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Valter Moretti

Spectral Theory and Quantum Mechanics

Mathematical Foundations of Quantum Theories, Symmetries and Introduction to the Algebraic Formulation

Second Edition

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To Bianca

Preface to the Second Edition

In this second English edition (third, if one includes the first Italian one), a large number of typos and errors of various kinds have been amended.

I have added more than 100 pages of fresh material, both mathematical and physical, in particular regarding the notion of *superselection rules*—addressed from several different angles—the machinery of *von Neumann algebras* and the abstract *algebraic formulation*. I have considerably expanded the lattice approach to Quantum Mechanics in Chap. 7, which now contains precise statements leading up to *Solèr's theorem* on the characterization of quantum lattices, as well as generalised versions of *Gleason's theorem*. As a matter of fact, Chap. 7 and the related Chap. 11 have been completely reorganised. I have incorporated a variety of results on the theory of von Neumann algebras and a broader discussion on the mathematical formulation of superselection rules, also in relation to the von Neumann algebra of observables. The corresponding preparatory material has been fitted into Chap. 3. Chapter 12 has been developed further, in order to include technical facts concerning groups of quantum symmetries and their strongly continuous unitary representations. I have examined in detail the relationship between *Nelson* domains and *Gårding* domains. Each chapter has been enriched by many new exercises, remarks, examples and references. I would like once again to thank my colleague Simon Chiossi for revising and improving my writing.

For having pointed out typos and other errors and for useful discussions, I am grateful to Gabriele Anzellotti, Alejandro Ascárate, Nicolò Cangiotti, Simon G. Chiossi, Claudio Dappiaggi, Nicolò Drago, Alan Garbarz, Riccardo Ghiloni, Igor Khavkine, Bruno Hideki F. Kimura, Sonia Mazzucchi, Simone Murro, Giuseppe Nardelli, Marco Oppio, Alessandro Perotti and Nicola Pinamonti.

Povo, Trento, Italy
September 2017

Valter Moretti

Preface to the First Edition

I must have been 8 or 9 when my father, a man of letters but well-read in every discipline and with a curious mind, told me this story: "A great scientist named Albert Einstein discovered that any object with a mass can't travel faster than the speed of light". To my bewilderment I replied, boldly: "This can't be true, if I run almost at that speed and then accelerate a little, surely I will run faster than light, right?" My father was adamant: "No, it's impossible to do what you say, it's a known physics fact". After a while I added: "That bloke, Einstein, must've checked this thing many times ... how do you say, he did many experiments*?" The answer I got was utterly unexpected: "Not even one I believe. He used maths!"*

What did numbers and geometrical figures have to do with the existence of an upper limit to speed? How could one stand by such an apparently nonsensical statement as the existence of a maximum speed, although certainly true (I trusted my father), just based on maths? How could mathematics have such big a control on the real world? And Physics *? What on earth was it, and what did it have to do with maths? This was one of the most beguiling and irresistible things I had ever heard till that moment... I had to find out more about it.*

This is an extended and enhanced version of an existing textbook written in Italian (and published by Springer-Verlag). That edition and this one are based on a common part that originated, in preliminary form, when I was a *Physics* undergraduate at the University of Genova. The third-year compulsory lecture course called *Theoretical Physics* was the second exam that had us pupils seriously climbing the walls (the first being the famous *Physics II*, covering thermodynamics and classical electrodynamics).

Quantum Mechanics, taught in *Institutions*, elicited a novel and involved way of thinking, a true challenge for craving students: for months we hesitantly faltered on a hazy and uncertain terrain, not understanding what was really key among the notions we were trying—struggling, I should say—to learn, together with a completely new formalism: linear operators on Hilbert spaces. At that time, actually, we did not realise we were using this mathematical theory, and for many mates of mine, the matter would have been, rightly perhaps, completely futile; Dirac's *bra* vectors were what they were, and that's it! They were certainly not elements in the topological dual of the Hilbert space. The notions of *Hilbert space* and *dual topological space* had no right of abode in the mathematical toolbox of the majority of my fellows, even if they would soon come back in through the back door, with the course *Mathematical Methods of Physics* taught by Prof. G. Cassinelli. Mathematics, and the mathematical formalisation of physics, had always been my flagship to overcome the difficulties that studying physics presented me with, to the point that eventually (after a Ph.D. in Theoretical Physics) I officially became a mathematician. Armed with a maths' background—learnt in an extracurricular course of study that I cultivated over the years, in parallel to academic physics—and eager to broaden my knowledge, I tried to formalise every notion I met in that new and riveting lecture course. At the same time, I was carrying along a similar project for the mathematical formalisation of General Relativity, unaware that the work put into Quantum Mechanics would have been incommensurably bigger.

The formulation of the spectral theorem as it is discussed in § 8, 9 is the same I learnt when taking the *Theoretical Physics* exam, which for this reason was a dialogue of the deaf. Later my interest turned to *Quantum Field Theory*, a subject I still work on today, though in the slightly more general framework of QFT *in curved spacetime*. Notwithstanding, my fascination with the elementary formulation of Quantum Mechanics never faded over the years, and time and again chunks were added to the opus I begun writing as a student.

Teaching this material to master's and doctoral students in mathematics and physics, thereby inflicting on them the result of my efforts to simplify the matter, has proved to be crucial for improving the text. It forced me to typeset in LaTeX the pile of loose notes and correct several sections, incorporating many people's remarks.

Concerning this, I would like to thank my colleagues, the friends from the *newsgroups it.scienza.fisica*, *it.scienza.matematica* and *free.it.scienza.fisica*, and the many students—some of which are now fellows of mine—who contributed to improve the preparatory material of the treatise, whether directly or not, in the course of time: S. Albeverio, G. Anzellotti, P. Armani, G. Bramanti, S. Bonaccorsi, A. Cassa, B. Cocciaro, G. Collini, M. Dalla Brida, S. Doplicher, L. Di Persio, E. Fabri, C. Fontanari, A. Franceschetti, R. Ghiloni, A. Giacomini, V. Marini, S. Mazzucchi, E. Pagani, E. Pelizzari, G. Tessaro, M. Toller, L. Tubaro, D. Pastorello, A. Pugliese, F. Serra Cassano, G. Ziglio and S. Zerbini. I am indebted, for various reasons also unrelated to the book, to my late colleague Alberto Tognoli. My greatest appreciation goes to R. Aramini, D. Cadamuro and C. Dappiaggi, who read various versions of the manuscript and pointed out a number of mistakes.

I am grateful to my friends and collaborators R. Brunetti, C. Dappiaggi and N. Pinamonti for lasting technical discussions, for suggestions on many topics covered in the book and for pointing out primary references.

At last, I would like to thank E. Gregorio for the invaluable and on-the-spot technical help with the LaTeX package.

In the transition from the original Italian to the expanded English version, a massive number of (uncountably many!) typos and errors of various kinds have been corrected. I owe to E. Annigoni, M. Caffini, G. Collini, R. Ghiloni, A. Iacopetti, M. Oppio and D. Pastorello in this respect. Fresh material was added,

both mathematical and physical, including a chapter, at the end, on the so-called *algebraic formulation*.

In particular, Chap. 4 contains the proof of Mercer's theorem for positive Hilbert–Schmidt operators. The analysis of the first two axioms of Quantum Mechanics in Chap. 7 has been deepened and now comprises the algebraic characterisation of quantum states in terms of positive functionals with unit norm on the C^*-algebra of compact operators. General properties of C^*-algebras and *-morphisms are introduced in Chap. 8. As a consequence, the statements of the spectral theorem and several results on functional calculus underwent a minor but necessary reshaping in Chaps. 8 and 9. I incorporated in Chap. 10 (Chap. 9 in the first edition) a brief discussion on abstract differential equations in Hilbert spaces. An important example concerning Bargmann's theorem was added in Chap. 12 (formerly Chap. 11). In the same chapter, after introducing the Haar measure, the Peter–Weyl theorem on unitary representations of compact groups is stated and partially proved. This is then applied to the theory of the angular momentum. I also thoroughly examined the superselection rule for the angular momentum. The discussion on POVMs in Chap.13 (ex Chap. 12) is enriched with further material, and I included a primer on the fundamental ideas of non-relativistic scattering theory. Bell's inequalities (Wigner's version) are given considerably more space. At the end of the first chapter, basic point-set topology is recalled together with abstract measure theory. The overall effort has been to create a text as self-contained as possible. I am aware that the material presented has clear limitations and gaps. Ironically—my own research activity is devoted to relativistic theories—the entire treatise unfolds at a non-relativistic level, and the quantum approach to Poincaré's symmetry is left behind.

I thank my colleagues F. Serra Cassano, R. Ghiloni, G. Greco, S. Mazzucchi, A. Perotti and L. Vanzo for useful technical conversations on this second version. For the same reason, and also for translating this elaborate opus into English, I would like to thank my colleague S. G. Chiossi.

Trento, Italy — Valter Moretti
October 2012

Contents

Chapter 1
Introduction and Mathematical Backgrounds

> *"O frati", dissi "che per cento milia perigli siete giunti a l'occidente, a questa tanto picciola vigilia d'i nostri sensi ch'è del rimanente non vogliate negar l'esperienza, di retro al sol, del mondo sanza gente".*
>
> Dante Alighieri, the Divine Comedy, Inferno, canto XXVI[1]

1.1 On the Book

1.1.1 Scope and Structure

One of the aims of the present book is to explain the mathematical foundations of Quantum Mechanics (QM), and Quantum Theories in general, in a mathematically rigorous way. This is a treatise on Mathematics (or Mathematical Physics) rather than a text on Quantum Mechanics. Except for a few cases, the physical phenomenology is left in the background in order to privilege the theory's formal and logical aspects. At any rate, several examples of the physical formalism are presented, lest one lose touch with the world of physics.

In alternative to, and irrespective of, the physical content, the book should be considered as an introductory text, albeit touching upon rather advanced topics, on functional analysis on Hilbert spaces, including a few elementary yet fundamental results on C^*-algebras. Special attention is given to a series of results in spectral theory, such as the various formulations of the spectral theorem for bounded normal operators and not necessarily bounded, self-adjoint ones. This is, as a matter of fact, one further scope of the text. The mathematical formulation of Quantum Theories is "confined" to Chaps. 6, 7, 11–13 and partly Chap. 14. The remaining chapters are

[1] ("Brothers" I said, "who through a hundred thousand dangers have reached the channel to the west, to the short evening watch which your own senses still must keep, do not choose to deny the experience of what lies past the Sun and of the world yet uninhabited." Dante Alighieri, The Divine Comedy, translated by J. Finn Cotter, edited by C. Franco, Forum Italicum Publishing, New York, 2006.)

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_1

logically independent of those, although the motivations for certain mathematical definitions are to be found in Chaps. 7, 10–14.

A third purpose is to collect in one place a number of rigorous and useful results on the mathematical structure of QM and Quantum Theories. These are more advanced than what is normally encountered in quantum physics' manuals. Many of these aspects have been known for a long time but are scattered in the specialistic literature. We should mention *Solèr's theorem*, *Gleason's theorem*, *the theorem of Kochen and Specker*, the *theorems of Stone–von Neumann* and *Mackey*, *Stone's theorem* and *von Neumann's theorem* about one-parameter unitary groups, *Kadison's theorem*, besides the better known *Wigner*, *Bargmann* and *GNS theorems*; or, more abstract results in operator theory such as *Fuglede's theorem*, or the *polar decomposition for closed unbounded operators* (which is relevant in the *Tomita–Takesaki theory* and statistical Quantum Mechanics in relationship to the KMS condition); furthermore, self-adjoint properties for symmetric operators, due to Nelson, that descend from the existence of dense sets of analytical vectors, and finally, Kato's work (but not only his) on the essential self-adjointness of certain kinds of operators and their limits from the bottom of the spectrum (mostly based on the *Kato–Rellich theorem*).

Some chapters suffice to cover a good part of the material suitable for advanced courses on Mathematical Methods in Physics; this is common for master's degrees in Physics or doctoral degrees, if we assume a certain familiarity with notions, results and elementary techniques of measure theory. The text may also be used for a higher-level course in Mathematical Physics that includes foundational material on QM. In the attempt to reach out to master or Ph.D. students, both in physics with an interest in mathematical methods or in mathematics with an inclination towards physical applications, the author has tried to prepare a self-contained text, as far as possible: hence a primer was included on general topology and abstract measure theory, together with an appendix on differential geometry. Most chapters are accompanied by exercises, many of which are solved explicitly.

The book could, finally, be useful to scholars to organise and present accurately the profusion of advanced material disseminated in the literature.

Results from topology and measure theory, much needed throughout the whole treatise, are recalled at the end of this introductory chapter. The rest of the book is ideally divided into three parts. The first part, up to Chap. 5, regards the general theory of operators on Hilbert spaces, and introduces several fairly general notions, like Banach spaces. Core results are proved, such as the theorems of Baire, Hahn–Banach and Banach–Steinhaus, as well as the fixed-point theorem of Banach–Caccioppoli, the Arzelà-Ascoli theorem and Fredholm's alternative, plus some elementary consequences. This part contains a summary of basic topological notions, in the belief that it might benefit physics' students. The latter's training on point-set topology is at times disparate and often presents gaps, because this subject is, alas, usually taught sporadically in physics' curricula, and not learnt in an organic way like students in mathematics do.

Part two ends with Chap. 10. Beside laying out the quantum formalism, it develops spectral theory, in terms of projector-valued measures, up to the spectral decomposition theorems for unbounded self-adjoint operators on Hilbert spaces. This includes the features of maps of operators (functional analysis) for measurable maps that are not necessarily bounded. General spectral aspects and the properties of their domains are investigated. A great emphasis is placed on C^*-algebras and the relative functional calculus, including an elementary study of the *Gelfand transform* and the *commutative Gelfand–Najmark theorem*. The technical results leading to the spectral theorem are stated and proven in a completely abstract manner in Chap. 8, forgetting that the algebras in question are actually operator algebras, and thus showing their broader validity. In Chap. 10 spectral theory is applied to several practical and completely abstract contexts, both quantum and not.

Chapter 6 treats, from a physical perspective, the motivation underlying the theory. The general mathematical formulation of QM concerns Chap. 7. The mathematical starting point is the idea, going back to von Neumann, that the propositions of physical quantum systems are described by the lattice of orthogonal projectors on a complex Hilbert space. Maximal sets of physically compatible propositions (in the quantum sense) are described by distributive, orthocomplemented, bounded, σ-complete lattices. From this standpoint the quantum definition of an observable in terms of a self-adjoint operator is extremely natural, as is, on the other hand, the formulation of the spectral decomposition theorem. Quantum states are defined as measures on the lattice of *all* orthogonal projectors, which is no longer distributive (due to the presence, in the quantum world, of *incompatible* propositions and observables). States are characterised as positive operators of trace class with unit trace under *Gleason's theorem*. Pure states (rays in the Hilbert space of the physical system) arise as extreme elements of the convex body of states. Generalisations of Gleason's statement are also discussed in a more advanced section of Chap. 7. The same chapter also discusses how to recover the Hilbert space starting from the lattice of elementary propositions, following the *theorems of Piron and Solèr*. The notion of *superselection rule* is also introduced here, and the discussion is expanded in Chap. 11 in terms of direct decomposition of *von Neumann factors* of observables. In that chapter the notion of von Neumann algebra of observables is exploited to present the mathematical formulation of quantum theories in more general situations, where not all self-adjoint operators represent observables.

The third part of the book is devoted to the mathematical axioms of QM, and more advanced topics like *quantum symmetries* and the *algebraic formulation of quantum theories*. Quantum symmetries and symmetry groups (both according to Wigner and to Kadison) are studied in depth. Dynamical symmetries and the quantum version of *Noether's theorem* are covered as well. The *Galilean group*, together with its subgroups and central extensions, is employed repeatedly as reference symmetry group, to explain the theory of projective unitary representations. *Bargmann's theorem* on the existence of unitary representations of simply connected Lie groups whose Lie algebra obeys a certain cohomology constraint is proved, and Bargmann's *rule of superselection of the mass* is discussed in detail. Then the useful theorems of Gårding and Nelson for projective unitary representations of Lie groups of symmetries are considered. Important topics are examined that are often neglected in manuals, like the uniqueness of unitary representations of the canonical commutation relations (theorems of Stone–von Neumann and Mackey), or the theoretical difficulties in defining time as the conjugate operator to energy (the Hamiltonian). The mathematical hurdles one must overcome in order to make the statement of *Ehrenfest's theorem* precise are briefly treated. Chapter 14 offers an introduction to the ideas and methods of the abstract formulation of observables and algebraic states via C^*-algebras. Here one finds the proof of the *GNS theorem* and some consequences of purely mathematical flavour, like the general *theorem of Gelfand–Najmark*. This closing chapter also contains material on quantum symmetries in an algebraic setting. As an example the *Weyl* C^*-algebra associated to a symplectic space (usually infinite-dimensional) is presented.

The appendices at the end of the book recap facts on partially ordered sets, groups and differential geometry.

The author has chosen not to include topics, albeit important, such as the theory of rigged Hilbert spaces (the famous *Gelfand triples*) [GeVi64], and the powerful formulation of QM based on the *path integral* approach [AH-KM08, Maz09]. Doing so would have meant adding further preparatory material, in particular regarding the theory of distributions, and extending measure theory to the infinite-dimensional case.

There are very valuable and recent textbooks similar to this one, at least in the mathematical approach. One of the most interesting and useful is the far-reaching [BEH07].

1.1.2 Prerequisites

Apart from a firm background on linear algebra, plus some group theory and representation theory, essential requisites are the basics of calculus in one and several real variables, measure theory on σ-algebras [Coh80, Rud86] (summarised at the end of this chapter), and a few notions on complex functions.

Imperative, on the physics' side, is the acquaintance with undergraduate physics. More precisely, analytical mechanics (the groundwork of Hamilton's formulation of dynamics) and electromagnetism (the key features of electromagnetic waves and the crucial wavelike phenomena like interference, diffraction, scattering).

Lesser elementary, yet useful, facts will be recalled where needed (including examples) to enable a robust understanding. One section of Chap. 12 will need elementary *Lie group* theory. For this we refer to the book's epilogue: the last appendix summarises tidbits of differential geometry rather thoroughly. Further details should be looked up in [War75, NaSt82].

1.1.3 General Conventions

1. The symbol $:=$ means "equal, by definition, to".
2. The *inclusion* symbols $\subset, \supset$ allow for equality $=$.

3. The symbol $\bigsqcup$ denotes the disjoint union.
4. $\mathbb{N}$ is the set of natural numbers *including zero*, and $\mathbb{R}_+ := [0, +\infty)$.
5. Unless otherwise stated, the *field of scalars of a* normed, Banach or Hilbert *vector space* is the field of complex numbers $\mathbb{C}$, and *inner product* always means *Hermitian* inner product.
6. The *complex conjugate* of a number c is denoted by $\overline{c}$. As the same symbol is used for the *closure* of a set of operators, should there be confusion we will comment on the meaning.
7. The *inner product* of two vectors ψ, ϕ in a Hilbert space H is written as $(\psi|\phi)$ to distinguish it from the *ordered pair* (ψ, ϕ). The product's *left* entry is antilinear: $(\alpha\psi|\phi) = \overline{\alpha}(\psi|\phi)$.
   If $\psi, \phi \in \mathsf{H}$, the symbols $\psi(\phi|\ \ )$ and $(\phi|\ \ )\psi$ denote the *same* linear operator $\mathsf{H} \ni \chi \mapsto (\phi|\chi)\psi$.
8. Complete orthonormal systems in Hilbert spaces are called *Hilbert bases*. When no confusion arises, a Hilbert basis is simply referred to as a *basis*.
9. The word *operator* tacitly implies it is *linear*.
10. An operator $U : \mathsf{H} \to \mathsf{H}'$ between Hilbert spaces H and H' that is isometric and surjective is called *unitary*, even if elsewhere in the literature the name is reserved for the case $\mathsf{H} = \mathsf{H}'$.
11. By *vector subspace* we mean a subspace *for the linear operations*, even in presence of additional structures on the ambient space (e.g. Hilbert, Banach *etc.*).
12. For the Hermitian conjugation we always use the symbol *. Note that *Hermitian operator*, *symmetric operator*, and *self-adjoint operator* are *not* considered synonyms.
13. When referring to maps, *one-to-one*, 1–1 and injective mean the same, just like *onto* and surjective. *Bijective* means simultaneously one-to-one and onto, and *invertible* is a synonym of bijective. One should beware that a *one-to-one correspondence* is a bijective mapping. An *isomorphism*, irrespective of the algebraic structures at stake, is a 1–1 map onto its image, hence a bijective homomorphism.
14. **Boldface** typeset (within a definition or elsewhere) is typically used when *defining* a term for the first time.
15. Corollaries, definitions, examples, lemmas, notations, remarks, propositions and theorems are labelled sequentially to simplify lookup.
16. At times we use the shorthand '*iff*', instead of 'if and only if', to say that two statements imply one another, i.e. they are logically equivalent.

Finally, if h denotes *Planck's constant*, we adopt the notation, widely used by physicists,

$$\hbar := \frac{h}{2\pi} = 1.054571800(13) \times 10^{-34}\ \mathrm{Js}\,.$$

1.2 On Quantum Theories

1.2.1 Quantum Mechanics as a Mathematical Theory

From a mathematical point of view Quantum Mechanics represents a rare blend of mathematical elegance and descriptive insight into the physical world. The theory essentially makes use of techniques of functional analysis mixed with incursions and overlaps with measure theory, probability and mathematical logic.

There are (at least) two possible ways to formulate precisely (i.e. mathematically) elementary QM. The eldest one, historically speaking, is due to von Neumann ([Neu32]) in essence, and is formulated using the language of Hilbert spaces and the spectral theory of unbounded operators. A more recent and mature formulation was developed by several authors in the attempt to solve quantum field theory problems in mathematical physics. It relies on the theory of abstract algebras (*-algebras and C^*-algebras) that are built mimicking the operator algebras defined and studied, again, by von Neumann (nowadays known as W^*-*algebras* or *von Neumann algebras*), but freed from the Hilbert-space structure (for instance, [BrRo02] is a classic on operator algebras). The core result is the celebrated *GNS theorem* (after Gelfand, Najmark and Segal) [Haa96, BrRo02], that we will prove in Chap. 14. The newer formulation can be considered an extension of the former one, in a very precise sense that we shall not go into here, also by virtue of the novel physical context it introduces and by the possibility of treating physical systems with infinitely many degrees of freedom, i.e. quantum fields. In particular, this second formulation makes precise sense of the demand for *locality* and *covariance* of relativistic quantum field theories [Haa96], and allows to extend quantum field theories to a curved spacetime.

The algebraic formulation of elementary QM, even though it can be achieved and despite its utmost finesse, is not a strict necessity (see for example [Str05a] and parts of [DA10]). Given the relatively basic nature of our book we shall treat almost exclusively the first formulation, which displays an impressive mathematical complexity together with a manifest formal elegance. We will introduce the algebraic formulation in the last chapter only, and stay within the general framework rather than consider QM as a physical object.

A crucial mathematical tool to develop a Hilbert-space formulation for QM is the *spectral theorem for self-adjoint operators* (unbounded, usually) defined on dense subspaces of a Hilbert space. This theorem, which can be extended to normal operators, was first proved by von Neumann in [Neu32] apropos the mathematical structure of QM: this fundamental work ought to be considered a XX century milestone of mathematical physics and pure mathematics. The definition of abstract Hilbert spaces and much of the corresponding theory, as we know it today, are also due to von Neumann and his formalisation of QM. Von Neumann built the modern, axiomatic notion of an abstract Hilbert space [Neu32, Sect. 1] by considering the two approaches to QM known at that time: the one relying on Heisenberg matrices, and the one using Schrödinger's wavefunctions.

The relationship between QM and *spectral theory* depends upon the following fact. The standard way of interpreting QM dictates that physical quantities that are measurable over quantum systems can be associated to unbounded self-adjoint operators on a suitable Hilbert space. The spectrum of each operator coincides with the collection of values the associated physical quantity can attain. The construction of a physical quantity from easy properties and propositions of the type "the value of the quantity falls in the interval $(a, b]$", which correspond to orthogonal projectors in the mathematical scheme one adopts, is nothing else but an integration procedure with respect to an appropriate projector-valued spectral measure. In practice, then, the spectral theorem is just a means to construct complicated operators starting from projectors or, conversely, decompose operators in terms of projector-valued measures.

The contemporary formulation of spectral theory is certainly different from that of von Neumann, although the latter already contained all basic constituents. Von Neumann's treatise (dating back to 1932) discloses an impressive depth still today, especially in the most difficult parts of the physical interpretation of the QM formalism. If we read that book it becomes clear that von Neumann was well aware of these issues, as opposed to many colleagues of his. It would be interesting to juxtapose his opus to the much more renowned book by Dirac [Dir30] on QM's fundamentals, a comparison that we leave to the interested reader. At any rate, the great interpretative strength von Neumann gave to QM begins to be recognised by experimental physicists as well, in particular those concerned with quantum measurements [BrKh95].

The so-called *quantum logics* arise from the attempt to formalise QM from the most radical stand: endowing the same *logic* used to treat quantum systems with properties different from those of ordinary logic, and modifying the semantic theory. For example, more than two truth values are allowed, and the Boolean lattice of propositions is replaced by a more complicated non-distributive structure. In the first formulation of quantum logic, known as *standard quantum logic* and introduced by Von Neumann and Birkhoff in 1936, the role of the Boolean algebra of propositions is taken by an *orthomodular lattice*: this is modelled, as a matter of fact, on the set of orthogonal projectors on a Hilbert space, or the collection of closed projection spaces [Bon97], plus some composition rules. Despite its sophistication, that model is known to contain many flaws when one tries to translate it in concrete (*operational*) physical terms. Beside the various formulations of quantum logic [Bon97, DCGi02, EGL09], there are also other foundational formulations based on alternative viewpoints (e.g., *topos* theory).

1.2.2 QM in the Panorama of Contemporary Physics

Quantum Mechanics and *General and Special Relativity* (GSR) represent the two paradigms by which the physics of the XX and XXI centuries developed. QM is, roughly speaking, the physical theory of the atomic and sub-atomic world, while GSR is the physical theory of gravity, the macroscopic world and cosmology (as recently as 2016, GSR received thunderous experimental confirmation with the detection of *gravitational waves*). These two paradigms coalesced, in several contexts, to give rise to relativistic quantum theories. *Relativistic Quantum Field Theory* [StWi00, Wei99], in particular, has witnessed a striking growth and a spectacular predictive and explanatory success relative to the theory of elementary particles and fundamental interactions. Two examples for all. In the so-called *standard model of elementary particles*, that theory predicted the unification of the *weak* and *electromagnetic forces* which was confirmed experimentally at the end of the 1980 s during a memorable experiment at C.E.R.N., in Geneva, where the particles Z_0 and $W^{\pm}$, expected by electro-weak unification, were first observed. More recently, another prediction was confirmed: the existence of the *Higgs boson*, suspected since the 1960 s and eventually detected 50 years after. On March 14, 2013, referring to the newly observed particle, C.E.R.N. confirmed that: '*CMS and ATLAS have compared a number of options for the spin-parity of this particle, and these all prefer no spin and even parity. This, coupled with the measurements of the interactions with other particles, strongly indicates that the new particle is a Higgs boson.*'

The best-ever accuracy in the measurement of a physical quantity in the whole history of physics was predicted by *quantum* electrodynamics. The quantity is the so-called gyro-magnetic ratio g of the electron, a dimensionless number. The value expected by quantum electrodynamics for $a := g/2 - 1$ was

$$0.001159652359 \pm 0.000000000282\ ,$$

and the experimental result turned out to be

$$0.001159652209 \pm 0.000000000031\ .$$

Many physicists believe QM to be the fundamental theory of the universe (more than relativistic theories), also owing to the impressive range of linear scales at which it holds: from 1 m (Bose–Einstein condensates) to *at least* 10^{-16} m (inside nucleons, at quark level). QM has had an enormous success, both theoretical and experimental, in materials' science, optics, electronics, with several key repercussions: every technological object of common use that is complex enough to contain a *semiconductor* (childrens' toys, mobile phones, remote controls…) exploits the quantum properties of matter.

Going back to the two major approaches of the past century – QM and GSR – there remain a number of obscure points where these paradigms seem to clash. In particular, the so-called "quantisation of gravity" and the structure of spacetime at *Planck scales* ($\sim 10^{-35}$ m, $\sim 10^{-43}$ s, the length and time scales obtained from the fundamental constants of the two theories: the speed of light, the universal constant of gravity and Planck's constant). The necessity of a discontinuous spacetime at ultra-microscopic scales is also reinforced by certain mathematical (and conceptual) hurdles that the so-called theory of quantum *Renormalisation* has yet to overcome, as consequence of the infinite values arising in computing processes due to the interaction of elementary particles. From a purely mathematical perspective the existence of infinite values is actually related to the problem, already intrinsically ambiguous, of defining the product of two distributions: infinites are not the root of the problem, but a mere manifestation of it.

These issues, whether unsolved or partially solved, have underpinned important theoretical advancements of late, which in turn influenced the developments of pure mathematics itself. Examples include (super-)*String* theory, the various *Noncommutative Geometries*, first of all Alain Connes' version, and *Loop Quantum Gravity*. The difficulty in deciding which of these theories makes any physical sense and is apt to describe the universe at very small scales is also practical: today's technology is not capable of preparing experiments that enable to distinguish among all available theories. However, it is relevant to note that recent experimental observations of the so-called γ-bursts, conducted with the telescope "Fermi Gamma-ray", have lowered the threshold for detecting quantum-gravity phenomena (e.g. the violation of Lorentz's symmetry) well below Planck's length [Abd09]. Other discrepancies between QM and GSR, about which the debate is more relaxed today than it was in the past, have to do with QM verses the notions of *locality of relativistic nature* (Einstein–Podolsky–Rosen paradox [Bon97]) in relationship to QM's *entanglement* phenomena. This is due in particular to Bell's work in the late 1960s, and to the famous experiments of Aspect. Both disproved Einstein's expectations, and secondly they confirmed the Copenhagen interpretation, eventually proving that nonlocality is a characteristic of Nature, independent of whether one accepts the standard interpretation of QM or not. The vast majority of physicists seems to agree that the existence of nonlocal physical processes, as QM forecasts, does not imply any concrete violation of the core of Relativity (quantum entanglement does not involve superluminal transmission of information, nor the violation of causality [Bon97]).

In the standard interpretation of QM that is called the *Copenhagen interpretation* there are parts that remain physically and mathematically unintelligible, yet still very interesting conceptually. In particular, and despite several appealing attempts, it still not clear how standard mechanics may be seen as a limit subcase of QM, nor how to demarcate (even roughly, or temporarily) the two worlds. Further, the question remains about the physical and mathematical description of the so-called *process of quantum measurement*, of which more later, which is strictly related to the classical limit of QM. From this fact, as well, other interpretations of the QM formalisms were born that differ deeply from the Copenhagen interpretation. Among these more recent interpretations, once considered heresies, Bohm's interpretation relies on *hidden variables* [Bon97, Des99] and is particularly intriguing.

Doubts are sometimes raised about the formulation of QM and about it being not truly clear, but just a list of procedures that "actually work", whereas its true nature is something inaccessible, sort of "noetic". In the author's opinion a dangerous epistemological mistake hides behind this point of view. The misconception is based on the belief that "explaining" a phenomenon means reducing it to the categories of daily life, as if everyday experience reached farther than reality itself. Quite the contrary: those categories were built upon conventional wisdom, and hence without any alleged metaphysical insight. There could be a deep philosophical landscape unfolding on the other side of that simple "actually works", and it may draw us closer to reality rather than pushing us away from it. Quantum Mechanics has taught us to think in a different fashion, and for this reason it has been (is, actually) an incredible opportunity for humanity. To turn our backs on QM and declare we do not understand it because it refuses to befit our familiar mental categories means locking the door that separates us from something huge. This is the author's stance, who does indeed consider *Heisenberg's uncertainty principle* (a theorem in today's formulation, despite the name) one of the highest achievements of the human enterprise.

Mathematics is the most accurate of languages invented by man. It allows to create formal structures corresponding to worlds that may or may not exist. The plausibility of these hypothetical realities is found solely in the logical or syntactical coherence of the corresponding mathematical structure. In this way semantic "chimeras" might arise, that turn out to be syntactically coherent nevertheless. Sometimes these creatures are consistent with worlds or states that do exist, although they have not been discovered yet. A feature that is attributable to an existing entity can only either be present or not, according to the classical ontological view. Quantum Mechanics, in particular, leads to say that any such property may not simply obey the true/false pattern, but also be "uncertain", despite being inherent to the object itself. This tremendous philosophical leap can be entirely managed within the mathematical foundations of QM, and represents the most profound philosophical legacy of Heisenberg's principle.

At least two general issues remain unanswered, both of gnoseological nature, essentially, and common to the entire formulation of modern science. The first is the relationship between theoretical entities and the objects we have experience of. The problem is particularly delicate in QM, where the notion of what a measuring instrument is has not yet been fully clarified. Generally speaking, the relationship of a theoretical entity with an experimental object is not direct, and still based on often understated theoretical assumptions. But this is also the case in classical theories, when one, for example, wants to tackle problems such as the geometry of the physical space. There, it is necessary to identify, inside the physical reality, objects that correspond to the idea of a point, a segment, and so on, and to do that we use other assumptions, like the fact that the geometry of the straightedge is the same as when inspecting space with light beams. The second issue is the hopelessness of trying to prove the syntactic coherence of a mathematical construction. We may attempt to reduce the latter to the coherence of set theory, or category theory. That this reduction should prove the construction's solidity has more to do with psychology than with it being a real fact, due to the profusion of well-known paradoxes disseminated along the history of the formalisation of mathematics, and eventually due to Gödel's famous theorem.

In spite of all, QM (but also other scientific theories) has been – and is – capable of predicting new facts and not yet observed phenomena that have been confirmed experimentally.

In this sense Quantum Mechanics must contain elements of reality.

1.3 Backgrounds on General Topology

For the reader's sake we collect here notions of point-set topology that will be used by and large in the book. All statements are elementary and classical, and can be found easily in any university treatise, so for brevity we will prove almost nothing. The practiced reader may skip this section completely and return to it at subsequent stages for reference.

1.3.1 Open/Closed Sets and Basic Point-Set Topology

Open and *closed sets* are defined as follows [Ser94II], in the greatest generality.

Definition 1.1 The pair $(\mathsf{X}, \mathscr{T})$, where X is a set and $\mathscr{T}$ a collection of subsets of X, is called a **topological space** if:

(i) $\emptyset, \mathsf{X} \in \mathscr{T}$,

(ii) the union of (arbitrarily many) elements of $\mathscr{T}$ is an element of $\mathscr{T}$,

(iii) the intersection of a finite number of elements of $\mathscr{T}$ belongs to $\mathscr{T}$.

$\mathscr{T}$ is called a **topology** on X and the elements of $\mathscr{T}$ are the **open sets** of X.

Definition 1.2 On a topological space $(\mathsf{X}, \mathscr{T})$:

(a) a **basis** for the topology of $(\mathsf{X}, \mathscr{T})$ is a subset $\mathscr{B} \subset \mathscr{T}$ such that each element in $\mathscr{T}$ is the union of elements of $\mathscr{B}$;

(b) an **open neighbourhood** of $p \in \mathsf{X}$ is an element $A \in \mathscr{T}$ such that $p \in A$;

(c) $x \in S \subset \mathsf{X}$ is an **interior point** of S if there exists an open neighbourhood A of x contained in S.

The **interior** of a set $S \subset \mathsf{X}$ is the set:

$$Int(S) := \{x \in \mathsf{X} \mid x \text{ is an interior point of } S\} .$$

The **exterior** of a set $S \subset \mathsf{X}$ is the set:

$$Ext(S) := \{x \in \mathsf{X} \mid x \text{ is an interior point of } \mathsf{X} \setminus S\} .$$

The **frontier** of a set $S \subset \mathsf{X}$ is the difference set:

$$\partial S := \mathsf{X} \setminus (Int(S) \cup Ext(S)) .$$

(d) $C \subset \mathsf{X}$ is called **closed** if $\mathsf{X} \setminus C$ is open.

A subset $S \subset \mathsf{X}$ in a topological space $(\mathsf{X}, \mathscr{T})$ inherits the structure of a topological space from $(\mathsf{X}, \mathscr{T})$ by defining a topology on S as $\mathscr{T}_S := \{S \cap A \mid A \in \mathscr{T}\}$. This topology (the definition is easily satisfied) is called the **induced topology** on S by $(\mathsf{X}, \mathscr{T})$.

Most of the topological spaces we will see in this text are *Hausdorff spaces*, in which open sets "separate" points.

Definition 1.3 A topological space $(\mathsf{X}, \mathscr{T})$ and its topology are called **Hausdorff** if they satisfy the **Hausdorff property**: for every $x, x' \in \mathsf{X}$ there exist $A, A' \in \mathscr{T}$, with $x \in A$, $x' \in A'$, such that $A \cap A' = \emptyset$.

Remark 1.4 **(1)** Both X and $\emptyset$ are open and closed sets.
(2) Closed sets satisfy properties that are "dual" to open sets, as follows straightforwardly from their definition. Hence:
(i) $\emptyset$, X are closed,
(ii) the intersection of (arbitrarily many) closed sets is closed,
(iii) the finite union of closed sets is a closed set.
(3) The simplest example of a Hausdorff topology is the collection of subsets of $\mathbb{R}$ containing the empty set and arbitrary unions of open intervals. This is a *basis* for the topology in the sense of Definition 1.1. It is called the **Euclidean topology** or **standard topology** of $\mathbb{R}$.
(4) A slightly more complicated example of Hausdorff topology is the **Euclidean topology**, or **standard topology**, of $\mathbb{R}^n$ and $\mathbb{C}^n$. It is the usual topology one refers to in elementary calculus, and is built as follows. If $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, the **standard norm** of $(c_1, \ldots, c_n) \in \mathbb{K}^n$ is, by definition:

$$||(c_1, \ldots, c_n)|| := \sqrt{\sum_{k=1}^{n} |c_k|^2}\,, \quad (c_1, \ldots, c_n) \in \mathbb{K}^n\,. \tag{1.1}$$

The set:

$$B_\delta(x_0) := \{x \in \mathbb{K}^n \mid ||x - x_0|| < \delta\} \tag{1.2}$$

is, hence, the usual **open ball** of $\mathbb{K}^n$ of radius $\delta > 0$ and centre $x_0 \in \mathbb{K}^n$. The open sets in the standard topology of $\mathbb{K}^n$ are, empty set aside, the unions of open balls of any given radius and centre. These balls constitute a basis for the standard topology of $\mathbb{R}^n$ and $\mathbb{C}^n$. ■

Here are notions that will come up often in the sequel.

Definition 1.5 If $(\mathsf{X}, \mathscr{T})$ is a topological space, the **closure** of $S \subset \mathsf{X}$ is the set:

$$\overline{S} := \cap\{C \supset S\,, C \subset \mathsf{X} \mid C \quad is\, closed\}\,. \tag{1.3}$$

The subset S is called **dense** in X if $\overline{S} = \mathsf{X}$.
The space $(\mathsf{X}, \mathscr{T})$ is said to be **separable** if there exists a dense and countable subset $S \subset \mathsf{X}$.

From the definition these properties follow.

Proposition 1.6 *If* $(\mathsf{X}, \mathscr{T})$ *is a topological space and* $S \subset \mathsf{X}$*:*
(a) $\overline{S}$ *is closed.*
(b) $\overline{\overline{S}} = \overline{S}$.
(c) *If* $T \subset \mathsf{X}$*, then* $S \subset T$ *implies* $\overline{S} \subset \overline{T}$.
(d) S *is closed if and only if* $\overline{S} = S$.

Definition 1.7 A topological space $(\mathsf{X}, \mathscr{T})$ has a **countable basis**, or is **second-countable**, if there is a countable subset $\mathscr{T}_0 \subset \mathscr{T}$ (the "countable basis") such that every open set is the union of elements of $\mathscr{T}_0$.

If $(\mathsf{X}, \mathscr{T})$ has a countable basis then *Lindelöf's lemma* holds:

Theorem 1.8 (Lindelöf's lemma) *Let* $(\mathsf{X}, \mathscr{T})$ *be a second-countable topological space. Then any open covering of a given subset in* X *admits a countable sub-covering: if* $B \subset \mathsf{X}$ *and* $\{A_i\}_{i\in I} \subset \mathscr{T}$ *with* $\cup_{i\in I} A_i \supset B$*, then* $\cup_{i\in J} A_i \supset B$ *for some countable subset* $J \subset I$.

Remark 1.9 $\mathbb{R}^n$ and $\mathbb{C}^n$, equipped with the standard topology, are second-countable: for $\mathbb{R}^n$, $\mathscr{T}_0$ can be taken to be the collection of open balls with rational radii and centred at rational points. The generalisation to $\mathbb{C}^n$ is obvious. ■

In conclusion, we recall the definition of product topology.

Definition 1.10 If $\{(\mathsf{X}_i, \mathscr{T}_i)\}_{i\in F}$ is a collection of topological spaces indexed by a finite set F, the **product topology** on $\times_{i\in F}\mathsf{X}_i$ is the topology whose open sets are $\emptyset$ and the unions of Cartesian products $\times_{i\in F} A_i$, with $A_i \in \mathscr{T}_i$ for any $i \in F$.

If F has arbitrary cardinality, the previous definition cannot be generalised directly. If we did so in the obvious way we would not maintain important properties, such as Tychonoff's theorem, that we will discuss later. Nevertheless, a natural topology on $\times_{i\in F}\mathsf{X}_i$ can be defined, still called **product topology** because is extends Definition 1.10.

Definition 1.11 If $\{(\mathsf{X}_i, \mathscr{T}_i)\}_{i\in F}$ is a collection of topological spaces with F of arbitrary cardinality, the **product topology** on $\times_{i\in F}\mathsf{X}_i$ has as open sets $\emptyset$ and the unions of Cartesian products $\times_{i\in F} A_i$, with $A_i \in \mathscr{T}_i$ for any $i \in F$, such that on each $\times_{i\in F} A_i$ we have $A_i = \mathsf{X}_i$ but for a finite number of indices i.

Remark 1.12 **(1)** The standard topology of $\mathbb{R}^n$ is the product topology obtained by endowing the single factors $\mathbb{R}$ with the standard topology. The same happens for $\mathbb{C}^n$.
(2) Either in case of finitely many factors, or infinitely many, the canonical projections:

$$\pi_i : \times_{j\in F}\mathsf{X}_j \ni \{x_j\} \mapsto x_i \in \mathsf{X}_i$$

are clearly continuous if we put the product topology on the domain. It can be proved that the product topology is the coarsest among all topologies making the canonical projections continuous (coarsest means it is contained in any such topology). ■

1.3.2 Convergence and Continuity

Let us pass to convergence and continuity. First of all we need to recall the notions of *convergence of a sequence* and *limit point*.

Definition 1.13 Let $(\mathsf{X}, \mathscr{T})$ be a topological space.
(a) A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ **converges** to a point $x \in \mathsf{X}$, called the **limit of the sequence**:

$$x = \lim_{n\to+\infty} x_n \quad \textit{and also written} \quad x_n \to x \quad \textit{as } n \to +\infty$$

if, for any open neighbourhood A of x there exists $N_A \in \mathbb{R}$ such that $x_n \in A$ whenever $n > N_A$.
(b) $x \in \mathsf{X}$ is a **limit point** of a subset $S \subset \mathsf{X}$ if any open neighbourhood A of x contains a point of $S \setminus \{x\}$.

Remark 1.14 It should be patent from the definitions that in a Hausdorff space the limit of a sequence is unique, if it exists. ■

The relationship between limit points and closure of a set is sanctioned by the following classical and elementary result:

Proposition 1.15 *Let* $(\mathsf{X}, \mathscr{T})$ *be a topological space and* $S \subset \mathsf{X}$.
$\overline{S}$ *coincides with the union of* S *and the set of its limit points.*

The definition of continuity and continuity at a point is recalled below.

Definition 1.16 Let $f : \mathsf{X} \to \mathsf{X}'$ be a function between topological spaces $(\mathsf{X}, \mathscr{T})$, $(\mathsf{X}', \mathscr{T}')$.
(a) f is called **continuous** if $f^{-1}(A') \in \mathscr{T}$ for any $A' \in \mathscr{T}'$.
(b) f is said **continuous at** $p \in \mathsf{X}$ if, for any open neighbourhood $A'_{f(p)}$ of $f(p)$, there is an open neighbourhood A_p of p such that $f(A_p) \subset A'_{f(p)}$.
(c) f is called a **homeomorphism** if:

(i) f is continuous,
(ii) f is bijective,
(iii) $f^{-1} : \mathsf{X}' \to \mathsf{X}$ is continuous.

In this case X and X' are said to be **homeomorphic** (under f).
(d) f is called **open** (respectively **closed**) if $f(A) \in \mathscr{T}'$ for $A \in \mathscr{T}$ (resp. $\mathsf{X}' \setminus f(C) \in \mathscr{T}'$ for $\mathsf{X} \setminus C \in \mathscr{T}$).

Remark 1.17 **(1)** It is easy to check that $f : \mathsf{X} \to \mathsf{X}'$ is continuous if and only if it is continuous at every point $p \in \mathsf{X}$.
(2) The notion of continuity at p as of (b) reduces to the more familiar "ϵ-δ" definition when the spaces X and X' are $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the standard topology. To see this bear in mind that: (a) open neighbourhoods can always be chosen to be open balls of radii δ and ϵ, centred at p and $f(p)$ respectively; (b) every open neighbourhood of a point contains an open ball centred at that point. ■

Let us mention a useful result concerning the standard real line $\mathbb{R}$. One defines the **limit supremum** (also **superior limit**, or simply **limsup**) and the **limit infimum** (**inferior limit** or just **liminf**) of a sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ as follows:

$$\limsup_{n\in\mathbb{N}} s_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} s_n \quad \left(= \lim_{k\to+\infty} \sup_{n\geq k} s_n\right), \quad \liminf_{n\in\mathbb{N}} s_n := \sup_{k\in\mathbb{N}} \inf_{n\geq k} s_n \quad \left(= \lim_{k\to+\infty} \inf_{n\geq k} s_n\right).$$

Note how these numbers exist for any given sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, possibly being infinite, as they arise as limits of monotone sequences, whereas the limit of $\{s_n\}_{n\in\mathbb{N}}$ might not exist (neither finite nor infinite). However, it is not hard to prove the following elementary fact.

Proposition 1.18 *If $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, then $\lim_{n\to+\infty} s_n$ exists, possibly infinite, if and only if*

$$\limsup_{n\in\mathbb{N}} s_n = \liminf_{n\in\mathbb{N}} s_n .$$

In such a case:

$$\lim_{n\to+\infty} s_n = \limsup_{n\in\mathbb{N}} s_n = \liminf_{n\in\mathbb{N}} s_n .$$

1.3.3 Compactness

Let us briefly recall some easy facts about *compact* sets.

Definition 1.19 Let $(\mathsf{X}, \mathscr{T})$ be a topological space and $K \subset \mathsf{X}$ a subset.
(a) K is called **compact** if any open covering of it admits a finite sub-covering: if $\{A_i\}_{i\in I} \subset \mathscr{T}$ with $\cup_{i\in I} A_i \supset K$ then $\cup_{i\in J} A_i \supset K$ for some finite subset $J \subset I$.
(b) K is said **relatively compact** if $\overline{K}$ is compact.
(c) X is **locally compact** if any point in X has a relatively compact open neighbourhood.

Compact sets satisfy a host of properties [Ser94II] and we will not be concerned with them much more (though we shall return to them in Chap. 4). Let us recall a few results, at any rate.
We begin with the relationship to calculus on $\mathbb{R}^n$. If X is $\mathbb{R}^n$ (or $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$), the celebrated *Heine–Borel theorem* holds [Ser94II].

Theorem 1.20 (Heine–Borel) *If $\mathbb{R}^n$ is equipped with the standard topology, $K \subset \mathbb{R}^n$ is compact if and only if K is simultaneously closed and* **bounded** *(meaning $K \subset B_\delta(x)$ for some $x \in \mathbb{R}^n$, $\delta > 0$).*

In calculus, the *Weierstrass theorem*, which deals with continuous maps defined on compact subsets of $\mathbb{R}^n$ (or $\mathbb{C}^n$), can be proved directly without the definition of compactness. Actually one can prove a more general statement on $\mathbb{R}^n$-valued ($\mathbb{C}^n$-valued) continuous maps defined on compact subsets.

Proposition 1.21 *If $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, let $|| \ ||$ denote the standard norm of $\mathbb{K}^n$ as in (1.1), and endow $\mathbb{K}^n$ with the standard topology.*
If $f : K \to \mathbb{K}^n$ is continuous on the compact subset K of a topological space, then it is **bounded**, *i.e. there exists $M \in \mathbb{R}$ such that $||f(x)|| \leq M$ for any $x \in K$.*

Proof Since f is continuous at any point $p \in K$, we have $||f(x)|| \leq M_p \in \mathbb{R}$ for all x in an open neighbourhood A_p of p. As K is compact, we may extract a finite sub-covering $\{A_{p_k}\}_{k=1,\dots,N}$ from $\{A_p\}_{p\in K}$ that covers K. The number $M := \max_{k=1,\dots,N} M_k$ satisfies the request. □

Remark 1.22 **(1)** It is easily proved that if X is a Hausdorff space and $K \subset \mathsf{X}$ is compact then K is closed.
(2) Similarly, if K is compact in X, then every closed subset $K' \subset K$ is compact.
(3) Continuous functions map compact sets to compact sets.
(4) By definition of compactness and of induced topology it is clear that a set $K \subset Y$, with the induced topology on $Y \subset \mathsf{X}$, is compact in Y if and only if it is compact in X. ■

The properties of being compact and Hausdorff bear an interesting relationship. One such property is expressed by the following useful statement.

Proposition 1.23 *Let $f : M \to N$ be a continuous map from the compact space M to the Hausdorff space N. If f is bijective then it is a homeomorphism.*

On locally compact Hausdorff spaces an important technical result, known as **Urysohn's lemma**, holds. To state it, we first need to define the **support of a map** $f : \mathsf{X} \to \mathbb{C}$:

$$supp(f) := \overline{\{x \in \mathsf{X} \mid f(x) \neq 0\}} ,$$

where the bar is the topological closure in the space X.

Theorem 1.24 (Urysohn's lemma) *If $(\mathsf{X}, \mathscr{T})$ is a Hausdorff, locally compact space, for any compact $K \subset \mathsf{X}$ and any open set $U \supset K$ there exists a continuous map $f : \mathsf{X} \to [0, 1]$ such that:*
(i) $supp(f) \subset U$,
(ii) $supp(f)$ is compact,
(iii) $f(x) = 1$ if $x \in K$.

Eventually, the following key theorem relates the product topology to compactness.

Theorem 1.25 (Tychonoff) *The Cartesian product of (arbitrarily many) compact spaces is compact in the product topology.*

1.3.4 Connectedness

Definition 1.26 A topological space X is said to be **connected** if it cannot be written as the union of two disjoint open sets different from $\emptyset$ and X.
A subset $A \subset \mathsf{X}$ is **connected** if it is connected in the induced topology.

By defining the equivalence relation:

$x \sim x' \Leftrightarrow x, x' \in C$, *where* C *is a connected set in* X,

the resulting equivalence classes are maximal connected subsets in X called the **connected components** of X. Consequently, connected components are disjoint, cover X and are clearly closed.

Definition 1.27 A subset A in a topological space X is **path-connected** if for any pair of points $p, q \in A$ there is a continuous map (a path) $\gamma : [0, 1] \to A$ such that $\gamma(0) = p, \gamma(1) = q$.

Definition 1.28 A subset A in a topological space X is called **simply connected** if, for any $p, q \in A$ and any (continuous) paths $\gamma_i : [0, 1] \to A$, $i = 0, 1$, such that $\gamma_i(0) = p, \gamma_i(1) = q$, there exists a continuous map $\gamma : [0, 1] \times [0, 1] \to A$, called a **homotopy**, satisfying $\gamma(s, 0) = p, \gamma(s, 1) = q$ for all $s \in [0, 1]$ and $\gamma(0, t) = \gamma_0(t)$, $\gamma(1, t) = \gamma_1(t)$ for all $t \in [0, 1]$.

Remark 1.29 **(1)** Directly from the definition we have that continuous functions map connected spaces to connected spaces and path-connected spaces to path-connected spaces.
(2) A path-connected space is connected, but not conversely in general. A non-empty, open connected subset of $\mathbb{R}^n$ is always path-connected. This is a general property that holds in **locally path-connected** spaces, in which each point has a path-connected open neighbourhood.
(3) It can be proved that the product of two simply connected spaces, if equipped with the product topology, is simply connected. ■

1.4 Round-Up on Measure Theory

This section contains, for the reader's sake, basic notions and elementary results on abstract measure theory, plus fundamental facts from real analysis on Lebesgue's measure on the real line. To keep the treatise short we will not prove any statements, for these can be found in the classics [Hal69, Coh80, Rud86]. Well-read users might want to skip this part entirely, and refer to it for explanations on the conventions and notations used throughout.

1.4.1 Measure Spaces

The modern theory of integration is rooted in the notion of a σ-*algebra* of sets: this is a collection $\Sigma(\mathsf{X})$ of subsets of a given 'universe' set X that can be "measured" by an arbitrary "measuring" function μ that we will fix later. The definition of a σ-algebra specifies which are the good properties that subsets should possess in relationship to the operations of union and intersection. The "σ" in the name points to the closure property (Definition 1.30(c)) of $\Sigma(\mathsf{X})$ under countable unions. The *integral of a function* defined on X with respect to a measure μ on the σ-algebra will be built step by step.

