Coupon collector's problem.** Let $X_1, X_2, \ldots$ be i.i.d. uniform on $\{1, 2, \ldots, n\}$. To motivate the name, think of collecting baseball cards (or coupons). Suppose that the $i$th item we collect is chosen at random from the set of possibilities and is independent of the previous choices. Let $\tau_k^n = \inf\{m : |\{X_1, \ldots, X_m\}| = k\}$ be the first time we have $k$ different items. In this problem, we are interested in the asymptotic behavior of $T_n = \tau_n^n$, the time to collect a complete set. It is easy to see that $\tau_1^n = 1$. To make later formulas work out nicely, we will set $\tau_0^n = 0$. For $1 \le k \le n$, $X_{n,k} \equiv \tau_k^n - \tau_{k-1}^n$ represents the time to get a choice different from our first $k-1$, so $X_{n,k}$ has a geometric distribution with parameter $1 - (k-1)/n$ and is independent of the earlier waiting times $X_{n,j}$, $1 \le j < k$. Example 1.6.5 tells us that if $X$ has a geometric distribution with parameter $p$, then $EX = 1/p$ and $\text{var}\,(X) \le 1/p^2$. Using the linearity of expected value, bounds on $\sum_{m=1}^n 1/m$ in (2.2.1), and Theorem 2.2.1, we see that

$$ET_n = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right)^{-1} = n \sum_{m=1}^{n} m^{-1} \sim n \log n$$

$$\text{var}\,(T_n) \le \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right)^{-2} = n^2 \sum_{m=1}^{n} m^{-2} \le n^2 \sum_{m=1}^{\infty} m^{-2}$$

Taking $b_n = n \log n$ and using Theorem 2.2.4, it follows that

$$\frac{T_n - n \sum_{m=1}^n m^{-1}}{n \log n} \to 0 \quad \text{in probability}$$

and hence $T_n/(n \log n) \to 1$ in probability.

For a concrete example, take $n = 365$, that is, we are interested in the number of people we need to meet until we have seen someone with every birthday. In this case the limit theorem says it will take about $365 \log 365 = 2153.46$ tries to get a complete set. Note that the number of trials is 5.89 times the number of birthdays.

**Example 2.2.4. Random permutations.** Let $\Omega_n$ consist of the $n!$ permutations (i.e., one-to-one mappings from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$) and make this into a probability space by assuming all the permutations are equally likely. This application of the weak law concerns the cycle structure of a random permutation $\pi$, so we begin by describing the decompostion of a permutation into cycles. Consider the sequence $1, \pi(1), \pi(\pi(1)), \ldots$ Eventually, $\pi^k(1) = 1$. When it does, we say the first cycle is completed and has length $k$. To start the second cycle, we pick the smallest integer $i$ not in the first cycle and look at $i, \pi(i), \pi(\pi(i)), \ldots$ until we come back to $i$. We repeat the construction until all the elements are accounted for. For example, if the permutation is

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\pi(i)$ | 3 | 9 | 6 | 8 | 2 | 1 | 5 | 4 | 7 |

then the cycle decomposition is (136) (2975) (48).

Let $X_{n,k} = 1$ if a right parenthesis occurs after the $k$th number in the decomposition, $X_{n,k} = 0$ otherwise and let $S_n = X_{n,1} + \cdots + X_{n,n}$ = the number of cycles. (In the example, $X_{9,3} = X_{9,7} = X_{9,9} = 1$, and the other $X_{9,m} = 0$.) I claim that

**Lemma 2.2.5.** *$X_{n,1}, \ldots, X_{n,n}$ are independent and $P(X_{n,j} = 1) = \frac{1}{n-j+1}$.*

Intuitively, this is true since, independent of what has happened so far, there are $n - j + 1$ values that have not appeared in the range, and only one of them will complete the cycle.

*Proof.* To prove this, it is useful to generate the permutation in a special way. Let $i_1 = 1$. Pick $j_1$ at random from $\{1, \ldots, n\}$ and let $\pi(i_1) = j_1$. If $j_1 \neq 1$, let $i_2 = j_1$. If $j_1 = 1$, let $i_2 = 2$. In either case, pick $j_2$ at random from $\{1, \ldots, n\} - \{j_1\}$. In general, if $i_1, j_1, \ldots, i_{k-1}, j_{k-1}$ have been selected and we have set $\pi(i_\ell) = j_\ell$ for $1 \le \ell < k$, then (a) if $j_{k-1} \in \{i_1, \ldots, i_{k-1}\}$ so a cycle has just been completed, we let $i_k = \inf(\{1, \ldots, n\} - \{i_1, \ldots, i_{k-1}\})$ and (b) if $j_{k-1} \notin \{i_1, \ldots, i_{k-1}\}$, we let $i_k = j_{k-1}$. In either case we pick $j_k$ at random from $\{1, \ldots, n\} - \{j_1, \ldots, j_{k-1}\}$ and let $\pi(i_k) = j_k$.

The construction above is tedious to write out, or to read, but now I can claim with a clear conscience that $X_{n,1}, \ldots, X_{n,n}$ are independent and $P(X_{n,k} = 1) = 1/(n - j + 1)$ because when we pick $j_k$, there are $n - j + 1$ values in $\{1, \ldots, n\} - \{j_1, \ldots, j_{k-1}\}$ and only one of them will complete the cycle. ■

To check the conditions of Theorem 2.2.4, now note

$$ES_n = 1/n + 1/(n-1) + \cdots + 1/2 + 1$$

$$\text{var}(S_n) = \sum_{k=1}^n \text{var}(X_{n,k}) \le \sum_{k=1}^n E(X_{n,k}^2) = \sum_{k=1}^n E(X_{n,k}) = ES_n$$

where the results on the second line follow from Theorem 2.2.1, the fact that $\text{var}(Y) \le EY^2$, and $X_{n,k}^2 = X_{n,k}$. Now $ES_n \sim \log n$, so if $b_n = (\log n)^{.5+\epsilon}$ with $\epsilon > 0$, the conditions of Theorem 2.2.4 are satisfied and it follows that

$$\frac{S_n - \sum_{m=1}^n m^{-1}}{(\log n)^{.5+\epsilon}} \to 0 \quad \text{in probability}$$

Taking $\epsilon = 0.5$, we have that $S_n/\log n \to 1$ in probability, but (∗) says more. We will see in Example 3.4.6 that (∗) is false if $\epsilon = 0$.

**Example 2.2.5. An occupancy problem**. Suppose we put $r$ balls at random in $n$ boxes, that is, all $n^r$ assignments of balls to boxes have equal probability. Let $A_i$ be the event that the $i$th box is empty and $N_n$ = the number of empty boxes. It is easy to see that

$$P(A_i) = (1 - 1/n)^r \qquad \text{and} \qquad EN_n = n(1 - 1/n)^r$$

A little calculus (take logarithms) shows that if $r/n \to c$, $EN_n/n \to e^{-c}$. (For a proof, see Lemma 3.1.1.) To compute the variance of $N_n$, we observe that

$$EN_n^2 = E\left(\sum_{m=1}^n 1_{A_m}\right)^2 = \sum_{1 \le k,m \le n} P(A_k \cap A_m)$$

$$\begin{aligned} \text{var}\,(N_n) = EN_n^2 - (EN_n)^2 &= \sum_{1 \le k,m \le n} P(A_k \cap A_m) - P(A_k)P(A_m) \\ &= n(n-1)\{(1-2/n)^r - (1-1/n)^{2r}\} + n\{(1-1/n)^r - (1-1/n)^{2r}\} \end{aligned}$$

The first term comes from $k \ne m$ and the second from $k = m$. Since $(1 - 2/n)^r \to e^{-2c}$ and $(1 - 1/n)^r \to e^{-c}$, it follows easily from the last formula that $\text{var}\,(N_n/n) = \text{var}\,(N_n)/n^2 \to 0$. Taking $b_n = n$ in Theorem 2.2.4 now we have

$$N_n/n \to e^{-c} \quad \text{in probability}$$

### 2.2.3 *Truncation*

To truncate a random variable $X$ at level $M$ means to consider

$$\bar{X} = X1_{(|X| \le M)} = \begin{cases} X & \text{if } |X| \le M \\ 0 & \text{if } |X| > M \end{cases}$$

To extend the weak law to random variables without a finite second moment, we will truncate and then use Chebyshev's inequality. We begin with a very general but also very useful result. Its proof is easy because we have assumed what we need for the proof. Later we will have to work a little to verify the assumptions in special cases, but the general result serves to identify the essential ingredients in the proof.

**Theorem 2.2.6. Weak law for triangular arrays.** *For each $n$ let $X_{n,k}$, $1 \le k \le n$, be independent. Let $b_n > 0$ with $b_n \to \infty$, and let $\bar{X}_{n,k} = X_{n,k}1_{(|X_{n,k}| \le b_n)}$. Suppose that as $n \to \infty$*
*(i)* $\sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$, *and*
*(ii)* $b_n^{-2} \sum_{k=1}^n E\bar{X}_{n,k}^2 \to 0$.

*If we let* $S_n = X_{n,1} + \cdots + X_{n,n}$ *and put* $a_n = \sum_{k=1}^n E\bar{X}_{n,k}$, *then*

$$(S_n - a_n)/b_n \to 0 \text{ in probability}$$

*Proof.* Let $\bar{S}_n = \bar{X}_{n,1} + \cdots + \bar{X}_{n,n}$. Clearly,

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \epsilon\right) \le P(S_n \ne \bar{S}_n) + P\left(\left|\frac{\bar{S}_n - a_n}{b_n}\right| > \epsilon\right)$$

To estimate the first term, we note that

$$P(S_n \ne \bar{S}_n) \le P\left(\cup_{k=1}^n \{\bar{X}_{n,k} \ne X_{n,k}\}\right) \le \sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$$

by (i). For the second term, we note that Chebyshev's inequality, $a_n = E\bar{S}_n$, Theorem 2.2.1, and $\text{var}(X) \le EX^2$ imply

$$P\left(\left|\frac{\bar{S}_n - a_n}{b_n}\right| > \epsilon\right) \le \epsilon^{-2} E\left|\frac{\bar{S}_n - a_n}{b_n}\right|^2 = \epsilon^{-2} b_n^{-2} \text{var}(\bar{S}_n)$$

$$= (b_n\epsilon)^{-2} \sum_{k=1}^n \text{var}(\bar{X}_{n,k}) \le (b_n\epsilon)^{-2} \sum_{k=1}^n E(\bar{X}_{n,k})^2 \to 0$$

by (ii), and the proof is complete. ■

From Theorem 2.2.6, we get the following result for a single sequence.

**Theorem 2.2.7. Weak law of large numbers.** *Let* $X_1, X_2, \ldots$ *be i.i.d. with*

$$xP(|X_i| > x) \to 0 \quad \text{as } x \to \infty$$

*Let* $S_n = X_1 + \cdots + X_n$ *and let* $\mu_n = E(X_1 1_{(|X_1| \le n)})$. *Then* $S_n/n - \mu_n \to 0$ *in probability.*

**Remark.** The assumption in the theorem is necessary for the existence of constants $a_n$ so that $S_n/n - a_n \to 0$. See Feller, Vol. II (1971), pp. 234–6, for a proof.

*Proof.* We will apply Theorem 2.2.6 with $X_{n,k} = X_k$ and $b_n = n$. To check (i), we note

$$\sum_{k=1}^n P(|X_{n,k}| > n) = nP(|X_i| > n) \to 0$$

by assumption. To check (ii), we need to show $n^{-2} \cdot nE\bar{X}_{n,1}^2 \to 0$. To do this, we need the following result, which will be useful several times below.

**Lemma 2.2.8.** *If* $Y \ge 0$ *and* $p > 0$ *then* $E(Y^p) = \int_0^\infty py^{p-1}P(Y > y)\,dy$.

*Proof.* Using the definition of expected value, Fubini's theorem (for nonnegative random variables), and then calculating the resulting integrals gives

$$\begin{aligned}\int_0^\infty py^{p-1}P(Y>y)\,dy &= \int_0^\infty\int_\Omega py^{p-1}1_{(Y>y)}\,dP\,dy\\ &= \int_\Omega\int_0^\infty py^{p-1}1_{(Y>y)}\,dy\,dP\\ &= \int_\Omega\int_0^Y py^{p-1}\,dy\,dP = \int_\Omega Y^p\,dP = EY^p\end{aligned}$$

which is the desired result. ■

Returning to the proof of Theorem 2.2.7, we observe that Lemma 2.2.8 and the fact that $\bar{X}_{n,1} = X_1 1_{(|X_1|\le n)}$ imply

$$E(\bar{X}_{n,1}^2) = \int_0^\infty 2yP(|\bar{X}_{n,1}|>y)\,dy \le \int_0^n 2yP(|X_1|>y)\,dy$$

since $P(|\bar{X}_{n,1}|>y)=0$ for $y\ge n$ and $= P(|X_1|>y) - P(|X_1|>n)$ for $y\le n$. We claim that $yP(|X_1|>y)\to 0$ implies

$$E(\bar{X}_{n,1}^2)/n = \frac{1}{n}\int_0^n 2yP(|X_1|>y)\,dy \to 0$$

as $n\to\infty$. Intuitively, this holds since the right-hand side is the average of $g(y) = 2yP(|X_1|>y)$ over $[0,n]$ and $g(y)\to 0$ as $y\to\infty$. To spell out the details, note that $0\le g(y)\le 2y$ and $g(y)\to 0$ as $y\to\infty$, so we must have $M=\sup g(y)<\infty$. If we let $\epsilon_K = \sup\{g(y) : y>K\}$, then by considering the integrals over $[0,K]$ and $[K,n]$ separately

$$\int_0^n 2yP(|X_1|>y)\,dy \le KM + (n-K)\epsilon_K$$

Dividing by $n$ and letting $n\to\infty$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\int_0^n 2yP(|X_1|>y)\,dy \le \epsilon_K$$

Since $K$ is arbitrary and $\epsilon_K\to 0$ as $K\to\infty$, the desired result follows. ■

Finally, we have the weak law in its most familiar form.

**Theorem 2.2.9.** *Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_i|<\infty$. Let $S_n = X_1+\cdots+X_n$ and let $\mu = EX_1$. Then $S_n/n\to\mu$ in probability.*

**Remark.** Applying Lemma 2.2.8 with $p=1-\epsilon$ and $\epsilon>0$, we see that $xP(|X_1|>x)\to 0$ implies $E|X_1|^{1-\epsilon}<\infty$, so the assumption in is not much weaker than finite mean.

*Proof.* Two applications of the dominated convergence theorem imply

$$x P(|X_1| > x) \le E(|X_1| 1_{(|X_1|>x)}) \to 0 \quad \text{as } x \to \infty$$
$$\mu_n = E(X_1 1_{(|X_1|\le n)}) \to E(X_1) = \mu \quad \text{as } n \to \infty$$

Using Theorem 2.2.7, we see that if $\epsilon > 0$ then $P(|S_n/n - \mu_n| > \epsilon/2) \to 0$. Since $\mu_n \to \mu$, it follows that $P(|S_n/n - \mu| > \epsilon) \to 0$. ■

**Example 2.2.6.** For an example where the weak law does not hold, suppose $X_1, X_2, \ldots$ are independent and have a **Cauchy distribution**:

$$P(X_i \le x) = \int_{-\infty}^{x} \frac{dt}{\pi(1+t^2)}$$

As $x \to \infty$,

$$P(|X_1| > x) = 2\int_x^\infty \frac{dt}{\pi(1+t^2)} \sim \frac{2}{\pi}\int_x^\infty t^{-2}\,dt = \frac{2}{\pi}x^{-1}$$

From the necessity of the condition above, we can conclude that there is no sequence of constants $\mu_n$ so that $S_n/n - \mu_n \to 0$. We will see later that $S_n/n$ always has the same distribution as $X_1$. (See Exercise 3.3.8.)

As the next example shows, we can have a weak law in some situations in which $E|X| = \infty$.

**Example 2.2.7. The "St. Petersburg paradox."** Let $X_1, X_2, \ldots$ be independent random variables with

$$P(X_i = 2^j) = 2^{-j} \quad \text{for } j \ge 1$$

In words, you win $2^j$ dollars if it takes $j$ tosses to get a heads. The paradox here is that $EX_1 = \infty$, but you clearly wouldn't pay an infinite amount to play this game. An application of Theorem 2.2.6 will tell us how much we should pay to play the game $n$ times.

In this example, $X_{n,k} = X_k$. To apply Theorem 2.2.6, we have to pick $b_n$. To do this, we are guided by the principle that in checking (ii) we want to take $b_n$ as small as we can and have (i) hold. With this in mind, we observe that if $m$ is an integer,

$$P(X_1 \ge 2^m) = \sum_{j=m}^{\infty} 2^{-j} = 2^{-m+1}$$

Let $m(n) = \log_2 n + K(n)$ where $K(n) \to \infty$ and is chosen so that $m(n)$ is an integer (and hence the displayed formula is valid). Letting $b_n = 2^{m(n)}$, we have

$$nP(X_1 \ge b_n) = n2^{-m(n)+1} = 2^{-K(n)+1} \to 0$$

proving (i). To check (ii), we observe that if $\bar{X}_{n,k} = X_k 1_{(|X_k| \le b_n)}$ then

$$E\bar{X}_{n,k}^2 = \sum_{j=1}^{m(n)} 2^{2j} \cdot 2^{-j} \le 2^{m(n)} \sum_{k=0}^{\infty} 2^{-k} = 2b_n$$

So the expression in (ii) is smaller than $2n/b_n$, which $\to 0$ since

$$b_n = 2^{m(n)} = n2^{K(n)} \text{ and } K(n) \to \infty$$

The last two steps are to evaluate $a_n$ and to apply Theorem 2.2.6.

$$E\bar{X}_{n,k} = \sum_{j=1}^{m(n)} 2^j 2^{-j} = m(n)$$

so $a_n = nm(n)$. We have $m(n) = \log n + K(n)$ (here and until the end of the example all logs are base 2), so if we pick $K(n)/\log n \to 0$, then $a_n/n \log n \to 1$ as $n \to \infty$. Using Theorem 2.2.6 now, we have

$$\frac{S_n - a_n}{n2^{K(n)}} \to 0 \quad \text{in probability}$$

If we suppose that $K(n) \le \log\log n$ for large $n$, then the last conclusion holds with the denominator replaced by $n \log n$, and it follows that $S_n/(n \log n) \to 1$ in probability.

Returning to our original question, we see that a fair price for playing $n$ times is \$ $\log_2 n$ per play. When $n = 1024$, this is \$10 per play. Nicolas Bernoulli wrote in 1713, "There ought not to exist any even halfway sensible person who would not sell the right of playing the game for 40 ducates (per play)." If the wager were 1 ducat, one would need $2^{40} \approx 10^{12}$ plays to start to break even.

## Exercises

**2.2.1.** Let $X_1, X_2, \ldots$ be uncorrelated with $EX_i = \mu_i$ and $\text{var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\nu_n = ES_n/n$ then as $n \to \infty$, $S_n/n - \nu_n \to 0$ in $L^2$ and in probability.

**2.2.2.** The $L^2$ weak law generalizes immediately to certain dependent sequences. Suppose $EX_n = 0$ and $EX_nX_m \le r(n-m)$ for $m \le n$ (no absolute value on the left-hand side!) with $r(k) \to 0$ as $k \to \infty$. Show that $(X_1 + \cdots + X_n)/n \to 0$ in probability.

**2.2.3. Monte Carlo integration**. (i) Let $f$ be a measurable function on [0, 1] with $\int_0^1 |f(x)| dx < \infty$. Let $U_1, U_2, \ldots$ be independent and uniformly distributed on [0, 1], and let

$$I_n = n^{-1}(f(U_1) + \cdots + f(U_n))$$

Show that $I_n \to I \equiv \int_0^1 f\, dx$ in probability. (ii) Suppose $\int_0^1 |f(x)|^2\, dx < \infty$. Use Chebyshev's inequality to estimate $P(|I_n - I| > a/n^{1/2})$.

**2.2.4.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = (-1)^k k) = C/k^2 \log k$ for $k \geq 2$ where $C$ is chosen to make the sum of the probabilities $= 1$. Show that $E|X_i| = \infty$, but there is a finite constant $\mu$ so that $S_n/n \to \mu$ in probability.

**2.2.5.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i > x) = e/x \log x$ for $x \geq e$. Show that $E|X_i| = \infty$, but there is a sequence of constants $\mu_n \to \infty$ so that $S_n/n - \mu_n \to 0$ in probability.

**2.2.6.** (i) Show that if $X \geq 0$ is integer valued $EX = \sum_{n \geq 1} P(X \geq n)$. (ii) Find a similar expression for $EX^2$.

**2.2.7.** Generalize Lemma 2.2.8 to conclude that if $H(x) = \int_{(-\infty,x]} h(y)\, dy$ with $h(y) \geq 0$, then

$$E\ H(X) = \int_{-\infty}^{\infty} h(y) P(X \geq y)\, dy$$

An important special case is $H(x) = \exp(\theta x)$ with $\theta > 0$.

**2.2.8. An unfair "fair game."** Let $p_k = 1/2^k k(k+1)$, $k = 1, 2, \ldots$ and $p_0 = 1 - \sum_{k \geq 1} p_k$.

$$\sum_{k=1}^{\infty} 2^k p_k = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots = 1$$

so if we let $X_1, X_2, \ldots$ be i.i.d. with $P(X_n = -1) = p_0$ and

$$P(X_n = 2^k - 1) = p_k \quad \text{for } k \geq 1$$

then $EX_n = 0$. Let $S_n = X_1 + \cdots + X_n$. Use (5.5) with $b_n = 2^{m(n)}$ where $m(n) = \min\{m : 2^{-m} m^{-3/2} \leq n^{-1}\}$ to conclude that

$$S_n/(n/\log_2 n) \to -1 \text{ in probability}$$

**2.2.9. Weak law for positive variables.** Suppose $X_1, X_2, \ldots$ are i.i.d., $P(0 \leq X_i < \infty) = 1$ and $P(X_i > x) > 0$ for all $x$. Let $\mu(s) = \int_0^s x\, dF(x)$ and $\nu(s) = \mu(s)/s(1 - F(s))$. It is known that there exist constants $a_n$ so that $S_n/a_n \to 1$ in probability, if and only if $\nu(s) \to \infty$ as $s \to \infty$. Pick $b_n \geq 1$ so that $n\mu(b_n) = b_n$ (this works for large $n$), and use Theorem 2.2.6 to prove that the condition is sufficient.

## 2.3 Borel-Cantelli Lemmas

If $A_n$ is a sequence of subsets of $\Omega$, we let

$$\limsup A_n = \lim_{m \to \infty} \cup_{n=m}^{\infty} A_n = \{\omega \text{ that are in infinitely many } A_n\}$$

(the limit exists since the sequence is decreasing in $m$) and let

$$\liminf A_n = \lim_{m \to \infty} \cap_{n=m}^{\infty} A_n = \{\omega \text{ that are in all but finitely many } A_n\}$$

(the limit exists since the sequence is increasing in $m$). The names lim sup and lim inf can be explained by noting that

$$\limsup_{n\to\infty} 1_{A_n} = 1_{(\limsup A_n)} \qquad \liminf_{n\to\infty} 1_{A_n} = 1_{(\liminf A_n)}$$

It is common to write $\limsup A_n = \{\omega : \omega \in A_n \text{ i.o.}\}$, where i.o. stands for infinitely often. An example which illustrates the use of this notation is "$X_n \to 0$ a.s. if and only if for all $\epsilon > 0$, $P(|X_n| > \epsilon \text{ i.o.}) = 0$." The reader will see many other examples below. The next result should be familiar from measure theory even though its name may not be.

**Theorem 2.3.1. Borel-Cantelli lemma.** *If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then*

$$P(A_n \text{ i.o.}) = 0.$$

*Proof.* Let $N = \sum_k 1_{A_k}$ be the number of events that occur. Fubini's theorem implies $EN = \sum_k P(A_k) < \infty$, so we must have $N < \infty$ a.s. ■

The next result is a typical application of the Borel-Cantelli lemma.

**Theorem 2.3.2.** *$X_n \to X$ in probability if and only if for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges almost surely to $X$.*

*Proof.* Let $\epsilon_k$ be a sequence of positive numbers that $\downarrow 0$. For each $k$, there is an $n(m_k) > n(m_{k-1})$ so that $P(|X_{n(m_k)} - X| > \epsilon_k) \le 2^{-k}$. Since

$$\sum_{k=1}^{\infty} P(|X_{n(m_k)} - X| > \epsilon_k) < \infty$$

the Borel-Cantelli lemma implies $P(|X_{n(m_k)} - X| > \epsilon_k \text{ i.o.}) = 0$, that is, $X_{n(m_k)} \to X$ a.s. To prove the second conclusion, we note that if for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges almost surely to $X$ then we can apply the next lemma to the sequence of numbers $y_n = P(|X_n - X| > \delta)$ for any $\delta > 0$ to get the desired result. ■

**Theorem 2.3.3.** *Let $y_n$ be a sequence of elements of a topological space. If every subsequence $y_{n(m)}$ has a further subsequence $y_{n(m_k)}$ that converges to $y$, then $y_n \to y$.*

*Proof.* If $y_n \not\to y$, then there is an open set $G$ containing $y$ and a subsequence $y_{n(m)}$ with $y_{n(m)} \notin G$ for all $m$, but clearly no subsequence of $y_{n(m)}$ converges to $y$. ■

**Remark.** Since there is a sequence of random variables that converges in probability but not a.s. (for an example, see Exercises 2.3.13 or 2.3.14), it follows from Theorem 2.3.3 that a.s. convergence does not come from a metric, or even from a topology. Exercise 2.3.8 will give a metric for convergence in probability, and Exercise 2.3.9 will show that the space of random variables is a complete space under this metric.

Theorem 2.3.2 allows us to upgrade convergence in probability to convergence almost surely. An example of the usefulness of this is

**Theorem 2.3.4.** *If $f$ is continuous and $X_n \to X$ in probability then $f(X_n) \to f(X)$ in probability. If, in addition, $f$ is bounded, then $Ef(X_n) \to Ef(X)$.*

*Proof.* If $X_{n(m)}$ is a subsequence then Theorem 2.3.2 implies there is a further subsequence $X_{n(m_k)} \to X$ almost surely. Since $f$ is continuous, Exercise 1.3.3 implies $f(X_{n(m_k)}) \to f(X)$ almost surely and Theorem 2.3.2 implies $f(X_n) \to f(X)$ in probability. If $f$ is bounded, then the bounded convergence theorem implies $Ef(X_{n(m_k)}) \to Ef(X)$, and applying Theorem 2.3.3 to $y_n = Ef(X_n)$ gives the desired result. ■

As our second application of the Borel-Cantelli lemma, we get our first strong law of large numbers:

**Theorem 2.3.5.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$ and $EX_i^4 < \infty$. If $S_n = X_1 + \cdots + X_n$ then $S_n/n \to \mu$ a.s.*

*Proof.* By letting $X_i' = X_i - \mu$, we can suppose without loss of generality that $\mu = 0$. Now

$$ES_n^4 = E\left(\sum_{i=1}^n X_i\right)^4 = E \sum_{1\le i,j,k,\ell\le n} X_i X_j X_k X_\ell$$

Terms in the sum of the form $E(X_i^3 X_j)$, $E(X_i^2 X_j X_k)$, and $E(X_i X_j X_k X_\ell)$ are 0 (if $i, j, k, \ell$ are distinct) since the expectation of the product is the product of the expectations, and in each case one of the terms has expectation 0. The only terms that do not vanish are those of the form $EX_i^4$ and $EX_i^2 X_j^2 = (EX_i^2)^2$. There are $n$ and $3n(n-1)$ of these terms, respectively. (In the second case we can pick the two indices in $n(n-1)/2$ ways, and with the indices fixed, the term can arise in a total of six ways.) The last observation implies

$$ES_n^4 = nEX_1^4 + 3(n^2 - n)(EX_1^2)^2 \le Cn^2$$

where $C < \infty$. Chebyshev's inequality gives us

$$P(|S_n| > n\epsilon) \le E(S_n^4)/(n\epsilon)^4 \le C/(n^2\epsilon^4)$$

Summing on $n$ and using the Borel-Cantelli lemma gives $P(|S_n| > n\epsilon \text{ i.o.}) = 0$. Since $\epsilon$ is arbitrary, the proof is complete. ■

The converse of the Borel-Cantelli lemma is trivially false.

**Example 2.3.1.** Let $\Omega = (0, 1)$, $\mathcal{F}$ = Borel sets, $P$ = Lebesgue measure. If $A_n = (0, a_n)$ where $a_n \to 0$ as $n \to \infty$, then $\limsup A_n = \emptyset$, but if $a_n \geq 1/n$, we have $\sum a_n = \infty$.

The example just given suggests that for general sets we cannot say much more than the next result.

**Exercise 2.3.1.** Prove that $P(\limsup A_n) \geq \limsup P(A_n)$ and
$P(\liminf A_n) \leq \liminf P(A_n)$

For independent events, however, the necessary condition for $P(\limsup A_n) > 0$ is sufficient for $P(\limsup A_n) = 1$.

**Theorem 2.3.6. The second Borel-Cantelli lemma**. *If the events $A_n$ are independent, then $\sum P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$.*

*Proof.* Let $M < N < \infty$. Independence and $1 - x \leq e^{-x}$ imply

$$P\left(\cap_{n=M}^N A_n^c\right) = \prod_{n=M}^N (1 - P(A_n)) \leq \prod_{n=M}^N \exp(-P(A_n))$$
$$= \exp\left(-\sum_{n=M}^N P(A_n)\right) \to 0 \quad \text{as } N \to \infty$$

So $P(\cup_{n=M}^\infty A_n) = 1$ for all $M$, and since $\cup_{n=M}^\infty A_n \downarrow \limsup A_n$ it follows that $P(\limsup A_n) = 1$. ∎

A typical application of the second Borel-Cantelli lemma is:

**Theorem 2.3.7.** *If $X_1, X_2, \ldots$ are i.i.d. with $E|X_i| = \infty$, then $P(|X_n| \geq n \text{ i.o.}) = 1$. So if $S_n = X_1 + \cdots + X_n$ then $P(\lim S_n/n \text{ exists} \in (-\infty, \infty)) = 0$.*

*Proof.* From Lemma 2.2.8, we get

$$E|X_1| = \int_0^\infty P(|X_1| > x)\, dx \leq \sum_{n=0}^\infty P(|X_1| > n)$$

Since $E|X_1| = \infty$ and $X_1, X_2, \ldots$ are i.i.d., it follows from the second Borel-Cantelli lemma that $P(|X_n| \geq n \text{ i.o.}) = 1$. To prove the second claim, observe that

$$\frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{S_n}{n(n+1)} - \frac{X_{n+1}}{n+1}$$

and on $C \equiv \{\omega : \lim_{n\to\infty} S_n/n \text{ exists} \in (-\infty, \infty)\}$, $S_n/(n(n+1)) \to 0$. So, on $C \cap \{\omega : |X_n| \ge n \text{ i.o.}\}$, we have

$$\left|\frac{S_n}{n} - \frac{S_{n+1}}{n+1}\right| > 2/3 \quad \text{i.o.}$$

contradicting the fact that $\omega \in C$. From the last observation, we conclude that

$$\{\omega : |X_n| \ge n \text{ i.o.}\} \cap C = \emptyset$$

and since $P(|X_n| \ge n \text{ i.o.}) = 1$, it follows that $P(C) = 0$. ■

Theorem 2.3.7 shows that $E|X_i| < \infty$ is necessary for the strong law of large numbers. The reader will have to wait until Theorem 2.4.1 to see that condition is also sufficient. The next result extends the second Borel-Cantelli lemma and sharpens its conclusion.

**Theorem 2.3.8.** *If $A_1, A_2, \ldots$ are pairwise independent and $\sum_{n=1}^\infty P(A_n) = \infty$, then as $n \to \infty$*

$$\sum_{m=1}^n 1_{A_m} \Big/ \sum_{m=1}^n P(A_m) \to 1 \quad a.s.$$

*Proof.* Let $X_m = 1_{A_m}$ and let $S_n = X_1 + \cdots + X_n$. Since the $A_m$ are pairwise independent, the $X_m$ are uncorrelated and hence Theorem 2.2.1 implies

$$\text{var}(S_n) = \text{var}(X_1) + \cdots + \text{var}(X_n)$$

$\text{var}(X_m) \le E(X_m^2) = E(X_m)$, since $X_m \in \{0, 1\}$, so $\text{var}(S_n) \le E(S_n)$. Chebyshev's inequality implies

$$(*) \qquad P(|S_n - ES_n| > \delta ES_n) \le \text{var}(S_n)/(\delta ES_n)^2 \le 1/(\delta^2 ES_n) \to 0$$

as $n \to \infty$ (since we have assumed $ES_n \to \infty$).

The last computation shows that $S_n/ES_n \to 1$ in probability. To get almost sure convergence, we have to take subsequences. Let $n_k = \inf\{n : ES_n \ge k^2\}$. Let $T_k = S_{n_k}$ and note that the definition and $EX_m \le 1$ imply $k^2 \le ET_k \le k^2 + 1$. Replacing $n$ by $n_k$ in $(*)$ and using $ET_k \ge k^2$ shows

$$P(|T_k - ET_k| > \delta ET_k) \le 1/(\delta^2 k^2)$$

So $\sum_{k=1}^\infty P(|T_k - ET_k| > \delta ET_k) < \infty$, and the Borel-Cantelli lemma implies $P(|T_k - ET_k| > \delta ET_k \text{ i.o.}) = 0$. Since $\delta$ is arbitrary, it follows that $T_k/ET_k \to 1$ a.s. To show $S_n/ES_n \to 1$ a.s., pick an $\omega$ so that $T_k(\omega)/ET_k \to 1$ and observe that if $n_k \le n < n_{k+1}$, then

$$\frac{T_k(\omega)}{ET_{k+1}} \le \frac{S_n(\omega)}{ES_n} \le \frac{T_{k+1}(\omega)}{ET_k}$$

To show that the terms at the left and right ends $\to 1$, we rewrite the last inequalities as

$$\frac{ET_k}{ET_{k+1}} \cdot \frac{T_k(\omega)}{ET_k} \le \frac{S_n(\omega)}{ES_n} \le \frac{T_{k+1}(\omega)}{ET_{k+1}} \cdot \frac{ET_{k+1}}{ET_k}$$

From this, we see it is enough to show $ET_{k+1}/ET_k \to 1$, but this follows from

$$k^2 \le ET_k \le ET_{k+1} \le (k+1)^2 + 1$$

and the fact that $\{(k+1)^2+1\}/k^2 = 1 + 2/k + 2/k^2 \to 1$. ■

The moral of the proof of Theorem 2.3.8 is that if you want to show that $X_n/c_n \to 1$ a.s. for sequences $c_n$, $X_n \ge 0$ that are increasing, it is enough to prove the result for a subsequence $n(k)$ that has $c_{n(k+1)}/c_{n(k)} \to 1$. For practice with this technique, try the following.

**Exercise 2.3.2.** Let $0 \le X_1 \le X_2 \dots$ be random variables with $EX_n \sim an^\alpha$ with $a, \alpha > 0$, and $\operatorname{var}(X_n) \le Bn^\beta$ with $\beta < 2\alpha$. Show that $X_n/n^\alpha \to a$ a.s.

**Exercise 2.3.3.** Let $X_n$ be independent Poisson r.v.'s with $EX_n = \lambda_n$, and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum \lambda_n = \infty$, then $S_n/ES_n \to 1$ a.s.

**Example 2.3.2. Record values.** Let $X_1, X_2, \dots$ be a sequence of random variables and think of $X_k$ as the distance for an individual's $k$th high jump or shot-put toss so that $A_k = \{X_k > \sup_{j<k} X_j\}$ is the event that a record occurs at time $k$. Ignoring the fact that an athelete's performance may get better with more experience or that injuries may occur, we will suppose that $X_1, X_2, \dots$ are i.i.d. with a distribution $F(x)$ that is continuous. Even though it may seem that the occurrence of a record at time $k$ will make it less likely that one will occur at time $k+1$, we

*Claim.* The $A_k$ are independent with $P(A_k) = 1/k$.

To prove this, we start by observing that since $F$ is continuous $P(X_j = X_k) = 0$ for any $j \ne k$ (see Exercise 2.1.8), so we can let $Y_1^n > Y_2^n > \cdots > Y_n^n$ be the random variables $X_1, \dots, X_n$ put into decreasing order and define a random permutation of $\{1, \dots, n\}$ by $\pi_n(i) = j$ if $X_i = Y_j^n$, that is, if the $i$th random variable has rank $j$. Since the distribution of $(X_1, \dots, X_n)$ is not affected by changing the order of the random variables, it is easy to see:

(a) The permutation $\pi_n$ is uniformly distributed over the set of $n!$ possibilities.

*Proof of (a).* This is "obvious" by symmetry, but if one wants to hear more, we can argue as follows. Let $\pi_n$ be the permutation induced by $(X_1, \dots, X_n)$, and let $\sigma_n$ be a randomly chosen permutation of $\{1, \dots, n\}$ independent of the $X$ sequence. Then we can say two things about the permutation induced by $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$: (i) it is $\pi_n \circ \sigma_n$, and (ii) it has the same distribution as $\pi_n$. The desired result follows now by noting that if $\pi$ is any permutation, $\pi \circ \sigma_n$, is uniform over the $n!$ possibilities. ■

Once you believe (a), the rest is easy:

(b) $P(A_n) = P(\pi_n(n) = 1) = 1/n$.
(c) If $m < n$ and $i_{m+1}, \ldots i_n$ are distinct elements of $\{1, \ldots, n\}$ then

$$P(A_m | \pi_n(j) = i_j \text{ for } m+1 \le j \le n) = 1/m$$

Intuitively, this is true since if we condition on the ranks of $X_{m+1}, \ldots, X_n$, then this determines the set of ranks available for $X_1, \ldots, X_m$, but all possible orderings of the ranks are equally likely and hence there is probability $1/m$ that the smallest rank will end up at $m$.

*Proof of (c).* If we let $\sigma_m$ be a randomly chosen permutation of $\{1, \ldots, m\}$, then (i) $\pi_n \circ \sigma_m$ has the same distribution as $\pi_n$, and (ii) since the application of $\sigma_m$ randomly rearranges $\pi_n(1), \ldots, \pi_n(m)$ the desired result follows. ■

If we let $m_1 < m_2 \ldots < m_k$, then it follows from (c) that

$$P(A_{m_1} | A_{m_2} \cap \ldots \cap A_{m_k}) = P(A_{m_1})$$

and the claim follows by induction.

Using Theorem 2.3.8 and the by now familiar fact that $\sum_{m=1}^n 1/m \sim \log n$, we have

**Theorem 2.3.9.** *If $R_n = \sum_{m=1}^n 1_{A_m}$ is the number of records at time n then as $n \to \infty$,*

$$R_n / \log n \to 1 \quad a.s.$$

The reader should note that the last result is independent of the distribution $F$ (as long as it is continuous).

**Remark.** Let $X_1, X_2, \ldots$ be i.i.d. with a distribution that is continuous. Let $Y_i$ be the number of $j \le i$ with $X_j > X_i$. It follows from (a) that $Y_i$ are independent random variables with $P(Y_i = j) = 1/i$ for $0 \le j < i - 1$.

**Comic relief.** Let $X_0, X_1, \ldots$ be i.i.d. and imagine they are the offers you get for a car you are going to sell. Let $N = \inf\{n \ge 1 : X_n > X_0\}$. Symmetry implies $P(N > n) \ge 1/(n+1)$. (When the distribution is continuous this probability is exactly $1/(n+1)$, but our distribution now is general and ties go to the first person who calls.) Using Exercise 2.2.7 now:

$$EN = \sum_{n=0}^{\infty} P(N > n) \ge \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$$

so the expected time you have to wait until you get an offer better than the first one is $\infty$. To avoid lawsuits, let me hasten to add that I am not suggesting that you should take the first offer you get!

**Example 2.3.3. Head runs.** Let $X_n$, $n \in \mathbf{Z}$, be i.i.d. with $P(X_n = 1) = P(X_n = -1) = 1/2$. Let $\ell_n = \max\{m : X_{n-m+1} = \cdots = X_n = 1\}$ be the length of the run of $+1$'s at time $n$, and let $L_n = \max_{1 \le m \le n} \ell_m$ be the longest run at time $n$. We use a two-sided sequence so that for all $n$, $P(\ell_n = k) = (1/2)^{k+1}$ for $k \ge 0$. Since $\ell_1 < \infty$, the result we are going to prove

$$L_n / \log_2 n \to 1 \quad \text{a.s.} \tag{2.3.1}$$

is also true for a one-sided sequence. To prove (2.3.1), we begin by observing

$$P(\ell_n \ge (1+\epsilon)\log_2 n) \le n^{-(1+\epsilon)}$$

for any $\epsilon > 0$, so it follows from the Borel-Cantelli lemma that $\ell_n \le (1+\epsilon)\log_2 n$ for $n \ge N_\epsilon$. Since $\epsilon$ is arbitrary, it follows that

$$\limsup_{n\to\infty} L_n / \log_2 n \le 1 \quad \text{a.s.}$$

To get a result in the other direction, we break the first $n$ trials into disjoint blocks of length $[(1-\epsilon)\log_2 n] + 1$, on which the variables are all 1 with probability

$$2^{-[(1-\epsilon)\log_2 n]-1} \ge n^{-(1-\epsilon)}/2,$$

to conclude that if $n$ is large enough so that $[n/\{[(1-\epsilon)\log_2 n] + 1\}] \ge n/\log_2 n$

$$P(L_n \le (1-\epsilon)\log_2 n) \le (1 - n^{-(1-\epsilon)}/2)^{n/(\log_2 n)} \le \exp(-n^\epsilon/2\log_2 n)$$

which is summable, so the Borel-Cantelli lemma implies

$$\liminf_{n\to\infty} L_n / \log_2 n \ge 1 \quad \text{a.s.}$$

**Exercise 2.3.4.** Show that $\limsup_{n\to\infty} \ell_n / \log_2 n = 1$, $\liminf_{n\to\infty} \ell_n = 0$ a.s.

## *Exercises*

**2.3.5.** Prove the first result in Theorem 2.3.4 directly from the definition.

**2.3.6. Fatou's lemma.** Suppose $X_n \ge 0$ and $X_n \to X$ in probability. Show that $\liminf_{n\to\infty} EX_n \ge EX$.

**2.3.7. Dominated convergence.** Suppose $X_n \to X$ in probability and (a) $|X_n| \le Y$ with $EY < \infty$ or (b) there is a continuous function $g$ with $g(x) > 0$ for large $x$ with $|x|/g(x) \to 0$ as $|x| \to \infty$ so that $Eg(X_n) \le C < \infty$ for all $n$. Show that $EX_n \to EX$.

**2.3.8.** Show (a) that $d(X,Y) = E(|X-Y|/(1+|X-Y|))$ defines a metric on the set of random variables, that is, (i) $d(X,Y) = 0$ if and only if $X = Y$ a.s., (ii)

$d(X, Y) = d(Y, X)$, (iii) $d(X, Z) \le d(X, Y) + d(Y, Z)$ and (b) that $d(X_n, X) \to 0$ as $n \to \infty$ if and only if $X_n \to X$ in probability.

**2.3.9.** Show that random variables are a complete space under the metric defined in the previous exercise, that is, if $d(X_m, X_n) \to 0$ whenever $m, n \to \infty$, then there is a r.v. $X_\infty$ so that $X_n \to X_\infty$ in probability.

**2.3.10.** If $X_n$ is any sequence of random variables, there are constants $c_n \to \infty$ so that $X_n/c_n \to 0$ a.s.

**2.3.11.** (i) If $P(A_n) \to 0$ and $\sum_{n=1}^\infty P(A_n^c \cap A_{n+1}) < \infty$ then $P(A_n \text{ i.o.}) = 0$. (ii) Find an example of a sequence $A_n$ to which the result in (i) can be applied but the Borel-Cantelli lemma cannot.

**2.3.12.** Let $A_n$ be a sequence of independent events with $P(A_n) < 1$ for all $n$. Show that $P(\cup A_n) = 1$ implies $P(A_n \text{ i.o.}) = 1$.

**2.3.13.** Let $X_1, X_2, \ldots$ be independent. Show that $\sup X_n < \infty$ a.s. if and only if $\sum_n P(X_n > A) < \infty$ for some $A$.

**2.3.14.** Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that (i) $X_n \to 0$ in probability if and only if $p_n \to 0$, and (ii) $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

**2.3.15.** Let $Y_1, Y_2, \ldots$ be i.i.d. Find necessary and sufficient conditions for
(i) $Y_n/n \to 0$ almost surely, (ii) $(\max_{m\le n} Y_m)/n \to 0$ almost surely,
(iii) $(\max_{m\le n} Y_m)/n \to 0$ in probability, and (iv) $Y_n/n \to 0$ in probability.

**2.3.16.** The last two exercises give examples with $X_n \to X$ in probability without $X_n \to X$ a.s. There is one situation in which the two notions are equivalent. Let $X_1, X_2, \ldots$ be a sequence of r.v.'s on $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a countable set and $\mathcal{F}$ consists of all subsets of $\Omega$. Show that $X_n \to X$ in probability implies $X_n \to X$ a.s.

**2.3.17.** Show that if $X_n$ is the outcome of the $n$th play of the St. Petersburg game (Example 2.2.7), then $\limsup_{n\to\infty} X_n/(n \log_2 n) = \infty$ a.s. and hence the same result holds for $S_n$. This shows that the convergence $S_n/(n \log_2 n) \to 1$ in probability proved in Section 2.2 does not occur a.s.

**2.3.18.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i > x) = e^{-x}$, let $M_n = \max_{1\le m\le n} X_m$. Show that (i) $\limsup_{n\to\infty} X_n/\log n = 1$ a.s. and (ii) $M_n/\log n \to 1$ a.s.

**2.3.19.** Let $X_1, X_2, \ldots$ be i.i.d. with distribution $F$, let $\lambda_n \uparrow \infty$, and let $A_n = \{\max_{1\le m\le n} X_m > \lambda_n\}$. Show that $P(A_n \text{ i.o.}) = 0$ or $1$ according as $\sum_{n\ge 1}(1 - F(\lambda_n)) < \infty$ or $= \infty$.

**2.3.20. Kochen-Stone lemma**. Suppose $\sum P(A_k) = \infty$. Use Exercises 1.6.6 and 2.3.1 to show that if

$$\limsup_{n\to\infty} \left(\sum_{k=1}^n P(A_k)\right)^2 \Bigg/ \left(\sum_{1\le j,k\le n} P(A_j \cap A_k)\right) = \alpha > 0$$

then $P(A_n \text{ i.o.}) \ge \alpha$. The case $\alpha = 1$ contains Theorem 2.3.6.

## 2.4 Strong Law of Large Numbers

We are now ready to give Etemadi's (1981) proof of

**Theorem 2.4.1. Strong law of large numbers.** *Let $X_1, X_2, \ldots$ be pairwise independent identically distributed random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.*

*Proof.* As in the proof of the weak law of large numbers, we begin by truncating.

**Lemma 2.4.2.** *Let $Y_k = X_k 1_{(|X_k|\le k)}$ and $T_n = Y_1 + \cdots + Y_n$. It is sufficient to prove that $T_n/n \to \mu$ a.s.*

*Proof.* $\sum_{k=1}^\infty P(|X_k| > k) \le \int_0^\infty P(|X_1| > t)\,dt = E|X_1| < \infty$ so $P(X_k \ne Y_k \text{ i.o.}) = 0$. This shows that $|S_n(\omega) - T_n(\omega)| \le R(\omega) < \infty$ a.s. for all $n$, from which the desired result follows. ∎

The second step is not so intuitive, but it is an important part of this proof and the one given in Section 2.5.

**Lemma 2.4.3.** $\sum_{k=1}^\infty$ *var*$(Y_k)/k^2 \le 4E|X_1| < \infty$.

*Proof.* To bound the sum, we observe

$$\text{var}\,(Y_k) \le E(Y_k^2) = \int_0^\infty 2yP(|Y_k| > y)\,dy \le \int_0^k 2yP(|X_1| > y)\,dy$$

so using Fubini's theorem (since everything is $\ge 0$ and the sum is just an integral with respect to counting measure on $\{1, 2, \ldots\}$)

$$\begin{aligned}\sum_{k=1}^\infty E(Y_k^2)/k^2 &\le \sum_{k=1}^\infty k^{-2} \int_0^\infty 1_{(y<k)}\, 2y\; P(|X_1| > y)\,dy \\ &= \int_0^\infty \left\{\sum_{k=1}^\infty k^{-2} 1_{(y<k)}\right\} 2yP(|X_1| > y)\,dy\end{aligned}$$

Since $E|X_1| = \int_0^\infty P(|X_1| > y)\,dy$, we can complete the proof by showing ∎

**Lemma 2.4.4.** *If* $y \geq 0$ *then* $2y \sum_{k>y} k^{-2} \leq 4$.

*Proof.* We being with the observation that if $m \geq 2$ then

$$\sum_{k \geq m} k^{-2} \leq \int_{m-1}^{\infty} x^{-2} dx = (m-1)^{-1}$$

When $y \geq 1$, the sum starts with $k = [y] + 1 \geq 2$, so

$$2y \sum_{k>y} k^{-2} \leq 2y/[y] \leq 4$$

since $y/[y] \leq 2$ for $y \geq 1$ (the worst case being $y$ close to 2). To cover $0 \leq y < 1$, we note that in this case

$$2y \sum_{k>y} k^{-2} \leq 2\left(1 + \sum_{k=2}^{\infty} k^{-2}\right) \leq 4$$

This establishes Lemma 2.4.4 which completes the proof of Lemma 2.4.3 and of the theorem. ■

The first two steps, Lemmas 2.4.2 and 2.4.3 above, are standard. Etemadi's inspiration was that since $X_n^+, n \geq 1$, and $X_n^-, n \geq 1$, satisfy the assumptions of the theorem and $X_n = X_n^+ - X_n^-$, we can without loss of generality suppose $X_n \geq 0$. As in the proof of Theorem 2.3.8, we will prove the result first for a subsequence and then use monotonicity to control the values in between. This time, however, we let $\alpha > 1$ and $k(n) = [\alpha^n]$. Chebyshev's inequality implies that if $\epsilon > 0$

$$\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) \leq \epsilon^{-2} \sum_{n=1}^{\infty} \text{var}\,(T_{k(n)})/k(n)^2$$

$$= \epsilon^{-2} \sum_{n=1}^{\infty} k(n)^{-2} \sum_{m=1}^{k(n)} \text{var}\,(Y_m) = \epsilon^{-2} \sum_{m=1}^{\infty} \text{var}\,(Y_m) \sum_{n:k(n) \geq m} k(n)^{-2}$$

where we have used Fubini's theorem to interchange the two summations of non-negative terms. Now $k(n) = [\alpha^n]$ and $[\alpha^n] \geq \alpha^n/2$ for $n \geq 1$, so summing the geometric series and noting that the first term is $\leq m^{-2}$:

$$\sum_{n:\alpha^n \geq m} [\alpha^n]^{-2} \leq 4 \sum_{n:\alpha^n \geq m} \alpha^{-2n} \leq 4(1 - \alpha^{-2})^{-1} m^{-2}$$

Combining our computations shows

$$\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) \leq 4(1 - \alpha^{-2})^{-1} \epsilon^{-2} \sum_{m=1}^{\infty} E(Y_m^2) m^{-2} < \infty$$

by Lemma 2.4.3. Since $\epsilon$ is arbitrary $(T_{k(n)} - ET_{k(n)})/k(n) \to 0$ a.s. The dominated convergence theorem implies $EY_k \to EX_1$ as $k \to \infty$, so $ET_{k(n)}/k(n) \to EX_1$

and we have shown $T_{k(n)}/k(n) \to EX_1$ a.s. To handle the intermediate values, we observe that if $k(n) \leq m < k(n+1)$

$$\frac{T_{k(n)}}{k(n+1)} \leq \frac{T_m}{m} \leq \frac{T_{k(n+1)}}{k(n)}$$

(here we use $Y_i \geq 0$), so recalling $k(n) = [\alpha^n]$, we have $k(n+1)/k(n) \to \alpha$ and

$$\frac{1}{\alpha} EX_1 \leq \liminf_{n\to\infty} T_m/m \leq \limsup_{m\to\infty} T_m/m \leq \alpha EX_1$$

Since $\alpha > 1$ is arbitrary, the proof is complete. ∎

The next result shows that the strong law holds whenever $EX_i$ exists.

**Theorem 2.4.5.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. If $S_n = X_1 + \cdots + X_n$ then $S_n/n \to \infty$ a.s.*

*Proof.* Let $M > 0$ and $X_i^M = X_i \wedge M$. The $X_i^M$ are i.i.d. with $E|X_i^M| < \infty$, so if $S_i^M = X_1^M + \cdots + X_n^M$ then Theorem 2.4.1 implies $S_n^M/n \to EX_i^M$. Since $X_i \geq X_i^M$, it follows that

$$\liminf_{n\to\infty} S_n/n \geq \lim_{n\to\infty} S_n^M/n = EX_i^M$$

The monotone convergence theorem implies $E(X_i^M)^+ \uparrow EX_i^+ = \infty$ as $M \uparrow \infty$, so $EX_i^M = E(X_i^M)^+ - E(X_i^M)^- \uparrow \infty$, and we have $\liminf_{n\to\infty} S_n/n \geq \infty$, which implies the desired result. ∎

The rest of this section is devoted to applications of the strong law of large numbers.

**Example 2.4.1. Renewal theory.** Let $X_1, X_2, \ldots$ be i.i.d. with $0 < X_i < \infty$. Let $T_n = X_1 + \cdots + X_n$ and think of $T_n$ as the time of $n$th occurrence of some event. For a concrete situation, consider a diligent janitor who replaces a light bulb the instant it burns out. Suppose the first bulb is put in at time 0 and let $X_i$ be the lifetime of the $i$th light bulb. In this interpretation, $T_n$ is the time the $n$th light bulb burns out and $N_t = \sup\{n : T_n \leq t\}$ is the number of light bulbs that have burned out by time $t$.

**Theorem 2.4.6.** *If $EX_1 = \mu \leq \infty$, then as $t \to \infty$,*

$$N_t/t \to 1/\mu \quad \text{a.s.} \qquad (1/\infty = 0).$$

*Proof.* By Theorems 2.4.1 and 2.4.5, $T_n/n \to \mu$ a.s. From the definition of $N_t$, it follows that $T(N_t) \leq t < T(N_t + 1)$, so dividing through by $N_t$ gives

$$\frac{T(N_t)}{N_t} \leq \frac{t}{N_t} \leq \frac{T(N_t+1)}{N_t+1} \cdot \frac{N_t+1}{N_t}$$

To take the limit, we note that since $T_n < \infty$ for all $n$, we have $N_t \uparrow \infty$ as $t \to \infty$. The strong law of large numbers implies that for $\omega \in \Omega_0$ with $P(\Omega_0) = 1$, we have $T_n(\omega)/n \to \mu$, $N_t(\omega) \uparrow \infty$, and hence

$$\frac{T_{N_t(\omega)}(\omega)}{N_t(\omega)} \to \mu \qquad \frac{N_t(\omega)+1}{N_t(\omega)} \to 1$$

From this it follows that for $\omega \in \Omega_0$ that $t/N_t(\omega) \to \mu$ a.s. ■

The last argument shows that if $X_n \to X_\infty$ a.s. and $N(n) \to \infty$ a.s. then $X_{N(n)} \to X_\infty$ a.s. We have written this out with care because the analogous result for convergence in probability is false.

**Exercise 2.4.1.** Give an example with $X_n \in \{0, 1\}$, $X_n \to 0$ in probability, $N(n) \uparrow \infty$ a.s., and $X_{N(n)} \to 1$ a.s.

**Example 2.4.2. Empirical distribution functions**. Let $X_1, X_2, \ldots$ be i.i.d. with distribution $F$ and let

$$F_n(x) = n^{-1} \sum_{m=1}^{n} 1_{(X_m \le x)}$$

$F_n(x) =$ the observed frequency of values that are $\le x$, hence the name given above. The next result shows that $F_n$ converges uniformly to $F$ as $n \to \infty$.

**Theorem 2.4.7. The Glivenko-Cantelli theorem.** *As $n \to \infty$,*

$$\sup_x |F_n(x) - F(x)| \to 0 \quad a.s.$$

*Proof.* Fix $x$ and let $Y_n = 1_{(X_n \le x)}$. Since the $Y_n$ are i.i.d. with $EY_n = P(X_n \le x) = F(x)$, the strong law of large numbers implies that $F_n(x) = n^{-1}\sum_{m=1}^n Y_m \to F(x)$ a.s. In general, if $F_n$ is a sequence of nondecreasing functions that converges pointwise to a bounded and continuous limit $F$, then $\sup_x |F_n(x) - F(x)| \to 0$. However, the distribution function $F(x)$ may have jumps, so we have to work a little harder.

Again, fix $x$ and let $Z_n = 1_{(X_n < x)}$. Since the $Z_n$ are i.i.d. with $EZ_n = P(X_n < x) = F(x-) = \lim_{y \uparrow x} F(y)$, the strong law of large numbers implies that $F_n(x-) = n^{-1}\sum_{m=1}^n Z_m \to F(x-)$ a.s. For $1 \le j \le k-1$ let $x_{j,k} = \inf\{y : F(y) \ge j/k\}$. The pointwise convergence of $F_n(x)$ and $F_n(x-)$ imply that we can pick $N_k(\omega)$ so that if $n \ge N_k(\omega)$, then

$$|F_n(x_{j,k}) - F(x_{j,k})| < k^{-1} \quad \text{and} \quad |F_n(x_{j,k}-) - F(x_{j,k}-)| < k^{-1}$$

for $1 \le j \le k-1$. If we let $x_{0,k} = -\infty$ and $x_{k,k} = \infty$, then the last two inequalities hold for $j = 0$ or $k$. If $x \in (x_{j-1,k}, x_{j,k})$ with $1 \le j \le k$ and $n \ge N_k(\omega)$, then using the monotonicity of $F_n$ and $F$, and $F(x_{j,k}-) - F(x_{j-1,k}) \le k^{-1}$, we have

$$F_n(x) \le F_n(x_{j,k}-) \le F(x_{j,k}-) + k^{-1} \le F(x_{j-1,k}) + 2k^{-1} \le F(x) + 2k^{-1}$$
$$F_n(x) \ge F_n(x_{j-1,k}) \ge F(x_{j-1,k}) - k^{-1} \ge F(x_{j,k}-) - 2k^{-1} \ge F(x) - 2k^{-1}$$

so $\sup_x |F_n(x) - F(x)| \le 2k^{-1}$, and we have proved the result. ■

**Example 2.4.3. Shannon's theorem**. Let $X_1, X_2, \ldots \in \{1, \ldots, r\}$ be independent with $P(X_i = k) = p(k) > 0$ for $1 \le k \le r$. Here we are thinking of $1, \ldots, r$ as the letters of an alphabet, and $X_1, X_2, \ldots$ are the successive letters produced by an information source. In this i.i.d. case, it is the proverbial monkey at a typewriter. Let $\pi_n(\omega) = p(X_1(\omega)) \cdots p(X_n(\omega))$ be the probability of the realization we observed in the first $n$ trials. Since $\log \pi_n(\omega)$ is a sum of independent random variables, it follows from the strong law of large numbers that

$$-n^{-1} \log \pi_n(\omega) \to H \equiv -\sum_{k=1}^{r} p(k) \log p(k) \text{ a.s.}$$

The constant $H$ is called the **entropy** of the source and is a measure of how random it is. The last result is the **asymptotic equipartition property**: If $\epsilon > 0$, then as $n \to \infty$,

$$P\{\exp(-n(H+\epsilon)) \le \pi_n(\omega) \le \exp(-n(H-\epsilon)\} \to 1$$

### *Exercises*

**2.4.2. Lazy janitor.** Suppose the $i$th light bulb burns for an amount of time $X_i$ and then remains burned out for time $Y_i$ before being replaced. Suppose the $X_i, Y_i$ are positive and independent with the $X$'s having distribution $F$ and the $Y$'s having distribution $G$, both of which have finite mean. Let $R_t$ be the amount of time in $[0, t]$ that we have a working light bulb. Show that $R_t/t \to EX_i/(EX_i + EY_i)$ almost surely.

**2.4.3.** Let $X_0 = (1, 0)$ and define $X_n \in \mathbf{R}^2$ inductively by declaring that $X_{n+1}$ is chosen at random from the ball of radius $|X_n|$ centered at the origin, i.e., $X_{n+1}/|X_n|$ is uniformly distributed on the ball of radius 1 and independent of $X_1, \ldots, X_n$. Prove that $n^{-1} \log |X_n| \to c$ a.s. and compute $c$.

**2.4.4. Investment problem.** We assume that at the beginning of each year you can buy bonds for \$1 that are worth \$ $a$ at the end of the year or stocks that are worth a random amount $V \ge 0$. If you always invest a fixed proportion $p$ of your wealth in bonds, then your wealth at the end of year $n+1$ is $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose $V_1, V_2, \ldots$ are i.i.d. with $EV_n^2 < \infty$ and $E(V_n^{-2}) < \infty$. (i) Show that $n^{-1} \log W_n \to c(p)$ a.s. (ii) Show that $c(p)$ is concave. (Use Theorem A.5.1 in the Appendix to justify differentiating under the expected value.) (iii) By investigating $c'(0)$ and $c'(1)$, give conditions on $V$ that guarantee that the optimal choice of $p$

is in (0,1). (iv) Suppose $P(V=1)=P(V=4)=1/2$. Find the optimal $p$ as a function of $a$.

## 2.5 Convergence of Random Series*

In this section, we will pursue a second approach to the strong law of large numbers based on the convergence of random series. This approach has the advantage that it leads to estimates on the rate of convergence under moment assumptions, Theorems 2.5.7 and 2.5.8, and to a negative result for the infinite mean case, Theorem 2.5.9, which is stronger than the one in Theorem 2.3.7. The first two results in this section are of considerable interest in their own right, although we will see more general versions in Lemma 3.1.1 and Theorem 3.4.2.

To state the first result, we need some notation. Let $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, \ldots) =$ the future after time $n$ = the smallest $\sigma$-field with respect to which all the $X_m, m \geq n$ are measurable. Let $\mathcal{T} = \cap_n \mathcal{F}'_n$ = the remote future, or **tail $\sigma$-field.** Intuitively, $A \in \mathcal{T}$ if and only if changing a finite number of values does not affect the occurrence of the event. As usual, we turn to examples to help explain the definition.

**Example 2.5.1.** If $B_n \in \mathcal{R}$ then $\{X_n \in B_n \text{ i.o.}\} \in \mathcal{T}$. If we let $X_n = 1_{A_n}$ and $B_n = \{1\}$, this example becomes $\{A_n \text{ i.o.}\}$.

**Example 2.5.2.** Let $S_n = X_1 + \cdots + X_n$. It is easy to check that

$\{\lim_{n\to\infty} S_n \text{ exists }\} \in \mathcal{T}$,
$\{\limsup_{n\to\infty} S_n > 0\} \notin \mathcal{T}$,
$\{\limsup_{n\to\infty} S_n/c_n > x\} \in \mathcal{T}$ if $c_n \to \infty$

The next result shows that all examples are trivial.

**Theorem 2.5.1. Kolmogorov's 0-1 law.** *If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$, then $P(A) = 0$ or 1.*

*Proof.* We will show that $A$ is independent of itself, that is, $P(A \cap A) = P(A)P(A)$, so $P(A) = P(A)^2$, and hence $P(A) = 0$ or 1. We will sneak up on this conclusion in two steps:

(a) $A \in \sigma(X_1, \ldots, X_k)$ and $B \in \sigma(X_{k+1}, X_{k+2}, \ldots)$ are independent.

*Proof of (a).* If $B \in \sigma(X_{k+1}, \ldots, X_{k+j})$ for some $j$, this follows from Theorem 2.1.5. Since $\sigma(X_1, \ldots, X_k)$ and $\cup_j \sigma(X_{k+1}, \ldots, X_{k+j})$ are $\pi$-systems that contain $\Omega$ (a) follows from Theorem 2.1.3.

(b) $A \in \sigma(X_1, X_2, \ldots)$ and $B \in \mathcal{T}$ are independent.

*Proof of (b).* Since $\mathcal{T} \subset \sigma(X_{k+1}, X_{k+2}, \ldots)$, if $A \in \sigma(X_1, \ldots, X_k)$ for some $k$, this follows from (a). $\cup_k \sigma(X_1, \ldots, X_k)$ and $\mathcal{T}$ are $\pi$-systems that contain $\Omega$, so (b) follows from Theorem 2.1.3.

Since $\mathcal{T} \subset \sigma(X_1, X_2, \ldots)$, (b) implies an $A \in \mathcal{T}$ is independent of itself, and Theorem 2.5.1 follows. ■

If $A_1, A_2, \ldots$ are independent, then Theorem 2.5.1 implies $P(A_n \text{ i.o.}) = 0$ or 1. Applying Theorem 2.5.1 to Example 2.5.2 gives $P(\lim_{n\to\infty} S_n \text{ exists}) = 0$ or 1. The next result will help us prove the probability is 1 in certain situations.

**Theorem 2.5.2. Kolmogorov's maximal inequality.** *Suppose $X_1, \ldots, X_n$ are independent with $EX_i = 0$ and $var(X_i) < \infty$. If $S_n = X_1 + \cdots + X_n$, then*

$$P\left(\max_{1\le k\le n} |S_k| \ge x\right) \le x^{-2}\, var(S_n)$$

**Remark.** Under the same hypotheses, Chebyshev's inequality (Theorem 1.6.4) gives only

$$P(|S_n| \ge x) \le x^{-2} \operatorname{var}(S_n)$$

*Proof.* Let $A_k = \{|S_k| \ge x \text{ but } |S_j| < x \text{ for } j < k\}$, that is, we break things down according to the time that $|S_k|$ first exceeds $x$. Since the $A_k$ are disjoint and $(S_n - S_k)^2 \ge 0$,

$$\begin{aligned} ES_n^2 &\ge \sum_{k=1}^n \int_{A_k} S_n^2\, dP = \sum_{k=1}^n \int_{A_k} S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2\, dP \\ &\ge \sum_{k=1}^n \int_{A_k} S_k^2\, dP + \sum_{k=1}^n \int 2S_k 1_{A_k} \cdot (S_n - S_k)\, dP \end{aligned}$$

$S_k 1_{A_k} \in \sigma(X_1, \ldots, X_k)$ and $S_n - S_k \in \sigma(X_{k+1}, \ldots, X_n)$ are independent by Theorem 2.1.6, so using Theorem 2.1.9 and $E(S_n - S_k) = 0$ shows

$$\int 2S_k 1_{A_k} \cdot (S_n - S_k)\, dP = E(2S_k 1_{A_k}) \cdot E(S_n - S_k) = 0$$

Now, using the fact that $|S_k| \ge x$ on $A_k$ and the $A_k$ are disjoint,

$$ES_n^2 \ge \sum_{k=1}^n \int_{A_k} S_k^2\, dP \ge \sum_{k=1}^n x^2 P(A_k) = x^2 P\left(\max_{1\le k\le n} |S_k| \ge x\right)$$ ■

**Exercise 2.5.1.** Suppose $X_1, X_2, \ldots$ are i.i.d. with $EX_i = 0$, $\operatorname{var}(X_i) = C < \infty$. Use Theorem 2.5.2 with $n = m^\alpha$ where $\alpha(2p - 1) > 1$ to conclude that if $S_n = X_1 + \cdots + X_n$ and $p > 1/2$, then $S_n/n^p \to 0$ almost surely.

We turn now to our results on convergence of series. To state them, we need a definition. We say that $\sum_{n=1}^\infty a_n$ converges if $\lim_{N\to\infty} \sum_{n=1}^N a_n$ exists.

**Theorem 2.5.3.** *Suppose $X_1, X_2, \ldots$ are independent and have $EX_n = 0$. If*

$$\sum_{n=1}^{\infty} var(X_n) < \infty$$

*then with probability one $\sum_{n=1}^{\infty} X_n(\omega)$ converges.*

*Proof.* Let $S_N = \sum_{n=1}^{N} X_n$. From Theorem 2.5.2, we get

$$P\left(\max_{M\le m\le N} |S_m - S_M| > \epsilon\right) \le \epsilon^{-2} \operatorname{var}(S_N - S_M) = \epsilon^{-2} \sum_{n=M+1}^{N} \operatorname{var}(X_n)$$

Letting $N \to \infty$ in the last result, we get

$$P\left(\sup_{m\ge M} |S_m - S_M| > \epsilon\right) \le \epsilon^{-2} \sum_{n=M+1}^{\infty} \operatorname{var}(X_n) \to 0 \quad \text{as } M \to \infty$$

If we let $w_M = \sup_{m,n\ge M} |S_m - S_n|$ then $w_M \downarrow$ as $M \uparrow$ and

$$P(w_M > 2\epsilon) \le P\left(\sup_{m\ge M} |S_m - S_M| > \epsilon\right) \to 0$$

as $M \to \infty$ so $w_M \downarrow 0$ almost surely. But $w_M(\omega) \downarrow 0$ implies $S_n(\omega)$ is a Cauchy sequence and hence $\lim_{n\to\infty} S_n(\omega)$ exists, so the proof is complete. ■

**Example 2.5.3.** Let $X_1, X_2, \ldots$ be independent with

$$P(X_n = n^{-\alpha}) = P(X_n = -n^{-\alpha}) = 1/2$$

$EX_n = 0$ and $\operatorname{var}(X_n) = n^{-2\alpha}$ so if $\alpha > 1/2$ it follows from Theorem 2.5.3 that $\sum X_n$ converges. Theorem 2.5.4 below shows that $\alpha > 1/2$ is also necessary for this conclusion. Notice that there is absolute convergence, that is, $\sum |X_n| < \infty$, if and only if $\alpha > 1$.

Theorem 2.5.3 is sufficient for all of our applications, but our treatment would not be complete if we did not mention the last word on convergence of random series.

**Theorem 2.5.4. Kolmogorov's three-series theorem.** *Let $X_1, X_2, \ldots$ be independent. Let $A > 0$ and let $Y_i = X_i 1_{(|X_i|\le A)}$. In order that $\sum_{n=1}^{\infty} X_n$ converges a.s., it is necessary and sufficient that*

$$(i) \sum_{n=1}^{\infty} P(|X_n| > A) < \infty, \quad (ii) \sum_{n=1}^{\infty} EY_n \text{ converges, and } \quad (iii) \sum_{n=1}^{\infty} var(Y_n) < \infty$$

*Proof.* We will prove the necessity in Example 3.4.7 as an application of the central limit theorem. To prove the sufficiency, let $\mu_n = EY_n$. (iii) and Theorem 2.5.3 imply that $\sum_{n=1}^{\infty}(Y_n - \mu_n)$ converges a.s. Using (ii) now gives that $\sum_{n=1}^{\infty} Y_n$

converges a.s. (i) and the Borel-Cantelli lemma imply $P(X_n \neq Y_n \text{ i.o.}) = 0$, so $\sum_{n=1}^{\infty} X_n$ converges a.s. ∎

The link between convergence of series and the strong law of large numbers is provided by

**Theorem 2.5.5. Kronecker's lemma.** *If $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} x_n/a_n$ converges then*

$$a_n^{-1} \sum_{m=1}^{n} x_m \to 0$$

*Proof.* Let $a_0 = 0$, $b_0 = 0$, and for $m \geq 1$, let $b_m = \sum_{k=1}^{m} x_k/a_k$. Then $x_m = a_m(b_m - b_{m-1})$ and so

$$\begin{aligned}
a_n^{-1} \sum_{m=1}^{n} x_m &= a_n^{-1} \left\{ \sum_{m=1}^{n} a_m b_m - \sum_{m=1}^{n} a_m b_{m-1} \right\} \\
&= a_n^{-1} \left\{ a_n b_n + \sum_{m=2}^{n} a_{m-1} b_{m-1} - \sum_{m=1}^{n} a_m b_{m-1} \right\} \\
&= b_n - \sum_{m=1}^{n} \frac{(a_m - a_{m-1})}{a_n} b_{m-1}
\end{aligned}$$

(Recall $a_0 = 0$.) By hypothesis, $b_n \to b_\infty$ as $n \to \infty$. Since $a_m - a_{m-1} \geq 0$, the last sum is an average of $b_0, \ldots, b_n$. Intuitively, if $\epsilon > 0$ and $M < \infty$ are fixed and $n$ is large, the average assigns mass $\geq 1 - \epsilon$ to the $b_m$ with $m \geq M$, so

$$\sum_{m=1}^{n} \frac{(a_m - a_{m-1})}{a_n} b_{m-1} \to b_\infty$$

To argue formally, let $B = \sup |b_n|$, pick $M$ so that $|b_m - b_\infty| < \epsilon/2$ for $m \geq M$, then pick $N$ so that $a_M/a_n < \epsilon/4B$ for $n \geq N$. Now if $n \geq N$, we have

$$\begin{aligned}
\left| \sum_{m=1}^{n} \frac{(a_m - a_{m-1})}{a_n} b_{m-1} - b_\infty \right| &\leq \sum_{m=1}^{n} \frac{(a_m - a_{m-1})}{a_n} |b_{m-1} - b_\infty| \\
&\leq \frac{a_M}{a_n} \cdot 2B + \frac{a_n - a_M}{a_n} \cdot \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

proving the desired result since $\epsilon$ is arbitrary. ∎

**Theorem 2.5.6. The strong law of large numbers.** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.*

*Proof.* Let $Y_k = X_k 1_{(|X_k| \leq k)}$ and $T_n = Y_1 + \cdots + Y_n$. By (a) in the proof of Theorem 2.4.1 it suffices to show that $T_n/n \to \mu$. Let $Z_k = Y_k - EY_k$, so $EZ_k = 0$.

Now $\text{var}(Z_k) = \text{var}(Y_k) \le EY_k^2$ and (b) in the proof of Theorem 2.4.1 imply

$$\sum_{k=1}^{\infty} \text{var}(Z_k)/k^2 \le \sum_{k=1}^{\infty} EY_k^2/k^2 < \infty$$

Applying Theorem 2.5.3 now, we conclude that $\sum_{k=1}^{\infty} Z_k/k$ converges a.s., so Theorem 2.5.5 implies

$$n^{-1}\sum_{k=1}^{n}(Y_k - EY_k) \to 0 \quad \text{and hence} \quad \frac{T_n}{n} - n^{-1}\sum_{k=1}^{n} EY_k \to 0 \text{ a.s.}$$

The dominated convergence theorem implies $EY_k \to \mu$ as $k \to \infty$. From this, it follows easily that $n^{-1}\sum_{k=1}^{n} EY_k \to \mu$ and hence $T_n/n \to \mu$. ∎

### 2.5.1 *Rates of Convergence*

As mentioned earlier, one of the advantages of the random series proof is that it provides estimates on the rate of convergence of $S_n/n \to \mu$. By subtracting $\mu$ from each random variable, we can and will suppose without loss of generality that $\mu = 0$.

**Theorem 2.5.7.** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_i = 0$ and $EX_i^2 = \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. If $\epsilon > 0$ then*

$$S_n/n^{1/2}(\log n)^{1/2+\epsilon} \to 0 \quad a.s.$$

**Remark.** Kolmogorov's test, Theorem 8.8.2, will show that

$$\limsup_{n\to\infty} S_n/n^{1/2}(\log\log n)^{1/2} = \sigma\sqrt{2} \quad \text{a.s.}$$

so the last result is not far from the best possible.

*Proof.* Let $a_n = n^{1/2}(\log n)^{1/2+\epsilon}$ for $n \ge 2$ and $a_1 > 0$.

$$\sum_{n=1}^{\infty} \text{var}(X_n/a_n) = \sigma^2\left(\frac{1}{a_1^2} + \sum_{n=2}^{\infty}\frac{1}{n(\log n)^{1+2\epsilon}}\right) < \infty$$

so applying Theorem 2.5.3 we get $\sum_{n=1}^{\infty} X_n/a_n$ converges a.s., and the indicated result follows from Theorem 2.5.5. ∎

The next result, due to Marcinkiewicz and Zygmund, treats the situation in which $EX_i^2 = \infty$ but $E|X_i|^p < \infty$ for some $1 < p < 2$.

**Theorem 2.5.8.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_1 = 0$ and $E|X_1|^p < \infty$ where $1 < p < 2$. If $S_n = X_1 + \cdots + X_n$ then $S_n/n^{1/p} \to 0$ a.s.*

*Proof.* Let $Y_k = X_k 1_{(|X_k| \le k^{1/p})}$ and $T_n = Y_1 + \cdots + Y_n$.

$$\sum_{k=1}^{\infty} P(Y_k \neq X_k) = \sum_{k=1}^{\infty} P(|X_k|^p > k) \le E|X_k|^p < \infty$$

so the Borel-Cantelli lemma implies $P(Y_k \neq X_k \text{ i.o.}) = 0$, and it suffices to show $T_n/n^{1/p} \to 0$. Using $\text{var}\,(Y_m) \le E(Y_m^2)$, Lemma 2.2.8 with $p = 2$, $P(|Y_m| > y) \le P(|X_1| > y)$, and Fubini's theorem (everything is $\ge 0$), we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \text{var}\,(Y_m/m^{1/p}) &\le \sum_{m=1}^{\infty} EY_m^2/m^{2/p} \\
&\le \sum_{m=1}^{\infty} \sum_{n=1}^{m} \int_{(n-1)^{1/p}}^{n^{1/p}} \frac{2y}{m^{2/p}} P(|X_1| > y)\, dy \\
&= \sum_{n=1}^{\infty} \int_{(n-1)^{1/p}}^{n^{1/p}} \sum_{m=n}^{\infty} \frac{2y}{m^{2/p}} P(|X_1| > y)\, dy
\end{aligned}$$

To bound the integral, we note that for $n \ge 2$ comparing the sum with the integral of $x^{-2/p}$

$$\sum_{m=n}^{\infty} m^{-2/p} \le \frac{p}{2-p}(n-1)^{(p-2)/p} \le C y^{p-2}$$

when $y \in [(n-1)^{1/p}, n^{1/p}]$. Since $E|X_i|^p = \int_0^\infty p x^{p-1} P(|X_i| > x)\, dx < \infty$, it follows that

$$\sum_{m=1}^{\infty} \text{var}\,(Y_m/m^{1/p}) < \infty$$

If we let $\mu_m = EY_m$ and apply Theorem 2.5.3 and Theorem 2.5.5, it follows that

$$n^{-1/p} \sum_{m=1}^{n} (Y_m - \mu_m) \to 0 \quad \text{a.s.}$$

To estimate $\mu_m$, we note that since $EX_m = 0$, $\mu_m = -E(X_i; |X_i| > m^{1/p})$, so

$$\begin{aligned}
|\mu_m| \le E(|X|; |X_i| > m^{1/p}) &= m^{1/p} E(|X|/m^{1/p}; |X_i| > m^{1/p}) \\
&\le m^{1/p} E((|X|/m^{1/p})^p; |X_i| > m^{1/p}) \\
&\le m^{-1+1/p} p^{-1} E(|X_i|^p; |X_i| > m^{1/p})
\end{aligned}$$

Now $\sum_{m=1}^{n} m^{-1+1/p} \le C n^{1/p}$ and $E(|X_i|^p; |X_i| > m^{1/p}) \to 0$ as $m \to \infty$, so $n^{-1/p} \sum_{m=1}^{n} \mu_m \to 0$, and the desired result follows. ∎

**Exercise 2.5.2.** The converse of the last result is much easier. Let $p > 0$. If $S_n/n^{1/p} \to 0$ a.s., then $E|X_1|^p < \infty$.

### 2.5.2 *Infinite Mean*

The St. Petersburg game, discussed in Example 2.2.7 and Exercise 2.3.17, is a situation in which $EX_i = \infty$, $S_n/n\log_2 n \to 1$ in probability but

$$\limsup_{n\to\infty} S_n/(n\log_2 n) = \infty \text{ a.s.}$$

The next result, due to Feller (1946), shows that when $E|X_1| = \infty$, $S_n/a_n$ cannot converge almost surely to a nonzero limit. In Theorem 2.3.7 we considered the special case $a_n = n$.

**Theorem 2.5.9.** *Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_1| = \infty$ and let $S_n = X_1 + \cdots + X_n$. Let $a_n$ be a sequence of positive numbers with $a_n/n$ increasing. Then $\limsup_{n\to\infty} |S_n|/a_n = 0$ or $\infty$ according as $\sum_n P(|X_1| \ge a_n) < \infty$ or $= \infty$.*

*Proof.* Since $a_n/n \uparrow$, $a_{kn} \ge ka_n$ for any integer $k$. Using this and $a_n \uparrow$,

$$\sum_{n=1}^{\infty} P(|X_1| \ge ka_n) \ge \sum_{n=1}^{\infty} P(|X_1| \ge a_{kn}) \ge \frac{1}{k}\sum_{m=k}^{\infty} P(|X_1| \ge a_m)$$

The last observation shows that if the sum is infinite, $\limsup_{n\to\infty} |X_n|/a_n = \infty$. Since $\max\{|S_{n-1}|, |S_n|\} \ge |X_n|/2$, it follows that $\limsup_{n\to\infty} |S_n|/a_n = \infty$.

To prove the other half, we begin with the identity

$$(*) \qquad \sum_{m=1}^{\infty} mP(a_{m-1} \le |X_i| < a_m) = \sum_{n=1}^{\infty} P(|X_i| \ge a_{n-1})$$

To see this, write $m = \sum_{n=1}^{m} 1$ and then use Fubini's theorem. We now let $Y_n = X_n 1_{(|X_n|<a_n)}$, and $T_n = Y_1 + \cdots + Y_n$. When the sum is finite, $P(Y_n \ne X_n \text{ i.o.}) = 0$, and it suffices to investigate the behavior of the $T_n$. To do this, we let $a_0 = 0$ and compute

$$\begin{aligned}
\sum_{n=1}^{\infty} \text{var}(Y_n/a_n) &\le \sum_{n=1}^{\infty} EY_n^2/a_n^2 \\
&= \sum_{n=1}^{\infty} a_n^{-2} \sum_{m=1}^{n} \int_{[a_{m-1},a_m)} y^2\, dF(y) \\
&= \sum_{m=1}^{\infty} \int_{[a_{m-1},a_m)} y^2\, dF(y) \sum_{n=m}^{\infty} a_n^{-2}
\end{aligned}$$

Since $a_n \ge na_m/m$, we have $\sum_{n=m}^{\infty} a_n^{-2} \le (m^2/a_m^2)\sum_{n=m}^{\infty} n^{-2} \le Cma_m^{-2}$, so

$$\le C\sum_{m=1}^{\infty} m \int_{[a_{m-1},a_m)} dF(y)$$

Using $(*)$ now, we conclude $\sum_{n=1}^{\infty} \text{var}(Y_n/a_n) < \infty$.

The last step is to show $ET_n/a_n \to 0$. To begin, we note that if $E|X_i| = \infty$, $\sum_{n=1}^\infty P(|X_i| > a_n) < \infty$, and $a_n/n \uparrow$ we must have $a_n/n \uparrow \infty$. To estimate $ET_n/a_n$ now, we observe that

$$\begin{aligned}
\left| a_n^{-1} \sum_{m=1}^n EY_m \right| &\le a_n^{-1} n \sum_{m=1}^n E(|X_m|; |X_m| < a_m) \\
&\le \frac{na_N}{a_n} + \frac{n}{a_n} E(|X_i|; a_N \le |X_i| < a_n)
\end{aligned}$$

where the last inequality holds for any fixed $N$. Since $a_n/n \to \infty$, the first term converges to 0. Since $m/a_m \downarrow$, the second is

$$\begin{aligned}
&\le \sum_{m=N+1}^n \frac{m}{a_m} E(|X_i|; a_{m-1} \le |X_i| < a_m) \\
&\le \sum_{m=N+1}^\infty mP(a_{m-1} \le |X_i| < a_m)
\end{aligned}$$

($*$) shows that the sum is finite, so it is small if $N$ is large and the desired result follows. ■

## *Exercises*

**2.5.3.** Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Show that for any $t$

$$\sum_{n=1}^\infty X_n \cdot \frac{\sin(n\pi t)}{n} \quad \text{converges a.s.}$$

We will see this series again at the end of Section 8.1.

**2.5.4.** Let $X_1, X_2, \ldots$ be independent with $EX_n = 0$, $\text{var}(X_n) = \sigma_n^2$. (i) Show that if $\sum_n \sigma_n^2/n^2 < \infty$ then $\sum_n X_n/n$ converges a.s. and hence $n^{-1}\sum_{m=1}^n X_m \to 0$ a.s. (ii) Suppose $\sum \sigma_n^2/n^2 = \infty$ and without loss of generality that $\sigma_n^2 \le n^2$ for all $n$. Show that there are independent random variables $X_n$ with $EX_n = 0$ and $\text{var}(X_n) \le \sigma_n^2$ so that $X_n/n$ and hence $n^{-1}\sum_{m\le n} X_m$ does not converge to 0 a.s.

**2.5.5.** Let $X_n \ge 0$ be independent for $n \ge 1$. The following are equivalent:
(i) $\sum_{n=1}^\infty X_n < \infty$ a.s. (ii) $\sum_{n=1}^\infty [P(X_n > 1) + E(X_n 1_{(X_n \le 1)})] < \infty$
(iii) $\sum_{n=1}^\infty E(X_n/(1 + X_n)) < \infty$.

**2.5.6.** Let $\psi(x) = x^2$ when $|x| \le 1$ and $= |x|$ when $|x| \ge 1$. Show that if $X_1, X_2, \ldots$ are independent with $EX_n = 0$ and $\sum_{n=1}^\infty E\psi(X_n) < \infty$, then $\sum_{n=1}^\infty X_n$ converges a.s.

**2.5.7.** Let $X_n$ be independent. Suppose $\sum_{n=1}^\infty E|X_n|^{p(n)} < \infty$ where $0 < p(n) \le 2$ for all $n$ and $EX_n = 0$ when $p(n) > 1$. Show that $\sum_{n=1}^\infty X_n$ converges a.s.

**2.5.8.** Let $X_1, X_2, \ldots$ be i.i.d. and not $\equiv 0$. Then the radius of convergence of the power series $\sum_{n\ge 1} X_n(\omega)z^n$ (i.e., $r(\omega) = \sup\{c : \sum |X_n(\omega)|c^n < \infty\}$) is 1 a.s. or 0 a.s., according as $E\log^+|X_1| < \infty$ or $= \infty$ where $\log^+ x = \max(\log x, 0)$.

**2.5.9.** Let $X_1, X_2, \ldots$ be independent and let $S_{m,n} = X_{m+1} + \cdots + X_n$. Then

$$(\star) \qquad P\left(\max_{m<j\le n} |S_{m,j}| > 2a\right) \min_{m<k\le n} P(|S_{k,n}| \le a) \le P(|S_{m,n}| > a)$$

**2.5.10.** Use $(\star)$ to prove a theorem of P. Lévy: Let $X_1, X_2, \ldots$ be independent and let $S_n = X_1 + \cdots + X_n$. If $\lim_{n\to\infty} S_n$ exists in probability, then it also exists a.s.

**2.5.11.** Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. Use $(\star)$ to conclude that if $S_n/n \to 0$ in probability, then $(\max_{1\le m\le n} S_m)/n \to 0$ in probability.

**2.5.12.** Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. Suppose $a_n \uparrow \infty$ and $a(2^n)/a(2^{n-1})$ is bounded. (i) Use $(\star)$ to show that if $S_n/a(n) \to 0$ in probability and $S_{2^n}/a(2^n) \to 0$ a.s., then $S_n/a(n) \to 0$ a.s. (ii) Suppose in addition that $EX_1 = 0$ and $EX_1^2 < \infty$. Use the previous exercise and Chebyshev's inequality to conclude that $S_n/n^{1/2}(\log_2 n)^{1/2+\epsilon} \to 0$ a.s.

## 2.6 Large Deviations*

Let $X_1, X_2, \ldots$ be i.i.d. and let $S_n = X_1 + \cdots + X_n$. In this section, we will investigate the rate at which $P(S_n > na) \to 0$ for $a > \mu = EX_i$. We will ultimately conclude that if the **moment-generating function** $\varphi(\theta) = E\exp(\theta X_i) < \infty$ for some $\theta > 0$, $P(S_n \ge na) \to 0$ exponentially rapidly and we will identify

$$\gamma(a) = \lim_{n\to\infty} \frac{1}{n} \log P(S_n \ge na)$$

Our first step is to prove that the limit exists. This is based on an observation that will be useful several times below. Let $\pi_n = P(S_n \ge na)$.

$$\pi_{m+n} \ge P(S_m \ge ma, S_{n+m} - S_m \ge na) = \pi_m \pi_n$$

since $S_m$ and $S_{n+m} - S_m$ are independent. Letting $\gamma_n = \log \pi_n$ transforms multiplication into addition.

**Lemma 2.6.1.** *If $\gamma_{m+n} \ge \gamma_m + \gamma_n$ then as $n \to \infty$, $\gamma_n/n \to \sup_m \gamma_m/m$.*

*Proof.* Clearly, $\limsup \gamma_n/n \le \sup \gamma_m/m$. To complete the proof, it suffices to prove that for any $m$ liminf $\gamma_n/n \ge \gamma_m/m$. Writing $n = km + \ell$ with $0 \le \ell < m$ and making repeated use of the hypothesis gives $\gamma_n \ge k\gamma_m + \gamma_\ell$. Dividing by $n = km + \ell$ gives

$$\frac{\gamma(n)}{n} \ge \left(\frac{km}{km+\ell}\right)\frac{\gamma(m)}{m} + \frac{\gamma(\ell)}{n}$$

Letting $n \to \infty$ and recalling $n = km + \ell$ with $0 \le \ell < m$ gives the desired result. ■

Lemma 2.6.1 implies that $\lim_{n\to\infty} \frac{1}{n} \log P(S_n \ge na) = \gamma(a)$ exists $\le 0$. It follows from the formula for the limit that

$$P(S_n \ge na) \le e^{n\gamma(a)} \tag{2.6.1}$$

The last two observations give us some useful information about $\gamma(a)$.

**Exercise 2.6.1.** The following are equivalent: (a) $\gamma(a) = -\infty$, (b) $P(X_1 \ge a) = 0$, and (c) $P(S_n \ge na) = 0$ for all $n$.

**Exercise 2.6.2.** Use the definition to conclude that if $\lambda \in [0, 1]$ is rational, then $\gamma(\lambda a + (1-\lambda)b) \ge \lambda\gamma(a) + (1-\lambda)\gamma(b)$. Use monotonicity to conclude that the last relationship holds for all $\lambda \in [0, 1]$ so $\gamma$ is concave and hence Lipschitz continuous on compact subsets of $\gamma(a) > -\infty$.

The conclusions above are valid for any distribution. For the rest of this section, we will suppose:

(H1) $\quad \varphi(\theta) = E\exp(\theta X_i) < \infty$ for some $\theta > 0$

Let $\theta_+ = \sup\{\theta : \phi(\theta) < \infty\}$, $\theta_- = \inf\{\theta : \phi(\theta) < \infty\}$ and note that $\phi(\theta) < \infty$ for $\theta \in (\theta_-, \theta_+)$. (H1) implies that $EX_i^+ < \infty$ so $\mu = EX^+ - EX^- \in [-\infty, \infty)$. If $\theta > 0$ Chebyshev's inequality implies

$$e^{\theta na} P(S_n \ge na) \le E\exp(\theta S_n) = \varphi(\theta)^n$$

or letting $\kappa(\theta) = \log\varphi(\theta)$

$$P(S_n \ge na) \le \exp(-n\{a\theta - \kappa(\theta)\}) \tag{2.6.2}$$

Our first goal is to show:

**Lemma 2.6.2.** *If $a > \mu$ and $\theta > 0$ is small, then $a\theta - \kappa(\theta) > 0$.*

*Proof.* $\kappa(0) = \log\varphi(0) = 0$, so it suffices to show that (i) $\kappa$ is continuous at 0, (ii) differentiable on $(0, \theta_+)$, and (iii) $\kappa'(\theta) \to \mu$ as $\theta \to 0$. For then

$$a\theta - \kappa(\theta) = \int_0^\theta a - \kappa'(x)\,dx > 0$$

for small $\theta$.

Let $F(x) = P(X_i \le x)$. To prove (i), we note that if $0 < \theta < \theta_0 < \theta_-$

$$e^{\theta x} \le 1 + e^{\theta_0 x} \tag{$*$}$$

so by the dominated convergence theorem as $\theta \to 0$

$$\int e^{\theta x}\,dF \to \int 1\,dF = 1$$

To prove (ii) we note that if $|h| < h_0$, then

$$|e^{hx} - 1| = \left|\int_0^{hx} e^y\,dy\right| \le |hx|e^{h_0 x}$$

so an application of the dominated convergence theorem shows that

$$\begin{aligned}
\varphi'(\theta) &= \lim_{h\to 0} \frac{\varphi(\theta+h) - \varphi(\theta)}{h} \\
&= \lim_{h\to 0} \int \frac{e^{hx} - 1}{h} e^{\theta x}\,dF(x) \\
&= \int x e^{\theta x} dF(x) \quad \text{for } \theta \in (0, \theta_+)
\end{aligned}$$

From the last equation, it follows that $\kappa(\theta) = \log \phi(\theta)$ has $\kappa'(\theta) = \phi'(\theta)/\phi(\theta)$. Using $(*)$ and the dominated convergence theorem gives (iii), and the proof is complete. ■

Having found an upper bound on $P(S_n \ge na)$, it is natural to optimize it by finding the maximum of $\theta a - \kappa(\theta)$:

$$\frac{d}{d\theta}\{\theta a - \log \varphi(\theta)\} = a - \varphi'(\theta)/\varphi(\theta)$$

so (assuming things are nice) the maximum occurs when $a = \varphi'(\theta)/\varphi(\theta)$. To turn the parenthetical clause into a mathematical hypothesis, we begin by defining

$$F_\theta(x) = \frac{1}{\varphi(\theta)} \int_{-\infty}^{x} e^{\theta y}\,dF(y)$$

whenever $\phi(\theta) < \infty$. It follows from the proof of Lemma 2.6.2 that if $\theta \in (\theta_-, \theta_+)$, $F_\theta$ is a distribution function with mean

$$\int x\,dF_\theta(x) = \frac{1}{\varphi(\theta)} \int_{-\infty}^{\infty} x e^{\theta x}\,dF(x) = \frac{\varphi'(\theta)}{\varphi(\theta)}$$

Repeating the proof in Lemma 2.6.2, it is easy to see that if $\theta \in (\theta_-, \theta_+)$, then

$$\phi''(\theta) = \int_{-\infty}^{\infty} x^2 e^{\theta x}\,dF(x)$$

So we have

$$\frac{d}{d\theta}\frac{\varphi'(\theta)}{\varphi(\theta)} = \frac{\varphi''(\theta)}{\varphi(\theta)} - \left(\frac{\varphi'(\theta)}{\varphi(\theta)}\right)^2 = \int x^2\,dF_\theta(x) - \left(\int x\,dF_\theta(x)\right)^2 \ge 0$$

since the last expression is the variance of $F_\theta$. If we assume

(H2) the distribution $F$ is not a point mass at $\mu$

then $\varphi'(\theta)/\varphi(\theta)$ is strictly increasing and $a\theta - \log\phi(\theta)$ is concave. Since we have $\varphi'(0)/\varphi(0) = \mu$, this shows that for each $a > \mu$ there is at most one $\theta_a \geq 0$ that solves $a = \varphi'(\theta_a)/\varphi(\theta_a)$, and this value of $\theta$ maximizes $a\theta - \log\varphi(\theta)$. Before discussing the existence of $\theta_a$, we will consider some examples.

**Example 2.6.1. Normal distribution.**

$$\int e^{\theta x}(2\pi)^{-1/2}\exp(-x^2/2)\,dx = \exp(\theta^2/2)\int(2\pi)^{-1/2}\exp(-(x-\theta)^2/2)\,dx$$

The integrand in the last integral is the density of a normal distribution with mean $\theta$ and variance 1, so $\varphi(\theta) = \exp(\theta^2/2)$, $\theta \in (-\infty, \infty)$. In this case, $\varphi'(\theta)/\varphi(\theta) = \theta$ and

$$F_\theta(x) = e^{-\theta^2/2}\int_{-\infty}^{x} e^{\theta y}(2\pi)^{-1/2}e^{-y^2/2}\,dy$$

is a normal distribution with mean $\theta$ and variance 1.

**Example 2.6.2. Exponential distribution with parameter $\lambda$.** If $\theta < \lambda$

$$\int_0^\infty e^{\theta x}\lambda e^{-\lambda x}\,dx = \lambda/(\lambda-\theta)$$

$\varphi'(\theta)\varphi(\theta) = 1/(\lambda-\theta)$ and

$$F_\theta(x) = \frac{\lambda}{\lambda-\theta}\int_0^x e^{\theta y}\lambda e^{-\lambda y}\,dy$$

is an exponential distribution with parameter $\lambda - \theta$ and hence mean $1/(\lambda-\theta)$.

**Example 2.6.3. Coin flips.** $P(X_i = 1) = P(X_i = -1) = 1/2$

$$\varphi(\theta) = (e^\theta + e^{-\theta})/2$$
$$\varphi'(\theta)/\varphi(\theta) = (e^\theta - e^{-\theta})/(e^\theta + e^{-\theta})$$

$F_\theta(\{x\})/F(\{x\}) = e^{\theta x}/\phi(\theta)$ so

$$F_\theta(\{1\}) = e^\theta/(e^\theta + e^{-\theta}) \quad\text{and}\quad F_\theta(\{-1\}) = e^{-\theta}/(e^\theta + e^{-\theta})$$

**Example 2.6.4. Perverted exponential.** Let $g(x) = Cx^{-3}e^{-x}$ for $x \geq 1$, $g(x) = 0$ otherwise, and choose $C$ so that $g$ is a probability density. In this case,

$$\varphi(\theta) = \int e^{\theta x}g(x)dx < \infty$$

if and only if $\theta \leq 1$, and when $\theta \leq 1$, we have

$$\frac{\varphi'(\theta)}{\varphi(\theta)} \leq \frac{\varphi'(1)}{\varphi(1)} = \int_1^\infty Cx^{-2}\,dx \bigg/ \int_1^\infty Cx^{-3}dx = 2$$

Recall $\theta_+ = \sup\{\theta : \varphi(\theta) < \infty\}$. In Examples 2.6.1 and 2.6.2, we have $\phi'(\theta)/\phi(\theta) \uparrow \infty$ as $\theta \uparrow \theta_+$ so we can solve $a = \phi'(\theta)/\phi(\theta)$ for any $a > \mu$. In

Example 2.6.3, $\phi'(\theta)/\phi(\theta) \uparrow 1$ as $\theta \to \infty$, but we cannot hope for much more since $F$ and hence $F_\theta$ is supported on $\{-1, 1\}$.

**Exercise 2.6.3.** Let $x_o = \sup\{x : F(x) < 1\}$. Show that if $x_o < \infty$ then $\phi(\theta) < \infty$ for all $\theta > 0$ and $\phi'(\theta)/\phi(\theta) \to x_o$ as $\theta \uparrow \infty$.

Example 2.6.4 presents a problem since we cannot solve $a = \varphi'(\theta)/\varphi(\theta)$ when $a > 2$. Theorem 2.6.5 will cover this problem case, but first we will treat the cases in which we can solve the equation.

**Theorem 2.6.3.** *Suppose in addition to (H1) and (H2) that there is a $\theta_a \in (0, \theta_+)$ so that $a = \varphi'(\theta_a)/\varphi(\theta_a)$. Then, as $n \to \infty$,*

$$n^{-1} \log P(S_n \geq na) \to -a\theta_a + \log \varphi(\theta_a)$$

*Proof.* The fact that the limsup of the left-hand side $\leq$ the right-hand side follows from (2.6.2). To prove the other inequality, pick $\lambda \in (\theta_a, \theta_+)$, let $X_1^\lambda, X_2^\lambda, \ldots$ be i.i.d. with distribution $F_\lambda$ and let $S_n^\lambda = X_1^\lambda + \cdots + X_n^\lambda$. Writing $dF/dF_\lambda$ for the Radon-Nikodym derivative of the associated measures, it is immediate from the definition that $dF/dF_\lambda = e^{-\lambda x}\varphi(\lambda)$. If we let $F_\lambda^n$ and $F^n$ denote the distributions of $S_n^\lambda$ and $S_n$, then

**Lemma 2.6.4.** $\dfrac{dF^n}{dF_\lambda^n} = e^{-\lambda x}\varphi(\lambda)^n.$

*Proof.* We will prove this by induction. The result holds when $n = 1$. For $n > 1$, we note that

$$\begin{aligned}
F^n = F^{n-1} * F(z) &= \int_{-\infty}^{\infty} dF^{n-1}(x) \int_{-\infty}^{z-x} dF(y) \\
&= \int dF_\lambda^{n-1}(x) \int dF_\lambda(y)\, 1_{(x+y\leq z)} e^{-\lambda(x+y)}\varphi(\lambda)^n \\
&= E\left(1_{(S_{n-1}^\lambda + X_n^\lambda \leq z)} e^{-\lambda(S_{n-1}^\lambda + X_n^\lambda)}\varphi(\lambda)^n\right) \\
&= \int_{-\infty}^{z} dF_\lambda^n(u) e^{-\lambda u}\varphi(\lambda)^n
\end{aligned}$$

where in the last two equalities we have used Theorem 1.6.9 for $(S_{n-1}^\lambda, X_n^\lambda)$ and $S_n^\lambda$. ■

If $\nu > a$, then the lemma and monotonicity imply

$$(*) \qquad P(S_n \geq na) \geq \int_{na}^{n\nu} e^{-\lambda x}\varphi(\lambda)^n dF_\lambda^n(x) \geq \varphi(\lambda)^n e^{-\lambda n\nu}(F_\lambda^n(n\nu) - F_\lambda^n(na))$$

$F_\lambda$ has mean $\varphi'(\lambda)/\varphi(\lambda)$, so if we have $a < \varphi'(\lambda)/\varphi(\lambda) < \nu$, then the weak law of large numbers implies

$$F_\lambda^n(n\nu) - F_\lambda^n(na) \to 1 \text{ as } n \to \infty$$

From the last conclusion and $(*)$ it follows that

$$\liminf_{n\to\infty} n^{-1} \log P(S_n > na) \geq -\lambda\nu + \log\phi(\lambda)$$

Since $\lambda > \theta_a$ and $\nu > a$ are arbitrary, the proof is complete. ■

To get a feel for what the answers look like, we consider our examples. To prepare for the computations, we recall some important information:

$$\kappa(\theta) = \log\phi(\theta) \quad \kappa'(\theta) = \phi'(\theta)/\phi(\theta) \quad \theta_a \text{ solves } \kappa'(\theta_a) = a$$

$$\gamma(a) = \lim_{n\to\infty}(1/n)\log P(S_n \geq na) = -a\theta_a + \kappa(\theta_a)$$

**Normal distribution.** (Example 2.6.1):

$$\kappa(\theta) = \theta^2/2 \qquad \kappa'(\theta) = \theta \qquad \theta_a = a$$

$$\gamma(a) = -a\theta_a + \kappa(\theta_a) = -a^2/2$$

**Exercise 2.6.4.** Check the last result by observing that $S_n$ has a normal distribution with mean 0 and variance $n$, and then using Theorem 1.2.3.

**Exponential distribution.** (Example 2.6.2) with $\lambda = 1$:

$$\kappa(\theta) = -\log(1-\theta) \qquad \kappa'(\theta) = 1/(1-\theta) \qquad \theta_a = 1 - 1/a$$

$$\gamma(a) = -a\theta_a + \kappa(\theta_a) = -a + 1 + \log a$$

With these two examples as models, the reader should be able to do

**Exercise 2.6.5.** Let $X_1, X_2, \ldots$ be i.i.d. Poisson with mean 1, and let $S_n = X_1 + \cdots + X_n$. Find $\lim_{n\to\infty}(1/n)\log P(S_n \geq na)$ for $a > 1$. The answer and another proof can be found in Exercise 3.1.4.

**Coin flips.** (Example 2.6.3). Here we take a different approach. To find the $\theta$ that makes the mean of $F_\theta = a$, we set $F_\theta(\{1\}) = e^\theta/(e^\theta + e^{-\theta}) = (1+a)/2$. Letting $x = e^\theta$ gives

$$2x = (1+a)(x + x^{-1}) \qquad (a-1)x^2 + (1+a) = 0$$

So $x = \sqrt{(1+a)/(1-a)}$ and $\theta_a = \log x = \{\log(1+a) - \log(1-a)\}/2$.

$$\phi(\theta_a) = \frac{e^{\theta_a} + e^{-\theta_a}}{2} = \frac{e^{\theta_a}}{1+a} = \frac{1}{\sqrt{(1+a)(1-a)}}$$

$$\gamma(a) = -a\theta_a + \kappa(\theta_a) = -\{(1+a)\log(1+a) + (1-a)\log(1-a)\}/2$$

In Exercise 3.1.3, this result will be proved by a direct computation. Since the formula for $\gamma(a)$ is rather ugly, the following simpler bound is useful.

**Exercise 2.6.6.** Show that for coin flips $\varphi(\theta) \le \exp(\varphi(\theta) - 1) \le \exp(\beta\theta^2)$ for $\theta \le 1$ where $\beta = \sum_{n=1}^{\infty} 1/(2n)! \approx 0.586$, and use (2.6.2) to conclude that $P(S_n \ge an) \le \exp(-na^2/4\beta)$ for all $a \in [0, 1]$. It is customary to simplify this further by using $\beta \le \sum_{n=1}^{\infty} 2^{-n} = 1$.

Turning now to the problematic values for which we cannot solve $a = \phi'(\theta_a)/\phi(\theta_a)$, we begin by observing that if $x_o = \sup\{x : F(x) < 1\}$ and $F$ is not a point mass at $x_o$, then $\phi'(\theta)/\phi(\theta) \uparrow x_0$ as $\theta \uparrow \infty$ but $\phi'(\theta)/\phi(\theta) < x_0$ for all $\theta < \infty$. However, the result for $a = x_o$ is trivial:

$$\frac{1}{n} \log P(S_n \ge nx_o) = \log P(X_i = x_o) \quad \text{for all } n$$

**Exercise 2.6.7.** Show that as $a \uparrow x_o$, $\gamma(a) \downarrow \log P(X_i = x_o)$.

When $x_o = \infty$, $\phi'(\theta)/\phi(\theta) \uparrow \infty$ as $\theta \uparrow \infty$, so the only case that remains is covered by

**Theorem 2.6.5.** *Suppose $x_o = \infty$, $\theta_+ < \infty$, and $\varphi'(\theta)/\varphi(\theta)$ increases to a finite limit $a_0$ as $\theta \uparrow \theta_+$. If $a_0 \le a < \infty$*

$$n^{-1} \log\ P(S_n \ge na) \to -a\theta_+ + \log \varphi(\theta_+)$$

*that is, $\gamma(a)$ is linear for $a \ge a_0$.*

*Proof.* Since $(\log \varphi(\theta))' = \varphi'(\theta)/\varphi(\theta)$, integrating from 0 to $\theta_+$ shows that $\log(\varphi(\theta_+)) < \infty$. Letting $\theta = \theta_+$ in (2.6.2) shows that the limsup of the left-hand side $\le$ the right-hand side. To get the other direction we will use the transformed distribution $F_\lambda$, for $\lambda = \theta_+$. Letting $\theta \uparrow \theta_+$ and using the dominated convergence theorem for $x \le 0$ and the monotone convergence theorem for $x \ge 0$, we see that $F_\lambda$ has mean $a_0$. From $(*)$ in the proof of Theorem 2.6.3, we see that if $a_0 \le a < \nu = a + 3\epsilon$

$$P(S_n \ge na) \ge \varphi(\lambda)^n e^{-n\lambda\nu}(F_\lambda^n(n\nu) - F_\lambda^n(na))$$

and hence

$$\frac{1}{n} \log P(S_n \ge na) \ge \log \varphi(\lambda) - \lambda\nu + \frac{1}{n} \log P(S_n^\lambda \in (na, n\nu])$$

Letting $X_1^\lambda, X_2^\lambda, \ldots$ be i.i.d. with distribution $F_\lambda$ and $S_n^\lambda = X_1^\lambda + \cdots + X_n^\lambda$, we have

$$\begin{aligned} P(S_n^\lambda \in (na, n\nu]) \geq\ & P\{S_{n-1}^\lambda \in ((a_0 - \epsilon)n, (a_0 + \epsilon)n]\} \\ & \cdot P\{X_n^\lambda \in ((a - a_0 + \epsilon)n, (a - a_0 + 2\epsilon)n]\} \\ \geq\ & \frac{1}{2} P\{X_n^\lambda \in ((a - a_0 + \epsilon)n, (a - a_0 + \epsilon)(n + 1)]\} \end{aligned}$$

for large $n$ by the weak law of large numbers. To get a lower bound on the right-hand side of the last equation, we observe that

$$\limsup_{n\to\infty} \frac{1}{n} \log P(X_1^\lambda \in ((a - a_0 + \epsilon)n, (a - a_0 + \epsilon)(n + 1)]) = 0$$

for if the lim sup was $< 0$, we would have $E \exp(\eta X_1^\lambda) < \infty$ for some $\eta > 0$ and hence $E \exp((\lambda + \eta)X_1) < \infty$, contradicting the definition of $\lambda = \theta_+$. To finish the argument now, we recall that Theorem 2.6.1 implies that

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq na) = \gamma(a)$$

exists, so our lower bound on the lim sup is good enough. ■

By adapting the proof of the last result, you can show that (H1) is necessary for exponential convergence:

**Exercise 2.6.8.** Suppose $EX_i = 0$ and $E \exp(\theta X_i) = \infty$ for all $\theta > 0$. Then

$$\frac{1}{n} \log P(S_n \geq na) \to 0 \text{ for all } a > 0$$

**Exercise 2.6.9.** Suppose $EX_i = 0$. Show that if $\epsilon > 0$ then

$$\liminf_{n\to\infty} P(S_n \geq na)/nP(X_1 \geq n(a + \epsilon)) \geq 1$$

Hint: Let $F_n = \{X_i \geq n(a + \epsilon) \text{ for exactly one } i \leq n\}$.

# 3

# Central Limit Theorems

The first four sections of this chapter develop the central limit theorem. The last five treat various extensions and complements. We begin this chapter by considering special cases of these results that can be treated by elementary computations.

## 3.1 The De Moivre-Laplace Theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. In words, we are betting \$1 on the flipping of a fair coin and $S_n$ is our winnings at time $n$. If $n$ and $k$ are integers

$$P(S_{2n} = 2k) = \binom{2n}{n+k} 2^{-2n}$$

since $S_{2n} = 2k$ if and only if there are $n+k$ flips that are $+1$ and $n-k$ flips that are $-1$ in the first $2n$. The first factor gives the number of such outcomes and the second the probability of each one. **Stirling's formula** (see Feller, Vol. I., 1968, p. 52) tells us

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \quad \text{as } n \to \infty \tag{3.1.1}$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$, so

$$\begin{aligned}\binom{2n}{n+k} &= \frac{(2n)!}{(n+k)!(n-k)!} \\ &\sim \frac{(2n)^{2n}}{(n+k)^{n+k}(n-k)^{n-k}} \cdot \frac{(2\pi(2n))^{1/2}}{(2\pi(n+k))^{1/2}(2\pi(n-k))^{1/2}}\end{aligned}$$

and we have

$$\begin{aligned}\binom{2n}{n+k} 2^{-2n} \sim \left(1 + \frac{k}{n}\right)^{-n-k} \cdot \left(1 - \frac{k}{n}\right)^{-n+k} \\ \cdot (\pi n)^{-1/2} \cdot \left(1 + \frac{k}{n}\right)^{-1/2} \cdot \left(1 - \frac{k}{n}\right)^{-1/2}\end{aligned} \tag{3.1.2}$$

The first two terms on the right are

$$= \left(1 - \frac{k^2}{n^2}\right)^{-n} \cdot \left(1 + \frac{k}{n}\right)^{-k} \cdot \left(1 - \frac{k}{n}\right)^{k}$$

A little calculus shows that:

**Lemma 3.1.1.** *If $c_j \to 0$, $a_j \to \infty$ and $a_j c_j \to \lambda$ then $(1 + c_j)^{a_j} \to e^{\lambda}$.*

*Proof.* As $x \to 0$, $\log(1 + x)/x \to 1$, so $a_j \log(1 + c_j) \to \lambda$, and the desired result follows. ∎

**Exercise 3.1.1.** Generalize the last proof to conclude that if $\max_{1 \le j \le n} |c_{j,n}| \to 0$, $\sum_{j=1}^{n} c_{j,n} \to \lambda$, and $\sup_n \sum_{j=1}^{n} |c_{j,n}| < \infty$ then $\prod_{j=1}^{n}(1 + c_{j,n}) \to e^{\lambda}$.

Using Lemma 3.1.1 now, we see that if $2k = x\sqrt{2n}$, that is, $k = x\sqrt{n/2}$, then

$$\begin{aligned}
\left(1 - \frac{k^2}{n^2}\right)^{-n} &= \left(1 - x^2/2n\right)^{-n} \to e^{x^2/2} \\
\left(1 + \frac{k}{n}\right)^{-k} &= \left(1 + x/\sqrt{2n}\right)^{-x\sqrt{n/2}} \to e^{-x^2/2} \\
\left(1 - \frac{k}{n}\right)^{k} &= \left(1 - x/\sqrt{2n}\right)^{x\sqrt{n/2}} \to e^{-x^2/2}
\end{aligned}$$

For this choice of $k$, $k/n \to 0$, so

$$\left(1 + \frac{k}{n}\right)^{-1/2} \cdot \left(1 - \frac{k}{n}\right)^{-1/2} \to 1$$

and putting things together gives:

**Theorem 3.1.2.** *If $2k/\sqrt{2n} \to x$ then $P(S_{2n} = 2k) \sim (\pi n)^{-1/2} e^{-x^2/2}$.*

Our next step is to compute

$$P(a\sqrt{2n} \le S_{2n} \le b\sqrt{2n}) = \sum_{m \in [a\sqrt{2n}, b\sqrt{2n}] \cap 2\mathbf{Z}} P(S_{2n} = m)$$

Changing variables $m = x\sqrt{2n}$, we have that the above is

$$\approx \sum_{x \in [a,b] \cap (2\mathbf{Z}/\sqrt{2n})} (2\pi)^{-1/2} e^{-x^2/2} \cdot (2/n)^{1/2}$$

where $2\mathbf{Z}/\sqrt{2n} = \{2z/\sqrt{2n} : z \in \mathbf{Z}\}$. We have multiplied and divided by $\sqrt{2}$ since the space between points in the sum is $(2/n)^{1/2}$, so if $n$ is large, the sum above is

$$\approx \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$$

The integrand is the density of the (standard) normal distribution, so, changing notation, we can write the last quantity as $P(a \le \chi \le b)$ where $\chi$ is a random variable with that distribution.

It is not hard to fill in the details to get:

**Theorem 3.1.3. The De Moivre-Laplace Theorem.** *If $a < b$ then as $m \to \infty$*

$$P(a \le S_m/\sqrt{m} \le b) \to \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$$

(To remove the restriction to even integers observe $S_{2n+1} = S_{2n} \pm 1$.) The last result is a special case of the central limit theorem given in Section 3.4, so further details are left to the reader.

### *Exercises*

The next three exercises illustrate the use of Stirling's formula. In them, $X_1, X_2, \ldots$ are i.i.d. and $S_n = X_1 + \cdots + X_n$.

**3.1.2.** If the $X_i$ have a Poisson distribution with mean 1, then $S_n$ has a Poisson distribution with mean $n$, i.e., $P(S_n = k) = e^{-n} n^k / k!$ Use Stirling's formula to show that if $(k - n)/\sqrt{n} \to x$ then

$$\sqrt{2\pi n} P(S_n = k) \to \exp(-x^2/2)$$

As in the case of coin flips it follows that

$$P(a \le (S_n - n)/\sqrt{n} \le b) \to \int_a^b (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

but proving the last conclusion is not part of the exercise.

In the next two examples you should begin by considering $P(S_n = k)$ when $k/n \to a$ and then relate $P(S_n = j + 1)$ to $P(S_n = j)$ to show $P(S_n \ge k) \le C P(S_n = k)$.

**3.1.3.** Suppose $P(X_i = 1) = P(X_i = -1) = 1/2$. Show that if $a \in (0, 1)$

$$\frac{1}{2n} \log P(S_{2n} \ge 2na) \to -\gamma(a)$$

where $\gamma(a) = \frac{1}{2}\{(1 + a)\log(1 + a) + (1 - a)\log(1 - a)\}$.

**3.1.4.** Suppose $P(X_i = k) = e^{-1}/k!$ for $k = 0, 1, \ldots$ Show that if $a > 1$

$$\frac{1}{n} \log P(S_n \ge na) \to a - 1 - a \log a$$

## 3.2 Weak Convergence

In this section, we will define the type of convergence that appears in the central limit theorem and explore some of its properties. A sequence of distribution functions is said to **converge weakly** to a limit $F$ (written $F_n \Rightarrow F$) if $F_n(y) \to F(y)$ for all $y$ that are continuity points of $F$. A sequence of random variables $X_n$ is said to **converge weakly** or **converge in distribution** to a limit $X_\infty$ (written $X_n \Rightarrow X_\infty$) if their distribution functions $F_n(x) = P(X_n \le x)$ converge weakly. To see that convergence at continuity points is enough to identify the limit, observe that $F$ is right continuous and by Exercise 1.2.3, the discontinuities of $F$ are at most a countable set.

### 3.2.1 *Examples*

Two examples of weak convergence that we have seen earlier are:

**Example 3.2.1.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 1) = P(X_i = -1) = 1/2$ and let $S_n = X_1 + \cdots + X_n$. Then Theorem 3.1.3 implies

$$F_n(y) = P(S_n/\sqrt{n} \le y) \to \int_{-\infty}^{y} (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

**Example 3.2.2.** Let $X_1, X_2, \ldots$ be i.i.d. with distribution $F$. The Glivenko-Cantelli theorem (Theorem 2.4.7) implies that for almost every $\omega$,

$$F_n(y) = n^{-1} \sum_{m=1}^{n} 1_{(X_m(\omega) \le y)} \to F(y) \text{ for all } y$$

In the last two examples convergence occurred for all $y$, even though in the second case the distribution function could have discontinuities. The next example shows why we restrict our attention to continuity points.

**Example 3.2.3.** Let $X$ have distribution $F$. Then $X + 1/n$ has distribution

$$F_n(x) = P(X + 1/n \le x) = F(x - 1/n)$$

As $n \to \infty$, $F_n(x) \to F(x-) = \lim_{y \uparrow x} F(y)$, so convergence only occurs at continuity points.

**Example 3.2.4. Waiting for rare events.** Let $X_p$ be the number of trials needed to get a success in a sequence of independent trials with success probability $p$. Then $P(X_p \ge n) = (1-p)^{n-1}$ for $n = 1, 2, 3, \ldots$, and it follows from Lemma 3.1.1 that as $p \to 0$,

$$P(pX_p > x) \to e^{-x} \quad \text{for all } x \ge 0$$

In words, $pX_p$ converges weakly to an exponential distribution.

**Example 3.2.5. Birthday problem.** Let $X_1, X_2, \ldots$ be independent and uniformly distributed on $\{1, \ldots, N\}$, and let $T_N = \min\{n : X_n = X_m \text{ for some } m < n\}$.

$$P(T_N > n) = \prod_{m=2}^{n} \left(1 - \frac{m-1}{N}\right)$$

When $N = 365$, this is the probability that two people in a group of size $n$ do not have the same birthday (assuming all birthdays are equally likely). Using Exercise 3.1.1, it is easy to see that

$$P(T_N/N^{1/2} > x) \to \exp(-x^2/2) \text{ for all } x \geq 0$$

Taking $N = 365$ and noting $22/\sqrt{365} = 1.1515$ and $(1.1515)^2/2 = 0.6630$, this says that

$$P(T_{365} > 22) \approx e^{-0.6630} \approx 0.515$$

This answer is 2% smaller than the true probability 0.524.

Before giving our sixth example, we need a simple result called **Scheffé's theorem**. Suppose we have probability densities $f_n$, $1 \leq n \leq \infty$, and $f_n \to f_\infty$ pointwise as $n \to \infty$. Then for all Borel sets $B$

$$\begin{aligned} \left| \int_B f_n(x)dx - \int_B f_\infty(x)dx \right| &\leq \int |f_n(x) - f_\infty(x)|dx \\ &= 2 \int (f_\infty(x) - f_n(x))^+ \, dx \to 0 \end{aligned}$$

by the dominated convergence theorem, the equality following from the fact that the $f_n \geq 0$ and have integral $= 1$. Writing $\mu_n$ for the corresponding measures, we have shown that the **total variation norm**

$$\|\mu_n - \mu_\infty\| \equiv \sup_B |\mu_n(B) - \mu_\infty(B)| \to 0$$

a conclusion stronger than weak convergence. (Take $B = (-\infty, x]$.) The example $\mu_n = a$ point mass at $1/n$ (with $1/\infty = 0$) shows that we may have $\mu_n \Rightarrow \mu_\infty$ with $\|\mu_n - \mu_\infty\| = 1$ for all $n$.

**Exercise 3.2.1.** Give an example of random variables $X_n$ with densities $f_n$ so that $X_n \Rightarrow$ a uniform distribution on (0,1) but $f_n(x)$ does not converge to 1 for any $x \in [0, 1]$.

**Example 3.2.6. Central order statistic.** Put $(2n + 1)$ points at random in (0,1), that is, with locations that are independent and uniformly distributed. Let $V_{n+1}$ be the $(n + 1)$th largest point. It is easy to see that

**Lemma 3.2.1.** *$V_{n+1}$ has density function*

$$f_{V_{n+1}}(x) = (2n+1)\binom{2n}{n} x^n (1-x)^n$$

*Proof.* There are $2n+1$ ways to pick the observation that falls at $x$, then we have to pick $n$ indices for observations $< x$, which can be done in $\binom{2n}{n}$ ways. Once we have decided on the indices that will land $< x$ and $> x$, the probability the corresponding random variables will do what we want is $x^n(1-x)^n$, and the probability density that the remaining one will land at $x$ is 1. If you don't like the previous sentence, compute the probability $X_1 < x - \epsilon, \dots, X_n < x - \epsilon$, $x - \epsilon < X_{n+1} < x + \epsilon$, $X_{n+2} > x + \epsilon, \dots X_{2n+1} > x + \epsilon$, then let $\epsilon \to 0$. ■

To compute the density function of $Y_n = 2(V_{n+1} - 1/2)\sqrt{2n}$, we use Exercise 1.2.5, or simply change variables $x = 1/2 + y/2\sqrt{2n}$, $dx = dy/2\sqrt{2n}$ to get

$$\begin{aligned} f_{Y_n}(y) &= (2n+1)\binom{2n}{n}\left(\frac{1}{2} + \frac{y}{2\sqrt{2n}}\right)^n \left(\frac{1}{2} - \frac{y}{2\sqrt{2n}}\right)^n \frac{1}{2\sqrt{2n}} \\ &= \binom{2n}{n} 2^{-2n} \cdot (1 - y^2/2n)^n \cdot \frac{2n+1}{2n} \cdot \sqrt{\frac{n}{2}} \end{aligned}$$

The first factor is $P(S_{2n} = 0)$ for a simple random walk, so Theorem 3.1.2 and Lemma 3.1.1 imply that

$$f_{Y_n}(y) \to (2\pi)^{-1/2} \exp(-y^2/2) \text{ as } n \to \infty$$

Here and in what follows we write $P(Y_n = y)$ for the density function of $Y_n$. Using Scheffé's theorem now, we conclude that $Y_n$ converges weakly to a standard normal distribution.

**Exercise 3.2.2. Convergence of maxima.** Let $X_1, X_2, \dots$ be independent with distribution $F$, and let $M_n = \max_{m \le n} X_m$. Then $P(M_n \le x) = F(x)^n$. Prove the following limit laws for $M_n$:

(i) If $F(x) = 1 - x^{-\alpha}$ for $x \ge 1$ where $\alpha > 0$, then for $y > 0$,

$$P(M_n/n^{1/\alpha} \le y) \to \exp(-y^{-\alpha})$$

(ii) If $F(x) = 1 - |x|^\beta$ for $-1 \le x \le 0$ where $\beta > 0$, then for $y < 0$,

$$P(n^{1/\beta} M_n \le y) \to \exp(-|y|^\beta)$$

(iii) If $F(x) = 1 - e^{-x}$ for $x \ge 0$ then for all $y \in (-\infty, \infty)$

$$P(M_n - \log n \le y) \to \exp(-e^{-y})$$

The limits that appear above are called the **extreme value distributions**. The last one is called the **double exponential** or **Gumbel distribution**. Necessary and sufficient conditions for $(M_n - b_n)/a_n$ to converge to these limits were obtained by Gnedenko (1943). For a recent treatment, see Resnick (1987).

**Exercise 3.2.3.** Let $X_1, X_2, \ldots$ be i.i.d. and have the standard normal distribution.
(i) From Theorem 1.2.3, we know

$$P(X_i > x) \sim \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2} \quad \text{as } x \to \infty$$

Use this to conclude that for any real number $\theta$

$$P(X_i > x + (\theta/x))/P(X_i > x) \to e^{-\theta}$$

(ii) Show that if we define $b_n$ by $P(X_i > b_n) = 1/n$

$$P(b_n(M_n - b_n) \le x) \to \exp(-e^{-x})$$

(iii) Show that $b_n \sim (2\log n)^{1/2}$ and conclude $M_n/(2\log n)^{1/2} \to 1$ in probability.

### 3.2.2 *Theory*

The next result is useful for proving things about weak convergence.

**Theorem 3.2.2.** *If $F_n \Rightarrow F_\infty$ then there are random variables $Y_n$, $1 \le n \le \infty$, with distribution $F_n$ so that $Y_n \to Y_\infty$ a.s.*

*Proof.* Let $\Omega = (0,1)$, $\mathcal{F} =$ Borel sets, $P =$ Lebesgue measure, and let $Y_n(x) = \sup\{y : F_n(y) < x\}$. By Theorem 1.2.2, $Y_n$ has distribution $F_n$. We will now show that $Y_n(x) \to Y_\infty(x)$ for all but a countable number of $x$. To do this, it is convenient to write $Y_n(x)$ as $F_n^{-1}(x)$ and drop the subscript when $n = \infty$. We begin by identifying the exceptional set. Let $a_x = \sup\{y : F(y) < x\}$, $b_x = \inf\{y : F(y) > x\}$, and $\Omega_0 = \{x : (a_x, b_x) = \emptyset\}$ where $(a_x, b_x)$ is the open interval with the indicated endpoints. $\Omega - \Omega_0$ is countable since the $(a_x, b_x)$ are disjoint and each nonempty interval contains a different rational number. If $x \in \Omega_0$ then $F(y) < x$ for $y < F^{-1}(x)$ and $F(z) > x$ for $z > F^{-1}(x)$. To prove that $F_n^{-1}(x) \to F^{-1}(x)$ for $x \in \Omega_0$, there are two things to show:

(a) $\liminf_{n\to\infty} F_n^{-1}(x) \ge F^{-1}(x)$

*Proof of (a).* Let $y < F^{-1}(x)$ be such that $F$ is continuous at $y$. Since $x \in \Omega_0$, $F(y) < x$ and if $n$ is sufficiently large $F_n(y) < x$, that is, $F_n^{-1}(x) \ge y$. Since this holds for all $y$ satisfying the indicated restrictions, the result follows.

(b) $\limsup_{n\to\infty} F_n^{-1}(x) \le F^{-1}(x)$

*Proof of (b).* Let $y > F^{-1}(x)$ be such that $F$ is continuous at $y$. Since $x \in \Omega_0$, $F(y) > x$ and if $n$ is sufficiently large $F_n(y) > x$, that is, $F_n^{-1}(x) \le y$. Since this holds for all $y$ satisfying the indicated restrictions, the result follows and we have completed the proof. ■

Theorem 3.2.2 allows us to immediately generalize some of our earlier results.

**Exercise 3.2.4. Fatou's lemma.** Let $g \geq 0$ be continuous. If $X_n \Rightarrow X_\infty$ then

$$\liminf_{n\to\infty} Eg(X_n) \geq Eg(X_\infty)$$

**Exercise 3.2.5. Integration to the limit.** Suppose $g, h$ are continuous with $g(x) > 0$, and $|h(x)|/g(x) \to 0$ as $|x| \to \infty$. If $F_n \Rightarrow F$ and $\int g(x)\,dF_n(x) \leq C < \infty$, then

$$\int h(x)\,dF_n(x) \to \int h(x)dF(x)$$

The next result illustrates the usefulness of Theorem 3.2.2 and gives an equivalent definition of weak convergence that makes sense in any topological space.

**Theorem 3.2.3.** $X_n \Rightarrow X_\infty$ *if and only if for every bounded continuous function* $g$ *we have* $Eg(X_n) \to Eg(X_\infty)$.

*Proof.* Let $Y_n$ have the same distribution as $X_n$ and converge a.s. Since $g$ is continuous, $g(Y_n) \to g(Y_\infty)$ a.s. and the bounded convergence theorem implies

$$Eg(X_n) = Eg(Y_n) \to Eg(Y_\infty) = Eg(X_\infty)$$

To prove the converse, let

$$g_{x,\epsilon}(y) = \begin{cases} 1 & y \leq x \\ 0 & y \geq x+\epsilon \\ \text{linear} & x \leq y \leq x+\epsilon \end{cases}$$

Since $g_{x,\epsilon}(y) = 1$ for $y \leq x$, $g_{x,\epsilon}$ is continuous, and $g_{x,\epsilon}(y) = 0$ for $y > x+\epsilon$,

$$\limsup_{n\to\infty} P(X_n \leq x) \leq \limsup_{n\to\infty} Eg_{x,\epsilon}(X_n) = Eg_{x,\epsilon}(X_\infty) \leq P(X_\infty \leq x+\epsilon)$$

Letting $\epsilon \to 0$ gives $\limsup_{n\to\infty} P(X_n \leq x) \leq P(X_\infty \leq x)$. The last conclusion is valid for any $x$. To get the other direction, we observe

$$\liminf_{n\to\infty} P(X_n \leq x) \geq \liminf_{n\to\infty} Eg_{x-\epsilon,\epsilon}(X_n) = Eg_{x-\epsilon,\epsilon}(X_\infty) \geq P(X_\infty \leq x-\epsilon)$$

Letting $\epsilon \to 0$ gives $\liminf_{n\to\infty} P(X_n \leq x) \geq P(X_\infty < x) = P(X_\infty \leq x)$ if $x$ is a continuity point. The results for the lim sup and the lim inf combine to give the desired result. ■

The next result is a trivial but useful generalization of Theorem 3.2.3.

**Theorem 3.2.4. Continuous mapping theorem.** *Let* $g$ *be a measurable function and* $D_g = \{x : g \text{ is discontinuous at } x\}$. *If* $X_n \Rightarrow X_\infty$ *and* $P(X_\infty \in D_g) = 0$ *then* $g(X_n) \Rightarrow g(X)$. *If in addition* $g$ *is bounded, then* $Eg(X_n) \to Eg(X_\infty)$.

**Remark.** $D_g$ is always a Borel set. See Exercise 1.3.6.

*Proof.* Let $Y_n =_d X_n$ with $Y_n \to Y_\infty$ a.s. If $f$ is continuous, then $D_{f\circ g} \subset D_g$, so $P(Y_\infty \in D_{f\circ g}) = 0$, and it follows that $f(g(Y_n)) \to f(g(Y_\infty)$ a.s. If, in addition, $f$ is bounded, then the bounded convergence theorem implies $Ef(g(Y_n)) \to Ef(g(Y_\infty)$. Since this holds for all bounded continuous functions, it follows from Theorem 3.2.3 that $g(X_n) \Rightarrow g(X_\infty)$.

The second conclusion is easier. Since $P(Y_\infty \in D_g) = 0$, $g(Y_n) \to g(Y_\infty)$ a.s., and the desired result follows from the bounded convergence theorem. ■

The next result provides a number of useful alternative definitions of weak convergence.

**Theorem 3.2.5.** *The following statements are equivalent:*
*(i)* $X_n \Rightarrow X_\infty$
*(ii) For all open sets* $G$, $\liminf_{n\to\infty} P(X_n \in G) \geq P(X_\infty \in G)$.
*(iii) For all closed sets* $K$, $\limsup_{n\to\infty} P(X_n \in K) \leq P(X_\infty \in K)$.
*(iv) For all sets* $A$ *with* $P(X_\infty \in \partial A) = 0$, $\lim_{n\to\infty} P(X_n \in A) = P(X_\infty \in A)$.

**Remark.** To help remember the directions of the inequalities in (ii) and (iii), consider the special case in which $P(X_n = x_n) = 1$. In this case, if $x_n \in G$ and $x_n \to x_\infty \in \partial G$, then $P(X_n \in G) = 1$ for all $n$ but $P(X_\infty \in G) = 0$. Letting $K = G^c$ gives an example for (iii).

*Proof.* We will prove four things and leave it to the reader to check that we have proved the result given above.

(i) implies (ii): Let $Y_n$ have the same distribution as $X_n$ and $Y_n \to Y_\infty$ a.s. Since $G$ is open,

$$\liminf_{n\to\infty} 1_G(Y_n) \geq 1_G(Y_\infty)$$

so Fatou's lemma implies

$$\liminf_{n\to\infty} P(Y_n \in G) \geq P(Y_\infty \in G)$$

(ii) is equivalent to (iii): This follows easily from: $A$ is open if and only if $A^c$ is closed and $P(A) + P(A^c) = 1$.

(ii) and (iii) imply (iv): Let $K = \bar{A}$ and $G = A^o$ be the closure and interior of $A$, respectively. The boundary of $A$, $\partial A = \bar{A} - A^o$ and $P(X_\infty \in \partial A) = 0$, so

$$P(X_\infty \in K) = P(X_\infty \in A) = P(X_\infty \in G)$$

Using (ii) and (iii) now

$$\limsup_{n\to\infty} P(X_n \in A) \le \limsup_{n\to\infty} P(X_n \in K) \le P(X_\infty \in K) = P(X_\infty \in A)$$

$$\liminf_{n\to\infty} P(X_n \in A) \ge \liminf_{n\to\infty} P(X_n \in G) \ge P(X_\infty \in G) = P(X_\infty \in A)$$

(iv) implies (i): Let $x$ be such that $P(X_\infty = x) = 0$, i.e., $x$ is a continuity point of $F$, and let $A = (-\infty, x]$. ∎

The next result is useful in studying limits of sequences of distributions.

**Theorem 3.2.6. Helly's selection theorem.** *For every sequence $F_n$ of distribution functions, there is a subsequence $F_{n(k)}$ and a right continuous nondecreasing function $F$ so that* $\lim_{k\to\infty} F_{n(k)}(y) = F(y)$ *at all continuity points $y$ of $F$.*

**Remark.** The limit may not be a distribution function. For example, if $a + b + c = 1$ and $F_n(x) = a\ 1_{(x\ge n)} + b\ 1_{(x\ge -n)} + c\ G(x)$ where $G$ is a distribution function, then $F_n(x) \to F(x) = b + cG(x)$,

$$\lim_{x\downarrow -\infty} F(x) = b \quad \text{and} \quad \lim_{x\uparrow\infty} F(x) = b + c = 1 - a$$

In words, an amount of mass $a$ escapes to $+\infty$, and mass $b$ escapes to $-\infty$. The type of convergence that occurs in Theorem 3.2.6 is sometimes called **vague convergence**, and will be denoted here by $\Rightarrow_v$.

*Proof.* The first step is a diagonal argument. Let $q_1, q_2, \ldots$ be an enumeration of the rationals. Since for each $k$, $F_m(q_k) \in [0, 1]$ for all $m$, there is a sequence $m_k(i) \to \infty$ that is a subsequence of $m_{k-1}(j)$ (let $m_0(j) \equiv j$) so that

$$F_{m_k(i)}(q_k) \text{ converges to } G(q_k) \text{ as } i \to \infty$$

Let $F_{n(k)} = F_{m_k(k)}$. By construction $F_{n(k)}(q) \to G(q)$ for all rational $q$. The function $G$ may not be right continuous, but $F(x) = \inf\{G(q) : q \in \mathbf{Q}, q > x\}$ is, since

$$\lim_{x_n\downarrow x} F(x_n) = \inf\{G(q) : q \in \mathbf{Q}, q > x_n \text{ for some } n\}$$

$$= \inf\{G(q) : q \in \mathbf{Q}, q > x\} = F(x)$$

To complete the proof, let $x$ be a continuity point of $F$. Pick rationals $r_1, r_2, s$ with $r_1 < r_2 < x < s$ so that

$$F(x) - \epsilon < F(r_1) \le F(r_2) \le F(x) \le F(s) < F(x) + \epsilon$$

Since $F_{n(k)}(r_2) \to G(r_2) \ge F(r_1)$, and $F_{n(k)}(s) \to G(s) \le F(s)$, it follows that if $k$ is large,

$$F(x) - \epsilon < F_{n(k)}(r_2) \le F_{n(k)}(x) \le F_{n(k)}(s) < F(x) + \epsilon$$

which is the desired conclusion. ∎

The last result raises a question: When can we conclude that no mass is lost in the limit in Theorem 3.2.6?

**Theorem 3.2.7.** *Every subsequential limit is the distribution function of a probability measure if and only if the sequence $F_n$ is* **tight**, *that is, for all $\epsilon > 0$ there is an $M_\epsilon$ so that*

$$\limsup_{n\to\infty} 1 - F_n(M_\epsilon) + F_n(-M_\epsilon) \le \epsilon$$

*Proof.* Suppose the sequence is tight and $F_{n(k)} \Rightarrow_v F$. Let $r < -M_\epsilon$ and $s > M_\epsilon$ be continuity points of $F$. Since $F_n(r) \to F(r)$ and $F_n(s) \to F(s)$, we have

$$\begin{aligned} 1 - F(s) + F(r) &= \lim_{k\to\infty} 1 - F_{n(k)}(s) + F_{n(k)}(r) \\ &\le \limsup_{n\to\infty} 1 - F_n(M_\epsilon) + F_n(-M_\epsilon) \le \epsilon \end{aligned}$$

The last result implies $\limsup_{x\to\infty} 1 - F(x) + F(-x) \le \epsilon$. Since $\epsilon$ is arbitrary, it follows that $F$ is the distribution function of a probability measure.

To prove the converse now suppose $F_n$ is not tight. In this case, there is an $\epsilon > 0$ and a subsequence $n(k) \to \infty$ so that

$$1 - F_{n(k)}(k) + F_{n(k)}(-k) \ge \epsilon$$

for all $k$. By passing to a further subsequence $F_{n(k_j)}$ we can suppose that $F_{n(k_j)} \Rightarrow_v F$. Let $r < 0 < s$ be continuity points of $F$.

$$\begin{aligned} 1 - F(s) + F(r) &= \lim_{j\to\infty} 1 - F_{n(k_j)}(s) + F_{n(k_j)}(r) \\ &\ge \liminf_{j\to\infty} 1 - F_{n(k_j)}(k_j) + F_{n(k_j)}(-k_j) \ge \epsilon \end{aligned}$$

Letting $s \to \infty$ and $r \to -\infty$, we see that $F$ is not the distribution function of a probability measure. ■

The following sufficient condition for tightness is often useful.

**Theorem 3.2.8.** *If there is a $\varphi \ge 0$ so that $\varphi(x) \to \infty$ as $|x| \to \infty$ and*

$$C = \sup_n \int \varphi(x) dF_n(x) < \infty$$

*then $F_n$ is tight.*

*Proof.* $1 - F_n(M) + F_n(-M) \le C / \inf_{|x|\ge M} \varphi(x)$. ■

The first two exercises below define metrics for convergence in distriubtion. The fact that convergence in distribution comes from a metric immediately implies

**Theorem 3.2.9.** *If each subsequence of $X_n$ has a further subsequence that converges to $X$, then $X_n \Rightarrow X$.*

We will prove this again at the end of the proof of Theorem 3.3.6.

### *Exercises*

**3.2.6. The Lévy Metric.** Show that

$$\rho(F, G) = \inf\{\epsilon : F(x - \epsilon) - \epsilon \le G(x) \le F(x + \epsilon) + \epsilon \text{ for all } x\}$$

defines a metric on the space of distributions and $\rho(F_n, F) \to 0$ if and only if $F_n \Rightarrow F$.

**3.2.7. The Ky Fan metric** on random variables is defined by

$$\alpha(X, Y) = \inf\{\epsilon \ge 0 : P(|X - Y| > \epsilon) \le \epsilon\}$$

Show that if $\alpha(X, Y) = \alpha$, then the corresponding distributions have Lévy distance $\rho(F, G) \le \alpha$.

**3.2.8.** Let $\alpha(X, Y)$ be the metric in the previous exercise and let $\beta(X, Y) = E(|X - Y|/(1 + |X - Y|))$ be the metric of Exercise 2.3.8. If $\alpha(X, Y) = a$, then

$$a^2/(1 + a) \le \beta(X, Y) \le a + (1 - a)a/(1 + a)$$

**3.2.9.** If $F_n \Rightarrow F$ and $F$ is continuous, then $\sup_x |F_n(x) - F(x)| \to 0$.

**3.2.10.** If $F$ is any distribution function, there is a sequence of distribution functions of the form $\sum_{m=1}^n a_{n,m} 1_{(x_{n,m} \le x)}$ with $F_n \Rightarrow F$. Hint: Use Theorem 2.4.7.

**3.2.11.** Let $X_n$, $1 \le n \le \infty$, be integer valued. Show that $X_n \Rightarrow X_\infty$ if and only if $P(X_n = m) \to P(X_\infty = m)$ for all $m$.

**3.2.12.** Show that if $X_n \to X$ in probability then $X_n \Rightarrow X$, and that, conversely, if $X_n \Rightarrow c$, where $c$ is a constant, then $X_n \to c$ in probability.

**3.2.13. Converging together lemma.** If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where $c$ is a constant, then $X_n + Y_n \Rightarrow X + c$. A useful consequence of this result is that if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$, then $Z_n \Rightarrow X$.

**3.2.14.** Suppose $X_n \Rightarrow X$, $Y_n \ge 0$, and $Y_n \Rightarrow c$, where $c > 0$ is a constant. Then $X_n Y_n \Rightarrow cX$. This result is true without the assumptions $Y_n \ge 0$ and $c > 0$. We have imposed these only to make the proof less tedious.

**3.2.15.** Show that if $X_n = (X_n^1, \dots, X_n^n)$ is uniformly distributed over the surface of the sphere of radius $\sqrt{n}$ in $\mathbf{R}^n$ then $X_n^1 \Rightarrow$ a standard normal. Hint: Let $Y_1, Y_2, \dots$ be i.i.d. standard normals and let $X_n^i = Y_i(n/\sum_{m=1}^n Y_m^2)^{1/2}$.

**3.2.16.** Suppose $Y_n \ge 0$, $EY_n^\alpha \to 1$ and $EY_n^\beta \to 1$ for some $0 < \alpha < \beta$. Show that $Y_n \to 1$ in probability.

**3.2.17.** For each $K < \infty$ and $y < 1$ there is a $c_{y,K} > 0$ so that $EX^2 = 1$ and $EX^4 \le K$ implies $P(|X| > y) \ge c_{y,K}$.

## 3.3 Characteristic Functions

This long section is divided into five parts. The first three are required reading, the last two are optional. In the first part, we show that the characteristic function $\varphi(t) = E\exp(itX)$ determines $F(x) = P(X \le x)$, and we give recipes for computing $F$ from $\varphi$. In the second part, we relate weak convergence of distributions to the behavior of the corresponding characteristic functions. In the third part, we relate the behavior of $\varphi(t)$ at 0 to the moments of $X$. In the fourth part, we prove Polya's criterion and use it to construct some famous and some strange examples of characteristic functions. Finally, in the fifth part, we consider the moment problem, that is, when is a distribution characterized by its moments.

### 3.3.1 *Definition, Inversion Formula*

If $X$ is a random variable we define its **characteristic function (ch.f.)** by

$$\varphi(t) = Ee^{itX} = E\cos tX + iE\sin tX$$

The last formula requires taking the expected value of a complex-valued random variable, but as the second equality may suggest, no new theory is required. If $Z$ is complex valued, we define $EZ = E(\operatorname{Re} Z) + iE(\operatorname{Im} Z)$ where $\operatorname{Re}(a + bi) = a$ is the **real part** and $\operatorname{Im}(a + bi) = b$ is the **imaginary part**. Some other definitions we will need are: the **modulus** of the complex number $z = a + bi$ is $|a + bi| = (a^2 + b^2)^{1/2}$, and the **complex conjugate** of $z = a + bi$, $\bar{z} = a - bi$.

**Theorem 3.3.1.** *All characteristic functions have the following properties:*
*(a)* $\varphi(0) = 1$,
*(b)* $\varphi(-t) = \overline{\varphi(t)}$,
*(c)* $|\varphi(t)| = |Ee^{itX}| \le E|e^{itX}| = 1$
*(d)* $|\varphi(t + h) - \varphi(t)| \le E|e^{ihX} - 1|$, *so* $\varphi(t)$ *is uniformly continuous on* $(-\infty, \infty)$.
*(e)* $Ee^{it(aX+b)} = e^{itb}\varphi(at)$

*Proof.* (a) is obvious. For (b) we note that

$$\varphi(-t) = E(\cos(-tX) + i\sin(-tX)) = E(\cos(tX) - i\sin(tX))$$

(c) follows from Exercise 1.6.2 since $\varphi(x, y) = (x^2 + y^2)^{1/2}$ is convex.

$$\begin{aligned}|\varphi(t + h) - \varphi(t)| &= |E(e^{i(t+h)X} - e^{itX})| \\ &\le E|e^{i(t+h)X} - e^{itX}| = E|e^{ihX} - 1|\end{aligned}$$

so uniform convergence follows from the bounded convergence theorem. For (e) we note $Ee^{it(aX+b)} = e^{itb}Ee^{i(ta)X} = e^{itb}\varphi(at)$. ∎

The main reason for introducing charactersitic functions is the following:

**Theorem 3.3.2.** *If $X_1$ and $X_2$ are independent and have ch.f.'s $\varphi_1$ and $\varphi_2$, then $X_1 + X_2$ has ch.f. $\varphi_1(t)\varphi_2(t)$.*

*Proof.*

$$Ee^{it(X_1+X_2)} = E(e^{itX_1}e^{itX_2}) = Ee^{itX_1}Ee^{itX_2}$$

since $e^{itX_1}$ and $e^{itX_2}$ are independent. ∎

The next order of business is to give some examples.

**Example 3.3.1. Coin flips.** If $P(X = 1) = P(X = -1) = 1/2$, then

$$Ee^{itX} = (e^{it} + e^{-it})/2 = \cos t$$

**Example 3.3.2. Poisson distribution.** If $P(X = k) = e^{-\lambda}\lambda^k/k!$ for $k = 0, 1, 2, \dots,$ then

$$Ee^{itX} = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k e^{itk}}{k!} = \exp(\lambda(e^{it} - 1))$$

**Example 3.3.3. Normal distribution**

| | |
|---|---|
| Density | $(2\pi)^{-1/2}\exp(-x^2/2)$ |
| Ch.f. | $\exp(-t^2/2)$ |

Combining this result with (e) of Theorem 3.3.1, we see that a normal distribution with mean $\mu$ and variance $\sigma^2$ has ch.f. $\exp(i\mu t - \sigma^2 t^2/2)$. Similar scalings can be applied to other examples, so we will often just give the ch.f. for one member of the family.

*Physics Proof.*

$$\int e^{itx}(2\pi)^{-1/2}e^{-x^2/2}\,dx = e^{-t^2/2}\int (2\pi)^{-1/2}e^{-(x-it)^2/2}\,dx$$

The integral is 1 since the integrand is the normal density with mean $it$ and variance 1. ∎

*Math Proof.* Now that we have cheated and figured out the answer, we can verify it by a formal calculation that gives very little insight into why it is true. Let

$$\varphi(t) = \int e^{itx}(2\pi)^{-1/2}e^{-x^2/2}dx = \int \cos tx\,(2\pi)^{-1/2}e^{-x^2/2}dx$$

since $i \sin tx$ is an odd function. Differentiating with respect to $t$ (referring to Theorem A.5.1 for the justification) and then integrating by parts gives

$$\begin{aligned}\varphi'(t) &= \int -x \sin tx \,(2\pi)^{-1/2} e^{-x^2/2} dx \\ &= -\int t \cos tx \,(2\pi)^{-1/2} e^{-x^2/2} dx = -t\varphi(t)\end{aligned}$$

This implies $\frac{d}{dt}\{\varphi(t)\exp(t^2/2)\} = 0$, so $\varphi(t)\exp(t^2/2) = \varphi(0) = 1$. ■

In the next three examples, the density is 0 outside the indicated range.

**Example 3.3.4. Uniform distribution on** $(a, b)$

| | | |
|---|---|---|
| Density | $1/(b-a)$ | $x \in (a, b)$ |
| Ch.f. | $(e^{itb} - e^{ita})/\, it(b-a)$ | |

In the special case $a = -c$, $b = c$, the ch.f. is $(e^{itc} - e^{-itc})/2cit = (\sin ct)/ct$.

*Proof.* Once you recall that $\int_a^b e^{\lambda x}\, dx = (e^{\lambda b} - e^{\lambda a})/\lambda$ holds for complex $\lambda$, this is immediate. ■

**Example 3.3.5. Triangular distribution**

| | | |
|---|---|---|
| Density | $1 - \vert x\vert$ | $x \in (-1, 1)$ |
| Ch.f. | $2(1 - \cos t)/t^2$ | |

*Proof.* To see this, notice that if $X$ and $Y$ are independent and uniform on $(-1/2, 1/2)$, then $X + Y$ has a triangular distribution. Using Example 3.3.4 now and Theorem 3.3.2, it follows that the desired ch.f. is

$$\{(e^{it/2} - e^{-it/2})/it\}^2 = \{2\sin(t/2)/t\}^2$$

Using the trig identity $\cos 2\theta = 1 - 2\sin^2\theta$ with $\theta = t/2$ converts the answer into the form given above. ■

**Example 3.3.6. Exponential distribution**

| | | |
|---|---|---|
| Density | $e^{-x}$ | $x \in (0, \infty)$ |
| Ch.f. | $1/(1 - it)$ | |

*Proof.* Integrating gives

$$\int_0^\infty e^{itx} e^{-x} dx = \left.\frac{e^{(it-1)x}}{it-1}\right|_0^\infty = \frac{1}{1-it}$$

since $\exp((it-1)x) \to 0$ as $x \to \infty$. ■

For the next result we need the following fact, which follows from the fact that $\int f\, d(\mu + \nu) = \int f\, d\mu + \int f\, d\nu$.

**Lemma 3.3.3.** *If $F_1, \ldots, F_n$ have ch.f. $\varphi_1, \ldots, \varphi_n$ and $\lambda_i \geq 0$ have $\lambda_1 + \ldots + \lambda_n = 1$, then $\sum_{i=1}^n \lambda_i F_i$ has ch.f. $\sum_{i=1}^n \lambda_i \varphi_i$.*

**Example 3.3.7. Bilateral exponential**

| | | |
|---|---|---|
| Density | $\frac{1}{2}e^{-\lvert x\rvert}$ | $x \in (-\infty, \infty)$ |
| Ch.f. | $1/(1+t^2)$ | |

*Proof.* This follows from Lemma 3.3.3 with $F_1$ the distribution of an exponential random variable $X$, $F_2$ the distribution of $-X$, and $\lambda_1 = \lambda_2 = 1/2$. Then using (b) of Theorem 3.3.1 we see the desired ch.f. is

$$\frac{1}{2(1-it)} + \frac{1}{2(1+it)} = \frac{(1+it)+(1-it)}{2(1+t^2)} = \frac{1}{(1+t^2)} \qquad \blacksquare$$

**Exercise 3.3.1.** Show that if $\varphi$ is a ch.f., then $\operatorname{Re}\varphi$ and $|\varphi|^2$ are also.

The first issue to be settled is that the characteristic function uniquely determines the distribution. This and more is provided by

**Theorem 3.3.4. The inversion formula.** *Let $\varphi(t) = \int e^{itx}\mu(dx)$ where $\mu$ is a probability measure. If $a < b$, then*

$$\lim_{T\to\infty} (2\pi)^{-1} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

**Remark.** The existence of the limit is part of the conclusion. If $\mu = \delta_0$, a point mass at 0, $\varphi(t) \equiv 1$. In this case, if $a = -1$ and $b = 1$, the integrand is $(2\sin t)/t$ and the integral does not converge absolutely.

*Proof.* Let

$$I_T = \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \int_{-T}^{T} \int \frac{e^{-ita} - e^{-itb}}{it} e^{itx}\mu(dx)\, dt$$

The integrand may look bad near $t = 0$, but if we observe that

$$\frac{e^{-ita} - e^{-itb}}{it} = \int_a^b e^{-ity}\, dy$$

we see that the modulus of the integrand is bounded by $b - a$. Since $\mu$ is a probability measure and $[-T, T]$ is a finite interval, it follows from Fubini's theorem, $\cos(-x) = \cos x$, and $\sin(-x) = -\sin x$ that

$$\begin{aligned} I_T &= \int \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} e^{itx}\, dt\, \mu(dx) \\ &= \int \left\{ \int_{-T}^{T} \frac{\sin(t(x-a))}{t}\, dt - \int_{-T}^{T} \frac{\sin(t(x-b))}{t}\, dt \right\} \mu(dx) \end{aligned}$$

Introducing $R(\theta, T) = \int_{-T}^{T} (\sin \theta t)/t \, dt$, we can write the last result as

$$(*) \qquad I_T = \int \{R(x-a, T) - R(x-b, T)\}\mu(dx)$$

If we let $S(T) = \int_0^T (\sin x)/x \, dx$, then for $\theta > 0$ changing variables $t = x/\theta$ shows that

$$R(\theta, T) = 2\int_0^{T\theta} \frac{\sin x}{x} \, dx = 2S(T\theta)$$

while for $\theta < 0$, $R(\theta, T) = -R(|\theta|, T)$. Introducing the function $\operatorname{sgn} x$, which is 1 if $x > 0$, $-1$ if $x < 0$, and 0 if $x = 0$, we can write the last two formulas together as

$$R(\theta, T) = 2(\operatorname{sgn} \theta)S(T|\theta|)$$

As $T \to \infty$, $S(T) \to \pi/2$ (see Exercise 1.7.5), so we have $R(\theta, T) \to \pi \operatorname{sgn} \theta$ and

$$R(x-a, T) - R(x-b, T) \to \begin{cases} 2\pi & a < x < b \\ \pi & x = a \text{ or } x = b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$|R(\theta, T)| \le 2 \sup_y S(y) < \infty$, so using the bounded convergence theorem with $(*)$ implies

$$(2\pi)^{-1} I_T \to \mu(a, b) + \frac{1}{2}\mu(\{a, b\})$$

proving the desired result. ∎

**Exercise 3.3.2.** (i) Imitate the proof of Theorem 3.3.4 to show that

$$\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t) \, dt$$

(ii) If $P(X \in h\mathbf{Z}) = 1$ where $h > 0$, then its ch.f. has $\varphi(2\pi/h + t) = \varphi(t)$, so

$$P(X = x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx}\varphi(t) \, dt \quad \text{for } x \in h\mathbf{Z}$$

(iii) If $X = Y + b$, then $E\exp(itX) = e^{itb}E\exp(itY)$. So if $P(X \in b + h\mathbf{Z}) = 1$, the inversion formula in (ii) is valid for $x \in b + h\mathbf{Z}$.

Two trivial consequences of the inversion formula are:

**Exercise 3.3.3.** If $\varphi$ is real then $X$ and $-X$ have the same distribution.

**Exercise 3.3.4.** If $X_i$, $i = 1, 2$ are independent and have normal distributions with mean 0 and variance $\sigma_i^2$, then $X_1 + X_2$ has a normal distribution with mean 0 and variance $\sigma_1^2 + \sigma_2^2$.

The inversion formula is simpler when $\varphi$ is integrable, but as the next result shows, this only happens when the underlying measure is nice.

**Theorem 3.3.5.** *If $\int |\varphi(t)|\, dt < \infty$, then $\mu$ has bounded continuous density*

$$f(y) = \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt$$

*Proof.* As we observed in the proof of Theorem 3.3.4

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-ity}\, dy \right| \le |b - a|$$

so the integral in Theorem 3.3.4 converges absolutely in this case and

$$\mu(a, b) + \frac{1}{2}\mu(\{a, b\}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt \le \frac{(b-a)}{2\pi} \int_{-\infty}^{\infty} |\varphi(t)| dt$$

The last result implies $\mu$ has no point masses and

$$\begin{aligned} \mu(x, x+h) &= \frac{1}{2\pi} \int \frac{e^{-itx} - e^{-it(x+h)}}{it} \varphi(t)\, dt \\ &= \frac{1}{2\pi} \int \left( \int_x^{x+h} e^{-ity}\, dy \right) \varphi(t)\, dt \\ &= \int_x^{x+h} \left( \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt \right) dy \end{aligned}$$

by Fubini's theorem, so the distribution $\mu$ has density function

$$f(y) = \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt$$

The dominated convergence theorem implies $f$ is continuous, and the proof is complete. ■

**Exercise 3.3.5.** Give an example of a measure $\mu$ with a density but for which $\int |\varphi(t)| dt = \infty$. Hint: Two of the examples above have this property.

**Exercise 3.3.6.** Show that if $X_1, \ldots, X_n$ are independent and uniformly distributed on $(-1, 1)$, then for $n \ge 2$, $X_1 + \cdots + X_n$ has density

$$f(x) = \frac{1}{\pi} \int_0^{\infty} (\sin t/t)^n \cos tx\, dt$$

Although it is not obvious from the formula, $f$ is a polynomial in each interval $(k, k+1)$, $k \in \mathbf{Z}$ and vanishes on $[-n, n]^c$.

Theorem 3.3.5 and the next result show that the behavior of $\varphi$ at infinity is related to the smoothness of the underlying measure.

**Exercise 3.3.7.** Suppose $X$ and $Y$ are independent and have ch.f. $\varphi$ and distribution $\mu$. Apply Exercise 3.3.2 to $X - Y$ and use Exercise 2.1.8 to get

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\, dt = P(X - Y = 0) = \sum_x \mu(\{x\})^2$$

**Remark.** The last result implies that if $\varphi(t) \to 0$ as $t \to \infty$, $\mu$ has no point masses. Exercise 3.3.13 gives an example to show that the converse is false. The Riemann-Lebesgue lemma (Exercise 1.4.4) shows that if $\mu$ has a density, $\varphi(t) \to 0$ as $t \to \infty$.

Applying the inversion formula Theorem 3.3.5 to the ch.f. in Examples 3.3.5 and 3.3.7 gives us two more examples of ch.f. The first one does not have an official name, so we gave it one to honor its role in the proof of Polya's criterion; see Theorem 3.3.10.

**Example 3.3.8. Polya's distribution**

| | |
|---|---|
| Density | $(1-\cos x)/\pi x^2$ |
| Ch.f. | $(1-|t|)^+$ |

*Proof.* Theorem 3.3.5 implies

$$\frac{1}{2\pi}\int \frac{2(1-\cos s)}{s^2} e^{-isy}\, ds = (1-|y|)^+$$

Now let $s = x$, $y = -t$. ∎

**Example 3.3.9. The Cauchy distribution**

| | |
|---|---|
| Density | $1/\pi(1+x^2)$ |
| Ch.f. | $\exp(-|t|)$ |

*Proof.* Theorem 3.3.5 implies

$$\frac{1}{2\pi}\int \frac{1}{1+s^2} e^{-isy}\, ds = \frac{1}{2} e^{-|y|}$$

Now let $s = x$, $y = -t$ and multiply each side by 2. ∎

**Exercise 3.3.8.** Use the last result to conclude that if $X_1, X_2, \ldots$ are independent and have the Cauchy distribution, then $(X_1 + \cdots + X_n)/n$ has the same distribution as $X_1$.

### 3.3.2 *Weak Convergence*

Our next step toward the central limit theorem is to relate convergence of characteristic functions to weak convergence.

**Theorem 3.3.6. Continuity theorem.** *Let $\mu_n$, $1 \le n \le \infty$ be probability measures with ch.f. $\varphi_n$. (i) If $\mu_n \Rightarrow \mu_\infty$ then $\varphi_n(t) \to \varphi_\infty(t)$ for all $t$. (ii) If $\varphi_n(t)$ converges pointwise to a limit $\varphi(t)$ that is continuous at 0, then the associated sequence of distributions $\mu_n$ is tight and converges weakly to the measure $\mu$ with characteristic function $\varphi$.*

**Remark.** To see why continuity of the limit at 0 is needed in (ii), let $\mu_n$ have a normal distribution with mean 0 and variance $n$. In this case $\varphi_n(t) = \exp(-nt^2/2) \to 0$ for $t \ne 0$, and $\varphi_n(0) = 1$ for all $n$, but the measures do not converge weakly since $\mu_n((-\infty, x]) \to 1/2$ for all $x$.

*Proof.* (i) is easy. $e^{itx}$ is bounded and continuous, so if $\mu_n \Rightarrow \mu_\infty$, then Theorem 3.2.3 implies $\varphi_n(t) \to \varphi_\infty(t)$. To prove (ii), our first goal is to prove tightness. We begin with some calculations that may look mysterious but will prove to be very useful.

$$\int_{-u}^{u} 1 - e^{itx}\,dt = 2u - \int_{-u}^{u} (\cos tx + i\sin tx)\,dt = 2u - \frac{2\sin ux}{x}$$

Dividing both sides by $u$, integrating $\mu_n(dx)$, and using Fubini's theorem on the left-hand side gives

$$u^{-1}\int_{-u}^{u} (1 - \varphi_n(t))\,dt = 2\int \left(1 - \frac{\sin ux}{ux}\right)\mu_n(dx)$$

To bound the right-hand side, we note that

$$|\sin x| = \left|\int_0^x \cos(y)\,dy\right| \le |x| \quad \text{for all } x$$

so we have $1 - (\sin ux/ux) \ge 0$. Discarding the integral over $(-2/u, 2/u)$ and using $|\sin ux| \le 1$ on the rest, the right-hand side is

$$\ge 2\int_{|x|\ge 2/u} \left(1 - \frac{1}{|ux|}\right)\mu_n(dx) \ge \mu_n(\{x : |x| > 2/u\})$$

Since $\varphi(t) \to 1$ as $t \to 0$,

$$u^{-1}\int_{-u}^{u} (1 - \varphi(t))\,dt \to 0 \text{ as } u \to 0$$

Pick $u$ so that the integral is $< \epsilon$. Since $\varphi_n(t) \to \varphi(t)$ for each $t$, it follows from the bounded convergence theorem that for $n \ge N$

$$2\epsilon \ge u^{-1}\int_{-u}^{u} (1 - \varphi_n(t))\,dt \ge \mu_n\{x : |x| > 2/u\}$$

Since $\epsilon$ is arbitrary, the sequence $\mu_n$ is tight.

To complete the proof now we observe that if $\mu_{n(k)} \Rightarrow \mu$, then it follows from the first sentence of the proof that $\mu$ has ch.f. $\varphi$. The last observation and tightness imply that every subsequence has a further subsequence that converges to $\mu$. I claim that this implies the whole sequence converges to $\mu$. To see this, observe that we have shown that if $f$ is bounded and continuous then every subsequence of $\int f\,d\mu_n$ has a further subsequence that converges to $\int f\,d\mu$, so Theorem 2.3.3 implies that the whole sequence converges to that limit. This shows $\int f\,d\mu_n \to \int f\,d\mu$ for all bounded continuous functions $f$, so the desired result follows from Theorem 3.2.3. ■

**Exercise 3.3.9.** Suppose that $X_n \Rightarrow X$ and $X_n$ has a normal distribution with mean 0 and variance $\sigma_n^2$. Prove that $\sigma_n^2 \to \sigma^2 \in [0, \infty)$.

**Exercise 3.3.10.** Show that if $X_n$ and $Y_n$ are independent for $1 \le n \le \infty$, $X_n \Rightarrow X_\infty$, and $Y_n \Rightarrow Y_\infty$, then $X_n + Y_n \Rightarrow X_\infty + Y_\infty$.

**Exercise 3.3.11.** Let $X_1, X_2, \ldots$ be independent and let $S_n = X_1 + \cdots + X_n$. Let $\varphi_j$ be the ch.f. of $X_j$ and suppose that $S_n \to S_\infty$ a.s. Then $S_\infty$ has ch.f. $\prod_{j=1}^\infty \varphi_j(t)$.

**Exercise 3.3.12.** Using the identity $\sin t = 2\sin(t/2)\cos(t/2)$ repeatedly leads to $(\sin t)/t = \prod_{m=1}^\infty \cos(t/2^m)$. Prove the last identity by interpreting each side as a characteristic function.

**Exercise 3.3.13.** Let $X_1, X_2, \ldots$ be independent taking values 0 and 1 with probability 1/2 each. $X = 2\sum_{j\ge1} X_j/3^j$ has the Cantor distribution. Compute the ch.f. $\varphi$ of $X$ and notice that $\varphi$ has the same value at $t = 3^k\pi$ for $k = 0, 1, 2, \ldots$

### 3.3.3 *Moments and Derivatives*

In the proof of Theorem 3.3.6, we derived the inequality

$$\mu\{x : |x| > 2/u\} \le u^{-1}\int_{-u}^{u} (1 - \varphi(t))\,dt \tag{3.3.1}$$

which shows that the smoothness of the characteristic function at 0 is related to the decay of the measure at $\infty$. The next result continues this theme. We leave the proof to the reader. (Use Theorem A.5.1.)

**Exercise 3.3.14.** If $\int |x|^n \mu(dx) < \infty$, then its characteristic function $\varphi$ has a continuous derivative of order $n$ given by $\varphi^{(n)}(t) = \int (ix)^n e^{itx}\mu(dx)$.

**Exercise 3.3.15.** Use the last exercise and the series expansion for $e^{-t^2/2}$ to show that the standard normal distribution has

$$EX^{2n} = (2n)!/2^n n! = (2n-1)(2n-3)\cdots 3\cdot 1 \equiv (2n-1)!!$$

The result in Exercise 3.3.14 shows that if $E|X|^n < \infty$, then its characteristic function is $n$ times differentiable at 0, and $\varphi^n(0) = E(iX)^n$. Expanding $\varphi$ in a

Taylor series about 0 leads to

$$\varphi(t) = \sum_{m=0}^{n} \frac{E(itX)^m}{m!} + o(t^n)$$

where $o(t^n)$ indicates a quantity $g(t)$ that has $g(t)/t^n \to 0$ as $t \to 0$. For our purposes below, it will be important to have a good estimate on the error term, so we will now derive the last result. The starting point is a little calculus.

**Lemma 3.3.7.**

$$\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \le \min\left( \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right) \tag{3.3.2}$$

The first term on the right is the usual order of magnitude we expect in the correction term. The second is better for large $|x|$ and will help us prove the central limit theorem without assuming finite third moments.

*Proof.* Integrating by parts gives

$$\int_0^x (x-s)^n e^{is}\, ds = \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is}\, ds$$

When $n = 0$, this says

$$\int_0^x e^{is}\, ds = x + i \int_0^x (x-s) e^{is}\, ds$$

The left-hand side is $(e^{ix} - 1)/i$, so rearranging gives

$$e^{ix} = 1 + ix + i^2 \int_0^x (x-s) e^{is} ds$$

Using the result for $n = 1$ now gives

$$e^{ix} = 1 + ix + \frac{i^2 x^2}{2} + \frac{i^3}{2} \int_0^x (x-s)^2 e^{is} ds$$

and iterating we arrive at

(a) $$e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} = \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds$$

To prove the result now it only remains to estimate the "error term" on the right-hand side. Since $|e^{is}| \le 1$ for all $s$,

(b) $$\left| \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds \right| \le |x|^{n+1}/(n+1)!$$

The last estimate is good when $x$ is small. The next is designed for large $x$. Integrating by parts

$$\frac{i}{n}\int_0^x (x-s)^n e^{is}\,ds = -\frac{x^n}{n} + \int_0^x (x-s)^{n-1} e^{is}\,ds$$

Noticing $x^n/n = \int_0^x (x-s)^{n-1}ds$ now gives

$$\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}\,ds = \frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)\,ds$$

and since $|e^{ix}-1| \le 2$, it follows that

(c) $$\left|\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}\,ds\right| \le \left|\frac{2}{(n-1)!}\int_0^x (x-s)^{n-1}\,ds\right| \le 2|x|^n/n!$$

Combining (a), (b), and (c) we have the desired result. ■

Taking expected values, using Jensen's inequality, applying Theorem 3.3.2 to $x = tX$, gives

$$\left|Ee^{itX} - \sum_{m=0}^n E\frac{(itX)^m}{m!}\right| \le E\left|e^{itX} - \sum_{m=0}^n \frac{(itX)^m}{m!}\right| \le E\min\left(|tX|^{n+1}, 2|tX|^n\right) \tag{3.3.3}$$

where in the second step we have dropped the denominators to make the bound simpler.

In the next section, the following special case will be useful.

**Theorem 3.3.8.** *If $E|X|^2 < \infty$, then*

$$\varphi(t) = 1 + itEX - t^2E(X^2)/2 + o(t^2)$$

*Proof.* The error term is $\le t^2E(|t|\cdot|X|^3 \wedge 2|X|^2)$. The variable in parentheses is smaller than $2|X|^2$ and converges to 0 as $t \to 0$, so the desired conclusion follows from the dominated convergence theorem. ■

**Remark.** The point of the estimate in (3.3.3), which involves the minimum of two terms rather than just the first one which would result from a naive application of Taylor series, is that we get the conclusion in Theorem 3.3.8 under the assumption $E|X|^2 < \infty$, that is, we do not have to assume $E|X|^3 < \infty$.

**Exercise 3.3.16.** (i) Suppose that the family of measures $\{\mu_i, i \in I\}$ is tight, that is, $\sup_i \mu_i([-M,M]^c) \to 0$ as $M \to \infty$. Use (d) in Theorem 3.3.1 and (3.3.3) with $n = 0$ to show that their ch.f.'s $\varphi_i$ are equicontinuous, that is, if $\epsilon > 0$ we can pick $\delta > 0$ so that if $|h| < \delta$, then $|\varphi_i(t+h) - \varphi_i(t)| < \epsilon$. (ii) Suppose $\mu_n \Rightarrow \mu_\infty$. Use Theorem 3.3.6 and equicontinuity to conclude that the ch.f.'s $\varphi_n \to \varphi_\infty$ uniformly on compact sets. [Argue directly. You don't need to go to AA.] (iii) Give an example to show that the convergence need not be uniform on the whole real line.

**Exercise 3.3.17.** Let $X_1, X_2, \ldots$ be i.i.d. with characteristic function $\varphi$. (i) If $\varphi'(0) = ia$ and $S_n = X_1 + \cdots + X_n$ then $S_n/n \to a$ in probability. (ii) If $S_n/n \to a$ in probability then $\varphi(t/n)^n \to e^{iat}$ as $n \to \infty$ through the integers. (iii) Use (ii) and the uniform continuity established in (d) of Theorem 3.3.1 to show that $(\varphi(h) - 1)/h \to -ia$ as $h \to 0$ through the positive reals. Thus the weak law holds if and only if $\varphi'(0)$ exists. This result is due to E. J. G. Pitman (1956), with a little help from John Walsh, who pointed out that we should prove (iii).

The last exercise in combination with Exercise 2.2.4 shows that $\varphi'(0)$ may exist when $E|X| = \infty$.

**Exercise 3.3.18.** $2\int_0^\infty (1 - \operatorname{Re}\varphi(t))/(\pi t^2)\, dt = \int |y|\, dF(y)$. Hint: Change variables $x = |y|t$ in the density function of Example 3.3.8, which integrates to 1.

The next result shows that the existence of second derivatives implies the existence of second moments.

**Theorem 3.3.9.** *If* $\limsup_{h\downarrow 0}\{\varphi(h) - 2\varphi(0) + \varphi(-h)\}/h^2 > -\infty$, *then* $E|X|^2 < \infty$.

*Proof.* $(e^{ihx} - 2 + e^{-ihx})/h^2 = -2(1 - \cos hx)/h^2 \le 0$ and $2(1 - \cos hx)/h^2 \to x^2$ as $h \to 0$, so Fatou's lemma and Fubini's theorem imply

$$\begin{aligned}\int x^2\, dF(x) &\le 2 \liminf_{h\to 0} \int \frac{1 - \cos hx}{h^2}\, dF(x) \\ &= -\limsup_{h\to 0} \frac{\varphi(h) - 2\varphi(0) + \varphi(-h)}{h^2} < \infty\end{aligned}$$

which proves the desired result. ∎

**Exercise 3.3.19.** Show that if $\lim_{t\downarrow 0}(\varphi(t) - 1)/t^2 = c > -\infty$ then $EX = 0$ and $E|X|^2 = -2c < \infty$. In particular, if $\varphi(t) = 1 + o(t^2)$, then $\varphi(t) \equiv 1$.

**Exercise 3.3.20.** If $Y_n$ are r.v.'s with ch.f.'s $\varphi_n$, then $Y_n \Rightarrow 0$ if and only if there is a $\delta > 0$ so that $\varphi_n(t) \to 1$ for $|t| \le \delta$.

**Exercise 3.3.21.** Let $X_1, X_2, \ldots$ be independent. If $S_n = \sum_{m\le n} X_m$ converges in distribution, then it converges in probability (and hence a.s. by Exercise 2.5.10). Hint: The last exercise implies that if $m, n \to \infty$ then $S_m - S_n \to 0$ in probability. Now use Exercise 2.5.11.

### 3.3.4 *Polya's Criterion**

The next result is useful for constructing examples of ch.f.'s.

**Theorem 3.3.10. Polya's criterion.** *Let $\varphi(t)$ be real nonnegative and have $\varphi(0) = 1$, $\varphi(t) = \varphi(-t)$, and $\varphi$ is decreasing and convex on $(0, \infty)$ with*

$$\lim_{t\downarrow 0} \varphi(t) = 1, \qquad \lim_{t\uparrow\infty} \varphi(t) = 0$$

*Then there is a probability measure $\nu$ on $(0, \infty)$, so that*

$$(*) \qquad \varphi(t) = \int_0^\infty \left(1 - \left|\frac{t}{s}\right|\right)^+ \nu(ds)$$

*and hence $\varphi$ is a characteristic function.*

**Remark.** Before we get lost in the details of the proof, the reader should note that $(*)$ displays $\varphi$ as a convex combination of ch.f.'s of the form given in Example 3.3.8, so an extension of Lemma 3.3.3 (to be proved below) implies that this is a ch.f.

The assumption that $\lim_{t\to 0}\varphi(t) = 1$ is necessary because the function $\varphi(t) = 1_{\{0\}}(t)$ which is 1 at 0 and 0 otherwise, satisfies all the other hypotheses. We could allow $\lim_{t\to\infty}\varphi(t) = c > 0$ by having a point mass of size $c$ at 0, but we leave this extension to the reader.

*Proof.* Let $\varphi'$ be the right derivative of $\phi$, that is,

$$\varphi'(t) = \lim_{h\downarrow 0} \frac{\varphi(t+h) - \varphi(t)}{h}$$

Since $\varphi$ is convex, this exists and is right continuous and increasing. So we can let $\mu$ be the measure on $(0, \infty)$ with $\mu(a, b] = \varphi'(b) - \varphi'(a)$ for all $0 \le a < b < \infty$, and let $\nu$ be the measure on $(0, \infty)$ with $d\nu/d\mu = s$.

Now $\varphi'(t) \to 0$ as $t \to \infty$ (for if $\varphi'(t) \downarrow -\epsilon$ we would have $\varphi(t) \le 1 - \epsilon t$ for all $t$), so Exercise A.4.7 implies

$$-\varphi'(s) = \int_s^\infty r^{-1}\nu(dr)$$

Integrating again and using Fubini's theorem we have for $t \ge 0$

$$\begin{aligned}\varphi(t) &= \int_t^\infty \int_s^\infty r^{-1}\nu(dr)\, ds = \int_t^\infty r^{-1}\int_t^r ds\, \nu(dr) \\ &= \int_t^\infty \left(1 - \frac{t}{r}\right)\nu(dr) = \int_0^\infty \left(1 - \frac{t}{r}\right)^+ \nu(dr)\end{aligned}$$

Using $\varphi(-t) = \varphi(t)$ to extend the formula to $t \le 0$, we have $(*)$. Setting $t = 0$ in $(*)$ shows $\nu$ has total mass 1.

If $\varphi$ is piecewise linear, $\nu$ has a finite number of atoms, and the result follows from Example 3.3.8 and Lemma 3.3.3. To prove the general result, let $\nu_n$ be a sequence of measures on $(0, \infty)$ with a finite number of atoms that converges weakly to $\nu$ (see Exercise 3.2.10) and let

$$\varphi_n(t) = \int_0^\infty \left(1 - \left|\frac{t}{s}\right|\right)^+ \nu_n(ds)$$

Since $s \to (1 - |t/s|)^+$ is bounded and continuous, $\varphi_n(t) \to \varphi(t)$, and the desired result follows from part (ii) of Theorem 3.3.6. ■

A classic application of Polya's criterion is:

**Exercise 3.3.22.** Show that $\exp(-|t|^\alpha)$ is a characteristic function for $0 < \alpha \le 1$.

(The case $\alpha = 1$ corresponds to the Cauchy distribution.) The next argument, which we learned from Frank Spitzer, proves that this is true for $0 < \alpha \le 2$. The case $\alpha = 2$ corresponds to a normal distribution, so that case can be safely ignored in the proof.

**Example 3.3.10.** $\exp(-|t|^\alpha)$ is a characteristic function for $0 < \alpha < 2$.

*Proof.* A little calculus shows that for any $\beta$ and $|x| < 1$

$$(1 - x)^\beta = \sum_{n=0}^\infty \binom{\beta}{n} (-x)^n$$

where

$$\binom{\beta}{n} = \frac{\beta(\beta - 1)\cdots(\beta - n + 1)}{1 \cdot 2 \cdots n}$$

Let $\psi(t) = 1 - (1 - \cos t)^{\alpha/2} = \sum_{n=1}^\infty c_n (\cos t)^n$ where

$$c_n = \binom{\alpha/2}{n} (-1)^{n+1}$$

$c_n \ge 0$ (here we use $\alpha < 2$), and $\sum_{n=1}^\infty c_n = 1$ (take $t = 0$ in the definition of $\psi$). $\cos t$ is a characteristic function (see Example 3.3.1), so an easy extension of Lemma 3.3.3 shows that $\psi$ is a ch.f. We have $1 - \cos t \sim t^2/2$ as $t \to 0$, so

$$1 - \cos(t \cdot 2^{1/2} \cdot n^{-1/\alpha}) \sim n^{-2/\alpha} t^2$$

Using Lemma 3.1.1 and (ii) of Theorem 3.3.6 now, it follows that

$$\exp(-|t|^\alpha) = \lim_{n \to \infty} \{\psi(t \cdot 2^{1/2} \cdot n^{-1/\alpha})\}^n$$

is a ch.f. ■

Exercise 3.3.19 shows that $\exp(-|t|^\alpha)$ is not a ch.f. when $\alpha > 2$. A reason for interest in these characteristic functions is explained by the following generalization of Exercise 3.3.8.

**Exercise 3.3.23.** If $X_1, X_2, \ldots$ are independent and have characteristic function $\exp(-|t|^\alpha)$, then $(X_1 + \cdots + X_n)/n^{1/\alpha}$ has the same distribution as $X_1$.

We will return to this topic in Section 3.7. Polya's criterion can also be used to construct some "pathological examples."

**Exercise 3.3.24.** Let $\varphi_1$ and $\varphi_2$ be ch.f.'s. Show that $A = \{t : \varphi_1(t) = \varphi_2(t)\}$ is closed, contains 0, and is symmetric about 0. Show that if $A$ is a set with these properties and $\varphi_1(t) = e^{-|t|}$, there is a $\varphi_2$ so that $\{t : \varphi_1(t) = \varphi_2(t)\} = A$.

**Example 3.3.11.** For some purposes, it is nice to have an explicit example of two ch.f.'s that agree on $[-1, 1]$. From Example 3.3.8, we know that $(1 - |t|)^+$ is the ch.f. of the density $(1 - \cos x)/\pi x^2$. Define $\psi(t)$ to be equal to $\varphi$ on $[-1, 1]$ and periodic with period 2, that is, $\psi(t) = \psi(t + 2)$. The Fourier series for $\psi$ is

$$\psi(u) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{2}{\pi^2(2n-1)^2} \exp(i(2n-1)\pi u)$$

The right-hand side is the ch.f. of a discrete distribution with

$$P(X = 0) = 1/2 \quad \text{and} \quad P(X = (2n-1)\pi) = 2\pi^{-2}(2n-1)^{-2} \quad n \in \mathbf{Z}.$$

**Exercise 3.3.25.** Find independent r.v.'s $X$, $Y$, and $Z$ so that $Y$ and $Z$ do not have the same distribution but $X + Y$ and $X + Z$ do.

**Exercise 3.3.26.** Show that if $X$ and $Y$ are independent and $X + Y$ and $X$ have the same distribution, then $Y = 0$ a.s.

For more curiosities, see Feller, Vol. II (1971), Section XV.2a.

### 3.3.5 *The Moment Problem**

Suppose $\int x^k dF_n(x)$ has a limit $\mu_k$ for each $k$. Then the sequence of distributions is tight by Theorem 3.2.8 and every subsequential limit has the moments $\mu_k$ by Exercise 3.2.5, so we can conclude the sequence converges weakly if there is only one distribution with these moments. It is easy to see that this is true if $F$ is concentrated on a finite interval $[-M, M]$ since every continuous function can be approximated uniformly on $[-M, M]$ by polynomials. The result is false in general.

**Counterexample 1.** Heyde (1963). Consider the **lognormal density**

$$f_0(x) = (2\pi)^{-1/2} x^{-1} \exp(-(\log x)^2/2) \qquad x \geq 0$$

and for $-1 \leq a \leq 1$ let

$$f_a(x) = f_0(x)\{1 + a \sin(2\pi \log x)\}$$

To see that $f_a$ is a density and has the same moments as $f_0$, it suffices to show that

$$\int_0^\infty x^r f_0(x)\sin(2\pi \log x)\,dx = 0 \text{ for } r = 0, 1, 2, \dots$$

Changing variables $x = \exp(s + r)$, $s = \log x - r$, $ds = dx/x$ the integral becomes

$$(2\pi)^{-1/2}\int_{-\infty}^\infty \exp(rs + r^2)\exp(-(s + r)^2/2)\sin(2\pi(s + r))\,ds$$
$$= (2\pi)^{-1/2}\exp(r^2/2)\int_{-\infty}^\infty \exp(-s^2/2)\sin(2\pi s)\,ds = 0$$

The two equalities holding because $r$ is an integer and the integrand is odd. From the proof, it should be clear that we could let

$$g(x) = f_0(x)\left\{1 + \sum_{k=1}^\infty a_k \sin(k\pi \log x)\right\} \quad \text{if } \sum_{k=1}^\infty |a_k| \le 1$$

to get a large family of densities having the same moments as the lognormal.

The moments of the lognormal are easy to compute. Recall that if $\chi$ has the standard normal distribution, then Exercise 1.2.6 implies $\exp(\chi)$ has the lognormal distribution.

$$EX^n = E\exp(n\chi) = \int e^{nx}(2\pi)^{-1/2}e^{-x^2/2}\,dx$$
$$= e^{n^2/2}\int (2\pi)^{-1/2}e^{-(x-n)^2/2}\,dx = \exp(n^2/2)$$

since the last integrand is the density of the normal with mean $n$ and variance 1. Somewhat remarkably, there is a family of discrete random variables with these moments. Let $a > 0$ and

$$P(Y_a = ae^k) = a^{-k}\exp(-k^2/2)/c_a \quad \text{for } k \in \mathbf{Z}$$

where $c_a$ is chosen to make the total mass 1.

$$\exp(-n^2/2)EY_a^n = \exp(-n^2/2)\sum_k (ae^k)^n a^{-k}\exp(-k^2/2)/c_a$$
$$= \sum_k a^{-(k-n)}\exp(-(k-n)^2/2)/c_a = 1$$

by the definition of $c_a$.

The lognormal density decays like $\exp(-(\log x)^2/2)$ as $|x| \to \infty$. The next counterexample has more rapid decay. Since the exponential distribution, $e^{-x}$ for $x \ge 0$, is determined by its moments (see Exercise 3.3.28 below), we cannot hope to do much better than this.

**Counterexample 2.** Let $\lambda \in (0, 1)$ and for $-1 \le a \le 1$ let

$$f_{a,\lambda}(x) = c_\lambda \exp(-|x|^\lambda)\{1 + a\sin(\beta|x|^\lambda \operatorname{sgn}(x))\}$$

where $\beta = \tan(\lambda\pi/2)$ and $1/c_\lambda = \int \exp(-|x|^\lambda)\,dx$. To prove that these are density functions and that for a fixed value of $\lambda$ they have the same moments, it suffices to show

$$\int x^n \exp(-|x|^\lambda)\sin(\beta|x|^\lambda \operatorname{sgn}(x))\,dx = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

This is clear for even $n$ since the integrand is odd. To prove the result for odd $n$, it suffices to integrate over $[0, \infty)$. Using the identity

$$\int_0^\infty t^{p-1} e^{-qt} dt = \Gamma(p)/q^p \quad \text{when } \operatorname{Re} q > 0$$

with $p = (n+1)/\lambda$, $q = 1 + \beta i$, and changing variables $t = x^\lambda$, we get

$$\begin{aligned}
&\Gamma((n+1)/\lambda)/(1+\beta\, i)^{(n+1)/\lambda} \\
&= \int_0^\infty x^{\lambda\{(n+1)/\lambda - 1\}} \exp(-(1+\beta i)x^\lambda)\lambda\, x^{\lambda-1}\,dx \\
&= \lambda \int_0^\infty x^n \exp(-x^\lambda)\cos(\beta x^\lambda) dx - i\lambda \int_0^\infty x^n \exp(-x^\lambda)\sin(\beta x^\lambda)\,dx
\end{aligned}$$

Since $\beta = \tan(\lambda\pi/2)$

$$(1+\beta i)^{(n+1)/\lambda} = (\cos\lambda\pi/2)^{-(n+1)/\lambda}(\exp(i\lambda\pi/2))^{(n+1)/\lambda}$$

The right-hand side is real since $\lambda < 1$ and $(n+1)$ is even, so

$$\int_0^\infty x^n \exp(-x^\lambda)\sin(\beta x^\lambda)\,dx = 0$$

A useful sufficient condition for a distribution to be determined by its moments is

**Theorem 3.3.11.** *If* $\limsup_{k\to\infty} \mu_{2k}^{1/2k}/2k = r < \infty$, *then there is at most one d.f.* $F$ *with* $\mu_k = \int x^k dF(x)$ *for all positive integers* $k$.

**Remark.** This is slightly stronger than **Carleman's condition**

$$\sum_{k=1}^\infty 1/\mu_{2k}^{1/2k} = \infty$$

which is also sufficient for the conclusion of Theorem 3.3.11.

*Proof.* Let $F$ be any d.f. with the moments $\mu_k$ and let $\nu_k = \int |x|^k dF(x)$. The Cauchy-Schwarz inequality implies $\nu_{2k+1}^2 \le \mu_{2k}\mu_{2k+2}$, so

$$\limsup_{k\to\infty} (\nu_k^{1/k})/k = r < \infty$$

Taking $x = tX$ in Lemma 3.3.2 and multiplying by $e^{i\theta X}$, we have

$$\left| e^{i\theta X} \left( e^{itX} - \sum_{m=0}^{n-1} \frac{(itX)^m}{m!} \right) \right| \le \frac{|tX|^n}{n!}$$

Taking expected values and using Exercise 3.3.14 gives

$$\left| \varphi(\theta + t) - \varphi(\theta) - t\varphi'(\theta) \ldots - \frac{t^{n-1}}{(n-1)!} \varphi^{(n-1)}(\theta) \right| \le \frac{|t|^n}{n!} \nu_n$$

Using the last result, the fact that $\nu_k \le (r + \epsilon)^k k^k$ for large $k$, and the trivial bound $e^k \ge k^k/k!$ (expand the left-hand side in its power series), we see that for any $\theta$

$$(*) \qquad \varphi(\theta + t) = \varphi(\theta) + \sum_{m=1}^{\infty} \frac{t^m}{m!} \varphi^{(m)}(\theta) \quad \text{for } |t| < 1/er$$

Let $G$ be another distribution with the given moments and $\psi$ its ch.f. Since $\varphi(0) = \psi(0) = 1$, it follows from $(*)$ and induction that $\varphi(t) = \psi(t)$ for $|t| \le k/3r$ for all $k$, so the two ch.f.'s coincide and the distributions are equal. ∎

Combining Theorem 3.3.11 with the discussion that began our consideration of the moment problem.

**Theorem 3.3.12.** *Suppose $\int x^k dF_n(x)$ has a limit $\mu_k$ for each $k$ and*

$$\limsup_{k \to \infty} \mu_{2k}^{1/2k} / 2k < \infty$$

*then $F_n$ converges weakly to the unique distribution with these moments.*

**Exercise 3.3.27.** Let $G(x) = P(|X| < x)$, $\lambda = \sup\{x : G(x) < 1\}$, and $\nu_k = E|X|^k$. Show that $\nu_k^{1/k} \to \lambda$, so the assumption of Theorem 3.3.12 holds if $\lambda < \infty$.

**Exercise 3.3.28.** Suppose $|X|$ has density $Cx^\alpha \exp(-x^\lambda)$ on $(0, \infty)$. Changing variables $y = x^\lambda$, $dy = \lambda x^{\lambda - 1}\, dx$

$$E|X|^n = \int_0^\infty C\lambda y^{(n+\alpha)/\lambda} \exp(-y) y^{1/\lambda - 1} dy = C\lambda\Gamma((n + \alpha + 1)/\lambda)$$

Use the identity $\Gamma(x + 1) = x\Gamma(x)$ for $x \ge 0$ to conclude that the assumption of Theorem 3.3.12 is satisfied for $\lambda \ge 1$ but not for $\lambda < 1$. This shows the normal ($\lambda = 2$) and gamma ($\lambda = 1$) distributions are determined by their moments.

Our results so far have been for the so-called **Hamburger moment problem**. If we assume *a priori* that the distribution is concentrated on $[0, \infty)$, we have the **Stieltjes moment problem**. There is a 1-1 correspondence between $X \ge 0$ and symmetric distributions on $\mathbf{R}$ given by $X \to \xi X^2$ where $\xi \in \{-1, 1\}$ is independent of $X$ and takes its two values with equal probability. From this we see that

$$\limsup_{k\to\infty} \nu_k^{1/2k}/2k < \infty$$

is sufficient for there to be a unique distribution on $[0, \infty)$ with the given moments. The next example shows that for nonnegative random variables, the last result is close to the best possible.

**Counterexample 3.** Let $\lambda \in (0, 1/2)$, $\beta = \tan(\lambda\pi)$, $-1 \le a \le 1$ and

$$f_a(x) = c_\lambda \exp(-x^\lambda)(1 + a\sin(\beta x^\lambda)) \quad \text{for } x \ge 0$$

where $1/c_\lambda = \int_0^\infty \exp(-x^\lambda)\, dx$.

By imitating the calculations in Counterexample 2, it is easy to see that the $f_a$ are probability densities that have the same moments. This example seems to be due to Stoyanov (1987), pp. 92–3. The special case $\lambda = 1/4$ is widely known.

## 3.4 Central Limit Theorems

We are now ready for the main business of the chapter. We will first prove the central limit theorem for

### 3.4.1 *i.i.d. Sequences*

**Theorem 3.4.1.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$, $var(X_i) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + \cdots + X_n$ then*

$$(S_n - n\mu)/\sigma n^{1/2} \Rightarrow \chi$$

*where $\chi$ has the standard normal distribution.*

This notation is non-standard but convenient. To see the logic, note that the square of a normal has a chi-squared distribution.

*Proof.* By considering $X_i' = X_i - \mu$, it suffices to prove the result when $\mu = 0$. From Theorem 3.3.8,

$$\varphi(t) = E\exp(itX_1) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

so

$$E\exp(itS_n/\sigma n^{1/2}) = \left(1 - \frac{t^2}{2n} + o(n^{-1})\right)^n$$

From Lemma 3.1.1 it should be clear that the last quantity $\to \exp(-t^2/2)$ as $n \to \infty$, which with Theorem 3.3.6 completes the proof. However, Lemma 3.1.1 is a fact about real numbers, so we need to extend it to the complex case to complete the proof.

**Theorem 3.4.2.** *If $c_n \to c \in \mathbf{C}$ then $(1 + c_n/n)^n \to e^c$.*

*Proof.* The proof is based on two simple facts:

**Lemma 3.4.3.** *Let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ be complex numbers of modulus $\le \theta$. Then*

$$\left| \prod_{m=1}^n z_m - \prod_{m=1}^n w_m \right| \le \theta^{n-1} \sum_{m=1}^n |z_m - w_m|$$

*Proof.* The result is true for $n = 1$. To prove it for $n > 1$ observe that

$$\begin{aligned} \left| \prod_{m=1}^n z_m - \prod_{m=1}^n w_m \right| &\le \left| z_1 \prod_{m=2}^n z_m - z_1 \prod_{m=2}^n w_m \right| + \left| z_1 \prod_{m=2}^n w_m - w_1 \prod_{m=2}^n w_m \right| \\ &\le \theta \left| \prod_{m=2}^n z_m - \prod_{m=2}^n w_m \right| + \theta^{n-1} |z_1 - w_1| \end{aligned}$$

and use induction. ■

**Lemma 3.4.4.** *If $b$ is a complex number with $|b| \le 1$ then $|e^b - (1 + b)| \le |b|^2$.*

*Proof.* $e^b - (1 + b) = b^2/2! + b^3/3! + b^4/4! + \ldots$, so if $|b| \le 1$, then

$$|e^b - (1 + b)| \le \frac{|b|^2}{2}(1 + 1/2 + 1/2^2 + \ldots) = |b|^2$$ ■

*Proof of Theorem 3.4.2.* Let $z_m = (1 + c_n/n)$, $w_m = \exp(c_n/n)$, and $\gamma > |c|$. For large $n$, $|c_n| < \gamma$. Since $1 + \gamma/n \le \exp(\gamma/n)$, it follows from Lemmas 3.4.3 and 3.4.4 that

$$|(1 + c_n/n)^n - e^{c_n}| \le \left(e^{\gamma/n}\right)^{n-1} n \left| \frac{c_n}{n} \right|^2 \le e^\gamma \frac{\gamma^2}{n} \to 0$$

as $n \to \infty$. ■

To get a feel for what the central limit theorem says, we will look at some concrete cases.

**Example 3.4.1. Roulette.** A roulette wheel has slots numbered 1–36 (18 red and 18 black) and two slots numbered 0 and 00 that are painted green. Players can bet \$1 that the ball will land in a red (or black) slot and win \$1 if it does. If we let $X_i$ be the winnings on the $i$th play, then $X_1, X_2, \ldots$ are i.i.d. with $P(X_i = 1) = 18/38$ and $P(X_i = -1) = 20/38$.

$$EX_i = -1/19 \quad \text{and} \quad \text{var}\,(X) = EX^2 - (EX)^2 = 1 - (1/19)^2 = 0.9972$$

We are interested in

$$P(S_n \geq 0) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \geq \frac{-n\mu}{\sigma\sqrt{n}}\right)$$

Taking $n = 361 = 19^2$ and replacing $\sigma$ by 1 to keep computations simple,

$$\frac{-n\mu}{\sigma\sqrt{n}} = \frac{361 \cdot (1/19)}{\sqrt{361}} = 1$$

So the central limit theorem and our table of the normal distribution in the back of the book tells us that

$$P(S_n \geq 0) \approx P(\chi \geq 1) = 1 - 0.8413 = 0.1587$$

In words, after 361 spins of the roulette wheel, the casino will have won \$19 of your money on the average, but there is a probability of about 0.16 that you will be ahead.

**Example 3.4.2. Coin flips.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 0) = P(X_i = 1) = 1/2$. If $X_i = 1$ indicates that a heads occured on the $i$th toss, then $S_n = X_1 + \cdots + X_n$ is the total number of heads at time $n$.

$$EX_i = 1/2 \quad \text{and} \quad \text{var}\,(X) = EX^2 - (EX)^2 = 1/2 - 1/4 = 1/4$$

So the central limit theorem tells us $(S_n - n/2)/\sqrt{n/4} \Rightarrow \chi$. Our table of the normal distribution tells us that

$$P(\chi > 2) = 1 - 0.9773 = 0.0227$$

so $P(|\chi| \leq 2) = 1 - 2(0.0227) = 0.9546$, or plugging into the central limit theorem

$$0.95 \approx P((S_n - n/2)/\sqrt{n/4} \in [-2, 2]) = P(S_n - n/2 \in [-\sqrt{n}, \sqrt{n}])$$

Taking $n = 10{,}000$, this says that 95% of the time the number of heads will be between 4900 and 5100.

**Example 3.4.3. Normal approximation to the binomial.** Let $X_1, X_2, \ldots$ and $S_n$ be as in the previous example. To estimate $P(S_{16} = 8)$ using the central limit theorem, we regard 8 as the interval [7.5, 8.5]. Since $\mu = 1/2$, and $\sigma\sqrt{n} = 2$ for $n = 16$

$$P(|S_{16} - 8| \leq 0.5) = P\left(\frac{|S_n - n\mu|}{\sigma\sqrt{n}} \leq 0.25\right)$$
$$\approx P(|\chi| \leq 0.25) = 2(0.5987 - 0.5) = 0.1974$$

Even though $n$ is small, this agrees well with the exact probability

$$\binom{16}{8} 2^{-16} = \frac{13 \cdot 11 \cdot 10 \cdot 9}{65,536} = 0.1964.$$

The computations above motivate the **histogram correction**, which is important in using the normal approximation for small $n$. For example, if we are going to approximate $P(S_{16} \le 11)$, then we regard this probability as $P(S_{16} \le 11.5)$. One obvious reason for doing this is to get the same answer if we regard $P(S_{16} \le 11) = 1 - P(S_{16} \ge 12)$.

**Exercise 3.4.1.** Suppose you roll a die 180 times. Use the normal approximation (with the histogram correction) to estimate the probability that you will get fewer than 25 sixes.

**Example 3.4.4. Normal approximation to the Poisson.** Let $Z_\lambda$ have a Poisson distribution with mean $\lambda$. If $X_1, X_2, \ldots$ are independent and have Poisson distributions with mean 1, then $S_n = X_1 + \cdots + X_n$ has a Poisson distribution with mean $n$. Since $\text{var}\,(X_i) = 1$, the central limit theorem implies

$$(S_n - n)/n^{1/2} \Rightarrow \chi \quad \text{as } n \to \infty$$

To deal with values of $\lambda$ that are not integers, let $N_1, N_2, N_3$ be independent Poisson with means $[\lambda]$, $\lambda - [\lambda]$, and $[\lambda] + 1 - \lambda$. If we let $S_{[\lambda]} = N_1$, $Z_\lambda = N_1 + N_2$ and $S_{[\lambda]+1} = N_1 + N_2 + N_3$ then $S_{[\lambda]} \le Z_\lambda \le S_{[\lambda]+1}$ and using the limit theorem for the $S_n$ it follows that

$$(Z_\lambda - \lambda)/\lambda^{1/2} \Rightarrow \chi \quad \text{as } \lambda \to \infty$$

**Example 3.4.5. Pairwise independence** is good enough for the strong law of large numbers (see Theorem 2.4.1). It is not good enough for the central limit theorem. Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. We will arrange things so that for $n \ge 1$,

$$S_{2^n} = \xi_1(1 + \xi_2) \cdots (1 + \xi_{n+1}) = \begin{cases} \pm 2^n & \text{with prob } 2^{-n-1} \\ 0 & \text{with prob } 1 - 2^{-n} \end{cases}$$

To do this we let $X_1 = \xi_1$, $X_2 = \xi_1 \xi_2$, and for $m = 2^{n-1} + j$, $0 < j \le 2^{n-1}$, $n \ge 2$ let $X_m = X_j \xi_{n+1}$. Each $X_m$ is a product of a different set of $\xi_j$'s, so they are pairwise independent.

## *Exercises*

**3.4.2.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$, $0 < \text{var}\,(X_i) < \infty$, and let $S_n = X_1 + \cdots + X_n$. (a) Use the central limit theorem and Kolmogorov's zero-one law to conclude that limsup $S_n/\sqrt{n} = \infty$ a.s. (b) Use an argument by contradiction to show that $S_n/\sqrt{n}$ does not converge in probability. Hint: Consider $n = m!$.

**3.4.3.** Let $X_1, X_2, \ldots$ be i.i.d. and let $S_n = X_1 + \cdots + X_n$. Assume that $S_n/\sqrt{n} \Rightarrow$ a limit and conclude that $EX_i^2 < \infty$. Sketch: Suppose $EX_i^2 = \infty$. Let $X_1'$, $X_2', \ldots$ be an independent copy of the original sequence. Let $Y_i = X_i - X_i'$,

$U_i = Y_i 1_{(|Y_i| \le A)}$, $V_i = Y_i 1_{(|Y_i|>A)}$, and observe that for any $K$

$$P\left(\sum_{m=1}^n Y_m \ge K\sqrt{n}\right) \ge P\left(\sum_{m=1}^n U_m \ge K\sqrt{n}, \sum_{m=1}^n V_m \ge 0\right)$$
$$\ge \frac{1}{2} P\left(\sum_{m=1}^n U_m \ge K\sqrt{n}\right) \ge \frac{1}{5}$$

for large $n$ if $A$ is large enough. Since $K$ is arbitrary, this is a contradiction.

**3.4.4.** Let $X_1, X_2, \ldots$ be i.i.d. with $X_i \ge 0$, $EX_i = 1$, and $\text{var}\,(X_i) = \sigma^2 \in (0, \infty)$. Show that $2(\sqrt{S_n} - \sqrt{n}) \Rightarrow \sigma\chi$.

**3.4.5. Self-normalized sums.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Then

$$\sum_{m=1}^n X_m \Big/ \left(\sum_{m=1}^n X_m^2\right)^{1/2} \Rightarrow \chi$$

**3.4.6. Random index central limit theorem.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$, and let $S_n = X_1 + \cdots + X_n$. Let $N_n$ be a sequence of nonnegative integer-valued random variables and $a_n$ a sequence of integers with $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$S_{N_n}/\sigma\sqrt{a_n} \Rightarrow \chi$$

Hint: Use Kolmogorov's inequality (Theorem 2.5.2) to conclude that if $Y_n = S_{N_n}/\sigma\sqrt{a_n}$ and $Z_n = S_{a_n}/\sigma\sqrt{a_n}$, then $Y_n - Z_n \to 0$ in probability.

**3.4.7. A central limit theorem in renewal theory.** Let $Y_1, Y_2, \ldots$ be i.i.d. positive random variables with $EY_i = \mu$ and $\text{var}\,(Y_i) = \sigma^2 \in (0, \infty)$. Let $S_n = Y_1 + \cdots + Y_n$ and $N_t = \sup\{m : S_m \le t\}$. Apply the previous exercise to $X_i = Y_i - \mu$ to prove that as $t \to \infty$,

$$(\mu N_t - t)/(\sigma^2 t/\mu)^{1/2} \Rightarrow \chi$$

**3.4.8. A second proof of the renewal CLT.** Let $Y_1, Y_2, \ldots, S_n$, and $N_t$ be as in the last exercise. Let $u = [t/\mu]$, $D_t = S_u - t$. Use Kolmogorov's inequality to show

$$P(|S_{u+m} - (S_u + m\mu)| > t^{2/5} \text{ for some } m \in [-t^{3/5}, t^{3/5}]) \to 0 \quad \text{as } t \to \infty$$

Conclude $|N_t - (t - D_t)/\mu|/t^{1/2} \to 0$ in probability, and then obtain the result in the previous exercise.

Our next step is to generalize the central limit theorem to:

### 3.4.2 *Triangular Arrays*

**Theorem 3.4.5. The Lindeberg-Feller theorem.** *For each $n$, let $X_{n,m}$, $1 \le m \le n$, be independent random variables with $EX_{n,m} = 0$. Suppose*
*(i)* $\sum_{m=1}^n EX_{n,m}^2 \to \sigma^2 > 0$
*(ii) For all* $\epsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^n E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) = 0$.
*Then* $S_n = X_{n,1} + \cdots + X_{n,n} \Rightarrow \sigma\chi$ *as* $n \to \infty$.

**Remarks.** In words, the theorem says that a sum of a large number of small independent effects has approximately a normal distribution. To see that Theorem 3.4.5 contains our first central limit theorem, let $Y_1, Y_2 \dots$ be i.i.d. with $EY_i = 0$ and $EY_i^2 = \sigma^2 \in (0, \infty)$, and let $X_{n,m} = Y_m/n^{1/2}$. Then $\sum_{m=1}^n EX_{n,m}^2 = \sigma^2$ and if $\epsilon > 0$

$$\sum_{m=1}^n E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) = nE(|Y_1/n^{1/2}|^2; |Y_1/n^{1/2}| > \epsilon)$$
$$= E(|Y_1|^2; |Y_1| > \epsilon n^{1/2}) \to 0$$

by the dominated convergence theorem since $EY_1^2 < \infty$.

*Proof.* Let $\varphi_{n,m}(t) = E\exp(itX_{n,m})$, $\sigma_{n,m}^2 = EX_{n,m}^2$. By Theorem 3.3.6, it suffices to show that

$$\prod_{m=1}^n \varphi_{n,m}(t) \to \exp(-t^2\sigma^2/2)$$

Let $z_{n,m} = \varphi_{n,m}(t)$ and $w_{n,m} = (1 - t^2\sigma_{n,m}^2/2)$. By (3.3.3)

$$\begin{aligned}|z_{n,m} - w_{n,m}| &\le E(|tX_{n,m}|^3 \wedge 2|tX_{n,m}|^2)\\ &\le E(|tX_{n,m}|^3; |X_{n,m}| \le \epsilon) + E(2|tX_{n,m}|^2; |X_{n,m}| > \epsilon)\\ &\le \epsilon t^3 E(|X_{n,m}|^2; |X_{n,m}| \le \epsilon) + 2t^2E(|X_{n,m}|^2; |X_{n,m}| > \epsilon)\end{aligned}$$

Summing $m = 1$ to $n$, letting $n \to \infty$, and using (i) and (ii) gives

$$\limsup_{n\to\infty} \sum_{m=1}^n |z_{n,m} - w_{n,m}| \le \epsilon t^3\sigma^2$$

Since $\epsilon > 0$ is arbitrary, it follows that the sequence converges to 0. Our next step is to use Lemma 3.4.3 with $\theta = 1$ to get

$$\left|\prod_{m=1}^n \varphi_{n,m}(t) - \prod_{m=1}^n (1 - t^2\sigma_{n,m}^2/2)\right| \to 0$$

To check the hypotheses of Lemma 3.4.3, note that since $\varphi_{n,m}$ is a ch.f. $|\varphi_{n,m}(t)| \le 1$ for all $n, m$. For the terms in the second product we note that

$$\sigma_{n,m}^2 \le \epsilon^2 + E(|X_{n,m}|^2 ; |X_{n,m}| > \epsilon)$$

and $\epsilon$ is arbitrary so (ii) implies $\sup_m \sigma_{n,m}^2 \to 0$ and thus if $n$ is large $1 \geq 1 - t^2\sigma_{n,m}^2/2 > -1$ for all $m$.

To complete the proof now, we apply Exercise 3.1.1 with $c_{m,n} = -t^2\sigma_{n,m}^2/2$. We have just shown $\sup_m \sigma_{n,m}^2 \to 0$. (i) implies

$$\sum_{m=1}^{n} c_{m,n} \to -\sigma^2 t^2/2$$

so $\prod_{m=1}^n (1 - t^2\sigma_{n,m}^2/2) \to \exp(-t^2\sigma^2/2)$ and the proof is complete. ∎

**Example 3.4.6. Cycles in a random permutation and record values.** Continuing the analysis of Examples 2.2.4 and 2.3.2, let $Y_1, Y_2, \ldots$ be independent with $P(Y_m = 1) = 1/m$, and $P(Y_m = 0) = 1 - 1/m$. $EY_m = 1/m$ and $\text{var}(Y_m) = 1/m - 1/m^2$. So if $S_n = Y_1 + \cdots + Y_n$ then $ES_n \sim \log n$ and $\text{var}(S_n) \sim \log n$. Let

$$X_{n,m} = (Y_m - 1/m)/(\log\ n)^{1/2}$$

$EX_{n,m} = 0$, $\sum_{m=1}^n EX_{n,m}^2 \to 1$, and for any $\epsilon > 0$

$$\sum_{m=1}^{n} E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) \to 0$$

since the sum is 0 as soon as $(\log n)^{-1/2} < \epsilon$. Applying Theorem 3.4.5 now gives

$$(\log n)^{-1/2}\left(S_n - \sum_{m=1}^{n}\frac{1}{m}\right) \Rightarrow \chi$$

Observing that

$$\sum_{m=1}^{n-1}\frac{1}{m} \geq \int_1^n x^{-1}\,dx = \log n \geq \sum_{m=2}^{n}\frac{1}{m}$$

shows $\left|\log n - \sum_{m=1}^n 1/m\right| \leq 1$ and the conclusion can be written as

$$(S_n - \log n)/(\log n)^{1/2} \Rightarrow \chi$$

**Example 3.4.7. The converse of the three series theorem.** Recall the setup of Theorem 2.5.4. Let $X_1, X_2, \ldots$ be independent, let $A > 0$, and let $Y_m = X_m 1_{(|X_m| \leq A)}$. In order that $\sum_{n=1}^\infty X_n$ converges (i.e., $\lim_{N\to\infty} \sum_{n=1}^N X_n$ exists) it is necessary that

$$\text{(i)} \sum_{n=1}^{\infty} P(|X_n| > A) < \infty, \ \text{(ii)} \sum_{n=1}^{\infty} EY_n \text{ converges, and (iii)} \sum_{n=1}^{\infty} \text{var}(Y_n) < \infty$$

*Proof.* The necessity of the first condition is clear. For if that sum is infinite, $P(|X_n| > A \text{ i.o.}) > 0$ and $\lim_{n\to\infty} \sum_{m=1}^n X_m$ cannot exist. Suppose next that the sum in (i) is finite but the sum in (iii) is infinite. Let

$$c_n = \sum_{m=1}^n \mathrm{var}\,(Y_m) \quad \text{and} \quad X_{n,m} = (Y_m - EY_m)/c_n^{1/2}$$

$EX_{n,m} = 0$, $\sum_{m=1}^n EX_{n,m}^2 = 1$, and for any $\epsilon > 0$

$$\sum_{m=1}^n E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) \to 0$$

since the sum is 0 as soon as $2A/c_n^{1/2} < \epsilon$. Applying Theorem 3.4.5 now gives that if $S_n = X_{n,1} + \cdots + X_{n,n}$ then $S_n \Rightarrow \chi$. Now

(i) if $\lim_{n\to\infty} \sum_{m=1}^n X_m$ exists, $\lim_{n\to\infty} \sum_{m=1}^n Y_m$ exists.
(ii) if we let $T_n = (\sum_{m\le n} Y_m)/c_n^{1/2}$ then $T_n \Rightarrow 0$.

The last two results and Exercise 3.2.13 imply $(S_n - T_n) \Rightarrow \chi$. Since

$$S_n - T_n = -\left(\sum_{m\le n} EY_m\right)/c_n^{1/2}$$

is not random, this is absurd.

Finally, assume the series in (i) and (iii) are finite. Theorem 2.5.3 implies that $\lim_{n\to\infty} \sum_{m=1}^n (Y_m - EY_m)$ exists, so if $\lim_{n\to\infty} \sum_{m=1}^n X_m$ and hence $\lim_{n\to\infty} \sum_{m=1}^n Y_m$ does, taking differences shows that (ii) holds. ∎

**Example 3.4.8. Infinite variance.** Suppose $X_1, X_2, \ldots$ are i.i.d. and have $P(X_1 > x) = P(X_1 < -x)$ and $P(|X_1| > x) = x^{-2}$ for $x \ge 1$.

$$E|X_1|^2 = \int_0^\infty 2x P(|X_1| > x)\, dx = \infty$$

but it turns out that when $S_n = X_1 + \cdots + X_n$ is suitably normalized it converges to a normal distribution. Let

$$Y_{n,m} = X_m 1_{(|X_m| \le n^{1/2} \log\log n)}$$

The truncation level $c_n = n^{1/2} \log\log n$ is chosen large enough to make

$$\sum_{m=1}^n P(Y_{n,m} \ne X_m) \le nP(|X_1| > c_n) \to 0$$

However, we want the variance of $Y_{n,m}$ to be as small as possible, so we keep the truncation close to the lowest possible level.

Our next step is to show $EY_{n,m}^2 \sim \log n$. For this we need upper and lower bounds. Since $P(|Y_{n,m}| > x) \le P(|X_1| > x)$ and is 0 for $x > c_n$, we have

$$\begin{aligned} EY_{n,m}^2 &\le \int_0^{c_n} 2yP(|X_1| > y)\, dy = 1 + \int_1^{c_n} 2/y\, dy \\ &= 1 + 2\log c_n = 1 + \log n + 2 \log\log\log n \sim \log n \end{aligned}$$

In the other direction, we observe $P(|Y_{n,m}| > x) = P(|X_1| > x) - P(|X_1| > c_n)$ and the right-hand side is $\geq (1 - (\log\log n)^{-2})P(|X_1| > x)$ when $x \leq \sqrt{n}$ so

$$EY_{n,m}^2 \geq (1 - (\log\log n)^{-2}) \int_1^{\sqrt{n}} 2/y\, dy \sim \log n$$

If $S'_n = Y_{n,1} + \cdots + Y_{n,n}$ then $\operatorname{var}(S'_n) \sim n \log n$, so we apply Theorem 3.4.5 to $X_{n,m} = Y_{n,m}/(n \log n)^{1/2}$. Things have been arranged so that (i) is satisfied. Since $|Y_{n,m}| \leq n^{1/2} \log\log n$, the sum in (ii) is 0 for large $n$, and it follows that $S'_n/(n \log n)^{1/2} \Rightarrow \chi$. Since the choice of $c_n$ guarantees $P(S_n \neq S'_n) \to 0$, the same result holds for $S_n$.

**Remark.** In Section 3.6, we will see that if we replace $P(|X_1| > x) = x^{-2}$ in Example 3.4.8 by $P(|X_1| > x) = x^{-\alpha}$ where $0 < \alpha < 2$, then $S_n/n^{1/\alpha} \Rightarrow$ to a limit which is not $\chi$. The last word on convergence to the normal distribution is the next result, due to Lévy.

**Theorem 3.4.6.** *Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. In order that there exist constants $a_n$ and $b_n > 0$ so that $(S_n - a_n)/b_n \Rightarrow \chi$, it is necessary and sufficient that*

$$y^2 P(|X_1| > y)/E(|X_1|^2; |X_1| \leq y) \to 0.$$

A proof can be found in Gnedenko and Kolmogorov (1954), a reference that contains the last word on many results about sums of independent random variables.

### *Exercises*

In the next five problems $X_1, X_2, \ldots$ are independent and $S_n = X_1 + \cdots + X_n$.

**3.4.9.** Suppose $P(X_m = m) = P(X_m = -m) = m^{-2}/2$, and for $m \geq 2$

$$P(X_m = 1) = P(X_m = -1) = (1 - m^{-2})/2$$

Show that $\operatorname{var}(S_n)/n \to 2$ but $S_n/\sqrt{n} \Rightarrow \chi$. The trouble here is that $X_{n,m} = X_m/\sqrt{n}$ does not satisfy (ii) of Theorem 3.4.5.

**3.4.10.** Show that if $|X_i| \leq M$ and $\sum_n \operatorname{var}(X_n) = \infty$, then

$$(S_n - ES_n)/\sqrt{\operatorname{var}(S_n)} \Rightarrow \chi$$

**3.4.11.** Suppose $EX_i = 0$, $EX_i^2 = 1$, and $E|X_i|^{2+\delta} \leq C$ for some $0 < \delta, C < \infty$. Show that $S_n/\sqrt{n} \Rightarrow \chi$.

**3.4.12.** Prove **Lyapunov's Theorem.** Let $\alpha_n = \{\operatorname{var}(S_n)\}^{1/2}$. If there is a $\delta > 0$ so that

$$\lim_{n\to\infty} \alpha_n^{-(2+\delta)} \sum_{m=1}^{n} E(|X_m - EX_m|^{2+\delta}) = 0$$

then $(S_n - ES_n)/\alpha_n \Rightarrow \chi$. Note that the previous exercise is a special case of this result.

**3.4.13.** Suppose $P(X_j = j) = P(X_j = -j) = 1/2j^{\beta}$ and $P(X_j = 0) = 1 - j^{-\beta}$ where $\beta > 0$. Show that (i) if $\beta > 1$ then $S_n \to S_\infty$ a.s., (ii) if $\beta < 1$ then $S_n/n^{(3-\beta)/2} \Rightarrow c\chi$, (iii) if $\beta = 1$ then $S_n/n \Rightarrow \aleph$ where

$$E\exp(it\aleph) = \exp\left(-\int_0^1 x^{-1}(1-\cos xt)\,dx\right)$$

### 3.4.3 *Prime Divisors (Erdös-Kac)**

Our aim here is to prove that an integer picked at random from $\{1, 2, \ldots, n\}$ has about

$$\log\log n + \chi(\log\log n)^{1/2}$$

prime divisors. Since $\exp(e^4) = 5.15 \times 10^{23}$, this result does not apply to most numbers we encounter in "everyday life." The first step in deriving this result is to give a

*Second proof of Theorem 3.4.5.* The first step is to let

$$h_n(\epsilon) = \sum_{m=1}^{n} E(X_{n,m}^2; |X_{n,m}| > \epsilon)$$

and observe

**Lemma 3.4.7.** $h_n(\epsilon) \to 0$ *for each fixed* $\epsilon > 0$ *so we can pick* $\epsilon_n \to 0$ *so that* $h_n(\epsilon_n) \to 0$.

*Proof.* Let $N_m$ be chosen so that $h_n(1/m) \le 1/m$ for $n \ge N_m$ and $m \to N_m$ is increasing. Let $\epsilon_n = 1/m$ for $N_m \le n < N_{m+1}$, and $= 1$ for $n < N_1$. When $N_m \le n < N_{m+1}$, $\epsilon_n = 1/m$, so $|h_n(\epsilon_n)| = |h_n(1/m)| \le 1/m$, and the desired result follows. ∎

Let $X'_{n,m} = X_{n,m} 1_{(|X_{n,m}| > \epsilon_n)}$, $Y_{n,m} = X_{n,m} 1_{(|X_{n,m}| \le \epsilon_n)}$, and $Z_{n,m} = Y_{n,m} - EY_{n,m}$. Clearly $|Z_{n,m}| \le 2\epsilon_n$. Using $X_{n,m} = X'_{n,m} + Y_{n,m}$, $Z_{n,m} = Y_{n,m} - EY_{n,m}$, $EY_{n,m} = -EX'_{n,m}$, the variance of the sum is the sum of the variances, and

$\text{var}(W) \le EW^2$, we have

$$E\left(\sum_{m=1}^n X_{n,m} - \sum_{m=1}^n Z_{n,m}\right)^2 = E\left(\sum_{m=1}^n X'_{n,m} - EX'_{n,m}\right)^2$$
$$= \sum_{m=1}^n E(X'_{n,m} - EX'_{n,m})^2 \le \sum_{m=1}^n E(X'_{n,m})^2 \to 0$$

as $n \to \infty$, by the choice of $\epsilon_n$.

Let $S_n = \sum_{m=1}^n X_{n,m}$ and $T_n = \sum_{m=1}^n Z_{n,m}$. The last computation shows $S_n - T_n \to 0$ in $L^2$ and hence in probability by Lemma 2.2.2. Thus, by Exercise 3.2.13, it suffices to show $T_n \Rightarrow \sigma\chi$. (i) implies $ES_n^2 \to \sigma^2$. We have just shown that $E(S_n - T_n)^2 \to 0$, so the triangle inequality for the $L^2$ norm implies $ET_n^2 \to \sigma^2$. To compute higher moments, we observe

$$T_n^r = \sum_{k=1}^r \sum_{r_i} \frac{r!}{r_1! \cdots r_k!} \frac{1}{k!} \sum_{i_j} Z_{n,i_1}^{r_1} \cdots Z_{n,i_k}^{r_k}$$

where $\sum_{r_i}$ extends over all $k$-tuples of positive integers with $r_1 + \cdots + r_k = r$ and $\sum_{i_j}$ extends over all $k$-tuples of distinct integers with $1 \le i \le n$. If we let

$$A_n(r_1, \ldots, r_k) = \sum_{i_j} EZ_{n,i_1}^{r_1} \cdots EZ_{n,i_k}^{r_k}$$

then

$$ET_n^r = \sum_{k=1}^r \sum_{r_i} \frac{r!}{r_1! \cdots r_k!} \frac{1}{k!} A_n(r_1, \ldots r_k)$$

To evaluate the limit of $ET_n^r$ we observe:

(a) If some $r_j = 1$, then $A_n(r_1, \ldots r_k) = 0$ since $EZ_{n,i_j} = 0$.
(b) If all $r_j = 2$, then

$$\sum_{i_j} EZ_{n,i_1}^2 \cdots EZ_{n,i_k}^2 \le \left(\sum_{m=1}^n EZ_{n,m}^2\right)^k \to \sigma^{2k}$$

To argue the other inequality, we note that for any $1 \le a < b \le k$ we can estimate the sum over all the $i_1, \ldots, i_k$ with $i_a = i_b$ by replacing $EZ_{n,i_a}^2$ by $(2\epsilon_n)^2$ to get (the factor $\binom{k}{2}$ giving the number of ways to pick $1 \le a < b \le k$)

$$\left(\sum_{m=1}^n EZ_{n,m}^2\right)^k - \sum_{i_j} EZ_{n,i_1}^2 \cdots EZ_{n,i_k}^2 \le \binom{k}{2}(2\epsilon_n)^2 \left(\sum_{m=1}^n EZ_{n,m}^2\right)^{k-1} \to 0$$

(c) If all the $r_i \ge 2$ but some $r_j > 2$ then using

$$E|Z_{n,i_j}|^{r_j} \le (2\epsilon_n)^{r_j - 2} EZ_{n,i_j}^2$$

we have

$$|A_n(r_1, \dots r_k)| \le \sum_{i_j} E|Z_{n,i_1}|^{r_1} \cdots E|Z_{n,i_k}|^{r_k}$$
$$\le (2\epsilon_n)^{r-2k} A_n(2, \dots 2) \to 0$$

When $r$ is odd, some $r_j$ must be $= 1$ or $\ge 3$ so $ET_n^r \to 0$ by (a) and (c). If $r = 2k$ is even, (a)–(c) imply

$$ET_n^r \to \frac{\sigma^{2k}(2k)!}{2^k k!} = E(\sigma\chi)^r$$

and the result follows from Theorem 3.3.12. ∎

Turning to the result for prime divisors, let $P_n$ denote the uniform distribution on $\{1, \dots, n\}$. If $P_\infty(A) \equiv \lim P_n(A)$ exists, the limit is called the density of $A \subset \mathbf{Z}$. Let $A_p$ be the set of integers divisible by $p$. Clearly, if $p$ is a prime $P_\infty(A_p) = 1/p$ and $q \ne p$ is another prime

$$P_\infty(A_p \cap A_q) = 1/pq = P_\infty(A_p)P_\infty(A_q)$$

Even though $P_\infty$ is not a probability measure (since $P(\{i\}) = 0$ for all $i$), we can interpret this as saying that the events of being divisible by $p$ and $q$ are independent. Let $\delta_p(n) = 1$ if $n$ is divisible by $p$, and $= 0$ otherwise, and

$$g(n) = \sum_{p \le n} \delta_p(n) \quad \text{be the number of prime divisors of } n$$

this and future sums on $p$ being over the primes. Intuitively, the $\delta_p(n)$ behave like $X_p$ that are i.i.d. with

$$P(X_p = 1) = 1/p \quad \text{and} \quad P(X_p = 0) = 1 - 1/p$$

The mean and variance of $\sum_{p \le n} X_p$ are

$$\sum_{p \le n} 1/p \quad \text{and} \quad \sum_{p \le n} 1/p(1 - 1/p)$$

respectively. It is known that

$$(*) \qquad \sum_{p \le n} 1/p = \log\log n + O(1)$$

(see Hardy and Wright, 1959, Chapter XXII), while anyone can see $\sum_p 1/p^2 < \infty$, so applying Theorem 3.4.5 to $X_p$ and making a small leap of faith gives us:

**Theorem 3.4.8. Erdös-Kac central limit theorem.** *As $n \to \infty$*

$$P_n\left(m \le n : g(m) - \log\log n \le x(\log\log n)^{1/2}\right) \to P(\chi \le x)$$

*Proof.* We begin by showing that we can ignore the primes "near" $n$. Let

$$\alpha_n = n^{1/\log\log n}$$
$$\log \alpha_n = \log n / \log\log n$$
$$\log\log \alpha_n = \log\log n - \log\log\log n$$

The sequence $\alpha_n$ has two nice properties:

(a) $\left(\sum_{\alpha_n < p \le n} 1/p\right) / (\log\log n)^{1/2} \to 0$ by $(*)$

*Proof of (a).* By $(*)$

$$\begin{aligned}\sum_{\alpha_n < p \le n} 1/p &= \sum_{p \le n} 1/p - \sum_{p \le \alpha_n} 1/p \\ &= \log\log n - \log\log \alpha_n + O(1) \\ &= \log\log\log n + O(1)\end{aligned}$$

(b) If $\epsilon > 0$, then $\alpha_n \le n^\epsilon$ for large $n$ and hence $\alpha_n^r / n \to 0$ for all $r < \infty$. ∎

*Proof of (b).* $1/\log\log n \to 0$ as $n \to \infty$.

Let $g_n(m) = \sum_{p \le \alpha_n} \delta_p(m)$ and let $E_n$ denote expected value w.r.t. $P_n$.

$$E_n\left(\sum_{\alpha_n < p \le n} \delta_p\right) = \sum_{\alpha_n < p \le n} P_n(m : \delta_p(m) = 1) \le \sum_{\alpha_n < p \le n} 1/p$$

so by (a) it is enough to prove the result for $g_n$. Let

$$S_n = \sum_{p \le \alpha_n} X_p$$

where the $X_p$ are the independent random variables introduced above. Let $b_n = ES_n$ and $a_n^2 = \text{var}(S_n)$. (a) tells us that $b_n$ and $a_n^2$ are both

$$\log\log n + o((\log\log n)^{1/2})$$

so it suffices to show

$$P_n(m : g_n(m) - b_n \le x a_n) \to P(\chi \le x)$$

An application of Theorem 3.4.5 shows $(S_n - b_n)/a_n \Rightarrow \chi$, and since $|X_p| \le 1$ it follows from the second proof of Theorem 3.4.5 that

$$E\left((S_n - b_n)/a_n\right)^r \to E\chi^r \quad \text{for all } r$$

Using notation from that proof (and replacing $i_j$ by $p_j$)

$$ES_n^r = \sum_{k=1}^r \sum_{r_i} \frac{r!}{r_1! \cdots r_k!} \frac{1}{k!} \sum_{p_j} E(X_{p_1}^{r_1} \cdots X_{p_k}^{r_k})$$

Since $X_p \in \{0, 1\}$, the summand is

$$E(X_{p_1} \cdots X_{p_k}) = 1/(p_1 \cdots p_k)$$

A little thought reveals that

$$E_n(\delta_{p_1} \cdots \delta_{p_k}) \le \frac{1}{n}[n/(p_1 \cdots p_k)]$$

The two moments differ by $\le 1/n$, so

$$|E(S_n^r) - E_n(g_n^r)| = \sum_{k=1}^{r} \sum_{r_i} \frac{r!}{r_1! \cdots r_k!} \frac{1}{k!} \sum_{p_j} \frac{1}{n}$$
$$\le 13n \Big( \sum_{p \le \alpha_n} 1 \Big)^r \le \frac{\alpha_n^r}{n} \to 0$$

by (b). Now

$$E(S_n - b_n)^r = \sum_{m=0}^{r} \binom{r}{m} E S_n^m (-b_n)^{r-m}$$
$$E(g_n - b_n)^r = \sum_{m=0}^{r} \binom{r}{m} E g_n^m (-b_n)^{r-m}$$

so subtracting and using our bound on $|E(S_n^r) - E_n(g_n^r)|$ with $r = m$

$$|E(S_n - b_n)^r - E(g_n - b_n)^r| \le \sum_{m=0}^{r} \binom{r}{m} \frac{1}{n} \alpha_n^m b_n^{r-m} = (\alpha_n + b_n)^r/n \to 0$$

since $b_n \le \alpha_n$. This is more than enough to conclude that

$$E\left((g_n - b_n)/a_n\right)^r \to E\chi^r$$

and the desired result follows from Theorem 3.3.12. ■

### 3.4.4 *Rates of Convergence (Berry-Esseen)**

**Theorem 3.4.9.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$, $EX_i^2 = \sigma^2$, and $E|X_i|^3 = \rho < \infty$. If $F_n(x)$ is the distribution of $(X_1 + \cdots + X_n)/\sigma\sqrt{n}$ and $\mathcal{N}(x)$ is the standard normal distribution, then*

$$|F_n(x) - \mathcal{N}(x)| \le 3\rho/\sigma^3\sqrt{n}$$

**Remarks.** The reader should note that the inequality holds for all $n$ and $x$, but since $\rho \ge \sigma^3$, it only has nontrivial content for $n \ge 10$. It is easy to see that the rate cannot be faster than $n^{-1/2}$. When $P(X_i = 1) = P(X_i = -1) = 1/2$, symmetry and (1.4) imply

$$F_{2n}(0) = \frac{1}{2}\{1 + P(S_{2n} = 0)\} = \frac{1}{2}(1 + (\pi n)^{-1/2}) + o(n^{-1/2})$$

The constant 3 is not the best known (van Beek, 1972, gets 0.8), but as Feller brags, “our streamlined method yields a remarkably good bound even though it avoids the usual messy numerical calculations.” The hypothesis $E|X|^3$ is needed to get the rate $n^{-1/2}$. Heyde (1967) has shown that for $0 < \delta < 1$

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x |F_n(x) - \mathcal{N}(x)| < \infty$$

if and only if $E|X|^{2+\delta} < \infty$. For this and more on rates of convergence, see Hall (1982).

*Proof.* Since neither side of the inequality is affected by scaling, we can suppose without loss of generality that $\sigma^2 = 1$. The first phase of the argument is to derive an inequality, Lemma 3.4.11, that relates the difference between the two distributions to the distance between their ch.f.'s. Polya's density (see Example 3.3.8 and use (e) of Theorem 3.3.1)

$$h_L(x) = \frac{1 - \cos Lx}{\pi L x^2}$$

has ch.f. $\omega_L(\theta) = (1 - |\theta/L|)^+$ for $|\theta| \le L$. We will use $H_L$ for its distribution function. We will convolve the distributions under consideration with $H_L$ to get ch.f. that have compact support. The first step is to show that convolution with $H_L$ does not reduce the difference between the distributions too much.

**Lemma 3.4.10.** *Let $F$ and $G$ be distribution functions with $G'(x) \le \lambda < \infty$. Let $\Delta(x) = F(x) - G(x)$, $\eta = \sup|\Delta(x)|$, $\Delta_L = \Delta * H_L$, and $\eta_L = \sup|\Delta_L(x)|$. Then*

$$\eta_L \ge \frac{\eta}{2} - \frac{12\lambda}{\pi L} \quad \textit{or} \quad \eta \le 2\eta_L + \frac{24\lambda}{\pi L}$$

*Proof.* $\Delta$ goes to 0 at $\pm\infty$, $G$ is continuous, and $F$ is a d.f., so there is an $x_0$ with $\Delta(x_0) = \eta$ or $\Delta(x_0-) = -\eta$. By looking at the d.f.'s of $(-1)$ times the r.v.'s in the second case, we can suppose without loss of generality that $\Delta(x_0) = \eta$. Since $G'(x) \le \lambda$ and $F$ is nondecreasing, $\Delta(x_0 + s) \ge \eta - \lambda s$. Letting $\delta = \eta/2\lambda$, and $t = x_0 + \delta$, we have

$$\Delta(t - x) \ge \begin{cases} (\eta/2) + \lambda x & \text{for } |x| \le \delta \\ -\eta & \text{otherwise} \end{cases}$$

To estimate the convolution $\Delta_L$, we observe

$$2\int_\delta^\infty h_L(x)\,dx \le 2\int_\delta^\infty 2/(\pi L x^2) dx = 4/(\pi L \delta)$$

Looking at $(-\delta, \delta)$ and its complement separately and noticing that symmetry implies $\int_{-\delta}^{\delta} x h_L(x)\,dx = 0$, we have

$$\eta_L \geq \Delta_L(t) \geq \frac{\eta}{2}\left(1 - \frac{4}{\pi L\delta}\right) - \eta\frac{4}{\pi L\delta} = \frac{\eta}{2} - \frac{6\eta}{\pi L\delta} = \frac{\eta}{2} - \frac{12\lambda}{\pi L}$$

which proves the lemma. ∎

**Lemma 3.4.11.** *Let $K_1$ and $K_2$ be d.f. with mean 0 whose ch.f. $\kappa_i$ are integrable*

$$K_1(x) - K_2(x) = (2\pi)^{-1} \int -e^{-itx} \frac{\kappa_1(t) - \kappa_2(t)}{it}\,dt$$

*Proof.* Since the $\kappa_i$ are integrable, the inversion formula, Theorem 3.3.4, implies that the density $k_i(x)$ has

$$k_i(y) = (2\pi)^{-1} \int e^{-ity} \kappa_i(t)\,dt$$

Subtracting the last expression with $i = 2$ from the one with $i = 1$, then integrating from $a$ to $x$ and letting $\Delta K = K_1 - K_2$ gives

$$\begin{aligned}\Delta K(x) - \Delta K(a) &= (2\pi)^{-1} \int_a^x \int e^{-ity}\{\kappa_1(t) - \kappa_2(t)\}\,dt\,dy \\ &= (2\pi)^{-1} \int \{e^{-ita} - e^{-itx}\} \frac{\kappa_1(t) - \kappa_2(t)}{it}\,dt\end{aligned}$$

the application of Fubini's theorem being justified since the $\kappa_i$ are integrable in $t$ and we are considering a bounded interval in $y$.

The factor $1/it$ could cause problems near zero, but we have supposed that the $K_i$ have mean 0, so $\{1 - \kappa_i(t)\}/t \to 0$ by Exercise 3.3.14, and hence $(\kappa_1(t) - \kappa_2(t))/it$ is bounded and continuous. The factor $1/it$ improves the integrability for large $t$ so $(\kappa_1(t) - \kappa_2(t))/it$ is integrable. Letting $a \to -\infty$ and using the Riemann-Lebesgue lemma (Exercise 1.4.4) proves the result. ∎

Let $\varphi_F$ and $\varphi_G$ be the ch.f.'s of $F$ and $G$. Applying Lemma 3.4.11 to $F_L = F * H_L$ and $G_L = G * H_L$, gives

$$\begin{aligned}|F_L(x) - G_L(x)| &\leq \frac{1}{2\pi} \int |\varphi_F(t)\omega_L(t) - \varphi_G(t)\omega_L(t)|\,\frac{dt}{|t|} \\ &\leq \frac{1}{2\pi} \int_{-L}^{L} |\varphi_F(t) - \varphi_G(t)|\,\frac{dt}{|t|}\end{aligned}$$

because $|\omega_L(t)| \leq 1$. Using Lemma 3.4.10 now, we have

$$|F(x) - G(x)| \leq \frac{1}{\pi} \int_{-L}^{L} |\varphi_F(\theta) - \varphi_G(\theta)|\,\frac{d\theta}{|\theta|} + \frac{24\lambda}{\pi L}$$

where $\lambda = \sup_x G'(x)$. Plugging in $F = F_n$ and $G = \mathcal{N}$ gives

$$|F_n(x) - \mathcal{N}(x)| \le \frac{1}{\pi}\int_{-L}^{L} |\varphi^n(\theta/\sqrt{n}) - \psi(\theta)|\,\frac{d\theta}{|\theta|} + \frac{24\lambda}{\pi L} \tag{3.4.1}$$

and it remains to estimate the right-hand side. This phase of the argument is fairly routine, but there is a fair amount of algebra. To save the reader from trying to improve the inequalities along the way in hopes of getting a better bound, we would like to observe that we have used the fact that $C = 3$ to get rid of the cases $n \le 9$, and we use $n \ge 10$ in (e).

To estimate the second term in (3.4.1), we observe that

(a) $$\sup_x G'(x) = G'(0) = (2\pi)^{-1/2} = 0.39894 < 2/5$$

For the first, we observe that if $|\alpha|, |\beta| \le \gamma$

(b) $$|\alpha^n - \beta^n| \le \sum_{m=0}^{n-1} |\alpha^{n-m}\beta^m - \alpha^{n-m-1}\beta^{m+1}| \le n|\alpha - \beta|\gamma^{n-1}$$

Using (3.3.3) now gives (recall we are supposing $\sigma^2 = 1$)

(c) $$|\varphi(t) - 1 + t^2/2| \le \rho|t|^3/6$$

so if $t^2 \le 2$

(d) $$|\varphi(t)| \le 1 - t^2/2 + \rho|t|^3/6$$

Let $L = 4\sqrt{n}/3\rho$. If $|\theta| \le L$, then by (d) and the fact $\rho|\theta|/\sqrt{n} \le 4/3$

$$\begin{aligned}|\varphi(\theta/\sqrt{n})| &\le 1 - \theta^2/2n + \rho|\theta|^3/6n^{3/2}\\ &\le 1 - 5\theta^2/18n \le \exp(-5\theta^2/18n)\end{aligned}$$

since $1 - x \le e^{-x}$. We will now apply (b) with

$$\alpha = \varphi(\theta/\sqrt{n}) \qquad \beta = \exp(-\theta^2/2n) \qquad \gamma = \exp(-5\theta^2/18n)$$

Since we are supposing $n \ge 10$

(e) $$\gamma^{n-1} \le \exp(-\theta^2/4)$$

For the other part of (b), we write

$$n|\alpha - \beta| \le n|\varphi(\theta/\sqrt{n}) - 1 + \theta^2/2n| + n|1 - \theta^2/2n - \exp(-\theta^2/2n)|$$

To bound the first term on the right-hand side, observe that (c) implies

$$n|\varphi(\theta/\sqrt{n}) - 1 + \theta^2/2n| \le \rho|\theta|^3/6n^{1/2}$$

For the second term, note that if $0 < x < 1$, then we have an alternating series with decreasing terms, so

$$|e^{-x} - (1 - x)| = \left|-\frac{x^2}{2!} + \frac{x^3}{3!} - \dots\right| \le \frac{x^2}{2}$$

Taking $x = \theta^2/2n$, it follows that for $|\theta| \le L \le \sqrt{2n}$

$$n|1 - \theta^2/2n - \exp(-\theta^2/2n)| \le \theta^4/8n$$

Combining this with our estimate on the first term gives

$$\text{(f)} \qquad n|\alpha - \beta| \le \rho|\theta|^3/6n^{1/2} + \theta^4/8n$$

Using (f) and (e) in (b), gives

$$\begin{aligned}\frac{1}{|\theta|}|\varphi^n(\theta/\sqrt{n}) - \exp(-\theta^2/2)| &\le \exp(-\theta^2/4)\left\{\frac{\rho\theta^2}{6n^{1/2}} + \frac{|\theta|^3}{8n}\right\} \\ &\le \frac{1}{L}\exp(-\theta^2/4)\left\{\frac{2\theta^2}{9} + \frac{|\theta|^3}{18}\right\}\end{aligned}$$

since $\rho/\sqrt{n} = 4/3L$, and $1/n = 1/\sqrt{n} \cdot 1/\sqrt{n} \le 4/3L \cdot 1/3$ since $\rho \ge 1$ and $n \ge 10$. Using the last result and (a) in Lemma 3.4.11 gives

$$\pi L|F_n(x) - \mathcal{N}(x)| \le \int \exp(-\theta^2/4)\left\{\frac{2\theta^2}{9} + \frac{|\theta|^3}{18}\right\} d\theta + 9.6$$

Recalling $L = 4\sqrt{n}/3\rho$, we see that the last result is of the form $|F_n(x) - \mathcal{N}(x)| \le C\rho/\sqrt{n}$. To evaluate the constant, we observe

$$\int (2\pi a)^{-1/2} x^2 \exp(-x^2/2a)dx = a$$

and writing $x^3 = 2x^2 \cdot x/2$ and integrating by parts

$$\begin{aligned}2\int_0^\infty x^3 \exp(-x^2/4)\,dx &= 2\int_0^\infty 4x\exp(-x^2/4)\,dx \\ &= -16e^{-x^2/4}\Big|_0^\infty = 16\end{aligned}$$

This gives us

$$|F_n(x) - \mathcal{N}(x)| \le \frac{1}{\pi}\cdot\frac{3}{4}\left(\frac{2}{9}\cdot 2\cdot\sqrt{4\pi} + \frac{16}{18} + 9.6\right)\frac{\rho}{\sqrt{n}} < 3\frac{\rho}{\sqrt{n}}$$

For the last step, you have to get out your calculator or trust Feller. ■

## 3.5 Local Limit Theorems*

In Section 3.1 we saw that if $X_1, X_2, \ldots$ are i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and $k_n$ is a sequence of integers with $2k_n/(2n)^{1/2} \to x$, then

$$P(S_{2n} = 2k_n) \sim (\pi n)^{-1/2}\exp(-x^2/2)$$

In this section, we will prove two theorems that generalize the last result. We begin with two definitions. A random variable $X$ has a **lattice distribution** if there are constants $b$ and $h > 0$ so that $P(X \in b + h\mathbf{Z}) = 1$, where $b + h\mathbf{Z} = \{b + hz : z \in \mathbf{Z}\}$.

The largest $h$ for which the last statement holds is called the **span** of the distribution.

**Example 3.5.1.** If $P(X = 1) = P(X = -1) = 1/2$, then $X$ has a lattice distribution with span 2. When $h$ is 2, one possible choice is $b = -1$.

The next result relates the last definition to the characteristic function. To check (ii) in its statement, note that in the last example $E(e^{itX}) = \cos t$ has $|\cos(t)| = 1$ when $t = n\pi$.

**Theorem 3.5.1.** *Let $\varphi(t) = Ee^{itX}$. There are only three possibilities.*
*(i) $|\varphi(t)| < 1$ for all $t \neq 0$.*
*(ii) There is a $\lambda > 0$ so that $|\varphi(\lambda)| = 1$ and $|\varphi(t)| < 1$ for $0 < t < \lambda$. In this case, $X$ has a lattice distribution with span $2\pi/\lambda$.*
*(iii) $|\varphi(t)| = 1$ for all $t$. In this case, $X = b$ a.s. for some $b$.*

*Proof.* We begin with (ii). It suffices to show that $|\varphi(t)| = 1$ if and only if $P(X \in b + (2\pi/t)\mathbf{Z}) = 1$ for some $b$. First, if $P(X \in b + (2\pi/t)\mathbf{Z}) = 1$, then

$$\varphi(t) = Ee^{itX} = e^{itb} \sum_{n \in \mathbf{Z}} e^{i2\pi n} P(X = b + (2\pi/t)n) = e^{itb}$$

Conversely, if $|\varphi(t)| = 1$, then there is equality in the inequality $|Ee^{itX}| \leq E|e^{itX}|$, so by Exercise 1.6.1 the distribution of $e^{itX}$ must be concentrated at some point $e^{itb}$, and $P(X \in b + (2\pi/t)\mathbf{Z}) = 1$.

To prove trichotomy now, we suppose that (i) and (ii) do not hold, that is, there is a sequence $t_n \downarrow 0$ so that $|\varphi(t_n)| = 1$. The first paragraph shows that there is a $b_n$ so that $P(X \in b_n + (2\pi/t_n)\mathbf{Z}) = 1$. Without loss of generality, we can pick $b_n \in (-\pi/t_n, \pi/t_n]$. As $n \to \infty$, $P(X \notin (-\pi/t_n, \pi/t_n]) \to 0$, so it follows that $P(X = b_n) \to 1$. This is only possible if $b_n = b$ for $n \geq N$, and $P(X = b) = 1$. ■

We call the three cases in Theorem 3.5.1 (i) **nonlattice**, (ii) **lattice**, and (iii) **degenerate**. The reader should notice that this means that lattice random variables are by definition nondegenerate. Before we turn to the main business of this section, we would like to introduce one more special case. If $X$ is a lattice distribution and we can take $b = 0$, i.e., $P(X \in h\mathbf{Z}) = 1$, then $X$ is said to be **arithmetic**. In this case, if $\lambda = 2\pi/h$ then $\varphi(\lambda) = 1$ and $\varphi$ is periodic: $\varphi(t + \lambda) = \varphi(t)$.

Our first local limit theorem is for the lattice case. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$, $EX_i^2 = \sigma^2 \in (0, \infty)$, and having a common lattice distribution with span $h$. If $S_n = X_1 + \cdots + X_n$ and $P(X_i \in b + h\mathbf{Z}) = 1$ then $P(S_n \in nb + h\mathbf{Z}) = 1$. We put

$$p_n(x) = P(S_n/\sqrt{n} = x) \quad \text{for } x \in \mathcal{L}_n = \{(nb + hz)/\sqrt{n} : z \in \mathbf{Z}\}$$

and

$$n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2) \quad \text{for } x \in (-\infty, \infty)$$

**Theorem 3.5.2.** *Under the hypotheses above, as $n \to \infty$*

$$\sup_{x \in \mathcal{L}_n} \left| \frac{n^{1/2}}{h} p_n(x) - n(x) \right| \to 0$$

**Remark.** To explain the statement, note that if we followed the approach in Example 3.4.3, then we would conclude that for $x \in \mathcal{L}_n$,

$$p_n(x) \approx \int_{x-h/2\sqrt{n}}^{x+h/2\sqrt{n}} n(y)\,dy \approx \frac{h}{\sqrt{n}} n(x)$$

*Proof.* Let $Y$ be a random variable with $P(Y \in a + \theta\mathbf{Z}) = 1$ and $\psi(t) = E\exp(itY)$. It follows from part (iii) of Exercise 3.3.2 that

$$P(Y = x) = \frac{1}{2\pi/\theta} \int_{-\pi/\theta}^{\pi/\theta} e^{-itx} \psi(t)\,dt$$

Using this formula with $\theta = h/\sqrt{n}$, $\psi(t) = E\exp(itS_n/\sqrt{n}) = \varphi^n(t/\sqrt{n})$, and then multiplying each side by $1/\theta$ gives

$$\frac{n^{1/2}}{h} p_n(x) = \frac{1}{2\pi} \int_{-\pi\sqrt{n}/h}^{\pi\sqrt{n}/h} e^{-itx} \varphi^n(t/\sqrt{n})\,dt$$

Using the inversion formula, Theorem 3.3.5, for $n(x)$, which has ch.f. $\exp(-\sigma^2 t^2/2)$, gives

$$n(x) = \frac{1}{2\pi} \int e^{-itx} \exp(-\sigma^2 t^2/2)\,dt$$

Subtracting the last two equations gives (recall $\pi > 1$, $|e^{-itx}| \le 1$)

$$\begin{aligned}\left| \frac{n^{1/2}}{h} p_n(x) - n(x) \right| \le & \int_{-\pi\sqrt{n}/h}^{\pi\sqrt{n}/h} |\varphi^n(t/\sqrt{n}) - \exp(-\sigma^2 t^2/2)|\,dt \\ & + \int_{\pi\sqrt{n}/h}^{\infty} \exp(-\sigma^2 t^2/2)\,dt\end{aligned}$$

The right-hand side is independent of $x$, so to prove Theorem 3.5.2 it suffices to show that it approaches 0. The second integral clearly $\to 0$. To estimate the first integral, we observe that $\varphi^n(t/\sqrt{n}) \to \exp(-\sigma^2 t^2/2)$, so the integrand goes to 0 and it is now just a question of "applying the dominated convergence theorem."

To do this, we will divide the integral into three pieces. The bounded convergence theorem implies that for any $A < \infty$ the integral over $(-A, A)$ approaches 0. To estimate the integral over $(-A, A)^c$, we observe that since $EX_i = 0$ and $EX_i^2 = \sigma^2$, formula (3.3.3) and the triangle inequality imply that

$$|\varphi(u)| \le |1 - \sigma^2 u^2/2| + \frac{u^2}{2} E(\min(|u| \cdot |X|^3, 6|X|^2))$$

The last expected value $\to 0$ as $u \to 0$. This means we can pick $\delta > 0$ so that if $|u| < \delta$, it is $\le \sigma^2/2$ and hence

$$|\varphi(u)| \le 1 - \sigma^2 u^2/2 + \sigma^2 u^2/4 = 1 - \sigma^2 u^2/4 \le \exp(-\sigma^2 u^2/4)$$

since $1 - x \le e^{-x}$. Applying the last result to $u = t/\sqrt{n}$, we see that for $t \le \delta\sqrt{n}$

$$(*) \qquad |\varphi(t/\sqrt{n})^n| \le \exp(-\sigma^2 t^2/4)$$

So the integral over $(-\delta\sqrt{n}, \delta\sqrt{n}) - (-A, A)$ is smaller than

$$2\int_A^{\delta\sqrt{n}} \exp(-\sigma^2 t^2/4)\, dt$$

which is small if $A$ is large.

To estimate the rest of the integral we observe that since $X$ has span $h$, Theorem 3.5.1 implies $|\varphi(u)| \ne 1$ for $u \in [\delta, \pi/h]$. $\varphi$ is continuous, so there is an $\eta < 1$ so that $|\varphi(u)| \le \eta < 1$ for $|u| \in [\delta, \pi/h]$. Letting $u = t/\sqrt{n}$ again, we see that the integral over $[-\pi\sqrt{n}/h, \pi\sqrt{n}/h] - (-\delta\sqrt{n}, \delta\sqrt{n})$ is smaller than

$$2\int_{\delta\sqrt{n}}^{\pi\sqrt{n}/h} \eta^n + \exp(-\sigma^2 t^2/2)\, dt$$

which $\to 0$ as $n \to \infty$. This completes the proof. ■

We turn now to the nonlattice case. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$, $EX_i^2 = \sigma^2 \in (0, \infty)$, and having a common characteristic function $\varphi(t)$ that has $|\varphi(t)| < 1$ for all $t \ne 0$. Let $S_n = X_1 + \cdots + X_n$ and $n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$.

**Theorem 3.5.3.** *Under the hypotheses above, if $x_n/\sqrt{n} \to x$ and $a < b$,*

$$\sqrt{n}P(S_n \in (x_n + a, x_n + b)) \to (b - a)n(x)$$

**Remark.** The proof of this result has to be a little devious because the assumption above does not give us much control over the behavior of $\varphi$. For a bad example, let $q_1, q_2, \ldots$ be an enumeration of the positive rationals that has $q_n \le n$. Suppose

$$P(X = q_n) = P(X = -q_n) = 1/2^{n+1}$$

In this case $EX = 0$, $EX^2 < \infty$, and the distribution is nonlattice. However, the characteristic function has $\limsup_{t\to\infty} |\varphi(t)| = 1$.

*Proof.* To tame bad ch.f.'s, we use a trick. Let $\delta > 0$

$$h_0(y) = \frac{1}{\pi} \cdot \frac{1 - \cos \delta y}{\delta y^2}$$

be the density of the Polya's distribution and let $h_\theta(x) = e^{i\theta x} h_0(x)$. If we introduce the Fourier transform

$$\hat{g}(u) = \int e^{iuy} g(y)\, dy$$

then it follows from Example 3.3.8 that

$$\hat{h}_0(u) = \begin{cases} 1 - |u/\delta| & \text{if } |u| \le \delta \\ 0 & \text{otherwise} \end{cases}$$

and it is easy to see that $\hat{h}_\theta(u) = \hat{h}_0(u+\theta)$. We will show that for any $\theta$

(a) $$\sqrt{n}\, Eh_\theta(S_n - x_n) \to n(x) \int h_\theta(y)\,dy$$

Before proving (a), we will show it implies Theorem 3.5.3. Let

$$\mu_n(A) = \sqrt{n} P(S_n - x_n \in A), \quad \text{and} \quad \mu(A) = n(x)|A|$$

where $|A|$ = the Lebesgue measure of $A$. Let

$$\alpha_n = \sqrt{n}\, Eh_0(S_n - x_n) \quad \text{and} \quad \alpha = n(x) \int h_0(y)\,dy = n(x)$$

Finally, define probability measures by

$$\nu_n(B) = \frac{1}{\alpha_n} \int_B h_0(y)\mu_n(dy), \quad \text{and} \quad \nu(B) = \frac{1}{\alpha} \int_B h_0(y)\mu(dy)$$

Taking $\theta = 0$ in (a) we see $\alpha_n \to \alpha$ and so (a) implies

(b) $$\int e^{i\theta y} \nu_n(dy) \to \int e^{i\theta y} \nu(dy)$$

Since this holds for all $\theta$, it follows from Theorem 3.3.6 that $\nu_n \Rightarrow \nu$. Now if $|a|, |b| < 2\pi/\delta$, then the function

$$k(y) = \frac{1}{h_0(y)} \cdot 1_{(a,b)}(y)$$

is bounded and continuous a.s. with respect to $\nu$ so it follows from Theorem 3.2.4 that

$$\int k(y)\nu_n(dy) \to \int k(y)\nu(dy)$$

Since $\alpha_n \to \alpha$, this implies

$$\sqrt{n} P(S_n \in (x_n + a, x_n + b)) \to (b-a)n(x)$$

which is the conclusion of Theorem 3.5.3.

Turning now to the proof of (a), the inversion formula, Theorem 3.3.5, implies

$$h_0(x) = \frac{1}{2\pi} \int e^{-iux} \hat{h}_0(u)\,du$$

Recalling the definition of $h_\theta$, using the last result, and changing variables $u = v + \theta$, we have

$$\begin{aligned} h_\theta(x) = e^{i\theta x} h_0(x) &= \frac{1}{2\pi} \int e^{-i(u-\theta)x} \hat{h}_0(u)\,du \\ &= \frac{1}{2\pi} \int e^{-ivx} \hat{h}_\theta(v)\,dv \end{aligned}$$

since $\hat{h}_\theta(v) = \hat{h}_0(v+\theta)$. Letting $F_n$ be the distribution of $S_n - x_n$ and integrating gives

$$Eh_\theta(S_n - x_n) = \frac{1}{2\pi}\int\int e^{-iux}\hat{h}_\theta(u)\,du\,dF_n(x)$$
$$= \frac{1}{2\pi}\int\int e^{-iux}\,dF_n(x)\hat{h}_\theta(u)\,du$$

by Fubini's theorem. (Recall that $\hat{h}_\theta(u)$ has compact support and $F_n$ is a distribution function.) Using (e) of Theorem 3.3.1, we see that the last expression

$$= \frac{1}{2\pi}\int \varphi(-u)^n e^{iux_n}\hat{h}_\theta(u)\,du$$

To take the limit as $n \to \infty$ of this integral, let $[-M, M]$ be an interval with $\hat{h}_\theta(u) = 0$ for $u \notin [-M, M]$. By $(*)$ above, we can pick $\delta$ so that for $|u| < \delta$

(c) $$|\varphi(u)| \le \exp(-\sigma^2 u^2/4)$$

Let $I = [-\delta, \delta]$ and $J = [-M, M] - I$. Since $|\varphi(u)| < 1$ for $u \neq 0$ and $\varphi$ is continuous, there is a constant $\eta < 1$ so that $|\varphi(u)| \le \eta < 1$ for $u \in J$. Since $|\hat{h}_\theta(u)| \le 1$, this implies that

$$\left|\frac{\sqrt{n}}{2\pi}\int_J \varphi(-u)^n e^{iux_n}\hat{h}_\theta(u)\,du\right| \le \frac{\sqrt{n}}{2\pi}\cdot 2M\eta^n \to 0$$

as $n \to \infty$. For the integral over $I$, change variables $u = t/\sqrt{n}$ to get

$$\frac{1}{2\pi}\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \varphi(-t/\sqrt{n})^n e^{itx_n/\sqrt{n}}\hat{h}_\theta(t/\sqrt{n})\,dt$$

The central limit theorem implies $\varphi(-t/\sqrt{n})^n \to \exp(-\sigma^2 t^2/2)$. Using (c) now and the dominated convergence theorem gives (recall $x_n/\sqrt{n} \to x$)

$$\frac{\sqrt{n}}{2\pi}\int_I \varphi(-u)^n e^{iux_n}\hat{h}_\theta(u)\,du \to \frac{1}{2\pi}\int \exp(-\sigma^2 t^2/2)e^{itx}\hat{h}_\theta(0)\,dt$$
$$= n(x)\hat{h}_\theta(0) = n(x)\int h_\theta(y)\,dy$$

by the inversion formula, Theorem 3.3.5, and the definition of $\hat{h}_\theta(0)$. This proves (a) and completes the proof of Theorem 3.5.3. ∎

## 3.6 Poisson Convergence

### 3.6.1 *The Basic Limit Theorem*

Our first result is sometimes facetiously called the "weak law of small numbers" or the "law of rare events." These names derive from the fact that the Poisson appears as the limit of a sum of indicators of events that have small probabilities.

**Theorem 3.6.1.** *For each $n$, let $X_{n,m}$, $1 \le m \le n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}$, $P(X_{n,m} = 0) = 1 - p_{n,m}$. Suppose*
*(i)* $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$, *and*
*(ii)* $\max_{1 \le m \le n} p_{n,m} \to 0$.
*If $S_n = X_{n,1} + \cdots + X_{n,n}$ then $S_n \Rightarrow Z$ where $Z$ is Poisson$(\lambda)$.*

Here Poisson($\lambda$) is shorthand for Poisson distribution with mean $\lambda$, that is,

$$P(Z = k) = e^{-\lambda}\lambda^k/k!$$

Note that in the spirit of the Lindeberg-Feller theorem, no single term contributes very much to the sum. In contrast to that theorem, the contributions, when positive, are not small.

*First proof.* Let $\varphi_{n,m}(t) = E(\exp(itX_{n,m})) = (1 - p_{n,m}) + p_{n,m}e^{it}$ and let $S_n = X_{n,1} + \cdots + X_{n,n}$. Then

$$E\exp(itS_n) = \prod_{m=1}^{n}(1 + p_{n,m}(e^{it} - 1))$$

Let $0 \le p \le 1$. $|\exp(p(e^{it} - 1))| = \exp(p\,\mathrm{Re}\,(e^{it} - 1)) \le 1$ and $|1 + p(e^{it} - 1)| \le 1$ since it is on the line segment connecting 1 to $e^{it}$. Using Lemma 3.4.3 with $\theta = 1$ and then Lemma 3.4.4, which is valid when $\max_m p_{n,m} \le 1/2$ since $|e^{it} - 1| \le 2$,

$$\begin{aligned}
&\left|\exp\left(\sum_{m=1}^{n} p_{n,m}(e^{it} - 1)\right) - \prod_{m=1}^{n}\{1 + p_{n,m}(e^{it} - 1)\}\right| \\
&\quad \le \sum_{m=1}^{n} \left|\exp(p_{n,m}(e^{it} - 1)) - \{1 + p_{n,m}(e^{it} - 1)\}\right| \\
&\quad \le \sum_{m=1}^{n} p_{n,m}^2 |e^{it} - 1|^2
\end{aligned}$$

Using $|e^{it} - 1| \le 2$ again, it follows that the last expression

$$\le 4\left(\max_{1 \le m \le n} p_{n,m}\right)\sum_{m=1}^{n} p_{n,m} \to 0$$

by assumptions (i) and (ii). The last conclusion and $\sum_{m=1}^{n} p_{n,m} \to \lambda$ imply

$$E\exp(itS_n) \to \exp(\lambda(e^{it} - 1))$$

To complete the proof now, we consult Example 3.3.2 for the ch.f. of the Poisson distribution and apply Theorem 3.3.6. ■

We will now consider some concrete situations in which Theorem 3.6.1 can be applied. In each case we are considering a situation in which $p_{n,m} = c/n$, so we approximate the distribution of the sum by a Poisson with mean $c$.

**Example 3.6.1.** In a calculus class with 400 students, the number of students who have their birthday on the day of the final exam has approximately a Poisson distribution with mean $400/365 = 1.096$. This means that the probability no one was born on that date is about $e^{-1.096} = 0.334$. Similar reasoning shows that the number of babies born on a given day or the number of people who arrive at a bank between 1:15 and 1:30 should have a Poisson distribution.

**Example 3.6.2.** Suppose we roll two dice 36 times. The probability of "double ones" (one on each die) is $1/36$, so the number of times this occurs should have approximately a Poisson distribution with mean 1. Comparing the Poisson approximation with exact probabilities shows that the agreement is good even though the number of trials is small.

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Poisson | 0.3678 | 0.3678 | 0.1839 | 0.0613 |
| exact | 0.3627 | 0.3730 | 0.1865 | 0.0604 |

After we give the second proof of Theorem 3.6.1, we will discuss rates of convergence. Those results will show that for large $n$ the largest discrepancy occurs for $k = 1$ and is about $1/2en$ ( $= 0.0051$ in this case).

**Example 3.6.3.** Let $\xi_{n,1}, \dots, \xi_{n,n}$ be independent and uniformly distributed over $[-n, n]$. Let $X_{n,m} = 1$ if $\xi_{n,m} \in (a, b)$, $= 0$ otherwise. $S_n$ is the number of points that land in $(a, b)$. $p_{n,m} = (b - a)/2n$ so $\sum_m p_{n,m} = (b - a)/2$. This shows that (i) and (ii) in Theorem 3.6.1 hold, and we conclude that $S_n \Rightarrow Z$, a Poisson r.v. with mean $(b - a)/2$. A two-dimensional version of the last theorem might explain why the statistics of flying bomb hits in the South of London during World War II fit a Poisson distribution. As Feller, Vol. I (1968), pp. 160–161 reports, the area was divided into 576 areas of 1/4 square kilometers each. The total number of hits was 537 for an average of 0.9323 per cell. The following table compares $N_k$ the number of cells with $k$ hits with the predictions of the Poisson approximation.

| $k$ | 0 | 1 | 2 | 3 | 4 | $\geq 5$ |
|---|---|---|---|---|---|---|
| $N_k$ | 229 | 211 | 93 | 35 | 7 | 1 |
| Poisson | 226.74 | 211.39 | 98.54 | 30.62 | 7.14 | 1.57 |

For other observations fitting a Poisson distribution, see Feller, Vol. I (1968), Section VI.7.

Our second proof of Theorem 3.6.1 requires a little more work but provides information about the rate of convergence. We begin by defining the **total variation distance** between two measures on a countable set $S$.

$$\|\mu - \nu\| \equiv \frac{1}{2} \sum_z |\mu(z) - \nu(z)| = \sup_{A \subset S} |\mu(A) - \nu(A)|$$

The first equality is a definition. To prove the second, note that for any $A$

$$\sum_z |\mu(z) - \nu(z)| \ge |\mu(A) - \nu(A)| + |\mu(A^c) - \nu(A^c)| = 2|\mu(A) - \nu(A)|$$

and there is equality when $A = \{z : \mu(z) \ge \nu(z)\}$.