We begin by defining σ-algebras, and a weaker version (*algebras of sets*) where unions are allowed only finite cardinality, which has an interest of its own.

Definition 1.30 A **σ-algebra** over the set X is a collection $\Sigma(\mathsf{X})$ of subsets of X such that:

(a) $\mathsf{X} \in \Sigma(\mathsf{X})$,
(b) $E \in \Sigma(\mathsf{X})$ implies $\mathsf{X} \setminus E \in \Sigma(\mathsf{X})$,
(c) if $\{E_k\}_{k\in\mathbb{N}} \subset \Sigma(\mathsf{X})$ then $\bigcup_{k\in\mathbb{N}} E_k \in \Sigma(\mathsf{X})$.

A **measurable space** is a pair $(\mathsf{X}, \Sigma(\mathsf{X}))$, where X is a set and $\Sigma(\mathsf{X})$ a σ-algebra on X.
A collection $\Sigma_0(\mathsf{X})$ of subsets of X is called an **algebra** (of sets) on X in case (a), (b) hold (replacing $\Sigma(\mathsf{X})$ by $\Sigma_0(\mathsf{X})$), and (c) is weakened to:
(c)' if $\{E_k\}_{k\in F} \subset \Sigma_0(\mathsf{X})$, with F finite, then $\bigcup_{k\in F} E_k \in \Sigma_0(\mathsf{X})$.

Remark 1.31 **(1)** From (a) and (b) it follows that $\emptyset \in \Sigma(\mathsf{X})$. Item (c) includes *finite* unions in $\Sigma(\mathsf{X})$: a σ-algebra is an algebra of sets. This is a consequence of (c) if one takes finitely many distinct E_k. Parts (b) and (c) imply $\Sigma(\mathsf{X})$ is also closed under intersections (at most countable).
(2) By definition of σ-algebra it follows that the intersection of σ-algebras on X is a σ-algebra on X. Moreover, the collection of all subsets of X is a σ-algebra on X. ■
Remark (2) prompts us to introduce a relevant technical notion. If $\mathscr{A}$ is a collection of subsets in X, there always is at least one σ-algebra containing all elements of $\mathscr{A}$. Since the intersection of all σ-algebras on X containing $\mathscr{A}$ is still a σ-algebra, the latter is well defined and called the σ-algebra **generated by** $\mathscr{A}$.-algebra generated by $\mathscr{A}$ Now let us define a notion, crucial for our purposes, where topology and measure theory merge.

Definition 1.32 If X is a topological space with topology $\mathscr{T}$, the σ-algebra on X generated by $\mathscr{T}$, denoted $\mathscr{B}(\mathsf{X})$, is called **Borel σ-algebra** on X.

Remark 1.33 **(1)** The notation $\mathscr{B}(\mathsf{X})$ is slightly ambiguous since $\mathscr{T}$ does not appear. We shall use that symbol anyway, unless confusion arises.
(2) If X coincides with $\mathbb{R}$ or $\mathbb{C}$ we shall assume in the sequel that $\Sigma(\mathsf{X})$ is the Borel σ-algebra $\mathscr{B}(\mathsf{X})$ determined by the standard topology on X (that of $\mathbb{R}^2$ if we are talking of $\mathbb{C}$).
(3) By definition of σ-algebra it follows immediately that $\mathscr{B}(\mathsf{X})$ contains in particular open and closed subsets, intersections of (at most countably many) open sets and unions of (at most countably many) closed sets. ■
The mathematical concept we are about to present is that of a *measurable* function. In a manner of speaking, this notion corresponds to that of a continuous function in topology.

Definition 1.34 Let $(\mathsf{X}, \Sigma(\mathsf{X}))$, $(\mathsf{Y}, \Sigma(\mathsf{Y}))$ be measurable spaces. A function $f : \mathsf{X} \to \mathsf{Y}$ is said to be **measurable** (with respect to the two σ-algebras) whenever $f^{-1}(E) \in \Sigma(\mathsf{X})$ for any $E \in \Sigma(\mathsf{Y})$.
In particular, if X is a topological space and we take $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$, and $Y = \mathbb{R}$ or $\mathbb{C}$, measurable functions from X to Y are called **(Borel) measurable functions**, respectively real or complex.

Remark 1.35 Let X and Y be topological spaces with topologies $\mathscr{T}(\mathsf{X})$ and $\mathscr{T}(\mathsf{Y})$. It is easily proved that $f : \mathsf{X} \to \mathsf{Y}$ is measurable with respect to the Borel σ-algebras $\mathscr{B}(\mathsf{X})$, $\mathscr{B}(\mathsf{Y})$ if and only if $f^{-1}(E) \in \mathscr{B}(\mathsf{X})$ for any $E \in \mathscr{T}(\mathsf{Y})$.
Immediately, then, *every continuous map* $f : \mathsf{X} \to \mathbb{C}$ or $f : \mathsf{X} \to \mathbb{R}$ *is Borel measurable*.

Let us summarise the main features of measurable maps from X to $\mathsf{Y} = \mathbb{R}, \mathbb{C}$.

Proposition 1.36 *Let* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *be a measurable space and* $M_{\mathbb{R}}(\mathsf{X})$, $M(\mathsf{X})$ *the sets of measurable maps from* X *to* $\mathbb{R}$, $\mathbb{C}$ *respectively. The following results hold.*

(a) $M_{\mathbb{R}}(\mathsf{X})$ *and* $M(\mathsf{X})$ *are vector spaces, respectively real and complex, with respect to pointwise linear combinations*

$$(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x), \quad x \in \mathsf{X},$$

for any measurable maps f, g *from* X *to* $\mathbb{R}$, $\mathbb{C}$ *and any real or complex numbers* α, β.

(b) *If* $f, g \in M_{\mathbb{R}}(\mathsf{X})$, $M(\mathsf{X})$ *then* $f \cdot g \in M_{\mathbb{R}}(\mathsf{X})$, $M(\mathsf{X})$, *with* $(f \cdot g)(x) := f(x)g(x)$ *for all* $x \in \mathsf{X}$.

(c) *The following facts are equivalent:*

(i) $f \in M(\mathsf{X})$,

(ii) $\overline{f} \in M(\mathsf{X})$,

(iii) $Re\, f, Im\, f \in M_{\mathbb{R}}(\mathsf{X})$,

where $\overline{f}(x) := \overline{f(x)}$, $(Re\, f)(x) := Re(f(x))$, *and* $(Im\, f)(x) := Im(f(x))$, *for all* $x \in \mathsf{X}$.

(d) *If* $f \in M_{\mathbb{R}}(\mathsf{X})$ *or* $f \in M(\mathsf{X})$ *then* $|f| \in M_{\mathbb{R}}(\mathsf{X})$, *where* $|f|(x) := |f(x)|$, $x \in \mathsf{X}$.

(e) *If* $f_n \in M(\mathsf{X})$, *or* $f_n \in M_{\mathbb{R}}(\mathsf{X})$, *for any* $n \in \mathbb{N}$ *and* $f_n(x) \to f(x)$ *for all* $x \in \mathsf{X}$ *as* $n \to +\infty$, *then* $f \in M(\mathsf{X})$, *or* $f \in M_{\mathbb{R}}(\mathsf{X})$.

(f) *If* $f_n \in M_{\mathbb{R}}(\mathsf{X})$ *and* $\sup_{n\in\mathbb{N}} f_n(x)$ *is finite for any* $x \in \mathsf{X}$, *then the function* $\mathsf{X} \ni x \mapsto \sup_{n\in\mathbb{N}} f_n(x)$ *belongs to* $M_{\mathbb{R}}(\mathsf{X})$.

(g) *If* $f_n \in M_{\mathbb{R}}(\mathsf{X})$ *and* $\limsup_{n\in\mathbb{N}} f_n(x)$ *is finite for all* $x \in \mathsf{X}$, *the function* $\mathsf{X} \ni x \mapsto \limsup_{n\in\mathbb{N}} f_n(x)$ *is an element of* $M_{\mathbb{R}}(\mathsf{X})$.

(h) *If* $f, g \in M_{\mathbb{R}}(\mathsf{X})$ *the map* $\mathsf{X} \ni x \mapsto \sup\{f(x), g(x)\}$ *is in* $M_{\mathbb{R}}(\mathsf{X})$.

(i) *If* $f \in M_{\mathbb{R}}(\mathsf{X})$ *and* $f \geq 0$, *then the map* $\mathsf{X} \ni x \mapsto \sqrt{f(x)}$ *is in* $M_{\mathbb{R}}(\mathsf{X})$.

From now on, as is customary in measure theory, we will work with the **extended real line**:

$$[-\infty, \infty] := \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\},$$

where $\mathbb{R}$ is enlarged by adding two symbols $\pm\infty$. The ordering of the reals is extended by declaring $-\infty < r < +\infty$ for any $r \in \mathbb{R}$ and defining on $\overline{\mathbb{R}}$ the topology whose basis consists of real open intervals and the sets $[-\infty, a)$, $(a, +\infty]$ for any $a \in \mathbb{R}$ (the notation should be obvious). Moreover one defines: $|-\infty| := |+\infty| := +\infty$.

Now a standard result.

Proposition 1.37 *If* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *is a measurable space, then* $f : \mathsf{X} \to \overline{\mathbb{R}}$ *is measurable if and only if* $f^{-1}((a, +\infty]) \in \Sigma(\mathsf{X})$ *for any* $a \in \mathbb{R}$.
Furthermore, statements (d), (e), (f), (g), (h) of Proposition 1.36 still hold when f_n *and* f *are* $\overline{\mathbb{R}}$*-valued, with the proviso that one drops finiteness in (f) and (g).*

Remark 1.38 **(1)** In (f), (g) and (h) of Proposition 1.36 we may substitute sup with inf and obtain valid statements.
(2) As far as the first statement of Proposition 1.37, the analogous statements with $(a, +\infty]$ replaced by $[a, +\infty]$, $[-\infty, a)$, or $[-\infty, a]$ hold. ■

1.4.2 Positive σ-Additive Measures

We pass to define *σ-additive, positive measures*.

Definition 1.39 (*Positive measure*) If $(\mathsf{X}, \Sigma(\mathsf{X}))$ is a measurable space, a **(σ-additive) positive measure** on X (with respect to $\Sigma(\mathsf{X})$) is a function $\mu : \Sigma(\mathsf{X}) \to [0, +\infty]$ satisfying:
(a) $\mu(\emptyset) = 0$
(b) $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$ if $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma(\mathsf{X})$, and $E_n \cap E_m = \emptyset$ if $n \neq m$ (**σ-additivity**).
The triple $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ is called a **measure space**.

Remark 1.40 **(1)** The series in (b), having non-negative terms, is well defined and can be rearranged at will.
(2) Easy consequences of the definition are the following properties.
Monotonicity: if $E \subset F$ with $E, F \in \Sigma(\mathsf{X})$,

$$\mu(E) \leq \mu(F)$$

Sub-additivity: if $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma(\mathsf{X})$:

$$\mu\left(\cup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu(E_n)$$

Inner continuity: if $E_1 \subset E_2 \subset E_3 \subset \cdots$ for $E_n \in \Sigma(\mathsf{X})$, $n = 1, 2, \ldots$, then:

$$\mu\left(\cup_{n=1}^{+\infty} E_n\right) = \lim_{n\to+\infty} \mu(E_n)$$

Outer continuity: if $E_1 \supset E_2 \supset E_3 \supset \cdots$ for $E_n \in \Sigma(\mathsf{X})$, $n = 1, 2, \ldots$, and $\mu(E_m) < +\infty$ for some m, then:

$$\mu\left(\cap_{n=1}^{+\infty} E_n\right) = \lim_{n \to +\infty} \mu(E_n)$$

■

Measures on σ-algebras can be constructed using extension techniques, by starting with measures on algebras (hence not closed under countable unions). We will employ such recipes later in the text. An important extension theorem for measures [Hal69] goes like this.

Theorem 1.41 *Let $\Sigma_0(\mathsf{X})$ be an algebra of sets on X and suppose $\mu_0 : \Sigma_0(\mathsf{X}) \to [0, +\infty]$ is a map satisfying:*
(i) Definition 1.39(a),
(ii) Definition 1.39(b) whenever $\cup_{n\in\mathbb{N}} E_n \in \Sigma_0(\mathsf{X})$ for $E_k \in \Sigma_0(\mathsf{X})$, $k \in \mathbb{N}$.
If $\Sigma(\mathsf{X})$ denotes the σ-algebra generated by $\Sigma_0(\mathsf{X})$, then we have
(i)

$$\Sigma(\mathsf{X}) \ni R \mapsto \mu(R) := \inf\left\{ \sum_{n\in\mathbb{N}} \mu_0(S_n) \;\middle|\; \{S_n\}_{n\in\mathbb{N}} \subset \Sigma_0(\mathsf{X}) \,, \cup_{n\in\mathbb{N}} S_n \supset R \right\} \tag{1.4}$$

is a σ-additive positive measure on X with respect to $\Sigma(\mathsf{X})$ that restricts to μ_0 on $\Sigma_0(\mathsf{X})$.
(ii) *If $\mathsf{X} = \cup_{n\in\mathbb{N}} \mathsf{X}_n$, with $\mathsf{X}_n \in \Sigma_0(\mathsf{X})$ and $\mu_0(\mathsf{X}_n) < +\infty$ for any $n \in \mathbb{N}$, then μ is the unique σ-additive positive measure on X, with respect to $\Sigma(\mathsf{X})$, restricting to μ_0 on $\Sigma_0(\mathsf{X})$.*

As we shall use several kinds of positive measures and measure spaces henceforth, we need to gather some special instances in one place.

Definition 1.42 (*Kinds of positive measures*) A measure space $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ and its (positive, σ-additive) measure μ are called:
(i) **finite**, if $\mu(\mathsf{X}) < +\infty$;
(ii) σ**-finite**, if $\mathsf{X} = \cup_{n\in\mathbb{N}} E_n$, $E_n \in \Sigma(\mathsf{X})$ and $\mu(E_n) < +\infty$ for any $n \in \mathbb{N}$;
(iii) a **probability space** and **probability measure**, if $\mu(\mathsf{X}) = 1$;
(iv) a **Borel space** and **Borel measure**, if $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ with X locally compact, Hausdorff.
In case μ is a Borel measure, and more generally if $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$, with X locally compact and Hausdorff, μ is called:
(v) **inner regular**, if :

$$\mu(E) = \sup\{\mu(K) \mid K \subset E\,, \quad K \quad is\ compact\}$$

for any $E \in \Sigma(\mathsf{X})$;

(vi) **outer regular**, if:

$$\mu(E) = \inf\{\mu(V) \mid V \supset E\,, \quad V \quad is\, open\}$$

for any $E \in \Sigma(\mathsf{X})$;

(vii) **regular**, when simultaneously inner and outer regular.

In the general case the measure μ is **concentrated on** $E \in \Sigma(\mathsf{X})$ when:

$$\mu(S) = \mu(E \cap S) \quad for\ any\ S \in \Sigma(\mathsf{X}).$$

Remark 1.43 Inner regularity requires that compact sets belong to the σ-algebra of sets on which the measure acts. In case of measures on σ-algebras including Borel's, this fact is true on locally compact Hausdorff spaces because compact sets are closed in Hausdorff spaces (Remark 1.22(1)) and hence they belong in the Borel σ-algebra.

A key notion, very often used in the sequel, is that of *support* of a measure on a Borel σ-algebra.

Definition 1.44 Let $(\mathsf{X}, \mathscr{T}(\mathsf{X}))$ be a topological space and $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$. The **support** of a (positive, σ-additive) measure μ on $\Sigma(\mathsf{X})$ is the closed subset of X:

$$supp(\mu) := \mathsf{X} \setminus \bigcup_{O \in \mathscr{T}(\mathsf{X}),\ \mu(O)=0} O\,.$$

Note how the open set $\mathsf{X} \setminus supp(\mu)$ does not necessarily have zero measure. Still, the following is useful.

Proposition 1.45 *If $\mu : \Sigma(\mathsf{X}) \to [0, +\infty]$ is a σ-additive positive measure on X and $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$, then μ is concentrated on $supp(\mu)$ if at least one of the following conditions holds:*

(i) X has a countable basis for its topology,

(ii) X is Hausdorff, locally compact and μ is inner regular.

Proof Let $A := \mathsf{X} \setminus supp(\mu)$ be the union (usually not countable) of all open sets in X with zero measure. Decompose $S \in \Sigma(\mathsf{X})$ into the disjoint union $S = (A \cap S) \cup (supp(\mu) \cap S)$. The additivity of μ implies $\mu(S) = \mu(A \cap S) + \mu(supp(\mu) \cap S)$. By positivity and monotonicity $0 \leq \mu(A \cap S) \leq \mu(A)$, so the result holds provided $\mu(A) = 0$. Let us then prove $\mu(A) = 0$. Under (i), Lindelöf's lemma guarantees we can write A as a countable union of open sets of zero measure $A = \cup_{i \in \mathbb{N}} A_i$, and positivity plus sub-additivity force $0 \leq \mu(A) \leq \sum_{i \in \mathbb{N}} \mu(A_i) = 0$. Therefore $\mu(A) = 0$.

In case (ii), by inner regularity we have $\mu(A) = 0$ if $\mu(K) = 0$, for any compact set $K \subset A$. Since A is a union of zero-measure sets by construction, K will be covered by open sets of zero measure. By compactness we may then extract a finite covering $A_1, \ldots, A_n$. Again by positivity and sub-additivity, $0 \leq \mu(K) \leq \mu(A_1) + \cdots + \mu(A_n) = 0$, whence $\mu(K) = 0$, as required. □

In conclusion we briefly survey zero-measure sets [Coh80, Rud86].

Definition 1.46 If $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ is a measure space, a set $E \in \Sigma(\mathsf{X})$ has **zero measure** if $\mu(E) = 0$. Then E is called a **zero-measure set**, (more rarely, a null or negligible set).
The measure space $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ and μ are called **complete** if, given any $E \in \Sigma(\mathsf{X})$ of zero measure, every subset in E belongs to $\Sigma(\mathsf{X})$ (so it has zero measure, by monotonicity).

A property P is said to hold **almost everywhere (with respect to** μ), shortened to **a.e.**, if P is true everywhere on X minus a set E of **zero measure**.

Remark 1.47 **(1)** Every measure space $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ can be made complete in the following manner.

Proposition 1.48 *If* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ *is a (σ-additive, positive) measure space, there is a measure space* $(\mathsf{X}, \Sigma'(\mathsf{X}), \mu')$*, called the* **completion** *of* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$*, such that :*

(i) $\Sigma'(\mathsf{X}) \supset \Sigma(\mathsf{X})$*,*

(ii) $\mu' \restriction_{\Sigma(\mathsf{X})} = \mu$*,*

(iii) $(\mathsf{X}, \Sigma'(\mathsf{X}), \mu')$ *is complete.*

The completion can be constructed in the two ensuing ways (yielding the same measure space).

(a) *Take the collection* $\Sigma'(\mathsf{X})$ *of* $E \subset \mathsf{X}$ *for which there exist* $A_E, B_E \in \Sigma(\mathsf{X})$ *with* $B_E \subset E \subset A_E$ *and* $\mu(A_E \setminus B_E) = 0$*. Then* $\mu'(E) := \mu(A_E)$*.*

(b) *Let* $\Sigma'(\mathsf{X})$ *be the collection of subsets of* X *of the form* $E \cup Z$*, where* $E \in \Sigma(\mathsf{X})$ *and* $Z \subset N_Z$ *for some* $N_Z \in \Sigma(\mathsf{X})$ *with* $\mu(N_Z) = 0$*. Then* $\mu'(E \cup Z) := \mu(E)$*.*

It is quite evident from (b) that if $(\mathsf{X}, \Sigma_1(\mathsf{X}), \mu_1)$ *is a complete measure space such that, once again,* $\Sigma_1(\mathsf{X}) \supset \Sigma(\mathsf{X})$*,* $\mu_1 \restriction_{\Sigma(\mathsf{X})} = \mu$*, then necessarily* $\Sigma_1(\mathsf{X}) \supset \Sigma'(\mathsf{X})$ *and* $\mu_1 \restriction_{\Sigma'(\mathsf{X})} = \mu'$*. In this sense the completion of a measure space is the* smallest *complete extension. When the initial measure space is already complete, the completion is the space itself.*

Notice that the completion depends heavily on μ*: in general, distinct measures on the same* σ*-algebra give rise to different completions.*

Moreover, measurable functions for the completed σ*-algebra are, generally speaking, no longer measurable for the initial one, whereas the converse is true: by completing the measurable space we enlarge the class of measurable functions.*

(2) *If* $(\mathsf{X}, \Sigma, \mu)$ *is a measure space and* $E \in \Sigma$*, we may restrict* Σ *and* μ *to* E *like this: define* $\Sigma\restriction_E := \{S \cap E \mid S \in \Sigma\}$ *and* $\mu\restriction_E (S) := \mu(S)$ *for any* $S \in \Sigma\restriction_E$*. It should be clear that* $(E, \Sigma\restriction_E, \mu\restriction_E)$ *is a measure space corresponding to the natural restriction of the initial measure on* E*.*

If $g : \mathsf{X} \to \mathbb{C}$ *(respectively* $\mathbb{R}$, $[-\infty, +\infty]$, $[0, +\infty]$*) is a measurable function with respect to* Σ*, then by construction the restriction* $g\restriction_E$ *of* g *to* E *is measurable with respect to* $\Sigma\restriction_E$*.*

Conversely, if $f : E \to \mathbb{C}$ *(*$\mathbb{R}$, $[-\infty, +\infty]$, $[0, +\infty]$*) is measurable with respect to* $\Sigma\restriction_E$*, it is simple to show that its extension* $\widetilde{f} : \mathsf{X} \to \mathbb{C}$ *(*$\mathbb{R}$, $[-\infty, +\infty]$, $[0, +\infty]$*), with* $\widetilde{f}(x) = f(x)$ *if* $x \in E$ *and* $\widetilde{f}(x) = 0$ *otherwise, is measurable with respect to* Σ*.*

(3) *One can prove [Rud86] that if every $f_n : \mathsf{X} \to \mathbb{R}$, or $\mathbb{C}$, is measurable for $n \in \mathbb{N}$, if $f(x) = \lim_{n\to+\infty} f_n(x)$ a.e. with respect to μ on X and we set $f(x) = c$ for some constant $c \in \mathbb{R}$, or $\mathbb{C}$, on the set N where the limit does not exist, then f is measurable.*

If μ is complete, f turns out to be measurable irrespective of how we define it on N. ∎

1.4.3 Integration of Measurable Functions

We are now ready to define the integral of a measurable function with respect to a σ-additive positive measure μ defined on a measurable space $(\mathsf{X}, \Sigma(\mathsf{X}))$. We proceed in steps, defining the integral on a special class of functions first, and then extending it to the measurable case.

The starting point are functions with values in $[0, +\infty] := [0, +\infty) \cup \{+\infty\}$. For technical reasons it is convenient to extend the notion of sum and product of non-negative real numbers so that $+\infty \cdot 0 := 0$, $+\infty \cdot r := +\infty$ if $r \in (0, +\infty]$, and $+\infty \pm r := +\infty$ if $r \in [0, +\infty)$.

A (non-negative) map $s : \mathsf{X} \to [0, +\infty]$ is called **simple** if it is measurable and its range is finite in $[0, +\infty]$. Such a function can be written, for certain $s_1, \ldots, s_n \in [0, +\infty) \cup \{+\infty\}$, as:

$$s = \sum_{i=1,\ldots,n} s_i \chi_{E_i}$$

where $E_1, E_2, \ldots, E_n$ are pairwise-disjoint elements of $\Sigma(\mathsf{X})$ and χ_{E_i} are the corresponding characteristic functions. The decomposition is not unique. Every function that can be written like this is simple. The **integral** of the simple map s with respect to μ is defined as the number in $[0, +\infty]$:

$$\int_{\mathsf{X}} s(x) d\mu(x) := \sum_{i=1,2,\ldots,n} s_i \mu(E_i) .$$

It is not difficult to show that the definition does not depend on the choice of decomposition of $s = \sum_{i=1,\ldots,n} s_i \chi_{E_i}$.

This notion can then be generalised to non-negative measurable functions in the obvious way: if $f : \mathsf{X} \to [0, +\infty]$ is measurable, let the **integral** of f with respect to μ be:

$$\int_{\mathsf{X}} f(x) d\mu(x) := \sup \left\{ \int_{\mathsf{X}} s(x) d\mu(x) \;\middle|\; s \geq 0 \text{ is simple and } s \leq f \right\} .$$

Note the integral may equal $+\infty$.

To justify the definition, we must remark that simple functions approximate with arbitrary accuracy non-negative measurable functions, as implied by the ensuing classical technical result [Rud86] (which we will state for complex functions and prove in Proposition 7.49).

Proposition 1.49 *If $f : \mathsf{X} \to [0, +\infty]$ is measurable, there exists a sequence of simple maps $0 \le s_1 \le s_2 \le \cdots \le s_n \le f$ with $s_n \to f$ pointwise. The convergence is uniform if there exists $C \in [0, +\infty)$ such that $f(x) \le C$ for all $x \in \mathsf{X}$.*

Note that the definition implies an elementary, yet important property of the integral.

Proposition 1.50 *If $f, g : \mathsf{X} \to [0, +\infty]$ are measurable and $f(x) \le g(x)$ a.e. on X with respect to μ, then the integrals (in $[0, +\infty]$) satisfy:*

$$\int_{\mathsf{X}} f(x) d\mu(x) \le \int_{\mathsf{X}} g(x) d\mu(x) .$$

To finish the construction we define the integral of a complex-valued measurable function in the most natural way: we write it as sum of its real and imaginary parts and then decompose the latter two real functions into their respective positive and negative parts. To overcome having to deal with awkward differences of infinite quantities we must restrict the class of functions, which we do now by introducing *μ-integrable functions*.

Definition 1.51 (*Integral*) If $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ is a (σ-additive, positive) measure space, a measurable map $f : \mathsf{X} \to \mathbb{C}$ is **integrable with respect to** μ (or **in** μ), or μ-**integrable**, if:

$$\int_{\mathsf{X}} |f(x)| d\mu(x) < +\infty .$$

Then the **integral of** f on X with respect to μ is the complex number:

$$\int_{\mathsf{X}} f(x)d\mu(x) = \int_{\mathsf{X}} Re(f)_+ d\mu(x) - \int_{\mathsf{X}} Re(f)_- d\mu(x) + i\left(\int_{\mathsf{X}} Im(f)_+ d\mu(x) - \int_{\mathsf{X}} Im(f)_- d\mu(x)\right) ,$$

where, if $g : \mathsf{X} \to \mathbb{R}$, we defined non-negative maps:

$$g_+(x) := \sup\{g(x), 0\} \quad and \quad g_-(x) := -\inf\{g(x), 0\} \quad for\, any\, x \in \mathbb{R}.$$

The set of μ-integrable functions on X will be indicated by $\mathscr{L}^1(\mathsf{X}, \mu)$.

If $f \in \mathscr{L}^1(\mathsf{X}, \mu)$ and $E \subset \mathsf{X}$ is in the σ-algebra of X, we set:

$$\int_E f(x)d\mu(x) := \int_{\mathsf{X}} f(x)\chi_E(x)\, d\mu(x) , \tag{1.5}$$

where χ_E is the characteristic function of E.

It is no problem to check that the integral of $f : \mathsf{X} \to \mathbb{C}$ on X with respect to μ generalises the integral of a measurable function $\mathsf{X} \to [0, +\infty)$. Also not hard is the following proposition, that clarifies the elementary features of the integral with respect to the measure μ.

Proposition 1.52 *If* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ *is a (σ-additive, positive) measure space, then the measurable maps* $f, g : \mathsf{X} \to \mathbb{C}$ *satisfy:*

(a) *if* $|f(x)| \leq |g(x)|$ *a.e. on* X*, then* $g \in \mathscr{L}^1(\mathsf{X}, \mu)$ *implies* $f \in \mathscr{L}^1(\mathsf{X}, \mu)$.

(b) *If* $f = g$ *a.e. on* X *then* f *and* g *are either both μ-integrable or neither is. In the former case*

$$\int_{\mathsf{X}} f(x)d\mu(x) = \int_{\mathsf{X}} g(x)d\mu(x) .$$

(c) *If* f, g *are μ-integrable, then* $af + bg$ *is μ-integrable for any chosen* $a, b \in \mathbb{C}$*; moreover,*

$$\int_{\mathsf{X}} af(x) + bg(x)d\mu(x) = a\int_{\mathsf{X}} f(x)d\mu(x) + b\int_{\mathsf{X}} g(x)d\mu(x) .$$

(d) *If* $f \geq 0$ *a.e. and* f *is μ-integrable, then:*

$$\int_{\mathsf{X}} f(x)d\mu(x) \geq 0$$

and the integral is null only if $f = 0$ *a.e.*

(e) *If* f *is μ-integrable, then:*

$$\left|\int_{\mathsf{X}} f(x)d\mu(x)\right| \leq \int_{\mathsf{X}} |f(x)|d\mu(x) .$$

Remark 1.53 Consider the restriction $(E, \Sigma\restriction_E, \mu\restriction_E)$ of the measure space $(\mathsf{X}, \Sigma, \mu)$ to the subset $E \in \Sigma$ as explained in Remark 1.47(2). The extension of $f \in \mathscr{L}(E, \mu\restriction_E)$ to X, say $\widetilde{f}$, defined as the zero map outside E, satisfies $\widetilde{f} \in \mathscr{L}^1(\mathsf{X}, \mu)$. Additionally,

$$\int_E f(x)d\mu\restriction_E (x) = \int_{\mathsf{X}} \widetilde{f}(x)d\mu(x) = \int_E \widetilde{f}(x)d\mu(x) .$$

■

The three central theorems of measure theory are listed below [Rud86].

Theorem 1.54 (Beppo Levi's monotone convergence) *Let* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ *be a (positive and σ-additive) measure space and* $\{f_n\}_{n\in\mathbb{N}}$ *a sequence of measurable functions* $\mathsf{X} \to [0, +\infty]$ *such that, a.e. at* $x \in \mathsf{X}$, $0 \leq f_n(x) \leq f_{n+1}(x)$ *for all* $n \in \mathbb{N}$.

Then:

$$\int_{\mathsf{X}} \lim_{n\to+\infty} f_n(x)d\mu(x) = \lim_{n\to+\infty}\int_{\mathsf{X}} f_n(x)d\mu(x) \leq +\infty ,$$

where the map $\lim_{n\to+\infty} f_n(x)$ *is zero where the limit does not exist, and the integral is the one defined for functions with values in* $[0, +\infty]$.

Theorem 1.55 (Fatou's lemma) *Let* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ *be a (σ-additive, positive) measure space and* $\{f_n\}_{n\in\mathbb{N}}$ *a sequence of measurable maps* $f_n : \mathsf{X} \to [0, +\infty]$.

Then:

$$\int_{\mathsf{X}} \liminf_{n\to+\infty} f_n(x)d\mu(x) \leq \liminf_{n\to+\infty} \int_{\mathsf{X}} f_n(x)d\mu(x) \leq +\infty \,,$$

the integral being the one defined for functions with values in $[0, +\infty]$.

Theorem 1.56 (Lebesgue's dominated convergence) *Let* $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ *be a (positive, σ-additive) measure space,* $\{f_n\}_{n\in\mathbb{N}}$ *a sequence of measurable maps* $f_n : \mathsf{X} \to \mathbb{C}$, *with* $f_n(x) \to f(x)$ *a.e. at* $x \in \mathsf{X}$ *as* $n \to +\infty$.

If there is a μ-integrable map $g : \mathsf{X} \to \mathbb{C}$ *such that* $|f_n(x)| \leq |g(x)|$ *a.e. at* $x \in \mathsf{X}$ *for any* $n \in \mathbb{N}$, *then* f *(set to zero where* $f_n(x) \not\to f(x)$*) is μ-integrable, and furthermore:*

$$\int_{\mathsf{X}} f(x)d\mu(x) = \lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x)d\mu(x) \;\; \textit{and} \;\; \lim_{n\to+\infty} \int_{\mathsf{X}} |f(x) - f_n(x)|\, d\mu(x) = 0 \,.$$

The next proposition (cf. Remark 1.47(1)) shows that the completion does not really affect integration, as we expect.

Proposition 1.57 *Let* $(\mathsf{X}, \Sigma, \mu)$ *be a measure space and* $(\mathsf{X}, \Sigma', \mu')$ *its completion. If* $f : \mathsf{X} \to \mathbb{C}$ *is measurable with respect to* Σ' *there exists a measurable map* $g : \mathsf{X} \to \mathbb{C}$ *with respect to* Σ *with* $f = g$ *almost everywhere with respect to* μ. *If, moreover,* $f \in \mathscr{L}^1(\mathsf{X}, \mu')$, *then* $g \in \mathscr{L}^1(\mathsf{X}, \mu)$ *and*

$$\int_{\mathsf{X}} f(x)d\mu'(x) = \int_{\mathsf{X}} g(x)d\mu(x) \,.$$

Proof Splitting f into real and imaginary parts and these into their positive and negative parts with the aid of Proposition 1.49, we can construct a sequence $s'_n := \sum_{i=1}^{M_n} c_i^{(n)} \chi_{E_i^{'(n)}}$ where $E_i^{'(n)} \in \Sigma'$, $E_i^{'(n)} \cap E_j^{'(n)} = \emptyset$ if $i \neq j$, $|s'_n(x)| \leq |s'_{n+1}(x)| \leq |f(x)|$ and $s'_n(x) \to f(x)$ everywhere on X as $n \to +\infty$. Because of Remark 1.47(1), we can write $E_i^{'(n)} = E_i^{(n)} \cup Z_i^{(n)}$ where $E_i^{(n)} \in \Sigma$, while $Z_i^{(n)} \subset N_i^{(n)} \in \Sigma$ with $\mu(N_i^{(n)}) = 0$. Define the maps, measurable with respect to Σ, $s_n := \sum_{i=1}^{M_n} c_i^{(n)} \chi_{E_i^{(n)} \setminus N_i^{(n)}}$. By construction $N := \cup_{n,i} N_i^{(n)}$ has zero μ-measure, being a countable union of zero-measure sets. Then set $g(x) = \lim_{n\to+\infty} s_n(x)$, measurable with respect to Σ as limit of measurable functions. The limit exists for any x, for it equals, by construction, 0 on N and $f(x)$ on $\mathsf{X} \setminus N$. Therefore we proved g is Σ-measurable and $g(x) = f(x)$ a.e. with respect to μ, as required.

Now to the last statement. By construction $|s_n(x)| \leq |s_{n+1}(x)| \leq |g(x)|$, $|s'_n(x)| \leq |s'_{n+1}(x)| \leq |f(x)|$, $|s_n(x)| \to |g(x)|$, $|s'_n(x)| \to |f(x)|$ and $\int |s_n| d\mu = \int |s_n| d\mu' = \int |s'_n| d\mu'$. Therefore the monotone convergence theorem applied to the sequence

$|s_n|$, with respect to both measures μ and μ', warrants that $g \in \mathscr{L}^1(\mathsf{X}, \mu)$ if $f \in \mathscr{L}^1(\mathsf{X}, \mu')$. By dominated convergence we finally have $\int_{\mathsf{X}} f d\mu' = \int_{\mathsf{X}} g d\mu$. ■

1.4.4 Riesz's Theorem for Positive Borel Measures

Moving on to Borel measures, we mention two important theorems. The first is the well-known *Riesz theorem for positive Borel measures* [Coh80], which we shall often use in the sequel: it tells that every linear and positive functional on the space of continuous maps with compact support on a locally compact, Hausdorff space is actually an integral. From now on, given a topological space X, $C_c(\mathsf{X})$ will be the complex vector space of continuous maps $f : X \to \mathbb{C}$ with compact support. The vector-space structure of $C_c(\mathsf{X})$ comes from pointwise linear combinations of $f, g \in C_c(\mathsf{X})$, with $\alpha, \beta \in \mathbb{C}$:

$$(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x) \quad \text{for all } x \in \mathsf{X}.$$

Theorem 1.58 (Riesz's theorem for positive Borel measures) *Take a locally compact Hausdorff space X and consider a linear functional $\Lambda : C_c(\mathsf{X}) \to \mathbb{C}$ such that $\Lambda f \geq 0$ whenever $f \in C_c(\mathsf{X})$ satisfies $f \geq 0$. Then there exists a σ-additive, positive measure μ_Λ on the Borel σ-algebra $\mathscr{B}(\mathsf{X})$ such that:*

$$\Lambda f = \int_{\mathsf{X}} f d\mu_\Lambda \ \text{ if } f \in C_c(X) \quad \text{and} \quad \mu_\Lambda(K) < +\infty \ \text{ when } K \subset \mathsf{X} \text{ is compact}.$$

The measure μ_Λ can be chosen to be regular, in which case it is uniquely determined.

This result can be strengthened to produce a complete measure representing Λ, by extending $(\mathsf{X}, \mathscr{B}(\mathsf{X}), \mu_\Lambda)$ to its completion, and in particular enlarging the Borel σ-algebra in a way that depends on μ_Λ (regular, we may assume). In this way it is far from evident that the extended measure is still regular. But this is precisely what happens, because of the following, useful, fact [Coh80].

Proposition 1.59 *Let $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ be a measure space, where X is locally compact and Hausdorff and $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$. If μ is regular, the measure obtained by completing $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ is regular.*

A second valuable comment is that under certain assumptions on X, μ_Λ becomes automatically regular and hence uniquely determined by Λ. This is a consequence of a technical fact [Rud86, Theorem 2.18], which we recall here.

Proposition 1.60 *If ν is a positive Borel measure on a locally compact Hausdorff space X, and each open set is a countable union of compact sets of finite measure, then ν is regular.*

The second pivotal result is *Luzin's theorem* [Rud86], according to which on locally compact Hausdorff spaces, the functions of $C_c(\mathsf{X})$ approximate, so to speak, measurable functions when we work with measures on σ-algebras large enough to contain $\mathscr{B}(\mathsf{X})$ and satisfy further conditions (this happens in spaces with Lebesgue measure, that we shall meet very soon).

Theorem 1.61 (Luzin) *Let* X *be a locally compact Hausdorff space,* μ *a measure on a* σ*-algebra* $\Sigma(\mathsf{X})$ *such that:*

(i) $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$,
(ii) $\mu(K) < +\infty$ *if* $K \subset \mathsf{X}$ *is compact,*
(iii) μ *is regular,*
(iv) μ *is complete.*

Suppose $f : \mathsf{X} \to \mathbb{C}$ *is measurable and such that* $f(x) = 0$ *if* $x \in \mathsf{X} \setminus A$, *for some* $A \in \Sigma(\mathsf{X})$ *with* $\mu(A) < +\infty$. *Then for any* $\epsilon > 0$ *there is a map* $g \in C_c(\mathsf{X})$ *such that:*

$$\mu\left(\{x \in \mathsf{X} \mid f(x) \neq g(x)\}\right) < \epsilon \,.$$

Moreover, g *can be chosen so that*

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in \mathsf{X}} |f(x)|.$$

Corollary 1.62 *Under the same assumptions of Theorem 1.61, if* $|f| \leq 1$ *there exists a sequence* $\{g_n\}_{n \in \mathbb{N}} \subset C_c(X)$ *with* $|g_n| \leq 1$ *for any* $n \in \mathbb{N}$ *and such that:*

$$f(x) = \lim_{n \to +\infty} g_n(x) \quad \textit{almost everywhere on } \mathsf{X}.$$

Remark 1.63 Generally speaking, it is not possible to replace "almost everywhere" with "everywhere" in the statement above. There are Borel measurable functions that are not the pointwise limit of a sequence of continuous functions. The simplest example is the *Dirichlet function*, $\chi : [0, 1] \to \mathbb{R}$, defined as $\chi(x) = 1$ if $x \in \mathbb{Q}$ and $\chi(x) = 1$ otherwise. However, χ is the pointwise limit of a sequence of functions which are pointwise limits of sequences of continuous functions.

As a matter of fact, given a Borel σ-algebra, and regardless of the choice of a measure thereon, there exist Borel-measurable functions which are not the pointwise limit of any sequence of continuous functions. These issues are properly treated by the theory of *Baire functions*: technically, χ is a Baire function of class 2 on the interval $[0, 1]$ that is not of class 1.

1.4.5 Differentiating Measures

Definition 1.64 If μ, ν are positive σ-additive measures defined on the same σ-algebra Σ:

(a) ν is called **absolutely continuous** with respect to μ (or **dominated** by μ), written $\nu \prec \mu$, whenever $\nu(E) = 0$ if $\mu(E) = 0$ with $E \in \Sigma$.

(b) ν is **singular** with respect to μ when there exist $A, B \in \Sigma$, $A \cap B = \emptyset$, such that μ is concentrated on A and ν concentrated on B.

Note μ is singular with respect to ν if and only if ν is singular with respect to μ.

The paramount Radon-Nikodym theorem holds [Coh80, Rud86]. Recall that given a subset A of B, $\chi_A : B \to \mathbb{R}$ is the **characteristic function** of A if $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ otherwise.

Theorem 1.65 (Radon–Nikodym) *Suppose μ and ν are positive, σ-additive and σ-finite measures on the same σ-algebra Σ over X. If $\nu \prec \mu$ there exists a measurable function $\frac{d\nu}{d\mu} : \mathsf{X} \to [0, +\infty]$ such that:*

$$\nu(E) = \int_{\mathsf{X}} \chi_E \frac{d\nu}{d\mu}\, d\mu \quad for\ any\ E \in \Sigma.$$

This map $\frac{d\nu}{d\mu}$ is called the **Radon–Nikodym derivative** *of ν in μ, and is determined by μ and ν up to sets of zero μ-measure.*

Furthermore, $f \in \mathscr{L}^1(\mathsf{X}, \nu) \Leftrightarrow f \cdot \frac{d\nu}{d\mu} \in \mathscr{L}^1(\mathsf{X}, \mu)$, in which case:

$$\int_{\mathsf{X}} f d\nu = \int_{\mathsf{X}} f \frac{d\nu}{d\mu} d\mu\ .$$

1.4.6 Lebesgue's Measure on $\mathbb{R}^n$

Lebesgue's measure on $\mathbb{R}^n$ is the prototype of all abstract positive measures. We define it, *a posteriori*, remembering what we proved in the previous sections. The starting point is the following proposition, itself a corollary of [Rud86, Theorem 2.20].

Proposition 1.66 *Fix $n = 1, 2, \ldots$. There exists a unique σ-additive, positive Borel measure μ_n on $\mathbb{R}^n$ satisfying:*

(i) $\mu_n \left(\times_{k=1}^n [a_k, b_k]\right) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$ *if* $a_k \leq b_k$, $a_k, b_k \in \mathbb{R}$,
(ii) μ_n *is invariant under translations:* $\mu_n(E + \mathbf{t}) = \mu_n(E)$ *for* $E \in \mathscr{B}(\mathbb{R}^n)$, $\mathbf{t} \in \mathbb{R}^n$.

It is possible to extend $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu_n)$ to a measure space $(\mathbb{R}^n, \mathscr{M}(\mathbb{R}^n), m_n)$, so that the measure m_n:

(i) maps compact sets to finite values,
(ii) is complete,
(iii) is regular,
(iv) is translation-invariant.

The extension is characterised as follows. If $A \subset \mathbb{R}^n$ then $A \in \mathscr{M}(\mathbb{R}^n)$ if and only if $F \subset A \subset G$ with $\mu_n(G \setminus F) = 0$, where $F, G \in \mathscr{B}(\mathbb{R}^n)$ are a countable (at most) union and intersection of closed and open sets respectively. In such a case $m_n(A) := \mu_n(G)$.

Remark 1.67 As a consequence, $\mathscr{M}(\mathbb{R}^n)$ is contained in the completion of $\mathscr{B}(\mathbb{R}^n)$ with respect to μ_n (cf. Remark 1.47(1)). Since $\mathscr{M}(\mathbb{R}^n)$ is complete and the completion is the smallest complete extension, we conclude that $(\mathbb{R}^n, \mathscr{M}(\mathbb{R}^n), m_n)$ is nothing but the completion of $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mu_n)$. ■

Definition 1.68 (*Lebesgue measure*) The measure m_n and the σ-algebra $\mathscr{M}(\mathbb{R}^n)$ determined by Proposition 1.66 are called **Lebesgue measure on** $\mathbb{R}^n$ and **Lebesgue σ-algebra on** $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}$) that is measurable with respect to $\mathscr{M}(\mathbb{R}^n)$ is said **Lebesgue measurable**.

Notation 1.69 From now on we shall often denote Lebesgue's measure by dx and not only m_n. For example,

$$m_n(E) = \int_{\mathbb{R}^n} \chi_E(x)dx \quad \text{if } E \in \mathscr{M}(\mathbb{R}^n)\ .$$

Sometimes we shall speak of **Lebesgue measure on a measurable subset**, like in *Lebesgue measure on* $[a, b]$. This will mean the restriction of Lebesgue's measure on $\mathbb{R}$ to $[a, b]$ in the sense of Remark 1.47(2). In such cases we shall tacitly follow Remark 1.53. In the restricted Lebesgue measure we will drop the symbol $\restriction_E$. For example, $\mathscr{L}^1([a, b], dx)$ will denote $\mathscr{L}^1([a, b], dx\restriction_{[a,b]})$. ■

Remark 1.70 **(1)** The Lebesgue measure m_n is invariant under the whole isometry group of $\mathbb{R}^n$, not just under translations: therefore it is also invariant under rotations, reflections and any composition of these, translations included.
(2) Borel measurable maps $f : \mathbb{R}^n \to \mathbb{C}$ are thus Lebesgue measurable, but the converse is generally false. Continuous maps $f : \mathbb{R}^n \to \mathbb{C}$ trivially belong to both categories.
(3) The restriction of m_n to $\mathscr{B}(\mathbb{R}^n)$ is just the measure μ_n of Proposition 1.66, hence a *regular* Borel measure.
(4) Condition (i) in Proposition 1.66 implies, on one hand, that already the Borel measure μ_n assigns finite values to compact sets, these being bounded in $\mathbb{R}^n$. On the other hand it immediately implies, by monotonicity, that both μ_n and m_n assign *non-zero measure* to non-empty open sets. This fact has an important consequence, expressed by the next useful, albeit simple, proposition.

Proposition 1.71 *Let $\mu : \mathscr{B}(\mathsf{X}) \to [0, +\infty]$ be a σ-additive positive measure on X such that $\mu(B) > 0$ if $B \neq \emptyset$ is open. (In particular μ can be the Lebesgue measure on $\mathbb{R}^n$ restricted to an open set $\mathsf{X} \subset \mathbb{R}^n$.) If $f : \mathsf{X} \to \mathbb{C}$ is continuous and $f(x) = 0$ a.e. with respect to μ, then $f(x) = 0$ for any $x \in \mathsf{X}$.*

Proof As f is continuous and $\mathbb{C} \setminus \{0\}$ is open, then $B := f^{-1}(\mathbb{C} \setminus \{0\})$ is open. If we had $\mu(B) > 0$, then f would not be zero almost everywhere. Hence $\mu(B) = 0$ and we must have $B = \emptyset$, i.e. $f(x) = 0$ for all $x \in \mathsf{X}$.

(5) Invariance under translations in Proposition 1.66 is extremely strong a requirement. One can prove [Rud86] that if $\nu : \mathscr{B}(\mathbb{R}^n) \to [0, +\infty]$ maps compact sets to finite values and is translation-invariant, there exists a constant $c \geq 0$ such that $\nu(E) = cm_n(E)$ for every $E \in \mathscr{B}(\mathbb{R}^n)$. ■

An established result, crucial in computations, is the following.

Proposition 1.72 *Given $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, with $-\infty < a_i < b_i < +\infty$ for $i = 1, \ldots, n$, consider $f : \mathbb{R}^n \to \mathbb{R}$ bounded on K with $f(x) = 0$ if $x \in \mathbb{R}^n \setminus K$.*

(a) *If $n = 1$, f is Riemann integrable on K if and only if it is continuous on K almost everywhere with respect to Lebesgue's measure on $\mathbb{R}$.*

(b) *If $n \geq 1$ and f is Riemann integrable on K, then it is Lebesgue measurable and Lebesgue integrable with respect to Lebesgue's measure on $\mathbb{R}^n$. Moreover,*

$$\int_{\mathbb{R}^n} f(x)dx = \int_K f(x)dx_R(x)\,,$$

where on the left is the Lebesgue integral, on the right the Riemann integral.

The two pivotal theorems of calculus, initially formulated for the Riemann integral, generalise to the Lebesgue integral on the real line as follows. Before that, we need some definitions.

Definition 1.73 If $a, b \in \mathbb{R}$, a map $f : [a, b] \to \mathbb{C}$ has **bounded variation** on $[a, b]$ if, however we choose a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ in the interval, we have:

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq C$$

where $C \in \mathbb{R}$ does *not* depend on the choice of points x_k, nor their number.

A subclass of functions of bounded variation is that of *absolutely continuous* maps.

Definition 1.74 If $a, b \in \mathbb{R}$, $f : [a, b] \to \mathbb{C}$ is **absolutely continuous** on $[a, b]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that, for any finite family of pairwise-disjoint, open subintervals (a_k, b_k), $k = 1, 2, \ldots, n$,

$$\sum_{k=1}^{n}(b_k - a_k) < \delta \quad implies \quad \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon\,.$$

Remark 1.75 **(1)** Absolutely continuous functions on $[a, b]$ have bounded variation and are uniformly continuous (not conversely).
(2) Maps with bounded variation on $[a, b]$, and absolutely continuous ones on $[a, b]$, form vector spaces. The product of absolutely continuous maps on the compact interval $[a, b]$ is absolutely continuous.
(3) It is not hard to see that a differentiable map $f : [a, b] \to \mathbb{C}$ (admitting, in particular, left and right derivatives at the endpoints) with bounded derivative is absolutely continuous, hence it also has bounded variation on $[a, b]$. A weaker version is this: $f : [a, b] \to \mathbb{C}$ is absolutely continuous if it is **Lipschitz**, i.e. if there exists $L > 0$ such that $|f(x) - f(y)| \le L|x - y|$, $x, y \in [a, b]$. ■

Now we are in the position to state [KoFo99] two classical results in real analysis due to Lebesgue, that generalise the fundamental theorems of Riemann integration to the Lebesgue integral. Below, dx and dt are Lebesgue measures.

Theorem 1.76 *Fix $a, b \in \mathbb{R}$, $a < b$.*
(a) *If $f : [a, b] \to \mathbb{C}$ is absolutely continuous then it admits derivative $f'(x)$ for almost every $x \in [a, b]$ with respect to Lebesgue's measure. By defining, say, $f'(x) := 0$ where the derivative does not exist, f' becomes Lebesgue measurable, $f' \in \mathscr{L}^1([a, b], dx)$ and*

$$\int_a^x f'(t)dt = f(x) - f(a) \quad for\, all\, x \in [a, b].$$

(b) *If $f \in \mathscr{L}^1([a, b], dx)$, the map $[a, b] \ni x \mapsto \int_a^x f(t)dt$ is absolutely continuous, and*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad a.e. on\, [a, b]\, with\, respect\, to\, Lebesgue's\, measure.$$

To end the section, we mention a famous decomposition theorem for Borel measures on $\mathbb{R}$ that plays a role in spectral theory [ReSi80].

Let μ be a (σ-additive, positive) regular Borel measure on $\mathbb{R}$ with $\mu(K) < +\infty$ for any compact set $K \subset \mathbb{R}$.

(i) The set $P_\mu := \{x \in \mathbb{R} \mid \mu(\{x\}) \neq 0\}$ is called the set of **atoms** of μ (note P_μ is either finite or countable);

(ii) μ is said **continuous** if $P_\mu = \emptyset$;

(iii) μ is a **purely atomic** measure if $\mu(S) = \sum_{p \in S} \mu(\{p\})$, $S \in \mathscr{B}(\mathbb{R})$.
A (σ-additive, positive) regular Borel measure μ on $\mathbb{R}$ with $\mu(K) < +\infty$ for any compact set $K \subset \mathbb{R}$ can be decomposed *uniquely* into a sum:

$$\mu = \mu_{pa} + \mu_c\,,$$

where μ_{pa} is purely atomic and μ_c continuous, by setting:

$$\mu_{pa}(S) := \mu(P_\mu \cap S) \quad \forall S \in \mathscr{B}(\mathbb{R}) \quad \text{and so}\ \mu_c := \mu - \mu_{pa}.$$

More precisely, a key decomposition theorem due to Lebesgue holds. Here is one of the most elementary versions [ReSi80].

Theorem 1.77 (Lebesgue) *Let μ be a σ-finite regular Borel (σ-additive, positive) measure on $\mathbb{R}$ with $\mu(K) < +\infty$ for any compact set $K \subset \mathbb{R}$. Then μ decomposes in a unique way as the sum of three (σ-additive, positive) measures on $\mathscr{B}(\mathbb{R})$*

$$\mu = \mu_{ac} + \mu_{pa} + \mu_{sing} \ .$$

For the Lebesgue measure *(restricted to $\mathscr{B}(\mathbb{R})$), μ_{ac} is an absolutely continuous measure, μ_{pa} a purely atomic singular measure and μ_{sing} is a continuous singular measure.*

1.4.7 The Product Measure

If $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$ and $(\mathsf{Y}, \Sigma(\mathsf{Y}), \nu)$ are measure spaces, we indicate with $\Sigma(\mathsf{X}) \otimes \Sigma(\mathsf{Y})$ the σ-algebra on $\mathsf{X} \times \mathsf{Y}$ generated by the family of rectangles $E \times F$ with $E \in \Sigma(\mathsf{X})$ and $F \in \Sigma(\mathsf{Y})$.