**Exercise 3.6.1.** Show that (i) $d(\mu, \nu) = \|\mu - \nu\|$ defines a metric on probability measures on $\mathbf{Z}$ and (ii) $\|\mu_n - \mu\| \to 0$ if and only if $\mu_n(x) \to \mu(x)$ for each $x \in \mathbf{Z}$, which by Exercise 3.2.11 is equivalent to $\mu_n \Rightarrow \mu$.

**Exercise 3.6.2.** Show that $\|\mu - \nu\| \le 2\delta$ if and only if there are random variables $X$ and $Y$ with distributions $\mu$ and $\nu$ so that $P(X \ne Y) \le \delta$.

The next three lemmas are the keys to our second proof.

**Lemma 3.6.2.** *If $\mu_1 \times \mu_2$ denotes the product measure on $\mathbf{Z} \times \mathbf{Z}$ that has $(\mu_1 \times \mu_2)(x, y) = \mu_1(x)\mu_2(y)$, then*

$$\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| \le \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$$

*Proof.* $2\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\| = \sum_{x,y} |\mu_1(x)\mu_2(y) - \nu_1(x)\nu_2(y)|$

$$\begin{aligned}
&\le \sum_{x,y} |\mu_1(x)\mu_2(y) - \nu_1(x)\mu_2(y)| + \sum_{x,y} |\nu_1(x)\mu_2(y) - \nu_1(x)\nu_2(y)| \\
&= \sum_y \mu_2(y) \sum_x |\mu_1(x) - \nu_1(x)| + \sum_x \nu_1(x) \sum_y |\mu_2(y) - \nu_2(y)| \\
&= 2\|\mu_1 - \nu_1\| + 2\|\mu_2 - \nu_2\|
\end{aligned}$$

which gives the desired result. ■

**Lemma 3.6.3.** *If $\mu_1 * \mu_2$ denotes the convolution of $\mu_1$ and $\mu_2$, that is,*

$$\mu_1 * \mu_2(x) = \sum_y \mu_1(x - y)\mu_2(y)$$

*then $\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| \le \|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|$*

*Proof.* $2\|\mu_1 * \mu_2 - \nu_1 * \nu_2\| = \sum_x \left| \sum_y \mu_1(x - y)\mu_2(y) - \sum_y \nu_1(x - y)\nu_2(y) \right|$

$$\begin{aligned}
&\le \sum_x \sum_y |\mu_1(x - y)\mu_2(y) - \nu_1(x - y)\nu_2(y)| \\
&= 2\|\mu_1 \times \mu_2 - \nu_1 \times \nu_2\|
\end{aligned}$$

which gives the desired result. ■

**Lemma 3.6.4.** *Let $\mu$ be the measure with $\mu(1) = p$ and $\mu(0) = 1 - p$. Let $\nu$ be a Poisson distribution with mean $p$. Then $\|\mu - \nu\| \le p^2$.*

*Proof.* $2\|\mu - \nu\| = |\mu(0) - \nu(0)| + |\mu(1) - \nu(1)| + \sum_{n\ge 2} \nu(n)$

$$= |1 - p - e^{-p}| + |p - p\, e^{-p}| + 1 - e^{-p}(1+p)$$

Since $1 - x \le e^{-x} \le 1$ for $x \ge 0$, the above

$$= e^{-p} - 1 + p + p(1 - e^{-p}) + 1 - e^{-p} - pe^{-p}$$
$$= 2p(1 - e^{-p}) \le 2p^2$$

which gives the desired result. ∎

*Second proof of Theorem 3.6.1.* Let $\mu_{n,m}$ be the distribution of $X_{n,m}$. Let $\mu_n$ be the distribution of $S_n$. Let $\nu_{n,m}$, $\nu_n$, and $\nu$ be Poisson distributions with means $p_{n,m}$, $\lambda_n = \sum_{m\le n} p_{n,m}$, and $\lambda$, respectively. Since $\mu_n = \mu_{n,1} * \cdots * \mu_{n,n}$ and $\nu_n = \nu_{n,1} * \cdots * \nu_{n,n}$, Lemmas 3.6.3, 3.6.2, and 3.6.4 imply

$$\|\mu_n - \nu_n\| \le \sum_{m=1}^{n} \|\mu_{n,m} - \nu_{n,m}\| \le 2\sum_{m=1}^{n} p_{n,m}^2 \tag{3.6.1}$$

Using the definition of total variation distance now gives

$$\sup_A |\mu_n(A) - \nu_n(A)| \le \sum_{m=1}^{n} p_{n,m}^2$$

Assumptions (i) and (ii) imply that the right-hand side $\to 0$. Since $\nu_n \Rightarrow \nu$ as $n \to \infty$, the result follows. ∎

**Remark.** The proof above is due to Hodges and Le Cam (1960). By different methods, C. Stein (1987) (see (43) on p. 89) has proved

$$\sup_A |\mu_n(A) - \nu_n(A)| \le (\lambda \vee 1)^{-1} \sum_{m=1}^{n} p_{n,m}^2$$

**Rates of convergence.** When $p_{n,m} = 1/n$, (3.6.1) becomes

$$\sup_A |\mu_n(A) - \nu_n(A)| \le 1/n$$

To assess the quality of this bound, we will compare the Poisson and binomial probabilities for $k$ successes.

| $k$ | Poisson | Binomial |
|---|---|---|
| 0 | $e^{-1}$ | $\left(1 - \frac{1}{n}\right)^n$ |
| 1 | $e^{-1}$ | $n \cdot n^{-1}\left(1 - \frac{1}{n}\right)^{n-1} = \left(1 - \frac{1}{n}\right)^{n-1}$ |
| 2 | $e^{-1}/2!$ | $\binom{n}{2} n^{-2}\left(1 - \frac{1}{n}\right)^{n-2} = \left(1 - \frac{1}{n}\right)^{n-1} / \, 2!$ |
| 3 | $e^{-1}/3!$ | $\binom{n}{3} n^{-3}\left(1 - \frac{1}{n}\right)^{n-3} = \left(1 - \frac{2}{n}\right)\left(1 - \frac{1}{n}\right)^{n-2} \Big/ \, 3!$ |

Since $(1-x) \le e^{-x}$, we have $\mu_n(0) - \nu_n(0) \le 0$. Expanding

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

gives

$$(n-1)\log\left(1 - \frac{1}{n}\right) = -\frac{n-1}{n} - \frac{n-1}{2n^2} - \dots = -1 + \frac{1}{2n} + O(n^{-2})$$

So

$$n\left(\left(1-\frac{1}{n}\right)^{n-1} - e^{-1}\right) = ne^{-1}\left(\exp\{1/2n + O(n^{-2})\} - 1\right) \to e^{-1}/2$$

and it follows that

$$n(\mu_n(1) - \nu_n(1)) \to e^{-1}/2$$
$$n(\mu_n(2) - \nu_n(2)) \to e^{-1}/4$$

For $k \ge 3$, using $(1-2/n) \le (1-1/n)^2$ and $(1-x) \le e^{-x}$ shows $\mu_n(k) - \nu_n(k) \le 0$, so

$$\sup_{A \subset \mathbf{Z}} |\mu_n(A) - \nu_n(A)| \approx 3/4en$$

There is a large literature on Poisson approximations for dependent events. Here we consider

### 3.6.2 *Two Examples with Dependence*

**Example 3.6.4. Matching.** Let $\pi$ be a random permutation of $\{1, 2, \dots, n\}$, let $X_{n,m} = 1$ if $m$ is a fixed point (0 otherwise), and let $S_n = X_{n,1} + \dots + X_{n,n}$ be the number of fixed points. We want to compute $P(S_n = 0)$. (For a more exciting story, consider men checking hats or wives swapping husbands.) Let $A_{n,m} = \{X_{n,m} = 1\}$. The inclusion-exclusion formula implies

$$\begin{aligned} P\left(\cup_{m=1}^n A_m\right) &= \sum_m P(A_m) - \sum_{\ell<m} P(A_\ell \cap A_m) \\ &\quad + \sum_{k<\ell<m} P(A_k \cap A_\ell \cap A_m) - \dots \\ &= n \cdot \frac{1}{n} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} - \dots \end{aligned}$$

since the number of permutations with $k$ specified fixed points is $(n-k)!$ Canceling some factorials gives

$$P(S_n > 0) = \sum_{m=1}^n \frac{(-1)^{m-1}}{m!} \quad \text{so} \quad P(S_n = 0) = \sum_{m=0}^n \frac{(-1)^m}{m!}$$

Recognizing the second sum as the first $n+1$ terms in the expansion of $e^{-1}$ gives

$$|P(S_n = 0) - e^{-1}| = \left| \sum_{m=n+1}^{\infty} \frac{(-1)^m}{m!} \right|$$
$$\le \frac{1}{(n+1)!} \left| \sum_{k=0}^{\infty} (n+2)^{-k} \right| = \frac{1}{(n+1)!} \cdot \left(1 - \frac{1}{n+2}\right)^{-1}$$

a much better rate of convergence than $1/n$. To compute the other probabilities, we observe that by considering the locations of the fixed points

$$P(S_n = k) = \binom{n}{k} \frac{1}{n(n-1)\cdots(n-k+1)} P(S_{n-k} = 0)$$
$$= \frac{1}{k!} P(S_{n-k} = 0) \to e^{-1}/k!$$

**Example 3.6.5. Occupancy problem.** Suppose that $r$ balls are placed at random into $n$ boxes. It follows from the Poisson approximation to the binomial that if $n \to \infty$ and $r/n \to c$, then the number of balls in a given box will approach a Poisson distribution with mean $c$. The last observation should explain why the fraction of empty boxes approached $e^{-c}$ in Example 2.2.5. Here we will show:

**Theorem 3.6.5.** *If $ne^{-r/n} \to \lambda \in [0, \infty)$ the number of empty boxes approaches a Poisson distribution with mean $\lambda$.*

*Proof.* To see where the answer comes from, notice that in the Poisson approximation the probability that a given box is empty is $e^{-r/n} \approx \lambda/n$, so if the occupancy of the various boxes were independent, the result would follow from Theorem 3.6.1. To prove the result, we begin by observing

$$P(\text{ boxes } i_1, i_2, \ldots, i_k \text{ are empty }) = \left(1 - \frac{k}{n}\right)^r$$

If we let $p_m(r,n) =$ the probability exactly $m$ boxes are empty when $r$ balls are put in $n$ boxes, then $P(\text{ no empty box }) = 1 - P(\text{at least one empty box})$. So by inclusion-exclusion

(a) $$p_0(r,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 - \frac{k}{n}\right)^r$$

By considering the locations of the empty boxes

(b) $$p_m(r,n) = \binom{n}{m} \left(1 - \frac{m}{n}\right)^r p_0(r, n-m)$$

To evaluate the limit of $p_m(r,n)$ we begin by showing that if $ne^{-r/n} \to \lambda$ then

(c) $$\binom{n}{m} \left(1 - \frac{m}{n}\right)^r \to \lambda^m/m!$$

One half of this is easy. Since $(1-x) \le e^{-x}$ and $ne^{-r/n} \to \lambda$

(d) $$\binom{n}{m}\left(1-\frac{m}{n}\right)^r \le \frac{n^m}{m!}e^{-mr/n} \to \lambda^m/m!$$

For the other direction, observe $\binom{n}{m} \ge (n-m)^m/m!$ so

$$\binom{n}{m}\left(1-\frac{m}{n}\right)^r \ge \left(1-\frac{m}{n}\right)^{m+r} n^m/m!$$

Now $(1-m/n)^m \to 1$ as $n \to \infty$ and $1/m!$ is a constant. To deal with the rest, we note that if $0 \le t \le 1/2$ then

$$\begin{aligned}\log(1-t) &= -t - t^2/2 - t^3/3\ldots\\ &\ge -t - \frac{t^2}{2}\left(1+2^{-1}+2^{-2}+\cdots\right) = -t-t^2\end{aligned}$$

so we have

$$\log\left(n^m\left(1-\frac{m}{n}\right)^r\right) \ge m\log n - rm/n - r(m/n)^2$$

Our assumption $ne^{-r/n} \to \lambda$ means

$$r = n\log n - n\log\lambda + o(n)$$

so $r(m/n)^2 \to 0$. Multiplying the last display by $m/n$ and rearranging gives $m\log n - rm/n \to m\log\lambda$. Combining the last two results shows

$$\liminf_{n\to\infty} n^m\left(1-\frac{m}{n}\right)^r \ge \lambda^m$$

and (c) follows. From (a), (c), and the dominated convergence theorem (using (d) to get the domination), we get

(e) if $ne^{-r/n} \to \lambda$ then $p_0(r,n) \to \sum_{k=0}^{\infty}(-1)^k\frac{\lambda^k}{k!} = e^{-\lambda}$

For fixed $m$, $(n-m)e^{-r/(n-m)} \to \lambda$, so it follows from (e) that $p_0(r,n-m) \to e^{-\lambda}$. Combining this with (b) and (c) completes the proof. ■

**Example 3.6.6. Coupon collector's problem.** Let $X_1, X_2, \ldots$ be i.i.d. uniform on $\{1,2,\ldots,n\}$ and $T_n = \inf\{m : \{X_1,\ldots X_m\} = \{1,2,\ldots,n\}\}$. Since $T_n \le m$ if and only if $m$ balls fill up all $n$ boxes, it follows from Theorem 3.6.5 that

$$P(T_n - n\log n \le nx) \to \exp(-e^{-x})$$

*Proof.* If $r = n\log n + nx$ then $ne^{-r/n} \to e^{-x}$. ■

Note that $T_n$ is the sum of $n$ independent random variables (see Example 2.2.3), but $T_n$ does not converge to the normal distribution. The problem is that the last few terms in the sum are of order $n$, so the hypotheses of the Lindeberg-Feller theorem are not satisfied.

For a concrete instance of the previous result consider: What is the probability that in a village of 2190 ($= 6 \cdot 365$) people all birthdays are represented? Do you think the answer is much different for 1825 ($= 5 \cdot 365$) people?

*Solution.* Here $n = 365$, so $365 \log 365 = 2153$, and

$$\begin{aligned}
P(T_{365} \le 2190) &= P((T_{365} - 2153)/365 \le 37/365) \\
&\approx \exp(-e^{-0.1014}) = \exp(-0.9036) = 0.4051 \\
P(T_{365} \le 1825) &= P((T_{365} - 2153)/365 \le -328/365) \\
&\approx \exp(-e^{0.8986}) = \exp(-2.4562) = 0.085
\end{aligned}$$

As we observed in Example 2.2.3, if we let

$$\tau_k^n = \inf\{m : |\{X_1, \ldots, X_m\}| = k\}$$

then $\tau_1^n = 1$ and for $2 \le k \le n$, $\tau_k^n - \tau_{k-1}^n$ are independent and have a geometric distribution with parameter $1 - (k-1)/n$.

**Exercise 3.6.3.** Suppose $k/n^{1/2} \to \lambda \in [0, \infty)$ and show that $\tau_k^n - k \Rightarrow$ Poisson$(\lambda^2/2)$. Hint: This is easy if you use Theorem 3.6.6 below.

**Exercise 3.6.4.** Let $\mu_{n,k} = E\tau_k^n$ and $\sigma_{n,k}^2 = \text{var}\,(\tau_k^n)$. Suppose $k/n \to a \in (0, 1)$, and use the Lindeberg-Feller theorem to show $(\tau_k^n - \mu_{n,k})/\sqrt{n} \Rightarrow \sigma\chi$.

The last result is true when $k/n^{1/2} \to \infty$ and $n - k \to \infty$; see Baum and Billingsley (1966). Results for $k = n - j$ can be obtained from Theorem 3.6.5, so we have examined all the possibilities.

### 3.6.3 *Poisson Processes*

Theorem 3.6.1 generalizes trivially to give the following result.

**Theorem 3.6.6.** *Let $X_{n,m}$, $1 \le m \le n$ be independent nonnegative integer valued random variables with $P(X_{n,m} = 1) = p_{n,m}$, $P(X_{n,m} \ge 2) = \epsilon_{n,m}$.*
*(i)* $\sum_{m=1}^n p_{n,m} \to \lambda \in (0, \infty)$,
*(ii)* $\max_{1 \le m \le n} p_{n,m} \to 0$, *and*
*(iii)* $\sum_{m=1}^n \epsilon_{n,m} \to 0$.
*If $S_n = X_{n,1} + \cdots + X_{n,n}$ then $S_n \Rightarrow Z$ where $Z$ is Poisson$(\lambda)$.*

*Proof.* Let $X'_{n,m} = 1$ if $X_{n,m} = 1$, and 0 otherwise. Let $S'_n = X'_{n,1} + \cdots + X'_{n,n}$. (i)–(ii) and Theorem 3.6.1 imply $S'_n \Rightarrow Z$, (iii) tells us $P(S_n \ne S'_n) \to 0$, and the result follows from the converging together lemma, Exercise 3.2.13. ■

The next result, which uses Theorem 3.6.6, explains why the Poisson distribution comes up so frequently in applications. Let $N(s,t)$ be the number of arrivals at a bank or an ice cream parlor in the time interval $(s,t]$. Suppose

(i) the numbers of arrivals in disjoint intervals are independent,
(ii) the distribution of $N(s,t)$ only depends on $t-s$,
(iii) $P(N(0,h)=1)=\lambda h+o(h)$, and
(iv) $P(N(0,h)\ge 2)=o(h)$.

Here, the two $o(h)$ stand for functions $g_1(h)$ and $g_2(h)$ with $g_i(h)/h\to 0$ as $h\to 0$.

**Theorem 3.6.7.** *If (i)–(iv) hold, then $N(0,t)$ has a Poisson distribution with mean $\lambda t$.*

*Proof.* Let $X_{n,m}=N((m-1)t/n,mt/n)$ for $1\le m\le n$ and apply Theorem 3.6.6. ∎

A family of random variables $N_t$, $t\ge 0$ satisfying

(i) if $0=t_0<t_1<\cdots<t_n$, $N(t_k)-N(t_{k-1})$, $1\le k\le n$ are independent,
(ii) $N(t)-N(s)$ is Poisson$(\lambda(t-s))$,

is called a **Poisson process with rate** $\lambda$. To understand how $N_t$ behaves, it is useful to have another method to construct it. Let $\xi_1,\xi_2,\ldots$ be independent random variables with $P(\xi_i>t)=e^{-\lambda t}$ for $t\ge 0$. Let $T_n=\xi_1+\cdots+\xi_n$ and $N_t=\sup\{n:T_n\le t\}$ where $T_0=0$. In the language of renewal theory (see Theorem 2.4.6), $T_n$ is the time of the nth arrival and $N_t$ is the number of arrivals by time $t$. To check that $N_t$ is a Poisson process, we begin by recalling (see Theorem 2.1.12):

$$f_{T_n}(s)=\frac{\lambda^n s^{n-1}}{(n-1)!}e^{-\lambda s}\text{ for } s\ge 0$$

that is, the distribution of $T_n$ has a density given by the right-hand side. Now

$$P(N_t=0)=P(T_1>t)=e^{-\lambda t}$$

and for $n\ge 1$

$$\begin{aligned}P(N_t=n)=P(T_n\le t<T_{n+1})&=\int_0^t P(T_n=s)P(\xi_{n+1}>t-s)\,ds\\&=\int_0^t\frac{\lambda^n s^{n-1}}{(n-1)!}e^{-\lambda s}e^{-\lambda(t-s)}\,ds=e^{-\lambda t}\frac{(\lambda t)^n}{n!}\end{aligned}$$

The last two formulas show that $N_t$ has a Poisson distribution with mean $\lambda t$. To check that the number of arrivals in disjoint intervals is independent, we observe

$$P(T_{n+1}\ge u|N_t=n)=P(T_{n+1}\ge u,T_n\le t)/P(N_t=n)$$

To compute the numerator, we observe

$$P(T_{n+1} \geq u, T_n \leq t) = \int_0^t f_{T_n}(s) P(\xi_{n+1} \geq u - s)\, ds$$
$$= \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} e^{-\lambda(u-s)} ds = e^{-\lambda u} \frac{(\lambda t)^n}{n!}$$

The denominator is $P(N_t = n) = e^{-\lambda t}(\lambda t)^n/n!$, so

$$P(T_{n+1} \geq u | N_t = n) = e^{-\lambda u}/e^{-\lambda t} = e^{-\lambda(u-t)}$$

or, rewriting things, $P(T_{n+1} - t \geq s | N_t = n) = e^{-\lambda s}$. Let $T_1' = T_{N(t)+1} - t$, and $T_k' = T_{N(t)+k} - T_{N(t)+k-1}$ for $k \geq 2$. The last computation shows that $T_1'$ is independent of $N_t$. If we observe that

$$P(T_n \leq t, T_{n+1} \geq u, T_{n+k} - T_{n+k-1} \geq v_k, k = 2, \ldots, K)$$
$$= P(T_n \leq t, T_{n+1} \geq u) \prod_{k=2}^{K} P(\xi_{n+k} \geq v_k)$$

then it follows that

(a) $T_1', T_2', \ldots$ are i.i.d. and independent of $N_t$.

The last observation shows that the arrivals after time $t$ are independent of $N_t$ and have the same distribution as the original sequence. From this it follows easily that

(b) If $0 = t_0 < t_1 \ldots < t_n$ then $N(t_i) - N(t_{i-1}), i = 1, \ldots, n$ are independent.

To see this, observe that the vector $(N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1}))$ is $\sigma(T_k', k \geq 1)$ measurable and hence is independent of $N(t_1)$. Then use induction to conclude

$$P(N(t_i) - N(t_{i-1}) = k_i, i = 1, \ldots, n) = \prod_{i=1}^{n} \exp(-\lambda(t_i - t_{i-1})) \frac{\lambda(t_i - t_{i-1}))^{k_i}}{k_i!}$$

**Remark.** The key to the proof of (a) is the lack of memory property of the exponential distribution:

$$(*) \qquad P(T > t + s | T > t) = P(T > s)$$

which implies that the location of the first arrival after $t$ is independent of what occurred before time $t$ and has an exponential distribution.

**Exercise 3.6.5.** Show that if $P(T > 0) = 1$ and $(*)$ holds, then there is a $\lambda > 0$ so that $P(T > t) = e^{-\lambda t}$ for $t \geq 0$. Hint: First show that this holds for $t = m2^{-n}$.

**Exercise 3.6.6.** Show that (iii) and (iv) in Theorem 3.6.7 can be replaced by

(v) If $N_{s-} = \lim_{r \uparrow s} N_r$ then $P(N_s - N_{s-} \geq 2$ for some $s) = 0$.

That is, if (i), (ii), and (v) hold, then there is a $\lambda \ge 0$ so that $N(0,t)$ has a Poisson distribution with mean $\lambda t$. Prove this by showing: (a) If $u(s) = P(N_s = 0)$ then (i) and (ii) imply $u(r)u(s) = u(r+s)$. It follows that $u(s) = e^{-\lambda s}$ for some $\lambda \ge 0$, so (iii) holds. (b) If $v(s) = P(N_s \ge 2)$ and $A_n = \{N_{k/n} - N_{(k-1)/n} \ge 2 \text{ for some } k \le n\}$ then (v) implies $P(A_n) \to 0$ as $n \to \infty$ and (iv) holds.

**Exercise 3.6.7.** Let $T_n$ be the time of the $n$th arrival in a rate $\lambda$ Poisson process. Let $U_1, U_2, \dots, U_n$ be independent uniform on (0,1) and let $V_k^n$ be the $k$th smallest number in $\{U_1, \dots, U_n\}$. Show that the vectors $(V_1^n, \dots, V_n^n)$ and $(T_1/T_{n+1}, \dots, T_n/T_{n+1})$ have the same distribution.

**Spacings.** The last result can be used to study the spacings between the order statistics of i.i.d. uniforms. We use notation of Exercise 3.6.7 in the next four exercises, taking $\lambda = 1$ and letting $V_0^n = 0$, and $V_{n+1}^n = 1$.

**Exercise 3.6.8.** Smirnov (1949) $nV_k^n \Rightarrow T_k$.

**Exercise 3.6.9.** Weiss (1955) $n^{-1} \sum_{m=1}^n 1_{(n(V_i^n - V_{i-1}^n) > x)} \to e^{-x}$ in probability.

**Exercise 3.6.10.** $(n/\log n) \max_{1 \le m \le n+1} V_m^n - V_{m-1}^n \to 1$ in probability.

**Exercise 3.6.11.** $P(n^2 \min_{1 \le m \le n} V_m^n - V_{m-1}^n > x) \to e^{-x}$.

For the rest of the section, we concentrate on the Poisson process itself.

**Exercise 3.6.12. Thinning.** Let $N$ have a Poisson distribution with mean $\lambda$ and let $X_1, X_2, \dots$ be an independent i.i.d. sequence with $P(X_i = j) = p_j$ for $j = 0, 1, \dots, k$. Let $N_j = |\{m \le N : X_m = j\}|$. Show that $N_0, N_1, \dots, N_k$ are independent and $N_j$ has a Poisson distribution with mean $\lambda p_j$.

In the important special case $X_i \in \{0, 1\}$, the result says that if we thin a Poisson process by flipping a coin with probability $p$ of heads to see if we keep the arrival, then the result is a Poisson process with rate $\lambda p$.

**Exercise 3.6.13. Poissonization and the occupancy problem.** If we put a Poisson number of balls with mean $r$ in $n$ boxes and let $N_i$ be the number of balls in box $i$, then the last exercise implies $N_1, \dots, N_n$ are independent and have a Poisson distribution with mean $r/n$. Use this observation to prove Theorem 3.6.5.
Hint: If $r = n\log n - (\log\lambda)n + o(n)$ and $s_i = n\log n - (\log\mu_i)n$ with $\mu_2 < \lambda < \mu_1$, then the normal approximation to the Poisson tells us $P(\text{Poisson}(s_1) < r < \text{Poisson}(s_2)) \to 1$ as $n \to \infty$.

**Example 3.6.7. Compound Poisson process.** At the arrival times $T_1, T_2, \dots$ of a Poisson process with rate $\lambda$, groups of customers of size $\xi_1, \xi_2, \dots$ arrive at an ice cream parlor. Suppose the $\xi_i$ are i.i.d. and independent of the $T_j's$. This is a **compound Poisson process**. The result of Exercise 3.6.12 shows that $N_t^k =$ the number of groups of size $k$ to arrive in $[0, t]$ are independent Poisson's with mean $p_k \lambda t$.

**Example 3.6.8. A Poisson process on a measure space** $(S, \mathcal{S}, \mu)$ is a random map $m : \mathcal{S} \to \{0, 1, \ldots\}$ that for each $\omega$ is a measure on $\mathcal{S}$ and has the following property: if $A_1, \ldots, A_n$ are disjoint sets with $\mu(A_i) < \infty$, then $m(A_1), \ldots, m(A_n)$ are independent and have Poisson distributions with means $\mu(A_i)$. $\mu$ is called the **mean measure** of the process. Exercise 3.6.12 implies that if $\mu(S) < \infty$ we can construct $m$ by the following recipe: let $X_1, X_2, \ldots$ be i.i.d. elements of $S$ with distribution $\nu(\cdot) = \mu(\cdot)/\mu(S)$, let $N$ be an independent Poisson random variable with mean $\mu(S)$, and let $m(A) = |\{j \le N : X_j \in A\}|$. To extend the construction to infinite measure spaces, e.g., $S = \mathbf{R}^d$, $\mathcal{S} =$ Borel sets, $\mu =$ Lebesgue measure, divide the space up into disjoint sets of finite measure and put independent Poisson processes on each set.

## 3.7 Stable Laws*

Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. Theorem 3.4.1 showed that if $EX_i = \mu$ and $\text{var}(X_i) = \sigma^2 \in (0, \infty)$, then

$$(S_n - n\mu)/\sigma n^{1/2} \Rightarrow \chi$$

In this section, we will investigate the case $EX_1^2 = \infty$ and give necessary and sufficient conditions for the existence of constants $a_n$ and $b_n$ so that

$$(S_n - b_n)/a_n \Rightarrow Y \quad \text{where } Y \text{ is nondegenerate}$$

We begin with an example. Suppose the distribution of $X_i$ has

$$P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2 \quad \text{for } x \ge 1 \tag{3.7.1}$$

where $0 < \alpha < 2$. If $\varphi(t) = E\exp(itX_1)$, then

$$\begin{aligned} 1 - \varphi(t) &= \int_1^\infty (1 - e^{itx}) \frac{\alpha}{2|x|^{\alpha+1}}\, dx + \int_{-\infty}^{-1} (1 - e^{itx}) \frac{\alpha}{2|x|^{\alpha+1}}\, dx \\ &= \alpha \int_1^\infty \frac{1 - \cos(tx)}{x^{\alpha+1}}\, dx \end{aligned}$$

Changing variables $tx = u$, $dx = du/t$, the last integral becomes

$$= \alpha \int_t^\infty \frac{1 - \cos u}{(u/t)^{\alpha+1}} \frac{du}{t} = t^\alpha \alpha \int_t^\infty \frac{1 - \cos u}{u^{\alpha+1}}\, du$$

As $u \to 0$, $1 - \cos u \sim u^2/2$. So $(1 - \cos u)/u^{\alpha+1} \sim u^{-\alpha+1}/2$, which is integrable, since $\alpha < 2$ implies $-\alpha + 1 > -1$. If we let

$$C = \alpha \int_0^\infty \frac{1 - \cos u}{u^{\alpha+1}} du < \infty$$

and observe (3.7.1) implies $\varphi(t) = \varphi(-t)$, then the results above show

$$1 - \varphi(t) \sim C|t|^\alpha \text{ as } t \to 0 \tag{3.7.2}$$

Let $X_1, X_2, \ldots$ be i.i.d. with the distribution given in (3.7.1) and let $S_n = X_1 + \cdots + X_n$.

$$E \exp(itS_n/n^{1/\alpha}) = \varphi(t/n^{1/\alpha})^n = (1 - \{1 - \varphi(t/n^{1/\alpha})\})^n$$

As $n \to \infty$, $n(1 - \varphi(t/n^{1/\alpha})) \to C|t|^\alpha$, so it follows from Theorem 3.4.2 that

$$E \exp(itS_n/n^{1/\alpha}) \to \exp(-C|t|^\alpha)$$

From part (ii) of Theorem 3.3.6, it follows that the expression on the right is the characteristic function of some $Y$ and

$$S_n/n^{1/\alpha} \Rightarrow Y \tag{3.7.3}$$

To prepare for our general result, we will now give another proof of (3.7.3). If $0 < a < b$ and $an^{1/\alpha} > 1$, then

$$P(an^{1/\alpha} < X_1 < bn^{1/\alpha}) = \frac{1}{2}(a^{-\alpha} - b^{-\alpha})n^{-1}$$

so it follows from Theorem 3.6.1 that

$$N_n(a, b) \equiv |\{m \le n : X_m/n^{1/\alpha} \in (a, b)\}| \Rightarrow N(a, b)$$

where $N(a, b)$ has a Poisson distribution with mean $(a^{-\alpha} - b^{-\alpha})/2$. An easy extension of the last result shows that if $A \subset \mathbf{R} - (-\delta, \delta)$ and $\delta n^{1/\alpha} > 1$, then

$$P(X_1/n^{1/\alpha} \in A) = n^{-1} \int_A \frac{\alpha}{2|x|^{\alpha+1}}\, dx$$

so $N_n(A) \equiv |\{m \le n : X_m/n^{1/\alpha} \in A\}| \Rightarrow N(A)$, where $N(A)$ has a Poisson distribution with mean

$$\mu(A) = \int_A \frac{\alpha}{2|x|^{\alpha+1}}\, dx < \infty$$

The limiting family of random variables $N(A)$ is called a **Poisson process on** $(-\infty, \infty)$ **with mean measure** $\mu$. (See Example 3.6.8 for more on this process.) Notice that for any $\epsilon > 0$, $\mu(\epsilon, \infty) = \epsilon^{-\alpha}/2 < \infty$, so $N(\epsilon, \infty) < \infty$.

The last paragraph describes the limiting behavior of the random set

$$\mathcal{X}_n = \{X_m/n^{1/\alpha} : 1 \le m \le n\}$$

To describe the limit of $S_n/n^{1/\alpha}$, we will "sum up the points." Let $\epsilon > 0$ and

$$I_n(\epsilon) = \{m \le n : |X_m| > \epsilon n^{1/\alpha}\}$$

$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m \qquad \bar{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon)$$

$I_n(\epsilon)$ = the indices of the "big terms," that is, those $> \epsilon n^{1/\alpha}$ in magnitude. $\hat{S}_n(\epsilon)$ is the sum of the big terms, and $\bar{S}_n(\epsilon)$ is the rest of the sum. The first thing we will do is show that the contribution of $\bar{S}_n(\epsilon)$ is small if $\epsilon$ is. Let

$$\bar{X}_m(\epsilon) = X_m 1_{(|X_m| \le \epsilon n^{1/\alpha})}$$

Symmetry implies $E\bar{X}_m(\epsilon) = 0$, so $E(\bar{S}_n(\epsilon)^2) = nE\bar{X}_1(\epsilon)^2$.

$$\begin{aligned} E\bar{X}_1(\epsilon)^2 &= \int_0^\infty 2yP(|\bar{X}_1(\epsilon)| > y)\,dy \le \int_0^1 2y\,dy + \int_1^{\epsilon n^{1/\alpha}} 2y\,y^{-\alpha}\,dy \\ &= 1 + \frac{2}{2-\alpha}\epsilon^{2-\alpha}n^{2/\alpha-1} - \frac{2}{2-\alpha} \le \frac{2\epsilon^{2-\alpha}}{2-\alpha}n^{2/\alpha-1} \end{aligned}$$

where we have used $\alpha < 2$ in computing the integral and $\alpha > 0$ in the final inequality. From this it follows that

$$E(\bar{S}_n(\epsilon)/n^{1/\alpha})^2 \le \frac{2\epsilon^{2-\alpha}}{2-\alpha} \tag{3.7.4}$$

To compute the limit of $\hat{S}_n(\epsilon)/n^{1/\alpha}$, we observe that $|I_n(\epsilon)|$ has a binomial distribution with success probability $p = \epsilon^{-\alpha}/n$. Given $|I_n(\epsilon)| = m$, $\hat{S}_n(\epsilon)/n^{1/\alpha}$ is the sum of $m$ independent random variables with a distribution $F_n^\epsilon$ that is symmetric about 0 and has

$$1 - F_n^\epsilon(x) = P(X_1/n^{1/\alpha} > x \,|\, |X_1|/n^{1/\alpha} > \epsilon) = x^{-\alpha}/2\epsilon^{-\alpha} \quad \text{for } x \ge \epsilon$$

The last distribution is the same as that of $\epsilon X_1$, so if $\varphi(t) = E\exp(itX_1)$, the distribution $F_n^\epsilon$ has characteristic function $\varphi(\epsilon t)$. Combining the observations in this paragraph gives

$$E\exp(it\hat{S}_n(\epsilon)/n^{1/\alpha}) = \sum_{m=0}^{n} \binom{n}{m}(\epsilon^{-\alpha}/n)^m(1-\epsilon^{-\alpha}/n)^{n-m}\varphi(\epsilon t)^m$$

Writing

$$\binom{n}{m}\frac{1}{n^m} = \frac{1}{m!}\frac{n(n-1)\cdots(n-m+1)}{n^m} \le \frac{1}{m!}$$

noting $(1-\epsilon^{-\alpha}/n)^n \le \exp(-\epsilon^{-\alpha})$ and using the dominated convergence theorem

$$\begin{aligned} E\exp(it\hat{S}_n(\epsilon)/n^{1/\alpha}) &\to \sum_{m=0}^{\infty}\exp(-\epsilon^{-\alpha})(\epsilon^{-\alpha})^m\varphi(\epsilon t)^m/m! \\ &= \exp(-\epsilon^{-\alpha}\{1-\varphi(\epsilon t)\}) \end{aligned} \tag{3.7.5}$$

To get (3.7.3) now, we use the following generalization of Lemma 3.4.7.

**Lemma 3.7.1.** *If $h_n(\epsilon) \to g(\epsilon)$ for each $\epsilon > 0$ and $g(\epsilon) \to g(0)$ as $\epsilon \to 0$, then we can pick $\epsilon_n \to 0$ so that $h_n(\epsilon_n) \to g(0)$.*

*Proof.* Let $N_m$ be chosen so that $|h_n(1/m) - g(1/m)| \le 1/m$ for $n \ge N_m$ and $m \to N_m$ is increasing. Let $\epsilon_n = 1/m$ for $N_m \le n < N_{m+1}$ and $= 1$ for $n < N_1$.

When $N_m \le n < N_{m+1}$, $\epsilon_n = 1/m$, so it follows from the triangle inequality and the definition of $\epsilon_n$ that

$$\begin{aligned}|h_n(\epsilon_n) - g(0)| &\le |h_n(1/m) - g(1/m)| + |g(1/m) - g(0)| \\ &\le 1/m + |g(1/m) - g(0)|\end{aligned}$$

When $n \to \infty$, we have $m \to \infty$ and the result follows. ∎

Let $h_n(\epsilon) = E\exp(it\hat{S}_n(\epsilon)/n^{1/\alpha})$ and $g(\epsilon) = \exp(-\epsilon^{-\alpha}\{1 - \varphi(\epsilon t)\})$. (3.7.2) implies $1 - \varphi(t) \sim C|t|^\alpha$ as $t \to 0$, so

$$g(\epsilon) \to \exp(-C|t|^\alpha) \quad \text{as } \epsilon \to 0$$

and Lemma 3.7.1 implies we can pick $\epsilon_n \to 0$ with $h_n(\epsilon_n) \to \exp(-C|t|^\alpha)$. Introducing $Y$ with $E\exp(itY) = \exp(-C|t|^\alpha)$, it follows that $\hat{S}_n(\epsilon_n)/n^{1/\alpha} \Rightarrow Y$. If $\epsilon_n \to 0$, then (3.7.4) implies

$$\bar{S}_n(\epsilon_n)/n^{1/\alpha} \Rightarrow 0$$

and (3.7.3) follows from the converging together lemma, Exercise 3.2.13. ∎

Once we give one final definition, we will state and prove the general result alluded to above. $L$ is said to be **slowly varying**, if

$$\lim_{x\to\infty} L(tx)/L(x) = 1 \quad \text{for all } t > 0$$

**Exercise 3.7.1.** Show that $L(t) = \log t$ is slowly varying but $t^\epsilon$ is not if $\epsilon \neq 0$.

**Theorem 3.7.2.** *Suppose $X_1, X_2, \ldots$ are i.i.d. with a distribution that satisfies*
*(i)* $\lim_{x\to\infty} P(X_1 > x)/P(|X_1| > x) = \theta \in [0, 1]$
*(ii)* $P(|X_1| > x) = x^{-\alpha}L(x)$
*where $\alpha < 2$ and $L$ is slowly varying. Let $S_n = X_1 + \cdots + X_n$*

$$a_n = \inf\{x : P(|X_1| > x) \le n^{-1}\} \quad \textit{and} \quad b_n = nE(X_1 1_{(|X_1| \le a_n)})$$

*As $n \to \infty$, $(S_n - b_n)/a_n \Rightarrow Y$ where $Y$ has a nondegenerate distribution.*

**Remark.** This is not much of a generalization of the example, but the conditions are necessary for the existence of constants $a_n$ and $b_n$ so that $(S_n - b_n)/a_n \Rightarrow Y$, where $Y$ is nondegenerate. Proofs of necessity can be found in Chapter 9 of Breiman (1968) or in Gnedenko and Kolmogorov (1954). (3.7.11) gives the ch.f. of $Y$. The reader has seen the main ideas in the second proof of (3.7.3) and so can skip to that point without much loss.

*Proof.* It is not hard to see that (ii) implies

$$nP(|X_1| > a_n) \to 1 \tag{3.7.6}$$

To prove this, note that $nP(|X_1| > a_n) \le 1$ and let $\epsilon > 0$. Taking $x = a_n/(1+\epsilon)$ and $t = 1 + 2\epsilon$, (ii) implies

$$(1+2\epsilon)^{-\alpha} = \lim_{n\to\infty} \frac{P(|X_1| > (1+2\epsilon)a_n/(1+\epsilon))}{P(|X_1| > a_n/(1+\epsilon))} \le \liminf_{n\to\infty} \frac{P(|X_1| > a_n)}{1/n}$$

proving (3.7.6) since $\epsilon$ is arbitrary. Combining (3.7.6) with (i) and (ii) gives

$$nP(X_1 > xa_n) \to \theta x^{-\alpha} \quad \text{for } x > 0 \tag{3.7.7}$$

so $|\{m \le n : X_m > xa_n\}| \Rightarrow \text{Poisson}(\theta x^{-\alpha})$. The last result leads, as before, to the conclusion that $\mathcal{X}_n = \{X_m/a_n : 1 \le m \le n\}$ converges to a Poisson process on $(-\infty, \infty)$ with mean measure

$$\mu(A) = \int_{A\cap(0,\infty)} \theta\alpha|x|^{-(\alpha+1)}\,dx + \int_{A\cap(-\infty,0)} (1-\theta)\alpha|x|^{-(\alpha+1)}\,dx$$

To sum up the points, let $I_n(\epsilon) = \{m \le n : |X_m| > \epsilon a_n\}$

$$\hat{\mu}(\epsilon) = EX_m 1_{(\epsilon a_n < |X_m| \le a_n)} \qquad \hat{S}_n(\epsilon) = \sum_{m\in I_n(\epsilon)} X_m$$

$$\bar{\mu}(\epsilon) = EX_m 1_{(|X_m| \le \epsilon a_n)}$$

$$\bar{S}_n(\epsilon) = (S_n - b_n) - (\hat{S}_n(\epsilon) - n\hat{\mu}(\epsilon)) = \sum_{m=1}^n \{X_m 1_{(|X_m|\le\epsilon a_n)} - \bar{\mu}(\epsilon)\}$$

If we let $\bar{X}_m(\epsilon) = X_m 1_{(|X_m|\le\epsilon a_n)}$, then

$$E(\bar{S}_n(\epsilon)/a_n)^2 = n \operatorname{var}(\bar{X}_1(\epsilon)/a_n) \le nE(\bar{X}_1(\epsilon)/a_n)^2$$

$$E(\bar{X}_1(\epsilon)/a_n)^2 \le \int_0^\epsilon 2yP(|X_1| > ya_n)\,dy$$

$$= P(|X_1| > a_n) \int_0^\epsilon 2y \frac{P(|X_1| > ya_n)}{P(|X_1| > a_n)}\,dy$$

We would like to use (3.7.7) and (ii) to conclude

$$nE(\bar{X}_1(\epsilon)/a_n)^2 \to \int_0^\epsilon 2y\,y^{-\alpha}\,dy = \frac{2}{2-\alpha}\epsilon^{2-\alpha}$$

and hence

$$\limsup_{n\to\infty} E(\bar{S}_n(\epsilon)/a_n)^2 \le \frac{2\epsilon^{2-\alpha}}{2-\alpha} \tag{3.7.8}$$

To justify interchanging the limit and the integral and complete the proof of (3.7.8), we show the following (take $\delta < 2-\alpha$):

**Lemma 3.7.3.** *For any $\delta > 0$ there is $C$ so that for all $t \ge t_0$ and $y \le 1$*

$$P(|X_1| > yt)/P(|X_1| > t) \le Cy^{-\alpha-\delta}$$

*Proof.* (ii) implies that as $t \to \infty$

$$P(|X_1| > t/2)/P(|X_1| > t) \to 2^\alpha$$

so for $t \geq t_0$ we have

$$P(|X_1| > t/2)/P(|X_1| > t) \leq 2^{\alpha+\delta}$$

Iterating and stopping the first time $t/2^m < t_0$, we have for all $n \geq 1$

$$P(|X_1| > t/2^n)/P(|X_1| > t) \leq C 2^{(\alpha+\delta)n}$$

where $C = 1/P(|X_1| > t_0)$. Applying the last result to the first $n$ with $1/2^n < y$ and noticing $y \leq 1/2^{n-1}$, we have

$$P(|X_1| > yt)/P(|X_1| > t) \leq C 2^{\alpha+\delta} y^{-\alpha-\delta}$$

which proves the lemma. ■

To compute the limit of $\hat{S}_n(\epsilon)$, we observe that $|I_n(\epsilon)| \Rightarrow \text{Poisson}(\epsilon^{-\alpha})$. Given $|I_n(\epsilon)| = m$, $\hat{S}_n(\epsilon)/a_n$ is the sum of $m$ independent random variables with distribution $F_n^\epsilon$ that has

$$\begin{aligned}
1 - F_n^\epsilon(x) &= P(X_1/a_n > x \,|\, |X_1|/a_n > \epsilon) \to \theta x^{-\alpha}/\epsilon^{-\alpha} \\
F_n^\epsilon(-x) &= P(X_1/a_n < -x \,|\, |X_1|/a_n > \epsilon) \to (1-\theta)|x|^{-\alpha}/\epsilon^{-\alpha}
\end{aligned}$$

for $x \geq \epsilon$. If we let $\psi_n^\epsilon(t)$ denote the characteristic function of $F_n^\epsilon$, then Theorem 3.3.6 implies

$$\psi_n^\epsilon(t) \to \psi^\epsilon(t) = \int_\epsilon^\infty e^{itx} \theta \epsilon^\alpha \alpha x^{-(\alpha+1)} dx + \int_{-\infty}^{-\epsilon} e^{itx}(1-\theta)\epsilon^\alpha \alpha |x|^{-(\alpha+1)}\, dx$$

as $n \to \infty$. So repeating the proof of (3.7.5) gives

$$\begin{aligned}
E \exp(it\hat{S}_n(\epsilon)/a_n) &\to \exp(-\epsilon^{-\alpha}\{1 - \psi^\epsilon(t)\}) \\
&= \exp\left(\int_\epsilon^\infty (e^{itx} - 1)\theta\alpha x^{-(\alpha+1)}\, dx \right.\\
&\qquad \left. + \int_{-\infty}^{-\epsilon} (e^{itx} - 1)(1-\theta)\alpha|x|^{-(\alpha+1)} dx\right)
\end{aligned}$$

where we have used $\epsilon^{-\alpha} = \int_\epsilon^\infty \alpha x^{-(\alpha+1)}\, dx$. To bring in

$$\hat{\mu}(\epsilon) = E X_m 1_{(\epsilon a_n < |X_m| \leq a_n)}$$

we observe that (3.7.7) implies $nP(xa_n < X_m \leq ya_n) \to \theta(x^{-\alpha} - y^{-\alpha})$. So

$$n\hat{\mu}(\epsilon)/a_n \to \int_\epsilon^1 x\theta\alpha x^{-(\alpha+1)}\, dx + \int_{-1}^{-\epsilon} x(1-\theta)\alpha|x|^{-(\alpha+1)}\, dx$$

From this it follows that $E\exp(it\{\hat{S}_n(\epsilon) - n\hat{\mu}(\epsilon)\}/a_n) \to$

$$\begin{aligned}\exp\Biggl(\int_1^\infty &(e^{itx} - 1)\theta\alpha x^{-(\alpha+1)}\,dx \\ &+ \int_\epsilon^1 (e^{itx} - 1 - itx)\theta\alpha x^{-(\alpha+1)}\,dx \\ &+ \int_{-1}^{-\epsilon} (e^{itx} - 1 - itx)(1-\theta)\alpha|x|^{-(\alpha+1)}\,dx \\ &+ \int_{-\infty}^{-1} (e^{itx} - 1)(1-\theta)\alpha|x|^{-(\alpha+1)}\,dx\Biggr)\end{aligned} \tag{3.7.9}$$

The last expression is messy, but $e^{itx} - 1 - itx \sim -t^2x^2/2$ as $t \to 0$, so we need to subtract the $itx$ to make

$$\int_0^1 (e^{itx} - 1 - itx)x^{-(\alpha+1)}dx \quad \text{converge when } \alpha \geq 1$$

To reduce the number of integrals from four to two, we can write the limit as $\epsilon \to 0$ of the right-hand side of (3.7.9) as

$$\begin{aligned}\exp\Biggl(itc + \int_0^\infty &\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\theta\alpha x^{-(\alpha+1)}\,dx \\ &+ \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)(1-\theta)\alpha|x|^{-(\alpha+1)}\,dx\Biggr)\end{aligned} \tag{3.7.10}$$

where $c$ is a constant. Combining (3.7.6) and (3.7.9) using Lemma 3.7.1, it follows easily that $(S_n - b_n)/a_n \Rightarrow Y$ where $Ee^{itY}$ is given in (3.7.10). ■

**Exercise 3.7.2.** Show that when $\alpha < 1$, centering is unnecessary, that is, we can let $b_n = 0$.

By doing some calculus (see Breiman, 1968, pp. 204–206) one can rewrite (3.7.10) as

$$\exp(itc - b|t|^\alpha\{1 + i\kappa\ \operatorname{sgn}(t) w_\alpha(t)\}) \tag{3.7.11}$$

where $-1 \leq \kappa \leq 1$, ($\kappa = 2\theta - 1$) and

$$w_\alpha(t) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ (2/\pi)\log|t| & \text{if } \alpha = 1 \end{cases}$$

The reader should note that while we have assumed $0 < \alpha < 2$ throughout the developments above, if we set $\alpha = 2$ then the term with $\kappa$ vanishes and (3.7.11) reduces to the characteristic function of the normal distribution with mean $c$ and variance $2b$.

The distributions whose characteristic functions are given in (3.7.11) are called **stable laws**. $\alpha$ is commonly called the **index**. When $\alpha = 1,\ c = 0$, and $\kappa = 0$, we have the Cauchy distribution. Apart from the Cauchy and the normal, there is only one other case in which the density is known: When $\alpha = 1/2$, $\kappa = 1$, $c = 0$, and $b = 1$, the density is

$$(2\pi y^3)^{-1/2} \exp(-1/2y) \quad \text{for } y \geq 0 \tag{3.7.12}$$

One can calculate the ch.f. and verify our claim. However, later (see Section 7.4) we will be able to check the claim without effort, so we leave the somewhat tedious calculation to the reader.

We are now finally ready to treat some examples.

**Example 3.7.1.** Let $X_1, X_2, \ldots$ be i.i.d. with a density that is symmetric about 0, and continuous and positive at 0. We claim that

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \Rightarrow \text{a Cauchy distribution } (\alpha = 1, \kappa = 0)$$

To verify this, note that

$$P(1/X_i > x) = P(0 < X_i < x^{-1}) = \int_0^{x^{-1}} f(y)\,dy \sim f(0)/x$$

as $x \to \infty$. A similar calculation shows $P(1/X_i < -x) \sim f(0)/x$, so in (i) in Theorem 3.7.2 holds with $\theta = 1/2$, and (ii) holds with $\alpha = 1$. The scaling constant $a_n \sim 2f(0)n$, whereas the centering constant vanishes because we have supposed the distribution of $X$ is symmetric about 0.

**Remark.** Readers who want a challenge should try to drop the symmetry assumption, assuming for simplicity that $f$ is differentiable at 0.

**Example 3.7.2.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 1) = P(X_i = -1) = 1/2$, let $S_n = X_1 + \cdots + X_n$, and let $\tau = \inf\{n \geq 1 : S_n = 1\}$. In Chapter 4 (see the discussion after (4.3.2)) we will show

$$P(\tau > 2n) \sim \pi^{-1/2} n^{-1/2} \quad \text{as } n \to \infty$$

Let $\tau_1, \tau_2, \ldots$ be independent with the same distribution as $\tau$, and let $T_n = \tau_1 + \cdots + \tau_n$. Results in Section 4.1 imply that $T_n$ has the same distribution as the $n$th time $S_m$ hits 0. We claim that $T_n/n^2$ converges to the stable law with $\alpha = 1/2$, $\kappa = 1$ and note that this is the key to the derivation of (3.7.12). To prove the claim, note that in (i) in Theorem 3.7.2 holds with $\theta = 1$ and (ii) holds with $\alpha = 1/2$. The scaling constant $a_n \sim Cn^2$. Since $\alpha < 1$, Exercise 3.7.2 implies the centering constant is unnecessary.

**Example 3.7.3.** Assume $n$ objects $X_{n,1}, \ldots, X_{n,n}$ are placed independently and at random in $[-n, n]$. Let

$$F_n = \sum_{m=1}^{n} \operatorname{sgn}(X_{n,m})/|X_{n,m}|^p$$

be the net force exerted on 0. We will now show that if $p > 1/2$, then

$$\lim_{n\to\infty} E\exp(itF_n) = \exp(-c|t|^{1/p})$$

To do this, it is convenient to let $X_{n,m} = nY_m$ where the $Y_i$ are i.i.d. on $[-1, 1]$. Then

$$F_n = n^{-p} \sum_{m=1}^{n} \operatorname{sgn}(Y_m)/|Y_m|^p$$

Letting $Z_m = \operatorname{sgn}(Y_m)/|Y_m|^p$, $Z_m$ is symmetric about 0 with $P(|Z_m| > x) = P(|Y_m| < x^{-1/p})$, so (i) in Theorem 3.7.2 holds with $\theta = 1/2$ and (ii) holds with $\alpha = 1/p$. The scaling constant $a_n \sim Cn^p$ and the centering constant is 0 by symmetry.

**Exercise 3.7.3.** Show that (i) If $p < 1/2$ then $F_n/n^{1/2-p} \Rightarrow c\chi$.
(ii) If $p = 1/2$ then $F_n/(\log\, n)^{1/2} \Rightarrow c\chi$.

**Example 3.7.4.** In the examples above, we have had $b_n = 0$. To get a feel for the centering constants, consider $X_1, X_2, \ldots$ i.i.d. with

$$P(X_i > x) = \theta x^{-\alpha} \qquad P(X_i < -x) = (1-\theta)x^{-\alpha}$$

where $0 < \alpha < 2$. In this case $a_n = n^{1/\alpha}$ and

$$b_n = n\int_1^{n^{1/\alpha}} (2\theta - 1)\alpha x^{-\alpha}\, dx \sim \begin{cases} cn & \alpha > 1 \\ cn\log n & \alpha = 1 \\ cn^{1/\alpha} & \alpha < 1 \end{cases}$$

When $\alpha < 1$ the centering is the same size as the scaling and can be ignored. When $\alpha > 1$, $b_n \sim n\mu$ where $\mu = EX_i$.

Our next result explains the name **stable laws**. A random variable $Y$ is said to have a **stable law** if for every integer $k > 0$ there are constants $a_k$ and $b_k$ so that if $Y_1, \ldots, Y_k$ are i.i.d. and have the same distribution as $Y$, then $(Y_1 + \cdots + Y_k - b_k)/a_k =_d Y$. The last definition makes half of the next result obvious.

**Theorem 3.7.4.** *$Y$ is the limit of $(X_1 + \cdots + X_k - b_k)/a_k$ for some i.i.d. sequence $X_i$ if and only if $Y$ has a stable law.*

*Proof.* If $Y$ has a stable law, we can take $X_1, X_2, \ldots$ i.i.d. with distribution $Y$. To go the other way, let

$$Z_n = (X_1 + \cdots + X_n - b_n)/a_n$$

and $S_n^j = X_{(j-1)n+1} + \cdots + X_{jn}$. A little arithmetic shows

$$\begin{aligned} Z_{nk} &= (S_n^1 + \cdots + S_n^k - b_{nk})/a_{nk} \\ a_{nk} Z_{nk} &= (S_n^1 - b_n) + \cdots + (S_n^k - b_n) + (kb_n - b_{nk}) \\ a_{nk} Z_{nk}/a_n &= (S_n^1 - b_n)/a_n + \cdots + (S_n^k - b_n)/a_n + (kb_n - b_{nk})/a_n \end{aligned}$$

The first $k$ terms on the right-hand side $\Rightarrow Y_1 + \cdots + Y_k$ as $n \to \infty$ where $Y_1, \ldots, Y_k$ are independent and have the same distribution as $Y$, and $Z_{nk} \Rightarrow Y$. Taking $W_n = Z_{nk}$ and

$$W_n' = \frac{a_{kn}}{a_n} Z_{nk} - \frac{kb_n - b_{nk}}{a_n}$$

gives the desired result. ■

**Theorem 3.7.5. Convergence of types theorem.** *If $W_n \Rightarrow W$ and there are constants $\alpha_n > 0$, $\beta_n$ so that $W_n' = \alpha_n W_n + \beta_n \Rightarrow W'$ where $W$ and $W'$ are nondegenerate, then there are constants $\alpha$ and $\beta$ so that $\alpha_n \to \alpha$ and $\beta_n \to \beta$.*

*Proof.* Let $\varphi_n(t) = E \exp(itW_n)$.

$$\psi_n(t) = E \exp(it(\alpha_n W_n + \beta_n)) = \exp(it\beta_n)\varphi_n(\alpha_n t)$$

If $\varphi$ and $\psi$ are the characteristic functions of $W$ and $W'$, then

(a) $$\varphi_n(t) \to \varphi(t) \qquad \psi_n(t) = \exp(it\beta_n)\varphi_n(\alpha_n t) \to \psi(t)$$

Take a subsequence $\alpha_{n(m)}$ that converges to a limit $\alpha \in [0, \infty]$. Our first step is to observe $\alpha = 0$ is impossible. If this happens, then using the uniform convergence proved in Exercise 3.3.16

(b) $$|\psi_n(t)| = |\varphi_n(\alpha_n t)| \to 1$$

$|\psi(t)| \equiv 1$, and the limit is degenerate by Theorem 3.5.1. Letting $t = u/\alpha_n$ and interchanging the roles of $\varphi$ and $\psi$ shows $\alpha = \infty$ is impossible. If $\alpha$ is a subsequential limit, then arguing as in (b) gives $|\psi(t)| = |\varphi(\alpha t)|$. If there are two subsequential limits $\alpha' < \alpha$, using the last equation for both limits implies $|\varphi(u)| = |\varphi(u\alpha'/\alpha)|$. Iterating gives $|\varphi(u)| = |\varphi(u(\alpha'/\alpha)^k)| \to 1$ as $k \to \infty$, contradicting our assumption that $W'$ is nondegenerate, so $\alpha_n \to \alpha \in [0, \infty)$.

To conclude that $\beta_n \to \beta$ now, we observe that (ii) of Exercise 3.3.16 implies $\varphi_n \to \varphi$ uniformly on compact sets so $\varphi_n(\alpha_n t) \to \varphi(\alpha t)$. If $\delta$ is small enough so that $|\varphi(\alpha t)| > 0$ for $|t| \le \delta$, it follows from (a) and another use of Exercise 3.3.16 that

$$\exp(it\beta_n) = \frac{\psi_n(t)}{\varphi_n(\alpha t)} \to \frac{\psi(t)}{\varphi(\alpha t)}$$

uniformly on $[-\delta, \delta]$. $\exp(it\beta_n)$ is the ch.f. of a point mass at $\beta_n$. Using (3.3.1) now as in the proof of Theorem 3.3.6, it follows that the sequence of distributions that are point masses at $\beta_n$ is tight, that is, $\beta_n$ is bounded. If $\beta_{n_m} \to \beta$, then $\exp(it\beta) = \psi(t)/\varphi(\alpha t)$ for $|t| \le \delta$, so there can only be one subsequential limit. ■

Theorem 3.7.4 justifies calling the distributions with characteristic functions given by (3.7.11) or (3.7.10) stable laws. To complete the story, we should mention that these are the only stable laws. Again, see Chapter 9 of Breiman (1968) or Gnedenko and Kolmogorov (1954). The next example shows that it is sometimes useful to know what all the possible limits are.

**Example 3.7.5. The Holtsmark distribution.** ($\alpha = 3/2$, $\kappa = 0$). Suppose stars are distributed in space according to a Poisson process with density $t$ and their masses are i.i.d. Let $X_t$ be the $x$-component of the gravitational force at 0 when the density is $t$. A change of density $1 \to t$ corresponds to a change of length $1 \to t^{-1/3}$, and gravitational attraction follows an inverse square law, so

$$X_t \stackrel{d}{=} t^{3/2} X_1 \tag{3.7.13}$$

If we imagine thinning the Poisson process by rolling an $n$-sided die, then Exercise 3.6.12 implies

$$X_t \stackrel{d}{=} X^1_{t/n} + \cdots + X^n_{t/n}$$

where the random variables on the right-hand side are independent and have the same distribution as $X_{t/n}$. It follows from Theorem 3.7.4 that $X_t$ has a stable law. The scaling property (3.7.13) implies $\alpha = 3/2$. Since $X_t =_d -X_t$, $\kappa = 0$.

## *Exercises*

**3.7.4.** Let $Y$ be a stable law with $\kappa = 1$. Use the limit theorem Theorem 3.7.2 to conclude that $Y \ge 0$ if $\alpha < 1$.

**3.7.5.** Let $X$ be symmetric stable with index $\alpha$. (i) Use (3.3.1) to show that $E|X|^p < \infty$ for $p < \alpha$. (ii) Use the second proof of (3.7.3) to show that $P(|X| \ge x) \ge Cx^{-\alpha}$, so $E|X|^\alpha = \infty$.

**3.7.6.** Let $Y, Y_1, Y_2, \ldots$ be independent and have a stable law with index $\alpha$. Theorem 3.7.4 implies there are constants $\alpha_k$ and $\beta_k$ so that $Y_1 + \cdots + Y_k$ and $\alpha_k Y + \beta_k$ have the same distribution. Use the proof of Theorem 3.7.4, Theorem 3.7.2, and Exercise 3.7.2 to conclude that (i) $\alpha_k = k^{1/\alpha}$, (ii) if $\alpha < 1$ then $\beta_k = 0$.

**3.7.7.** Let $Y$ be a stable law with index $\alpha < 1$ and $\kappa = 1$. Exercise 3.7.4 implies that $Y \ge 0$, so we can define its Laplace transform $\psi(\lambda) = E\exp(-\lambda Y)$. The previous exercise implies that for any integer $n \ge 1$ we have $\psi(\lambda)^n = \psi(n^{1/\alpha}\lambda)$. Use this to conclude $E\exp(-\lambda Y) = \exp(-c\lambda^\alpha)$.

**3.7.8.** (i) Show that if $X$ is symmetric stable with index $\alpha$ and $Y \geq 0$ is an independent stable with index $\beta < 1$, then $XY^{1/\alpha}$ is symmetric stable with index $\alpha\beta$. (ii) Let $W_1$ and $W_2$ be independent standard normals. Check that $1/W_2^2$ has the density given in (3.7.12) and use this to conclude that $W_1/W_2$ has a Cauchy distribution.

## 3.8 Infinitely Divisible Distributions*

In the last section, we identified the distributions that can appear as the limit of normalized sums of i.i.d.r.v.'s. In this section, we will describe those that are limits of sums

$$(*) \qquad S_n = X_{n,1} + \cdots + X_{n,n}$$

where the $X_{n,m}$ are i.i.d. Note the verb "describe." We will prove almost nothing in this section, just state some of the most important facts to bring the reader up to cocktail-party literacy.

A sufficient condition for $Z$ to be a limit of sums of the form $(*)$ is that $Z$ has an **infinitely divisible distribution**, that is, for each $n$ there is an i.i.d. sequence $Y_{n,1}, \ldots, Y_{n,n}$ so that

$$Z \stackrel{d}{=} Y_{n,1} + \cdots + Y_{n,n}$$

Our first result shows that this condition is also necessary.

**Theorem 3.8.1.** *$Z$ is a limit of sums of type $(*)$ if and only if $Z$ has an infinitely divisible distribution.*

*Proof.* As remarked above, we only have to prove necessity. Write

$$S_{2n} = (X_{2n,1} + \cdots + X_{2n,n}) + (X_{2n,n+1} + \cdots + X_{2n,2n}) \equiv Y_n + Y_n'$$

The random variables $Y_n$ and $Y_n'$ are independent and have the same distribution. If $S_n \Rightarrow Z$, then the distributions of $Y_n$ are a tight sequence since

$$P(Y_n > y)^2 = P(Y_n > y)P(Y_n' > y) \leq P(S_{2n} > 2y)$$

and similarly $P(Y_n < -y)^2 \leq P(S_{2n} < -2y)$. If we take a subsequence $n_k$ so that $Y_{n_k} \Rightarrow Y$ (and hence $Y'_{n_k} \Rightarrow Y'$), then $Z =_d Y + Y'$. A similar argument shows that $Z$ can be divided into $n > 2$ pieces, and the proof is complete. ∎

With Theorem 3.8.1 established, we turn now to examples. In the first three cases, the distribution is infinitely divisible because it is a limit of sums of the form $(*)$. The number gives the relevant limit theorem.

**Example 3.8.1. Normal distribution.** Theorem 3.4.1

**Example 3.8.2. Stable laws.** Theorem 3.7.2

**Example 3.8.3. Poisson distribution.** Theorem 3.6.1

**Example 3.8.4. Compound Poisson distribution.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. and $N(\lambda)$ be an independent Poisson r.v. with mean $\lambda$. Then $Z = \xi_1 + \cdots + \xi_{N(\lambda)}$ has an infinitely divisible distribution. (Let $X_{n,j} =_d \xi_1 + \cdots + \xi_{N(\lambda/n)}$.) For developments below, we would like to observe that if $\varphi(t) = E\exp(it\xi_i)$ then

$$E\exp(itZ) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\varphi(t)^n = \exp(-\lambda(1-\varphi(t))) \tag{3.8.1}$$

**Exercise 3.8.1.** Show that the gamma distribution is infinitely divisible.

The next two exercises give examples of distributions that are not infinitely divisible.

**Exercise 3.8.2.** Show that the distribution of a bounded r.v. $Z$ is infinitely divisible if and only if $Z$ is constant. Hint: Show $\operatorname{var}(Z) = 0$.

**Exercise 3.8.3.** Show that if $\mu$ is infinitely divisible, its ch.f. $\varphi$ never vanishes. Hint: Look at $\psi = |\varphi|^2$, which is also infinitely divisible; to avoid taking $n$th roots of complex numbers, then use Exercise 3.3.20.

Example 2.8.4 is a son of 2.8.3 but a father of 2.8.1 and 2.8.2. To explain this remark, we observe that if $\xi = \epsilon$ and $-\epsilon$ with probability $1/2$ each then $\varphi(t) = (e^{i\epsilon t} + e^{-i\epsilon t})/2 = \cos(\epsilon t)$. So if $\lambda = \epsilon^{-2}$, then (3.8.1) implies

$$E\exp(itZ) = \exp(-\epsilon^{-2}(1-\cos(\epsilon t))) \to \exp(-t^2/2)$$

as $\epsilon \to 0$. In words, the normal distribution is a limit of compound Poisson distributions. To see that stable laws are also a special case (using the notation from the proof of Theorem 3.7.2), let

$$I_n(\epsilon) = \{m \le n : |X_m| > \epsilon a_n\}$$

$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m$$

$$\bar{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon)$$

If $\epsilon_n \to 0$ then $\bar{S}_n(\epsilon_n)/a_n \Rightarrow 0$. If $\epsilon$ is fixed then as $n \to \infty$ we have $|I_n(\epsilon)| \Rightarrow$ Poisson$(\epsilon^{-\alpha})$ and $\hat{S}_n(\epsilon)/a_n \Rightarrow$ a compound Poisson distribution:

$$E\exp(it\hat{S}_n(\epsilon)/a_n) \to \exp(-\epsilon^{-\alpha}\{1 - \psi^{\epsilon}(t)\})$$

Combining the last two observations and using the proof of Theorem 3.7.2 shows that stable laws are limits of compound Poisson distributions. The formula (3.7.10) for the limiting ch.f.

$$\exp\left(itc + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\theta\alpha x^{-(\alpha+1)}\,dx \right.$$
$$\left. + \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)(1-\theta)\alpha|x|^{-(\alpha+1)}\,dx\right) \tag{3.8.2}$$

helps explain:

**Theorem 3.8.2. Lévy-Khinchin theorem.** *Z has an infinitely divisible distribution if and only if its characteristic function has*

$$\log\varphi(t) = ict - \frac{\sigma^2 t^2}{2} + \int \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\mu(dx)$$

*where $\mu$ is a measure with $\mu(\{0\}) = 0$ and $\int \frac{x^2}{1+x^2}\mu(dx) < \infty$.*

For a proof, see Breiman (1968), Section 9.5., or Feller II (1971), Section XVII.2. $\mu$ is called the **Lévy measure** of the distribution. Comparing with (3.8.2) and recalling the proof of Theorem 3.7.2 suggests the following interpretation of $\mu$: If $\sigma^2 = 0$ then $Z$ can be built up by making a Poisson process on $\mathbf{R}$ with mean measure $\mu$ and then summing up the points. As in the case of stable laws, we have to sum the points in $[-\epsilon, \epsilon]^c$, subtract an appropriate constant, and let $\epsilon \to 0$.

**Exercise 3.8.4.** What is the Lévy measure for the limit $\aleph$ in part (iii) of Exercise 3.4.13?

The theory of infinitely divisible distributions is simpler in the case of finite variance. In this case, we have:

**Theorem 3.8.3. Kolmogorov's theorem.** *Z has an infinitely divisible distribution with mean 0 and finite variance if and only if its ch.f. has*

$$\log\varphi(t) = \int (e^{itx} - 1 - itx)x^{-2}\,\nu(dx)$$

*Here the integrand is $-t^2/2$ at 0, $\nu$ is called the* **canonical measure**, *and var$(Z) = \nu(\mathbf{R})$.*

To explain the formula, note that if $Z_\lambda$ has a Poisson distribution with mean $\lambda$,

$$E\exp(itx(Z_\lambda - \lambda)) = \exp(\lambda(e^{itx} - 1 - itx))$$

so the measure for $Z = x(Z_\lambda - \lambda)$ has $\nu(\{x\}) = \lambda x^2$.

## 3.9 Limit Theorems in $\mathbf{R}^d$

Let $X = (X_1, \dots, X_d)$ be a random vector. We define its **distribution function** by $F(x) = P(X \le x)$. Here $x \in \mathbf{R}^d$, and $X \le x$ means $X_i \le x_i$ for $i = 1, \dots, d$. As in one dimension, $F$ has three obvious properties:

(i) It is nondecreasing, that is, if $x \le y$ then $F(x) \le F(y)$.
(ii) $\lim_{x\to\infty} F(x) = 1, \quad \lim_{x_i\to-\infty} F(x) = 0$.
(iii) $F$ is right continuous, that is, $\lim_{y\downarrow x} F(y) = F(x)$.

Here $x \to \infty$ means each coordinate $x_i$ goes to $\infty$, $x_i \to -\infty$ means we let $x_i \to -\infty$ keeping the other coordinates fixed, and $y \downarrow x$ means each coordinate $y_i \downarrow x_i$.

As discussed in Section 1.1, an additional condition is needed to guarantee that $F$ is the distribution function of a probability measure. Let

$$A = (a_1, b_1] \times \cdots \times (a_d, b_d]$$
$$V = \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}$$

$V$ = the vertices of the rectangle $A$. If $v \in V$, let

$$\operatorname{sgn}(v) = (-1)^{\#\text{ of } a\text{'s in } v}$$

The inclusion-exclusion formula implies

$$P(X \in A) = \sum_{v\in V} \operatorname{sgn}(v) F(v)$$

So if we use $\Delta_A F$ to denote the right-hand side, we need

(iv) $\Delta_A F \ge 0$ for all rectangles $A$.

The last condition guarantees that the measure assigned to each rectangle is $\ge 0$. At this point we have defined the measure on the semialgebra $\mathcal{S}_d$ defined in Example 1.1.3. Theorem 1.1.6 now implies that there is a unique probability measure with distribution $F$.

**Exercise 3.9.1.** If $F$ is the distribution of $(X_1, \dots, X_d)$, then $F_i(x) = P(X_i \le x)$ are its **marginal distributions**. How can they be obtained from $F$?

**Exercise 3.9.2.** Let $F_1, \dots, F_d$ be distributions on $\mathbf{R}$. Show that for any $\alpha \in [-1, 1]$

$$F(x_1, \dots, x_d) = \left\{1 + \alpha \prod_{i=1}^{d}(1 - F_i(x_i))\right\} \prod_{j=1}^{d} F_j(x_j)$$

is a d.f. with the given marginals. The case $\alpha = 0$ corresponds to independent r.v.'s.

**Exercise 3.9.3.** A distribution $F$ is said to have a **density** $f$ if

$$F(x_1, \dots, x_k) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(y)\, dy_k \dots dy_1$$

Show that if $f$ is continuous, $\partial^k F/\partial x_1 \dots \partial x_k = f$.

If $F_n$ and $F$ are distribution functions on $\mathbf{R}^d$, we say that $F_n$ **converges weakly** to $F$, and write $F_n \Rightarrow F$, if $F_n(x) \to F(x)$ at all continuity points of $F$. Our first task is to show that there are enough continuity points for this to be a sensible definition. For a concrete example, consider

$$F(x, y) = \begin{cases} 1 & \text{if } x \geq 0, \ y \geq 1 \\ y & \text{if } x \geq 0, \ 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$F$ is the distribution function of $(0, Y)$ where $Y$ is uniform on (0,1). Notice that this distribution has no atoms, but $F$ is discontinuous at $(0, y)$ when $y > 0$.

Keeping the last example in mind, observe that if $x_n < x$, that is, $x_{n,i} < x_i$ for all coordinates $i$, and $x_n \uparrow x$ as $n \to \infty$ then

$$F(x) - F(x_n) = P(X \leq x) - P(X \leq x_n) \downarrow P(X \leq x) - P(X < x)$$

In $d = 2$, the last expression is the probability $X$ lies in

$$\{(a, x_2) : a \leq x_1\} \cup \{(x_1, b) : b \leq x_2\}$$

Let $H_c^i = \{x : x_i = c\}$ be the hyperplane where the $i$th coordinate is $c$. For each $i$, the $H_c^i$ are disjoint so $D^i = \{c : P(X \in H_c^i) > 0\}$ is at most countable. It is easy to see that if $x$ has $x_i \notin D^i$ for all $i$ then $F$ is continuous at $x$. This gives us more than enough points to reconstruct $F$.

As in Section 3.2, it will be useful to have several equivalent definitions of weak convergence. In Chapter 8, we will need to know that this is valid for an arbitrary metric space $(S, \rho)$, so we will prove the result in that generality and insert another equivalence that will be useful there. $f$ is said to be **Lipschitz continuous** if there is a constant $C$ so that $|f(x) - f(y)| \leq C\rho(x, y)$.

**Theorem 3.9.1.** *The following statements are equivalent to $X_n \Rightarrow X_\infty$.*
*(i) $Ef(X_n) \to Ef(X_\infty)$ for all bounded continuous $f$.*
*(ii) $Ef(X_n) \to Ef(X_\infty)$ for all bounded Lipschitz continuous $f$.*
*(iii) For all closed sets $K$, $\limsup_{n\to\infty} P(X_n \in K) \leq P(X_\infty \in K)$.*
*(iv) For all open sets $G$, $\liminf_{n\to\infty} P(X_n \in G) \geq P(X_\infty \in G)$.*
*(v) For all sets $A$ with $P(X_\infty \in \partial A) = 0$, $\lim_{n\to\infty} P(X_n \in A) = P(X_\infty \in A)$.*
*(vi) Let $D_f$ = the set of discontinuities of $f$. For all bounded functions $f$ with $P(X_\infty \in D_f) = 0$, we have $Ef(X_n) \to Ef(X_\infty)$.*

*Proof.* We will begin by showing that (i)–(vi) are equivalent.

*(i) implies (ii):* Trivial.

*(ii) implies (iii):* Let $\rho(x, K) = \inf\{\rho(x, y) : y \in K\}$, $\varphi_j(r) = (1 - jr)^+$, and $f_j(x) = \varphi_j(\rho(x, K))$. $f_j$ is Lipschitz continuous, has values in [0,1], and $\downarrow 1_K(x)$ as $j \uparrow \infty$. So

$$\limsup_{n\to\infty} P(X_n \in K) \leq \lim_{n\to\infty} Ef_j(X_n) = Ef_j(X_\infty) \downarrow P(X_\infty \in K) \text{ as } j \uparrow \infty$$

*(iii) is equivalent to (iv):* As in the proof of Theorem 3.2.5, this follows easily from two facts: $A$ is open if and only if $A^c$ is closed; $P(A) + P(A^c) = 1$.

*(iii) and (iv) imply (v):* Let $K = \bar{A}$, $G = A^o$, and reason as in the proof of Theorem 3.2.5.

*(v) implies (vi):* Suppose $|f(x)| \le K$ and pick $\alpha_0 < \alpha_1 < \cdots < \alpha_\ell$ so that $P(f(X_\infty) = \alpha_i) = 0$ for $0 \le i \le \ell$, $\alpha_0 < -K < K < \alpha_\ell$, and $\alpha_i - \alpha_{i-1} < \epsilon$. This is always possible since $\{\alpha : P(f(X_\infty) = \alpha) > 0\}$ is a countable set. Let $A_i = \{x : \alpha_{i-1} < f(x) \le \alpha_i\}$. $\partial A_i \subset \{x : f(x) \in \{\alpha_{i-1}, \alpha_i\}\} \cup D_f$, so $P(X_\infty \in \partial A_i) = 0$, and it follows from (v) that

$$\sum_{i=1}^{\ell} \alpha_i P(X_n \in A_i) \to \sum_{i=1}^{\ell} \alpha_i P(X_\infty \in A_i)$$

The definition of the $\alpha_i$ implies

$$0 \le \sum_{i=1}^{\ell} \alpha_i P(X_n \in A_i) - Ef(X_n) \le \epsilon \quad \text{for } 1 \le n \le \infty$$

Since $\epsilon$ is arbitrary, it follows that $Ef(X_n) \to Ef(X_\infty)$.

*(vi) implies (i):* Trivial.

It remains to show that the six conditions are equivalent to weak convergence ($\Rightarrow$).

*(v) implies* ($\Rightarrow$) : If $F$ is continuous at $x$, then $A = (-\infty, x_1] \times \cdots \times (-\infty, x_d]$ has $\mu(\partial A) = 0$, so $F_n(x) = P(X_n \in A) \to P(X_\infty \in A) = F(x)$.

($\Rightarrow$) *implies (iv):* Let $D^i = \{c : P(X_\infty \in H_c^i) > 0\}$ where $H_c^i = \{x : x^i = c\}$. We say a rectangle $A = (a_1, b_1] \times \cdots \times (a_d, b_d]$ is good if $a_i, b_i \notin D^i$ for all $i$. ($\Rightarrow$) implies that for all good rectangles $P(X_n \in A) \to P(X_\infty \in A)$. This is also true for $B$ that are a finite disjoint union of good rectangles. Now any open set $G$ is an increasing limit of $B_k$'s that are a finite disjoint union of good rectangles, so

$$\liminf_{n\to\infty} P(X_n \in G) \ge \liminf_{n\to\infty} P(X_n \in B_k) = P(X_\infty \in B_k) \uparrow P(X_\infty \in G)$$

as $k \to \infty$. The proof is complete. ■

**Remark.** In Section 3.2, we proved that (i)–(v) are consequences of weak convergence by constructing r.v.'s with the given distributions so that $X_n \to X_\infty$ a.s. This can be done in $\mathbf{R}^d$ (or any complete separable metric space), but the construction is rather messy. See Billingsley (1979), pp. 337–340, for a proof in $\mathbf{R}^d$.

**Exercise 3.9.4.** Let $X_n$ be random vectors. Show that if $X_n \Rightarrow X$, then the coordinates $X_{n,i} \Rightarrow X_i$.

A sequence of probability measures $\mu_n$ is said to be **tight** if for any $\epsilon > 0$, there is an $M$ so that $\liminf_{n\to\infty} \mu_n([-M, M]^d) \ge 1 - \epsilon$.

**Theorem 3.9.2.** *If $\mu_n$ is tight, then there is a weakly convergent subsequence.*

*Proof.* Let $F_n$ be the associated distribution functions, and let $q_1, q_2, \dots$ be an enumeration of $\mathbf{Q}^d$ = the points in $\mathbf{R}^d$ with rational coordinates. By a diagonal argument like the one in the proof of Theorem 3.2.6, we can pick a subsequence so that $F_{n(k)}(q) \to G(q)$ for all $q \in \mathbf{Q}^d$. Let

$$F(x) = \inf\{G(q) : q \in \mathbf{Q}^d, q > x\}$$

where $q > x$ means $q_i > x_i$ for all $i$. It is easy to see that $F$ is right continuous. To check that it is a distribution function, we observe that if $A$ is a rectangle with vertices in $\mathbf{Q}^d$, then $\Delta_A F_n \ge 0$ for all $n$, so $\Delta_A G \ge 0$, and taking limits we see that the last conclusion holds for $F$ for all rectangles $A$. Tightness implies that $F$ has properties (i) and (ii) of a distribution $F$. We leave it to the reader to check that $F_n \Rightarrow F$. The proof of Theorem 3.2.6 works if you read inequalities such as $r_1 < r_2 < x < s$ as the corresponding relations between vectors. ∎

The **characteristic function** of $(X_1, \dots, X_d)$ is $\varphi(t) = E\exp(it \cdot X)$ where $t \cdot X = t_1X_1 + \cdots + t_dX_d$ is the usual dot product of two vectors.

**Theorem 3.9.3. Inversion formula.** *If $A = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $\mu(\partial A) = 0$ then*

$$\mu(A) = \lim_{T\to\infty} (2\pi)^{-d} \int_{[-T,T]^d} \prod_{j=1}^d \psi_j(t_j)\varphi(t)\,dt$$

*where $\psi_j(s) = (\exp(-isa_j) - \exp(-isb_j))/is$.*

*Proof.* Fubini's theorem implies

$$\int_{[-T,T]^d} \int \prod_{j=1}^d \psi_j(t_j)\exp(it_jx_j)\,\mu(dx)\,dt$$

$$= \int \prod_{j=1}^d \int_{-T}^T \psi_j(t_j)\exp(it_jx_j)\,dt_j\,\mu(dx)$$

It follows from the proof of Theorem 3.3.4 that

$$\int_{-T}^T \psi_j(t_j)\exp(it_jx_j)\,dt_j \to \pi\left(1_{(a_j,b_j)}(x) + 1_{[a_j,b_j]}(x)\right)$$

so the desired conclusion follows from the bounded convergence theorem. ∎

**Exercise 3.9.5.** Let $\varphi$ be the ch.f. of a distribution $F$ on $\mathbf{R}$. What is the distribution on $\mathbf{R}^d$ that corresponds to the ch.f. $\psi(t_1, \dots, t_d) = \varphi(t_1 + \cdots + t_d)$?

**Exercise 3.9.6.** Show that random variables $X_1, \ldots, X_k$ are independent if and only if

$$\varphi_{X_1,\ldots X_k}(t) = \prod_{j=1}^{k} \varphi_{X_j}(t_j)$$

**Theorem 3.9.4. Convergence theorem.** *Let $X_n$, $1 \le n \le \infty$ be random vectors with ch.f. $\varphi_n$. A necessary and sufficient condition for $X_n \Rightarrow X_\infty$ is that $\varphi_n(t) \to \varphi_\infty(t)$.*

*Proof.* $\exp(it \cdot x)$ is bounded and continuous, so if $X_n \Rightarrow X_\infty$ then $\varphi_n(t) \to \varphi_\infty(t)$. To prove the other direction it suffices, as in the proof of Theorem 3.3.6, to prove that the sequence is tight. To do this, we observe that if we fix $\theta \in \mathbf{R}^d$, then for all $s \in \mathbf{R}$, $\varphi_n(s\theta) \to \varphi_\infty(s\theta)$, so it follows from Theorem 3.3.6 that the distributions of $\theta \cdot X_n$ are tight. Applying the last observation to the $d$ unit vectors $e_1, \ldots, e_d$ shows that the distributions of $X_n$ are tight and completes the proof. ∎

**Remark.** As before, if $\varphi_n(t) \to \varphi_\infty(t)$ with $\varphi_\infty(t)$ continuous at 0, then $\varphi_\infty(t)$ is the ch.f. of some $X_\infty$ and $X_n \Rightarrow X_\infty$.

Theorem 3.9.4 has an important corollary.

**Theorem 3.9.5. Cramér-Wold device.** *A sufficient condition for $X_n \Rightarrow X_\infty$ is that $\theta \cdot X_n \Rightarrow \theta \cdot X_\infty$ for all $\theta \in \mathbf{R}^d$.*

*Proof.* The indicated condition implies $E \exp(i\theta \cdot X_n) \to E \exp(i\theta \cdot X_\infty)$ for all $\theta \in \mathbf{R}^d$. ∎

Theorem 3.9.5 leads immediately to

**Theorem 3.9.6. The central limit theorem in $\mathbf{R}^d$.** *Let $X_1, X_2, \ldots$ be i.i.d. random vectors with $EX_n = \mu$, and finite covariances*

$$\Gamma_{ij} = E((X_{n,i} - \mu_i)(X_{n,j} - \mu_j))$$

*If $S_n = X_1 + \cdots + X_n$ then $(S_n - n\mu)/n^{1/2} \Rightarrow \chi$, where $\chi$ has a multivariate normal distribution with mean 0 and covariance $\Gamma$, that is,*

$$E \exp(i\theta \cdot \chi) = \exp\left( -\sum_i \sum_j \theta_i \theta_j \Gamma_{ij}/2 \right)$$

*Proof.* By considering $X'_n = X_n - \mu$, we can suppose without loss of generality that $\mu = 0$. Let $\theta \in \mathbf{R}^d$. $\theta \cdot X_n$ is a random variable with mean 0 and variance

$$E\Big(\sum_i \theta_i X_{n,i}\Big)^2 = \sum_i \sum_j E\left(\theta_i \theta_j X_{n,i} X_{n,j}\right) = \sum_i \sum_j \theta_i \theta_j \Gamma_{ij}$$

so it follows from the one-dimensional central limit theorem and Theorem 3.9.5 that $S_n/n^{1/2} \Rightarrow \chi$ where

$$E\exp(i\theta\cdot\chi) = \exp\left(-\sum_i\sum_j \theta_i\theta_j\Gamma_{ij}/2\right)$$

which proves the desired result. ∎

To illustrate the use of Theorem 3.9.6, we consider two examples. In each $e_1, \ldots, e_d$ are the $d$ unit vectors.

**Example 3.9.1. Simple random walk on $\mathbf{Z}^d$.** Let $X_1, X_2, \ldots$ be i.i.d. with

$$P(X_n = +e_i) = P(X_n = -e_i) = 1/2d \qquad \text{for } i = 1, \ldots, d$$

$EX_n^i = 0$ and if $i \neq j$ then $EX_n^iX_n^j = 0$ since both components cannot be nonzero simultaneously. So the covariance matrix is $\Gamma_{ij} = (1/2d)I$.

**Example 3.9.2.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_n = e_i) = 1/6$ for $i = 1, 2, \ldots, 6$. In words, we are rolling a die and keeping track of the numbers that come up. $EX_{n,i} = 1/6$ and $EX_{n,i}X_{n,j} = 0$ for $i \neq j$, so $\Gamma_{ij} = (1/6)(5/6)$ when $i = j$ and $= -(1/6)^2$ when $i \neq j$. In this case, the limiting distribution is concentrated on $\{x : \sum_i x_i = 0\}$.

Our treatment of the central limit theorem would not be complete without some discussion of the multivariate normal distribution. We begin by observing that $\Gamma_{ij} = \Gamma_{ji}$, and if $EX_i = 0$ and $EX_iX_j = \Gamma_{i,j}$,

$$\sum_i\sum_j \theta_i\theta_j\Gamma_{ij} = E\left(\sum_i \theta_iX_i\right)^2 \geq 0$$

so $\Gamma$ is symmetric and nonnegative definite. A well-known result implies that there is an orthogonal matrix $U$ (i.e., one with $U^tU = I$, the identity matrix) so that $\Gamma = U^tVU$, where $V \geq 0$ is a diagonal matrix. Let $W$ be the nonnegative diagonal matrix with $W^2 = V$. If we let $A = WU$, then $\Gamma = A^tA$. Let $Y$ be a $d$-dimensional vector whose components are independent and have normal distributions with mean 0 and variance 1. If we view vectors as $1 \times d$ matrices and let $\chi = YA$, then $\chi$ has the desired normal distribution. To check this, observe that

$$\theta\cdot YA = \sum_i \theta_i \sum_j Y_jA_{ji}$$

has a normal distribution with mean 0 and variance

$$\sum_j\left(\sum_i A_{ji}\theta_i\right)^2 = \sum_j\left(\sum_i \theta_iA^t_{ij}\right)\left(\sum_k A_{jk}\theta_k\right) = \theta A^tA\theta^t = \theta\Gamma\theta^t$$

so $E(\exp(i\theta\cdot\chi)) = \exp(-(\theta\Gamma\theta^t)/2)$.

If the covariance matrix has rank $d$, we say that the normal distribution is **nondegenerate.** In this case, its density function is given by

$$(2\pi)^{-d/2}(\det\Gamma)^{-1/2}\exp\left(-\sum_{i,j} y_i\Gamma_{ij}^{-1}y_j/2\right)$$

The joint distribution in degenerate cases can be computed by using a linear transformation to reduce to the nondegenerate case. For instance, in Example 3.9.2 we can look at the distribution of $(X_1, \dots, X_5)$.

**Exercise 3.9.7.** Suppose $(X_1, \dots, X_d)$ has a multivariate normal distribution with mean vector $\theta$ and covariance $\Gamma$. Show $X_1, \dots, X_d$ are independent if and only if $\Gamma_{ij} = 0$ for $i \neq j$. In words, uncorrelated random variables with a joint normal distribution are independent.

**Exercise 3.9.8.** Show that $(X_1, \dots, X_d)$ has a multivariate normal distribution with mean vector $\theta$ and covariance $\Gamma$ if and only if every linear combination $c_1X_1 + \cdots + c_dX_d$ has a normal distribution with mean $c\theta^t$ and variance $c\Gamma c^t$.

# 4

# Random Walks

Let $X_1, X_2, \ldots$ be i.i.d. taking values in $\mathbf{R}^d$ and let $S_n = X_1 + \cdots + X_n$. $S_n$ is a **random walk**. In the previous chapter, we were primarily concerned with the distribution of $S_n$. In this one, we will look at properties of the sequence $S_1(\omega), S_2(\omega), \ldots$ For example, does the last sequence return to (or near) 0 infinitely often? The first section introduces stopping times, a concept that will be very important in this and the next two chapters. After the first section is completed, the remaining three can be read in any order or skipped without much loss. The second section is not starred since it contains some basic facts about random walks.

## 4.1 Stopping Times

Most of the results in this section are valid for i.i.d. $X$'s taking values in some nice measurable space $(S, \mathcal{S})$ and will be proved in that generality. For several reasons, it is convenient to use the special probability space from the proof of Kolmogorov's extension theorem:

$$\begin{aligned}
\Omega &= \{(\omega_1, \omega_2, \ldots) : \omega_i \in S\} \\
\mathcal{F} &= \mathcal{S} \times \mathcal{S} \times \ldots \\
P &= \mu \times \mu \times \ldots \qquad \mu \text{ is the distribution of } X_i \\
X_n(\omega) &= \omega_n
\end{aligned}$$

So, throughout this section, we will suppose (without loss of generality) that our random variables are constructed on this special space.

Before taking up our main topic, we will prove a 0-1 law that, in the i.i.d. case, generalizes Kolmogorov's. To state the new 0-1 law, we need two definitions. A **finite permutation** of $\mathbf{N} = \{1, 2, \ldots\}$ is a map $\pi$ from $\mathbf{N}$ onto $\mathbf{N}$ so that $\pi(i) \neq i$ for only finitely many $i$. If $\pi$ is a finite permutation of $\mathbf{N}$ and $\omega \in S^{\mathbf{N}}$, we define $(\pi\omega)_i = \omega_{\pi(i)}$. In words, the coordinates of $\omega$ are rearranged according to $\pi$. Since $X_i(\omega) = \omega_i$, this is the same as rearranging the random variables. An event $A$ is **permutable** if $\pi^{-1}A \equiv \{\omega : \pi\omega \in A\}$ is equal to $A$ for any finite permutation $\pi$, or in other words, if its occurrence is not affected by rearranging finitely many of the random variables. The collection of permutable events is a $\sigma$-field. It is called the **exchangeable** $\sigma$-field and denoted by $\mathcal{E}$.

To see the reason for interest in permutable events, suppose $S = \mathbf{R}$ and let $S_n(\omega) = X_1(\omega) + \cdots + X_n(\omega)$. Two examples of permutable events are

(i) $\{\omega : S_n(\omega) \in B \text{ i.o.}\}$
(ii) $\{\omega : \limsup_{n\to\infty} S_n(\omega)/c_n \ge 1\}$

In each case, the event is permutable because $S_n(\omega) = S_n(\pi\omega)$ for large $n$. The list of examples can be enlarged considerably by observing:

(iii) All events in the tail $\sigma$-field $\mathcal{T}$ are permutable.

To see this, observe that if $A \in \sigma(X_{n+1}, X_{n+2}, \ldots)$, then the occurrence of $A$ is unaffected by a permutation of $X_1, \ldots, X_n$. (i) shows that the converse of (iii) is false. The next result shows that for an i.i.d. sequence, there is no difference between $\mathcal{E}$ and $\mathcal{T}$. They are both trivial.

**Theorem 4.1.1. Hewitt-Savage 0-1 law.** *If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{E}$ then $P(A) \in \{0, 1\}$.*

*Proof.* Let $A \in \mathcal{E}$. As in the proof of Kolmogorov's 0-1 law, we will show that $A$ is independent of itself, that is, $P(A) = P(A \cap A) = P(A)P(A)$ so $P(A) \in \{0, 1\}$. Let $A_n \in \sigma(X_1, \ldots, X_n)$ so that

(a) $$P(A_n \Delta A) \to 0$$

Here $A\Delta B = (A - B) \cup (B - A)$ is the symmetric difference. The existence of the $A_n$'s is proved in part ii of Lemma A.2.1. $A_n$ can be written as $\{\omega : (\omega_1, \ldots, \omega_n) \in B_n\}$ with $B_n \in \mathcal{S}^n$. Let

$$\pi(j) = \begin{cases} j + n & \text{if } 1 \le j \le n \\ j - n & \text{if } n + 1 \le j \le 2n \\ j & \text{if } j \ge 2n + 1 \end{cases}$$

Observing that $\pi^2$ is the identity (so we don't have to worry about whether to write $\pi$ or $\pi^{-1}$) and the coordinates are i.i.d. (so the permuted coordinates are) gives

(b) $$P(\omega : \omega \in A_n \Delta A) = P(\omega : \pi\omega \in A_n \Delta A)$$

Now $\{\omega : \pi\omega \in A\} = \{\omega : \omega \in A\}$, since $A$ is permutable, and

$$\{\omega : \pi\omega \in A_n\} = \{\omega : (\omega_{n+1}, \ldots, \omega_{2n}) \in B_n\}$$

If we use $A_n'$ to denote the last event then we have

(c) $$\{\omega : \pi\omega \in A_n \Delta A\} = \{\omega : \omega \in A_n' \Delta A\}$$

Combining (b) and (c) gives

(d) $$P(A_n \Delta A) = P(A_n' \Delta A)$$

It is easy to see that

$$|P(B) - P(C)| \leq |P(B\Delta C)|$$

so (d) implies $P(A_n), P(A'_n) \to P(A)$. Now $A - C \subset (A - B) \cup (B - C)$ and, with a similar inequality for $C - A$, implies $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$. The last inequality, (d), and (a) imply

$$P(A_n \Delta A'_n) \leq P(A_n \Delta A) + P(A\Delta A'_n) \to 0$$

The last result implies

$$\begin{aligned} 0 &\leq P(A_n) - P(A_n \cap A'_n) \\ &\leq P(A_n \cup A'_n) - P(A_n \cap A'_n) = P(A_n \Delta A'_n) \to 0 \end{aligned}$$

so $P(A_n \cap A'_n) \to P(A)$. But $A_n$ and $A'_n$ are independent, so

$$P(A_n \cap A'_n) = P(A_n)P(A'_n) \to P(A)^2$$

This shows $P(A) = P(A)^2$ and proves Theorem 4.1.1. ■

A typical application of Theorem 4.1.1 is

**Theorem 4.1.2.** *For a random walk on* **R**, *there are only four possibilities, one of which has probability 1.*

*(i)* $S_n = 0$ *for all* $n$.
*(ii)* $S_n \to \infty$.
*(iii)* $S_n \to -\infty$.
*(iv)* $-\infty = \liminf S_n < \limsup S_n = \infty$.

*Proof.* Theorem 4.1.1 implies $\limsup S_n$ is a constant $c \in [-\infty, \infty]$. Let $S'_n = S_{n+1} - X_1$. Since $S'_n$ has the same distribution as $S_n$, it follows that $c = c - X_1$. If $c$ is finite, subtracting $c$ from both sides we conclude $X_1 \equiv 0$ and (i) occurs. Turning the last statement around, we see that if $X_1 \not\equiv 0$, then $c = -\infty$ or $\infty$. The same analysis applies to the liminf. Discarding the impossible combination $\limsup S_n = -\infty$ and $\liminf S_n = +\infty$, we have proved the result. ■

**Exercise 4.1.1. Symmetric random walk.** Let $X_1, X_2, \ldots \in \mathbf{R}$ be i.i.d. with a distribution that is symmetric about 0 and nondegenerate (i.e., $P(X_i = 0) < 1$). Show that we are in case (iv) of Theorem 4.1.2.

**Exercise 4.1.2.** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Use the central limit theorem to conclude that we are in case (iv) of Theorem 4.1.2. Later in Exercise 4.1.11 you will show that $EX_i = 0$ and $P(X_i = 0) < 1$ is sufficient.

The special case in which $P(X_i = 1) = P(X_i = -1) = 1/2$ is called **simple random walk**. Since a simple random walk cannot skip over any integers, it follows from either exercise above that with probability 1 it visits every integer infinitely many times.

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n) =$ the information known at time $n$. A random variable $N$ taking values in $\{1, 2, \dots\} \cup \{\infty\}$ is said to be a **stopping time** or an **optional random variable** if for every $n < \infty$, $\{N = n\} \in \mathcal{F}_n$. If we think of $S_n$ as giving the (logarithm of the) price of a stock at time $n$, and $N$ as the time we sell it, then the last definition says that the decision to sell at time $n$ must be based on the information known at that time. The last interpretation gives one explanation for the second name. $N$ is a time at which we can exercise an option to buy a stock. Chung prefers the second name because $N$ is "usually rather a momentary pause after which the process proceeds again: time marches on!"

The canonical example of a stopping time is $N = \inf\{n : S_n \in A\}$, the **hitting time of A**. To check that this is a stopping time, we observe that

$$\{N = n\} = \{S_1 \in A^c, \dots, S_{n-1} \in A^c, S_n \in A\} \in \mathcal{F}_n$$

Two concrete examples of hitting times that have appeared above are

**Example 4.1.1.** $N = \inf\{k : |S_k| \geq x\}$ from the proof of Theorem 2.5.2.

**Example 4.1.2.** If the $X_i \geq 0$ and $N_t = \sup\{n : S_n \leq t\}$ is the random variable that first appeared in Example 2.4.1, then $N_t + 1 = \inf\{n : S_n > t\}$ is a stopping time.

The next result allows us to construct new examples from the old ones.

**Exercise 4.1.3.** If $S$ and $T$ are stopping times, then $S \wedge T$ and $S \vee T$ are stopping times. Since constant times are stopping times, it follows that $S \wedge n$ and $S \vee n$ are stopping times.

**Exercise 4.1.4.** Suppose $S$ and $T$ are stopping times. Is $S + T$ a stopping time? Give a proof or a counterexample.

Associated with each stopping time $N$ is a $\sigma$-field $\mathcal{F}_N =$ the information known at time $N$. Formally, $\mathcal{F}_N$ is the collection of sets $A$ that have $A \cap \{N = n\} \in \mathcal{F}_n$ for all $n < \infty$, that is, when $N = n$, $A$ must be measurable with respect to the information known at time $n$. Trivial but important examples of sets in $\mathcal{F}_N$ are $\{N \leq n\}$, that is, $N$ is measurable with respect to $\mathcal{F}_N$.

**Exercise 4.1.5.** Show that if $Y_n \in \mathcal{F}_n$ and $N$ is a stopping time, $Y_N \in \mathcal{F}_N$. As a corollary of this result, we see that if $f : S \to \mathbf{R}$ is measurable, $T_n = \sum_{m \leq n} f(X_m)$, and $M_n = \max_{m \leq n} T_m$, then $T_N$ and $M_N \in \mathcal{F}_N$. An important special case is $S = \mathbf{R}$, $f(x) = x$.

**Exercise 4.1.6.** Show that if $M \le N$ are stopping times, then $\mathcal{F}_M \subset \mathcal{F}_N$.

**Exercise 4.1.7.** Show that if $L \le M$ and $A \in \mathcal{F}_L$, then

$$N = \begin{cases} L & \text{on } A \\ M & \text{on } A^c \end{cases} \qquad \text{is a stopping time}$$

Our first result about $\mathcal{F}_N$ is

**Theorem 4.1.3.** *Let $X_1, X_2, \ldots$ be i.i.d., $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $N$ be a stopping time with $P(N < \infty) > 0$. Conditional on $\{N < \infty\}$, $\{X_{N+n}, n \ge 1\}$ is independent of $\mathcal{F}_N$ and has the same distribution as the original sequence.*

*Proof.* By Theorem A.1.5, it is enough to show that if $A \in \mathcal{F}_N$ and $B_j \in \mathcal{S}$ for $1 \le j \le k$, then

$$P(A, N < \infty, X_{N+j} \in B_j, 1 \le j \le k) = P(A \cap \{N < \infty\}) \prod_{j=1}^{k} \mu(B_j)$$

where $\mu(B) = P(X_i \in B)$. The method ("divide and conquer") is one that we will see many times below. We break things down according to the value of $N$ in order to replace $N$ by $n$ and reduce to the case of a fixed time.

$$\begin{aligned} P(A, N = n, X_{N+j} \in B_j, 1 \le j \le k) &= P(A, N = n, X_{n+j} \in B_j, 1 \le j \le k) \\ &= P(A \cap \{N = n\}) \prod_{j=1}^{k} \mu(B_j) \end{aligned}$$

since $A \cap \{N = n\} \in \mathcal{F}_n$ and that $\sigma$-field is independent of $X_{n+1}, \ldots, X_{n+k}$. Summing over $n$ now gives the desired result. ∎

To delve further into properties of stopping times, we recall that we have supposed $\Omega = S^{\mathbf{N}}$ and define the **shift** $\theta : \Omega \to \Omega$ by

$$(\theta\omega)(n) = \omega(n+1) \qquad n = 1, 2, \ldots$$

In words, we drop the first coordinate and shift the others one place to the left. The iterates of $\theta$ are defined by composition. Let $\theta^1 = \theta$, and for $k \ge 2$, let $\theta^k = \theta \circ \theta^{k-1}$. Clearly, $(\theta^k\omega)(n) = \omega(n+k)$, $n = 1, 2, \ldots$ To extend the last definition to stopping times, we let

$$\theta^N \omega = \begin{cases} \theta^n \omega & \text{on } \{N = n\} \\ \Delta & \text{on } \{N = \infty\} \end{cases}$$

Here $\Delta$ is an extra point that we add to $\Omega$. According to the only joke in Blumenthal and Getoor (1968), $\Delta$ is a "cemetery or heaven depending upon your point of view." Seriously, $\Delta$ is a convenience in making definitions like the next one.

**Example 4.1.3. Returns to 0.** For a concrete example of the use of $\theta$, suppose $S = \mathbf{R}^d$ and let

$$\tau(\omega) = \inf\{n : \omega_1 + \cdots + \omega_n = 0\}$$

where $\inf \emptyset = \infty$, and we set $\tau(\Delta) = \infty$. If we let $\tau_2(\omega) = \tau(\omega) + \tau(\theta^\tau \omega)$, then on $\{\tau < \infty\}$,

$$\begin{aligned} \tau(\theta^\tau \omega) &= \inf\{n : (\theta^\tau \omega)_1 + \cdots + (\theta^\tau \omega)_n = 0\} \\ &= \inf\{n : \omega_{\tau+1} + \cdots + \omega_{\tau+n} = 0\} \end{aligned}$$

$$\tau(\omega) + \tau(\theta^\tau \omega) = \inf\{m > \tau : \omega_1 + \cdots + \omega_m = 0\}$$

So $\tau_2$ is the time of the second visit to 0 (and thanks to the conventions $\theta^\infty \omega = \Delta$ and $\tau(\Delta) = \infty$, this is true for all $\omega$). The last computation generalizes easily to show that if we let

$$\tau_n(\omega) = \tau_{n-1}(\omega) + \tau(\theta^{\tau_{n-1}} \omega)$$

then $\tau_n$ is the time of the $n$th visit to 0.

If we have any stopping time $T$, we can define its iterates by $T_0 = 0$ and

$$T_n(\omega) = T_{n-1}(\omega) + T(\theta^{T_{n-1}} \omega) \quad \text{for } n \geq 1$$

If we assume $P = \mu \times \mu \times \ldots$ then

$$P(T_n < \infty) = P(T < \infty)^n \tag{4.1.1}$$

*Proof.* We will prove this by induction. The result is trivial when $n = 1$. Suppose now that it is valid for $n - 1$. Applying Theorem 4.1.3 to $N = T_{n-1}$, we see that $T(\theta^{T_{n-1}}) < \infty$ is independent of $T_{n-1} < \infty$ and has the same probability as $T < \infty$, so

$$\begin{aligned} P(T_n < \infty) &= P(T_{n-1} < \infty, T(\theta^{T_{n-1}} \omega) < \infty) \\ &= P(T_{n-1} < \infty) P(T < \infty) = P(T < \infty)^n \end{aligned}$$

by the induction hypothesis. ■

Letting $t_n = T(\theta^{T_{n-1}})$, we can extend Theorem 4.1.3 to

**Theorem 4.1.4.** *Suppose $P(T < \infty) = 1$. Then the "random vectors"*

$$V_n = (t_n, X_{T_{n-1}+1}, \ldots, X_{T_n})$$

*are independent and identically distributed.*

*Proof.* It is clear from Theorem 4.1.3 that $V_n$ and $V_1$ have the same distribution. The independence follows from Theorem 4.1.3 and induction since $V_1, \ldots, V_{n-1} \in \mathcal{F}(T_{n-1})$. ■

**Example 4.1.4. Ladder variables.** Let $\alpha(\omega) = \inf\{n : \omega_1 + \cdots + \omega_n > 0\}$ where $\inf \emptyset = \infty$, and set $\alpha(\Delta) = \infty$. Let $\alpha_0 = 0$ and let

$$\alpha_k(\omega) = \alpha_{k-1}(\omega) + \alpha(\theta^{\alpha_{k-1}}\omega)$$

for $k \geq 1$. At time $\alpha_k$, the random walk is at a record high value.

The next three exercises investigate these times.

**Exercise 4.1.8.** (i) If $P(\alpha < \infty) < 1$ then $P(\sup S_n < \infty) = 1$.
(ii) If $P(\alpha < \infty) = 1$, then $P(\sup S_n = \infty) = 1$.

**Exercise 4.1.9.** Let $\beta = \inf\{n : S_n < 0\}$. Prove that the four possibilities in Theorem 4.1.2 correspond to the four combinations of $P(\alpha < \infty) < 1$ or $= 1$, and $P(\beta < \infty) < 1$ or $= 1$.

**Exercise 4.1.10.** Let $S_0 = 0$, $\bar{\beta} = \inf\{n \geq 1 : S_n \leq 0\}$ and

$$A_m^n = \{0 \geq S_m, S_1 \geq S_m, \ldots, S_{m-1} \geq S_m, \ S_m < S_{m+1}, \ldots, S_m < S_n\}$$

(i) Show $1 = \sum_{m=0}^n P(A_m^n) = \sum_{m=0}^n P(\alpha > m) P(\bar{\beta} > n - m)$.
(ii) Let $n \to \infty$ and conclude $E\alpha = 1/P(\bar{\beta} = \infty)$.

**Exercise 4.1.11.** (i) Combine the last exercise with the proof of (ii) in Exercise 4.1.8 to conclude that if $EX_i = 0$, then $P(\bar{\beta} = \infty) = 0$. (ii) Show that if we assume in addition that $P(X_i = 0) < 1$, then $P(\beta = \infty) = 0$, and Exercise 4.1.9 implies we are in case (iv) of Theorem 4.1.2.

A famous result about stopping times for random walks is:

**Theorem 4.1.5. Wald's equation.** *Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_i| < \infty$. If $N$ is a stopping time with $EN < \infty$, then $ES_N = EX_1 EN$.*

*Proof.* First suppose the $X_i \geq 0$.

$$ES_N = \int S_N \, dP = \sum_{n=1}^{\infty} \int S_n 1_{\{N=n\}} \, dP = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int X_m 1_{\{N=n\}} \, dP$$

Since the $X_i \geq 0$, we can interchange the order of summation (i.e., use Fubini's theorem) to conclude that the last expression

$$= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int X_m 1_{\{N=n\}} \, dP = \sum_{m=1}^{\infty} \int X_m 1_{\{N \geq m\}} \, dP$$

Now $\{N \geq m\} = \{N \leq m-1\}^c \in \mathcal{F}_{m-1}$ and is independent of $X_m$, so the last expression

$$= \sum_{m=1}^{\infty} EX_m P(N \geq m) = EX_1 EN$$

To prove the result in general, we run the last argument backwards. If we have $EN < \infty$ then

$$\infty > \sum_{m=1}^{\infty} E|X_m| P(N \geq m) = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int |X_m| 1_{\{N=n\}} \, dP$$

The last formula shows that the double sum converges absolutely in one order, so Fubini's theorem gives

$$\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \int X_m 1_{\{N=n\}} dP = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int X_m 1_{\{N=n\}} dP$$

Using the independence of $\{N \geq m\} \in \mathcal{F}_{m-1}$ and $X_m$, and rewriting the last identity, it follows that

$$\sum_{m=1}^{\infty} EX_m P(N \geq m) = ES_N$$

Since the left-hand side is $EN\, EX_1$, the proof is complete. ■

**Exercise 4.1.12.** Let $X_1, X_2, \ldots$ be i.i.d. uniform on (0,1), let $S_n = X_1 + \cdots + X_n$, and let $T = \inf\{n : S_n > 1\}$. Show that $P(T > n) = 1/n!$, so $ET = e$ and $ES_T = e/2$.

**Example 4.1.5. Simple random walk.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_i = 1) = 1/2$ and $P(X_i = -1) = 1/2$. Let $a < 0 < b$ be integers and let $N = \inf\{n : S_n \notin (a, b)\}$. To apply Theorem 4.1.5, we have to check that $EN < \infty$. To do this, we observe that if $x \in (a, b)$, then

$$P(x + S_{b-a} \notin (a, b)) \geq 2^{-(b-a)}$$

since $b - a$ steps of size $+1$ in a row will take us out of the interval. Iterating the last inequality, it follows that

$$P(N > n(b-a)) \leq \left(1 - 2^{-(b-a)}\right)^n$$

so $EN < \infty$. Applying Theorem 4.1.5 now gives $ES_N = 0$ or

$$bP(S_N = b) + aP(S_N = a) = 0$$

Since $P(S_N = b) + P(S_N = a) = 1$, it follows that $(b-a)P(S_N = b) = -a$, so

$$P(S_N = b) = \frac{-a}{b-a} \qquad P(S_N = a) = \frac{b}{b-a}$$

Letting $T_a = \inf\{n : S_n = a\}$, we can write the last conclusion as

$$P(T_a < T_b) = \frac{b}{b-a} \quad \text{for } a < 0 < b \tag{4.1.2}$$

Setting $b = M$ and letting $M \to \infty$ gives

$$P(T_a < \infty) \geq P(T_a < T_M) \to 1$$

for all $a < 0$. From symmetry (and the fact that $T_0 \equiv 0$), it follows that

$$P(T_x < \infty) = 1 \quad \text{for all } x \in \mathbf{Z} \tag{4.1.3}$$

Our final fact about $T_x$ is that $ET_x = \infty$ for $x \neq 0$. To prove this, note that if $ET_x < \infty$ then Theorem 4.1.5 would imply

$$x = ES_{T_x} = EX_1 ET_x = 0$$

In Section 4.3, we will compute the distribution of $T_1$ and show that

$$P(T_1 > t) \sim C\, t^{-1/2}$$

**Exercise 4.1.13. Asymmetric simple random walk.** Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = 1) = p > 1/2$ and $P(X_1 = -1) = 1 - p$, and let $S_n = X_1 + \cdots + X_n$. Let $\alpha = \inf\{m : S_m > 0\}$ and $\beta = \inf\{n : S_n < 0\}$.

(i) Use Exercise 4.1.9 to conclude that $P(\alpha < \infty) = 1$ and $P(\beta < \infty) < 1$.

(ii) If $Y = \inf S_n$, then $P(Y \leq -k) = P(\beta < \infty)^k$.

(iii) Apply Wald's equation to $\alpha \wedge n$ and let $n \to \infty$ to get $E\alpha = 1/EX_1 = 1/(2p-1)$. Comparing with Exercise 4.1.10 shows $P(\bar{\beta} = \infty) = 2p - 1$.

**Exercise 4.1.14. An optimal stopping problem.** Let $X_n$, $n \geq 1$ be i.i.d. with $EX_1^+ < \infty$ and let

$$Y_n = \max_{1 \leq m \leq n} X_m - cn$$

That is, we are looking for a large value of $X$, but we have to pay $c > 0$ for each observation. (i) Let $T = \inf\{n : X_n > a\}$, $p = P(X_n > a)$, and compute $EY_T$. (ii) Let $\alpha$ (possibly $< 0$) be the unique solution of $E(X_1 - \alpha)^+ = c$. Show that $EY_T = \alpha$ in this case and use the inequality

$$Y_n \leq \alpha + \sum_{m=1}^{n} ((X_m - \alpha)^+ - c)$$

for $n \geq 1$ to conclude that if $\tau \geq 1$ is a stopping time with $E\tau < \infty$, then $EY_\tau \leq \alpha$. The analysis above assumes that you have to play at least once. If the optimal $\alpha < 0$, then you shouldn't play at all.

**Theorem 4.1.6. Wald's second equation.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_n = 0$ and $EX_n^2 = \sigma^2 < \infty$. If $T$ is a stopping time with $ET < \infty$, then $ES_T^2 = \sigma^2 ET$.*

*Proof.* Using the definitions and then taking expected value

$$S_{T\wedge n}^2 = S_{T\wedge(n-1)}^2 + (2X_nS_{n-1} + X_n^2)1_{(T\geq n)}$$
$$ES_{T\wedge n}^2 = ES_{T\wedge(n-1)}^2 + \sigma^2 P(T \geq n)$$

since $EX_n = 0$ and $X_n$ is independent of $S_{n-1}$ and $1_{(T\geq n)} \in \mathcal{F}_{n-1}$. [The expectation of $S_{n-1}X_n$ exists since both random variables are in $L^2$.] From the last equality and induction we get

$$ES_{T\wedge n}^2 = \sigma^2 \sum_{m=1}^{n} P(T \geq m)$$

$$E(S_{T\wedge n} - S_{T\wedge m})^2 = \sigma^2 \sum_{k=m+1}^{n} P(T \geq n)$$

The second equality follows from the first applied to $X_{m+1}, X_{m+2}, \ldots$. The second equality implies that $S_{T\wedge n}$ is a Cauchy sequence in $L^2$, so letting $n \to \infty$ in the first, it follows that $ES_T^2 = \sigma^2 ET$. ■

**Example 4.1.6. Simple random walk, II.** Continuing Example 4.1.5 we investigate $N = \inf\{S_n \notin (a, b)\}$. We have shown that $EN < \infty$. Since $\sigma^2 = 1$, it follows from Theorem 4.1.6 and (4.1.2) that

$$EN = ES_N^2 = a^2\frac{b}{b-a} + b^2\frac{-a}{b-a} = -ab$$

If $b = L$ and $a = -L$, $EN = L^2$.

An amusing consequence of Theorem 4.1.6 is

**Theorem 4.1.7.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_n = 0$ and $EX_n^2 = 1$, and let $T_c = \inf\{n \geq 1 : |S_n| > cn^{1/2}\}$.*

$$ET_c \quad \begin{cases} < \infty & \text{for } c < 1 \\ = \infty & \text{for } c \geq 1 \end{cases}$$

*Proof.* One half of this is easy. If $ET_c < \infty$ then, the previous exercise implies $ET_c = E(S_{T_c}^2) > c^2 ET_c$, a contradiction if $c \geq 1$. To prove the other direction, we let $\tau = T_c \wedge n$ and observe $S_{\tau-1}^2 \leq c^2(\tau - 1)$, so using the Cauchy-Schwarz inequality

$$E\tau = ES_\tau^2 = ES_{\tau-1}^2 + 2E(S_{\tau-1}X_\tau) + EX_\tau^2 \quad \leq c^2 E\tau + 2c(E\tau\, EX_\tau^2)^{1/2} + EX_\tau^2$$

To complete the proof now, we will show

**Lemma 4.1.8.** *If $T$ is a stopping time with $ET = \infty$, then*

$$EX_{T\wedge n}^2 / E(T \wedge n) \to 0$$

Theorem 4.1.7 follows, for if $\epsilon < 1 - c^2$ and $n$ is large, we will have $E\tau \le (c^2 + \epsilon) E\tau$, a contradiction.

*Proof.* We begin by writing

$$E(X_{T\wedge n}^2) = E(X_{T\wedge n}^2; X_{T\wedge n}^2 \le \epsilon(T \wedge n)) + \sum_{j=1}^{n} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j)$$

The first term is $\le \epsilon E(T \wedge n)$. To bound the second, choose $N \ge 1$ so that for $n \ge N$

$$\sum_{j=1}^{n} E(X_j^2; X_j^2 > \epsilon j) < n\epsilon$$

This is possible since the dominated convergence theorem implies $E(X_j^2; X_j^2 > \epsilon j) \to 0$ as $j \to \infty$. For the first part of the sum, we use a trivial bound

$$\sum_{j=1}^{N} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j) \le N E X_1^2$$

To bound the remainder of the sum, we note (i) $X_j^2 \ge 0$; (ii) $\{T \wedge n \ge j\}$ is $\in \mathcal{F}_{j-1}$ and hence is independent of $X_j^2 1_{(X_j^2 > \epsilon j)}$, (iii) use some trivial arithmetic, (iv) use Fubini's theorem and enlarge the range of $j$, (v) use the choice of $N$ and a trivial inequality

$$\sum_{j=N}^{n} E(X_j^2; T \wedge n = j, X_j^2 > \epsilon j) \le \sum_{j=N}^{n} E(X_j^2; T \wedge n \ge j, X_j^2 > \epsilon j)$$

$$= \sum_{j=N}^{n} P(T \wedge n \ge j) E(X_j^2; X_j^2 > \epsilon j) = \sum_{j=N}^{n} \sum_{k=j}^{\infty} P(T \wedge n = k) E(X_j^2; X_j^2 > \epsilon j)$$

$$\le \sum_{k=N}^{\infty} \sum_{j=1}^{k} P(T \wedge n = k) E(X_j^2; X_j^2 > \epsilon j) \le \sum_{k=N}^{\infty} \epsilon k P(T \wedge n = k) \le \epsilon E(T \wedge n)$$

Combining our estimates shows

$$E X_{T\wedge n}^2 \le 2\epsilon E(T \wedge n) + N E X_1^2$$

Letting $n \to \infty$ and noting $E(T \wedge n) \to \infty$, we have

$$\limsup_{n\to\infty} E X_{T\wedge n}^2 / E(T \wedge n) \le 2\epsilon$$

where $\epsilon$ is arbitrary. ∎

## 4.2 Recurrence

Throughout this section, $S_n$ will be a random walk, that is, $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are i.i.d., and we will investigate the question mentioned at the beginning of the chapter. Does the sequence $S_1(\omega), S_2(\omega), \ldots$ return to (or near) 0 infinitely often? The answer to the last question is either Yes or No, and the random walk is called recurrent or transient accordingly. We begin with some definitions that formulate the question precisely and a result that establishes a dichotomy between the two cases.

The number $x \in \mathbf{R}^d$ is said to be a **recurrent value** for the random walk $S_n$ if for every $\epsilon > 0$, $P(\|S_n - x\| < \epsilon \text{ i.o.}) = 1$. Here $\|x\| = \sup |x_i|$. The reader will see the reason for this choice of norm in the proof of Lemma 4.2.5. The Hewitt-Savage 0-1 law, Theorem 4.1.1, implies that if the last probability is $< 1$, it is 0. Our first result shows that to know the set of recurrent values, it is enough to check $x = 0$. A number $x$ is said to be a **possible value** of the random walk if for any $\epsilon > 0$, there is an $n$ so that $P(\|S_n - x\| < \epsilon) > 0$.

**Theorem 4.2.1.** *The set $\mathcal{V}$ of recurrent values is either $\emptyset$ or a closed subgroup of $\mathbf{R}^d$. In the second case, $\mathcal{V} = \mathcal{U}$, the set of possible values.*

*Proof.* Suppose $\mathcal{V} \neq \emptyset$. It is clear that $\mathcal{V}^c$ is open, so $\mathcal{V}$ is closed. To prove that $\mathcal{V}$ is a group, we will first show that

$(*)$ if $x \in \mathcal{U}$ and $y \in \mathcal{V}$ then $y - x \in \mathcal{V}$.

This statement has been formulated so that once it is established, the result follows easily. Let

$$p_{\delta,m}(z) = P(\|S_n - z\| \geq \delta \text{ for all } n \geq m)$$

If $y - x \notin \mathcal{V}$, there is an $\epsilon > 0$ and $m \geq 1$ so that $p_{2\epsilon,m}(y - x) > 0$. Since $x \in \mathcal{U}$, there is a $k$ so that $P(\|S_k - x\| < \epsilon) > 0$. Since

$$P(\|S_n - S_k - (y - x)\| \geq 2\epsilon \text{ for all } n \geq k + m) = p_{2\epsilon,m}(y - x)$$

and is independent of $\{\|S_k - x\| < \epsilon\}$, it follows that

$$p_{\epsilon,m+k}(y) \geq P(\|S_k - x\| < \epsilon) p_{2\epsilon,m}(y - x) > 0$$

contradicting $y \in \mathcal{V}$, so $y - x \in \mathcal{V}$.

To conclude that $\mathcal{V}$ is a group when $\mathcal{V} \neq \emptyset$, let $q, r \in \mathcal{V}$, and observe: (i) taking $x = y = r$ in $(*)$ shows $0 \in \mathcal{V}$, (ii) taking $x = r$, $y = 0$ shows $-r \in \mathcal{V}$, and (iii) taking $x = -r$, $y = q$ shows $q + r \in \mathcal{V}$. To prove that $\mathcal{V} = \mathcal{U}$ now, observe that if $u \in \mathcal{U}$ taking $x = u$, $y = 0$ shows $-u \in \mathcal{V}$, and since $\mathcal{V}$ is a group, it follows that $u \in \mathcal{V}$. ∎

If $\mathcal{V} = \emptyset$, the random walk is said to be **transient**; otherwise it is called **recurrent**. Before plunging into the technicalities needed to treat a general random walk, we begin by analyzing the special case Polya considered in 1921. Legend has it that Polya thought of this problem while wandering around in a park near Zürich when he noticed that he kept encountering the same young couple. History does not record what the young couple thought.

**Example 4.2.1. Simple random walk on $\mathbf{Z}^d$.**

$$P(X_i = e_j) = P(X_i = -e_j) = 1/2d$$

for each of the $d$ unit vectors $e_j$. To analyze this case, we begin with a result that is valid for any random walk. Let $\tau_0 = 0$ and $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ be the time of the $n$th return to 0. From (4.1.1), it follows that

$$P(\tau_n < \infty) = P(\tau_1 < \infty)^n$$

a fact that leads easily to:

**Theorem 4.2.2.** *For any random walk, the following are equivalent:*
*(i) $P(\tau_1 < \infty) = 1$, (ii) $P(S_m = 0$ i.o.$) = 1$, and (iii) $\sum_{m=0}^{\infty} P(S_m = 0) = \infty$.*

*Proof.* If $P(\tau_1 < \infty) = 1$, then $P(\tau_n < \infty) = 1$ for all $n$ and $P(S_m = 0 \text{ i.o.}) = 1$. Let

$$V = \sum_{m=0}^{\infty} 1_{(S_m=0)} = \sum_{n=0}^{\infty} 1_{(\tau_n<\infty)}$$

be the number of visits to 0, counting the visit at time 0. Taking expected value and using Fubini's theorem to put the expected value inside the sum:

$$\begin{aligned} EV &= \sum_{m=0}^{\infty} P(S_m = 0) = \sum_{n=0}^{\infty} P(\tau_n < \infty) \\ &= \sum_{n=0}^{\infty} P(\tau_1 < \infty)^n = \frac{1}{1 - P(\tau_1 < \infty)} \end{aligned}$$

The second equality shows that (ii) implies (iii) and, in combination with the last two, shows that if (i) is false, then (iii) is false (i.e., (iii) implies (i)). ∎

**Theorem 4.2.3.** *Simple random walk is recurrent in $d \le 2$ and transient in $d \ge 3$.*

To steal a joke from Kakutani (UCLA colloquium talk): "A drunk man will eventually find his way home, but a drunk bird may get lost forever."

*Proof.* Let $\rho_d(m) = P(S_m = 0)$. $\rho_d(m)$ is 0 if $m$ is odd. From Theorem 3.1.3, we get $\rho_1(2n) \sim (\pi n)^{-1/2}$ as $n \to \infty$. This and Theorem 4.2.2 gives the result in one dimension. Our next step is

*Simple random walk is recurrent in two dimensions.* Note that in order for $S_{2n} = 0$, we must for some $0 \le m \le n$ have $m$ up steps, $m$ down steps, $n - m$ to the left, and $n - m$ to the right, so

$$\rho_2(2n) = 4^{-2n} \sum_{m=0}^{n} \frac{2n!}{m!\,m!\,(n-m)!\,(n-m)!}$$
$$= 4^{-2n}\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = 4^{-2n}\binom{2n}{n}^2 = \rho_1(2n)^2$$

To see the next-to-last equality, consider choosing $n$ students from a class with $n$ boys and $n$ girls and observe that for some $0 \le m \le n$, you must choose $m$ boys and $n - m$ girls. Using the asymptotic formula $\rho_1(2n) \sim (\pi n)^{-1/2}$, we get $\rho_2(2n) \sim (\pi n)^{-1}$. Since $\sum n^{-1} = \infty$, the result follows from Theorem 4.2.2.

**Remark.** For a direct proof of $\rho_2(2n) = \rho_1(2n)^2$, note that if $T_n^1$ and $T_n^2$ are independent, one-dimensional random walks, then $T_n$ jumps from $x$ to $x + (1, 1)$, $x + (1, -1)$, $x + (-1, 1)$, and $x + (-1, -1)$ with equal probability, so rotating $T_n$ by 45 degrees and dividing by $\sqrt{2}$ gives $S_n$.

*Simple random walk is transient in three dimensions.* Intuitively, this holds since the probability of being back at 0 after $2n$ steps is $\sim cn^{-3/2}$, and this is summable. We will not compute the probability exactly but will get an upper bound of the right order of magnitude. Again, since the number of steps in the directions $\pm e_i$ must be equal for $i = 1, 2, 3$,

$$\rho_3(2n) = 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-j-k)!)^2}$$
$$= 2^{-2n}\binom{2n}{n}\sum_{j,k}\left(3^{-n}\frac{n!}{j!k!(n-j-k)!}\right)^2$$
$$\le 2^{-2n}\binom{2n}{n}\max_{j,k} 3^{-n}\frac{n!}{j!k!(n-j-k)!}$$

where in the last inequality we have used the fact that if $a_{j,k}$ are $\ge 0$ and sum to 1, then $\sum_{j,k} a_{j,k}^2 \le \max_{j,k} a_{j,k}$. Our last step is to show

$$\max_{j,k} 3^{-n}\frac{n!}{j!k!(n-j-k)!} \le Cn^{-1}$$

To do this, we note that (a) if any of the numbers $j$, $k$ or $n - j - k$ is $< [n/3]$, increasing the smallest number and decreasing the largest number decreases the denominator (since $x(1 - x)$ is maximized at 1/2), so the maximum occurs when all three numbers are as close as possible to $n/3$; (b) Stirling's formula implies

$$\frac{n!}{j!k!(n-j-k)!} \sim \frac{n^n}{j^j k^k (n-j-k)^{n-j-k}} \cdot \sqrt{\frac{n}{jk(n-j-k)}} \cdot \frac{1}{2\pi}$$

Taking $j$ and $k$ within 1 of $n/3$ the first term on the right is $\le C3^n$, and the desired result follows.

*Simple random walk is transient in $d > 3$.* Let $T_n = (S_n^1, S_n^2, S_n^3)$, $N(0) = 0$ and $N(n) = \inf\{m > N(n-1) : T_m \neq T_{N(n-1)}\}$. It is easy to see that $T_{N(n)}$ is a three-dimensional simple random walk. Since $T_{N(n)}$ returns infinitely often to 0 with probability 0 and the first three coordinates are constant in between the $N(n)$, $S_n$ is transient. ■

**Remark.** Let $\pi_d = P(S_n = 0 \text{ for some } n \geq 1)$ be the probability that simple random walk on $\mathbf{Z}^d$ returns to 0. The last display in the proof of Theorem 4.2.2 implies

$$\sum_{n=0}^{\infty} P(S_{2n} = 0) = \frac{1}{1 - \pi_d} \tag{4.2.1}$$

In $d = 3$, $P(S_{2n} = 0) \sim Cn^{-3/2}$ so $\sum_{n=N}^{\infty} P(S_{2n} = 0) \sim C'N^{-1/2}$, and the series converges rather slowly. For example, if we want to compute the return probability to five decimal places, we would need $10^{10}$ terms. At the end of the section, we will give another formula that leads very easily to accurate results.

The rest of this section is devoted to proving the following facts about random walks:

- $S_n$ is recurrent in $d = 1$ if $S_n/n \to 0$ in probability.
- $S_n$ is recurrent in $d = 2$ if $S_n/n^{1/2} \Rightarrow$ a nondegenerate normal distribution.
- $S_n$ is transient in $d \geq 3$ if it is "truly three-dimensional."

To prove the last result, we will give a necessary and sufficient condition for recurrence.

The first step in deriving these results is to generalize Theorem 4.2.2.

**Lemma 4.2.4.** *If $\sum_{n=1}^{\infty} P(\|S_n\| < \epsilon) < \infty$, then $P(\|S_n\| < \epsilon \text{ i.o.}) = 0$. If $\sum_{n=1}^{\infty} P(\|S_n\| < \epsilon) = \infty$ then $P(\|S_n\| < 2\epsilon \text{ i.o.}) = 1$.*

*Proof.* The first conclusion follows from the Borel-Cantelli lemma. To prove the second, let $F = \{\|S_n\| < \epsilon \text{ i.o.}\}^c$. Breaking things down according to the last time $\|S_n\| < \epsilon$,

$$\begin{aligned}
P(F) &= \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon, \|S_n\| \geq \epsilon \text{ for all } n \geq m+1) \\
&\geq \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon, \|S_n - S_m\| \geq 2\epsilon \text{ for all } n \geq m+1) \\
&= \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon)\rho_{2\epsilon,1}
\end{aligned}$$

where $\rho_{\delta,k} = P(\|S_n\| \geq \delta \text{ for all } n \geq k)$. Since $P(F) \leq 1$, and

$$\sum_{m=0}^{\infty} P(\|S_m\| < \epsilon) = \infty$$

it follows that $\rho_{2\epsilon,1} = 0$. To extend this conclusion to $\rho_{2\epsilon,k}$ with $k \geq 2$, let

$$A_m = \{\|S_m\| < \epsilon, \|S_n\| \geq \epsilon \text{ for all } n \geq m+k\}$$

Since any $\omega$ can be in at most $k$ of the $A_m$, repeating the argument above gives

$$k \geq \sum_{m=0}^{\infty} P(A_m) \geq \sum_{m=0}^{\infty} P(\|S_m\| < \epsilon)\rho_{2\epsilon,k}$$

So $\rho_{2\epsilon,k} = P(\|S_n\| \geq 2\epsilon \text{ for all } j \geq k) = 0$, and since $k$ is arbitrary, the desired conclusion follows. ■

Our second step is to show that the convergence or divergence of the sums in Lemma 4.2.4 is independent of $\epsilon$. The previous proof works for any norm. For the next one, we need $\|x\| = \sup_i |x_i|$.

**Lemma 4.2.5.** *Let $m$ be an integer $\geq 2$.*

$$\sum_{n=0}^{\infty} P(\|S_n\| < m\epsilon) \leq (2m)^d \sum_{n=0}^{\infty} P(\|S_n\| < \epsilon)$$

*Proof.* We begin by observing

$$\sum_{n=0}^{\infty} P(\|S_n\| < m\epsilon) \leq \sum_{n=0}^{\infty} \sum_{k} P(S_n \in k\epsilon + [0,\epsilon)^d)$$

where the inner sum is over $k \in \{-m, \ldots, m-1\}^d$. If we let

$$T_k = \inf\{\ell \geq 0 : S_\ell \in k\epsilon + [0,\epsilon)^d\}$$

then breaking things down according to the value of $T_k$ and using Fubini's theorem gives

$$\begin{aligned}\sum_{n=0}^{\infty} P(S_n \in k\epsilon + [0,\epsilon)^d) &= \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} P(S_n \in k\epsilon + [0,\epsilon)^d, T_k = \ell) \\ &\leq \sum_{\ell=0}^{\infty}\sum_{n=\ell}^{\infty} P(\|S_n - S_\ell\| < \epsilon, T_k = \ell)\end{aligned}$$

Since $\{T_k = \ell\}$ and $\{\|S_n - S_\ell\| < \epsilon\}$ are independent, the last sum

$$= \sum_{m=0}^{\infty} P(T_k = m) \sum_{j=0}^{\infty} P(\|S_j\| < \epsilon) \leq \sum_{j=0}^{\infty} P(\|S_j\| < \epsilon)$$

Since there are $(2m)^d$ values of $k$ in $\{-m, \ldots, m-1\}^d$, the proof is complete. ■

Combining Lemmas 4.2.4 and 4.2.5 gives:

**Theorem 4.2.6.** *The convergence (resp. divergence) of* $\sum_n P(\|S_n\| < \epsilon)$ *for a single value of* $\epsilon > 0$ *is sufficient for transience (resp. recurrence).*

In $d = 1$, if $EX_i = \mu \neq 0$, then the strong law of large numbers implies $S_n/n \to \mu$, so $|S_n| \to \infty$ and $S_n$ is transient. As a converse, we have

**Theorem 4.2.7. Chung-Fuchs theorem.** *Suppose* $d = 1$. *If the weak law of large numbers holds in the form* $S_n/n \to 0$ *in probability, then* $S_n$ *is recurrent.*

*Proof.* Let $u_n(x) = P(|S_n| < x)$ for $x > 0$. Lemma 4.2.5 implies

$$\sum_{n=0}^{\infty} u_n(1) \geq \frac{1}{2m} \sum_{n=0}^{\infty} u_n(m) \geq \frac{1}{2m} \sum_{n=0}^{Am} u_n(n/A)$$

for any $A < \infty$ since $u_n(x) \geq 0$ and is increasing in $x$. By hypothesis $u_n(n/A) \to 1$, so letting $m \to \infty$ and noticing the right-hand side is $A/2$ times the average of the first $Am$ terms

$$\sum_{n=0}^{\infty} u_n(1) \geq A/2$$

Since $A$ is arbitrary, the sum must be $\infty$, and the desired conclusion follows from Theorem 4.2.6. ■

**Theorem 4.2.8.** *If* $S_n$ *is a random walk in* $\mathbf{R}^2$ *and* $S_n/n^{1/2} \Rightarrow$ *a nondegenerate normal distribution, then* $S_n$ *is recurrent.*

**Remark.** The conclusion is also true if the limit is degenerate, but in that case the random walk is essentially one- (or zero)-dimensional, and the result follows from the Chung-Fuchs theorem.

*Proof.* Let $u(n, m) = P(\|S_n\| < m)$. Lemma 4.2.5 implies

$$\sum_{n=0}^{\infty} u(n, 1) \geq (4m^2)^{-1} \sum_{n=0}^{\infty} u(n, m)$$

If $m/\sqrt{n} \to c$, then

$$u(n, m) \to \int_{[-c,c]^2} n(x)\, dx$$

where $n(x)$ is the density of the limiting normal distribution. If we use $\rho(c)$ to denote the right-hand side and let $n = [\theta m^2]$, it follows that $u([\theta m^2], m) \to \rho(\theta^{-1/2})$. If we write

$$m^{-2} \sum_{n=0}^{\infty} u(n, m) = \int_0^{\infty} u([\theta m^2], m)\, d\theta$$

let $m \to \infty$, and use Fatou's lemma, we get

$$\liminf_{m\to\infty} (4m^2)^{-1} \sum_{n=0}^{\infty} u(n, m) \geq 4^{-1} \int_0^{\infty} \rho(\theta^{-1/2})\, d\theta$$

Since the normal density is positive and continuous at 0,

$$\rho(c) = \int_{[-c,c]^2} n(x)\, dx \sim n(0)(2c)^2$$

as $c \to 0$. So $\rho(\theta^{-1/2}) \sim 4n(0)/\theta$ as $\theta \to \infty$, the integral diverges, and backtracking to the first inequality in the proof, it follows that $\sum_{n=0}^{\infty} u(n, 1) = \infty$, proving the result. ■

We come now to the promised necessary and sufficient condition for recurrence. Here $\phi = E \exp(it \cdot X_j)$ is the ch.f. of one step of the random walk.

**Theorem 4.2.9.** *Let $\delta > 0$. $S_n$ is recurrent if and only if*

$$\int_{(-\delta,\delta)^d} Re\, \frac{1}{1 - \varphi(y)}\, dy = \infty$$

We will prove a weaker result:

**Theorem 4.2.10.** *Let $\delta > 0$. $S_n$ is recurrent if and only if*

$$\sup_{r<1} \int_{(-\delta,\delta)^d} Re\, \frac{1}{1 - r\varphi(y)}\, dy = \infty$$

**Remark.** Half of the work needed to get the first result from the second is trivial.

$$0 \leq \text{Re}\, \frac{1}{1 - r\varphi(y)} \to \text{Re}\, \frac{1}{1 - \varphi(y)} \quad \text{as } r \to 1$$

so Fatou's lemma shows that if the integral is infinite, the walk is recurrent. The other direction is rather difficult: the second result is in Chung and Fuchs (1951), but a proof of the first result had to wait for Ornstein (1969) and Stone (1969) to solve the problem independently. Their proofs use a trick to reduce to the case where the increments have a density and then a second trick to deal with that case, so we will not give the details here. The reader can consult either of the sources cited or Port and Stone (1969), where the result is demonstrated for random walks on Abelian groups.

*Proof.* The first ingredient in the solution is the

**Lemma 4.2.11. Parseval relation.** *Let $\mu$ and $\nu$ be probability measures on $\mathbf{R}^d$ with ch.f.'s $\varphi$ and $\psi$.*

$$\int \psi(t)\,\mu(dt) = \int \varphi(x)\,\nu(dx)$$

*Proof.* Since $e^{it\cdot x}$ is bounded, Fubini's theorem implies

$$\int \psi(t)\mu(dt) = \iint e^{itx}\nu(dx)\mu(dt) = \iint e^{itx}\mu(dt)\nu(dx) = \int \varphi(x)\nu(dx) \quad \blacksquare$$

Our second ingredient is a little calculus.

**Lemma 4.2.12.** *If* $|x| \le \pi/3$ *then* $1 - \cos x \ge x^2/4$.

*Proof.* It suffices to prove the result for $x > 0$. If $z \le \pi/3$, then $\cos z \ge 1/2$,

$$\sin y = \int_0^y \cos z\,dz \ge \frac{y}{2}$$
$$1 - \cos x = \int_0^x \sin y\,dy \ge \int_0^x \frac{y}{2}\,dy = \frac{x^2}{4}$$

which proves the desired result. $\blacksquare$

From Example 3.3.5, we see that the density

$$\frac{\delta - |x|}{\delta^2} \quad \text{when} \quad |x| \le \delta, \qquad 0 \quad \text{otherwise}$$

has ch.f. $2(1 - \cos\delta t)/(\delta t)^2$. Let $\mu_n$ denote the distribution of $S_n$. Using Lemma 4.2.12 (note $\pi/3 \ge 1$) and then Lemma 4.2.11, we have

$$P(\|S_n\| < 1/\delta) \le 4^d \int \prod_{i=1}^d \frac{1 - \cos(\delta t_i)}{(\delta t_i)^2}\mu_n(dt)$$
$$= 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^d \frac{\delta - |x_i|}{\delta^2}\,\varphi^n(x)\,dx$$

Our next step is to sum from 0 to $\infty$. To be able to interchange the sum and the integral, we first multiply by $r^n$, where $r < 1$:

$$\sum_{n=0}^\infty r^n P(\|S_n\| < 1/\delta) \le 2^d \int_{(-\delta,\delta)^d} \prod_{i=1}^d \frac{\delta - |x_i|}{\delta^2}\,\frac{1}{1 - r\varphi(x)}\,dx$$

Symmetry dictates that the integral on the right is real, so we can take the real part without affecting its value. Letting $r \uparrow 1$ and using $(\delta - |x|)/\delta \le 1$

$$\sum_{n=0}^{\infty} P(\|S_n\| < 1/\delta) \le \left(\frac{2}{\delta}\right)^d \sup_{r<1} \int_{(-\delta,\delta)^d} \mathrm{Re}\, \frac{1}{1 - r\varphi(x)}\, dx$$

and using Theorem 4.2.6 gives half of Theorem 4.2.10.

To prove the other direction, we begin by noting that Example 3.3.8 shows that the density $(1 - \cos(x/\delta))/\pi x^2/\delta$ has ch.f. $1 - |\delta t|$ when $|t| \le 1/\delta$, 0 otherwise. Using $1 \ge \prod_{i=1}^d (1 - |\delta x_i|)$ and then Lemma 4.2.11,

$$\begin{aligned} P(\|S_n\| < 1/\delta) &\ge \int_{(-1/\delta,1/\delta)^d} \prod_{i=1}^d (1 - |\delta x_i|)\, \mu_n(dx) \\ &= \int \prod_{i=1}^d \frac{1 - \cos(t_i/\delta)}{\pi t_i^2/\delta}\, \varphi^n(t)\, dt \end{aligned}$$

Multiplying by $r^n$ and summing gives

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \ge \int \prod_{i=1}^d \frac{1 - \cos(t_i/\delta)}{\pi t_i^2/\delta}\, \frac{1}{1 - r\varphi(t)}\, dt$$

The last integral is real, so its value is unaffected if we integrate only the real part of the integrand. If we do this and apply Lemma 4.2.12, we get

$$\sum_{n=0}^{\infty} r^n P(\|S_n\| < 1/\delta) \ge (4\pi\delta)^{-d} \int_{(-\delta,\delta)^d} \mathrm{Re}\, \frac{1}{1 - r\varphi(t)}\, dt$$

Letting $r \uparrow 1$ and using Theorem 4.2.6 now completes the proof of Theorem 4.2.10. ■

We will now consider some examples. Our goal in $d = 1$ and $d = 2$ is to convince you that the conditions in Theorems 4.2.7 and 4.2.8 are close to the best possible.

*$d = 1$.* Consider the symmetric stable laws that have ch.f. $\varphi(t) = \exp(-|t|^\alpha)$. To avoid using facts that we have not proved, we will obtain our conclusions from Theorem 4.2.10. It is not hard to use that form of the criterion in this case since

$$\begin{aligned} 1 - r\varphi(t) &\downarrow 1 - \exp(-|t|^\alpha) &\quad &\text{as } r \uparrow 1 \\ 1 - \exp(-|t|^\alpha) &\sim |t|^\alpha &\quad &\text{as } t \to 0 \end{aligned}$$

From this, it follows that the corresponding random walk is transient for $\alpha < 1$ and recurrent for $\alpha \ge 1$. The case $\alpha > 1$ is covered by Theorem 4.2.7 since these random walks have mean 0. The result for $\alpha = 1$ is new because the Cauchy distribution does not satisfy $S_n/n \to 0$ in probability. The random walks with $\alpha < 1$ are interesting because Theorem 4.1.2 implies (see Exercise 4.1.1)

$$-\infty = \liminf S_n < \limsup S_n = \infty$$

but $P(|S_n| < M \text{ i.o.}) = 0$ for any $M < \infty$.

**Remark.** The stable law examples are misleading in one respect. Shepp (1964) proved that recurrent random walks may have arbitrarily large tails. To be precise, given a function $\epsilon(x) \downarrow 0$ as $x \uparrow \infty$, there is a recurrent random walk with $P(|X_1| \geq x) \geq \epsilon(x)$ for large $x$.

*d = 2.* Let $\alpha < 2$, and let $\varphi(t) = \exp(-|t|^\alpha)$ where $|t| = (t_1^2 + t_2^2)^{1/2}$. $\varphi$ is the characteristic function of a random vector $(X_1, X_2)$ that has two nice properties:

(i) the distribution of $(X_1, X_2)$ is invariant under rotations,
(ii) $X_1$ and $X_2$ have symmetric stable laws with index $\alpha$.

Again, $1 - r\varphi(t) \downarrow 1 - \exp(-|t|^\alpha)$ as $r \uparrow 1$ and $1 - \exp(-|t|^\alpha) \sim |t|^\alpha$ as $t \to 0$. Changing to polar coordinates and noticing

$$2\pi \int_0^\delta dx\, x\, x^{-\alpha} < \infty$$

when $1 - \alpha > -1$ shows that the random walks with ch.f. $\exp(-|t|^\alpha)$, $\alpha < 2$ are transient. When $p < \alpha$, we have $E|X_1|^p < \infty$ by Exercise 3.7.5, so these examples show that Theorem 4.2.8 is reasonably sharp.

*d ≥ 3.* The integral $\int_0^\delta dx\, x^{d-1}\, x^{-2} < \infty$, so if a random walk is recurrent in $d \geq 3$, its ch.f. must $\to 1$ faster than $t^2$. In Exercise 3.3.19, we observed that (in one dimension) if $\varphi(r) = 1 + o(r^2)$, then $\varphi(r) \equiv 1$. By considering $\varphi(r\theta)$ where $r$ is real and $\theta$ is a fixed vector, the last conclusion generalizes easily to $\mathbf{R}^d$, $d > 1$, and suggests that once we exclude walks that stay on a plane through 0, no three-dimensional random walks are recurrent.

A random walk in $\mathbf{R}^3$ is **truly three-dimensional** if the distribution of $X_1$ has $P(X_1 \cdot \theta \neq 0) > 0$ for all $\theta \neq 0$.

**Theorem 4.2.13.** *No truly three-dimensional random walk is recurrent.*

*Proof.* We will deduce the result from Theorem 4.2.10. We begin with some arithmetic. If $z$ is complex, the conjugate of $1 - z$ is $1 - \bar{z}$, so

$$\frac{1}{1-z} = \frac{1-\bar{z}}{|1-z|^2} \qquad \text{and} \qquad \operatorname{Re}\frac{1}{1-z} = \frac{\operatorname{Re}(1-z)}{|1-z|^2}$$

If $z = a + bi$ with $a \leq 1$, then using the previous formula and dropping the $b^2$ from the denominator,

$$\operatorname{Re}\frac{1}{1-z} = \frac{1-a}{(1-a)^2 + b^2} \leq \frac{1}{1-a}$$

Taking $z = r\phi(t)$ and supposing for the second inequality that $0 \leq \operatorname{Re}\phi(t) \leq 1$, we have

$$\text{(a)} \qquad \operatorname{Re}\frac{1}{1-r\varphi(t)} \leq \frac{1}{\operatorname{Re}(1-r\varphi(t))} \leq \frac{1}{\operatorname{Re}(1-\varphi(t))}$$

The last calculation shows that it is enough to estimate

$$\operatorname{Re}(1-\varphi(t)) = \int \{1-\cos(x\cdot t)\}\mu(dx) \ge \int_{|x\cdot t|<\pi/3} \frac{|x\cdot t|^2}{4}\,\mu(dx)$$

by Lemma 4.2.12. Writing $t = \rho\theta$ where $\theta \in S = \{x : |x| = 1\}$ gives

(b) $$\operatorname{Re}(1-\varphi(\rho\theta)) \ge \frac{\rho^2}{4}\int_{|x\cdot\theta|<\pi/3\rho} |x\cdot\theta|^2\mu(dx)$$

Fatou's lemma implies that if we let $\rho \to 0$ and $\theta(\rho) \to \theta$, then

(c) $$\liminf_{\rho\to 0}\int_{|x\cdot\theta(\rho)|<\pi/3\rho} |x\cdot\theta(\rho)|^2\mu(dx) \ge \int |x\cdot\theta|^2\mu(dx) > 0$$

I claim that this implies that for $\rho < \rho_0$

(d) $$\inf_{\theta\in S}\int_{|x\cdot\theta|<\pi/3\rho} |x\cdot\theta|^2\mu(dx) = C > 0$$

To get the last conclusion, observe that if it is false, then for $\rho = 1/n$ there is a $\theta_n$ so that

$$\int_{|x\cdot\theta_n|<n\pi/3} |x\cdot\theta_n|^2\mu(dx) \le 1/n$$

All the $\theta_n$ lie in $S$, a compact set, so if we pick a convergent subsequence, we contradict (c). Combining (b) and (d) gives

$$\operatorname{Re}(1-\varphi(\rho\theta)) \ge C\rho^2/4$$

Using the last result and (a) then changing to polar coordinates, we see that if $\delta$ is small (so $\operatorname{Re}\phi(y) \ge 0$ on $(-\delta,\delta)^d$)

$$\begin{aligned}\int_{(-\delta,\delta)^d} \operatorname{Re}\frac{1}{1-r\phi(y)}\,dy &\le \int_0^{\delta\sqrt{d}} d\rho\,\rho^{d-1}\int d\theta\,\frac{1}{\operatorname{Re}(1-\phi(\rho\theta))}\\ &\le C'\int_0^1 d\rho\,\rho^{d-3} < \infty\end{aligned}$$

when $d > 2$, so the desired result follows from Theorem 4.2.10. ∎

**Remark.** The analysis becomes much simpler when we consider random walks on $\mathbf{Z}^d$. The inversion formula given in Exercise 3.3.2 implies

$$P(S_n = 0) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} \varphi^n(t)\,dt$$

Multiplying by $r^n$ and summing gives

$$\sum_{n=0}^{\infty} r^n P(S_n = 0) = (2\pi)^{-d}\int_{(-\pi,\pi)^d} \frac{1}{1-r\varphi(t)}\,dt$$

In the case of simple random walk in $d = 3$, $\phi(t) = \frac{1}{3}\sum_{j=1}^{3} \cos t_j$ is real.

$$\frac{1}{1 - r\phi(t)} \uparrow \frac{1}{1 - \phi(t)} \quad \text{when } \phi(t) > 0$$

$$0 \leq \frac{1}{1 - r\phi(t)} \leq 1 \quad \text{when } \phi(t) \leq 0$$

So, using the monotone and bounded convergence theorems

$$\sum_{n=0}^{\infty} P(S_n = 0) = (2\pi)^{-3} \int_{(-\pi,\pi)^3} \left(1 - \frac{1}{3}\sum_{i=1}^{3} \cos x_i\right)^{-1} dx$$

This integral was first evaluated by Watson in 1939 in terms of elliptic integrals, which could be found in tables. Glasser and Zucker (1977) showed that it was

$$(\sqrt{6}/32\pi^3)\Gamma(1/24)\Gamma(5/24)\Gamma(7/24)\Gamma(11/24) = 1.516386059137\ldots$$

so it follows from (4.2.1) that

$$\pi_3 = 0.340537329544 \ldots$$

For numerical results in $4 \leq d \leq 9$, see Kondo and Hara (1987).

## 4.3 Visits to 0, Arcsine Laws*

In the last section, we took a broad look at the recurrence of random walks. In this section, we will take a deep look at one example: simple random walk (on $\mathbf{Z}$). To steal a line from Chung, "We shall treat this by combinatorial methods as an antidote to the analytic skulduggery above." The developments here follow Chapter III of Feller, vol. I. To facilitate discussion, we will think of the sequence $S_1, S_2, \ldots, S_n$ as being represented by a polygonal line with segments $(k-1, S_{k-1}) \to (k, S_k)$. A **path** is a polygonal line that is a possible outcome of simple random walk. To count the number of paths from (0,0) to $(n, x)$, it is convenient to introduce $a$ and $b$ defined as follows: $a = (n + x)/2$ is the number of positive steps in the path and $b = (n - x)/2$ is the number of negative steps. Notice that $n = a + b$ and $x = a - b$. If $-n \leq x \leq n$ and $n - x$ is even, the $a$ and $b$ defined above are nonnegative integers, and the number of paths from (0,0) to $(n, x)$ is

$$N_{n,x} = \binom{n}{a} \tag{4.3.1}$$

Otherwise, the number of paths is 0.

**Theorem 4.3.1. Reflection principle.** *If $x, y > 0$, then the number of paths from $(0, x)$ to $(n, y)$ that are 0 at some time is equal to the number of paths from $(0, -x)$ to $(n, y)$.*

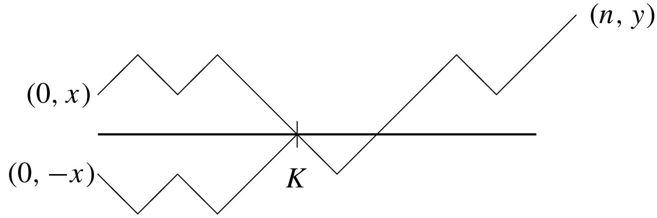



Figure 4.1. Reflection principle.

*Proof.* Suppose $(0, s_0), (1, s_1), \ldots, (n, s_n)$ is a path from $(0, x)$ to $(n, y)$. Let $K = \inf\{k : s_k = 0\}$. Let $s'_k = -s_k$ for $k \le K$, $s'_k = s_k$ for $K \le k \le n$. Then $(k, s'_k)$, $0 \le k \le n$, is a path from $(0, -x)$ to $(n, y)$. Conversely, if $(0, t_0), (1, t_1), \ldots, (n, t_n)$ is a path from $(0, -x)$ to $(n, y)$, then it must cross 0. Let $K = \inf\{k : t_k = 0\}$. Let $t'_k = -t_k$ for $k \le K$, $t'_k = t_k$ for $K \le k \le n$. Then $(k, t'_k)$, $0 \le k \le n$, is a path from $(0, -x)$ to $(n, y)$ that is 0 at time $K$. The last two observations set up a one-to-one correspondence between the two classes of paths, so their numbers must be equal. ■

From Theorem 4.3.1 we get a result first proved in 1878.

**Theorem 4.3.2. Ballot theorem.** *Suppose that in an election candidate A gets $\alpha$ votes, and candidate B gets $\beta$ votes where $\beta < \alpha$. The probability that throughout the counting A always leads B is $(\alpha - \beta)/(\alpha + \beta)$.*

*Proof.* Let $x = \alpha - \beta$, $n = \alpha + \beta$. Clearly, there are as many such outcomes as there are paths from (1,1) to $(n, x)$ that are never 0. The reflection principle implies that the number of paths from (1,1) to $(n, x)$ that are 0 at some time the number of paths from (1,−1) to $(n, x)$, so by (4.3.1) the number of paths from (1,1) to $(n, x)$ that are never 0 is

$$\begin{aligned} N_{n-1,x-1} - N_{n-1,x+1} &= \binom{n-1}{\alpha-1} - \binom{n-1}{\alpha} \\ &= \frac{(n-1)!}{(\alpha-1)!(n-\alpha)!} - \frac{(n-1)!}{\alpha!(n-\alpha-1)!} \\ &= \frac{\alpha-(n-\alpha)}{n} \cdot \frac{n!}{\alpha!(n-\alpha)!} = \frac{\alpha-\beta}{\alpha+\beta} N_{n,x} \end{aligned}$$

since $n = \alpha + \beta$, this proves the desired result. ■

Using the ballot theorem, we can compute the distribution of the time to hit 0 for simple random walk.

**Lemma 4.3.3.** $P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0)$.

*Proof.* $P(S_1 > 0, \ldots, S_{2n} > 0) = \sum_{r=1}^{\infty} P(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 2r)$. From the proof of Theorem 4.3.2, we see that the number of paths from (0,0) to $(2n, 2r)$ that are never 0 at positive times (= the number of paths from (1,1) to $(2n, 2r)$ that are never 0) is

$$N_{2n-1,2r-1} - N_{2n-1,2r+1}$$

If we let $p_{n,x} = P(S_n = x)$, then this implies

$$P(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = 2r) = \frac{1}{2}(p_{2n-1,2r-1} - p_{2n-1,2r+1})$$

Summing from $r = 1$ to $\infty$ gives

$$P(S_1 > 0, \ldots, S_{2n} > 0) = \frac{1}{2} p_{2n-1,1} = \frac{1}{2} P(S_{2n} = 0)$$

Symmetry implies $P(S_1 < 0, \ldots, S_{2n} < 0) = (1/2)P(S_{2n} = 0)$, and the proof is complete. ■

Let $R = \inf\{m \geq 1 : S_m = 0\}$. Combining Lemma 4.3.2 with Theorem 3.1.2 gives

$$P(R > 2n) = P(S_{2n} = 0) \sim \pi^{-1/2} n^{-1/2} \tag{4.3.2}$$

Since $P(R > x)/\ P(|R| > x) = 1$, it follows from Theorem 3.7.4 that $R$ is in the domain of attraction of the stable law with $\alpha = 1/2$ and $\kappa = 1$. This implies that if $R_n$ is the time of the $n$th return to 0 then $R_n/n^2 \Rightarrow Y$, the indicated stable law. In Example 3.7.2, we considered $\tau = T_1$ where $T_x = \inf\{n : S_n = x\}$. Since $S_1 \in \{-1, 1\}$ and $T_1 =_d T_{-1}$, $R =_d 1 + T_1$, and it follows that $T_n/n^2 \Rightarrow Y$, the same stable law. In Example 8.6.6, we will use this observation to show that the limit has the same distribution as the hitting time of 1 for Brownian motion, which has a density given in (8.4.8).

This completes our discussion of visits to 0. We turn now to the arcsine laws. The first one concerns

$$L_{2n} = \sup\{m \leq 2n : S_m = 0\}$$

It is remarkably easy to compute the distribution of $L_{2n}$.

**Lemma 4.3.4.** *Let $u_{2m} = P(S_{2m} = 0)$. Then $P(L_{2n} = 2k) = u_{2k} u_{2n-2k}$.*

*Proof.* $P(L_{2n} = 2k) = P(S_{2k} = 0, S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0)$, so the desired result follows from Lemma 4.3.3. ■

**Theorem 4.3.5. Arcsine law for the last visit to 0.** *For $0 < a < b < 1$,*

$$P(a \leq L_{2n}/2n \leq b) \to \int_a^b \pi^{-1}(x(1-x))^{-1/2}\, dx$$

To see the reason for the name, substitute $y = x^{1/2}$, $dy = (1/2)x^{-1/2}\,dx$ in the integral to obtain

$$\int_{\sqrt{a}}^{\sqrt{b}} \frac{2}{\pi}(1-y^2)^{-1/2}\,dy = \frac{2}{\pi}\{\arcsin(\sqrt{b}) - \arcsin(\sqrt{a})\}$$

Since $L_{2n}$ is the time of the last zero before $2n$, it is surprising that the answer is symmetric about 1/2. The symmetry of the limit distribution implies

$$P(L_{2n}/2n \le 1/2) \to 1/2$$

In gambling terms, if two people were to bet \$1 on a coin flip every day of the year, then with probability $1/2$, one of the players will be ahead from July 1 to the end of the year, an event that would undoubtedly cause the other player to complain about his bad luck.

*Proof of Theorem 4.3.5.* From the asymptotic formula for $u_{2n}$, it follows that if $k/n \to x$, then

$$nP(L_{2n} = 2k) \to \pi^{-1}(x(1-x))^{-1/2}$$

To get from this to the desired result, we let $2na_n =$ the smallest even integer $\ge 2na$, let $2nb_n =$ the largest even integer $\le 2nb$, and let $f_n(x) = nP(L_{2n} = k)$ for $2k/2n \le x < 2(k+1)/2n$, so we can write

$$P(a \le L_{2n}/2n \le b) = \sum_{k=na_n}^{nb_n} P(L_{2n} = 2k) = \int_{a_n}^{b_n+1/n} f_n(x)\,dx$$

Our first result implies that uniformly on compact sets

$$f_n(x) \to f(x) = \pi^{-1}(x(1-x))^{-1/2}$$

The uniformity of the convergence implies

$$\sup_{a_n \le x \le b_n+1/n} f_n(x) \to \sup_{a \le x \le b} f(x) < \infty$$

if $0 < a \le b < 1$, so the bounded convergence theorem gives

$$\int_{a_n}^{b_n+1/n} f_n(x)\,dx \to \int_a^b f(x)\,dx \qquad \blacksquare$$

The next result deals directly with the amount of time one player is ahead.

**Theorem 4.3.6. Arcsine law for time above 0.** *Let $\pi_{2n}$ be the number of segments $(k-1, S_{k-1}) \to (k, S_k)$ that lie above the axis (i.e., in $\{(x,y) : y \ge 0\}$), and let $u_m = P(S_m = 0)$.*

$$P(\pi_{2n} = 2k) = u_{2k}u_{2n-2k}$$

*and consequently, if* $0 < a < b < 1$,

$$P(a \le \pi_{2n}/2n \le b) \to \int_a^b \pi^{-1}(x(1-x))^{-1/2}\,dx$$

**Remark.** Since $\pi_{2n} =_d L_{2n}$, the second conclusion follows from the proof of Theorem 4.3.5. The reader should note that the limiting density $\pi^{-1}(x(1-x))^{-1/2}$ has a minimum at $x = 1/2$, and $\to \infty$ as $x \to 0$ or 1. An equal division of steps between the positive and negative side is therefore the least likely possibility, and completely one-sided divisions have the highest probability.

*Proof.* Let $\beta_{2k,2n}$ denote the probability of interest. We will prove $\beta_{2k,2n} = u_{2k}u_{2n-2k}$ by induction. When $n = 1$, it is clear that

$$\beta_{0,2} = \beta_{2,2} = 1/2 = u_0 u_2$$

For a general $n$, first suppose $k = n$. From the proof of Lemma 4.3.3, we have

$$\begin{aligned}
\frac{1}{2}u_{2n} &= P(S_1 > 0, \ldots, S_{2n} > 0) \\
&= P(S_1 = 1, S_2 - S_1 \ge 0, \ldots, S_{2n} - S_1 \ge 0) \\
&= \frac{1}{2}P(S_1 \ge 0, \ldots, S_{2n-1} \ge 0) \\
&= \frac{1}{2}P(S_1 \ge 0, \ldots, S_{2n} \ge 0) = \frac{1}{2}\beta_{2n,2n}
\end{aligned}$$

The next-to-last equality follows from the observation that if $S_{2n-1} \ge 0$, then $S_{2n-1} \ge 1$, and hence $S_{2n} \ge 0$.

The last computation proves the result for $k = n$. Since $\beta_{0,2n} = \beta_{2n,2n}$, the result is also true when $k = 0$. Suppose now that $1 \le k \le n-1$. In this case, if $R$ is the time of the first return to 0, then $R = 2m$ with $0 < m < n$. Letting $f_{2m} = P(R = 2m)$ and breaking things up according to whether the first excursion was on the positive or negative side gives

$$\beta_{2k,2n} = \frac{1}{2}\sum_{m=1}^{k} f_{2m}\beta_{2k-2m,2n-2m} + \frac{1}{2}\sum_{m=1}^{n-k} f_{2m}\beta_{2k,2n-2m}$$

Using the induction hypothesis, it follows that

$$\beta_{2k,2n} = \frac{1}{2}u_{2n-2k}\sum_{m=1}^{k} f_{2m}u_{2k-2m} + \frac{1}{2}u_{2k}\sum_{m=1}^{n-k} f_{2m}u_{2n-2k-2m}$$

By considering the time of the first return to 0, we see

$$u_{2k} = \sum_{m=1}^{k} f_{2m}u_{2k-2m} \qquad u_{2n-2k} = \sum_{m=1}^{n-k} f_{2m}u_{2n-2k-2m}$$

and the desired result follows. ■

Our derivation of Theorem 4.3.6 relied heavily on special properties of simple random walk. There is a closely related result due to E. Sparre-Andersen that is valid for very general random walks. However, notice that the hypothesis (ii) in the next result excludes simple random walk.

**Theorem 4.3.7.** *Let* $\nu_n = |\{k : 1 \le k \le n, S_k > 0\}|$. *Then*

*(i)* $P(\nu_n = k) = P(\nu_k = k)P(\nu_{n-k} = 0)$

*(ii) If the distribution of* $X_1$ *is symmetric and* $P(S_m = 0) = 0$ *for all* $m \ge 1$, *then*

$$P(\nu_n = k) = u_{2k}u_{2n-2k}$$

*where* $u_{2m} = 2^{-2m}\binom{2m}{m}$ *is the probability simple random walk is 0 at time* $2m$.

*(iii) Under the hypotheses of (ii),*

$$P(a \le \nu_n/n \le b) \to \int_a^b \pi^{-1}(x(1-x))^{-1/2}\,dx \quad \text{for } 0 < a < b < 1$$

*Proof.* Taking things in reverse order, (iii) is an immediate consequence of (ii) and the proof of Theorem 4.3.5. Our next step is to show that (ii) follows from (i) by induction. When $n = 1$, our assumptions imply $P(\nu_1 = 0) = 1/2 = u_0u_2$. If $n > 1$ and $1 \le k < n$, then (i) and the induction hypothesis imply

$$P(\nu_n = k) = u_{2k}u_0 \cdot u_0u_{2n-2k} = u_{2k}u_{2n-2k}$$

since $u_0 = 1$. To handle the cases $k = 0$ and $k = n$, we note that Lemma 4.3.4 implies

$$\sum_{k=0}^{n} u_{2k}u_{2n-2k} = 1$$

We have $\sum_{k=0}^{n} P(\nu_n = k) = 1$ and our assumptions imply $P(\nu_n = 0) = P(\nu_n = n)$, so these probabilities must be equal to $u_0u_{2n}$.

The proof of (i) is tricky and requires careful definitions since we are not supposing $X_1$ is symmetric or that $P(S_m = 0) = 0$. Let $\nu'_n = |\{k : 1 \le k \le n, S_k \le 0\}| = n - \nu_n$.

$$M_n = \max_{0 \le j \le n} S_j \qquad \ell_n = \min\{j : 0 \le j \le n, S_j = M_n\}$$

$$M'_n = \min_{0 \le j \le n} S_j \qquad \ell'_n = \max\{j : 0 \le j \le n, S_j = M'_n\}$$

The first symmetry is straightforward.

**Lemma 4.3.8.** $(\ell_n, S_n)$ *and* $(n - \ell'_n, S_n)$ *have the same distribution.*

*Proof.* If we let $T_k = S_n - S_{n-k} = X_n + \cdots + X_{n-k+1}$, then $T_k$ $0 \le k \le n$ has the same distribution as $S_k$, $0 \le k \le n$. Clearly,

$$\max_{0 \le k \le n} T_k = S_n - \min_{0 \le k \le n} S_{n-k}$$

and the set of $k$ for which the extrema are attained are the same. ∎

The second symmetry is much less obvious.

**Lemma 4.3.9.** $(\ell_n, S_n)$ *and* $(\nu_n, S_n)$ *have the same distribution.* $(\ell'_n, S_n)$ *and* $(\nu'_n, S_n)$ *have the same distribution.*

**Remark.** (i) follows from Lemma 4.3.8 and the trivial observation

$$P(\ell_n = k) = P(\ell_k = k)P(\ell_{n-k} = 0)$$

so, once Lemma 4.3.9 is established, the proof of Theorem 4.3.7 will be complete.

*Proof.* When $n = 1$, $\{\ell_1 = 0\} = \{S_1 \le 0\} = \{\nu_1 = 0\}$, and $\{\ell'_1 = 0\} = \{S_1 > 0\} = \{\nu'_1 = 0\}$. We shall prove the general case by induction, supposing that both statements have been proved when $n$ is replaced by $n - 1$. Let

$$G(y) = P(\ell_{n-1} = k, S_{n-1} \le y)$$
$$H(y) = P(\nu_{n-1} = k, S_{n-1} \le y)$$

On $\{S_n \le 0\}$, we have $\ell_{n-1} = \ell_n$, and $\nu_{n-1} = \nu_n$, so if $F(y) = P(X_1 \le y)$, then for $x \le 0$,

$$P(\ell_n = k, S_n \le x) = \int F(x-y)\,dG(y) \tag{4.3.3}$$
$$= \int F(x-y)\,dH(y) = P(\nu_n = k, S_n \le x)$$

On $\{S_n > 0\}$, we have $\ell'_{n-1} = \ell'_n$, and $\nu'_{n-1} = \nu'_n$, so repeating the last computation shows that for $x \ge 0$

$$P(\ell'_n = n-k, S_n > x) = P(\nu'_n = n-k, S_n > x)$$

Since $(\ell_n, S_n)$ has the same distribution as $(n - \ell'_n, S_n)$ and $\nu'_n = n - \nu_n$, it follows that for $x \ge 0$

$$P(\ell_n = k, S_n > x) = P(\nu_n = k, S_n > x)$$

Setting $x = 0$ in the last result and (4.3.3) and adding gives

$$P(\ell_n = k) = P(\nu_n = k)$$

Subtracting the last two equations and combining the result with (4.3.3) gives

$$P(\ell_n = k, S_n \le x) = P(\nu_n = k, S_n \le x)$$

for all $x$. Since $(\ell_n, S_n)$ has the same distribution as $(n - \ell'_n, S_n)$ and $\nu'_n = n - \nu_n$, it follows that

$$P(\ell'_n = n-k, S_n > x) = P(\nu'_n = n-k, S_n > x)$$

for all $x$. This completes the proof of Lemma 4.3.9 and hence of Theorem 4.3.7. ∎

## 4.4 Renewal Theory*

Let $\xi_1, \xi_2, \ldots$ be i.i.d. positive random variables with distribution $F$ and define a sequence of times by $T_0 = 0$, and $T_k = T_{k-1} + \xi_k$ for $k \geq 1$. As explained in Section 2.4, we think of $\xi_i$ as the lifetime of the $i$th light bulb, and $T_k$ is the time the $k$th bulb burns out. A second interpretation from Section 3.6 is that $T_k$ is the time of arrival of the $k$th customer. To have a neutral terminology, we will refer to the $T_k$ as **renewals**. The term refers to the fact that the process "starts afresh" at $T_k$, that is, $\{T_{k+j} - T_k, j \geq 1\}$ has the same distribution as $\{T_j, j \geq 1\}$.

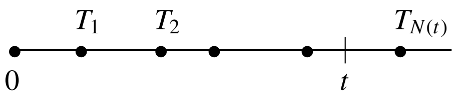



Figure 4.2. Renewal sequence.

Departing slightly from the notation in Sections 2.4 and 3.6, we let $N_t = \inf\{k : T_k > t\}$. $N_t$ is the number of renewals in $[0, t]$, counting the renewal at time 0 (see Figure 4.2). In Theorem 2.4.6, we showed that

**Theorem 4.4.1.** *As $t \to \infty$, $N_t/t \to 1/\mu$ a.s. where $\mu = E\xi_i \in (0, \infty]$ and $1/\infty = 0$.*

Our first result concerns the asymptotic behavior of $U(t) = EN_t$.

**Theorem 4.4.2.** *As $t \to \infty$, $U(t)/t \to 1/\mu$.*

*Proof.* We will apply Wald's equation to the stopping time $N_t$. The first step is to show that $P(\xi_i > 0) > 0$ implies $EN_t < \infty$. To do this, pick $\delta > 0$ so that $P(\xi_i > \delta) = \epsilon > 0$ and pick $K$ so that $K\delta \geq t$. Since $K$ consecutive $\xi_i's$ that are $> \delta$ will make $T_n > t$, we have

$$P(N_t > mK) \leq (1 - \epsilon^K)^m$$

and $EN_t < \infty$. If $\mu < \infty$, applying Wald's equation now gives

$$\mu EN_t = ET_{N_t} \geq t$$

so $U(t) \geq t/\mu$. The last inequality is trivial when $\mu = \infty$, so it holds in general.

Turning to the upper bound, we observe that if $P(\xi_i \leq c) = 1$, then repeating the last argument shows $\mu EN_t = ES_{N_t} \leq t + c$, and the result holds for bounded distributions. If we let $\bar{\xi}_i = \xi_i \wedge c$ and define $\bar{T}_n$ and $\bar{N}_t$ in the obvious way then

$$EN_t \leq E\bar{N}_t \leq (t + c)/E(\bar{\xi}_i)$$

Letting $t \to \infty$ and then $c \to \infty$ gives $\limsup_{t\to\infty} EN_t/t \leq 1/\mu$, and the proof is complete. ∎

**Exercise 4.4.1.** Show that $t/E(\xi_i \wedge t) \leq U(t) \leq 2t/E(\xi_i \wedge t)$.

**Exercise 4.4.2.** Deduce Theorem 4.4.2 from Theorem 4.4.1 by showing

$$\limsup_{t\to\infty} E(N_t/t)^2 < \infty.$$

Hint: Use a comparison like the one in the proof of Theorem 4.4.2.

**Exercise 4.4.3.** Customers arrive at times of a Poisson process with rate 1. If the server is occupied, they leave. (Think of a public telephone or prostitute.) If not, they enter service and require a service time with a distribution $F$ that has mean $\mu$. Show that the times at which customers enter service are a renewal process with mean $\mu + 1$, and use Theorem 4.4.1 to conclude that the asymptotic fraction of customers served is $1/(\mu + 1)$.

To take a closer look at when the renewals occur, we let

$$U(A) = \sum_{n=0}^{\infty} P(T_n \in A)$$

$U$ is called the **renewal measure**. We absorb the old definition, $U(t) = EN_t$, into the new one by regarding $U(t)$ as shorthand for $U([0, t])$. This should not cause problems, since $U(t)$ is the distribution function for the renewal measure. The asymptotic behavior of $U(t)$ depends on whether the distribution $F$ is **arithmetic**, that is, concentrated on $\{\delta, 2\delta, 3\delta, \ldots\}$ for some $\delta > 0$, or **nonarithmetic**, that is, not arithmetic. We will treat the first case in Chapter 5 as an application of Markov chains, so we will restrict our attention to the second case here.

**Theorem 4.4.3. Blackwell's renewal theorem.** *If $F$ is nonarithmetic, then*

$$U([t, t + h]) \to h/\mu \quad \text{as } t \to \infty.$$

We will prove the result in the case $\mu < \infty$ by "coupling" following Lindvall (1977) and Athreya, McDonald, and Ney (1978). To set the stage for the proof, we need a definition and some preliminary computations. If $T_0 \geq 0$ is independent of $\xi_1, \xi_2, \ldots$ and has distribution $G$, then $T_k = T_{k-1} + \xi_k$, $k \geq 1$ defines a **delayed renewal process**, and $G$ is the **delay distribution**. If we let $N_t = \inf\{k : T_k > t\}$ as before and set $V(t) = EN_t$, then breaking things down according to the value of $T_0$ gives

$$V(t) = \int_0^t U(t - s)\, dG(s) \tag{4.4.1}$$

The last integral, and all similar expressions below, is intended to include the contribution of any mass $G$ has at 0. If we let $U(r) = 0$ for $r < 0$, then the last equation can be written as $V = U * G$, where $*$ denotes convolution.

Applying similar reasoning to $U$ gives

$$U(t) = 1 + \int_0^t U(t - s)\, dF(s) \tag{4.4.2}$$

or, introducing convolution notation,

$$U = 1_{[0,\infty)}(t) + U * F.$$

Convolving each side with $G$ (and recalling $G * U = U * G$) gives

$$V = G * U = G + V * F \tag{4.4.3}$$

We know $U(t) \sim t/\mu$. Our next step is to find a $G$ so that $V(t) = t/\mu$. Plugging what we want into (4.4.3) gives

$$t/\mu = G(t) + \int_0^t \frac{t-y}{\mu}\, dF(y)$$

$$\text{so} \qquad G(t) = t/\mu - \int_0^t \frac{t-y}{\mu}\, dF(y)$$

The integration-by-parts formula is

$$\int_0^t K(y)\, dH(y) = H(t)K(t) - H(0)K(0) - \int_0^t H(y)\, dK(y)$$

If we let $H(y) = (y-t)/\mu$ and $K(y) = 1 - F(y)$, then

$$\frac{1}{\mu}\int_0^t 1 - F(y)\, dy = \frac{t}{\mu} - \int_0^t \frac{t-y}{\mu}\, dF(y)$$

so we have

$$G(t) = \frac{1}{\mu}\int_0^t 1 - F(y)\, dy \tag{4.4.4}$$

It is comforting to note that $\mu = \int_{[0,\infty)} 1 - F(y)\, dy$, so the last formula defines a probability distribution. When the delay distribution $G$ is the one given in (4.4.4), we call the result the **stationary renewal process**. Something very special happens when $F(t) = 1 - \exp(-\lambda t)$, $t \geq 0$ where $\lambda > 0$ (i.e., the renewal process is a rate $\lambda$ Poisson process). In this case, $\mu = 1/\lambda$ so $G(t) = F(t)$.

*Proof of Theorem 4.4.3 for* $\mu < \infty$. Let $T_n$ be a renewal process (with $T_0 = 0$) and $T'_n$ be an independent stationary renewal process. Our first goal is to find $J$ and $K$ so that $|T_J - T'_K| < \epsilon$ and the increments $\{T_{J+i} - T_J, i \geq 1\}$ and $\{T'_{K+i} - T'_K, i \geq 1\}$ are i.i.d. sequences independent of what has come before.

Let $\eta_1, \eta_2, \ldots$ and $\eta'_1, \eta'_2, \ldots$ be i.i.d. independent of $T_n$ and $T'_n$, and take the values 0 and 1 with probability 1/2 each. Let $\nu_n = \eta_1 + \cdots + \eta_n$ and $\nu'_n = 1 + \eta'_1 + \cdots + \eta'_n$, $S_n = T_{\nu_n}$ and $S'_n = T'_{\nu'_n}$. The increments of $S_n - S'_n$ are 0 with probability at least 1/4, and the support of their distribution is symmetric and contains the support of the $\xi_k$, so if the distribution of the $\xi_k$ is nonarithmetic, the random walk $S_n - S'_n$ is irreducible. Since the increments of $S_n - S'_n$ have mean 0, $N = \inf\{n : |S_n - S'_n| < \epsilon\}$ has $P(N < \infty) = 1$, and we can let $J = \nu_N$ and $K = \nu'_N$. Let (see Figure 4.3 for a picture)

$$T''_n = \begin{cases} T_n & \text{if } J \geq n \\ T_J + T'_{K+(n-J)} - T'_K & \text{if } J < n \end{cases}$$

In other words, the increments $T''_{J+i} - T''_J$ are the same as $T'_{K+i} - T'_K$ for $i \geq 1$.

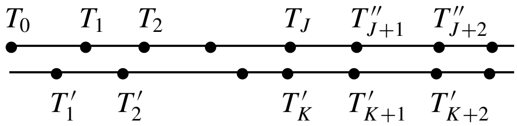



Figure 4.3. Coupling of renewal processes.

It is easy to see from the construction that $T_n$ and $T''_n$ have the same distribution. If we let

$$N'[s,t] = |\{n : T'_n \in [s,t]\}| \quad \text{and} \quad N''[s,t] = |\{n : T''_n \in [s,t]\}|$$

be the number of renewals in $[s,t]$ in the two processes, then on $\{T_J \le t\}$

$$N''[t,t+h] = N'[t + T'_K - T_J, t+h+T'_K - T_J] \begin{cases} \ge N'[t+\epsilon, t+h-\epsilon] \\ \le N'[t-\epsilon, t+h+\epsilon] \end{cases}$$

To relate the expected number of renewals in the two processes, we observe that even if we condition on the location of all the renewals in $[0,s]$, the expected number of renewals in $[s,s+t]$ is at most $U(t)$, since the worst thing that could happen is to have a renewal at time $s$. Combining the last two observations, we see that if $\epsilon < h/2$ (so $[t+\epsilon, t+h-\epsilon]$ has positive length)

$$\begin{aligned} U([t,t+h]) = EN''[t,t+h] &\ge E(N'[t+\epsilon, t+h-\epsilon]; T_J \le t) \\ &\ge \frac{h-2\epsilon}{\mu} - P(T_J > t)U(h) \end{aligned}$$

since $EN'[t+\epsilon, t+h-\epsilon] = (h-2\epsilon)/\mu$ and $\{T_J > t\}$ is determined by the renewals of $T$ in $[0,t]$ and the renewals of $T'$ in $[0, t+\epsilon]$. For the other direction, we observe

$$\begin{aligned} U([t,t+h]) &\le E(N'[t-\epsilon, t+h+\epsilon]; T_J \le t) + E(N''[t,t+h]; T_J > t) \\ &\le \frac{h+2\epsilon}{\mu} + P(T_J > t)U(h) \end{aligned}$$

The desired result now follows from the fact that $P(T_J > t) \to 0$ and $\epsilon < h/2$ is arbitrary. ∎

*Proof of Theorem 4.4.3 for* $\mu = \infty$. In this case, there is no stationary renewal process, so we have to resort to other methods. Let

$$\beta = \limsup_{t\to\infty} U(t, t+1] = \lim_{k\to\infty} U(t_k, t_k + 1]$$

for some sequence $t_k \to \infty$. We want to prove that $\beta = 0$, for then by addition the previous conclusion holds with 1 replaced by any integer $n$ and, by monotonicity, with $n$ replaced by any $h < n$, and this gives us the result in Theorem 4.4.3. Fix $i$ and let

$$a_{k,j} = \int_{(j-1,j]} U(t_k - y, t_k + 1 - y]\, dF^{i*}(y)$$

By considering the location of $T_i$ we get

(a) $$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{k,j} = \lim_{k\to\infty} \int U(t_k - y, t_k + 1 - y]\, dF^{i*}(y) = \beta$$

Since $\beta$ is the lim sup, we must have

(b) $$\limsup_{k\to\infty} a_{k,j} \le \beta \cdot P(T_i \in (j-1, j])$$

We want to conclude from (a) and (b) that

(c) $$\liminf_{k\to\infty} a_{k,j} \ge \beta \cdot P(T_i \in (j-1, j])$$

To do this, we observe that by considering the location of the first renewal in $(j-1, j]$

(d) $$0 \le a_{k,j} \le U(1) P(T_i \in (j-1, j])$$

(c) is trivial when $\beta = 0$, so we can suppose $\beta > 0$. To argue by contradiction, suppose there exist $j_0$ and $\epsilon > 0$ so that

$$\liminf_{k\to\infty} a_{k,j_0} \le \beta \cdot \{P(T_i \in (j_0 - 1, j_0]) - \epsilon\}$$

Pick $k_n \to \infty$ so that

$$a_{k_n,j_0} \to \beta \cdot \{P(T_i \in (j_0 - 1, j_0]) - \epsilon\}$$

Using (d), we can pick $J \ge j_0$ so that

$$\limsup_{n\to\infty} \sum_{j=J+1}^{\infty} a_{k_n,j} \le U(1) \sum_{j=J+1}^{\infty} P(T_i \in (j-1, j]) \le \beta\epsilon/2$$

Now an easy argument shows

$$\limsup_{n\to\infty} \sum_{j=1}^{J} a_{k_n,j} \le \sum_{j=1}^{J} \limsup_{n\to\infty} a_{k_n,j} \le \beta \left( \sum_{j=1}^{J} P(T_i \in (j-1, j]) - \epsilon \right)$$

by (b) and our assumption. Adding the last two results shows

$$\limsup_{n\to\infty} \sum_{j=1}^{\infty} a_{k_n,j} \le \beta(1 - \epsilon/2)$$

which contradicts (a), and proves (c).

Now, if $j - 1 < y \le j$, we have

$$U(t_k - y, t_k + 1 - y] \le U(t_k - j, t_k + 2 - j]$$

so using (c), it follows that for $j$ with $P(T_i \in (j-1, j]) > 0$, we must have

$$\liminf_{k\to\infty} U(t_k - j, t_k + 2 - j] \ge \beta$$

Summing over $i$, we see that the last conclusion is true when $U(j-1, j] > 0$.

The support of $U$ is closed under addition. (If $x$ is in the support of $F^{m*}$ and $y$ is in the support of $F^{n*}$, then $x + y$ is in the support of $F^{(m+n)*}$.) We have assumed $F$ is nonarithmetic, so $U(j-1, j] > 0$ for $j \geq j_0$. Letting $r_k = t_k - j_0$ and considering the location of the last renewal in $[0, r_k]$ and the index of the $T_i$ gives

$$1 = \sum_{i=0}^{\infty} \int_0^{r_k} (1 - F(r_k - y))\, dF^{i*}(y) = \int_0^{r_k} (1 - F(r_k - y))\, dU(y)$$
$$\geq \sum_{n=1}^{\infty} (1 - F(2n))\, U(r_k - 2n, r_k + 2 - 2n]$$

Since $\liminf_{k\to\infty} U(r_k - 2n, r_k + 2 - 2n] \geq \beta$ and

$$\sum_{n=0}^{\infty} (1 - F(2n)) \geq \mu/2 = \infty$$

$\beta$ must be 0, and the proof is complete. ■

**Remark.** Following Lindvall (1977), we have based the proof for $\mu = \infty$ on part of Feller's (1961) proof of the discrete renewal theorem (i.e., for arithmetic distributions). See Freedman (1971b), pp. 22–25, for an account of Feller's proof. Purists can find a proof that does everything by coupling in Thorisson (1987).

Our next topic is the **renewal equation**: $H = h + H * F$. Two cases we have seen in (4.4.2) and (4.4.3) are:

**Example 4.4.1.** $h \equiv 1$: $U(t) = 1 + \int_0^t U(t-s)\, dF(s)$

**Example 4.4.2.** $h(t) = G(t)$: $V(t) = G(t) + \int_0^t V(t-s)\, dF(s)$

The last equation is valid for an arbitrary delay distribution. If we let $G$ be the distribution in (4.4.4) and subtract the last two equations, we get

**Example 4.4.3.** $H(t) = U(t) - t/\mu$ satisfies the renewal equation with $h(t) = \frac{1}{\mu}\int_t^\infty 1 - F(s)\, ds$.

Last but not least, we have an example that is a typical application of the renewal equation.

**Example 4.4.4.** Let $x > 0$ be fixed, and let $H(t) = P(T_{N(t)} - t > x)$. By considering the value of $T_1$, we get

$$H(t) = (1 - F(t + x)) + \int_0^t H(t-s)\, dF(s)$$

The examples above should provide motivation for:

**Theorem 4.4.4.** *If $h$ is bounded then the function*

$$H(t) = \int_0^t h(t-s)\,dU(s)$$

*is the unique solution of the renewal equation that is bounded on bounded intervals.*

*Proof.* Let $U_n(A) = \sum_{m=0}^n P(T_m \in A)$ and

$$H_n(t) = \int_0^t h(t-s)\,dU_n(s) = \sum_{m=0}^n \left(h * F^{m*}\right)(t)$$

Here, $F^{m*}$ is the distribution of $T_m$, and we have extended the definition of $h$ by setting $h(r) = 0$ for $r < 0$. From the last expression, it should be clear that

$$H_{n+1} = h + H_n * F$$

The fact that $U(t) < \infty$ implies $U(t) - U_n(t) \to 0$. Since $h$ is bounded,

$$|H_n(t) - H(t)| \le \|h\|_\infty |U(t) - U_n(t)|$$

and $H_n(t) \to H(t)$ uniformly on bounded intervals. To estimate the convolution, we note that

$$\begin{aligned} |H_n * F(t) - H * F(t)| &\le \sup_{s \le t} |H_n(s) - H(s)| \\ &\le \|h\|_\infty |U(t) - U_n(t)| \end{aligned}$$

since $U - U_n = \sum_{m=n+1}^\infty F^{m*}$ is increasing in $t$. Letting $n \to \infty$ in $H_{n+1} = h + H_n * F$, we see that $H$ is a solution of the renewal equation that is bounded on bounded intervals.

To prove uniqueness, we observe that if $H_1$ and $H_2$ are two solutions, then $K = H_1 - H_2$ satisfies $K = K * F$. If $K$ is bounded on bounded intervals, iterating gives $K = K * F^{n*} \to 0$ as $n \to \infty$, so $H_1 = H_2$. ■

The proof of Theorem 4.4.4 is valid when $F(\infty) = P(\xi_i < \infty) < 1$. In this case, we have a **terminating renewal process**. After a geometric number of trials with mean $1/(1 - F(\infty))$, $T_n = \infty$. This "trivial case" has some interesting applications.

**Example 4.4.5. Pedestrian delay.** A chicken wants to cross a road (we won't ask why) on which the traffic is a Poisson process with rate $\lambda$. She needs one unit of time with no arrival to safely cross the road. Let $M = \inf\{t \ge 0 :$ there are no arrivals in $(t, t+1]\}$ be the waiting time until she starts to cross the street. By considering the time of the first arrival, we see that $H(t) = P(M \le t)$ satisfies

$$H(t) = e^{-\lambda} + \int_0^1 H(t-y)\,\lambda e^{-\lambda y}\,dy$$

Comparing with Example 4.4.1 and using Theorem 4.4.4, we see that

$$H(t) = e^{-\lambda} \sum_{n=0}^{\infty} F^{n*}(t)$$

We could have gotten this answer without renewal theory by noting

$$P(M \le t) = \sum_{n=0}^{\infty} P(T_n \le t, T_{n+1} = \infty)$$

The last representation allows us to compute the mean of $M$. Let $\mu$ be the mean of the interarrival time given that it is $< 1$, and note that the lack of memory property of the exponential distribution implies

$$\mu = \int_0^1 x\lambda e^{-\lambda x}\, dx = \int_0^\infty - \int_1^\infty = \frac{1}{\lambda} - \left(1 + \frac{1}{\lambda}\right) e^{-\lambda}$$

Then, by considering the number of renewals in our terminating renewal process,

$$EM = \sum_{n=0}^{\infty} e^{-\lambda}(1 - e^{-\lambda})^n n\mu = (e^{\lambda} - 1)\mu$$

since if $X$ is a geometric with success probability $e^{-\lambda}$, then $EM = \mu E(X - 1)$.

**Example 4.4.6. Cramér's estimates of ruin.** Consider an insurance company that collects money at rate $c$ and experiences i.i.d. claims at the arrival times of a Poisson process $N_t$ with rate 1. If its initial capital is $x$, its wealth at time $t$ is

$$W_x(t) = x + ct - \sum_{m=1}^{Nt} Y_i$$

Here $Y_1, Y_2, \ldots$ are i.i.d. with distribution $G$ and mean $\mu$. Let

$$R(x) = P(W_x(t) \ge 0 \text{ for all } t)$$

be the probability of never going bankrupt starting with capital $x$. By considering the time and size of the first claim:

(a) $$R(x) = \int_0^\infty e^{-s} \int_0^{x+cs} R(x + cs - y)\, dG(y)\, ds$$

This does not look much like a renewal equation, but with some ingenuity it can be transformed into one. Changing variables $t = x + cs$,

$$R(x)e^{-x/c} = \int_x^\infty e^{-t/c} \int_0^t R(t - y)\, dG(y)\, \frac{dt}{c}$$

Differentiating w.r.t. $x$ and then multiplying by $e^{x/c}$,

$$R'(x) = \frac{1}{c} R(x) - \int_0^x R(x - y)\, dG(y) \cdot \frac{1}{c}$$

Integrating $x$ from 0 to $w$,

(b) $$R(w) - R(0) = \frac{1}{c}\int_0^w R(x)\,dx - \frac{1}{c}\int_0^w\int_0^x R(x-y)\,dG(y)\,dx$$

Interchanging the order of integration in the double integral, letting

$$S(w) = \int_0^w R(x)\,dx$$

using $dG = -d(1-G)$, and then integrating by parts,

$$\begin{aligned}
-\frac{1}{c}\int_0^w\int_y^w R(x-y)dx\,dG(y) &= -\frac{1}{c}\int_0^w S(w-y)\,dG(y)\\
&= \frac{1}{c}\int_0^w S(w-y)\,d(1-G)(y)\\
&= \frac{1}{c}\left\{-S(w) + \int_0^w (1-G(y))R(w-y)\,dy\right\}
\end{aligned}$$

Plugging this into (b), we finally have a renewal equation:

(c) $$R(w) = R(0) + \int_0^w R(w-y)\frac{1-G(y)}{c}\,dy$$

It took some cleverness to arrive at the last equation, but it is straightforward to analyze. First, we dismiss a trivial case. If $\mu > c$,

$$\frac{1}{t}\left(ct - \sum_{m=1}^{Nt} Y_i\right) \to c - \mu < 0 \quad \text{a.s.}$$

so $R(x) \equiv 0$. When $\mu < c$,

$$F(x) = \int_0^x \frac{1-G(y)}{c}\,dy$$

is a defective probability distribution with $F(\infty) = \mu/c$. Our renewal equation can be written as

(d) $$R = R(0) + R * F$$

so comparing with Example 4.4.1 and using Theorem 4.4.4 tells us $R(w) = R(0)U(w)$. To complete the solution, we have to compute the constant $R(0)$. Letting $w \to \infty$ and noticing $R(w) \to 1, U(w) \to (1 - F(\infty))^{-1} = (1 - \mu/c)^{-1}$, we have $R(0) = 1 - \mu/c$.

The basic fact about solutions of the renewal equation (in the nonterminating case) is:

**Theorem 4.4.5. The renewal theorem.** *If $F$ is nonarithmetic and $h$ is directly Riemann integrable then as $t \to \infty$*

$$H(t) \to \frac{1}{\mu}\int_0^\infty h(s)\,ds$$

Intuitively, this holds since Theorem 4.4.4 implies

$$H(t) = \int_0^t h(t-s)\,dU(s)$$

and Theorem 4.4.3 implies $dU(s) \to ds/\mu$ as $s \to \infty$. We will define directly Riemann integrable in a minute. We will start doing the proof and then figure out what we need to assume.

*Proof.* Suppose

$$h(s) = \sum_{k=0}^{\infty} a_k 1_{[k\delta,(k+1)\delta)}(s)$$

where $\sum_{k=0}^{\infty} |a_k| < \infty$. Since $U([t, t+\delta]) \le U([0, \delta]) < \infty$, it follows easily from Theorem 4.4.3 that

$$\int_0^t h(t-s)dU(s) = \sum_{k=0}^{\infty} a_k U((t-(k+1)\delta, t-k\delta]) \to \frac{1}{\mu}\sum_{k=0}^{\infty} a_k\delta$$

(Pick $K$ so that $\sum_{k\ge K} |a_k| \le \epsilon/2U([0,\delta])$ and then $T$ so that

$$|a_k| \cdot |U((t-(k+1)\delta, t-k\delta]) - \delta/\mu| \le \frac{\epsilon}{2K}$$

for $t \ge T$ and $0 \le k < K$.) If $h$ is an arbitrary function on $[0, \infty)$, we let

$$I^\delta = \sum_{k=0}^{\infty} \delta \ \sup\{h(x) : x \in [k\delta, (k+1)\delta)\}$$

$$I_\delta = \sum_{k=0}^{\infty} \delta \ \inf\{h(x) : x \in [k\delta, (k+1)\delta)\}$$

be upper and lower Riemann sums approximating the integral of $h$ over $[0, \infty)$. Comparing $h$ with the obvious upper and lower bounds that are constant on $[k\delta, (k+1)\delta)$ and using the result for the special case,

$$\frac{I_\delta}{\mu} \le \liminf_{t\to\infty} \int_0^t h(t-s)\,dU(s) \le \limsup_{t\to\infty} \int_0^t h(t-s)\,dU(s) \le \frac{I^\delta}{\mu}$$

If $I^\delta$ and $I_\delta$ both approach the same finite limit $I$ as $\delta \to 0$, then $h$ is said to be **directly Riemann integrable**, and it follows that

$$\int_0^t h(t-s)\,dU(y) \to I/\mu \qquad \blacksquare$$

**Remark.** The word "direct" in the name refers to the fact that although the Riemann integral over $[0, \infty)$ is usually defined as the limit of integrals over $[0, a]$, we are approximating the integral over $[0, \infty)$ directly.

In checking the new hypothesis in Theorem 4.4.5, the following result is useful.

**Lemma 4.4.6.** *If $h(x) \ge 0$ is decreasing with $h(0) < \infty$ and $\int_0^\infty h(x)\,dx < \infty$, then $h$ is directly Riemann integrable.*

*Proof.* Because $h$ is decreasing, $I^\delta = \sum_{k=0}^\infty \delta h(k\delta)$ and $I_\delta = \sum_{k=0}^\infty \delta h((k+1)\delta)$. So

$$I^\delta \geq \int_0^\infty h(x)\,dx \geq I_\delta = I^\delta - h(0)\delta$$

proving the desired result. ■

The last result suffices for all our applications, so we leave it to the reader to do.

**Exercise 4.4.4.** If $h \geq 0$ is continuous, then $h$ is directly Riemann integrable if and only if $I^\delta < \infty$ for some $\delta > 0$ (and hence for all $\delta > 0$).

Returning now to our examples, we skip the first two because, in those cases, $h(t) \to 1$ as $t \to \infty$, so $h$ is not integrable in any sense.

**Example 4.4.7. Continuation of Example 4.4.3.** $h(t) = \frac{1}{\mu}\int_{[t,\infty)} 1 - F(s)\,ds$. $h$ is decreasing, $h(0) = 1$, and

$$\begin{aligned}\mu \int_0^\infty h(t)\,dt &= \int_0^\infty \int_t^\infty 1 - F(s)\,ds\,dt \\ &= \int_0^\infty \int_0^s 1 - F(s)\,dt\,ds = \int_0^\infty s(1-F(s))\,ds = E(\xi_i^2/2)\end{aligned}$$

So, if $\nu \equiv E(\xi_i^2) < \infty$, it follows from Lemma 4.4.6, Theorem 4.4.5, and the formula in Example 4.4.3 that

$$0 \leq U(t) - t/\mu \to \nu/2\mu^2 \quad \text{as } t \to \infty$$

When the renewal process is a rate $\lambda$ Poisson process, that is, $P(\xi_i > t) = e^{-\lambda t}$, $N(t) - 1$ has a Poisson distribution with mean $\lambda t$, so $U(t) = 1 + \lambda t$. According to Feller, Vol. II (1971), p. 385, if the $\xi_i$ are uniform on (0,1), then

$$U(t) = \sum_{k=0}^n (-1)^k e^{t-k}(t-k)^k/k! \quad \text{for } n \leq t \leq n+1$$

As he says, the exact expression "reveals little about the nature of $U$. The asymptotic formula $0 \leq U(t) - 2t \to 2/3$ is much more interesting."

**Example 4.4.8. Continuation of Example 4.4.4.** $h(t) = 1 - F(t+x)$. Again, $h$ is decreasing, but this time $h(0) \leq 1$ and the integral of $h$ is finite when $\mu = E(\xi_i) < \infty$. Applying Lemma 4.4.6 and Theorem 4.4.5 now gives

$$P(T_{N(t)} - t > x) \to \frac{1}{\mu}\int_0^\infty h(s)\,ds = \frac{1}{\mu}\int_x^\infty 1 - F(t)\,dt$$

so (when $\mu < \infty$) the distribution of the **residual waiting time** $T_{N(t)} - t$ converges to the delay distribution that produces the stationary renewal process. This fact also follows from our proof of 4.4.3.

Using the method employed to study Example 4.4.4, one can analyze various other aspects of the asymptotic behavior of renewal processes. To avoid repeating ourselves:

*We assume throughout that $F$ is nonarithmetic, and in problems where the mean appears we assume it is finite.*

**Exercise 4.4.5.** Let $A_t = t - T_{N(t)-1}$ be the "age" at time $t$, that is, the amount of time since the last renewal. If we fix $x > 0$, then $H(t) = P(A_t > x)$ satisfies the renewal equation

$$H(t) = (1 - F(t)) \cdot 1_{(x,\infty)}(t) + \int_0^t H(t-s)\,dF(s)$$

so $P(A_t > x) \to \frac{1}{\mu}\int_{(x,\infty)}(1 - F(t))dt$, which is the limit distribution for the residual lifetime $B_t = T_{N(t)} - t$.

**Remark.** The last result can be derived from Example 4.4.4 by noting that if $t > x$, then $P(A_t \geq x) = P(B_{t-x} > x) = P($ no renewal in $(t-x, t])$. To check the placement of the strict inequality, recall that $N_t = \inf\{k : T_k > t\}$, so we always have $A_s \geq 0$ and $B_s > 0$.

**Exercise 4.4.6.** Use the renewal equation in the last problem and Theorem 4.4.4 to conclude that if $T$ is a rate $\lambda$ Poisson process $A_t$ has the same distribution as $\xi_i \wedge t$.

**Exercise 4.4.7.** Let $A_t = t - T_{N(t)-1}$ and $B_t = T_{N(t)} - t$. Show that

$$P(A_t > x, B_t > y) \to \frac{1}{\mu}\int_{x+y}^{\infty}(1 - F(t))\,dt$$

**Exercise 4.4.8. Alternating renewal process.** Let $\xi_1, \xi_2, \ldots > 0$ be i.i.d. with distribution $F_1$, and let $\eta_1, \eta_2, \ldots > 0$ be i.i.d. with distribution $F_2$. Let $T_0 = 0$, and for $k \geq 1$, let $S_k = T_{k-1} + \xi_k$ and $T_k = S_k + \eta_k$. In words, we have a machine that works for an amount of time $\xi_k$, breaks down, and then requires $\eta_k$ units of time to be repaired. Let $F = F_1 * F_2$, and let $H(t)$ be the probability the machine is working at time $t$. Show that if $F$ is nonarithmetic then, as $t \to \infty$

$$H(t) \to \mu_1/(\mu_1 + \mu_2)$$

where $\mu_i$ is the mean of $F_i$.

**Exercise 4.4.9.** Write a renewal equation for $H(t) = P($number of renewals in $[0,t]$ is odd$)$ and use the renewal theorem to show that $H(t) \to 1/2$. Note: This is a special case of the previous exercise.

**Exercise 4.4.10. Renewal densities.** Show that if $F(t)$ has a directly Riemann integrable density function $f(t)$, then the $V = U - 1_{[0,\infty)}$ has a density $v$ that satisfies

$$v(t) = f(t) + \int_0^t v(t-s)\,dF(s)$$

Use the renewal theorem to conclude that if $f$ is directly Riemann integrable, then $v(t) \to 1/\mu$ as $t \to \infty$.

Finally, we have an example that would have been given right after Theorem 4.4.1 but was delayed because we had not yet defined a delayed renewal process.

**Example 4.4.9. Patterns in coin tossing.** Let $X_n, n \geq 1$ take values $H$ and $T$ with probability $1/2$ each. Let $T_0 = 0$ and $T_m = \inf\{n > T_{m-1} : (X_n, \ldots, X_{n+k-1}) = (i_1, \ldots, i_k)\}$, where $(i_1, \ldots, i_k)$ is some pattern of heads and tails. It is easy to see that the $T_j$ form a delayed renewal process, that is, $t_j = T_j - T_{j-1}$ are independent for $j \geq 1$ and identically distributed for $j \geq 2$. To see that the distribution of $t_1$ may be different, let $(i_1, i_2, i_3) = (H, H, H)$. In this case, $P(t_1 = 1) = 1/8$, $P(t_2 = 1) = 1/2$.

**Exercise 4.4.11.**

(i) Show that for any pattern of length $k$, $Et_j = 2^k$ for $j \geq 2$.

(ii) Compute $Et_1$ when the pattern is HH, and when it is HT. Hint: For HH, observe

$$Et_1 = P(HH) + P(HT)E(t_1 + 2) + P(T)E(t_1 + 1)$$

# 5

# Martingales

A martingale $X_n$ can be thought of as the fortune at time $n$ of a player who is betting on a fair game; submartingales (supermartingales) as the outcome of betting on a favorable (unfavorable) game. There are two basic facts about martingales. The first is that you cannot make money betting on them (see Theorem 5.2.5), and in particular if you choose to stop playing at some bounded time $N$, then your expected winnings $EX_N$ are equal to your initial fortune $X_0$. (We are supposing for the moment that $X_0$ is not random.) Our second fact, Theorem 5.2.8, concerns submartingales. To use a heuristic we learned from Mike Brennan, "They are the stochastic analogues of nondecreasing sequences and so if they are bounded above (to be precise, $\sup_n EX_n^+ < \infty$) they converge almost surely." As the material in Section 5.3 shows, this result has diverse applications. Later sections give sufficient conditions for martingales to converge in $L^p$, $p > 1$ (Section 5.4) and in $L^1$ (Section 5.5); consider martingales indexed by $n \le 0$ (Section 5.6); and give sufficient conditions for $EX_N = EX_0$ to hold for unbounded stopping times (Section 5.7). The last result is quite useful for studying the behavior of random walks and other systems.

## 5.1 Conditional Expectation

We begin with a definition that is important for this chapter and the next one. After giving the definition, we will consider several examples to explain it. Given are a probability space $(\Omega, \mathcal{F}_o, P)$, a $\sigma$-field $\mathcal{F} \subset \mathcal{F}_o$, and a random variable $X \in \mathcal{F}_o$ with $E|X| < \infty$. We define the **conditional expectation of $X$ given $\mathcal{F}$**, $E(X|\mathcal{F})$, to be any random variable $Y$ that has

(i) $Y \in \mathcal{F}$, that is, is $\mathcal{F}$ measurable, and
(ii) for all $A \in \mathcal{F}$, $\int_A X\,dP = \int_A Y\,dP$.

Any $Y$ satisfying (i) and (ii) is said to be a **version of** $E(X|\mathcal{F})$. The first thing to be settled is that the conditional expectation exists and is unique. We tackle the second claim first, but start with a technical point.

**Lemma 5.1.1.** *If $Y$ satisfies (i) and (ii), then it is integrable.*

*Proof.* Letting $A = \{Y > 0\} \in \mathcal{F}$, using (ii) twice, and then adding

$$\int_A Y\,dP = \int_A X\,dP \le \int_A |X|\,dP$$
$$\int_{A^c} -Y\,dP = \int_{A^c} -X\,dP \le \int_{A^c} |X|\,dP$$

So we have $E|Y| \le E|X|$. ■

**Uniqueness.** If $Y'$ also satisfies (i) and (ii), then

$$\int_A Y\,dP = \int_A Y'\,dP \quad \text{for all } A \in \mathcal{F}$$

Taking $A = \{Y - Y' \ge \epsilon > 0\}$, we see

$$0 = \int_A X - X\,dP = \int_A Y - Y'\,dP \ge \epsilon P(A)$$

so $P(A) = 0$. Since this holds for all $\epsilon$, we have $Y \le Y'$ a.s., and interchanging the roles of $Y$ and $Y'$, we have $Y = Y'$ a.s. Technically, all equalities such as $Y = E(X|\mathcal{F})$ should be written as $Y = E(X|\mathcal{F})$ a.s., but we have ignored this point in previous chapters and will continue to do so.

**Exercise 5.1.1.** Generalize the last argument to show that if $X_1 = X_2$ on $B \in \mathcal{F}$ then $E(X_1|\mathcal{F}) = E(X_2|\mathcal{F})$ a.s. on $B$.

**Existence.** To start, we recall $\nu$ is said to be **absolutely continuous with respect to** $\mu$ (abbreviated $\nu \ll \mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$, and we use Theorem A.4.6:

**Radon-Nikodym theorem.** Let $\mu$ and $\nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. If $\nu \ll \mu$, there is a function $f \in \mathcal{F}$ so that for all $A \in \mathcal{F}$,

$$\int_A f\,d\mu = \nu(A)$$

$f$ is usually denoted $d\nu/d\mu$ and called the **Radon-Nikodym derivative**.

The last theorem easily gives the existence of conditional expectation. Suppose first that $X \ge 0$. Let $\mu = P$ and

$$\nu(A) = \int_A X\,dP \quad \text{for } A \in \mathcal{F}$$

The dominated convergence theorem implies $\nu$ is a measure (see Exercise 1.5.4), and the definition of the integral implies $\nu \ll \mu$. The Radon- Nikodym derivative $d\nu/d\mu \in \mathcal{F}$ and for any $A \in \mathcal{F}$ has

$$\int_A X\,dP = \nu(A) = \int_A \frac{d\nu}{d\mu}\,dP$$

Taking $A = \Omega$, we see that $d\nu/d\mu \geq 0$ is integrable, and we have shown that $d\nu/d\mu$ is a version of $E(X|\mathcal{F})$.

To treat the general case now, write $X = X^+ - X^-$, let $Y_1 = E(X^+|\mathcal{F})$ and $Y_2 = E(X^-|\mathcal{F})$. Now $Y_1 - Y_2 \in \mathcal{F}$ is integrable, and for all $A \in \mathcal{F}$ we have

$$\begin{aligned}\int_A X\,dP &= \int_A X^+\,dP - \int_A X^-\,dP \\ &= \int_A Y_1\,dP - \int_A Y_2\,dP = \int_A (Y_1 - Y_2)\,dP\end{aligned}$$

This shows $Y_1 - Y_2$ is a version of $E(X|\mathcal{F})$ and completes the proof. ∎

### 5.1.1 *Examples*

Intuitively, we think of $\mathcal{F}$ as describing the information we have at our disposal – for each $A \in \mathcal{F}$, we know whether or not $A$ has occurred. $E(X|\mathcal{F})$ is then our "best guess" of the value of $X$ given the information we have. Some examples should help to clarify this and connect $E(X|\mathcal{F})$ with other definitions of conditional expectation.

**Example 5.1.1.** If $X \in \mathcal{F}$, then $E(X|\mathcal{F}) = X$; that is, if we know $X$, then our "best guess" is $X$ itself. Since $X$ always satisfies (ii), the only thing that can keep $X$ from being $E(X|\mathcal{F})$ is condition (i). A special case of this example is $X = c$, where $c$ is a constant.

**Example 5.1.2.** At the other extreme from perfect information is no information. Suppose $X$ is independent of $\mathcal{F}$, that is, for all $B \in \mathcal{R}$ and $A \in \mathcal{F}$,

$$P(\{X \in B\} \cap A) = P(X \in B)P(A)$$

We claim that, in this case, $E(X|\mathcal{F}) = EX$; that is, if you don't know anything about $X$, then the best guess is the mean $EX$. To check the definition, note that $EX \in \mathcal{F}$ so (i). To verify (ii), we observe that if $A \in \mathcal{F}$, then since $X$ and $1_A \in \mathcal{F}$ are independent, Theorem 2.1.9 implies

$$\int_A X\,dP = E(X1_A) = EX\,E1_A = \int_A EX\,dP$$

The reader should note that here and in what follows the game is "guess and verify." We come up with a formula for the conditional expectation and then check that it satisfies (i) and (ii).

**Example 5.1.3.** In this example, we relate the new definition of conditional expectation to the first one taught in an undergraduate probability course. Suppose $\Omega_1, \Omega_2, \ldots$ is a finite or infinite partition of $\Omega$ into disjoint sets, each of which has positive probability, and let $\mathcal{F} = \sigma(\Omega_1, \Omega_2, \dots)$ be the $\sigma$-field generated by these sets. Then

$$E(X|\mathcal{F}) = \frac{E(X;\Omega_i)}{P(\Omega_i)} \quad \text{on } \Omega_i$$

In words, the information in $\Omega_i$ tells us which element of the partition our outcome lies in, and given this information, the best guess for $X$ is the average value of $X$ over $\Omega_i$. To prove our guess is correct, observe that the proposed formula is constant on each $\Omega_i$, so it is measurable with respect to $\mathcal{F}$. To verify (ii), it is enough to check the equality for $A = \Omega_i$, but this is trivial:

$$\int_{\Omega_i} \frac{E(X;\Omega_i)}{P(\Omega_i)}\, dP = E(X;\Omega_i) = \int_{\Omega_i} X\, dP$$

A degenerate but important special case is $\mathcal{F} = \{\emptyset, \Omega\}$, the trivial $\sigma$-field. In this case, $E(X|\mathcal{F}) = EX$.

To continue the connection with undergraduate notions, let

$$P(A|\mathcal{G}) = E(1_A|\mathcal{G})$$
$$P(A|B) = P(A \cap B)/P(B)$$

and observe that in the last example $P(A|\mathcal{F}) = P(A|\Omega_i)$ on $\Omega_i$.

**Exercise 5.1.2. Bayes' formula.** Let $G \in \mathcal{G}$ and show that

$$P(G|A) = \int_G P(A|\mathcal{G})\, dP \Big/ \int_\Omega P(A|\mathcal{G})\, dP$$

When $\mathcal{G}$ is the $\sigma$-field generated by a partition, this reduces to the usual Bayes' formula

$$P(G_i|A) = P(A|G_i)P(G_i) \Big/ \sum_j P(A|G_j)P(G_j)$$

The definition of conditional expectation given a $\sigma$-field contains conditioning on a random variable as a special case. We define

$$E(X|Y) = E(X|\sigma(Y))$$

where $\sigma(Y)$ is the $\sigma$-field generated by $Y$.

**Example 5.1.4.** To continue making connection with definitions of conditional expectation from undergraduate probability, suppose $X$ and $Y$ have joint density $f(x, y)$, that is,

$$P((X,Y) \in B) = \int_B f(x, y)\, dx\, dy \quad \text{for } B \in \mathcal{R}^2$$

and suppose for simplicity that $\int f(x, y)\,dx > 0$ for all $y$. We claim that in this case, if $E|g(X)| < \infty$ then $E(g(X)|Y) = h(Y)$, where

$$h(y) = \int g(x) f(x, y)\,dx \Big/ \int f(x, y)\,dx$$

To "guess" this formula, note that treating the probability densities $P(Y = y)$ as if they were real probabilities

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f(x, y)}{\int f(x, y)\,dx}$$

so, integrating against the conditional probability density, we have

$$E(g(X)|Y = y) = \int g(x) P(X = x|Y = y)\,dx$$

To "verify" the proposed formula now, observe $h(Y) \in \sigma(Y)$ so (i) holds. To check (ii), observe that if $A \in \sigma(Y)$ then $A = \{\omega : Y(\omega) \in B\}$ for some $B \in \mathcal{R}$, so

$$\begin{aligned} E(h(Y); A) &= \int_B \int h(y) f(x, y)\,dx\,dy = \int_B \int g(x) f(x, y)\,dx\,dy \\ &= E(g(X) 1_B(Y)) = E(g(X); A) \end{aligned}$$

**Remark.** To drop the assumption that $\int f(x, y)\,dx > 0$, define $h$ by

$$h(y) \int f(x, y)\,dx = \int g(x) f(x, y)\,dx$$

(i.e., $h$ can be anything where $\int f(x, y)\,dx = 0$), and observe that this is enough for the proof.

**Example 5.1.5.** Suppose $X$ and $Y$ are independent. Let $\varphi$ be a function with $E|\varphi(X, Y)| < \infty$ and let $g(x) = E(\varphi(x, Y))$. We will now show that

$$E(\varphi(X, Y)|X) = g(X)$$

*Proof.* It is clear that $g(X) \in \sigma(X)$. To check (ii), note that if $A \in \sigma(X)$, then $A = \{X \in C\}$, so using the change of variables formula (Theorem 1.6.9) and the fact that the distribution of $(X, Y)$ is product measure (Theorem 2.1.7), then the definition of $g$, and change of variables again,

$$\begin{aligned} \int_A \phi(X, Y)\,dP &= E\{\phi(X, Y) 1_C(X)\} \\ &= \iint \phi(x, y) 1_C(x)\,\nu(dy)\,\mu(dx) \\ &= \int 1_C(x) g(x)\,\mu(dx) = \int_A g(X)\,dP \end{aligned}$$

which proves the desired result. ■

**Example 5.1.6. Borel's paradox.** Let $X$ be a randomly chosen point on the earth, let $\theta$ be its longitude, and $\varphi$ be its latitude. It is customary to take $\theta \in [0, 2\pi)$ and $\varphi \in (-\pi/2, \pi/2]$ but we can equally well take $\theta \in [0, \pi)$ and $\varphi \in (-\pi, \pi]$. In words, the new longitude specifies the great circle on which the point lies and then $\varphi$ gives the angle.

At first glance it might seem that if $X$ is uniform on the globe, then $\theta$ and the angle $\varphi$ on the great circle should both be uniform over their possible values. $\theta$ is uniform, but $\varphi$ is not. The paradox completely evaporates once we realize that in the new or in the traditional formulation $\varphi$ is independent of $\theta$, so the conditional distribution is the unconditional one, which is not uniform since there is more land near the equator than near the North Pole.

### 5.1.2 *Properties*

Conditional expectation has many of the same properties that ordinary expectation does.

**Theorem 5.1.2.** *(a) Conditional expectation is linear:*

$$E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F}) \tag{5.1.1}$$

*(b) If $X \le Y$, then*

$$E(X|\mathcal{F}) \le E(Y|\mathcal{F}). \tag{5.1.2}$$

*(c) If $X_n \ge 0$ and $X_n \uparrow X$ with $EX < \infty$, then*

$$E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F}) \tag{5.1.3}$$

**Remark.** By applying the last result to $Y_1 - Y_n$, we see that if $Y_n \downarrow Y$ and we have $E|Y_1|, E|Y| < \infty$, then $E(Y_n|\mathcal{F}) \downarrow E(Y|\mathcal{F})$.

*Proof.* To prove (a), we need to check that the right-hand side is a version of the left. It clearly is $\mathcal{F}$-measurable. To check (ii), we observe that if $A \in \mathcal{F}$, then by linearity of the integral and the defining properties of $E(X|\mathcal{F})$ and $E(Y|\mathcal{F})$,

$$\begin{aligned}\int_A \{aE(X|\mathcal{F}) + E(Y|\mathcal{F})\}\,dP &= a\int_A E(X|\mathcal{F})\,dP + \int_A E(Y|\mathcal{F})\,dP \\ &= a\int_A X\,dP + \int_A Y\,dP = \int_A aX + Y\,dP\end{aligned}$$

which proves (5.1.1).

Using the definition

$$\int_A E(X|\mathcal{F})\,dP = \int_A X\,dP \le \int_A Y\,dP = \int_A E(Y|\mathcal{F})\,dP$$

Letting $A = \{E(X|\mathcal{F}) - E(Y|\mathcal{F}) \ge \epsilon > 0\}$, we see that the indicated set has probability 0 for all $\epsilon > 0$, and we have proved (5.1.2).

Let $Y_n = X - X_n$. It suffices to show that $E(Y_n|\mathcal{F}) \downarrow 0$. Since $Y_n \downarrow$, (5.1.2) implies $Z_n \equiv E(Y_n|\mathcal{F}) \downarrow$ a limit $Z_\infty$. If $A \in \mathcal{F}$, then

$$\int_A Z_n \, dP = \int_A Y_n \, dP$$

Letting $n \to \infty$, noting $Y_n \downarrow 0$, and using the dominated convergence theorem gives that $\int_A Z_\infty \, dP = 0$ for all $A \in \mathcal{F}$, so $Z_\infty \equiv 0$. ■

**Exercise 5.1.3.** Prove **Chebyshev's inequality.** If $a > 0$, then

$$P(|X| \geq a|\mathcal{F}) \leq a^{-2} E(X^2|\mathcal{F})$$

**Exercise 5.1.4.** Suppose $X \geq 0$ and $EX = \infty$. (There is nothing to prove when $EX < \infty$.) Show there is a unique $\mathcal{F}$-measurable $Y$ with $0 \leq Y \leq \infty$ so that

$$\int_A X \, dP = \int_A Y \, dP \quad \text{for all } A \in \mathcal{F}$$

Hint: Let $X_M = X \wedge M$, $Y_M = E(X_M|\mathcal{F})$, and let $M \to \infty$.

**Theorem 5.1.3.** *If $\varphi$ is convex and $E|X|$, $E|\varphi(X)| < \infty$, then*

$$\varphi(E(X|\mathcal{F})) \leq E(\varphi(X)|\mathcal{F}) \tag{5.1.4}$$

*Proof.* If $\varphi$ is linear, the result is trivial, so we will suppose $\varphi$ is not linear. We do this so that if we let $S = \{(a,b) : a, b \in \mathbf{Q},\ ax + b \leq \varphi(x) \text{ for all } x\}$, then $\varphi(x) = \sup\{ax + b : (a,b) \in S\}$. See the proof of Theorem 1.6.2 for more details. If $\varphi(x) \geq ax + b$, then (5.1.2) and (5.1.1) imply

$$E(\varphi(X)|\mathcal{F}) \geq a\, E(X|\mathcal{F}) + b \quad \text{a.s.}$$

Taking the sup over $(a,b) \in S$ gives

$$E(\varphi(X)|\mathcal{F}) \geq \varphi(E(X|\mathcal{F})) \quad \text{a.s.}$$

which proves the desired result. ■

**Remark.** Here we have written a.s. by the inequalities to stress that there is an exceptional set for each $a, b$, so we have to take the sup over a countable set.

**Exercise 5.1.5.** Imitate the proof in the remark after Theorem 1.5.2 to prove the conditional Cauchy-Schwarz inequality.

$$E(XY|\mathcal{G})^2 \leq E(X^2|\mathcal{G})E(Y^2|\mathcal{G})$$

**Theorem 5.1.4.** *Conditional expectation is a contraction in $L^p$, $p \geq 1$.*

*Proof.* (5.1.4) implies $|E(X|\mathcal{F})|^p \leq E(|X|^p|\mathcal{F})$. Taking expected values gives

$$E(|E(X|\mathcal{F})|^p) \leq E(E(|X|^p|\mathcal{F})) = E|X|^p$$ ■

In the last equality, we have used an identity that is an immediate consequence of the definition (use property (ii) in the definition with $A = \Omega$).

$$E(E(Y|\mathcal{F})) = E(Y) \tag{5.1.5}$$

Conditional expectation also has properties, like (5.1.5), that have no analogue for "ordinary" expectation.

**Theorem 5.1.5.** *If $\mathcal{F} \subset \mathcal{G}$ and $E(X|\mathcal{G}) \in \mathcal{F}$, then $E(X|\mathcal{F}) = E(X|\mathcal{G})$.*

*Proof.* By assumption, $E(X|\mathcal{G}) \in \mathcal{F}$. To check the other part of the definition, we note that if $A \in \mathcal{F} \subset \mathcal{G}$, then

$$\int_A X\, dP = \int_A E(X|\mathcal{G}\, dP \qquad \blacksquare$$

**Theorem 5.1.6.** *If $\mathcal{F}_1 \subset \mathcal{F}_2$, then (i) $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$, (ii) $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.*

In words, the smaller $\sigma$-field always wins. As the proof will show, the first equality is trivial. The second is easy to prove, but in combination with Theorem 5.1.7 is a powerful tool for computing conditional expectations. I have seen it used several times to prove results that are false.

*Proof.* Once we notice that $E(X|\mathcal{F}_1) \in \mathcal{F}_2$, (i) follows from Example 5.1.1. To prove (ii), notice that $E(X|\mathcal{F}_1) \in \mathcal{F}_1$, and if $A \in \mathcal{F}_1 \subset \mathcal{F}_2$, then

$$\int_A E(X|\mathcal{F}_1)\, dP = \int_A X\, dP = \int_A E(X|\mathcal{F}_2)\, dP \qquad \blacksquare$$

**Exercise 5.1.6.** Give an example on $\Omega = \{a, b, c\}$ in which

$$E(E(X|\mathcal{F}_1)|\mathcal{F}_2) \neq E(E(X|\mathcal{F}_2)|\mathcal{F}_1)$$

The next result shows that for conditional expectation with respect to $\mathcal{F}$, random variables $X \in \mathcal{F}$ are like constants. They can be brought outside the "integral."

**Theorem 5.1.7.** *If $X \in \mathcal{F}$ and $E|Y|$, $E|XY| < \infty$, then*

$$E(XY|\mathcal{F}) = XE(Y|\mathcal{F}).$$

*Proof.* The right-hand side $\in \mathcal{F}$, so we have to check (ii). To do this, we use the usual four-step procedure. First, suppose $X = 1_B$ with $B \in \mathcal{F}$. In this case, if $A \in \mathcal{F}$,

$$\int_A 1_B E(Y|\mathcal{F})\, dP = \int_{A\cap B} E(Y|\mathcal{F})\, dP = \int_{A\cap B} Y\, dP = \int_A 1_B Y\, dP$$

so (ii) holds. The last result extends to simple $X$ by linearity. If $X, Y \geq 0$, let $X_n$ be simple random variables that $\uparrow X$, and use the monotone convergence theorem to conclude that

$$\int_A X E(Y|\mathcal{F})\, dP = \int_A XY\, dP$$

To prove the result in general, split $X$ and $Y$ into their positive and negative parts. ■

**Exercise 5.1.7.** Show that when $E|X|$, $E|Y|$, and $E|XY|$ are finite, each statement implies the next one, and give examples with $X, Y \in \{-1, 0, 1\}$ a.s. that show the reverse implications are false: (i) $X$ and $Y$ are independent, (ii) $E(Y|X) = EY$, (iii) $E(XY) = EXEY$.

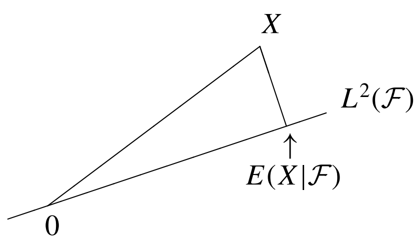



Figure 5.1. Conditional expectation as projection in $L^2$.

**Theorem 5.1.8.** *Suppose $EX^2 < \infty$. $E(X|\mathcal{F})$ is the variable $Y \in \mathcal{F}$ that minimizes the "mean square error" $E(X - Y)^2$.*

**Remark.** This result gives a "geometric interpretation" of $E(X|\mathcal{F})$ (see Figure 5.1). $L^2(\mathcal{F}_o) = \{Y \in \mathcal{F}_o : EY^2 < \infty\}$ is a Hilbert space, and $L^2(\mathcal{F})$ is a closed subspace. In this case, $E(X|\mathcal{F})$ is the projection of $X$ onto $L^2(\mathcal{F})$. That is, the point in the subspace closest to $X$.

*Proof.* We begin by observing that if $Z \in L^2(\mathcal{F})$, then Theorem 5.1.7 implies

$$ZE(X|\mathcal{F}) = E(ZX|\mathcal{F})$$

($E|XZ| < \infty$ by the Cauchy-Schwarz inequality.) Taking expected values gives

$$E(ZE(X|\mathcal{F})) = E(E(ZX|\mathcal{F})) = E(ZX)$$

or, rearranging,

$$E[Z(X - E(X|\mathcal{F}))] = 0 \quad \text{for } Z \in L^2(\mathcal{F})$$

If $Y \in L^2(\mathcal{F})$ and $Z = E(X|\mathcal{F}) - Y$, then

$$E(X-Y)^2 = E\{X - E(X|\mathcal{F}) + Z\}^2 = E\{X - E(X|\mathcal{F})\}^2 + EZ^2$$

since the cross-product term vanishes. From the last formula, it is easy to see $E(X-Y)^2$ is minimized when $Z=0$. ■

**Exercise 5.1.8.** Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 < \infty$, then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2)$$

Dropping the second term on the left, we get an inequality that says geometrically, the larger the subspace, the closer the projection is, or statistically, more information means a smaller mean square error. An important special case occurs when $\mathcal{G} = \{\emptyset, \Omega\}$.

**Exercise 5.1.9.** Let $\operatorname{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\operatorname{var}(X) = E(\operatorname{var}(X|\mathcal{F})) + \operatorname{var}(E(X|\mathcal{F}))$$

**Exercise 5.1.10.** Let $Y_1, Y_2, \ldots$ be i.i.d. with mean $\mu$ and variance $\sigma^2$, $N$ an independent positive integer valued r.v. with $EN^2 < \infty$ and $X = Y_1 + \cdots + Y_N$. Show that $\operatorname{var}(X) = \sigma^2 EN + \mu^2 \operatorname{var}(N)$. To understand and help remember the formula, think about the two special cases in which $N$ or $Y$ is constant.

**Exercise 5.1.11.** Show that if $X$ and $Y$ are random variables with $E(Y|\mathcal{G}) = X$ and $EY^2 = EX^2 < \infty$, then $X = Y$ a.s.

**Exercise 5.1.12.** The result in the last exercise implies that if $EY^2 < \infty$ and $E(Y|\mathcal{G})$ has the same distribution as $Y$, then $E(Y|\mathcal{G}) = Y$ a.s. Prove this under the assumption $E|Y| < \infty$. Hint: The trick is to prove that $\operatorname{sgn}(X) = \operatorname{sgn}(E(X|\mathcal{G}))$ a.s., and then take $X = Y - c$ to get the desired result.

### 5.1.3 *Regular Conditional Probabilities**

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ a measurable map, and $\mathcal{G}$ a $\sigma$-field $\subset \mathcal{F}$. $\mu : \Omega \times \mathcal{S} \to [0,1]$ is said to be a **regular conditional distribution** for $X$ given $\mathcal{G}$ if

(i) For each $A$, $\omega \to \mu(\omega, A)$ is a version of $P(X \in A|\mathcal{G})$.
(ii) For a.e. $\omega$, $A \to \mu(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.

When $S = \Omega$ and $X$ is the identity map, $\mu$ is called a **regular conditional probability**.

**Exercise 5.1.13. Continuation of Example 1.4.** Suppose $X$ and $Y$ have a joint density $f(x,y) > 0$. Let

$$\mu(y, A) = \int_A f(x,y)\,dx \Big/ \int f(x,y)\,dx$$

Show that $\mu(Y(\omega), A)$ is a r.c.d. for $X$ given $\sigma(Y)$.

Regular conditional distributions are useful because they allow us to simultaneously compute the conditional expectation of all functions of $X$ and to generalize properties of ordinary expectation in a more straightforward way.

**Exercise 5.1.14.** Let $\mu(\omega, A)$ be a r.c.d. for $X$ given $\mathcal{F}$, and let $f : (S, \mathcal{S}) \to (\mathbf{R}, \mathcal{R})$ have $E|f(X)| < \infty$. Start with simple functions and show that

$$E(f(X)|\mathcal{F}) = \int \mu(\omega, dx) f(x) \quad \text{a.s.}$$

**Exercise 5.1.15.** Use regular conditional probability to get the conditional Hölder inequality from the unconditional one, that is, show that if $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ then

$$E(|XY||\mathcal{G}) \le E(|X|^p|\mathcal{G})^{1/p} E(|Y|^q|\mathcal{G})^{1/q}$$

Unfortunately, r.c.d.'s do not always exist. The first example was due to Dieudonné (1948). See Doob (1953), p. 624, or Faden (1985) for more recent developments. Without going into the details of the example, it is easy to see the source of the problem. If $A_1, A_2, \ldots$ are disjoint, then (5.1.1) and (5.1.3) imply

$$P(X \in \cup_n A_n|\mathcal{G}) = \sum_n P(X \in A_n|\mathcal{G}) \quad \text{a.s.}$$

but if $\mathcal{S}$ contains enough countable collections of disjoint sets, the exceptional sets may pile up. Fortunately,

**Theorem 5.1.9.** *r.c.d.'s exist if $(S, \mathcal{S})$ is nice.*

*Proof.* By definition, there is a 1-1 map $\varphi : S \to \mathbf{R}$ so that $\varphi$ and $\varphi^{-1}$ are measurable. Using monotonicity (5.1.2) and throwing away a countable collection of null sets, we find there is a set $\Omega_o$ with $P(\Omega_o) = 1$ and a family of random variables $G(q, \omega)$, $q \in \mathbf{Q}$ so that $q \to G(q, \omega)$ is nondecreasing and $\omega \to G(q, \omega)$ is a version of $P(\varphi(X) \le q|\mathcal{G})$. Let $F(x, \omega) = \inf\{G(q, \omega) : q > x\}$. The notation may remind the reader of the proof of Theorem 3.2.6. The argument given there shows $F$ is a distribution function. Since $G(q_n, \omega) \downarrow F(x, \omega)$, the remark after Theorem 5.1.2 implies that $F(x, \omega)$ is a version of $P(\varphi(X) \le x|\mathcal{G})$.

Now, for each $\omega \in \Omega_o$, there is a unique measure $\nu(\omega, \cdot)$ on $(\mathbf{R}, \mathcal{R})$ so that $\nu(\omega, (-\infty, x]) = F(x, \omega)$. To check that for each $B \in \mathcal{R}$, $\nu(\omega, B)$ is a version of $P(\varphi(X) \in B|\mathcal{G})$, we observe that the class of $B$ for which this statement is true (this includes the measurability of $\omega \to \nu(\omega, B)$) is a $\lambda$-system that contains all sets of the form $(a_1, b_1] \cup \cdots (a_k, b_k]$ where $-\infty \le a_i < b_i \le \infty$, so the desired result follows from the $\pi - \lambda$ theorem. To extract the desired r.c.d., notice that if $A \in \mathcal{S}$ and $B = \varphi(A)$, then $B = (\varphi^{-1})^{-1}(A) \in \mathcal{R}$, and set $\mu(\omega, A) = \nu(\omega, B)$. ■

The following generalization of Theorem 5.1.9 will be needed in Section 6.1.

**Exercise 5.1.16.** Suppose $X$ and $Y$ take values in a nice space $(S, \mathcal{S})$ and $\mathcal{G} = \sigma(Y)$. There is a function $\mu : S \times \mathcal{S} \to [0, 1]$ so that
(i) for each A, $\mu(Y(\omega), A)$ is a version of $P(X \in A|\mathcal{G})$
(ii) for a.e. $\omega$, $A \to \mu(Y(\omega), A)$ is a probability measure on $(S, \mathcal{S})$.

## 5.2 Martingales, Almost Sure Convergence

In this section we will define martingales and their cousins supermartingales and submartingales, and take the first steps in developing their theory. Let $\mathcal{F}_n$ be a **filtration**, that is, an increasing sequence of $\sigma$-fields. A sequence $X_n$ is said to be **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$ for all $n$. If $X_n$ is sequence with

(i) $E|X_n| < \infty$,
(ii) $X_n$ is adapted to $\mathcal{F}_n$,
(iii) $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all $n$,

then $X$ is said to be a **martingale** (with respect to $\mathcal{F}_n$). If in the last definition, $=$ is replaced by $\le$ or $\ge$, then $X$ is said to be a **supermartingale** or **submartingale**, respectively.

**Example 5.2.1. Simple random walk.** Consider the successive tosses of a fair coin and let $\xi_n = 1$ if the $n$th toss is heads and $\xi_n = -1$ if the $n$th toss is tails. Let $X_n = \xi_1 + \cdots + \xi_n$ and $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ for $n \ge 1$, $X_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. I claim that $X_n, n \ge 0$, is a martingale with respect to $\mathcal{F}_n$. To prove this, we observe that $X_n \in \mathcal{F}_n$, $E|X_n| < \infty$, and $\xi_{n+1}$ is independent of $\mathcal{F}_n$, so using the linearity of conditional expectation, (5.1.1), and Example 5.1.2,

$$E(X_{n+1}|\mathcal{F}_n) = E(X_n|\mathcal{F}_n) + E(\xi_{n+1}|\mathcal{F}_n) = X_n + E\xi_{n+1} = X_n$$

Note that, in this example, $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and $\mathcal{F}_n$ is the smallest filtration that $X_n$ is adapted to. In what follows, when the filtration is not mentioned, we will take $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

**Exercise 5.2.1.** Suppose $X_n$ is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and $X_n$ is a martingale w.r.t. $\mathcal{F}_n$.

**Example 5.2.2. Superharmonic functions.** If the coin tosses considered above have $P(\xi_n = 1) \le 1/2$ then the computation just completed shows $E(X_{n+1}|\mathcal{F}_n) \le X_n$, i.e., $X_n$ is a supermartingale. In this case, $X_n$ corresponds to betting on an unfavorable game so there is nothing “super” about a supermartingale. The name comes from the fact that if $f$ is superharmonic (i.e., $f$ has continuous derivatives of order $\le 2$ and $\partial^2 f/\partial x_1^2 + \cdots + \partial^2 f/\partial x_d^2 \le 0$), then

$$f(x) \ge \frac{1}{|B(0, r)|} \int_{B(x,r)} f(y)\, dy$$

where $B(x, r) = \{y : |x - y| \le r\}$ is the ball of radius $r$, and $|B(0, r)|$ is the volume of the ball of radius $r$.

**Exercise 5.2.2.** Suppose $f$ is superharmonic on $\mathbf{R}^d$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniform on $B(0, 1)$, and define $S_n$ by $S_n = S_{n-1} + \xi_n$ for $n \ge 1$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Our first result is an immediate consequence of the definition of a supermartingale. We could take the conclusion of the result as the definition of supermartingale, but then the definition would be harder to check.

**Theorem 5.2.1.** *If $X_n$ is a supermartingale then for $n > m$, $E(X_n|\mathcal{F}_m) \le X_m$.*

*Proof.* The definition gives the result for $n = m + 1$. Suppose $n = m + k$ with $k \ge 2$. By Theorem 5.1.2,

$$E(X_{m+k}|\mathcal{F}_m) = E(E(X_{m+k}|\mathcal{F}_{m+k-1})|\mathcal{F}_m) \le E(X_{m+k-1}|\mathcal{F}_m)$$

by the definition and (5.1.2). The desired result now follows by induction. ■

**Theorem 5.2.2.** *(i) If $X_n$ is a submartingale, then for $n > m$, $E(X_n|\mathcal{F}_m) \ge X_m$. (ii) If $X_n$ is a martingale then for $n > m$, $E(X_n|\mathcal{F}_m) = X_m$.*

*Proof.* To prove (i), note that $-X_n$ is a supermartingale and use (5.1.1). For (ii), observe that $X_n$ is a supermartingale and a submartingale. ■

**Remark.** The idea in the proof of Theorem 5.2.2 can be used many times below. To keep from repeating ourselves, we will just state the result for either supermartingales or submartingales and leave it to the reader to translate the result for the other two.

**Theorem 5.2.3.** *If $X_n$ is a martingale w.r.t. $\mathcal{F}_n$ and $\varphi$ is a convex function with $E|\varphi(X_n)| < \infty$ for all $n$, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently, if $p \ge 1$ and $E|X_n|^p < \infty$ for all $n$, then $|X_n|^p$ is a submartingale w.r.t. $\mathcal{F}_n$.*

*Proof.* By Jensen's inequality and the definition,

$$E(\varphi(X_{n+1})|\mathcal{F}_n) \ge \varphi(E(X_{n+1}|\mathcal{F}_n)) = \varphi(X_n)$$ ■

**Theorem 5.2.4.** *If $X_n$ is a submartingale w.r.t. $\mathcal{F}_n$ and $\varphi$ is an increasing convex function with $E|\varphi(X_n)| < \infty$ for all $n$, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently (i) If $X_n$ is a submartingale, then $(X_n - a)^+$ is a submartingale. (ii) If $X_n$ is a supermartingale, then $X_n \wedge a$ is a supermartingale.*

*Proof.* By Jensen's inequality and the assumptions,

$$E(\varphi(X_{n+1})|\mathcal{F}_n) \geq \varphi(E(X_{n+1}|\mathcal{F}_n)) \geq \varphi(X_n) \qquad \blacksquare$$

**Exercise 5.2.3.** Give an example of a submartingale $X_n$ so that $X_n^2$ is a supermartingale. Hint: $X_n$ does not have to be random.

Let $\mathcal{F}_n$, $n \geq 0$ be a filtration. $H_n$, $n \geq 1$ is said to be a **predictable sequence** if $H_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$. In words, the value of $H_n$ may be predicted (with certainty) from the information available at time $n-1$. In this section, we will be thinking of $H_n$ as the amount of money a gambler will bet at time $n$. This can be based on the outcomes at times $1, \ldots, n-1$, but not on the outcome at time $n$!

Once we start thinking of $H_n$ as a gambling system, it is natural to ask how much money we would make if we used it. For concreteness, let us suppose that the game consists of flipping a coin and that for each dollar you bet, you win one dollar when the coin comes up heads and lose your dollar when the coin comes up tails. Let $X_n$ be the net amount of money you would have won at time $n$ if you had bet one dollar each time. If you bet according to a gambling system $H$, then your winnings at time $n$ would be

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$$

since $X_m - X_{m-1} = +1$ or $-1$ when the $m$th toss results in a win or loss, respectively.

Let $\xi_m = X_m - X_{m-1}$. A famous gambling system called the "martingale" is defined by $H_1 = 1$ and for $n \geq 2$, $H_n = 2H_{n-1}$ if $\xi_{n-1} = -1$ and $H_n = 1$ if $\xi_{n-1} = 1$. In words, we double our bet when we lose, so that if we lose $k$ times and then win, our net winnings will be $-1 - 2 \ldots - 2^{k-1} + 2^k = 1$. This system seems to provide us with a "sure thing" as long as $P(\xi_m = 1) > 0$. However, the next result says there is no system for beating an unfavorable game.

**Theorem 5.2.5.** *Let $X_n$, $n \geq 0$, be a supermartingale. If $H_n \geq 0$ is predictable and each $H_n$ is bounded then $(H \cdot X)_n$ is a supermartingale.*

*Proof.* Using the fact that conditional expectation is linear, $(H \cdot X)_n \in \mathcal{F}_n$, $H_n \in \mathcal{F}_{n-1}$, and (5.1.7), we have

$$\begin{aligned} E((H \cdot X)_{n+1}|\mathcal{F}_n) &= (H \cdot X)_n + E(H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n) \\ &= (H \cdot X)_n + H_{n+1}E((X_{n+1} - X_n)|\mathcal{F}_n) \leq (H \cdot X)_n \end{aligned}$$

since $E((X_{n+1} - X_n)|\mathcal{F}_n) \leq 0$ and $H_{n+1} \geq 0$. $\blacksquare$

**Remark.** The same result is obviously true for submartingales and for martingales (in the last case, without the restriction $H_n \geq 0$).

The notion of a stopping time, introduced in Section 4.1, is closely related to the concept of a gambling system. Recall that a random variable $N$ is said to be a **stopping time** if $\{N = n\} \in \mathcal{F}_n$ for all $n < \infty$. If you think of $N$ as the time a gambler stops gambling, then the condition above says that the decision to stop at time $n$ must be measurable with respect to the information he has at that time. If we let $H_n = 1_{\{N \ge n\}}$, then $\{N \ge n\} = \{N \le n-1\}^c \in \mathcal{F}_{n-1}$, so $H_n$ is predictable, and it follows from Theorem 5.2.5 that $(H \cdot X)_n = X_{N\wedge n} - X_0$ is a supermartingale. Since the constant sequence $Y_n = X_0$ is a supermartingale and the sum of two supermartingales is also, we have:

**Theorem 5.2.6.** *If $N$ is a stopping time and $X_n$ is a supermartingale, then $X_{N\wedge n}$ is a supermartingale.*

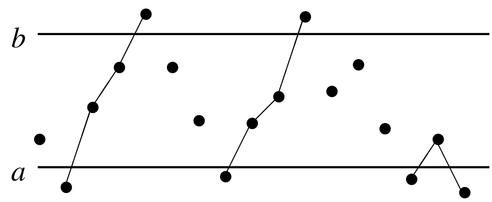



Figure 5.2. Upcrossings of $(a, b)$. Lines indicate increments that are included in $(H \cdot X)_n$. In $Y_n$ the points $< a$ are moved up to $a$.

Although you cannot make money with gambling systems, you can prove theorems with them. Suppose $X_n$, $n \ge 0$, is a submartingale. Let $a < b$, let $N_0 = -1$, and for $k \ge 1$ let (see Figure 5.2 for a picture)

$$N_{2k-1} = \inf\{m > N_{2k-2} : X_m \le a\}$$
$$N_{2k} = \inf\{m > N_{2k-1} : X_m \ge b\}$$

The $N_j$ are stopping times, and $\{N_{2k-1} < m \le N_{2k}\} = \{N_{2k-1} \le m-1\} \cap \{N_{2k} \le m-1\}^c \in \mathcal{F}_{m-1}$, so

$$H_m = \begin{cases} 1 & \text{if } N_{2k-1} < m \le N_{2k} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

defines a predictable sequence. $X(N_{2k-1}) \le a$ and $X(N_{2k}) \ge b$, so between times $N_{2k-1}$ and $N_{2k}$, $X_m$ crosses from below $a$ to above $b$. $H_m$ is a gambling system that tries to take advantage of these "upcrossings." In stock market terms, we buy when $X_m \le a$ and sell when $X_m \ge b$, so every time an upcrossing is completed, we make a profit of $\ge (b-a)$. Finally, $U_n = \sup\{k : N_{2k} \le n\}$ is the number of upcrossings completed by time $n$.

**Theorem 5.2.7. Upcrossing inequality.** *If $X_m$, $m \ge 0$, is a submartingale, then*

$$(b-a)EU_n \le E(X_n - a)^+ - E(X_0 - a)^+$$

*Proof.* Let $Y_m = a + (X_m - a)^+$. By Theorem 5.2.4, $Y_m$ is a submartingale. Clearly, it upcrosses $[a, b]$ the same number of times that $X_m$ does, and we have $(b - a)U_n \le (H \cdot Y)_n$, since each upcrossing results in a profit $\ge (b - a)$, and a final incomplete upcrossing (if there is one) makes a nonnegative contribution to the right-hand side. Let $K_m = 1 - H_m$. Clearly, $Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n$, and it follows from Theorem 5.2.5 that $E(K \cdot Y)_n \ge E(K \cdot Y)_0 = 0$, so $E(H \cdot Y)_n \le E(Y_n - Y_0)$, proving the desired inequality. ■

We have proved the result in its classical form, even though this is a little misleading. The key fact is that $E(K \cdot Y)_n \ge 0$, that is, no matter how hard you try, you can't lose money betting on a submartingale. From the upcrossing inequality, we easily get

**Theorem 5.2.8. Martingale convergence theorem.** *If $X_n$ is a submartingale with $\sup EX_n^+ < \infty$, then as $n \to \infty$, $X_n$ converges a.s. to a limit $X$ with $E|X| < \infty$.*

*Proof.* Since $(X - a)^+ \le X^+ + |a|$, Theorem 5.2.7 implies that

$$EU_n \le (|a| + EX_n^+)/(b - a)$$

As $n \uparrow \infty$, $U_n \uparrow U$ the number of upcrossings of $[a, b]$ by the whole sequence, so if $\sup EX_n^+ < \infty$, then $EU < \infty$ and hence $U < \infty$ a.s. Since the last conclusion holds for all rational $a$ and $b$,

$$\cup_{a,b\in\mathbf{Q}}\{\liminf X_n < a < b < \limsup X_n\} \quad \text{has probability } 0$$

and hence $\limsup X_n = \liminf X_n$ a.s., that is, $\lim X_n$ exists a.s. Fatou's lemma guarantees $EX^+ \le \liminf EX_n^+ < \infty$, so $X < \infty$ a.s. To see $X > -\infty$, we observe that

$$EX_n^- = EX_n^+ - EX_n \le EX_n^+ - EX_0$$

(since $X_n$ is a submartingale), so another application of Fatou's lemma shows

$$EX^- \le \liminf_{n\to\infty} EX_n^- \le \sup_n EX_n^+ - EX_0 < \infty$$

and completes the proof. ■

**Remark.** To prepare for the proof of Theorem 5.6.1, the reader should note that we have shown that if the number of upcrossings of $(a, b)$ by $X_n$ is finite for all $a, b \in \mathbf{Q}$, then the limit of $X_n$ exists.

An important special case of Theorem 5.2.8 is

**Theorem 5.2.9.** *If $X_n \ge 0$ is a supermartingale, then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.*

*Proof.* $Y_n = -X_n \le 0$ is a submartingale with $EY_n^+ = 0$. Since $EX_0 \ge EX_n$, the inequality follows from Fatou's lemma. ∎

In the next section, we will give several applications of the last two results. We close this one by giving two "counterexamples."

**Example 5.2.3.** The first shows that the assumptions of Theorem 5.2.9 (or 5.2.8) do not guarantee convergence in $L^1$. Let $S_n$ be a symmetric simple random walk with $S_0 = 1$, that is, $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots$ are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Let $N = \inf\{n : S_n = 0\}$ and let $X_n = S_{N \wedge n}$. Theorem 5.2.6 implies that $X_n$ is a nonnegative martingale. Theorem 5.2.9 implies $X_n$ converges to a limit $X_\infty < \infty$ that must be $\equiv 0$, since convergence to $k > 0$ is impossible. (If $X_n = k > 0$, then $X_{n+1} = k \pm 1$.) Since $EX_n = EX_0 = 1$ for all $n$ and $X_\infty = 0$, convergence cannot occur in $L^1$.

Example 5.2.3 is an important counterexample to keep in mind as you read the rest of this chapter. The next two are not as important.

**Example 5.2.4.** We will now give an example of a martingale with $X_k \to 0$ in probability but not a.s. Let $X_0 = 0$. When $X_{k-1} = 0$, let $X_k = 1$ or $-1$ with probability $1/2k$ and $= 0$ with probability $1 - 1/k$. When $X_{k-1} \ne 0$, let $X_k = kX_{k-1}$ with probability $1/k$ and $= 0$ with probability $1 - 1/k$. From the construction, $P(X_k = 0) = 1 - 1/k$, so $X_k \to 0$ in probability. On the other hand, the second Borel-Cantelli lemma implies $P(X_k = 0 \text{ for } k \ge K) = 0$, and values in $(-1, 1) - \{0\}$ are impossible, so $X_k$ does not converge to 0 a.s.

**Exercise 5.2.4.** Give an example of a martingale $X_n$ with $X_n \to -\infty$ a.s. Hint: Let $X_n = \xi_1 + \cdots + \xi_n$, where the $\xi_i$ are independent (but not identically distributed) with $E\xi_i = 0$.

Our final result is useful in reducing questions about submartingales to questions about martingales.

**Theorem 5.2.10. Doob's decomposition.** *Any submartingale $X_n$, $n \ge 0$, can be written in a unique way as $X_n = M_n + A_n$, where $M_n$ is a martingale and $A_n$ is a predictable increasing sequence with $A_0 = 0$.*

*Proof.* We want $X_n = M_n + A_n$, $E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$, and $A_n \in \mathcal{F}_{n-1}$. So we must have

$$\begin{aligned} E(X_n|\mathcal{F}_{n-1}) &= E(M_n|\mathcal{F}_{n-1}) + E(A_n|\mathcal{F}_{n-1}) \\ &= M_{n-1} + A_n = X_{n-1} - A_{n-1} + A_n \end{aligned}$$

and it follows that

(a) $A_n - A_{n-1} = E(X_n|\mathcal{F}_{n-1}) - X_{n-1}$
(b) $M_n = X_n - A_n$

Now $A_0 = 0$ and $M_0 = X_0$ by assumption, so we have $A_n$ and $M_n$ defined for all time, and we have proved uniqueness. To check that our recipe works, we observe that $A_n - A_{n-1} \geq 0$ since $X_n$ is a submartingale and induction shows $A_n \in \mathcal{F}_{n-1}$. To see that $M_n$ is a martingale, we use (b), $A_n \in \mathcal{F}_{n-1}$ and (a):

$$\begin{aligned} E(M_n|\mathcal{F}_{n-1}) &= E(X_n - A_n|\mathcal{F}_{n-1}) \\ &= E(X_n|\mathcal{F}_{n-1}) - A_n = X_{n-1} - A_{n-1} = M_{n-1} \end{aligned}$$

which completes the proof. ■

**Exercise 5.2.5.** Let $X_n = \sum_{m \leq n} 1_{B_m}$ and suppose $B_n \in \mathcal{F}_n$. What is the Doob decomposition for $X_n$?

### *Exercises*

**5.2.6.** Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_i = 0$ and $\text{var}\,(\xi_m) = \sigma_m^2 < \infty$, and let $s_n^2 = \sum_{m=1}^n \sigma_m^2$. Then $S_n^2 - s_n^2$ is a martingale.

**5.2.7.** If $\xi_1, \xi_2, \ldots$ are independent and have $E\xi_i = 0$, then

$$X_n^{(k)} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \xi_{i_1} \cdots \xi_{i_k}$$

is a martingale. When $k = 2$ and $S_n = \xi_1 + \cdots + \xi_n$, $2X_n^{(2)} = S_n^2 - \sum_{m \leq n} \xi_m^2$.

**5.2.8.** Generalize (i) of Theorem 5.2.4 by showing that if $X_n$ and $Y_n$ are submartingales w.r.t. $\mathcal{F}_n$ then $X_n \vee Y_n$ is also.

**5.2.9.** Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. (i) Show that $X_n = \prod_{m \leq n} Y_m$ defines a martingale. (ii) Use Theorem 5.2.9 and an argument by contradiction to show $X_n \to 0$ a.s. (iii) Use the strong law of large numbers to conclude $(1/n) \log X_n \to c < 0$.

**5.2.10.** Suppose $y_n > -1$ for all $n$ and $\sum |y_n| < \infty$. Show that $\prod_{m=1}^\infty (1 + y_m)$ exists.

**5.2.11.** Let $X_n$ and $Y_n$ be positive integrable and adapted to $\mathcal{F}_n$. Suppose

$$E(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n) X_n$$

with $\sum Y_n < \infty$ a.s. Prove that $X_n$ converges a.s. to a finite limit by finding a closely related supermartingale to which Theorem 5.2.9 can be applied.

**5.2.12.** Use the random walks in Exercise 5.2.2 to conclude that in $d \le 2$, nonnegative superharmonic functions must be constant. The example $f(x) = |x|^{2-d}$ shows this is false in $d > 2$.

**5.2.13. The switching principle.** Suppose $X_n^1$ and $X_n^2$ are supermartingales with respect to $\mathcal{F}_n$, and $N$ is a stopping time so that $X_N^1 \ge X_N^2$. Then

$$Y_n = X_n^1 1_{(N>n)} + X_n^2 1_{(N\le n)} \text{ is a supermartingale.}$$
$$Z_n = X_n^1 1_{(N\ge n)} + X_n^2 1_{(N<n)} \text{ is a supermartingale.}$$

**5.2.14. Dubins' inequality.** For every positive supermartingale $X_n$, $n \ge 0$, the number of upcrossings $U$ of $[a, b]$ satisfies

$$P(U \ge k) \le \left(\frac{a}{b}\right)^k E\min(X_0/a, 1)$$

To prove this, we let $N_0 = -1$ and for $j \ge 1$, let

$$N_{2j-1} = \inf\{m > N_{2j-2} : X_m \le a\}$$
$$N_{2j} = \inf\{m > N_{2j-1} : X_m \ge b\}$$

Let $Y_n = 1$ for $0 \le n < N_1$ and for $j \ge 1$,

$$Y_n = \begin{cases} (b/a)^{j-1}(X_n/a) & \text{for } N_{2j-1} \le n < N_{2j} \\ (b/a)^j & \text{for } N_{2j} \le n < N_{2j+1} \end{cases}$$

(i) Use the switching principle in the previous exercise and induction to show that $Z_n^j = Y_{n\wedge N_j}$ is a supermartingale. (ii) Use $EY_{n\wedge N_{2k}} \le EY_0$ and let $n \to \infty$ to get Dubins' inequality.

## 5.3 Examples

In this section, we will apply the martingale convergence theorem to generalize the second Borel-Cantelli lemma and to study Polya's urn scheme, Radon Nikodym derivatives, and branching processes. The four topics are independent of each other and are taken up in the order indicated.

### 5.3.1 *Bounded Increments*

Our first result shows that martingales with bounded increments either converge or oscillate between $+\infty$ and $-\infty$.

**Theorem 5.3.1.** *Let $X_1, X_2, \ldots$ be a martingale with $|X_{n+1} - X_n| \le M < \infty$. Let*

$$C = \{\lim X_n \text{ exists and is finite}\}$$
$$D = \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}$$

*Then $P(C \cup D) = 1$.*

*Proof.* Since $X_n - X_0$ is a martingale, we can without loss of generality suppose that $X_0 = 0$. Let $0 < K < \infty$ and let $N = \inf\{n : X_n \le -K\}$. $X_{n\wedge N}$ is a martingale with $X_{n\wedge N} \ge -K - M$ a.s. so applying Theorem 5.2.9 to $X_{n\wedge N} + K + M$ shows $\lim X_n$ exists on $\{N = \infty\}$. Letting $K \to \infty$, we see that the limit exists on $\{\liminf X_n > -\infty\}$. Applying the last conclusion to $-X_n$, we see that $\lim X_n$ exists on $\{\limsup X_n < \infty\}$ and the proof is complete. ∎

**Exercise 5.3.1.** Let $X_n$, $n \ge 0$, be a submartingale with $\sup X_n < \infty$. Let $\xi_n = X_n - X_{n-1}$, and suppose $E(\sup \xi_n^+) < \infty$. Show that $X_n$ converges a.s.

**Exercise 5.3.2.** Give an example of a martingale $X_n$ with $\sup_n |X_n| < \infty$ and $P(X_n = a \text{ i.o.}) = 1$ for $a = -1, 0, 1$. This example shows that it is not enough to have $\sup |X_{n+1} - X_n| < \infty$ in Theorem 5.3.1.

**Exercise 5.3.3.** (Assumes familiarity with finite state Markov chains.) Fine tune the example for the previous problem so that $P(X_n = 0) \to 1 - 2p$ and $P(X_n = -1)$, $P(X_n = 1) \to p$, where $p$ is your favorite number in $(0, 1)$, that is, you are asked to do this for one value of $p$ that you may choose. This example shows that a martingale can converge in distribution without converging a.s. (or in probability).

**Exercise 5.3.4.** Let $X_n$ and $Y_n$ be positive integrable and adapted to $\mathcal{F}_n$. Suppose $E(X_{n+1}|\mathcal{F}_n) \le X_n + Y_n$, with $\sum Y_n < \infty$ a.s. Prove that $X_n$ converges a.s. to a finite limit. Hint: Let $N = \inf_k \sum_{m=1}^k Y_m > M$, and stop your supermartingale at time $N$.

**Theorem 5.3.2. Second Borel-Cantelli lemma, II.** *Let $\mathcal{F}_n$, $n \ge 0$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $A_n$, $n \ge 1$ a sequence of events with $A_n \in \mathcal{F}_n$. Then*

$$\{A_n \ i.o.\} = \left\{ \sum_{n=1}^{\infty} P(A_n|\mathcal{F}_{n-1}) = \infty \right\}$$

*Proof.* If we let $X_0 = 0$ and $X_n = \sum_{m=1}^n 1_{A_m} - P(A_m|\mathcal{F}_{m-1})$ for $n \ge 1$, then $X_n$ is a martingale with $|X_n - X_{n-1}| \le 1$. Using the notation of Theorem 5.3.1, we have

$$\text{on } C, \quad \sum_{n=1}^{\infty} 1_{A_n} = \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} P(A_n|\mathcal{F}_{n-1}) = \infty$$

$$\text{on } D, \quad \sum_{n=1}^{\infty} 1_{A_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} P(A_n|\mathcal{F}_{n-1}) = \infty$$

Since $P(C \cup D) = 1$, the result follows. ∎

**Exercise 5.3.5.** Let $p_m \in [0, 1)$. Use the Borel-Cantelli lemmas to show that

$$\prod_{m=1}^{\infty}(1 - p_m) = 0 \quad \text{if and only if} \sum_{m=1}^{\infty} p_m = \infty.$$

**Exercise 5.3.6.** Show $\sum_{n=2}^{\infty} P(A_n | \cap_{m=1}^{n-1} A_m^c) = \infty$ implies $P(\cap_{m=1}^{\infty} A_m^c) = 0$.

### 5.3.2 *Polya's Urn Scheme*

An urn contains $r$ red and $g$ green balls. At each time we draw a ball out, then replace it, and add $c$ more balls of the color drawn. Let $X_n$ be the fraction of green balls after the $n$th draw. To check that $X_n$ is a martingale, note that if there are $i$ red balls and $j$ green balls at time $n$, then

$$X_{n+1} = \begin{cases} (j+c)/(i+j+c) & \text{with probability } j/(i+j) \\ j/(i+j+c) & \text{with probability } i/(i+j) \end{cases}$$

and we have

$$\frac{j+c}{i+j+c} \cdot \frac{j}{i+j} + \frac{j}{i+j+c} \cdot \frac{i}{i+j} = \frac{(j+c+i)j}{(i+j+c)(i+j)} = \frac{j}{i+j}$$

Since $X_n \geq 0$, Theorem 5.2.9 implies that $X_n \to X_\infty$ a.s. To compute the distribution of the limit, we observe (a) the probability of getting green on the first $m$ draws then red on the next $\ell = n - m$ draws is

$$\frac{g}{g+r} \cdot \frac{g+c}{g+r+c} \cdots \frac{g+(m-1)c}{g+r+(m-1)c} \cdot \frac{r}{g+r+mc} \cdots \frac{r+(\ell-1)c}{g+r+(n-1)c}$$

and (b) any other outcome of the first $n$ draws with $m$ green balls drawn and $\ell$ red balls drawn has the same probability since the denominator remains the same and the numerator is permuted. Consider the special case $c = 1, g = 1, r = 1$. Let $G_n$ be the number of green balls after the $n$th draw has been completed and the new ball has been added. It follows from (a) and (b) that

$$P(G_n = m+1) = \binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1}$$

so $X_\infty$ has a uniform distribution on (0,1).

If we suppose that $c = 1$, $g = 2$, and $r = 1$, then

$$P(G_n = m+2) = \frac{n!}{m!(n-m)!} \frac{(m+1)!(n-m)!}{(n+2)!/2} \to 2x$$

if $n \to \infty$ and $m/n \to x$. In general, the distribution of $X_\infty$ has density

$$\frac{\Gamma((g+r)/c)}{\Gamma(g/c)\Gamma(r/c)} x^{(g/c)-1}(1-x)^{(r/c)-1}$$

This is the **beta distribution** with parameters $g/c$ and $r/c$. In Example 5.4.5 we will see that the limit behavior changes drastically if, in addition to the $c$ balls of the color chosen, we always add one ball of the opposite color.

### 5.3.3 *Radon-Nikodym Derivatives*

Let $\mu$ be a finite measure and $\nu$ a probability measure on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_n \uparrow \mathcal{F}$ be $\sigma$-fields (i.e., $\sigma(\cup \mathcal{F}_n) = \mathcal{F}$). Let $\mu_n$ and $\nu_n$ be the restrictions of $\mu$ and $\nu$ to $\mathcal{F}_n$.

**Theorem 5.3.3.** *Suppose $\mu_n \ll \nu_n$ for all $n$. Let $X_n = d\mu_n/d\nu_n$ and let $X = \limsup X_n$. Then*

$$\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\})$$

**Remark.** $\mu_r(A) \equiv \int_A X\, d\nu$ is a measure $\ll \nu$. Since Theorem 5.2.9 implies $\nu(X = \infty) = 0$, $\mu_s(A) \equiv \mu(A \cap \{X = \infty\})$ is singular w.r.t. $\nu$. Thus $\mu = \mu_r + \mu_s$ gives the Lebesgue decomposition of $\mu$ (see Theorem A.4.5), and $X_\infty = d\mu_r/d\nu$, $\nu$-a.s. Here and in the proof we need to keep track of the measure to which the a.s. refers.