If μ, ν are σ-finite, one can define uniquely a σ-finite measure on $\Sigma(\mathsf{X}) \otimes \Sigma(\mathsf{Y})$, written $\mu \otimes \nu$, by imposing

$$\mu \otimes \nu(E \times F) = \mu(E)\nu(F) \quad \text{if } E \in \Sigma(\mathsf{X}) \text{ and } F \in \Sigma(\mathsf{Y}).$$

This measure $\mu \otimes \nu$ is called the **product** measure of μ, ν.

Remark 1.78 **(1)** We have the following fact [Rud86].

Proposition 1.79 *If f is measurable with respect to $\Sigma(\mathsf{X}) \otimes \Sigma(\mathsf{Y})$, then $\mathsf{Y} \ni y \mapsto f(x, y)$ and $\mathsf{X} \ni x \mapsto f(x, y)$ are measurable for any $x \in \mathsf{X}$ and $y \in \mathsf{Y}$, respectively.*
(2) *The completion of the product of the Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$ coincides with the Lebesgue measure on $\mathbb{R}^{n+m}$ [Rud86].* ■

The theorems of Fubini and Tonelli, which we state as one, hold.

Theorem 1.80 (Fubini and Tonelli) *Let $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$, $(\mathsf{Y}, \Sigma(\mathsf{Y}), \nu)$ be spaces with σ-finite measures, and consider a map $f : \mathsf{X} \times \mathsf{Y} \to \mathbb{C}$.*
(a) *If f is $\mu \otimes \nu$-integrable:*

(i) $\mathsf{Y} \ni y \mapsto f(x, y)$ is ν-integrable for almost every $x \in \mathsf{X}$, and $\mathsf{X} \ni x \mapsto f(x, y)$ is μ-integrable for almost every $y \in \mathsf{Y}$,

(ii) $\mathsf{X} \ni x \mapsto \int_Y f(x, y) d\nu(y)$ and $\mathsf{Y} \ni y \mapsto \int_{\mathsf{X}} f(x, y) d\mu(x)$ (set to zero where the integrals do not exist) are integrable on X and on Y respectively. Moreover:

$$\int_{\mathsf{X}\times\mathsf{Y}} f(x, y) d\mu \otimes d\nu(x, y) = \int_{\mathsf{X}} \left(\int_{\mathsf{Y}} f(x, y) d\nu(y) \right) d\mu(x) = \int_{\mathsf{Y}} \left(\int_{\mathsf{X}} f(x, y) d\mu(x) \right) d\nu(y) \ .$$

(b) *Suppose f is measurable with respect to $\Sigma(\mathsf{X}) \otimes \Sigma(\mathsf{Y})$. Then:*

(i) if $\mathsf{Y} \ni y \mapsto f(x, y)$ *is* ν*-integrable for almost every* $x \in \mathsf{X}$*, or* $\mathsf{X} \ni x \mapsto f(x, y)$ *is* μ*-integrable for almost every* $y \in \mathsf{Y}$*, then the corresponding maps* $\mathsf{X} \ni x \mapsto \int_Y |f(x, y)| d\nu(y)$ *and* $\mathsf{Y} \ni y \mapsto \int_{\mathsf{X}} |f(x, y)| d\mu(x)$ *(null where the integrals are not defined) are measurable;*

(ii) if, additionally:

$$\int_{\mathsf{X}} \left(\int_{\mathsf{Y}} |f(x, y)| d\nu(y) \right) d\mu(x) < +\infty \quad or \quad \int_{\mathsf{Y}} \left(\int_{\mathsf{X}} |f(x, y)| d\mu(x) \right) d\nu(y) < +\infty$$

respectively, then f *is* $\mu \otimes \nu$*-integrable.*

1.4.8 Complex (and Signed) Measures

We recall a few definitions and elementary results from the theory of complex functions [Rud86].

Definition 1.81 (*Complex measure*) A **complex measure** on X is a map $\mu : \Sigma \to \mathbb{C}$ associating a complex number to every element in a σ-algebra Σ on X so that:

(i) $\mu(\emptyset) = 0$ and
(ii) $\mu(\cup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{+\infty} \mu(E_n)$, *independently of the summing order*, for any collection $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ with $E_n \cap E_m = \emptyset$ if $n \neq m$.

Under (i)–(ii), if $\mu(\Sigma) \subset \mathbb{R}$, then μ is called a **signed measure** or **charge** on X.

Remark 1.82 **(1)** Requirement (ii) is equivalent to asking absolute convergence of the series $\sum_{n=0}^{+\infty} \mu(E_n)$ to $\mu(\cup_{n\in\mathbb{N}} E_n)$, by virtue of a generalisation of a classical result of Riemann on rearranging real series that do not converge absolutely. It easy to prove that if a series of complex numbers converges absolutely, then it can be rearranged arbitrarily, to give always the same sum. When, instead, the convergence is not absolute, the *Lévy-Steinitz rearrangement theorem* says the sum depends on the rearrangement, so there are several possible sums. So we conclude that [Rud64]:

Theorem 1.83 *If* $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$*, the series* $\sum_{n=0}^{+\infty} z_n$ *converges absolutely (*$\sum_{n=0}^{+\infty} |z_n| < +\infty$*) if and only if there exists* $S \in \mathbb{C}$ *such that* $\sum_{n=0}^{+\infty} z_{P(n)} = S$ *for any bijection* $P : \mathbb{N} \to \mathbb{N}$*.*

(2) There is a way to generate a finite positive measure starting from any complex (or signed) measure, that goes as follows. If $E \in \Sigma$, we shall say $\{E_i\}_{i\in I} \subset \Sigma$ is a **partition** of E if I is finite or countable, $\cup_{i\in I} E_i = E$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. The σ*-additive positive measure* $|\mu|$ on Σ, called the **total variation of** μ, is by definition:

$$|\mu|(E) := \sup \left\{ \sum_{i\in I} |\mu(E_i)| \;\middle|\; \{E_i\}_{i\in I} \text{ partition of} E \right\} \quad \text{for any } E \in \Sigma.$$

It clearly satisfies $|\mu|(E) \geq |\mu(E)|$ if $E \in \Sigma$. Moreover, $|\mu|(\mathsf{X}) < +\infty$ [Rud86]. Therefore $|\mu|$ is a (σ-additive, positive) *finite* measure on Σ for any given complex measure μ. ■

In analogy to the real case, the *support* of a complex (or signed) measure on a Borel σ-algebra is defined below.

Definition 1.84 If $(\mathsf{X}, \mathscr{T}(\mathsf{X}))$ is a topological space and $\Sigma(\mathsf{X}) \supset \mathscr{B}(\mathsf{X})$, the **support** of a complex (or signed) measure μ on $\Sigma(\mathsf{X})$ is the closed subset of X:

$$supp(\mu) := \mathsf{X} \setminus \bigcup_{O \in \mathscr{T}(\mathsf{X}),\ |\mu|(O)=0} O\,.$$

The definition of absolutely continuous measure with respect to a given measure generalises straightforwardly to complex measures.

Definition 1.85 A complex (or signed) measure ν is **absolutely continuous** with respect to a given σ-additive, positive measure μ, or is **dominated by** μ, $\nu \prec \mu$, whenever both are defined over one σ-algebra Σ on X and $\mu(E) = 0$ implies $\nu(E) = 0$ for $E \in \Sigma$.

The theorem of Radon–Nikodym (Theorem 1.65) can be generalised to the case of complex/signed measures [Rud86]:

Theorem 1.86 (Radon–Nikodym theorem for complex and signed measures) *Let ν be a complex (or signed) measure, μ a σ-additive, positive and σ-finite measure, both defined on the σ-algebra Σ over X. If $\nu \prec \mu$ there exists a map $\frac{d\nu}{d\mu} \in \mathscr{L}^1(\mathsf{X}, \mu)$ such that:*

$$\nu(E) = \int_{\mathsf{X}} \chi_E \frac{d\nu}{d\mu}\, d\mu \quad for\ any\ E \in \Sigma$$

where χ_E is the characteristic function of $E \subset \mathsf{X}$.
Such map $\frac{d\nu}{d\mu}$ is unique up to sets of zero μ-measure, and is called the **Radon–Nikodym derivative** *of ν in μ.*

The following important result is a corollary of the above [Coh80, Rud86].

Theorem 1.87 (Characterisation of complex measures) *For any complex measure μ on a σ-algebra Σ on X, there exists a measurable function $h : \mathsf{X} \to \mathbb{C}$ with $|h| = 1$ on X, unique up to redefinition on zero-measure sets, that belongs in $\mathscr{L}^1(\mathsf{X}, |\mu|)$ and such that $\mu(E) = \int_E h\, d|\mu|$ for all $E \in \Sigma$.*

The same result holds, with the obvious changes, for signed measures.

According to it, if $f \in \mathscr{L}^1(\mathsf{X}, |\mu|)$ we define the **integral of f with respect to the complex measure μ** by:

$$\int_{\mathsf{X}} f d\mu := \int_{\mathsf{X}} f h \, d|\mu| \, .$$

In Chap. 2 (Example 2.48(1)) we shall state a general version of Riesz's representation theorem for complex measures.

1.4.9 Exchanging Derivatives and Integrals

In this final section we state the pivotal theorem that allows to differentiate inside an integral for a general positive measure. The proof is an easy consequence of the dominated convergence theorem plus Lagrange's mean value theorem.

Theorem 1.88 (Differentiation inside an integral) *In relation to the (σ-additive, positive) measure space $(\mathsf{X}, \Sigma(\mathsf{X}), \mu)$, consider the family of maps $\{h_t\}_{t\in A} \subset \mathscr{L}^1(\mathsf{X}, \mu)$ where $A \subset \mathbb{R}^m$ is open and $t = (t_1, \ldots, t_m)$. Assume that*

(i) for some $k \in \{1, 2, \ldots, m\}$ the derivative

$$\frac{\partial h_t(x)}{\partial t_k}$$

exists for any $x \in \mathsf{X}$ and $t \in A$;

(ii) there is a map $g \in \mathscr{L}^1(\mathsf{X}, \mu)$ such that:

$$\left|\frac{\partial h_t(x)}{\partial t_k}\right| \leq |g(x)| \; for \; any \; t \in A, \, a.e. on \, \mathsf{X} \, .$$

Then:
(a) $\mathsf{X} \ni x \mapsto \frac{\partial h_t}{\partial t_k} \in \mathscr{L}^1(\mathsf{X}, \mu)$*,*
(b) *for any $t \in A$, integral and derivative can be swapped:*

$$\frac{\partial}{\partial t_k} \int_{\mathsf{X}} h_t(x) d\mu(x) = \int_{\mathsf{X}} \frac{\partial h_t(x)}{\partial t_k} d\mu(x) \, . \tag{1.6}$$

Furthermore:

(iii) if, for a given g, condition (ii) holds simultaneously for all $k = 1, 2, \ldots, m$, almost everywhere at $x \in \mathsf{X}$, and every function (for any fixed $t \in A$)

$$A \ni t \mapsto \frac{\partial h_t(x)}{\partial t_k}$$

is continuous, then

(c) *the function:*

$$A \ni t \mapsto \int_{\mathsf{X}} h_t(x) d\mu(x)$$

belongs to $C^1(A)$.

Remark 1.89 Theorem 1.88 is true also when we take a complex (or signed) measure μ and replace $\mathscr{L}^1(\mathsf{X}, \mu)$ by $\mathscr{L}^1(\mathsf{X}, |\mu|)$ in the statement. The proof is direct, and relies on Theorem 1.87. ■

Chapter 2
Normed and Banach Spaces, Examples and Applications

I'm convinced mathematics is the most important investigating tool of the legacy of the human enterprise, it being the source of everything.

René Descartes

In the book's first proper chapter, we will discuss the fundamental notions and theorems about normed and Banach spaces. We will introduce certain algebraic structures modelled on natural algebras of operators on Banach spaces. Banach operator algebras play a relevant role in modern formulations of Quantum Mechanics.

The chapter will, in essence, introduce the working language and the elementary topological instruments of the theory of linear operators. Even if mostly self-contained, the chapter is by no means exhaustive if compared to the immense literature on the basic properties of normed and Banach spaces. The texts [Rud86, Rud91] should be consulted in this respect. In due course we shall specialise to operators on complex Hilbert spaces, with a short detour in Chap. 4 into the more general features of compact operators.

The most important notions of the present chapter are without any doubt *bounded operators* and the various *topologies (induced by norms or seminorms) on spaces of operators*. The relevance of these mathematical tools descends from the fact that the language of linear operators on linear spaces is the language used to formulate QM. Here the class of bounded operators plays a central technical part, even though in QM one is forced, on physical grounds, to introduce and work with unbounded operators too, as we shall see in the second part of the book.

The chapter's first section is devoted to the elementary concepts of normed space, Banach space and their basic topological properties. We shall discuss examples, like the space of continuous maps $C(\mathsf{K})$ over a compact space K, and prove the crucial theorem of Arzelà–Ascoli in this setup. In the examples we will also prove key results such as the completeness of L^p spaces (Fischer–Riesz theorem).

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_2

The norm of an operator is defined in the second section, and we will establish its main features.

Section three presents the fundamental results of Banach spaces, in their simplest versions. These are the theorems of Hahn–Banach, Banach–Steinhaus, plus the corollary to Baire's category theorem known as the open mapping theorem. We will prove a few useful technical consequences (the inverse operator theorem and the closed graph theorem). Then we will introduce the various operator topologies that come into play, prove the theorem of Banach–Alaoglu and recall briefly the Krein–Milman theorem and Fréchet spaces.

Section four is devoted to projection operators in normed spaces. This notion will be specialised in the subsequent chapter to that of an orthogonal projector, which will be more useful for our purposes.

In the final two sections we will treat two elementary but important topics: equivalent norms (including the proof that n-dimensional normed spaces are Banach and homeomorphic to the standard $\mathbb{C}^n$) and the theory of contractions in complete normed spaces (including, in the examples, the proof of the local existence and uniqueness of solutions to first-order ODEs on $\mathbb{R}^n$ or $\mathbb{C}^n$). The latter will be the only instance of *nonlinear* functional analysis present in the book.

From now onwards we shall assume that the reader is familiar with vector spaces and linear mappings (a standard reference text for which is [Ser94I]).

2.1 Normed and Banach Spaces and Algebras

After we have adapted the notions of the previous chapter to normed spaces we shall introduce Banach spaces. Then, by augmenting the algebraic structure with an inner product, we will study normed and Banach algebras.

2.1.1 Normed Spaces and Essential Topological Properties

The first definitions we present are those of *norm*, *normed space* and *continuous map* between normed spaces.

Examples of normed spaces, very common in functional analysis and its physical applications, will be provided later, especially in the next section.

Definition 2.1 (*Normed space*) Let X be a vector space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. A map $N : \mathsf{X} \to \mathbb{R}$ is called a **norm** on X, and (X, N) is a **normed space**, if:
N0. $N(u) \geq 0$ for any $u \in \mathsf{X}$,
N1. $N(\lambda u) = |\lambda| N(u)$ for any $\lambda \in \mathbb{K}$ and $u \in \mathsf{X}$,
N2. $N(u + v) \leq N(u) + N(v)$, for any $u, v \in \mathsf{X}$,
N3. $N(u) = 0 \Rightarrow u = \mathbf{0}$, for any $u \in \mathsf{X}$.
When **N0**, **N1**, **N2** are valid but **N3** does not necessarily hold, N is called a **seminorm**.

Remarks 2.2 **(1)** Clearly, from **N1** descends $N(\mathbf{0}) = 0$, while **N2** implies the inequality:

$$|N(u) - N(v)| \le N(u - v) \quad \text{se } u, v \in \mathsf{X}. \tag{2.1}$$

(2) N1 is called *homogeneity* property, **N2** is known as *triangle inequality* or *subadditivity*. Together, **N0** and **N3** are referred to as *positive definiteness*, whereas **N0** alone is sometimes called *semi-definiteness*. ■

Notation 2.3 Henceforth the symbols $|| \ ||$ and $p(\ )$, with subscripts if necessary, will denote a norm and a seminorm respectively. Other symbols might be used as well.

An elementary yet fundamental notion is that of *open ball*.

Definition 2.4 Let $(\mathsf{X}, || \ ||)$ be a normed space.
The **open ball of centre** $x_0 \in \mathsf{X}$ **and radius** $r > 0$ is the set:

$$B_r(x_0) := \{x \in \mathsf{X} \mid ||x - x_0|| < r\} .$$

A set $A \subset \mathsf{X}$ is **bounded** if there exists an open ball $B_r(x_0)$ (of finite radius!) such that $B_r(x_0) \supset A$.

Later on we shall define the same object using a seminorm p instead of a norm $|| \ ||$. Two useful properties of open balls (valid if using seminorms too), that follow immediately from **N2** and the definition, are:

$$B_\delta(y) \subset B_r(x) \qquad \text{if } y \in B_r(x) \text{ and } 0 < \delta + ||y - x|| < r, \tag{2.2}$$

$$B_r(x) \cap B_{r'}(x') = \emptyset \quad \text{if } 0 < r + r' < ||x - x'||. \tag{2.3}$$

Let us introduce the natural topology of a normed space.

Definition 2.5 Consider a normed space $(\mathsf{X}, || \ ||)$.
(a) A subset $A \subset \mathsf{X}$ is **open** if $A = \emptyset$ or A is the union of open balls.
(b) The **norm topology** of X is the family of open sets in X.

Remarks 2.6 **(1)** By (2.2) we have:

$$A \subset \mathsf{X}\, is\ open \quad \Leftrightarrow \quad \forall x \in A, \exists r_x > 0\, such\ that\, B_{r_x}(x) \subset A. \tag{2.4}$$

(2) By definition of open set and formulas (2.2), (2.4), it follows that open sets as defined in Definition 2.5 are also open according to Definition 1.1. Hence the collection of open sets in a normed space is indeed an honest *topology*. The normed space X equipped with the above family of open sets is a true *topological space*. The collection of open balls with arbitrary centres and radii is a *basis* for the norm topology of the normed space $(\mathsf{X}, || \ ||)$.

(3) Each normed space $(\mathsf{X}, ||\ ||)$ satisfies the *Hausdorff property*, cf. Definition 1.3 (and as such is a *Hausdorff space*). The proof follows from (2.3) by choosing $A = B_r(x)$, $A' = B_{r'}(x')$ with $r + r' < ||x - x'||$; the latter is non-zero if $x \neq x'$, by property **N3**. Had we defined the topology using a *seminorm* (rather than a norm), the Hausdorff property would not have been guaranteed. ■

Consider the following statements, valid in any normed space: (a) open neighbourhoods can be chosen to be open balls (of radii ε and δ); (b) each open neighbourhood of a point contains an open ball centred at that point (this follows from the definition of open set in a normed space and (2.2)). A straightforward consequence of (a) and (b) is that continuity, see (1.16), can be equivalently expressed as follows in normed spaces.

Definition 2.7 A map $f : (\mathsf{X}, ||\ ||_\mathsf{X}) \to (\mathsf{Y}, ||\ ||_\mathsf{Y})$ between normed spaces is **continuous at** $x_0 \in \mathsf{X}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(x) - f(x_0)||_\mathsf{Y} < \varepsilon$ whenever $||x - x_0||_\mathsf{X} < \delta$.

A map $f : \mathsf{X} \to \mathsf{Y}$ is **continuous** if it is continuous at each point of X.

Analogously, in normed spaces, convergent sequences (Definition 1.13) become:

Definition 2.8 If $(\mathsf{X}, ||\ ||)$ is a normed space, the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ **converges** to $x \in \mathsf{X}$:

$$x_n \to x \quad \text{as} \quad n \to +\infty \quad \text{or} \quad \lim_{n\to+\infty} x_n = x$$

if and only if for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{R}$ such that $||x_n - x|| < \varepsilon$ whenever $n > N_\varepsilon$. Equivalently

$$\lim_{n\to+\infty} ||x_n - x|| = 0\,.$$

The point x is the **limit of the sequence**.

Remark 2.9 If $(\mathsf{X}, ||\ ||)$ is a normed space and $A \subset \mathsf{X}$ a subset, a point $x \in X$ is a limit point of A if and only if there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A \setminus \{x\}$ converging to x. In fact if x is a limit point for A, every open ball $B_{1/n}(x)$, $n = 1, 2, \ldots$, contains at least one point $x_n \in A \setminus \{x\}$, and by construction $x_n \to x$ as $n \to +\infty$. Conversely, let $\{x_n\}_{n\in\mathbb{N}} \subset A \setminus \{x\}$ tend to x. Since every open neighbourhood B of x contains a ball $B_\varepsilon(x)$ centred at x by (2.4), the definition of convergence implies $B_\varepsilon(x)$, and so B, contains every point x_n with $n > N_\varepsilon$ for some $N_\varepsilon \in \mathbb{R}$. Thus x is a limit point. ■

A nice class of *continuous* linear functions is that of *isometries*.

Definition 2.10 If $(\mathsf{X}, ||\ ||_\mathsf{X})$, $(\mathsf{Y}, ||\ ||_\mathsf{Y})$ are normed spaces over the same field $\mathbb{C}$ or $\mathbb{R}$, a linear map $L : \mathsf{X} \to \mathsf{Y}$ is called **isometric**, or an **isometry**, if $||L(x)||_\mathsf{Y} = ||x||_\mathsf{X}$ for all $x \in \mathsf{X}$.

If the isometry $L : \mathsf{X} \to \mathsf{Y}$ is onto, it is an **isomorphism of normed spaces**.

Given an isomorphism L of normed spaces, the domain and codomain are called **isomorphic (under L)**.

Remarks 2.11 **(1)** It is obvious that an isometry $L : \mathsf{X} \to \mathsf{Y}$ is injective, by **N3**, but it may *not* be onto. If $\mathsf{X} = \mathsf{Y}$ and L is not surjective, then X must be infinite-dimensional.
(2) Since the pre-image of an open ball under an isometry is an open ball, each isometry $f : \mathsf{X} \to \mathsf{Y}$ between normed spaces X, Y is continuous in the two topologies.
(3) If an isometry $f : \mathsf{X} \to \mathsf{Y}$ is onto (an isomorphism), its inverse $f^{-1} : \mathsf{Y} \to \mathsf{X}$ is still linear and isometric, hence an isomorphism. An isomorphism of normed spaces is clearly a (linear) homeomorphisms of the two topological spaces.
(4) Other textbooks may provide a different definition, *not equivalent* to ours, of isomorphism of normed spaces. They typically require an isomorphism be only a linear continuous map with continuous inverse (i.e. a linear homeomorphism). An isomorphism according to Definition 2.10 is also such in this second meaning, but not conversely. Having f, f^{-1} both continuous is much weaker a condition than preserving norms. For instance $f : \mathsf{X} \ni x \mapsto ax \in \mathsf{X}$, with $a \neq 0$ fixed, is an isomorphism from X to itself for the second definition, but not in our sense. ■

A further technical result that we wish to present is the direct analogue of something that happens in the space $\mathbb{R}$ normed by the absolute value.

Proposition 2.12 *A function $f : \mathsf{X} \to \mathsf{Y}$ between normed spaces X, Y is continuous at $x \in \mathsf{X}$ if and only if it is* **sequentially continuous** *at x, i.e. $f(x_n) \to f(x)$ as $n \to +\infty$, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ such that $x_n \to x$ as $n \to +\infty$.*

Proof If f is continuous at x, let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ tend to x. By continuity, for any $\varepsilon > 0$ there is $\delta > 0$ such that $||f(x_n) - f(x)||_{\mathsf{Y}} < \varepsilon$ when $||x_n - x||_{\mathsf{X}} < \delta$. Since $||x_n - x|| \to 0$, then for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{R}$ such that $||f(x_n) - f(x)||_{\mathsf{Y}} < \varepsilon$ whenever $n > N_\varepsilon$. Thus f is sequentially continuous at x. Now assume f is *not* continuous at x_0, and let us show it cannot be sequentially continuous at x. With these assumptions there must be $\varepsilon > 0$ such that for any $n = 1, 2, \ldots$, there exists $x_n \in \mathsf{X}$ with $||x_n - x||_{\mathsf{X}} < 1/n$ but $||f(x) - f(x_n)||_{\mathsf{Y}} > \varepsilon$. The sequence $\{x_n\}_{n=1,2,\ldots}$ tends to x, but the corresponding images $\{f(x_n)\}_{n=1,2,\ldots}$ do *not* converge to $f(x)$ in Y. Therefore f is not sequentially continuous at x. □

At last we want to discuss *continuity properties* of the *vector-space operations* with respect to the *norm topologies* on normed spaces.

If $(\mathsf{X}, ||\ ||_{\mathsf{X}})$, $(\mathsf{Y}, ||\ ||_{\mathsf{Y}})$ are normed spaces over the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, we can form the Cartesian product $\mathsf{Y} \times \mathsf{X}$ and its *product topology*, induced by the topologies of the factors X, Y (cf. 1.10). This topology has as open sets the empty set and the unions of Cartesian products of open balls in X and Y. In case $\mathsf{Y} = \mathsf{X}$, we can study the continuity of the *sum of two vectors* in $\mathsf{X} \times \mathsf{X}$:

$$+ : \mathsf{X} \times \mathsf{X} \ni (u, v) \mapsto u + v \in \mathsf{X}\,,$$

where $\mathsf{X} \times \mathsf{X}$ has the product topology. From **N2**

$$||u + v|| \leq ||u|| + ||v||,$$

making the operation $+$ **jointly continuous** in its two arguments with respect to the norm topologies. Said otherwise, the addition is *continuous* in the product topology of the domain and the standard topology of the range.

In fact, the triangle inequality implies that given $(u_0, v_0) \in \mathsf{X} \times \mathsf{X}$ and $\varepsilon > 0$, then $u + v \in B_\varepsilon(u_0 + v_0)$ provided $(u, v) \in B_\delta(u_0) \times B_\delta(v_0)$ with $0 < \delta < \varepsilon/2$.
If Y is the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, thought of as normed by the modulus, we can consider the continuity of the *product between a scalar and a vector* in $\mathbb{K} \times \mathsf{X}$:

$$\mathbb{K} \times \mathsf{X} \ni (\alpha, u) \mapsto \alpha u \in \mathsf{X} ,$$

where the left-hand side has the product topology. From **N2** and **N1**,

$$||\alpha u|| = |\alpha| \, ||u||$$

implies that the multiplication by scalars is a **jointly continuous** operation in its arguments in the two norm topologies; that is to say, it is *continuous* with respect to the product topology on the domain and the standard one on the range. Here too the proof is easy: from the above identity and **N2**, given $(\alpha_0, u_0) \in \mathbb{K} \times \mathsf{X}$ and $\varepsilon > 0$, then $\alpha u \in B_\varepsilon(\alpha_0 u_0)$ if we take $(\alpha, u) \in B^{(\mathbb{K})}_{\delta'}(\alpha) \times B_\delta(u_0)$ with $0 < \delta = \varepsilon/(2|\alpha_0| + 1)$ and $0 < \delta' < \varepsilon/(2(||u_0|| + \delta))$. ($B^{(\mathbb{K})}_{\delta'}(\alpha)$ denotes an open ball in the normed space $\mathbb{K}$.)

2.1.2 Banach Spaces

Some of the above material can be adapted to completely general topological spaces. At the same time there are properties, like *completeness* (which we treat below), that befit the theory of normed spaces (and more generally *metrisable spaces*, which we will only mention elsewhere, in passing).

A well-known fact from the elementary theory on $\mathbb{R}^n$ is that convergent sequences $\{x_n\}_{n\in\mathbb{N}}$ in a normed space $(\mathsf{X}, || \ ||)$ satisfy the *Cauchy property*:

Definition 2.13 (*Cauchy property*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space $(\mathsf{X}, || \ ||)$ satisfies the **Cauchy property** if, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{R}$ such that $||x_n - x_m|| < \varepsilon$ whenever $n, m > N_\varepsilon$.

Such a sequence is called a **Cauchy sequence**.

In fact, suppose $\{x_n\}_{n\in\mathbb{N}}$ converges. This means $||x_n - x|| \to 0$ as $n \to +\infty$. Then $||x_k - x|| < \varepsilon$ for $k > N_\varepsilon$, so $||x_n - x_m|| \le ||x_n - x|| + ||x_m - x|| < \varepsilon$ for $n, m > N_{\varepsilon/2}$.

The idea of the above argument is that if a sequence converges to some point, its terms get closer to one another. It is interesting to see whether the converse holds as well: does a sequence of vectors that become closer always admit a limit?

As is well known from elementary calculus, the answer is yes on $\mathsf{X} = \mathbb{R}$ with the absolute value norm. Therefore it is true also on $\mathbb{C}$ and on any vector space built over

Cartesian products of standard copies of $\mathbb{R}^n$, $\mathbb{C}^n$. This is guaranteed by the fact that $\mathbb{R}$ satisfies the *completeness axiom*.

Normed spaces in which Cauchy sequences are convergent are called **complete** normed spaces. A general normed space is not complete, and complete normed spaces are scarce, hence interesting by default. They have relevant and useful features, especially for the physical applications that will be the object of the book.

Definition 2.14 (*Banach space*) A normed space is called a **Banach space** if it is **complete**, i.e. if any Cauchy sequence inside the space converges to a point of the space.

Remarks 2.15 **(1)** The property of being complete is invariant under isomorphisms of normed spaces, but not under homeomorphisms (continuous maps with continuous inverses, not necessarily linear). A counterexample is provided by the pair $\mathbb{R}$ and $(0, 1)$, both normed by the absolute value. Although they are homeomorphic, the line is complete, the interval is not.
(2) It is easy to prove that any closed subspace M in a Banach space B is itself a Banach space for the restricted norm: each Cauchy sequence in M is Cauchy for B too, so it must converge to a point in B. But this point must belong to M because M is closed and contains its limit points. ■

The spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ with standard norm:

$$||(c_1, \ldots, c_n)|| = \sqrt{\sum_{k=1}^{n} |c_k|^2}$$

are the simplest instances of *finite-dimensional* Banach spaces, respectively complex and real. As a matter of fact we shall prove in Sect. 2.5 that every *finite-dimensional* complex Banach space is homeomorphic to a standard $\mathbb{C}^n$, and show explicit examples of Banach spaces starting from the next section. At any rate, any normed space satisfies a nice property: it can be *completed* to a Banach space determined by it, in which it is moreover *dense*.

Theorem 2.16 (Completion of Banach spaces) *Let* $(\mathsf{X}, || \ ||)$ *be a normed vector space over* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$.
(a) *There exists a Banach space* (Y, N) *over* $\mathbb{K}$*, called* **completion** *of* X*, such that* X *is isometrically identified with a dense subspace of* Y *under a linear injective mapping* $J : \mathsf{X} \to \mathsf{Y}$.

Put otherwise, there is a linear 1-1 *map* $J : \mathsf{X} \to \mathsf{Y}$ *with*

$$\overline{J(\mathsf{X})} = \mathsf{Y} \quad \textit{and} \quad N(J(x)) = ||x|| \quad \textit{for any } x \in \mathsf{X}.$$

(b) *If the triple* (J_1, Y_1, N_1)*, with* $J_1 : \mathsf{X} \to \mathsf{Y}_1$ *a linear isometry and* (Y_1, N_1) *a Banach space on* $\mathbb{K}$*, is such that* $(\mathsf{X}, || \ ||)$ *is isometric to a dense subspace of* Y_1 *under* J_1*, then there is a unique isomorphism of normed spaces* $\phi : \mathsf{Y} \to \mathsf{Y}_1$ *such that* $J_1 = \phi \circ J$.

Sketch of the proof. The idea is similar to the procedure that generates the real numbers by completing the rationals.
(a) Let C denote the space of Cauchy sequences in X and define the equivalence relation on C:

$$x_n \sim x'_n \quad \Leftrightarrow \quad \lim_{n\to\infty} ||x_n - x'_n|| = 0 .$$

Clearly we can think that $\mathsf{X} \subset C/\!\sim$ by identifying each x of X with the equivalence class of the constant sequence $x_n = x$. Let J be the identification map. Then $C/\!\sim$ is easily a $\mathbb{K}$-vector space with norm induced by the structure of X. Now one should prove that $C/\!\sim$ is complete, that J is linear and isometric (hence 1-1) and that $J(\mathsf{X})$ is dense in $\mathsf{Y} := C/\!\sim$.
(b) $J_1 \circ J^{-1} : J(\mathsf{X}) \to \mathsf{Y}_1$ is a linear and continuous isometry from a dense set $J(\mathsf{X}) \subset \mathsf{Y}$ to a Banach space Y_1, so it extends uniquely to a linear, continuous isometry ϕ on Y (see Proposition 2.47). As ϕ is isometric, it is injective. The same is true about the extension ϕ' of $J \circ J_1^{-1} : J_1(\mathsf{X}) \to \mathsf{Y}$, and by construction $(J \circ J_1^{-1}) \circ (J_1 \circ J^{-1}) = id_{J(\mathsf{X})}$. Extending to $\overline{J(\mathsf{X})} = \mathsf{Y}$ by continuity, we see $\phi' \circ \phi = id_{\mathsf{Y}}$, and similarly $\phi \circ \phi' = id_{\mathsf{Y}_1}$. In conclusion ϕ and ϕ' are onto, so in particular ϕ is an isomorphism of normed spaces and by construction $J_1 = \phi \circ J$. The uniqueness of an isomorphism $\phi : \mathsf{Y} \to \mathsf{Y}'$ satisfying $J_1 = \phi \circ J$ is easy, once one notices that each such map $\psi : \mathsf{Y} \to \mathsf{Y}_1$ fulfils $J - J = (\phi - \psi) \circ J$ by linearity, hence $(\phi - \psi) \restriction_{J(\mathsf{X})} = 0$. The uniqueness of the extension of $(\phi - \psi) \restriction_{J(\mathsf{X})}$, continuous and with dense domain $J(\mathsf{X})$, to $\overline{J(\mathsf{X})} = \mathsf{Y}$, eventually warrants that $\phi = \psi$. $\square$

The next proposition is a useful criterion to check if a normed space is Banach.

Proposition 2.17 *Let* $(\mathsf{X}, ||\ ||)$ *be a normed space, and assume every absolutely convergent series* $\sum_{n=0}^{+\infty} x_n$ *of elements of* X *(i.e.* $\sum_{n=0}^{+\infty} ||x_n|| < +\infty$*) converges in* X*. Then* $(\mathsf{X}, ||\ ||)$ *is a Banach space.*

Proof Take a Cauchy sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ and let us show that if the above property holds, the sequence converges in X. Since the sequence is Cauchy, for any $k = 0, 1, 2, \ldots$ there is N_k such that $||v_n - v_m|| < 2^{-k}$ whenever $n, m \geq N_k$. Choose N_k so that $N_{k+1} > N_k$ and extract the subsequence $\{v_{N_k}\}_{k\in\mathbb{N}}$. Now define vectors $z_0 := v_{N_1}$, $z_k := v_{N_{k+1}} - v_{N_k}$ and consider the series $\sum_{k=0}^{+\infty} z_k$. Notice $v_{N_k} = \sum_{k'=0}^{k} z_{k'}$. By construction $||z_k|| < 2^{-k}$, so the series converges absolutely. Under the assumptions made, there will exist $v \in \mathsf{X}$ such that:

$$\lim_{k\to+\infty} v_{N_k} = \lim_{k\to+\infty} \sum_{k'=0}^{k} z_{k'} = v .$$

Hence the subsequence $\{v_{N_k}\}_{k\in\mathbb{N}}$ of the Cauchy sequence $\{v_n\}_{n\in\mathbb{N}}$ converges to $v \in \mathsf{X}$. To finish it suffices to show that the whole $\{v_n\}_{n\in\mathbb{N}}$ converges to v. As

$$||v_n - v|| \leq ||v_n - v_{N_k}|| + ||v_{N_k} - v|| ,$$

for a given $\varepsilon > 0$ we can find N_ε such that $||v_n - v_{N_k}|| < \varepsilon/2$ whenever $n, N_k > N_\varepsilon$, because $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. On the other hand we can also find M_ε such that $||v_{N_k} - v|| < \varepsilon/2$ whenever $k > M_\varepsilon$, since $v_{N_k} \to v$. Therefore taking $k > M_\varepsilon$ large enough, so that $N_k > N_\varepsilon$, we have $||v_n - v|| < \varepsilon$ for $n > N_\varepsilon$. As $\varepsilon > 0$ was arbitrary, we have $v_n \to v$ as $n \to +\infty$. $\square$

2.1.3 Example: The Banach Space $C(\mathsf{K}; \mathbb{K}^n)$, The Theorems of Dini and Arzelà–Ascoli

One of the simplest examples of a non-trivial (and generically, infinite-dimensional) Banach space is $C(\mathsf{K}; \mathbb{K}^n)$, the space of continuous maps from a compact space K to $\mathbb{K}^n$, with $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The chosen norm is the supremum norm $||f||_\infty := \sup_{x\in\mathsf{K}} ||f(x)||$. This is always finite for $f \in C(\mathsf{K}; \mathbb{K}^n)$ (Proposition 1.21).

Proposition 2.18 *Let $\mathbb{K} = \mathbb{C}$ (or $\mathbb{R}$) and consider the normed space $(\mathbb{K}^n, ||\ ||)$ with norm (1.1). If K is compact, the vector space $C(\mathsf{K}; \mathbb{K}^n)$ of continuous maps from K to $\mathbb{K}^n$, equipped with the norm:*

$$||f||_\infty := \sup_{x\in\mathsf{K}} ||f(x)|| ,$$

is a complex (or real) Banach space.

Proof Let $\{f_n\}_{n\in\mathbb{N}} \subset C(\mathsf{K}; \mathbb{K})$ be a Cauchy sequence. We want to show there is $f \in C(\mathsf{K}; \mathbb{K})$ such that $||f_n - f||_\infty \to 0$ as $n \to +\infty$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, for any given $x \in \mathsf{K}$, also n-tuples $f_n(x) \in \mathbb{K}^n$ are a Cauchy sequence. Thus, since $\mathbb{K}^n$ is complete, we have a pointwise-defined map:

$$f(x) := \lim_{n\to+\infty} f_n(x) .$$

The claim is that $f \in C(\mathsf{K}; \mathbb{K})$ and $||f_n - f||_\infty \to 0$ as $n \to +\infty$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, for any $\varepsilon > 0$ there is N_ε such that, if $n, m > N_\varepsilon$,

$$||f_n(x) - f_m(x)|| < \varepsilon , \quad \text{for } every\ x \in \mathsf{K}.$$

By definition of f, on the other hand, for a *given* $x \in \mathsf{K}$ and any $\varepsilon'_x > 0$, there is N_{x,ε'_x} such that $||f_m(x) - f(x)|| < \varepsilon'_x$ whenever $m > N_{x,\varepsilon'_x}$. Using these two facts we have

$$||f_n(x) - f(x)|| \le ||f_n(x) - f_m(x)|| + ||f_m(x) - f(x)|| < \varepsilon + \varepsilon'_x$$

provided $n > N_\varepsilon$ and choosing $m > \max(N_\varepsilon, N_{x,\varepsilon'_x})$. Overall, if $n > N_\varepsilon$, then

$$||f_n(x) - f(x)|| < \varepsilon + \varepsilon'_x , \quad \text{for any } \varepsilon'_x > 0.$$

Since $\varepsilon'_x > 0$ is arbitrary, the inequality holds when $\varepsilon'_x = 0$, possibly becoming an equality. Thus the dependency on x disappears, and in conclusion, for any $\varepsilon > 0$ we have found $N_\varepsilon \in \mathbb{N}$ such that

$$||f_n(x) - f(x)|| \leq \varepsilon\,, \quad \text{for } all\ x \in \mathsf{K} \tag{2.5}$$

when $n > N_\varepsilon$. Hence $\{f_n\}$ converges to f *uniformly*. Since (2.5) holds for any $x \in \mathsf{K}$, it holds for the supremum on K: for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that $\sup_{x\in\mathsf{K}} ||f_n(x) - f(x)|| < \varepsilon$, whenever $n > N_\varepsilon$. Put differently,

$$||f_n - f||_\infty \to 0\,, \quad \text{as } n \to +\infty.$$

To finish we must prove f is continuous. Given $x \in \mathsf{K}$, for any $\varepsilon > 0$ we will find $\delta > 0$ such that $||f(x') - f(x)|| < \varepsilon$ when $||x' - x|| < \delta$. For that, we exploit uniform convergence and choose n such that $||f(z) - f_n(z)|| < \varepsilon/3$ for the given ε and any $z \in \mathsf{K}$. Furthermore, as f_n is continuous, there is $\delta > 0$ such that $||f_n(x') - f_n(x)|| < \varepsilon/3$ whenever $||x' - x|| < \delta$. Putting everything together and using the triangle inequality allows to conclude the following: if $||x' - x|| < \delta$,

$$\begin{aligned} ||f(x') - f(x)|| &\leq ||f(x') - f_n(x')|| + ||f_n(x') - f_n(x)|| + ||f_n(x) - f(x)|| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

as claimed, so $f \in C(\mathsf{K}; \mathbb{K})$. □

Notation 2.19 From now on we will write $C(\mathsf{K}) := C(\mathsf{K}; \mathbb{C})$. ■

A useful analytical result about the uniform convergence of monotone sequences of real functions on compact sets is a classical result of Dini.

Theorem 2.20 (Dini's theorem on uniform convergence) *Let* K *be a compact space and take* $\{f_n\}_{n\in\mathbb{N}} \subset C(\mathsf{K}; \mathbb{R})$ *such that:*
(i) each f_n *is continuous,*
(ii) $f_n(x) \leq f_{n+1}(x)$ *for* $n = 1, 2, \ldots$ *and* $x \in \mathsf{K}$*,*
(iii) $f_n \to f$ *pointwise as* $n \to +\infty$*.*
Then, if f *is continuous,* $||f - f_n||_\infty \to 0$ *as* $n \to +\infty$*.*
The same is true if (ii) is replaced with: $f_n(x) \geq f_{n+1}(x)$*.*

Proof Fix $\varepsilon > 0$ and define $g_n := f - f_n$ for any $n \in \mathbb{N}$. Denote by B_n the set of $x \in \mathsf{K}$ for which $g_n(x) < \varepsilon$. As g_n is continuous, the set B_n is open, and $B_{n+1} \supset B_n$ since $g_{n+1}(x) \leq g_n(x)$, by construction. Since $g_n(x) \to 0$, then necessarily $\cup_{n\in\mathbb{N}} B_n = \mathsf{K}$. But K is compact, so we can choose sets $B_{n_1}, B_{n_2}, \ldots B_{n_N}$ so that $B_{n_{k+1}} \supset B_{n_k}$ and $B_{n_1} \cup B_{n_2} \cup \cdots \cup B_{n_N} \supset \mathsf{K}$. As $\mathsf{K} \supset B_{n_{k+1}} \supset B_{n_k}$, we have $B_{n_N} = \mathsf{K}$. Hence for the given $\varepsilon > 0$, there exists n_N such that $|f(x) - f_n(x)| < \varepsilon$ for $n > n_N$, $x \in \mathsf{K}$. Therefore $||f - f_n||_\infty < \varepsilon$, as claimed. The case $f_n(x) \geq f_{n+1}(x)$ is completely analogous. □

In the special case K is a compact set containing a dense and countable subset, the Banach space $C(\mathsf{K})$ has an interesting property by the *theorem of Arzelà–Ascoli*. We state below the simplest version of this result: even if it is not strictly related to the contents of this book, its importance (especially in the more general form) and the stereotypical strategy of proof make it worthy of attention.

Definition 2.21 A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n : \mathsf{X} \to \mathbb{C}$ on a normed space[1] $(\mathsf{X}, ||\ ||)$ is **equicontinuous** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_n(x) - f_n(x')| < \varepsilon$ whenever $||x - x'|| < \delta$ for every $n \in \mathbb{N}$ and every $x, x' \in \mathsf{X}$.

Theorem 2.22 (Arzelà–Ascoli) *Let K be a compact and separable (cf. Definition 1.5) space. Suppose a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C(\mathsf{K})$ is:*

(a) equicontinuous

and

(b) bounded by some $C \in \mathbb{R}$, i.e. $||f_n||_\infty < C$ for any $n \in \mathbb{N}$.

Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converging to some map $f \in C(\mathsf{K})$ in the topology induced by the norm $||\ ||_\infty$.

Proof Consider the points q of a dense and countable set $Q \subset \mathsf{K}$ and label them by $\mathbb{N}$. If q_1 denotes the first point, consider the values $|f_n(q_1)|$ as n varies. They lie in a compact set $[0, C]$, so either there are finitely many, and $f_n(q_1) = x_1 \in \mathbb{C}$ for a single x_1 and infinitely many n, or the $f_n(q_1)$ accumulate at $x_1 \in \mathbb{C}$. In either case there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k}(q_1) \to x_1 \in \mathbb{C}$ for some $x_1 \in \mathbb{C}$. Call the elements of $\{f_{n_k}\}_{k\in\mathbb{N}}$ by f_{1n}, where $n \in \mathbb{N}$. Now repeat the procedure and consider $|f_{1n}(q_2)|$, where q_2 is the second point of Q, and extract a subsequence $\{f_{2n}\}_{n\in\mathbb{N}}$ from $\{f_{1n}\}_{n\in\mathbb{N}}$. By construction, $f_{2n}(q_1) \to x_1$ and $f_{2n}(q_2) \to x_2 \in \mathbb{C}$, as $n \to +\infty$. Continuing in this way for every $k \in \mathbb{N}$ we end up building a subsequence $\{f_{kn}\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ that converges to $x_1, x_2, \ldots, x_k \in \mathbb{C}$ when evaluated at the points $q_1, q_2, \ldots, q_k \in Q$. Take the subsequence of $\{f_n\}_{n\in\mathbb{N}}$ formed by all diagonal terms in the various subsequences, $\{f_{nn}\}_{n\in\mathbb{N}}$. We claim this is a Cauchy sequence for $||\ ||_\infty$. So let us fix $\varepsilon > 0$ and find the $\delta > 0$ corresponding to $\varepsilon/3$ by equicontinuity, then cover K with balls of radius δ centred at every point of K. Using the compactness of K we extract a finite covering of balls with radius δ, say $B^{(1)}_\delta, B^{(2)}_\delta, \ldots, B^{(N)}_\delta$, and choose $q^{(j)} \in B^{(j)}_\delta \cap Q$, for any $j = 1, \ldots, N$. For any $x \in B^{(j)}_\delta$ we have:

$$|f_{nn}(s) - f_{mm}(s)|$$

$$\leq |f_{nn}(s) - f_{nn}(q^{(j)})| + |f_{nn}(q^{(j)}) - f_{mm}(q^{(j)})| + |f_{mm}(q^{(j)}) - f_{mm}(s)|\,.$$

The first and third terms are smaller than $\varepsilon/3$ by construction. Since $f_{nn}(q^{(j)})$ converges in $\mathbb{C}$ as $n \to +\infty$, the second term is less than $\varepsilon/3$ provided $n, m > M^{(j)}_\varepsilon$ for some $M^{(j)}_\varepsilon \geq 0$. Hence if $M_\varepsilon = \max_{j=1,\ldots,N} M^{(j)}_\varepsilon$:

$$|f_{nn}(s) - f_{mm}(s)| < \varepsilon \quad \text{for } n, m > M_\varepsilon, \text{ and any } s \in \mathsf{K}.$$

[1]The definition generalises to metric spaces in the obvious way.

In other words

$$||f_{nn} - f_{mm}||_\infty < \varepsilon \quad \text{if } n, m > M_\varepsilon$$

as claimed. □

Remarks 2.23 **(1)** The theorem applies in particular when K is the closure of a non-empty open and bounded set $A \subset \mathbb{R}^n$, because points with rational coordinates form a countable dense subset in K. Moreover, the same proof holds (trivial changes apart) if we replace $C(\mathsf{K})$ with $C(\mathsf{K}; \mathbb{K}^n)$.
(2) We will prove in Chap. 4, Proposition 4.3, that in a normed space $(\mathsf{X}, || \ ||)$ a subset $A \subset \mathsf{X}$ is relatively compact (its closure is compact) if we can extract a convergent subsequence from any sequence of A. By virtue of this fact, the theorem of Arzelà–Ascoli actually says the following:
if K is a compact separable space, every equicontinuous subset of $C(\mathsf{K})$ that is bounded for $|| \ ||_\infty$ is relatively compact in $(C(\mathsf{K}), || \ ||_\infty)$.
(3) An important result in functional analysis [Mrr01], which we will not prove, is the *Banach–Mazur theorem*: any complex separable Banach space is isometrically isomorphic to a closed subspace of $(C([0, 1]), || \ ||_\infty)$. ■

Several examples of Banach spaces will be given at the end of the next section, after we have talked about normed and Banach algebras.

2.1.4 Normed Algebras, Banach Algebras and Examples

As we shall see in a moment, in many applications there is a tight connection between *algebras* and normed spaces, which goes through linear operators on a normed space. The most important normed algebras in physics are, as a matter of fact, operator algebras.

But the notions of algebra and normed algebra are completely independent of operators. An algebra arises by enriching a vector space with a product that is associative, distributes over the sum and behaves associatively for the multiplication by scalars. A normed algebra is an algebra equipped with a norm that renders the vector space normed and behaves "properly" with respect to the product. Here are the main definitions.

Definition 2.24 (*Algebra*) An **(associative) algebra** $\mathfrak{A}$ over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ is a $\mathbb{K}$-vector space with an operation $\circ : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, called **product**, that is **associative**:

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \text{for each triple } a, b, c \in \mathfrak{A} \tag{2.6}$$

and distributes over the vector space operations:
A1. $a \circ (b + c) = a \circ b + a \circ c \quad \forall\, a, b, c \in \mathfrak{A}$,
A2. $(b + c) \circ a = b \circ a + c \circ a \quad \forall\, a, b, c \in \mathfrak{A}$,

A3. $\alpha(a \circ b) = (\alpha a) \circ b = a \circ (\alpha b) \quad \forall \alpha \in \mathbb{K}$ and $\forall\, a, b \in \mathfrak{A}$.
The algebra $(\mathfrak{A}, \circ)$ is called:
commutative, or **Abelian**, if
A4. $a \circ b = b \circ a$ for any pair $a, b \in \mathfrak{A}$;
unital (or **with unit**) if it contains an element $\mathbb{I}$, called **unit** of the algebra, such that:
A5. $\mathbb{I} \circ u = u \circ \mathbb{I} = u$ for any $a \in \mathfrak{A}$;
a **normed algebra** or **normed unital algebra** if it is a normed vector space with norm $|| \ \ ||$ satisfying
A6. $||a \circ b|| \leq ||a|| ||b||$ for $a, b \in \mathfrak{A}$,
and in presence of a unit $\mathbb{I}$ also:
A7. $||\mathbb{I}|| = 1$;
a **Banach algebra** or **Banach algebra with unit** if $\mathfrak{A}$ is a Banach space plus a normed algebra, or normed unital algebra, for the same norm.

A **homomorphism** $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ between algebras, whether unital, normed or Banach, is a linear map preserving products, and units if present (but not necessarily the norms if present):

$$\phi(a \circ_1 b) = \phi(a) \circ_2 \phi(b) \quad \text{if } a, b \in \mathfrak{A}_1, \qquad \phi(\mathbb{I}_1) = \mathbb{I}_2 \, ,$$

using the obvious notations. A bijective algebra homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}'$ (between normed or Banach algebras, with or without unit) is an **algebra isomorphism**, and an **algebra automorphism** if $\mathfrak{A} = \mathfrak{A}'$.

If there is an isomorphism $\phi : \mathfrak{A} \to \mathfrak{A}'$, the algebras $\mathfrak{A}, \mathfrak{A}'$ (normed/Banach/with unit) are said to be **isomorphic**, and **isometrically isomorphic** if the isomorphism also preserves the norms.

A (**normed/Banach/unital**) **subalgebra** is a subset $\mathfrak{A}_1 \subset \mathfrak{A}$ in a (normed/Banach/unital) algebra $\mathfrak{A}$ that inherits the algebra structure by *restricting* the algebra operations (if present: the same unit of $\mathfrak{A}$, the restricted norm of $\mathfrak{A}$, and completeness if $\mathfrak{A}$ is Banach).

Remarks 2.25 **(1)** The norm does *not* show up in the definition of homomorphism and isomorphism between the various kinds of algebra.
(2) If a unit exists it must be unique: if both $\mathbb{I}$ and $\mathbb{I}'$ satisfy **A5**, then $\mathbb{I}' = \mathbb{I}' \circ \mathbb{I} = \mathbb{I}$. ■

For normed algebras, the above axioms easily imply that all operations are *continuous* in the norm topologies involved. We showed in Sect. 2.1.1 that this is true for the sum and the multiplication by scalars. The product $\circ$, too, is jointly continuous in its arguments (i.e. in the product topology of $\mathfrak{A} \times \mathfrak{A}$) and hence also continuous in each argument alone.

Proposition 2.26 *If $\mathfrak{A}$ is a normed algebra (with product $\circ$), the map*

$$\mathfrak{A} \times \mathfrak{A} \ni (a, b) \mapsto a \circ b \in \mathfrak{A}$$

is continuous using the product topology in the domain.