*Proof.* As the reader can probably anticipate:

**Lemma 5.3.4.** $X_n$ *(defined on* $(\Omega, \mathcal{F}, \nu)$*) is a martingale w.r.t.* $\mathcal{F}_n$.

*Proof.* We observe that, by definition, $X_n \in \mathcal{F}_n$. Let $A \in \mathcal{F}_n$. Since $X_n \in \mathcal{F}_n$ and $\nu_n$ is the restriction of $\nu$ to $\mathcal{F}_n$

$$\int_A X_n\, d\nu = \int_A X_n\, d\nu_n$$

Using the definition of $X_n$ and Exercise A.4.7

$$\int_A X_n\, d\nu_n = \mu_n(A) = \mu(A)$$

the last equality holding since $A \in \mathcal{F}_n$ and $\mu_n$ is the restriction of $\mu$ to $\mathcal{F}_n$. If $A \in \mathcal{F}_{m-1} \subset \mathcal{F}_m$, using the last result for $n = m$ and $n = m - 1$ gives

$$\int_A X_m d\nu = \mu(A) = \int_A X_{m-1} d\nu$$

so $E(X_m|\mathcal{F}_{m-1}) = X_{m-1}$. ∎

Since $X_n$ is a nonnegative martingale, Theorem 5.2.9 implies that $X_n \to X$ $\nu$-a.s. We want to check that the equality in the theorem holds. Dividing $\mu(A)$ by $\mu(\Omega)$, we can without loss of generality suppose $\mu$ is a probability measure. Let $\rho = (\mu + \nu)/2$, $\rho_n = (\mu_n + \nu_n)/2 =$ the restriction of $\rho$ to $\mathcal{F}_n$. Let $Y_n = d\mu_n/d\rho_n$, $Z_n = d\nu_n/d\rho_n$. $Y_n, Z_n \geq 0$ and $Y_n + Z_n = 2$ (by Exercise A.4.6), so $Y_n$ and $Z_n$ are bounded martingales with limits $Y$ and Z. As the reader can probably guess,

$$(*) \qquad Y = d\mu/d\rho \qquad Z = d\nu/d\rho$$

It suffices to prove the first equality. From the proof of Lemma 5.3.4, if $A \in \mathcal{F}_m \subset \mathcal{F}_n$,

$$\mu(A) = \int_A Y_n \, d\rho \to \int_A Y \, d\rho$$

by the bounded convergence theorem. The last computation shows that

$$\mu(A) = \int_A Y \, d\rho \quad \text{for all } A \in \mathcal{G} = \cup_m \mathcal{F}_m$$

$\mathcal{G}$ is a $\pi$-system, so the $\pi - \lambda$ theorem implies the equality is valid for all $A \in \mathcal{F} = \sigma(\mathcal{G})$ and $(*)$ is proved.

It follows from Exercises A.4.8 and A.4.9 that $X_n = Y_n/Z_n$. At this point, the reader can probably leap to the conclusion that $X = Y/Z$. To get there carefully, note that $Y + Z = 2$ $\rho$-a.s., so $\rho(Y = 0, Z = 0) = 0$. Having ruled out $0/0$, we have $X = Y/Z$ $\rho$-a.s. (Recall $X \equiv \limsup X_n$.) Let $W = (1/Z) \cdot 1_{(Z>0)}$. Using $(*)$, then $1 = ZW + 1_{(Z=0)}$, we have

(a) $$\mu(A) = \int_A Y \, d\rho = \int_A YWZ \, d\rho + \int_A 1_{(Z=0)} Y \, d\rho$$

Now $(*)$ implies $d\nu = Z \, d\rho$, and it follows from the definitions that

$$YW = X1_{(Z>0)} = X \quad \nu\text{-a.s.}$$

the second equality holding since $\nu(\{Z = 0\}) = 0$. Combining things, we have

(b) $$\int_A YWZ \, d\rho = \int_A X \, d\nu$$

To handle the other term, we note that $(*)$ implies $d\mu = Y \, d\rho$, and it follows from the definitions that $\{X = \infty\} = \{Z = 0\}$ $\mu$-a.s. so

(c) $$\int_A 1_{(Z=0)} Y \, d\rho = \int_A 1_{(X=\infty)} \, d\mu$$

Combining (a), (b), and (c) gives the desired result. ∎

**Example 5.3.1.** Suppose $\mathcal{F}_n = \sigma(I_{k,n} : 0 \le k < K_n)$ where for each $n$, $I_{k,n}$ is a partition of $\Omega$, and the $(n+1)$th partition is a refinement of the $n$th. In this case, the condition $\mu_n \ll \nu_n$ is $\nu(I_{k,n}) = 0$ implies $\mu(I_{k,n}) = 0$, and the martingale $X_n = \mu(I_{k,n})/\nu(I_{k,n})$ on $I_{k,n}$ is an approximation to the Radon-Nikodym derivative. For a concrete example, consider $\Omega = [0, 1)$, $I_{k,n} = [k2^{-n}, (k+1)2^{-n})$ for $0 \le k < 2^n$, and $\nu =$ Lebesgue measure.

**Exercise 5.3.7.** Check by direct computation that the $X_n$ in Example 5.3.1 is a martingale. Show that if we drop the condition $\mu_n \ll \nu_n$ and set $X_n = 0$ when $\nu(I_{k,n}) = 0$, then $E(X_{n+1}|\mathcal{F}_n) \le X_n$.

**Exercise 5.3.8.** Apply Theorem 5.3.3 to Example 5.3.1 to get a "probabilistic" proof of the Radon-Nikodym theorem. To be precise, suppose $\mathcal{F}$ is **countably**

**generated** (i.e., there is a sequence of sets $A_n$ so that $\mathcal{F} = \sigma(A_n : n \geq 1)$) and show that if $\mu$ and $\nu$ are $\sigma$-finite measures and $\mu \ll \nu$, then there is a function $g$ so that $\mu(A) = \int_A g\, d\nu$.

**Remark.** Before you object to this as circular reasoning (the Radon-Nikodym theorem was used to define conditional expectation!), observe that the conditional expectations that are needed for Example 5.3.1 have elementary definitions.

**Kakutani dichotomy for infinite product measures.** Let $\mu$ and $\nu$ be measures on sequence space $(\mathbf{R}^{\mathbf{N}}, \mathcal{R}^{\mathbf{N}})$ that make the coordinates $\xi_n(\omega) = \omega_n$ independent. Let $F_n(x) = \mu(\xi_n \leq x)$, $G_n(x) = \nu(\xi_n \leq x)$. Suppose $F_n \ll G_n$ and let $q_n = dF_n/dG_n$. Let $\mathcal{F}_n = \sigma(\xi_m : m \leq n)$, let $\mu_n$ and $\nu_n$ be the restrictions of $\mu$ and $\nu$ to $\mathcal{F}_n$, and let

$$X_n = \frac{d\mu_n}{d\nu_n} = \prod_{m=1}^{n} q_m.$$

Theorem 5.3.3 implies that $X_n \to X$ $\nu$-a.s. $\sum_{m=1}^{\infty} \log(q_m) > -\infty$ is a tail event, so the Kolmogorov 0-1 law implies

$$\nu(X = 0) \in \{0, 1\} \tag{5.3.1}$$

and it follows from Theorem 5.3.3 that either $\mu \ll \nu$ or $\mu \perp \nu$. The next result gives a concrete criterion for which of the two alternatives occurs.

**Theorem 5.3.5.** *$\mu \ll \nu$ or $\mu \perp \nu$, according as $\prod_{m=1}^{\infty} \int \sqrt{q_m}\, dG_m > 0$ or $= 0$.*

*Proof.* Jensen's inequality and Exercise A.4.7 imply

$$\left(\int \sqrt{q_m}\, dG_m\right)^2 \leq \int q_m\, dG_m = \int dF_m = 1$$

so the infinite product of the integrals is well defined and $\leq 1$. Let

$$X_n = \prod_{m \leq n} q_m(\omega_m)$$

as above, and recall that $X_n \to X$ $\nu$-a.s. If the infinite product is 0, then

$$\int X_n^{1/2}\, d\nu = \prod_{m=1}^{n} \int \sqrt{q_m}\, dG_m \to 0$$

Fatou's lemma implies

$$\int X^{1/2}\, d\nu \leq \liminf_{n \to \infty} \int X_n^{1/2}\, d\nu = 0$$

so $X = 0$ $\nu$-a.s., and Theorem 5.3.3 implies $\mu \perp \nu$. To prove the other direction, let $Y_n = X_n^{1/2}$. Now $\int q_m\, dG_m = 1$, so if we use $E$ to denote expected value with respect to $\nu$, then $EY_m^2 = EX_m = 1$, so

$$E(Y_{n+k} - Y_n)^2 = E(X_{n+k} + X_n - 2X_n^{1/2}X_{n+k}^{1/2}) = 2\left(1 - \prod_{m=n+1}^{n+k} \int \sqrt{q_m}\, dG_m\right)$$

Now $|a - b| = |a^{1/2} - b^{1/2}| \cdot (a^{1/2} + b^{1/2})$, so using Cauchy-Schwarz and the fact $(a+b)^2 \le 2a^2 + 2b^2$ gives

$$\begin{aligned} E|X_{n+k} - X_n| &= E(|Y_{n+k} - Y_n|(Y_{n+k} + Y_n)) \\ &\le \left(E(Y_{n+k} - Y_n)^2 E(Y_{n+k} + Y_n)^2\right)^{1/2} \\ &\le \left(4E(Y_{n+k} - Y_n)^2\right)^{1/2} \end{aligned}$$

From the last two equations, it follows that if the infinite product is $> 0$, then $X_n$ converges to $X$ in $L^1(\nu)$, so $\nu(X = 0) < 1$, (5.3.1) implies the probability is 0, and the desired result follows from Theorem 5.3.3. ■

**Bernoulli product measures.** For the next three exercises, suppose $F_n$, $G_n$ are concentrated on $\{0, 1\}$ and have $F_n(0) = 1 - \alpha_n$, $G_n(0) = 1 - \beta_n$.

**Exercise 5.3.9.** (i) Use Theorem 5.3.5 to find a necessary and sufficient condition for $\mu \ll \nu$. (ii) Suppose that $0 < \epsilon \le \alpha_n, \beta_n \le 1 - \epsilon < 1$. Show that in this case the condition is simply $\sum(\alpha_n - \beta_n)^2 < \infty$.

**Exercise 5.3.10.** Show that if $\sum \alpha_n < \infty$ and $\sum \beta_n = \infty$ in the previous exercise then $\mu \perp \nu$. This shows that the condition $\sum(\alpha_n - \beta_n)^2 < \infty$ is not sufficient for $\mu \ll \nu$ in general.

**Exercise 5.3.11.** Suppose $0 < \alpha_n, \beta_n < 1$. Show that $\sum |\alpha_n - \beta_n| < \infty$ is sufficient for $\mu \ll \nu$ in general.

### 5.3.4 *Branching Processes*

Let $\xi_i^n$, $i, n \ge 1$, be i.i.d. nonnegative integer-valued random variables. Define a sequence $Z_n$, $n \ge 0$ by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases} \tag{5.3.2}$$

$Z_n$ is called a **Galton-Watson process**. The idea behind the definitions is that $Z_n$ is the number of individuals in the $n$th generation, and each member of the $n$th generation gives birth independently to an identically distributed number of children. $p_k = P(\xi_i^n = k)$ is called the **offspring distribution**.

**Lemma 5.3.6.** *Let $\mathcal{F}_n = \sigma(\xi_i^m : i \ge 1, 1 \le m \le n)$ and $\mu = E\xi_i^m \in (0, \infty)$. Then $Z_n/\mu^n$ is a martingale w.r.t. $\mathcal{F}_n$.*

*Proof.* Clearly, $Z_n \in \mathcal{F}_n$.

$$E(Z_{n+1}|\mathcal{F}_n) = \sum_{k=1}^{\infty} E(Z_{n+1}1_{\{Z_n=k\}}|\mathcal{F}_n)$$

by the linearity of conditional expectation, (5.1.1), and the monotone convergence theorem, (5.1.3). On $\{Z_n = k\}$, $Z_{n+1} = \xi_1^{n+1} + \cdots + \xi_k^{n+1}$, so the sum is

$$\sum_{k=1}^{\infty} E((\xi_1^{n+1} + \cdots + \xi_k^{n+1})1_{\{Z_n=k\}}|\mathcal{F}_n) = \sum_{k=1}^{\infty} 1_{\{Z_n=k\}} E(\xi_1^{n+1} + \cdots + \xi_k^{n+1}|\mathcal{F}_n)$$

by Theorem 5.1.7. Since each $\xi_j^{n+1}$ is independent of $\mathcal{F}_n$, the last expression

$$= \sum_{k=1}^{\infty} 1_{\{Z_n=k\}} k\mu = \mu Z_n$$

Dividing both sides by $\mu^{n+1}$ now gives the desired result. ∎

**Remark.** The reader should notice that in the proof of Lemma 5.3.6 we broke things down according to the value of $Z_n$ to get rid of the random index. A simpler way of doing the last argument (that we will use in the future) is to use Exercise 5.1.1 to conclude that on $\{Z_n = k\}$

$$E(Z_{n+1}|\mathcal{F}_n) = E(\xi_1^{n+1} + \cdots + \xi_k^{n+1}|\mathcal{F}_n) = k\mu = \mu Z_n$$

$Z_n/\mu^n$ is a nonnegative martingale, so Theorem 5.2.9 implies $Z_n/\mu^n \to$ a limit a.s. We begin by identifying cases when the limit is trivial.

**Theorem 5.3.7.** *If $\mu < 1$ then $Z_n = 0$ for all $n$ sufficiently large, so $Z_n/\mu^n \to 0$.*

*Proof.* $E(Z_n/\mu^n) = E(Z_0) = 1$, so $E(Z_n) = \mu^n$. Now $Z_n \geq 1$ on $\{Z_n > 0\}$ so

$$P(Z_n > 0) \leq E(Z_n; Z_n > 0) = E(Z_n) = \mu^n \to 0$$

exponentially fast if $\mu < 1$. ∎

The last answer should be intuitive. If each individual on the average gives birth to less than one child, the species will die out. The next result shows that after we exclude the trivial case in which each individual has exactly one child, the same result holds when $\mu = 1$.

**Theorem 5.3.8.** *If $\mu = 1$ and $P(\xi_i^m = 1) < 1$ then $Z_n = 0$ for all $n$ sufficiently large.*

*Proof.* When $\mu = 1$, $Z_n$ is itself a nonnegative martingale. Since $Z_n$ is integer valued and by Theorem 5.2.9 converges to an a.s. finite limit $Z_\infty$, we must have $Z_n = Z_\infty$ for large $n$. If $P(\xi_i^m = 1) < 1$ and $k > 0$, then $P(Z_n = k$ for all $n \geq N) = 0$ for any $N$, so we must have $Z_\infty \equiv 0$. ∎

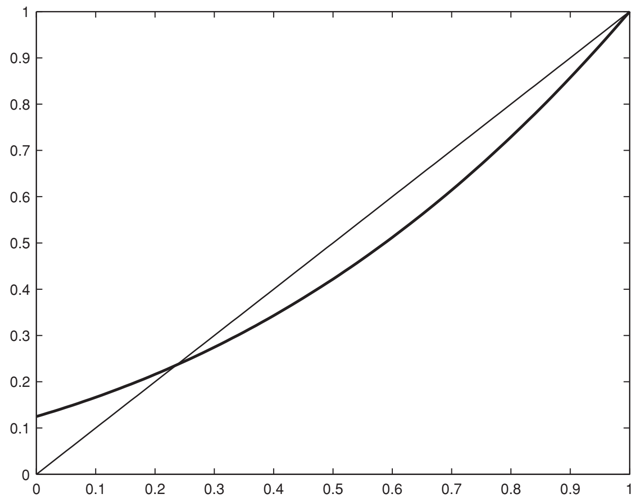



Figure 5.3. Generating function for Binomial(3,1/2).

When $\mu \le 1$, the limit of $Z_n/\mu^n$ is 0 because the branching process dies out. Our next step is to show that if $\mu > 1$, then $P(Z_n > 0 \text{ for all } n) > 0$. For $s \in [0, 1]$, let $\varphi(s) = \sum_{k\ge 0} p_k s^k$ where $p_k = P(\xi_i^m = k)$. $\varphi$ is the **generating function** for the offspring distribution $p_k$. See Figure 5.3 for an example.

**Theorem 5.3.9.** *$P(Z_n = 0$ for some $n) = \rho$ the unique fixed point of $\phi$ in* $[0, 1)$.

*Proof.* Differentiating and referring to Theorem A.5.2 for the justification gives for $s < 1$

$$\varphi'(s) = \sum_{k=1}^{\infty} k\, p_k s^{k-1} \ge 0$$

$$\varphi''(s) = \sum_{k=2}^{\infty} k(k-1) p_k s^{k-2} \ge 0$$

So $\varphi$ is increasing and convex, and $\lim_{s\uparrow 1} \varphi'(s) = \sum_{k=1}^{\infty} k p_k = \mu$.

Our interest in $\varphi$ stems from the following facts.

(a) If $\theta_m = P(Z_m = 0)$ then $\theta_m = \sum_{k=0}^{\infty} p_k (\theta_{m-1})^k$.

*Proof of (a).* If $Z_1 = k$, an event with probability $p_k$, then $Z_m = 0$ if and only if all $k$ families die out in the remaining $m - 1$ units of time, an independent event with probability $\theta_{m-1}^k$. Summing over the disjoint possibilities for each $k$ gives the desired result. ■

(b) If $\varphi'(1) = \mu > 1$, there is a unique $\rho < 1$ so that $\varphi(\rho) = \rho$.

*Proof of (b).* $\varphi(0) \ge 0$, $\varphi(1) = 1$, and $\varphi'(1) > 1$, so $\varphi(1-\epsilon) < 1-\epsilon$ for small $\epsilon$. The last two observations imply the existence of a fixed point. To see it is unique, observe that $\mu > 1$ implies $p_k > 0$ for some $k > 1$, so $\varphi''(\theta) > 0$ for $\theta > 0$. Since

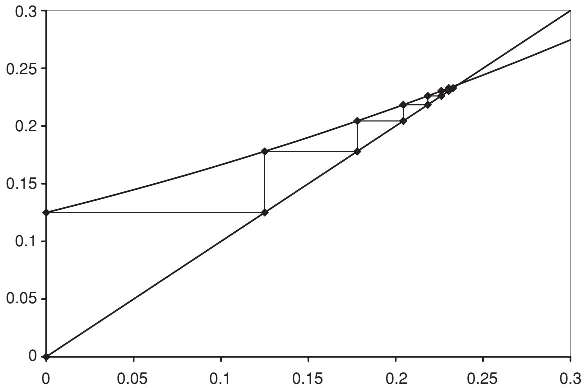



Figure 5.4. Iteration as in part (c) for the Binomial (3,1/2) generating function.

$\varphi$ is strictly convex, it follows that if $\rho < 1$ is a fixed point, then $\varphi(x) < x$ for $x \in (\rho, 1)$. ■

(c) As $m \uparrow \infty$, $\theta_m \uparrow \rho$.

*Proof of (c).* $\theta_0 = 0$, $\varphi(\rho) = \rho$, and $\varphi$ is increasing, so induction implies $\theta_m$ is increasing and $\theta_m \le \rho$. Let $\theta_\infty = \lim \theta_m$. Taking limits in $\theta_m = \varphi(\theta_{m-1})$, we see $\theta_\infty = \varphi(\theta_\infty)$. Since $\theta_\infty \le \rho$, it follows that $\theta_\infty = \rho$. ■

Combining (a)–(c) shows $P(Z_n = 0 \text{ for some } n) = \lim \theta_n = \rho < 1$ and proves Theorem 5.3.9. ■

The last result shows that when $\mu > 1$, the limit of $Z_n/\mu^n$ has a chance of being nonzero. The best result on this question is due to Kesten and Stigum:

**Theorem 5.3.10.** $W = \lim Z_n/\mu^n$ *is not* $\equiv 0$ *if and only if* $\sum p_k k \log k < \infty$.

For a proof, see Athreya and Ney (1972), pp. 24–29. In the next section, we will show that $\sum k^2 p_k < \infty$ is sufficient for a nontrivial limit.

**Exercise 5.3.12.** Show that if $P(\lim Z_n/\mu^n = 0) < 1$, then it is $= \rho$, and hence

$$\{\lim Z_n/\mu^n > 0\} = \{Z_n > 0 \text{ for all } n\} \quad \text{a.s.}$$

**Exercise 5.3.13.** Galton and Watson, who invented the process that bears their names, were interested in the survival of family names. Suppose each family has exactly three children but coin flips determine their sex. In the 1800s, only male children kept the family name, so following the male offspring leads to a branching process with $p_0 = 1/8$, $p_1 = 3/8$, $p_2 = 3/8$, $p_3 = 1/8$. Compute the probability $\rho$ that the family name will die out when $Z_0 = 1$.

## 5.4 Doob's Inequality, Convergence in $L^p$

We begin by proving a consequence of Theorem 5.2.6.

**Theorem 5.4.1.** *If $X_n$ is a submartingale and $N$ is a stopping time with $P(N \leq k) = 1$ then*

$$EX_0 \leq EX_N \leq EX_k$$

**Remark.** Let $S_n$ be a simple random walk with $S_0 = 1$ and let $N = \inf\{n : S_n = 0\}$. (See Example 5.2.3 for more details.) $ES_0 = 1 > 0 = ES_N$, so the first inequality need not hold for unbounded stopping times. In Section 5.7 we will give conditions that guarantee $EX_0 \leq EX_N$ for unbounded $N$.

*Proof.* Theorem 5.2.6 implies $X_{N\wedge n}$ is a submartingale, so it follows that

$$EX_0 = EX_{N\wedge 0} \leq EX_{N\wedge k} = EX_N$$

To prove the other inequality, let $K_n = 1_{\{N<n\}} = 1_{\{N\leq n-1\}}$. $K_n$ is predictable, so Theorem 5.2.5 implies $(K \cdot X)_n = X_n - X_{N\wedge n}$ is a submartingale, and it follows that

$$EX_k - EX_N = E(K \cdot X)_k \geq E(K \cdot X)_0 = 0 \qquad \blacksquare$$

**Exercise 5.4.1.** Show that if $j \leq k$, then $E(X_j; N = j) \leq E(X_k; N = j)$ and sum over $j$ to get a second proof of $EX_N \leq EX_k$.

**Exercise 5.4.2.** Generalize the proof of Theorem 5.4.1 to show that if $X_n$ is a submartingale and $M \leq N$ are stopping times with $P(N \leq k) = 1$, then $EX_M \leq EX_N$.

**Exercise 5.4.3.** Use the stopping times from the Exercise 4.1.7 to strengthen the conclusion of the previous exercise to $E(X_N|\mathcal{F}_M) \geq X_M$.

We will see below that Theorem 5.4.1 is very useful. The first indication of this is:

**Theorem 5.4.2. Doob's inequality.** *Let $X_m$ be a submartingale,*

$$\bar{X}_n = \max_{0\leq m\leq n} X_m^+$$

*$\lambda > 0$, and $A = \{\bar{X}_n \geq \lambda\}$. Then*

$$\lambda P(A) \leq EX_n 1_A \leq EX_n^+$$

*Proof.* Let $N = \inf\{m : X_m \geq \lambda \text{ or } m = n\}$. Since $X_N \geq \lambda$ on $A$,

$$\lambda P(A) \leq EX_N 1_A \leq EX_n 1_A$$

The second inequality follows from the fact that Theorem 5.4.1 implies $EX_N \le EX_n$, and we have $X_N = X_n$ on $A^c$. The second inequality is trivial, so the proof is complete. ■

**Example 5.4.1. Random walks.** If we let $S_n = \xi_1 + \cdots + \xi_n$ where the $\xi_m$ are independent and have $E\xi_m = 0$, $\sigma_m^2 = E\xi_m^2 < \infty$, then Theorem 5.2.3 implies $X_n = S_n^2$ is a submartingale. If we let $\lambda = x^2$ and apply Theorem 5.4.2 to $X_n$, we get Kolmogorov's maximal inequality, Theorem 2.5.2:

$$P\left(\max_{1\le m\le n} |S_m| \ge x\right) \le x^{-2}\,\mathrm{var}\,(S_n)$$

Using martingales, one can also prove a lower bound on the maximum that can be used instead of the central limit theorem in our proof of the necessity of the conditions in the three series theorem. (See Example 3.4.7.)

**Exercise 5.4.4.** Suppose in addition to the conditions introduced above that $|\xi_m| \le K$ and let $s_n^2 = \sum_{m\le n} \sigma_m^2$. Exercise 5.2.6 implies that $S_n^2 - s_n^2$ is a martingale. Use this and Theorem 5.4.1 to conclude

$$P\left(\max_{1\le m\le n} |S_m| \le x\right) \le (x+K)^2/\,\mathrm{var}\,(S_n)$$

**Exercise 5.4.5.** Let $X_n$ be a martingale with $X_0 = 0$ and $EX_n^2 < \infty$. Show that

$$P\left(\max_{1\le m\le n} X_m \ge \lambda\right) \le EX_n^2/(EX_n^2 + \lambda^2)$$

Hint: Use the fact that $(X_n + c)^2$ is a submartingale and optimize over $c$.

Integrating the inequality in Theorem 5.4.2 gives:

**Theorem 5.4.3. $L^p$ maximum inequality**. *If $X_n$ is a submartingale, then for* $1 < p < \infty$,

$$E(\bar{X}_n^p) \le \left(\frac{p}{p-1}\right)^p E(X_n^+)^p$$

*Consequently, if $Y_n$ is a martingale and $Y_n^* = \max_{0\le m\le n} |Y_m|$,*

$$E|Y_n^*|^p \le \left(\frac{p}{p-1}\right)^p E(|Y_n|^p)$$

*Proof.* The second inequality follows by applying the first to $X_n = |Y_n|$. To prove the first we will, for reasons that will become clear in a moment, work with $\bar{X}_n \wedge M$ rather than $\bar{X}_n$. Since $\{\bar{X}_n \wedge M \ge \lambda\}$ is always $\{\bar{X}_n \ge \lambda\}$ or $\emptyset$, this does not change the application of Theorem 5.4.2. Using Lemma 2.2.8, Theorem 5.4.2, Fubini's theorem, and a little calculus gives

$$\begin{aligned}
E((\bar{X}_n \wedge M)^p) &= \int_0^\infty p\lambda^{p-1} P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda \\
&\leq \int_0^\infty p\lambda^{p-1} \left( \lambda^{-1} \int X_n^+ 1_{(\bar{X}_n \wedge M \geq \lambda)}\, dP \right) d\lambda \\
&= \int X_n^+ \int_0^{\bar{X}_n \wedge M} p\lambda^{p-2}\, d\lambda\, dP \\
&= \frac{p}{p-1} \int X_n^+ (\bar{X}_n \wedge M)^{p-1}\, dP
\end{aligned}$$

If we let $q = p/(p-1)$ be the exponent conjugate to $p$ and apply Hölder's inequality, Theorem 1.6.3, we see that the above

$$\leq q(E|X_n^+|^p)^{1/p}(E|\bar{X}_n \wedge M|^p)^{1/q}$$

If we divide both sides of the last inequality by $(E|\bar{X}_n \wedge M|^p)^{1/q}$, we get

$$E(|\bar{X}_n \wedge M|^p) \leq \left( \frac{p}{p-1} \right)^p E(X_n^+)^p$$

Letting $M \to \infty$ and using the monotone convergence theorem gives the desired result. ∎

**Example 5.4.2. Theorem 5.4.3 is false when p = 1.** Again, the counterexample is provided by Example 5.2.3. Let $S_n$ be a simple random walk starting from $S_0 = 1$, $N = \inf\{n : S_n = 0\}$, and $X_n = S_{N \wedge n}$. Theorem 5.4.1 implies $EX_n = ES_{N\wedge n} = ES_0 = 1$ for all $n$. Using hitting probabilities for simple random walk, (4.1.2) $a = -1$, $b = M - 1$, we have

$$P\left( \max_m X_m \geq M \right) = \frac{1}{M}$$

so $E(\max_m X_m) = \sum_{M=1}^\infty P(\max_m X_m \geq M) = \sum_{M=1}^\infty 1/M = \infty$. The monotone convergence theorem implies that $E \max_{m \leq n} X_m \uparrow \infty$ as $n \uparrow \infty$.

The next result gives an extension of Theorem 5.4.2 to $p = 1$. Since this is not one of the most important results, the proof is left to the reader.

**Theorem 5.4.4.** *Let $X_n$ be a submartingale and* $\log^+ x = \max(\log x, 0)$.

$$E\bar{X}_n \leq (1 - e^{-1})^{-1}\{1 + E(X_n^+ \log^+(X_n^+))\}$$

**Remark.** The last result is almost the best possible condition for $\sup |X_n| \in L^1$. Gundy has shown that if $X_n$ is a positive martingale that has $X_{n+1} \leq CX_n$ and $EX_0 \log^+ X_0 < \infty$, then $E(\sup X_n) < \infty$ implies $\sup\ E(X_n \log^+ X_n) < \infty$. For a proof, see Neveu (1975), pp. 71–73.

**Exercise 5.4.6.** Prove Theorem 5.4.4 by carrying out the following steps: (i) Imitate the proof of 5.4.2 but use the trivial bound $P(A) \le 1$ for $\lambda \le 1$ to show

$$E(\bar{X}_n \wedge M) \le 1 + \int X_n^+ \log(\bar{X}_n \wedge M)\, dP$$

(ii) Use calculus to show $a \log b \le a \log a + b/e \le a \log^+ a + b/e$.

From Theorem 5.4.2, we get the following:

**Theorem 5.4.5. $L^p$ convergence theorem.** *If $X_n$ is a martingale with* $\sup E|X_n|^p < \infty$ *where $p > 1$, then $X_n \to X$ a.s. and in $L^p$.*

*Proof.* $(EX_n^+)^p \le (E|X_n|)^p \le E|X_n|^p$, so it follows from the martingale convergence theorem (5.2.8) that $X_n \to X$ a.s. The second conclusion in Theorem 5.4.3 implies

$$E\left(\sup_{0\le m\le n} |X_m|\right)^p \le \left(\frac{p}{p-1}\right)^p E|X_n|^p$$

Letting $n \to \infty$ and using the monotone convergence theorem implies $\sup |X_n| \in L^p$. Since $|X_n - X|^p \le (2 \sup |X_n|)^p$, it follows from the dominated convergence theorem that $E|X_n - X|^p \to 0$. ■

The most important special case of the results in this section occurs when $p = 2$. To treat this case, the next two results are useful.

**Theorem 5.4.6. Orthogonality of martingale increments.** *Let $X_n$ be a martingale with $EX_n^2 < \infty$ for all $n$. If $m \le n$ and $Y \in \mathcal{F}_m$ has $EY^2 < \infty$, then*

$$E((X_n - X_m)Y) = 0$$

*Proof.* The Cauchy-Schwarz inequality implies $E|(X_n - X_m)Y| < \infty$. Using (5.1.5), Theorem 5.1.7, and the definition of a martingale,

$$E((X_n - X_m)Y) = E[E((X_n - X_m)Y|\mathcal{F}_m)] = E[YE((X_n - X_m)|\mathcal{F}_m)] = 0 \quad ■$$

**Theorem 5.4.7. Conditional variance formula.** *If $X_n$ is a martingale with $EX_n^2 < \infty$ for all $n$,*

$$E((X_n - X_m)^2|\mathcal{F}_m) = E(X_n^2|\mathcal{F}_m) - X_m^2.$$

**Remark.** This is the conditional analogue of $E(X - EX)^2 = EX^2 - (EX)^2$ and is proved in exactly the same way.

*Proof.* Using the linearity of conditional expectation and then Theorem 5.1.7, we have

$$\begin{aligned} E(X_n^2 - 2X_nX_m + X_m^2|\mathcal{F}_m) &= E(X_n^2|\mathcal{F}_m) - 2X_mE(X_n|\mathcal{F}_m) + X_m^2 \\ &= E(X_n^2|\mathcal{F}_m) - 2X_m^2 + X_m^2 \end{aligned}$$

which gives the desired result. ■

**Exercise 5.4.7.** Let $X_n$ and $Y_n$ be martingales with $EX_n^2 < \infty$ and $EY_n^2 < \infty$.

$$EX_nY_n - EX_0Y_0 = \sum_{m=1}^{n} E(X_m - X_{m-1})(Y_m - Y_{m-1})$$

The next two results generalize Theorems 2.5.3 and 2.5.7. Let $X_n$, $n \ge 0$, be a martingale and let $\xi_n = X_n - X_{n-1}$ for $n \ge 1$.

**Exercise 5.4.8.** If $EX_0^2, \sum_{m=1}^\infty E\xi_m^2 < \infty$ then $X_n \to X_\infty$ a.s. and in $L^2$.

**Exercise 5.4.9.** If $b_m \uparrow \infty$ and $\sum_{m=1}^\infty E\xi_m^2/b_m^2 < \infty$, then $X_n/b_n \to 0$ a.s. In particular, if $E\xi_n^2 \le K < \infty$ and $\sum_{m=1}^\infty b_m^{-2} < \infty$, then $X_n/b_n \to 0$ a.s.

**Example 5.4.3. Branching processes.** We continue the study begun at the end of the last section. Using the notation introduced there, we suppose $\mu = E(\xi_i^m) > 1$ and $\text{var}(\xi_i^m) = \sigma^2 < \infty$. Let $X_n = Z_n/\mu^n$. Taking $m = n-1$ in Theorem 5.4.7 and rearranging, we have

$$E(X_n^2|\mathcal{F}_{n-1}) = X_{n-1}^2 + E((X_n - X_{n-1})^2|\mathcal{F}_{n-1})$$

To compute the second term, we observe

$$\begin{aligned} E((X_n - X_{n-1})^2|\mathcal{F}_{n-1}) &= E((Z_n/\mu^n - Z_{n-1}/\mu^{n-1})^2|\mathcal{F}_{n-1}) \\ &= \mu^{-2n}E((Z_n - \mu Z_{n-1})^2|\mathcal{F}_{n-1}) \end{aligned}$$

It follows from Exercise 5.1.1 that on $\{Z_{n-1} = k\}$,

$$E((Z_n - \mu Z_{n-1})^2|\mathcal{F}_{n-1}) = E\left(\left(\sum_{i=1}^{k} \xi_i^n - \mu k\right)^2 \middle| \mathcal{F}_{n-1}\right) = k\sigma^2 = Z_{n-1}\sigma^2$$

Combining the last three equations gives

$$EX_n^2 = EX_{n-1}^2 + E(Z_{n-1}\sigma^2/\mu^{2n}) = EX_{n-1}^2 + \sigma^2/\mu^{n+1}$$

since $E(Z_{n-1}/\mu^{n-1}) = EZ_0 = 1$. Now $EX_0^2 = 1$, so $EX_1^2 = 1 + \sigma^2/\mu^2$, and induction gives

$$EX_n^2 = 1 + \sigma^2 \sum_{k=2}^{n+1} \mu^{-k}$$

This shows $\sup EX_n^2 < \infty$, so $X_n \to X$ in $L^2$, and hence $EX_n \to EX$. $EX_n = 1$ for all $n$, so $EX = 1$ and $X$ is not $\equiv 0$. It follows from Exercise 5.3.12 that $\{X > 0\} = \{Z_n > 0 \text{ for all n }\}$.

### 5.4.1 *Square Integrable Martingales**

For the rest of this section, we will suppose

$$X_n \text{ is a martingale with } X_0 = 0 \text{ and } EX_n^2 < \infty \text{ for all } n$$

Theorem 5.2.3 implies $X_n^2$ is a submartingale. It follows from Doob's decomposition Theorem 5.2.10 that we can write $X_n^2 = M_n + A_n$, where $M_n$ is a martingale, and from formulas in Theorems 5.2.10 and 5.4.7 that

$$A_n = \sum_{m=1}^{n} E(X_m^2|\mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_{m=1}^{n} E((X_m - X_{m-1})^2|\mathcal{F}_{m-1})$$

$A_n$ is called the **increasing process** associated with $X_n$. $A_n$ can be thought of as a path by path measurement of the variance at time $n$, and $A_\infty = \lim A_n$ as the total variance in the path. Theorems 5.4.9 and 5.4.10 describe the behavior of the martingale on $\{A_n < \infty\}$ and $\{A_n = \infty\}$, respectively. The key to the proof of the first result is the following:

**Theorem 5.4.8.** $E\left(\sup_m |X_m|^2\right) \le 4EA_\infty$.

*Proof.* Applying the $L^2$ maximum inequality (Theorem 5.4.3) to $X_n$ gives

$$E\left(\sup_{0\le m\le n} |X_m|^2\right) \le 4EX_n^2 = 4EA_n$$

since $EX_n^2 = EM_n + EA_n$ and $EM_n = EM_0 = EX_0^2 = 0$. Using the monotone convergence theorem now gives the desired result. ■

**Theorem 5.4.9.** $\lim_{n\to\infty} X_n$ *exists and is finite a.s. on* $\{A_\infty < \infty\}$.

*Proof.* Let $a > 0$. Since $A_{n+1} \in \mathcal{F}_n$, $N = \inf\{n : A_{n+1} > a^2\}$ is a stopping time. Applying Theorem 5.4.8 to $X_{N\wedge n}$ and noticing $A_{N\wedge n} \le a^2$ gives

$$E\left(\sup_n |X_{N\wedge n}|^2\right) \le 4a^2$$

so the $L^2$ convergence theorem, 5.4.5, implies that $\lim X_{N\wedge n}$ exists and is finite a.s. Since $a$ is arbitrary, the desired result follows. ■

The next result is a variation on the theme of Exercise 5.4.9.

**Theorem 5.4.10.** *Let* $f \ge 1$ *be increasing with* $\int_0^\infty f(t)^{-2}\,dt < \infty$. *Then* $X_n/f(A_n) \to 0$ *a.s. on* $\{A_\infty = \infty\}$.

*Proof.* $H_m = f(A_m)^{-1}$ is bounded and predictable, so Theorem 5.2.5 implies

$$Y_n \equiv (H \cdot X)_n = \sum_{m=1}^{n} \frac{X_m - X_{m-1}}{f(A_m)} \quad \text{is a martingale}$$

If $B_n$ is the increasing process associated with $Y_n$, then

$$\begin{aligned} B_{n+1} - B_n &= E((Y_{n+1} - Y_n)^2 | \mathcal{F}_n) \\ &= E\left( \left. \frac{(X_{n+1} - X_n)^2}{f(A_{n+1})^2} \right| \mathcal{F}_n \right) = \frac{A_{n+1} - A_n}{f(A_{n+1})^2} \end{aligned}$$

since $f(A_{n+1}) \in \mathcal{F}_n$. Our hypotheses on $f$ imply that

$$\sum_{n=0}^{\infty} \frac{A_{n+1} - A_n}{f(A_{n+1})^2} \le \sum_{n=0}^{\infty} \int_{[A_n, A_{n+1})} f(t)^{-2}\, dt < \infty$$

so it follows from Theorem 5.4.9 that $Y_n \to Y_\infty$, and the desired conclusion follows from Kronecker's lemma, Theorem 2.5.5. ∎

**Example 5.4.4.** Let $\epsilon > 0$ and $f(t) = (t \log^{1+\epsilon} t)^{1/2} \vee 1$. Then $f$ satisfies the hypotheses of Theorem 5.4.10. Let $\xi_1, \xi_2, \dots$ be independent with $E\xi_m = 0$ and $E\xi_m^2 = \sigma_m^2$. In this case, $X_n = \xi_1 + \cdots + \xi_n$ is a square integrable martingale with $A_n = \sigma_1^2 + \cdots + \sigma_n^2$, so if $\sum_{i=1}^\infty \sigma_i^2 = \infty$, Theorem 5.4.10 implies $X_n / f(A_n) \to 0$, generalizing Theorem 2.5.7.

From Theorem 5.4.10 we get a result due to Dubins and Freedman (1965) that extends our two previous versions in Theorems 2.3.6 and 5.3.2.

**Theorem 5.4.11. Second Borel-Cantelli Lemma, III.** *Suppose $B_n$ is adapted to $\mathcal{F}_n$ and let $p_n = P(B_n | \mathcal{F}_{n-1})$. Then*

$$\sum_{m=1}^{n} 1_{B(m)} \Bigg/ \sum_{m=1}^{n} p_m \to 1 \quad a.s. \text{ on} \quad \left\{ \sum_{m=1}^{\infty} p_m = \infty \right\}$$

*Proof.* Define a martingale by $X_0 = 0$ and $X_n - X_{n-1} = 1_{B_n} - P(B_n | \mathcal{F}_{n-1})$ for $n \ge 1$ so that we have

$$\left( \sum_{m=1}^{n} 1_{B(m)} \Bigg/ \sum_{m=1}^{n} p_m \right) - 1 = X_n \Bigg/ \sum_{m=1}^{n} p_m$$

The increasing process associated with $X_n$ has

$$\begin{aligned} A_n - A_{n-1} &= E((X_n - X_{n-1})^2 | \mathcal{F}_{n-1}) \\ &= E\left( (1_{B_n} - p_n)^2 \big| \mathcal{F}_{n-1} \right) = p_n - p_n^2 \le p_n \end{aligned}$$

On $\{A_\infty < \infty\}$, $X_n \to$ a finite limit by Theorem 5.4.9, so on $\{A_\infty < \infty\} \cap \{\sum_m p_m = \infty\}$

$$X_n \bigg/ \sum_{m=1}^n p_m \to 0$$

$\{A_\infty = \infty\} = \{\sum_m p_m(1-p_m) = \infty\} \subset \{\sum_m p_m = \infty\}$, so on $\{A_\infty = \infty\}$ the desired conclusion follows from Theorem 5.4.10 with $f(t) = t \vee 1$. ■

**Remark.** The trivial example $B_n = \Omega$ for all $n$ shows we may have $A_\infty < \infty$ and $\sum p_m = \infty$ a.s.

**Example 5.4.5. Bernard Friedman's urn.** Consider a variant of Polya's urn (see Section 5.3) in which we add $a$ balls of the color drawn and $b$ balls of the opposite color where $a \geq 0$ and $b > 0$. We will show that if we start with $g$ green balls and $r$ red balls, where $g, r > 0$, then the fraction of green balls $g_n \to 1/2$. Let $G_n$ and $R_n$ be the number of green and red balls after the $n$th draw is completed. Let $B_n$ be the event that the $n$th ball drawn is green, and let $D_n$ be the number of green balls drawn in the first $n$ draws. It follows from Theorem 5.4.11 that

$$(\star) \qquad D_n \bigg/ \sum_{m=1}^n g_{m-1} \to 1 \quad \text{a.s. on} \quad \sum_{m=1}^\infty g_{m-1} = \infty$$

which always holds since $g_m \geq g/(g + r + (a+b)m)$. At this point, the argument breaks into three cases.

*Case 1.* $a = b = c$. In this case, the result is trivial since we always add $c$ balls of each color.

*Case 2.* $a > b$. We begin with the observation

$$(*) \qquad g_{n+1} = \frac{G_{n+1}}{G_{n+1} + R_{n+1}} = \frac{g + aD_n + b(n - D_n)}{g + r + n(a+b)}$$

If $\text{limsup}_{n\to\infty} g_n \leq x$ then $(\star)$ implies $\text{limsup}_{n\to\infty} D_n/n \leq x$ and (since $a > b$)

$$\limsup_{n\to\infty} g_{n+1} \leq \frac{ax + b(1-x)}{a+b} = \frac{b + (a-b)x}{a+b}$$

The right-hand side is a linear function with slope $< 1$ and fixed point at $1/2$, so starting with the trivial upper bound $x = 1$ and iterating, we conclude that $\limsup g_n \leq 1/2$. Interchanging the roles of red and green shows $\liminf_{n\to\infty} g_n \geq 1/2$, and the result follows.

*Case 3.* $a < b$. The result is easier to believe in this case, since we are adding more balls of the type not drawn, but is a little harder to prove. The trouble is that when $b > a$ and $D_n \leq xn$, the right-hand side of $(*)$ is maximized by taking $D_n = 0$, so we need to also use the fact that if $r_n$ is fraction of red balls, then

$$r_{n+1} = \frac{R_{n+1}}{G_{n+1} + R_{n+1}} = \frac{r + bD_n + a(n - D_n)}{g + r + n(a + b)}$$

Combining this with the formula for $g_{n+1}$, it follows that if $\limsup_{n\to\infty} g_n \le x$ and $\limsup_{n\to\infty} r_n \le y$, then

$$\limsup_{n\to\infty} g_n \le \frac{a(1-y)+by}{a+b} = \frac{a+(b-a)y}{a+b}$$
$$\limsup_{n\to\infty} r_n \le \frac{bx+a(1-x)}{a+b} = \frac{a+(b-a)x}{a+b}$$

Starting with the trivial bounds $x = 1$, $y = 1$ and iterating (observe that the two upper bounds are always the same), we conclude as in Case 2 that both limsups are $\le 1/2$. ■

**Remark.** B. Friedman (1949) considered a number of different urn models. The result above is due to Freedman (1965), who proved the result by different methods. The proof above is due to Ornstein and comes from a remark in Freedman's paper.

Theorem 5.4.8 came from using Theorem 5.4.3. If we use Theorem 5.4.2 instead, we get a slightly better result.

**Theorem 5.4.12.** $E(\sup_n |X_n|) \le 3EA_\infty^{1/2}$.

*Proof.* As in the proof of Theorem 5.4.9 we let $a > 0$ and let $N = \inf\{n : A_{n+1} > a^2\}$. This time, however, our starting point is

$$P\left(\sup_m |X_m| > a\right) \le P(N < \infty) + P\left(\sup_m |X_{N\wedge m}| > a\right)$$

$P(N < \infty) = P(A_\infty > a^2)$. To bound the second term, we apply Theorem 5.4.2 to $X^2_{N\wedge m}$ with $\lambda = a^2$ to get

$$P\left(\sup_{m\le n} |X_{N\wedge m}| > a\right) \le a^{-2}EX^2_{N\wedge n} = a^{-2}EA_{N\wedge n} \le a^{-2}E(A_\infty \wedge a^2)$$

Letting $n \to \infty$ in the last inequality, substituting the result in the first one, and integrating gives

$$\int_0^\infty P\left(\sup_m |X_m| > a\right) da \le \int_0^\infty P(A_\infty > a^2)\, da + \int_0^\infty a^{-2}E(A_\infty \wedge a^2)\, da$$

Since $P(A_\infty > a^2) = P(A_\infty^{1/2} > a)$, the first integral is $EA_\infty^{1/2}$. For the second, we use Lemma 2.2.8 (in the first and fourth steps), Fubini's theorem, and calculus to get

$$\int_0^\infty a^{-2} E(A_\infty \wedge a^2)\, da = \int_0^\infty a^{-2} \int_0^{a^2} P(A_\infty > b)\, db\, da$$
$$= \int_0^\infty P(A_\infty > b) \int_{\sqrt{b}}^\infty a^{-2}\, da\, db = \int_0^\infty b^{-1/2} P(A_\infty > b)\, db = 2EA_\infty^{1/2}$$

which completes the proof. ■

**Exercise 5.4.10.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $E\xi_i = 0$ and $E\xi_i^2 < \infty$. Let $S_n = \xi_1 + \cdots + \xi_n$. Theorem 5.4.1 implies that for any stopping time $N$, $ES_{N\wedge n} = 0$. Use Theorem 5.4.12 to conclude that if $EN^{1/2} < \infty$ then $ES_N = 0$.

**Remark.** Let $\xi_i$ in Exercise 5.4.10 take the values $\pm 1$ with equal probability, and let $T = \inf\{n : S_n = -1\}$. Since $S_T = -1$ does not have mean 0, it follows that $ET^{1/2} = \infty$. If we recall from (4.3.2) that $P(T > t) \sim Ct^{-1/2}$, we see that the result in Exercise 5.4.10 is almost the best possible.

## 5.5 Uniform Integrability, Convergence in $L^1$

In this section, we will give necessary and sufficient conditions for a martingale to converge in $L^1$. The key to this is the following definition. A collection of random variables $X_i$, $i \in I$, is said to be **uniformly integrable** if

$$\lim_{M\to\infty} \left( \sup_{i\in I} E(|X_i|; |X_i| > M) \right) = 0$$

If we pick $M$ large enough so that the sup $< 1$, it follows that

$$\sup_{i\in I} E|X_i| \le M + 1 < \infty$$

This remark will be useful several times below.

A trivial example of a uniformly integrable family is a collection of random variables that are dominated by an integrable random variable, that is, $|X_i| \le Y$ where $EY < \infty$. Our first result gives an interesting example that shows that uniformly integrable families can be very large.

**Theorem 5.5.1.** *Given a probability space $(\Omega, \mathcal{F}_o, P)$ and an $X \in L^1$, then $\{E(X|\mathcal{F}) : \mathcal{F}$ is a $\sigma$-field $\subset \mathcal{F}_o\}$ is uniformly integrable.*

*Proof.* If $A_n$ is a sequence of sets with $P(A_n) \to 0$, then the dominated convergence theorem implies $E(|X|; A_n) \to 0$. From the last result, it follows that if $\epsilon > 0$, we can pick $\delta > 0$ so that if $P(A) \le \delta$ then $E(|X|; A) \le \epsilon$. (If not, there are sets $A_n$ with $P(A_n) \le 1/n$ and $E(|X|; A_n) > \epsilon$, a contradiction.)

Pick $M$ large enough so that $E|X|/M \le \delta$. Jensen's inequality and the definition of conditional expectation imply

$$E(\,|E(X|\mathcal{F})|\,;\,|E(X|\mathcal{F})| > M) \le E(\,E(|X||\mathcal{F})\,;\,E(|X||\mathcal{F}) > M)$$
$$= E(\,|X|\,;\,E(|X||\mathcal{F}) > M)$$

since $\{E(|X||\mathcal{F}) > M\} \in \mathcal{F}$. Using Chebyshev's inequality and recalling the definition of $M$, we have

$$P\{E(|X||\mathcal{F}) > M\} \le E\{E(|X||\mathcal{F})\}/M = E|X|/M \le \delta$$

So, by the choice of $\delta$, we have

$$E(|E(X|\mathcal{F})|; |E(X|\mathcal{F})| > M) \le \epsilon \qquad \text{for all } \mathcal{F}$$

Since $\epsilon$ was arbitrary, the collection is uniformly integrable. ∎

A common way to check uniform integrability is to use:

**Exercise 5.5.1.** Let $\varphi \ge 0$ be any function with $\varphi(x)/x \to \infty$ as $x \to \infty$, for example, $\varphi(x) = x^p$ with $p > 1$ or $\varphi(x) = x \log^+ x$. If $E\varphi(|X_i|) \le C$ for all $i \in I$, then $\{X_i : i \in I\}$ is uniformly integrable.

The relevance of uniform integrability to convergence in $L^1$ is explained by:

**Theorem 5.5.2.** *If $X_n \to X$ in probability, then the following are equivalent:*
*(i) $\{X_n : n \ge 0\}$ is uniformly integrable.*
*(ii) $X_n \to X$ in $L^1$.*
*(iii) $E|X_n| \to E|X| < \infty$.*

*Proof. (i) implies (ii).* Let

$$\varphi_M(x) = \begin{cases} M & \text{if } x \ge M \\ x & \text{if } |x| \le M \\ -M & \text{if } x \le -M \end{cases}$$

The triangle inequality implies

$$|X_n - X| \le |X_n - \varphi_M(X_n)| + |\varphi_M(X_n) - \varphi_M(X)| + |\varphi_M(X) - X|$$

Since $|\varphi_M(Y) - Y)| = (|Y| - M)^+ \le |Y|1_{(|Y|>M)}$, taking expected value gives

$$E|X_n - X| \le E|\varphi_M(X_n) - \varphi_M(X)| + E(|X_n|; |X_n| > M) + E(|X|; |X| > M)$$

Theorem 2.3.4 implies that $\varphi_M(X_n) \to \varphi_M(X)$ in probability, so the first term $\to 0$ by the bounded convergence theorem. (See Exercise 2.3.7.) If $\epsilon > 0$ and $M$ is large, uniform integrability implies that the second term $\le \epsilon$. To bound the third term, we observe that uniform integrability implies $\sup E|X_n| < \infty$, so Fatou's lemma (in the form given in Exercise 2.3.6) implies $E|X| < \infty$, and by making

$M$ larger we can make the third term $\le \epsilon$. Combining the last three facts shows $\limsup E|X_n - X| \le 2\epsilon$. Since $\epsilon$ is arbitrary, this proves (ii).

*(ii) implies (iii).* Jensen's inequality implies

$$|E|X_n| - E|X|| \le E||X_n| - |X|| \le E|X_n - X| \to 0$$

*(iii) implies (i).* Let

$$\psi_M(x) = \begin{cases} x & \text{on } [0, M-1], \\ 0 & \text{on } [M, \infty) \\ \text{linear} & \text{on } [M-1, M] \end{cases} .$$

The dominated convergence theorem implies that if $M$ is large, $E|X| - E\psi_M(|X|) \le \epsilon/2$. As in the first part of the proof, the bounded convergence theorem implies $E\psi_M(|X_n|) \to E\psi_M(|X|)$, so using (iii) we get that if $n \ge n_0$

$$\begin{aligned} E(|X_n|; |X_n| > M) &\le E|X_n| - E\psi_M(|X_n|) \\ &\le E|X| - E\psi_M(|X|) + \epsilon/2 < \epsilon \end{aligned}$$

By choosing $M$ larger, we can make $E(|X_n|; |X_n| > M) \le \epsilon$ for $0 \le n < n_0$, so $X_n$ is uniformly integrable. ■

We are now ready to state the main theorems of this section. We have already done all the work, so the proofs are short.

**Theorem 5.5.3.** *For a submartingale, the following are equivalent:*
*(i) It is uniformly integrable.*
*(ii) It converges a.s. and in $L^1$.*
*(iii) It converges in $L^1$.*

*Proof.* *(i) implies (ii).* Uniform integrability implies $\sup E|X_n| < \infty$ so the martingale convergence theorem implies $X_n \to X$ a.s., and Theorem 5.5.2 implies $X_n \to X$ in $L^1$. *(ii) implies (iii).* Trivial. *(iii) implies (i).* $X_n \to X$ in $L^1$ implies $X_n \to X$ in probability, (see Lemma 2.2.2) so this follows from Theorem 5.5.2. ■

Before proving the analogue of Theorem 5.5.3 for martingales, we will isolate two parts of the argument that will be useful later.

**Lemma 5.5.4.** *If integrable random variables $X_n \to X$ in $L^1$ then*

$$E(X_n; A) \to E(X; A)$$

*Proof.* $|EX_m 1_A - EX 1_A| \le E|X_m 1_A - X 1_A| \le E|X_m - X| \to 0$. ■

**Lemma 5.5.5.** *If a martingale $X_n \to X$ in $L^1$, then $X_n = E(X|\mathcal{F}_n)$.*

*Proof.* The martingale property implies that if $m > n$, $E(X_m|\mathcal{F}_n) = X_n$, so if $A \in \mathcal{F}_n$, $E(X_n; A) = E(X_m; A)$. Lemma 5.5.4 implies $E(X_m; A) \to E(X; A)$, so we have $E(X_n; A) = E(X; A)$ for all $A \in \mathcal{F}_n$. Recalling the definition of conditional expectation, it follows that $X_n = E(X|\mathcal{F}_n)$. ■

**Theorem 5.5.6.** *For a martingale, the following are equivalent:*
*(i) It is uniformly integrable.*
*(ii) It converges a.s. and in $L^1$.*
*(iii) It converges in $L^1$.*
*(iv) There is an integrable random variable $X$ so that $X_n = E(X|\mathcal{F}_n)$.*

*Proof.* *(i) implies (ii).* Since martingales are also submartingales, this follows from Theorem 5.5.3. *(ii) implies (iii).* Trivial. *(iii) implies (iv).* Follows from Lemma 5.5.5. *(iv) implies (i).* This follows from Theorem 5.5.1. ■

The next result is related to Lemma 5.5.5, but goes in the other direction.

**Theorem 5.5.7.** *Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, i.e., $\mathcal{F}_n$ is an increasing sequence of $\sigma$-fields and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. As $n \to \infty$,*

$$E(X|\mathcal{F}_n) \to E(X|\mathcal{F}_\infty) \quad \text{a.s. and in } L^1$$

*Proof.* The first step is to note that if $m > n$ then Theorem 5.1.6 implies

$$E(E(X|\mathcal{F}_m)|\mathcal{F}_n) = E(X|\mathcal{F}_n)$$

so $Y_n = E(X|\mathcal{F}_n)$ is a martingale. Theorem 5.5.1 implies that $Y_n$ is uniformly integrable, so Theorem 5.5.6 implies that $Y_n$ converges a.s. and in $L^1$ to a limit $Y_\infty$. The definition of $Y_n$ and Lemma 5.5.5 imply $E(X|\mathcal{F}_n) = Y_n = E(Y_\infty|\mathcal{F}_n)$, and hence

$$\int_A X\,dP = \int_A Y_\infty\,dP \qquad \text{for all } A \in \mathcal{F}_n$$

Since $X$ and $Y_\infty$ are integrable, and $\cup_n \mathcal{F}_n$ is a $\pi$-system, the $\pi - \lambda$ theorem implies that the last result holds for all $A \in \mathcal{F}_\infty$. Since $Y_\infty \in \mathcal{F}_\infty$, it follows that $Y_\infty = E(X|\mathcal{F}_\infty)$. ■

**Exercise 5.5.2.** Let $Z_1, Z_2, \ldots$ be i.i.d. with $E|Z_i| < \infty$, let $\theta$ be an independent r.v. with finite mean, and let $Y_i = Z_i + \theta$. If $Z_i$ is normal(0,1), then in statistical terms we have a sample from a normal population with variance 1 and unknown mean. The distribution of $\theta$ is called the **prior distribution**, and $P(\theta \in \cdot|Y_1, \ldots, Y_n)$ is called the **posterior distribution** after $n$ observations. Show that $E(\theta|Y_1, \ldots, Y_n) \to \theta$ a.s.

In the next two exercises, $\Omega = [0, 1)$, $I_{k,n} = [k2^{-n}, (k+1)2^{-n})$, and $\mathcal{F}_n = \sigma(I_{k,n} : 0 \le k < 2^n)$.

**Exercise 5.5.3.** $f$ is said to be **Lipschitz continuous** if $|f(t) - f(s)| \le K|t - s|$ for $0 \le s, t < 1$. Show that $X_n = (f((k+1)2^{-n}) - f(k2^{-n}))/2^{-n}$ on $I_{k,n}$ defines a martingale, $X_n \to X_\infty$ a.s. and in $L^1$, and

$$f(b) - f(a) = \int_a^b X_\infty(\omega)\, d\omega$$

**Exercise 5.5.4.** Suppose $f$ is integrable on [0,1). $E(f|\mathcal{F}_n)$ is a step function and $\to f$ in $L^1$. From this it follows immediately that if $\epsilon > 0$, there is a step function $g$ on [0,1] with $\int |f - g|\, dx < \epsilon$. This approximation is much simpler than the bare-hands approach we used in Exercise 1.4.3, but of course we are using a lot of machinery.

An immediate consequence of Theorem 5.5.7 is:

**Theorem 5.5.8. Lévy's 0-1 law.** *If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$, then $E(1_A|\mathcal{F}_n) \to 1_A$ a.s.*

To steal a line from Chung: *"The reader is urged to ponder over the meaning of this result and judge for himself whether it is obvious or incredible."* We will now argue for the two points of view.

*"It is obvious."* $1_A \in \mathcal{F}_\infty$, and $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, so our best guess of $1_A$ given the information in $\mathcal{F}_n$ should approach $1_A$ (the best guess given $\mathcal{F}_\infty$).

*"It is incredible."* Let $X_1, X_2, \ldots$ be independent and suppose $A \in \mathcal{T}$, the tail $\sigma$-field. For each $n$, $A$ is independent of $\mathcal{F}_n$, so $E(1_A|\mathcal{F}_n) = P(A)$. As $n \to \infty$, the left-hand side converges to $1_A$ a.s., so $P(A) = 1_A$ a.s., and it follows that $P(A) \in \{0, 1\}$, that is, we have proved Kolmogorov's 0-1 law.

The last argument may not show that Theorem 5.5.8 is "too unusual or improbable to be possible," but this and other applications of Theorem 5.5.8 below show that it is a very useful result.

**Exercise 5.5.5.** Let $X_n$ be r.v.'s taking values in $[0, \infty)$. Let $D = \{X_n = 0 \text{ for some } n \ge 1\}$ and assume

$$P(D|X_1, \ldots, X_n) \ge \delta(x) > 0 \quad \text{a.s. on } \{X_n \le x\}$$

Use Theorem 5.5.8 to conclude that $P(D \cup \{\lim_n X_n = \infty\}) = 1$.

**Exercise 5.5.6.** Let $Z_n$ be a branching process with offspring distribution $p_k$ (see the end of Section 5.3 for definitions). Use the last result to show that if $p_0 > 0$, then $P\,(\lim_n Z_n = 0 \text{ or } \infty) = 1$.

**Exercise 5.5.7.** Let $X_n \in [0, 1]$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and suppose

$$P(X_{n+1} = \alpha + \beta X_n | \mathcal{F}_n) = X_n \qquad P(X_{n+1} = \beta X_n | \mathcal{F}_n) = 1 - X_n$$

Show $P(\lim_n X_n = 0 \text{ or } 1) = 1$ and if $X_0 = \theta$ then $P(\lim_n X_n = 1) = \theta$.

A more technical consequence of Theorem 5.5.7 is:

**Theorem 5.5.9. Dominated convergence theorem for conditional expectations.** *Suppose $Y_n \to Y$ a.s. and $|Y_n| \le Z$ for all $n$ where $EZ < \infty$. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ then*

$$E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty) \quad a.s.$$

*Proof.* Let $W_N = \sup\{|Y_n - Y_m| : n, m \ge N\}$. $W_N \le 2Z$, so $EW_N < \infty$. Using monotonicity (5.1.2) and applying Theorem 5.5.7 to $W_N$ gives

$$\limsup_{n\to\infty} E(|Y_n - Y||\mathcal{F}_n) \le \lim_{n\to\infty} E(W_N|\mathcal{F}_n) = E(W_N|\mathcal{F}_\infty)$$

The last result is true for all $N$ and $W_N \downarrow 0$ as $N \uparrow \infty$, so (5.1.3) implies $E(W_N|\mathcal{F}_\infty) \downarrow 0$, and Jensen's inequality gives us

$$|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \le E(|Y_n - Y||\mathcal{F}_n) \to 0 \quad \text{a.s. as } n \to \infty$$

Theorem 5.5.7 implies $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ a.s. The desired result follows from the last two conclusions and the triangle inequality. ∎

**Exercise 5.5.8.** Show that if $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y$ in $L^1$, then $E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ in $L^1$.

**Example 5.5.1.** Suppose $X_1, X_2, \ldots$ are uniformly integrable and $\to X$ a.s. Theorem 5.5.2 implies $X_n \to X$ in $L^1$, and combining this with Exercise 5.5.8 shows $E(X_n|\mathcal{F}) \to E(X|\mathcal{F})$ in $L^1$. We will now show that $E(X_n|\mathcal{F})$ need not converge a.s. Let $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ be independent r.v.'s with

$$P(Y_n = 1) = 1/n \qquad P(Y_n = 0) = 1 - 1/n$$
$$P(Z_n = n) = 1/n \qquad P(Z_n = 0) = 1 - 1/n$$

Let $X_n = Y_n Z_n$. $P(X_n > 0) = 1/n^2$ so the Borel-Cantelli lemma implies $X_n \to 0$ a.s. $E(X_n; |X_n| \ge 1) = n/n^2$, so $X_n$ is uniformly integrable. Let $\mathcal{F} = \sigma(Y_1, Y_2, \ldots)$.

$$E(X_n|\mathcal{F}) = Y_n E(Z_n|\mathcal{F}) = Y_n E Z_n = Y_n$$

Since $Y_n \to 0$ in $L^1$ but not a.s., the same is true for $E(X_n|\mathcal{F})$.

## 5.6 Backwards Martingales

A **backwards martingale** (some authors call them reversed) is a martingale indexed by the negative integers, that is, $X_n$, $n \le 0$, adapted to an increasing sequence of $\sigma$-fields $\mathcal{F}_n$ with

$$E(X_{n+1}|\mathcal{F}_n) = X_n \quad \text{for } n \le -1$$

Because the $\sigma$-fields decrease as $n \downarrow -\infty$, the convergence theory for backwards martingales is particularly simple.

**Theorem 5.6.1.** *$X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in $L^1$.*

*Proof.* Let $U_n$ be the number of upcrossings of $[a, b]$ by $X_{-n}, \dots, X_0$. The upcrossing inequality, Theorem 5.2.7, implies $(b - a)EU_n \le E(X_0 - a)^+$. Letting $n \to \infty$ and using the monotone convergence theorem, we have $EU_\infty < \infty$, so by the remark after the proof of Theorem 5.2.8, the limit exists a.s. The martingale property implies $X_n = E(X_0|\mathcal{F}_n)$, so Theorem 5.5.1 implies $X_n$ is uniformly integrable and Theorem 5.5.2 tells us that the convergence occurs in $L^1$. ■

**Exercise 5.6.1.** Show that if $X_0 \in L^p$ the convergence occurs in $L^p$.

The next result identifies the limit in Theorem 5.6.1.

**Theorem 5.6.2.** *If $X_{-\infty} = \lim_{n\to-\infty} X_n$ and $\mathcal{F}_{-\infty} = \cap_n \mathcal{F}_n$, then $X_{-\infty} = E(X_0|\mathcal{F}_{-\infty})$.*

*Proof.* Clearly, $X_{-\infty} \in \mathcal{F}_{-\infty}$. $X_n = E(X_0|\mathcal{F}_n)$, so if $A \in \mathcal{F}_{-\infty} \subset \mathcal{F}_n$ then

$$\int_A X_n \, dP = \int_A X_0 \, dP$$

Theorem 5.6.1 and Lemma 5.5.4 imply $E(X_n; A) \to E(X_{-\infty}; A)$, so

$$\int_A X_{-\infty} \, dP = \int_A X_0 \, dP$$

for all $A \in \mathcal{F}_{-\infty}$, proving the desired conclusion. ■

The next result is Theorem 5.5.7 backwards.

**Theorem 5.6.3.** *If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n \downarrow -\infty$ (i.e., $\mathcal{F}_{-\infty} = \cap_n \mathcal{F}_n$), then*

$$E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_{-\infty}) \quad \textit{a.s. and in } L^1$$

*Proof.* $X_n = E(Y|\mathcal{F}_n)$ is a backwards martingale, so Theorem 5.6.1 and 5.6.2 imply that as $n \downarrow -\infty$, $X_n \to X_{-\infty}$ a.s. and in $L^1$, where

$$X_{-\infty} = E(X_0|\mathcal{F}_{-\infty}) = E(E(Y|\mathcal{F}_0)|\mathcal{F}_{-\infty}) = E(Y|\mathcal{F}_{-\infty})$$ ■

**Exercise 5.6.2.** Prove the backwards analogue of Theorem 5.5.9. Suppose $Y_n \to Y_{-\infty}$ a.s. as $n \to -\infty$ and $|Y_n| \le Z$ a.s. where $EZ < \infty$. If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$, then $E(Y_n|\mathcal{F}_n) \to E(Y_{-\infty}|\mathcal{F}_{-\infty})$ a.s.

Even though the convergence theory for backwards martingales is easy, there are some nice applications. For the rest of the section, we return to the special space utilized in Section 4.1, so we can utilize definitions given there. That is, we suppose

$$\Omega = \{(\omega_1, \omega_2, \dots) : \omega_i \in S\}$$
$$\mathcal{F} = \mathcal{S} \times \mathcal{S} \times \dots$$
$$X_n(\omega) = \omega_n$$

Let $\mathcal{E}_n$ be the $\sigma$-field generated by events that are invariant under permutations that leave $n+1, n+2, \dots$ fixed and let $\mathcal{E} = \cap_n \mathcal{E}_n$ be the exchangeable $\sigma$-field.

**Example 5.6.1. Strong law of large numbers.** Let $\xi_1, \xi_2, \dots$ be i.i.d. with $E|\xi_i| < \infty$. Let $S_n = \xi_1 + \dots + \xi_n$, let $X_{-n} = S_n/n$, and let

$$\mathcal{F}_{-n} = \sigma(S_n, S_{n+1}, S_{n+2}, \dots) = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, \dots)$$

To compute $E(X_{-n}|\mathcal{F}_{-n-1})$, we observe that if $j, k \le n+1$, symmetry implies $E(\xi_j|\mathcal{F}_{-n-1}) = E(\xi_k|\mathcal{F}_{-n-1})$, so

$$\begin{aligned} E(\xi_{n+1}|\mathcal{F}_{-n-1}) &= \frac{1}{n+1}\sum_{k=1}^{n+1} E(\xi_k|\mathcal{F}_{-n-1}) \\ &= \frac{1}{n+1} E(S_{n+1}|\mathcal{F}_{-n-1}) = \frac{S_{n+1}}{n+1} \end{aligned}$$

Since $X_{-n} = (S_{n+1} - \xi_{n+1})/n$, it follows that

$$\begin{aligned} E(X_{-n}|\mathcal{F}_{-n-1}) &= E(S_{n+1}/n|\mathcal{F}_{-n-1}) - E(\xi_{n+1}/n|\mathcal{F}_{-n-1}) \\ &= \frac{S_{n+1}}{n} - \frac{S_{n+1}}{n(n+1)} = \frac{S_{n+1}}{n+1} = X_{-n-1} \end{aligned}$$

The last computation shows that $X_{-n}$ is a backwards martingale, so it follows from Theorems 5.6.1 and 5.6.2 that $\lim_{n\to\infty} S_n/n = E(X_{-1}|\mathcal{F}_{-\infty})$. Since $\mathcal{F}_{-n} \subset \mathcal{E}_n$, $\mathcal{F}_{-\infty} \subset \mathcal{E}$. The Hewitt-Savage 0-1 law (Theorem 4.1.1) says $\mathcal{E}$ is trivial, so we have

$$\lim_{n\to\infty} S_n/n = E(X_{-1}) \quad \text{a.s.}$$

**Example 5.6.2. Ballot theorem.** Let $\{\xi_j, 1 \le j \le n\}$ be i.i.d. nonnegative integer-valued r.v.'s, let $S_k = \xi_1 + \dots + \xi_k$, and let $G = \{S_j < j \text{ for } 1 \le j \le n\}$. Then

$$P(G|S_n) = (1 - S_n/n)^+ \tag{5.6.1}$$

**Remark.** To explain the name, let $\xi_1, \xi_2, \ldots, \xi_n$ be i.i.d. and take values 0 or 2 with probability 1/2 each. Interpreting 0's and 2's as votes for candidates $A$ and $B$, we see that $G = \{A \text{ leads } B \text{ throughout the counting}\}$ so if $n = \alpha + \beta$

$$P(G|B \text{ gets } \beta \text{ votes}) = \left(1 - \frac{2\beta}{n}\right)^+ = \frac{\alpha - \beta}{\alpha + \beta}$$

the result in Theorem 4.3.2.

*Proof.* The result is trivial when $S_n \geq n$, so suppose $S_n < n$. Computations in Example 5.6.1 show that $X_{-j} = S_j/j$ is a martingale w.r.t. $\mathcal{F}_{-j} = \sigma(S_j, \ldots, S_n)$. Let $T = \inf\{k \geq -n : X_k \geq 1\}$ and set $T = -1$ if the set is $\emptyset$. We claim that $X_T = 1$ on $G^c$. To check this, note that if $S_{j+1} < j + 1$, then $S_j \leq S_{j+1} \leq j$. Since $G \subset \{T = -1\}$ and $S_1 < 1$ implies $S_1 = 0$, we have $X_T = 0$ on $G$. Noting $\mathcal{F}_{-n} = \sigma(S_n)$ and using Exercise 5.4.3, we see that on $\{S_n < n\}$,

$$P(G^c|S_n) = E(X_T|\mathcal{F}_{-n}) = X_{-n} = S_n/n \qquad \blacksquare$$

**Example 5.6.3. Hewitt-Savage 0-1 law.** If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{E}$, then $P(A) \in \{0, 1\}$.

The key to the new proof is:

**Lemma 5.6.4.** *Suppose $X_1, X_2, \ldots$ are i.i.d. and let*

$$A_n(\varphi) = \frac{1}{(n)_k} \sum_i \varphi(X_{i_1}, \ldots, X_{i_k})$$

*where the sum is over all sequences of distinct integers $1 \leq i_1, \ldots, i_k \leq n$ and*

$$(n)_k = n(n-1)\cdots(n-k+1)$$

*is the number of such sequences. If $\varphi$ is bounded, $A_n(\varphi) \to E\varphi(X_1, \ldots, X_k)$ a.s.*

*Proof.* $A_n(\varphi) \in \mathcal{E}_n$, so

$$A_n(\varphi) = E(A_n(\varphi)|\mathcal{E}_n) = \frac{1}{(n)_k} \sum_i E(\varphi(X_{i_1}, \ldots, X_{i_k})|\mathcal{E}_n)$$
$$= E(\varphi(X_1, \ldots, X_k)|\mathcal{E}_n)$$

since all the terms in the sum are the same. Theorem 5.6.3 with $\mathcal{F}_{-m} = \mathcal{E}_m$ for $m \geq 1$ implies that

$$E(\varphi(X_1, \ldots, X_k)|\mathcal{E}_n) \to E(\varphi(X_1, \ldots, X_k)|\mathcal{E})$$

We want to show that the limit is $E(\varphi(X_1, \ldots, X_k))$. The first step is to observe that there are $k(n-1)_{k-1}$ terms in $A_n(\varphi)$ involving $X_1$ and $\varphi$ is bounded, so if we let $1 \in i$ denote the sum over sequences that contain 1.

$$\frac{1}{(n)_k} \sum_{1 \in i} \varphi(X_{i_1}, \ldots, X_{i_k}) \leq \frac{k(n-1)_{k-1}}{(n)_k} \sup \phi \to 0$$

This shows that

$$E(\varphi(X_1, \ldots, X_k)|\mathcal{E}) \in \sigma(X_2, X_3, \ldots)$$

Repeating the argument for $2, 3, \ldots, k$ shows

$$E(\varphi(X_1, \ldots, X_k)|\mathcal{E}) \in \sigma(X_{k+1}, X_{k+2}, \ldots)$$

Intuitively, if the conditional expectation of a r.v. is independent of the r.v. then

(a) $$E(\varphi(X_1, \ldots, X_k)|\mathcal{E}) = E(\varphi(X_1, \ldots, X_k))$$

To show this, we prove:

(b) If $EX^2 < \infty$ and $E(X|\mathcal{G}) \in \mathcal{F}$ with $X$ independent of $\mathcal{F}$ then $E(X|\mathcal{G}) = EX$.

*Proof.* Let $Y = E(X|\mathcal{G})$ and note that Theorem 5.1.4 implies $EY^2 \le EX^2 < \infty$. By independence, $EXY = EX\,EY = (EY)^2$ since $EY = EX$. From the geometric interpretation of conditional expectation, Theorem 5.1.8, $E((X - Y)Y) = 0$, so $EY^2 = EXY = (EY)^2$ and $\text{var}\,(Y) = EY^2 - (EY)^2 = 0$. ■

(a) holds for all bounded $\varphi$, so $\mathcal{E}$ is independent of $\mathcal{G}_k = \sigma(X_1, \ldots, X_k)$. Since this holds for all $k$, and $\cup_k \mathcal{G}_k$ is a $\pi$-system that contains $\Omega$, Theorem 2.1.2 implies that $\mathcal{E}$ is independent of $\sigma(\cup_k \mathcal{G}_k) \supset \mathcal{E}$, and we get the usual 0-1 law punch line. If $A \in \mathcal{E}$, it is independent of itself, and hence $P(A) = P(A \cap A) = P(A)P(A)$, that is, $P(A) \in \{0, 1\}$. ■

**Example 5.6.4. de Finetti's Theorem.** A sequence $X_1, X_2, \ldots$ is said to be **exchangeable** if for each $n$ and permutation $\pi$ of $\{1, \ldots, n\}$, $(X_1, \ldots, X_n)$ and $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ have the same distribution.

**Theorem 5.6.5. de Finetti's Theorem.** *If $X_1, X_2, \ldots$ are exchangeable, then conditional on $\mathcal{E}$, $X_1, X_2, \ldots$ are independent and identically distributed.*

*Proof.* Repeating the first calculation in the proof of Lemma 5.6.4 and using the notation introduced there shows that for any exchangeable sequence,

$$\begin{aligned} A_n(\varphi) = E(A_n(\varphi)|\mathcal{E}_n) &= \frac{1}{(n)_k} \sum_i E(\varphi(X_{i_1}, \ldots, X_{i_k})|\mathcal{E}_n) \\ &= E(\varphi(X_1, \ldots, X_k)|\mathcal{E}_n) \end{aligned}$$

since all the terms in the sum are the same. Again, Theorem 5.6.3 implies that

$$A_n(\varphi) \to E(\varphi(X_1, \ldots, X_k)|\mathcal{E}) \tag{5.6.2}$$

This time, however, $\mathcal{E}$ may be nontrivial, so we cannot hope to show that the limit is $E(\varphi(X_1, \ldots, X_k))$.

Let $f$ and $g$ be bounded functions on $\mathbf{R}^{k-1}$ and $\mathbf{R}$, respectively. If we let $I_{n,k}$ be the set of all sequences of distinct integers $1 \le i_1, \dots, i_k \le n$, then

$$\begin{aligned}(n)_{k-1} A_n(f)\, n A_n(g) &= \sum_{i \in I_{n,k-1}} f(X_{i_1}, \dots, X_{i_{k-1}}) \sum_m g(X_m) \\ &= \sum_{i \in I_{n,k}} f(X_{i_1}, \dots, X_{i_{k-1}}) g(X_{i_k}) \\ &\quad + \sum_{i \in I_{n,k-1}} \sum_{j=1}^{k-1} f(X_{i_1}, \dots, X_{i_{k-1}}) g(X_{i_j})\end{aligned}$$

If we let $\varphi(x_1, \dots, x_k) = f(x_1, \dots, x_{k-1}) g(x_k)$, note that

$$\frac{(n)_{k-1} n}{(n)_k} = \frac{n}{(n-k+1)} \quad \text{and} \quad \frac{(n)_{k-1}}{(n)_k} = \frac{1}{(n-k+1)}$$

then rearrange, we have

$$A_n(\varphi) = \frac{n}{n-k+1} A_n(f) A_n(g) - \frac{1}{n-k+1} \sum_{j=1}^{k-1} A_n(\varphi_j)$$

where $\varphi_j(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_{k-1}) g(x_j)$. Applying (5.6.2) to $\varphi$, $f$, $g$, and all the $\varphi_j$ gives

$$E(f(X_1, \dots, X_{k-1}) g(X_k) | \mathcal{E}) = E(f(X_1, \dots, X_{k-1}) | \mathcal{E}) E(g(X_k) | \mathcal{E})$$

It follows by induction that

$$E\left( \prod_{j=1}^k f_j(X_j) \middle| \mathcal{E} \right) = \prod_{j=1}^k E(f_j(X_j) | \mathcal{E}) \qquad \blacksquare$$

When the $X_i$ take values in a nice space, there is a regular conditional distribution for $(X_1, X_2, \dots)$ given $\mathcal{E}$, and the sequence can be represented as a mixture of i.i.d. sequences. Hewitt and Savage (1956) call the sequence **presentable** in this case. For the usual measure theoretic problems, the last result is not valid when the $X_i$ take values in an arbitrary measure space. See Dubins and Freedman (1979) and Freedman (1980) for counterexamples.

The simplest special case of Theorem 5.6.5 occurs when the $X_i \in \{0, 1\}$. In this case,

**Theorem 5.6.6.** *If $X_1, X_2, \dots$ are exchangeable and take values in $\{0, 1\}$ then there is a probability distribution on $[0, 1]$ so that*

$$P(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) = \int_0^1 \theta^k (1-\theta)^{n-k} \, dF(\theta)$$

This result is useful for people concerned about the foundations of statistics (see Section 3.7 of Savage (1972)), since from the palatable assumption of symmetry one gets the powerful conclusion that the sequence is a mixture of i.i.d. sequences.

Theorem 5.6.6 has been proved in a variety of different ways. See Feller, Vol. II (1971), pp. 228–9, for a proof that is related to the moment problem. Diaconis and Freedman (1980) have a nice proof that starts with the trivial observation that the distribution of a finite exchangeable sequence $X_m$, $1 \le m \le n$ has the form $p_0 H_{0,n} + \cdots + p_n H_{n,n}$ where $H_{m,n}$ is "drawing without replacement from an urn with $m$ ones and $n - m$ zeros." If $m \to \infty$ and $m/n \to p$ then $H_{m,n}$ approaches product measure with density $p$. Theorem 5.6.6 follows easily from this, and one can get bounds on the rate of convergence.

### *Exercises*

**5.6.3.** Prove directly from the definition that if $X_1, X_2, \ldots \in \{0, 1\}$ are exchangeable,

$$P(X_1 = 1, \ldots, X_k = 1 | S_n = m) = \binom{n-k}{n-m} \Big/ \binom{n}{m}$$

**5.6.4.** If $X_1, X_2, \ldots \in \mathbf{R}$ are exchangeable with $EX_i^2 < \infty$ then $E(X_1 X_2) \ge 0$.

**5.6.5.** Use the first few lines of the proof of Lemma 5.6.4 to conclude that if $X_1, X_2, \ldots$ are i.i.d. with $EX_i = \mu$ and $\text{var}(X_i) = \sigma^2 < \infty$ then

$$\binom{n}{2}^{-1} \sum_{1 \le i < j \le n} (X_i - X_j)^2 \to 2\sigma^2$$

## 5.7 Optional Stopping Theorems

In this section, we will prove a number of results that allow us to conclude that if $X_n$ is a submartingale and $M \le N$ are stopping times, then $EX_M \le EX_N$. Example 5.2.3 shows that this is not always true, but Exercise 5.4.2 shows this is true if $N$ is bounded, so our attention will be focused on the case of unbounded $N$.

**Theorem 5.7.1.** *If $X_n$ is a uniformly integrable submartingale, then for any stopping time $N$, $X_{N \wedge n}$ is uniformly integrable.*

*Proof.* $X_n^+$ is a submartingale, so Theorem 5.4.1 implies $EX_{N \wedge n}^+ \le EX_n^+$. Since $X_n^+$ is uniformly integrable, it follows from the remark after the definition that

$$\sup_n EX_{N \wedge n}^+ \le \sup_n EX_n^+ < \infty$$

Using the martingale convergence theorem (5.2.8) now gives $X_{N \wedge n} \to X_N$ a.s. (here $X_\infty = \lim_n X_n$) and $E|X_N| < \infty$. With this established, the rest is easy. We write

$$\begin{aligned} E(|X_{N \wedge n}|; |X_{N \wedge n}| > K) &= E(|X_N|; |X_N| > K, N \le n) \\ &\quad + E(|X_n|; |X_n| > K, N > n) \end{aligned}$$

Since $E|X_N| < \infty$ and $X_n$ is uniformly integrable, if $K$ is large then each term is $< \epsilon/2$. ■

From the last computation in the proof of Theorem 5.7.1, we get:

**Theorem 5.7.2.** *If $E|X_N| < \infty$ and $X_n 1_{(N>n)}$ is uniformly integrable, then $X_{N\wedge n}$ is uniformly integrable.*

From Theorem 5.7.1, we immediately get:

**Theorem 5.7.3.** *If $X_n$ is a uniformly integrable submartingale, then for any stopping time $N \le \infty$, we have $EX_0 \le EX_N \le EX_\infty$, where $X_\infty = \lim X_n$.*

*Proof.* Theorem 5.4.1 implies $EX_0 \le EX_{N\wedge n} \le EX_n$. Letting $n \to \infty$ and observing that Theorem 5.7.1 and 5.5.3 imply $X_{N\wedge n} \to X_N$ and $X_n \to X_\infty$ in $L^1$ gives the desired result. ■

From Theorem 5.7.3, we get the following useful corollary.

**Theorem 5.7.4. Optional stopping theorem.** *If $L \le M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale, then $EY_L \le EY_M$ and*

$$Y_L \le E(Y_M|\mathcal{F}_L)$$

*Proof.* Use the inequality $EX_N \le EX_\infty$ in Theorem 5.7.3 with $X_n = Y_{M\wedge n}$ and $N = L$. To prove the second result, let $A \in \mathcal{F}_L$ and

$$N = \begin{cases} L & \text{on } A \\ M & \text{on } A^c \end{cases}$$

is a stopping time by Exercise 4.1.7. Using the first result now shows $EY_N \le EY_M$. Since $N = M$ on $A^c$, it follows from the last inequality and the definition of conditional expectation that

$$E(Y_L; A) \le E(Y_M; A) = E(E(Y_M|\mathcal{F}_L); A)$$

Taking $A_\epsilon = \{Y_L - E(Y_M|\mathcal{F}_L) > \epsilon\}$, we conclude $P(A_\epsilon) = 0$ for all $\epsilon > 0$ and the desired result follows. ■

The last result is the one we use the most (usually the first inequality with $L = 0$). Theorem 5.7.2 is useful in checking the hypothesis. A typical application is the following generalization of Wald's equation, Theorem 4.1.5.

**Theorem 5.7.5.** *Suppose $X_n$ is a submartingale and $E(|X_{n+1} - X_n||\mathcal{F}_n) \le B$ a.s. If $N$ is a stopping time with $EN < \infty$, then $X_{N\wedge n}$ is uniformly integrable and hence $EX_N \ge EX_0$.*

**Remark.** As usual, using the last result twice shows that if $X$ is a martingale, then $EX_N = EX_0$. To recover Wald's equation, let $S_n$ be a random walk, let $\mu = E(S_n - S_{n-1})$, and apply the martingale result to $X_n = S_n - n\mu$.

*Proof.* We begin by observing that

$$|X_{N\wedge n}| \le |X_0| + \sum_{m=0}^{\infty} |X_{m+1} - X_m| 1_{(N>m)}$$

To prove uniform integrability, it suffices to show that the right-hand side has finite expectation for then $|X_{N\wedge n}|$ is dominated by an integrable r.v. Now, $\{N > m\} \in \mathcal{F}_m$, so

$$E(|X_{m+1} - X_m|; N > m) = E(E(|X_{m+1} - X_m||\mathcal{F}_m); N > m) \le BP(N > m)$$

and $E\sum_{m=0}^{\infty} |X_{m+1} - X_m| 1_{(N>m)} \le B \sum_{m=0}^{\infty} P(N > m) = BEN < \infty$. ■

Before we delve further into applications, we pause to prove one last stopping theorem that does not require uniform integrability.

**Theorem 5.7.6.** *If $X_n$ is a nonnegative supermartingale and $N \le \infty$ is a stopping time, then $EX_0 \ge EX_N$ where $X_\infty = \lim X_n$, which exists by Theorem 5.2.9.*

*Proof.* By Theorem 5.4.1, $EX_0 \ge EX_{N\wedge n}$. The monotone convergence theorem implies

$$E(X_N; N < \infty) = \lim_{n\to\infty} E(X_N; N \le n)$$

and Fatou's lemma implies

$$E(X_N; N = \infty) \le \liminf_{n\to\infty} E(X_n; N > n)$$

Adding the last two lines and using our first observation,

$$EX_N \le \liminf_{n\to\infty} EX_{N\wedge n} \le EX_0$$ ■

**Exercise 5.7.1.** If $X_n \ge 0$ is a supermartingale, then $P(\sup X_n > \lambda) \le EX_0/\lambda$.

**Applications to random walks.** For the rest of the section, including all the exercises below, $\xi_1, \xi_2, \ldots$ are i.i.d., $S_n = \xi_1 + \cdots + \xi_n$, and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

**Theorem 5.7.7. Asymmetric simple random walk** *refers to the special case in which $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = q \equiv 1 - p$ with $p \ne q$. Without loss of generality we assume $1/2 < p < 1$.*
*(a) If $\varphi(x) = \{(1-p)/p\}^x$, then $\varphi(S_n)$ is a martingale.*

*(b) If we let* $T_x = \inf\{n : S_n = x\}$, *then for* $a < 0 < b$

$$P(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}$$

*(c) If* $a < 0$, *then* $P(\min_n S_n \le a) = P(T_a < \infty) = \{(1-p)/p\}^{-a}$.
*(d) If* $b > 0$, *then* $P(T_b < \infty) = 1$ *and* $ET_b = b/(2p-1)$.

*Proof.* Since $S_n$ and $\xi_{n+1}$ are independent, Example 5.1.5 implies that on $\{S_n = m\}$,

$$E(\phi(S_{n+1})|\mathcal{F}_n) = p \cdot \left(\frac{1-p}{p}\right)^{m+1} + (1-p)\left(\frac{1-p}{p}\right)^{m-1}$$
$$= \{1 - p + p\}\left(\frac{1-p}{p}\right)^m = \phi(S_n)$$

which proves (a).

Let $N = T_a \wedge T_b$. We showed in Example 4.1.5 that $N < \infty$. Since $\phi(S_{N\wedge n})$ is bounded, it is uniformly integrable, and Theorem 5.7.4 with $L = 0$, $M = N$ implies

$$\phi(0) = E\phi(S_N) = P(T_a < T_b)\phi(a) + P(T_b < T_a)\phi(b)$$

Using $P(T_a < T_b) + P(T_b < T_a) = 1$ and solving gives (b).

Letting $b \to \infty$ and noting $\phi(b) \to 0$ gives the result in (c), since $T_a < \infty$ if and only if $T_a < T_b$ for some $b$. To start to prove $(d)$ we note that $\phi(a) \to \infty$ as $a \to -\infty$, so $P(T_b < \infty) = 1$. For the second conclusion, we note that $X_n = S_n - (p-q)n$ is a martingale. Since $T_b \wedge n$ is a bounded stopping time, Theorem 5.4.1 implies

$$0 = E\left(S_{T_b \wedge n} - (p-q)(T_b \wedge n)\right)$$

Now $b \ge S_{T_b\wedge n} \ge \min_m S_m$ and (c) implies $E(\inf_m S_m) > -\infty$, so the dominated convergence theorem implies $ES_{T_b\wedge n} \to ES_{T_b}$ as $n \to \infty$. The monotone convergence theorem implies $E(T_b \wedge n) \uparrow ET_b$, so we have $b = (p-q)ET_b$. ■

**Remark.** The reader should study the technique in this proof of (d) because it is useful in a number of situations (e.g., the exercises below). We apply Theorem 5.4.1 to the bounded stopping time $T_b \wedge n$, then let $n \to \infty$, and use appropriate convergence theorems. Here this is an alternative to showing that $X_{T_b\wedge n}$ is uniformly integrable.

## *Exercises*

**5.7.2.** Let $S_n$ be an asymmetric simple random walk with $1/2 < p < 1$, and let $\sigma^2 = pq$. Use the fact that $X_n = (S_n - (p-q)n)^2 - \sigma^2 n$ is a martingale to show $\text{var}(T_b) = b\sigma^2/(p-q)^3$.

**5.7.3.** Let $S_n$ be a symmetric simple random walk starting at 0, and let $T = \inf\{n : S_n \notin (-a, a)\}$ where $a$ is an integer. (i) Use the fact that $S_n^2 - n$ is a martingale to show that $ET = a^2$. (ii) Find constants $b$ and $c$ so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale, and use this to compute $ET^2$.

The last five exercises are devoted to the study of exponential martingales.

**5.7.4.** Suppose $\xi_i$ is not constant. Let $\varphi(\theta) = E\exp(\theta\xi_1) < \infty$ for $\theta \in (-\delta, \delta)$, and let $\psi(\theta) = \log\varphi(\theta)$. (i) $X_n^\theta = \exp(\theta S_n - n\psi(\theta))$ is a martingale. (ii) $\psi$ is strictly convex. (iii) Show $E\sqrt{X_n^\theta} \to 0$ and conclude that $X_n^\theta \to 0$ a.s.

**5.7.5.** Let $S_n$ be asymmetric simple random walk with $p \geq 1/2$. Let $T_1 = \inf\{n : S_n = 1\}$. Use the martingale of Exercise 7.4 to conclude (i) if $\theta > 0$ then $1 = e^\theta E\varphi(\theta)^{-T_1}$, where $\varphi(\theta) = pe^\theta + qe^{-\theta}$ and $q = 1 - p$. (ii) Set $pe^\theta + qe^{-\theta} = 1/s$ and then solve for $x = e^{-\theta}$ to get

$$Es^{T_1} = (1 - \{1 - 4pqs^2\}^{1/2})/2qs$$

**5.7.6.** Suppose $\varphi(\theta_o) = E\exp(\theta_o\xi_1) = 1$ for some $\theta_o < 0$ and $\xi_i$ is not constant. It follows from the result in Exercise 5.7.4 that $X_n = \exp(\theta_o S_n)$ is a martingale. Let $T = \inf\{n : S_n \notin (a, b)\}$ and $Y_n = X_{n\wedge T}$. Use Theorem 5.7.4 to conclude that $EX_T = 1$ and $P(S_T leqa) \leq \exp(-\theta_o a)$.

**5.7.7.** Suppose the $\xi_i$ are integer valued with $P(\xi_i < -1) = 0$ and $EX_i > 0$. Show that $\varphi(\theta_o) = E\exp(\theta_o\xi_1) = 1$ for some $\theta_o < 0$. Use the martingale $X_n = \exp(\theta_o S_n)$ to conclude that $P(S_T \leq a) = \exp(-\theta_o a)$.

**5.7.8.** Let $S_n$ be the total assets of an insurance company at the end of year $n$. In year $n$, premiums totaling $c > 0$ are received and claims $\zeta_n$ are paid where $\zeta_n$ is Normal$(\mu, \sigma^2)$ and $\mu < c$. To be precise, if $\xi_n = c - \zeta_n$ then $S_n = S_{n-1} + \xi_n$. The company is ruined if its assets drop to 0 or less. Show that if $S_0 > 0$ is nonrandom, then

$$P(\text{ ruin }) \leq \exp(-2(c - \mu)S_0/\sigma^2)$$

**5.7.9.** Let $Z_n$ be a branching process with offspring distribution $p_k$, defined in part d of Section 4.3, and let $\varphi(\theta) = \sum p_k\theta^k$. Suppose $\rho < 1$ has $\varphi(\rho) = \rho$. Show that $\rho^{Z_n}$ is a martingale and use this to conclude $P(Z_n = 0 \text{ for some } n \geq 1 | Z_0 = x) = \rho^x$.

# 6
# Markov Chains

The main object of study in this chapter is (temporally homogeneous) Markov chains on a countable state space $S$. That is, a sequence of r.v.'s $X_n$, $n \geq 0$, with

$$P(X_{n+1} = j|\mathcal{F}_n) = p(X_n, j)$$

where $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $p(i, j) \geq 0$ and $\sum_j p(i, j) = 1$. The theory focuses on the asymptotic behavior of $p^n(i, j) \equiv P(X_n = j|X_0 = i)$. The basic results are that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p^m(i, j) \quad \text{exists always}$$

and under a mild assumption called aperiodicity:

$$\lim_{n\to\infty} p^n(i, j) \quad \text{exists}$$

In nice situations, that is, $X_n$ is irreducible and positive recurrent, the limits above are a probability distribution that is independent of the starting state $i$. In words, the chain converges to equilibrium as $n \to \infty$. One of the attractions of Markov chain theory is that these powerful conclusions come out of assumptions that are satisfied in a large number of examples.

## 6.1 Definitions

Let $(S, \mathcal{S})$ be a measurable space.

A function $p : S \times \mathcal{S} \to \mathbf{R}$ is said to be a **transition probability** if:

(i) For each $x \in S$, $A \to p(x, A)$ is a probability measure on $(S, \mathcal{S})$.
(ii) For each $A \in \mathcal{S}$ , $x \to p(x, A)$ is a measurable function.

We say $X_n$ is a Markov chain (w.r.t. $\mathcal{F}_n$) with transition probability $p$ if

$$P(X_{n+1} \in B|\mathcal{F}_n) = p(X_n, B)$$

Given a transition probability $p$ and an **initial distribution** $\mu$ on $(S, \mathcal{S})$, we can define a consistent set of finite dimensional distributions by

$$P(X_j \in B_j, 0 \le j \le n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n) \tag{6.1.1}$$

If we suppose that $(S, \mathcal{S})$ is nice, Kolmogorov's extenson theorem, Theorem 2.1.14, allows us to construct a probability measure $P_\mu$ on **sequence space** $(S^{\{0,1,\ldots\}}, \mathcal{S}^{\{0,1,\ldots\}})$ so that the coordinate maps $X_n(\omega) = \omega_n$ have the desired distributions.

**Notation.** When $\mu = \delta_x$, a point mass at $x$, we use $P_x$ as an abbreviation for $P_{\delta_x}$. The measures $P_x$ are the basic objects because, once they are defined, we can define the $P_\mu$ (even for infinite measures $\mu$) by

$$P_\mu(A) = \int \mu(dx)\, P_x(A)$$

Our next step is to show

**Theorem 6.1.1.** *$X_n$ is a Markov chain (with respect to $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$) with transition probability $p$.*

*Proof.* To prove this, we let $A = \{X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n\}$, $B_{n+1} = B$, and observe that using the definition of the integral, the definition of $A$, and the definition of $P_\mu$

$$\begin{aligned}
\int_A 1_{(X_{n+1} \in B)}\, dP_\mu &= P_\mu(A, X_{n+1} \in B) \\
&= P_\mu(X_0 \in B_0, X_1 \in B_1, \ldots, X_n \in B_n, X_{n+1} \in B) \\
&= \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)\, p(x_n, B_{n+1})
\end{aligned}$$

We would like to assert that the last expression is

$$= \int_A p(X_n, B)\, dP_\mu$$

To do this, replace $p(x_n, B_n)$ by a general function $f(x_n)$. If $f$ is an indicator function, the desired equality is true. Linearity implies that it is valid for simple functions, and the bounded convergence theorem implies that it is valid for bounded measurable $f$, for example, $f(x) = p(x, B_{n+1})$.

The collection of sets for which

$$\int_A 1_{(X_{n+1} \in B)}\, dP_\mu = \int_A p_n(X_n, B)\, dP_\mu$$

holds is a $\lambda$-system, and the collection for which it has been proved is a $\pi$-system, so it follows from the $\pi - \lambda$ theorem, Theorem 2.1.2, that the equality is true for all $A \in \mathcal{F}_n$. This shows that

$$P(X_{n+1} \in B|\mathcal{F}_n) = p(X_n, B)$$

and proves the desired result. ■

At this point, we have shown that given a sequence of transition probabilities and an initial distribution, we can construct a Markov chain. Conversely,

**Theorem 6.1.2.** *If $X_n$ is a Markov chain with transition probabilities p and initial distribution $\mu$, then the finite dimensional distributions are given by (6.1.1).*

*Proof.* Our first step is to show that if $X_n$ has transition probability $p$, then for any bounded measurable $f$

$$E(f(X_{n+1})|\mathcal{F}_n) = \int p(X_n, dy)f(y) \tag{6.1.2}$$

The desired conclusion is a consequence of the next result. Let $\mathcal{H} =$ the collection of bounded functions for which the identity holds.

**Theorem 6.1.3. Monotone class theorem.** *Let $\mathcal{A}$ be a $\pi$-system that contains $\Omega$ and let $\mathcal{H}$ be a collection of real-valued functions that satisfies:*
*(i) If $A \in \mathcal{A}$, then $1_A \in \mathcal{H}$.*
*(ii) If $f, g \in \mathcal{H}$, then $f + g$, and $cf \in \mathcal{H}$ for any real number $c$.*
*(iii) If $f_n \in \mathcal{H}$ are nonnegative and increase to a bounded function $f$, then $f \in \mathcal{H}$.*
*Then $\mathcal{H}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{A})$.*

*Proof.* The assumption $\Omega \in \mathcal{A}$, (ii), and (iii) imply that $\mathcal{G} = \{A : 1_A \in \mathcal{H}\}$ is a $\lambda$-system, so by (i) and the $\pi - \lambda$ theorem, Theorem 2.1.2, $\mathcal{G} \supset \sigma(\mathcal{A})$. (ii) implies that $\mathcal{H}$ contains all simple functions, and (iii) implies that $\mathcal{H}$ contains all bounded measurable functions. ■

Returning to our main topic, we observe that familiar properties of conditional expectation and (6.1.2) imply

$$\begin{aligned} E\left(\prod_{m=0}^{n} f_m(X_m)\right) &= E\,E\left(\prod_{m=0}^{n} f_m(X_m)\middle|\mathcal{F}_{n-1}\right) \\ &= E\left(\prod_{m=0}^{n-1} f_m(X_m)E(f_n(X_n)|\mathcal{F}_{n-1})\right) \\ &= E\left(\prod_{m=0}^{n-1} f_m(X_m)\int p_{n-1}(X_{n-1}, dy)f_n(y)\right) \end{aligned}$$

The last integral is a bounded measurable function of $X_{n-1}$, so it follows by induction that if $\mu$ is the distribution of $X_0$, then

$$E\left(\prod_{m=0}^{n} f_m(X_m)\right) = \int \mu(dx_0) f_0(x_0) \int p_0(x_0, dx_1) f_1(x_1) \cdots \int p_{n-1}(x_{n-1}, dx_n) f_n(x_n) \tag{6.1.3}$$

that is, the finite dimensional distributions coincide with those in (6.1.1). ∎

With Theorem 6.1.2 established, it follows that we can describe a Markov chain by giving a transition probabilities $p$. Having done this, we can and will suppose that the random variables $X_n$ are the coordinate maps ($X_n(\omega) = \omega_n$) on sequence space

$$(\Omega_o, \mathcal{F}) = (S^{\{0,1,\ldots\}}, \mathcal{S}^{\{0,1,\ldots\}})$$

We choose this representation because it gives us two advantages in investigating the Markov chain: (i) For each initial distribution $\mu$ we have a measure $P_\mu$ defined by (6.1.1) that makes $X_n$ a Markov chain with $P_\mu(X_0 \in A) = \mu(A)$. (ii) We have the shift operators $\theta_n$ defined in Section 4.1: $(\theta_n\omega)(m) = \omega_{m+n}$.

## 6.2 Examples

Having introduced on the framework in which we will investigate things, we can finally give some more examples.

**Example 6.2.1. Random walk.** Let $\xi_1, \xi_2, \ldots \in \mathbf{R}^d$ be independent with distribution $\mu$. Let $X_0 = x \in \mathbf{R}^d$ and let $X_n = X_0 + \xi_1 + \cdots + \xi_n$. Then $X_n$ is a Markov chain with transition probability.

$$p(x, A) = \mu(A - x)$$

where $A - x = \{y - x : y \in A\}$.

To prove this, we will use an extension of Example 5.1.5.

**Lemma 6.2.1.** *Let $X$ and $Y$ take values in $(S, \mathcal{S})$. Suppose $\mathcal{F}$ and $Y$ are independent. Let $X \in \mathcal{F}$, $\varphi$ be a function with $E|\varphi(X,Y)| < \infty$ and let $g(x) = E(\varphi(x, Y))$.*

$$E(\varphi(X, Y)|\mathcal{F}) = g(X)$$

*Proof.* Suppose first that $\phi(x, y) = 1_A(x)1_B(y)$ and let $C \in \mathcal{F}$.

$$\begin{aligned} E(\varphi(X, Y); C) &= P(\{X \in A\} \cap C \cap \{Y \in B\}) \\ &= P(\{X \in A\} \cap C) P(\{Y \in B\}) \end{aligned}$$

since $\{X \in A\} \cap C \in \mathcal{F}$ and $\{Y \in B\}$ are independent. $g(x) = 1_A(x)P(Y \in B)$, so the above

$$= E(g(X); C)$$

We now apply the monotone class theorem, Theorem 6.1.3. Let $\mathcal{A}$ be the subsets of $S \times S$ of the form $A \times B$ with $A, B \in \mathcal{S}$. $\mathcal{A}$ is a $\pi$-system that contains $\Omega$. Let $\mathcal{H}$ be the collection of $\phi$ for which the result holds. We have shown (i). Properties (ii) and (iii) follow from the bounded convergence theorem which completes the proof. ■

To get the desired result from Lemma 6.2.1, we let $\mathcal{F} = \mathcal{F}_n$, $X = X_n$, $Y = \xi_{n+1}$, and $\phi(x, y) = 1_{\{x+y\in A\}}$. In this case $g(x) = \mu(A - x)$ and the desired result follows.

In the next four examples, $S$ is a countable set and $\mathcal{S}$ = all subsets of $S$. Let $p(i, j) \geq 0$ and suppose $\sum_j p(i, j) = 1$ for all $i$. Intuitively, $p(i, j) = P(X_{n+1} = j | X_n = i)$. From $p(i, j)$ we can define a transition probability by

$$p(i, A) = \sum_{j\in A} p(i, j)$$

In each case, we will not be as formal in checking the Markov property, but simply give the transition probability and leave the rest to the reader. The details are much simpler because all we have to show is that

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots X_0 = i_0) = p(i, j)$$

and these are elementary conditional probabilities.

**Example 6.2.2. Branching processes.** $S = \{0, 1, 2, \ldots\}$

$$p(i, j) = P\left(\sum_{m=1}^{i} \xi_m = j\right)$$

where $\xi_1, \xi_2, \ldots$ are i.i.d. nonnegative integer-valued random variables. In words each of the $i$ individuals at time $n$ (or in generation $n$) gives birth to an independent and identically distributed number of offspring.

To make the connection with our earlier discussion of branching processes, do:

**Exercise 6.2.1.** Let $Z_n$ be the process defined in (5.3.2). Check that $Z_n$ is a Markov chain with the indicated transition probability.

**Example 6.2.3. Renewal chain.** $S = \{0, 1, 2, \ldots\}$, $f_k \geq 0$, and $\sum_{k=1}^{\infty} f_k = 1$.

$$\begin{aligned} p(0, j) &= f_{j+1} && \text{for } j \geq 0 \\ p(i, i-1) &= 1 && \text{for } i \geq 1 \\ p(i, j) &= 0 && \text{otherwise} \end{aligned}$$

To explain the definition, let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_m = j) = f_j$, let $T_0 = i_0$ and for $k \geq 1$ let $T_k = T_{k-1} + \xi_k$. $T_k$ is the time of the $k$th arrival in a renewal process that has its first arrival at time $i_0$. Let

$$Y_m = \begin{cases} 1 & \text{if } m \in \{T_0, T_1, T_2, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

and let $X_n = \inf\{m - n : m \geq n, Y_m = 1\}$. $Y_m = 1$ if a renewal occurs at time $m$, and $X_n$ is the amount of time until the first renewal $\geq n$.

An example should help clarify the definition:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_n$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| $X_n$ | 3 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 4 | 3 | 2 | 1 | 0 |

It is clear that if $X_n = i > 0$ then $X_{n+1} = i - 1$. When $X_n = 0$, we have $T_{N_n} = n$, where $N_n = \inf\{k : T_k \geq n\}$ is a stopping time, so Theorem 4.1.3 implies $\xi_{N_n+1}$ is independent of $\sigma(X_0, \xi_1, \ldots, \xi_{N_n}) \supset \sigma(X_0, \ldots, X_n)$. We have $p(0, j) = f_{j+1}$ since $\xi_{N_n+1} = j + 1$ implies $X_{n+1} = j$.

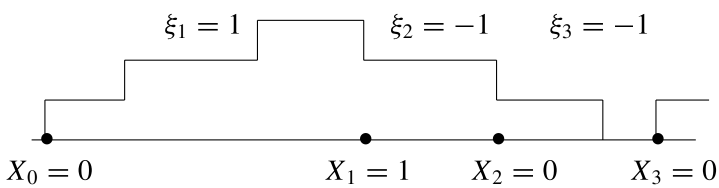



Figure 6.1. Realization of the M/G/1 queue. Black dots indicate the times at which the customers enter service.

**Example 6.2.4. M/G/1 queue.** In this model, customers arrive according to a Poisson process with rate $\lambda$. (M is for Markov and refers to the fact that in a Poisson process the number of arrivals in disjoint time intervals is independent.) Each customer requires an independent amount of service with distribution $F$. (G is for general service distribution. 1 indicates that there is one server.) Let $X_n$ be the number of customers waiting in the queue at the time the $n$th customer enters service. To be precise, when $X_0 = x$, the chain starts with $x$ people waiting in line and customer 0 just beginning her service.

To understand the definitions that follow, Figure 6.1 is useful. To define our Markov chain $X_n$, let

$$a_k = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} \, dF(t)$$

be the probability that $k$ customers arrive during a service time. Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_i = k - 1) = a_k$. We think of $\xi_i$ as the net number of customers to arrive during the $i$th service time, subtracting 1 for the customer who completed service, so we define $X_n$ by

$$X_{n+1} = (X_n + \xi_{n+1})^+ \tag{6.2.1}$$

The positive part only takes effect when $X_n = 0$ and $\xi_{n+1} = -1$ (e.g., $X_2 = 0$, $\xi_3 = -1$) and reflects the fact that when the queue has size 0 and no one arrives during the service time, the next queue size is 0, since we do not start counting until the next customer arrives and then the queue length will be 0.

It is easy to see that the sequence defined in (6.2.1) is a Markov chain with transition probability

$$p(0,0) = a_0 + a_1$$
$$p(j, j-1+k) = a_k \qquad \text{if } j \ge 1 \text{ or } k > 1$$

The formula for $a_k$ is rather complicated, and its exact form is not important, so we will simplify things by assuming only that $a_k > 0$ for all $k \ge 0$ and $\sum_{k\ge 0} a_k = 1$.

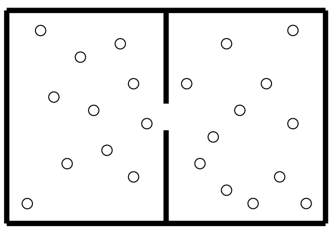


Figure 6.2. Physical motivation for the Ehrenfest chain.

**Example 6.2.5. Ehrenfest chain.** $S = \{0, 1, \dots, r\}$

$$p(k, k+1) = (r-k)/r$$
$$p(k, k-1) = k/r$$
$$p(i, j) = 0 \qquad \text{otherwise}$$

In words, there is a total of $r$ balls in two urns; $k$ in the first and $r-k$ in the second. We pick one of the $r$ balls at random and move it to the other urn. See Figure 6.2 for a picture. Ehrenfest used this to model the division of air molecules between two chambers (of equal size and shape) that are connected by a small hole. For an interesting account of this chain, see Kac (1947a).

**Example 6.2.6. Birth and death chains.** $S = \{0, 1, 2, \dots\}$ These chains are defined by the restriction $p(i, j) = 0$ when $|i - j| > 1$. The fact that these processes cannot jump over any integers makes it particularly easy to compute things for them.

That should be enough examples for the moment. We conclude this section with some simple calculations. For a Markov chain on a countable state space, (6.1.1) says

$$P_\mu(X_k = i_k, 0 \le k \le n) = \mu(i_0) \prod_{m=1}^{n} p(i_{m-1}, i_m)$$

When $n = 1$

$$P_\mu(X_1 = j) = \sum_i \mu(i) p(i, j) = \mu p(j)$$

that is, the product of the row vector $\mu$ with the matrix $p$. When $n = 2$,

$$P_i(X_2 = k) = \sum_j p(i, j) p(j, k) = p^2(i, k)$$

that is, the second power of the matrix $p$. Combining the two formulas and generalizing,

$$P_\mu(X_n = j) = \sum_i \mu(i) p^n(i, j) = \mu p^n(j)$$

### *Exercises*

**6.2.2.** Suppose $S = \{1, 2, 3\}$ and

$$p = \begin{pmatrix} .1 & 0 & .9 \\ .7 & .3 & 0 \\ 0 & .4 & .6 \end{pmatrix}$$

Compute $p^2(1, 2)$ and $p^3(2, 3)$ by considering the different ways to get from 1 to 2 in two steps and from 2 to 3 in three steps.

**6.2.3.** Suppose $S = \{0, 1\}$ and

$$p = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Use induction to show that

$$P_\mu(X_n = 0) = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha+\beta} \right\}$$

**6.2.4.** Let $\xi_0, \xi_1, \ldots$ be i.i.d. $\in \{H, T\}$, taking each value with probability $1/2$. Show that $X_n = (\xi_n, \xi_{n+1})$ is a Markov chain and compute its transition probability $p$. What is $p^2$?

**6.2.5. Brother-sister mating.** In this scheme, two animals are mated, and among their direct descendants two individuals of opposite sex are selected at random. These animals are mated and the process continues. Suppose each individual can be one of three genotypes $AA$, $Aa$, $aa$, and suppose that the type of the offspring is determined by selecting a letter from each parent. With these rules, the pair of genotypes in the $n$th generation is a Markov chain with six states:

$$AA, AA \quad AA, Aa \quad AA, aa \quad Aa, Aa \quad Aa, aa \quad aa, aa$$

Compute its transition probability.

**6.2.6. Bernoulli-Laplace model of diffusion.** Suppose two urns, which we will call left and right, have $m$ balls each. $b$ (which we will assume is $\le m$) balls are black, and $2m - b$ are white. At each time, we pick one ball from each urn and interchange them. Let the state at time $n$ be the number of black balls in the left urn. Compute the transition probability.

**6.2.7.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\in \{1, 2, \ldots, N\}$ and taking each value with probability $1/N$. Show that $X_n = |\{\xi_1, \ldots, \xi_n\}|$ is a Markov chain and compute its transition probability.

**6.2.8.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\in \{-1, 1\}$, taking each value with probability $1/2$. Let $S_0 = 0$, $S_n = \xi_1 + \cdots \xi_n$ and $X_n = \max\{S_m : 0 \le m \le n\}$. Show that $X_n$ is not a Markov chain.

**6.2.9.** Let $\theta, U_1, U_2, \ldots$ be independent and uniform on $(0, 1)$. Let $X_i = 1$ if $U_i \le \theta$, $= -1$ if $U_i > \theta$, and let $S_n = X_1 + \cdots + X_n$. In words, we first pick $\theta$ according to the uniform distribution and then flip a coin with probability $\theta$ of heads to generate a random walk. Compute $P(X_{n+1} = 1 | X_1, \ldots, X_n)$ and conclude $S_n$ is a temporally inhomogeneous Markov chain. This is due to the fact that "$S_n$ is a sufficient statistic for estimating $\theta$."

## 6.3 Extensions of the Markov Property

If $X_n$ is a Markov chain with transition probability $p$, then by definition,

$$P(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B)$$

In this section, we will prove two extensions of the last equality in which $\{X_{n+1} \in B\}$ is replaced by a bounded function of the future, $h(X_n, X_{n+1}, \ldots)$, and $n$ is replaced by a stopping time $N$. These results, especially the second, will be the keys to developing the theory of Markov chains.

As mentioned in Section 6.1, we can and will suppose that the $X_n$ are the coordinate maps on sequence space

$$(\Omega_o, \mathcal{F}) = (S^{\{0,1,\ldots\}}, \mathcal{S}^{\{0,1,\ldots\}})$$

$\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, and for each initial distribution $\mu$ we have a measure $P_\mu$ defined by (6.1.1) that makes $X_n$ a Markov chain with $P_\mu(X_0 \in A) = \mu(A)$. Define the shift operators $\theta_n : \Omega_o \to \Omega_o$ by $(\theta_n \omega)(m) = \omega(m + n)$.

**Theorem 6.3.1. The Markov property.** *Let $Y : \Omega_o \to \mathbf{R}$ be bounded and measurable.*

$$E_\mu(Y \circ \theta_n | \mathcal{F}_n) = E_{X_n} Y$$

**Remark.** Here the subscript $\mu$ on the left-hand side indicates that the conditional expectation is taken with respect to $P_\mu$. The right-hand side is the function $\varphi(x) = E_x Y$ evaluated at $x = X_n$. To make the connection with the introduction of this section, let

$$Y(\omega) = h(\omega_0, \omega_1, \dots)$$

We denote the function by $Y$, a letter usually used for random variables, because that's exactly what $Y$ is, a measurable function defined on our probability space $\Omega_o$.

*Proof.* We begin by proving the result in a special case and then use the $\pi - \lambda$ and monotone class theorems to get the general result. Let $A = \{\omega : \omega_0 \in A_0, \dots, \omega_m \in A_m\}$ and $g_0, \dots g_n$ be bounded and measurable. Applying (6.1.3) with $f_k = 1_{A_k}$ for $k < m$, $f_m = 1_{A_m} g_0$, and $f_k = g_{k-m}$ for $m < k \le m + n$ gives

$$\begin{aligned} E_\mu\left(\prod_{k=0}^n g_k(X_{m+k}); A\right) &= \int_{A_0} \mu(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_m} p(x_{m-1}, dx_m) \\ &\quad \cdot g_0(x_m) \int p(x_m, dx_{m+1}) g_1(x_{m+1}) \\ &\quad \cdots \int p(x_{m+n-1}, dx_{m+n}) g_n(x_{m+n}) \\ &= E_\mu\left(E_{X_m}\left(\prod_{k=0}^n g_k(X_k)\right); A\right) \end{aligned}$$

The collection of sets for which the last formula holds is a $\lambda$-system, and the collection for which it has been proved is a $\pi$-system, so using the $\pi - \lambda$ theorem, Theorem 2.1.2, shows that the last identity holds for all $A \in \mathcal{F}_m$.

Fix $A \in \mathcal{F}_m$ and let $\mathcal{H}$ be the collection of bounded measurable $Y$ for which

$$(*) \qquad E_\mu(Y \circ \theta_m; A) = E_\mu(E_{X_m} Y; A)$$

The last computation shows that $(*)$ holds when

$$Y(\omega) = \prod_{0 \le k \le n} g_k(\omega_k)$$

To finish the proof, we will apply the monotone class theorem, Theorem 6.1.3. Let $\mathcal{A}$ be the collection of sets of the form $\{\omega : \omega_0 \in A_0, \dots, \omega_k \in A_k\}$. $\mathcal{A}$ is a $\pi$-system, so taking $g_k = 1_{A_k}$ shows (i) holds. $\mathcal{H}$ clearly has properties (ii) and (iii), so Theorem 6.1.3 implies that $\mathcal{H}$ contains the bounded functions measurable w.r.t $\sigma(\mathcal{A})$, and the proof is complete. ■

**Exercise 6.3.1.** Use the Markov property to show that if $A \in \sigma(X_0, \dots, X_n)$ and $B \in \sigma(X_n, X_{n+1}, \dots)$, then for any initial distribution $\mu$

$$P_\mu(A \cap B | X_n) = P_\mu(A|X_n) P_\mu(B|X_n)$$

In words, the past and future are conditionally independent given the present. Hint: Write the left-hand side as $E_\mu(E_\mu(1_A 1_B|\mathcal{F}_n)|X_n)$.

The next two results illustrate the use of Theorem 6.3.1. We will see many other applications below.

**Theorem 6.3.2. Chapman-Kolmogorov equation.**

$$P_x(X_{m+n} = z) = \sum_y P_x(X_m = y) P_y(X_n = z)$$

*Proof.* $P_x(X_{n+m}=z) = E_x(P_x(X_{n+m}=z|\mathcal{F}_m)) = E_x(P_{X_m}(X_n=z))$ by the Markov property, Theorem 6.3.1 since $1_{(X_n=z)} \circ \theta_m = 1_{(X_{n+m}=z)}$. ■

**Theorem 6.3.3.** *Let $X_n$ be a Markov chain and suppose*

$$P\left(\cup_{m=n+1}^\infty \{X_m \in B_m\} \middle| X_n\right) \ge \delta > 0 \quad \text{on } \{X_n \in A_n\}$$

*Then $P(\{X_n \in A_n \text{ i.o.}\} - \{X_n \in B_n \text{ i.o.}\}) = 0$.*

**Remark.** To quote Chung, "The intuitive meaning of the preceding theorem has been given by Doeblin as follows: if the chance of a pedestrian's getting run over is greater than $\delta > 0$ each time he crosses a certain street, then he will not be crossing it indefinitely (since he will be killed first)!"

*Proof.* Let $\Lambda_n = \{X_{n+1} \in B_{n+1}\} \cup \{X_{n+2} \in B_{n+2}\} \cup \dots$

$$\Lambda = \cap \Lambda_n = \{X_n \in B_n \text{ i.o.}\}$$

and $\Gamma = \{X_n \in A_n \text{ i.o.}\}$. Let $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ and $\mathcal{F}_\infty = \sigma(\cup \mathcal{F}_n)$. Using the Markov property and the dominated convergence theorem for conditional expectations, Theorem 5.5.9,

$$E(1_{\Lambda_n}|X_n) = E(1_{\Lambda_n}|\mathcal{F}_n) \to E(1_\Lambda|\mathcal{F}_\infty) = 1_\Lambda$$

On $\Gamma$, the left-hand side is $\ge \delta$ i.o. This is only possible if $\Gamma \subset \Lambda$. ■

**Exercise 6.3.2.** A state $a$ is called **absorbing** if $P_a(X_1 = a) = 1$. Let $D = \{X_n = a$ for some $n \ge 1\}$ and let $h(x) = P_x(D)$. (i) Use Theorem 6.3.3 to conclude that $h(X_n) \to 0$ a.s. on $D^c$. Here a.s. means $P_\mu$ a.s. for any initial distribution $\mu$. (ii) Obtain the result in Exercise 5.5.5 as a special case.

We are now ready for our second extension of the Markov property. Recall $N$ is said to be a stopping time if $\{N = n\} \in \mathcal{F}_n$. As in Chapter 4, let

$$\mathcal{F}_N = \{A : A \cap \{N = n\} \in \mathcal{F}_n \text{ for all } n\}$$

be the information known at time $N$, and let

$$\theta_N \omega = \begin{cases} \theta_n \omega & \text{on } \{N = n\} \\ \Delta & \text{on } \{N = \infty\} \end{cases}$$

where $\Delta$ is an extra point that we add to $\Omega_o$. In the next result and its applications, we will explicitly restrict our attention to $\{N < \infty\}$, so the reader does not have to worry about the second part of the definition of $\theta_N$.

**Theorem 6.3.4. Strong Markov property.** *Suppose that for each n, $Y_n : \Omega \to \mathbf{R}$ is measurable and $|Y_n| \le M$ for all n. Then*

$$E_\mu(Y_N \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y_N \text{ on } \{N < \infty\}$$

*where the right-hand side is $\varphi(x, n) = E_x Y_n$ evaluated at $x = X_N$, $n = N$.*

*Proof.* Let $A \in \mathcal{F}_N$. Breaking things down according to the value of $N$,

$$E_\mu(Y_N \circ \theta_N; A \cap \{N < \infty\}) = \sum_{n=0}^{\infty} E_\mu(Y_n \circ \theta_n; A \cap \{N = n\})$$

Since $A \cap \{N = n\} \in \mathcal{F}_n$, using Theorem 6.3.1 now converts the right side into

$$\sum_{n=0}^{\infty} E_\mu(E_{X_n} Y_n; A \cap \{N = n\}) = E_\mu(E_{X_N} Y_N; A \cap \{N < \infty\}) \qquad \blacksquare$$

**Remark.** The reader should notice that the proof is trivial. All we do is break things down according to the value of $N$, replace $N$ by $n$, apply the Markov property, and reverse the process. This is the standard technique for proving results about stopping times.

The next example illustrates the use of Theorem 6.3.4 and explains why we want to allow the $Y$ that we apply to the shifted path to depend on $n$.

**Theorem 6.3.5. Reflection principle.** *Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed with a distribution that is symmetric about 0. Let $S_n = \xi_1 + \cdots + \xi_n$. If $a > 0$, then*

$$P\left(\sup_{m \le n} S_m > a\right) \le 2P(S_n > a)$$

**Remark.** First, a trivial comment: The strictness of the inequality is not important. If the result holds for $>$, it holds for $\ge$ and vice versa.

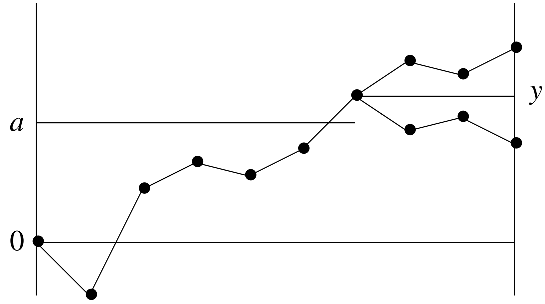



Figure 6.3. Proof by picture of the reflection principle.

A second more important one: We do the proof in two steps because that is how formulas like this are derived in practice. First, one computes intuitively and then figures out how to extract the desired formula from Theorem 6.3.4.

*Proof in words.* First note that if $Z$ has a distribution that is symmetric about 0, then

$$P(Z \geq 0) \geq P(Z > 0) + \frac{1}{2}P(Z = 0) = \frac{1}{2}$$

If we let $N = \inf\{m \leq n : S_m > a\}$ (with $\inf \emptyset = \infty$), then on $\{N < \infty\}$, $S_n - S_N$ is independent of $S_N$ and has $P(S_n - S_N \geq 0) \geq 1/2$. So (see Figure 6.3 for a picture)

$$P(S_n > a) \geq \frac{1}{2}P(N \leq n)$$

*Proof.* Let $Y_m(\omega) = 1$ if $m \leq n$ and $\omega_{n-m} > a$, $Y_m(\omega) = 0$ otherwise. The definition of $Y_m$ is chosen so that $(Y_N \circ \theta_N)(\omega) = 1$ if $\omega_n > a$ (and hence $N \leq n$), and $= 0$ otherwise. The strong Markov property implies

$$E_0(Y_N \circ \theta_N | \mathcal{F}_N) = E_{S_N} Y_N \quad \text{on } \{N < \infty\} = \{N \leq n\}$$

To evaluate the right-hand side, we note that if $y > a$, then

$$E_y Y_m = P_y(S_{n-m} > a) \geq P_y(S_{n-m} \geq y) \geq 1/2$$

So integrating over $\{N \leq n\}$ and using the definition of conditional expectation gives

$$\frac{1}{2}P(N \leq n) \leq E_0(E_0(Y_N \circ \theta_N | \mathcal{F}_N); N \leq n) = E_0(Y_N \circ \theta_N; N \leq n)$$

since $\{N \leq n\} \in \mathcal{F}_N$. Recalling that $Y_N \circ \theta_N = 1_{\{S_n > a\}}$, the last quantity

$$= E_0(1_{\{S_n > a\}}; N \leq n) = P_0(S_n > a)$$

since $\{S_n > a\} \subset \{N \leq n\}$. ∎

**Exercises**

The next five exercises concern the hitting times

$$\tau_A = \inf\{n \geq 0 : X_n \in A\} \qquad \tau_y = \tau_{\{y\}}$$
$$T_A = \inf\{n \geq 1 : X_n \in A\} \qquad T_y = T_{\{y\}}$$

To keep the two definitions straight, note that the symbol $\tau$ is smaller than $T$. Some of the results below are valid for a general $S$, but for simplicity,

*We will suppose throughout that $S$ is countable.*

**6.3.3. First entrance decomposition.** Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m)p^{n-m}(y, y)$$

**6.3.4.** Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

**6.3.5.** Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \leq (1 - \epsilon)^k$.

**6.3.6.** Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$$(*) \qquad h(x) = \sum_y p(x, y)h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if $h$ satisfies $(*)$ then $h(X(n \wedge \tau_{A\cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on $A$ and 0 on $B$.

**6.3.7.** Let $X_n$ be a Markov chain with $S = \{0, 1, \dots, N\}$ and suppose that $X_n$ is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all $x$. (i) Show that 0 and $N$ are absorbing states, that is, $p(0, 0) = p(N, N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

**6.3.8. Wright-Fisher model.** Suppose $S = \{0, 1, \dots, N\}$ and consider

$$p(i, j) = \binom{N}{j}(i/N)^j(1 - i/N)^{N-j}$$

Show that this chain satisfies the hypotheses of Exercise 6.3.7.

**6.3.9.** In brother-sister mating described in Exercise 6.2.5, $AA, AA$ and $aa, aa$ are absorbing states. Show that the number of $A$'s in the pair is a martingale and use this to compute the probability of getting absorbed in $AA, AA$ starting from each of the states.

**6.3.10.** Let $\tau_A = \inf\{n \geq 0 : X_n \in A\}$ and $g(x) = E_x\tau_A$. Suppose that $S - A$ is finite and for each $x \in S - A$, $P_x(\tau_A < \infty) > 0$. (i) Show that

$$(*) \qquad g(x) = 1 + \sum_y p(x, y)g(y) \quad \text{for } x \notin A$$

(ii) Show that if $g$ satisfies $(*)$, $g(X(n \wedge \tau_A)) + n \wedge \tau_A$ is a martingale. (iii) Use this to conclude that $g(x) = E_x\tau_A$ is the only solution of $(*)$ that is 0 on $A$.

**6.3.11.** Let $\xi_0, \xi_1, \ldots$ be i.i.d. $\in \{H, T\}$, taking each value with probability $1/2$, and let $X_n = (\xi_n, \xi_{n+1})$ be the Markov chain from Exercise 6.2.4. Let $N_1 = \inf\{n \geq 0 : (\xi_n, \xi_{n+1}) = (H, H)\}$. Use the results in the last exercise to compute $EN_1$. [No, there is no missing subscript on $E$, but you will need to first compute $g(x)$.]

**6.3.12.** Consider simple random walk $S_n$, the Markov chain with $p(x, x+1) = 1/2$, and $p(x, x-1) = 1/2$. Let $\tau = \min\{n : S_n \notin (0, N)\}$. Use the result from Exercise 6.3.10 to show that $E_x\tau = x(N - x)$.

## 6.4 Recurrence and Transience

In this section and the next two, we will consider only Markov chains on a countable state space. Let $T_y^0 = 0$, and for $k \geq 1$, let

$$T_y^k = \inf\{n > T_y^{k-1} : X_n = y\}$$

$T_y^k$ is the time of the $k$th return to $y$. The reader should note that $T_y^1 > 0$ so any visit at time 0 does not count. We adopt this convention so that if we let $T_y = T_y^1$ and $\rho_{xy} = P_x(T_y < \infty)$, then

**Theorem 6.4.1.** $P_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}$.

Intuitively, in order to make $k$ visits to $y$, we first have to go from $x$ to $y$ and then return $k - 1$ times to $y$.

*Proof.* When $k = 1$, the result is trivial, so we suppose $k \geq 2$. Let $Y(\omega) = 1$ if $\omega_n = y$ for some $n \geq 1$, $Y(\omega) = 0$ otherwise. If $N = T_y^{k-1}$, then $Y \circ \theta_N = 1$ if $T_y^k < \infty$. The strong Markov property, Theorem 6.3.4, implies

$$E_x(Y \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y \quad \text{on } \{N < \infty\}$$

On $\{N < \infty\}$, $X_N = y$, so the right-hand side is $P_y(T_y < \infty) = \rho_{yy}$, and it follows that

$$\begin{aligned}
P_x(T_y^k < \infty) &= E_x(Y \circ \theta_N; N < \infty) \\
&= E_x(E_x(Y \circ \theta_N | \mathcal{F}_N); N < \infty) \\
&= E_x(\rho_{yy}; N < \infty) = \rho_{yy} P_x(T_y^{k-1} < \infty)
\end{aligned}$$

The result now follows by induction. ■

A state $y$ is said to be **recurrent** if $\rho_{yy} = 1$ and **transient** if $\rho_{yy} < 1$. If $y$ is recurrent, Theorem 6.4.1 implies $P_y(T_y^k < \infty) = 1$ for all $k$, so $P_y(X_n = y \text{ i.o.}) = 1$.

**Exercise 6.4.1.** Suppose $y$ is recurrent and for $k \geq 0$, let $R_k = T_y^k$ be the time of the $k$th return to $y$, and for $k \geq 1$ let $r_k = R_k - R_{k-1}$ be the $k$th interarrival time. Use the strong Markov property to conclude that under $P_y$, the vectors $v_k = (r_k, X_{R_{k-1}}, \dots, X_{R_k - 1})$, $k \geq 1$ are i.i.d.