Proof Dropping the symbol $\circ$, we have

$$||ab - a_0 b_0|| = ||ab - a_0 b + a_0 b - a_0 b_0|| \leq ||a - a_0||\,||b|| + ||a_0||\,||b - b_0||\,.$$

Fixing $\varepsilon > 0$, since the norm is continuous in its own-induced topology, there is $\delta_0 > 0$ such that $||b - b_0|| < \delta_0$ implies $-1 < ||b|| - ||b_0|| < 1$. Therefore:

$$||ab - a_0 b_0|| = ||ab - a_0 b + a_0 b - a_0 b_0|| \leq ||b - b_0||(1 + ||b_0|| + ||a_0||)\,.$$

Choosing now $\delta \leq \min(\delta_0, \varepsilon/(1 + ||b_0|| + ||a_0||))$ and considering elements $(a, b) \in B_\delta(a_0) \times B_\delta(b_0)$:

$$||ab - a_0 b_0|| < \varepsilon\,.$$

This proves the continuity of $\circ$ in the product topology of $\mathfrak{A} \times \mathfrak{A}$. $\square$

We now pass to show that, in a unital Banach algebra, the map $a \mapsto a^{-1}$ is continuous if its domain is suitably defined.

Proposition 2.27 *Let $\mathfrak{A}$ be a unital Banach algebra (with unit $\mathbb{I}$ and product $\circ$). The group*

$$G\mathfrak{A} := \{a \in \mathfrak{A} \mid \exists a^{-1} \in \mathfrak{A} \quad \textit{such that} \quad a \circ a^{-1} = a^{-1} \circ a = \mathbb{I}\}$$

is open in $\mathfrak{A}$ and the map $G\mathfrak{A} \ni a \mapsto a^{-1} \in \mathfrak{A}$ is continuous.

Proof First we shall show that if $\mathfrak{A}$ is a Banach algebra, then $G\mathfrak{A}$ is open so that it makes sense to invert elements in a neighbourhood of any $a_0 \in G\mathfrak{A}$. Next we will prove that the map $G\mathfrak{A} \ni a \mapsto a^{-1}$ is continuous. With $a \in \mathfrak{A}$, the series

$$\sum_{n=0}^{+\infty} (-1)^n a^n$$

converges in the norm topology when $||a|| < 1$, because its partial sums are Cauchy sequences and the space is complete by hypothesis. The proof now is the same as for the convergence of the geometric series. Moreover, since the product is continuous:

$$(\mathbb{I} + a) \sum_{n=0}^{+\infty} (-1)^n a^n = \sum_{n=0}^{+\infty} (-1)^n (\mathbb{I} + a) a^n = \mathbb{I} + \lim_{n \to +\infty} (-1)^{n+1} a^n = \mathbb{I}\,.$$

Similarly:

$$\left(\sum_{n=0}^{+\infty}(-1)^n a^n\right)(\mathbb{I}+a)=\mathbb{I}\,.$$

Hence if $||a|| < 1$ we have $\mathbb{I}+a \in G\mathfrak{A}$ and:

$$(\mathbb{I}+a)^{-1}=\sum_{n=0}^{+\infty}(-1)^n a^n\,.$$

At this point, if $b \in G\mathfrak{A}$ we can write $c = b + c - b = b(\mathbb{I} + b^{-1}(c - b))$. Therefore $||b^{-1}(c - b)|| < 1$ implies c has an inverse:

$$c^{-1}=\sum_{n=0}^{+\infty}(-1)^n((c-b)b^{-1})^n b^{-1}\,.$$

In particular, if $b \in G\mathfrak{A}$ and we fix $0 < \delta < 1/||b^{-1}||$, then $c \in B_\delta(b)$ gives $c \in G\mathfrak{A}$, because $||b^{-1}(c - b)|| \leq ||b^{-1}||\,||(c - b)|| < 1$. Thus we have proved $G\mathfrak{A}$ open.

Now to the continuity of $G\mathfrak{A} \ni a \mapsto a^{-1}$. Fix $a_0 \in G\mathfrak{A}$ and δ with $0 < \delta < ||a_0^{-1}||^{-1}$, and note that $||a - a_0|| < \delta$ forces

$$||a^{-1}-a_0^{-1}|| \leq ||a^{-1}(a_0-a)a_0^{-1}|| \leq ||a^{-1}||\,||a-a_0||\,||a_0^{-1}||$$

$$\leq (||a^{-1}-a_0^{-1}||+||a_0^{-1}||)\delta||a_0^{-1}||\,.$$

Therefore (the first factor is positive by construction)

$$(1-\delta||a_0^{-1}||)\,||a^{-1}-a_0^{-1}|| \leq \delta||a_0^{-1}||^2\,.$$

We conclude that if $||a - a_0|| < \delta$,

$$||a^{-1}-a_0^{-1}|| \leq \frac{\delta}{1-\delta||a_0^{-1}||}||a_0^{-1}||^2\,.$$

Defining $\varepsilon := \frac{\delta}{1-\delta||a_0^{-1}||}||a_0^{-1}||^2$ we have $\delta = \frac{\varepsilon}{\varepsilon||a_0^{-1}||+||a_0^{-1}||^2}$. The conclusion, as claimed, is that for any $\varepsilon > 0$ (satisfying the starting constraint) $||a^{-1} - a_0^{-1}|| < \varepsilon$ with $a \in B_\delta(a_0)$ and $\delta > 0$ above, so $a \mapsto a^{-1}$ is continuous. □

Notation 2.28 In the sequel we will conventionally denote the product of two elements of an algebra by juxtaposition, as in ab, rather than by $a \circ b$. In other contexts a dot might be used: $f \cdot g$, especially when working with functions. ■

Examples 2.29 Let us see examples of Banach spaces and Banach algebras, a few of which will require some abstract measure theory.

(1) The number fields $\mathbb{C}$ and $\mathbb{R}$ are commutative Banach algebras with unit. For both the norm is the modulus/absolute value.

(2) Given any set X, and $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, let $L(\mathsf{X})$ be the set of **bounded** maps $f : \mathsf{X} \to \mathbb{K}$, i.e. $\sup_{x\in\mathsf{X}} |f(x)| < \infty$. Then $L(\mathsf{X})$ is naturally a $\mathbb{K}$-vector space for the usual linear combinations: if $\alpha, \beta \in \mathbb{C}$ and $f, g \in L(\mathsf{X})$,

$$(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x) \quad \text{for all } x \in \mathsf{X}.$$

We can define a product making $L(\mathsf{X})$ an algebra: for $f, g \in L(\mathsf{X})$,

$$(f \cdot g)(x) := f(x)g(x) \quad \text{for any } x \in \mathsf{X}.$$

The algebra is commutative and has a unit (the constant map 1). A norm that renders $L(\mathsf{X})$ Banach is the sup norm: $||f||_\infty := \sup_{x\in\mathsf{X}} |f(x)|$. The proof is simple (it uses the completeness of $\mathbb{C}$, and goes pointwise on X) and can be found in the exercises at the end of the chapter.

(3) Define on the above X a σ-algebra Σ. The subalgebra of Σ-measurable functions $M_b(\mathsf{X}) \subset L(\mathsf{X})$ is closed in $L(\mathsf{X})$ in the topology of the sup norm. Thus $M_b(\mathsf{X})$ is a commutative Banach algebra. This is immediate from the previous example, because the pointwise limit of measurable maps is measurable.

(4) The vector space of continuous maps from a topological space X to $\mathbb{C}$ is written $C(\mathsf{X})$; the symbol already appeared for X compact in Sect. 2.1.3.

We indicate by $C_b(\mathsf{X}) \subset C(\mathsf{X})$ the subspace of bounded continuous maps and by $C_c(\mathsf{X}) \subset C_b(\mathsf{X})$ the space of continuous maps with compact support.

These all coincide if X compact, and are clearly commutative algebras for the operations of example (2). The algebras $C(\mathsf{X})$ and $C_b(\mathsf{X})$ have the constant map 1 as unit, whereas $C_c(\mathsf{X})$ has no unit when X is not compact. Here is a list of general properties:

(a) $C_b(\mathsf{X})$ is a Banach algebra for the sup norm $||\ ||_\infty$.

(b) If $\mathsf{X} = \mathsf{K}$ is compact, $C_c(\mathsf{K}) = C(\mathsf{K})$ is a Banach algebra with unit for the sup norm $||\ ||_\infty$, as we saw in Sect. 2.1.3. An important result in the theory of Banach algebras [Rud91] states that *any commutative Banach algebra with unit over $\mathbb{C}$ is isomorphic to an algebra $C(\mathsf{K})$ for some compact space K.*

(c) If the space X is

1. *Hausdorff* and
2. *locally compact*,

then the completion of the normed space $C_c(\mathsf{X})$ is a commutative Banach algebra $C_0(\mathsf{X})$ (without unit), called algebra of continuous maps $f : \mathsf{X} \to \mathbb{C}$ that **vanish at infinity** [Rud86]: this means that for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathsf{X}$ (depending on f in general) such that $|f(x)| < \varepsilon$ for any $x \in \mathsf{X} \setminus K_\varepsilon$.

(d) Irrespective of compactness, $C(\mathsf{X})$, $C_c(\mathsf{X})$, $C_0(\mathsf{X})$ are in general *not* dense in $M_b(\mathsf{X})$ (using $|| \; ||_\infty$ and the Borel σ-algebra on X). If we take $\mathsf{X} = [0, 1]$ for instance, the bounded map $f : [0, 1] \to \mathbb{C}$ equal to 0 except for $f(1/2) = 2$ is Borel measurable (with the standard topologies on $\mathbb{C}$ and the induced one on $[0, 1]$), hence $f \in M_b([0, 1])$. However, there cannot exist any sequence of continuous maps $f_n : [0, 1] \to \mathbb{C}$ converging to f uniformly. The same can be said if $\mathsf{X} \subset \mathbb{R}^n$ is a compact set with non-empty interior and we take $f : \mathsf{X} \to \mathbb{C}$ to be $f(q) = 0$ for $q \in \mathsf{X} \setminus \{p\}$, $p \in Int(\mathsf{X})$, and $f(p) = 1$.

(5) If X is Hausdorff and compact, consider a subalgebra A in $C(\mathsf{X})$ that contains the unit (the function 1) and is closed under complex conjugation: $f \in A \Rightarrow f^* \in A$, where $f^*(x) := \overline{f(x)}$ for any $x \in \mathsf{X}$ and the bar denotes complex conjugation. Then A is said to **separate points** in X if, given any $x, y \in \mathsf{X}$ with $x \neq y$, there is a map $f \in A$ satisfying $f(x) \neq f(y)$. The *Stone-Weierstrass theorem* [Rud91] says the following.

Theorem 2.30 (Stone-Weierstrass) *Let X be a compact Hausdorff space and consider the Banach algebra with unit $(C(\mathsf{X}), || \; ||_\infty)$. Then any subalgebra $\mathfrak{A} \subset C(\mathsf{X})$ containing the unit, closed under complex conjugation and that separates points has $C(\mathsf{X})$ as closure with respect to $|| \; ||_\infty$.*

A typical example is the algebra $\mathfrak{A}$ of complex polynomials in n variables (the standard coordinates of $\mathbb{R}^n$) restricted to a compact subset X in $\mathbb{R}^n$. The theorem asserts that these polynomials approximate uniformly any continuous complex function on X. This is useful to construct bases in Hilbert spaces, as we shall explain later.

(6) Let $(\mathsf{X}, \Sigma, \mu)$ be a positive, σ-additive *measure space*. Recall this means a set X, a σ-algebra Σ of subsets in X and a positive and σ-additive measure $\mu : \Sigma \to [0, +\infty]$.

Then we have **Hölder's inequality** and **Minkowski's inequality**, respectively:

$$\int_{\mathsf{X}} |f(x)g(x)| d\mu(x) \leq \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x)\right)^{1/p} \left(\int_{\mathsf{X}} |g(x)|^q d\mu(x)\right)^{1/q} \tag{2.7}$$

$$\left(\int_{\mathsf{X}} |f(x) + g(x)|^p d\mu(x)\right)^{1/p} \leq \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x)\right)^{1/p} + \left(\int_{\mathsf{X}} |g(x)|^p d\mu(x)\right)^{1/p} \tag{2.8}$$

for any $f, g : \mathsf{X} \to \mathbb{C}$ measurable, $p, q > 0$ subject to $1/p + 1/q = 1$ in the former, $p \geq 1$ in the latter [Rud86]. These inequalities are proved in two exercises at the end of the chapter.

Let $\mathscr{L}^p(\mathsf{X}, \Sigma, \mu)$, or henceforth $\mathscr{L}^p(\mathsf{X}, \mu)$ by dropping the σ-algebra, be the set of Σ-measurable maps $f : \mathsf{X} \to \mathbb{C}$ such that $\int_{\mathsf{X}} |f(x)|^p d\mu(x) < \infty$. Using Minkowski's inequality one sees easily $\mathscr{L}^p(\mathsf{X}, \mu)$ is a vector space under linear composition of functions, and

$$P_p(f) := \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x) \right)^{1/p} \tag{2.9}$$

is a *seminorm*. Since $P_p(f) = 0$ if and only if $f = 0$ a.e. for μ, in order to obtain a norm (i.e., to have **N3**) we must identify the zero map and any function that differs from it by a zero-measure set. To this end we define an equivalence relation on $\mathscr{L}^p(\mathsf{X}, \mu)$: $f \sim g \;\Leftrightarrow\; f - g$ is zero a.e. for μ. The quotient space $\mathscr{L}^p(\mathsf{X}, \mu)/\sim$, written $L^p(\mathsf{X}, \mu)$, inherits a vector-space structure over $\mathbb{C}$ from $\mathscr{L}^p(\mathsf{X}, \mu)$ by setting:

$$[f] + [g] := [f + g] \quad \text{and} \quad \alpha[f] := [\alpha f] \quad \text{for any } \alpha \in \mathbb{C},\ f, g \in \mathscr{L}^p(\mathsf{X}, \mu).$$

It is not hard to show both left-hand sides are independent of the representatives chosen in the equivalence classes on the right.

It can also be proved that $L^p(\mathsf{X}, \mu)$ is a Banach space for the norm:

$$||[f]||_p := \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x) \right)^{1/p} , \tag{2.10}$$

where f is any representative of $[f] \in L^p(\mathsf{X}, \mu)$. We shall slightly abuse the notation in the sequel, and write $||f||_p$ instead of $P_p(f)$ when dealing with functions and not equivalence classes.

If $(\mathsf{X}, \Sigma', \mu')$ is the completion of $(\mathsf{X}, \Sigma, \mu)$ (cf. Remark 1.47(1)), in general $\mathscr{L}^p(\mathsf{X}, \mu')$ is larger than $\mathscr{L}^p(\mathsf{X}, \mu)$. But if we pass to the quotient then $L^p(\mathsf{X}, \mu') = L^p(\mathsf{X}, \mu)$ by way of Proposition 1.57.

Theorem 2.31 (Fischer–Riesz) *If $(\mathsf{X}, \Sigma, \mu)$ is a positive, σ-additive measure space, the associated normed space $L^p(\mathsf{X}, \mu)$ is, for any $1 \le p < +\infty$, a Banach space.*

Proof Throughout this proof we shall omit the square brackets for the elements of $L^p(\mathsf{X}, \mu)$, and identify cosets with functions (up to null sets). To prove the claim, thanks to Proposition 2.17 it is sufficient to verify that if the series $\sum_{n=0}^{+\infty} f_n$ in $\mathscr{L}^p(\mathsf{X}, \mu)$ converges absolutely, $\sum_{n=0}^{+\infty} ||f_n||_p \le K < +\infty$, then $\sum_{n=0}^{+\infty} f_n = f$ a.e. for some $f \in \mathscr{L}^p(\mathsf{X}, \mu)$ in the topology of $||\ \ ||_p$. We will need the auxiliary sequence $g_N(x) := \sum_{n=1}^{N} |f_n(x)|$, $N = 1, 2, \ldots$. By construction $||g_N||_p \le \sum_{n=1}^{N} ||f_n||_p \le K$ for any $N = 1, 2, \ldots$. We claim the limit $\lim_{N\to+\infty} g_N(x)$ is finite for almost all $x \in \mathsf{X}$. The sequence of integrable functions g_N^p is non-negative and non-decreasing by construction, and $\int_{\mathsf{X}} g_N(x)^p d\mu(x) < K^p$ for any N. By *monotone convergence* the limit g^p of g_N^p exists, as a map in $[0, +\infty]$, because the sequence of the given $g_N^p \ge 0$ is non-decreasing, and must have finite integral. Thus $g^p \ge 0$ is finite up to possible zero-measure sets. As $p \ge 0$, at points $x \in \mathsf{X}$ where $g(x)^p < +\infty$ we have $\lim_{N\to+\infty} g_N(x) = g(x) < +\infty$. By construction, where $g(x)$ is finite the series $\sum_{n=0}^{+\infty} f_n(x)$ converges absolutely. Therefore it converges to certain values $f(x) \in \mathbb{C}$. Defining $f(x) = 0$ where the series of f_n does not converge, we obtain a series $\sum_{n=0}^{+\infty} f_n$ that converges a.e. to a map $f : \mathsf{X} \to \mathbb{C}$ (measurable since limit, a.e., of measurable functions, and, say, null on the zero-measure set where the series does not converge). The map f belongs to $\mathscr{L}^p(\mathsf{X}, \mu)$: if $f_N(x) := \sum_n^N f_n(x)$, the sequence

$|f_N|^p$ is non-negative and $\int_X |f_N(x)|^p d\mu(x) < K^p$ for any N. By *Fatou's lemma* $f \in \mathscr{L}^p(X, \mu)$. Now we prove $\int_X |f_N(x) - f(x)|^p d\mu(x) \to 0$ as $n \to +\infty$. Easily (see the footnote in Exercise 2.14) $|f_N(x) - f(x)|^p \leq 2^p(|f_N(x)|^p + |f(x)|^p)$. Since, by construction, $|f_N|^p + |f|^p \leq |g|^p + |f|^p \in \mathscr{L}^1(X, \mu)$, we can invoke the *dominated convergence theorem* for the sequence $|f_N - f|^p$, known to converge a.e. to 0, and obtain $\int_X |f_N(x) - f(x)|^p d\mu(x) \to 0$ as $n \to +\infty$. We have thus proved that the initial series $\sum_{n=0}^{+\infty} f_n$, assumed absolutely convergent in $\mathscr{L}^p(X, \mu)$, satisfies $\sum_{n=0}^{+\infty} f_n = f$ a.e. for the above $f \in \mathscr{L}^p(X, \mu)$ in norm $|| \ ||_p$. This ends the proof. □

This argument implies a technical fact, extremely useful in the applications, that deserves separate mentioning.

Proposition 2.32 *Take $1 \leq p < +\infty$ and let (X, Σ, μ) be a σ-additive, positive measure space. If $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{L}^p(X, \mu)$ converges to f in $|| \ ||_p$ as $n \to +\infty$, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ a.e. for μ.*

Proof The sequence $\{f_n\}_{n\in\mathbb{N}}$ is convergent hence Cauchy, and we can extract a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $||f_{n_{k+1}} - f_{n_k}|| \leq 2^{-k}$. Define the telescopic sequence $s_k := f_{n_{k+1}} - f_{n_k}$. The series $f_{n_0} + \sum_{k=1}^{+\infty} s_k$ is absolutely convergent, for $\sum_{k=1}^{+\infty} ||s_k||_p < \sum_{k=1}^{+\infty} 2^{-k} < +\infty$. As in the proof of Theorem 2.31, we conclude: (a) the sum $s \in \mathscr{L}^p(X, \mu)$ of the series exists, in the sense of $|| \ ||_p$ convergence; (b) the series converges pointwise to s almost everywhere, that is $f_{n_0}(x) + \sum_{k\in\mathbb{N}} s_k(x) = s(x)$. Since $f_{n_0}(x) + \sum_{k=0}^{+\infty} s_k(x) = f_{n_k}(x)$, what we have found is that $f_{n_k} \to s \in \mathscr{L}^p(X, \mu)$ both pointwise μ-almost everywhere, and with respect to $|| \ ||_p$ as well. But by assumption $f_{n_k} \to f \in \mathscr{L}^p(X, \mu)$ with respect to $|| \ ||_p$, so $||f - s||_p = 0$ and hence $f(x) = s(x)$ a.e. for μ. Eventually, then, $f_{n_k}(x) \to f(x)$ a.e. for μ. □

To wrap up the example notice that the Banach space $L^p(X, \mu)$ is not, in general, an algebra (for the usual pointwise product of functions), because the pointwise product in $\mathscr{L}^p(X, \mu)$ does not normally belong to the space.

(7) With reference to example (6), consider the special case where X is not countable, Σ is the power set of X and μ the **counting measure**:

$$\mu(S) = \text{number of elements of } S \subset X, \text{ with } \mu(S) = \infty \text{ if } S \text{ is infinite.}$$

Given a measurable space Y, any map $f : X \to Y$ is measurable, and $L^p(X, \mu)$ is simply denoted by $\ell^p(X)$. Its elements are "sequences" $\{z_x\}_{x\in X}$ of complex numbers, labelled by X, such that:

$$\sum_{x\in X} |z_x|^p < \infty ,$$

where the sum is given by:

$$\sup \left\{ \sum_{x\in X_0} |z_x|^p \ \middle| \ X_0 \subset X, \quad X_0 \text{ finite} \right\} .$$

If X is countable, $\mathsf{X} = \mathbb{N}$ or $\mathbb{Z}$ in particular, the above definition of sum of *positive* numbers indexed by X is the usual sum of a series. For example, $\ell^p(\mathbb{N})$ is the space of sequences $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ with:

$$\sum_{n=0}^{+\infty} |c_n|^p < +\infty \, .$$

(8) Take a measure space $(\mathsf{X}, \Sigma, \mu)$ and consider the class $\mathscr{L}^\infty(\mathsf{X}, \mu)$ of complex measurable maps $f : \mathsf{X} \to \mathbb{C}$ such that $|f(x)| < M_f$ a.e. for μ, for some $M_f \in \mathbb{R}$ (depending on f). Then $\mathscr{L}^\infty(\mathsf{X}, \mu)$ has a natural structure of a vector space and of a commutative algebra with unit (the function 1) if we use the ordinary product and linear combinations as in example (2). We can equip $\mathscr{L}^\infty(\mathsf{X}, \mu)$ with the seminorm:

$$P_\infty(f) := \text{ess sup}|f|$$

defined by the **essential supremum of** $f \in \mathscr{L}^\infty(\mathsf{X}, \mu)$:

$$\text{ess sup}|f| := \inf \{r \in \mathbb{R} \;\mid\; \mu\left(\{x \in \mathsf{X} \mid |f(x)| \geq r\}\right) = 0\} \, . \tag{2.11}$$

Naïvely speaking, the latter is the "smallest" upper bound of $|f|$ when we ignore what happens on zero-measure sets.

In particular (exercise):

$$P_\infty(f \cdot g) \leq P_\infty(f) P_\infty(g) \quad \text{if } f, g \in \mathscr{L}^\infty(\mathsf{X}, \mu) \, .$$

As we did for $\mathscr{L}^p$, if we identify maps that differ only on zero-measure sets, we can form the quotient space $L^\infty(\mathsf{X}, \mu)$, with well-defined product:

$$[f] \cdot [g] := [f \cdot g] \quad \text{for } f, g \in \mathscr{L}^\infty(\mathsf{X}, \mu) \, .$$

Exactly as for the L^p spaces, the seminorm P_∞ is (clearly) a norm on $L^\infty(\mathsf{X}, \mu)$:

$$||[f]||_\infty := \text{ess sup}|f| \, .$$

In analogy to the spaces L^p, also $L^\infty(\mathsf{X}, \mu)$ is a Banach space. Moreover, being closed under products means it is a Banach algebra as well.

Theorem 2.33 (Fischer–Riesz, L^∞ case) *If* $(\mathsf{X}, \Sigma, \mu)$ *is a σ-additive, positive measure space, the associated normed space* $L^\infty(\mathsf{X}, \mu)$ *is a Banach space.*

Proof As customary, we indicate with f (no brackets) the generic element of $L^\infty(\mathsf{X}, \mu)$, and identify it with a function (up to null sets) when necessary. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(\mathsf{X}, \mu)$ be a Cauchy sequence for $|| \; ||_\infty$. Define, for $k, m, n \in \mathbb{N}$, sets $A_k := \{x \in \mathsf{X} \mid |f_k(x)| > ||f_k||_\infty\}$ and $B_{n,m} := \{x \in \mathsf{X} \mid |f_n(x) - f_m(x)| > ||f_n - f_m||_\infty\}$. By construction $E := \cup_{k\in\mathbb{N}} \cup_{n,m\in\mathbb{N}} A_k \cup B_{n,m}$ must have zero measure, and the sequence of the f_n converges uniformly in $\mathsf{X} \setminus E$ to some f, which is therefore bounded. Extend f to the entire X by setting it to zero on $\mathsf{X} \setminus E$. Thus $f \in L^\infty(\mathsf{X}, \mu)$ and $||f_n - f||_\infty \to 0$ as $n \to +\infty$. $\square$

(9) Going back to example (8), in case Σ is the power set of X and μ the counting measure, the space $L^\infty(\mathsf{X}, \mu)$ is written simply $\ell^\infty(\mathsf{X})$. Its points are "sequences" $\{z_x\}_{x \in \mathsf{X}}$ of complex numbers indexed by X such that $\sup_{x \in \mathsf{X}} |z_x| < +\infty$.

With the notation of example (2), $\ell^\infty(\mathsf{X}) = L(\mathsf{X})$. ■

Notation 2.34 The literature prefers to use the bare letter f to indicate the equivalence class $[f] \in L^p(\mathsf{X}, \mu)$, $1 \leq p \leq \infty$. We shall stick to this convention when no confusion arises. ■

2.2 Operators, Spaces of Operators, Operator Norms

With the next definition we introduce *linear operators* and *linear functionals*, whose importance is paramount in the whole book. We shall assume from now on familiarity with linear operators (matrices) on finite-dimensional vector spaces, and we shall freely use results from that theory without explicit mention.

Definition 2.35 (*Operator and functional*) Let X, Y be vector spaces over the same field $\mathbb{K} := \mathbb{R}, \mathbb{C}$.
(a) $T : \mathsf{X} \to \mathsf{Y}$ is a **linear operator** (simply, an **operator**) **from** X to Y if it is linear:

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \quad \text{for any } \alpha, \beta \in \mathbb{K},\ f, g \in \mathsf{X}.$$

The set of linear operators from X to Y is indicated by $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$.

If $\mathsf{S} \subset \mathsf{X}$ is a subspace such that $T(\mathsf{S}) \subset \mathsf{S}$ for some linear operator $T : \mathsf{X} \to \mathsf{X}$, we say that S is an **invariant** subspace for/under T (or shorter, T-invariant).

If X and Y are normed, $\mathfrak{B}(\mathsf{X}, \mathsf{Y}) \subset \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ is the subset of continuous operators. In particular $\mathfrak{L}(\mathsf{X}) := \mathfrak{L}(\mathsf{X}, \mathsf{X})$ and $\mathfrak{B}(\mathsf{X}) := \mathfrak{B}(\mathsf{X}, \mathsf{X})$.
(b) $T : \mathsf{X} \to \mathbb{K}$ is a **linear functional** (a **functional**) **on** X if it is linear.
(c) We call the space $\mathsf{X}^* := \mathfrak{L}(\mathsf{X}, \mathbb{K})$ the **algebraic dual** of X, whereas $\mathsf{X}' := \mathfrak{B}(\mathsf{X}, \mathbb{K})$ is the **topological dual** (the **dual**) **of** X, with $\mathbb{K}$ normed by the absolute value.

Notation 2.36 Linear algebra textbooks usually write Tu for $T(u)$ when $T : \mathsf{X} \to \mathsf{Y}$ is a linear operator and $u \in \mathsf{X}$, and we shall adhere to this convention. ■

If $T, S \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ and $\alpha, \beta \in \mathbb{K}$, the linear combination $\alpha T + \beta S$ is the expected map: $(\alpha T + \beta S)(u) := \alpha(Tu) + \beta(Su)$ for any $u \in \mathsf{X}$.

Thus $\alpha T + \beta S$ is still in $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$. As linear combinations preserve continuity, we have the following.

Proposition 2.37 *Let* X, Y *be vector spaces over the same field* $\mathbb{K} := \mathbb{R}$, *or* $\mathbb{C}$. *Then* $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$, $\mathfrak{L}(\mathsf{X})$, X^*, $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$, $\mathfrak{B}(\mathsf{X})$ *and* X' *are* $\mathbb{K}$*-vector spaces.*

Another fundamental notion we introduce is that of a *bounded* operator (or functional). We begin with an elementary, yet important, fact.

Theorem 2.38 *Let* $(\mathsf{X}, ||\ ||_{\mathsf{X}})$, $(\mathsf{Y}, ||\ ||_{\mathsf{Y}})$ *be normed spaces over the same field* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$ *and take* $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$.
(a) *The following conditions are equivalent:*
 (i) there exists $K \in \mathbb{R}$ *such that* $||Tu||_{\mathsf{Y}} \leq K||u||_{\mathsf{X}}$ *for all* $u \in \mathsf{X}$;
 (ii) $\sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} < +\infty$.
(b) *If either of (i), (ii) hold:*

$$\sup\left\{ \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} \ \middle|\ u \in \mathsf{X} \setminus \{\mathbf{0}\} \right\} = \inf\left\{K \in \mathbb{R} \mid ||Tu||_{\mathsf{Y}} \leq K||u||_{\mathsf{X}} \quad \textit{for any } u \in \mathsf{X}\right\} .$$

Proof (a) Under (i), $\sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} \leq K < +\infty$ by construction. If (ii) holds, set $A := \sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}}$, and then $K := A$ satisfies (i).
(b) Call I the greatest lower bound of the numbers K fulfilling (i). Since

$$\sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} \leq K$$

we have $\sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} \leq I$. If the two sides of the equality to be proven are different, there exists K_0 with $\sup_{u\in\mathsf{X}\setminus\{\mathbf{0}\}} \frac{||Tu||_{\mathsf{Y}}}{||u||_{\mathsf{X}}} < K_0 < I$, whence $||Tu||_{\mathsf{Y}} < K_0||u||_{\mathsf{X}}$ for any $u \neq \mathbf{0}$, so $||Tu||_{\mathsf{Y}} \leq K_0||u||_{\mathsf{X}}$ for all $u \in \mathsf{X}$. Therefore K_0 satisfies (i), and $I \leq K_0$ by definition of I, in contradiction to $K_0 < I$. □

Notation 2.39 We will start omitting subscripts in norms when the corresponding spaces are clear from the context. ■

Definition 2.40 (*Operator norm*) Let X, Y be normed spaces over $\mathbb{C}$ or $\mathbb{R}$. The operator $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ is **bounded** if any one condition in Theorem 2.38(a) holds. The number

$$||T|| := \sup_{||u||\neq 0} \frac{||Tu||}{||u||} . \tag{2.12}$$

is called **(operator) norm** of T.

Remarks 2.41 **(1)** From the definition of $||T||$, if $T : \mathsf{X} \to \mathsf{Y}$ is bounded then:

$$||Tu|| \leq ||T||\ ||u|| , \quad \text{for any } u \in \mathsf{X} . \tag{2.13}$$

(2) The notion of bounded linear operator cannot clearly correspond to that of a bounded function. That is because the image of a linear map, in a vector space, cannot be bounded precisely because of linearity. Proposition 2.42 shows, though, that it still makes sense to view “boundedness” in terms of the bounded image of an operator, provided one restricts the domain to a bounded set. ■

The operator norm can be computed in alternative ways, which at times are useful in proofs. In this respect,

Proposition 2.42 *Let* X, Y *be normed spaces over* $\mathbb{C}$ *or* $\mathbb{R}$.

The operator $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *is bounded if and only if the right-hand side of any of the identities below exists and is finite, in which case:*

$$||T|| = \sup_{||u||=1} ||Tu|| \,, \tag{2.14}$$

$$||T|| = \sup_{||u||\leq 1} ||Tu|| \,, \tag{2.15}$$

$$||T|| = \inf \{K \in \mathbb{R} \mid ||Tu|| \leq K||u|| \quad \textit{for any } u \in \mathsf{X}\} \ . \tag{2.16}$$

Proof That T is bounded if and only if the right-hand side of (2.14) is finite, and the validity of (2.14) too, follow from the linearity of T and **N1**.

As for the second line (2.15), the set of vectors u with $||u|| \leq 1$ contains those for which $||u|| = 1$, so $\sup_{||u||\leq 1} ||Tu|| \geq \sup_{||u||=1} ||Tu||$. On the other hand, $||u|| \leq 1$ implies $||Tu|| \leq ||Tv||$ for some v with $||v|| = 1$ (any such v if $u = \mathbf{0}$, and $v = u/||u||$ otherwise). Hence $\sup_{||u||\leq 1} ||Tu|| \leq \sup_{||u||=1} ||Tu||$, from which $\sup_{||u||\leq 1} ||Tu|| = \sup_{||u||=1} ||Tu||$, as claimed.

That T is bounded iff the right-hand side of (2.16) is finite, and (2.16) itself, are consequences of Theorem 2.38(b). □

There is a relationship between *continuity* and *boundedness* of linear operators and functionals, which makes boundedness very important. The following simple theorem shows, amongst other things, that *bounded* operators are precisely the *continuous* ones.

Theorem 2.43 *Consider* $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *with* X, Y *normed over the same field* $\mathbb{R}$ *or* $\mathbb{C}$.

The following are equivalent facts:
(i) T *is continuous at* $\mathbf{0}$*;*
(ii) T *is continuous;*
(iii) T *is bounded.*

Proof (i) $\Leftrightarrow$ (ii). Since continuity trivially implies continuity at $\mathbf{0}$, we will show (i) $\Rightarrow$ (ii). As $(Tu) - (Tv) = T(u - v)$ we have $(\lim_{u\to v} Tu) - Tv = \lim_{u\to v}(Tu - Tv) = \lim_{(u-v)\to\mathbf{0}} T(u - v) = \mathbf{0}$ by continuity at $\mathbf{0}$.
(i) $\Rightarrow$ (iii). By the continuity at $\mathbf{0}$ there is $\delta > 0$ such that $||u|| < \delta$ implies $||Tu|| < 1$. Fixing $\delta' > 0$ with $\delta' < \delta$, if $v \in \mathsf{X} \setminus \{\mathbf{0}\}$, then $u = \delta' v/||v||$ has norm smaller than δ, so $||Tu|| < 1$, i.e. $||Tv|| < (1/\delta')||v||$. Therefore Theorem 2.38(a) holds with $K = 1/\delta'$, and by Definition 2.40 T is bounded.

(iii) $\Rightarrow$ (i). This is obvious: if T is bounded then $||Tu|| \leq ||T|| ||u||$, hence the continuity at $\mathbf{0}$ follows. □

The name "norm" for $||T||$ is not accidental: the operator norm renders $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$, *hence also* $\mathfrak{B}(\mathsf{X})$ *and* X', normed spaces, as we shall shortly see. More precisely, $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ is a Banach space if Y is Banach, so in particular X' is always a Banach space.

The next result is about the algebraic structure. Let us start by saying that the vector spaces $\mathfrak{L}(\mathsf{X})$ and $\mathfrak{B}(\mathsf{X})$ are closed under composition of maps (since this preserves continuity). Furthermore, it is immediate that $\mathfrak{L}(\mathsf{X})$, $\mathfrak{B}(\mathsf{X})$ satisfy the algebra axioms **A1**, **A2**, **A3** whenever the product of two operators is the *composite*. Therefore $\mathfrak{L}(\mathsf{X})$ and $\mathfrak{B}(\mathsf{X})$ possess a natural structure of *algebras with unit*, where the unit is the identity map $I : \mathsf{X} \to \mathsf{X}$, and $\mathfrak{B}(\mathsf{X})$ is a subalgebra in $\mathfrak{L}(\mathsf{X})$.

The final part of the theorem is a stronger statement, for it says $\mathfrak{B}(\mathsf{X})$ is a *normed unital algebra* for the operator norm, and a *Banach algebra* if X is Banach.

Theorem 2.44 *Let* X, Y *be normed spaces over* $\mathbb{C}$, *or* $\mathbb{R}$.
(a) *The map* $||\ || : T \mapsto ||T||$, *where* $||T||$ *is as in (2.12), is a norm on* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$.
(b) *On the unital algebra* $\mathfrak{B}(\mathsf{X})$ *the following properties hold, which turn it into a normed algebra with unit:*

(i) $||TS|| \leq ||T|| ||S||$, $T, S \in \mathfrak{B}(\mathsf{X})$,

(ii) $||I|| = 1$.

(c) *If* Y *is complete,* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *is a Banach space.*
In particular:

(i) if X *is a Banach space,* $\mathfrak{B}(\mathsf{X})$ *is a Banach algebra with unit (the identity operator);*

(ii) X' *is always a Banach space with the functionals' norm, even if* X *is not complete.*

Proof (a) is a direct consequence of the definition of operator norm: properties **N0**, **N1**, **N2**, **N3** can be checked for the operator norm by using them on the norm of Y, together with formula (2.14) and the definition of supremum.
(b) Part (i) is immediate from (2.13) and (2.14), and (ii) is straightforward if we use (2.14).

Let us see to (c). We claim that Y complete $\Rightarrow \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ Banach. Take a Cauchy sequence $\{T_n\} \subset \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ for the operator norm. By (2.13) we have

$$||T_n u - T_m u|| \leq ||T_n - T_m|| ||u|| ,$$

As $\{T_n\}$ is Cauchy, $\{T_n u\}$ is too. Since Y is complete, for any given $u \in \mathsf{X}$ there is a vector in Y:

$$Tu := \lim_{n\to\infty} T_n u .$$

Because every T_n is a linear operator, so is $\mathsf{X} \ni u \mapsto Tu$. There remains to show $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ and $||T - T_n|| \to 0$ as $n \to \infty$.

As $\{T_n\}$ is a Cauchy sequence, if $\varepsilon > 0$ then $||T_n - T_m|| \le \varepsilon$ for n, m sufficiently large, hence $||T_n u - T_m u|| \le ||T_n - T_m|| ||u|| \le \varepsilon ||u||$. Therefore, since the norm is continuous (it defines the topology of Y),

$$||Tu - T_m u|| = || \lim_{n\to+\infty} T_n u - T_m u|| = \lim_{n\to+\infty} ||T_n u - T_m u|| \le \varepsilon ||u||$$

if m is big enough. From this estimate, since $||Tu|| \le ||Tu - T_m u|| + ||T_m u||$ and by (2.13), we have

$$||Tu|| \le (\varepsilon + ||T_m||)||u|| .$$

This proves T is bounded, so $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ by Theorem 2.43. Now, since $||Tu - T_m u|| \le \varepsilon ||u||$ we also have $||T - T_m|| \le \varepsilon$ where ε can be arbitrarily small so long as m is large enough. That is to say, $||T - T_n|| \to 0$ as $n \to \infty$.

The proof of subcases (i), (ii) is quick: (i) follows from $\mathfrak{B}(\mathsf{X}) = \mathfrak{B}(\mathsf{X}, \mathsf{X})$, while (ii) holds because $\mathsf{X}' := \mathfrak{B}(\mathsf{X}, \mathbb{K})$ and the field $\mathbb{K}$ is a complete normed space. □

One last notion we need to define is *conjugate* or *adjoint operators* in normed spaces. Beware that there is a different notion of conjugate operator specific to Hilbert spaces, which we will address in the next chapter.

Take $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$, with X, Y normed. We can build an operator $T' \in \mathfrak{L}(\mathsf{Y}', \mathsf{X}')$ between the dual spaces (swapped), by imposing:

$$(T'f)(x) = f(T(x)) \quad \text{for any } x \in \mathsf{X},\ f \in \mathsf{Y}'.$$

This is well defined, and for every $f \in \mathsf{Y}'$ it produces a function $T'f : \mathsf{X} \to \mathbb{C}$ that is linear by construction, because it coincides with the composite of linear maps f and T. Furthermore, $T' : \mathsf{Y}' \ni f \to T'f \in \mathsf{X}'$ is linear:

$$(T'(af + bf))(x) = (af + bg)(T(x)) = af(T(x)) + bg(T(x))$$

$$= a(T'f)(x) + b(T'g)(x) \quad \text{for any } x \in \mathsf{X} .$$

Eventually, T' is bounded, in the obvious sense:

$$|(T'f)(x)| = |f(T(x))| \le ||f|| ||T|| ||x|| ,$$

and so:

$$||T'f|| = \sup_{||x||=1} |T'f(x)| \le ||f|| ||T|| .$$

Taking, on the left, the supremum over the collection of $f \in \mathsf{Y}'$ with $||f|| = 1$ gives:

$$||T'|| \le ||T|| . \tag{2.17}$$

After proving the *Hahn–Banach theorem*, we will show that $||T'|| = ||T||$ if X, Y are Banach spaces.

Definition 2.45 Let X, Y be normed spaces over the same field $\mathbb{C}$, or $\mathbb{R}$, and $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$. The **conjugate**, or **adjoint operator** to T, in the sense of normed spaces, is the operator $T' \in \mathfrak{B}(\mathsf{Y}', \mathsf{X}')$ defined by:

$$(T'f)(x) = f(T(x)) \quad \text{for any } x \in \mathsf{X},\ f \in \mathsf{Y}'. \tag{2.18}$$

Remark 2.46 The map $\mathfrak{B}(\mathsf{X}, \mathsf{Y}) \ni T \mapsto T' \in \mathfrak{B}(\mathsf{Y}', \mathsf{X}')$ is linear:

$$(aT + bS)' = aT' + bS' \quad \text{for any } a, b \in \mathbb{C},\ T, S \in \mathfrak{B}(\mathsf{X}, \mathsf{Y}).$$

■

Before we move on to the examples, we shall state an elementary result, very important in the applications, about the uniqueness of extensions of bounded operators and functionals defined on dense subsets.

Proposition 2.47 (Extension of bounded operators) *Let* X, Y *be normed spaces over* $\mathbb{C}$*, or* $\mathbb{R}$*, with* Y *Banach. Suppose* $S \subset \mathsf{X}$ *is a dense subspace of* X *and* $T : S \to \mathsf{Y}$ *is a bounded linear operator on* S*. Then*
(a) *there is a unique bounded linear operator* $\tilde{T} : \mathsf{X} \to \mathsf{Y}$ *such that* $\tilde{T}\restriction_S = T$*;*
(b) $||\tilde{T}|| = ||T||$.

Proof (a) Given $x \in \mathsf{X}$, there is a sequence $\{x_n\}$ in S converging to x. By hypothesis $||Tx_n - Tx_m|| \leq K||x_n - x_m||$ for $K < +\infty$. Since $x_n \to x$, the sequence of the x_n is Cauchy, and so is Tx_n. But Y is complete so there exists $\tilde{T}x := \lim_{n\to\infty} Tx_n \in \mathsf{Y}$. The limit depends only on x and not upon the sequence in S used to approximate: if $S \ni z_n \to x$ then by the norms' continuity

$$|| \lim_{n\to+\infty} Tx_n - \lim_{n\to+\infty} Tz_n|| = \lim_{n\to+\infty} ||Tx_n - Tz_n|| \leq \lim_{n\to+\infty} K||x_n - z_n|| = K||x - x|| = 0\,.$$

Clearly $\tilde{T}\restriction_S = T$, i.e. $\tilde{T}$ extends T, by choosing for any $x \in S$ the constant sequence $x_n := x$, that tends to x trivially. The linearity of $\tilde{T}$ is straightforward from the definition. Eventually, taking the limit as $n \to +\infty$ of $||Tx_n|| \leq K||x_n||$ gives $||\tilde{T}x|| \leq K||x||$, so $\tilde{T}$ is bounded. About uniqueness: if U is another bounded extension of T on X, then for any $x \in \mathsf{X}$, $\tilde{T}x - Ux = \lim_{n\to+\infty}(\tilde{T}x_n - Ux_n)$ by continuity, where the x_n belong to S (dense in X). As $\tilde{T}\restriction_S = T = U\restriction_S$, the limit is trivial and gives $\tilde{T}x = Ux$ for all $x \in \mathsf{X}$, i.e. $\tilde{T} = U$.
(b) Let $x \in \mathsf{X}$ and suppose $\{x_n\} \subset S$ converges to x: then

$$||\tilde{T}x|| = \lim_{n\to+\infty} ||Tx_n|| \leq \lim_{n\to+\infty} ||T||||x_n|| = ||T||||x||\,,$$

so $||\tilde{T}|| \leq ||T||$. But since $S \subset \mathsf{X}$ and $\tilde{T}\restriction_S = T$,

$$||\tilde{T}|| = \sup\left\{ \frac{||\tilde{T}x||}{||x||} \;\middle|\; \mathbf{0} \neq x \in \mathsf{X} \right\} \geq \sup\left\{ \frac{||\tilde{T}x||}{||x||} \;\middle|\; \mathbf{0} \neq x \in S \right\} = ||T|| \,.$$

Hence $||\tilde{T}|| \geq ||T||$, and so $||\tilde{T}|| = ||T||$. □

Examples 2.48 **(1)** *Complex measures* (see Sect. 1.4.8) allow to construct every bounded linear functional on $C_0(\mathsf{X})$, where X is locally compact and Hausdorff.

To see this, consider a locally compact Hausdorff space X equipped with a complex measure μ defined on the Borel σ-algebra of X. We know that the normed algebra $(C_c(\mathsf{X}), ||\ ||_\infty)$ completes to the Banach algebra $(C_0(\mathsf{X}), ||\ ||_\infty)$ of maps that vanish at infinity (Example 2.29(4)). Under the assumptions made, then, $||\mu|| := |\mu|(\mathsf{X})$, where the positive, σ-additive and finite measure $|\mu|$ is the *total variation* of μ (cf. Sect. 1.4.8). Easily then, $||\ ||$ is a norm on the space of complex Borel measures on X. Moreover, if $f \in C_0(\mathsf{X})$,

$$|\Lambda_\mu f| \leq ||\mu||||f||_\infty \quad \text{where } \Lambda_\mu f := \int_{\mathsf{X}} f\, d\mu \,,$$

and, as usual, $||f||_\infty = \sup_{x\in\mathsf{X}} |f(x)|$.

Consequently, every complex Borel measure μ defines an element Λ_μ in the (topological) dual of $C_0(\mathsf{X})$. *Riesz's theorem for complex measures* [Rud86] guarantees that this is a general fact, and even more.

In order to state it, recall that

Definition 2.49 (*Regular complex Borel measure*) A complex Borel measure μ is called **regular** if the finite positive Borel measure given by the total variation $|\mu|$ is regular (Definition 1.81).

Theorem 2.50 (Riesz's theorem for complex measures) *Let X be a locally compact Hausdorff space, and $\Lambda : C_0(\mathsf{X}) \to \mathbb{C}$ a continuous linear functional. Then there exists a unique regular complex Borel measure μ_Λ such that, for every $f \in C_0(\mathsf{X})$:*

$$\Lambda(f) = \int_{\mathsf{X}} f\, d\mu_\Lambda \,.$$

Moreover, $||\Lambda|| = ||\mu_\Lambda||$.

Since every regular complex Borel measure determines a bounded functional on $C_0(\mathsf{X})$ by integration, Riesz's theorem has the following consequence.

Corollary 2.51 *If X is locally compact and Hausdorff, the topological dual $C_0(\mathsf{X})'$ of the Banach space $(C_0(\mathsf{X}), ||\ ||_\infty)$ is identified with the real vector space of regular complex Borel measures μ on X, endowed with norm $||\mu|| := |\mu|(\mathsf{X})$. The function mapping μ to the functional $\Lambda_\mu : C_0(\mathsf{X}) \to \mathbb{R}$, with $\Lambda_\mu f := \int_{\mathsf{X}} f d\mu$, is an isomorphism of normed spaces.*

Also note, $C_c(\mathsf{X})$ being dense in $C_0(\mathsf{X})$, that a continuous functional on the former determines a unique functional on the latter, so Riesz's theorem characterises as well continuous functionals on $C_c(\mathsf{X})$ for the sup norm.

Furthermore, suppose every open set in X is the countable union of compact sets (as in $\mathbb{R}^n$, where each open set is the union of countably many closed balls of finite radius). Then the word *regular* can be dropped in Theorem 2.50, by way of Proposition 1.60, because compact sets have finite measure with respect to the finite $|\mu|$. In particular we have:

Theorem 2.52 (Riesz's theorem for complex measures on $\mathbb{R}^n$) *Let $K \subset \mathbb{R}^n$ be a compact set and $\Lambda : C(K) \to \mathbb{C}$ a continuous linear functional. Then there is a unique complex Borel measure μ_Λ on K such that*

$$\Lambda(f) = \int_K f d\mu_\Lambda$$

for any $f \in C(K)$. Additionally, μ_Λ is regular.

(2) Another nice class of dual Banach spaces is that of L^p spaces, cf. Example 2.29(6). In this respect [Rud86],

Proposition 2.53 *Let (X, Σ, μ) be a positive measure space. If $1 \leq p < +\infty$ the dual to the Banach space $L^p(\mathsf{X}, \mu)$ is $L^q(\mathsf{X}, \mu)$, where $1/p + 1/q = 1$, in the sense that the linear map:*

$$L^q(\mathsf{X}, \mu) \ni [g] \mapsto \Lambda_g \quad \textit{where } \Lambda_g(f) := \int_{\mathsf{X}} fg \, d\mu \ , \ f \in L^p(\mathsf{X}, \mu)$$

is an isomorphism $L^q(\mathsf{X}, \mu) \to (L^p(\mathsf{X}, \mu))'$ of normed spaces.

In the same way the dual to $L^1(\mathsf{X}, \mu)$ is identified with $L^\infty(\mathsf{X}, \mu)$, because the linear map

$$L^\infty(\mathsf{X}, \mu) \ni [g] \mapsto \Lambda'_g \quad \textit{where } \Lambda'_g(f) := \int_{\mathsf{X}} fg \, d\mu \ , \ f \in L^1(\mathsf{X}, \mu)$$

is an isomorphism $L^\infty(\mathsf{X}, \mu) \to (L^1(\mathsf{X}, \mu))'$ of normed spaces. ■

2.3 The Fundamental Theorems of Banach Spaces

This section is devoted to the most prominent theorems on normed and Banach spaces, in their simplest versions, and we will study their main consequences. These are the theorems of Hahn–Banach, Banach–Steinhaus and the open mapping theorem. The applications of the theorem of Banach–Steinhaus call forth several kinds of topologies, which play a major role in QM when the domain space is the Hilbert space of the theory, bounded operators are (certain) observables, and the basic features of the quantum system associated to measurement processes are a subclass of orthogonal projectors. In order to pass with continuity from the algebra of observables to that of projectors we need topologies that are weaker than the standard one. This sort of issues, that we shall address later, lead to the notion of *von Neumann algebra (of operators)*.

2.3.1 The Hahn–Banach Theorem and Its Immediate Consequences

The first result we present is the celebrated Hahn–Banach theorem, which deals with extending a continuous linear functional from a subspace to the ambient space in a continuous and norm-preserving way. More elaborated and stronger versions can be found in [Rud91]. We shall restrict to the simplest situation possible.

First of all, we remark that if X is normed and $\mathsf{M} \subset \mathsf{X}$ is a subspace, the norm of X restricted to M defines a normed space. In this sense we can talk of continuous operators and functionals on M, meaning they are bounded for the induced norm.

Theorem 2.54 (Hahn–Banach theorem for normed spaces) *Let* M *be a subspace (not necessarily closed) in a normed space* X *over* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$.

If $g : \mathsf{M} \to \mathbb{K}$ *is a continuous linear functional, there exists a continuous linear functional* $f : \mathsf{X} \to \mathbb{K}$ *such that* $f \restriction_M = g$ *and* $||f||_{\mathsf{X}} = ||g||_{\mathsf{M}}$.

Proof We shall follow the proof of [Rud86]. Start with $\mathbb{K} = \mathbb{R}$. If $g = 0$, an extension as required is $f = 0$. So let us suppose $g \neq 0$ and without loss of generality set $||g|| = 1$. To build the extension f we take $x_0 \in \mathsf{X} \setminus \mathsf{M}$ and call

$$\mathsf{M}_1 := \{x + \lambda x_0 \mid x \in \mathsf{M}\,, \lambda \in \mathbb{R}\}\,.$$

If we set $g_1 : \mathsf{M}_1 \to \mathbb{R}$ to be

$$g_1(x + \lambda x_0) = g(x) + \lambda \nu$$

for any given $\nu \in \mathbb{R}$, we obtain an extension of g to M_1. We claim ν can be taken so that $||g_1|| = 1$. For this it suffices to have ν such that:

$$|g(x) + \lambda \nu| \leq ||x + \lambda x_0||\,, \quad \text{for any } x \in \mathsf{M} \text{ and } \lambda \in \mathbb{R} \setminus \{0\}. \tag{2.19}$$

Substitute $-\lambda x$ to x and divide (2.19) by $|\lambda|$, obtaining the equivalent relation:

$$|g(x) - \nu| \leq ||x - x_0||\,, \quad \text{for any } x \in \mathsf{M}. \tag{2.20}$$

Set

$$a_x := g(x) - ||x - x_0|| \quad \text{and} \quad b_x := g(x) + ||x - x_0||. \tag{2.21}$$

Inequality (2.20), hence $||g_1|| = 1$, holds if ν satisfies $a_x \le \nu \le b_x$ for all $x \in \mathsf{M}$. So it is enough to prove that the intervals $[a_x, b_x]$, $x \in \mathsf{M}$, have a common point; in other words, that for all $x, y \in \mathsf{M}$:

$$a_x \le b_y \,. \tag{2.22}$$

But:

$$g(x) - g(y) = g(x - y) \le ||x - y|| \le ||x - x_0|| + ||y - x_0||$$

and (2.22) follows from (2.21). Hence, we managed to fix ν so that $||g_1|| = 1$.

Now consider the family $\mathscr{P}$ of pairs (M', g') where $\mathsf{M}' \supset \mathsf{M}$ is a subspace in X and $g' : \mathsf{M}' \to \mathbb{R}$ is a linear extension of g with $||g'|| = 1$. The family is not empty since (M_1, g_1) belongs in it. We can define a partial order on $\mathscr{P}$ (see Appendix A, also for the sequel) by setting $(\mathsf{M}', g') \le (\mathsf{M}'', g'')$ if $\mathsf{M}'' \supset \mathsf{M}'$, g'' extends g' and $||g'|| = ||g''|| = 1$. It is easy to show that any totally ordered subset of $\mathscr{P}$ admits an upper bound in $\mathscr{P}$. Then Zorn's lemma provides a maximal element in $\mathscr{P}$, say (M^1, f^1). Now we must have $\mathsf{M}^1 = \mathsf{X}$, for otherwise there would exist $x_0 \in \mathsf{X} \setminus \mathsf{M}^1$, and using the initial argument we could construct a non-trivial, norm-preserving extension f^1 to the subspace generated by x_0 and M^1, contradicting maximality. Therefore $f := f^1$ is the required extension.

Before passing to the case $\mathbb{K} = \mathbb{C}$ we need a lemma.

Lemma 2.55 *On a complex vector space* Y
(a) *if* $u(x) = Re\, g(x)$ *for all* $x \in \mathsf{Y}$ *for some complex linear functional* $g : \mathsf{Y} \to \mathbb{C}$, *the map* $u : \mathsf{Y} \to \mathbb{R}$ *is a real linear functional on* Y, *and:*

$$g(x) := u(x) - iu(ix) \quad \textit{for any } x \in \mathsf{Y}; \tag{2.23}$$

(b) *if* $u : \mathsf{Y} \to \mathbb{R}$ *is a real linear functional on* Y *and* g *is defined by (2.23), then* g *is a complex linear functional on* Y;
(c) *if* Y *is normed and* g, u *are related by (2.23), then* $||g|| = ||u||$.

Proof (a) are (b) are proved simultaneously by direct computation. As for (c), under the assumptions made: $|u(x)| \le |g(x)| = \sqrt{|u(x)|^2 + |u(ix)|^2}$, so $||u|| \le ||g||$. On the other hand taking $x \in \mathsf{Y}$, there is $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha g(x) = |g(x)|$. Consequently $|g(x)| = g(\alpha x) = u(\alpha x) \le ||u||\, ||\alpha x|| = ||u||\, ||x||$ and $||g|| \le ||u||$. □

Now back to the main proof. If $u : \mathsf{M} \to \mathbb{R}$ is the real part of g, then $g(x) = u(x) - iu(ix)$ and $||g|| = ||u||$ by the lemma. From the real case seen earlier we know there exists a linear extension $U : \mathsf{X} \to \mathbb{R}$ of u with $||U|| = ||u|| = ||g||$. Therefore if we put

$$f(x) := U(x) - iU(ix)\ , \quad \text{for any } x \in \mathsf{X},$$

$f : \mathsf{X} \to \mathbb{C}$ extends g to X, and $||f|| = ||U|| = ||g||$. □

Here is one of the most useful corollaries to Hahn–Banach. We remind that the topological dual $\mathfrak{B}(\mathsf{X}, \mathbb{C})$ of a normed space is indicated by X'.

Corollary 2.56 (to the Hahn–Banach theorem) *Let* X *be a normed space over* $\mathbb{K} = \mathbb{C}$*, or* $\mathbb{R}$*, and fix* $x_0 \in \mathsf{X}$*,* $x_0 \neq \mathbf{0}$*. Then there exists* $f \in \mathsf{X}'$*,* $||f|| = 1$*, such that* $f(x_0) = ||x_0||$*.*

Proof Choose $\mathsf{M} := \{\lambda x_0 \mid \lambda \in \mathbb{K}\}$ and $g : \lambda x_0 \to \lambda ||x_0||$. Let $f \in \mathsf{X}'$ denote the bounded functional extending g according to Hahn–Banach. By construction $f(x_0) = g(x_0) = ||x_0||$ and $||f||_{\mathsf{X}} = ||g||_{\mathsf{M}} = 1$. □

An immediate consequence of this is a statement about the norm of the conjugate operator $T' \in \mathfrak{B}(\mathsf{Y}', \mathsf{X}')$ to $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ (cf. Definition 2.45).

Proposition 2.57 *If* $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$*, with* X*,* Y *normed over* $\mathbb{C}$ *or* $\mathbb{R}$*, then:*

$$||T'|| = ||T||\ .$$

Proof In general we have (cf. (2.17)) $||T|| \geq ||T'||$, so we need only prove $||T|| \leq ||T'||$. Take $x \in \mathsf{X}$ and $Tx \neq \mathbf{0}$, define $y_0 := \frac{Tx}{||Tx||} \in \mathsf{Y}$. Clearly $||y_0|| = 1$, and by Corollary 2.56 there is $g \in \mathsf{Y}'$ such that $||g|| = 1$, $g(y_0) = 1$ hence $g(Tx) = ||Tx||$. But:

$$||Tx|| = g(Tx) = |(T'g)(x))| \leq ||T'g||\,||x|| \leq ||T'||\,||g||\,||x|| = ||T'||\,||x||\ ,$$

so eventually $||T|| \leq ||T'||$ as required. □

Next comes another fact, with important consequences for Banach algebras.

Corollary 2.58 (to the Hahn–Banach theorem) *Let* $\mathsf{X} \neq \{\mathbf{0}\}$ *be a normed space over* $\mathbb{C}$ *or* $\mathbb{R}$*.*

Then the elements of X' *separate* X*, i.e. for any* $x_1 \neq x_2$ *in* X *there exists* $f \in \mathsf{X}'$ *for which* $f(x_1) \neq f(x_2)$*.*

Proof It suffices to have $x_0 := x_1 - x_2$ in Corollary 2.56, for then $f(x_1) - f(x_2) = f(x_1 - x_2) = ||x_1 - x_2|| \neq 0$. □

Take $x \in \mathsf{X}$ and $f \in \mathsf{X}'$, $||f|| = 1$; then $|f(x)| \leq 1||x||$ and

$$\sup\{|f(x)| \mid f \in \mathsf{X}'\ ,\ \ ||f|| = 1\} \leq ||x||\ .$$

Corollary 2.56 allows to strengthen this fact by showing

$$\sup\{|f(x)| \mid f \in \mathsf{X}'\ ,\ \ ||f|| = 1\} = \max\{|f(x)| \mid f \in \mathsf{X}'\ ,\ \ ||f|| = 1\} = ||x||$$

directly. This may not seem so striking at first, but has a certain weight when comparing infinite-dimensional to finite-dimensional normed spaces.

From the elementary theory of *finite-dimensional* vector spaces X, the algebraic dual of the algebraic dual $(\mathsf{X}^*)^*$ has the nice property of being *naturally isomorphic* to X. The isomorphism is the linear function mapping $x \in \mathsf{X}$ to the linear functional $\mathfrak{I}(x)$ on X^* defined by $(\mathfrak{I}(x))(f) := f(x)$ for all $f \in \mathsf{X}^*$.

In infinite dimensions $\mathfrak{I}$ identifies X only to a subspace of $(\mathsf{X}^*)^*$, not the whole $(\mathsf{X}^*)^*$ in general. Is there a similar general statement about *topological duals* of infinite-dimensional normed spaces?

Note $(\mathsf{X}')'$ is the dual to a normed space (X' with the operator norm). Consequently $(\mathsf{X}')'$ is a normed space, under the operator norm.

Let us go back to the natural linear transformation $\mathfrak{I} : \mathsf{X} \to (\mathsf{X}')^*$ mapping $x \in \mathsf{X}$ to $\mathfrak{I}(x) \in (\mathsf{X}')^*$, where the linear function $\mathfrak{I}(x) : \mathsf{X}' \to \mathbb{K}$ is given by

$$(\mathfrak{I}(x))(f) := f(x) \quad \text{for any } f \in \mathsf{X}' \text{ and } x \in \mathsf{X} .$$

This is well defined, since $\mathfrak{I}(x)$ is a linear functional on X' so that $\mathfrak{I}(x) \in (\mathsf{X}')^*$. Now

$$\sup\{|f(x)| \mid f \in \mathsf{X}' \ , \ ||f|| = 1\} = ||x||$$

implies: (1) $\mathfrak{I}(x)$ is a *bounded* functional, so it belongs to $(\mathsf{X}')'$, and (2) $||\mathfrak{I}(x)|| = ||x||$. Therefore the linear mapping $\mathfrak{I} : \mathsf{X} \to (\mathsf{X}')'$ is an *isometry*, in particular *injective*. This gives an isometric inclusion $\mathsf{X} \subset (\mathsf{X}')'$ under the linear map $\mathfrak{I} : \mathsf{X} \to (\mathsf{X}')'$. Overall we have proved:

Corollary 2.59 (to the Hahn–Banach theorem) *Let X be a normed space over $\mathbb{C}$ or $\mathbb{R}$. The linear map $\mathfrak{I} : \mathsf{X} \to (\mathsf{X}')'$:*

$$(\mathfrak{I}(x))(f) := f(x) \quad \textit{for any } x \in \mathsf{X} \textit{ and } f \in \mathsf{X}', \tag{2.24}$$

is an isometry, and X is thus identified isometrically with a subspace of $(\mathsf{X}')'$.

There are infinite-dimensional examples where X does not fill $(\mathsf{X}')'$, and these justify the next notion.

Definition 2.60 A normed space X over $\mathbb{C}$ or $\mathbb{R}$ is **reflexive** if the isometry (2.24) is onto (an isomorphism of normed spaces).

Otherwise said, X is reflexive when X and $(\mathsf{X}')'$ are isometrically isomorphic under the natural map $\mathfrak{I}$. In Chap. 3 we will show that Hilbert spaces are reflexive.

Example 2.61 The Banach spaces $L^p(\mathsf{X}, \mu)$ of Examples 2.29 are reflexive for $1 < p < \infty$. The proof is straightforward: $L^p(\mathsf{X}, \mu)' = L^q(\mathsf{X}, \mu)$ for $1/p + 1/q = 1$, and swapping q with p gives $L^q(\mathsf{X}, \mu)' = L^p(\mathsf{X}, \mu)$. Hence: $(L^p(\mathsf{X}, \mu)')' = L^p(\mathsf{X}, \mu)$. ■

2.3.2 The Banach–Steinhaus Theorem or Uniform Boundedness Principle

Let us get to the second core result, the theorem of Banach–Steinhaus, and present the simplest formulation and consequences. It is also known as *uniform boundedness principle*, because it essentially – and remarkably – declares that pointwise equi-boundedness implies uniform boundedness for families of operators on a Banach space.

Theorem 2.62 (Banach–Steinhaus) *Let* X *be a Banach space,* Y *a normed space defined over the same field,* $\mathbb{C}$ *or* $\mathbb{R}$*. If* $\{T_\alpha\}_{\alpha\in A} \subset \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *is a family of operators such that:*

$$\sup_{\alpha\in A} ||T_\alpha x|| < +\infty \quad \text{for any } x \in \mathsf{X}\ ,$$

then there exists $K \geq 0$ *that bounds the family uniformly:*

$$||T_\alpha|| \leq K \quad \text{for any } \alpha \in A\ .$$

Proof The proof relies on finding an open ball $B_\rho(z) \subset \mathsf{X}$ for which there is $M \geq 0$ with $||T_\alpha(x)|| \leq M$ for all $\alpha \in A$ and any $x \in B_\rho(z)$. In fact, since $x = (x + z) - z$, we would then have:

$$||T_\alpha(x)|| \leq ||T_\alpha(x + z)|| + ||T_\alpha(z)|| \leq 2M\ , \quad \text{for any } \alpha \in A, x \in B_\rho(\mathbf{0})\ ,$$

so $||T_\alpha|| \leq 2M/\rho$ for all $\alpha \in A$, and the claim would follow.

We shall prove that $B_\rho(z)$ and M exist by contradiction. If such a ball did not exist, for some arbitrary open $B_{r_0}(x_0)$, there would be $x_1 \in B_{r_0}(x_0)$ for which $||T_{\alpha_1}(x_1)|| > 1$, with $\alpha_1 \in A$. As T_{α_1} is continuous, we could find a second open ball $B_{r_1}(x_1)$ with $\overline{B_{r_1}(x_1)} \subset B_{r_0}(x_0)$ and $0 < r_1 < r_0$ such that $||T_{\alpha_1}(x)|| \geq 1$, provided $x \in \overline{B_{\alpha_1}(x_1)}$. This recipe can be iterated to give rise to a sequence of open balls in X, $\{B_{r_k}(x_k)\}_{k\in\mathbb{N}}$, satisfying:

(i) $B_{r_k}(x_k) \supset \overline{B_{r_{k+1}}(x_{k+1})}$,

(ii) $r_k \to 0$ as $k \to +\infty$,

(iii) for every $k \in \mathbb{N}$ there is $\alpha_k \in A$ such that $||T_{\alpha_k}(x)|| \geq k$ if $x \in \overline{B_{r_k}(x_k)}$.

Now, (i) and (ii) imply the sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy, so it converges to some $y \in \mathsf{X}$ by completeness, and by construction $y \in \cap_{k\in\mathbb{N}}\overline{B_{r_k}(x_k)}$. But (iii) tells $||T_{\alpha_k}(y)|| \geq k$ for all $k \in \mathbb{N}$, contradicting the assumption that $\sup_{n\in\mathbb{N}} ||T_\alpha x|| < +\infty$ for any $x \in \mathsf{X}$. □

Here is a straightforward and useful corollary.

Corollary 2.63 (to the Banach–Steinhaus theorem) *Under the assumptions of the Banach–Steinhaus theorem the family of operators* $\{T_\alpha\}_{\alpha\in A}$ *is* **equicontinuous***: given any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $||x - x'|| < \delta$ *for* $x, x' \in \mathsf{X}$*, then* $||T_\alpha x - T_\alpha x'|| < \varepsilon$ *for any* $\alpha \in A$*.*

Proof Set $C_\gamma := \{x \in \mathsf{X} \mid ||x|| \leq \gamma\}$, for any $\gamma > 0$. Fix $\varepsilon > 0$, so we must find the $\delta > 0$ of the conclusion. By Banach–Steinhaus and Proposition 2.42, $||T_\alpha x|| \leq K < +\infty$ for any $\alpha \in A$ and $x \in C_1$. If $K = 0$ there is nothing to prove, so assume $K > 0$. Choose $\delta > 0$ for which $C_\delta \subset C_{\varepsilon/K}$. Then if $||x - x'|| < \delta$, we have $K(x - x')/\varepsilon \in C_{K\delta/\varepsilon} \subset C_1$ and so:

$$||T_\alpha x - T_\alpha x'|| = ||T_\alpha(x - x')|| = \frac{\varepsilon}{K}\left|\left|T_\alpha \frac{K(x - x')}{\varepsilon}\right|\right| < \frac{\varepsilon}{K}K = \varepsilon \quad \text{for any } \alpha \in A.$$

□

And here is another consequence about topological duals.

Corollary 2.64 (to the Banach–Steinhaus theorem) *Let* X *be a normed space over* $\mathbb{C}$ *or* $\mathbb{R}$. *If* $S \subset \mathsf{X}$ *is weakly bounded, i.e.*

$$\textit{for any } f \in \mathsf{X}' \textit{ there exists } c_f \geq 0 \textit{ such that } |f(x)| \leq c_f \textit{ for all } x \in S,$$

then S *is bounded in the norm of* X.

Proof Consider the elements $x \in S \subset \mathsf{X}$ as functionals in the dual $(\mathsf{X}')'$ to X' (using the isometry $\mathfrak{J} : \mathsf{X} \to (\mathsf{X}')'$ of Corollary 2.59). The family $S \subset (\mathsf{X}')'$ of functionals on X' is bounded on every $f \in \mathsf{X}'$, since by assumption $|x(f)| = |f(x)| \leq c_f$ (where we have written x for $\mathfrak{J}(x)$). Since X' is complete the theorem of Banach–Steinhaus guarantees $\sup\{|x(f)| \mid ||f|| = 1\} \leq K < +\infty$ for all $x \in S$. But $\mathfrak{J}$ is an isometry, so $||x|| \leq K < +\infty$ for all $x \in S$. □

2.3.3 *Weak Topologies. *-Weak Completeness of X′*

To state the last corollary to Banach–Steinhaus we need to introduce a new section on general topology and apply it to the operator spaces encountered so far. This will allow to see different types of convergence for sequences of operators, useful for the applications. They will help us prove a simple and useful result known as the *Banach–Alaoglu theorem*.

We begin with basic facts concerning *convexity*.

Definition 2.65 A subset $\emptyset \neq K \subset \mathsf{X}$ in a vector space is **convex** when:

$$\lambda x + (1 - \lambda)y \in K \quad \text{for any } \lambda \in [0, 1] \text{ and } x, y \in K.$$

A point $e \in K$ is **extremal** if it cannot be written as:

$$e = \lambda x + (1 - \lambda)y \quad \text{for some } \lambda \in (0, 1) \text{ and } x, y \in K \setminus \{e\}.$$

It should be clear that the intersection of convex sets is convex, because the segment joining two points lying in the intersection belongs to each set. So we are lead to the notion of *convex hull*.

Definition 2.66 The **convex hull** of a subset E in a vector space X is the convex set

$$co(E) := \bigcap \{K \supset E \mid K \subset \mathsf{X}\,,\, K \text{ convex}\}.$$

Let us go back open balls defined through seminorms.

Notation 2.67 Take $\delta > 0$, a seminorm p on the vector space X over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, and a point $x \in \mathsf{X}$. We denote by $B_{p,\delta}(x)$ the **open ball associated to the seminorm** p, centred at x and of radius δ:

$$B_{p,\delta}(x) := \{z \in \mathsf{X} \mid p(z - x) < \delta\}\,.$$

If $x = \mathbf{0}$ we will just write $B_{p,\delta}$ instead of $B_{p,\delta}(\mathbf{0})$.

If $A \subset \mathsf{X}$, $B \subset \mathsf{X}$, $x \in \mathsf{X}$ and $\alpha, \beta \in \mathbb{K}$, we will also abbreviate:

$$x + \beta A := \{x + \beta u \mid u \in A\} \quad \text{and} \quad \alpha A + \beta B := \{\alpha u + \beta v \mid u \in A\,, v \in B\}\,.$$

■

Immediately, then, for $\delta > 0$ the balls $B_{p,\delta}$ are:

(i) **convex**, since $x, y \in B_{p,\delta}$ implies trivially $(1 - \lambda)x + \lambda y \in B_{p,\delta}$ with $\lambda \in [0, 1]$,

(ii) **balanced**, i.e. $\lambda x \in B_{p,\delta}$ if $x \in B_{p,\delta}$ and $0 \leq \lambda \leq 1$,

(iii) **absorbing**, i.e. $x \in \mathsf{X}$ implies $\lambda^{-1}x \in B_{p,\delta}$ for some $\lambda > 0$.

All these properties are patently invariant under intersections. Hence also intersections of balls *centred at the origin* but defined by different seminorms enjoy the property.

Definition 2.68 Let $\mathscr{P} := \{p_i\}_{i \in I}$ be a family of seminorms on the vector space X over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The **topology** $\mathscr{T}(\mathscr{P})$ **on** X **generated, or induced, by** $\mathscr{P}$, is the unique topology admitting as **basis** (Definition 1.1) the collection:

$$x + \left(B_{p_{i_1},\delta_1} \cap \cdots \cap B_{p_{i_n},\delta_n}\right) \tag{2.25}$$

for any choice of: centres $x \in \mathsf{X}$, numbers $n = 1, 2, \ldots$, indices $i_1, \ldots, i_n \in I$ and radii $\delta_1 > 0, \ldots \delta_n > 0$.

The pair $(\mathsf{X}, \mathscr{P})$, where X is simultaneously a vector space with topology induced by the seminorms $\mathscr{P}$ and a topological space, is called a **locally convex space**.

Let us put it differently: $\mathscr{T}(\mathscr{P})$ has as open sets $\emptyset$ and all possible unions of sets of type (2.25), with any centre $x \in \mathsf{X}$, for any $n = 1, 2, \ldots$, any index $i_1, \ldots, i_n \in I$ and any $\delta_1 > 0, \ldots \delta_n > 0$.

Remark 2.69 If $\mathscr{P}$ reduces to a single *norm*, the corresponding topology is the topology induced by a norm discussed at the beginning of the chapter. If this sole element is a seminorm, we still have a topology, with the crucial difference that the Hausdorff property might be no longer valid. ■

Since adding vectors and multiplying vectors by scalars are continuous operations in any seminorm (the proof is the same as what we gave for a norm), they are continuous in the topology generated by a family $\mathscr{P}$ of seminorms as well. This means the vector space structure is **compatible** with the topology generated by $\mathscr{P}$. A vector space with a compatible topology as above is a **topological vector space**. A locally convex space is therefore a topological vector space.

Keeping Definition 1.13 in mind we can prove the next fact without effort.

Proposition 2.70 *A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ converges to $x_0 \in \mathsf{X}$ in the topology $\mathscr{T}(\mathscr{P})$ if and only if $p_i(x_n - x_0) \to 0$, for all $p_i \in \mathscr{P}$, as $n \to +\infty$.*

Our first example of topology induced by seminorms arises from the dual X' of a normed space.

Definition 2.71 If X is a normed space, the **weak topology** on X is the topology induced by the collection of seminorms p_f on X:

$$p_f(x) := |f(x)| \quad \text{with } x \in \mathsf{X}$$

for $f \in \mathsf{X}'$.

Consider pairs of normed spaces and the corresponding sets of operators between them. Using the topology induced by seminorms we can define certain "standard" topologies on the vector spaces $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$, $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ and the dual X', thus making them locally convex topological vector spaces. One such topology (and the corresponding dual one) is already known to us, namely the topology induced by the operator norm.

Definition 2.72 Let X, Y be normed spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.
(a) Define on $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$ (respectively $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$) the following operator topologies.

(i) The topology induced by the family of seminorms $p_{x,f}$:

$$p_{x,f}(T) := |f(T(x))| \quad \text{with } T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})(\mathfrak{B}(\mathsf{X}, \mathsf{Y})),$$

for given $x \in \mathsf{X}$ and $f \in \mathsf{Y}'$, is called **weak topology** on $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$ ($\mathfrak{B}(\mathsf{X}, \mathsf{Y})$);

(ii) The topology induced by the seminorms p_x:

$$p_x(T) := ||T(x)||_{\mathsf{Y}} \quad \text{with } T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})(\mathfrak{B}(\mathsf{X}, \mathsf{Y})),$$

for given $x \in \mathsf{X}$, is the **strong topology** on $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$ ($\mathfrak{B}(\mathsf{X}, \mathsf{Y})$);

(iii) The topology induced on $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ by the operator norm (2.12) is the **uniform topology** on $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$.

(b) In case $\mathsf{Y} = \mathbb{K}$ (we are talking about X') the uniform topology of (iii) goes under the name of **(dual) strong topology** of X', and the topologies of (i) and (ii), now coinciding, are called ***-weak topology** of X'. The *-weak topology on X' is thus induced by the seminorms p_x^*:

$$p_x^*(f) := |f(x)| \quad \text{with } f \in \mathsf{X}'$$

for a given $x \in \mathsf{X}$.

Remarks 2.73 **(1)** It is not hard to see that the open sets, in a normed space, of the weak topology are also open for the standard topology, not vice versa. Likewise, in $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$, open sets in the weak topology are open for the strong topology but not conversely. We can rephrase this better by saying that *the standard topology on* X *and the strong topology on* $\mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *are* **finer** *than the corresponding weak topologies*.

In the same way, when talking of operator spaces it is not hard to show that *the uniform topology is finer than the strong topology*.

For dual spaces an analogous property obviously holds: *the strong topology is finer than the* **-weak topology*.

(2) Proposition 2.70 has a number of immediate consequences:

Proposition 2.74 *Take* $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ *with* X *normed. Then* $x_n \to x \in \mathsf{X}$*,* $n \to +\infty$*, in the weak topology if and only if:*

$$f(x_n) \to f(x)\,, \quad as\, n \to +\infty, \, for\ any\ f \in \mathsf{X}'.$$

Proposition 2.75 *If* $\{T_n\}_{n\in\mathbb{N}} \subset \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *(o* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$*) and* $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *(resp.* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$*), then* $T_n \to T$*,* $n \to +\infty$*, in weak topology if and only if:*

$$f(T_n(x)) \to f(T(x))\,, \quad as\, n \to +\infty, \, for\ any\ x \in \mathsf{X},\ f \in \mathsf{Y}'.$$

Proposition 2.76 $T_n \to T$*,* $n \to +\infty$*, in the strong topology if and only if:*

$$||T_n(x) - T(x)||_{\mathsf{Y}} \to 0\,, \quad as\, n \to +\infty, \ for\ any\ x \in \mathsf{X}.$$

Now it is clear that:

(a) *Convergence of a sequence in a normed space* X *in the standard sense (for the norm topology) implies weak convergence (convergence in the weak topology).*

(b) *Uniform convergence of a sequence of operators in* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *(in the uniform topology) implies strong convergence (for the strong topology).*

(c) *Strong convergence of a sequence of operators in* $\mathfrak{L}(\mathsf{X}, , \mathsf{Y})$ or $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *implies weak convergence*.

(3) Proposition 2.70 also gives:

Proposition 2.77 *Let* $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{X}'$ *be a sequence, and take a functional* $f \in \mathsf{X}'$*. Then* $f_n \to f$*,* $n \to +\infty$*, in the* **-weak topology if and only if:*

$$f_n(x) \to f(x)\,, \quad \textit{as}\, n \to +\infty \textit{ for any chosen } x \in \mathsf{X}.$$

Now we also know that *the strong convergence of a sequence of functionals of* X' *(for the dual strong topology) implies* **-weak convergence*.

(4) We can put on X' yet a further weak topology, by viewing X' as acted upon by $(\mathsf{X}')'$. The seminorms inducing the topology are

$$p_s(f) := |s(f)|$$

for any $s \in (\mathsf{X}')'$. If X is not reflexive, this weak topology does *not* coincide, in general, with the *-weak topology seen above, because X is identified with a proper subspace in $(\mathsf{X}')'$, and so the seminorms of the *-weak topology are fewer than the weak topology ones. The weak topology is finer than the *-weak one: a weakly open set is *-weakly open, but the converse may not hold. Analogously, weak convergence of sequences in X' implies *-weak convergence, not the other way around. ■

Notation 2.78 To distinguish strong limits from weak limits in operator spaces, it is customary to use these special symbols:

$$T = s\text{-}\lim T_n$$

means T is the limit of the sequence of operators $\{T_n\}_{n\in\mathbb{N}}$ in the strong topology; the same notation goes if the operators are functionals and the topology is the dual strong one. Similarly,

$$T = w\text{-}\lim T_n$$

denotes the limit in the weak topology of the sequence of operators $\{T_n\}_{n\in\mathbb{N}}$, and one writes

$$f = w^*\text{-}\lim f_n$$

if f is the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ in the *-weak topology. ■

All the theory learnt so far eventually enables us to prove the last corollary to Banach–Steinhaus. If X is normed we know X' is complete in the strong topology, see Theorem 2.44(c)(ii). We can also prove completeness, as explained below, for the *-weak topology too, as long as X is a Banach space.

Corollary 2.79 (to the Banach–Steinhaus theorem) *If* X *is a Banach space over* $\mathbb{K} = \mathbb{C}$*, or* $\mathbb{R}$*, then* X' *is* **-weak complete: if* $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{X}'$ *is such that* $\{f_n(x)\}_{n\in\mathbb{N}}$ *is a Cauchy sequence for any* $x \in \mathsf{X}$*, then there exists* $f = w^*\text{-}\lim f_n \in \mathsf{X}'$.

Proof The field over which X is defined is complete by assumption, so for any $x \in \mathsf{X}$ there is $f(x) \in \mathbb{K}$ with $f_n(x) \to f(x)$. Immediately $f : \mathsf{X} \ni x \mapsto f(x)$ defines a linear functional. To end the proof we have to prove f is continuous. For any $x \in \mathsf{X}$ the sequence $f_n(x)$ is bounded (as Cauchy), so Banach–Steinhaus implies $|f_n(x)| \leq K < +\infty$ for all $x \in \mathsf{X}$ with $||x|| \leq 1$. Taking the limit as $n \to +\infty$ gives

$|f(x)| \leq K$ if $||x|| = 1$, hence $||f|| \leq K < +\infty$. Therefore, by Theorem 2.43 f is continuous. □

As last topic of this section, related to the topological facts just seen, we state and prove a useful technical tool, the *theorem of Banach–Alaoglu*: according to it, the unit ball in X', defined via the natural norm of X', is compact (Definition 1.19) in the *-weak topology of X'.

Theorem 2.80 (Banach–Alaoglu) *Let X be a normed space over $\mathbb{C}$. The closed unit ball $B := \{f \in X' \mid ||f|| \leq 1\}$ in the dual X' is compact in the *-weak topology.*

Proof For any $x \in X$ define $B_x := \{c \in \mathbb{C} \mid |c| \leq ||x||\} \subset \mathbb{C}$. As B_x is obviously compact, Tychonoff's Theorem 1.25 forces $P := \times_{x\in X} B_x$ to be compact in the product topology. A point p in P is, for each $x \in X$, just a complex number $p(x)$ with $|p(x)| \leq ||x||$. Elements in P are therefore functions $p : X \to \mathbb{C}$ (not necessarily linear!) such that $|p(x)| \leq ||x||$ for any $x \in X$. By construction $B \subset P$, and the topology induced by P on B is precisely the *-weak topology, as the definitions confirm. To finish the proof we need to prove B is closed, because closed subsets in a compact space are compact. Suppose, then, $B \ni p_n \to p \in P$ as $n \to +\infty$, in the topology of P. Since $|p(x)| \leq ||x||$, to prove that $p \in B$ it suffices to show that p is linear. This is evident by arguing pointwise: if $a, b \in \mathbb{C}$ and $x, y \in X$, then

$$p(ax + by) = \lim_{n\to+\infty} p_n(ax + by) = a \lim_{n\to+\infty} p_n(x) + b \lim_{n\to+\infty} p_n(y) = ap(x) + bp(y) ,$$

and the proof is concluded. □

We will see in Chap. 4 that B is never compact for the natural norm of X' if the space X' is infinite-dimensional. The same holds for any infinite-dimensional normed space.

2.3.4 Excursus: The Theorem of Krein–Milman, Locally Convex Metrisable Spaces and Fréchet Spaces

With this part we take a short break to digress on important properties of locally convex spaces in relationship to the issue of metrisability.

Let X be a locally convex space. In general, the topology induced by a seminorm or a family of seminorms $\mathscr{P} = \{p_i\}_{i\in I}$ on X will not be Hausdorff. It is easy to see the Hausdorff property holds if and only if $\cap_{i\in I} p_i^{-1}(0)$ is the null vector in X. This happens in particular if at least one p_i is a norm.

Locally convex Hausdorff spaces have this very relevant feature: not only extremal elements always exist in convex and compact subsets, but a convex subset is actually characterised by its extremal points. This is the content of the known *Krein–Milman theorem*, which we only state.

Theorem 2.81 (Krein–Milman) *Let* X *be a locally convex Hausdorff space and* $K \subset \mathsf{X}$ *a compact convex set. Then*
(a) *the set* E_K *of extremal elements of* K *is not empty.*
(b) $K = \overline{co(E_K)}$*, where the bar denotes the closure in the ambient topology of* X.

And now to metrisable spaces. Let us recall a notion that should be familiar from basic courses.

Definition 2.82 A **metric space** is a set M equipped with a function $d : \mathsf{M} \times \mathsf{M} \to \mathbb{R}$, called **distance** or **metric**, such that, for every $x, y, z \in \mathsf{M}$:
D1. $d(x, y) = d(y, x)$,
D2. $d(x, y) \geq 0$, and $d(x, y) = 0 \ \Leftrightarrow \ x = y$,
D3. $d(x, z) \leq d(x, y) + d(y, x)$.

Remarks 2.83 **(1)** Property **D1** is known as *symmetry* of the metric, **D2** is called *positive definiteness* and **D3** is the *triangle inequality*.
(2) Any normed space $(\mathsf{X}, || \ ||)$ (hence also $\mathbb{R}^n$ and $\mathbb{C}^n$) admits a natural metric structure (X, d) by setting $d(x, x') := ||x - x'||$, $x, x' \in \mathsf{X}$. Then clearly

$$d(x + z, y + z) = d(x, y) \text{ for any } x, y, z \in \mathsf{X} \,,$$

and the distance d is **translation-invariant**. ■

Generally speaking the structure of a metric space is much simpler than that of a normed space, because the former lacks the vector space operations. We have, nevertheless, the following notion, in complete analogy to normed spaces.

Definition 2.84 Given a metric space (M, d), an **open (metric) ball** centred at x of radius $r > 0$ is the set:

$$B_\delta(x) := \{y \in \mathsf{M} \mid d(x, y) < \delta\} \,. \tag{2.26}$$

Like normed spaces, metric spaces have a *natural topology* whose open sets are the empty set $\emptyset$ and the unions of *open metric balls* with arbitrary centres and radii.

Definition 2.85 Let (M, d) be a metric space.
(a) $A \subset \mathsf{M}$ is **open** if $A = \emptyset$ or A is the union of open balls.
(b) The **metric topology** of M is the norm topology of the open sets of (a).

Remarks 2.86 **(1)** Exactly as for normed spaces, by checking the axioms we see that the metric topology is an honest topology, with *basis* given by open metric balls. The metric topology is trivially *Hausdorff*, as in normed spaces.
(2) If the metric space (X, d) is separable, i.e. it has a dense countable subset $S \subset \mathsf{X}$, then it is **second countable**: it has a countable basis $\mathscr{B}$ for the topology. The latter is the family of open balls centred on S with rational radii. One can prove the converse holds too [KoFo99]:

Proposition 2.87 *A metric space is second countable if and only if it is separable.*

(3) In a normed space $(\mathsf{X}, || \ ||)$ the open balls defined by $|| \ ||$ coincide with the open balls of the norm distance $d(x, x') := ||x - x'||$. Thus the two topologies of X, viewed either as a normed space or as metric space, coincide.
(4) The previous remark applies in particular to $\mathbb{R}^n$ and $\mathbb{C}^n$, which are both metric spaces if we use the **Euclidean** or **standard distance**:

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} \,.$$

As observed above, the balls defined by the standard distance on $\mathbb{R}^n$ and $\mathbb{C}^n$ are precisely those associated to the standard norm (1.2) generating the standard topology. Therefore the topology defined by the Euclidean distance on $\mathbb{R}^n$ and $\mathbb{C}^n$ is just the standard topology.
(5) The metric spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete, for they are complete as normed spaces and the metric is the norm distance. ∎

Like normed spaces, metric spaces too admit a characterisation of continuity equivalent to 1.16.

Definition 2.88 Given metric spaces $(\mathsf{M}, d_\mathsf{M})$, $(\mathsf{N}, d_\mathsf{N})$, a map $f : \mathsf{M} \to \mathsf{N}$ is **continuous at** $x_0 \in \mathsf{M}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d_\mathsf{N}(f(x), f(x_0)) < \varepsilon$ whenever $d_\mathsf{M}(x, x_0) < \delta$.

A function $f : \mathsf{M} \to \mathsf{N}$ is **continuous** if it is continuous at every point in M.

The concept of a convergent sequence (Definition 1.13) specialises in a metric space, as it did in a normed space.

Definition 2.89 In a metric space (M, d) a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{M}$ **converges** to a point $x \in \mathsf{M}$, the **limit of the sequence**:

$$x_n \to x \quad \text{as} \quad n \to +\infty \quad \text{or} \quad \lim_{n\to+\infty} x_n = x \,,$$

if, for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{R}$ such that $d(x_n, x) < \varepsilon$ for $n > N_\varepsilon$, i.e.

$$\lim_{n\to+\infty} d(x_n, x) = 0 \,.$$

It turns out, here as well, that convergent sequences in the metric topology are Cauchy sequences (see below), but not conversely.

Definition 2.90 Let (M, d) be a metric space.
(a) A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{M}$ is a **Cauchy sequence** if for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{R}$ such that $d(x_n, x_m) < \varepsilon$ when $n, m > N_\varepsilon$.
(a) (M, d) is **complete** if every Cauchy sequence converges somewhere in the space.

A technically relevant problem is to tell whether a topological space, especially a locally convex space, admits a distance function whose metric topology coincides with the pre-existing one (note that in general distances do not exist if the topology is induced by seminorms). When that happens the space is called **metrisable**.

Going back to topological vector spaces, one can prove that any locally convex space $(\mathsf{X}, \mathscr{P})$ satisfying:

(a) $\mathscr{P} = \{p_n\}_{n=1,2,\ldots}$, i.e. $\mathscr{P}$ is *countable*,

(b) $\cap_{n=1,2,\ldots} p_n^{-1}(0) = \{\mathbf{0}\}$

is not just Hausdorff but even *metrisable*: the seminorm topology coincides with the metric topology of (X, d), provided we pick $d : \mathsf{X} \times \mathsf{X} \to \mathbb{R}_+$ suitably. One such choice is:

$$d(x, y) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

(which is invariant under translations). This is not the only possible distance that recovers the seminorm topology of X. Multiplying d by a given positive constant, for instance, will give a distance yielding the same topology as d.

A **Fréchet space** is a locally convex space X whose topology is Hausdorff, induced by a *finite or countable* number of seminorms, and *complete* as a metric space (X, d). A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ is Cauchy for a distance d in a locally convex metrisable space X if and only if it is Cauchy for every seminorm p generating the topology: for every $\varepsilon > 0$ there is $N_\varepsilon^{(p)} \in \mathbb{R}$ such that $p(x_n - x_m) < \varepsilon$ whenever $n, m > N_\varepsilon^{(p)}$. Consequently, completeness does not actually depend on the distance used to generate the locally convex topology.

Fréchet spaces, which we will not treat in this book, are of the highest interest in theoretical and mathematical physics as far as quantum field theories are concerned. Banach spaces are simple instances of Fréchet spaces, of course.

Example 2.91 A good example of a Fréchet space is the *Schwartz space*. To define it we need some notation, which will come in handy at the end of Chap. 3 as well. Points in $\mathbb{R}^n$ will be denoted by letters and their components by subscripts, as in $x = (x_1, \ldots, x_n)$.

A **multi-index** is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i = 0, 1, 2, \ldots$, and $|\alpha|$ is conventionally the sum $|\alpha| := \sum_{i=1}^n \alpha_i$. Moreover,

$$\partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \, .$$

Let $C^\infty(\mathbb{R}^n)$ denote the complex vector space of smooth complex functions on $\mathbb{R}^n$ (differentiable with continuity infinitely many times). The **Schwartz space** $\mathscr{S}(\mathbb{R}^n)$, seen as complex vector space, is the subspace in $C^\infty(\mathbb{R}^n)$ of functions f that *vanish at infinity, together with every derivative, faster than any inverse power of* $|x| := \sqrt{\sum_{i=1}^n x_i^2}$. Define

$$p_N(f) := \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |(\partial_x^\alpha f)(x)| < +\infty \quad N = 0, 1, 2, \ldots$$

The maps $p_N : \mathscr{S}(\mathbb{R}^n) \to \mathbb{R}_+$ are seminorms, and clearly satisfy $\cap_{N\in\mathbb{N}} p_N^{-1}(0) = \{\mathbf{0}\}$ because $p_0 = ||\ ||_\infty$ is a norm. Therefore $\mathscr{S}(\mathbb{R}^n)$, with the topology induced by the seminorms $\{p_N\}_{N\in\mathbb{N}}$, becomes a locally convex space. It is easy to show that $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space [Rud91]. The points in the dual $\mathscr{S}(\mathbb{R})'$, i.e. the linear functionals $\mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ that are continuous for the topology generated by the seminorms $\{p_N\}_{N\in\mathbb{N}}$, are the famous Schwartz distributions. ■

2.3.5 Baire's Category Theorem and Its Consequences: The Open Mapping Theorem and the Inverse Operator Theorem

We wish to discuss a general theorem about Banach spaces, the open mapping theorem, which counts among its consequences the continuity of inverse operators.

To prove these facts we shall introduce as little as possible on *Baire spaces*.

Definition 2.92 Let $(\mathsf{X}, \mathscr{T})$ be a topological space and $S \subset \mathsf{X}$ a subset.
(a) The **interior** $Int(S)$ of S is the set:

$$Int(S) := \{x \in \mathsf{X} \mid \exists A \subset \mathsf{X}\,,\, A \text{ open and } x \in A \subset S\}\,.$$

(b) S is **nowhere dense** if $Int(\overline{S}) = \emptyset$.
(c) S is a set of the **first category**, or a **meagre set**, if it is the countable union of nowhere dense sets.
(d) S is a set of the **second category**, or **non-meagre**, if it is not of the first category.

The following are immediate to prove.
(1) Countable unions of sets of the first category are of the first category.
(2) If $h : \mathsf{X} \to \mathsf{X}'$ is a homeomorphism, $S \subset \mathsf{X}$ is of the first/second category if and only if $h(S)$ is of the first/second category respectively.
(3) If $A \subset B \subset \mathsf{X}$ and B is of the first category in X, then A is of the first category.
(4) If $B \subset \mathsf{X}$ is closed and $Int(B) = \emptyset$, then B is of the first category in X.
We have the following important result.

Theorem 2.93 (Baire's category theorem) *Let* (X, d) *be a complete metric space.*
(a) *If* $\{U_n\}_{n\in\mathbb{N}}$ *is a countable family of open dense subsets in* X*, then* $\cap_{n\in\mathbb{N}} U_n$ *is dense in* X.
(b) X *is of the second category.*

Proof (a) Let $A \subset \mathsf{X}$ be open. If $U_0 \cap A = \emptyset$, then $z \in A$ admits an open neighbourhood disjoint from U_0, and hence not dense in X. Therefore $U_0 \cap A$ is open (intersection of open sets) and non-empty. Then there is an open metric ball $B_{r_0}(x_0)$

of radius $r_0 > 0$ and centre $x_0 \in \mathsf{X}$ (2.26) such that $\overline{B_{r_0}(x_0)} \subset U_0 \cap A$. We may repeat the procedure with $B_{r_0}(x_0)$ replacing A, U_1 replacing U_0, to find an open ball $B_{r_1}(x_1)$ with $\overline{B_{r_1}(x_1)} \subset U_1 \cap B_{r_0}(x_0)$. Iterating, we construct a countable collection of open balls $B_{r_n}(x_n)$ with $0 < r_n < 1/n$, such that $\overline{B_{r_n}(x_n)} \subset U_n \cap B_{r_{n+1}}(x_{n-1})$. Since $x_n \in \overline{B_{r_m}(x_m)}$ for $n \geq m$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. And since X is complete, $x_n \to x \in \mathsf{X}$ as $n \to +\infty$. By construction $x \in \overline{B_{r_{n-1}}(x_{n-1})} \subset B_{r_n}(x_n) \subset \cdots \subset U_0 \cap A \subset A$ for any $n \in \mathbb{N}$. Hence $x \in A \cap U_n$ for every $n \in \mathbb{N}$, and so $(\cap_{n\in\mathbb{N}} U_n) \cap A \neq \emptyset$ for every open subset $A \subset \mathsf{X}$. This implies $\cap_{n\in\mathbb{N}} U_n$ is dense in X, for it meets every open neighbourhood of any point in X.

(b) Assume $\{E_k\}_{k\in\mathbb{N}}$ is a collection of nowhere dense sets $E_k \subset \mathsf{X}$. If V_k is the complement of $\overline{E_k}$, it is open (its complement is closed) and dense in X (it is open and the complement's interior is empty). Part (a) then tells $\cap_{k\in\mathbb{N}} V_n \neq \emptyset$, so $\mathsf{X} \neq \cup_{k\in\mathbb{N}} \overline{E_k}$ by taking complements. A fortiori then $\mathsf{X} \neq \cup_{k\in\mathbb{N}} E_k$, so X is not of the first category. □

Remarks 2.94 **(1)** Baire's category theorem states, among other things, that *any collection, finite or countable, of dense open sets in a complete metric space always has non-empty (dense) intersection*. In the finite case it suffices to adapt the statement to $U_n = U_m$ for some $N \leq n, m$.
(2) Baire's theorem holds when X is a locally compact Hausdorff space. The first part is proved in analogy to the previous situation [Rud91], the second is identical.
(3) Baire's theorem applies, obviously, to Banach spaces, using the norm distance. ■

We can pass to the *open mapping theorem*. Remember that a map $f : \mathsf{X} \to \mathsf{Y}$ between normed spaces (or topological spaces) is **open** if $f(A)$ is open in Y whenever $A \subset \mathsf{X}$ is open. As usual, $B_r^{(\mathsf{Z})}(z)$ denotes the open ball of radius r and centre z in the normed space $(\mathsf{Z}, ||\ ||_{\mathsf{Z}})$.

Theorem 2.95 (Banach–Schauder's open mapping theorem) *Let* X *and* Y *be Banach spaces over* $\mathbb{C}$ *or* $\mathbb{R}$. *If the operator* $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *is surjective, then*

$$T(B_1^{(\mathsf{X})}(\mathbf{0})) \supset B_\delta^{(\mathsf{Y})}(\mathbf{0}) \quad \textit{for } \delta > 0 \textit{ small enough}\,. \tag{2.27}$$

As a consequence, T is an open map.

Proof Define in X the open ball $B_n := B_{2^{-n}}^{(\mathsf{X})}(\mathbf{0})$ at the origin, of radius 2^{-n}. We will show there is an open neighbourhood W_0 of the origin $\mathbf{0} \in \mathsf{Y}$ with:

$$W_0 \subset \overline{T(B_1)} \subset T(B_0)\,, \tag{2.28}$$

which will imply (2.27).

To prove (2.28), note $B_1 \supset B_2 - B_2$ (from now on we use Notations 2.67), so

$$T(B_1) \supset T(B_2) - T(B_2)$$

and $\overline{T(B_1)} \supset \overline{T(B_2) - T(B_2)}$. On the other hand, since $\overline{A+B} \supset \overline{A} + \overline{B}$, $A, B \subset \mathsf{Y}$ with Y normed (prove it as exercise), we have:

$$\overline{T(B_1)} \supset \overline{T(B_2) - T(B_2)} \supset \overline{T(B_2)} - \overline{T(B_2)} \,. \tag{2.29}$$

The first inclusion of (2.28) is therefore true if $\overline{T(B_2)}$ has non-empty interior: if $z \in Int(\overline{T(B_2)})$ then $z \in A \subset \overline{T(B_2)})$ with A open, so that $\mathbf{0} \in W_0 := A - A \subset \overline{T(B_2)} - \overline{T(B_2)} \subset \overline{T(B_1)}$ with W_0 open.

To show $Int(\overline{T(B_2)}) \neq \emptyset$, notice that the assumptions imply

$$\mathsf{Y} = T(\mathsf{X}) = \bigcup_{k=1}^{+\infty} kT(B_2) \,, \tag{2.30}$$

because B_2 is an open neighbourhood of $\mathbf{0}$. But Y is of the second category, so at least one $kT(B_2)$ is of the second category (otherwise Y would be of the first category, which is impossible by the second statement in Baire's category Theorem 2.93, for Y is complete). Since $y \mapsto ky$ is a homeomorphism of Y, $T(B_2)$ is of the second category in Y. Hence the closure of $T(B_2)$ has non-empty interior, proving one inclusion of (2.28).

For the other inclusion (the second from the left), we build a sequence of elements $y_n \in \mathsf{Y}$ inductively. Fix $y_1 \in \overline{T(B_1)}$, suppose that y_n is in $\overline{T(B_n)}$ for $n \geq 1$ and let us define y_{n+1} as follows. What was proved for $\overline{T(B_1)}$ holds for $\overline{T(B_{n+1})}$ too, so $\overline{T(B_{n+1})}$ contains an open neighbourhood of the origin. Now:

$$\left(y_n - \overline{T(B_{n+1})}\right) \cap T(B_n) \neq \emptyset \,, \tag{2.31}$$

implying there exists $x_n \in B_n$ such that:

$$T(x_n) \in y_n - \overline{T(B_{n+1})} \,. \tag{2.32}$$

Define: $y_{n+1} := y_n - Tx_n$ and note it belongs to $\overline{T(B_{n+1})}$. This is the inductive step. Since $||x_n|| < 2^{-n}$, $n = 1, 2, \ldots$, the sum $x_1 + \cdots + x_n$ gives a Cauchy sequence converging to some $x \in \mathsf{X}$ by completeness, and $||x|| < 1$. Hence $x \in B_0$. Since:

$$\sum_{n=1}^{m} Tx_n = \sum_{n=1}^{m} (y_n - y_{n+1}) = y_1 - y_{m+1} \,, \tag{2.33}$$

and because $y_{m+1} \to \mathbf{0}$ as $m \to +\infty$ (by continuity of T), we conclude $y_1 = Tx \in T(B_0)$. Now as y_1 was generic in $\overline{T(B_1)}$, that proves the second inclusion of (2.28) and ends the first part.

As for the second statement, (2.27) and the linearity of T imply that the image under T of any open ball $B_\varepsilon^{(\mathsf{X})}(x) = x + \varepsilon B_1(\mathbf{0})$, centred at any $x \in \mathsf{X}$, contains the open ball in Y centred at Tx: $B_{\delta_\varepsilon}^{(\mathsf{Y})}(\mathbf{0}) := Tx + \varepsilon B_\delta^{(\mathsf{Y})}(\mathbf{0})$ ($\delta > 0$ sufficiently small). Therefore the image under T of an open set $A = \cup_{x \in A} B_{\varepsilon_x}^{(\mathsf{X})}(x)$ is open in Y: $T(A) = \cup_{x \in A} B_{\delta_{\varepsilon_x}}^{(\mathsf{Y})}(Tx)$. This means T is *open*. $\square$

The most important elementary corollary of this theorem is without doubt *Banach's inverse operator theorem* for Banach spaces (there is a version for complete metric vector spaces as well).

Theorem 2.96 (Banach's inverse operator theorem) *Let* X *and* Y *be Banach spaces over* $\mathbb{C}$ *or* $\mathbb{R}$, *and suppose* $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *is injective and surjective. Then*
(a) $T^{-1} : \mathsf{Y} \to \mathsf{X}$ *is bounded, i.e.* $T^{-1} \in \mathfrak{B}(\mathsf{Y}, \mathsf{X})$*;*
(b) *there exists* $c > 0$ *such that:*

$$||Tx|| \geq c||x|| \,, \quad \textit{for any}\, x \in \mathsf{X}. \tag{2.34}$$

Proof (a) That T^{-1} is linear is straightforward, for we need only prove it is continuous. As T is open, the pre-image under T^{-1} of an open set in X is open, making T^{-1} continuous. (b) Since T^{-1} is bounded, there is $K \geq 0$ with $||T^{-1}y|| \leq K||y||$, for any $y \in \mathsf{Y}$. Notice that $K > 0$, for otherwise $T^{-1} = 0$ would not be invertible. For any $x \in \mathsf{X}$ we set $y = Tx$. Substituting in $||T^{-1}y|| \leq K||y||$ gives back, for $c = 1/K$, relation (2.34). $\square$

2.3.6 The Closed Graph Theorem

Now we discuss a very useful theorem in operator theory, known as the *closed graph theorem*.

Notation 2.97 **(1)** If X is a vector space and $\emptyset \neq X_1, \ldots, X_n \subset \mathsf{X}$, then:

$$< X_1, \ldots, X_n >$$

will denote the linear **span** of the sets X_i, i.e. the vector subspace of X containing all *finite* linear combinations of elements of any X_i.
(2) Take $\emptyset \neq \mathsf{X}_1, \ldots, \mathsf{X}_n$ *subspaces* of a vector space X. Then

$$\mathsf{Y} = \mathsf{X}_1 \oplus \cdots \oplus \mathsf{X}_n$$

denotes the **direct sum** $\mathsf{Y} \subset \mathsf{X}$ of the X_i, i.e.:
(i) $\mathsf{Y} =< \mathsf{X}_1, \cdots, \mathsf{X}_n >$ (so Y is a subspace in X) and
(ii) $\mathsf{X}_i \cap \mathsf{X}_j = \{\mathbf{0}\}$ for any pair $i, j = 1, \ldots, n, i \neq j$.
As is well known, (i) and (ii) are equivalent to demanding

$$x \in \mathsf{Y} \Rightarrow x = x_1 + \cdots + x_n \quad \text{with}\, x_k \in \mathsf{Y}_k \,\text{determined}\, uniquely \,\text{by}\, x, k = 1, \ldots, n.$$

(3) If $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are *vector spaces* over the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, we may furnish $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ with the structure of a $\mathbb{K}$-vector space by:

$$\alpha(x_1, \ldots, x_n) := (\alpha x_1, \ldots, \alpha x_n) \quad \text{and} \quad (x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n)$$

for any $\alpha \in \mathbb{K}$, $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathsf{X}_1 \times \cdots \times \mathsf{X}_n$. Calling

$$\Pi_{\mathsf{X}_k} : (x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \mapsto (\mathbf{0}, \ldots, \mathbf{0}, x_k, \mathbf{0}, \ldots, \mathbf{0})$$

the kth **canonical projection**, the vector space built on $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ coincides with $Ran(\Pi_{\mathsf{X}_1}) \oplus \cdots \oplus Ran(\Pi_{\mathsf{X}_n})$. As each X_k is naturally identified with the corresponding $Ran(\Pi_{\mathsf{X}_k})$, we will write $\mathsf{X}_1 \oplus \cdots \oplus \mathsf{X}_n$ to denote the natural vector space $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ above, even when the X_k are not all contained in one common space. ■

To prove the closed graph theorem we need some preliminaries. First of all, if $(\mathsf{X}, N_{\mathsf{X}})$ and $(\mathsf{Y}, N_{\mathsf{Y}})$ are normed spaces over $\mathbb{K} = \mathbb{C}$, or $\mathbb{R}$, we can consider $\mathsf{X} \oplus \mathsf{Y}$, in the Notation 2.97(3). The space $\mathsf{X} \oplus \mathsf{Y}$ has the *product topology* of X and Y, seen in Definition 1.10. The operations of the vector space $\mathsf{X} \oplus \mathsf{Y}$ are continuous in the product topology, as one proves with ease (the proof is the same as the one used for the operations on a normed space). And the canonical projections $\Pi_{\mathsf{X}} : \mathsf{X} \oplus \mathsf{Y} \to \mathsf{X}$, $\Pi_{\mathsf{Y}} : \mathsf{X} \oplus \mathsf{Y} \to \mathsf{Y}$ are continuous in the product topology on the domain and the topologies of X and Y on the codomains, another easy fact.