If $y$ is transient and we let $N(y) = \sum_{n=1}^{\infty} 1_{(X_n = y)}$ be the number of visits to $y$ at positive times, then

$$E_x N(y) = \sum_{k=1}^{\infty} P_x(N(y) \geq k) = \sum_{k=1}^{\infty} P_x(T_y^k < \infty)$$

$$= \sum_{k=1}^{\infty} \rho_{xy} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1 - \rho_{yy}} < \infty \tag{6.4.1}$$

Combining the last computation with our result for recurrent states gives a result that generalizes Theorem 4.2.2.

**Theorem 6.4.2.** *$y$ is recurrent if and only if $E_y N(y) = \infty$.*

**Exercise 6.4.2.** Let $a \in S$, $f_n = P_a(T_a = n)$, and $u_n = P_a(X_n = a)$. (i) Show that $u_n = \sum_{1 \leq m \leq n} f_m u_{n-m}$. (ii) Let $u(s) = \sum_{n \geq 0} u_n s^n$, $f(s) = \sum_{n \geq 1} f_n s^n$, and show $u(s) = 1/(1 - f(s))$. Setting $s = 1$ gives (6.4.1) for $x = y = a$.

**Exercise 6.4.3.** Consider asymmetric simple random walk on $\mathbf{Z}$, that is, we have $p(i, i+1) = p$, $p(i, i-1) = q = 1 - p$. In this case,

$$p^{2m}(0,0) = \binom{2m}{m} p^m q^m \quad \text{and} \quad p^{2m+1}(0,0) = 0$$

(i) Use the Taylor series expansion for $h(x) = (1-x)^{-1/2}$ to show $u(s) = (1 - 4pqs^2)^{-1/2}$ and use the last exercise to conclude $f(s) = 1 - (1 - 4pqs^2)^{1/2}$. (ii) Set $s = 1$ to get the probability the random walk will return to 0 and check that this is the same as the answer given in part (c) of Theorem 5.7.7.

The next result shows that recurrence is contagious.

**Theorem 6.4.3.** *If $x$ is recurrent and $\rho_{xy} > 0$ then $y$ is recurrent and $\rho_{yx} = 1$.*

*Proof.* We will first show $\rho_{yx} = 1$ by showing that if $\rho_{xy} > 0$ and $\rho_{yx} < 1$, then $\rho_{xx} < 1$. Let $K = \inf\{k : p^k(x, y) > 0\}$. There is a sequence $y_1, \ldots, y_{K-1}$ so that

$$p(x, y_1)p(y_1, y_2) \cdots p(y_{K-1}, y) > 0$$

Since $K$ is minimal, $y_i \neq x$ for $1 \leq i \leq K - 1$. If $\rho_{yx} < 1$, we have

$$P_x(T_x = \infty) \geq p(x, y_1)p(y_1, y_2) \cdots p(y_{K-1}, y)(1 - \rho_{yx}) > 0$$

a contradiction. So $\rho_{yx} = 1$.

To prove that $y$ is recurrent, observe that $\rho_{yx} > 0$ implies there is an $L$ so that $p^L(y, x) > 0$. Now

$$p^{L+n+K}(y, y) \geq p^L(y, x)p^n(x, x)p^K(x, y)$$

Summing over $n$, we see

$$\sum_{n=1}^{\infty} p^{L+n+K}(y, y) \geq p^L(y, x)p^K(x, y) \sum_{n=1}^{\infty} p^n(x, x) = \infty$$

so Theorem 6.4.2 implies $y$ is recurrent. ∎

**Exercise 6.4.4.** Use the strong Markov property to show that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$.

The next fact will help us identify recurrent states in examples. First we need two definitions. $C$ is **closed** if $x \in C$ and $\rho_{xy} > 0$ implies $y \in C$. The name comes from the fact that if $C$ is closed and $x \in C$ then $P_x(X_n \in C) = 1$ for all $n$. $D$ is **irreducible** if $x, y \in D$ implies $\rho_{xy} > 0$.

**Theorem 6.4.4.** *Let $C$ be a finite closed set. Then $C$ contains a recurrent state. If $C$ is irreducible then all states in $C$ are recurrent.*

*Proof.* In view of Theorem 6.4.3, it suffices to prove the first claim. Suppose it is false. Then for all $y \in C$, $\rho_{yy} < 1$ and $E_x N(y) = \rho_{xy}/(1 - \rho_{yy})$, but this is ridiculous since it implies

$$\infty > \sum_{y \in C} E_x N(y) = \sum_{y \in C} \sum_{n=1}^{\infty} p^n(x, y) = \sum_{n=1}^{\infty} \sum_{y \in C} p^n(x, y) = \sum_{n=1}^{\infty} 1$$

The first inequality follows from the fact that $C$ is finite and the last equality from the fact that $C$ is closed. ∎

To illustrate the use of the last result, consider:

**Example 6.4.1. A seven-state chain.** Consider the transition probability:

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | .3 | 0 | 0 | 0 | .7 | 0 | 0 |
| **2** | .1 | .2 | .3 | .4 | 0 | 0 | 0 |
| **3** | 0 | 0 | .5 | .5 | 0 | 0 | 0 |
| **4** | 0 | 0 | 0 | .5 | 0 | .5 | 0 |
| **5** | .6 | 0 | 0 | 0 | .4 | 0 | 0 |
| **6** | 0 | 0 | 0 | 0 | 0 | .2 | .8 |
| **7** | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

To identify the states that are recurrent and those that are transient, we begin by drawing a graph that will contain an arc from $i$ to $j$ if $p(i, j) > 0$ and $i \neq j$. We do not worry about drawing the self-loops corresponding to states with $p(i, i) > 0$ since such transitions cannot help the chain get somewhere new.

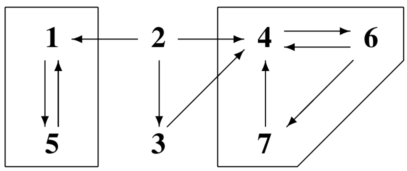



Figure 6.4. Graph for the seven-state chain.

In the case under consideration, we draw arcs from $1 \to 5$, $2 \to 1$, $2 \to 3$, $2 \to 4$, $3 \to 4$, $4 \to 6$, $4 \to 7$, $5 \to 1$, $6 \to 4$, $6 \to 7$, $7 \to 4$ (see Figure 6.4 for a picture).

(i) $\rho_{21} > 0$ and $\rho_{12} = 0$, so 2 must be transient, or we would contradict Theorem 6.4.3. Similarly, $\rho_{43} > 0$ and $\rho_{34} = 0$, so 4 must be transient

(ii) $\{1, 5\}$ and $\{4, 6, 7\}$ are irreducible closed sets, so Theorem 6.4.4 implies these states are recurrent.

The last reasoning can be used to identify transient and recurrent states when $S$ is finite, since for $x \in S$ either (i) there is a $y$ with $\rho_{xy} > 0$ and $\rho_{yx} = 0$ and $x$ must be transient, or (ii) $\rho_{xy} > 0$ implies $\rho_{yx} > 0$ . In case (ii), Exercise 6.4.4 implies $C_x = \{y : \rho_{xy} > 0\}$ is an irreducible closed set. (If $y, z \in C_x$ then $\rho_{yz} \geq \rho_{yx}\rho_{xz} > 0$. If $\rho_{yw} > 0$ then $\rho_{xw} \geq \rho_{xy}\rho_{yw} > 0$, so $w \in C_x$.) So Theorem 6.4.4 implies that $x$ is recurrent.

**Exercise 6.4.5.** Show that in the Ehrenfest chain (Example 6.2.5), all states are recurrent.

Example 6.4.1 motivates the following:

**Theorem 6.4.5. Decomposition theorem.** *Let $R = \{x : \rho_{xx} = 1\}$ be the recurrent states of a Markov chain. $R$ can be written as $\cup_i R_i$, where each $R_i$ is closed and irreducible.*

**Remark.** This result shows that for the study of recurrent states we can, without loss of generality, consider a single irreducible closed set.

*Proof.* If $x \in R$ let $C_x = \{y : \rho_{xy} > 0\}$. By Theorem 6.4.3, $C_x \subset R$, and if $y \in C_x$ then $\rho_{yx} > 0$. From this it follows easily that either $C_x \cap C_y = \emptyset$ or $C_x = C_y$. To prove the last claim, suppose $C_x \cap C_y \neq \emptyset$. If $z \in C_x \cap C_y$ then $\rho_{xy} \geq \rho_{xz}\rho_{zy} > 0$, so if $w \in C_y$, we have $\rho_{xw} \geq \rho_{xy}\rho_{yw} > 0$, and it follows that $C_x \supset C_y$. Interchanging the roles of $x$ and $y$ gives $C_y \supset C_x$, and we have proved our claim. If we let $R_i$ be a listing of the sets that appear as some $C_x$, we have the desired decomposition. ■

The rest of this section is devoted to examples. Specifically, we concentrate on the question: how do we tell whether a state is recurrent or transient? Reasoning based on Theorem 6.4.3 works occasionally when $S$ is infinite.

**Example 6.4.2. Branching process.** If the probability of no children is positive, then $\rho_{k0} > 0$ and $\rho_{0k} = 0$ for $k \geq 1$, so Theorem 6.4.4 implies that all states $k \geq 1$ are transient. The state 0 has $p(0, 0) = 1$ and is recurrent. It is called an **absorbing state** to reflect the fact that once the chain enters 0, it remains there for all time.

If $S$ is infinite and irreducible, all that Theorem 6.4.3 tells us is that either all the states are recurrent or all are transient, and we are left to figure out which case occurs.

**Example 6.4.3. Renewal chain.** Since $p(i, i-1) = 1$ for $i \geq 1$, it is clear that $\rho_{i0} = 1$ for all $i \geq 1$ and hence also for $i = 0$, that is, 0 is recurrent. If we recall that $p(0, j) = f_{j+1}$ and suppose that $\{k : f_k > 0\}$ is unbounded, then $\rho_{0i} > 0$ for all $i$ and all states are recurrent. If $K = \sup\{k : f_k > 0\} < \infty$, then $\{0, 1, \ldots, K-1\}$ is an irreducible closed set of recurrent states and all states $k \geq K$ are transient.

**Example 6.4.4. Birth and death chains on $\{0, 1, 2, \ldots\}$.** Let

$$p(i, i+1) = p_i \quad p(i, i-1) = q_i \quad p(i, i) = r_i$$

where $q_0 = 0$. Let $N = \inf\{n : X_n = 0\}$. To analyze this example, we are going to define a function $\varphi$ so that $\varphi(X_{N\wedge n})$ is a martingale. We start by setting $\varphi(0) = 0$ and $\varphi(1) = 1$. For the martingale property to hold when $X_n = k \geq 1$, we must have

$$\varphi(k) = p_k\varphi(k+1) + r_k\varphi(k) + q_k\varphi(k-1)$$

Using $r_k = 1 - (p_k + q_k)$, we can rewrite the last equation as

$$q_k(\varphi(k) - \varphi(k-1)) = p_k(\varphi(k+1) - \varphi(k))$$

$$\text{or} \quad \varphi(k+1) - \varphi(k) = \frac{q_k}{p_k}(\varphi(k) - \varphi(k-1))$$

Here and in what follows, we suppose that $p_k, q_k > 0$ for $k \geq 1$. Otherwise, the chain is not irreducible. Since $\varphi(1) - \varphi(0) = 1$, iterating the last result gives

$$\varphi(m+1) - \varphi(m) = \prod_{j=1}^{m} \frac{q_j}{p_j} \quad \text{for } m \geq 1$$

$$\varphi(n) = \sum_{m=0}^{n-1} \prod_{j=1}^{m} \frac{q_j}{p_j} \quad \text{for } n \geq 1$$

if we interpret the product as 1 when $m = 0$. Let $T_c = \inf\{n \geq 1 : X_n = c\}$. Now I claim that:

**Theorem 6.4.6.** *If $a < x < b$, then*

$$P_x(T_a < T_b) = \frac{\varphi(b) - \varphi(x)}{\varphi(b) - \varphi(a)} \qquad P_x(T_b < T_a) = \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)}$$

*Proof.* If we let $T = T_a \wedge T_b$ then $\varphi(X_{n \wedge T})$ is a bounded martingale and $T < \infty$ a.s. by Theorem 6.3.3, so $\varphi(x) = E_x \varphi(X_T)$ by Theorem 5.7.4. Since $X_T \in \{a, b\}$ a.s.,

$$\varphi(x) = \varphi(a) P_x(T_a < T_b) + \varphi(b)[1 - P_x(T_a < T_b)]$$

and solving gives the indicated formula. ∎

**Remark.** The answer and the proof should remind the reader of Example 4.1.5 and Theorem 5.7.7. To help remember the formula, observe that for any $\alpha$ and $\beta$, if we let $\psi(x) = \alpha\varphi(x) + \beta$ then $\psi(X_{n \wedge T})$ is also a martingale and the answer we get using $\psi$ must be the same. The last observation explains why the answer is a ratio of differences. To help remember which one, observe that the answer is 1 if $x = a$ and 0 if $x = b$.

Letting $a = 0$ and $b = M$ in Theorem 6.4.6 gives

$$P_x(T_0 > T_M) = \varphi(x)/\varphi(M)$$

Letting $M \to \infty$ and observing that $T_M \geq M - x$, $P_x$ a.s. we have proved:

**Theorem 6.4.7.** *0 is recurrent if and only if $\varphi(M) \to \infty$ as $M \to \infty$, that is,*

$$\varphi(\infty) \equiv \sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty$$

*If $\varphi(\infty) < \infty$ then $P_x(T_0 = \infty) = \varphi(x)/\varphi(\infty)$.*

We will now see what Theorem 6.4.7 says about some concrete cases.

**Example 6.4.5. Asymmetric simple random walk.** Suppose $p_j = p$ and $q_j = 1 - p$ for $j \geq 1$. In this case,

$$\varphi(n) = \sum_{m=0}^{n-1} \left(\frac{1-p}{p}\right)^m$$

From Theorem 6.4.7, it follows that 0 is recurrent if and only if $p \leq 1/2$, and if $p > 1/2$, then

$$P_x(T_0 < \infty) = \frac{\varphi(\infty) - \varphi(x)}{\varphi(\infty)} = \left(\frac{1-p}{p}\right)^x$$

**Exercise 6.4.6.** A gambler is playing roulette and betting \$1 on black each time. The probability she wins \$1 is 18/38, and the probability she loses \$1 is 20/38. (i) Calculate the probability that starting with \$20 she reaches \$40 before losing her money. (ii) Use the fact that $X_n + 2n/38$ is a martingale to calculate $E(T_{40} \wedge T_0)$.

**Example 6.4.6.** To probe the boundary between recurrence and transience, suppose $p_j = 1/2 + \epsilon_j$ where $\epsilon_j \sim Cj^{-\alpha}$ as $j \to \infty$, and $q_j = 1 - p_j$. A little arithmetic shows

$$\frac{q_j}{p_j} = \frac{1/2 - \epsilon_j}{1/2 + \epsilon_j} = 1 - \frac{2\epsilon_j}{1/2 + \epsilon_j} \approx 1 - 4Cj^{-\alpha} \quad \text{for large } j$$

CASE 1. $\alpha > 1$. It is easy to show that if $0 < \delta_j < 1$, then $\prod_j (1 - \delta_j) > 0$ if and only if $\sum_j \delta_j < \infty$, (see Exercise 5.3.5), so if $\alpha > 1$, $\prod_{j \leq k}(q_j/p_j) \downarrow$ a positive limit, and 0 is recurrent.

CASE 2. $\alpha < 1$. Using the fact that $\log(1 - \delta) \sim -\delta$ as $\delta \to 0$, we see that

$$\log \prod_{j=1}^{k} q_j/p_j \sim -\sum_{j=1}^{k} 4Cj^{-\alpha} \sim -\frac{4C}{1-\alpha} k^{1-\alpha} \quad \text{as } k \to \infty$$

so, for $k \geq K$, $\prod_{j=1}^k q_j/p_j \leq \exp(-2Ck^{1-\alpha}/(1-\alpha))$ and $\sum_{k=0}^{\infty} \prod_{j=1}^k \frac{q_j}{p_j} < \infty$ and hence 0 is transient.

CASE 3. $\alpha = 1$. Repeating the argument for Case 2 shows $\log \prod_{j=1}^k \frac{q_j}{p_j} \sim -4C \log k$. So, if $C > 1/4$, 0 is transient, and if $C < 1/4$, 0 is recurrent. The case $C = 1/4$ can go either way.

**Example 6.4.7. $M/G/1$ queue.** Let $\mu = \sum k a_k$ be the mean number of customers who arrive during one service time. We will now show that if $\mu > 1$, the chain is transient (i.e., all states are), but if $\mu \leq 1$, it is recurrent. For the case $\mu > 1$, we observe that if $\xi_1, \xi_2, \ldots$ are i.i.d. with $P(\xi_m = j) = a_{j+1}$ for $j \geq -1$ and $S_n = \xi_1 + \cdots + \xi_n$, then $X_0 + S_n$ and $X_n$ behave the same until time $N = \inf\{n : X_0 + S_n = 0\}$. When $\mu > 1$, $E\xi_m = \mu - 1 > 0$, so $S_n \to \infty$ a.s., and

$\inf S_n > -\infty$ a.s. It follows from the last observation that if $x$ is large, $P_x(N < \infty) < 1$, and the chain is transient.

To deal with the case $\mu \le 1$, we observe that it follows from arguments in the last paragraph that $X_{n\wedge N}$ is a supermartingale. Let $T = \inf\{n : X_n \ge M\}$. Since $X_{n\wedge N}$ is a nonnegative supermartingale, using Theorem 5.7.6 at time $\tau = T \wedge N$, and observing $X_\tau \ge M$ on $\{T < N\}$, $X_\tau = 0$ on $\{N < T\}$ gives

$$x \ge M P_x(T < N)$$

Letting $M \to \infty$ shows $P_x(N < \infty) = 1$, so the chain is recurrent.

**Remark.** There is another way of seeing that the $M/G/1$ queue is transient when $\mu > 1$. If we consider the customers who arrive during a person's service time to be her children, then we get a branching process. Results in Section 5.3 imply that when $\mu \le 1$ the branching process dies out with probability 1 (i.e., the queue becomes empty), so the chain is recurrent. When $\mu > 1$, Theorem 5.3.9 implies $P_x(T_0 < \infty) = \rho^x$, where $\rho$ is the unique fixed point $\in (0, 1)$ of the function $\varphi(\theta) = \sum_{k=0}^{\infty} a_k \theta^k$.

The next result encapsulates the techniques we used for birth and death chains and the $M/G/1$ queue.

**Theorem 6.4.8.** *Suppose $S$ is irreducible, and $\varphi \ge 0$ with $E_x\varphi(X_1) \le \varphi(x)$ for $x \notin F$, a finite set, and $\varphi(x) \to \infty$ as $x \to \infty$, that is, $\{x : \varphi(x) \le M\}$ is finite for any $M < \infty$, then the chain is recurrent.*

*Proof.* Let $\tau = \inf\{n > 0 : X_n \in F\}$. Our assumptions imply that $Y_n = \varphi(X_{n\wedge\tau})$ is a supermartingale. Let $T_M = \inf\{n > 0 : X_n \in F \text{ or } \varphi(X_n) > M\}$. Since $\{x : \varphi(x) \le M\}$ is finite and the chain is irreducible, $T_M < \infty$ a.s. Using Theorem 5.7.6 now, we see that

$$\varphi(x) \ge E_x\varphi(X_{T_M}) \ge M P_x(T_M < \tau)$$

since $\varphi(X_{T_M}) \ge M$ when $T_M < \tau$. Letting $M \to \infty$, we see that $P_x(\tau < \infty) = 1$ for all $x \notin F$. So $P_y(X_n \in F \text{ i.o.}) = 1$ for all $y \in S$, and since $F$ is finite, $P_y(X_n = z \text{ i.o.}) = 1$ for some $z \in F$. ∎

**Exercise 6.4.7.** Show that if we replace "$\varphi(x) \to \infty$" by "$\varphi(x) \to 0$" in the last theorem and assume that $\varphi(x) > 0$ for $x \in F$, then we can conclude that the chain is transient.

**Exercise 6.4.8.** Let $X_n$ be a birth and death chain with $p_j - 1/2 \sim C/j$ as $j \to \infty$ and $q_j = 1 - p_j$. (i) Show that if we take $C < 1/4$, then we can pick $\alpha > 0$ so that $\varphi(x) = x^\alpha$ satisfies the hypotheses of Theorem 6.4.8. (ii) Show that when $C > 1/4$, we can take $\alpha < 0$ and apply Exercise 6.4.7.

**Remark.** An advantage of the method of Exercise 6.4.8 over that of Example 6.4.6 is that it applies if we assume $P_x(|X_1 - x| \le M) = 1$ and $E_x(X_1 - x) \sim 2C/x$.

**Exercise 6.4.9.** $f$ is said to be **superharmonic** if $f(x) \ge \sum_y p(x, y)f(y)$, or equivalently $f(X_n)$ is a supermartingale. Suppose $p$ is irreducible. Show that $p$ is recurrent if and only if every nonnegative superharmonic function is constant.

**Exercise 6.4.10. M/M/$\infty$ queue.** Consider a telephone system with an infinite number of lines. Let $X_n =$ the number of lines in use at time $n$, and suppose

$$X_{n+1} = \sum_{m=1}^{X_n} \xi_{n,m} + Y_{n+1}$$

where the $\xi_{n,m}$ are i.i.d. with $P(\xi_{n,m} = 1) = p$ and $P(\xi_{n,m} = 0) = 1 - p$, and $Y_n$ is an independent i.i.d. sequence of Poisson mean $\lambda$ r.v.'s. In words, for each conversation we flip a coin with probability $p$ of heads to see if it continues for another minute. Meanwhile, a Poisson mean $\lambda$ number of conversations start between time $n$ and $n + 1$. Use Theorem 6.4.8 with $\varphi(x) = x$ to show that the chain is recurrent for any $p < 1$.

## 6.5 Stationary Measures

A measure $\mu$ is said to be a **stationary measure** if

$$\sum_x \mu(x)p(x, y) = \mu(y)$$

The last equation says $P_\mu(X_1 = y) = \mu(y)$. Using the Markov property and induction, it follows that $P_\mu(X_n = y) = \mu(y)$ for all $n \ge 1$. If $\mu$ is a probability measure, we call $\mu$ a **stationary distribution**, and it represents a possible equilibrium for the chain. That is, if $X_0$ has distribution $\mu$, then so does $X_n$ for all $n \ge 1$. If we stretch our imagination a little, we can also apply this interpretation when $\mu$ is an infinite measure. (When the total mass is finite, we can divide by $\mu(S)$ to get a stationary distribution.) Before getting into the theory, we consider some examples.

**Example 6.5.1. Random walk.** $S = \mathbf{Z}^d$. $p(x, y) = f(y - x)$, where $f(z) \ge 0$ and $\sum f(z) = 1$. In this case, $\mu(x) \equiv 1$ is a stationary measure since

$$\sum_x p(x, y) = \sum_x f(y - x) = 1$$

A transition probability that has $\sum_x p(x, y) = 1$ is called **doubly stochastic**. This is obviously a necessary and sufficient condition for $\mu(x) \equiv 1$ to be a stationary measure.

**Example 6.5.2. Asymmetric simple random walk.** $S = \mathbf{Z}$.

$$p(x, x + 1) = p \qquad p(x, x - 1) = q = 1 - p$$

By the last example, $\mu(x) \equiv 1$ is a stationary measure. When $p \neq q$, $\mu(x) = (p/q)^x$ is a second one. To check this, we observe that

$$\sum_x \mu(x)p(x, y) = \mu(y+1)p(y+1, y) + \mu(y-1)p(y-1, y)$$
$$= (p/q)^{y+1}q + (p/q)^{y-1}p = (p/q)^y[p+q] = (p/q)^y$$

**Example 6.5.3. The Ehrenfest chain.** $S = \{0, 1, \ldots, r\}$**.**

$$p(k, k+1) = (r-k)/r \qquad p(k, k-1) = k/r$$

In this case, $\mu(x) = 2^{-r}\binom{r}{x}$ is a stationary distribution. One can check this without pencil and paper by observing that $\mu$ corresponds to flipping $r$ coins to determine which urn each ball is to be placed in, and the transitions of the chain correspond to picking a coin at random and turning it over. Alternatively, you can pick up your pencil and check that $\mu(k+1)p(k+1, k) + \mu(k-1)p(k-1, k) = \mu(k)$.

**Example 6.5.4. Birth and death chains.** $S = \{0, 1, 2, \ldots\}$

$$p(x, x+1) = p_x \quad p(x, x) = r_x \quad p(x, x-1) = q_x$$

with $q_0 = 0$ and $p(i, j) = 0$ otherwise. In this case, there is the measure

$$\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$$

which has

$$\mu(x)p(x, x+1) = p_x \prod_{k=1}^{x} \frac{p_{k-1}}{q_k} = \mu(x+1)p(x+1, x)$$

Since $p(x, y) = 0$ when $|x - y| > 1$, it follows that

$$\mu(x)p(x, y) = \mu(y)p(y, x) \quad \text{for all } x, y \tag{6.5.1}$$

Summing over $x$ gives

$$\sum_x \mu(x)p(x, y) = \mu(y)$$

so (6.5.1) is stronger than being a stationary measure. (6.5.1) asserts that the amount of mass that moves from $x$ to $y$ in one jump is exactly the same as the amount that moves from $y$ to $x$. A measure $\mu$ that satisfies (6.5.1) is said to be a **reversible measure**. Since Examples 6.5.2 and 6.5.3 are birth and death chains, they have reversible measures. In Example 6.5.1 (random walks), $\mu(x) \equiv 1$ is a reversible measure if and only if $p(x, y) = p(y, x)$.

The next exercise explains the name "reversible."

**Exercise 6.5.1.** Let $\mu$ be a stationary measure and suppose $X_0$ has "distribution" $\mu$. Then $Y_m = X_{n-m}$, $0 \le m \le n$ is a Markov chain with initial measure $\mu$ and transition probability

$$q(x, y) = \mu(y)p(y, x)/\mu(x)$$

$q$ is called the **dual transition probability**. If $\mu$ is a reversible measure, then $q = p$.

**Exercise 6.5.2.** Find the stationary distribution for the Bernoulli-Laplace model of diffusion from Exercise 6.2.6.

**Example 6.5.5. Random walks on graphs.** A graph is described by giving a countable set of vertices $S$ and an adjacency matrix $a_{ij}$ that has $a_{ij} = 1$ if $i$ and $j$ are adjacent and 0 otherwise. To have an undirected graph with no loops, we suppose $a_{ij} = a_{ji}$ and $a_{ii} = 0$. If we suppose that

$$\mu(i) = \sum_j a_{ij} < \infty \quad \text{and let} \quad p(i, j) = a_{ij}/\mu(i)$$

then $p$ is a transition probability that corresponds to picking an edge at random and jumping to the other end. It is clear from the definition that

$$\mu(i)p(i, j) = a_{ij} = a_{ji} = \mu(j)p(j, i)$$

so $\mu$ is a reversible measure for $p$. A little thought reveals that if we assume only that

$$a_{ij} = a_{ji} \ge 0, \quad \mu(i) = \sum_j a_{ij} < \infty \quad \text{and} \quad p(i, j) = a_{ij}/\mu(i)$$

the same conclusion is valid. This is the most general example because if $\mu$ is a reversible measure for $p$, we can let $a_{ij} = \mu(i)p(i, j)$.

Reviewing the last five examples might convince you that most chains have reversible measures. This is a false impression. The $M/G/1$ queue has no reversible measures because if $x > y + 1$, $p(x, y) = 0$ but $p(y, x) > 0$. The renewal chain has similar problems.

**Theorem 6.5.1.** *Suppose p is irreducible. A necessary and sufficient condition for the existence of a reversible measure is that (i) $p(x, y) > 0$ implies $p(y, x) > 0$, and (ii) for any loop $x_0, x_1, \ldots, x_n = x_0$ with $\prod_{1\le i\le n} p(x_i, x_{i-1}) > 0$,*

$$\prod_{i=1}^n \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = 1$$

*Proof.* To prove the necessity of this **cycle condition**, due to Kolmogorov, we note that irreducibility implies that any stationary measure has $\mu(x) > 0$ for all

$x$, so (6.5.1) implies (i) holds. To check (ii), note that (6.5.1) implies that for the sequences considered above,

$$\prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = \prod_{i=1}^{n} \frac{\mu(x_i)}{\mu(x_{i-1})} = 1$$

To prove sufficiency, fix $a \in S$, set $\mu(a) = 1$, and if $x_0 = a, x_1, \ldots, x_n = x$ is a sequence with $\prod_{1 \le i \le n} p(x_i, x_{i-1}) > 0$ (irreducibility implies such a sequence will exist), we let

$$\mu(x) = \prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})}$$

The cycle condition guarantees that the last definition is independent of the path. To check (6.5.1) now, observe that if $p(y, x) > 0$, then, adding $x_{n+1} = y$ to the end of a path to $x$, we have

$$\mu(x) \frac{p(x, y)}{p(y, x)} = \mu(y) \qquad \blacksquare$$

Only special chains have reversible measures, but as the next result shows, many Markov chains have stationary measures.

**Theorem 6.5.2.** *Let $x$ be a recurrent state, and let $T = \inf\{n \ge 1 : X_n = x\}$. Then*

$$\mu_x(y) = E_x\left(\sum_{n=0}^{T-1} 1_{\{X_n = y\}}\right) = \sum_{n=0}^{\infty} P_x(X_n = y, T > n)$$

*defines a stationary measure.*

*Proof.* This is called the "cycle trick." The proof in words is simple. $\mu_x(y)$ is the expected number of visits to $y$ in $\{0, \ldots, T-1\}$. $\mu_x p(y) \equiv \sum \mu_x(z) p(z, y)$ is the expected number of visits to $y$ in $\{1, \ldots, T\}$, which is $= \mu_x(y)$ since $X_T = X_0 = x$. See Figure 6.5 for a picture.

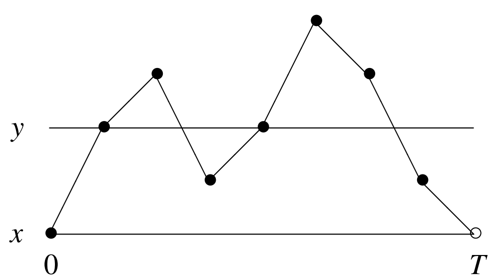



Figure 6.5. Picture of the cycle trick.

To translate this intuition into a proof, let $\bar{p}_n(x,y) = P_x(X_n = y, T > n)$ and use Fubini's theorem to get

$$\sum_y \mu_x(y)p(y,z) = \sum_{n=0}^{\infty} \sum_y \bar{p}_n(x,y)p(y,z)$$

CASE 1. $z \neq x$.

$$\begin{aligned}\sum_y \bar{p}_n(x,y)p(y,z) &= \sum_y P_x(X_n = y, T > n, X_{n+1} = z) \\ &= P_x(T > n+1, X_{n+1} = z) = \bar{p}_{n+1}(x,z)\end{aligned}$$

so $\sum_{n=0}^{\infty} \sum_y \bar{p}_n(x,y)p(y,z) = \sum_{n=0}^{\infty} \bar{p}_{n+1}(x,z) = \mu_x(z)$ since $\bar{p}_0(x,z) = 0$.

CASE 2. $z = x$.

$$\sum_y \bar{p}_n(x,y)p(y,x) = \sum_y P_x(X_n = y, T > n, X_{n+1} = x) = P_x(T = n+1)$$

so $\sum_{n=0}^{\infty} \sum_y \bar{p}_n(x,y)p(y,x) = \sum_{n=0}^{\infty} P_x(T = n+1) = 1 = \mu_x(x)$ since by definition $P_x(T = 0) = 0$. ∎

**Remark.** If $x$ is transient, then we have $\mu_x p(z) \leq \mu_x(z)$ with equality for all $z \neq x$.

**Technical note.** To show that we are not cheating, we should prove that $\mu_x(y) < \infty$ for all $y$. First, observe that $\mu_x p = \mu_x$ implies $\mu_x p^n = \mu_x$ for all $n \geq 1$, and $\mu_x(x) = 1$, so if $p^n(y,x) > 0$, then $\mu_x(y) < \infty$. Since the last result is true for all $n$, we see that $\mu_x(y) < \infty$ whenever $\rho_{yx} > 0$, but this is good enough. By Theorem 6.4.3, when $x$ is recurrent, $\rho_{xy} > 0$ implies $\rho_{yx} > 0$, and it follows from the argument above that $\mu_x(y) < \infty$. If $\rho_{xy} = 0$, then $\mu_x(y) = 0$.

**Exercise 6.5.3.** Use the construction in the proof of Theorem 6.5.2 to show that $\mu(j) = \sum_{k \geq j} f_{k+1}$ defines a stationary measure for the renewal chain (Example 6.2.3).

Theorem 6.5.2 allows us to construct a stationary measure for each closed set of recurrent states. Conversely, we have:

**Theorem 6.5.3.** *If $p$ is irreducible and recurrent (i.e., all states are) then the stationary measure is unique up to constant multiples.*

*Proof.* Let $\nu$ be a stationary measure and let $a \in S$.

$$\nu(z) = \sum_y \nu(y)p(y,z) = \nu(a)p(a,z) + \sum_{y \neq a} \nu(y)p(y,z)$$

Using the last identity to replace $\nu(y)$ on the right-hand side,

$$\begin{aligned}\nu(z) &= \nu(a)p(a,z) + \sum_{y\neq a} \nu(a)p(a,y)p(y,z) \\ &\qquad + \sum_{x\neq a}\sum_{y\neq a} \nu(x)p(x,y)p(y,z) \\ &= \nu(a)P_a(X_1 = z) + \nu(a)P_a(X_1 \neq a, X_2 = z) \\ &\qquad + P_\nu(X_0 \neq a, X_1 \neq a, X_2 = z)\end{aligned}$$

Continuing in the obvious way, we get

$$\nu(z) = \nu(a)\sum_{m=1}^{n} P_a(X_k \neq a, 1 \le k < m, X_m = z)$$
$$+ P_\nu(X_j \neq a, 0 \le j < n, X_n = z)$$

The last term is $\ge 0$. Letting $n \to \infty$ gives $\nu(z) \ge \nu(a)\mu_a(z)$, where $\mu_a$ is the measure defined in Theorem 6.5.2 for $x = a$. It follows from Theorem 6.5.2 that $\mu_a$ is a stationary distribution with $\mu_a(a) = 1$. (Here we are summing from 1 to $T$ rather than from 0 to $T-1$.) To turn the $\ge$ in the last equation into $=$, we observe

$$\nu(a) = \sum_x \nu(x)p^n(x,a) \ge \nu(a)\sum_x \mu_a(x)p^n(x,a) = \nu(a)\mu_a(a) = \nu(a)$$

Since $\nu(x) \ge \nu(a)\mu_a(x)$ and the left- and right-hand sides are equal, we must have $\nu(x) = \nu(a)\mu_a(x)$ whenever $p^n(x,a) > 0$. Since $p$ is irreducible, it follows that $\nu(x) = \nu(a)\mu_a(x)$ for all $x \in S$, and the proof is complete. ∎

Theorems 6.5.2 and 6.5.3 make a good team. The first result gives us a formula for a stationary distribution we call $\mu_x$, and the second shows it is unique up to constant multiples. Together they allow us to derive a lot of formulas.

**Exercise 6.5.4.** Let $w_{xy} = P_x(T_y < T_x)$. Show that $\mu_x(y) = w_{xy}/w_{yx}$.

**Exercise 6.5.5.** Show that if $p$ is irreducible and recurrent, then

$$\mu_x(y)\mu_y(z) = \mu_x(z)$$

**Exercise 6.5.6.** Use Theorems 6.5.2 and 6.5.3 to show that for simple random walk, (i) the expected number of visits to $k$ between successive visits to 0 is 1 for all $k$, and (ii) if we start from $k$, the expected number of visits to $k$ before hitting 0 is $2k$.

**Exercise 6.5.7. Another proof of Theorem 6.5.3.** Suppose $p$ is irreducible and recurrent and let $\mu$ be the stationary measure constructed in Theorem 6.5.2. $\mu(x) > 0$ for all $x$, and

$$q(x, y) = \mu(y)p(y, x)/\mu(x) \geq 0$$

defines a "dual" transition probability. (See Exercise 6.5.1.) (i) Show that $q$ is irreducible and recurrent. (ii) Suppose $\nu(y) \geq \sum_x \nu(x)p(x, y)$ (i.e, $\nu$ is an **excessive measure**) and let $h(x) = \nu(x)/\mu(x)$. Verify that $h(y) \geq \sum q(y, x)h(x)$ and use Exercise 6.4.9 to conclude that $h$ is constant, that is, $\nu = c\mu$.

**Remark.** The last result is stronger than Theorem 6.5.3 since it shows that in the recurrent case any excessive measure is a constant multiple of one stationary measure. The remark after the proof of Theorem 6.5.3 shows that if $p$ is irreducible and transient, there is an excessive measure for each $x \in S$.

Having examined the existence and uniqueness of stationary measures, we turn our attention now to **stationary distributions**, that is, probability measures $\pi$ with $\pi p = \pi$. Stationary measures may exist for transient chains, such as, random walks in $d \geq 3$, but

**Theorem 6.5.4.** *If there is a stationary distribution, then all states $y$ that have $\pi(y) > 0$ are recurrent.*

*Proof.* Since $\pi p^n = \pi$, Fubini's theorem implies

$$\sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x, y) = \sum_{n=1}^{\infty} \pi(y) = \infty$$

when $\pi(y) > 0$. Using Theorem 6.4.2 now gives

$$\infty = \sum_x \pi(x) \frac{\rho_{xy}}{1 - \rho_{yy}} \leq \frac{1}{1 - \rho_{yy}}$$

since $\rho_{xy} \leq 1$ and $\pi$ is a probability measure. So $\rho_{yy} = 1$. ■

**Theorem 6.5.5.** *If $p$ is irreducible and has stationary distribution $\pi$, then*

$$\pi(x) = 1/E_x T_x$$

**Remark.** Recycling Chung's quote regarding Theorem 5.5.8, we note that the proof will make $\pi(x) = 1/E_x T_x$ obvious, but it seems incredible that

$$\sum_x \frac{1}{E_x T_x} p(x, y) = \frac{1}{E_y T_y}$$

*Proof.* Irreducibility implies $\pi(x) > 0$ so all states are recurrent by Theorem 6.5.4. From Theorem 6.5.2,

$$\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$$

defines a stationary measure with $\mu_x(x) = 1$, and Fubini's theorem implies

$$\sum_y \mu_x(y) = \sum_{n=0}^{\infty} P_x(T_x > n) = E_x T_x$$

By Theorem 6.5.3, the stationary measure is unique up to constant multiples, so $\pi(x) = \mu_x(x)/E_x T_x$. Since $\mu_x(x) = 1$ by definition, the desired result follows. ∎

**Exercise 6.5.8.** Compute the expected number of moves it takes a knight to return to its initial position if it starts in a corner of the chessboard, assuming there are no other pieces on the board, and each time it chooses a move at random from its legal moves. (Note: A chessboard is $\{0, 1, \ldots, 7\}^2$. A knight's move is $L$-shaped; two steps in one direction followed by one step in a perpendicular direction.)

If a state $x$ has $E_x T_x < \infty$, it is said to be **positive recurrent**. A recurrent state with $E_x T_x = \infty$ is said to be **null recurrent**. Theorem 6.6.1 will explain these names. The next result helps us identify positive recurrent states.

**Theorem 6.5.6.** *If $p$ is irreducible, then the following are equivalent:*
*(i) Some $x$ is positive recurrent.*
*(ii) There is a stationary distribution.*
*(iii) All states are positive recurrent.*

*Proof.* *(i) implies (ii).* If $x$ is positive recurrent then

$$\pi(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)/E_x T_x$$

defines a stationary distribution.

*(ii) implies (iii).* Theorem 6.5.5 implies $\pi(y) = 1/E_y T_y$, and irreducibility tells us $\pi(y) > 0$ for all $y$, so $E_y T_y < \infty$.

*(iii) implies (i).* Trivial. ∎

**Exercise 6.5.9.** Suppose $p$ is irreducible and positive recurrent. Then $E_x T_y < \infty$ for all $x, y$.

**Exercise 6.5.10.** Suppose $p$ is irreducible and has a stationary measure $\mu$ with $\sum_x \mu(x) = \infty$. Then $p$ is not positive recurrent.

Theorem 6.5.6 shows that being positive recurrent is a **class property**. If it holds for one state in an irreducible set, then it is true for all. Turning to our examples, since $\mu(x) \equiv 1$ is a stationary measure, Exercise 6.5.10 implies that random walks (Example 6.5.1) are never positive recurrent. Random walks on graphs (Example 6.5.5) are irreducible if and only if the graph is connected. Since $\mu(i) \geq 1$ in the connected case, we have positive recurrence if and only if the graph is finite. The Ehrenfest chain (Example 6.5.3) is positive recurrent. To see this, note that the state space is finite, so there is a stationary distribution, and the conclusion follows from Theorem 6.5.4. A renewal chain is irreducible if $\{k : f_k > 0\}$ is unbounded (see Example 6.4.3); it is positive recurrent (i.e., all the states are) if and only if $E_0 T_0 = \sum k f_k < \infty$.

Birth and death chains (Example 6.5.4) have a stationary distribution if and only if

$$\sum_x \prod_{k=1}^{x} \frac{p_{k-1}}{q_k} < \infty$$

By Theorem 6.4.7, the chain is recurrent if and only if

$$\sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty$$

When $p_j = p$ and $q_j = (1-p)$ for $j \geq 1$, there is a stationary distribution if and only if $p < 1/2$, and the chain is transient when $p > 1/2$. In Section 6.4, we probed the boundary between recurrence and transience by looking at examples with $p_j = 1/2 + \epsilon_j$, where $\epsilon_j \sim C\, j^{-\alpha}$ as $j \to \infty$ and $C, \alpha \in (0, \infty)$. Since $\epsilon_j \geq 0$ and hence $p_{j-1}/q_j \geq 1$ for large $j$, none of these chains have stationary distributions. If we look at chains with $p_j = 1/2 - \epsilon_j$, then all we have done is interchange the roles of $p$ and $q$, and results from the last section imply that the chain is positive recurrent when $\alpha < 1$, or $\alpha = 1$ and $C > 1/4$.

**Example 6.5.6.** *$M/G/1$* **queue.** Let $\mu = \sum k a_k$ be the mean number of customers who arrive during one service time. In Example 6.4.7, we showed that the chain is recurrent if and only if $\mu \leq 1$. We will now show that the chain is positive recurrent if and only if $\mu < 1$. First, suppose that $\mu < 1$. When $X_n > 0$, the chain behaves like a random walk that has jumps with mean $\mu - 1$, so if $N = \inf\{n \geq 0 : X_n = 0\}$, then $X_{N \wedge n} - (\mu - 1)(N \wedge n)$ is a martingale. If $X_0 = x > 0$, then the martingale property implies

$$x = E_x X_{N \wedge n} + (1-\mu) E_x (N \wedge n) \geq (1-\mu) E_x (N \wedge n)$$

since $X_{N \wedge n} \geq 0$, and it follows that $E_x N \leq x/(1-\mu)$.

To prove that there is equality, observe that $X_n$ decreases by at most 1 each time and for $x \ge 1$, $E_x T_{x-1} = E_1 T_0$, so $E_x N = cx$. To identify the constant, observe that

$$E_1 N = 1 + \sum_{k=0}^{\infty} a_k E_k N$$

so $c = 1 + \mu c$ and $c = 1/(1-\mu)$. If $X_0 = 0$ then $p(0,0) = a_0 + a_1$ and $p(0, k-1) = a_k$ for $k \ge 2$. By considering what happens on the first jump, we see that (the first term may look wrong, but recall $k - 1 = 0$ when $k = 1$)

$$E_0 T_0 = 1 + \sum_{k=1}^{\infty} a_k \frac{k-1}{1-\mu} = 1 + \frac{\mu - (1-a_0)}{1-\mu} = \frac{a_0}{1-\mu} < \infty$$

This shows that the chain is positive recurrent if $\mu < 1$. To prove the converse, observe that the arguments above show that if $E_0 T_0 < \infty$ then $E_k N < \infty$ for all $k$, $E_k N = ck$, and $c = 1/(1-\mu)$, which is impossible if $\mu \ge 1$.

The last result when combined with Theorem 6.5.2 and 6.5.5 allows us to conclude that the stationary distribution has $\pi(0) = (1-\mu)/a_0$. This may not seem like much, but the equations in $\pi p = \pi$ are

$$\begin{aligned}
\pi(0) &= \pi(0)(a_0 + a_1) + \pi(1)a_0 \\
\pi(1) &= \pi(0)a_2 + \pi(1)a_1 + \pi(2)a_0 \\
\pi(2) &= \pi(0)a_3 + \pi(1)a_2 + \pi(2)a_1 + \pi(3)a_0
\end{aligned}$$

or, in general, for $j \ge 1$,

$$\pi(j) = \sum_{i=0}^{j+1} \pi(i) a_{j+1-i}$$

The equations have a "triangular" form, so knowing $\pi(0)$, we can solve for $\pi(1), \pi(2), \ldots$ The first expression,

$$\pi(1) = \pi(0)(1 - (a_0 + a_1))/a_0$$

is simple, but the formulas get progressively messier, and there is no nice closed-form solution.

**Exercise 6.5.11.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_m = k) = a_{k+1}$ for $k \ge -1$, let $S_n = x + \xi_1 + \cdots + \xi_n$, where $x \ge 0$, and let

$$X_n = S_n + \left(\min_{m \le n} S_m\right)^-$$

(6.2.1) shows that $X_n$ has the same distribution as the $M/G/1$ queue starting from $X_0 = x$. Use this representation to conclude that if $\mu = \sum k a_k < 1$, then as $n \to \infty$,

$$\frac{1}{n} |\{m \le n : X_{m-1} = 0, \xi_m = -1\}| \to (1-\mu) \quad \text{a.s.}$$

and hence $\pi(0) = (1-\mu)/a_0$ as proved above.

**Example 6.5.7.** $M/M/\infty$ **queue**. In this chain, introduced in Exercise 6.4.10,

$$X_{n+1} = \sum_{m=1}^{Xn} \xi_{n,m} + Y_{n+1}$$

where $\xi_{n,m}$ are i.i.d. Bernoulli with mean $p$ and $Y_{n+1}$ is an independent Poisson mean $\lambda$. It follows from properties of the Poisson distribution that if $X_n$ is Poisson with mean $\mu$, then $X_{n+1}$ is Poisson with mean $\mu p + \lambda$. Setting $\mu = \mu p + \lambda$, we find that a Poisson distribution with mean $\mu = \lambda/(1-p)$ is a stationary distribution.

There is a general result that handles Examples 6.5.6 and 6.5.7 and is useful in a number of other situations. This will be developed in the next two exercises.

**Exercise 6.5.12.** Let $X_n \geq 0$ be a Markov chain and suppose $E_x X_1 \leq x - \epsilon$ for $x > K$, where $\epsilon > 0$. Let $Y_n = X_n + n\epsilon$ and $\tau = \inf\{n : X_n \leq K\}$. $Y_{n\wedge\tau}$ is a positive supermartingale, and the optional stopping theorem implies $E_x\tau \leq x/\epsilon$.

**Exercise 6.5.13.** Suppose that $X_n$ has state space $\{0, 1, 2, \ldots\}$, the conditions of the last exercise hold when $K = 0$, and $E_0 X_1 < \infty$. Then 0 is positive recurrent. We leave it to the reader to formulate and prove a similar result when $K > 0$.

To close the section, we will give a self-contained proof of

**Theorem 6.5.7.** *If $p$ is irreducible and has a stationary distribution $\pi$, then any other stationary measure is a multiple of $\pi$.*

**Remark.** This result is a consequence of Theorems 6.5.4 and Theorem 6.5.3, but we find the method of proof amusing.

*Proof.* Since $p$ is irreducible, $\pi(x) > 0$ for all $x$. Let $\varphi$ be a concave function that is bounded on $(0, \infty)$, for example, $\varphi(x) = x/(x+1)$. Define the **entropy** of $\mu$ by

$$\mathcal{E}(\mu) = \sum_y \varphi\left(\frac{\mu(y)}{\pi(y)}\right)\pi(y)$$

The reason for the name will become clear during the proof.

$$\begin{aligned}
\mathcal{E}(\mu p) &= \sum_y \varphi\left(\sum_x \frac{\mu(x)p(x,y)}{\pi(y)}\right)\pi(y) = \sum_y \varphi\left(\sum_x \frac{\mu(x)}{\pi(x)} \cdot \frac{\pi(x)p(x,y)}{\pi(y)}\right)\pi(y) \\
&\geq \sum_y \sum_x \varphi\left(\frac{\mu(x)}{\pi(x)}\right)\frac{\pi(x)p(x,y)}{\pi(y)}\pi(y)
\end{aligned}$$

since $\varphi$ is concave, and $\nu(x) = \pi(x)p(x,y)/\pi(y)$ is a probability distribution. Since the $\pi(y)$'s cancel and $\sum_y p(x,y) = 1$, the last expression $= \mathcal{E}(\mu)$, and we have shown $\mathcal{E}(\mu p) \geq \mathcal{E}(\mu)$, that is, the entropy of an arbitrary initial measure $\mu$ is increased by an application of $p$.

If $p(x, y) > 0$ for all $x$ and $y$, and $\mu p = \mu$, it follows that $\mu(x)/\pi(x)$ must be constant, for otherwise there would be strict inequality in the application of Jensen's inequality. To get from the last special case to the general result, observe that if $p$ is irreducible,

$$\bar{p}(x, y) = \sum_{n=1}^{\infty} 2^{-n} p^n(x, y) > 0 \quad \text{for all } x, y$$

and $\mu p = \mu$ implies $\mu \bar{p} = \mu$. ■

## 6.6 Asymptotic Behavior

The first topic in this section is to investigate the asymptotic behavior of $p^n(x, y)$. If $y$ is transient, $\sum_n p^n(x, y) < \infty$, so $p^n(x, y) \to 0$ as $n \to \infty$. To deal with the recurrent states, we let

$$N_n(y) = \sum_{m=1}^{n} 1_{\{X_m = y\}}$$

be the number of visits to $y$ by time $n$.

**Theorem 6.6.1.** *Suppose $y$ is recurrent. For any $x \in S$, as $n \to \infty$*

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} 1_{\{T_y < \infty\}} \quad P_x\text{-a.s.}$$

*Here* $1/\infty = 0$.

*Proof.* Suppose first that we start at $y$. Let $R(k) = \min\{n \geq 1 : N_n(y) = k\}$ = the time of the $k$th return to $y$. Let $t_k = R(k) - R(k-1)$, where $R(0) = 0$. Since we have assumed $X_0 = y$, $t_1, t_2, \ldots$ are i.i.d. and the strong law of large numbers implies

$$R(k)/k \to E_y T_y \quad P_y\text{-a.s.}$$

Since $R(N_n(y)) \leq n < R(N_n(y) + 1)$,

$$\frac{R(N_n(y))}{N_n(y)} \leq \frac{n}{N_n(y)} < \frac{R(N_n(y)+1)}{N_n(y)+1} \cdot \frac{N_n(y)+1}{N_n(y)}$$

Letting $n \to \infty$, and recalling $N_n(y) \to \infty$ a.s. since $y$ is recurrent, we have

$$\frac{n}{N_n(y)} \to E_y T_y \quad P_y\text{-a.s.}$$

To generalize now to $x \neq y$, observe that if $T_y = \infty$ then $N_n(y) = 0$ for all $n$, and hence

$$N_n(y)/n \to 0 \quad \text{on } \{T_y = \infty\}$$

The strong Markov property implies that conditional on $\{T_y < \infty\}$, $t_2, t_3, \dots$ are i.i.d. and have $P_x(t_k = n) = P_y(T_y = n)$, so

$$R(k)/k = t_1/k + (t_2 + \cdots + t_k)/k \to 0 + E_y T_y \quad P_x\text{-a.s.}$$

Repeating the proof for the case $x = y$ shows

$$N_n(y)/n \to 1/E_y T_y \quad P_x\text{-a.s. on } \{T_y < \infty\}$$

and combining this with the result for $\{T_y = \infty\}$ completes the proof. ■

**Remark.** Theorem 6.6.1 should help explain the terms positive and null recurrent. If we start from $x$, then in the first case the asymptotic fraction of time spent at $x$ is positive and in the second case it is 0.

Since $0 \le N_n(y)/n \le 1$, it follows from the bounded convergence theorem that $E_x N_n(y)/n \to E_x(1_{\{T_y<\infty\}}/E_y T_y)$, so

$$\frac{1}{n}\sum_{m=1}^{n} p^m(x, y) \to \rho_{xy}/E_y T_y \tag{6.6.1}$$

The last result was proved for recurrent $y$ but also holds for transient $y$, since in that case, $E_y T_y = \infty$, and the limit is 0, since $\sum_m p^m(x, y) < \infty$.

(6.6.1) shows that the sequence $p^n(x, y)$ always converges in the Cesaro sense. The next example shows that $p^n(x, y)$ need not converge.

**Example 6.6.1.**

$$p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad p^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad p^3 = p, \quad p^4 = p^2, \dots$$

A similar problem also occurs in the Ehrenfest chain. In that case, if $X_0$ is even, then $X_1$ is odd, $X_2$ is even, … so $p^n(x, x) = 0$ unless $n$ is even. It is easy to construct examples with $p^n(x, x) = 0$ unless $n$ is a multiple of 3 or 17 or …

Theorem 6.6.4 below will show that this "periodicity" is the only thing that can prevent the convergence of the $p^n(x, y)$. First, we need a definition and two preliminary results. Let $x$ be a recurrent state, let $I_x = \{n \ge 1 : p^n(x, x) > 0\}$, and let $d_x$ be the greatest common divisor of $I_x$. $d_x$ is called the **period** of $x$. The first result says that the period is a class property.

**Lemma 6.6.2.** *If $\rho_{xy} > 0$, then $d_y = d_x$.*

*Proof.* Let $K$ and $L$ be such that $p^K(x, y) > 0$ and $p^L(y, x) > 0$. ($x$ is recurrent, so $\rho_{yx} > 0$.)

$$p^{K+L}(y, y) \ge p^L(y, x)p^K(x, y) > 0$$

so $d_y$ divides $K + L$, abbreviated $d_y|(K + L)$. Let $n$ be such that $p^n(x, x) > 0$.

$$p^{K+n+L}(y, y) \geq p^L(y, x)p^n(x, x)p^K(x, y) > 0$$

so $d_y|(K + n + L)$, and hence $d_y|n$. Since $n \in I_x$ is arbitrary, $d_y|d_x$. Interchanging the roles of $y$ and $x$ gives $d_x|d_y$, and hence $d_x = d_y$. ∎

If a chain is irreducible and $d_x = 1$, it is said to be aperiodic. The easiest way to check this is to find a state with $p(x, x) > 0$. The $M/G/1$ queue has $a_k > 0$ for all $k \geq 0$, so it has this property. The renewal chain is aperiodic if $g.c.d.\{k : f_k > 0\} = 1$.

**Lemma 6.6.3.** *If* $d_x = 1$ *then* $p^m(x, x) > 0$ *for* $m \geq m_0$.

*Proof by example.* Suppose $4, 7 \in I_x$. $p^{m+n}(x, x) \geq p^m(x, x)p^n(x, x)$, so $I_x$ is closed under addition, that is, if $m, n \in I_x$, then $m + n \in I_x$. A little calculation shows that in the example

$$I_x \supset \{4, \quad 7, 8, \quad 11, 12, \quad 14, 15, 16, \quad 18, 19, 20, 21, \ \ldots\}$$

so the result is true with $m_0 = 18$. (Once $I_x$ contains four consecutive integers, it will contain all the rest.)

*Proof.* Our first goal is to prove that $I_x$ contains two consecutive integers. Let $n_0$, $n_0 + k \in I_x$. If $k = 1$, we are done. If not, then since the greatest common divisor of $I_x$ is 1, there is an $n_1 \in I_x$ so that $k$ is not a divisor of $n_1$. Write $n_1 = mk + r$ with $0 < r < k$. Since $I_x$ is closed under addition, $(m + 1)(n_0 + k) > (m + 1)n_0 + n_1$ are both in $I_x$. Their difference is

$$k(m + 1) - n_1 = k - r < k$$

Repeating the last argument (at most $k$ times), we eventually arrive at a pair of consecutive integers $N, N + 1 \in I_x$. It is now easy to show that the result holds for $m_0 = N^2$. Let $m \geq N^2$ and write $m - N^2 = kN + r$ with $0 \leq r < N$. Then $m = r + N^2 + kN = r(1 + N) + (N - r + k)N \in I_x$. ∎

**Theorem 6.6.4. Convergence theorem.** *Suppose $p$ is irreducible, aperiodic (i.e., all states have $d_x = 1$), and has stationary distribution $\pi$. Then, as $n \to \infty$, $p^n(x, y) \to \pi(y)$.*

*Proof.* Let $S^2 = S \times S$. Define a transition probability $\bar{p}$ on $S \times S$ by

$$\bar{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2)p(y_1, y_2)$$

that is, each coordinate moves independently. Our first step is to check that $\bar{p}$ is irreducible. This may seem like a silly thing to do first, but this is the only step that requires aperiodicity. Since $p$ is irreducible, there are $K, L$, so that

$p^K(x_1, x_2) > 0$ and $p^L(y_1, y_2) > 0$. From Lemma 6.6.3, it follows that if $M$ is large, $p^{L+M}(x_2, x_2) > 0$ and $p^{K+M}(y_2, y_2) > 0$, so

$$\bar{p}^{K+L+M}((x_1, y_1), (x_2, y_2)) > 0$$

Our second step is to observe that since the two coordinates are independent, $\bar{\pi}(a, b) = \pi(a)\pi(b)$ defines a stationary distribution for $\bar{p}$, and Theorem 6.5.4 implies that for $\bar{p}$ all states are recurrent. Let $(X_n, Y_n)$ denote the chain on $S \times S$, and let $T$ be the first time that this chain hits the diagonal $\{(y, y) : y \in S\}$. Let $T_{(x,x)}$ be the hitting time of $(x, x)$. Since $\bar{p}$ is irreducible and recurrent, $T_{(x,x)} < \infty$ a.s. and hence $T < \infty$ a.s. The final step is to observe that on $\{T \le n\}$, the two coordinates $X_n$ and $Y_n$ have the same distribution. By considering the time and place of the first intersection and then using the Markov property,

$$\begin{aligned}
P(X_n = y, T \le n) &= \sum_{m=1}^{n} \sum_x P(T = m, X_m = x, X_n = y) \\
&= \sum_{m=1}^{n} \sum_x P(T = m, X_m = x) P(X_n = y | X_m = x) \\
&= \sum_{m=1}^{n} \sum_x P(T = m, Y_m = x) P(Y_n = y | Y_m = x) \\
&= P(Y_n = y, T \le n)
\end{aligned}$$

To finish up, we observe that

$$\begin{aligned}
P(X_n = y) &= P(Y_n = y, T \le n) + P(X_n = y, T > n) \\
&\le P(Y_n = y) + P(X_n = y, T > n)
\end{aligned}$$

and similarly, $P(Y_n = y) \le P(X_n = y) + P(Y_n = y, T > n)$. So

$$|P(X_n = y) - P(Y_n = y)| \le P(X_n = y, T > n) + P(Y_n = y, T > n)$$

and summing over $y$ gives

$$\sum_y |P(X_n = y) - P(Y_n = y)| \le 2P(T > n)$$

If we let $X_0 = x$ and let $Y_0$ have the stationary distribution $\pi$, then $Y_n$ has distribution $\pi$, and it follows that

$$\sum_y |p^n(x, y) - \pi(y)| \le 2P(T > n) \to 0$$

proving the desired result. If we recall the definition of the total variation distance given in Section 3.6, the last conclusion can be written as

$$\|p^n(x, \cdot) - \pi(\cdot)\| \le P(T > n) \to 0 \qquad \blacksquare$$

At first glance, it may seem strange to prove the convergence theorem by running independent copies of the chain. An approach that is slightly more complicated but explains better what is happening is to define

$$q((x_1, y_1), (x_2, y_2)) = \begin{cases} p(x_1, x_2)p(y_1, y_2) & \text{if } x_1 \neq y_1 \\ p(x_1, x_2) & \text{if } x_1 = y_1, x_2 = y_2 \\ 0 & \text{otherwise} \end{cases}$$

In words, the two coordinates move independently until they hit and then move together. It is easy to see from the definition that each coordinate is a copy of the original process. If $T'$ is the hitting time of the diagonal for the new chain $(X_n', Y_n')$, then $X_n' = Y_n'$ on $T' \leq n$, so it is clear that

$$\sum_y |P(X_n' = y) - P(Y_n' = y)| \leq 2\, P(X_n' \neq Y_n') = 2P(T' > n)$$

On the other hand, $T$ and $T'$ have the same distribution, so $P(T' > n) \to 0$, and the conclusion follows as before. The technique used in the last proof is called **coupling**. Generally, this term refers to building two sequences $X_n$ and $Y_n$ on the same space to conclude that $X_n$ converges in distribution by showing $P(X_n \neq Y_n) \to 0$, or more generally, that for some metric $\rho$, $\rho(X_n, Y_n) \to 0$ in probability.

## *Finite State Space*

The convergence theorem is much easier when the state space is finite.

**Exercise 6.6.1.** Show that if $S$ is finite and $p$ is irreducible and aperiodic, then there is an $m$ so that $p^m(x, y) > 0$ for all $x, y$.

**Exercise 6.6.2.** Show that if $S$ is finite, $p$ is irreducible and aperiodic, and $T$ is the coupling time defined in the proof of Theorem 6.6.4 then $P(T > n) \leq Cr^n$ for some $r < 1$ and $C < \infty$. So the convergence to equilibrium occurs exponentially rapidly in this case. Hint: First consider the case in which $p(x, y) > 0$ for all $x$ and $y$ and reduce the general case to this one by looking at a power of $p$.

**Exercise 6.6.3.** For any transition matrix $p$, define

$$\alpha_n = \sup_{i,j} \frac{1}{2} \sum_k |p^n(i, k) - p^n(j, k)|$$

The 1/2 is there because for any $i$ and $j$ we can define r.v.'s $X$ and $Y$ so that $P(X = k) = p^n(i, k)$, $P(Y = k) = p^n(j, k)$, and

$$P(X \neq Y) = (1/2) \sum_k |p^n(i, k) - p^n(j, k)|$$

Show that $\alpha_{m+n} \le \alpha_n \alpha_m$. Here you may find that the coupling interpretation may help you from getting lost in the algebra. Using Lemma 2.6.1, we can conclude that

$$\frac{1}{n} \log \alpha_n \to \inf_{m \ge 1} \frac{1}{m} \log \alpha_m$$

so if $\alpha_m < 1$ for some $m$, it approaches 0 exponentially fast.

As the last two exercises show, Markov chains on finite state spaces converge exponentially fast to their stationary distributions. In applications, however, it is important to have rates of convergence. The next two problems are a taste of an exciting research area.

**Example 6.6.2. Shuffling cards.** The state of a deck of $n$ cards can be represented by a permutation, $\pi(i)$ giving the location of the $i$th card. Consider the following method of mixing the deck up. The top card is removed and inserted under one of the $n-1$ cards that remain. I claim that by following the bottom card of the deck, we can see that it takes about $n \log n$ moves to mix up the deck. This card stays at the bottom until the first time ($T_1$) a card is inserted below it. It is easy to see that when the $k$th card is inserted below the original bottom card (at time $T_k$), all $k!$ arrangements of the cards below are equally likely, so at time $\tau_n = T_{n-1} + 1$ all $n!$ arrangements are equally likely. If we let $T_0 = 0$ and $t_k = T_k - T_{k-1}$ for $1 \le k \le n-1$, then these r.v.'s are independent, and $t_k$ has a geometric distribution with success probability $k/(n-1)$. These waiting times are the same as the ones in the coupon collector's problem (Example 2.2.3), so $\tau_n/(n \log n) \to 1$ in probability as $n \to \infty$. For more on card shuffling, see Aldous and Diaconis (1986).

**Example 6.6.3. Random walk on the hypercube.** Consider $\{0, 1\}^d$ as a graph with edges connecting each pair of points that differ in only one coordinate. Let $X_n$ be a random walk on $\{0, 1\}^d$ that stays put with probability $1/2$ and jumps to one of its $d$ neighbors with probability $1/2d$ each. Let $Y_n$ be another copy of the chain in which $Y_0$ (and hence $Y_n$, $n \ge 1$) is uniformly distributed on $\{0, 1\}^d$. We construct a coupling of $X_n$ and $Y_n$ by letting $U_1, U_2, \ldots$ be i.i.d. uniform on $\{1, 2, \ldots, d\}$, and letting $V_1, V_2, \ldots$ be independent i.i.d. uniform on $\{0, 1\}$ At time $n$, the $U_n$th coordinates of $X$ and $Y$ are each set equal to $V_n$. The other coordinates are unchanged. Let $T_d = \inf\{m : \{U_1, \ldots, U_m\} = \{1, 2, \ldots, d\}\}$. When $n \ge T_d$, $X_n = Y_n$. Results for the coupon collector's problem (Example 2.2.3) show that $T_d/(d \log d) \to 1$ in probability as $d \to \infty$.

## *Exercises*

**6.6.4. Strong law for additive functionals.** Suppose $p$ is irreducible and has stationary distribution $\pi$. Let $f$ be a function that has $\sum |f(y)|\pi(y) < \infty$. Let $T_x^k$

be the time of the $k$th return to $x$. (i) Show that

$$V_k^f = f(X(T_x^k)) + \cdots + f(X(T_x^{k+1} - 1)), \quad k \geq 1 \text{ are i.i.d.}$$

with $E|V_k^f| < \infty$. (ii) Let $K_n = \inf\{k : T_x^k \geq n\}$ and show that

$$\frac{1}{n} \sum_{m=1}^{K_n} V_m^f \to \frac{EV_1^f}{E_x T_x^1} = \sum f(y)\pi(y) \quad P_\mu - \text{a.s.}$$

(iii) Show that $\max_{1 \leq m \leq n} V_m^{|f|}/n \to 0$ and conclude

$$\frac{1}{n} \sum_{m=1}^{n} f(X_m) \to \sum_y f(y)\pi(y) \quad P_\mu - \text{a.s.}$$

for any initial distribution $\mu$.

**6.6.5. Central limit theorem for additive functionals.** Suppose in addition to the conditions in the Exercise 6.6.4 that $\sum f(y)\pi(y) = 0$, and $E_x(V_k^{|f|})^2 < \infty$. (i) Use the random index central limit theorem (Exercise 3.4.6) to conclude that for any initial distribution $\mu$,

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{K_n} V_m^f \Rightarrow c\chi \quad \text{under } P_\mu$$

(ii) Show that $\max_{1 \leq m \leq n} V_m^{|f|}/\sqrt{n} \to 0$ in probability and conclude

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} f(X_m) \Rightarrow c\chi \quad \text{under } P_\mu$$

**6.6.6. Ratio limit theorems.** Theorem 6.6.1 does not say much in the null recurrent case. To get a more informative limit theorem, suppose that $y$ is recurrent and $m$ is the (unique up to constant multiples) stationary measure on $C_y = \{z : \rho_{yz} > 0\}$. Let $N_n(z) = |\{m \leq n : X_n = z\}|$. Break up the path at successive returns to $y$ and show that $N_n(z)/N_n(y) \to m(z)/m(y)$ $P_x$-a.s. for all $x, z \in C_y$. Note that $n \to N_n(z)$ is increasing, so this is much easier than the previous problem.

**6.6.7.** We got (6.6.1) from Theorem 6.6.1 by taking expected value. This does not work for the ratio in the previous exercise, so we need another approach. Suppose $z \neq y$. (i) Let $\bar{p}_n(x, z) = P_x(X_n = z, T_y > n)$ and decompose $p^m(x, z)$ according to the value of $J = \sup\{j \in [1, m) : X_j = y\}$ to get

$$\sum_{m=1}^{n} p^m(x, z) = \sum_{m=1}^{n} \bar{p}_m(x, z) + \sum_{j=1}^{n-1} p^j(x, y) \sum_{k=1}^{n-j} \bar{p}_k(y, z)$$

(ii) Show that

$$\sum_{m=1}^{n} p^m(x, z) \Bigg/ \sum_{m=1}^{n} p^m(x, y) \to \frac{m(z)}{m(y)}$$

## 6.7 Periodicity, Tail $\sigma$-field*

**Lemma 6.7.1.** *Suppose $p$ is irreducible, recurrent, and all states have period $d$. Fix $x \in S$, and for each $y \in S$, let $K_y = \{n \geq 1 : p^n(x, y) > 0\}$. (i) There is an $r_y \in \{0, 1, \dots, d-1\}$ so that if $n \in K_y$ then $n = r_y \bmod d$, that is, the difference $n - r_y$ is a multiple of $d$. (ii) Let $S_r = \{y : r_y = r\}$ for $0 \leq r < d$. If $y \in S_i$, $z \in S_j$, and $p^n(y, z) > 0$, then $n = (j - i) \bmod d$. (iii) $S_0, S_1, \dots, S_{d-1}$ are irreducible classes for $p^d$, and all states have period 1.*

*Proof.* (i) Let $m(y)$ be such that $p^{m(y)}(y, x) > 0$. If $n \in K_y$, then $p^{n+m(y)}(x, x)$ is positive, so $d|(n + m)$. Let $r_y = (d - m(y)) \bmod d$. (ii) Let $m, n$ be such that $p^n(y, z)$, $p^m(x, y) > 0$. Since $p^{n+m}(x, z) > 0$, it follows from (i) that $n + m = j \bmod d$. Since $m = i \bmod d$, the result follows. The irreducibility in (iii) follows immediately from (ii). The aperiodicity follows from the definition of the period as the g.c.d. $\{x : p^n(x, x) > 0\}$. ■

A partition of the state space $S_0, S_1, \dots, S_{d-1}$ satisfying (ii) in Lemma 6.7.1 is called a **cyclic decomposition** of the state space. Except for the choice of the set to put first, it is unique. (Pick an $x \in S$. It lies in some $S_j$, but once the value of $j$ is known, irreducibility and (ii) allow us to calculate all the sets.)

**Exercise 6.7.1.** Find the decomposition for the Markov chain with transition probability

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | .5 | .5 | 0 | 0 |
| **2** | .3 | 0 | 0 | 0 | 0 | 0 | .7 |
| **3** | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **4** | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **5** | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **6** | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| **7** | 0 | 0 | 0 | .4 | 0 | .6 | 0 |

**Theorem 6.7.2. Convergence theorem, periodic case.** *Suppose $p$ is irreducible and has a stationary distribution $\pi$, and all states have period $d$. Let $x \in S$, and let $S_0, S_1, \dots, S_{d-1}$ be the cyclic decomposition of the state space with $x \in S_0$. If $y \in S_r$ then*

$$\lim_{m \to \infty} p^{md+r}(x, y) = \pi(y)d$$

*Proof.* If $y \in S_0$ then using (iii) in Lemma 6.7.1 and applying Theorem 6.6.4 to $p^d$ shows

$$\lim_{m \to \infty} p^{md}(x, y) \text{ exists}$$

To identify the limit, we note that (6.6.1) implies

$$\frac{1}{n}\sum_{m=1}^{n} p^m(x,y) \to \pi(y)$$

and (ii) of Lemma 6.7.1 implies $p^m(x,y) = 0$ unless $d|m$, so the limit in the first display must be $\pi(y)d$. If $y \in S_r$ with $1 \le r < d$, then

$$p^{md+r}(x,y) = \sum_{z\in S_r} p^r(x,z)p^{md}(z,y)$$

Since $y, z \in S_r$, it follows from the first case in the proof that $p^{md}(z,y) \to \pi(y)d$ as $m \to \infty$. $p^{md}(z,y) \le 1$, and $\sum_z p^r(x,z) = 1$, so the result follows from the dominated convergence theorem. ■

Let $\mathcal{F}'_n = \sigma(X_{n+1}, X_{n+2}, \ldots)$ and $\mathcal{T} = \cap_n \mathcal{F}'_n$ be the tail $\sigma$-field. The next result is due to Orey. The proof we give is from Blackwell and Freedman (1964).

**Theorem 6.7.3.** *Suppose $p$ is irreducible, recurrent, and all states have period $d$, $\mathcal{T} = \sigma(\{X_0 \in S_r\} : 0 \le r < d)$.*

**Remark.** To be precise, if $\mu$ is any initial distribution and $A \in \mathcal{T}$, then there is an $r$ so that $A = \{X_0 \in S_r\}$ $P_\mu$-a.s.

*Proof.* We build up to the general result in three steps.

CASE 1. Suppose $P(X_0 = x) = 1$. Let $T_0 = 0$, and for $n \ge 1$, let $T_n = \inf\{m > T_{n-1} : X_m = x\}$ be the time of the $n$th return to $x$. Let

$$V_n = (X(T_{n-1}), \ldots, X(T_n - 1))$$

The vectors $V_n$ are i.i.d. by Exercise 6.4.1, and the tail $\sigma$-field is contained in the exchangeable field of the $V_n$, so the Hewitt-Savage 0-1 law (Theorem 4.1.1, proved there for r.v's taking values in a general measurable space) implies that $\mathcal{T}$ is trivial in this case.

CASE 2. Suppose that the initial distribution is concentrated on one cyclic class, say $S_0$. If $A \in \mathcal{T}$, then $P_x(A) \in \{0,1\}$ for each $x$ by case 1. If $P_x(A) = 0$ for all $x \in S_0$, then $P_\mu(A) = 0$. Suppose $P_y(A) > 0$, and hence $= 1$, for some $y \in S_0$. Let $z \in S_0$. Since $p^d$ is irreducible and aperiodic on $S_0$, there is an $n$ so that $p^n(z,y) > 0$ and $p^n(y,y) > 0$. If we write $1_A = 1_B \circ \theta_n$, then the Markov property implies

$$1 = P_y(A) = E_y(E_y(1_B \circ \theta_n|\mathcal{F}_n)) = E_y(E_{X_n} 1_B)$$

so $P_y(B) = 1$. Another application of the Markov property gives

$$P_z(A) = E_z(E_{X_n} 1_B) \ge p^n(z,y) > 0$$

so $P_z(A) = 1$, and since $z \in S_0$ is arbitrary, $P_\mu(A) = 1$.

*General case.* From case 2, we see that $P(A|X_0 = y) \equiv 1$ or $\equiv 0$ on each cyclic class. This implies that either $\{X_0 \in S_r\} \subset A$ or $\{X_0 \in S_r\} \cap A = \emptyset$ $P_\mu$ a.s. Conversely, it is clear that $\{X_0 \in S_r\} = \{X_{nd} \in S_r \text{ i.o.}\} \in \mathcal{T}$, and the proof is complete. ∎

The next result will help us identify the tail $\sigma$-field in transient examples.

**Theorem 6.7.4.** *Suppose $X_0$ has initial distribution $\mu$. The equations*

$$h(X_n, n) = E_\mu(Z|\mathcal{F}_n) \quad \text{and} \quad Z = \lim_{n\to\infty} h(X_n, n)$$

*set up a 1-1 correspondence between bounded $Z \in \mathcal{T}$ and bounded* **space-time harmonic functions**, *that is, bounded $h : S \times \{0, 1, \ldots\} \to \mathbf{R}$, so that $h(X_n, n)$ is a martingale.*

*Proof.* Let $Z \in \mathcal{T}$, write $Z = Y_n \circ \theta_n$, and let $h(x, n) = E_x Y_n$.

$$E_\mu(Z|\mathcal{F}_n) = E_\mu(Y_n \circ \theta_n|\mathcal{F}_n) = h(X_n, n)$$

by the Markov property, so $h(X_n, n)$ is a martingale. Conversely, if $h(X_n, n)$ is a bounded martingale, using Theorems 5.2.8 and 5.5.6 shows $h(X_n, n) \to Z \in \mathcal{T}$ as $n \to \infty$, and $h(X_n, n) = E_\mu(Z|\mathcal{F}_n)$. ∎

**Exercise 6.7.2.** A random variable $Z$ with $Z = Z \circ \theta$, and hence $= Z \circ \theta_n$ for all $n$, is called **invariant**. Show that there is a 1-1 correspondence between bounded invariant random variables and bounded harmonic functions. We will have more to say about invariant r.v.'s in Section 7.1.

**Example 6.7.1. Simple random walk in d dimensions.** We begin by constructing a coupling for this process. Let $i_1, i_2, \ldots$ be i.i.d. uniform on $\{1, \ldots, d\}$. Let $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ be i.i.d. uniform on $\{-1, 1\}$. Let $e_j$ be the $j$th unit vector. Construct a coupled pair of $d$-dimensional simple random walks by

$$X_n = X_{n-1} + e(i_n)\xi_n$$

$$Y_n = \begin{cases} Y_{n-1} + e(i_n)\xi_n & \text{if } X^{i_n}_{n-1} = Y^{i_n}_{n-1} \\ Y_{n-1} + e(i_n)\eta_n & \text{if } X^{i_n}_{n-1} \neq Y^{i_n}_{n-1} \end{cases}$$

In words, the coordinate that changes is always the same in the two walks, and once they agree in one coordinate, future movements in that direction are the same. It is easy to see that if $X^i_0 - Y^i_0$ is even for $1 \le i \le d$, then the two random walks will hit with probability one.

Let $L_0 = \{z \in \mathbf{Z}^d : z^1 + \cdots + z^d \text{ is even }\}$ and $L_1 = \mathbf{Z}^d - L_0$. Although we have defined the notion only for the recurrent case, it should be clear that $L_0, L_1$ is the cyclic decomposition of the state space for simple random walk. If $S_n \in L_i$, then $S_{n+1} \in L_{1-i}$, and $p^2$ is irreducible on each $L_i$. To couple two random walks starting from $x, y \in L_i$, let them run independently until the first time all the coordinate differences are even, and then use the last coupling. In the remaining case, $x \in L_0$, $y \in L_1$ coupling is impossible.

The next result should explain our interest in coupling two $d$-dimensional simple random walks.

**Theorem 6.7.5.** *For $d$-dimensional simple random walk,*

$$\mathcal{T} = \sigma(\{X_0 \in L_i\}, i = 0, 1)$$

*Proof.* Let $x, y \in L_i$, and let $X_n, Y_n$ be a realization of the coupling defined above for $X_0 = x$ and $Y_0 = y$. Let $h(x, n)$ be a bounded space-time harmonic function. The martingale property implies $h(x, 0) = E_x h(X_n, n)$. If $|h| \le C$, it follows from the coupling that

$$|h(x, 0) - h(y, 0)| = |Eh(X_n, n) - Eh(Y_n, n)| \le 2CP(X_n \ne Y_n) \to 0$$

so $h(x, 0)$ is constant on $L_0$ and $L_1$. Applying the last result to $h'(x, m) = h(x, n + m)$, we see that $h(x, n) = a_n^i$ on $L_i$. The martingale property implies $a_n^i = a_{n+1}^{1-i}$, and the desired result follows from Theorem 6.7.4. ∎

**Example 6.7.2. Ornstein's coupling.** Let $p(x, y) = f(y - x)$ be the transition probability for an irreducible aperiodic random walk on $\mathbf{Z}$. To prove that the tail $\sigma$-field is trivial, pick $M$ large enough so that the random walk generated by the probability distribution $f_M(x)$ with $f_M(x) = c_M f(x)$ for $|x| \le M$ and $f_M(x) = 0$ for $|x| > M$ is irreducible and aperiodic. Let $Z_1, Z_2, \ldots$ be i.i.d. with distribution $f$, and let $W_1, W_2, \ldots$ be i.i.d. with distribution $f_M$. Let $X_n = X_{n-1} + Z_n$ for $n \ge 1$. If $X_{n-1} = Y_{n-1}$, we set $X_n = Y_n$. Otherwise, we let

$$Y_n = \begin{cases} Y_{n-1} + Z_n & \text{if } |Z_n| > m \\ Y_{n-1} + W_n & \text{if } |Z_n| \le m \end{cases}$$

In words, the big jumps are taken in parallel and the small jumps are independent. The recurrence of one-dimensional random walks with mean 0 implies $P(X_n \ne Y_n) \to 0$. Repeating the proof of Theorem 6.7.5, we see that $\mathcal{T}$ is trivial.

The tail $\sigma$-field in Theorem 6.7.5 is essentially the same as in Theorem 6.7.3. To get a more interesting $\mathcal{T}$, we look at:

**Example 6.7.3. Random walk on a tree.** To facilitate definitions, we will consider the system as a random walk on a group with three generators $a, b, c$ that have $a^2 = b^2 = c^2 = e$, the identity element. To form the random walk, let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_n = x) = 1/3$ for $x = a, b, c$, and let $X_n = X_{n-1}\xi_n$. (This is equivalent to a random walk on the tree in which each vertex has degree 3, but the algebraic formulation is convenient for computations.) Let $L_n$ be the length of the word $X_n$ when it has been reduced as much as possible, with $L_n = 0$ if $X_n = e$.

The reduction can be done as we go along. If the last letter of $X_{n-1}$ is the same as $\xi_n$, we erase it; otherwise we add the new letter. It is easy to see that $L_n$ is a Markov chain with a transition probability that has $p(0, 1) = 1$ and

$$p(j, j-1) = 1/3 \qquad p(j, j+1) = 2/3 \quad \text{for } j \geq 1$$

As $n \to \infty$, $L_n \to \infty$. From this, it follows easily that the word $X_n$ has a limit in the sense that the $i$th letter $X_n^i$ stays the same for large $n$. Let $X_\infty$ be the limiting word, that is, $X_\infty^i = \lim X_n^i$. $\mathcal{T} \supset \sigma(X_\infty^i, i \geq 1)$, but it is easy to see that this is not all. If $S_0 =$ the words of even length, and $S_1 = S_0^c$, then $X_n \in S_i$ implies $X_{n+1} \in S_{1-i}$, so $\{X_0 \in S_0\} \in \mathcal{T}$. Can the reader prove that we have now found all of $\mathcal{T}$? As Fermat once said, "I have a proof, but it won't fit in the margin."

**Remark.** This time the solution does not involve elliptic curves but uses "$h$-paths." See Furstenburg (1970) or decode the following: "Condition on the exit point (the infinite word). Then the resulting RW is an $h$-process, which moves closer to the boundary with probability 2/3 and farther with probability 1/3 (1/6 each to the two possibilities). Two such random walks couple, provided they have same parity." The quote is from Robin Pemantle, who says he consulted Itai Benajamini and Yuval Peres.

## 6.8 General State Space*

In this section, we will generalize the results from Sections 6.4–6.6 to a collection of Markov chains with uncountable state space called Harris chains. The developments here are motivated by three ideas. First, the proofs for countable state space if there is one point in the state space that the chain hits with probability 1. (Think, for example, about the construction of the stationary measure via the cycle trick.) Second, a recurrent Harris chain can be modified to contain such a point. Third, the collection of Harris chains is a comfortable level of generality; broad enough to contain a large number of interesting examples, yet restrictive enough to allow for a rich theory.

We say that a Markov chain $X_n$ is a **Harris chain** if we can find sets $A, B \in \mathcal{S}$, a function $q$ with $q(x, y) \geq \epsilon > 0$ for $x \in A$, $y \in B$, and a probability measure $\rho$ concentrated on $B$ so that:

(i) If $\tau_A = \inf\{n \geq 0 : X_n \in A\}$, then $P_z(\tau_A < \infty) > 0$ for all $z \in S$.
(ii) If $x \in A$ and $C \subset B$ then $p(x, C) \geq \int_C q(x, y)\, \rho(dy)$.

To explain the definition, we turn to some examples:

**Example 6.8.1. Countable state space.** If $S$ is countable and there is a point $a$ with $\rho_{xa} > 0$ for all $x$ (a condition slightly weaker than irreducibility), then we can take $A = \{a\}$, $B = \{b\}$, where $b$ is any state with $p(a, b) > 0$, $\mu = \delta_b$ the point mass at $b$, and $q(a, b) = p(a, b)$.

Conversely, if $S$ is countable and $(A', B')$ is a pair for which (i) and (ii) hold, then we can without loss of generality reduce $B'$ to a single point $b$. Having done this, if we set $A = \{b\}$, pick $c$ so that $p(b, c) > 0$, and set $B = \{c\}$, then (i) and (ii) hold with $A$ and $B$ both singletons.

**Example 6.8.2. Chains with continuous densities.** Suppose $X_n \in \mathbf{R}^d$ is a Markov chain with a transition probability that has $p(x, dy) = p(x, y)\,dy$ where $(x, y) \to p(x, y)$ is continuous. Pick $(x_0, y_0)$ so that $p(x_0, y_0) > 0$. Let $A$ and $B$ be open sets around $x_0$ and $y_0$ that are small enough so that $p(x, y) \geq \epsilon > 0$ on $A \times B$. If we let $\rho(C) = |B \cap C|/|B|$, where $|B|$ is the Lebesgue measure of $B$, then (ii) holds. If (i) holds, then $X_n$ is a Harris chain.

For concrete examples, consider:

(a) **Diffusion processes** are a large class of examples that lie outside the scope of this book, but are too important to ignore. When things are nice, specifically, if the generator of $X$ has Hölder continuous coefficients satisfying suitable growth conditions, see the Appendix of Dynkin (1965), then $P(X_1 \in dy) = p(x, y)\,dy$, and $p$ satisfies the conditions above.

(b) **Armaps.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. and $V_n = \theta V_{n-1} + \xi_n$. $V_n$ is called an **autoregressive moving average process** or **armap** for short. We call $V_n$ a **smooth armap** if the distribution of $\xi_n$ has a continuous density $g$. In this case $p(x, dy) = g(y - \theta x)\,dy$ with $(x, y) \to g(y - \theta x)$ continuous.

In analyzing the behavior of armaps there are a number of cases to consider depending on the nature of the support of $\xi_n$. We call $V_n$ a **simple armap** if the density function for $\xi_n$ is positive for at all points in $\mathbf{R}$. In this case we can take $A = B = [-1/2, 1/2]$ with $\rho =$ the restriction of Lebesgue measure.

(c) The **discrete Ornstein-Uhlenbeck process** is a special case of (a) and (b). Let $\xi_1, \xi_2, \ldots$ be i.i.d. standard normals and let $V_n = \theta V_{n-1} + \xi_n$. The Ornstein-Uhlenbeck (O.U.) process is a diffusion process $\{V_t, t \in [0, \infty)\}$ that models the velocity of a particle suspended in a liquid. See, for example, Breiman (1968), Section 16.1. Looking at $V_t$ at integer times (and dividing by a constant to make the variance 1) gives a Markov chain with the indicated distributions.

**Example 6.8.3. GI/G/1 queue, or storage model.** Let $\xi_1, \xi_2, \ldots$ be i.i.d. and define $W_n$ inductively by $W_n = (W_{n-1} + \xi_n)^+$. If $P(\xi_n < 0) > 0$, then we can take $A = B = \{0\}$, and (i) and (ii) hold. To explain the first name in the title, consider a queueing system in which customers arrive at times of a renewal process, that is, at times $0 = T_0 < T_1 < T_2 \ldots$ with $\zeta_n = T_n - T_{n-1}$, $n \geq 1$ i.i.d. Let $\eta_n$, $n \geq 0$, be the amount of service time the $n$th customer requires, and let $\xi_n = \eta_{n-1} - \zeta_n$. I claim that $W_n$ is the amount of time the $n$th customer has to wait to enter service. To see this, notice that the $(n-1)$th customer adds $\eta_{n-1}$ to the server's workload, and if the server is busy at all times in $[T_{n-1}, T_n)$, he reduces his workload by $\zeta_n$. If $W_{n-1} + \eta_{n-1} < \zeta_n$, then the server has enough time to finish his work and the next arriving customer will find an empty queue.

The second name in the title refers to the fact that $W_n$ can be used to model the contents of a storage facility. For an intuitive description, consider water reservoirs. We assume that rainstorms occur at times of a renewal process $\{T_n : n \geq 1\}$, that the $n$th rainstorm contributes an amount of water $\eta_n$, and that water is consumed at constant rate $c$. If we let $\zeta_n = T_n - T_{n-1}$ as before, and $\xi_n = \eta_{n-1} - c\zeta_n$, then $W_n$ gives the amount of water in the reservoir just before the $n$th rainstorm.

**History lesson.** Doeblin was the first to prove results for Markov chains on general state space. He supposed that there was an $n$ so that $p^n(x, C) \geq \epsilon\rho(C)$ for all $x \in S$ and $C \subset S$. See Doob (1953), Section V.5, for an account of his results. Harris (1956) generalized Doeblin's result by observing that it was enough to have a set $A$ so that (i) holds and the chain viewed on $A$ ($Y_k = X(T_A^k)$, where $T_A^k = \inf\{n > T_A^{k-1} : X_n \in A\}$ and $T_A^0 = 0$) satisfies Doeblin's condition. Our formulation, as well as most of the proofs in this section, follows Athreya and Ney (1978). For a nice description of the "traditional approach," see Revuz (1984).

Given a Harris chain on $(S, \mathcal{S})$, we will construct a Markov chain $\bar{X}_n$ with transition probability $\bar{p}$ on $(\bar{S}, \bar{\mathcal{S}})$, where $\bar{S} = S \cup \{\alpha\}$ and $\bar{\mathcal{S}} = \{B, B \cup \{\alpha\} : B \in \mathcal{S}\}$. The aim, as advertised earlier, is to manufacture a point $\alpha$ that the process hits with probability 1 in the recurrent case.

$$\begin{array}{ll} \text{If } x \in S - A & \bar{p}(x, C) = p(x, C) \text{ for } C \in \mathcal{S} \\ \text{If } x \in A & \bar{p}(x, \{\alpha\}) = \epsilon \\ & \bar{p}(x, C) = p(x, C) - \epsilon\rho(C) \text{ for } C \in \mathcal{S} \\ \text{If } x = \alpha & \bar{p}(\alpha, D) = \int \rho(dx)\bar{p}(x, D) \text{ for } D \in \bar{\mathcal{S}} \end{array}$$

Intuitively, $\bar{X}_n = \alpha$ corresponds to $X_n$ being distributed on $B$ according to $\rho$. Here, and in what follows, we will reserve $A$ and $B$ for the special sets that occur in the definition and use $C$ and $D$ for generic elements of $\mathcal{S}$. We will often simplify notation by writing $\bar{p}(x, \alpha)$ instead of $\bar{p}(x, \{\alpha\})$, $\mu(\alpha)$ instead of $\mu(\{\alpha\})$, and so forth.

Our next step is to prove three technical lemmas that will help us develop the theory below. Define a transition probability $v$ by

$$v(x, \{x\}) = 1 \quad \text{if} \quad x \in S \qquad v(\alpha, C) = \rho(C)$$

In words, $V$ leaves mass in $S$ alone but returns the mass at $\alpha$ to $S$ and distributes it according to $\rho$.

**Lemma 6.8.1.** *$v\bar{p} = \bar{p}$ and $\bar{p}v = p$.*

*Proof.* Before giving the proof, we would like to remind the reader that measures multiply the transition probability on the left, that is, in the first case we want to show $\mu v\bar{p} = \mu\bar{p}$. If we first make a transition according to $v$ and then one according to $\bar{p}$, this amounts to one transition according to $\bar{p}$, since only mass at $\alpha$ is affected by $v$ and

$$\bar{p}(\alpha, D) = \int \rho(dx)\bar{p}(x, D)$$

The second equality also follows easily from the definition. In words, if $\bar{p}$ acts first and then $v$, then $v$ returns the mass at $\alpha$ to where it came from. ■

From Lemma 6.8.1, it follows easily that we have:

**Lemma 6.8.2.** *Let $Y_n$ be an inhomogeneous Markov chain with $p_{2k} = v$ and $p_{2k+1} = \bar{p}$. Then $\bar{X}_n = Y_{2n}$ is a Markov chain with transition probability $\bar{p}$, and $X_n = Y_{2n+1}$ is a Markov chain with transition probability $p$.*

Lemma 6.8.2 shows that there is an intimate relationship between the asymptotic behavior of $X_n$ and that of $\bar{X}_n$. To quantify this, we need a definition. If $f$ is a bounded measurable function on $S$, let $\bar{f} = vf$, that is, $\bar{f}(x) = f(x)$ for $x \in S$ and $\bar{f}(\alpha) = \int f\, d\rho$.

**Lemma 6.8.3.** *If $\mu$ is a probability measure on $(S, \mathcal{S})$, then*

$$E_\mu f(X_n) = E_\mu \bar{f}(\bar{X}_n)$$

*Proof.* Observe that if $X_n$ and $\bar{X}_n$ are constructed as in Lemma 6.8.2, and $P(\bar{X}_0 \in S) = 1$, then $X_0 = \bar{X}_0$, and $X_n$ is obtained from $\bar{X}_n$ by making a transition according to $v$. ■

The last three lemmas will allow us to obtain results for $X_n$ from those for $\bar{X}_n$. We turn now to the task of generalizing the results of Sections 6.4–6.6 to $\bar{X}_n$. To facilitate comparison with the results for countable state space, we will break this section into four subsections, the first three of which correspond to Sections 6.4–6.6. In the fourth subsection, we take an in-depth look at the GI/G/1 queue. Before developing the theory, we will give one last example that explains why some of the statements are messy.

**Example 6.8.4. Perverted O.U. process.** Take the discrete O.U. process from part (c) of Example 6.8.2 and modify the transition probability at the integers $x \ge 2$ so that

$$p(x, \{x+1\}) = 1 - x^{-2}$$
$$p(x, A) = x^{-2}|A| \quad \text{for } A \subset (0, 1)$$

$p$ is the transition probability of a Harris chain, but

$$P_2(X_n = n + 2 \text{ for all } n) > 0$$

I can sympathize with the reader who thinks that such chains will not arise "in applications," but it seems easier (and better) to adapt the theory to include them than to modify the assumptions to exclude them.

### 6.8.1 *Recurrence and Transience*

We begin with the dichotomy between recurrence and transience. Let $R = \inf\{n \geq 1 : \bar{X}_n = \alpha\}$. If $P_\alpha(R < \infty) = 1$, then we call the chain **recurrent**; otherwise we call it **transient**. Let $R_1 = R$ and, for $k \geq 2$, let $R_k = \inf\{n > R_{k-1} : \bar{X}_n = \alpha\}$ be the time of the $k$th return to $\alpha$. The strong Markov property implies $P_\alpha(R_k < \infty) = P_\alpha(R < \infty)^k$, so $P_\alpha(\bar{X}_n = \alpha \text{ i.o.}) = 1$ in the recurrent case and $= 0$ in the transient case. It is easy to generalize Theorem 6.4.2 to the current setting.

**Exercise 6.8.1.** $\bar{X}_n$ is recurrent if and only if $\sum_{n=1}^\infty \bar{p}^n(\alpha, \alpha) = \infty$.

The next result generalizes Lemma 6.4.3.

**Theorem 6.8.4.** *Let $\lambda(C) = \sum_{n=1}^\infty 2^{-n} \bar{p}^n(\alpha, C)$. In the recurrent case, if $\lambda(C) > 0$ then $P_\alpha(\bar{X}_n \in C \text{ i.o.}) = 1$. For $\lambda$-a.e. $x$, $P_x(R < \infty) = 1$.*

*Proof.* The first conclusion follows from Lemma 6.3.3. For the second, let $D = \{x : P_x(R < \infty) < 1\}$ and observe that if $p^n(\alpha, D) > 0$ for some $n$, then

$$P_\alpha(\bar{X}_m = \alpha \text{ i.o.}) \leq \int \bar{p}^n(\alpha, dx) P_x(R < \infty) < 1 \qquad \blacksquare$$

**Remark.** Example 6.8.4 shows that we cannot expect to have $P_x(R < \infty) = 1$ for all $x$. To see that even when the state space is countable, we need not hit every point starting from $\alpha$, do

**Exercise 6.8.2.** If $X_n$ is a recurrent Harris chain on a countable state space, then $S$ can only have one irreducible set of recurrent states but may have a nonempty set of transient states. For a concrete example, consider a branching process in which the probability of no children $p_0 > 0$ and set $A = B = \{0\}$.

**Exercise 6.8.3.** Suppose $X_n$ is a recurrent Harris chain. Show that if $(A', B')$ is another pair satisfying the conditions of the definition, then Theorem 6.8.4 implies $P_\alpha(\bar{X}_n \in A' \text{ i.o.}) = 1$, so the recurrence or transience does not depend on the choice of $(A, B)$.

As in Section 6.4, we need special methods to determine whether an example is recurrent or transient.

**Exercise 6.8.4.** In the $GI/G/1$ queue, the waiting time $W_n$ and the random walk $S_n = X_0 + \xi_1 + \cdots + \xi_n$ agree until $N = \inf\{n : S_n < 0\}$, and at this time $W_N = 0$. Use this observation as we did in Example 6.4.7 to show that Example 6.8.3 is recurrent when $E\xi_n \le 0$ and transient when $E\xi_n > 0$.

**Exercise 6.8.5.** Let $V_n$ be a simple smooth armap with $E|\xi_i| < \infty$. Show that if $\theta < 1$, then $E_x|V_1| \le |x|$ for $|x| \ge M$. Use this and ideas from the proof of Theorem 6.4.8 to show that the chain is recurrent in this case.

**Exercise 6.8.6.** Let $V_n$ be an armap (not necessarily smooth or simple) and suppose $\theta > 1$. Let $\gamma \in (1, \theta)$ and observe that if $x > 0$, then $P_x(V_1 < \gamma x) \le C/((\theta - \gamma)x)$, so if $x$ is large, $P_x(V_n \ge \gamma^n x$ for all $n) > 0$.

**Remark.** In the case $\theta = 1$, the chain $V_n$ discussed in the last two exercises is a random walk with mean 0 and hence recurrent.

**Exercise 6.8.7.** In the discrete O.U. process, $X_{n+1}$ is normal with mean $\theta X_n$ and variance 1. What happens to the recurrence and transience if instead $Y_{n+1}$ is normal with mean 0 and variance $\beta^2|Y_n|$?

### 6.8.2 *Stationary Measures*

**Theorem 6.8.5.** *In the recurrent case, there is a stationary measure.*

*Proof.* Let $R = \inf\{n \ge 1 : \bar{X}_n = \alpha\}$, and let

$$\bar{\mu}(C) = E_\alpha\left(\sum_{n=0}^{R-1} 1_{\{\bar{X}_n \in C\}}\right) = \sum_{n=0}^{\infty} P_\alpha(\bar{X}_n \in C, R > n)$$

Repeating the proof of Theorem 6.5.2 shows that $\bar{\mu}\bar{p} = \bar{\mu}$. If we let $\mu = \bar{\mu}v$, then it follows from Lemma 6.8.1 that $\bar{\mu}v\, p = \bar{\mu}\bar{p}v = \bar{\mu}v$, so $\mu\, p = \mu$. ■

**Exercise 6.8.8.** Let $G_{k,\delta} = \{x : \bar{p}^k(x, \alpha) \ge \delta\}$. Show that $\bar{\mu}(G_{k,\delta}) \le 2k/\delta$ and use this to conclude that $\bar{\mu}$ and hence $\mu$ is $\sigma$-finite.

**Exercise 6.8.9.** Let $\lambda$ be the measure defined in Theorem 6.8.5. Show that $\bar{\mu} \ll \lambda$ and $\lambda \ll \bar{\mu}$.

**Exercise 6.8.10.** Let $V_n$ be an armap (not necessarily smooth or simple) with $\theta < 1$ and $E \log^+ |\xi_n| < \infty$. Show that $\sum_{m\ge 0} \theta^m \xi_m$ converges a.s. and defines a stationary distribution for $V_n$.

**Exercise 6.8.11.** In the $GI/G/1$ queue, the waiting time $W_n$ and the random walk $S_n = X_0 + \xi_1 + \cdots + \xi_n$ agree until $N = \inf\{n : S_n < 0\}$, and at this time

$W_N = 0$. Use this observation as we did in Example 6.5.6 to show that if $E\xi_n < 0$, $EN < \infty$ and hence there is a stationary distribution.

To investigate uniqueness of the stationary measure, we begin with:

**Lemma 6.8.6.** *If $\nu$ is a $\sigma$-finite stationary measure for $p$, then $\nu(A) < \infty$ and $\bar{\nu} = \nu\bar{p}$ is a stationary measure for $\bar{p}$ with $\bar{\nu}(\alpha) < \infty$.*

*Proof.* We will first show that $\nu(A) < \infty$. If $\nu(A) = \infty$, then part (ii) of the definition implies $\nu(C) = \infty$ for all sets $C$ with $\rho(C) > 0$. If $B = \cup_i B_i$ with $\nu(B_i) < \infty$, then $\rho(B_i) = 0$ by the last observation and $\rho(B) = 0$ by countable subadditivity, a contradiction. So $\nu(A) < \infty$ and $\bar{\nu}(\alpha) = \nu\bar{p}(\alpha) = \epsilon\nu(A) < \infty$. Using the fact that $\nu p = \nu$, we find

$$\nu\bar{p}(C) = \nu(C) - \epsilon\nu(A)\rho(B \cap C)$$

the last subtraction being well defined since $\nu(A) < \infty$, and it follows that $\bar{\nu}v = \nu$. To check $\bar{\nu}\bar{p} = \bar{\nu}$, we observe that Lemma 6.8.1 and the last result imply $\bar{\nu}\bar{p} = \bar{\nu}v\bar{p} = \nu\bar{p} = \bar{\nu}$. ∎

**Theorem 6.8.7.** *Suppose $p$ is recurrent. If $\nu$ is a $\sigma$-finite stationary measure then $\nu = \bar{\nu}(\alpha)\mu$, where $\mu$ is the measure constructed in the proof of Theorem 6.8.5.*

*Proof.* By Lemma 6.8.6, it suffices to prove that if $\bar{\nu}$ is a stationary measure for $\bar{p}$ with $\bar{\nu}(\alpha) < \infty$, then $\bar{\nu} = \bar{\nu}(\alpha)\bar{\mu}$. Repeating the proof of Theorem 6.5.3 with $a = \alpha$, it is easy to show that $\bar{\nu}(C) \geq \bar{\nu}(\alpha)\bar{\mu}(C)$. Continuing to compute as in that proof:

$$\bar{\nu}(\alpha) = \int \bar{\nu}(dx)\bar{p}^n(x, \alpha) \geq \bar{\nu}(\alpha) \int \bar{\mu}(dx)\bar{p}^n(x, \alpha) = \bar{\nu}(\alpha)\bar{\mu}(\alpha) = \bar{\nu}(\alpha)$$

Let $S_n = \{x : p^n(x, \alpha) > 0\}$. By assumption, $\cup_n S_n = S$. If $\bar{\nu}(D) > \bar{\nu}(\alpha)\bar{\mu}(D)$ for some $D$, then $\bar{\nu}(D \cap S_n) > \bar{\nu}(\alpha)\bar{\mu}(D \cap S_n)$, and it follows that $\bar{\nu}(\alpha) > \bar{\nu}(\alpha)$, a contradiction. ∎

### 6.8.3 *Convergence Theorem*

We say that a recurrent Harris chain $X_n$ is **aperiodic** if g.c.d. $\{n \geq 1 : p^n(\alpha, \alpha) > 0\} = 1$. This occurs, for example, if we can take $A = B$ in the definition, for then $p(\alpha, \alpha) > 0$.

**Theorem 6.8.8.** *Let $X_n$ be an aperiodic recurrent Harris chain with stationary distribution $\pi$. If $P_x(R < \infty) = 1$ then as $n \to \infty$,*

$$\|p^n(x, \cdot) - \pi(\cdot)\| \to 0$$

**Note.** Here $\| \, \|$ denotes the total variation distance between the measures. Lemma 6.8.4 guarantees that $\pi$ a.e. $x$ satisfies the hypothesis.

*Proof.* In view of Lemma 6.8.3, it suffices to prove the result for $\bar{p}$. We begin by observing that the existence of a stationary probability measure and the uniqueness result in Theorem 6.8.7 imply that the measure constructed in Theorem 6.8.5 has $E_\alpha R = \bar{\mu}(S) < \infty$. As in the proof of Theorem 6.6.4, we let $X_n$ and $Y_n$ be independent copies of the chain with initial distributions $\delta_x$ and $\pi$, respectively, and let $\tau = \inf\{n \geq 0 : X_n = Y_n = \alpha\}$. For $m \geq 0$, let $S_m$ (resp. $T_m$) be the times at which $X_n$ (resp. $Y_n$) visit $\alpha$ for the $(m+1)$th time. $S_m - T_m$ is a random walk with mean 0 steps, so $M = \inf\{m \geq 1 : S_m = T_m\} < \infty$ a.s., and it follows that this is true for $\tau$ as well. The computations in the proof of Theorem 6.6.4 show $|P(X_n \in C) - P(Y_n \in C)| \leq P(\tau > n)$. Since this is true for all $C$, $\|p^n(x, \cdot) - \pi(\cdot)\| \leq P(\tau > n)$, and the proof is complete. ∎

**Exercise 6.8.12.** Use Exercise 6.8.1 and imitate the proof of Theorem 6.5.4 to show that a Harris chain with a stationary distribution must be recurrent.

**Exercise 6.8.13.** Show that an armap with $\theta < 1$ and $E \log^+ |\xi_n| < \infty$ converges in distribution as $n \to \infty$. Hint: Recall the construction of $\pi$ in Exercise 6.8.10.

### 6.8.4 *GI/G/1 Queue*

For the rest of the section, we will concentrate on the $GI/G/1$ queue. Let $\xi_1, \xi_2, \ldots$ be i.i.d., let $W_n = (W_{n-1} + \xi_n)^+$, and let $S_n = \xi_1 + \cdots + \xi_n$. Recall $\xi_n = \eta_{n-1} - \zeta_n$, where the $\eta$'s are service times and $\zeta$'s are the interarrival times, and suppose $E\xi_n < 0$ so that Exercise 6.11 implies there is a stationary distribution.

**Exercise 6.8.14.** Let $m_n = \min(S_0, S_1, \ldots, S_n)$, where $S_n$ is the random walk defined above. (i) Show that $S_n - m_n =_d W_n$. (ii) Let $\xi'_m = \xi_{n+1-m}$ for $1 \leq m \leq n$. Show that $S_n - m_n = \max(S'_0, S'_1, \ldots, S'_n)$. (iii) Conclude that as $n \to \infty$ we have $W_n \Rightarrow M \equiv \max(S'_0, S'_1, S'_2, \ldots)$.

Explicit formulas for the distribution of $M$ are in general difficult to obtain. However, this can be done if either the arrival or service distribution is exponential. One reason for this is:

**Exercise 6.8.15.** Suppose $X, Y \geq 0$ are independent and $P(X > x) = e^{-\lambda x}$. Show that $P(X - Y > x) = ae^{-\lambda x}$, where $a = P(X - Y > 0)$.

**Example 6.8.5. Exponential service time.** Suppose $P(\eta_n > x) = e^{-\beta x}$ and $E\zeta_n > E\eta_n$. Let $T = \inf\{n : S_n > 0\}$ and $L = S_T$, setting $L = -\infty$ if $T = \infty$. The lack of memory property of the exponential distribution implies that $P(L > x) = re^{-\beta x}$, where $r = P(T < \infty)$. To compute the distribution of the maximum, $M$, let $T_1 = T$

and let $T_k = \inf\{n > T_{k-1} : S_n > S_{T_{k-1}}\}$ for $k \geq 2$. Theorem 4.1.3 implies that if $T_k < \infty$, then $S(T_{k+1}) - S(T_k) =_d L$ and is independent of $S(T_k)$. Using this and breaking things down according to the value of $K = \inf\{k : L_{k+1} = -\infty\}$, we see that for $x > 0$, the density function

$$P(M = x) = \sum_{k=1}^{\infty} r^k (1-r) e^{-\beta x} \beta^k x^{k-1}/(k-1)! = \beta r(1-r)e^{-\beta x(1-r)}$$

To complete the calculation, we need to calculate $r$. To do this, let

$$\phi(\theta) = E\exp(\theta \xi_n) = E\exp(\theta \eta_{n-1}) E\exp(-\theta \zeta_n)$$

which is finite for $0 < \theta < \beta$ since $\zeta_n \geq 0$ and $\eta_{n-1}$ has an exponential distribution. It is easy to see that

$$\phi'(0) = E\xi_n < 0 \qquad \lim_{\theta \uparrow \beta} \phi(\theta) = \infty$$

so there is a $\theta \in (0, \beta)$ with $\phi(\theta) = 1$. Exercise 5.7.4 implies that $\exp(\theta S_n)$ is a martingale. Theorem 5.4.1 implies $1 = E\exp(\theta S_{T \wedge n})$. Letting $n \to \infty$ and noting that $(S_n | T = n)$ has an exponential distribution and $S_n \to -\infty$ on $\{T = \infty\}$, we have

$$1 = r \int_0^\infty e^{\theta x} \beta e^{-\beta x}\, dx = \frac{r\beta}{\beta - \theta}$$

**Example 6.8.6. Poisson arrivals.** Suppose $P(\zeta_n > x) = e^{-\alpha x}$ and $E\zeta_n > E\eta_n$. Let $\bar{S}_n = -S_n$. Reversing time as in (ii) of Exercise 6.8.14, we see (for $n \geq 1$)

$$P\left(\max_{0 \leq k < n} \bar{S}_k < \bar{S}_n \in A\right) = P\left(\min_{1 \leq k \leq n} \bar{S}_k > 0, \bar{S}_n \in A\right)$$

Let $\psi_n(A)$ be the common value of the last two expressions, and let $\psi(A) = \sum_{n \geq 0} \psi_n(A)$. $\psi_n(A)$ is the probability the random walk reaches a new maximum (or ladder height; see Example 4.1.4) in $A$ at time $n$, so $\psi(A)$ is the number of ladder points in $A$ with $\psi(\{0\}) = 1$. Letting the random walk take one more step

$$P\left(\min_{1 \leq k \leq n} \bar{S}_k > 0, \bar{S}_{n+1} \leq x\right) = \int F(x - z)\, d\psi_n(z)$$

The last identity is valid for $n = 0$ if we interpret the left-hand side as $F(x)$. Let $\tau = \inf\{n \geq 1 : \bar{S}_n \leq 0\}$ and $x \leq 0$. Integrating by parts on the right-hand side and then summing over $n \geq 0$ gives

$$\begin{aligned} P(\bar{S}_\tau \leq x) &= \sum_{n=0}^{\infty} P\left(\min_{1 \leq k \leq n} \bar{S}_k > 0, \bar{S}_{n+1} \leq x\right) \\ &= \int_{y \leq x} \psi[0, x - y]\, dF(y) \end{aligned} \tag{6.8.1}$$

The limit $y \leq x$ comes from the fact that $\psi((-\infty, 0)) = 0$.

Let $\bar{\xi}_n = \bar{S}_n - \bar{S}_{n-1} = -\xi_n$. Exercise 6.8.15 implies $P(\bar{\xi}_n > x) = ae^{-\alpha x}$. Let $\bar{T} = \inf\{n : \bar{S}_n > 0\}$. $E\bar{\xi}_n > 0$, so $P(\bar{T} < \infty) = 1$. Let $J = \bar{S}_T$. As in the previous example, $P(J > x) = e^{-\alpha x}$. Let $V_n = J_1 + \cdots + J_n$. $V_n$ is a rate $\alpha$ Poisson process, so $\psi[0, x - y] = 1 + \alpha(x - y)$ for $x - y \geq 0$. Using (6.8.1) now and integrating by parts gives

$$
\begin{aligned}
P(\bar{S}_\tau \leq x) &= \int_{y \leq x} (1 + \alpha(x - y))\, dF(y) \\
&= F(x) + \alpha \int_{-\infty}^{x} F(y)\, dy \qquad \text{for } x \leq 0 \qquad (6.8.2)
\end{aligned}
$$

Since $P(\bar{S}_n = 0) = 0$ for $n \geq 1$, $-\bar{S}_\tau$ has the same distribution as $S_T$, where $T = \inf\{n : S_n > 0\}$. Combining this with part (ii) of Exercise 6.8.14 gives a "formula" for $P(M > x)$. Straightforward but somewhat tedious calculations show that if $B(s) = E\exp(-s\eta_n)$, then

$$
E\exp(-sM) = \frac{(1 - \alpha \cdot E\eta)s}{s - \alpha + \alpha B(s)}
$$

a result known as the **Pollaczek-Khintchine formula**. The computations we omitted can be found in Billingsley (1979) on p. 277 or several times in Feller, Vol. II (1971).

# 7
# Ergodic Theorems

$X_n, n \geq 0$, is said to be a stationary sequence if for each $k \geq 1$ it has the same distribution as the shifted sequence $X_{n+k}, n \geq 0$. The basic fact about these sequences, called the ergodic theorem, is that if $E|f(X_0)| < \infty$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) \quad \text{exists a.s.}$$

If $X_n$ is ergodic (a generalization of the notion of irreducibility for Markov chains) then the limit is $Ef(X_0)$. Sections 7.1 and 7.2 develop the theory needed to prove the ergodic theorem. In Section 7.3, we apply the ergodic theorem to study the recurrence of random walks with increments that are stationary sequences finding remarkable generalizations of the i.i.d. case. In Section 7.4, we prove a subadditive ergodic theorem. As the examples in Sections 7.4 and 7.5 should indicate, this is a useful generalization of the ergodic theorem.

## 7.1 Definitions and Examples

$X_0, X_1, \ldots$ is said to be a **stationary sequence** if for every $k$, the shifted sequence $\{X_{k+n}, n \geq 0\}$ has the same distribution, that is, for each $m$, $(X_0, \ldots, X_m)$ and $(X_k, \ldots, X_{k+m})$ have the same distribution. We begin by giving four examples that will be our constant companions.

**Example 7.1.1.** $X_0, X_1, \ldots$ are i.i.d.

**Example 7.1.2.** Let $X_n$ be a Markov chain with transition probability $p(x, A)$ and stationary distribution $\pi$, that is, $\pi(A) = \int \pi(dx)\, p(x, A)$. If $X_0$ has distribution $\pi$ then $X_0, X_1, \ldots$ is a stationary sequence. A special case to keep in mind for counterexamples is the chain with state space $S = \{0, 1\}$ and transition probability $p(x, \{1 - x\}) = 1$. In this case, the stationary distribution has $\pi(0) = \pi(1) = 1/2$ and $(X_0, X_1, \ldots) = (0, 1, 0, 1, \ldots)$ or $(1, 0, 1, 0, \ldots)$ with probability $1/2$ each.

**Example 7.1.3. Rotation of the circle.** Let $\Omega = [0, 1)$, $\mathcal{F} =$ Borel subsets, $P =$ Lebesgue measure. Let $\theta \in (0, 1)$, and for $n \geq 0$, let $X_n(\omega) = (\omega + n\theta) \bmod 1$, where $x \bmod 1 = x - [x]$, $[x]$ being the greatest integer $\leq x$. To see the reason for the name, map $[0, 1)$ into $\mathbf{C}$ by $x \to \exp(2\pi i x)$. This example is a special case of the last one. Let $p(x, \{y\}) = 1$ if $y = (x + \theta) \bmod 1$.

To make new examples from old, we can use:

**Theorem 7.1.1.** *If $X_0, X_1, \ldots$ is a stationary sequence and $g : \mathbf{R}^{\{0,1,\ldots\}} \to \mathbf{R}$ is measurable then $Y_k = g(X_k, X_{k+1}, \ldots)$ is a stationary sequence.*

*Proof.* If $x \in \mathbf{R}^{\{0,1,\ldots\}}$, let $g_k(x) = g(x_k, x_{k+1}, \ldots)$, and if $B \in \mathcal{R}^{\{0,1,\ldots\}}$, let

$$A = \{x : (g_0(x), g_1(x), \ldots) \in B\}$$

To check stationarity now, we observe

$$\begin{aligned} P(\omega : (Y_0, Y_1, \ldots) \in B) &= P(\omega : (X_0, X_1, \ldots) \in A) \\ &= P(\omega : (X_k, X_{k+1}, \ldots) \in A) \\ &= P(\omega : (Y_k, Y_{k+1}, \ldots) \in B) \end{aligned}$$

which proves the desired result. ∎

**Example 7.1.4. Bernoulli shift.** $\Omega = [0, 1)$, $\mathcal{F} =$ Borel subsets, $P =$ Lebesgue measure. $Y_0(\omega) = \omega$ and for $n \geq 1$, let $Y_n(\omega) = (2\,Y_{n-1}(\omega)) \bmod 1$. This example is a special case of (1.1). Let $X_0, X_1, \ldots$ be i.i.d. with $P(X_i = 0) = P(X_i = 1) = 1/2$, and let $g(x) = \sum_{i=0}^{\infty} x_i 2^{-(i+1)}$. The name comes from the fact that multiplying by 2 shifts the $X$'s to the left. This example is also a special case of Example 7.1.2. Let $p(x, \{y\}) = 1$ if $y = (2x) \bmod 1$.

Examples 7.1.3 and 7.1.4 are special cases of the following situation.

**Example 7.1.5.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable map $\varphi : \Omega \to \Omega$ is said to be **measure preserving** if $P(\varphi^{-1}A) = P(A)$ for all $A \in \mathcal{F}$. Let $\varphi^n$ be the $n$th iterate of $\varphi$ defined inductively by $\varphi^n = \varphi(\varphi^{n-1})$ for $n \geq 1$, where $\varphi^0(\omega) = \omega$. We claim that if $X \in \mathcal{F}$, then $X_n(\omega) = X(\varphi^n \omega)$ defines a stationary sequence. To check this, let $B \in \mathcal{R}^{n+1}$ and $A = \{\omega : (X_0(\omega), \ldots, X_n(\omega)) \in B\}$. Then

$$P((X_k, \ldots, X_{k+n}) \in B) = P(\varphi^k \omega \in A) = P(\omega \in A) = P((X_0, \ldots, X_n) \in B)$$

The last example is more than an important example. In fact, it is the only example! If $Y_0, Y_1, \ldots$ is a stationary sequence taking values in a nice space, Kolmogorov's extension theorem, Theorem A.3.1, allows us to construct a measure $P$ on sequence space $(S^{\{0,1,\ldots\}}, \mathcal{S}^{\{0,1,\ldots\}})$, so that the sequence $X_n(\omega) = \omega_n$ has the same distribution as that of $\{Y_n, n \geq 0\}$. If we let $\varphi$ be the shift operator, that is, $\varphi(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$, and let $X(\omega) = \omega_0$, then $\varphi$ is measure preserving and $X_n(\omega) = X(\varphi^n \omega)$.

In some situations, such as in the proof of Theorem 7.3.3 below, it is useful to observe:

**Theorem 7.1.2.** *Any stationary sequence* $\{X_n \ , \ n \geq 0\}$ *can be embedded in a two-sided stationary sequence* $\{Y_n : n \in \mathbf{Z}\}$.

*Proof.* We observe that

$$P(Y_{-m} \in A_0, \ldots, Y_n \in A_{m+n}) = P(X_0 \in A_0, \ldots, X_{m+n} \in A_{m+n})$$

is a consistent set of finite dimensional distributions, so a trivial generalization of the Kolmogorov extension theorem implies there is a measure $P$ on $(S^{\mathbf{Z}}, \mathcal{S}^{\mathbf{Z}})$ so that the variables $Y_n(\omega) = \omega_n$ have the desired distributions. ■

In view of the observations above, it suffices to give our definitions and prove our results in the setting of Example 7.1.5. Thus, our basic setup consists of

| | |
|---|---|
| $(\Omega, \mathcal{F}, P)$ | a probability space |
| $\varphi$ | a map that preserves $P$ |
| $X_n(\omega) = X(\varphi^n \omega)$ | where $X$ is a random variable |

We will now give some important definitions. Here and in what follows we assume $\varphi$ is measure-preserving. A set $A \in \mathcal{F}$ is said to be **invariant** if $\varphi^{-1} A = A$. (Here, as usual, two sets are considered to be equal if their symmetric difference has probability 0.) Some authors call $A$ **almost invariant** if $P(A \Delta \varphi^{-1}(A)) = 0$. We call such sets invariant and call $B$ **invariant in the strict sense** if $B = \varphi^{-1}(B)$.

**Exercise 7.1.1.** Show that the class of invariant events $\mathcal{I}$ is a $\sigma$-field, and $X \in \mathcal{I}$ if and only if $X$ is **invariant**, that is, $X \circ \varphi = X$ a.s.

**Exercise 7.1.2.** (i) Let $A$ be any set, let $B = \cup_{n=0}^{\infty} \varphi^{-n}(A)$. Show $\varphi^{-1}(B) \subset B$. (ii) Let $B$ be any set with $\varphi^{-1}(B) \subset B$ and let $C = \cap_{n=0}^{\infty} \varphi^{-n}(B)$. Show that $\varphi^{-1}(C) = C$. (iii) Show that $A$ is almost invariant if and only if there is a $C$ invariant in the strict sense with $P(A \Delta C) = 0$.

A measure-preserving transformation on $(\Omega, \mathcal{F}, P)$ is said to be **ergodic** if $\mathcal{I}$ is trivial, that is, for every $A \in \mathcal{I}$, $P(A) \in \{0, 1\}$. If $\varphi$ is not ergodic, then the space can be split into two sets $A$ and $A^c$, each having positive measure so that $\varphi(A) = A$ and $\varphi(A^c) = A^c$. In words, $\varphi$ is not "irreducible."

To investigate further the meaning of ergodicity, we return to our examples, renumbering them because the new focus is on checking ergodicity.

**Example 7.1.6. i.i.d. sequence.** We begin by observing that if $\Omega = \mathbf{R}^{\{0,1,\ldots\}}$ and $\varphi$ is the shift operator, then an invariant set $A$ has $\{\omega : \omega \in A\} = \{\omega : \varphi\omega \in A\} \in \sigma(X_1, X_2, \ldots)$. Iterating gives

$$A \in \cap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots) = \mathcal{T}, \quad \text{the tail } \sigma\text{-field}$$

so $\mathcal{I} \subset \mathcal{T}$. For an i.i.d. sequence, Kolmogorov's 0-1 law implies $\mathcal{T}$ is trivial, so $\mathcal{I}$ is trivial, and the sequence is ergodic (i.e., when the corresponding measure is put on sequence space $\Omega = \mathbf{R}^{\{0,1,2,,\ldots\}}$ the shift is).

**Example 7.1.7. Markov chains.** Suppose the state space $S$ is countable and the stationary distribution has $\pi(x) > 0$ for all $x \in S$. By Theorems 6.5.4 and 6.4.5, all states are recurrent, and we can write $S = \cup R_i$, where the $R_i$ are disjoint irreducible closed sets. If $X_0 \in R_i$, then with probability 1, $X_n \in R_i$ for all $n \geq 1$ so $\{\omega : X_0(\omega) \in R_i\} \in \mathcal{I}$ . The last observation shows that if the Markov chain is not irreducible, then the sequence is not ergodic. To prove the converse, observe that if $A \in \mathcal{I}$ , $1_A \circ \theta_n = 1_A$ where $\theta_n(\omega_0, \omega_1, \ldots) = (\omega_n, \omega_{n+1}, \ldots)$. So if we let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, the shift invariance of $1_A$ and the Markov property imply

$$E_\pi(1_A|\mathcal{F}_n) = E_\pi(1_A \circ \theta_n|\mathcal{F}_n) = h(X_n)$$

where $h(x) = E_x 1_A$. Lévy's 0-1 law implies that the left-hand side converges to $1_A$ as $n \to \infty$. If $X_n$ is irreducible and recurrent, then for any $y \in S$, the right-hand side $= h(y)$ i.o., so either $h(x) \equiv 0$ or $h(x) \equiv 1$, and $P_\pi(A) \in \{0, 1\}$. This example also shows that $\mathcal{I}$ and $\mathcal{T}$ may be different. When the transition probability $p$ is irreducible $\mathcal{I}$ is trivial, but if all the states have period $d > 1$, $\mathcal{T}$ is not. In Theorem 6.7.3, we showed that if $S_0, \ldots, S_{d-1}$ is the cyclic decomposition of $S$, then $\mathcal{T} = \sigma(\{X_0 \in S_r\} : 0 \leq r < d)$.

**Example 7.1.8. Rotation of the circle** is not ergodic if $\theta = m/n$ where $m < n$ are positive integers. If $B$ is a Borel subset of $[0, 1/n)$ and

$$A = \cup_{k=0}^{n-1}(B + k/n)$$

then $A$ is invariant. Conversely, if $\theta$ is irrational, then $\varphi$ is ergodic. To prove this, we need a fact from Fourier analysis. If $f$ is a measurable function on $[0, 1)$ with $\int f^2(x)\,dx < \infty$, then $f$ can be written as $f(x) = \sum_k c_k e^{2\pi ikx}$ where the equality is in the sense that as $K \to \infty$

$$\sum_{k=-K}^{K} c_k e^{2\pi ikx} \to f(x) \text{ in } L^2[0, 1)$$

and this is possible for only one choice of the coefficients $c_k = \int f(x) e^{-2\pi ikx}\,dx$. Now

$$f(\varphi(x)) = \sum_k c_k e^{2\pi ik(x+\theta)} = \sum_k (c_k e^{2\pi ik\theta}) e^{2\pi ikx}$$

The uniqueness of the coefficients $c_k$ implies that $f(\varphi(x)) = f(x)$ if and only if $c_k(e^{2\pi ik\theta} - 1) = 0$. If $\theta$ is irrational, this implies $c_k = 0$ for $k \neq 0$, so $f$ is constant. Applying the last result to $f = 1_A$ with $A \in \mathcal{I}$ shows that $A = \emptyset$ or $[0, 1)$ a.s.

**Exercise 7.1.3.** *A direct proof of ergodicity.* (i) Show that if $\theta$ is irrational, $x_n = n\theta \bmod 1$ is dense in [0,1). Hint: All the $x_n$ are distinct, so for any $N < \infty$, $|x_n - x_m| \leq 1/N$ for some $m < n \leq N$. (ii) Use Exercise A.2.1 to show that if $A$ is a Borel set with $|A| > 0$, then for any $\delta > 0$ there is an interval $J = [a, b)$ so that $|A \cap J| > (1 - \delta)|J|$. (iii) Combine this with (i) to conclude $P(A) = 1$.

**Example 7.1.9. Bernoulli shift** is ergodic. To prove this, we recall that the stationary sequence $Y_n(\omega) = \varphi^n(\omega)$ can be represented as

$$Y_n = \sum_{m=0}^{\infty} 2^{-(m+1)} X_{n+m}$$

where $X_0, X_1, \ldots$ are i.i.d. with $P(X_k = 1) = P(X_k = 0) = 1/2$, and use the following fact:

**Theorem 7.1.3.** *Let $g : \mathbf{R}^{\{0,1,\ldots\}} \to \mathbf{R}$ be measurable. If $X_0, X_1, \ldots$ is an ergodic stationary sequence, then $Y_k = g(X_k, X_{k+1}, \ldots)$ is ergodic.*

*Proof.* Suppose $X_0, X_1, \ldots$ is defined on sequence space with $X_n(\omega) = \omega_n$. If $B$ has $\{\omega : (Y_0, Y_1, \ldots) \in B\} = \{\omega : (Y_1, Y_2, \ldots) \in B\}$ then $A = \{\omega : (Y_0, Y_1, \ldots) \in B\}$ is shift invariant. ∎

**Exercise 7.1.4.** Use Fourier analysis as in Example 7.1.3 to prove that Example 7.1.4 is ergodic.

## *Exercises*

**7.1.5. Continued fractions.** Let $\varphi(x) = 1/x - [1/x]$ for $x \in (0, 1)$ and $A(x) = [1/x]$, where $[1/x]$ = the largest integer $\leq 1/x$. $a_n = A(\varphi^n x), n = 0, 1, 2, \ldots$ gives the continued fraction representation of $x$, that is,

$$x = 1/(a_0 + 1/(a_1 + 1/(a_2 + 1/\ldots)))$$

Show that $\varphi$ preserves $\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}$ for $A \subset (0, 1)$.

**Remark.** In his 1959 monograph, Kac claimed that it was "entirely trivial" to check that $\varphi$ is ergodic, but retracted his claim in a later footnote. We leave it to the reader to construct a proof or look up the answer in Ryll-Nardzewski (1951). Chapter 9 of Lévy (1937) is devoted to this topic and is still interesting reading today.

**7.1.6. Independent blocks.** Let $X_1, X_2, \ldots$ be a stationary sequence. Let $n < \infty$ and let $Y_1, Y_2, \ldots$ be a sequence so that $(Y_{nk+1}, \ldots, Y_{n(k+1)})$, $k \geq 0$ are i.i.d. and $(Y_1, \ldots, Y_n) = (X_1, \ldots, X_n)$. Finally, let $\nu$ be uniformly distributed on $\{1, 2, \ldots, n\}$, independent of $Y$, and let $Z_m = Y_{\nu+m}$ for $m \geq 1$. Show that $Z$ is stationary and ergodic.

## 7.2 Birkhoff's Ergodic Theorem

Throughout this section, $\varphi$ is a measure-preserving transformation on $(\Omega, \mathcal{F}, P)$. See Example 7.1.5 for details. We begin by proving a result that is usually referred to as:

**Theorem 7.2.1. The ergodic theorem.** *For any $X \in L^1$,*

$$\frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m \omega) \to E(X|\mathcal{I}) \quad \textit{a.s. and in } L^1$$

This result, due to Birkhoff (1931), is sometimes called the pointwise or individual ergodic theorem because of the a.s. convergence in the conclusion. When the sequence is ergodic, the limit is the mean $EX$. In this case, if we take $X = 1_A$, it follows that the asymptotic fraction of time $\varphi^m \in A$ is $P(A)$.

The proof we give is based on an odd integration inequality due to Yosida and Kakutani (1939). We follow Garsia (1965). The proof is not intuitive, but none of the steps are difficult.

**Lemma 7.2.2. Maximal ergodic lemma.** *Let $X_j(\omega) = X(\varphi^j\omega)$, $S_k(\omega) = X_0(\omega) + \cdots + X_{k-1}(\omega)$, and $M_k(\omega) = \max(0, S_1(\omega), \ldots, S_k(\omega))$. Then $E(X; M_k > 0) \geq 0$.*

*Proof.* If $j \leq k$, then $M_k(\varphi\omega) \geq S_j(\varphi\omega)$, so adding $X(\omega)$ gives

$$X(\omega) + M_k(\varphi\omega) \geq X(\omega) + S_j(\varphi\omega) = S_{j+1}(\omega)$$

and rearranging we have

$$X(\omega) \geq S_{j+1}(\omega) - M_k(\varphi\omega) \text{ for } j = 1, \ldots, k$$

Trivially, $X(\omega) \geq S_1(\omega) - M_k(\varphi\omega)$, since $S_1(\omega) = X(\omega)$ and $M_k(\varphi\omega) \geq 0$. Therefore

$$\begin{aligned} E(X(\omega); M_k > 0) &\geq \int_{\{M_k>0\}} \max(S_1(\omega), \ldots, S_k(\omega)) - M_k(\varphi\omega)\, dP \\ &= \int_{\{M_k>0\}} M_k(\omega) - M_k(\varphi\omega)\, dP \end{aligned}$$

Now $M_k(\omega) = 0$ and $M_k(\varphi\omega) \geq 0$ on $\{M_k > 0\}^c$, so the last expression is

$$\geq \int M_k(\omega) - M_k(\varphi\omega)\, dP = 0$$

since $\varphi$ is measure preserving. ∎

*Proof of Theorem 7.2.1.* $E(X|\mathcal{I})$ is invariant under $\varphi$ (see Exercise 7.1.1), so letting $X' = X - E(X|\mathcal{I})$ we can assume without loss of generality that $E(X|\mathcal{I}) = 0$. Let $\bar{X} = \limsup S_n/n$, let $\epsilon > 0$, and let $D = \{\omega : \bar{X}(\omega) > \epsilon\}$. Our goal is to prove that $P(D) = 0$. $\bar{X}(\varphi\omega) = \bar{X}(\omega)$, so $D \in \mathcal{I}$ . Let

$$X^*(\omega) = (X(\omega) - \epsilon)1_D(\omega) \qquad S_n^*(\omega) = X^*(\omega) + \ldots + X^*(\varphi^{n-1}\omega)$$

$$M_n^*(\omega) = \max(0, S_1^*(\omega), \ldots, S_n^*(\omega)) \qquad F_n = \{M_n^* > 0\}$$

$$F = \cup_n F_n = \left\{\sup_{k\geq 1} S_k^*/k > 0\right\}$$

Since $X^*(\omega) = (X(\omega) - \epsilon)1_D(\omega)$ and $D = \{\limsup S_k/k > \epsilon\}$, it follows that

$$F = \left\{\sup_{k\geq 1} S_k/k > \epsilon\right\} \cap D = D$$

Lemma 7.2.2 implies that $E(X^*; F_n) \geq 0$. Since $E|X^*| \leq E|X| + \epsilon < \infty$, the dominated convergence theorem implies $E(X^*; F_n) \to E(X^*; F)$, and it follows that $E(X^*; F) \geq 0$. The last conclusion looks innocent, but $F = D \in \mathcal{I}$, so it implies

$$0 \leq E(X^*; D) = E(X - \epsilon; D) = E(E(X|\mathcal{I}); D) - \epsilon P(D) = -\epsilon P(D)$$

since $E(X|\mathcal{I}) = 0$. The last inequality implies that

$$0 = P(D) = P(\limsup S_n/n > \epsilon)$$

and since $\epsilon > 0$ is arbitrary, it follows that $\limsup S_n/n \leq 0$. Applying the last result to $-X$ shows that $S_n/n \to 0$ a.s.