The product topology of $\mathsf{X} \oplus \mathsf{Y}$ admits **compatible norms**: there exist norms on $\mathsf{X} \oplus \mathsf{Y}$ inducing the product topology. One possibility is:

$$||(x, y)|| := \max\{N_{\mathsf{X}}(x), N_{\mathsf{Y}}(y)\} \quad \text{for any } (x, y) \in \mathsf{X} \oplus \mathsf{Y}\,. \tag{2.35}$$

That this norm generates the product topology, i.e. open sets are unions of products of open balls in X and Y, is proved as follows. Take the open neighbourhood of (x_0, y_0) product of two open balls $B_\delta^{(\mathsf{X})}(x_0) \times B_\mu^{(\mathsf{Y})}(y_0)$ in X and Y respectively. The open ball in $\mathsf{X} \oplus \mathsf{Y}$

$$\{(x, y) \in \mathsf{X} \times \mathsf{Y} \mid ||(x, y) - (x_0, y_0)|| < \min\{\delta, \mu\}/2\}$$

centred at (x_0, y_0) is contained in $B_\delta^{(\mathsf{X})}(x_0) \times B_\mu^{(\mathsf{Y})}(y_0)$. *Vice versa*, the product $B_\delta^{(\mathsf{X})}(x_0) \times B_\delta^{(\mathsf{Y})}(y_0)$, to which (x_0, y_0) belongs, is contained in the open ball

$$\{(x, y) \in \mathsf{X} \times \mathsf{Y} \mid ||(x, y) - (x_0, y_0)|| < \varepsilon\}$$

centred at (x_0, y_0), if $\varepsilon > \delta$. This implies that unions of products of metric balls in X and Y are unions of metric balls for norm (2.35), and conversely too. Hence the two topologies coincide and the proof ends.

Immediately we can prove $(\mathsf{X} \oplus \mathsf{Y}, || \ ||)$ *is a Banach space if both* $(\mathsf{X}, N_\mathsf{X})$ and $(\mathsf{Y}, N_\mathsf{Y})$ *are Banach.* (By Proposition 2.105, proved later, this fact will guarantee that any norm generating the product topology makes $\mathsf{X} \oplus \mathsf{Y}$ a Banach space.) In fact let $\{(x_n, y_n)\}$ be a Cauchy sequence in $\mathsf{X} \oplus \mathsf{Y}$. Then $\{x_n\}$ and $\{y_n\}$ are both Cauchy in X and Y respectively, by the above definition of norm on $\mathsf{X} \oplus \mathsf{Y}$. Call $x \in \mathsf{X}$ and $y \in \mathsf{Y}$ the limits of those sequences, which exist for X and Y are Banach spaces. If $\varepsilon > 0$, there are positive integers N_x and N_y satisfying

$$||(x, y) - (x_n, y_n)|| < \varepsilon$$

if $n > \max\{N_x, N_y\}$. Therefore $(x_n, y_n) \to (x, y)$ as $n \to +\infty$ in the norm topology of $\mathsf{X} \oplus \mathsf{Y}$, and the latter is a Banach space.

Definition 2.98 Let X, Y be normed spaces on $\mathbb{C}$ or $\mathbb{R}$. One says $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ is **closed** if the **graph** of the operator T,

$$G(T) := \{(x, Tx) \in \mathsf{X} \oplus \mathsf{Y} \mid x \in \mathsf{X}\}\ , \ ,$$

is a closed subspace in the product topology.

Equivalently, T is closed iff for any converging sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ such that $\{Tx_n\}_{n\in\mathbb{N}}$ converges in Y, we have:

$$\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n)\ .$$

The last equivalence relies on a general fact: a set ($G(T)$ in our case) is closed if and only if it coincides with its closure, if and only if it contains its limit points. Spelling out this fact in terms of the product topology gives our proof. We are ready for the *closed graph theorem.*

Theorem 2.99 (Closed graph theorem) *Let* $(\mathsf{X}, || \ ||_\mathsf{X})$ *and* $(\mathsf{Y}, || \ ||_\mathsf{Y})$ *be Banach spaces over* $\mathbb{K} = \mathbb{C}$.

Then $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ *is bounded if and only if it is closed.*

Proof Suppose T is bounded. Then it is banally closed by the definition of closed operator. Assume conversely that T is closed. Consider the linear bijective map $M : G(T) \ni (x, Tx) \mapsto x \in \mathsf{X}$. By hypothesis $G(T)$ is a closed subspace in the Banach space $\mathsf{X} \oplus \mathsf{Y}$, hence it becomes Banach for the restricted norm $|| \ ||$ of (2.35). The latter's definition implies $||M(x, Tx)||_\mathsf{X} = ||x||_\mathsf{X} \leq ||(x, Tx)||$, so M is bounded. Banach's bounded-inverse theorem tells $M^{-1} : \mathsf{X} \to G(T) \subset \mathsf{X} \oplus \mathsf{Y}$ is bounded. As the canonical projection $\Pi_\mathsf{Y} : \mathsf{X} \oplus \mathsf{Y} \to \mathsf{Y}$ is continuous, we infer that the linear map $\Pi_\mathsf{Y} \circ M^{-1} : x \mapsto Tx$ is continuous, hence bounded. $\square$

2.4 Projectors

Using the closed graph theorem we define a class of continuous operators, called *projectors*. This notion plays the leading role in formulating QM when the normed space is a Hilbert space.

Definition 2.100 (*Projector*) Let $(\mathsf{X}, || \; ||)$ be a normed space over $\mathbb{C}$ or $\mathbb{R}$. The operator $P \in \mathfrak{B}(\mathsf{X})$ is a **projector** if it is **idempotent**, i.e.

$$PP = P\,. \tag{2.36}$$

The target $\mathsf{M} := P(\mathsf{X})$ is called **projection space** of P, and we say P **projects** onto M.

Projectors are naturally associated to a direct sum decomposition of X into a pair of closed subspaces.

Proposition 2.101 *Let $P \in \mathfrak{B}(\mathsf{X})$ be a projector onto the normed space $(\mathsf{X}, || \; ||)$.*
(a) *If $Q : \mathsf{X} \to \mathsf{X}$ is the linear map such that*

$$Q + P = I\,, \tag{2.37}$$

then Q is a projector and:

$$PQ = QP = 0\,, \tag{2.38}$$

where 0 is the null operator (transforming any vector into the null vector $\mathbf{0} \in \mathsf{X}$).
(b) *The projection spaces $\mathsf{M} := P(\mathsf{X})$ and $\mathsf{N} := Q(\mathsf{X})$ are closed subspaces satisfying:*

$$\mathsf{X} = \mathsf{M} \oplus \mathsf{N}\,. \tag{2.39}$$

Proof (a) Q is continuous as sum of continuous operators. Moreover $QQ = (I - P)(I - P) = I - 2P + PP = I - 2P + P = I - P = Q$, $\quad PQ = P(I - P) = P - PP = P - P = 0$ and $(I - P)P = P - PP = P - P = 0$.
(b) If $P(x_n) \to y$ as $n \to +\infty$, by continuity of P we have $PP(x_n) \to P(y)$. Using Eq. (2.36) we rephrase this as $P(x_n) \to P(y)$, whence $y = P(y)$ by uniqueness of the limit (X is Hausdorff). So, $y \in \overline{\mathsf{M}}$ implies $y \in \mathsf{M} (\subset \overline{\mathsf{M}})$, and $\mathsf{M} = \overline{\mathsf{M}}$ is closed. The same argument proves N is closed. That M, N are subspaces is immediate from the linearity of P and Q. If we take $x \in \mathsf{X}$, then

$$x = P(x) + Q(x)\,,$$

and $\mathsf{X} = < \mathsf{M}, \mathsf{N} >$. To finish we need to have $\mathsf{M} \cap \mathsf{N} = \{\mathbf{0}\}$. Pick $x \in \mathsf{M} \cap \mathsf{N}$. Then $x = P(x)$, so $x = Q(x)$ by (2.36) ($x \in \mathsf{M}$ implies $x = Pz$ for some $z \in \mathsf{X}$, but then $Px = PPz = Pz = x$). Using Q on $x = Px$, and recalling $x = Qx$, gives $x = Q(x) = QP(x) = \mathbf{0}$ by (2.38), i.e. $x = \mathbf{0}$. $\square$

The closed graph theorem explains that Proposition 2.101 can be reversed, provided we further suppose the ambient space is Banach.

Proposition 2.102 *Let* $(\mathsf{X}, || \ ||)$ *be a Banach space,* $\mathsf{M}, \mathsf{N} \subset \mathsf{X}$ *closed subspaces such that* $\mathsf{X} = \mathsf{M} \oplus \mathsf{N}$. *Consider the functions* $P : \mathsf{X} \to \mathsf{M}$ *and* $Q : \mathsf{X} \to \mathsf{N}$ *that map* $x \in \mathsf{X}$ *to the respective elements in* M *and* N *according to* $\mathsf{X} = \mathsf{M} \oplus \mathsf{N}$. *Then*
(a) *P and Q are projectors onto* M *and* N *respectively.*
(b) *Properties (2.37) and (2.38) hold.*

Proof By assumption $x \in \mathsf{X}$ decomposes as $x = u_\mathsf{M} + u_\mathsf{N}$ for certain $u_\mathsf{M} \in \mathsf{M}$, $u_\mathsf{N} \in \mathsf{N}$, and the sum is unique once the subspaces are fixed. Uniqueness, and the fact that M and N are closed under linear combinations, imply that $P : x \mapsto u_\mathsf{M}$ and $Q : x \mapsto u_\mathsf{N}$ are linear, $PP = P$ and $QQ = Q$. Note that $P(\mathsf{X}) = \mathsf{M}$ and $Q(\mathsf{X}) = \mathsf{N}$ by construction. Moreover (2.37) holds since $\mathsf{X} = < \mathsf{M}, \mathsf{N} >$, while (2.38) is true by $\mathsf{M} \cap \mathsf{N} = \{\mathbf{0}\}$. To finish we need to show P and Q are continuous. Let us prove P is closed, and the closed graph theorem will then force continuity. The strategy for Q is analogous. So let $\{x_n\} \subset \mathsf{X}$ be a sequence converging to $x \in \mathsf{X}$, and such that also $\{Px_n\}$ converges in X. We claim that

$$Px = \lim_{n \to +\infty} Px_n \,.$$

As N is closed,

$$\mathsf{N} \ni Qx_n = x_n - Px_n \to x - \lim_{n \to +\infty} Px_n = z \in \mathsf{N}\,.$$

So we have

$$x = \lim_{n \to +\infty} Px_n + z\,,$$

with $z \in \mathsf{N}$, but $\lim_{n \to +\infty} Px_n \in \mathsf{M}$ as well, because M is closed and $Px_n \in \mathsf{M}$ for all n. On the other hand we know that

$$x = Px + Qx.$$

Since the decomposition is unique, necessarily

$$Px = \lim_{n \to +\infty} Px_n$$

and $z = Qx$. Therefore P is closed and so continuous. □

2.5 Equivalent Norms

One interesting consequence of Banach's inverse operator theorem is a criterion to establish when two norms on a complete vector space (for both) induce the same topology. Before stating the criterion (Proposition 2.105), let us prepare the ground. The section will end with the proof that all norms on a vector space of finite dimension are equivalent, and make the space Banach.

Definition 2.103 Two norms N_1, N_2 defined on one vector space X (over $\mathbb{C}$ or $\mathbb{R}$) are **equivalent** if there are constants $c, c' > 0$ such that:

$$cN_2(x) \leq N_1(x) \leq c'N_2(x)\,, \qquad \text{for any } x \in \mathsf{X}. \tag{2.40}$$

Remarks 2.104 **(1)** Note how (2.40) is equivalent to the similar inequality obtained by swapping N_1, N_2, and writing $1/c'$, $1/c$ in place of c, c' respectively.
(2) By this observation, it is straightforward that *if a given normed space is complete, then it is complete for any equivalent norm.*
(3) Two equivalent norms on a vector space generate the same topology, as is easy to prove. The next proposition discusses the converse.
(4) Equivalent norms define an equivalence relation on the space of norms on a given vector space. The proof is immediate from the definitions. ■

Proposition 2.105 *Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$. The norms N_1 and N_2 on X are equivalent if and only if the identity map $I : (\mathsf{X}, N_2) \ni x \mapsto x \in (\mathsf{X}, N_1)$ is a homeomorphism (which is to say, the metric topologies generated by the norms are the same).*

Proof It suffices to prove the 'if' part, for the sufficient condition is trivial by definition of equivalent norms. If I is a homeomorphism it is continuous at the origin, and in particular the unit open ball (for N_1) centred at $\mathbf{0}$ must contain an entire open ball (for N_2) at $\mathbf{0}$ of small enough radius $\delta > 0$. That is to say, $N_2(x) \leq \delta \Rightarrow N_1(x) < 1$. In particular, for $x \neq \mathbf{0}$, $N_2(\delta x/N_2(x)) \leq \delta$, so $N_1(\delta x/N_2(x)) < 1$, i.e. $\delta N_1(x) \leq N_2(x)$. For $x = \mathbf{0}$ the equality is trivial. Hence we have proved that there is $c' = 1/\delta > 0$ for which $N_1(x) \leq c'N_2(x)$, for any $x \in \mathsf{X}$. The other half of (2.40) is similar if we swap spaces. □

Proposition 2.106 *Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$ and suppose the norms N_1, N_2 both make X Banach. If there is a constant $c > 0$ such that:*

$$cN_2(x) \leq N_1(x)$$

for any $x \in \mathsf{X}$, the norms are equivalent.

Proof Consider the identity $I : x \mapsto x$, a linear and continuous map when thought of as $I : (\mathsf{X}, N_1) \to (\mathsf{X}, N_2)$, since $N_2(x) \leq (1/c)N_1(x)$ for all $x \in \mathsf{X}$. Banach's inverse function theorem, part (b), guarantees the existence of $c' > 0$ such that $N_1(x) \leq c'N_2(x)$ for all $x \in \mathsf{X}$. Then N_1 and N_2 satisfy (2.40). □

The important, and final, proposition in this section holds also on real vector spaces (writing $\mathbb{R}$ instead of $\mathbb{C}$ in the proof).

Proposition 2.107 *Let* X *be a* $\mathbb{C}$*-vector space of finite dimension. Then all norms are equivalent, and any one defines a Banach structure on* X.

Proof We can simply study $\mathbb{C}^n$, given that any complex vector space of finite dimension n is isomorphic to $\mathbb{C}^n$. Owing to remarks (2) and (4) above, it is sufficient to prove that any norm on $\mathbb{C}^n$ is equivalent to the standard norm. Keep in mind the fact, known from elementary analysis, that the standard $\mathbb{C}^n$ is complete, so any other equivalent norm makes it a Banach space, by Remark 2.104(2).
Let N be a norm on $\mathbb{C}^n$ and $e_1, \cdots, e_n$ the canonical basis. If $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i$ are generic points in $\mathbb{C}^n$, from properties **N0**, **N2** and **N1** (see the definition of norm) we have

$$0 \le N(x-y) \le \sum_{i=1}^{n} |x_i - y_i| N(e_i) \le Q \sum_{i=1}^{n} |x_i - y_i| \, ,$$

where $Q := \sum_i N(e_i)$. At the same time, trivially, if $||\cdot||$ is the standard norm then

$$|x_j - y_j| \le \max\{|x_i - y_i| \mid i = 1, 2, \cdots, n\} \le \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = ||x - y|| \, ,$$

whence

$$0 \le N(x-y) \le nQ||x-y|| \, .$$

This shows N is continuous in the standard topology. If $\mathbb{S} := \{x \in \mathbb{C}^n \mid ||x|| = 1\}$, and N' is a second norm on $\mathbb{C}^n$ that is continuous in the standard topology, then the map

$$\mathbb{S} \ni x \mapsto f(x) := \frac{N(x)}{N'(x)}$$

is continuous, being a quotient of continuous maps with non-zero denominator. But $\mathbb{S}$ is compact in the standard topology, so f has a minimum m and a maximum M. In particular, $M \ge m > 0$ because N, N' are strictly positive on $\mathbb{S}$ and m, M are attained at suitable points x_m, x_M in $\mathbb{S}$. By construction

$$mN'(x) \le N(x) \le MN'(x) \, , \quad \text{for any } x \in \mathbb{S} \, .$$

We claim that this inequality actually holds for any $x \in \mathbb{C}^n$. Write $x = \lambda x_0$ with $x_0 \in \mathbb{S}$ and $\lambda \ge 0$. Multiplying by $\lambda \ge 0$ the inequality, evaluating it at x_0 and using property **N1** gives precisely:

$$mN'(x) \le N(x) \le MN'(x) \quad \text{for any } x \in \mathbb{C}^n .$$

Now by choosing $N' := ||\cdot||$ we conclude that any norm on $\mathbb{C}^n$ is equivalent to the standard one. $\square$

2.6 The Fixed-Point Theorem and Applications

In this last section of the chapter, we present an elementary theorem with crucial consequences in analysis, especially in the theory of differential equations: the *fixed-point theorem*. We will state it for complete metric spaces and then examine it on Banach spaces.

2.6.1 The Fixed-Point Theorem of Banach–Caccioppoli

Let us start with a definition about metric spaces, cf. Definition 2.82.

Definition 2.108 Let (M, d) be a metric space. A map $G : \mathsf{M} \to \mathsf{M}$ is a **contraction (map)** in case there exists a real number $\lambda \in [0, 1)$ for which:

$$d(G(x), G(y)) \leq \lambda d(x, y) \quad \text{for any } x, y \in \mathsf{M}\,. \tag{2.41}$$

Remember that normed spaces $(\mathsf{X}, ||\ ||)$ are metric spaces once we specify the norm distance $d(x, y) := ||x - y||$ (and the metric topology induced by d coincides with the topology induced by $||\ ||$, as we saw in Sect. 2.3.4). Hence we can specialise the definition to normed spaces.

Definition 2.109 Let $(\mathsf{Y}, ||\ ||)$ be a normed space and $\mathsf{X} \subset \mathsf{Y}$ a subset (possibly the whole Y). A function $G : \mathsf{X} \to \mathsf{X}$ is a **contraction** if there exists a real number $\lambda \in [0, 1)$ for which:

$$||G(x) - G(y)|| \leq \lambda ||x - y|| \quad \text{for } x, y \in \mathsf{X}\,. \tag{2.42}$$

Remarks 2.110 **(1)** Note that the value $\lambda = 1$ is explicitly *excluded*.
(2) The demand of (2.41) implies immediately that any contraction is continuous in the metric topology of (M, d).

Similarly, (2.42) tells that any contraction on the set X is continuous in the induced norm topology of $(\mathsf{Y}, ||\ ||)$.
(3) We stress that, in Definition 2.109, (a) the function G is *not* requested to be linear, and (b) X is not necessarily a subspace of Y, but only a subset. Linear structures play no interesting role. $\blacksquare$

Let us state and prove the *fixed-point theorem* (of Banach and Caccioppoli) for metric spaces.

Theorem 2.111 (Fixed-point theorem for metric spaces) *Let $G : \mathsf{M} \to \mathsf{M}$ be a contraction on the complete metric space (M, d). Then there exists a unique element $z \in \mathsf{M}$, called* **fixed point**, *such that*

$$G(z) = z\,. \tag{2.43}$$

If $G : \mathsf{M} \to \mathsf{M}$ is not a contraction, but the n-fold composite $G^n = G \circ \cdots \circ G$ is a contraction for a given $n = 1, 2, \ldots$, then G admits a unique fixed point.

Proof Let us begin by proving the existence of z. Consider, for $x_0 \in \mathsf{M}$ arbitrary, the sequence defined recursively by $x_{n+1} = G(x_n)$. We claim this is a Cauchy sequence, and that its limit is a fixed point of G. Without loss of generality we may suppose $m \geq n$.

If $m = n$, trivially $d(x_m, x_n) = 0$. If $m > n$ we employ the triangle inequality repeatedly to get:

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)\,. \tag{2.44}$$

The generic summand on the right equals

$$\begin{aligned} d(x_{p+1}, x_p) = d(G(x_p), G(x_{p-1})) \leq \lambda d(x_p, x_{p-1}) &= \lambda d(G(x_{p-1}), G(x_{p-2})) \\ &\leq \lambda^2 d(x_{p-1}, x_{p-2}) \\ &\leq \cdots \leq \lambda^p d(x_1, x_0)\,. \end{aligned}$$

Hence, for $p = 1, 2, \ldots$ we have $d(x_{p+1}, x_p) \leq \lambda^p d(x_1, x_0)$. Inserting the latter inequality in the right-hand side of (2.44) produces the estimate:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_1, x_0)\sum_{p=n}^{m-1} \lambda^p = d(x_1, x_0)\lambda^n \sum_{p=0}^{m-n-1} \lambda^p \\ &\leq \lambda^n d(x_1, x_0)\sum_{p=0}^{+\infty} \lambda^p \leq d(x_1, x_0)\frac{\lambda^n}{1-\lambda} \end{aligned}$$

where we used the fact that $\sum_{p=0}^{+\infty} \lambda^p = (1-\lambda)^{-1}$ if $0 \leq \lambda < 1$. In conclusion:

$$d(x_m, x_n) \leq d(x_1, x_0)\frac{\lambda^n}{1-\lambda}\,. \tag{2.45}$$

For us $|\lambda| < 1$, so $d(x_1, x_0)\lambda^n/(1-\lambda) \to 0$ as $n \to +\infty$. Hence $d(x_m, x_n)$ can be rendered as small as we like by picking the minimum between m and n to be arbitrarily large. Therefore the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. But M is complete, so

$\lim_{n\to+\infty} x_n = x \in \mathsf{M}$ for a certain x. Moreover, G is a contraction, so continuous, and

$$G(x) = G\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} G(x_n) = \lim_{n\to+\infty} x_{n+1} = x\ ,$$

as claimed.

Let us see to uniqueness. For that, assume x and x' satisfy $G(x) = x$ and $G(x') = x'$. Then

$$d(x, x') = d(G(x), G(x')) \le \lambda d(x, x')\ .$$

If $d(x, x') \neq 0$, dividing by $d(x, x')$ would give $1 \le \lambda$, absurd by assumption. Hence $d(x, x') = 0$, so $x = x'$, because d is positive definite.

Now let us prove the theorem in case $B := G^n$ is a contraction. By the previous part B has a unique fixed point z. Clearly, if G admits a fixed point, this must be z. There remains to show that z is fixed under G as well. As B is a contraction, the sequence $B(z_0), B^2(z_0), B^3(z_0), \ldots$ converges to z, irrespective of $z_0 \in \mathsf{M}$, as we saw earlier in the proof. Therefore

$$G(z) = G(B^k(z)) = B^k(G(z)) = B^k(z_0) \to z \quad \text{as}\, k \to +\infty,$$

and $G(z) = z$. □

Moving to normed spaces, the theorem has as corollary the next fact, obtained using the norm distance $d(x, y) := ||x - y||$.

Theorem 2.112 (Fixed-point theorem for normed spaces) *Let $G : \mathsf{X} \to \mathsf{X}$ be a contraction on the closed set $\mathsf{X} \subset \mathsf{B}$, with B a Banach space over $\mathbb{R}$ or $\mathbb{C}$. Then there exists a unique element $z \in \mathsf{X}$, called* **fixed point***:*

$$G(z) = z\ . \tag{2.46}$$

If $G : \mathsf{X} \to \mathsf{X}$ is not a contraction, but the n-fold composite $G^n = G \circ \cdots \circ G$ is a contraction for some $n = 1, 2, \ldots$, then G admits a unique fixed point.

Proof Define $\mathsf{M} := \mathsf{X}$ and $d(x, y) := ||x - y||$, $x, y \in \mathsf{X}$. Thus X is a metric space. Actually (X, d) is complete. In fact, a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ for d is Cauchy for $||\ ||$ too, as is easy to verify. As $(\mathsf{B}, ||\ ||)$ is complete, the limit $x \in \mathsf{B}$ of $\{x_n\}_{n\in\mathbb{N}}$ exists. And since X is closed inside B, the point x belongs to X. Hence any Cauchy sequence of (X, d) converges in X, making (X, d) complete. At this point we invoke the previous theorem for the metric space (X, d) and conclude. □

The significance of the fixed-point theorem, by the way, depends on its role in proving existence and uniqueness theorems for equations of all sorts, especially integral and differential equations. The gist is to show that the equation to which we seek a solution z can be written as a fixed-point relation $G(z) = z$ in a suitable Banach space (or complete metric space). Example (1) below is a relatively simple case (G is linear), while the ensuing (2) typically pertains nonlinear contractions.

Example 2.113 Let us present two elementary instances of how the fixed-point theory is used. A more important situation will be treated in the following section.

(1) The **homogeneous Volterra equation on** $C([a,b])$ in the unknown $f \in C([a,b])$ reads:

$$f(x) = \int_a^x K(x,y) f(y) dy \,, \tag{2.47}$$

where K is a continuous function bounded by $M \geq 0$. We equip the Banach space $C([a,b])$ with the sup norm $|| \;\; ||_\infty$. The equation may be written in the form $f = Af$, where $A : C([a,b]) \to C([a,b])$ is the integral operator determined by the integral kernel K:

$$(Af)(x) := \int_a^x K(x,y) f(y) dy \,, \quad f \in C([a,b]). \tag{2.48}$$

If a solution exists, then clearly it is the fixed point of A. Not only this: the solution is also fixed under every operator A^n whichever power $n = 1, 2, \ldots$ we take. Let us show that we can fix n so to make A^n a contraction. By virtue of Theorem 2.112 this would guarantee that the homogeneous Volterra equation admits one, and one only, solution: this is necessarily the trivial one, because A is linear. A direct computation shows:

$$|(Af)(x)| = \left| \int_a^x K(x,y) f(y) dy \right| \leq M(x-a)||f||_\infty \,.$$

The first iteration gives

$$|(A^2 f)(x)| \leq M^2 \frac{(x-a)^2}{2} ||f||_\infty \,,$$

and, after $n-1$ steps,

$$|(A^n f)(x)| \leq M^n \frac{(x-a)^n}{n!} ||f||_\infty \,.$$

Hence:

$$||A^n f||_\infty \leq M^n \frac{(b-a)^n}{n!} ||f||_\infty \,,$$

and so:

$$||A^n|| \leq M^n \frac{(b-a)^n}{n!} \,.$$

For n large enough then, whatever a, b, M, are, we have:

$$M^n \frac{(b-a)^n}{n!} < 1 \,.$$

Therefore for some positive $\lambda < 1$:

$$||A^n f - A^n f'||_\infty \leq \lambda ||f - f'||_\infty ,$$

and, by the fixed-point theorem, the homogeneous Volterra equation on $C([a, b])$ only admits the trivial solution.

Consequently *the operator A of* (2.48) *cannot admit eigenvalues different from zero*.
In fact the characteristic equation for A,

$$A\psi = \lambda\psi \quad \text{for some } \lambda \in \mathbb{C} \text{ and some } \psi \neq \mathbf{0}, \tag{2.49}$$

is equivalent to:

$$\frac{1}{\lambda} A\psi = \psi \quad \lambda \in \mathbb{C} \setminus \{0\}, \psi \neq \mathbf{0}$$

if $\lambda \neq 0$. And $\lambda^{-1} A$ is a Volterra operator associated to the integral kernel $\lambda^{-1} K(x, y)$. Therefore the theorem may be used on A to give $\psi = \mathbf{0}$. Since the eigenvalue was not allowed to vanish, (2.49) has no solution.

This result will be generalised in Chap. 4 to Hilbert spaces. It will bear an important consequence in the study of Volterra's *inhomogeneous* equation, once we have proved *Fredholm's theorem* on integral equations.

(2) Consider the existence and uniqueness problem for a continuous map $y = f(x)$ when we only know an implicit relation of the type $F(x, y(x)) = 0$, for some given and sufficiently regular function F. We discuss a simplified version of a result that is conventionally known as either *Dini's theorem*, *implicit function theorem* or *inverse function theorem* [CoFr98II, Ser94II]. The point is to see the Banach–Caccioppoli theorem in action. Suppose we are given a function $F : [a, b] \times \mathbb{R} \to \mathbb{R}$, $a < b$, that is continuous and admits partial y-derivative such that $0 < m \leq |\frac{\partial F}{\partial y}| \leq M < +\infty$, $(x, y) \in [a, b] \times \mathbb{R}$.

We want to show that there exists a unique continuous map $f : [a, b] \to \mathbb{R}$ such that:

$$F(x, f(x)) = 0 \quad \text{for any } x \in [a, b].$$

The idea is to define a contraction $G : C([a, b]) \to C([a, b])$ having f as fixed point. To this end set:

$$(G(\psi))(x) := \psi(x) - \frac{1}{M} F(x, \psi(x)) \quad \text{for any } \psi \in C([a, b]), x \in [a, b].$$

This is well defined on $C([a, b])$, and if it contracts then its unique fixed point f satisfies:

$$f(x) = f(x) - \frac{1}{M} F(x, f(x)) \quad \text{for any } x \in [a, b].$$

In other words:

$$F(x, f(x)) = 0 \quad \text{for any } x \in [a, b],$$

so G is what we are after. But G is easily a contraction by the mean value theorem:

$$(G(\psi))(x) - (G(\psi'))(x) = \psi(x) - \psi'(x) - \frac{1}{M}\left(F(x, \psi(x)) - F(x, \psi'(x))\right),$$

so for some number z between $\psi(x)$ and $\psi'(x)$:

$$(G(\psi))(x) - (G(\psi'))(x) = \psi(x) - \psi'(x) - \frac{1}{M}(\psi(x) - \psi'(x))\frac{\partial F}{\partial y}|_{(x,z)},$$

and therefore:

$$|(G(\psi))(x) - (G(\psi'))(x)| \leq |\psi(x) - \psi'(x)| \left|1 - \frac{1}{M}\frac{\partial F}{\partial y}|_{(x,z)}\right| .$$

Because the derivative's range lies inside the positive interval $[m, M]$, we have:

$$||G(\psi) - G(\psi')||_\infty \leq ||\psi - \psi'||_\infty (1 - \frac{m}{M}) .$$

Now, by assumption $(1 - \frac{m}{M}) < 1$, so G is indeed a contraction. ■

2.6.2 *Application of the Fixed-Point Theorem: Local Existence and Uniqueness for Systems of Differential Equations*

The most important application, by far, of the fixed-point theorem is certainly the theorem of local existence and uniqueness for first-order systems of differential equations in *normal form* (where the highest derivative, here the first, is isolated on one side of the equation, as in (2.51) below). This result extends easily to global solutions and higher-order systems [CoFr98I, CoFr98II].

For this we need a preliminary notion. From now on $\mathbb{K}$ will be the complete field $\mathbb{R}$, or possibly $\mathbb{C}$, and $|| \ ||_{\mathbb{K}^p}$ the standard norm on $\mathbb{K}^p$.

Definition 2.114 Let $r \geq 0$ and $n, m > 0$ be given natural numbers, $\Omega \subset \mathbb{K}^r \times \mathbb{K}^n$ a non-empty open set. A function $F : \Omega \to \mathbb{K}^m$ is **locally Lipschitz** (**in the variable** $x \in \mathbb{K}^n$ for $r > 0$), if for any $p \in \Omega$ there exists a constant $L_p \geq 0$ such that:

$$||F(z, x) - F(z, x')||_{\mathbb{K}^m} \leq L_p ||x - x'||_{\mathbb{K}^n} , \quad \text{for any pair } (z, x), (z, x') \in O_p, \tag{2.50}$$

$O_p \ni p$ being an open set.

Any C^1 map $F : \Omega \to \mathbb{K}^m$ is locally Lipschitz in the variable x, as we shall shortly see, but first the theorem.

Theorem 2.115 (Local existence and uniqueness for systems of ODEs of order one) *Let $f : \Omega \to \mathbb{K}^n$ be a continuous and locally Lipschitz map in $x \in \mathbb{K}^n$ on the open set $\Omega \subset \mathbb{R} \times \mathbb{K}^n$. Given the first-order initial value problem (in normal form):*

$$\begin{cases} \dfrac{dx}{dt} = f\,(t, x(t))\,, \\ x(t_0) = x_0 \end{cases} \tag{2.51}$$

with $(t_0, x_0) \in \Omega$, there exists an open interval $I \ni t_0$ on which (2.51) has a unique solution, necessarily belonging in $C^1(I)$.

Proof Notice, to being with, that any solution $x = x(t)$ to (2.51) is of class C^1. Namely, it is continuous as the derivative exists, and directly from $\frac{dx}{dt} = f\,(t, x(t))$ we infer $\frac{dx}{dt}$ must be continuous, because the equation's right-hand side is a composite of continuous maps in t.

Now, suppose $x : I \to \mathbb{K}^n$ is differentiable and that (2.51) holds. By the fundamental theorem of calculus, by integrating (2.51) (the derivative of $x(t)$ is continuous) $x : I \to \mathbb{K}^n$ must satisfy

$$x(t) = x_0 + \int_{t_0}^{t} f\,(\tau, x(\tau))\,d\tau\,, \quad \text{for any} t \in I. \tag{2.52}$$

Conversely, if $x : I \to \mathbb{K}^n$ is continuous and satisfies (2.52), again the fundamental theorem of calculus (f is continuous) tells $x = x(t)$ is differentiable and implies (2.51). Therefore the continuous maps $x = x(t)$ defined on an open interval $I \ni t_0$ that solve the integral equation (2.52) are precisely the solutions to (2.51) defined over I. So instead of solving (2.51) we can solve the equivalent integral problem (2.52).

To prove existence, fix once and for all a relatively compact open set $Q \ni (t_0, x_0)$ with $\overline{Q} \subset \Omega$. Take $\overline{Q}$ small enough to have f locally Lipschitz in x. The standard norm on $\mathbb{K}^n$ will be written $||\ ||$, and we shall use:

(i) $0 \leq M := \max\{||f(t, x)|| \mid (t, x) \in \overline{Q}\}$;

(ii) the constant $L \geq 0$ such that $||f(t, x) - f(t, x')|| \leq L||x - x'||$, (t, x), $(t, x') \in \overline{Q}$;

(iii) $B_\varepsilon(x_0) := \{x \in \mathbb{K}^n \mid ||x - x_0|| \leq \varepsilon\}$ for $\varepsilon > 0$.

Consider the closed interval $J_\delta = [t_0 - \delta, t_0 + \delta]$, $\delta > 0$ and the Banach space $(C(J_\delta; \mathbb{K}^n), ||\ ||_\infty)$ of continuous maps $X : J_\delta \to \mathbb{K}^n$ (Proposition 2.18). On this space define the map G that assigns to any function X a function $G(X)$ by

$$G(X)(t) := x_0 + \int_{t_0}^{t} f\,(\tau, X(\tau))\,d\tau\,, \quad \text{for any } t \in J_\delta\,.$$

Note $G(X) \in C(J_\delta; \mathbb{K}^n)$ for $X \in C(J_\delta; \mathbb{K}^n)$ by the continuity of the integral in the upper limit, when the integrand is continuous. We claim G is a contraction map on a closed subset of $C(J_\delta; \mathbb{K}^n)$: [2]

$$\mathsf{M}_\varepsilon^{(\delta)} := \{X \in C(J_\delta; \mathbb{K}^n) \mid ||X(t) - x_0|| \leq \varepsilon \,, \forall t \in J_\delta\}$$

if $0 < \delta < \min\{\varepsilon/M, 1/L\}$, and $\delta, \varepsilon > 0$ are chosen small enough in order to have $J_\delta \times B_\varepsilon(x_0) \subset Q$. (Henceforth $\varepsilon > 0$ and $\delta > 0$ will be assumed to satisfy $J_\delta \times B_\varepsilon(x_0) \subset Q$.) With $X \in \mathsf{M}_\varepsilon^{(\delta)}$ we have:

$$||G(X)(t) - x_0|| \leq \left|\left|\int_{t_0}^t f(\tau, X(\tau))\, d\tau\right|\right| \leq \int_{t_0}^t ||f(\tau, X(\tau))||\, d\tau \leq \int_{t_0}^t M d\tau \leq \delta M \,.$$

Therefore $G(\mathsf{M}_\varepsilon^{(\delta)}) \subset \mathsf{M}_\varepsilon^{(\delta)}$ for $0 < \delta < \varepsilon/M$. If $X, X' \in \mathsf{M}_\varepsilon^{(\delta)}$ then for all $t \in J_\delta$:

$$G(X)(t) - G(X')(t) = \int_{t_0}^t \left[f(\tau, X(\tau)) - f\left(\tau, X'(\tau)\right)\right] d\tau \,,$$

$$\left|\left|G(X)(t) - G(X')(t)\right|\right| \leq \left|\left|\int_{t_0}^t \left[f(\tau, X(\tau)) - f\left(\tau, X'(\tau)\right)\right] d\tau\right|\right|$$

$$\leq \int_{t_0}^t \left|\left|f(\tau, X(\tau)) - f\left(\tau, X'(\tau)\right)\right|\right| d\tau \,.$$

But we have the Lipschitz bound

$$||f(t, x) - f(t, x')|| < L||x - x'|| \,,$$

so:

$$\left|\left|G(X)(t) - G(X')(t)\right|\right| \leq L \int_{t_0}^t \left|\left|X(\tau) - X'(\tau)\right|\right| d\tau \leq \delta L||X - X'||_\infty \,.$$

Taking the supremum on the left:

$$\left|\left|G(X) - G(X')\right|\right|_\infty \leq \delta L||X - X'||_\infty \,.$$

If, additionally, $\delta < 1/L$, it follows that $G : \mathsf{M}_\varepsilon^{(\delta)} \to \mathsf{M}_\varepsilon^{(\delta)}$ is a contraction on the closed set $\mathsf{M}_\varepsilon^{(\delta)}$. By Theorem 2.112 G has a fixed point, which is a continuous map $x = x(t) \in \mathbb{K}^n, t \in J_\delta$, that solves (2.52) by definition of G. Restricting x to the open interval $I := (t_0 - \delta, t_0 + \delta)$ gives a solution to the initial value problem (2.51).

[2] $\mathsf{M}_\varepsilon^{(\delta)} = \{X \in C(J_\delta; \mathbb{K}^n) \mid ||X - X_0||_\infty \leq \varepsilon\}$, where X_0 is here the constant *map* equal to x_0 on J_δ. Thus $\mathsf{M}_\varepsilon^{(\delta)}$ is the closure of the open ball of radius ε centred at X_0 inside $C(J_\delta; \mathbb{K}^n)$.

As for uniqueness, take another solution $x' = x'(t)$ to (2.52) on $I := (t_0 - \delta, t_0 + \delta)$, a priori distinct from $x = x(t)$. For any closed interval $J_{\delta'} := [t_0 - \delta', t_0 + \delta']$, $0 < \delta' < \delta$, $G : \mathsf{M}_\varepsilon^{(J_{\delta'})} \to \mathsf{M}_\varepsilon^{(J_{\delta'})}$ is by construction still a contracting map, and $x' = x'(t)$ a fixed point of it. Therefore x' coincides with $x = x(t)$ restricted to $J_{\delta'}$, by uniqueness. (In particular, the restriction of $\mathbf{x}'$ to $J_{\delta'}$ belongs to the complete metric space $\mathsf{M}_\varepsilon^{(J_{\delta'})}$ because we saw $||G(\mathbf{x}') - \mathbf{x}_0||_\infty \leq \delta' M < \varepsilon$, since $G(\mathbf{x}') = \mathbf{x}'$.) But since δ' is arbitrary in $(0, \delta)$, the two solutions coincide on $I = (t_0 - \delta, t_0 + \delta)$. $\square$

Just for the sake of completeness we remark that the previous theorem holds when f is C^1, because of the following elementary fact.

We adopt the usual notation

$$x = (x_1, \ldots, x_n)\,, z = (z_1, \ldots, z_l) \quad \text{and} \quad F(z, x) = (F_1(z, x), \ldots, F_m(z, x))$$

for an arbitrary map $F : \Omega \to \mathbb{R}^m$ with $\Omega = A \times B$, $A \subset \mathbb{R}^l$ and $B \subset \mathbb{R}^n$ non-empty open sets.

Proposition 2.116 *Consider $\Omega = A \times B$, $A \subset \mathbb{R}^l$ and $B \subset \mathbb{R}^n$ non-empty open sets. The map $F : \Omega \to \mathbb{R}^m$ is locally Lipschitz in x if, for every $z \in A$, the functions $B \ni x \mapsto F_k(z, x)$ admit first derivative, and if the partial derivatives, as (z, x) varies, are continuous on Ω.*

Proof Take $q = (z_0, x_0) \in \Omega$ and let $B' \subset \mathbb{R}^l$, $B \subset \mathbb{R}^n$ be open balls centred at z_0, x_0, with $\overline{B'} \times \overline{B} \subset \Omega$. Then $x(t) = p + t(r - p) \in \overline{B}$, for $t \in [0, 1]$ and $p, r \in \overline{B}$. Fix $z \in B'$. The mean value theorem applied to $[0, 1] \ni t \mapsto F_k(z, x(t))$ results in

$$F_k(z, r) - F_k(z, p) = F_k(z, x(1)) - F_k(z, x(0)) = \sum_{j=1}^{n} (r_j - p_j) \left.\frac{\partial F_k}{\partial x_j}\right|_{(z, x(\xi))} ,$$

where $(z, x(\xi)) \in \overline{B'} \times \overline{B}$. Schwarz's inequality then gives:

$$|F_k(z, r) - F_k(z, p)| \leq \sqrt{\sum_{j=1}^{n} |r_j - p_j|^2} \sqrt{\sum_{i=1}^{n} \left| \left.\frac{\partial F_k}{\partial x_i}\right|_{(z, x(\xi))} \right|^2}$$

$$\leq ||r - p|| \sqrt{\sum_{i=1}^{n} \left| \left.\frac{\partial F_k}{\partial x_i}\right|_{(z, x(\xi))} \right|^2} \leq M_k ||r - p|| \quad \text{for } (z, r), (z, p) \in \overline{B'} \times \overline{B}\,,$$

and such $M_k < +\infty$ exists since the radicand is continuous on the compact set $\overline{B'} \times \overline{B}$. Since $B' \times B$ is an open neighbourhood of the generic point $(z_0, x_0) \in \Omega$, the map F is locally Lipschitz in x:

$$||F(z, x_1) - F(z, x_2)|| \leq \sqrt{\sum_{k=1}^{m} M_k^2} \; ||x_1 - x_2|| \quad \text{for } (z, x_1), (z, x_2) \in B' \times B \, .$$

□

Remark 2.117 This particular proof of the theorem requires the local Lipschitz condition for f in (2.51) in order to use the fixed-point theorem. As a matter of fact, this is not necessary to grant *existence*. A more general existence result, due to Peano, can be proved (using the Arzelà–Ascoli Theorem 2.22) if one only assumes the continuity of f [KoFo99]. In general, though, the absence of the Lipschitz condition undermines the solution's uniqueness, as the following classical counterexample makes clear: consider

$$\frac{dx}{dt} = f(x(t)) \, , \quad x(0) = 0$$

where $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = 0$ on $x \leq 0$ and $f(x) = \sqrt{x}$ for $x > 0$, is continuous but not locally Lipschitz at $x = 0$. The Cauchy problem admits at least *two* solutions (both maximal):

(1) $x_1(t) = 0$, for any $t \in \mathbb{R}$
(2) $x_2(t) = 0$ for $t \leq 0$ and $x_2(t) = t^2/4$ on $t > 0$. ■

Exercises

2.1 Prove that any seminorm p satisfies $p(x) = p(-x)$.

2.2 Let K be a compact set, X a normed space and $f : K \to \mathsf{X}$ a continuous map. Show f is bounded, i.e. there exists $M \geq 0$ such that $||f(k)|| \leq M$ for any $k \in K$.

Hint. Adapt the proof of Proposition 1.21.

2.3 Prove that if S denotes a vector space of bounded maps from X to $\mathbb{C}$ (or to $\mathbb{R}$), then

$$\mathsf{S} \ni f \mapsto ||f||_\infty := \sup_{x \in \mathsf{X}} |f(x)|$$

defines a norm on S.

2.4 Let X be a topological space. Prove that the spaces of bounded complex functions $L(\mathsf{X})$, and of measurable and bounded complex functions $M_b(\mathsf{X})$ (cf. Examples 2.29), are Banach spaces for the norm $||\ ||_\infty$.

Solution. We shall prove the claim for $M_b(\mathsf{X})$, the other one being exactly the same. The claim is that an arbitrary Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subset M_b(\mathsf{X})$ converges uniformly to some $f \in M_b(\mathsf{X})$. By assumption the numerical sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ is Cauchy, for any $x \in \mathsf{X}$. Therefore there exists $f : \mathsf{X} \to \mathbb{C}$ such that $f_n(x) \to f(x)$, as

$n \to +\infty$, for any $x \in \mathsf{X}$. This function will be measurable because it arises as limit of measurable maps. We are left to prove that f is bounded and $f_n \to f$ uniformly. Start from the latter. Since

$$|f(x) - f_m(x)| = \lim_{n\to+\infty} |f_n(x) - f_m(x)| \le \lim_{n\to+\infty} ||f_n - f_m||_\infty \,,$$

and using the fact that the initial sequence is Cauchy for $||\ \ ||_\infty$, we have that for any $\varepsilon > 0$ there is N_ε such that:

$$\lim_{n\to+\infty} ||f_n - f_m||_\infty < \varepsilon \quad \text{for} m > N_\varepsilon .$$

Hence:

$$|f(x) - f_m(x)| < \varepsilon \quad \text{for}\, m > N_\varepsilon \text{ and any } x \in \mathsf{X}.$$

In other words $||f - f_m||_\infty \to 0$ as $m \to +\infty$, as required. Now the boundedness is obvious:

$$\sup_{x\in\mathsf{X}} |f(x)| \le \sup_{x\in\mathsf{X}} |f(x) - f_m(x)| + \sup_{x\in\mathsf{X}} |f_m(x)| < \varepsilon + ||f_m||_\infty < +\infty \,.$$

2.5 Show that the Banach spaces $(L(\mathsf{X}), ||\ \ ||_\infty)$ and $(M_b(\mathsf{X}), ||\ \ ||_\infty)$ (cf. Examples 2.29) are Banach algebras with unit.

Sketch. The unit is clearly the constant map 1. The property $||f \cdot g||_\infty \le ||f||_\infty ||g||_\infty$ follows from the definition of $||\ \ ||_\infty$, and the remaining conditions are easy.

2.6 Prove that the space $C_0(\mathsf{X})$ of continuous, complex functions on X that vanish at infinity (cf. Examples 2.29) is a Banach algebra for $||\ \ ||_\infty$. Explain in which circumstances the algebra has a unit.

Solution. We take a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0(\mathsf{X})$ and prove it converges uniformly to $f \in C_0(\mathsf{X})$. By hypothesis the numerical sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for any $x \in \mathsf{X}$. Therefore there exists a function $f : \mathsf{X} \to \mathbb{C}$ such that $f_n(x) \to f(x)$ for any $x \in \mathsf{X}$, as $n \to +\infty$. The proof that $||f - f_n||_\infty \to 0, n \to +\infty$, goes exactly as in Exercise 2.4. Since continuity is preserved by uniform limits, there remains to show $f \in C_0(\mathsf{X})$. Given $\varepsilon > 0$, pick n such that $||f - f_n|| < \varepsilon/2$, and choose a compact set $K_\varepsilon \subset \mathsf{X}$ so that $|f_n(x)| < \varepsilon/2$ for $x \in \mathsf{X} \setminus K_\varepsilon$. By construction

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < \varepsilon, \quad x \in \mathsf{X} \setminus K_\varepsilon .$$

The Banach space thus found is a Banach algebra for the familiar operations, as one proves without difficulty.

If the unit is present, it must be the constant map 1. If X is compact, the function 1 belongs to the space. But if X is not compact, then 1 cannot be in X, because the elements of $C_0(\mathsf{X})$ can be shrunk arbitrarily outside compact subsets, and no constant map does that.

2.7 Prove the space $C_b(\mathsf{X})$ of continuous and bounded complex functions on X (see Examples 2.29) is a Banach space for $|| \ ||_\infty$ and a Banach algebra with unit.

2.8 Prove that in Proposition 2.17 the converse implication holds as well. In other words, the proposition may be rephrased like this:

Proposition. *Let* $(\mathsf{X}, || \ ||)$ *be a normed space. Every absolutely convergent series* $\sum_{n=0}^{+\infty} x_n$ (i.e. $\sum_{n=0}^{+\infty} ||x_n|| < +\infty$) *converges in* X *iff* $(\mathsf{X}, || \ ||)$ *is a Banach space.*

Solution. Take an absolutely convergent series $\sum_{n=0}^{+\infty} x_n$ in X. The partial sums of the norms have to be a Cauchy sequence, i.e. for any $\varepsilon > 0$ there is $M_\varepsilon > 0$ with

$$\left| \sum_{j=0}^{n} ||x_j|| - \sum_{j=0}^{m} ||x_j|| \right| < \varepsilon \ , \quad \text{for } n, m > M_\varepsilon .$$

Supposing $n \geq m$:

$$\left| \sum_{j=m+1}^{n} ||x_j|| \right| < \varepsilon \ , \quad n, m > M_\varepsilon .$$

Therefore:

$$\left\| \sum_{j=0}^{n} x_j - \sum_{j=0}^{m} x_j \right\| = \left\| \sum_{j=m+1}^{n} x_j \right\| \leq \sum_{j=m+1}^{n} ||x_j|| < \varepsilon, \quad n, m > M_\varepsilon .$$

We proved the sequence of partial sums $\sum_{j=0}^{n} x_n$ is Cauchy. But the space is complete, so the series converges to a point in X.

2.9 Show that the space $C_c(\mathsf{X})$ of complex functions with compact support on X (cf. Examples 2.29) is *not*, in general, a Banach space for $|| \ ||_\infty$, nor is it dense in $C_b(\mathsf{X})$ if X is not compact.

Outline of proof. For the first statement we need to exhibit a counterexample for $\mathsf{X} = \mathbb{R}$. Consider for instance the continuous maps $f_n : \mathbb{R} \to \mathbb{C}$, $n = 1, 2, \ldots,$: $f_n(x) := \frac{\sin x}{x}$ for $0 < |x| < 2n\pi$, $f_n(0) = 1$ and $f_n(x) = 0$ at other points of $\mathbb{R}$. The sequence evidently converges pointwise to the continuous map defined as $\frac{\sin x}{x}$ on $\mathbb{R} \setminus \{0\}$ and set to 1 at the origin. It is easy to convince ourselves that the convergence is uniform. But the limit function does not have compact support. As for the second part, note that any constant map $c \neq 0$ belongs in $C_c(\mathsf{X})$. But if X is not compact, then $||f - c||_\infty \geq |c| > 0$ for any function $f \in C_c(\mathsf{X})$ because of the values attained outside the support of f.

2.10 Given a compact space K and a Banach space B, let $C(K; \mathsf{B})$ be the space of continuous maps $f : K \to \mathsf{B}$ in the norm topologies of domain and codomain. Define

$$||f||_\infty := \sup_{x\in K} ||f(x)|| \quad f \in C(K; \mathsf{B}) ,$$

where the norm on the right is the one on B. Show this norm is well defined, and that it turns $C(K; \mathsf{B})$ into a Banach space.

Hint. Keep in mind Exercise 2.2 and adjust the proof of Proposition 2.18.

2.11 Let $(\mathfrak{A}, \circ)$ be a Banach algebra without unit. Consider the direct sum $\mathfrak{A} \oplus \mathbb{C}$ and define the product:

$$(x, c) \cdot (y, c') := (x \circ y + cy + c'x, cc'), \quad (x', c'), (x, c) \in \mathfrak{A} \oplus \mathbb{C}$$

and the norm:

$$||(x, c)|| := ||x|| + |c|, \quad (x, c) \in \mathfrak{A} \oplus \mathbb{C}.$$

Show that the vector space $\mathfrak{A} \oplus \mathbb{C}$ with this product and norm becomes a Banach algebra with unit.