The clever part of the proof is over, and the rest is routine. To prove that convergence occurs in $L^1$, let

$$X'_M(\omega) = X(\omega)1_{(|X(\omega)|\leq M)} \quad \text{and} \quad X''_M(\omega) = X(\omega) - X'_M(\omega)$$

The part of the ergodic theorem we have proved implies

$$\frac{1}{n}\sum_{m=0}^{n-1} X'_M(\varphi^m\omega) \to E(X'_M|\mathcal{I}) \quad \text{a.s.}$$

Since $X'_M$ is bounded, the bounded convergence theorem implies

$$E\left|\frac{1}{n}\sum_{m=0}^{n-1} X'_M(\varphi^m\omega) - E(X'_M|\mathcal{I})\right| \to 0$$

To handle $X''_M$, we observe

$$E\left|\frac{1}{n}\sum_{m=0}^{n-1} X''_M(\varphi^m\omega)\right| \leq \frac{1}{n}\sum_{m=0}^{n-1} E|X''_M(\varphi^m\omega)| = E|X''_M|$$

and $E|E(X''_M|\mathcal{I})| \le EE(|X''_M||\mathcal{I}) = E|X''_M|$. So

$$E\left|\frac{1}{n}\sum_{m=0}^{n-1} X''_M(\varphi^m\omega) - E(X''_M|\mathcal{I})\right| \le 2E|X''_M|$$

and it follows that

$$\limsup_{n\to\infty} E\left|\frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m\omega) - E(X|\mathcal{I})\right| \le 2E|X''_M|$$

As $M \to \infty$, $E|X''_M| \to 0$ by the dominated convergence theorem, which completes the proof. ■

**Exercise 7.2.1.** Show that if $X \in L^p$ with $p > 1$, then the convergence in Theorem 7.2.1 occurs in $L^p$.

**Exercise 7.2.2.** (i) Show that if $g_n(\omega) \to g(\omega)$ a.s. and $E(\sup_k |g_k(\omega)|) < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} g_m(\varphi^m\omega) = E(g|\mathcal{I}) \quad \text{a.s.}$$

(ii) Show that if we suppose only that $g_n \to g$ in $L^1$, we get $L^1$ convergence.

Before turning to examples, we would like to prove a useful result that is a simple consequence of Lemma 7.2.2:

**Theorem 7.2.3. Wiener's maximal inequality.** *Let $X_j(\omega) = X(\varphi^j\omega)$, $S_k(\omega) = X_0(\omega) + \cdots + X_{k-1}(\omega)$, $A_k(\omega) = S_k(\omega)/k$, and $D_k = \max(A_1, \ldots, A_k)$. If $\alpha > 0$, then*

$$P(D_k > \alpha) \le \alpha^{-1}E|X|$$

*Proof.* Let $B = \{D_k > \alpha\}$. Applying Lemma 7.2.2 to $X' = X - \alpha$, with $X'_j(\omega) = X'(\varphi^j\omega)$, $S'_k = X'_0(\omega) + \cdots + X'_{k-1}$, and $M'_k = \max(0, S'_1, \ldots, S'_k)$, we conclude that $E(X'; M'_k > 0) \ge 0$. Since $\{M'_k > 0\} = \{D_k > \alpha\} \equiv B$, it follows that

$$E|X| \ge \int_B X\,dP \ge \int_B \alpha\,dP = \alpha P(B)$$ ■

**Exercise 7.2.3.** Use Lemma 7.2.3 and the truncation argument at the end of the proof of Theorem 7.2.2 to conclude that if Theorem 7.2.2 holds for bounded r.v.'s, then it holds whenever $E|X| < \infty$.

Our next step is to see what Theorem 7.2.2 says about our examples.

**Example 7.2.1. i.i.d. sequences.** Since $\mathcal{I}$ is trivial, the ergodic theorem implies that

$$\frac{1}{n}\sum_{m=0}^{n-1} X_m \to EX_0 \quad \text{a.s. and in } L^1$$

The a.s. convergence is the strong law of large numbers.

**Remark.** We can prove the $L^1$ convergence in the law of large numbers without invoking the ergodic theorem. To do this, note that

$$\frac{1}{n}\sum_{m=1}^{n} X_m^+ \to EX^+ \quad \text{a.s.} \qquad E\left(\frac{1}{n}\sum_{m=1}^{n} X_m^+\right) = EX^+$$

and use Theorem 5.5.2 to conclude that $\frac{1}{n}\sum_{m=1}^{n} X_m^+ \to EX^+$ in $L^1$. A similar result for the negative part and the triangle inequality now give the desired result.

**Example 7.2.2. Markov chains.** Let $X_n$ be an irreducible Markov chain on a countable state space that has a stationary distribution $\pi$. Let $f$ be a function with

$$\sum_x |f(x)|\pi(x) < \infty$$

In Example 7.1.7, we showed that $\mathcal{I}$ is trivial, so applying the ergodic theorem to $f(X_0(\omega))$ gives

$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \to \sum_x f(x)\pi(x) \quad \text{a.s. and in } L^1$$

For another proof of the almost sure convergence, see Exercise 6.6.4.

**Example 7.2.3. Rotation of the circle.** $\Omega = [0, 1)$ $\varphi(\omega) = (\omega + \theta) \bmod 1$. Suppose that $\theta \in (0, 1)$ is irrational, so that by a result in Section 7.1 $\mathcal{I}$ is trivial. If we set $X(\omega) = 1_A(\omega)$, with $A$ a Borel subset of [0,1), then the ergodic theorem implies

$$\frac{1}{n}\sum_{m=0}^{n-1} 1_{(\varphi^m\omega\in A)} \to |A| \quad \text{a.s.}$$

where $|A|$ denotes the Lebesgue measure of $A$. The last result for $\omega = 0$ is usually called **Weyl's equidistribution theorem**, although Bohl and Sierpinski should also get credit. For the history and a nonprobabilistic proof, see Hardy and Wright (1959), pp. 390–393.

To recover the number theoretic result, we will now show that:

**Theorem 7.2.4.** *If $A = [a, b)$ then the exceptional set is $\emptyset$.*

*Proof.* Let $A_k = [a + 1/k, b - 1/k)$. If $b - a > 2/k$, the ergodic theorem implies

$$\frac{1}{n} \sum_{m=0}^{n-1} 1_{A_k}(\varphi^m \omega) \to b - a - \frac{2}{k}$$

for $\omega \in \Omega_k$ with $P(\Omega_k) = 1$. Let $G = \cap \Omega_k$, where the intersection is over integers $k$ with $b - a > 2/k$. $P(G) = 1$, so $G$ is dense in [0,1). If $x \in [0, 1)$ and $\omega_k \in G$ with $|\omega_k - x| < 1/k$, then $\varphi^m \omega_k \in A_k$ implies $\varphi^m x \in A$, so

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} 1_A(\varphi^m x) \geq b - a - \frac{2}{k}$$

for all large enough $k$. Noting that $k$ is arbitrary and applying similar reasoning to $A^c$ shows

$$\frac{1}{n} \sum_{m=0}^{n-1} 1_A(\varphi^m x) \to b - a$$ ■

**Example 7.2.4. Benford's law.** As Gelfand first observed, the equidistribution theorem says something interesting about $2^m$. Let $\theta = \log_{10} 2$, $1 \leq k \leq 9$, and $A_k = [\log_{10} k, \log_{10}(k+1))$, where $\log_{10} y$ is the logarithm of $y$ to the base 10. Taking $x = 0$ in the last result, we have

$$\frac{1}{n} \sum_{m=0}^{n-1} 1_A(\varphi^m 0) \to \log_{10}\left(\frac{k+1}{k}\right)$$

A little thought reveals that the first digit of $2^m$ is $k$ if and only if $m\theta \bmod 1 \in A_k$. The numerical values of the limiting probabilities are

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| .3010 | .1761 | .1249 | .0969 | .0792 | .0669 | .0580 | .0512 | .0458 |

The limit distribution on $\{1, \ldots, 9\}$ is called Benford's (1938) law, although it was discovered by Newcomb (1881). As Raimi (1976) explains, in many tables the observed frequency with which $k$ appears as a first digit is approximately $\log_{10}((k+1)/k)$. Some of the many examples that are supposed to follow Benford's law are census populations of 3259 counties, 308 numbers from *Reader's Digest*, areas of 335 rivers, and 342 addresses of *American Men of Science*. The next table compares the percentages of the observations in the first five categories to Benford's law:

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Census | 33.9 | 20.4 | 14.2 | 8.1 | 7.2 |
| *Reader's Digest* | 33.4 | 18.5 | 12.4 | 7.5 | 7.1 |
| Rivers | 31.0 | 16.4 | 10.7 | 11.3 | 7.2 |
| Benford's Law | 30.1 | 17.6 | 12.5 | 9.7 | 7.9 |
| Addresses | 28.9 | 19.2 | 12.6 | 8.8 | 8.5 |

The fits are far from perfect, but in each case Benford's law matches the general shape of the observed distribution.

**Example 7.2.5. Bernoulli shift.** $\Omega = [0, 1)$, $\varphi(\omega) = (2\omega) \bmod 1$. Let $i_1, \ldots, i_k \in \{0, 1\}$, let $r = i_1 2^{-1} + \cdots + i_k 2^{-k}$, and let $X(\omega) = 1$ if $r \le \omega < r + 2^{-k}$. In words, $X(\omega) = 1$ if the first $k$ digits of the binary expansion of $\omega$ are $i_1, \ldots, i_k$. The ergodic theorem implies that

$$\frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m \omega) \to 2^{-k} \quad \text{a.s.}$$

that is, in almost every $\omega \in [0, 1)$ the pattern $i_1, \ldots, i_k$ occurs with its expected frequency. Since there are only a countable number of patterns of finite length, it follows that almost every $\omega \in [0, 1)$ is **normal**, that is, all patterns occur with their expected frequency. This is the binary version of Borel's (1909) normal number theorem.

## 7.3 Recurrence

In this section, we will study the recurrence properties of stationary sequences. Our first result is an application of the ergodic theorem. Let $X_1, X_2, \ldots$ be a stationary sequence taking values in $\mathbf{R}^d$, let $S_k = X_1 + \cdots + X_k$, let $A = \{S_k \ne 0 \text{ for all } k \ge 1\}$, and let $R_n = |\{S_1, \ldots, S_n\}|$ be the number of points visited at time $n$. Kesten, Spitzer, and Whitman (see Spitzer, 1964, p. 40) proved the next result when the $X_i$ are i.i.d. In that case, $\mathcal{I}$ is trivial, so the limit is $P(A)$.

**Theorem 7.3.1.** *As $n \to \infty$, $R_n/n \to E(1_A|\mathcal{I})$ a.s.*

*Proof.* Suppose $X_1, X_2, \ldots$ are constructed on $(\mathbf{R}^d)^{\{0,1,\ldots\}}$ with $X_n(\omega) = \omega_n$, and let $\varphi$ be the shift operator. It is clear that

$$R_n \ge \sum_{m=1}^{n} 1_A(\varphi^m \omega)$$

since the right-hand side $= |\{m : 1 \le m \le n, S_\ell \ne S_m \text{ for all } \ell > m\}|$. Using the ergodic theorem now gives

$$\liminf_{n\to\infty} R_n/n \ge E(1_A|\mathcal{I}) \quad \text{a.s.}$$

To prove the opposite inequality, let $A_k = \{S_1 \ne 0, S_2 \ne 0, \ldots, S_k \ne 0\}$. It is clear that

$$R_n \le k + \sum_{m=1}^{n-k} 1_{A_k}(\varphi^m \omega)$$

since the sum on the right-hand side $= |\{m : 1 \le m \le n-k, S_\ell \ne S_m$ for $m < \ell \le m+k\}|$. Using the ergodic theorem now gives

$$\limsup_{n\to\infty} R_n/n \le E(1_{A_k}|\mathcal{I})$$

As $k \uparrow \infty$, $A_k \downarrow A$, so the monotone convergence theorem for conditional expectations, (c) in Theorem 5.1.2, implies

$$E(1_{A_k}|\mathcal{I}) \downarrow E(1_A|\mathcal{I}) \quad \text{as } k \uparrow \infty$$

and the proof is complete. ■

**Exercise 7.3.1.** Let $g_n = P(S_1 \ne 0, \dots, S_n \ne 0)$ for $n \ge 1$ and $g_0 = 1$. Show that $ER_n = \sum_{m=1}^n g_{m-1}$.

From Theorem 7.3.1, we get a result about the recurrence of random walks with stationary increments that is (for integer-valued random walks) a generalization of the Chung-Fuchs theorem, 4.2.7.

**Theorem 7.3.2.** *Let $X_1, X_2, \dots$ be a stationary sequence taking values in $\mathbf{Z}$ with $E|X_i| < \infty$. Let $S_n = X_1 + \cdots + X_n$, and let $A = \{S_1 \ne 0, S_2 \ne 0, \dots\}$. (i) If $E(X_1|\mathcal{I}) = 0$, then $P(A) = 0$. (ii) If $P(A) = 0$, then $P(S_n = 0$ i.o.$) = 1$.*

**Remark.** In words, mean zero implies recurrence. The condition $E(X_1|\mathcal{I}) = 0$ is needed to rule out trivial examples that have mean 0 but are a combination of a sequence with positive and negative means, for example, $P(X_n = 1$ for all $n) = P(X_n = -1$ for all $n) = 1/2$.

*Proof.* If $E(X_1|\mathcal{I}) = 0$, then the ergodic theorem implies $S_n/n \to 0$ a.s. Now

$$\limsup_{n\to\infty}\left(\max_{1\le k\le n} |S_k|/n\right) = \limsup_{n\to\infty}\left(\max_{K\le k\le n} |S_k|/n\right) \le \left(\max_{k\ge K} |S_k|/k\right)$$

for any $K$ and the right-hand side $\downarrow 0$ as $K \uparrow \infty$. The last conclusion leads easily to

$$\lim_{n\to\infty}\left(\max_{1\le k\le n} |S_k|\right)\Big/ n = 0$$

Since $R_n \le 1 + 2\max_{1\le k\le n} |S_k|$, it follows that $R_n/n \to 0$, and Theorem 7.3.1 implies $P(A) = 0$.

Let $F_j = \{S_i \ne 0$ for $i < j, S_j = 0\}$ and $G_{j,k} = \{S_{j+i} - S_j \ne 0$ for $i < k$, $S_{j+k} - S_j = 0\}$. $P(A) = 0$ implies that $\sum P(F_k) = 1$. Stationarity implies

$P(G_{j,k}) = P(F_k)$, and for fixed $j$ the $G_{j,k}$ are disjoint, so $\cup_k G_{j,k} = \Omega$ a.s. It follows that

$$\sum_k P(F_j \cap G_{j,k}) = P(F_j) \quad \text{and} \quad \sum_{j,k} P(F_j \cap G_{j,k}) = 1$$

On $F_j \cap G_{j,k}$, $S_j = 0$ and $S_{j+k} = 0$, so we have shown $P(S_n = 0$ at least two times$) = 1$. Repeating the last argument shows $P(S_n = 0$ at least $k$ times$) = 1$ for all $k$, and the proof is complete. ■

**Exercise 7.3.2.** Imitate the proof of (i) in Theorem 7.3.2 to show that if we assume $P(X_i > 1) = 0$, $EX_i > 0$, and the sequence $X_i$ is ergodic in addition to the hypotheses of Theorem 7.3.2, then $P(A) = EX_i$.

**Remark.** This result was proved for asymmetric simple random walk in Exercise 4.1.13. It is interesting to note that we can use martingale theory to prove a result for random walks that do not skip over integers on the way down; see Exercise 5.7.7.

Extending the reasoning in the proof of part (ii) of Theorem 7.3.2 gives a result of Kac (1947b). Let $X_0, X_1, \ldots$ be a stationary sequence taking values in $(S, \mathcal{S})$. Let $A \in \mathcal{S}$, let $T_0 = 0$, and for $n \geq 1$, let $T_n = \inf\{m > T_{n-1} : X_m \in A\}$ be the time of the $n$th return to $A$.

**Theorem 7.3.3.** *If $P(X_n \in A$ at least once$) = 1$, then under $P(\cdot|X_0 \in A)$, $t_n = T_n - T_{n-1}$ is a stationary sequence with $E(T_1|X_0 \in A) = 1/P(X_0 \in A)$.*

**Remark.** If $X_n$ is an irreducible Markov chain on a countable state space $S$ starting from its stationary distribution $\pi$, and $A = \{x\}$, then Theorem 7.3.3 says $E_x T_x = 1/\pi(x)$, which is Theorem 6.5.5. Theorem 7.3.3 extends that result to an arbitrary $A \subset S$ and drops the assumption that $X_n$ is a Markov chain.

*Proof.* We first show that under $P(\cdot|X_0 \in A)$, $t_1, t_2, \ldots$ is stationary. To cut down on $\ldots$'s, we will only show that

$$P(t_1 = m, t_2 = n|X_0 \in A) = P(t_2 = m, t_3 = n|X_0 \in A)$$

It will be clear that the same proof works for any finite dimensional distribution. Our first step is to extend $\{X_n, n \geq 0\}$ to a two-sided stationary sequence $\{X_n, n \in \mathbf{Z}\}$ using Theorem 7.1.2. Let $C_k = \{X_{-1} \notin A, \ldots, X_{-k+1} \notin A, X_{-k} \in A\}$.

$$\left(\cup_{k=1}^K C_k\right)^c = \{X_k \notin A \text{ for } -K \leq k \leq -1\}$$

The last event has the same probability as $\{X_k \notin A$ for $1 \leq k \leq K\}$, so letting $K \to \infty$, we get $P\left(\cup_{k=1}^\infty C_k\right) = 1$. To prove the desired stationarity, we let

$I_{j,k} = \{i \in [j,k] : X_i \in A\}$ and observe that

$$\begin{aligned}
P(t_2 = m, t_3 = n, X_0 \in A) &= \sum_{\ell=1}^{\infty} P(X_0 \in A, t_1 = \ell, t_2 = m, t_3 = n) \\
&= \sum_{\ell=1}^{\infty} P(I_{0,\ell+m+n} = \{0, \ell, \ell+m, \ell+m+n\}) \\
&= \sum_{\ell=1}^{\infty} P(I_{-\ell,m+n} = \{-\ell, 0, m, m+n\}) \\
&= \sum_{\ell=1}^{\infty} P(C_\ell, X_0 \in A, t_1 = m, t_2 = n)
\end{aligned}$$

To complete the proof, we compute

$$\begin{aligned}
E(t_1 | X_0 \in A) &= \sum_{k=1}^{\infty} P(t_1 \geq k | X_0 \in A) = P(X_0 \in A)^{-1} \sum_{k=1}^{\infty} P(t_1 \geq k, X_0 \in A) \\
&= P(X_0 \in A)^{-1} \sum_{k=1}^{\infty} P(C_k) = 1/P(X_0 \in A)
\end{aligned}$$

since the $C_k$ are disjoint and their union has probability 1. ∎

In the next two exercises, we continue to use the notation of Theorem 7.3.3.

**Exercise 7.3.3.** Show that if $P(X_n \in A \text{ at least once}) = 1$ and $A \cap B = \emptyset$, then

$$E\left( \sum_{1 \leq m \leq T_1} 1_{(X_m \in B)} \middle| X_0 \in A \right) = \frac{P(X_0 \in B)}{P(X_0 \in A)}$$

When $A = \{x\}$ and $X_n$ is a Markov chain, this is the "cycle trick" for defining a stationary measure. See Theorem 6.5.2.

**Exercise 7.3.4.** Consider the special case in which $X_n \in \{0, 1\}$, and let $\bar{P} = P(\cdot | X_0 = 1)$. Here $A = \{1\}$ and so $T_1 = \inf\{m > 0 : X_m = 1\}$. Show $P(T_1 = n) = \bar{P}(T_1 \geq n)/\bar{E} T_1$. When $t_1, t_2, \ldots$ are i.i.d., this reduces to the formula for the first waiting time in a stationary renewal process.

In checking the hypotheses of Kac's theorem, a result Poincaré proved in 1899 is useful. First, we need a definition. Let $T_A = \inf\{n \geq 1 : \varphi^n(\omega) \in A\}$.

**Theorem 7.3.4.** *Suppose $\varphi : \Omega \to \Omega$ preserves $P$, that is, $P \circ \varphi^{-1} = P$. (i) $T_A < \infty$ a.s. on $A$, that is, $P(\omega \in A, T_A = \infty) = 0$. (ii) $\{\varphi^n(\omega) \in A \text{ i.o.}\} \supset A$. (iii) If $\varphi$ is ergodic and $P(A) > 0$, then $P(\varphi^n(\omega) \in A \text{ i.o.}) = 1$.*

**Remark.** Note that in (i) and (ii) we assume only that $\varphi$ is measure-preserving. Extrapolating from Markov chain theory, the conclusions can be "explained" by noting that (i) the existence of a stationary distribution implies the sequence is recurrent, and (ii) since we start in $A$, we do not have to assume irreducibility. Conclusion (iii) is, of course, a consequence of the ergodic theorem, but as the self-contained proof below indicates, it is a much simpler fact.

*Proof.* Let $B = \{\omega \in A, T_A = \infty\}$. A little thought shows that if $\omega \in \varphi^{-m}B$, then $\varphi^m(\omega) \in A$, but $\varphi^n(\omega) \notin A$ for $n > m$, so the $\varphi^{-m}B$ are pairwise disjoint. The fact that $\varphi$ is measure-preserving implies $P(\varphi^{-m}B) = P(B)$, so we must have $P(B) = 0$ (or $P$ would have infinite mass). To prove (ii), note that for any $k$, $\varphi^k$ is measure-preserving, so (i) implies

$$\begin{aligned} 0 &= P(\omega \in A, \varphi^{nk}(\omega) \notin A \text{ for all } n \geq 1) \\ &\geq P(\omega \in A, \varphi^{m}(\omega) \notin A \text{ for all } m \geq k) \end{aligned}$$

Since the last probability is 0 for all $k$, (ii) follows. Finally, for (iii), note that $B \equiv \{\omega : \varphi^n(\omega) \in A \text{ i.o.}\}$ is invariant and $\supset A$ by (b), so $P(B) > 0$, and it follows from ergodicity that $P(B) = 1$. ∎

## 7.4 A Subadditive Ergodic Theorem*

In this section we will prove Liggett's (1985) version of Kingman's (1968)

**Theorem 7.4.1. Subadditive ergodic theorem.** *Suppose $X_{m,n}$, $0 \leq m < n$ satisfy:*
*(i)* $X_{0,m} + X_{m,n} \geq X_{0,n}$
*(ii)* $\{X_{nk,(n+1)k}, n \geq 1\}$ *is a stationary sequence for each $k$.*
*(iii) The distribution of* $\{X_{m,m+k}, k \geq 1\}$ *does not depend on $m$.*
*(iv)* $EX_{0,1}^+ < \infty$ *and for each $n$,* $EX_{0,n} \geq \gamma_0 n$, *where* $\gamma_0 > -\infty$.

*Then*
*(a)* $\lim_{n\to\infty} EX_{0,n}/n = \inf_m EX_{0,m}/m \equiv \gamma$
*(b)* $X = \lim_{n\to\infty} X_{0,n}/n$ *exists a.s. and in $L^1$, so $EX = \gamma$.*
*(c) If all the stationary sequences in (ii) are ergodic then $X = \gamma$ a.s.*

**Remark.** Kingman assumed (iv), but instead of (i)–(iii) he assumed that $X_{\ell,m} + X_{m,n} \geq X_{\ell,n}$ for all $\ell < m < n$ and that the distribution of $\{X_{m+k,n+k}, 0 \leq m < n\}$ does not depend on $k$. In two of the four applications in the next section, these stronger conditions do not hold.

Before giving the proof, which is somewhat lengthy, we will consider several examples for motivation. Since the validity of (ii) and (iii) in each case is clear, we will only check (i) and (iv). The first example shows that Theorem 7.4.1 contains the ergodic theorem, 7.2.1, as a special case.

**Example 7.4.1. Stationary sequences.** Suppose $\xi_1, \xi_2, \ldots$ is a stationary sequence with $E|\xi_k| < \infty$, and let $X_{m,n} = \xi_{m+1} + \cdots + \xi_n$. Then $X_{0,n} = X_{0,m} + X_{m,n}$, and (iv) holds.

**Example 7.4.2. Range of random walk.** Suppose $\xi_1, \xi_2, \ldots$ is a stationary sequence and let $S_n = \xi_1 + \cdots + \xi_n$. Let $X_{m,n} = |\{S_{m+1}, \ldots, S_n\}|$. It is clear that $X_{0,m} + X_{m,n} \geq X_{0,n}$. $0 \leq X_{0,n} \leq n$, so (iv) holds. Applying (6.1) now gives $X_{0,n}/n \to X$ a.s. and in $L^1$, but it does not tell us what the limit is.

**Example 7.4.3. Longest common subsequences.** Given are ergodic stationary sequences $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$ Let $L_{m,n} = \max\{K : X_{i_k} = Y_{j_k}$ for $1 \leq k \leq K$, where $m < i_1 < i_2 \cdots < i_K \leq n$ and $m < j_1 < j_2 \cdots < j_K \leq n\}$. It is clear that

$$L_{0,m} + L_{m,n} \geq L_{0,n}$$

so $X_{m,n} = -L_{m,n}$ is subadditive. $0 \leq L_{0,n} \leq n$ so (iv) holds. Applying Theorem 7.4.1 now, we conclude that

$$L_{0,n}/n \to \gamma = \sup_{m \geq 1} E(L_{0,m}/m)$$

**Exercise 7.4.1.** Suppose that in the last exercise $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are i.i.d. and take the values 0 and 1 with probability 1/2 each. (a) Compute $EL_1$ and $EL_2/2$ to get lower bounds on $\gamma$. (b) Show $\gamma < 1$ by computing the expected number of $i$ and $j$ sequences of length $K = an$ with the desired property.

**Remark.** Chvátal and Sankoff (1975) have shown $0.727273 \leq \gamma \leq 0.866595$

**Example 7.4.4. Slow convergence.** Our final example shows that the convergence in (a) of Theorem 7.4.1 may occur arbitrarily slowly. Suppose $X_{m,m+k} = f(k) \geq 0$, where $f(k)/k$ is decreasing.

$$\begin{aligned} X_{0,n} = f(n) &= m\frac{f(n)}{n} + (n-m)\frac{f(n)}{n} \\ &\leq m\frac{f(m)}{m} + (n-m)\frac{f(n-m)}{n-m} = X_{0,m} + X_{m,n} \end{aligned}$$

The examples above should provide enough motivation for now. In the next section, we will give four more applications of Theorem 7.4.1.

*Proof of Theorem 7.4.1.* There are four steps. The first, second, and fourth date back to Kingman (1968). The half-dozen proofs of subadditive ergodic theorems that exist all do the crucial third step in a different way. Here we use the approach of S. Leventhal (1988), who in turn based his proof on Katznelson and Weiss (1982).

*Step 1.* The first thing to check is that $E|X_{0,n}| \le Cn$. To do this, we note that (i) implies $X_{0,m}^+ + X_{m,n}^+ \ge X_{0,n}^+$. Repeatedly using the last inequality and invoking (iii) gives $EX_{0,n}^+ \le nEX_{0,1}^+ < \infty$. Since $|x| = 2x^+ - x$, it follows from (iv) that

$$E|X_{0,n}| \le 2EX_{0,n}^+ - EX_{0,n} \le Cn < \infty$$

Let $a_n = EX_{0,n}$. (i) and (iii) imply that

$$a_m + a_{n-m} \ge a_n \tag{7.4.1}$$

From this, it follows easily that

$$a_n/n \to \inf_{m\ge1} a_m/m \equiv \gamma \tag{7.4.2}$$

To prove this, we observe that the liminf is clearly $\ge \gamma$, so all we have to show is that the limsup $\le a_m/m$ for any $m$. The last fact is easy, for if we write $n = km + \ell$ with $0 \le \ell < m$, then repeated use of (7.4.1) gives $a_n \le ka_m + a_\ell$. Dividing by $n = km + \ell$ gives

$$\frac{a_n}{n} \le \frac{km}{km+\ell}\cdot\frac{a_m}{m} + \frac{a_\ell}{n}$$

Letting $n \to \infty$ and recalling $0 \le \ell < m$ gives 7.4.2 and proves (a) in Theorem 7.4.1.

*Step 2.* Making repeated use of (i), we get

$$X_{0,n} \le X_{0,km} + X_{km,n}$$
$$X_{0,n} \le X_{0,(k-1)m} + X_{(k-1)m,km} + X_{km,n}$$

and so on until the first term on the right is $X_{0,m}$. Dividing by $n = km + \ell$ then gives

$$\frac{X_{0,n}}{n} \le \frac{k}{km+\ell}\cdot\frac{X_{0,m} + \cdots + X_{(k-1)m,km}}{k} + \frac{X_{km,n}}{n} \tag{7.4.3}$$

Using (ii) and the ergodic theorem now gives that

$$\frac{X_{0,m} + \cdots + X_{(k-1)m,km}}{k} \to A_m \quad \text{a.s. and in } L^1$$

where $A_m = E(X_{0,m}|\mathcal{I}_m)$ and the subscript indicates that $\mathcal{I}_m$ is the shift invariant $\sigma$-field for the sequence $X_{(k-1)m,km}$, $k \ge 1$. The exact formula for the limit is not important, but we will need to know later that $EA_m = EX_{0,m}$.

If we fix $\ell$ and let $\epsilon > 0$, then (iii) implies

$$\sum_{k=1}^{\infty} P(X_{km,km+\ell} > (km+\ell)\epsilon) \le \sum_{k=1}^{\infty} P(X_{0,\ell} > k\epsilon) < \infty$$

since $EX_{0,\ell}^+ < \infty$ by the result at the beginning of Step 1. The last two observations imply

$$\overline{X} \equiv \limsup_{n\to\infty} X_{0,n}/n \le A_m/m \tag{7.4.4}$$

Taking expected values now gives $E\overline{X} \le E(X_{0,m}/m)$, and taking the infimum over $m$, we have $E\overline{X} \le \gamma$. Note that if all the stationary sequences in (ii) are ergodic, we have $\overline{X} \le \gamma$.

**Remark.** If (i)–(iii) hold, $EX_{0,1}^+ < \infty$, and $\inf EX_{0,m}/m = -\infty$, then it follows from the last argument that as $X_{0,n}/n \to -\infty$ a.s. as $n \to \infty$.

*Step 3.* The next step is to let

$$\underline{X} = \liminf_{n\to\infty} X_{0,n}/n$$

and show that $E\underline{X} \ge \gamma$. Since $\infty > EX_{0,1} \ge \gamma \ge \gamma_0 > -\infty$, and we have shown in Step 2 that $E\overline{X} \le \gamma$, it will follow that $\underline{X} = \overline{X}$, that is, the limit of $X_{0,n}/n$ exists a.s. Let

$$\underline{X}_m = \liminf_{n\to\infty} X_{m,m+n}/n$$

(i) implies

$$X_{0,m+n} \le X_{0,m} + X_{m,m+n}$$

Dividing both sides by $n$ and letting $n \to \infty$ gives $\underline{X} \le \underline{X}_m$ a.s. However, (iii) implies that $\underline{X}_m$ and $\underline{X}$ have the same distribution so $\underline{X} = \underline{X}_m$ a.s.

Let $\epsilon > 0$ and let $Z = \epsilon + (\underline{X} \vee -M)$. Since $\underline{X} \le \overline{X}$ and $E\overline{X} \le \gamma < \infty$ by Step 2, $E|Z| < \infty$. Let

$$Y_{m,n} = X_{m,n} - (n-m)Z$$

$Y$ satisfies (i)–(iv), since $Z_{m,n} = -(n-m)Z$ does, and has

$$\underline{Y} \equiv \liminf_{n\to\infty} Y_{0,n}/n \le -\epsilon \tag{7.4.5}$$

Let $T_m = \min\{n \ge 1 : Y_{m,m+n} \le 0\}$. (iii) implies $T_m =_d T_0$ and

$$E(Y_{m,m+1}; T_m > N) = E(Y_{0,1}; T_0 > N)$$

(7.4.5) implies that $P(T_0 < \infty) = 1$, so we can pick $N$ large enough so that

$$E(Y_{0,1}; T_0 > N) \le \epsilon$$

Let

$$S_m = \begin{cases} T_m & \text{on } \{T_m \le N\} \\ 1 & \text{on } \{T_m > N\} \end{cases}$$

This is not a stopping time, but there is nothing special about stopping times for a stationary sequence! Let

$$\xi_m = \begin{cases} 0 & \text{on } \{T_m \le N\} \\ Y_{m,m+1} & \text{on } \{T_m > N\} \end{cases}$$

Since $Y(m, m+T_m) \le 0$ always and we have $S_m = 1$, $Y_{m,m+1} > 0$ on $\{T_m > N\}$, we have $Y(m, m+S_m) \le \xi_m$ and $\xi_m \ge 0$. Let $R_0 = 0$, and for $k \ge 1$, let

$R_k = R_{k-1} + S(R_{k-1})$. Let $K = \max\{k : R_k \le n\}$. From (i), it follows that

$$Y(0,n) \le Y(R_0, R_1) + \cdots + Y(R_{K-1}, R_K) + Y(R_K, n)$$

Since $\xi_m \ge 0$ and $n - R_K \le N$, the last quantity is

$$\le \sum_{m=0}^{n-1} \xi_m + \sum_{j=1}^{N} |Y_{n-j,n-j+1}|$$

Here we have used (i) on $Y(R_K, n)$. Dividing both sides by $n$, taking expected values, and letting $n \to \infty$ gives

$$\limsup_{n\to\infty} EY_{0,n}/n \le E\xi_0 \le E(Y_{0,1}; T_0 > N) \le \epsilon$$

It follows from (a) and the definition of $Y_{0,n}$ that

$$\gamma = \lim_{n\to\infty} EX_{0,n}/n \le 2\epsilon + E(\underline{X} \vee -M)$$

Since $\epsilon > 0$ and $M$ are arbitrary, it follows that $E\underline{X} \ge \gamma$, and Step 3 is complete.

*Step 4.* It only remains to prove convergence in $L^1$. Let $\Gamma_m = A_m/m$ be the limit in (7.4.4), recall $E\Gamma_m = E(X_{0,m}/m)$, and let $\Gamma = \inf \Gamma_m$. Observing that $|z| = 2z^+ - z$ (consider two cases $z \ge 0$ and $z < 0$), we can write

$$E|X_{0,n}/n - \Gamma| = 2E(X_{0,n}/n - \Gamma)^+ - E(X_{0,n}/n - \Gamma) \le 2E(X_{0,n}/n - \Gamma)^+$$

since

$$E(X_{0,n}/n) \ge \gamma = \inf E\Gamma_m \ge E\Gamma$$

Using the trivial inequality $(x+y)^+ \le x^+ + y^+$ and noticing $\Gamma_m \ge \Gamma$ now gives

$$E(X_{0,n}/n - \Gamma)^+ \le E(X_{0,n}/n - \Gamma_m)^+ + E(\Gamma_m - \Gamma)$$

Now $E\Gamma_m \to \gamma$ as $m \to \infty$ and $E\Gamma \ge E\bar{X} \ge E\underline{X} \ge \gamma$ by steps 2 and 3, so $E\Gamma = \gamma$, and it follows that $E(\Gamma_m - \Gamma)$ is small if $m$ is large. To bound the other term, observe that (i) implies

$$E(X_{0,n}/n - \Gamma_m)^+ \le E\left(\frac{X(0,m) + \cdots + X((k-1)m, km)}{km + \ell} - \Gamma_m\right)^+$$
$$+ E\left(\frac{X(km, n)}{n}\right)^+$$

The second term $= E(X^+_{0,\ell}/n) \to 0$ as $n \to \infty$. For the first, we observe $y^+ \le |y|$, and the ergodic theorem implies

$$E\left|\frac{X(0,m) + \cdots + X((k-1)m, km)}{k} - \Gamma_m\right| \to 0$$

so the proof of Theorem 7.4.1 is complete. ■

## 7.5 Applications*

In this section, we will give four applications of our subadditive ergodic theorem, 7.4.1. These examples are independent of each other and can be read in any order. In the last two, we encounter situations to which Liggett's version applies but Kingman's version does not.

**Example 7.5.1. Products of random matrices.** Suppose $A_1, A_2, \dots$ is a stationary sequence of $k \times k$ matrices with positive entries, and let

$$\alpha_{m,n}(i,j) = (A_{m+1} \cdots A_n)(i,j),$$

that is, the entry in row $i$ of column $j$ of the product. It is clear that

$$\alpha_{0,m}(1,1)\alpha_{m,n}(1,1) \le \alpha_{0,n}(1,1)$$

so if we let $X_{m,n} = -\log \alpha_{m,n}(1,1)$, then $X_{0,m} + X_{m,n} \ge X_{0,n}$. To check (iv), we observe that

$$\prod_{m=1}^{n} A_m(1,1) \le \alpha_{0,n}(1,1) \le k^{n-1} \prod_{m=1}^{n} \left( \sup_{i,j} A_m(i,j) \right)$$

or taking logs

$$-\sum_{m=1}^{n} \log A_m(1,1) \ge X_{0,n} \ge -(n \log k) - \sum_{m=1}^{n} \log \left( \sup_{i,j} A_m(i,j) \right)$$

So if $E \log A_m(1,1) > -\infty$, then $EX_{0,1}^+ < \infty$, and if

$$E \log \left( \sup_{i,j} A_m(i,j) \right) < \infty$$

then $EX_{0,n}^- \le \gamma_0 n$. If we observe that

$$P\left( \log \left( \sup_{i,j} A_m(i,j) \right) \ge x \right) \le \sum_{i,j} P\left( \log A_m(i,j) \ge x \right)$$

we see that it is enough to assume that

$$(*) \qquad E|\log A_m(i,j)| < \infty \quad \text{for all } i, j$$

When $(*)$ holds, applying Theorem 7.4.1 gives $X_{0,n}/n \to X$ a.s. Using the strict positivity of the entries, it is easy to improve that result to

$$\frac{1}{n} \log \alpha_{0,n}(i,j) \to -X \quad \text{a.s. for all } i, j \tag{7.5.1}$$

a result first proved by Furstenberg and Kesten (1960). ∎

The key to the proof above was the fact that $\alpha_{0,n}(1,1)$ was supermultiplicative. An alternative approach is to let

$$\|A\| = \max_i \sum_j |A(i,j)| = \max\{\|xA\|_1 : \|x\|_1 = 1\}$$

where $(xA)_j = \sum_i x_i A(i,j)$ and $\|x\|_1 = |x_1| + \cdots + |x_k|$. From the second definition, it is clear that $\|AB\| \le \|A\| \cdot \|B\|$, so if we let

$$\beta_{m,n} = \|A_{m+1} \cdots A_n\|$$

and $Y_{m,n} = \log \beta_{m,n}$, then $Y_{m,n}$ is subadditive. It is easy to use (7.5.1) to show that

$$\frac{1}{n} \log \|A_{m+1} \cdots A_n\| \to -X \quad \text{a.s.}$$

where $X$ is the limit of $X_{0,n}/n$. To see the advantage in having two proofs of the same result, we observe that if $A_1, A_2, \dots$ is an i.i.d. sequence, then $X$ is constant, and we can get upper and lower bounds by observing

$$\sup_{m\ge 1} (E \log \alpha_{0,m})/m = -X = \inf_{m \ge 1} (E \log \beta_{0,m})/m$$

**Remark.** Oseleděc (1968) proved a result which gives the asymptotic behavior of all of the eigenvalues of A. As Ragunathan (1979) and Ruelle (1979) have observed, this result can also be obtained from Theorem 7.4.1. See Krengel (1985) or the papers cited for details. Furstenberg and Kesten (1960) and later Ishitani (1977) have proved central limit theorems:

$$(\log \alpha_{0,n}(1,1) - \mu n)/n^{1/2} \Rightarrow \sigma \chi$$

where $\chi$ has the standard normal distribution. For more about products of random matrices, see Cohen, Kesten, and Newman (1985).

**Example 7.5.2. Increasing sequences in random permutations.** Let $\pi$ be a permutation of $\{1, 2, \dots, n\}$ and let $\ell(\pi)$ be the length of the longest increasing sequence in $\pi$, that is, the largest $k$ for which there are integers $i_1 < i_2 \cdots < i_k$ so that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Hammersley (1970) attacked this problem by putting a rate one Poisson process in the plane, and for $s < t \in [0, \infty)$, letting $Y_{s,t}$ denote the length of the longest increasing path lying in the square $R_{s,t}$ with vertices $(s,s)$, $(s,t)$, $(t,t)$, and $(t,s)$. That is, the largest $k$ for which there are points $(x_i, y_i)$ in the Poisson process with $s < x_1 < \cdots < x_k < t$ and $s < y_1 < \cdots < y_k < t$. It is clear that $Y_{0,m} + Y_{m,n} \le Y_{0,n}$. Applying Theorem 7.4.1 to $-Y_{0,n}$ shows

$$Y_{0,n}/n \to \gamma \equiv \sup_{m \ge 1} EY_{0,m}/m \quad \text{a.s.}$$

For each $k$, $Y_{nk,(n+1)k}$, $n \ge 0$ is i.i.d., so the limit is constant. We will show that $\gamma < \infty$ in Exercise 7.5.2.

To get from the result about the Poisson process back to the random permutation problem, let $\tau(n)$ be the smallest value of $t$ for which there are $n$ points in $R_{0,t}$. Let the $n$ points in $R_{0,\tau(n)}$ be written as $(x_i, y_i)$ where $0 < x_1 < x_2 \cdots < x_n \le \tau(n)$ and let $\pi_n$ be the unique permutation of $\{1, 2, \ldots, n\}$ so that $y_{\pi_n(1)} < y_{\pi_n(2)} \cdots < y_{\pi_n(n)}$. It is clear that $Y_{0,\tau(n)} = \ell(\pi_n)$. An easy argument shows:

**Lemma 7.5.1.** $\tau(n)/\sqrt{n} \to 1$ *a.s.*

*Proof.* Let $S_n$ be the number of points in $R_{0,\sqrt{n}}$. $S_n - S_{n-1}$ are independent Poisson r.v.'s with mean 1, so the strong law of large numbers implies $S_n/n \to 1$ a.s. If $\epsilon > 0$ then for large $n$, $S_{n(1-\epsilon)} < n < S_{n(1+\epsilon)}$ and hence $\sqrt{(1-\epsilon)n} \le \tau(n) \le \sqrt{(1+\epsilon)n}$. ■

It follows from Lemma 7.5.1 and the monotonicity of $m \to Y_{0,m}$ that

$$n^{-1/2}\ell(\pi_n) \to \gamma \quad \text{a.s.}$$

Hammersley (1970) has a proof that $\pi/2 \le \gamma \le e$, and Kingman (1973) shows that $1.59 < \gamma < 2.49$. See Exercises 7.5.1 and 7.5.2. Subsequent work on the random permutation problem, see Logan and Shepp (1977) and Vershik and Kerov (1977), has shown that $\gamma = 2$.

**Exercise 7.5.1.** Given a rate one Poisson process in $[0,\infty) \times [0,\infty)$, let $(X_1, Y_1)$ be the point that minimizes $x + y$. Let $(X_2, Y_2)$ be the point in $[X_1,\infty) \times [Y_1,\infty)$ that minimizes $x + y$, and so on. Use this construction to show that $\gamma \ge (8/\pi)^{1/2} > 1.59$.

**Exercise 7.5.2.** Let $\pi_n$ be a random permutation of $\{1, \ldots, n\}$ and let $J_k^n$ be the number of subsets of $\{1, \ldots n\}$ of size $k$ so that the associated $\pi_n(j)$ form an increasing subsequence. Compute $EJ_k^n$ and take $k \sim \alpha n^{1/2}$ to conclude $\gamma \le e$.

**Remark.** Kingman improved this by observing that $\ell(\pi_n) \ge \ell$ then $J_k^n \ge \binom{\ell}{k}$. Using this with the bound on $EJ_k^n$ and taking $\ell \sim \beta n^{1/2}$ and $k \sim \alpha n^{1/2}$, he showed $\gamma < 2.49$.

**Example 7.5.3. Age-dependent branching processes.** This is a variation of the branching process introduced in Subsection 5.3.4 in which each individual lives for an amount of time with distribution $F$ before producing $k$ offspring with probability $p_k$. The description of the process is completed by supposing that the process starts with one individual in generation 0 who is born at time 0, and when this particle dies, its offspring start independent copies of the original process.

Suppose $p_0 = 0$, let $X_{0,m}$ be the birth time of the first member of generation $m$, and let $X_{m,n}$ be the time lag necessary for that individual to have an offspring in generation $n$. In case of ties, pick an individual at random from those in generation $m$ born at time $X_{0,m}$. It is clear that $X_{0,n} \le X_{0,m} + X_{m,n}$. Since $X_{0,n} \ge 0$, (iv) holds if we assume $F$ has finite mean. Applying Theorem 7.4.1 now, it follows that

$$X_{0,n}/n \to \gamma \quad \text{a.s.}$$

The limit is constant because the sequences $\{X_{nk,(n+1)k}, n \ge 0\}$ are i.i.d.

**Remark.** The inequality $X_{\ell,m} + X_{m,n} \ge X_{\ell,n}$ is false when $\ell > 0$, because if we call $i_m$ the individual that determines the value of $X_{m,n}$ for $n > m$, then $i_m$ may not be a descendant of $i_\ell$.

As usual, one has to use other methods to identify the constant. Let $t_1, t_2, \dots$ be i.i.d. with distribution $F$, let $T_n = t_1 + \cdots + t_n$, and $\mu = \sum k p_k$. Let $Z_n(an)$ be the number of individuals in generation $n$ born by time $an$. Each individual in generation $n$ has probability $P(T_n \le an)$ to be born by time $an$, and the times are independent of the offspring numbers so

$$EZ_n(an) = EE(Z_n(an)|Z_n) = E(Z_n P(T_n \le an)) = \mu^n P(T_n \le an)$$

By results in Section 2.6, $n^{-1} \log P(T_n \le an) \to -c(a)$ as $n \to \infty$. If $\log \mu - c(a) < 0$ then Chebyshev's inequality and the Borel-Cantelli lemma imply $P(Z_n(an) \ge 1 \text{ i.o.}) = 0$. Conversely, if $EZ_n(an) > 1$ for some $n$, then we can define a supercritical branching process $Y_m$ that consists of the offspring in generation $mn$ that are descendants of individuals in $Y_{m-1}$ in generation $(m-1)n$ that are born less than $an$ units of time after their parents. This shows that with positive probability, $X_{0,mn} \le mna$ for all $m$. Combining the last two observations with the fact that $c(a)$ is strictly increasing gives

$$\gamma = \inf\{a : \log \mu - c(a) > 0\}$$

The last result is from Biggins (1977). See his 1978 and 1979 papers for extensions and refinements. Kingman (1975) has an approach to the problem via martingales:

**Exercise 7.5.3.** Let $\varphi(\theta) = E \exp(-\theta t_i)$ and

$$Y_n = (\mu\varphi(\theta))^{-n} \sum_{i=1}^{Z_n} \exp(-\theta T_n(i))$$

where the sum is over individuals in generation $n$ and $T_n(i)$ is the $i$th person's birth time. Show that $Y_n$ is a nonnegative martingale and use this to conclude that if $\exp(-\theta a)/\mu\varphi(\theta) > 1$, then $P(X_{0,n} \le an) \to 0$. A little thought reveals that this bound is the same as the answer in the last exercise.

**Example 7.5.4. First-passage percolation.** Consider $\mathbf{Z}^d$ as a graph with edges connecting each $x, y \in \mathbf{Z}^d$ with $|x - y| = 1$. Assign an independent nonnegative random variable $\tau(e)$ to each edge that represents the time required to traverse the edge going in either direction. If $e$ is the edge connecting $x$ and $y$, let $\tau(x, y) = \tau(y, x) = \tau(e)$. If $x_0 = x, x_1, \dots, x_n = y$ is a path from $x$ to $y$, that is, a sequence with $|x_m - x_{m-1}| = 1$ for $1 \le m \le n$, we define the **travel time** for the path to be $\tau(x_0, x_1) + \cdots + \tau(x_{n-1}, x_n)$. Define the **passage time** from $x$ to $y$, $t(x, y) =$ the infimum of the travel times over all paths from $x$ to $y$. Let $z \in \mathbf{Z}^d$, and let $X_{m,n} = t(mu, nu)$, where $u = (1, 0, \dots, 0)$.

Clearly $X_{0,m} + X_{m,n} \ge X_{0,n}$. $X_{0,n} \ge 0$, so if $E\tau(x, y) < \infty$, then (iv) holds, and Theorem 7.4.1 implies that $X_{0,n}/n \to X$ a.s. To see that the limit is constant, enumerate the edges in some order $e_1, e_2, \dots$ and observe that $X$ is measurable with respect to the tail $\sigma$-field of the i.i.d. sequence $\tau(e_1), \tau(e_2), \dots$

**Remark.** It is not hard to see that the assumption of finite first moment can be weakened. If $\tau$ has distribution $F$ with

$$(*) \qquad \int_0^\infty (1 - F(x))^{2d}\, dx < \infty$$

that is, the minimum of $2d$ independent copies has finite mean, then by finding $2d$ disjoint paths from 0 to $u = (1, 0, \dots, 0)$, one concludes that $E\tau(0, u) < \infty$ and (6.1) can be applied. The condition $(*)$ is also necessary for $X_{0,n}/n$ to converge to a finite limit. If $(*)$ fails and $Y_n$ is the minimum of $t(e)$ over all the edges from $\nu$, then

$$\limsup_{n\to\infty} X_{0,n}/n \ge \limsup_{n\to\infty} Y_n/n = \infty \quad \text{a.s.}$$

Above we considered the **point-to-point passage time**. A second object of interest is the **point-to-line passage time**:

$$a_n = \inf\{t(0, x) : x_1 = n\}$$

Unfortunately, it does not seem to be possible to embed this sequence in a subadditive family. To see the difficulty, let $\bar{t}(0, x)$ be infimum of travel times over paths from 0 to $x$ that lie in $\{y : y_1 \ge 0\}$, let

$$\bar{a}_m = \inf\{\bar{t}(0, x) : x_1 = m\}$$

and let $x^m$ be a point at which the infimum is achieved. We leave to the reader the highly nontrivial task of proving that such a point exists; see Smythe and Wierman (1978) for a proof. If we let $\bar{a}_{m,n}$ be the infimum of travel times over all paths that start at $x^m$, stay in $\{y : y_1 \ge m\}$, and end on $\{y : y_1 = n\}$, then $\bar{a}_{m,n}$ is independent of $\bar{a}_m$ and

$$\bar{a}_m + \bar{a}_{m,n} \ge \bar{a}_n$$

The last inequality is true without the half-space restriction, but the independence is not, and without the half-space restriction, we cannot get the stationarity properties needed to apply Theorem 7.4.1.

**Remark.** The family $\bar{a}_{m,n}$ is another example where $\bar{a}_{\ell,m} + \bar{a}_{m,n} \ge \bar{a}_{\ell,n}$ need not hold for $\ell > 0$.

A second approach to limit theorems for $a_m$ is to prove a result about the set of points that can be reached by time $t$: $\xi_t = \{x : t(0, x) \le t\}$. Cox and Durrett (1981) have shown

**Theorem 7.5.2.** *For any passage time distribution $F$ with $F(0) = 0$, there is a convex set $A$ so that for any $\epsilon > 0$ we have with probability one*

$$\xi_t \subset (1+\epsilon)tA \text{ for all } t \text{ sufficiently large}$$

*and* $|\xi_t^\epsilon \cap (1-\epsilon)tA \cap \mathbf{Z}^d|/t^d \to 0$ *as* $t \to \infty$.

Ignoring the boring details of how to state things precisely, the last result says $\xi_t/t \to A$ a.s. It implies that $a_n/n \to \gamma$ a.s., where $\gamma = 1/\sup\{x_1 : x \in A\}$. (Use the convexity and reflection symmetry of $A$.) When the distribution has finite mean (or satisfies the weaker condition in the remark above), $\gamma$ is the limit of $t(0, nu)/n$. Without any assumptions, $t(0, nu)/n \to \gamma$ in probability. For more details, see the paper cited above. Kesten's 1986 and 1987 papers are good sources for more about first-passage percolation.

**Exercise 7.5.4. Oriented first-passage percolation.** Consider a graph with vertices $\{(m, n) \in \mathbf{Z}^2 : m + n \text{ is even and } n \le 0\}$, and oriented edges connecting $(m, n)$ to $(m + 1, n - 1)$ and $(m, n)$ to $(m - 1, n - 1)$. Assign i.i.d. exponential mean one r.v.'s to each edge. Thinking of the number on edge $e$ as giving the time it takes water to travel down the edge, define $t(m, n) =$ the time at which the fluid first reaches $(m, n)$, and $a_n = \inf\{t(m, -n)\}$. Show that as $n \to \infty$, $a_n/n$ converges to a limit $\gamma$ a.s.

**Exercise 7.5.5.** Continuing with the setup in the last exercise: (i) Show $\gamma \le 1/2$ by considering $a_1$. (ii) Get a positive lower bound on $\gamma$ by looking at the expected number of paths down to $\{(m, -n) : -n \le m \le n\}$ with passage time $\le an$ and using results from Section 2.6.

**Remark.** If we replace the graph in Exercise 7.5.4 by a binary tree, then we get a problem equivalent to the first birth problem (Example 7.5.3) for $p_2 = 2$, $P(t_i > x) = e^{-x}$. In that case, the lower bound obtained by the methods of part (ii) Exercise 7.5.5 was sharp, but in this case it is not.

# 8
# Brownian Motion

Brownian motion is a process of tremendous practical and theoretical significance. It originated (a) as a model of the phenomenon observed by Robert Brown in 1828 that "pollen grains suspended in water perform a continual swarming motion," and (b) in Bachelier's (1900) work as a model of the stock market. These are just two of many systems that Brownian motion has been used to model. On the theoretical side, Brownian motion is a Gaussian Markov process with stationary independent increments. It lies in the intersection of three important classes of processes and is a fundamental example in each theory.

The first part of this chapter develops properties of Brownian motion. In Section 8.1, we define Brownian motion and investigate continuity properties of its paths. In Section 8.2, we prove the Markov property and a related 0-1 law. In Section 8.3, we define stopping times and prove the strong Markov property. In Section 8.4, we take a close look at the zero set of Brownian motion. In Section 8.5, we introduce some martingales associated with Brownian motion and use them to obtain information about its properties.

The second part of this chapter applies Brownian motion to some of the problems considered in Chapters 2 and 3. In Section 8.6, we embed random walks into Brownian motion to prove Donsker's theorem, a far-reaching generalization of the central limit theorem. In Section 8.7, we show that the discrepancy between the empirical distribution and the true distribution when suitably magnified converges to Brownian bridge. In Section 8.8, we prove laws of the iterated logarithm for Brownian motion and random walks with finite variance.

## 8.1 Definition and Construction

A one-dimensional **Brownian motion** is a real-valued process $B_t$, $t \geq 0$ that has the following properties:

(a) If $t_0 < t_1 < \cdots < t_n$, then $B(t_0)$, $B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ are independent.

(b) If $s, t \geq 0$, then

$$P(B(s+t) - B(s) \in A) = \int_A (2\pi t)^{-1/2} \exp(-x^2/2t)\, dx$$

(c) With probability 1, $t \to B_t$ is continuous.

(a) says that $B_t$ has independent increments. (b) says that the increment $B(s+t) - B(s)$ has a normal distribution with mean 0 and variance $t$. (c) is self-explanatory.

Thinking of Brown's pollen grain, (c) is certainly reasonable. (a) and (b) can be justified by noting that the movement of the pollen grain is due to the net effect of the bombardment of millions of water molecules, so by the central limit theorem, the displacement in any one interval should have a normal distribution, and the displacements in two disjoint intervals should be independent. Figure 8.1 shows a simulation of two dimentional Brownian motion.

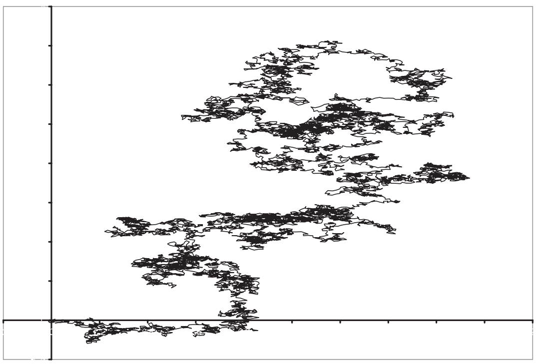


Figure 8.1. Simulation of two-dimensional Brownian motion.

Two immediate consequences of the definition that will be useful many times are:

**Translation invariance.** $\{B_t - B_0, t \geq 0\}$ is independent of $B_0$ and has the same distribution as a Brownian motion with $B_0 = 0$.

*Proof.* Let $\mathcal{A}_1 = \sigma(B_0)$ and $\mathcal{A}_2$ be the events of the form

$$\{B(t_1) - B(t_0) \in A_1, \dots, B(t_n) - B(t_{n-1}) \in A_n\}$$

The $\mathcal{A}_i$ are $\pi$-systems that are independent, so the desired result follows from the $\pi - \lambda$ theorem 2.1.2. ∎

**The Brownian scaling relation.** If $B_0 = 0$ then for any $t > 0$,

$$\{B_{st}, s \geq 0\} \stackrel{d}{=} \{t^{1/2} B_s, s \geq 0\} \tag{8.1.1}$$

To be precise, the two families of r.v.'s have the same finite dimensional distributions, that is, if $s_1 < \cdots < s_n$, then

$$(B_{s_1 t}, \dots, B_{s_n t}) \stackrel{d}{=} (t^{1/2} B_{s_1}, \dots t^{1/2} B_{s_n})$$

*Proof.* To check this when $n = 1$, we note that $t^{1/2}$ times a normal with mean 0 and variance $s$ is a normal with mean 0 and variance $st$. The result for $n > 1$ follows from independent increments. ∎

A second equivalent definition of Brownian motion starting from $B_0 = 0$, that we will occasionally find useful is that $B_t, t \geq 0$, is a real-valued process satisfying

(a′) $B(t)$ is a **Gaussian process** (i.e., all its finite dimensional distributions are multivariate normal).
(b′) $EB_s = 0$ and $EB_sB_t = s \wedge t$.
(c′) With probability one, $t \to B_t$ is continuous.

It is easy to see that (a) and (b) imply (a′). To get (b′) from (a) and (b), suppose $s < t$ and write

$$EB_sB_t = E(B_s^2) + E(B_s(B_t - B_s)) = s$$

The converse is even easier. (a′) and (b′) specify the finite dimensional distributions of $B_t$, which by the last calculation must agree with the ones defined in (a) and (b).

The first question that must be addressed in any treatment of Brownian motion is, "Is there a process with these properties?" The answer is "Yes," of course, or this chapter would not exist. For pedagogical reasons, we will pursue an approach that leads to a dead end and then retreat a little to rectify the difficulty. Fix an $x \in \mathbf{R}$ and for each $0 < t_1 < \cdots < t_n$, define a measure on $\mathbf{R}^n$ by

$$\mu_{x,t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) = \int_{A_1} dx_1 \cdots \int_{A_n} dx_n \prod_{m=1}^{n} p_{t_m - t_{m-1}}(x_{m-1}, x_m) \quad (8.1.2)$$

where $A_i \in \mathcal{R}$, $x_0 = x$, $t_0 = 0$, and

$$p_t(a, b) = (2\pi t)^{-1/2} \exp(-(b-a)^2/2t)$$

From the formula above, it is easy to see that for fixed $x$ the family $\mu$ is a consistent set of finite dimensional distributions (f.d.d.'s), that is, if $\{s_1, \ldots, s_{n-1}\} \subset \{t_1, \ldots, t_n\}$ and $t_j \notin \{s_1, \ldots, s_{n-1}\}$ then

$$\mu_{x,s_1,\ldots,s_{n-1}}(A_1 \times \cdots \times A_{n-1}) = \mu_{x,t_1,\ldots,t_n}(A_1 \times \cdots \times A_{j-1} \times \mathbf{R} \times A_j \times \cdots \times A_{n-1})$$

This is clear when $j = n$. To check the equality when $1 \leq j < n$, it is enough to show that

$$\int p_{t_j - t_{j-1}}(x, y) p_{t_{j+1} - t_j}(y, z)\, dy = p_{t_{j+1} - t_{j-1}}(x, z)$$

By translation invariance, we can without loss of generality assume $x = 0$, but all this says is that the sum of independent normals with mean 0 and variances $t_j - t_{j-1}$ and $t_{j+1} - t_j$ has a normal distribution with mean 0 and variance $t_{j+1} - t_{j-1}$.

With the consistency of f.d.d.'s verified, we get our first construction of Brownian motion:

**Theorem 8.1.1.** *Let $\Omega_o = \{$functions $\omega : [0, \infty) \to \mathbf{R}\}$ and $\mathcal{F}_o$ be the $\sigma$-field generated by the finite dimensional sets $\{\omega : \omega(t_i) \in A_i$ for $1 \le i \le n\}$, where $A_i \in \mathcal{R}$. For each $x \in \mathbf{R}$, there is a unique probability measure $\nu_x$ on $(\Omega_o, \mathcal{F}_o)$ so that $\nu_x\{\omega : \omega(0) = x\} = 1$ and when $0 < t_1 < \cdots < t_n$*

$$\nu_x\{\omega : \omega(t_i) \in A_i\} = \mu_{x,t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) \tag{8.1.3}$$

This follows from a generalization of Kolmogorov's extension theorem, (7.1) in the Appendix. We will not bother with the details since at this point we are at the dead end referred to above. If $C = \{\omega : t \to \omega(t)$ is continuous$\}$, then $C \notin \mathcal{F}_o$, that is, $C$ is not a measurable set. The easiest way of proving $C \notin \mathcal{F}_o$ is to do:

**Exercise 8.1.1.** $A \in \mathcal{F}_o$ if and only if there is a sequence of times $t_1, t_2, \ldots$ in $[0, \infty)$ and a $B \in \mathcal{R}^{\{1,2,\ldots\}}$ so that $A = \{\omega : (\omega(t_1), \omega(t_2), \ldots) \in B\}$. In words, all events in $\mathcal{F}_o$ depend on only countably many coordinates.

The above problem is easy to solve. Let $\mathbf{Q}_2 = \{m2^{-n} : m, n \ge 0\}$ be the **dyadic rationals**. If $\Omega_q = \{\omega : \mathbf{Q}_2 \to \mathbf{R}\}$ and $\mathcal{F}_q$ is the $\sigma$-field generated by the finite dimensional sets, then enumerating the rationals $q_1, q_2, \ldots$ and applying Kolmogorov's extension theorem shows that we can construct a probability $\nu_x$ on $(\Omega_q, \mathcal{F}_q)$ so that $\nu_x\{\omega : \omega(0) = x\} = 1$ and (8.1.3) holds when the $t_i \in \mathbf{Q}_2$. To extend $B_t$ to a process defined on $[0, \infty)$, we will show:

**Theorem 8.1.2.** *Let $T < \infty$ and $x \in \mathbf{R}$. $\nu_x$ assigns probability one to paths $\omega : \mathbf{Q}_2 \to \mathbf{R}$ that are uniformly continuous on $\mathbf{Q}_2 \cap [0, T]$.*

**Remark.** It will take quite a bit of work to prove Theorem 8.1.2. Before taking on that task, we will attend to the last measure theoretic detail: We tidy things up by moving our probability measures to $(C, \mathcal{C})$, where $C = \{$continuous $\omega : [0, \infty) \to \mathbf{R}\}$ and $\mathcal{C}$ is the $\sigma$-field generated by the coordinate maps $t \to \omega(t)$. To do this, we observe that the map $\psi$ that takes a uniformly continuous point in $\Omega_q$ to its unique continuous extension in $C$ is measurable, and we set

$$P_x = \nu_x \circ \psi^{-1}$$

Our construction guarantees that $B_t(\omega) = \omega_t$ has the right finite dimensional distributions for $t \in \mathbf{Q}_2$. Continuity of paths and a simple limiting argument show that this is true when $t \in [0, \infty)$. Finally, the reader should note that, as in the case of Markov chains, we have one set of random variables $B_t(\omega) = \omega(t)$, and a family of probability measures $P_x$, $x \in \mathbf{R}$, so that under $P_x$, $B_t$ is a Brownian motion with $P_x(B_0 = x) = 1$.

*Proof.* By translation invariance and scaling (8.1.1), we can without loss of generality suppose $B_0 = 0$ and prove the result for $T = 1$. In this case, part (b) of the definition and the scaling relation imply

$$E_0(|B_t - B_s|)^4 = E_0|B_{t-s}|^4 = C(t-s)^2$$

where $C = E_0|B_1|^4 < \infty$. From the last observation, we get the desired uniform continuity by using the following result due to Kolmogorov. Thanks to Robin Pemantle, the proof is now much simpler than in previous editions.

**Theorem 8.1.3.** *Suppose $E|X_s - X_t|^\beta \le K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. If $\gamma < \alpha/\beta$ then with probability 1 there is a constant $C(\omega)$ so that*

$$|X(q) - X(r)| \le C|q-r|^\gamma \quad \textit{for all } q, r \in \mathbf{Q}_2 \cap [0,1]$$

*Proof.* Let $G_n = \{|X(i/2^n) - X((i-1)/2^n)| \le 2^{-\gamma n} \text{ for all } 0 < i \le 2^n\}$. Chebyshev's inequality implies $P(|Y| > a) \le a^{-\beta} E|Y|^\beta$, so if we let $\lambda = \alpha - \beta\gamma > 0$, then

$$P(G_n^c) \le 2^n \cdot 2^{n\beta\gamma} \cdot E|X(j2^{-n}) - X(i2^{-n})|^\beta = K2^{-n\lambda}$$

**Lemma 8.1.4.** *On $H_N = \cap_{n=N}^\infty G_n$ we have*

$$|X(q) - X(r)| \le \frac{3}{1-2^{-\gamma}}|q-r|^\gamma$$

*for $q, r \in \mathbf{Q}_2 \cap [0,1]$ with $|q-r| < 2^{-N}$.*


*Proof of Lemma 8.1.4.* Let $q, r \in \mathbf{Q}_2 \cap [0,1]$ with $0 < r - q < 2^{-N}$. For some $m \ge N$ we can write

$$r = i2^{-m} + 2^{-r(1)} + \cdots + 2^{-r(\ell)}$$
$$q = (i-1)2^{-m} - 2^{-q(1)} - \cdots - 2^{-q(k)}$$

where $m < r(1) < \cdots < r(\ell)$ and $m < q(1) < \cdots < q(k)$. On $H_N$

$$|X(i2^{-m}) - X((i-1)2^{-m})| \le 2^{-\gamma m}$$

$$|X(q) - X((i-1)2^{-m})| \le \sum_{h=1}^{k} (2^{-q(h)})^\gamma \le \sum_{h=m}^{\infty} (2^{-\gamma})^h = \frac{2^{-\gamma m}}{1-2^{-\gamma}}$$

$$|X(r) - X(i2^{-m})| \le \frac{2^{-\gamma m}}{1-2^{-\gamma}}$$

Combining the last three inequalities with $2^{-m} \le |q-r|$ and $1 - 2^{-\gamma} > 1$ completes the proof of Lemma 8.1.4. ■

To prove Theorem 8.1.3 now, we note that

$$P(H_N^c) \le \sum_{n=N}^{\infty} P(G_n^c) \le K \sum_{n=N}^{\infty} 2^{-n\lambda} = K 2^{-N\lambda}/(1-2^{-\lambda})$$

Since $\sum_{N=1}^{\infty} P(H_N^c) < \infty$, the Borel-Cantelli lemma, Theorem 2.3.1, implies

$$|X(q) - X(r)| \le A|q-r|^{\gamma} \quad \text{for } q, r \in \mathbf{Q}_2 \text{ with } |q-r| < \delta(\omega).$$

To extend this to $q, r \in \mathbf{Q}_2 \cap [0,1]$, let $s_0 = q < s_1 < \cdots < s_n = r$ with $|s_i - s_{i-1}| < \delta(\omega)$, and use the triangle inequality to conclude $|X(q) - X(r)| \le C(\omega)|q-r|^{\gamma}$ where $C(\omega) = 1 + \delta(\omega)^{-1}$. ■

The scaling relation, (8.1.1), implies

$$E|B_t - B_s|^{2m} = C_m |t-s|^m \quad \text{where } C_m = E|B_1|^{2m}$$

so using Theorem 8.1.3 with $\beta = 2m, \alpha = m - 1$ and letting $m \to \infty$ gives a result of Wiener (1923).

**Theorem 8.1.5.** *Brownian paths are Hölder continuous for any exponent* $\gamma < 1/2$.

It is easy to show:

**Theorem 8.1.6.** *With probability one, Brownian paths are not Lipschitz continuous (and hence not differentiable) at any point.*

**Remark.** The nondifferentiability of Brownian paths was discovered by Paley, Wiener, and Zygmund (1933). Paley died in 1933 at the age of 26 in a skiing accident while the paper was in press. The proof we are about to give is due to Dvoretsky, Erdös, and Kakutani (1961).

*Proof.* Fix a constant $C < \infty$ and let $A_n = \{\omega :$ there is an $s \in [0,1]$ so that $|B_t - B_s| \le C|t-s|$ when $|t-s| \le 3/n\}$. For $1 \le k \le n-2$, let

$$Y_{k,n} = \max\left\{ \left| B\left(\frac{k+j}{n}\right) - B\left(\frac{k+j-1}{n}\right) \right| : j = 0, 1, 2 \right\}$$

$$B_n = \{ \text{ at least one } Y_{k,n} \le 5C/n \}$$

The triangle inequality implies $A_n \subset B_n$. The worst case is $s = 1$. We pick $k = n-2$ and observe

$$\begin{aligned}\left| B\left(\frac{n-3}{n}\right) - B\left(\frac{n-2}{n}\right) \right| &\le \left| B\left(\frac{n-3}{n}\right) - B(1) \right| + \left| B(1) - B\left(\frac{n-2}{n}\right) \right| \\ &\le C(3/n + 2/n)\end{aligned}$$

Using $A_n \subset B_n$ and the scaling relation (8.1.1) now gives

$$P(A_n) \le P(B_n) \le nP(|B(1/n)| \le 5C/n)^3 = nP(|B(1)| \le 5C/n^{1/2})^3$$
$$\le n\{(10C/n^{1/2}) \cdot (2\pi)^{-1/2}\}^3$$

since $\exp(-x^2/2) \le 1$. Letting $n \to \infty$ shows $P(A_n) \to 0$. Noticing $n \to A_n$ is increasing shows $P(A_n) = 0$ for all $n$ and completes the proof. ■

**Exercise 8.1.2.** Looking at the proof of Theorem 8.1.6 carefully shows that if $\gamma > 5/6$ then $B_t$ is not Hölder continuous with exponent $\gamma$ at any point in [0,1]. Show, by considering $k$ increments instead of 3, that the last conclusion is true for all $\gamma > 1/2 + 1/k$.

The next result is more evidence that the sample paths of Brownian motion behave locally like $\sqrt{t}$.

**Exercise 8.1.3.** Fix $t$ and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$. Compute

$$E\left(\sum_{m \le 2^n} \Delta_{m,n}^2 - t\right)^2$$

and use Borel-Cantelli to conclude that $\sum_{m \le 2^n} \Delta_{m,n}^2 \to t$ a.s. as $n \to \infty$.

**Remark.** The last result is true if we consider a sequence of partitions $\Pi_1 \subset \Pi_2 \subset \ldots$ with mesh $\to 0$. See Freedman (1971a), pp. 42–46. However, the true quadratic variation, defined as the sup over all partitions, is $\infty$.

*Multidimensional Brownian motion*

All of the result in this section have been for one-dimensional Brownian motion. To define a $d$-dimensional Brownian motion starting at $x \in \mathbf{R}^d$, we let $B_t^1, \ldots B_t^d$ be independent Brownian motions with $B_0^i = x_i$. As in the case $d = 1$, these are realized as probability measures $P_x$ on $(C, \mathcal{C})$ where $C = \{\text{continuous } \omega : [0, \infty) \to \mathbf{R}^d\}$ and $\mathcal{C}$ is the $\sigma$-field generated by the coordinate maps. Since the coordinates are independent, it is easy to see that the finite dimensional distributions satisfy (8.1.2) with transition probability

$$p_t(x, y) = (2\pi t)^{-d/2} \exp(-|y - x|^2/2t) \tag{8.1.4}$$

## 8.2 Markov Property, Blumenthal's 0-1 Law

Intuitively, the Markov property says, "If $s \ge 0$ then $B(t+s) - B(s)$, $t \ge 0$ is a Brownian motion that is independent of what happened before time $s$." The first step in making this into a precise statement is to explain what we mean by "what happened before time $s$." The first thing that comes to mind is

$$\mathcal{F}_s^o = \sigma(B_r : r \le s)$$

For reasons that will become clear as we go along, it is convenient to replace $\mathcal{F}_s^o$ by

$$\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t^o$$

The fields $\mathcal{F}_s^+$ are nicer because they are **right continuous**:

$$\cap_{t>s} \mathcal{F}_t^+ = \cap_{t>s} \left( \cap_{u>t} \mathcal{F}_u^o \right) = \cap_{u>s} \mathcal{F}_u^o = \mathcal{F}_s^+$$

In words, the $\mathcal{F}_s^+$ allow us an "infinitesimal peek at the future," that is, $A \in \mathcal{F}_s^+$ if it is in $\mathcal{F}_{s+\epsilon}^o$ for any $\epsilon > 0$. If $f(u) > 0$ for all $u > 0$, then in $d = 1$ the random variable

$$\limsup_{t \downarrow s} \frac{B_t - B_s}{f(t-s)}$$

is measurable with respect to $\mathcal{F}_s^+$ but not $\mathcal{F}_s^o$. We will see below that there are no interesting examples, that is, $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ are the same (up to null sets).

To state the Markov property, we need some notation. Recall that we have a family of measures $P_x$, $x \in \mathbf{R}^d$, on $(C, \mathcal{C})$ so that under $P_x$, $B_t(\omega) = \omega(t)$ is a Brownian motion starting at $x$. For $s \ge 0$, we define the **shift transformation** $\theta_s : C \to C$ by

$$(\theta_s \omega)(t) = \omega(s+t) \quad \text{for } t \ge 0$$

In words, we cut off the part of the path before time $s$ and then shift the path so that time $s$ becomes time 0.

**Theorem 8.2.1. Markov property.** *If $s \ge 0$ and $Y$ is bounded and $\mathcal{C}$ measurable, then for all $x \in \mathbf{R}^d$*

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B_s} Y$$

*where the right-hand side is the function $\varphi(x) = E_x Y$ evaluated at $x = B_s$.*

*Proof.* By the definition of conditional expectation, what we need to show is that

$$E_x(Y \circ \theta_s; A) = E_x(E_{B_s} Y; A) \quad \text{for all } A \in \mathcal{F}_s^+ \tag{8.2.1}$$

We will begin by proving the result for a carefully chosen special case and then use the monotone class theorem (MCT) to get the general case. Suppose $Y(\omega) = \prod_{1 \le m \le n} f_m(\omega(t_m))$, where $0 < t_1 < \cdots < t_n$ and the $f_m$ are bounded and measurable. Let $0 < h < t_1$, let $0 < s_1 \cdots < s_k \le s + h$, and let $A = \{\omega : \omega(s_j) \in A_j, 1 \le j \le k\}$, where $A_j \in \mathcal{R}$ for $1 \le j \le k$. From the definition of Brownian motion, it follows that

$$E_x(Y \circ \theta_s; A) = \int_{A_1} dx_1\, p_{s_1}(x, x_1) \int_{A_2} dx_2\, p_{s_2 - s_1}(x_1, x_2) \cdots$$
$$\int_{A_k} dx_k\, p_{s_k - s_{k-1}}(x_{k-1}, x_k) \int dy\, p_{s+h-s_k}(x_k, y)\varphi(y, h)$$

where

$$\varphi(y, h) = \int dy_1\, p_{t_1 - h}(y, y_1) f_1(y_1) \ldots \int dy_n\, p_{t_n - t_{n-1}}(y_{n-1}, y_n) f_n(y_n)$$

For more details, see the proof of (6.1.3), which applies without change here. Using that identity on the right-hand side, we have

$$E_x(Y \circ \theta_s; A) = E_x(\varphi(B_{s+h}, h); A) \tag{8.2.2}$$

The last equality holds for all finite dimensional sets $A$, so the $\pi - \lambda$ theorem, Theorem 2.1.2, implies that it is valid for all $A \in \mathcal{F}^o_{s+h} \supset \mathcal{F}^+_s$.

It is easy to see by induction on $n$ that

$$\psi(y_1) = f_1(y_1) \int dy_2\, p_{t_2 - t_1}(y_1, y_2) f_2(y_2)$$
$$\ldots \int dy_n\, p_{t_n - t_{n-1}}(y_{n-1}, y_n) f_n(y_n)$$

is bounded and measurable. Letting $h \downarrow 0$ and using the dominated convergence theorem shows that if $x_h \to x$, then

$$\phi(x_h, h) = \int dy_1\, p_{t_1 - h}(x_h, y_1)\psi(y_1) \to \phi(x, 0)$$

as $h \downarrow 0$. Using (8.2.2) and the bounded convergence theorem now gives

$$E_x(Y \circ \theta_s; A) = E_x(\varphi(B_s, 0); A)$$

for all $A \in \mathcal{F}^+_s$. This shows that (8.2.1) holds for $Y = \prod_{1 \le m \le n} f_m(\omega(t_m))$ and the $f_m$ are bounded and measurable.

The desired conclusion now follows from the monotone class theorem, 6.1.3. Let $\mathcal{H}$ = the collection of bounded functions for which (8.2.1) holds. $\mathcal{H}$ clearly has properties (ii) and (iii). Let $\mathcal{A}$ be the collection of sets of the form $\{\omega : \omega(t_j) \in A_j\}$, where $A_j \in \mathcal{R}$. The special case treated above shows (i) holds and the desired conclusion follows. ■

The next two exercises give typical applications of the Markov property. In Section 8.4, we will use these equalities to compute the distributions of $L$ and $R$.

**Exercise 8.2.1.** Let $T_0 = \inf\{s > 0 : B_s = 0\}$ and let $R = \inf\{t > 1 : B_t = 0\}$. $R$ is for right or return. Use the Markov property at time 1 to get

$$P_x(R > 1 + t) = \int p_1(x, y) P_y(T_0 > t)\, dy \tag{8.2.3}$$

**Exercise 8.2.2.** Let $T_0 = \inf\{s > 0 : B_s = 0\}$ and let $L = \sup\{t \le 1 : B_t = 0\}$. $L$ is for left or last. Use the Markov property at time $0 < t < 1$ to conclude

$$P_0(L \le t) = \int p_t(0, y) P_y(T_0 > 1 - t)\, dy \tag{8.2.4}$$

The reader will see many applications of the Markov property below, so we turn our attention now to a "triviality" that has surprising consequences. Since

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_{B(s)} Y \in \mathcal{F}_s^o$$

it follows from Theorem 5.1.5 that

$$E_x(Y \circ \theta_s | \mathcal{F}_s^+) = E_x(Y \circ \theta_s | \mathcal{F}_s^o)$$

From the last equation, it is a short step to:

**Theorem 8.2.2.** *If $Z \in \mathcal{C}$ is bounded then for all $s \ge 0$ and $x \in \mathbf{R}^d$,*

$$E_x(Z | \mathcal{F}_s^+) = E_x(Z | \mathcal{F}_s^o)$$

*Proof.* As in the proof of Theorem 8.2.1, it suffices to prove the result when

$$Z = \prod_{m=1}^{n} f_m(B(t_m))$$

and the $f_m$ are bounded and measurable. In this case, $Z$ can be written as $X(Y \circ \theta_s)$, where $X \in \mathcal{F}_s^o$ and $Y$ is $\mathcal{C}$ measurable, so

$$E_x(Z | \mathcal{F}_s^+) = X E_x(Y \circ \theta_s | \mathcal{F}_s^+) = X E_{B_s} Y \in \mathcal{F}_s^o$$

and the proof is complete. ∎

If we let $Z \in \mathcal{F}_s^+$, then Theorem 8.2.2 implies $Z = E_x(Z | \mathcal{F}_s^o) \in \mathcal{F}_s^o$, so the two $\sigma$-fields are the same up to null sets. At first glance, this conclusion is not exciting. The fun starts when we take $s = 0$ in Theorem 8.2.2 to get:

**Theorem 8.2.3. Blumenthal's 0-1 law.** *If $A \in \mathcal{F}_0^+$ then for all $x \in \mathbf{R}^d$,*

$$P_x(A) \in \{0, 1\}.$$

*Proof.* Using $A \in \mathcal{F}_0^+$, Theorem 8.2.2, and $\mathcal{F}_0^o = \sigma(B_0)$ is trivial under $P_x$ gives

$$1_A = E_x(1_A | \mathcal{F}_0^+) = E_x(1_A | \mathcal{F}_0^o) = P_x(A) \quad P_x \text{ a.s.}$$

This shows that the indicator function $1_A$ is a.s. equal to the number $P_x(A)$, and the result follows. ∎

In words, the last result says that the **germ field**, $\mathcal{F}_0^+$, is trivial. This result is very useful in studying the local behavior of Brownian paths. For the rest of the section we restrict our attention to $d = 1$.

**Theorem 8.2.4.** *If* $\tau = \inf\{t \geq 0 : B_t > 0\}$ *then* $P_0(\tau = 0) = 1$.

*Proof.* $P_0(\tau \leq t) \geq P_0(B_t > 0) = 1/2$ since the normal distribution is symmetric about 0. Letting $t \downarrow 0$, we conclude

$$P_0(\tau = 0) = \lim_{t \downarrow 0} P_0(\tau \leq t) \geq 1/2$$

so it follows from Theorem 8.2.3 that $P_0(\tau = 0) = 1$. ■

Once Brownian motion must hit $(0, \infty)$ immediately starting from 0, it must also hit $(-\infty, 0)$ immediately. Since $t \to B_t$ is continuous, this forces:

**Theorem 8.2.5.** *If* $T_0 = \inf\{t > 0 : B_t = 0\}$ *then* $P_0(T_0 = 0) = 1$.

A corollary of Theorem 8.2.5 is:

**Exercise 8.2.3.** If $a < b$, then with probability 1 there is a local maximum of $B_t$ in $(a, b)$. So the set of local maxima of $B_t$ is almost surely a dense set.

Another typical application of Theorem 8.2.3 is:

**Exercise 8.2.4.** (i) Suppose $f(t) > 0$ for all $t > 0$. Use Theorem 8.2.3 to conclude that $\limsup_{t \downarrow 0} B(t)/f(t) = c$, $P_0$ a.s., where $c \in [0, \infty]$ is a constant. (ii) Show that if $f(t) = \sqrt{t}$ then $c = \infty$, so with probability 1, Brownian paths are not Hölder continuous of order 1/2 at 0.

**Remark.** Let $\mathcal{H}_\gamma(\omega)$ be the set of times at which the path $\omega \in C$ is Hölder continuous of order $\gamma$. Theorem 8.1.5 shows that $P(\mathcal{H}_\gamma = [0, \infty)) = 1$ for $\gamma < 1/2$. Exercise 8.1.2 shows that $P(\mathcal{H}_\gamma = \emptyset) = 1$ for $\gamma > 1/2$. The last exercise shows $P(t \in \mathcal{H}_{1/2}) = 0$ for each $t$, but B. Davis (1983) has shown $P(\mathcal{H}_{1/2} \neq \emptyset) = 1$. Perkins (1983) has computed the Hausdorff dimension of

$$\left\{ t \in (0, 1) : \limsup_{h \downarrow 0} \frac{|B_{t+h} - B_t|}{h^{1/2}} \leq c \right\}$$

Theorem 8.2.3 concerns the behavior of $B_t$ as $t \to 0$. By using a trick, we can use this result to get information about the behavior as $t \to \infty$.

**Theorem 8.2.6.** *If $B_t$ is a Brownian motion starting at 0, then so is the process defined by* $X_0 = 0$ *and* $X_t = tB(1/t)$ *for* $t > 0$.

*Proof.* Here we will check the second definition of Brownian motion. To do this, we note: (i) If $0 < t_1 < \ldots < t_n$, then $(X(t_1), \ldots, X(t_n))$ has a multivariate normal distribution with mean 0. (ii) $EX_s = 0$ and if $s < t$ then

$$E(X_s X_t) = st E(B(1/s)B(1/t)) = s$$

For (iii) we note that $X$ is clearly continuous at $t \neq 0$.

To handle $t = 0$, we begin by observing that the strong law of large numbers implies $B_n/n \to 0$ as $n \to \infty$ through the integers. To handle values in between integers, we note that Kolmogorov's inequality, Theorem 2.5.2, implies

$$P\left(\sup_{0<k\le 2^m} |B(n+k2^{-m}) - B_n| > n^{2/3}\right) \le n^{-4/3} E(B_{n+1} - B_n)^2$$

Letting $m \to \infty$, we have

$$P\left(\sup_{u\in[n,n+1]} |B_u - B_n| > n^{2/3}\right) \le n^{-4/3}$$

Since $\sum_n n^{-4/3} < \infty$, the Borel-Cantelli lemma implies $B_u/u \to 0$ as $u \to \infty$. Taking $u = 1/t$, we have $X_t \to 0$ as $t \to 0$. ∎

Theorem 8.2.6 allows us to relate the behavior of $B_t$ as $t \to \infty$ and as $t \to 0$. Combining this idea with Blumenthal's 0-1 law leads to a very useful result. Let

$$\mathcal{F}'_t = \sigma(B_s : s \ge t) = \text{ the future at time } t$$
$$\mathcal{T} = \cap_{t\ge 0}\mathcal{F}'_t = \text{ the tail } \sigma\text{-field}$$

**Theorem 8.2.7.** *If $A \in \mathcal{T}$ then either $P_x(A) \equiv 0$ or $P_x(A) \equiv 1$.*

**Remark.** Notice that this is stronger than the conclusion of Blumenthal's 0-1 law. The examples $A = \{\omega : \omega(0) \in D\}$ show that for $A$ in the germ $\sigma$-field $\mathcal{F}_0^+$, the value of $P_x(A)$, $1_D(x)$ in this case, may depend on $x$.

*Proof.* Since the tail $\sigma$-field of $B$ is the same as the germ $\sigma$-field for $X$, it follows that $P_0(A) \in \{0, 1\}$. To improve this to the conclusion given, observe that $A \in \mathcal{F}'_1$, so $1_A$ can be written as $1_D \circ \theta_1$. Applying the Markov property gives

$$P_x(A) = E_x(1_D \circ \theta_1) = E_x(E_x(1_D \circ \theta_1|\mathcal{F}_1)) = E_x(E_{B_1} 1_D)$$
$$= \int (2\pi)^{-1/2} \exp(-(y-x)^2/2) P_y(D)\, dy$$

Taking $x = 0$, we see that if $P_0(A) = 0$, then $P_y(D) = 0$ for a.e. $y$ with respect to Lebesgue measure, and using the formula again shows $P_x(A) = 0$ for all $x$. To handle the case $P_0(A) = 1$, observe that $A^c \in \mathcal{T}$ and $P_0(A^c) = 0$, so the last result implies $P_x(A^c) = 0$ for all $x$. ∎

The next result is a typical application of Theorem 8.2.7.

**Theorem 8.2.8.** *Let $B_t$ be a one-dimensional Brownian motion starting at 0. Then with probability 1,*

$$\limsup_{t\to\infty} B_t/\sqrt{t} = \infty \qquad \liminf_{t\to\infty} B_t/\sqrt{t} = -\infty$$

*Proof.* Let $K < \infty$. By Exercise 2.3.1 and scaling

$$P_0(B_n/\sqrt{n} \geq K \text{ i.o.}) \geq \limsup_{n\to\infty} P_0(B_n \geq K\sqrt{n}) = P_0(B_1 \geq K) > 0$$

so the 0-1 law in Theorem 8.2.7 implies that the probability is 1. Since $K$ is arbitrary, this proves the first result. The second one follows from symmetry. ■

From Theorem 8.2.8, translation invariance, and the continuity of Brownian paths it follows that we have:

**Theorem 8.2.9.** *Let $B_t$ be a one-dimensional Brownian motion and let $A = \cap_n\{B_t = 0$ for some $t \geq n\}$. Then $P_x(A) = 1$ for all $x$.*

In words, one-dimensional Brownian motion is recurrent. For any starting point $x$, it will return to 0 "infinitely often," that is, there is a sequence of times $t_n \uparrow \infty$ so that $B_{t_n} = 0$. We have to be careful with the interpretation of the phrase in quotes since, starting from 0, $B_t$ will hit 0 infinitely many times by time $\epsilon > 0$.

**Last rites.** With our discussion of Blumenthal's 0-1 law complete, the distinction between $\mathcal{F}_s^+$ and $\mathcal{F}_s^o$ is no longer important, so we will make one final improvement in our $\sigma$-fields and remove the superscripts. Let

$$\begin{aligned}
\mathcal{N}_x &= \{A : A \subset D \text{ with } P_x(D) = 0\} \\
\mathcal{F}_s^x &= \sigma(\mathcal{F}_s^+ \cup \mathcal{N}_x) \\
\mathcal{F}_s &= \cap_x \mathcal{F}_s^x
\end{aligned}$$

$\mathcal{N}_x$ are the **null sets** and $\mathcal{F}_s^x$ are the completed $\sigma$-fields for $P_x$. Since we do not want the filtration to depend on the initial state, we take the intersection of all the $\sigma$-fields. The reader should note that it follows from the definition that the $\mathcal{F}_s$ are right-continuous.

## 8.3 Stopping Times, Strong Markov Property

Generalizing a definition in Section 4.1, we call a random variable $S$ taking values in $[0, \infty]$ a **stopping time** if for all $t \geq 0$, $\{S < t\} \in \mathcal{F}_t$. In the last definition, we have obviously made a choice between $\{S < t\}$ and $\{S \leq t\}$. This makes a big difference in discrete time but none in continuous time (for a right continuous filtration $\mathcal{F}_t$) :

If $\{S \leq t\} \in \mathcal{F}_t$ then $\{S < t\} = \cup_n\{S \leq t - 1/n\} \in \mathcal{F}_t$.

If $\{S < t\} \in \mathcal{F}_t$ then $\{S \leq t\} = \cap_n\{S < t + 1/n\} \in \mathcal{F}_t$.

The first conclusion requires only that $t \to \mathcal{F}_t$ is increasing. The second relies on the fact that $t \to \mathcal{F}_t$ is right continuous. Theorem 8.3.2 and 8.3.3 below show that when checking something is a stopping time, it is nice to know that the two definitions are equivalent.

**Theorem 8.3.1.** *If $G$ is an open set and $T = \inf\{t \geq 0 : B_t \in G\}$, then $T$ is a stopping time.*

*Proof.* Since $G$ is open and $t \to B_t$ is continuous, $\{T < t\} = \cup_{q<t}\{B_q \in G\}$, where the union is over all rational $q$, so $\{T < t\} \in \mathcal{F}_t$. Here we need to use the rationals to get a countable union, and hence a measurable set. ∎

**Theorem 8.3.2.** *If $T_n$ is a sequence of stopping times and $T_n \downarrow T$, then $T$ is a stopping time.*

*Proof.* $\{T < t\} = \cup_n \{T_n < t\}$. ∎

**Theorem 8.3.3.** *If $T_n$ is a sequence of stopping times and $T_n \uparrow T$, then $T$ is a stopping time.*

*Proof.* $\{T \leq t\} = \cap_n \{T_n \leq t\}$. ∎

**Theorem 8.3.4.** *If $K$ is a closed set and $T = \inf\{t \geq 0 : B_t \in K\}$, then $T$ is a stopping time.*

*Proof.* Let $B(x, r) = \{y : |y - x| < r\}$, let $G_n = \cup_{x \in K} B(x, 1/n)$, and let $T_n = \inf\{t \geq 0 : B_t \in G_n\}$. Since $G_n$ is open, it follows from Theorem 8.3.1 that $T_n$ is a stopping time. I claim that as $n \uparrow \infty$, $T_n \uparrow T$. To prove this, notice that $T \geq T_n$ for all $n$, so $\lim T_n \leq T$. To prove $T \leq \lim T_n$, we can suppose that $T_n \uparrow t < \infty$. Since $B(T_n) \in \bar{G}_n$ for all $n$ and $B(T_n) \to B(t)$, it follows that $B(t) \in K$ and $T \leq t$. ∎

**Exercise 8.3.1.** Let $S$ be a stopping time and let $S_n = ([2^n S] + 1)/2^n$ where $[x] =$ the largest integer $\leq x$. That is,

$$S_n = (m + 1)2^{-n} \text{ if } m2^{-n} \leq S < (m + 1)2^{-n}$$

In words, we stop at the first time of the form $k2^{-n}$ after $S$ (i.e., $> S$). From the verbal description, it should be clear that $S_n$ is a stopping time. Prove that it is.

**Exercise 8.3.2.** If $S$ and $T$ are stopping times, then $S \wedge T = \min\{S, T\}$, $S \vee T = \max\{S, T\}$, and $S + T$ are also stopping times. In particular, if $t \geq 0$, then $S \wedge t$, $S \vee t$, and $S + t$ are stopping times.

**Exercise 8.3.3.** Let $T_n$ be a sequence of stopping times. Show that

$$\sup_n T_n, \quad \inf_n T_n, \quad \limsup_n T_n, \quad \liminf_n T_n$$

are stopping times.

Theorems 8.3.4 and 8.3.1 will take care of all the hitting times we will consider. Our next goal is to state and prove the strong Markov property. To do this, we need to generalize two definitions from Section 4.1. Given a nonnegative random variable $S(\omega)$ we define the random shift $\theta_S$, which "cuts off the part of $\omega$ before $S(\omega)$ and then shifts the path so that time $S(\omega)$ becomes time 0." In symbols, we set

$$(\theta_S\omega)(t) = \begin{cases} \omega(S(\omega)+t) & \text{on } \{S < \infty\} \\ \Delta & \text{on } \{S = \infty\} \end{cases}$$

where $\Delta$ is an extra point we add to $C$. As in Section 6.3, we will usually explicitly restrict our attention to $\{S < \infty\}$, so the reader does not have to worry about the second half of the definition.

The second quantity $\mathcal{F}_S$, "the information known at time $S$," is a little more subtle. Imitating the discrete time definition from Section 4.1, we let

$$\mathcal{F}_S = \{A : A \cap \{S \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0\}$$

In words, this makes the reasonable demand that the part of $A$ that lies in $\{S \le t\}$ should be measurable with respect to the information available at time $t$. Again we have made a choice between $\le t$ and $< t$, but as in the case of stopping times, this makes no difference, and it is useful to know that the two definitions are equivalent.

**Exercise 8.3.4.** Show that when $\mathcal{F}_t$ is right continuous, the last definition is unchanged if we replace $\{S \le t\}$ by $\{S < t\}$.

For practice with the definition of $\mathcal{F}_S$, do:

**Exercise 8.3.5.** Let $S$ be a stopping time, let $A \in \mathcal{F}_S$, and let $R = S$ on $A$ and $R = \infty$ on $A^c$. Show that $R$ is a stopping time.

**Exercise 8.3.6.** Let $S$ and $T$ be stopping times.
(i) $\{S < t\}$, $\{S > t\}$, $\{S = t\}$ are in $\mathcal{F}_S$.
(ii) $\{S < T\}$, $\{S > T\}$, and $\{S = T\}$ are in $\mathcal{F}_S$ (and in $\mathcal{F}_T$).

Most of the properties of $\mathcal{F}_N$ derived in Section 4.1 carry over to continuous time. The next two will be useful below. The first is intuitively obvious: at a later time we have more information.

**Theorem 8.3.5.** *If $S \le T$ are stopping times, then $\mathcal{F}_S \subset \mathcal{F}_T$.*

*Proof.* If $A \in \mathcal{F}_S$, then $A \cap \{T \le t\} = (A \cap \{S \le t\}) \cap \{T \le t\} \in \mathcal{F}_t$. ∎

**Theorem 8.3.6.** *If $T_n \downarrow T$ are stopping times, then $\mathcal{F}_T = \cap \mathcal{F}(T_n)$.*

*Proof.* Theorem 8.3.5 implies $\mathcal{F}(T_n) \supset \mathcal{F}_T$ for all $n$. To prove the other inclusion, let $A \in \cap \mathcal{F}(T_n)$. Since $A \cap \{T_n < t\} \in \mathcal{F}_t$ and $T_n \downarrow T$, it follows that $A \cap \{T < t\} \in \mathcal{F}_t$. ■

The last result allows you to prove something that is obvious from the verbal definition.

**Exercise 8.3.7.** $B_S \in \mathcal{F}_S$, that is, the value of $B_S$ is measurable with respect to the information known at time $S$! To prove this, let $S_n = ([2^n S] + 1)/2^n$ be the stopping times defined in Exercise 8.3.1. Show $B(S_n) \in \mathcal{F}_{S_n}$, then let $n \to \infty$ and use Theorem 8.3.6.

We are now ready to state the strong Markov property, which says that the Markov property holds at stopping times. It is interesting that the notion of Brownian motion dates to the the very beginning of the 20th century, but the first proofs of the strong Markov property were given independently by Hunt (1956) and Dynkin and Yushkevich (1956). Hunt writes, “Although mathematicians use this extended Markoff property, at least as a heuristic principle, I have nowhere found it discussed with rigor.”

**Theorem 8.3.7. Strong Markov property.** *Let $(s, \omega) \to Y_s(\omega)$ be bounded and $\mathcal{R} \times \mathcal{C}$ measurable. If $S$ is a stopping time, then for all $x \in \mathbf{R}^d$*

$$E_x(Y_S \circ \theta_S | \mathcal{F}_S) = E_{B(S)} Y_S \text{ on } \{S < \infty\}$$

*where the right-hand side is the function $\varphi(x, t) = E_x Y_t$ evaluated at $x = B(S)$, $t = S$.*

**Remark.** The only facts about Brownian motion used here are that (i) it is a Markov process, and (ii) if $f$ is bounded and continuous then $x \to E_x f(B_t)$ is continuous. In Markov process theory, (ii) is called the Feller property. While Hunt’s proof only applies to Brownian motion, Dynkin and Yushkevich proved the result in this generality.

*Proof.* We first prove the result under the assumption that there is a sequence of times $t_n \uparrow \infty$, so that $P_x(S < \infty) = \sum P_x(S = t_n)$. In this case, the proof is basically the same as the proof of Theorem 6.3.4. We break things down according to the value of $S$, apply the Markov property, and put the pieces back together. If we let $Z_n = Y_{t_n}(\omega)$ and $A \in \mathcal{F}_S$, then

$$E_x(Y_S \circ \theta_S; A \cap \{S < \infty\}) = \sum_{n=1}^{\infty} E_x(Z_n \circ \theta_{t_n}; A \cap \{S = t_n\})$$

Now if $A \in \mathcal{F}_S$, $A \cap \{S = t_n\} = (A \cap \{S \le t_n\}) - (A \cap \{S \le t_{n-1}\}) \in \mathcal{F}_{t_n}$, so it follows from the Markov property that the above sum is

$$= \sum_{n=1}^{\infty} E_x(E_{B(t_n)} Z_n; A \cap \{S = t_n\}) = E_x(E_{B(S)} Y_S; A \cap \{S < \infty\})$$

To prove the result in general, we let $S_n = ([2^n S] + 1)/2^n$ be the stopping time defined in Exercise 8.3.1. To be able to let $n \to \infty$, we restrict our attention to $Y$'s of the form

$$Y_s(\omega) = f_0(s) \prod_{m=1}^{n} f_m(\omega(t_m)) \tag{8.3.1}$$

where $0 < t_1 < \cdots < t_n$ and $f_0, \ldots, f_n$ are bounded and continuous. If $f$ is bounded and continuous then the dominated convergence theorem implies that

$$x \to \int dy\, p_t(x, y) f(y)$$

is continuous. From this and induction, it follows that

$$\varphi(x, s) = E_x Y_s = f_0(s) \int dy_1\, p_{t_1}(x, y_1) f_1(y_1) \ldots \int dy_n\, p_{t_n - t_{n-1}}(y_{n-1}, y_n) f_n(y_n)$$

is bounded and continuous.

Having assembled the necessary ingredients, we can now complete the proof. Let $A \in \mathcal{F}_S$. Since $S \le S_n$, Theorem 8.3.5 implies $A \in \mathcal{F}(S_n)$. Applying the special case proved above to $S_n$ and observing that $\{S_n < \infty\} = \{S < \infty\}$ gives

$$E_x(Y_{S_n} \circ \theta_{S_n}; A \cap \{S < \infty\}) = E_x(\varphi(B(S_n), S_n); A \cap \{S < \infty\})$$

Now, as $n \to \infty$, $S_n \downarrow S$, $B(S_n) \to B(S)$, $\varphi(B(S_n), S_n) \to \varphi(B(S), S)$ and

$$Y_{S_n} \circ \theta_{S_n} \to Y_S \circ \theta_S$$

so the bounded convergence theorem implies that the result holds when $Y$ has the form given in (8.3.1).

To complete the proof now, we will apply the monotone class theorem. As in the proof of Theorem 8.2.1, we let $\mathcal{H}$ be the collection of $Y$ for which

$$E_x(Y_S \circ \theta_S; A) = E_x(E_{B(S)} Y_S; A) \quad \text{for all } A \in \mathcal{F}_S$$

and it is easy to see that (ii) and (iii) hold. This time, however, we take $\mathcal{A}$ to be the sets of the form $A = G_0 \times \{\omega : \omega(s_j) \in G_j, 1 \le j \le k\}$, where the $G_j$ are open sets. To verify (i), we note that if $K_j = G_j^c$ and $f_j^n(x) = 1 \wedge n\rho(x, K_j)$, where

$\rho(x, K) = \inf\{|x - y| : y \in K\}$ then $f_j^n$ are continuous functions with $f_j^n \uparrow 1_{G_j}$ as $n \uparrow \infty$. The facts that

$$Y_s^n(\omega) = f_0^n(s) \prod_{j=1}^{k} f_j^n(\omega(s_j)) \in \mathcal{H}$$

and (iii) holds for $\mathcal{H}$ imply that $1_A \in \mathcal{H}$. This verifies (i) in the monotone class theorem and completes the proof. ■

## 8.4 Path Properties

In this section, we will use the strong Markov property to derive properties of the zero set $\{t : B_t = 0\}$, the hitting times $T_a = \inf\{t : B_t = a\}$, and $\max_{0 \le s \le t} B_s$ for one-dimensional Brownian motion.

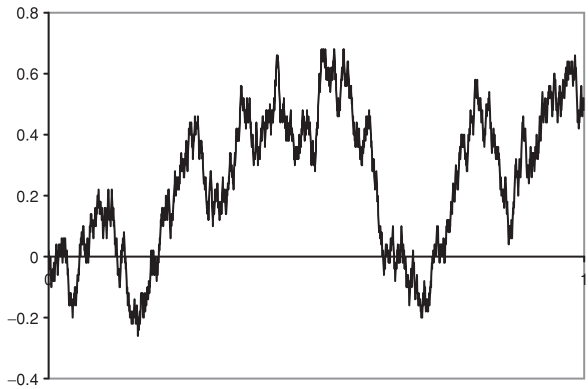



Figure 8.2. Simulation of one-dimensional Brownian motion.

### 8.4.1 *Zeros of Brownian Motion*

Let $R_t = \inf\{u > t : B_u = 0\}$ and let $T_0 = \inf\{u > 0 : B_u = 0\}$. Now Theorem 8.2.9 implies $P_x(R_t < \infty) = 1$, so $B(R_t) = 0$, and the strong Markov property and Theorem 8.2.5 imply

$$P_x(T_0 \circ \theta_{R_t} > 0 | \mathcal{F}_{R_t}) = P_0(T_0 > 0) = 0$$

Taking expected value of the last equation, we see that

$$P_x(T_0 \circ \theta_{R_t} > 0 \text{ for some rational } t) = 0$$

From this, it follows that if a point $u \in \mathcal{Z}(\omega) \equiv \{t : B_t(\omega) = 0\}$ is isolated on the left (i.e., there is a rational $t < u$ so that $(t, u) \cap \mathcal{Z}(\omega) = \emptyset$), then it is, with probability one, a decreasing limit of points in $\mathcal{Z}(\omega)$. This shows that the closed set $\mathcal{Z}(\omega)$ has no isolated points and hence must be uncountable. For the last step, see Hewitt and Stromberg (1965), p. 72.

If we let $|\mathcal{Z}(\omega)|$ denote the Lebesgue measure of $\mathcal{Z}(\omega)$, then Fubini's theorem implies

$$E_x(|\mathcal{Z}(\omega)| \cap [0,T]) = \int_0^T P_x(B_t = 0)\,dt = 0$$

So $\mathcal{Z}(\omega)$ is a set of measure zero.

The last four observations show that $\mathcal{Z}$ is like the Cantor set that is obtained by removing $(1/3, 2/3)$ from $[0, 1]$ and then repeatedly removing the middle third from the intervals that remain. The Cantor set is bigger however. Its Hausdorff dimension is $\log 2/\log 3$, whereas $\mathcal{Z}$ has dimension 1/2.

### 8.4.2 *Hitting Times*

**Theorem 8.4.1.** *Under $P_0$, $\{T_a, a \geq 0\}$ has stationary independent increments.*

*Proof.* The first step is to notice that if $0 < a < b$, then

$$T_b \circ \theta_{T_a} = T_b - T_a,$$

so if $f$ is bounded and measurable, the strong Markov property, 8.3.7, and translation invariance imply

$$\begin{aligned} E_0\left(f(T_b - T_a)\,\middle|\mathcal{F}_{T_a}\right) &= E_0\left(f(T_b)\circ\theta_{T_a}\,\middle|\mathcal{F}_{T_a}\right) \\ &= E_a f(T_b) = E_0 f(T_{b-a}) \end{aligned}$$

To show that the increments are independent, let $a_0 < a_1 \cdots < a_n$, let $f_i$, $1 \leq i \leq n$ be bounded and measurable, and let $F_i = f_i(T_{a_i} - T_{a_{i-1}})$. Conditioning on $\mathcal{F}_{T_{a_{n-1}}}$ and using the preceding calculation, we have

$$E_0\left(\prod_{i=1}^n F_i\right) = E_0\left(\prod_{i=1}^{n-1} F_i \cdot E_0(F_n|\mathcal{F}_{T_{a_{n-1}}})\right) = E_0\left(\prod_{i=1}^{n-1} F_i\right) E_0 F_n$$

By induction, it follows that $E_0 \prod_{i=1}^n F_i = \prod_{i=1}^n E_0 F_i$, which implies the desired conclusion. ∎

The scaling relation (8.1.1) implies

$$T_a \overset{d}{=} a^2 T_1 \tag{8.4.1}$$

Combining Theorem 8.4.1 and (8.4.1), we see that $t_k = T_k - T_{k-1}$ are i.i.d. and

$$\frac{t_1 + \cdots + t_n}{n^2} \to T_1$$

so using Theorem 3.7.4, we see that $T_a$ has a stable law. Since we are dividng by $n^2$ and $T_a \geq 0$, the index $\alpha = 1/2$ and the skewness parameter $\kappa = 1$; see (3.7.11).

Without knowing the theory mentioned in the previous paragraph, it is easy to determine the Laplace transform

$$\varphi_a(\lambda) = E_0 \exp(-\lambda T_a) \quad \text{for } a \geq 0$$

and reach the same conclusion. To do this, we start by observing that Theorem 8.4.1 implies

$$\varphi_x(\lambda)\varphi_y(\lambda) = \varphi_{x+y}(\lambda).$$

It follows easily from this that

$$\varphi_a(\lambda) = \exp(-ac(\lambda)) \tag{8.4.2}$$

*Proof.* Let $c(\lambda) = -\log \varphi_1(\lambda)$ so (8.4.2) holds when $a = 1$. Using the previous identity with $x = y = 2^{-m}$ and induction gives the result for $a = 2^{-m}$, $m \geq 1$. Then, letting $x = k2^{-m}$ and $y = 2^{-m}$, we get the result for $a = (k+1)2^{-m}$ with $k \geq 1$. Finally, to extend to $a \in [0, \infty)$, note that $a \to \phi_a(\lambda)$ is decreasing. ■

To identify $c(\lambda)$, we observe that (8.4.1) implies

$$E \exp(-T_a) = E \exp(-a^2 T_1)$$

so $ac(1) = c(a^2)$, i.e., $c(\lambda) = c(1)\sqrt{\lambda}$. Since all of our arguments also apply to $\sigma B_t$, we cannot hope to compute $c(1)$. Theorem 8.5.7 will show

$$E_0(\exp(-\lambda T_a)) = \exp(-a\sqrt{2\lambda}) \tag{8.4.3}$$

Our next goal is to compute the distribution of the hitting times $T_a$. This application of the strong Markov property shows why we want to allow the function $Y$ that we apply to the shifted path to depend on the stopping time $S$.

**Example 8.4.1. Reflection principle.** Let $a > 0$ and let $T_a = \inf\{t : B_t = a\}$. Then

$$P_0(T_a < t) = 2P_0(B_t \geq a) \tag{8.4.4}$$

*Intuitive proof.* We observe that if $B_s$ hits $a$ at some time $s < t$, then the strong Markov property implies that $B_t - B(T_a)$ is independent of what happened before time $T_a$. The symmetry of the normal distribution and $P_a(B_u = a) = 0$ for $u > 0$ then imply (see Figure 8.3 for a picture)

$$P_0(T_a < t, B_t > a) = \frac{1}{2} P_0(T_a < t) \tag{8.4.5}$$

Rearranging the last equation and using $\{B_t > a\} \subset \{T_a < t\}$ gives

$$P_0(T_a < t) = 2P_0(T_a < t, B_t > a) = 2P_0(B_t > a)$$

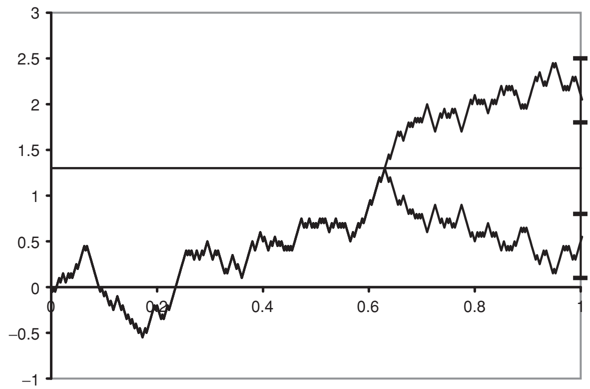



Figure 8.3. Proof by picture of the reflection principle.

*Proof.* To make the intuitive proof rigorous, we only have to prove (8.4.5). To extract this from the strong Markov property, Theorem 8.3.7, we let

$$Y_s(\omega) = \begin{cases} 1 & \text{if } s < t, \omega(t-s) > a \\ 0 & \text{otherwise} \end{cases}$$

We do this so that if we let $S = \inf\{s < t : B_s = a\}$ with $\inf \emptyset = \infty$, then

$$Y_S(\theta_S \omega) = \begin{cases} 1 & \text{if } S < t, B_t > a \\ 0 & \text{otherwise} \end{cases}$$

and the strong Markov property implies

$$E_0(Y_S \circ \theta_S | \mathcal{F}_S) = \varphi(B_S, S) \quad \text{on } \{S < \infty\} = \{T_a < t\}$$

where $\varphi(x, s) = E_x Y_s$. $B_S = a$ on $\{S < \infty\}$ and $\varphi(a, s) = 1/2$ if $s < t$, so taking expected values gives

$$\begin{aligned} P_0(T_a < t, B_t \geq a) &= E_0(Y_S \circ \theta_S; S < \infty) \\ &= E_0(E_0(Y_S \circ \theta_S | \mathcal{F}_S); S < \infty) = E_0(1/2; T_a < t) \end{aligned}$$

which proves (8.4.5). ■

**Exercise 8.4.1.** Generalize the proof of (8.4.5) to conclude that if $u < v \leq a$, then

$$P_0(T_a < t, u < B_t < v) = P_0(2a - v < B_t < 2a - u) \tag{8.4.6}$$

This should be obvious from the picture in Figure 8.3. Your task is to extract this from the strong Markov property.

Letting $(u, v)$ shrink down to $x$ in (8.4.6), we have for $a < x$

$$\begin{aligned} P_0(T_a < t, B_t = x) &= p_t(0, 2a - x) \\ P_0(T_a > t, B_t = x) &= p_t(0, x) - p_t(0, 2a - x) \end{aligned} \tag{8.4.7}$$

that is, the (subprobability) density for $B_t$ on the two indicated events. Since $\{T_a < t\} = \{M_t > a\}$, differentiating with respect to $a$ gives the joint density

$$f_{(M_t,B_t)}(a,x) = \frac{2(2a-x)}{\sqrt{2\pi t^3}} e^{-(2a-x)^2/2t}$$

Using (8.4.4), we can compute the probability density of $T_a$. We begin by noting that

$$P(T_a \le t) = 2\ P_0(B_t \ge a) = 2\int_a^\infty (2\pi t)^{-1/2} \exp(-x^2/2t) dx$$

then change variables $x = (t^{1/2}a)/s^{1/2}$ to get

$$\begin{aligned} P_0(T_a \le t) &= 2\int_t^0 (2\pi t)^{-1/2} \exp(-a^2/2s) \left(-t^{1/2}a/2s^{3/2}\right)\, ds \\ &= \int_0^t (2\pi s^3)^{-1/2} a \exp(-a^2/2s)\, ds \end{aligned} \tag{8.4.8}$$

Using the last formula, we can compute:

**Example 8.4.2.** The distribution of $L = \sup\{t \le 1 : B_t = 0\}$. By (8.2.4),

$$\begin{aligned} P_0(L \le s) &= \int_{-\infty}^\infty p_s(0,x) P_x(T_0 > 1-s)\, dx \\ &= 2\int_0^\infty (2\pi s)^{-1/2} \exp(-x^2/2s) \int_{1-s}^\infty (2\pi r^3)^{-1/2} x \exp(-x^2/2r)\, dr\, dx \\ &= \frac{1}{\pi}\int_{1-s}^\infty (sr^3)^{-1/2} \int_0^\infty x \exp(-x^2(r+s)/2rs)\, dx\, dr \\ &= \frac{1}{\pi}\int_{1-s}^\infty (sr^3)^{-1/2} rs/(r+s)\, dr \end{aligned}$$

Our next step is to let $t = s/(r+s)$ to convert the integral over $r \in [1-s, \infty)$ into one over $t \in [0, s]$. $dt = -s/(r+s)^2 dr$, so to make the calculations easier we first rewrite the integral as

$$= \frac{1}{\pi}\int_{1-s}^\infty \left(\frac{(r+s)^2}{rs}\right)^{1/2} \frac{s}{(r+s)^2}\, dr$$

and then change variables to get

$$P_0(L \le s) = \frac{1}{\pi}\int_0^s (t(1-t))^{-1/2}\, dt = \frac{2}{\pi} \arcsin(\sqrt{s}) \tag{8.4.9}$$

The arcsin may remind the reader of the limit theorem for $L_{2n} = \sup\{m \le 2n : S_m = 0\}$ given in Theorem 4.3.5. We will see in Section 8.6 that our new result is a consequence of the old one.

**Exercise 8.4.2.** Use (8.2.3) to show that $R = \inf\{t > 1 : B_t = 0\}$ has probability density

$$P_0(R = 1 + t) = 1/(\pi t^{1/2}(1 + t))$$

### 8.4.3 *Lévy's Modulus of Continuity*

Let $\text{osc}(\delta) = \sup\{|B_s - B_t| : s, t \in [0, 1], |t - s| < \delta\}$.

**Theorem 8.4.2.** *With probability 1,*

$$\limsup_{\delta \to 0} osc(\delta)/(\delta \log(1/\delta))^{1/2} \le 6$$

**Remark.** The constant 6 is not the best possible because the end of the proof is sloppy. Lévy (1937) showed

$$\limsup_{\delta \to 0} \text{osc}(\delta)/(\delta \log(1/\delta))^{1/2} = \sqrt{2}$$

See McKean (1969), pp. 14–16, or Itô and McKean (1965), pp. 36–38, where a sharper result due to Chung, Erdös, and Sirao (1959) is proved. In contrast, if we look at the behavior at a single point, Theorem 8.8.7 below shows

$$\limsup_{t \to 0} |B_t|/\sqrt{2t \log\log(1/t)} = 1 \quad \text{a.s.}$$

*Proof.* Let $I_{m,n} = [m2^{-n}, (m+1)2^{-n}]$, and $\Delta_{m,n} = \sup\{|B_t - B(m2^{-n})| : t \in I_{m,n}\}$. From (8.4.4) and the scaling relation, it follows that

$$\begin{aligned} P(\Delta_{m,n} \ge a2^{-n/2}) &\le 4P(B(2^{-n}) \ge a2^{-n/2}) \\ &= 4P(B(1) \ge a) \le 4\exp(-a^2/2) \end{aligned}$$

by Theorem 1.2.3 if $a \ge 1$. If $\epsilon > 0$, $b = 2(1+\epsilon)(\log\ 2)$, and $a_n = (bn)^{1/2}$, then the last result implies

$$P(\Delta_{m,n} \ge a_n 2^{-n/2} \text{ for some } m \le 2^n) \le 2^n \cdot 4\exp(-bn/2) = 4 \cdot 2^{-n\epsilon}$$

so the Borel-Cantelli lemma implies that if $n \ge N(\omega)$, $\Delta_{m,n} \le (bn)^{1/2}2^{-n/2}$. Now if $s \in I_{m,n}$, $s < t$ and $|s - t| < 2^{-n}$, then $t \in I_{m,n}$ or $I_{m+1,n}$. I claim that in either case the triangle inequality implies

$$|B_t - B_s| \le 3(bn)^{1/2}2^{-n/2}$$

To see this, note that the worst case is $t \in I_{m+1,n}$, but even in this case

$$\begin{aligned} |B_t - B_s| \le\ & |B_t - B((m+1)2^{-n})| \\ & + |B((m+1)2^{-n}) - B(m2^{-n})| + |B(m2^{-n}) - B_s| \end{aligned}$$

It follows from the last estimate that for $2^{-(n+1)} \le \delta < 2^{-n}$

$$\text{osc}(\delta) \le 3(bn)^{1/2}2^{-n/2} \le 3(b\log_2(1/\delta))^{1/2}(2\delta)^{1/2} = 6((1+\epsilon)\delta\log(1/\delta))^{1/2}$$

Recall $b = 2(1+\epsilon)\log 2$ and observe $\exp((\log 2)(\log_2 1/\delta)) = 1/\delta$. ∎

## 8.5 Martingales

At the end of Section 5.7 we used martingales to study the hitting times of random walks. The same methods can be used on Brownian motion, once we prove

**Theorem 8.5.1.** *Let $X_t$ be a right continuous martingale adapted to a right continuous filtration. If $T$ is a bounded stopping time, then $EX_T = EX_0$.*

*Proof.* Let $n$ be an integer so that $P(T \le n-1) = 1$. As in the proof of the strong Markov property, let $T_m = ([2^m T]+1)/2^m$. $Y_k^m = X(k2^{-m})$ is a martingale with respect to $\mathcal{F}_k^m = \mathcal{F}(k2^{-m})$ and $S_m = 2^m T_m$ is a stopping time for $(Y_k^m, \mathcal{F}_k^m)$, so by Exercise 5.4.3,

$$X(T_m) = Y_{S_m}^m = E(Y_{n2^m}^m | \mathcal{F}_{S_m}^m) = E(X_n | \mathcal{F}(T_m))$$

As $m \uparrow \infty$, $X(T_m) \to X(T)$ by right continuity and $\mathcal{F}(T_m) \downarrow \mathcal{F}(T)$ by Theorem 8.3.6, so it follows from Theorem 5.6.3 that

$$X(T) = E(X_n | \mathcal{F}(T))$$

Taking expected values gives $EX(T) = EX_n = EX_0$, since $X_n$ is a martingale. ∎

**Theorem 8.5.2.** *$B_t$ is a martingale w.r.t. the $\sigma$-fields $\mathcal{F}_t$ defined in Section 8.2.*

**Note.** We will use these $\sigma$-fields in all of the martingale results but will not mention them explicitly in the statements.

*Proof.* The Markov property implies that

$$E_x(B_t | \mathcal{F}_s) = E_{B_s}(B_{t-s}) = B_s$$

since symmetry implies $E_y B_u = y$ for all $u \ge 0$. ∎

From Theorem 8.5.2, it follows immediately that we have

**Theorem 8.5.3.** *If $a < x < b$, then $P_x(T_a < T_b) = (b-x)/(b-a)$.*

*Proof.* Let $T = T_a \wedge T_b$. Theorem 8.2.8 implies that $T < \infty$ a.s. Using Theorems 8.5.1 and 8.5.2, it follows that $x = E_x B(T \wedge t)$. Letting $t \to \infty$ and using the bounded convergence theorem, it follows that

$$x = aP_x(T_a < T_b) + b(1 - P_x(T_a < T_b))$$

Solving for $P_x(T_a < T_b)$ now gives the desired result. ∎

**Example 8.5.1. Optimal doubling in backgammon** (Keeler and Spencer, 1975). In our idealization, backgammon is a Brownian motion starting at $1/2$ run until it hits 1 or 0, and $B_t$ is the probability you will win given the events up to time $t$. Initially, the "doubling cube" sits in the middle of the board and either player can "double," that is, tell the other player to play on for twice the stakes or give up and pay the current wager. If a player accepts the double (that is, decides to play on), she gets possession of the doubling cube and is the only one who can offer the next double.

A doubling strategy is given by two numbers $b < 1/2 < a$, that is, offer a double when $B_t \geq a$ and give up if the other player doubles and $B_t < b$. It is not hard to see that for the optimal strategy, $b^* = 1 - a^*$, and that when $B_t = b^*$, accepting and giving up must have the same payoff. If you accept when your probability of winning is $b^*$, then you lose 2 dollars when your probability hits 0, but you win 2 dollars when your probability of winning hits $a^*$, since at that moment you can double and the other player gets the same payoff if they give up or play on. If giving up or playing on at $b^*$ is to have the same payoff, we must have

$$-1 = \frac{b^*}{a^*} \cdot 2 + \frac{a^* - b^*}{a^*} \cdot (-2)$$

Writing $b^* = c$ and $a^* = 1 - c$ and solving, we have $-(1 - c) = 2c - 2(1 - 2c)$ or $1 = 5c$. Thus $b^* = 1/5$ and $a^* = 4/5$. In words you should offer a double if your odds of winning are 80% and accept if they are $\geq$20%.

**Theorem 8.5.4.** *$B_t^2 - t$ is a martingale.*

*Proof.* Writing $B_t^2 = (B_s + B_t - B_2)^2$ we have

$$\begin{aligned}
E_x(B_t^2|\mathcal{F}_s) &= E_x(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2|\mathcal{F}_s) \\
&= B_s^2 + 2B_s E_x(B_t - B_s|\mathcal{F}_s) + E_x((B_t - B_s)^2|\mathcal{F}_s) \\
&= B_s^2 + 0 + (t - s)
\end{aligned}$$

since $B_t - B_s$ is independent of $\mathcal{F}_s$ and has mean 0 and variance $t - s$. ∎

**Theorem 8.5.5.** *Let $T = \inf\{t : B_t \notin (a, b)\}$, where $a < 0 < b$.*

$$E_0 T = -ab$$

*Proof.* Theorem 8.5.1 and 8.5.4 imply $E_0(B^2(T \wedge t)) = E_0(T \wedge t))$. Letting $t \to \infty$ and using the monotone convergence theorem gives $E_0(T \wedge t) \uparrow E_0 T$. Using the bounded convergence theorem and Theorem 8.5.3, we have

$$E_0 B^2(T \wedge t) \to E_0 B_T^2 = a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} = ab \frac{a-b}{b-a} = -ab \qquad ∎$$

**Theorem 8.5.6.** $\exp(\theta B_t - (\theta^2 t/2))$ *is a martingale.*

*Proof.* Bringing $\exp(\theta B_s)$ outside

$$E_x(\exp(\theta B_t)|\mathcal{F}_s) = \exp(\theta B_s)E(\exp(\theta(B_t - B_s))|\mathcal{F}_s)$$
$$= \exp(\theta B_s)\exp(\theta^2(t-s)/2)$$

since $B_t - B_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean 0 and variance $t - s$. ■

**Theorem 8.5.7.** *If $T_a = \inf\{t : B_t = a\}$ then $E_0 \exp(-\lambda T a) = \exp(-a\sqrt{2\lambda})$.*

*Proof.* Theorems 8.5.1 and 8.5.6 imply that $1 = E_0 \exp(\theta B(T \wedge t) - \theta^2(T_a \wedge t)/2)$. Taking $\theta = \sqrt{2\lambda}$, letting $t \to \infty$, and using the bounded convergence theorem gives $1 = E_0 \exp(a\sqrt{2\lambda} - \lambda T_a)$. ■

**Exercise 8.5.1.** Let $T = \inf\{B_t \notin (-a, a)\}$. Show that

$$E \exp(-\lambda T) = 1/\cosh(a\sqrt{2\lambda}).$$

**Exercise 8.5.2.** The point of this exercise is to get information about the amount of time it takes Brownian motion with drift $-b$, $X_t \equiv B_t - bt$ to hit level $a$. Let $\tau = \inf\{t : B_t = a + bt\}$, where $a > 0$. (i) Use the martingale $\exp(\theta B_t - \theta^2 t/2)$ with $\theta = b + (b^2 + 2\lambda)^{1/2}$ to show

$$E_0 \exp(-\lambda\tau) = \exp(-a\{b + (b^2 + 2\lambda)^{1/2}\})$$

Letting $\lambda \to 0$ gives $P_0(\tau < \infty) = \exp(-2ab)$.

**Exercise 8.5.3.** Let $\sigma = \inf\{t : B_t \notin (a, b)\}$ and let $\lambda > 0$. Use the strong Markov property to show

$$E_x \exp(-\lambda T_a) = E_x(e^{-\lambda\sigma}; T_a < T_b) + E_x(e^{-\lambda\sigma}; T_b < T_a)E_b \exp(-\lambda T_a)$$

(ii) Interchange the roles of a and $b$ to get a second equation, use Theorem 8.5.7, and solve to get

$$E_x(e^{-\lambda T}; T_a < T_b) = \sinh(\sqrt{2\lambda}(b-x))/\sinh(\sqrt{2\lambda}(b-a))$$
$$E_x(e^{-\lambda T}; T_b < T_a) = \sinh(\sqrt{2\lambda}(x-a))/\sinh(\sqrt{2\lambda}(b-a))$$

**Theorem 8.5.8.** *If $u(t, x)$ is a polynomial in t and x with*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0 \tag{8.5.1}$$

*then $u(t, B_t)$ is a martingale.*

*Proof.* Let $p_t(x, y) = (2\pi t)^{-1/2} \exp(-(y - x)^2/2t)$. The first step is to check that $p_t$ satisfies the heat equation: $\partial p_t/\partial t = (1/2)\partial^2 p_t/\partial y^2$.

$$\frac{\partial p}{\partial t} = -\frac{1}{2}2\pi(2\pi t)^{-1/2} \exp(-(y-x)^2/2t) + (2\pi t)^{-1/2}\frac{(y-x)^2}{2t^2} \exp(-(y-x)^2/2t)$$
$$\frac{\partial p}{\partial y} = -(2\pi t)^{-1/2} \cdot \frac{y-x}{2t} \exp(-(y-x)^2/2t)$$
$$\frac{\partial^2 p}{\partial y^2} = -\frac{1}{2t}(2\pi t)^{-1/2} \exp(-(y-x)^2/2t) + (2\pi t)^{-1/2}\frac{(y-x)^2}{4t^2} \exp(-(y-x)^2/2t)$$

Interchanging $\partial/\partial t$ and $\int$, and using the heat equation

$$\begin{aligned}\frac{\partial}{\partial t} E_x u(t, B_t) &= \int \frac{\partial}{\partial t}(p_t(x, y)u(t, y))\, dy \\ &= \int \frac{1}{2}\frac{\partial}{\partial y^2} p_t(x, y)u(t, y) + p_t(x, y)\frac{\partial}{\partial t}u(t, y)\, dy\end{aligned}$$

Integrating by parts twice the above

$$= \int p_t(x, y)\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial y^2}\right) u(t, y)\, dy = 0$$

Since $u(t, y)$ is a polynomial, there is no question about the convergence of integrals and there is no contribution from the boundary terms when we integrate by parts. ■

Examples of functions that satisfy (8.5.1) are $\exp(\theta x - \theta^2 t/2)$, $x$, $x^2 - t$, $x^3 - 3tx$, $x^4 - 6x^2 t + 3t^2 \ldots$

**Theorem 8.5.9.** *If* $T = \inf\{t : B_t \notin (-a, a)\}$ *then* $ET^2 = 5a^4/3$.

*Proof.* Theorem 8.5.1 implies

$$E(B(T \wedge t)^4 - 6(T \wedge t)B(T \wedge t)^2) = -3E(T \wedge t)^2.$$

From Theorem 8.5.5, we know that $ET = a^2 < \infty$. Letting $t \to \infty$ and using the dominated convergence theorem on the left-hand side and the monotone convergence theorem on the right gives

$$a^4 - 6a^2 ET = -3E(T^2)$$

Plugging in $ET = a^2$ gives the desired result. ■

**Exercise 8.5.4.** If $T = \inf\{t : B_t \notin (a, b)\}$, where $a < 0 < b$ and $a \neq -b$, then $T$ and $B_T^2$ are not independent, so we cannot calculate $ET^2$ as we did in the proof of Theorem 8.5.9. Use the Cauchy-Schwarz inequality to estimate $E(TB_T^2)$ and conclude $ET^2 \le C\ E(B_T^4)$, where $C$ is independent of $a$ and $b$.

**Exercise 8.5.5.** Find a martingale of the form $B_t^6 - c_1 t B_t^4 + c_2 t^2 B_t^2 - c_3 t^3$ and use it to compute the third moment of $T = \inf\{t : B_t \notin (-a, a)\}$.