2.12 Take a Banach algebra $\mathfrak{A}$ with unit $\mathbb{I}$ and an element $a \in \mathfrak{A}$ with $||a|| < 1$. Prove that the series $\sum_{n=0}^{+\infty}(-1)^n a^{2n}$, $a^0 := \mathbb{I}$, converges in the topology of $\mathfrak{A}$. What is the sum?

Hint. Show the series of partial sums is a Cauchy series. The sum is $(\mathbb{I} + a^2)^{-1}$.

2.13 (*Hard*.) Prove Hölder's inequality:

$$\int_X |f(x)g(x)|d\mu(x) \leq \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x)\right)^{1/p} \left(\int_{\mathsf{X}} |g(x)|^q d\mu(x)\right)^{1/q}$$

where $p, q > 0$ satisfy $1 = \frac{1}{p} + \frac{1}{q}$, f and g are measurable and μ is a positive measure on X.

Solution. Define $I := \int_X |f(x)|\, |g(x)|d\mu(x)$, $A := \left(\int_{\mathsf{X}} |f(x)|^p d\mu(x)\right)^{1/p}$ and $B := \left(\int_{\mathsf{X}} |g(x)|^q d\mu(x)\right)^{1/q}$. If either A or B is zero or infinite (conventionally, $\infty \cdot 0 = 0 \cdot \infty = 0$), the inequality is trivial. So let us assume $0 < A, B < +\infty$ and define $F(x) := |f(x)|/A$, $G(x) := |g(x)|/B$. Thus

$$\ln(F(x)G(x)) = \frac{1}{p}\ln(F(x)^p) + \frac{1}{q}\ln(G(x)^q) \leq \ln\left(\frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q\right) ,$$

because the logarithm is a convex function. Exponentiating gives

$$F(x)G(x) \leq \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q .$$

Integrating the above, and noting that the right-hand-side integral is $1/p + 1/q = 1$ we recover Hölder's inequality in the form:

$$\frac{\int_X |f(x)g(x)|d\mu(x)}{\left(\int_\mathsf{X} |f(x)|^p d\mu(x)\right)^{1/p} \left(\int_\mathsf{X} |g(x)|^q d\mu(x)\right)^{1/q}} \leq 1 \, .$$

2.14 (*Hard.*) Making use of Hölder's inequality, prove Minkowski's inequality:

$$\left(\int_X |f(x)+g(x)|^p d\mu(x)\right)^{1/p} \leq \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p} + \left(\int_X |g(x)|^p d\mu(x)\right)^{1/p}$$

where $p \geq 1$, f and g are measurable and μ a positive measure on X.

Solution. Define $I := \int_X |f(x)+g(x)|^p d\mu(x)$, $A := \left(\int_\mathsf{X} |f(x)|^p d\mu(x)\right)^{1/p}$ and $B := \left(\int_\mathsf{X} |g(x)|^p d\mu(x)\right)^{1/p}$. The inequality is trivial in case $p = 1$ or if one of A, B is infinite. So we assume $p > 1$, $A, B < +\infty$. Then I must be finite too, because $(a+b)^p \leq 2^p(a^p + b^p)$ for any $a, b \geq 0$ and $p \geq 1$.[3] Minkowski's inequality is trivial also when $I = 0$, so we consider only $p > 1$, $A, B < +\infty$, $0 < I < +\infty$. Note $|f+g|^p = |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$. Using Hölder's inequality on each summand on the right we have:

$$\int_\mathsf{X} |f(x)+g(x)|^p d\mu(x) \leq \left(\left(|f(x)+g(x)|^{(p-1)q} d\mu(x)\right)^{1/q}\right)$$
$$\times\left(\left(\int_\mathsf{X} |f(x)|^p d\mu(x)\right)^{1/p} + \left(\int_\mathsf{X} |g(x)|^p d\mu(x)\right)^{1/p}\right),$$

where $1 = \frac{1}{p} + \frac{1}{q}$. This last inequality can be written as $I \leq I^{1/q}(A+B)$, dividing which by $I^{1/q}$ produces $I^{1/p} \leq A + B$, i.e. Minkowski's inequality.

2.15 Take two finite-dimensional normed spaces X, Y and consider $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y}) = \mathfrak{B}(\mathsf{X}, \mathsf{Y})$. Fix bases in X and Y and represent T by the matrix $M(T)$. Show that one can choose bases for the dual spaces X', Y' so that the operator T' is determined by the transpose matrix $M(T)^t$.

2.16 Prove Proposition 2.70.

2.17 Consider the space $\mathfrak{B}(\mathsf{X})$ for some normed space X. Prove the strong topology is finer than the weak topology (put loosely: weakly open sets are strongly open), and the uniform topology is finer than the strong one.

2.18 Prove Propositions 2.74–2.77.

[3] This inequality descends from $(a+b) \leq 2\max\{a, b\}$, whose pth power reads $(a+b)^p \leq 2^p \max\{a^p, b^p\} \leq 2^p(a^p + b^p)$.

2.19 In a normed space X prove that if $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ tends to $x \in \mathsf{X}$ *weakly* (cf. Proposition 2.74), the set $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

Hint. Use Corollary 2.64.

2.20 If B is a Banach space and $T, S \in \mathfrak{B}(\mathsf{B})$, show:
(i) $(TS)' = S'T'$,
(ii) $(T')^{-1} = (T^{-1})'$, provided T is bijective.

2.21 Prove that if X and Y are reflexive Banach spaces, and $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$, then $(T')' = T$.

2.22 If X is normed, the function that maps $(T, S) \in \mathfrak{B}(\mathsf{X}) \times \mathfrak{B}(\mathsf{X})$ to $TS \in \mathfrak{B}(\mathsf{X})$ is continuous in the uniform topology. What can be said regarding the strong and weak topologies?

Solution. For both topologies the map is separately continuous in either argument, but not continuous as a function of two variables, in general.

2.23 If we define an isomorphism of normed spaces as a continuous linear map with continuous inverse, does an isomorphism preserve completeness?

Hint. Extend Proposition 2.105 to the case of a continuous linear map between normed spaces with continuous inverse.

2.24 Using weak equi-boundedness, prove this variant of the Banach–Steinhaus Theorem 2.62.

Proposition. *Let* X *be a Banach space and* Y *a normed space over the same field* $\mathbb{C}$, or $\mathbb{R}$. *Suppose the family of operators* $\{T_\alpha\}_{\alpha\in A} \subset \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ *satisfies*:

$$\sup_{\alpha\in A} |f(T_\alpha x)| < +\infty \quad \text{for any } x \in \mathsf{X},\ f \in \mathsf{Y}'\ .$$

Then there exists a uniform bound $K \geq 0$:

$$||T_\alpha|| \leq K \quad \text{for any } \alpha \in A\ .$$

Solution. Referring to Corollary 2.59, for any given $x \in \mathsf{X}$ define $F_{\alpha,x} := \mathfrak{I}(T_\alpha x) \in (\mathsf{Y}')'$. Then

$$\sup_{\alpha\in A} |F_{\alpha,x}(f)| < +\infty \quad \text{for any } f \in \mathsf{Y}'\ .$$

As Y' is complete, we can use Theorem 2.62 to infer the existence, for any $x \in \mathsf{X}$, of $K_x \geq 0$ that bounds uniformly the family $F_{\alpha,x} : \mathsf{Y}' \to \mathbb{C}$:

$$||F_{\alpha,x}|| \leq K_x \quad \text{for any } \alpha \in A\ .$$

But $\mathfrak{J}$ is isometric, so:

$$||T_\alpha(x)|| \leq K_x \quad \text{for any}\, \alpha \in A$$

and hence

$$\sup_{\alpha \in A} ||T_\alpha x|| < +\infty \quad \text{for any}\, x \in \mathsf{X}\,.$$

The Banach–Steinhaus Theorem 2.62 ends the proof.

2.25 Let K be compact, X a Banach space, and equip $\mathfrak{B}(\mathsf{X})$ with the strong topology. Prove that any continuous map $f : K \to \mathfrak{B}(\mathsf{X})$ belongs to $C(K; \mathfrak{B}(\mathsf{X}))$. (The latter is a Banach space, defined in Exercise 2.10, if we view $\mathfrak{B}(\mathsf{X})$ as a Banach space.)

Solution. We must prove

$$\sup_{k \in K} ||f(k)|| < +\infty$$

where on the left we used the operator norm of $\mathfrak{B}(\mathsf{X})$. As f is continuous in the strong topology, for any given $x \in \mathsf{X}$ the map $K \ni k \mapsto f(k)x \in \mathsf{X}$ is continuous. If we fix $x \in \mathsf{X}$, by Exercise 2.2 there exists $M_x \geq 0$ such that:

$$\sup_{k \in K} ||f(k)x||_{\mathsf{X}} < M_x\,.$$

The Banach–Steinhaus Theorem 2.62 ends the proof.

Chapter 3
Hilbert Spaces and Bounded Operators

There's no such thing as a deep theorem, but only theorems we haven't understood very well.

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With this chapter we introduce the first mathematical notions relative to Hilbert spaces that we will use to build the mathematical foundations of Quantum Mechanics. A good part is devoted to *Hilbert bases* (*complete orthonormal systems*), which we treat in full generality without assuming the Hilbert space be separable. Before that, we discuss a paramount result in the theory of Hilbert spaces: *Riesz's representation theorem*, according to which there is a natural anti-isomorphism between a Hilbert space and its dual.

The third section studies *adjoint operators* (to bounded operators), introduced by means of Riesz's theorem, and their place at the heart of the theory of bounded operators. In particular, we introduce *-algebras, C^**-algebras* (and operator C^*-algebras) and their representations. Here we define *self-adjoint*, *unitary* and *normal* operators, and examine their basic properties.

Section four is entirely dedicated to *orthogonal projectors* and their main features. We also introduce the useful notion of *partial isometry*.

The fifth section is concerned with the important *polar decomposition theorem* for bounded operators defined on the whole Hilbert space. The *positive square root* of a bounded operator is used as technical tool.

Section six introduces *von Neumann algebras*, with attention to the famous *double commutant theorem*.

The elementary theory of the *Fourier* and *Fourier–Plancherel transforms*, object of the last section, is introduced very rapidly and with, alas, no mention to Schwartz's distributions (for this see [Rud91, ReSi80, Vla02]).

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3.1 Elementary Notions, Riesz's Theorem and Reflexivity

The present section deals with the basics of Hilbert spaces, starting from the elementary definitions of *Hermitian inner product* and *Hermitian inner product space*.

3.1.1 Inner Product Spaces and Hilbert Spaces

Definition 3.1 If X is a complex vector space, a map $S : \mathsf{X} \times \mathsf{X} \to \mathbb{C}$ is called a **Hermitian inner product**, and (X, S) an **inner product space**, when:
H0. $S(u, u) \geq 0$ for any $u \in \mathsf{X}$;
H1. $S(u, \alpha v + \beta w) = \alpha S(u, v) + \beta S(u, w)$ for any $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in \mathsf{X}$;
H2. $S(u, v) = \overline{S(v, u)}$ for any $u, v \in \mathsf{X}$;
H3. $S(u, u) = 0 \Rightarrow u = \mathbf{0}$, for any $u \in \mathsf{X}$.
If **H0**, **H1**, **H2** hold and **H3** does not, S is a **Hermitian semi-inner product**.

Two vectors $u, v \in \mathsf{X}$ are **orthogonal**, written $u \perp v$, if $S(u, v) = 0$. In this case u and v are called **orthogonal** (or **normal**) to one another.

The **orthogonal space** to a non-empty subspace $K \subset \mathsf{X}$ is:

$$K^{\perp} := \{u \in \mathsf{X} \mid u \perp v \text{ for any } v \in K\} .$$

Remarks 3.2 **(1)** **H1** and **H2** imply that S is **antilinear** in the first argument:

$$S(\alpha v + \beta w, u) = \overline{\alpha} S(v, u) + \overline{\beta} S(w, u) \quad \text{for any } \alpha, \beta \in \mathbb{C}, u, v, w \in \mathsf{X}.$$

(2) It is immediate that $K^{\perp}$ is a vector subspace in X by **H1**, so the name orthogonal *space* is not accidental.
(3) In an inner product space (X, S), the definition of orthogonality implies a useful property we will use often:

$$K \subset K_1 \quad \Rightarrow \quad K_1^{\perp} \subset K^{\perp} \quad \text{for } K, K_1 \subset \mathsf{X}.$$

(4) From now, lest we misunderstand, "(semi-)inner product" will always stand for "Hermitian (semi-)inner product". ■

Proposition 3.3 *Let* X *be a* $\mathbb{C}$*-vector space with semi-inner product* S.
(a) *The* **Cauchy–Schwarz inequality** *holds:*

$$|S(x, y)|^2 \leq S(x, x) S(y, y), \quad x, y \in \mathsf{X}\,; \tag{3.1}$$

moreover

(i) there is equality in (3.1) if x, y are linearly dependent;

(ii) there is equality in (3.1) if and only if x, y are linearly dependent, provided S is an inner product.

(b) *As $x \in \mathsf{X}$ varies,*

$$p(x) := \sqrt{S(x, x)} \tag{3.2}$$

defines the seminorm **induced by** *S (a norm if S is an inner product).*

(c) *the seminorm p satisfies the* **parallelogram rule***:*

$$p(x+y)^2 + p(x-y)^2 = 2(p(x)^2 + p(y)^2), \quad x, y \in \mathsf{X}\,. \tag{3.3}$$

(d) *the* **polarisation formula** *holds:*

$$S(x, y) = \frac{1}{4}\left(p(x+y)^2 - p(x-y)^2 - ip(x+iy)^2 + ip(x-iy)^2\right), \quad x, y \in \mathsf{X} \tag{3.4}$$

Proof (a) If $\alpha \in \mathbb{C}$, using the properties of the semi-inner product,

$$0 \le S(x - \alpha y, x - \alpha y) = S(x, x) - \alpha S(x, y) - \overline{\alpha} S(y, x) + |\alpha|^2 S(y, y)\,. \tag{3.5}$$

Suppose $S(y, y) \neq 0$. Then setting $\alpha := \overline{S(x, y)}/S(y, y)$, (3.5) implies:

$$0 \le S(x, x) - |S(x, y)|^2/S(y, y)\,,$$

as claimed. If $S(y, y) = 0$, from (3.5) we find, for any $\alpha \in \mathbb{C}$:

$$0 \le S(x, x) - \alpha S(x, y) - \overline{\alpha} S(y, x)\,. \tag{3.6}$$

By choosing $\alpha \in \mathbb{R}$ large enough in absolute value we see that inequality (3.6) is not satisfied unless $S(x, y) + S(y, x) = 0$. Choosing now $\alpha = i\lambda$ with $\lambda \in \mathbb{R}$ large enough in absolute value, we find that (3.5) can hold only if $S(x, y) - S(y, x) = 0$. With the previous $S(x, y) = -S(y, x)$ it gives $S(x, y) = 0$. Summing up, $S(y, y) = 0$ implies (3.1) because $S(x, y) = 0$. If x, y are linearly dependent then $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{C}$. If so, the two sides of (3.1) are equal. Now assume S is an inner product and $|S(x, y)|^2 = S(x, x)S(y, y)$, and let us prove there are $\alpha, \beta \in \mathbb{C}$, not both zero, so that $\alpha x + \beta y = \mathbf{0}$. If at least one of x, y is null, the claim is true. So suppose neither vanishes, so $S(x, x) > 0 < S(y, y)$ by **H3**. Then redefining $u = x/\sqrt{S(x, x)}$, $v = y/\sqrt{S(y, y)}$, we have $|S(u, v)| = 1$ and so $S(u, v) = e^{i\eta}$ for some $\eta \in \mathbb{R}$. By **H3**, $\alpha' u + \beta' v = \mathbf{0}$ is equivalent to $S(\alpha' u + \beta' v, \alpha' u + \beta' v) = 0$, i.e.

$$|\alpha'|^2 + |\beta'|^2 + \overline{\alpha'}\beta' e^{i\eta} + \overline{\beta'}\alpha' e^{-i\eta} = 0\,,$$

as $S(u, v) = e^{i\eta}$. Choose $\alpha' = e^{i\mu}$, $\beta' = e^{i\nu}$, so that $-\mu + \nu + \eta = \pi$. Then the above identity holds, and setting $\alpha := e^{i\mu}\sqrt{S(y, y)}$, $\beta := e^{i\nu}\sqrt{S(x, x)}$ we have $\alpha, \beta \neq 0$ and $\alpha x + \beta y = \mathbf{0}$.

(b) Verifying that p is a seminorm is easy knowing S is a (semi-)inner product, except perhaps the triangle inequality **N2** which we prove now. By the properties of the inner product

$$p(x+y)^2 = p(x)^2 + 2ReS(x,y) + p(y)^2 ,$$

with Re denoting the real part of a complex number. As $ReS(x,y) \leq |S(x,y)|$, by (3.1), we also have $ReS(x,y) \leq p(x)p(y)$. Substituting above gives:

$$p(x+y)^2 \leq p(x)^2 + 2p(x)p(x) + p(y)^2 ,$$

i.e.

$$p(x,y)^2 \leq (p(x)+p(y))^2 ,$$

which in turn implies **N2**. Property **N3** is immediate from **H3**, in case S is an inner product.

Statements (c) and (d) are straightforward from the definition of p and the properties of inner product. □

Remarks 3.4 **(1)** The Cauchy–Schwarz inequality immediately implies that an inner product $S : \mathsf{X} \times \mathsf{X} \to \mathbb{C}$ is a *continuous* map on $\mathsf{X} \times \mathsf{X}$ in the product topology, when X has the *topology of the norm induced by the inner product*, i.e. (3.2). In particular the inner product is continuous in its arguments separately.
(2) If the ground field is $\mathbb{R}$ instead of $\mathbb{C}$, we have analogous Definition 3.1, by declaring a **real inner product** $S : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ to fulfil **H0**, **H1**, **H3** and replacing **H2** with the *symmetry property*:

H2'. $S(u,v) = S(v,u)$ for any $u, v \in \mathsf{X}$.

A **real semi-inner product** is a real inner product without **H3**, so to speak.
(3) Proposition 3.3 is still true for real (semi-)inner products, with the proviso that the new polarisation formula reads:

$$S(x,y) = \frac{1}{4}\left(p(x+y)^2 - p(x-y)^2\right) \tag{3.7}$$

over the field $\mathbb{R}$.
(4) The polarisation identity (3.4) and the corresponding Eq. (3.7) for real vector spaces are valid even if S is not an inner product, provided it is re-written into a suitable form. In the complex case linearity in the left entry and antilinearity in the right entry are sufficient. In the real case bilinearity and symmetry are enough. This matter is sorted out in Exercises 3.1 and 3.2. ■

A known result – rarely proved explicitly – is the following, due to Fréchet, von Neumann and Jordan. The proof is carried out in Exercises 3.3–3.5.

Theorem 3.5 *Let X be a complex vector space and $p : \mathsf{X} \to \mathbb{R}$ a norm (or seminorm). Then p satisfies the parallelogram rule (3.3) if and only if there exists a unique inner product (or semi-inner product) S inducing p by way of (3.2).*

Proof If the norm (seminorm) is induced by a Hermitian inner product, the parallelogram rule (3.3) is valid by Proposition 3.3(c). The proof that (3.3) implies the existence of an inner product (semi-inner product) S inducing p via (3.2) can be found in Exercises 3.3–3.5. □

Let us pass to *isomorphisms* of inner product spaces.

Definition 3.6 Let $(\mathsf{X}, S_{\mathsf{X}})$, $(\mathsf{Y}, S_{\mathsf{Y}})$ be inner product spaces. A linear map $L : \mathsf{X} \to \mathsf{Y}$ is called an **isometry** if:

$$S_{\mathsf{Y}}(L(x), L(y)) = S_{\mathsf{X}}(x, y) \quad \text{for any } x, y \in \mathsf{X}.$$

If the isometry $L : \mathsf{X} \to \mathsf{Y}$ is onto we call it an **isomorphism of inner product spaces**.

If one such L exists, the spaces X and Y are said to be **isomorphic (under** L**)**.

Remark 3.7 Every isometry $L : \mathsf{X} \to \mathsf{Y}$ is clearly 1-1 by **H3**, but may *not* be onto, even when $\mathsf{X} = \mathsf{Y}$, if the dimension of X is not finite. Every isometry is moreover continuous in the norm topologies induced by inner products. If surjective (an isomorphism), its inverse is an isometry (isomorphism). ■

Since an inner product space is also normed, we have two notions of isometry for a linear transformation $L : \mathsf{X} \to \mathsf{Y}$. The first refers to the preservation of inner products (as above), the second was given in Definition 2.10, and corresponds to the requirement: $||Lx||_{\mathsf{Y}} = ||x||_{\mathsf{X}}$ for any $x \in \mathsf{X}$, with reference to the norms induced by the inner products. The former type also satisfies the second definition. Using the polarisation formula (3.4) it can actually be proved that the two notions are equivalent.

Proposition 3.8 *A linear operator* $L : \mathsf{X} \to \mathsf{Y}$ *between inner product spaces is an isometry in the sense of Definition 3.6 if and only if:*

$$||Lx||_{\mathsf{Y}} = ||x||_{\mathsf{X}} \quad \textit{for any } x \in \mathsf{X},$$

where the norms are induced by the corresponding inner products.

Notation 3.9 Unless we say otherwise, from now on $(\ |\ )$ will indicate an inner product and $||\ \ ||$ the induced norm, as in Proposition 3.3. ■

Now to the truly central notion of the entire book, that of *Hilbert space*.

Definition 3.10 (*Hilbert space*) A complex vector space equipped with a Hermitian inner product is called a **Hilbert space** if the norm induced by the inner product makes it a Banach space.

An isomorphism of inner product spaces between Hilbert spaces is called:
(i) **isomorphism of Hilbert spaces**, or
(ii) **unitary transformation**, or
(iii) **unitary operator**.

It must be clear that under an isomorphism of inner product spaces $U : \mathsf{H} \to \mathsf{H}_1$, H_1 is a Hilbert space if and only if H is. Then U is a unitary transformation.

There is a result about completions similar to the one seen for Banach spaces.

Theorem 3.11 (Completion of Hilbert spaces) *Let* X *be a* $\mathbb{C}$*-vector space with inner product* S.
(a) *There exists a Hilbert space* $(\mathsf{H}, (\ |\ ))$*, called* **completion** *of* X*, such that* X *is identified with a dense subspace of* H *(for the norm induced by* $(\ |\ )$*) under a* 1-1 *linear map* $J : \mathsf{X} \to \mathsf{H}$ *that extends* S *to* $(\ |\ )$*:*

$$\overline{J(\mathsf{X})} = \mathsf{H} \quad \textit{and} \quad (J(x)|J(y)) = S(x, y) \quad \textit{for any } x, y \in \mathsf{X}.$$

(b) *If the triple* $(J_1, \mathsf{H}_1, (\ |\ )_1)$*, with* $J_1 : \mathsf{X} \to \mathsf{H}_1$ *a linear isometry and* $(\mathsf{H}_1, (\ |\ )_1)$ *a Hilbert space, is such that* J_1 *identifies* X *with a dense subspace in* H_1 *by extending* S *to* $(\ |\ )_1$*, then there is a unique unitary transformation* $\phi : \mathsf{H} \to \mathsf{H}_1$ *such that* $J_1 = \phi \circ J$.

Sketch of proof. (a) It is convenient to use the completion theorem for Banach spaces and then construct the Banach completion of the normed space (X, N), where $N(x) := \sqrt{S(x, x)}$. Since S is continuous and X is dense in the completion under the linear map J, S induces a semi-inner product $(\ |\ )$ on the Banach completion H. Actually, $(\ |\ )$ is an inner product on H because, still by continuity, it induces the same norm of the Banach space. Thus H is a Hilbert space and the map J sending X to a dense subspace in H satisfies all the requirements. (b) The proof is essentially the same as in the Banach case. $\square$

Examples 3.12 **(1)** $\mathbb{C}^n$ with inner product $(u|v) := \sum_{i=1}^{n} \overline{u_i} v_i$, where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, is a Hilbert space.
(2) A crucial Hilbert space in physics was defined in Example 2.29(6), namely $L^2(\mathsf{X}, \mu)$. Recall that if X is a measure space with positive, σ-additive measure μ on a σ-algebra Σ of subsets in X, then $L^2(\mathsf{X}, \mu)$ is a Banach space with norm $||\ ||_2$:

$$||[f]||_2^2 := \int_{\mathsf{X}} \overline{f(x)} f(x) d\mu(x) \,,$$

f being any representative in the equivalence class $[f] \in L^2(\mathsf{X}, \mu)$ (as usual, we shall write f instead of $[f]$).

If $f, g \in L^2(\mathsf{X}, \mu)$ then $\overline{f(x)}g(x) \in L^1(\mathsf{X}, \mu)$, for $(|f(x)|-|g(x)|)^2 \geq 0$ implies

$$2|f(x)||g(x)| \leq |f(x)|^2 + |g(x)|^2 .$$

Hence:

$$(f|g) := \int_{\mathsf{X}} \overline{f(x)}g(x)d\mu(x), \quad f, g \in L^2(\mathsf{X}, \mu) \tag{3.8}$$

is well defined (which follows also from Hölder's inequality, cf. Example 2.29(6)). Elementary features of integrals guarantee the right-hand side of (3.8) is a Hermitian inner product on $L^2(\mathsf{X}, \mu)$, which clearly induces $|| \; ||_2$. Therefore $L^2(\mathsf{X}, \mu)$ is a Hilbert space with inner product (3.8).
(3) If one takes $\mathsf{X} = \mathbb{N}$ and the counting measure μ (Example 2.29(7)), as a subcase of the previous situation we obtain the Hilbert space $\ell^2(\mathbb{N})$ of square-integrable complex sequences, where

$$(\{x_n\}_{n\in\mathbb{N}} | \{y_n\}_{n\in\mathbb{N}}) := \sum_{n=0}^{+\infty} \overline{x_n} y_n . \qquad \blacksquare$$

3.1.2 Riesz's Theorem and Its Consequences

Our next aim is to prove that Hilbert spaces are *reflexive*. In order to do so we need to develop a few tools related to the notion of orthogonal spaces, and prove the celebrated *Riesz theorem*.

Let us recall the definition of *convex set* (Definition 2.65).
Definition *A set* $\varnothing \neq K$ *in a vector space* X *is* **convex** *if*:

$$\lambda u + (1-\lambda)v \in K, \quad for\ any\, \lambda \in [0, 1]\, and\, u, v \in K.$$

Clearly any subspace in X is convex, but not all convex subsets of X are subspaces in X: open balls (with finite radius) in normed spaces are convex as sets but not subspaces. For the next theorem we remind that $< K >$ denotes the subspace in X generated by $K \subset \mathsf{X}$, and $\overline{K}$ is the closure of K.

Theorem 3.13 *Let* $(\mathsf{H}, (\;|\;))$ *be a Hilbert space and* $K \subset \mathsf{H}$ *a non-empty subset. Then*
(a) $K^\perp$ *is a closed subspace of* H.
(b) $K^\perp = < K >^\perp = \overline{< K >}^\perp = \overline{< K >}^\perp$.
(c) *If* K *is closed and convex, for any* $x \in \mathsf{H}$ *there is a unique* $P_K(x) \in K$ *such that* $||x - P_K(x)|| = \min\{||x-y|| \mid y \in K\}$, *where* $|| \; ||$ *is the norm induced by* $(\,|\,)$.
(d) *If* K *is a closed subspace, any vector* $x \in \mathsf{H}$ *decomposes in a unique fashion as* $z_x + y_x$ *with* $z_x \in \mathsf{K}$ *and* $y_x \in K^\perp$, *so that:*

$$\mathsf{H} = K \oplus K^{\perp} . \tag{3.9}$$

Moreover, $z_x := P_K(x)$.
(e) $(K^{\perp})^{\perp} = \overline{< K >}$.

Remark 3.14 Actually, (a) and (b) hold also on more general spaces than Hilbert spaces; it is enough to have an inner product space with the associated topology. ∎

Proof of Theorem 3.13. (a) $K^{\perp}$ is a subspace by the linearity of the inner product. By its continuity it follows that if $\{x_n\} \subset K^{\perp}$ converges to $x \in \mathsf{H}$ then $(x|y) = 0$ for any $y \in K$, hence $x \in K^{\perp}$. So $K^{\perp}$, containing all its limit points, is closed.
(b) The first identity is trivial by definition of orthogonality and by linearity of the inner product. The second relation follows immediately from (a). As for the third one, since $< K > \subset \overline{< K >}$ we have $< K >^{\perp} \supset \overline{< K >}^{\perp}$. But $< K >^{\perp} \subset \overline{< K >}^{\perp}$ by continuity, so $< K >^{\perp} = \overline{< K >}^{\perp}$, ending the chain of equalities, because we know $< K >^{\perp} = \overline{< K >^{\perp}}$.
(c) Let $0 \leq d = \inf_{y \in K} ||x - y||$ (this exists since the set of distances $||x - y||$ with $y \in K$ is bounded below and non-empty). Define a sequence $\{y_n\} \subset K$ such that $||x - y_n|| \to d$. We will show it is a Cauchy sequence. From the parallelogram rule (3.3), where x, y are replaced by $x - y_n$ and $x - y_m$, we have

$$||y_n - y_m||^2 = 2||x - y_n||^2 + 2||x - y_m||^2 - ||2x - y_n - y_n||^2 .$$

As $y_n/2 + y_m/2 \in K$ under the convexity assumption, and since d is the infimum of the numbers $||x - y||$ when $y \in K$, it follows $||2x - y_n - y_n||^2 = 4||x - (y_n + y_m)/2||^2 \geq 4d^2$. Given $\varepsilon > 0$, and taking n, m large enough, we have: $||x - y_n||^2 \leq d^2 + \varepsilon$, $||x - y_m||^2 \leq d^2 + \varepsilon$, whence

$$||y_n - y_m||^2 \leq 4(d^2 + \varepsilon) - 4d^2 = 4\varepsilon .$$

So the sequence is Cauchy. As H is complete, $\{y_n\}$ converges to some $y \in K$ because K is closed. The norm is continuous, so $d = ||x - y||$. We claim $y \in K$ is the unique point satisfying $d = ||x - y||$. For any other $y' \in K$ with the same property:

$$||y - y'||^2 = 2||x - y||^2 + 2||x - y'||^2 - ||2x - y - y'||^2 \leq 2d^2 + 2d^2 - 4d^2 = 0 ,$$

by the parallelogram rule; we have used, above, the fact that $||2x - y - y'||^2 = 4||x - (y + y')/2||^2 \geq 4d^2$ (K is convex, d is the infimum of $||x - z||$ when $z \in K$, so $y/2 + y'/2 \in K$). As $||y - y'|| = 0$ we have $y = y'$. Thus $P_K(x) := y$ fulfils all requirements.
(d) Take $x \in \mathsf{H}$, and $x_1 \in K$ with smallest distance from x. Set $x_2 := x - x_1$, and we will show $x_2 \in K^{\perp}$. Pick $y \in K$, so the map $\mathbb{R} \ni t \mapsto f(t) := ||x - x_1 + ty||^2$ has a minimum at $t = 0$. This is true if K is a subspace, so that $-x_1 + ty \in K$ for any $t \in \mathbb{R}$ if $x_1, y \in K$. Hence its derivative vanishes at $t = 0$:

$$f'(0) = \lim_{t\to 0} \frac{||x_2 + ty||^2 - ||x_2||^2}{t} = (x_2|y) + (y|x_2) = 2Re(x_2|y) \, .$$

Therefore $Re(x_2|y) = 0$. Replacing y by iy tells that the imaginary part of $(x_2|y)$ is zero too, so $(x_2|y) = 0$ and $x_2 \in K^\perp$. We have proved $< K, K^\perp >= \mathsf{H}$. There remains to show $K \cap K^\perp = \{\mathbf{0}\}$. But this is obvious because if $x \in K \cap K^\perp$, x must be orthogonal to itself, so $||x||^2 = (x|x) = 0$ and $x = \mathbf{0}$.
(e) Any $y \in K$ is orthogonal to every element of $K^\perp$; by linearity and continuity of the inner product this is true also when $y \in \overline{< K >}$. In other words,

$$\overline{< K >} \subset (K^\perp)^\perp \, . \tag{3.10}$$

Using (d) and replacing K with the closed subspace $\overline{< K >}$ we obtain $\mathsf{H} = \overline{< K >} \oplus \overline{< K >}^\perp$. By (b) that decomposition is equivalent to

$$\mathsf{H} = \overline{< K >} \oplus K^\perp \, . \tag{3.11}$$

If $u \in (K^\perp)^\perp$, by (3.11) there is a splitting into orthogonal ($(u_0|v) = 0$) vectors $u = u_0 + v$, with $u_0 \in \overline{< K >}$ and $v \in K^\perp$, and hence $(u|v) = (v|v)$. But $(u|v) = 0$ ($u \in (K^\perp)^\perp$ and $v \in K^\perp$) so $(v|v) = 0$ and therefore $(K^\perp)^\perp \ni u = u_0 \in \overline{< K >}$. We conclude $\overline{< K >} \supset (K^\perp)^\perp$, and hence the claim, by (3.10). $\square$

From (b) and (d) descends an immediate corollary.

Corollary 3.15 (to Theorem 3.13) *If S is a subset in a Hilbert space H, $< S >$ is dense in H if and only if $S^\perp = \{\mathbf{0}\}$.*

We are ready to state and prove a theorem due to F. Riesz, by far the most important theorem in the theory of Hilbert spaces.

Theorem 3.16 (Riesz) *Let $(\mathsf{H}, (\;|\;))$ be a Hilbert space. For any continuous linear functional $f : \mathsf{H} \to \mathbb{C}$ there exists a unique element $y_f \in \mathsf{H}$ such that:*

$$f(x) = (y_f|x) \quad \textit{for any } x \in \mathsf{H} \, .$$

The map $\mathsf{H}' \ni f \mapsto y_f \in \mathsf{H}$ is a bijection.

Proof Let us prove that for any $f \in \mathsf{H}'$ such a vector $y_f \in \mathsf{H}$ exists. The *null space* of f, $Kerf := \{x \in \mathsf{H} \mid f(x) = 0\}$, is a closed subspace since f is continuous. As H is a Hilbert space, $\mathsf{H} = Kerf \oplus (Kerf)^\perp$ by Theorem 3.13. If $Kerf = \mathsf{H}$ we define $y_f = \mathbf{0}$ and the proof ends. If $Kerf \neq \mathsf{H}$ we shall show $(Kerf)^\perp$ has dimension 1. Let $\mathbf{0} \neq y \in (Kerf)^\perp$. Then $f(y) \neq 0$ ($y \notin Kerf$!). For any $z \in (Kerf)^\perp$, the vector $z - \frac{f(z)}{f(y)} y$ belongs to $(Kerf)^\perp$, being a linear combination of elements in $(Kerf)^\perp$. But $z - \frac{f(z)}{f(y)} y \in Kerf$ as well, by the linearity of f. Therefore $z - \frac{f(z)}{f(y)} y \in Kerf \cap (Kerf)^\perp$, and $z - \frac{f(z)}{f(y)} y = \mathbf{0}$. So any other vector $z \in (Kerf)^\perp$ is a linear combination $z = \frac{f(z)}{f(y)} y$ of y, meaning y is a basis for $(Kerf)^\perp$. If y is as above, define:

$$y_f := \frac{\overline{f(y)}}{(y|y)} y$$

and we now show y_f represents f in the sense required. If $x \in \mathsf{H}$, we decompose $x = n + x^\perp$ along $Ker f \oplus (Ker f)^\perp$, where

$$x^\perp = \frac{f(x^\perp)}{f(y)} y = \frac{f(x)}{f(y)} y,$$

because $f(x^\perp) = f(x)$ (by linearity, since $f(n) = 0$). So

$$(y_f|x) = \left(\frac{\overline{f(y)}}{(y|y)} y \middle| n + \frac{f(x)}{f(y)} y \right) = 0 + \frac{f(y)}{f(y)} \frac{(y|y)}{(y|y)} f(x) = f(x)\,.$$

The function $\mathsf{H}' \ni f \mapsto y_f \in \mathsf{H}$ is well defined, i.e. f determines y_f uniquely: if $(y|x) = (y'|x)$ for any $x \in K$ then $(y - y'|x) = 0$ for any $x \in K$; and choosing $x = y - y'$ gives $||y - y'||^2 = (y - y'|y - y') = 0$, so $y = y'$. Injectivity is an easy consequence of having $f(x) = (y_f|x)$. The map $\mathsf{H}' \ni f \mapsto y_f \in \mathsf{H}$ is further onto, because, for any $y \in \mathsf{H}$, $\mathsf{H} \ni x \mapsto (y|x)$ is a point in H' by linearity and continuity of the inner product. □

Corollary 3.17 (to Riesz's theorem) *Every Hilbert space is reflexive.*

Proof First of all we can endow H' with an inner product $(f|g)' := (y_g|y_f)$, where $f, g \in \mathsf{H}'$ with $f(x) = (y_f|x)$ and $g(x) = (y_g|x)$, $x \in \mathsf{H}$. The norm induced by $(\ |\ )'$ on H' coincides with the norm of H'

$$||f|| := \sup_{||x||=1} |f(x)|\,,$$

for which H' is complete (Theorem 2.44). By Theorem 3.16 we may write, in fact,

$$||f|| = \sup_{||x||=1} |(y_f|x)|\,,$$

and the Cauchy–Schwarz inequality implies $||f|| \leq ||y_f||$. We also have $|(y_f|x)| = ||y_f||$ for $x = y_f/||y_f||$, hence $||f|| = ||y_f||$, which is precisely the norm induced by $(\ |\ )'$.

Therefore $(\mathsf{H}', (|)')$ is a Hilbert space and $(\mathsf{H}')'$ its dual. Theorem 3.16 guarantees that for any element in $(\mathsf{H}')'$, say F, there exists $g_F \in \mathsf{H}'$ such that $F(f) = (g_F|f)'$ for any $f \in \mathsf{H}'$. But $(g_F|f)' = (y_f|y_{g_F}) = f(y_{g_F})$. We have thus shown, for any $F \in (\mathsf{H}')'$, the existence (and uniqueness, by Corollary 2.59 to Hahn–Banach) of a vector $y_{g_F} \in \mathsf{H}$ such that:

$$F(f) = f(y_{g_F})$$

for any $f \in \mathsf{H}'$. This is the reflexivity of H. □

Remark 3.18 From this proof we see that the topological dual H', equipped with inner product $(\,|\,)'$, $(f|g)' := (y_g|y_f)$, is a Hilbert space. The map $\mathsf{H}' \ni f \mapsto y_f \in \mathsf{H}$ is antilinear, 1-1, onto and it preserves the inner product by construction. In this sense H and H' are *anti-isomorphic*. ■

3.2 Hilbert Bases

Now we can introduce the mathematical arsenal attached to the notion of a *Hilbert basis*. This is a well-known generalisation, to infinite dimensions, of an orthonormal basis. We shall work in the most general setting, where Hilbert spaces are *not necessarily separable* and a basis can have any cardinality, even uncountable.

First we have to explain the meaning of an infinite sum of non-negative numbers, often over an uncountable set. An **indexed set** $\{\alpha_i\}_{i\in I}$ is a function $I \ni i \mapsto \alpha_i$. The set I is the **set of indices** and α_i is the i**th** element of the indexed set. Note that it can happen that $\alpha_i = \alpha_j$ for $i \neq j$.

Definition 3.19 If $A = \{\alpha_i\}_{i\in I}$ is a non-empty set of non-negative reals indexed by a set I of arbitrary cardinality, the **sum of the indexed set** A is the number, in $[0, +\infty) \cup \{+\infty\}$, defined by :

$$\sum_{i\in I} \alpha_i := \sup \left\{ \sum_{j\in F} \alpha_j \;\middle|\; F \subset I\,, \quad F \text{finite} \right\} . \tag{3.12}$$

Remark 3.20 From now on a set will be called **countable** when it can be mapped bijectively to the natural numbers $\mathbb{N}$. Thus, here, a *finite* set is *not* countable. ■

Proposition 3.21 *In relation to Definition 3.19 we have:*
(a) *the sum of the set A coincides with the sum $\sum_{i\in I} \alpha_i$ if I is finite, and with the sum of the series $\sum_{n=0}^{+\infty} \alpha_{i_n}$ if I is countable, irrespective of the ordering, i.e. independently of the bijection $\mathbb{N} \ni n \mapsto i_n \in I$. (The series always converges, possibly to $+\infty$, because its terms are non-negative.)*
(b) *If the sum of the set A is finite, the subset of I for which $\alpha_i \neq 0$ is finite or countable. By restricting to this subset, the sum of A equals either the sum of finitely many terms or the sum of the series, as in (a).*
(c) *If μ is the counting measure on I, defined by the σ-algebra of the power set of I (if $J \subset I$ then $\mu(J) \leq +\infty$ is the cardinality of J by definition):*

$$\sum_{i\in I} \alpha_i = \int_A \alpha_i \, d\mu(i) . \tag{3.13}$$

Proof (a) The case when I is finite is obvious, so we look at I countable and suppose to have chosen a particular ordering on I, so that we can write $A = \{\alpha_{i_n}\}_{n\in\mathbb{N}}$. We will show that the sum $\sum_{n=0}^{+\infty} \alpha_{i_n}$ of $\{\alpha_{i_n}\}_{n\in\mathbb{N}}$ coincides with the sum of (3.12), which by definition does not depend on the chosen ordering. Because of (3.12) we have:

$$\sum_{n=0}^{N} \alpha_{i_n} \leq \sum_{i\in I} \alpha_i \ .$$

The limit as $N \to +\infty$ exists and equals the supremum of the set of partial sums, since the latter are non-decreasing. Therefore:

$$\sum_{n=0}^{+\infty} \alpha_{i_n} \leq \sum_{i\in I} \alpha_i \ . \tag{3.14}$$

On the other hand, if $F \subset I$ is finite, we may fix N_F large enough so that $\{\alpha_i\}_{i\in F} \subset \{\alpha_{i_0}, \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{N_F}}\}$. Thus

$$\sum_{i\in F} \alpha_i \leq \sum_{n=0}^{N_F} \alpha_{i_n} \ ,$$

Taking now the supremum over finite sets $F \subset I$, and remembering that the supremum of the partial sums is the sum of the series, gives

$$\sum_{i\in I} \alpha_i \leq \sum_{n=0}^{+\infty} \alpha_{i_n} \ . \tag{3.15}$$

Then (3.14) and (3.15) produce the claim.
(b) Suppose $S < +\infty$, $S \geq 0$, is the sum of the set A. If $S = 0$ all elements of A are zero and the proof ends, so assume $S > 0$. Any α_i is contained in $[0, S]$, for otherwise the sum would be larger than S, and in particular $\alpha_i \neq 0$ implies $\alpha_i \in (0, S]$. Define $S_n := S/n$, $n = 1, 2, \ldots$. If N_k denotes the number of indices $i \in I$ for which α_i belongs in $(S_{k+1}, S_k]$, then surely $S \geq S_{k+1}N_k$, hence N_k is finite for any k. But $\cup_{k=1}^{+\infty}(S_{k+1}, S_k] = (0, S]$, so all $\alpha_i \neq 0$ are accounted for. There can be at most countably many of these values, since: (i) there are countably many intervals $(S_{k+1}, S_k]$ and (ii) each interval contains a finite number of $\alpha_i \neq 0$.
(c) Since any function is measurable with respect to the given measure (the σ-algebra is the power set), identity (3.13) is an immediate consequence of the definition of integral of a positive function (cf. Sect. 1.4.3). $\square$

Now we can define, step by step, *complete orthonormal systems*, also known as *Hilbert bases*.

Definition 3.22 (*Hilbert basis*) Let $(\mathsf{X}, (\ |\ ))$ be an inner product space and $\varnothing \neq N \subset \mathsf{X}$ a subset.

(a) N is an **orthogonal system** (of vectors) if (i) $N \not\ni \mathbf{0}$ and (ii) $x \perp y$ for any $x, y \in N, x \neq y$.
(b) N is an **orthonormal system** if its elements are mutually orthogonal unit vectors, $(x|x) = 1$ for any $x \in N$.
If $(\mathsf{H}, (\ |\ ))$ is a Hilbert space, $N \subset \mathsf{H}$ is a **complete orthonormal system**, or a **Hilbert basis**, if it is orthonormal and:

$$N^{\perp} = \{\mathbf{0}\}\,. \tag{3.16}$$

When no confusion arises, a Hilbert basis will be simply referred to as a **basis**.

Remark 3.23 Any orthogonal system N is made of *linearly independent* vectors: if $F \subset N$ is finite and $\mathbf{0} = \sum_{x \in F} \alpha_x x$, then

$$0 = \left(\sum_{x \in F} \alpha_x x \,\middle|\, \sum_{y \in F} \alpha_y y \right) = \sum_{x \in F} \sum_{y \in F} \overline{\alpha_x} \alpha_y (x|y) = \sum_{x \in F} |\alpha_x|^2 ||x||^2\,.$$

As $||x|| > 0$ and $|\alpha_x|^2 \geq 0$, necessarily $|\alpha_x| = 0$, so $\alpha_x = 0$, for any $x \in F$. ■

Theorem 3.24 (Bessel's inequality) *For any orthonormal system $N \subset \mathsf{X}$ in an inner product space $(\mathsf{X}, (\ |\ ))$,*

$$\sum_{z \in N} |(x|z)|^2 \leq ||x||^2 \quad \textit{for any } x \in \mathsf{X}\,. \tag{3.17}$$

In particular, only a countable number of products $(x|z)$ are non-zero, at most.

Proof By Definition 3.19 and Proposition 3.21(b) the claim holds if inequality (3.17) is true for all finite $F \subset N$. So let $F = \{z_1, \ldots, z_n\}$, $x \in \mathsf{X}$ and take $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Expanding $||x - \sum_{k=1}^{n} \alpha_k z_k||^2$ in terms of the inner product of X and because of the orthonormality of z_p and z_q, plus the inner product's linearity, we obtain:

$$\left|\left| x - \sum_{k=1}^{n} \alpha_k z_k \right|\right|^2 = ||x||^2 + \sum_{k=1}^{n} |\alpha_k|^2 - \sum_{k=1}^{n} \alpha_k (x|z_k) - \sum_{k=1}^{n} \overline{\alpha_k (x|z_k)}\,.$$

The right-hand side equals:

$$||x||^2 - \sum_{k=1}^{n} |(x|z_k)|^2 + \sum_{k=1}^{n} \left(|(x|z_k)|^2 - \alpha_k (x|z_k) - \overline{\alpha_k (x|z_k)} + |\alpha_k|^2 \right)\,.$$

So,

$$\left|\left| x - \sum_{k=1}^{n} \alpha_k z_k \right|\right|^2 = ||x||^2 - \sum_{k=1}^{n} |(x|z_k)|^2 + \sum_{k=1}^{n} |\, (z_k|x) - \alpha_k \,|^2\,.$$

On the right there is only one absolute minimum point $\alpha_k = (z_k|x)$, $k = 1, \ldots, n$. Therefore

$$0 \leq \left|\left| x - \sum_{k=1}^{n} (z_k|x) z_k \right|\right|^2 = ||x||^2 - \sum_{k=1}^{n} |(x|z_k)|^2 ,$$

and finally:

$$\sum_{k=1}^{n} |(x|z_k)|^2 \leq ||x||^2 .$$

□

Lemma 3.25 *Let* $\{x_n\}_{n \in \mathbb{N}}$ *be a countable orthogonal system indexed by* $\mathbb{N}$ *in the Hilbert space* $(\mathsf{H}, (\ |\ ))$*, and let* $||\ \ ||$ *be the norm induced by* $(\ |\ )$*. If*

$$\sum_{n=0}^{+\infty} ||x_n||^2 < +\infty , \tag{3.18}$$

then:
(a) *there exists a unique vector* $x \in \mathsf{H}$ *such that*

$$\sum_{n=0}^{+\infty} x_n = x , \tag{3.19}$$

where convergence is understood as convergence of partial sums in the topology induced by $||\ \ ||$*;*
(b) *the series (3.19) can be rearranged, i.e.*

$$\sum_{n=0}^{+\infty} x_{f(n)} = x \tag{3.20}$$

for any bijection $f : \mathbb{N} \to \mathbb{N}$*.*

Proof (a) Take $A_n := \sum_{k=0}^{n} x_k$. By the orthonormality of the x_k and the definition of norm via the inner product we have, for $n > m$:

$$||A_n - A_m||^2 = \sum_{k=m+1}^{n} ||x_k||^2 .$$

Since the series converges,

$$||A_n - A_m||^2 = \sum_{k=m+1}^{n} ||x_k||^2 \leq \sum_{k=m+1}^{+\infty} ||x_k||^2 \to 0 \quad \text{as } m \to +\infty,$$

which in turn implies that the partial sums $\{A_n\}$ are a Cauchy sequence. Since H is complete, the sequence has a limit point $x \in \mathsf{H}$, and so does the series. But H is normed, and the Hausdorff property tells that limits, x included, are unique.
(b) Fix a bijective map $f : \mathbb{N} \to \mathbb{N}$. Set, as above, $A_n := \sum_{k=0}^{n} x_k$ and $\sigma_n := \sum_{k=0}^{n} x_{f(k)}$. The positive-term series $\sum_{k=0}^{+\infty} ||x_{f(k)}||^2$ converges because its partial sums are smaller than the converging series $\sum_{k=0}^{+\infty} ||x_k||^2$. From part (a) the limit in H of σ_n exists, and the rearranged series will converge in H, too. We claim this limit is precisely x.

Define $r_n := \max\{f(0), f(1), \ldots, f(n)\}$, so

$$||A_{r_n} - \sigma_n||^2 \leq \sum_{k \in J_n} ||x_k||^2$$

where J_n arises from

$$\{0, 1, 2, \ldots, \max\{f(0), f(1), \ldots, f(n)\}\}$$

by erasing $f(0), f(1), \ldots, f(n)$. By bijectivity the remaining elements correspond to certain points of the infinite set

$$\{f(n+1), f(n+2), \ldots\} .$$

Therefore

$$||A_{r_n} - \sigma_n||^2 \leq \sum_{k \in J_n} ||x_k||^2 \leq \sum_{k=n+1}^{+\infty} ||x_{f(k)}||^2 . \tag{3.21}$$

As $\sum_{k=0}^{+\infty} ||x_{f(k)}||^2 < +\infty$, relation (3.21) implies:

$$\lim_{n \to +\infty} (A_{r_n} - \sigma_n) = 0 .$$

On the other hand $r_n \geq n$ (f is injective, and if we had

$$\max\{f(0), f(1), \ldots, f(n)\} < n ,$$

the various $f(n)$ should be $n+1$ non-negative integers smaller than n, a contradiction). Therefore $\lim_{n \to +\infty} r_n = +\infty$, so:

$$x = \lim_{n \to +\infty} A_n = \lim_{n \to +\infty} A_{r_n} = \lim_{n \to +\infty} \sigma_n . \qquad \square$$

We can now state, and prove, the fundamental theorem about Hilbert bases, according to which Hilbert bases generalise orthonormal bases in inner product spaces *of finite dimension*. The novelty is that, at present, also *infinite linear combinations* are allowed, using the topology of H: any element of a Hilbert space can be written in a unique fashion as an infinite linear combination of basis elements.

Irrespective of the existence of bases, by Zorn's lemma (or equivalently, the axiom of choice) there are also 'algebraic' bases that require no topology. The difference between a Hilbert basis and an algebraic basis is that the latter concerns *finite combinations* only: despite the basis has infinite cardinality, any vector in the (Hilbert) space can be decomposed, uniquely, as a finite linear combination of the basis' elements.

Theorem 3.26 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space and* $N \subset \mathsf{H}$ *an orthonormal system. The following facts are equivalent:*
(a) *N is a Hilbert basis (an orthonormal system with $N^\perp = \{\mathbf{0}\}$).*
(b) *Given $x \in \mathsf{H}$, (at most) countably many products $(z|x)$ are non-zero for all $z \in N$, and:*

$$x = \sum_{z \in N} (z|x) z \,, \tag{3.22}$$

where the series converges in the sense that partial sums converge in the inner product topology.
(c) *Given $x, y \in \mathsf{H}$, (at most) countably many products $(z|x)$, $(y|z)$ are non-zero for all $z \in N$, and*

$$(x|y) = \sum_{z \in N} (x|z)(z|y) \,. \tag{3.23}$$

(d) *If $x \in \mathsf{H}$:*

$$||x||^2 = \sum_{z \in N} |(z|x)|^2 \,. \tag{3.24}$$

in the sense of Definition 3.19.
(e) $\overline{< N >} = \mathsf{H}$*, i.e. the span of N is dense in H.*
Under any of the above properties, in (3.22) and (3.23) the indexing order of non-null coefficients of $(x|z)$, $(z|x) = \overline{(x|z)}$ and $(z|y)$ is irrelevant.