**Exercise 8.5.6.** Show that $(1+t)^{-1/2} \exp(B_t^2/2(1+t))$ is a martingale and use this to conclude that $\limsup_{t\to\infty} B_t/((1+t)\log(1+t))^{1/2} \le 1/\sqrt{2}$ a.s.

### 8.5.1 *Multidimensional Brownian Motion*

Let $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_i^2$ be the Laplacian of $f$. The starting point for our investigation is to note that repeating the calculation from the proof of Theorem 8.5.8 shows that in $d > 1$ dimensions,

$$p_t(x, y) = (2\pi t)^{-d/2} \exp(-|y - x|^2/2t)$$

satisfies the heat equation $\partial p_t/\partial t = (1/2)\Delta_y p_t$, where the subscript $y$ on $\delta$ indicates that the Laplacian acts in the $y$ variable.

**Theorem 8.5.10.** *Suppose $v \in C^2$, that is, all first- and second-order partial derivatives exist and are continuous, and $v$ has compact support. Then*

$$v(B_t) - \int_0^t \frac{1}{2}\Delta v(B_s)\, ds \quad \textit{is a martingale.}$$

*Proof.* Repeating the proof of Theorem 8.5.8,

$$\begin{aligned}
\frac{\partial}{\partial t} E_x v(B_t) &= \int v(y) \frac{\partial}{\partial t} p_t(x, y)\, dy \\
&= \int \frac{1}{2} v(y)(\Delta_y p_t(x, y))\, dy \\
&= \int \frac{1}{2} p_t(x, y)\Delta_y v(y)\, dy
\end{aligned}$$

the calculus steps being justified by our assumptions. ∎

We will use this result for two special cases:

$$\varphi(x) = \begin{cases} \log|x| & d = 2 \\ |x|^{2-d} & d \ge 3 \end{cases}$$

We leave it to the reader to check that in each case $\Delta\varphi = 0$. Let $S_r = \inf\{t : |B_t| = r\}$, $r < R$, and $\tau = S_r \wedge S_R$. The first detail is to note that Theorem 8.2.8 implies that if $|x| < R$, then $P_x(S_R < \infty)$. Once we know this, we can conclude

**Theorem 8.5.11.** *If $|x| < R$, then $E_x S_R = (R^2 - |x|^2)/d$.*

*Proof.* It follows from Theorem 8.5.4 that $|B_t|^2 - dt = \sum_{i=1}^d (B_t^i)^2 - t$ is a martingale. Theorem 8.5.1 implies $|x|^2 = E|B_{S_R \wedge t}|^2 - dE(S_R \wedge t)$. Letting $t \to \infty$ gives the desired result. ■

**Lemma 8.5.12.** $\varphi(x) = E_x \varphi(B_\tau)$

*Proof.* Define $\psi(x) = g(|x|)$ to be $C^2$ and have compact support, and have $\psi(x) = \phi(x)$ when $r < |x| < R$. Theorem 8.5.10 implies that $\psi(x) = E_x \psi(B_{t\wedge\tau})$. Letting $t \to \infty$ now gives the desired result. ■

Lemma 8.5.12 implies that

$$\varphi(x) = \varphi(r)P_x(S_r < S_R) + \varphi(R)(1 - P_x(S_r < S_R))$$

where $\varphi(r)$ is short for the value of $\varphi(x)$ on $\{x : |x| = r\}$. Solving now gives

$$P_x(S_r < S_R) = \frac{\varphi(R) - \varphi(x)}{\varphi(R) - \varphi(r)} \tag{8.5.2}$$

In $d = 2$, the last formula says

$$P_x(S_r < S_R) = \frac{\log R - \log |x|}{\log R - \log r} \tag{8.5.3}$$

If we fix $r$ and let $R \to \infty$ in (8.5.3), the right-hand sidegoes to 1. So

$$P_x(S_r < \infty) = 1 \quad \text{for any } x \text{ and any } r > 0$$

It follows that two-dimensional Brownian motion is **recurrent** inthe sense that if $G$ is any open set, then $P_x(B_t \in G \text{ i.o.}) \equiv 1$.

If we fix $R$, let $r \to 0$ in (8.5.3), and let $S_0 = \inf\{t > 0 : B_t = 0\}$, then for $x \neq 0$

$$P_x(S_0 < S_R) \le \lim_{r\to 0} P_x(S_r < S_R) = 0$$

Since this holds for all $R$ and since the continuity of Brownian pathsimplies $S_R \uparrow \infty$ as $R \uparrow \infty$, we have $P_x(S_0 < \infty) = 0$ for all $x \neq 0$. To extend the last result to $x = 0$, we note that the Markov property implies

$$P_0(B_t = 0 \text{ for some } t \ge \epsilon) = E_0[P_{B_\epsilon}(T_0 < \infty)] = 0$$

for all $\epsilon > 0$, so $P_0(B_t = 0 \text{ for some } t > 0) = 0$, and thanks to our definition of $S_0 = \inf\{t > 0 : B_t = 0\}$, we have

$$P_x(S_0 < \infty) = 0 \quad \text{for all } x \tag{8.5.4}$$

Thus, in $d \ge 2$ Brownian motion will not hit 0 at a positive time even if it starts there.

For $d \ge 3$, formula (8.5.2) says

$$P_x(S_r < S_R) = \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}} \tag{8.5.5}$$

There is no point in fixing $R$ and letting $r \to 0$, here. The fact that two-dimensional Brownian motion does not hit 0 implies that three-dimensional Brownian motion does not hit 0 and indeed will not hit the line $\{x : x_1 = x_2 = 0\}$. If we fix $r$ and let $R \to \infty$ in (8.5.5), we get

$$P_x(S_r < \infty) = (r/|x|)^{d-2} < 1 \quad \text{if } |x| > r \tag{8.5.6}$$

From the last result it follows easily that for $d \geq 3$, Brownian motionis **transient**, that is, it does not return infinitely often to any bounded set.

**Theorem 8.5.13.** *As $t \to \infty$, $|B_t| \to \infty$ a.s.*

*Proof.* Let $A_n = \{|B_t| > n^{1-\epsilon}$ for all $t \geq S_n\}$. The strong Markov property implies

$$P_x(A_n^c) = E_x(P_{B(S_n)}(S_{n^{1-\epsilon}} < \infty)) = (n^{1-\epsilon}/n)^{d-2} \to 0$$

as $n \to \infty$. Now $\limsup A_n = \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n$ has

$$P(\limsup A_n) \geq \limsup P(A_n) = 1$$

So infinitely often the Brownian path never returns to $\{x : |x| \leq n^{1-\epsilon}\}$ after time $S_n$, and this implies the desired result. ∎

The scaling relation (8.1.1) implies that $S_{\sqrt{t}} =_d tS_1$, so the proof of Theorem 8.5.13 suggests that

$$|B_t|/t^{(1-\epsilon)/2} \to \infty$$

Dvoretsky and Erdös (1951) have proved the following result about how fast Brownian motion goes to $\infty$ in $d \geq 3$.

**Theorem 8.5.14.** *Suppose $g(t)$ is positive and decreasing. Then*

$$P_0(|B_t| \leq g(t)\sqrt{t} \text{ i.o. as } t \uparrow \infty) = 1 \text{ or } 0$$

*according as $\int^{\infty} g(t)^{d-2}/t\,dt = \infty$ or $< \infty$.*

Here the absence of the lower limit implies that we are only concerned with the behavior of the integral "near $\infty$." A little calculus shows that

$$\int^{\infty} t^{-1} \log^{-\alpha} t\,dt = \infty \text{ or } < \infty$$

according as $\alpha \leq 1$ or $\alpha > 1$, so $B_t$ goes to $\infty$ faster than $\sqrt{t}/(\log t)^{\alpha/d-2}$ for any $\alpha > 1$. Note that in view of the Brownian scaling relationship $B_t =_d t^{1/2}B_1$, we could not sensibly expect escape at a faster rate than $\sqrt{t}$. The last result shows that the escape rate is not much slower.

## 8.6 Donsker's Theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $EX = 0$ and $EX^2 = 1$, and let $S_n = X_1 + \cdots + X_n$. In this section, we will show that as $n \to \infty$, $S(nt)/n^{1/2}$, $0 \leq t \leq 1$ converges in distribution to $B_t$, $0 \le t \le 1$, a Brownian motion starting from $B_0 = 0$. We will say precisely what the last sentence means below. The key to its proof is:

**Theorem 8.6.1. Skorokhod's representation theorem.** *If $EX = 0$ and $EX^2 < \infty$, then there is a stopping time $T$ for Brownian motion so that $B_T =_d X$ and $ET = EX^2$.*

**Remark.** The Brownian motion in the statement and all the Brownian motions in this section have $B_0 = 0$.

*Proof.* Suppose first that $X$ is supported on $\{a, b\}$, where $a < 0 < b$. Since $EX = 0$, we must have

$$P(X = a) = \frac{b}{b-a} \qquad P(X = b) = \frac{-a}{b-a}$$

If we let $T = T_{a,b} = \inf\{t : B_t \notin (a, b)\}$, then Theorem 8.5.3 implies $B_T =_d X$ and Theorem 8.5.5 tells us that

$$ET = -ab = EB_T^2$$

To treat the general case, we will write $F(x) = P(X \le x)$ as a mixture of two point distributions with mean 0. Let

$$c = \int_{-\infty}^0 (-u)\, dF(u) = \int_0^\infty v\, dF(v)$$

If $\varphi$ is bounded and $\varphi(0) = 0$, then using the two formulas for $c$,

$$\begin{aligned} c \int \varphi(x)\, dF(x) &= \left(\int_0^\infty \varphi(v)\, dF(v)\right) \int_{-\infty}^0 (-u) dF(u) \\ &\quad + \left(\int_{-\infty}^0 \varphi(u)\, dF(u)\right) \int_0^\infty v\, dF(v) \\ &= \int_0^\infty dF(v) \int_{-\infty}^0 dF(u)\, (v\varphi(u) - u\varphi(v)) \end{aligned}$$

So we have

$$\int \varphi(x)\, dF(x) = c^{-1} \int_0^\infty dF(v) \int_{-\infty}^0 dF(u)(v-u) \left\{ \frac{v}{v-u}\varphi(u) + \frac{-u}{v-u}\varphi(v) \right\}$$

The last equation gives the desired mixture. If we let $(U, V) \in \mathbf{R}^2$ have

$$\begin{aligned} &P\{(U, V) = (0, 0)\} = F(\{0\}) \\ &P((U, V) \in A) = c^{-1} \iint_{(u,v)\in A} dF(u)\, dF(v)\, (v-u) \end{aligned} \tag{8.6.1}$$

for $A \subset (-\infty, 0) \times (0, \infty)$ and define probability measures by $\mu_{0,0}(\{0\}) = 1$ and

$$\mu_{u,v}(\{u\}) = \frac{v}{v-u} \qquad \mu_{u,v}(\{v\}) = \frac{-u}{v-u} \quad \text{for } u < 0 < v$$

then

$$\int \varphi(x)\,dF(x) = E\left(\int \varphi(x)\,\mu_{U,V}(dx)\right)$$

We proved the last formula when $\varphi(0) = 0$, but it is easy to see that it is true in general. Letting $\varphi \equiv 1$ in the last equation shows that the measure defined in (8.6.1) has total mass 1.

From the calculations above it follows that if we have $(U, V)$ with distribution given in (8.6.1) and an independent Brownian motion defined on the same space, then $B(T_{U,V}) =_d X$. Sticklers for detail will notice that $T_{U,V}$ is not a stopping time for $B_t$ since $(U, V)$ is independent of the Brownian motion. This is not a serious problem since if we condition on $U = u$ and $V = v$, then $T_{u,v}$ is a stopping time, and this is good enough for all the calculations below. For instance, to compute $E(T_{U,V})$ we observe

$$E(T_{U,V}) = E\{E(T_{U,V}|(U, V))\} = E(-UV)$$

by Theorem 8.5.5. (8.6.1) implies

$$\begin{aligned} E(-UV) &= \int_{-\infty}^{0} dF(u)(-u) \int_{0}^{\infty} dF(v)v(v-u)c^{-1} \\ &= \int_{-\infty}^{0} dF(u)(-u) \left\{-u + \int_{0}^{\infty} dF(v)c^{-1}v^2\right\} \end{aligned}$$

since

$$c = \int_0^\infty v\,dF(v) = \int_{-\infty}^{0} (-u)\,dF(u)$$

Using the second expression for $c$ now gives

$$E(T_{U,V}) = E(-UV) = \int_{-\infty}^{0} u^2 dF(u) + \int_0^\infty v^2 dF(v) = EX^2 \qquad \blacksquare$$

**Exercise 8.6.1.** Use Exercise 8.5.4 to conclude that $E(T_{U,V}^2) \le CEX^4$.

**Remark.** One can embed distributions in Brownian motion without adding random variables to the probability space: see Dubins (1968), Root (1969), or Sheu (1986).

From Theorem 8.6.1, it is only a small step to:

**Theorem 8.6.2.** *Let $X_1, X_2, \ldots$ be i.i.d. with a distribution $F$, which has mean 0 and variance 1, and let $S_n = X_1 + \cdots + X_n$. There is a sequence of stopping times $T_0 = 0, T_1, T_2, \ldots$ such that $S_n =_d B(T_n)$ and $T_n - T_{n-1}$ are independent and identically distributed.*

*Proof.* Let $(U_1, V_1), (U_2, V_2), \ldots$ be i.i.d. and have distribution given in (8.6.1), and let $B_t$ be an independent Brownian motion. Let $T_0 = 0$, and for $n \geq 1$, let

$$T_n = \inf\{t \geq T_{n-1} : B_t - B(T_{n-1}) \notin (U_n, V_n)\}$$ ∎

As a corollary of Theorem 8.6.2, we get:

**Theorem 8.6.3. Central limit theorem.** *Under the hypotheses of Theorem 8.6.2, $S_n/\sqrt{n} \Rightarrow \chi$, where $\chi$ has the standard normal distribution.*

*Proof.* If we let $W_n(t) = B(nt)/\sqrt{n} =_d B_t$ by Brownian scaling, then

$$S_n/\sqrt{n} \stackrel{d}{=} B(T_n)/\sqrt{n} = W_n(T_n/n)$$

The weak law of large numbers implies that $T_n/n \to 1$ in probability. It should be clear from this that $S_n/\sqrt{n} \Rightarrow B_1$. To fill in the details, let $\epsilon > 0$, pick $\delta$ so that

$$P(|B_t - B_1| > \epsilon \text{ for some } \ t \in (1 - \delta, 1 + \delta)) < \epsilon/2$$

then pick $N$ large enough so that for $n \geq N$, $P(|T_n/n - 1| > \delta) < \epsilon/2$. The last two estimates imply that for $n \geq N$

$$P(|W_n(T_n/n) - W_n(1)| > \epsilon) < \epsilon$$

Since $\epsilon$ is arbitrary, it follows that $W_n(T_n/n) - W_n(1) \to 0$ in probability. Applying the converging together lemma, Exercise 3.2.13, with $X_n = W_n(1)$ and $Z_n = W_n(T_n/n)$, the desired result follows. ∎

Our next goal is to prove a strengthening of the central limit theorem that allows us to obtain limit theorems for functionals of $\{S_m : 0 \leq m \leq n\}$, for example: $\max_{0 \leq m \leq n} S_m$ or $|\{m \leq n : S_m > 0\}|$. Let $C[0, 1] = \{\text{continuous } \omega : [0, 1] \to \mathbf{R}\}$. When equipped with the norm $\|\omega\| = \sup\{|\omega(s)| : s \in [0, 1]\}$, $C[0, 1]$ becomes a complete separable metric space. To fit $C[0, 1]$ into the framework of Section 3.9, we want our measures defined on $\mathcal{B}$ = the $\sigma$-field generated by the open sets. Fortunately,

**Lemma 8.6.4.** *$\mathcal{B}$ is the same as $\mathcal{C}$ the $\sigma$-field generated by the finite dimensional sets $\{\omega : \omega(t_i) \in A_i\}$.*

*Proof.* Observe that if $\xi$ is a given continuous function,

$$\{\omega : \|\omega - \xi\| \leq r - 1/n\} = \cap_q \{\omega : |\omega(q) - \xi(q)| \leq r - 1/n\}$$

where the intersection is over all rationals in [0,1]. Letting $n \to \infty$ shows $\{\omega : \|\omega - \xi\| < r\} \in \mathcal{C}$ and $\mathcal{B} \subset \mathcal{C}$. To prove the reverse inclusion, observe that if the $A_i$ are open, the finite dimensional set $\{\omega : \omega(t_i) \in A_i\}$ is open, so the $\pi - \lambda$ theorem implies $\mathcal{B} \supset \mathcal{C}$. ∎

A sequence of probability measures $\mu_n$ on $C[0, 1]$ is said to **converge weakly** to a limit $\mu$ if for all bounded continuous functions $\varphi : C[0, 1] \to \mathbf{R}$, $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$. Let $\mathbf{N}$ be the nonnegative integers and let

$$S(u) = \begin{cases} S_k & \text{if } u = k \in \mathbf{N} \\ \text{linear on } [k, k+1] & \text{for } k \in \mathbf{N} \end{cases}$$

We will prove:

**Theorem 8.6.5. Donsker's theorem.** *Under the hypotheses of Theorem 8.6.3,*

$$S(n\cdot)/\sqrt{n} \Rightarrow B(\cdot),$$

*that is, the associated measures on* $C[0, 1]$ *converge weakly.*

To motivate ourselves for the proof, we will begin by extracting several corollaries. The key to each one is a consequence of the following result, which follows from Theorem 3.9.1.

**Theorem 8.6.6.** *If* $\psi : C[0, 1] \to \mathbf{R}$ *has the property that it is continuous* $P_0$*-a.s., then*

$$\psi(S(n\cdot)/\sqrt{n}) \Rightarrow \psi(B(\cdot))$$

**Example 8.6.1.** Let $\psi(\omega) = \omega(1)$. In this case, $\psi : C[0, 1] \to \mathbf{R}$ is continuous and Theorem 8.6.6 gives the central limit theorem.

**Example 8.6.2. Maxima.** Let $\psi(\omega) = \max\{\omega(t) : 0 \le t \le 1\}$. Again, $\psi : C[0, 1] \to \mathbf{R}$ is continuous. This time Theorem 8.6.6 implies

$$\max_{0 \le m \le n} S_m/\sqrt{n} \Rightarrow M_1 \equiv \max_{0 \le t \le 1} B_t$$

To complete the picture, we observe that by (8.4.4), the distribution of the right-hand side is

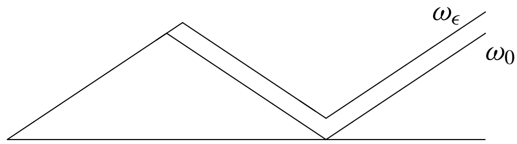
$$P_0(M_1 \ge a) = P_0(T_a \le 1) = 2P_0(B_1 \ge a)$$

**Exercise 8.6.2.** Suppose $S_n$ is one-dimensional simple random walk and let

$$R_n = 1 + \max_{m \le n} S_m - \min_{m \le n} S_m$$

be the number of points visited by time $n$. Show that $R_n/\sqrt{n} \Rightarrow a$ limit.

**Example 8.6.3. Last 0 before time $n$.** Let $\psi(\omega) = \sup\{t \le 1 : \omega(t) = 0\}$. This time, $\psi$ is not continuous, for if $\omega_\epsilon$ with $\omega_\epsilon(0) = 0$ is piecewise linear with slope 1 on $[0, 1/3 + \epsilon]$, $-1$ on $[1/3 + \epsilon, 2/3]$, and slope 1 on $[2/3, 1]$, then $\psi(\omega_0) = 2/3$ but $\psi(\omega_\epsilon) = 0$ for $\epsilon > 0$.




It is easy to see that if $\psi(\omega) < 1$ and $\omega(t)$ has positive and negative values in each interval $(\psi(\omega) - \delta, \psi(\omega))$, then $\psi$ is continuous at $\omega$. By arguments in Subsection 8.4.1, the last set has $P_0$ measure 1. (If the zero at $\psi(\omega)$ was isolated on the left, it would not be isolated on the right.) It follows that

$$\sup\{m \le n : S_{m-1} \cdot S_m \le 0\}/n \Rightarrow L = \sup\{t \le 1 : B_t = 0\}$$

The distribution of $L$ is given in (8.4.9). The last result shows that the arcsine law, Theorem 4.3.5, proved for simple random walks holds when the mean is 0 and variance is finite.

**Example 8.6.4. Occupation times of half-lines.** Let

$$\psi(\omega) = |\{t \in [0,1] : \omega(t) > a\}|.$$

The point $\omega \equiv a$ shows that $\psi$ is not continuous, but it is easy to see that $\psi$ is continuous at paths $\omega$ with $|\{t \in [0,1] : \omega(t) = a\}| = 0$. Fubini's theorem implies that

$$E_0|\{t \in [0,1] : B_t = a\}| = \int_0^1 P_0(B_t = a)\,dt = 0$$

so $\psi$ is continuous $P_0$-a.s. With a little work, Theorem 8.6.6 implies

$$|\{m \le n : S_m > a\sqrt{n}\}|/n \Rightarrow |\{t \in [0,1] : B_t > a\}|$$

*Proof.* Application of Theorem 8.6.6 gives that for any $a$,

$$|\{t \in [0,1] : S(nt) > a\sqrt{n}\}| \Rightarrow |\{t \in [0,1] : B_t > a\}|$$

To convert this into a result about $|\{m \le n : S_m > a\sqrt{n}\}|$, we note that on $\{\max_{m \le n} |X_m| \le \epsilon\sqrt{n}\}$, which by Chebyshev's inequality has a probability $\to 1$, we have

$$\begin{aligned}|\{t \in [0,1] : S(nt) > (a+\epsilon)\sqrt{n}\}| &\le \frac{1}{n}|\{m \le n : S_m > a\sqrt{n}\}| \\ &\le |\{t \in [0,1] : S(nt) > (a-\epsilon)\sqrt{n}\}|\end{aligned}$$

Combining this with the first conclusion of the proof and using the fact that $b \to |\{t \in [0,1] : B_t > b\}|$ is continuous at $b = a$ with probability 1, one arrives easily at the desired conclusion. ∎

To compute the distribution of $|\{t \in [0,1] : B_t > 0\}|$, observe that we proved in Theorem 4.3.7 that if $S_n =_d -S_n$ and $P(S_m = 0) = 0$ for all $m \ge 1$, for example, the $X_i$ have a symmetric continuous distribution, then the left-hand side converges to the arcsine law, so the right-hand side has that distribution and is the limit for any random walk with mean 0 and finite variance. The last argument uses an idea called the "invariance principle" that originated with Erdös and Kac (1946, 1947): the asymptotic behavior of functionals of $S_n$ should be the same as long as the central limit theorem applies. Our final application is from the original paper of Donsker (1951). Erdös and Kac (1946) give the limit distribution for the case $k = 2$.

**Example 8.6.5.** Let $\psi(\omega) = \int_{[0,1]} \omega(t)^k dt$ where $k > 0$ is an integer. $\psi$ is continuous, so applying Theorem 8.6.6 gives

$$\int_0^1 (S(nt)/\sqrt{n})^k \, dt \Rightarrow \int_0^1 B_t^k \, dt$$

To convert this into a result about the original sequence, we begin by observing that if $x < y$ with $|x - y| \le \epsilon$ and $|x|, |y| \le M$, then

$$|x^k - y^k| \le \int_x^y \frac{|z|^{k+1}}{k+1} \, dz \le \frac{\epsilon M^{k+1}}{k+1}$$

From this, it follows that on

$$G_n(M) = \left\{ \max_{m \le n} |X_m| \le M^{-(k+2)}\sqrt{n}, \ \max_{m \le n} |S_m| \le M\sqrt{n} \right\}$$

we have

$$\left| \int_0^1 (S(nt)/\sqrt{n})^k \, dt - n^{-1-(k/2)} \sum_{m=1}^n S_m^k \right| \le \frac{1}{(k+1)M}$$

For fixed $M$, it follows from Chebyshev's inequality, Example 8.6.2, and Theorem 3.2.5 that

$$\liminf_{n \to \infty} P(G_n(M)) \ge P\left( \max_{0 \le t \le 1} |B_t| < M \right)$$

The right-hand side is close to 0 if $M$ is large, so

$$\int_0^1 (S(nt)/\sqrt{n})^k \, dt - n^{-1-(k/2)} \sum_{m=1}^n S_m^k \to 0$$

in probability, and it follows from the converging together lemma (Exercise 3.2.13) that

$$n^{-1-(k/2)} \sum_{m=1}^n S_m^k \Rightarrow \int_0^1 B_t^k \, dt$$

It is remarkable that the last result holds under the assumption that $EX_i = 0$ and $EX_i^2 = 1$, that is, we do not need to assume that $E|X_i^k| < \infty$.

**Exercise 8.6.3.** When $k = 1$, the last result says that if $X_1, X_2, \dots$ are i.i.d. with $EX_i = 0$ and $EX_i^2 = 1$, then

$$n^{-3/2} \sum_{m=1}^{n} (n + 1 - m) X_m \Rightarrow \int_0^1 B_t \, dt$$

(i) Show that the right-hand side has a normal distribution with mean 0 and variance $1/3$. (ii) Deduce this result from the Lindeberg-Feller theorem.

*Proof of Theorem 8.6.5.* To simplify the proof and prepare for generalizations in the next section, let $X_{n,m}$, $1 \le m \le n$, be a triangular array of random variables, $S_{n,m} = X_{n,1} + \cdots + X_{n,m}$ and suppose $S_{n,m} = B(\tau_m^n)$. Let

$$S_{n,(u)} = \begin{cases} S_{n,m} & \text{if } u = m \in \{0, 1, \dots, n\} \\ \text{linear for } u \in [m-1, m] & \text{when } m \in \{1, \dots, n\} \end{cases}$$

**Lemma 8.6.7.** *If $\tau_{[ns]}^n \to s$ in probability for each $s \in [0,1]$ then*

$$\| S_{n,(n\cdot)} - B(\cdot) \| \to 0 \quad \textit{in probability}$$

To make the connection with the original problem, let $X_{n,m} = X_m/\sqrt{n}$ and define $\tau_1^n, \dots, \tau_n^n$ so that $(S_{n,1}, \dots, S_{n,n}) =_d (B(\tau_1^n), \dots, B(\tau_n^n))$. If $T_1, T_2, \dots$ are the stopping times defined in the proof of Theorem 8.6.3, Brownian scaling implies $\tau_m^n =_d T_m/n$, so the hypothesis of Lemma 8.6.7 is satisfied.

*Proof.* The fact that $B$ has continuous paths (and hence is uniformly continuous on [0,1]) implies that if $\epsilon > 0$ then there is a $\delta > 0$ so that $1/\delta$ is an integer and

(a) $$P(|B_t - B_s| < \epsilon \text{ for all } 0 \le s \le 1, |t - s| < 2\delta) > 1 - \epsilon$$

The hypothesis of Lemma 8.6.7 implies that if $n \ge N_\delta$, then

$$P(|\tau_{[nk\delta]}^n - k\delta| < \delta \quad \text{for } k = 1, 2, \dots, 1/\delta) \ge 1 - \epsilon$$

Since $m \to \tau_m^n$ is increasing, it follows that if $s \in ((k-1)\delta, k\delta)$,

$$\tau_{[ns]}^n - s \ge \tau_{[n(k-1)\delta]}^n - k\delta$$
$$\tau_{[ns]}^n - s \le \tau_{[nk\delta]}^n - (k+1)\delta$$

so if $n \ge N_\delta$,

(b) $$P\left( \sup_{0 \le s \le 1} |\tau_{[ns]}^n - s| < 2\delta \right) \ge 1 - \epsilon$$

When the events in (a) and (b) occur

(c) $$|S_{n,m} - B_{m/n}| < \epsilon \text{ for all } m \le n$$

To deal with $t = (m+\theta)/n$ with $0 < \theta < 1$, we observe that

$$|S_{n,(nt)} - B_t| \le (1-\theta)|S_{n,m} - B_{m/n}| + \theta|S_{n,m+1} - B_{(m+1)/n}|$$
$$+ (1-\theta)|B_{m/n} - B_t| + \theta|B_{(m+1)/n} - B_t|$$

Using (c) on the first two terms and (a) on the last two, we see that if $n \ge N_\delta$ and $1/n < 2\delta$, then $\|S_{n,(n\cdot)} - B(\cdot)\| < 2\epsilon$ with probability $\ge 1 - 2\epsilon$. Since $\epsilon$ is arbitrary, the proof of Lemma 8.6.7 is complete. ■

To get Theorem 8.6.5 now, we have to show:

**Lemma 8.6.8.** *If $\varphi$ is bounded and continuous, then $E\varphi(S_{n,(n\cdot)}) \to E\varphi(B(\cdot))$.*

*Proof.* For fixed $\epsilon > 0$, let $G_\delta = \{\omega : \text{if } \|\omega - \omega'\| < \delta \text{ then } |\varphi(\omega) - \varphi(\omega')| < \epsilon\}$. Since $\varphi$ is continuous, $G_\delta \uparrow C[0,1]$ as $\delta \downarrow 0$. Let $\Delta = \|S_{n,(n\cdot)} - B(\cdot)\|$. The desired result now follows from Lemma 8.6.7 and the trivial inequality

$$|E\varphi(S_{n,(n\cdot)}) - E\varphi(B(\cdot))| \le \epsilon + (2\sup|\varphi(\omega)|)\{P(G_\delta^c) + P(\Delta \ge \delta)\} \quad ■$$

To accommodate our final example, we need a trivial generalization of Theorem 8.6.5. Let $C[0,\infty) = \{\text{continuous } \omega : [0,\infty) \to \mathbf{R}\}$ and let $\mathcal{C}[0,\infty)$ be the $\sigma$-field generated by the finite dimensional sets. Given a probability measure $\mu$ on $C[0,\infty)$, there is a corresponding measure $\pi_M\mu$ on $C[0,M] = \{\text{continuous } \omega : [0,M] \to \mathbf{R}\}$ (with $\mathcal{C}[0,M]$ the $\sigma$-field generated by the finite dimensional sets) obtained by "cutting off the paths at time M." Let $(\psi_M\omega)(t) = \omega(t)$ for $t \in [0,M]$ and let $\pi_M\mu = \mu \circ \psi_M^{-1}$. We say that a sequence of probability measures $\mu_n$ on $C[0,\infty)$ converges weakly to $\mu$ if for all $M$, $\pi_M\mu_n$ converges weakly to $\pi_M\mu$ on $C[0,M]$, the last concept being defined by a trivial extension of the definitions for $M = 1$. With these definitions, it is easy to conclude:

**Theorem 8.6.9.** *$S(n\cdot)/\sqrt{n} \Rightarrow B(\cdot)$, that is, the associated measures on $C[0,\infty)$ converge weakly.*

*Proof.* By definition, all we have to show is that weak convergence occurs on $C[0,M]$ for all $M < \infty$. The proof of Theorem 8.6.5 works in the same way when 1 is replaced by $M$. ■

**Example 8.6.6.** Let $N_n = \inf\{m : S_m \ge \sqrt{n}\}$ and $T_1 = \inf\{t : B_t \ge 1\}$. Since $\psi(\omega) = T_1(\omega) \wedge 1$ is continuous $P_0$ a.s. on $C[0,1]$ and the distribution of $T_1$ is continuous, it follows from Theorem 8.6.6 that for $0 < t < 1$

$$P(N_n \le nt) \to P(T_1 \le t)$$

Repeating the last argument with 1 replaced by $M$ and using Theorem 8.6.9 shows that the last conclusion holds for all $t$.

## 8.7 Empirical Distributions, Brownian Bridge

Let $X_1, X_2, \ldots$ be i.i.d. with distribution $F$. Theorem 2.4.7 shows that with probability 1, the empirical distribution

$$\hat{F}_n(x) = \frac{1}{n}|\{m \le n : X_m \le x\}|$$

converges uniformly to $F(x)$. In this section, we will investigate the rate of convergence when $F$ is continuous. We impose this restriction so we can reduce to the case of a uniform distribution on (0,1) by setting $Y_n = F(X_n)$. (See Exercise 1.2.4.) Since $x \to F(x)$ is nondecreasing and continuous and no observations land in intervals of constancy of $F$, it is easy to see that if we let

$$\hat{G}_n(y) = \frac{1}{n}|\{m \le n : Y_m \le y\}|$$

then

$$\sup_x |\hat{F}_n(x) - F(x)| = \sup_{0<y<1} |\hat{G}_n(y) - y|$$

For the rest of the section, then, we will assume $Y_1, Y_2, \ldots$ is i.i.d. uniform on (0,1). To be able to apply Donsker's theorem, we will transform the problem. Put the observations $Y_1, \ldots, Y_n$ in increasing order: $U_1^n < U_2^n < \cdots < U_n^n$. I claim that

$$\sup_{0<y<1} \hat{G}_n(y) - y = \sup_{1\le m\le n} \frac{m}{n} - U_m^n$$

$$\inf_{0<y<1} \hat{G}_n(y) - y = \inf_{1\le m\le n} \frac{m-1}{n} - U_m^n \tag{8.7.1}$$

since the sup occurs at a jump of $\hat{G}_n$ and the inf right before a jump. For a picture, see Figure 8.4.

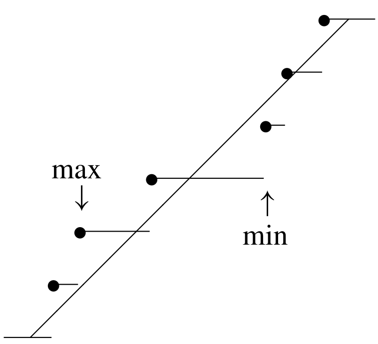



Figure 8.4. Picture proof of formulas in (8.7.1).

We will show that

$$D_n \equiv n^{1/2} \sup_{0<y<1} |\hat{G}_n(y) - y|$$

has a limit, so the extra $-1/n$ in the inf does not make any difference.

Our third and final maneuver is to give a special construction of the order statistics $U_1^n < U_2^n \cdots < U_n^n$. Let $W_1, W_2, \ldots$ be i.i.d. with $P(W_i > t) = e^{-t}$ and let $Z_n = W_1 + \cdots + W_n$.

**Lemma 8.7.1.** $\{U_k^n : 1 \le k \le n\} \stackrel{d}{=} \{Z_k/Z_{n+1} : 1 \le k \le n\}$

*Proof.* We change variables $v = r(t)$, where $v_i = t_i/t_{n+1}$ for $i \le n$, $v_{n+1} = t_{n+1}$. The inverse function is

$$s(v) = (v_1 v_{n+1}, \ldots, v_n v_{n+1}, v_{n+1})$$

which has matrix of partial derivatives $\partial s_i/\partial v_j$ given by

$$\begin{pmatrix} v_{n+1} & 0 & \ldots & 0 & v_1 \\ 0 & v_{n+1} & \ldots & 0 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & v_{n+1} & v_n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

The determinant of this matrix is $v_{n+1}^n$, so if we let $W = (V_1, \ldots, V_{n+1}) = r(Z_1, \ldots, Z_{n+1})$, the change of variables formula implies $W$ has joint density

$$f_W(v_1, \ldots, v_n, v_{n+1}) = \left( \prod_{m=1}^n \lambda e^{-\lambda v_{n+1}(v_m - v_{m-1})} \right) \lambda e^{-\lambda v_{n+1}(1-v_n)} v_{n+1}^n$$

To find the joint density of $V = (V_1, \ldots, V_n)$, we simplify the preceding formula and integrate out the last coordinate to get

$$f_V(v_1, \ldots, v_n) = \int_0^\infty \lambda^{n+1} v_{n+1}^n e^{-\lambda v_{n+1}}\, dv_{n+1} = n!$$

for $0 < v_1 < v_2 \cdots < v_n < 1$, which is the desired joint density. ■

We turn now to the limit law for $D_n$. As argued above, it suffices to consider

$$\begin{aligned} D_n' &= n^{1/2} \max_{1 \le m \le n} \left| \frac{Z_m}{Z_{n+1}} - \frac{m}{n} \right| \\ &= \frac{n}{Z_{n+1}} \max_{1 \le m \le n} \left| \frac{Z_m}{n^{1/2}} - \frac{m}{n} \cdot \frac{Z_{n+1}}{n^{1/2}} \right| \\ &= \frac{n}{Z_{n+1}} \max_{1 \le m \le n} \left| \frac{Z_m - m}{n^{1/2}} - \frac{m}{n} \cdot \frac{Z_{n+1} - n}{n^{1/2}} \right| \end{aligned} \tag{8.7.2}$$

If we let

$$B_n(t) = \begin{cases} (Z_m - m)/n^{1/2} & \text{if } t = m/n \text{ with } m \in \{0, 1, \ldots, n\} \\ \text{linear} & \text{on } [(m-1)/n, m/n] \end{cases}$$

then

$$D'_n = \frac{n}{Z_{n+1}} \max_{0 \le t \le 1} \left| B_n(t) - t\left\{ B_n(1) + \frac{Z_{n+1} - Z_n}{n^{1/2}} \right\} \right|$$

The strong law of large numbers implies $Z_{n+1}/n \to 1$ a.s., so the first factor will disappear in the limit. To find the limit of the second, we observe that Donsker's theorem, Theorem 8.6.5, implies $B_n(\cdot) \Rightarrow B(\cdot)$, a Brownian motion, and computing second moments shows

$$(Z_{n+1} - Z_n)/n^{1/2} \to 0 \text{ in probability}$$

$\psi(\omega) = \max_{0 \le t \le 1} |\omega(t) - t\omega(1)|$ is a continuous function from $C[0, 1]$ to $\mathbf{R}$, so it follows from Donsker's theorem that:

**Theorem 8.7.2.** *$D_n \Rightarrow \max_{0 \le t \le 1} |B_t - tB_1|$, where $B_t$ is a Brownian motion starting at 0.*

**Remark.** Doob (1949) suggested this approach to deriving results of Kolmogorov and Smirnov, which was later justified by Donsker (1952). Our proof follows Breiman (1968).

To identify the distribution of the limit in Theorem 8.7.2, we will first prove

$$\{B_t - tB_1, 0 \le t \le 1\} \stackrel{d}{=} \{B_t, 0 \le t \le 1 | B_1 = 0\} \tag{8.7.3}$$

a process we will denote by $B_t^0$ and call the **Brownian bridge**. The event $B_1 = 0$ has probability 0, but it is easy to see what the conditional probability should mean. If $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, $x_0 = 0$, $x_{n+1} = 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\begin{aligned} &P(B(t_1) = x_1, \ldots, B(t_n) = x_n | B(1) = 0) \\ &\qquad = \frac{1}{p_1(0,0)} \prod_{m=1}^{n+1} p_{t_m - t_{m-1}}(x_{m-1}, x_m) \end{aligned} \tag{8.7.4}$$

where $p_t(x, y) = (2\pi t)^{-1/2} \exp(-(y-x)^2/2t)$.

*Proof of (8.7.3).* Formula (8.7.4) shows that the f.d.d.'s of $B_t^0$ are multivariate normal and have mean 0. Since $B_t - tB_1$ also has this property, it suffices to show that the covariances are equal. We begin with the easier computation. If $s < t$, then

$$E((B_s - sB_1)(B_t - tB_1)) = s - st - st + st = s(1-t) \tag{8.7.5}$$

For the other process, $P(B_s^0 = x, B_t^0 = y)$ is

$$\begin{aligned} &\frac{\exp(-x^2/2s)}{(2\pi s)^{1/2}} \cdot \frac{\exp(-(y-x)^2/2(t-s))}{(2\pi(t-s))^{1/2}} \cdot \frac{\exp(-y^2/2(1-t))}{(2\pi(1-t))^{1/2}} \cdot (2\pi)^{1/2} \\ &\quad = (2\pi)^{-1}(s(t-s)(1-t))^{-1/2} \exp(-(ax^2 + 2bxy + cy^2)/2) \end{aligned}$$

where

$$a = \frac{1}{s} + \frac{1}{t-s} = \frac{t}{s(t-s)} \qquad b = -\frac{1}{t-s}$$
$$c = \frac{1}{t-s} + \frac{1}{1-t} = \frac{1-s}{(t-s)(1-t)}$$

Recalling the discussion at the end of Section 3.9 and noticing

$$\begin{pmatrix} \frac{t}{s(t-s)} & \frac{-1}{(t-s)} \\ \frac{-1}{(t-s)} & \frac{1-s}{(t-s)(1-t)} \end{pmatrix}^{-1} = \begin{pmatrix} s(1-s) & s(1-t) \\ s(1-t) & t(1-t) \end{pmatrix}$$

(multiply the matrices!) shows (8.7.3) holds. ■

Our final step in investigating the limit distribution of $D_n$ is to compute the distribution of $\max_{0 \le t \le 1} |B_t^0|$. To do this, we first prove

**Theorem 8.7.3.** *The density function of $B_t$ on $\{T_a \wedge T_b > t\}$ is*

$$P_x(T_a \wedge T_b > t, B_t = y) = \sum_{n=-\infty}^{\infty} P_x(B_t = y + 2n(b-a)) \tag{8.7.6}$$
$$- P_x(B_t = 2b - y + 2n(b-a))$$

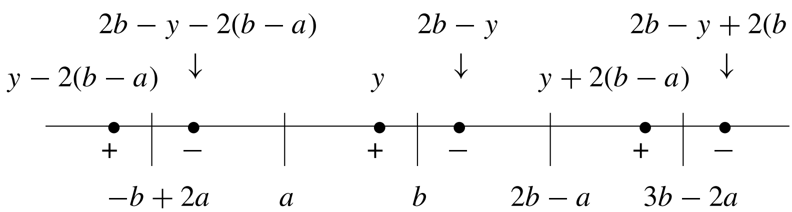



Figure 8.5. Picture of the infinite series in (8.7.6). Note that the array of + and − is antisymmetric when seen from $a$ or $b$.

*Proof.* We begin by observing that if $A \subset (a, b)$,

$$P_x(T_a \wedge T_b > t, B_t \in A) = P_x(B_t \in A) - P_x(T_a < T_b, T_a < t, B_t \in A)$$
$$- P_x(T_b < T_a, T_b < t, B_t \in A) \tag{8.7.7}$$

If we let $\rho_a(y) = 2a - y$ be reflection through $a$ and observe that $\{T_a < T_b\}$ is $\mathcal{F}(T_a)$ measurable, then it follows from the proof of (8.4.5) that

$$P_x(T_a < T_b, T_a < t, B_t \in A) = P_x(T_a < T_b, B_t \in \rho_a A)$$

where $\rho_a A = \{\rho_a(y) : y \in A\}$. To get rid of the $T_a < T_b$, we observe that

$$P_x(T_a < T_b, B_t \in \rho_a A) = P_x(B_t \in \rho_a A) - P_x(T_b < T_a, B_t \in \rho_a A)$$

Noticing that $B_t \in \rho_a A$ and $T_b < T_a$ imply $T_b < t$ and using the reflection principle again gives

$$\begin{aligned} P_x(T_b < T_a, B_t \in \rho_a A) &= P_x(T_b < T_a, B_t \in \rho_b \rho_a A) \\ &= P_x(B_t \in \rho_b \rho_a A) - P_x(T_a < T_b, B_t \in \rho_b \rho_a A) \end{aligned}$$

Repeating the last two calculations $n$ more times gives

$$\begin{aligned} P_x(T_a < T_b, B_t \in \rho_a A) = &\sum_{m=0}^{n} P_x(B_t \in \rho_a (\rho_b \rho_a)^m A) - P_x(B_t \in (\rho_b \rho_a)^{m+1} A) \\ &+ P_x(T_a < T_b, B_t \in (\rho_b \rho_a)^{n+1} A) \end{aligned}$$

Each pair of reflections pushes $A$ further away from 0, so letting $n \to \infty$ shows

$$P_x(T_a < T_b, B_t \in \rho_a A) = \sum_{m=0}^{\infty} P_x(B_t \in \rho_a (\rho_b \rho_a)^m A) - P_x(B_t \in (\rho_b \rho_a)^{m+1} A)$$

Interchanging the roles of a and $b$ gives

$$P_x(T_b < T_a, B_t \in \rho_b A) = \sum_{m=0}^{\infty} P_x(B_t \in \rho_b (\rho_a \rho_b)^m A) - P_x(B_t \in (\rho_a \rho_b)^{m+1} A)$$

Combining the last two expressions with (8.7.7) and using $\rho_c^{-1} = \rho_c$, $(\rho_a \rho_b)^{-1} = \rho_b^{-1} \rho_a^{-1}$ gives

$$P_x(T_a \wedge T_b > t, B_t \in A) = \sum_{m=-\infty}^{\infty} P_x(B_t \in (\rho_b \rho_a)^n A) - P_x(B_t \in \rho_a (\rho_b \rho_a)^n A)$$

To prepare for applications, let $A = (u, v)$ where $a < u < v < b$, notice that $\rho_b \rho_a (y) = y + 2(b - a)$, and change variables in the second sum to get

$$\begin{aligned} &P_x(T_a \wedge T_b > t, u < B_t < v) = \\ &\sum_{n=-\infty}^{\infty} \{P_x(u + 2n(b-a) < B_t < v + 2n(b-a)) \\ &\quad - P_x(2b - v + 2n(b-a) < B_t < 2b - u + 2n(b-a))\} \end{aligned} \tag{8.7.8}$$

Letting $u = y - \epsilon, v = y + \epsilon$, dividing both sides by $2\epsilon$, and letting $\epsilon \to 0$ (leaving it to the reader to check that the dominated convergence theorem applies) gives the desired result. ∎

Setting $x = y = 0$, $t = 1$, and dividing by $(2\pi)^{-1/2} = P_0(B_1 = 0)$, we get a result for the Brownian bridge $B_t^0$:

$$P_0\left(a < \min_{0\le t\le 1} B_t^0 < \max_{0\le t\le 1} B_t^0 < b\right) \tag{8.7.9}$$
$$= \sum_{n=-\infty}^{\infty} e^{-(2n(b-a))^2/2} - e^{-(2b+2n(b-a))^2/2}$$

Taking $a = -b$, we have

$$P_0\left(\max_{0\le t\le 1} |B_t^0| < b\right) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-2m^2b^2} \tag{8.7.10}$$

This formula gives the distribution of the Kolmogorov-Smirnov statistic, which can be used to test if an i.i.d. sequence $X_1, \ldots, X_n$ has distribution $F$. To do this, we transform the data to $F(X_n)$ and look at the maximum discrepancy between the empirical distribution and the uniform. (8.7.10) tells us the distribution of the error when the $X_i$ have distribution $F$.

(8.7.9) gives the joint distribution of the maximum and minimum of Brownian bridge. In theory, one can let $a \to -\infty$ in this formula to find the distribution of the maximum, but in practice it is easier to start over again.

**Exercise 8.7.1.** Use Exercise 8.4.6 and the reasoning that led to (8.7.9) to conclude

$$P\left(\max_{0\le t\le 1} B_t^0 > b\right) = \exp(-2b^2)$$

## 8.8 Laws of the Iterated Logarithm*

Our first goal is to show

**Theorem 8.8.1. LIL for Brownian motion.**

$$\limsup_{t\to\infty} B_t/(2t \log\log t)^{1/2} = 1 \quad a.s.$$

Here LIL is short for "law of the iterated logarithm," a name that refers to the $\log\log t$ in the denominator. Once Theorem 8.8.1 is established, we can use the Skorokhod representation to prove the analogous result for random walks with mean 0 and finite variance.

*Proof.* The key to the proof is (8.4.4).

$$P_0\left(\max_{0\le s\le 1} B_s > a\right) = P_0(T_a \le 1) = 2\ P_0(B_1 \ge a) \tag{8.8.1}$$

To bound the right-hand side, we use Theorem 1.2.3.

$$\int_x^\infty \exp(-y^2/2)\,dy \le \frac{1}{x}\exp(-x^2/2) \tag{8.8.2}$$

$$\int_x^\infty \exp(-y^2/2)\,dy \sim \frac{1}{x}\exp(-x^2/2) \quad \text{as } x \to \infty \tag{8.8.3}$$

where $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to \infty$. The last result and Brownian scaling imply that

$$P_0(B_t > (tf(t))^{1/2}) \sim \kappa f(t)^{-1/2} \exp(-f(t)/2)$$

where $\kappa = (2\pi)^{-1/2}$ is a constant that we will try to ignore below. The last result implies that if $\epsilon > 0$, then

$$\sum_{n=1}^\infty P_0(B_n > (nf(n))^{1/2}) \begin{cases} < \infty & \text{when } f(n) = (2+\epsilon)\log n \\ = \infty & \text{when } f(n) = (2-\epsilon)\log n \end{cases}$$

and hence by the Borel-Cantelli lemma that

$$\limsup_{n\to\infty} B_n/(2n\log n)^{1/2} \le 1 \quad \text{a.s.}$$

To replace $\log n$ by $\log\log n$, we have to look along exponentially growing sequences. Let $t_n = \alpha^n$, where $\alpha > 1$.

$$\begin{aligned} P_0\left(\max_{t_n \le s \le t_{n+1}} B_s > (t_n f(t_n))^{1/2}\right) &\le P_0\left(\max_{0\le s\le t_{n+1}} B_s/t_{n+1}^{1/2} > \left(\frac{f(t_n)}{\alpha}\right)^{1/2}\right) \\ &\le 2\kappa (f(t_n)/\alpha)^{-1/2}\exp(-f(t_n)/2\alpha) \end{aligned}$$

by (8.8.1) and (8.8.2). If $f(t) = 2\alpha^2 \log\log t$, then

$$\log\log t_n = \log(n\log\alpha) = \log n + \log\log\alpha$$

so $\exp(-f(t_n)/2\alpha) \le C_\alpha n^{-\alpha}$, where $C_\alpha$ is a constant that depends only on $\alpha$, and hence

$$\sum_{n=1}^\infty P_0\left(\max_{t_n\le s\le t_{n+1}} B_s > (t_n f(t_n))^{1/2}\right) < \infty$$

Since $t \to (tf(t))^{1/2}$ is increasing and $\alpha > 1$ is arbitrary, it follows that

$$\limsup B_t/(2t\log\log t)^{1/2} \le 1 \tag{8.8.4}$$

To prove the other half of Theorem 8.8.1, again let $t_n = \alpha^n$, but this time $\alpha$ will be large, since to get independent events, we look at

$$P_0\left(B(t_{n+1}) - B(t_n) > (t_{n+1}f(t_{n+1}))^{1/2}\right) = P_0\left(B_1 > (\beta f(t_{n+1}))^{1/2}\right)$$

where $\beta = t_{n+1}/(t_{n+1} - t_n) = \alpha/(\alpha - 1) > 1$. The last quantity is

$$\ge \frac{\kappa}{2}(\beta f(t_{n+1}))^{-1/2}\exp(-\beta f(t_{n+1})/2)$$

if $n$ is large by (8.8.3). If $f(t) = (2/\beta^2)\log\log t$, then $\log\log t_n = \log n + \log\log\alpha$, so

$$\exp(-\beta f(t_{n+1})/2) \geq C_\alpha n^{-1/\beta}$$

where $C_\alpha$ is a constant that depends only on $\alpha$, and hence

$$\sum_{n=1}^{\infty} P_0\left(B(t_{n+1}) - B(t_n) > (t_{n+1} f(t_{n+1}))^{1/2}\right) = \infty$$

Since the events in question are independent, it follows from the second Borel-Cantelli lemma that

$$B(t_{n+1}) - B(t_n) > ((2/\beta^2)t_{n+1}\log\log t_{n+1})^{1/2} \quad \text{i.o.} \tag{8.8.5}$$

From (8.8.4), we get

$$\limsup_{n\to\infty} B(t_n)/(2t_n \log\log t_n)^{1/2} \leq 1 \tag{8.8.6}$$

Since $t_n = t_{n+1}/\alpha$ and $t \to \log\log t$ is increasing, combining (8.8.5) and (8.8.6) and recalling $\beta = \alpha/(\alpha - 1)$ gives

$$\limsup_{n\to\infty} B(t_{n+1})/(2t_{n+1}\log\log t_{n+1})^{1/2} \geq \frac{\alpha - 1}{\alpha} - \frac{1}{\alpha^{1/2}}$$

Letting $\alpha \to \infty$ now gives the desired lower bound, and the proof of Theorem 8.8.1 is complete. ■

**Exercise 8.8.1.** Let $t_k = \exp(e^k)$. Show that

$$\limsup_{k\to\infty} B(t_k)/(2t_k \log\log\log t_k)^{1/2} = 1 \quad \text{a.s.}$$

Theorem 8.2.6 implies that $X_t = tB(1/t)$ is a Brownian motion. Changing variables and using Theorem 8.8.1, we conclude

$$\limsup_{t\to 0} |B_t|/(2t\log\log(1/t))^{1/2} = 1 \quad \text{a.s.} \tag{8.8.7}$$

To take a closer look at the local behavior of Brownian paths, we note that Blumenthal's 0-1 law, Theorem 8.2.3, implies $P_0(B_t < h(t)$ for all $t$ sufficiently small$) \in \{0, 1\}$. $h$ is said to belong to the **upper class** if the probability is 1, the **lower class** if it is 0.

**Theorem 8.8.2. Kolmogorov's test.** *If $h(t) \uparrow$ and $t^{-1/2}h(t) \downarrow$, then $h$ is upper or lower class according as*

$$\int_0^1 t^{-3/2} h(t) \exp(-h^2(t)/2t)\, dt \quad \textit{converges or diverges}$$

Recalling (8.4.8), we see that the integrand is the probability of hitting $h(t)$ at time $t$. To see what Theorem 8.8.2 says, define $\lg_k(t) = \log(\lg_{k-1}(t))$ for $k \geq 2$ and

$t > a_k = \exp(a_{k-1})$, where $\lg_1(t) = \log(t)$ and $a_1 = 0$. A little calculus shows that when $n \geq 4$,

$$h(t) = \left(2t\left\{\lg_2(1/t) + \frac{3}{2}\lg_3(1/t) + \sum_{m=4}^{n-1}\lg_m(1/t) + (1+\epsilon)\lg_n(1/t)\right\}\right)^{1/2}$$

is upper or lower class according as $\epsilon > 0$ or $\epsilon \leq 0$.

Approximating $h$ from above by piecewise constant functions, it is easy to show that if the integral in Theorem 8.8.2 converges, $h(t)$ is an upper class function. The proof of the other direction is much more difficult; see Motoo (1959) or Section 4.12 of Itô and McKean (1965).

Turning to random walk, we will prove a result due to Hartman and Wintner (1941):

**Theorem 8.8.3.** *If $X_1, X_2, \ldots$ are i.i.d. with $EX_i = 0$ and $EX_i^2 = 1$, then*

$$\limsup_{n\to\infty} S_n/(2n\log\log n)^{1/2} = 1$$

*Proof.* By Theorem 8.6.2, we can write $S_n = B(T_n)$ with $T_n/n \to 1$ a.s. As in the proof of Donsker's theorem, this is all we will use in the argument below. Theorem 8.8.3 will follow from Theorem 8.8.1 once we show

$$(S_{[t]} - B_t)/(t\log\log t)^{1/2} \to 0 \quad \text{a.s.} \tag{8.8.8}$$

To do this, we begin by observing that if $\epsilon > 0$ and $t \geq t_o(\omega)$

$$T_{[t]} \in [t/(1+\epsilon), t(1+\epsilon)] \tag{8.8.9}$$

To estimate $S_{[t]} - B_t$, we let $M(t) = \sup\{|B(s) - B(t)| : t/(1+\epsilon) \leq s \leq t(1+\epsilon)\}$. To control the last quantity, we let $t_k = (1+\epsilon)^k$ and notice that if $t_k \leq t \leq t_{k+1}$,

$$\begin{aligned} M(t) &\leq \sup\{|B(s) - B(t)| : t_{k-1} \leq s, t \leq t_{k+2}\} \\ &\leq 2\ \sup\{|B(s) - B(t_{k-1})| : t_{k-1} \leq s \leq t_{k+2}\} \end{aligned}$$

Noticing $t_{k+2} - t_{k-1} = \delta t_{k-1}$, where $\delta = (1+\epsilon)^3 - 1$, scaling implies

$$\begin{aligned} &P\left(\max_{t_{k-1}\leq s\leq t_{k+2}} |B(s) - B(t)| > (3\delta t_{k-1}\log\log t_{k-1})^{1/2}\right) \\ &= P\left(\max_{0\leq r\leq 1} |B(r)| > (3\log\log t_{k-1})^{1/2}\right) \\ &\leq 2\kappa(3\log\log t_{k-1})^{-1/2}\exp(-3\log\log t_{k-1}/2) \end{aligned}$$

by a now-familiar application of (8.8.1) and (8.8.2). Summing over $k$ and using (b) gives

$$\limsup_{t\to\infty} (S_{[t]} - B_t)/(t\log\log t)^{1/2} \leq (3\delta)^{1/2}$$

If we recall $\delta = (1+\epsilon)^3 - 1$ and let $\epsilon \downarrow 0$, (a) follows and the proof is complete. ■

**Exercise 8.8.2.** Show that if $E|X_i|^\alpha = \infty$ for some $\alpha < 2$, then

$$\limsup_{n\to\infty} |X_n|/n^{1/\alpha} = \infty \quad \text{a.s.}$$

so the law of the iterated logarithm fails.

Strassen (1965) has shown an exact converse. If Theorem 8.8.3 holds then $EX_i = 0$ and $EX_i^2 = 1$. Another one of his contributions to this subject is

**Theorem 8.8.4. Strassen's (1964) invariance principle.** *Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = 1$, let $S_n = X_1 + \cdots + X_n$, and let $S_{(n\cdot)}$ be the usual linear interpolation. The limit set (i.e., the collection of limits of convergent subsequences) of*

$$Z_n(\cdot) = (2n \log\log n)^{-1/2} S(n\cdot) \quad \textit{for } n \geq 3$$

*is $\mathcal{K} = \{f : f(x) = \int_0^x g(y)\,dy \text{ with } \int_0^1 g(y)^2\,dy \leq 1\}$.*

Jensen's inequality implies $f(1)^2 \leq \int_0^1 g(y)^2\,dy \leq 1$ with equality if and only if $f(t) = t$, so Theorem 8.8.4 contains Theorem 8.8.3 as a special case and provides some information about how the large value of $S_n$ came about.

**Exercise 8.8.3.** Give a direct proof that, under the hypotheses of Theorem 8.8.4, the limit set of $\{S_n/(2n \log\log n)^{1/2}\}$ is $[-1, 1]$.

# Appendix A
## Measure Theory Details

This Appendix proves the results from measure theory that were stated but not proved in the text.

### A.1 Carathéodory's Extension Theorem

This section is devoted to the proof of:

**Theorem A.1.1.** *Let $\mathcal{S}$ be a semialgebra, and let $\mu$ defined on $\mathcal{S}$ have $\mu(\emptyset) = 0$. Suppose (i) if $S \in \mathcal{S}$ is a finite disjoint union of sets $S_i \in \mathcal{S}$, then $\mu(S) = \sum_i \mu(S_i)$, and (ii) if $S_i, S \in \mathcal{S}$ with $S = +_{i \geq 1} S_i$, then $\mu(S) \leq \sum_i \mu(S_i)$. Then $\mu$ has a unique extension $\bar{\mu}$ that is a measure on $\bar{\mathcal{S}}$ the algebra generated by $\mathcal{S}$. If the extension is $\sigma$-finite, then there is a unique extension $\nu$ that is a measure on $\sigma(\mathcal{S})$.*

*Proof.* Lemma 1.1.3 shows that $\bar{\mathcal{S}}$ is the collection of finite disjoint unions of sets in $S$. We define $\bar{\mu}$ on $\bar{\mathcal{S}}$ by $\bar{\mu}(A) = \sum_i \mu(S_i)$ whenever $A = +_i S_i$. To check that $\bar{\mu}$ is well defined, suppose that $A = +_j T_j$ and observe $S_i = +_j (S_i \cap T_j)$ and $T_j = +_i (S_i \cap T_j)$, so (i) implies

$$\sum_i \mu(S_i) = \sum_{i,j} \mu(S_i \cap T_j) = \sum_j \mu(T_j) \qquad \blacksquare$$

In Section 1.1 we proved:

**Lemma A.1.2.** *Suppose only that (i) holds.*
*(a) If $A, B_i \in \bar{\mathcal{S}}$ with $A = +_{i=1}^n B_i$ then $\bar{\mu}(A) = \sum_i \bar{\mu}(B_i)$.*
*(b) If $A, B_i \in \bar{\mathcal{S}}$ with $A \subset \cup_{i=1}^n B_i$ then $\bar{\mu}(A) \leq \sum_i \bar{\mu}(B_i)$.*

To extend the additivity property to $A \in \bar{\mathcal{S}}$ that are countable disjoint unions $A = +_{i \geq 1} B_i$, where $B_i \in \bar{\mathcal{S}}$, we observe that each $B_i = +_j S_{i,j}$ with $S_{i,j} \in \mathcal{S}$ and $\sum_{i \geq 1} \bar{\mu}(B_i) = \sum_{i \geq 1, j} \mu(S_{i,j})$, so replacing the $B_i$'s by $S_{i,j}$'s, we can without loss of generality suppose that the $B_i \in \mathcal{S}$. Now $A \in \bar{\mathcal{S}}$ implies $A = +_j T_j$ (a finite disjoint union) and $T_j = +_{i \geq 1} T_j \cap B_i$, so (ii) implies

$$\mu(T_j) \leq \sum_{i \geq 1} \mu(T_j \cap B_i)$$

Summing over $j$ and observing that nonnegative numbers can be summed in any order,

$$\bar{\mu}(A) = \sum_j \mu(T_j) \leq \sum_{i \geq 1} \sum_j \mu(T_j \cap B_i) = \sum_{i \geq 1} \mu(B_i)$$

the last equality following from (i). To prove the opposite inequality, let $A_n = B_1 + \cdots + B_n$, and $C_n = A \cap A_n^c$. $C_n \in \bar{\mathcal{S}}$, since $\bar{\mathcal{S}}$ is an algebra, so finite additivity of $\bar{\mu}$ implies

$$\bar{\mu}(A) = \bar{\mu}(B_1) + \cdots + \bar{\mu}(B_n) + \bar{\mu}(C_n) \geq \bar{\mu}(B_1) + \cdots + \bar{\mu}(B_n)$$

and letting $n \to \infty$, $\bar{\mu}(A) \geq \sum_{i \geq 1} \bar{\mu}(B_i)$.

Having defined a measure on the algebra $\bar{\mathcal{S}}$, we now complete the proof by establishing

**Theorem A.1.3. Carathéodory's Extension Theorem.** *Let $\mu$ be a $\sigma$-finite measure on an algebra $\mathcal{A}$. Then $\mu$ has a unique extension to $\sigma(\mathcal{A})$ = the smallest $\sigma$-algebra containing $\mathcal{A}$.*

**Uniqueness.** We will prove that the extension is unique before tackling the more difficult problem of proving its existence. The key to our uniqueness proof is Dynkin's $\pi - \lambda$ theorem, a result that we will use many times in the book. As usual, we need a few definitions before we can state the result. $\mathcal{P}$ is said to be a **$\pi$-system** if it is closed under intersection, that is, if $A, B \in \mathcal{P}$ then $A \cap B \in \mathcal{P}$. For example, the collection of rectangles $(a_1, b_1] \times \cdots \times (a_d, b_d]$ is a $\pi$-system. $\mathcal{L}$ is said to be a **$\lambda$-system** if it satisfies: (i) $\Omega \in \mathcal{L}$. (ii) If $A, B \in \mathcal{L}$ and $A \subset B$, then $B - A \in \mathcal{L}$ . (iii) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$, then $A \in \mathcal{L}$ . The reader will see in a moment that the next result is just what we need to prove uniqueness of the extension.

**Theorem A.1.4. $\pi - \lambda$ Theorem.** *If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*

*Proof.* We will show that

(a) if $\ell(\mathcal{P})$ is the smallest $\lambda$-system containing $\mathcal{P}$, then $\ell(\mathcal{P})$ is a $\sigma$-field.

The desired result follows from (a). To see this, note that since $\sigma(\mathcal{P})$ is the smallest $\sigma$-field and $\ell(\mathcal{P})$ is the smallest $\lambda$-system containing $\mathcal{P}$, we have

$$\sigma(\mathcal{P}) \subset \ell(\mathcal{P}) \subset \mathcal{L}$$

To prove (a) we begin by noting that a $\lambda$-system that is closed under intersection is a $\sigma$-field since

$$\text{if } A \in \mathcal{L} \text{ then } A^c = \Omega - A \in \mathcal{L}$$
$$A \cup B = (A^c \cap B^c)^c$$
$$\cup_{i=1}^n A_i \uparrow \cup_{i=1}^\infty A_i \text{ as } n \uparrow \infty$$

Thus, it is enough to show

(b) $\ell(\mathcal{P})$ is closed under intersection.

To prove (b), we let $\mathcal{G}_A = \{B : A \cap B \in \ell(\mathcal{P})\}$ and prove

(c) if $A \in \ell(\mathcal{P})$, then $\mathcal{G}_A$ is a $\lambda$-system.

To check this, we note:

(i) $\Omega \in \mathcal{G}_A$ since $A \in \ell(\mathcal{P})$.
(ii) if $B, C \in \mathcal{G}_A$ and $B \supset C$, then $A \cap (B - C) = (A \cap B) - (A \cap C) \in \ell(\mathcal{P})$ since $A \cap B$, $A \cap C \in \ell(\mathcal{P})$ and $\ell(\mathcal{P})$ is a $\lambda$-system.
(iii) if $B_n \in \mathcal{G}_A$ and $B_n \uparrow B$, then $A \cap B_n \uparrow A \cap B \in \ell(\mathcal{P})$ since $A \cap B_n \in \ell(\mathcal{P})$ and $\ell(\mathcal{P})$ is a $\lambda$-system.

To get from (c) to (b), we note that since $\mathcal{P}$ is a $\pi$-system,

$$\text{if } A \in \mathcal{P} \text{ then } \mathcal{G}_A \supset \mathcal{P}, \text{ and so (c) implies } \mathcal{G}_A \supset \ell(\mathcal{P})$$

that is, if $A \in \mathcal{P}$ and $B \in \ell(\mathcal{P})$, then $A \cap B \in \ell(\mathcal{P})$. Interchanging $A$ and $B$ in the last sentence: if $A \in \ell(\mathcal{P})$ and $B \in \mathcal{P}$ then $A \cap B \in \ell(\mathcal{P})$ but this implies

$$\text{if } A \in \ell(\mathcal{P}) \text{ then } \mathcal{G}_A \supset \mathcal{P} \text{ and so (c) implies } \mathcal{G}_A \supset \ell(\mathcal{P}).$$

This conclusion implies that if $A, B \in \ell(\mathcal{P})$, then $A \cap B \in \ell(\mathcal{P})$, which proves (b) and completes the proof. ■

To prove that the extension in Theorem A.1.3 is unique, we will show:

**Theorem A.1.5.** *Let $\mathcal{P}$ be a $\pi$-system. If $\nu_1$ and $\nu_2$ are measures (on $\sigma$-fields $\mathcal{F}_1$ and $\mathcal{F}_2$) that agree on $\mathcal{P}$ and there is a sequence $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$ and $\nu_i(A_n) < \infty$, then $\nu_1$ and $\nu_2$ agree on $\sigma(\mathcal{P})$.*

*Proof.* Let $A \in \mathcal{P}$ have $\nu_1(A) = \nu_2(A) < \infty$. Let

$$\mathcal{L} = \{B \in \sigma(\mathcal{P}) : \nu_1(A \cap B) = \nu_2(A \cap B)\}$$

We will now show that $\mathcal{L}$ is a $\lambda$-system. Since $A \in \mathcal{P}$, $\nu_1(A) = \nu_2(A)$ and $\Omega \in \mathcal{L}$. If $B, C \in \mathcal{L}$ with $C \subset B$, then

$$\begin{aligned}\nu_1(A \cap (B - C)) &= \nu_1(A \cap B) - \nu_1(A \cap C)\\ &= \nu_2(A \cap B) - \nu_2(A \cap C) = \nu_2(A \cap (B - C))\end{aligned}$$

Here we use the fact that $\nu_i(A) < \infty$ to justify the subtraction. Finally, if $B_n \in \mathcal{L}$ and $B_n \uparrow B$, then part (iii) of Theorem 1.1.1 implies

$$\nu_1(A \cap B) = \lim_{n\to\infty} \nu_1(A \cap B_n) = \lim_{n\to\infty} \nu_2(A \cap B_n) = \nu_2(A \cap B)$$

Since $\mathcal{P}$ is closed under intersection by assumption, the $\pi - \lambda$ theorem implies $\mathcal{L} \supset \sigma(\mathcal{P})$, that is, if $A \in \mathcal{P}$ with $\nu_1(A) = \nu_2(A) < \infty$ and $B \in \sigma(\mathcal{P})$, then $\nu_1(A \cap B) = \nu_2(A \cap B)$. Letting $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$, $\nu_1(A_n) = \nu_2(A_n) < \infty$, and using the last result and part (iii) of Theorem 1.1.1, we have the desired conclusion. ■

**Exercise A.1.1.** Give an example of two probability measures $\mu \neq \nu$ on $\mathcal{F} =$ all subsets of $\{1, 2, 3, 4\}$ that agree on a collection of sets $\mathcal{C}$ with $\sigma(\mathcal{C}) = \mathcal{F}$, that is, the smallest $\sigma$-algebra containing $\mathcal{C}$ is $\mathcal{F}$.

**Existence.** Our next step is to show that a measure (not necessarily $\sigma$-finite) defined on an algebra $\mathcal{A}$ has an extension to the $\sigma$-algebra generated by $\mathcal{A}$. If $E \subset \Omega$, we let $\mu^*(E) = \inf \sum_i \mu(A_i)$ where the infimum is taken over all sequences from $\mathcal{A}$ so that $E \subset \cup_i A_i$. Intuitively, if $\nu$ is a measure that agrees with $\mu$ on $\mathcal{A}$, then it follows from part (ii) of Theorem 1.1.1 that

$$\nu(E) \le \nu(\cup_i A_i) \le \sum_i \nu(A_i) = \sum_i \mu(A_i)$$

so $\mu^*(E)$ is an upper bound on the measure of $E$. Intuitively, the measurable sets are the ones for which the upper bound is tight. Formally, we say that $E$ is **measurable** if

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c) \quad \text{for all sets } F \subset \Omega \tag{A.1.1}$$

The last definition is not very intuitive, but we will see in the proofs below that it works very well.

It is immediate from the definition that $\mu^*$ has the following properties:

(i) **Monotonicity**. If $E \subset F$ then $\mu^*(E) \le \mu^*(F)$.
(ii) **Subadditivity**. If $F \subset \cup_i F_i$, a countable union, then $\mu^*(F) \le \sum_i \mu^*(F_i)$.

Any set function with $\mu^*(\emptyset) = 0$ that satisfies (i) and (ii) is called an **outer measure**. Using (ii) with $F_1 = F \cap E$ and $F_2 = F \cap E^c$ (and $F_i = \emptyset$ otherwise), we see that to prove a set is measurable, it is enough to show

$$\mu^*(F) \ge \mu^*(F \cap E) + \mu^*(F \cap E^c) \tag{A.1.2}$$

We begin by showing that our new definition extends the old one.

**Lemma A.1.6.** *If $A \in \mathcal{A}$, then $\mu^*(A) = \mu(A)$ and $A$ is measurable.*

*Proof.* Part (ii) of Theorem 1.1.1 implies that if $A \subset \cup_i A_i$, then

$$\mu(A) \le \sum_i \mu(A_i)$$

so $\mu(A) \le \mu^*(A)$. Of course, we can always take $A_1 = A$ and the other $A_i = \emptyset$ so $\mu^*(A) \le \mu(A)$.

To prove that any $A \in \mathcal{A}$ is measurable, we begin by noting that the inequality is (A.1.2) trivial when $\mu^*(F) = \infty$, so we can without loss of generality assume $\mu^*(F) < \infty$. To prove that (A.1.2) holds when $E = A$, we observe that since $\mu^*(F) < \infty$ there is a sequence $B_i \in \mathcal{A}$ so that $\cup_i B_i \supset F$ and

$$\sum_i \mu(B_i) \le \mu^*(F) + \epsilon$$

Since $\mu$ is additive on $\mathcal{A}$, and $\mu = \mu^*$ on $\mathcal{A}$, we have

$$\mu(B_i) = \mu^*(B_i \cap A) + \mu^*(B_i \cap A^c)$$

Summing over $i$ and using the subadditivity of $\mu^*$ gives

$$\mu^*(F) + \epsilon \ge \sum_i \mu^*(B_i \cap A) + \sum_i \mu^*(B_i \cap A^c) \ge \mu^*(F \cap A) + \mu^*(F^c \cap A)$$

which proves the desired result since $\epsilon$ is arbitrary. ■

**Lemma A.1.7.** *The class $\mathcal{A}^*$ of measurable sets is a $\sigma$-field, and the restriction of $\mu^*$ to $\mathcal{A}^*$ is a measure.*

**Remark.** This result is true for any outer measure.

*Proof.* It is clear from the definition that

(a) If $E$ is measurable, then $E^c$ is.

Our first nontrivial task is to prove

(b) If $E_1$ and $E_2$ are measurable, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are.

*Proof of (b).* To prove the first conclusion, let $G$ be any subset of $\Omega$. Using subadditivity, the measurability of $E_2$ (let $F = G \cap E_1^c$ in (A.1.1), and the measurability of $E_1$, we get

$$\begin{aligned}
\mu^*(G \cap (E_1 \cup E_2)) &+ \mu^*(G \cap (E_1^c \cap E_2^c)) \\
&\le \mu^*(G \cap E_1) + \mu^*(G \cap E_1^c \cap E_2) + \mu^*(G \cap E_1^c \cap E_2^c) \\
&= \mu^*(G \cap E_1) + \mu^*(G \cap E_1^c) = \mu^*(G)
\end{aligned}$$

To prove that $E_1 \cap E_2$ is measurable, we observe $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$ and use (a). ■

(c) Let $G \subset \Omega$ and $E_1, \ldots, E_n$ be disjoint measurable sets. Then

$$\mu^* \left(G \cap \cup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu^*(G \cap E_i)$$

*Proof of (c).* Let $F_m = \cup_{i \le m} E_i$. $E_n$ is measurable, $F_n \supset E_n$, and $F_{n-1} \cap E_n = \emptyset$, so

$$\begin{aligned}\mu^*(G \cap F_n) &= \mu^*(G \cap F_n \cap E_n) + \mu^*(G \cap F_n \cap E_n^c) \\ &= \mu^*(G \cap E_n) + \mu^*(G \cap F_{n-1})\end{aligned}$$

The desired result follows from this by induction. ■

(d) If the sets $E_i$ are measurable, then $E = \cup_{i=1}^\infty E_i$ is measurable.

*Proof of (d).* Let $E_i' = E_i \cap (\cap_{j<i} E_j^c)$. (a) and (b) imply $E_i'$ is measurable, so we can suppose without loss of generality that the $E_i$ are pairwise disjoint. Let $F_n = E_1 \cup \cdots \cup E_n$. $F_n$ is measurable by (b), so using monotonicity and (c) we have

$$\begin{aligned}\mu^*(G) = \mu^*(G \cap F_n) + \mu^*(G \cap F_n^c) &\ge \mu^*(G \cap F_n) + \mu^*(G \cap E^c) \\ &= \sum_{i=1}^n \mu^*(G \cap E_i) + \mu^*(G \cap E^c)\end{aligned}$$

Letting $n \to \infty$ and using subadditivity

$$\mu^*(G) \ge \sum_{i=1}^\infty \mu^*(G \cap E_i) + \mu^*(G \cap E^c) \ge \mu^*(G \cap E) + \mu^*(G \cap E^c)$$

which is (A.1.2). ■

The last step in the proof of Theorem A.1.7 is

(e) If $E = \cup_i E_i$ where $E_1, E_2, \ldots$ are disjoint and measurable, then

$$\mu^*(E) = \sum_{i=1}^\infty \mu^*(E_i)$$

*Proof of (e).* Let $F_n = E_1 \cup \cdots \cup E_n$. By monotonicity and (c)

$$\mu^*(E) \ge \mu^*(F_n) = \sum_{i=1}^n \mu^*(E_i)$$

Letting $n \to \infty$ now and using subadditivity gives the desired conclusion. ■

## A.2 Which Sets Are Measurable?

The proof of Theorem A.1.3 given in the last section defines an extension to $\mathcal{A}^* \supset \sigma(\mathcal{A})$. Our next goal is to describe the relationship between these two $\sigma$-algebras. Let $\mathcal{A}_\sigma$ denote the collection of countable unions of sets in $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ denote the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Our first goal is to show that every measurable set is almost a set in $\mathcal{A}_{\sigma\delta}$.

Define the symetric difference by $A\Delta B = (A - B) \cup (B - A)$.

**Lemma A.2.1.** *Let $E$ be any set with $\mu^*(E) < \infty$.*
*(i) For any $\epsilon > 0$, there is an $A \in \mathcal{A}_\sigma$ with $A \supset E$ and $\mu^*(A) \le \mu^*(E) + \epsilon$.*
*(ii) For any $\epsilon > 0$, there is a $B \in \mathcal{A}$ with $\mu(B\Delta E) \le 2\epsilon$, where*
*(ii) There is a $C \in \mathcal{A}_{\sigma\delta}$ with $C \supset E$ and $\mu^*(C) = \mu^*(E)$.*

*Proof.* By the definition of $\mu^*$, there is a sequence $A_i$ so that $A \equiv \cup_i A_i \supset E$ and $\sum_i \mu(A_i) \le \mu^*(E) + \epsilon$. The definition of $\mu^*$ implies $\mu^*(A) \le \sum_i \mu(A_i)$, establishing (i).

For (ii) we note that there is a finite union $B = \cup i = 1^n A_i$ so that $\mu(A - B) \le \epsilon$, and hence $\mu(E - B) \le \epsilon$. Since $\mu(B - E) \le \mu(A - E) \le \epsilon$ the desired result follows.

For (iii), let $A_n \in \mathcal{A}_\sigma$ with $A_n \supset E$ and $\mu^*(A_n) \le \mu^*(E) + 1/n$, and let $C = \cap_n A_n$. Clearly, $C \in \mathcal{A}_{\sigma\delta}$, $B \supset E$, and hence by monotonicity, $\mu^*(C) \ge \mu^*(E)$. To prove the other inequality, notice that $B \subset A_n$ and hence $\mu^*(C) \le \mu^*(A_n) \le \mu^*(E) + 1/n$ for any $n$. ∎

**Theorem A.2.2.** *Suppose $\mu$ is $\sigma$-finite on $\mathcal{A}$. $B \in \mathcal{A}^*$ if and only if there is an $A \in \mathcal{A}_{\sigma\delta}$ and a set $N$ with $\mu^*(N) = 0$ so that $B = A - N (= A \cap N^c)$.*

*Proof.* It follows from Lemma A.1.6 and A.1.7 if $A \in \mathcal{A}_{\sigma\delta}$ then $A \in \mathcal{A}^*$. A.1.2 in Section A.1 and monotonicity imply sets with $\mu^*(N) = 0$ are measurable, so using Lemma A.1.7 again it follows that $A \cap N^c \in \mathcal{A}^*$. To prove the other direction, let $\Omega_i$ be a disjoint collection of sets with $\mu(\Omega_i) < \infty$ and $\Omega = \cup_i \Omega_i$. Let $B_i = B \cap \Omega_i$ and use Lemma A.2.1 to find $A_i^n \in \mathcal{A}_\sigma$ so that $A_i^n \supset B_i$ and $\mu(A_i^n) \le \mu^*(E_i) + 1/n2^i$. Let $A_n = \cup_i A_i^n$. $B \subset A_n$ and

$$A_n - B \subset \sum_{i=1}^{\infty} (A_i^n - B_i)$$

so, by subadditivity,

$$\mu^*(A_n - B) \le \sum_{i=1}^{\infty} \mu^*(A_i^n - B_i) \le 1/n$$

Since $A_n \in \mathcal{A}_\sigma$, the set $A = \cap_n A_n \in \mathcal{A}_{\sigma\delta}$. Clearly, $A \supset B$. Since $N \equiv A - B \subset A_n - B$ for all $n$, monotonicity implies $\mu^*(N) = 0$, and the proof of is complete. ∎

A measure space $(\Omega, \mathcal{F}, \mu)$ is said to be **complete** if $\mathcal{F}$ contains all subsets of sets of measure 0. In the proof of Theorem A.2.2, we showed that $(\Omega, \mathcal{A}^*, \mu^*)$ is complete. Our next result shows that $(\Omega, \mathcal{A}^*, \mu^*)$ is the completion of $(\Omega, \sigma(\mathcal{A}), \mu)$.

**Theorem A.2.3.** *If $(\Omega, \mathcal{F}, \mu)$ is a measure space, then there is a complete measure space $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$, called the* **completion** *of $(\Omega, \mathcal{F}, \mu)$, so that (i) $E \in \bar{\mathcal{F}}$ if and only if $E = A \cup B$, where $A \in \mathcal{F}$ and $B \subset N \in \mathcal{F}$ with $\mu(N) = 0$, and (ii) $\bar{\mu}$ agrees with $\mu$ on $\mathcal{F}$.*

*Proof.* The first step is to check that $\bar{\mathcal{F}}$ is a $\sigma$-algebra. If $E_i = A_i \cup B_i$ where $A_i \in \mathcal{F}$ and $B_i \subset N_i$ where $\mu(N_i) = 0$, then $\cup_i A_i \in \mathcal{F}$, and subadditivity implies $\mu(\cup_i N_i) \le \sum_i \mu(N_i) = 0$, so $\cup_i E_i \in \bar{\mathcal{F}}$ . As for complements, if $E = A \cup B$ and $B \subset N$, then $B^c \supset N^c$, so

$$E^c = A^c \cap B^c = (A^c \cap N^c) \cup (A^c \cap B^c \cap N)$$

$A^c \cap N^c$ is in $\mathcal{F}$ and $A^c \cap B^c \cap N \subset N$, so $E^c \in \bar{\mathcal{F}}$.

We define $\bar{\mu}$ in the obvious way: If $E = A \cup B$ where $A \in \mathcal{F}$ and $B \subset N$ where $\mu(N) = 0$, then we let $\bar{\mu}(E) = \mu(A)$. The first thing to show is that $\bar{\mu}$ is well defined, that is, if $E = A_i \cup B_i$, $i = 1, 2$, are two decompositions, then $\mu(A_1) = \mu(A_2)$. Let $A_0 = A_1 \cap A_2$ and $B_0 = B_1 \cup B_2$. $E = A_0 \cup B_0$ is a third decomposition with $A_0 \in \mathcal{F}$ and $B_0 \subset N_1 \cup N_2$, and has the pleasant property that if $i = 1$ or 2,

$$\mu(A_0) \le \mu(A_i) \le \mu(A_0) + \mu(N_1 \cup N_2) = \mu(A_0)$$

The last detail is to check that $\bar{\mu}$ is measure, but that is easy. If $E_i = A_i \cup B_i$ are disjoint, then $\cup_i E_i$ can be decomposed as $\cup_i A_i \cup (\cup_i B_i)$, and the $A_i \subset E_i$ are disjoint, so

$$\bar{\mu}(\cup_i E_i) = \mu(\cup_i A_i) = \sum_i \mu(A_i) = \sum_i \bar{\mu}(E_i) \qquad \blacksquare$$

Theorem 1.1.6 allows us to construct Lebesgue measure $\lambda$ on $(\mathbf{R}^d, \mathcal{R}^d)$. Using Theorem A.2.3, we can extend $\lambda$ to be a measure on $(\mathbf{R}, \bar{\mathcal{R}}^d)$, where $\bar{\mathcal{R}}^d$ is the completion of $\mathcal{R}^d$. Having done this, it is natural (if somewhat optimistic) to ask: Are there any sets that are not in $\bar{\mathcal{R}}^d$? The answer is "Yes," and we will now give an example of a nonmeasurable $B$ in $\mathbf{R}$.

*A nonmeasurable subset of [0,1)*

The key to our construction is the observation that $\lambda$ is translation invariant: that is, if $A \in \bar{\mathcal{R}}$ and $x + A = \{x + y : y \in A\}$, then $x + A \in \bar{\mathcal{R}}$ and $\lambda(A) = \lambda(x + A)$. We say that $x, y \in [0, 1)$ are equivalent and write $x \sim y$ if $x - y$ is a rational number. By the axiom of choice, there is a set $B$ that contains exactly one element from each equivalence class. $B$ is our nonmeasurable set, that is,

**Theorem A.2.4.** *$B \notin \bar{\mathcal{R}}$.*

*Proof.* The key is the following:

**Lemma A.2.5.** *If $E \subset [0, 1)$ is in $\bar{\mathcal{R}}$, $x \in (0, 1)$, and $x +' E = \{(x + y) \bmod 1 : y \in E\}$, then $\lambda(E) = \lambda(x +' E)$.*

*Proof.* Let $A = E \cap [0, 1 - x)$ and $B = E \cap [1 - x, 1)$. Let $A' = x + A = \{x + y : y \in A\}$ and $B' = x - 1 + B$. $A, B \in \bar{\mathcal{R}}$, so by translation invariance $A', B' \in \bar{\mathcal{R}}$ and $\lambda(A) = \lambda(A')$, $\lambda(B) = \lambda(B')$. Since $A' \subset [x, 1)$ and $B' \subset [0, x)$ are disjoint,

$$\lambda(E) = \lambda(A) + \lambda(B) = \lambda(A') + \lambda(B') = \lambda(x +' E) \qquad \blacksquare$$

From Lemma A.2.5, it follows easily that $B$ is not measurable; if it were, then $q +' B$, $q \in \mathbf{Q} \cap [0, 1)$ would be a countable disjoint collection of measurable subsets of [0,1), all with the same measure $\alpha$ and having

$$\cup_{q \in \mathbf{Q} \cap [0,1)} (q +' B) = [0, 1)$$

If $\alpha > 0$, then $\lambda([0, 1)) = \infty$, and if $\alpha = 0$, then $\lambda([0, 1)) = 0$. Neither conclusion is compatible with the fact that $\lambda([0, 1)) = 1$, so $B \notin \bar{\mathcal{R}}$. $\blacksquare$

**Exercise A.2.1.** Let $B$ be the nonmeasurable set constructed in Theorem A.2.4. (i) Let $B_q = q +' B$ and show that if $D_q \subset B_q$ is measurable, then $\lambda(D_q) = 0$. (ii) Use (i) to conclude that if $A \subset \mathbf{R}$ has $\lambda(A) > 0$, there is a nonmeasurable $S \subset A$.

Letting $B' = B \times [0, 1]^{d-1}$ where $B$ is our nonmeasurable subset of (0,1), we get a nonmeasurable set in $d > 1$. In $d = 3$, there is a much more interesting example, but we need the reader to do some preliminary work. In Euclidean geometry, two subsets of $\mathbf{R}^d$ are said to be **congruent** if one set can be mapped onto the other by translations and rotations.

**Claim.** Two congruent measurable sets must have the same Lebesgue measure.

**Exercise A.2.2.** Prove the claim in $d = 2$ by showing (i) if $B$ is a rotation of a rectangle $A$ then $\lambda^*(B) = \lambda(A)$. (ii) If $C$ is congruent to $D$ then $\lambda^*(C) = \lambda^*(D)$.

### *Banach-Tarski theorem*

Banach and Tarski (1924) used the axiom of choice to show that it is possible to partition the sphere $\{x : |x| \leq 1\}$ in $\mathbf{R}^3$ into a finite number of sets $A_1, \ldots, A_n$ and find congruent sets $B_1, \ldots, B_n$ whose union is two disjoint spheres of radius 1! Since congruent sets have the same Lebesgue measure, at least one of the sets $A_i$ must be nonmeasurable. The construction relies on the fact that the group generated by rotations in $\mathbf{R}^3$ is not Abelian. Lindenbaum (1926) showed that this cannot be done with any bounded set in $\mathbf{R}^2$. For a popular account of the Banach-Tarski theorem, see French (1988).

*Solovay's theorem*

The axiom of choice played an important role in the last two constructions of nonmeasurable sets. Solovay (1970) proved that its use is unavoidable. In his own words, "We show that the existence of a non-Lebesgue measurable set cannot be proved in Zermelo-Frankel set theory if the use of the axiom of choice is disallowed." This should convince the reader that all subsets of $\mathbf{R}^d$ that arise "in practice" are in $\bar{\mathcal{R}}^d$.

## A.3 Kolmogorov's Extension Theorem

To construct some of the basic objects of study in probability theory, we will need an existence theorem for measures on infinite product spaces. Let $\mathbf{N} = \{1, 2, \ldots\}$ and

$$\mathbf{R}^{\mathbf{N}} = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \mathbf{R}\}$$

We equip $\mathbf{R}^{\mathbf{N}}$ with the product $\sigma$-algebra $\mathcal{R}^{\mathbf{N}}$, which is generated by the **finite dimensional rectangles** = sets of the form $\{\omega : \omega_i \in (a_i, b_i] \text{ for } i = 1, \ldots, n\}$, where $-\infty \le a_i < b_i \le \infty$.

**Theorem A.3.1. Kolmogorov's extension theorem.** *Suppose we are given probability measures $\mu_n$ on $(\mathbf{R}^n, \mathcal{R}^n)$ that are consistent, that is,*

$$\mu_{n+1}((a_1, b_1] \times \cdots \times (a_n, b_n] \times \mathbf{R}) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

*Then there is a unique probability measure $P$ on $(\mathbf{R}^{\mathbf{N}}, \mathcal{R}^{\mathbf{N}})$ with*

$$(*) \qquad P(\omega : \omega_i \in (a_i, b_i], 1 \le i \le n) = \mu_n((a_1, b_1] \times \cdots \times (a_n, b_n])$$

An important example of a consistent sequence of measures is

**Example A.3.1.** Let $F_1, F_2, \ldots$ be distribution functions, and let $\mu_n$ be the measure on $\mathbf{R}^n$ with

$$\mu_n((a_1, b_1] \times \cdots \times (a_n, b_n]) = \prod_{m=1}^{n} (F_m(b_m) - F_m(a_m))$$

In this case, if we let $X_n(\omega) = \omega_n$, then the $X_n$ are independent and $X_n$ has distribution $F_n$.

*Proof of Theorem A.3.1.* Let $\mathcal{S}$ be the sets of the form $\{\omega : \omega_i \in (a_i, b_i], 1 \le i \le n\}$, and use $(*)$ to define $P$ on $\mathcal{S}$. $\mathcal{S}$ is a semialgebra, so by Theorem A.1.1 it is enough to show that if $A \in \mathcal{S}$ is a disjoint union of $A_i \in \mathcal{S}$, then $P(A) \le \sum_i P(A_i)$. If the union is finite, then all the $A_i$ are determined by the values of a finite number of coordinates and the conclusion follows from the proof of Theorem 1.1.6.

Suppose now that the union is infinite. Let $\mathcal{A} = \{$ finite disjoint unions of sets in $\mathcal{S}\}$ be the algebra generated by $\mathcal{S}$. Since $\mathcal{A}$ is an algebra (by Lemma 1.1.3),

$$B_n \equiv A - \cup_{i=1}^n A_i$$

is a finite disjoint union of rectangles, and by the result for finite unions,

$$P(A) = \sum_{i=1}^n P(A_i) + P(B_n)$$

It suffices then to show

**Lemma A.3.2.** *If $B_n \in \mathcal{A}$ and $B_n \downarrow \emptyset$ then $P(B_n) \downarrow 0$.*

*Proof.* Suppose $P(B_n) \downarrow \delta > 0$. By repeating sets in the sequence, we can suppose

$$B_n = \cup_{k=1}^{K_n} \{\omega : \omega_i \in (a_i^k, b_i^k], 1 \le i \le n\} \quad \text{where } -\infty \le a_i^k < b_i^k \le \infty$$

The strategy of the proof is to approximate the $B_n$ from within by compact rectangles with almost the same probability and then use a diagonal argument to show that $\cap_n B_n \neq \emptyset$. There is a set $C_n \subset B_n$ of the form

$$C_n = \cup_{k=1}^{K_n} \{\omega : \omega_i \in [\bar{a}_i^k, \bar{b}_i^k], 1 \le i \le n\} \quad \text{with } -\infty < \bar{a}_k^i < \bar{b}_k^i < \infty$$

that has $P(B_n - C_n) \le \delta/2^{n+1}$. Let $D_n = \cap_{m=1}^n C_m$.

$$P(B_n - D_n) \le \sum_{m=1}^n P(B_m - C_m) \le \delta/2$$

so $P(D_n) \downarrow$ a limit $\ge \delta/2$. Now there are sets $C_n^*, D_n^* \subset \mathbf{R}^n$ so that

$$C_n = \{\omega : (\omega_1, \ldots, \omega_n) \in C_n^*\} \quad \text{and} \quad D_n = \{\omega : (\omega_1, \ldots, \omega_n) \in D_n^*\}$$

Note that

$$C_n = C_n^* \times \mathbf{R} \times \mathbf{R} \times \cdots \quad \text{and} \quad D_n = D_n^* \times \mathbf{R} \times \mathbf{R} \times \cdots$$

so $C_n$ and $C_n^*$ (and $D_n$ and $D_n^*$) are closely related but $C_n \subset \Omega$ and $C_n^* \subset \mathbf{R}^n$.

$C_n^*$ is a finite union of closed rectangles, so

$$D_n^* = C_n^* \cap_{m=1}^{n-1} (C_m^* \times \mathbf{R}^{n-m})$$

is a compact set. For each $m$, let $\omega_m \in D_m$. $D_m \subset D_1$ so $\omega_{m,1}$ (i.e., the first coordinate of $\omega_m$) is in $D_1^*$ Since $D_1^*$ is compact, we can pick a subsequence $m(1, j) \ge j$ so that as $j \to \infty$,

$$\omega_{m(1,j),1} \to \text{ a limit } \theta_1$$

For $m \ge 2$, $D_m \subset D_2$ and hence $(\omega_{m,1}, \omega_{m,2}) \in D_2^*$. Since $D_2^*$ is compact, we can pick a subsequence of the previous subsequence (i.e., $m(2, j) = m(1, i_j)$ with $i_j \ge j$) so that as $j \to \infty$

$$\omega_{m(2,j),2} \to \text{ a limit } \theta_2$$

Continuing in this way, we define $m(k, j)$ a subsequence of $m(k-1, j)$ so that as $j \to \infty$,

$$\omega_{m(k,j),k} \to \text{ a limit } \theta_k$$

Let $\omega'_i = \omega_{m(i,i)}$. $\omega'_i$ is a subsequence of all the subsequences so $\omega'_{i,k} \to \theta_k$ for all $k$. Now $\omega'_{i,1} \in D_1^*$ for all $i \geq 1$ and $D_1^*$ is closed so $\theta_1 \in D_1^*$. Turning to the second set, $(\omega'_{i,1}, \omega'_{i,2}) \in D_2^*$ for $i \geq 2$ and $D_2^*$ is closed, so $(\theta_1, \theta_2) \in D_2^*$. Repeating the last argument, we conclude that $(\theta_1, \dots, \theta_k) \in D_k^*$ for all $k$, so $\omega = (\theta_1, \theta_2, \dots) \in D_k$ (no star here since we are now talking about subsets of $\Omega$) for all $k$ and

$$\emptyset \neq \cap_k D_k \subset \cap_k B_k$$

a contraction that proves the desired result. ■

## A.4 Radon-Nikodym Theorem

In this section, we prove the Radon-Nikodym theorem. To develop that result, we begin with a topic that at first may appear to be unrelated. Let $(\Omega, \mathcal{F})$ be a measurable space. $\alpha$ is said to be a **signed measure** on $(\Omega, \mathcal{F})$ if (i) $\alpha$ takes values in $(-\infty, \infty]$, (ii) $\alpha(\emptyset) = 0$, and (iii) if $E = +_i E_i$ is a disjoint union then $\alpha(E) = \sum_i \alpha(E_i)$, in the following sense:

If $\alpha(E) < \infty$, the sum converges absolutely and $= \alpha(E)$.

If $\alpha(E) = \infty$, then $\sum_i \alpha(E_i)^- < \infty$ and $\sum_i \alpha(E_i)^+ = \infty$.

Clearly, a signed measure cannot be allowed to take both the values $\infty$ and $-\infty$, since $\alpha(A) + \alpha(B)$ might not make sense. In most formulations, a signed measure is allowed to take values in either $(-\infty, \infty]$ or $[-\infty, \infty)$. We will ignore the second possibility to simplify statements later. As usual, we turn to examples to help explain the definition.

**Example A.4.1.** Let $\mu$ be a measure, $f$ be a function with $\int f^- \, d\mu < \infty$, and let $\alpha(A) = \int_A f \, d\mu$. Exercise 5.8 implies that $\alpha$ is a signed measure.

**Example A.4.2.** Let $\mu_1$ and $\mu_2$ be measures with $\mu_2(\Omega) < \infty$, and let $\alpha(A) = \mu_1(A) - \mu_2(A)$.

The Jordan decomposition, (A.4.4) below, will show that Example A.4.2 is the general case. To derive that result, we begin with two definitions. A set $A$ is **positive** if every measurable $B \subset A$ has $\alpha(B) \geq 0$. A set $A$ is **negative** if every measurable $B \subset A$ has $\alpha(B) \leq 0$.

**Exercise A.4.1.** In Example A.4.1, $A$ is positive if and only if $\mu(A \cap \{x : f(x) < 0\}) = 0$.

**Lemma A.4.1.** *(i) Every measurable subset of a positive set is positive. (ii) If the sets $A_n$ are positive, then $A = \cup_n A_n$ is also positive.*

*Proof.* (i) is trivial. To prove (ii), observe that

$$B_n = A_n \cap \left(\cap_{m=1}^{n-1} A_m^c\right) \subset A_n$$

are positive, disjoint, and $\cup_n B_n = \cup_n A_n$. Let $E \subset A$ be measurable, and let $E_n = E \cap B_n$. $\alpha(E_n) \geq 0$ since $B_n$ is positive, so $\alpha(E) = \sum_n \alpha(E_n) \geq 0$. ■

The conclusions in Lemma A.4.1 remain valid if positive is replaced by negative. The next result is the key to the proof of Theorem A.4.3.

**Lemma A.4.2.** *Let $E$ be a measurable set with $\alpha(E) < 0$. Then there is a negative set $F \subset E$ with $\alpha(F) < 0$.*

*Proof.* If $E$ is negative, this is true. If not, let $n_1$ be the smallest positive integer so that there is an $E_1 \subset E$ with $\alpha(E_1) \geq 1/n_1$. Let $k \geq 2$. If $F_k = E - (E_1 \cup \cdots \cup E_{k-1})$ is negative, we are done. If not, we continue the construction letting $n_k$ be the smallest positive integer so that there is an $E_k \subset F_k$ with $\alpha(E_k) \geq 1/n_k$. If the construction does not stop for any $k < \infty$, let

$$F = \cap_k F_k = E - (\cup_k E_k)$$

Since $0 > \alpha(E) > -\infty$ and $\alpha(E_k) \geq 0$, it follows from the definition of signed measure that

$$\alpha(E) = \alpha(F) + \sum_{k=1}^{\infty} \alpha(E_k)$$

$\alpha(F) \leq \alpha(E) < 0$, and the sum is finite. From the last observation and the construction, it follows that $F$ can have no subset $G$ with $\alpha(G) > 0$, for then $\alpha(G) \geq 1/N$ for some $N$ and we would have a contradiction. ■

**Theorem A.4.3. Hahn decompositon.** *Let $\alpha$ be a signed measure. Then there is a positive set $A$ and a negative set $B$ so that $\Omega = A \cup B$ and $A \cap B = \emptyset$.*

*Proof.* Let $c = \inf\{\alpha(B) : B \text{ is negative}\} \leq 0$. Let $B_i$ be negative sets with $\alpha(B_i) \downarrow c$. Let $B = \cup_i B_i$. By Lemma A.4.1, $B$ is negative, so by the definition of $c$, $\alpha(B) \geq c$. To prove $\alpha(B) \leq c$, we observe that $\alpha(B) = \alpha(B_i) + \alpha(B - B_i) \leq \alpha(B_i)$, since $B$ is negative, and let $i \to \infty$. The last two inequalities show that $\alpha(B) = c$, and it follows from our definition of a signed measure that $c > -\infty$. Let $A = B^c$. To show $A$ is positive, observe that if $A$ contains a set with $\alpha(E) < 0$, then by Lemma A.4.2, it contains a negative set $F$ with $\alpha(F) < 0$, but then $B \cup F$ would be a negative set that has $\alpha(B \cup F) = \alpha(B) + \alpha(F) < c$, a contradiction. ■

The Hahn decomposition is not unique. In Example A.4.1, $A$ can be any set with

$$\{x : f(x) > 0\} \subset A \subset \{x : f(x) \geq 0\} \quad \text{a.e.}$$

where $B \subset C$ a.e. means $\mu(B \cap C^c) = 0$. The last example is typical of the general situation. Suppose $\Omega = A_1 \cup B_1 = A_2 \cup B_2$ are two Hahn decompositions. $A_2 \cap B_1$ is positive and negative, so it is a **null set**: All its subsets have measure 0. Similarly, $A_1 \cap B_2$ is a null set.

Two measures $\mu_1$ and $\mu_2$ are said to be **mutually singular** if there is a set $A$ with $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$. In this case, we also say $\mu_1$ is **singular with respect to** $\mu_2$ and write $\mu_1 \perp \mu_2$.

**Exercise A.4.2.** Show that the uniform distribution on the Cantor set (Example 1.2.4) is singular with respect to Lebesgue measure.

**Theorem A.4.4. Jordan decomposition.** *Let $\alpha$ be a signed measure. There are mutually singular measures $\alpha_+$ and $\alpha_-$ so that $\alpha = \alpha_+ - \alpha_-$. Moreover, there is only one such pair.*

*Proof.* Let $\Omega = A \cup B$ be a Hahn decomposition. Let

$$\alpha_+(E) = \alpha(E \cap A) \quad \text{and} \quad \alpha_-(E) = -\alpha(E \cap B)$$

Since $A$ is positive and $B$ is negative, $\alpha_+$ and $\alpha_-$ are measures. $\alpha_+(A^c) = 0$ and $\alpha_-(A) = 0$, so they are mutually singular. To prove uniqueness, suppose $\alpha = \nu_1 - \nu_2$ and $D$ is a set with $\nu_1(D) = 0$ and $\nu_2(D^c) = 0$. If we set $C = D^c$, then $\Omega = C \cup D$ is a Hahn decomposition, and it follows from the choice of $D$ that

$$\nu_1(E) = \alpha(C \cap E) \quad \text{and} \quad \nu_2(E) = -\alpha(D \cap E)$$

Our uniqueness result for the Hahn decomposition shows that $A \cap D = A \cap C^c$ and $B \cap C = A^c \cap C$ are null sets, so $\alpha(E \cap C) = \alpha(E \cap (A \cup C)) = \alpha(E \cap A)$ and $\nu_1 = \alpha_+$. ∎

**Exercise A.4.3.** Show that $\alpha_+(E) = \sup\{\alpha(F) : F \subset E\}$.

**Remark.** Let $\alpha$ be a **finite signed measure** (i.e., one that does not take the value $\infty$ or $-\infty$) on $(\mathbf{R}, \mathcal{R})$. Let $\alpha = \alpha_+ - \alpha_-$ be its Jordan decomposition. Let $A(x) = \alpha((-\infty, x])$, $F(x) = \alpha_+((-\infty, x])$, and $G(x) = \alpha_-((-\infty, x])$. $A(x) = F(x) - G(x)$, so the distribution function for a finite signed measure can be written as a difference of two bounded increasing functions. It follows from Example A.4.2 that the converse is also true. Let $|\alpha| = \alpha^+ + \alpha^-$. $|\alpha|$ is called the **total variation of** $\alpha$, since in this example $|\alpha|((a, b])$ is the total variation of $A$ over $(a, b]$ as defined in analysis textbooks. See, for example, Royden (1988), p. 103. We exclude the left endpoint of the interval since a jump there makes no contribution to the total variation on $[a, b]$, but it does appear in $|\alpha|$.

Our third and final decomposition is:

**Theorem A.4.5. Lebesgue decomposition.** *Let $\mu$, $\nu$ be $\sigma$-finite measures. $\nu$ can be written as $\nu_r + \nu_s$, where $\nu_s$ is singular with respect to $\mu$ and*

$$\nu_r(E) = \int_E g\, d\mu$$

*Proof.* By decomposing $\Omega = +_i \Omega_i$, we can suppose without loss of generality that $\mu$ and $\nu$ are finite measures. Let $\mathcal{G}$ be the set of $g \geq 0$ so that $\int_E g\, d\mu \leq \nu(E)$ for all $E$.

(a) If $g, h \in \mathcal{G}$ then $g \vee h \in \mathcal{G}$.

*Proof of (a).* Let $A = \{g > h\}$, $B = \{g \leq h\}$.

$$\int_E g \vee h\, d\mu = \int_{E\cap A} g\, d\mu + \int_{E\cap B} h\, d\mu \leq \nu(E \cap A) + \nu(E \cap B) = \nu(E)$$

Let $\kappa = \sup\{\int g\, d\mu : g \in \mathcal{G}\} \leq \nu(\Omega) < \infty$. Pick $g_n$ so that $\int g_n\, d\mu > \kappa - 1/n$ and let $h_n = g_1 \vee \cdots \vee g_n$. By (a), $h_n \in \mathcal{G}$. As $n \uparrow \infty$, $h_n \uparrow h$. The definition of $\kappa$, the monotone convergence theorem, and the choice of $g_n$ imply that

$$\kappa \geq \int h\, d\mu = \lim_{n\to\infty} \int h_n\, d\mu \geq \lim_{n\to\infty} \int g_n\, d\mu = \kappa$$

Let $\nu_r(E) = \int_E h\, d\mu$ and $\nu_s(E) = \nu(E) - \nu_r(E)$. The last detail is to show

(b) $\nu_s$ is singular with respect to $\mu$.

*Proof of (b).* Let $\epsilon > 0$ and let $\Omega = A_\epsilon \cup B_\epsilon$ be a Hahn decomposition for $\nu_s - \epsilon\mu$. Using the definition of $\nu_r$ and then the fact that $A_\epsilon$ is positive for $\nu_s - \epsilon\mu$ (so $\epsilon\mu(A_\epsilon \cap E) \leq \nu_s(A_\epsilon \cap E)$),

$$\int_E (h + \epsilon 1_{A_\epsilon})\, d\mu = \nu_r(E) + \epsilon\mu(A_\epsilon \cap E) \leq \nu(E)$$

This holds for all $E$, so $k = h + \epsilon 1_{A_\epsilon} \in \mathcal{G}$. It follows that $\mu(A_\epsilon) = 0$, for if not, then $\int k\, d\mu > \kappa$ a contradiction. Letting $A = \cup_n A_{1/n}$, we have $\mu(A) = 0$. To see that $\nu_s(A^c) = 0$, observe that if $\nu_s(A^c) > 0$, then $(\nu_s - \epsilon\mu)(A^c) > 0$ for small $\epsilon$, a contradiction since $A^c \subset B_\epsilon$, a negative set. ■

**Exercise A.4.4.** Prove that the Lebesgue decomposition is unique. Note that you can suppose without loss of generality that $\mu$ and $\nu$ are finite.

We are finally ready for the main business of the section. We say a measure $\nu$ is **absolutely continuous with respect to** $\mu$ (and write $\nu \ll \mu$) if $\mu(A) = 0$ implies that $\nu(A) = 0$.

**Exercise A.4.5.** If $\mu_1 \ll \mu_2$ and $\mu_2 \perp \nu$, then $\mu_1 \perp \nu$.

**Theorem A.4.6. Radon-Nikodym theorem.** *If $\mu$, $\nu$ are $\sigma$-finite measures and $\nu$ is absolutely continuous with respect to $\mu$, then there is a $g \geq 0$ so that $\nu(E) = \int_E g\, d\mu$. If $h$ is another such function, then $g = h$ $\mu$ a.e.*

*Proof.* Let $\nu = \nu_r + \nu_s$ be any Lebesgue decomposition. Let $A$ be chosen so that $\nu_s(A^c) = 0$ and $\mu(A) = 0$. Since $\nu \ll \mu$, $0 = \nu(A) \geq \nu_s(A)$ and $\nu_s \equiv 0$. To prove uniqueness, observe that if $\int_E g\, d\mu = \int_E h\, d\mu$ for all $E$, then letting $E \subset \{g > h,\ g \leq n\}$ be any subset of finite measure, we conclude $\mu(g > h, g \leq n) = 0$ for all $n$, so $\mu(g > h) = 0$, and, similarly, $\mu(g < h) = 0$. ∎

**Example A.4.3.** Theorem A.4.6 may fail if $\mu$ is not $\sigma$-finite. Let $(\Omega, \mathcal{F}) = (\mathbf{R}, \mathcal{R})$, $\mu =$ counting measure and $\nu =$ Lebesgue measure.

The function $g$ whose existence is proved in Theorem A.4.6 is often denoted $d\nu/d\mu$. This notation suggests the following properties, whose proofs are left to the reader.

**Exercise A.4.6.** If $\nu_1, \nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$

$$d(\nu_1 + \nu_2)/d\mu = d\nu_1/d\mu + d\nu_2/d\mu$$

**Exercise A.4.7.** If $\nu \ll \mu$ and $f \geq 0$, then $\int f\, d\nu = \int f \frac{d\nu}{d\mu}\, d\mu$.

**Exercise A.4.8.** If $\pi \ll \nu \ll \mu$, then $d\pi/d\mu = (d\pi/d\nu) \cdot (d\nu/d\mu)$.

**Exercise A.4.9.** If $\nu \ll \mu$ and $\mu \ll \nu$, then $d\mu/d\nu = (d\nu/d\mu)^{-1}$.

## A.5 Differentiating under the Integral

At several places in the text, we need to interchange differentiate inside a sum or an integral. This section is devoted to results that can be used to justify those computations.

**Theorem A.5.1.** *Let $(S, \mathcal{S}, \mu)$ be a measure space. Let $f$ be a complex-valued function defined on $\mathbf{R} \times S$. Let $\delta > 0$, and suppose that for $x \in (y - \delta, y + \delta)$ we have*
*(i) $u(x) = \int_S f(x,s)\, \mu(ds)$ with $\int_S |f(x,s)|\, \mu(ds) < \infty$*
*(ii) for fixed $s$, $\partial f/\partial x(x,s)$ exists and is a continuous function of $x$,*
*(iii) $v(x) = \int_S \frac{\partial f}{\partial x}(x,s)\, \mu(ds)$ is continuous at $x = y$,*
*and (iv) $\int_S \int_{-\delta}^{\delta} \left| \frac{\partial f}{\partial x}(y + \theta, s) \right|\, d\theta\, \mu(ds) < \infty$*

*then $u'(y) = v(y)$.*

*Proof.* Letting $|h| \le \delta$ and using (i), (ii), (iv), and Fubini's theorem in the form given in Exercise 1.7.4, we have

$$\begin{aligned} u(y+h) - u(y) &= \int_S f(y+h,s) - f(y,s)\,\mu(ds) \\ &= \int_S \int_0^h \frac{\partial f}{\partial x}(y+\theta,s)\,d\theta\,\mu(ds) \\ &= \int_0^h \int_S \frac{\partial f}{\partial x}(y+\theta,s)\,\mu(ds)\,d\theta \end{aligned}$$

The last equation implies

$$\frac{u(y+h)-u(y)}{h} = \frac{1}{h}\int_0^h v(y+\theta)\,d\theta$$

Since $v$ is continuous at $y$ by (iii), letting $h \to 0$ gives the desired result. ■

**Example A.5.1.** For a result in Section 3.3, we need to know that we can differentiate under the integral sign in

$$u(x) = \int \cos(xs)e^{-s^2/2}\,ds$$

For convenience, we have dropped a factor $(2\pi)^{-1/2}$ and changed variables to match Theorem A.5.1. Clearly, (i) and (ii) hold. The dominated convergence theorem implies (iii)

$$x \to \int -s\sin(sx)e^{-s^2/2}\,ds$$

is continuous. For (iv), we note

$$\int \left|\frac{\partial f}{\partial x}(x,s)\right|\,ds = \int |s|e^{-s^2/2}\,ds < \infty$$

and the value does not depend on $x$, so (iv) holds.

For some examples the following form is more convenient:

**Theorem A.5.2.** *Let $(S, \mathcal{S}, \mu)$ be a measure space. Let $f$ be a complex valued function defined on $\mathbf{R} \times S$. Let $\delta > 0$, and suppose that for $x \in (y-\delta, y+\delta)$ we have*

*(i) $u(x) = \int_S f(x,s)\,\mu(ds)$ with $\int_S |f(x,s)|\,\mu(ds) < \infty$*

*(ii) for fixed $s$, $\partial f/\partial x(x,s)$ exists and is a continuous function of $x$,*

*(iii′)* $$\int_S \sup_{\theta\in[-\delta,\delta]} \left|\frac{\partial f}{\partial x}(y+\theta,s)\right|\,\mu(ds) < \infty$$

*then $u'(y) = v(y)$.*

*Proof.* In view of Theorem A.5.1 it is enough to show that (iii) and (iv) of that result hold. Since

$$\int_{-\delta}^{\delta} \left| \frac{\partial f}{\partial x}(y+\theta, s) \right| d\theta \leq 2\delta \sup_{\theta \in [-\delta, \delta]} \left| \frac{\partial f}{\partial x}(y+\theta, s) \right|$$

it is clear that (iv) holds. To check (iii), we note that

$$|v(x) - v(y)| \leq \int_S \left| \frac{\partial f}{\partial x}(x, s) - \frac{\partial f}{\partial x}(y, s) \right| \mu(ds)$$

(ii) implies that the integrand $\to 0$ as $x \to y$. The desired result follows from (iii$'$) and the dominated convergence theorem. ■

To indicate the usefulness of the new result, we prove:

**Example A.5.2.** If $\phi(\theta) = Ee^{\theta Z} < \infty$ for $\theta \in [-\epsilon, \epsilon]$ then $\phi'(0) = EZ$.

*Proof.* Here $\theta$ plays the role of $x$, and we take $\mu$ to be the distribution of $Z$. Let $\delta = \epsilon/2$. $f(x,s) = e^{xs} \geq 0$, so (i) holds by assumption. $\partial f/\partial x = se^{xs}$ is clearly a continuous function, so (ii) holds. To check (iii$'$), we note that there is a constant $C$ so that if $x \in (-\delta, \delta)$, then $|s|e^{xs} \leq C\left(e^{-\epsilon s} + e^{\epsilon s}\right)$. ■

Taking $S = \mathbf{Z}$ with $\mathcal{S}$ = all subsets of $S$ and $\mu$ = counting measure in Theorem A.5.2 gives the following:

**Theorem A.5.3.** *Let $\delta > 0$. Suppose that for $x \in (y-\delta, y+\delta)$ we have*
*(i) $u(x) = \sum_{n=1}^{\infty} f_n(x)$ with $\sum_{n=1}^{\infty} |f_n(x)| < \infty$*
*(ii) for each $n$, $f_n'(x)$ exists and is a continuous function of $x$,*

*and (iii) $\sum_{n=1}^{\infty} \sup_{\theta \in (-\delta, \delta)} |f_n'(y+\theta)| < \infty$*

*then $u'(x) = v(x)$.*

**Example A.5.3.** In Section 2.6 we want to show that if $p \in (0,1)$ then

$$\left( \sum_{n=1}^{\infty} (1-p)^n \right)' = -\sum_{n=1}^{\infty} n(1-p)^{n-1}$$

*Proof.* Let $f_n(x) = (1-x)^n$, $y = p$, and pick $\delta$ so that $[y-\delta, y+\delta] \subset (0,1)$. Clearly (i) $\sum_{n=1}^{\infty} |(1-x)^n| < \infty$ and (ii) $f_n'(x) = n(1-x)^{n-1}$ is continuous for $x$ in $[y-\delta, y+\delta]$. To check (iii), we note that if we let $2\eta = y - \delta$ then there is a constant $C$ so that if $x \in [y-\delta, y+\delta]$ and $n \geq 1$, then

$$n(1-x)^{n-1} = \frac{n(1-x)^{n-1}}{(1-\eta)^{n-1}} \cdot (1-\eta)^{n-1} \leq C(1-\eta)^{n-1}$$ ■

# References


D. Aldous and P. Diaconis (1986) Shuffling cards and stopping times. *Amer. Math. Monthly.* 93, 333–348

E. S. Andersen and B. Jessen (1948) On the introduction of measures in infinite product spaces. *Danske Vid. Selsk. Mat.-Fys. Medd.* 25, No. 4

K. Athreya and P. Ney (1972) *Branching processes.* Springer-Verlag, New York

K. Athreya and P. Ney (1978) A new approach to the limit theory of recurrent Markov chains. *Trans. AMS.* 245, 493–501

K. Athreya, D. McDonald, and P. Ney (1978) Coupling and the renewal theorem. *Am. Math. Monthly.* 85, 809–814

L. Bachelier (1900) Théorie de la spéculation. *Ann. Sci. École Norm. Sup.* 17, 21–86

S. Banach and A. Tarski (1924) Sur la décomposition des ensembles de points en parties respectivements congruent. *Fund. Math.* 6, 244–277

L. E. Baum and P. Billingsley (1966) Asymptotic distributions for the coupon collector's problem. *Ann. Math. Statist.* 36, 1835–1839

F. Benford (1938) The law of anomalous numbers. *Proc. Am. Phil. Soc.* 78, 552–572

J. D. Biggins (1977) Chernoff's theorem in branching random walk. *J. Appl. Probab.* 14, 630–636

J. D. Biggins (1978) The asymptotic shape of branching random walk. *Adv. Appl. Probab.* 10, 62–84

J. D. Biggins (1979) Growth rates in branching random walk. *Z. Warsch. verw. Gebiete.* 48, 17–34

P. Billingsley (1979) *Probability and measure.* John Wiley and Sons, New York

G. D. Birkhoff (1931) Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA.* 17, 656–660

D. Blackwell and D. Freedman (1964) The tail $\sigma$-field of a Markov chain and a theorem of Orey. *Ann. Math. Statist.* 35, 1291–1295

R. M. Blumenthal and R. K. Getoor (1968) *Markov processes and their potential theory.* Academic Press, New York

E. Borel (1909) Les probabilités dénombrables et leur applications arithmétiques. *Rend. Circ. Mat. Palermo.* 27, 247–271

L. Breiman (1968) *Probability.* Addison-Wesley, Reading, MA

K. L. Chung (1974) *A course in probability theory*, second edition. Academic Press, New York.

K. L. Chung, P. Erdös, and T. Sirao (1959) On the Lipschitz's condition for Brownian motion. *J. Math. Soc. Japan.* 11, 263–274

K. L. Chung and W. H. J. Fuchs (1951) On the distribution of values of sums of independent random variables. *Memoirs of the AMS*, No. 6

V. Chvátal and D. Sankoff (1975) Longest common subsequences of two random sequences. *J. Appl. Probab.* 12, 306–315

J. Cohen, H. Kesten, and C. Newman (1985) *Random matrices and their applications.* AMS Contemporary Math. 50, Providence, RI

J. T. Cox and R. Durrett (1981) Limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* 9, 583–603

B. Davis (1983) On Brownian slow points. *Z. Warsch. verw. Gebiete.* 64, 359–367

P. Diaconis and D. Freedman (1980) Finite exchangeable sequences. *Ann. Prob.* 8, 745–764

J. Dieudonné (1948) Sur la théorème de Lebesgue-Nikodym, II. *Ann. Univ. Grenoble.* 23, 25–53

M. Donsker (1951) An invariance principle for certain probability limit theorems. *Memoirs of the AMS*, No. 6

M. Donsker (1952) Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* 23, 277–281

J. L. Doob (1949) A heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* 20, 393–403

J. L. Doob (1953) *Stochastic processes.* John Wiley and Sons, New York

L. E. Dubins (1968) On a theorem of Skorkhod. *Ann. Math. Statist.* 39, 2094–2097

L. E. Dubins and D. A. Freedman (1965) A sharper form of the Borel-Cantelli lemma and the strong law. *Ann. Math. Statist.* 36, 800–807

L. E. Dubins and D. A. Freedman (1979) Exchangeable processes need not be distributed mixtures of independent and identically distributed random variables. *Z. Warsch. verw. Gebiete.* 48, 115–132

R. M. Dudley (1989) *Real Analysis and Probability.* Wadsworth Pub. Co., Pacific Grove, CA

A. Dvoretsky and P. Erdös (1951) Some problems on random walk in space. *Proc. 2nd Berkeley Symp.* 353–367

A. Dvoretsky, P. Erdös, and S. Kakutani (1961) Nonincrease everywhere of the Brownian motion process. *Proc. 4th Berkeley Symp.*, Vol. II, 103–116

E. B. Dynkin (1965) *Markov processes.* Springer-Verlag, New York

E. B. Dynkin and A. A. Yushkevich (1956) Strong Markov processes. *Theory Prob. Appl.* 1, 134–139

P. Erdös and M. Kac (1946) On certain limit theorems of the theory of probability. *Bull. AMS.* 52, 292–302

P. Erdös and M. Kac (1947) On the number of positive sums of independent random variables. *Bull. AMS.* 53, 1011–1020

N. Etemadi (1981) An elementary proof of the strong law of large numbers. *Z. Warsch. verw. Gebiete.* 55, 119–122

A. M. Faden (1985) The existence of regular conditional probabilities: necessary and sufficient conditions. *Ann. Probab.* 13, 288–298

W. Feller (1946) A limit theorem for random variables with infinite moments. *Am. J. Math.* 68, 257–262

W. Feller (1961) A simple proof of renewal theorems. *Comm. Pure Appl. Math.* 14, 285–293

W. Feller (1968) *An introduction to probability theory and its applications, Vol. I*, third edition. John Wiley and Sons, New York

W. Feller (1971) *An introduction to probability theory and its applications, Vol. II,* second edition. John Wiley and Sons, New York

D. Freedman (1965) Bernard Friedman's urn. *Ann. Math. Statist.* 36, 956–970

D. Freedman (1971a) *Brownian motion and diffusion.* Originally published by Holden Day, San Francisco, CA. Second edition by Springer-Verlag, New York

D. Freedman (1971b) *Markov chains.* Originally published by Holden Day, San Francisco, CA. Second edition by Springer-Verlag, New York

D. Freedman (1980) A mixture of independent and identically distributed random variables need not admit a regular conditional probability given the exchangeable $\sigma$-field. *Z. Warsch. verw. Gebiete.* 51, 239–248

R. M. French (1988) The Banach-Tarski theorem. *Math. Intelligencer.* 10, No. 4, 21–28

B. Friedman (1949) A simple urn model. *Comm. Pure Appl. Math.* 2, 59–70

H. Furstenburg (1970) Random walks in discrete subgroups of Lie Groups. In *Advances in Probability*, edited by P. E. Ney. Marcel Dekker, New York

H. Furstenburg and H. Kesten (1960) Products of random matrices. *Ann. Math. Statist.* 31, 451–469

A. Garsia (1965) A simple proof of E. Hopf's maximal ergodic theorem. *J. Math. Mech.* 14, 381–382

M. L. Glasser and I. J. Zucker (1977) Extended Watson integrals for the cubic lattice. *Proc. Natl. Acad. Sci. USA.* 74, 1800–1801

B. V. Gnedenko (1943) Sur la distribution limité du terme maximum d'une série aléatoire. *Ann. Math.* 44, 423–453

B. V. Gnedenko and A.V. Kolmogorov (1954) *Limit distributions for sums of independent random variables.* Addison-Wesley, Reading, MA

P. Hall (1982) *Rates of convergence in the central limit theorem.* Pitman Pub. Co., Boston, MA

P. R. Halmos (1950) *Measure theory.* Van Nostrand, New York

J. M. Hammersley (1970) A few seedlings of research. *Proc. 6th Berkeley Symp.*, Vol. I, 345–394

G. H. Hardy and J. E. Littlewood (1914) Some problems of Diophantine approximation. *Acta Math.* 37, 155–239

G. H. Hardy and E. M. Wright (1959) *An introduction to the theory of numbers*, fourth edition. Oxford University Press, London

T. E. Harris (1956) The existence of stationary measures for certain Markov processes. *Proc. 3rd Berkeley Symp.*, Vol. II, 113–124

P. Hartman and A. Wintner (1941) On the law of the iterated logarithm. *Am. J. Math.* 63, 169–176

E. Hewitt and L. J. Savage (1956) Symmetric measures on Cartesian products. *Trans. AMS.* 80, 470–501

E. Hewitt and K. Stromberg (1965) *Real and abstract analysis.* Springer-Verlag, New York

C. C. Heyde (1963) On a property of the lognormal distribution. *J. R. Statist. Soc. B.* 29, 392–393

C. C. Heyde (1967) On the influence of moments on the rate of convergence to the normal distribution. *Z. Warsch. verw. Gebiete.* 8, 12–18

J. L. Hodges, Jr. and L. Le Cam (1960) The Poisson approximation to the binomial distribution. *Ann. Math. Statist.* 31, 737–740

G. Hunt (1956) Some theorems concerning Brownian motion. *Trans. AMS.* 81, 294–319

H. Ishitani (1977) A central limit theorem for the subadditive process and its application to products of random matrices. *RIMS, Kyoto.* 12, 565–575

K. Itô and H. P. McKean (1965). *Diffusion processes and their sample paths.* Springer-Verlag, New York

M. Kac (1947a) Brownian motion and the theory of random walk. *Am. Math. Monthly.* 54, 369–391

M. Kac (1947b) On the notion of recurrence in discrete stochastic processes. *Bull. AMS.* 53, 1002–1010

M. Kac (1959) *Statistical independence in probability, analysis, and number theory.* Carus Monographs, Math. Assoc. of America

Y. Katznelson and B. Weiss (1982) A simple proof of some ergodic theorems. *Israel J. Math.* 42, 291–296

E. Keeler and J. Spencer (1975) Optimal doubling in backgammon. *Operations Research.* 23, 1063–1071

H. Kesten (1986) Aspects of first passage percolation. In *École d'été de probabilités de Saint-Flour XIV*. Lecture Notes in Math 1180, Springer-Verlag, New York

H. Kesten (1987) Percolation theory and first passage percolation. *Ann. Probab.* 15, 1231–1271

J. F. C. Kingman (1968) The ergodic theory of subadditive processes. *J. R. Statist. Soc. B.* 30, 499–510

J. F. C. Kingman (1973) Subadditive ergodic theory. *Ann. Probab.* 1, 883–909

J. F. C. Kingman (1975) The first birth problem for age dependent branching processes. *Ann. Probab.* 3, 790–801

K. Kondo and T. Hara (1987) Critical exponent of susceptibility for a general class of ferromagnets in $d > 4$ dimensions. *J. Math. Phys.* 28, 1206–1208

U. Krengel (1985) *Ergodic theorems.* deGruyter, New York

S. Leventhal (1988) A proof of Liggett's version of the subadditive ergodic theorem. *Proc. AMS.* 102, 169–173

P. Lévy (1937) *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris

T. M. Liggett (1985) An improved subadditive ergodic theorem. *Ann. Probab.* 13, 1279–1285

A. Lindenbaum (1926) Contributions à l'étude de l'espace metrique. *Fund. Math.* 8, 209–222

T. Lindvall (1977) A probabilistic proof of Blackwell's renewal theorem. *Ann. Probab.* 5, 482–485

B. F. Logan and L. A. Shepp (1977) A variational problem for random Young tableaux. *Adv. Math.* 26, 206–222

H. P. McKean (1969) *Stochastic integrals.* Academic Press, New York

M. Motoo (1959) Proof of the law of the iterated logarithm through diffusion equation. *Ann. Inst. Stat. Math.* 10, 21–28

J. Neveu (1965) *Mathematical foundations of the calculus of probabilities.* Holden-Day, San Francisco, CA

J. Neveu (1975) *Discrete parameter martingales.* North Holland, Amsterdam

S. Newcomb (1881) Note on the frequency of use of the different digits in natural numbers. *Am. J. Math.* 4, 39–40

D. Ornstein (1969) Random walks. *Trans. AMS.* 138, 1–60

V. I. Oseledĕc (1968) A multiplicative ergodic theorem. Lyapunov characteristic numbers for synmaical systems. *Trans. Moscow Math. Soc.* 19, 197–231

R. E. A. C. Paley, N. Wiener, and A. Zygmund (1933) Notes on random functions. *Math. Z.* 37, 647–668

E. J. G. Pitman (1956) On derivatives of characteristic functions at the origin. *Ann. Math. Statist.* 27, 1156–1160

E. Perkins (1983) On the Hausdorff dimension of the Brownian slow points. *Z. Warsch. verw. Gebiete.* 64, 369–399

S. C. Port and C. J. Stone (1969) Potential theory of random walks on abelian groups. *Acta Math.* 122, 19–114

M. S. Ragunathan (1979) A proof of Oseledĕc's multiplicative ergodic theorem. *Israel J. Math.* 32, 356–362

R. Raimi (1976) The first digit problem. *Amer. Math. Monthly.* 83, 521–538

S. Resnick (1987) *Extreme values, regular variation, and point processes.* Springer-Verlag, New York

D. Revuz (1984) *Markov chains*, second edition. North Holland, Amsterdam

D. H. Root (1969) The existence of certain stopping times on Brownian motion. *Ann. Math. Statist.* 40, 715–718

H. Royden (1988) *Real analysis*, third edition. McMillan, New York

D. Ruelle (1979) Ergodic theory of differentiable dynamical systems. *IHES Pub. Math.* 50, 275–306

C. Ryll-Nardzewski (1951) On the ergodic theorems, II. *Studia Math.* 12, 74–79

L. J. Savage (1972) *The foundations of statistics*, second edition. Dover, New York

L. A. Shepp (1964) Recurrent random walks may take arbitrarily large steps. *Bull. AMS.* 70, 540–542

S. Sheu (1986) Representing a distribution by stopping Brownian motion: Root's construction. *Bull. Austral. Math. Soc.* 34, 427–431

N. V. Smirnov (1949) Limit distributions for the terms of a variational series. *AMS Transl. Series.* 1, No. 67

R. Smythe and J. C. Wierman (1978) *First passage percolation on the square lattice.* Lecture Notes in Math 671, Springer-Verlag, New York

R. M. Solovay (1970) A model of set theory in which every set of reals is Lebesgue measurable. *Ann. Math.* 92, 1–56

F. Spitzer (1964) *Principles of random walk.* Van Nostrand, Princeton, NJ

C. Stein (1987) *Approximate computation of expectations.* IMS Lecture Notes, Vol. 7

C. J. Stone (1969) On the potential operator for one dimensional recurrent random walks. *Trans. AMS.* 136, 427–445

J. Stoyanov (1987) *Counterexamples in probability.* John Wiley and Sons, New York

V. Strassen (1964) An invariance principle for the law of the iterated logarithm. *Z. Warsch. verw. Gebiete.* 3, 211–226

V. Strassen (1965) A converse to the law of the iterated logarithm. *Z. Warsch. verw. Gebiete.* 4, 265–268

H. Thorisson (1987) A complete coupling proof of Blackwell's renewal theorem. *Stoch. Proc. Appl.* 26, 87–97

P. van Beek (1972) An application of Fourier methods to the problem of sharpening the Berry-Esseen inequality. *Z. Warsch. verw. Gebiete.* 23, 187–196

A. M. Vershik and S. V. Kerov (1977) Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of random Young tableau. *Dokl. Akad. Nauk SSR.* 233, 1024–1027

H. Wegner (1973) On consistency of probability measures. *Z. Warsch. verw. Gebiete.* 27, 335–338

L. Weiss (1955) The stochastic convergence of a function of sample successive differences. *Ann. Math. Statist.* 26, 532–536

N. Wiener (1923) Differential space. *J. Math. Phys.* 2, 131–174

K. Yosida and S. Kakutani (1939) Birkhoff's ergodic theorem and the maximal ergodic theorem. *Proc. Imp. Acad. Tokyo.* 15, 165–168

# Index