Proof (a) $\Rightarrow$ (b). By Theorem 3.24 only countably many coefficients $(z|x)$ are non-null, at most. Indicate by $(z_n|x)$, $n \in \mathbb{N}$, these numbers and fix $S_N := \sum_{n=0}^{N} (z_n|x) z_n$. The system $\{(z_n|x)z_n\}_{n\in\mathbb{N}}$ is by construction orthogonal, and because $||z_n|| = 1$ Bessel's inequality implies $\sum_{n=0}^{+\infty} ||(z_n|x)z_n||^2 < +\infty$. By Lemma 3.25(a) the series (3.22) converges to a unique $x' \in \mathsf{H}$, $x' = \sum_{n=0}^{+\infty} (z_n|x) z_n$. Moreover, the series can be rearranged, with the same limit x' by Lemma 3.25(b). We claim $x' = x$. The linearity and continuity of the inner product force, for $z' \in N$:

$$(x - x'|z') = (x|z') - \sum_{z \in N} (x|z)(z|z') = (x|z') - (x|z') = 0$$

where we have used the fact that the set of coefficients z is an orthonormal system. Since $z' \in N$ is arbitrary, $x - x' \in N^\perp$ and so $x - x' = \mathbf{0}$, as $N^\perp = \{\mathbf{0}\}$ by assumption.

This proves that (3.22) holds independently from the way we index the coefficients $(z|x) \neq 0$.
(b) $\Rightarrow$ (c). If (b) holds, (c) is an obvious consequence, due to continuity and linearity of the inner product, plus the fact N is orthonormal.
(c) $\Rightarrow$ (d). When $y = x$, (d) is a special case of (c).
(d) $\Rightarrow$ (a). If (d) is true and $x \in \mathsf{H}$ is such that $(x|z) = 0$ for any $z \in N$, then $||x|| = 0$, i.e. $x = \mathbf{0}$. In other words $N^\perp = \{\mathbf{0}\}$, that is to say (a) holds.
So, we have proved (a), (b), (c) and (d) are equivalent. To finish notice that (b) implies immediately (e), while (e) implies (a): if $x \in N^\perp$, the inner product's linearity gives $x \in < N >^\perp \subset \overline{< N >}^\perp$. But Theorem 3.13(b) says $\overline{< N >}^\perp = \overline{< N >}^\perp$. Since $\overline{< N >} = \mathsf{H}$ by hypothesis, $x \in \mathsf{H}^\perp = \{\mathbf{0}\}$. But this is (a), given that $N^\perp = \{\mathbf{0}\}$.

The fact that the complex series in (3.23) can be rearranged to have the same sum relies on the following argument. Consider the set

$$A := \{z \;\;|\;\; (x|z) \neq 0 \text{ or } (y|z) \neq 0\}\,,$$

which is countable. The Cauchy–Schwarz inequality in $\ell^2(A)$ produces:

$$\sum_{z\in A} |(x|z)||(z|y)| \leq \left(\sum_{z\in A} |(x|z)|^2\right)^{1/2} \left(\sum_{z\in A} |(y|z)|^2\right)^{1/2} < +\infty$$

by (d). Hence the series $\sum_{z\in N}(x|z)(z|y) = \sum_{z\in A}(x|z)(z|y)$ can be rearranged as one likes, because it converges absolutely. □

Zorn's lemma now guarantees each Hilbert space admits a complete orthonormal system.

Theorem 3.27 *Every Hilbert space* $\mathsf{H} \neq \{\mathbf{0}\}$ *admits a Hilbert basis.*

Proof Let $\mathsf{H} \neq \{\mathbf{0}\}$ be a Hilbert space and consider the collection $\mathscr{A}$ of orthonormal systems in H. Define on $\mathscr{A}$ the partial order relation given by set-theoretical inclusion. By construction any ordered subset $\mathscr{E} \subset \mathscr{A}$ is bounded above by the union of all elements of $\mathscr{E}$. Zorn's lemma tells us that $\mathscr{A}$ has maximal element N. Therefore there are in H no vectors that are normal to every element in N, non-zero and not belonging to N itself. This means N is a complete orthonormal system. □

Before moving on to separable Hilbert spaces, let us give another important result from the general theory.

Theorem 3.28 *Let* H *be a Hilbert space with Hilbert basis* N*. Then*
(a) H *is isomorphic, as Hilbert space, to* $L^2(N, \mu)$*, where* μ *is the positive counting measure of* N *(see Examples 2.29(6, 7) and 3.12(2)); the unitary transformation that identifies the two spaces is*

$$\mathsf{H} \ni x \mapsto \{(z|x)\}_{z\in N} \in L^2(N, \mu)\;; \tag{3.25}$$

(b) *all Hilbert bases of* H *have the same cardinality (that of* N*), called the* **dimension** *of the Hilbert space. (If* $\mathsf{H} = \{\mathbf{0}\}$ *the dimension of* H *is assumed to be* 0.*)*
(c) *If* H_1 *is a Hilbert space with the same dimension as* H*, the two spaces are isomorphic as Hilbert spaces.*

Proof (a) The map $U : \mathsf{H} \ni x \mapsto \{(z|x)\}_{z\in N} \in L^2(N, \mu)$ is well defined because if $x \in \mathsf{H}$ and N is a basis, then property (d) of Theorem 3.26 holds, according to which $\{(z|x)\}_{z\in N} \in L^2(N, \mu)$. This function is definitely 1-1: if $x, x' \in \mathsf{H}$ give equal coefficients $(z|x) = (z|x')$ for any $z \in N$, then $x = x'$ by Theorem 3.26(b). The map is onto as well: if $\{\alpha_z\}_{z\in N} \in L^2(N, \mu)$, so $\sum_{z\in N} |\alpha_z|^2 < +\infty$, by Lemma 3.25 there is $x := \sum_{z\in N} \alpha_z z$ and $(z|x) = \alpha_z$ by inner product continuity and orthonormality of N. Now Theorem 3.26(c) implies U is isometric. Therefore $U : \mathsf{H} \to L^2(N, \mu)$ is a unitary operator, making H and $L^2(N, \mu)$ isomorphic Hilbert spaces.
(b) If one Hilbert basis has finite cardinality c, it must be an algebraic basis for H. Elementary geometric techniques allow to prove that if a basis of finite cardinality c exists, then any other set of linearly independent vectors M has cardinality $\leq c$, and the maximum is reached if and only if M spans the whole space. Since a basis, being an orthogonal system, is made of linearly independent vectors, we conclude that any basis of H has cardinality $\leq c$, hence $= c$ because it spans H finitely. This forbids the situation where one basis is finite and another infinite. So let N and M be bases di H of infinite cardinality. If $x \in M$, define $N_x := \{z \in N \mid (x|z) \neq 0\}$. As $1 = (z|z) = \sum_{x\in M} |(z|x)|^2$, we must have, for any $z \in N$, an element $x \in M$ such that $z \in N_x$. Therefore $N \subset \cup_{x\in M} N_x$ and then the cardinality of N will be less than or equal to that of $\cup_{x\in M} N_x$. But the latter is the cardinality of M because any N_x is at most countable by Theorem 3.26(b). So the cardinality of N does not exceed the cardinality of M. Swapping the roles of N and M we obtain that the cardinality of M is not larger than that of N, and the theorem of Schröder–Bernstein ensures the two cardinalities are equal.
(c) Let N and N_1 be bases of H and H_1 respectively, and suppose they have the same cardinality. Then there is a bijective map taking points in N to points in N_1 that induces a natural isomorphism V of inner product spaces between the L^2 space on N and the L^2 space on N_1 with respect to the counting measure. Therefore V is an isomorphism of Hilbert spaces. If $U_1 : \mathsf{H}_1 \to L^2(N_1, \mu)$ is the isomorphism analogous to the aforementioned $U : \mathsf{H} \to L^2(N, \mu)$, then $UVU_1^{-1} : \mathsf{H}_1 \to \mathsf{H}$ is a unitary transformation, by construction, making H and H_1 isomorphic spaces. □

So-called *separable* Hilbert spaces are particularly interesting in physics.

Definition 3.29 A Hilbert space is **separable** if it admits a countable dense subset.

There is a well-known characterisation of separability.

Theorem 3.30 *Let* $\mathsf{H} \neq \{\mathbf{0}\}$ *be a Hilbert space.*
(a) H *is separable if and only if either* $\dim \mathsf{H} < \infty$ *or it has a countable Hilbert basis.*
(b) *If* H *is separable, then every basis is either finite, with cardinality equal to* $\dim \mathsf{H}$*, or countable.*
(c) *If* H *is separable then it is isomorphic either to* $\ell^2(\mathbb{N})$*, or to the standard* $\mathbb{C}^n$*, where* $n = \dim \mathsf{H}$.

Proof (a) If the Hilbert space has a finite or countable basis, Theorem 3.26(b) ensures that a countable dense set exists, because rational numbers are dense in the reals. This set consists clearly of finite linear combinations of basis elements with complex coefficients having rational real and imaginary parts. The proof is easy and left to the reader. Conversely, suppose a Hilbert space is separable. By Theorem 3.27 we know bases exist, and we want to show that any basis must be countable at most.

Suppose, by contradiction, that N is an uncountable basis for the separable Hilbert space H. For any chosen $z, z' \in N$, $z \neq z'$, any point $x \in \mathsf{H}$ satisfies $||z - z'|| \leq ||x - z'|| + ||z - x||$, by the triangle inequality induced by the inner product. At the same time $\{z, z'\}$ is an orthonormal system, so $||z - z'||^2 = (z - z'|z - z') = ||z||^2 + ||z'||^2 + 0 = 1 + 1 = 2$. Hence $||x - z|| + ||x - z'|| \geq \sqrt{2}$. This implies that two open balls of radius $\varepsilon < \sqrt{2}/2$ centred at z and z' are disjoint, irrespective of how we pick $z, z' \in N$ with $z \neq z'$. Call $\{B(z)\}_{z \in N}$ a family of such balls parametrised by their centres $z \in N$. If $D \subset \mathsf{H}$ is a countable dense set (the space is separable), then for any $z \in N$ there exists $x \in D$ with $x \in B(z)$. The balls are pairwise disjoint, so there will be one x for each ball, all different from one another. But the cardinality of $\{B(z)\}_{z \in N}$ is not countable, hence neither D can be countable, a contradiction.

Although (b) and (c) are straightforward consequences of Theorem 3.28, for the sake of the argument let us outline a proof.
(b) From the basic theory, if a (Hilbert or algebraic) basis is finite, the cardinality of any other basis equals the dimension of the space. Moreover, any linearly independent set (viewed as basis) cannot contain a number of vectors exceeding the dimension. From this, if a Hilbert space is separable and one of its bases is finite, then all bases are finite and have cardinality $\dim \mathsf{H}$. Under the same hypotheses, if a basis is countable then any other is countable by (a).
(c) Fix a basis N. Using Theorem 3.26 one verifies quickly that the map sending $\mathsf{H} \ni x = \sum_{u \in N} \alpha_u u$ to the (finite or infinite) family $\{\alpha_u\}_{u \in N}$ is an isomorphism sending H to $\mathbb{C}^n$ (if $\dim \mathsf{H}$ is finite) or to $\ell^2(\mathbb{N})$ (if $\dim \mathsf{H}$ is infinite). $\square$

Here is another useful proposition about separable Hilbert spaces.

Proposition 3.31 *Let* $(\mathsf{H}, (\ |\ ))$ *be a Hilbert space with* $\mathsf{H} \neq \{\mathbf{0}\}$.
(a) *If* $Y := \{y_n\}_{n \in \mathbb{N}}$ *is a set of linearly independent vectors and* $Y^\perp = \{\mathbf{0}\}$*, or equivalently* $\overline{< Y >} = \mathsf{H}$*, then* H *is separable and there exists a basis* $X := \{x_n\}_{n \in \mathbb{N}}$ *in* H *such that, for any* $p \in \mathbb{N}$*, the span of* $y_0, y_1, \ldots, y_p$ *coincides with the span of* $x_0, x_1, \ldots, x_p$.
(b) *If* H *is separable and* $\mathsf{S} \subset \mathsf{H}$ *is a (non-closed) dense subspace of* H*, then* S *contains a basis of* H.

Proof (a) We shall only sketch the proof, since the argument essentially duplicates the *Gram–Schmidt orthonormalisation process* known from basic geometry courses [Ser94I].

Since $y_0 \neq 0$, we set $x_0 := y_0/||y_0||$. Consider the non-null vector $z_1 := y_1 - (x_0|y_1)x_0$ (recall y_0, y_1 are linearly independent). Clearly x_0, z_1 are not zero, they are orthogonal (so linearly independent) and span the same subspace as y_0, y_1. Setting $x_1 := z_1/||z_1||$ produces an orthonormal set $\{x_0, x_1\}$ spanning the same space as y_0, y_1. The procedure can be iterated inductively, by defining:

$$z_n := y_n - \sum_{k=0}^{n-1}(x_k|y_n)x_k \ ,$$

and considering the set of $x_n := z_n/||z_n||$. By induction it is easy to see $z_0, \ldots, z_k$ are non-null, orthogonal (hence linearly independent) and span the same space generated by the linearly independent $y_0, \ldots, y_k$. If $u \perp y_n$ for any $n \in \mathbb{N}$, then $u \perp x_n$ for any $n \in \mathbb{N}$ (it is enough to express x_n as a linear combination of $y_0, \ldots, y_n$), and conversely (writing y_n as combination of $x_0, \ldots, x_n$). Therefore $X^\perp = Y^\perp = \{\mathbf{0}\}$ and X is a basis for H.

(b) We claim S must contain a subset S_0 that is countable and dense in H. In fact, let $\{y_n\}_{n\in\mathbb{N}}$ be countable and dense in H. For any y_n there is a sequence $\{x_{nm}\}_{m\in\mathbb{N}} \subset \mathsf{S}$ such that $x_{nm} \to y_n$ as $m \to +\infty$. It is straightforward that the countable subset $S_0 := \{x_{nm}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ of S is dense in H. Relabelling the elements of S_0 over the naturals so that $x_1 \neq 0$ we have $S_0 = \{x_q\}_{q\in\mathbb{N}}$. Now we can decompose S_0 in two subsets S_1 and S_2 as follows. The set S_1 contains x_1. Further, if x_2 is linearly independent from x_1 we put x_2 in S_1, otherwise in S_2. If x_3 is linearly independent from $x_1, x_2 \in S_1$ we put it in S_1, otherwise in S_2, and we continue like this until we exhaust S_0. Then by construction S_1 contains a set of linearly independent generators of S_0. Thus $\overline{< S_1 >} \supset \overline{S_0} = \mathsf{H}$. This process builds a complete orthonormal system by *finite* linear combinations of $Y := S_1$, as explained in (a), and so it gives a basis made of elements of S since $\mathsf{S} \supset S_1$ is a subspace. $\square$

Examples 3.32 **(1)** Consider the Hilbert space $L^2([-L/2, L/2], dx)$ (cf. Examples 3.12(2)) where dx is the usual Lebesgue measure on $\mathbb{R}$ and $L > 0$. Take measurable functions (they are continuous)

$$f_n(x) := \frac{e^{i\frac{2\pi n}{L}x}}{\sqrt{L}} \tag{3.26}$$

for $n \in \mathbb{Z}$ and $x \in [-L/2, L/2]$. It is immediate to see that the f_n belong to the space and form an orthonormal system for the inner product:

$$(f|g) := \int_{-L/2}^{L/2} \overline{f(x)}g(x)dx \tag{3.27}$$

of $L^2([-L/2, L/2], dx)$. Consider the Banach algebra $C([-L/2, L/2])$ (a vector subspace of $L^2([-L/2, L/2], dx)$) of continuous maps with supremum norm (Examples 2.29(4, 5)). The subspace $S \subset C([-L/2, L/2])$ spanned by all f_n, $n \in \mathbb{Z}$, is a subalgebra of $C([-L/2, L/2])$. Now, S contains 1, it is closed under complex conjugation and it is not hard to see that it separates points in $[-L/2, L/2]$ (the maps f_n already separate points), so the Stone–Weierstrass Theorem 2.30 guarantees S is dense in $C([-L/2, L/2])$. On the other hand it is well known that continuous maps on $[-L/2, L/2]$ form a dense space in $L^2([-L/2, L/2], dx)$ in the latter's topology [Rud86, p. 85]. At last, the topology of $C([-L/2, L/2])$ is finer than the topology of $L^2([-L/2, L/2], dx)$, because $(f|f) \leq L \sup |f|^2 = L(\sup |f|)^2$ if $f \in C([-L/2, L/2])$. Therefore S is dense in $L^2([-L/2, L/2], dx)$. By Theorem 3.26(e), the vectors f_n form a basis in $L^2([-L/2, L/2], dx)$, making the latter separable.

(2) Consider the Hilbert space $L^2([-1, 1], dx)$, dx being the Lebesgue measure. As in the previous example the Banach algebra $C([-1, 1])$ is dense in $L^2([-1, 1], dx)$ in the latter's topology. In contrast to what we had previously, let

$$g_n(x) := x^n \,, \tag{3.28}$$

for $n = 0, 1, 2, \ldots$, $x \in [-1, 1]$. It can be proved that these vectors are linearly independent. Their span S in $C([-1, 1])$ is a subalgebra in $C([-1, 1])$ that contains 1, is closed under conjugation and separates points. Hence the Stone–Weierstrass Theorem 2.30 implies it is dense in $C([-1, 1])$. By arguing as in the above example S is also dense in $L^2([-1, 1], dx)$. The novelty is that now the functions g_n do not constitute an orthonormal system. However, using Proposition 3.31 we can immediately build a complete orthonormal system on $L^2([-1, 1], dx)$. These basis elements, up to a normalisation, are called **Legendre polynomials**:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n, \quad n = 0, 1, 2, \ldots \,.$$

By definition they obey orthogonality relations:

$$\int_{[-1,1]} P_n(x) P_m(x) dx = \frac{2\delta_{nm}}{2n + 1} \,.$$

(3) In the previous examples we exhibited two separable L^2 spaces. It can be proved that $L^p(\mathsf{X}, \mu)$ $(1 \leq p < +\infty)$ is separable if and only if the measure μ is **separable**, in the following sense. Take the subset of the σ-algebra Σ of μ made of all finite-measure sets and mod out zero-measure sets. The quotient is a *metric space* (cf. Definition 2.82) with distance:

$$d(A, B) := \mu((A \setminus B) \cup (B \setminus A)) \,.$$

The measure μ is said to be separable if this metric space admits a dense and countable subset. Concerning separable measures we have the following result [Hal69].

Proposition 3.33 (On separable L^p measures and spaces) *A σ-additive positive measure μ, and hence also $L^p(\mathsf{X}, \mu)$, is separable if the following conditions hold: (i) μ is σ-finite (X is the union of at most countably many sets of finite measure) and (ii) the σ-algebra of the measure space of μ is generated by a countable collection of measurable sets at most.*

As consequence we have

Proposition 3.34 (On separable Borel measures and L^p spaces) *Every σ-finite Borel measure on a second-countable topological space produces a separable L^p space.*

Remark 3.35 This is the case, in particular, of the L^p space relative to the Lebesgue measure on $\mathbb{R}^n$ restricted to Borel sets in $\mathbb{R}^n$. Note, though, that this L^p space is the same we find by using the entire Lebesgue σ-algebra, since the latter is the completion of the Borel σ-algebra for the Lebesgue measure restricted to Borel subsets, by Proposition 1.57 (see the remark following Proposition 1.66). ■

Positive and σ-additive Borel measures on locally compact Hausdorff spaces are called **Radon measures** if they are regular and if compact sets have finite measure. A Radon measure is σ-finite if the space on which it is defined is σ-compact, i.e. the union of (at most) countably many compact sets.

(4) Consider the space $L^2((a, b), dx)$, with $-\infty \leq a < b \leq +\infty$ and dx being the usual Lebesgue measure on $\mathbb{R}$. From the definitions of the Fourier and Fourier–Plancherel transforms (Proposition 3.115) we infer an extremely useful result:

Let $f : (a, b) \to \mathbb{C}$ be measurable and such that: (1) the set $\{x \in (a, b) \mid f(x) = 0\}$ has zero measure, and (2) there exist constants $C, \delta > 0$ such that $|f(x)| < Ce^{-\delta|x|}$ for any $x \in (a, b)$.

Then the finite span of the maps $x \mapsto x^n f(x)$, $n = 0, 1, 2, \ldots$, is dense in $L^2((a, b), dx)$.

The importance of this fact lies in that it allows to construct with ease bases in $L^2((a, b), dx)$ even when a or b are infinite (a case in which we cannot apply the Stone–Weierstrass Theorem 2.30). In fact, the Gram–Schmidt process applied to $f_n(x) := x^n f(x)$ yields a basis as explained in Proposition 3.31.

For instance, applying Gram–Schmidt to $f(x) := e^{-x^2/2}$ gives, normalisation apart, the basis of $L^2(\mathbb{R}, dx)$ of so-called (normalised) **Hermite functions**:

$$\psi_0(x) = \pi^{-1/4} e^{-x^2/2}$$

and, recursively,

$$\psi_{n+1} = (2(n+1))^{-1/2}\left(x - \frac{d}{dx}\right)\psi_n \quad n = 0, 1, 2, \ldots$$

A computation, still relying on Gram–Schmidt essentially, shows that ψ_n can be obtained alternatively as:

$$\psi_n(x) := (2^n n!\sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2} \quad n = 0, 1, 2, \ldots$$

where H_n is a polynomial of degree $n = 0, 1, 2, \ldots$ called nth **Hermite polynomial**:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad n = 0, 1, 2, \ldots$$

There are orthogonality relations:

$$\int_{\mathbb{R}} e^{-x^2} H_n(x) H_m(x) dx = \delta_{nm} 2^n n! \sqrt{\pi} \,.$$

In QM this particular basis is important when one studies the physical system known as the *one-dimensional harmonic oscillator*.

Applying the same procedure to $f(x) := e^{-x/2}$ produces a basis of $L^2((0, +\infty), dx)$ given, up to normalisation, by **Laguerre's functions** $e^{-x} L_n(x)$, $n = 0, 1, \ldots$. The polynomial L_n has degree n and goes under the name of nth **Laguerre polynomial**. Laguerre polynomials are obtained from the formula:

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) \quad n = 0, 1, 2, \ldots$$

Again, we have normalising relations:

$$\int_{[0,+\infty)} e^{-x} L_n(x) L_m(x) dx = \delta_{nm} (n!)^2 \,.$$

In QM the basis of Laguerre functions is important when working with physical systems having a spherical symmetry, like the hydrogen atom, for instance.

(5) Consider the separable Hilbert space $L^2(\mathbb{R}^n, dx)$ (dx being the usual Lebesgue measure on $\mathbb{R}^n$). It is a renowned fact [Vla02] that real-valued smooth functions on $\mathbb{R}^n$ with compact support (or complex-valued functions that decrease at infinity faster than any negative power of $|x|$) are dense in $L^p(\mathbb{R}^n, dx)$, $1 \le p < \infty$. It falls out of Proposition 3.31(b) that such subspaces contain bases of $L^2(\mathbb{R}^n, dx)$.

(6) We will now construct the so-called *Bargmann–Hilbert space*, also known as *Bargmann–Fock space*. This is a Hilbert space with a host of applications in QM and Quantum Field Theory. Consider the following positive σ-additive measure defined on Borel sets $E \subset \mathbb{C}$, where χ_E is the **characteristic function** of E ($\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$):

$$\mu(E) := \frac{1}{\pi} \int_{\mathbb{C}} \chi_E(z) e^{-|z|^2} dz d\overline{z} \,.$$

Here, as is customary in this formalism, we denoted by $dzd\overline{z}$ the Lebesgue measure of $\mathbb{R}^2$ identified with $\mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ is **entire** if it is holomorphic everywhere on $\mathbb{C}$. Let $\mathsf{H}(\mathbb{C})$ be the space of entire functions. Take the subspace of $L^2(\mathbb{C}, \mu)$ given by the intersection $L^2(\mathbb{C}, \mu) \cap \mathsf{H}(\mathbb{C})$ – where the elements of $\mathsf{H}(\mathbb{C})$ represent equivalence classes of maps (as for L^p spaces). It is far from obvious that $L^2(\mathbb{C}, \mu) \cap \mathsf{H}(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C}, \mu)$, because it is not evident that a sequence of entire functions converges, *in* $L^2(\mathbb{C}, \mu)$ *sense*, to an entire function (i.e. the limit will be entire up to zero-measure sets). Bargmann, however, managed to prove [Bar61] that

$$\text{if } f \in \mathsf{H}(\mathbb{C}), \text{ then } \quad \int_{\mathbb{C}} |f(z)|^2 d\mu(z) = \sum_{n=0}^{+\infty} |f_n|^2 \leq +\infty \tag{3.29}$$

where:

$$f_n = \sqrt{n!}a_n \quad \text{with} \quad f(z) = \sum_{n=0}^{+\infty} a_n z^n \,. \tag{3.30}$$

The power series in (3.30) is just the Taylor expansion of f: it converges absolutely for any $z \in \mathbb{C}$ and uniformly on any compact set in $\mathbb{C}$, and it exists by the mere fact that f is entire. Notice that (3.29) establishes in particular that the positive-term series on the right converges iff the integral of the left-hand-side function converges. Hence $f, g \in L^2(\mathbb{C}, \mu) \cap \mathsf{H}(\mathbb{C})$ if and only if $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n=1,2...} \in \ell^2(\mathbb{N})$ (Example 2.29(7)), in which case the polarisation formula (3.4) and (3.29) give:

$$\int_{\mathbb{C}} \overline{f(z)} g(z) d\mu(z) = \sum_{n=0}^{+\infty} \overline{f_n} g_n \,. \tag{3.31}$$

In the notation of (3.30), let us consider the map:

$$J : L^2(\mathbb{C}, \mu) \cap \mathsf{H}(\mathbb{C}) \ni f \mapsto \{f_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \,.$$

This linear, isometric (hence 1-1) transformation is actually surjective as well. In fact, since the series $\sum_{n\in\mathbb{N}} \frac{|z|^{2n}}{(n!)^2}$ converges for any $z \in \mathbb{C}$, the Cauchy–Schwarz inequality implies that the series:

$$\sum_{n\in\mathbb{N}} \frac{c_n}{\sqrt{n!}} \frac{z^n}{\sqrt{n!}} =: f(z)$$

converges absolutely for any $z \in \mathbb{C}$, provided $\{c_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and if we define an entire map f and $J(f) = \{c_n\}_{n\in\mathbb{N}}$. Since $\ell^2(\mathbb{N})$ is complete we conclude that:

(a) the complex vector space $\mathsf{B}_1 := L^2(\mathbb{C}, \mu) \cap \mathsf{H}(\mathbb{C})$ is a Hilbert space, i.e. a *closed* subspace of $L^2(\mathbb{C}, \mu)$,

(b) B_1 is isomorphic to $\ell^2(\mathbb{N})$ under J (hence in particular separable),

(c) the system of entire maps $\{u_n\}_{n\in\mathbb{N}}$:

$$u_n(z) = \frac{z^n}{\sqrt{n!}} \quad \text{for any } z \in \mathbb{C}, n \in \mathbb{N} \tag{3.32}$$

is a basis for B_1.

B_1 is called the **Bargmann–Hilbert** or **Bargmann–Fock space**.

To conclude we observe that all constructions have a straightforward generalisation to n copies of $\mathbb{C}$, giving the n-dimensional Bargmann space $\mathsf{B}_n := L^2(\mathbb{C}^n, d\mu_n) \cap \mathsf{H}(\mathbb{C}^n)$ where, for any Borel set $E \in \mathbb{C}^n$:

$$\mu_n(E) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \chi_E(z) e^{-\sum_{k=1}^n |z_k|^2} dz_1 d\overline{z_1} \otimes \cdots \otimes dz_n d\overline{z_n} \,,$$

$\mathsf{H}(\mathbb{C}^n)$ is the space of holomorphic maps in n variables on $\mathbb{C}^n$ and the integral is computed in the product of the measures μ on each copy of $\mathbb{C}$. ■

3.3 Hermitian Adjoints and Applications

We examine here one of the most important notions of the theory of operators on a Hilbert space that derives from Riesz's Theorem 3.16: *(Hermitian) adjoint operators*. We have to stress that this concept can be extended to unbounded operators, but in this section we consider *only* the bounded case. The general situation will be dealt with extensively in a subsequent chapter. It is also worth recalling that a (related) notion of adjoint operator (or conjugate operator) was given in Definition 2.45, without the need of Hilbert structures. In the sequel we will not use the latter, exception made for the occasional remark.

From a more abstract viewpoint, the Hermitian conjugation will give us the opportunity for introducing relevant mathematical notions in advanced formulations of QM: we are talking about **-algebras*, *C^*-algebras* and their representations.

3.3.1 Hermitian Conjugation, or Adjunction

Let $(\mathsf{H}_1, (\ |\ )_1)$, $(\mathsf{H}_2, (\ |\ )_2)$ be Hilbert spaces and $T \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$ an operator. For a given $u \in \mathsf{H}_2$, consider:

$$\mathsf{H}_1 \ni v \mapsto (u|Tv)_2 \in \mathbb{C} \,. \tag{3.33}$$

This map is certainly linear and bounded:

$$|(u|Tv)_2| \leq ||u||_2 \, ||Tv||_2 \leq ||u||_2 \, ||T|| \, ||v||_1 \,.$$

Hence it belongs to H_1'. By Riesz's Theorem 3.16 there exists $w_{T,u} \in \mathsf{H}_1$ such that

$$(u|Tv)_2 = (w_{T,u}|v)_1 \,, \quad \text{for any } v \in \mathsf{H}_1 \,. \tag{3.34}$$

Moreover, the map $\mathsf{H}_2 \ni u \mapsto w_{T,u} \in \mathsf{H}_1$ is linear. In fact:

$$(w_{T,\alpha u+\beta u'}|v)_1 = (\alpha u + \beta u'|Tv)_2 = \overline{\alpha}(u|Tv)_2 + \overline{\beta}(u'|Tv)_2 = (\alpha w_{T,u} + \beta w_{T,u'}|v)_1 \,,$$

so, for any $v \in \mathsf{H}_1$,

$$0 = (w_{T,\alpha u+\beta u'} - \alpha w_{T,u} - \beta w_{T,u'}|v)_1 \,.$$

Choosing $v := w_{T,\alpha u+\beta u'} - \alpha w_{T,u} - \beta w_{T,u'}$, we have $w_{T,\alpha u+\beta u'} - \alpha w_{T,u} - \beta w_{T,u'} = \mathbf{0}$ hence

$$w_{T,\alpha u+\beta u'} = \alpha w_{T,u} + \beta w_{T,u'}$$

for any $\alpha, \beta \in \mathbb{C}$, $u, u' \in \mathsf{H}_2$. Therefore there exists a linear operator:

$$T^* : \mathsf{H}_2 \ni u \mapsto w_{T,u} \in \mathsf{H}_1 \,.$$

By construction T^* satisfies $(u|Tv)_2 = (T^*u|v)_1$ for any pair $u \in \mathsf{H}_2$, $v \in \mathsf{H}_1$, and actually it is the unique linear operator with such property. If there were another such $B \in \mathfrak{L}(\mathsf{H}_2, \mathsf{H}_1)$, then $(T^*u|v)_1 = (Bu|v)_1$ for any $v \in \mathsf{H}_1$. Consequently $((T^* - B)u|v)_1 = 0$ for any $v \in \mathsf{H}_1$. Choosing $v := (T^* - B)u$ would give $||(T^* - B)u||_1^2 = 0$, so $T^*u - Bu = \mathbf{0}$. Since $u \in \mathsf{H}_2$ is arbitrary, $T^* = B$. Overall, we have proved the following fact.

Proposition 3.36 *Let* $(\mathsf{H}_1, (\,|\,)_1)$, $(\mathsf{H}_2, (\,|\,)_2)$ *be Hilbert spaces, and* $T \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$. *There exists a unique linear operator* $T^* : \mathsf{H}_2 \to \mathsf{H}_1$ *such that:*

$$(u|Tv)_2 = (T^*u|v)_1 \,, \quad \textit{for any pair } u \in \mathsf{H}_2, v \in \mathsf{H}_1. \tag{3.35}$$

We are ready to define *adjoint Hermitian operators*. From now on we will drop the adjective "Hermitian", given that this textbook will never use non-Hermitian adjoint operators as we said at the beginning.

Definition 3.37 Let $(\mathsf{H}_1, (\,|\,)_1)$, $(\mathsf{H}_2, (\,|\,)_2)$ be Hilbert spaces and $T \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$. The unique linear operator $T^* \in \mathfrak{L}(\mathsf{H}_2, \mathsf{H}_1)$ fulfilling (3.35) is called the **(Hermitian) adjoint**, or **Hermitian conjugate** to the operator T.

Recall that given a linear operator $T : \mathsf{X} \to \mathsf{Y}$ between vector spaces, $Ran(T) := \{T(x) \mid x \in \mathsf{X}\}$ and $Ker(T) := \{x \in \mathsf{X} \mid T(x) = \mathbf{0}\}$ denote the subspaces of Y and X called **range** (or **image**) and **kernel** (or **null space**) of T.

The operation of Hermitian conjugation enjoys the following elementary properties.

Proposition 3.38 *Let* $(\mathsf{H}_1, (\,|\,)_1)$, $(\mathsf{H}_2, (\,|\,)_2)$ *be Hilbert spaces and* $T \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$. *Then*

(a) $T^* \in \mathfrak{B}(\mathsf{H}_2, \mathsf{H}_1)$*, and more precisely:*

$$||T^*|| = ||T|| \,, \tag{3.36}$$
$$||T^*T|| = ||T||^2 = ||TT^*|| \,. \tag{3.37}$$

(b) *The Hermitian conjugation is* **involutive***:*

$$(T^*)^* = T \,.$$

(c) *If $S \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$ and $\alpha, \beta \in \mathbb{C}$:*

$$(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^* \,, \tag{3.38}$$

and if $S \in \mathfrak{B}(\mathsf{H}, \mathsf{H}_1)$, with H a Hilbert space:

$$(TS)^* = S^* T^* \,. \tag{3.39}$$

(d) *We have:*

$$Ker(T) = [Ran(T^*)]^{\perp} \,, \quad Ker(T^*) = [Ran(T)]^{\perp} \,. \tag{3.40}$$

(e) *T is bijective if and only if T^* is bijective, in which case $(T^*)^{-1} = (T^{-1})^*$.*

Proof From now on we will write $|| \ ||$ to denote both $|| \ ||_1$ and $|| \ ||_2$, and similarly for inner products. It will be clear from the context which is which.
(a) For any pair $u \in \mathsf{H}_2$, $x \in \mathsf{H}_1$ we have $|(T^*u|x)| = |(u|Tx)| \leq ||u|| \, ||T|| \, ||x||$. By choosing $x := T^*u$ we have in particular $||T^*u||^2 \leq ||T|| \, ||u|| \, ||T^*u||$, so $||T^*u|| \leq ||T|| \, ||u||$. Hence T^* is bounded and $||T^*|| \leq ||T||$. Therefore it makes sense to define $(T^*)^*$, so $||(T^*)^*|| \leq ||T^*||$. This inequality becomes $||T|| \leq ||T^*||$ by (b) (whose proof only uses the boundedness of T^*). As $||T^*|| \leq ||T||$ and $||T|| \leq ||T^*||$, equation (3.36) follows. Let us pass to (3.37). It suffices to prove the first identity, since the second descends from the first one and (3.36), by (b) (which does not depend on (a)). By Theorem 2.44(b), case (i), whose conclusion holds for $S \in \mathfrak{B}(\mathsf{Y}, \mathsf{X})$, $T \in \mathfrak{B}(\mathsf{Z}, \mathsf{Y})$ with $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ normed, we have $||T^*T|| \leq ||T^*|| \, ||T|| = ||T||^2$. At the same time:

$$||T||^2 = (\sup_{||x|| \leq 1} ||Tx||)^2 = \sup_{||x|| \leq 1} ||Tx||^2 = \sup_{||x|| \leq 1} (Tx|Tx) \,.$$

By definition of adjoint and by Cauchy–Schwarz (on the last term) we obtain:

$$||T||^2 = \sup_{||x|| \leq 1} (Tx|Tx) = \sup_{||x|| \leq 1} |(T^*Tx|x)| \leq \sup_{||x|| \leq 1} ||T^*Tx|| = ||T^*T|| \,.$$

Therefore $||T^*T|| \leq ||T||^2$ and $||T||^2 \leq ||T^*T||$, so $||T^*T|| = ||T||^2$.
(b) This follows immediately from the uniqueness of the adjoint operator. By known properties of the inner product and the definition of adjoint to T, in fact, we have:

$$(v|T^*u) = \overline{(T^*u|v)} = \overline{(u|Tv)} = (Tv|u) .$$

(c) If $u \in \mathsf{H}_2$, $v \in \mathsf{H}_1$ then

$$(u|(\alpha T + \beta S)v) = \alpha(u|Tv) + \beta(u|Sv) = \alpha(T^*u|v) + \beta(S^*u|v) = ((\overline{\alpha}T^* + \overline{\beta}S^*)u|v) .$$

The adjoint's uniqueness gives (3.38). If $v \in \mathsf{H}$, $u \in \mathsf{H}_2$,

$$(u|(TS)v) = (T^*u|Sv) = ((S^*T^*)u|v) .$$

By uniqueness (3.39) holds.
(d) It is enough to prove the first identity, as the second one is a consequence of it and of part (b). Since $(T^*u|v) = (u|Tv)$, if $v \in Ker(T)$ then $(T^*u|v) = 0$ for any $u \in \mathsf{H}_2$, so $v \in [Ran(T^*)]^\perp$. Conversely, still by $(T^*u|v) = (u|Tv)$, if $v \in [Ran(T^*)]^\perp$ then $(u|Tv) = 0$ for any $u \in \mathsf{H}_2$. If we choose $u := Tv$ then $Tv = \mathbf{0}$ and so $v \in Ker(T)$.
(e) If T is bijective then T^{-1} is bounded by Banach's inverse operator theorem. Therefore $(T^{-1})^*$ exists. We have: $T^{-1}T = TT^{-1} = I$. Using the second property of (c) and remembering $I^* = I$, let us compute the adjoint of both sides: $T^*(T^{-1})^* = (T^{-1})^*T^* = I$. These are equivalent to saying T^* is bijective and $(T^*)^{-1} = (T^{-1})^*$. Eventually, if T^* is bijective, then also $(T^*)^* = T$ is bijective, by what we have just seen and by (b). □

Remark 3.39 The relationship between Hermitian adjoints and conjugate operators seen in Definition 2.45 goes as follows. Start with $T \in \mathfrak{B}(\mathsf{H}_1, \mathsf{H}_2)$ and compute the conjugate $T' \in \mathfrak{B}(\mathsf{H}_2', \mathsf{H}_1')$ and the adjoint $T^* \in \mathfrak{B}(\mathsf{H}_2, \mathsf{H}_1)$. Then:

$$(T^*y_f|x)_1 = (y_f|Tx)_2 = (T'f)(x) \quad \text{for any } f \in \mathsf{H}_2', x \in \mathsf{H}_1,$$

where $f \in \mathsf{H}_2'$, while $y_f \in \mathsf{H}_2$ is the element in H_2 representing f under Riesz's Theorem 3.16. As $x \in \mathsf{H}_1$ is arbitrary, we may write:

$$T'f = (T^*y_f|\ \ )_1 \quad \text{for any } f \in \mathsf{H}_2'. \tag{3.41}$$

Given that the Riesz map $\mathsf{H}_2' \ni f \mapsto y_f \in \mathsf{H}_2$ is bijective, the above equation determines T' completely whenever T^* is given, and conversely. ■

*3.3.2 *-Algebras, C*-Algebras, and *-Representations*

Hermitian conjugation gives us an excuse for introducing one of the most useful mathematical concepts in advanced formulations of QM, namely C^**-algebras* (also known as B^**-algebras*). We shall return to this notion in Chap. 8 to discuss the spectral decomposition theorem, and in Chap. 14, when we will deal with the so-called algebraic formulation of quantum theories.

Definition 3.40 (C^**-algebra*) Let $\mathfrak{A}$ be a (commutative, Banach and unital) algebra (normed by $|| \ ||$) over the field $\mathbb{C}$. A map $^* : \mathfrak{A} \to \mathfrak{A}$ such that:

I1. (antilinearity) $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for any $x, y \in \mathfrak{A}$, $\alpha, \beta \in \mathbb{C}$,

I2. (involutivity) $(x^*)^* = x$ for any $x \in \mathfrak{A}$,

I3. $(xy)^* = y^* x^*$ for any $x, y \in \mathfrak{A}$,

is called an **involution**, and the structure $(\mathfrak{A}, ^*)$ is a *-algebra (respectively commutative, Banach, unital, normed).

A Banach *-algebra (with unit) is a C^***-algebra** (with unit) if, further:

$$||x^* x|| = ||x||^2 . \tag{3.42}$$

Given (unital) *-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, an algebra homomorphism $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a ***-homomorphism** if it preserves the involution: $f(x^{*_1}) = f(x)^{*_2}$ for any $x \in \mathfrak{A}_1$ (^{*_1} is the involution of $\mathfrak{A}_1$ and ^{*_2} the involution in $\mathfrak{A}_2$) and also the units, if present. We call f a ***-isomorphism** if it is additionally bijective. A *-isomorphism $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ is called ***-automorphism** when $\mathfrak{A}_1 = \mathfrak{A}_2$.

An element x in a *-algebra $\mathfrak{A}$ (with unit $\mathbb{I}$ in cases (iii), (iv) below) is called:

(i) **normal** if $xx^* = x^* x$,

(ii) **Hermitian** or **self-adjoint** if $x^* = x$,

(iii) **isometric** if $x^* x = \mathbb{I}$,

(iv) **unitary** if $x^* x = xx^* = \mathbb{I}$.

A ***-subalgebra** (C^***-subalgebra**) of a *-algebra (C^*-algebra) $\mathfrak{A}$ is the natural object: a subalgebra that is a *-algebra (C^*-algebra) for the restricted involution (and for the restricted Banach structure in case of a C^*-subalgebra). If the *-algebra (C^*-algebra) has a unit, any *-subalgebra (C^*-subalgebra) with unit is required to have the same unit of the *-algebra.

Remarks 3.41 **(1)** If $\mathfrak{A}$ is a *-algebra (with unit), and $\{\mathfrak{A}_i\}_{i \in I}$ is a collection of *-subalgebras (with unit), it is easy to see $\bigcap_{i \in \mathrm{i}} \mathfrak{A}_i$ is a *-subalgebra (with unit) of $\mathfrak{A}$. If we add the topological structure and $\{\mathfrak{A}_i\}_{i \in I}$ are C^*-subalgebras (with unit) of the C^*-algebra (with unit) $\mathfrak{A}$, then $\bigcap_{i \in \mathrm{i}} \mathfrak{A}_i$ is a C^*-subalgebra (with unit) of $\mathfrak{A}$. Everything is completely obvious except possibly for the completeness of $\bigcap_{i \in \mathrm{i}} \mathfrak{A}_i$. This follows directly from the fact it is closed, hence complete, being an intersection of closed (complete) sets $\mathfrak{A}_i$.

(2) If $S \subset \mathfrak{A}$ is a subset in a (unital) *-algebra $\mathfrak{A}$, the (unital) *-algebra **generated** by S is the intersection of all (unital) *-subalgebras in $\mathfrak{A}$ that contain S. The same holds for (unital) C^*-algebras, *mutatis mutandis*.

(3) It is easy to prove that the inverse map of a *-isomorphism is a *-isomorphism as well.

(4) The definition of involution * and C^*-algebra can be stated, almost identically, for (associative) algebras over $\mathbb{R}$ instead of $\mathbb{C}$. The only difference is that the involution is $\mathbb{R}$-linear and satisfies **I2** and **I3**. This book will only deal with the complex versions. ■

Sometimes a *-homomorphism is defined as a linear map preserving products and involutions, *but not necessarily units* (when present), contrarily to our assumption. However the following elementary result holds.

Proposition 3.42 *Let $\mathfrak{A}_1, \mathfrak{A}_2$ be unital *-algebras and consider a linear map $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ that preserves the products and the involutions. If ϕ is surjective, then it is a *-homomorphism.*

Proof It is sufficient to establish that ϕ preserves unit elements. With the obvious notation, $\phi(a) = \phi(\mathbb{I}_1 a) = \phi(\mathbb{I}_1)\phi(a)$ for every $a \in \mathfrak{A}_1$. Since $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is surjective, there exists $a_1 \in \mathfrak{A}_1$ such that $\phi(a_1) = \mathbb{I}_2$, so $\mathbb{I}_2 = \phi(a_1) = \phi(\mathbb{I}_1)\phi(a_1) = \phi(\mathbb{I}_1)\mathbb{I}_2 = \phi(\mathbb{I}_1)$. □

Remark 3.43 Note that the proof remains valid irrespective of the involutions: *a product-preserving surjective linear map between unital algebras must also preserve the units.* ■

Before we return to $\mathfrak{B}(\mathsf{H})$, let us see a few general features of *-algebras (and C^*-algebras) that descend from the definition.

Proposition 3.44 *Let $\mathfrak{A}$ be a *-algebra with involution *.*
(a) *If $(\mathfrak{A}, ||\ ||)$ is a C^*-algebra and $x \in \mathfrak{A}$ is normal, then for any $m = 1, 2, \ldots$:*

$$||x^m|| = ||x||^m \ .$$

(b) *If $(\mathfrak{A}, ||\ ||)$ is a C^*-algebra and $x \in \mathfrak{A}$,*

$$||x^*|| = ||x|| \ .$$

(c) *If $\mathfrak{A}$ has unit $\mathbb{I}$, then $\mathbb{I}^* = \mathbb{I}$. Moreover, $x \in \mathfrak{A}$ has an inverse if and only if x^* has an inverse, in which case $(x^{-1})^* = (x^*)^{-1}$.*

Proof (a) If $||x|| = 0$ the claim is trivial, so assume $x \neq \mathbf{0}$. A repeated use of (3.42), **I2**, **I3** and the fact that $xx^* = x^*x$ gives:

$$||x^2||^2 = ||(x^2)^*x^2|| = ||(x^*)^2x^2|| = ||(x^*x)^*(x^*x)|| = ||x^*x||^2 = (||x||^2)^2$$

whence $||x^2|| = ||x||^2$ by norm positivity. Iterating we obtain $||x^{2^k}|| = ||x||^{2^k}$ for any natural number k. If $m = 3, 4, \ldots$ there exist two natural numbers n, k with $m + n = 2^k$, so:

$$||x||^m||x||^n = ||x||^{n+m} = ||x^{n+m}|| \leq ||x^m||\,||x^n|| \leq ||x^m||\,||x||^n \leq ||x||^m||x||^n \ .$$

But then all inequalities are equalities, so in particular:

$$||x^m|| \, ||x||^n = ||x||^m ||x||^n ,$$

dividing which by $||x||^n$ (non-zero since $x \neq \mathbf{0}$ and $|| \cdot ||$ is a norm) proves the claim.
(b) Equation (3.42) implies $||x||^2 = ||xx^*|| \leq ||x|| \, ||x^*||$ so $||x|| \leq ||x^*||$. Similarly $||x^*|| \leq ||(x^*)^*||$, and then $(x^*)^* = x$ concludes this part.
(c) $\mathbb{I}\mathbb{I}^* = \mathbb{I}^*$ by definition of unit; on the other hand $\mathbb{I}\mathbb{I}^* = (\mathbb{I}^*)^*\mathbb{I}^* = (\mathbb{I}^*\mathbb{I})^*$. From these two descends $\mathbb{I}^* = (\mathbb{I}^*\mathbb{I})^* = (\mathbb{I}^*)^* = \mathbb{I}$. The other statement follows from this, **I2** and the uniqueness of the inverse. □

It is easy to construct a C^*-algebra out of a family of C^*-algebras with the following procedure.

Proposition 3.45 *Let $\{\mathfrak{A}_j\}_{j\in J}$ be a family of C^*-algebras, not necessarily unital, where J has arbitrary cardinality. Consider the set $\bigoplus_{j\in J} \mathfrak{A}_j \subset \times_{j\in J}\mathfrak{A}_j$ of families $\{a_j\}_{j\in J}$, such that*

$$\left|\left|\{a_j\}_{j\in J}\right|\right| := \sup_{j\in J} ||a_j||_j < +\infty . \tag{3.43}$$

Equip $\bigoplus_{j\in J} \mathfrak{A}_j$ with a *-algebra structure by declaring (with obvious notation)

(i) $\alpha\{a_j\}_{j\in J} + \beta\{a'_j\}_{j\in J} := \{\alpha a_j + \beta a'_j\}_{j\in J}$ with $\alpha, \beta \in \mathbb{C}$,
(ii) $\{a_j\}_{j\in J} \circ \{a'_j\}_{j\in J} := \{a_j \circ_j a'_j\}_{j\in J}$,
(iii) $\{a_j\}^*_{j\in J} := \{a_j^{*_j}\}_{j\in J}$,

Under these assumptions, (3.43) defines a norm making $\bigoplus_{j\in J} \mathfrak{A}_j$ a C^*-algebra. If every $\mathfrak{A}_j$ is unital, $\mathbb{I} := \{\mathbb{I}_j\}_{j\in J}$ is the unit of this C^*-algebra.

Proof The argument is straightforward. The completeness of $\bigoplus_{j\in J} \mathfrak{A}_j$ as a Banach space is essentially the same as the completeness of $C(\mathsf{K}, \mathbb{K}^n)$ (Proposition 2.18) or any other space of $\mathbb{C}$-valued, bounded functions. The C^* relation

$$\left|\left|\{a_k\}^*_{k\in J} \circ \{a_j\}_{j\in J}\right|\right| = \left|\left|\{a_j\}_{j\in J}\right|\right|^2$$

descends immediately from the C^* property of each C^*-algebra $\mathfrak{A}_j$ and the given definitions. □

Definition 3.46 The C^*-algebra $\bigoplus_{j\in J} \mathfrak{A}_j$ constructed in Proposition 3.45 is called **direct sum** of the family of C^*-algebras $\{\mathfrak{A}_j\}_{j\in J}$. An element of $\bigoplus_{j\in J} \mathfrak{A}_j$ is denoted by $\oplus_j a_j := \{a_j\}_{j\in J}$.

Remark 3.47 The structure of a C^*-algebra is remarkable in that its topological and algebraic properties are deeply intertwined. We will prove later (Corollary 8.18) that a *-homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ between unital C^*-algebras is automatically continuous, because $||\phi(a)||_{\mathfrak{B}} \leq ||a||_{\mathfrak{A}}$ for any $a \in \mathfrak{A}$. Moreover, ϕ is isometric, i.e. $||\phi(a)||_{\mathfrak{B}} = ||a||_{\mathfrak{A}}$ for any $a \in \mathfrak{A}$, if and only if it is injective (Theorem 8.22). ■

Examples 3.48 **(1)** The Banach algebras of complex-valued functions seen in Examples 2.29(2), (3), (4), (8) and (9) are all instances of commutative C^*-algebras whose involution is the complex conjugation of functions.
(2) By virtue of (a), (b), (c) in Proposition 3.38 we have this result.

Theorem 3.49 *If* H *is Hilbert space,* $\mathfrak{B}(\mathsf{H})$ *is a* C^**-algebra with unit if the involution is defined as the Hermitian conjugation.*

(3) The algebra $\mathbb{H}$ of **quaternions** is a 4-dimensional real vector space with a privileged basis $\{1, i, j, k\}$. It is equipped with a product that turns it into an $\mathbb{R}$-algebra with unit, given by the basis element 1. The product is determined, keeping in mind Definition 2.24, by the relations $ii = jj = kk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

As $\mathbb{R}$ identifies naturally with the Abelian subalgebra of elements the form $a1$, $a \in \mathbb{R}$, the algebra $\mathbb{H}$ can be viewed as a real normed algebra with unit: it is enough to think real numbers as quaternions, and define the product of a real scalar by a quaternion using the product in $\mathbb{H}$. The norm is the usual Euclidean norm for the standard basis of $\mathbb{H}$, namely $||a1 + bi + cj + dk|| := \sqrt{a^2 + b^2 + c^2 + d^2}$. It is easy to check that $\mathbb{H}$ becomes thus a real Banach algebra with unit. Although the ground field is $\mathbb{R}$, it is possible to define an involution on $\mathbb{H}$ via quaternionic conjugation: $(a1+bi+cj+dk)^* = a1-bi-cj-dk$, with $a, b, c, d \in \mathbb{R}$. Then the usual properties of involutions hold (the field is real, so the involution is $\mathbb{R}$-linear), in relation to the norm too, and also the property typical of C^*-algebras: $||a^*a|| = ||a||^2$. Product and norm are linked by the rule $||ab|| = ||a||\,||b||$, $a, b \in \mathbb{H}$, reminiscent of the modulus on $\mathbb{C}$ and the absolute value on $\mathbb{R}$. A further property, shared by $\mathbb{R}$ and $\mathbb{C}$ as well, is that the quaternion algebra is a real associative **division algebra**: *an associative algebra with multiplicative unit different from the additive neutral element where any non-zero element is invertible*.

A concrete representation of $\mathbb{H}$ is given by the real subalgebra of $M(2, \mathbb{C})$ (2×2 complex matrices) spanned over $\mathbb{R}$ by the identity I and the three **Pauli matrices** $-i\sigma_1, -i\sigma_2, -i\sigma_3$: these correspond to the quaternionic units 1 and i, j, k (see Remark 7.28(3)).

Hence $\mathbb{H}$ is also a (non-commutative) **division ring**, that is a non-trivial ring where every non-zero element admits a multiplicative inverse. A commutative division ring is obviously a *field*, just like $\mathbb{R}$ and $\mathbb{C}$.

In 1887 Frobenius proved what has become a classical result.

Theorem 3.50 (Frobenius) *A finite-dimensional associative division algebra over* $\mathbb{R}$ *is necessarily isomorphic to* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$.

A much more recent result [UrWr60], extending a previous classical result by Hurwitz (where also a finite dimension was assumed), replaces finite-dimensionality for a demand on the norm.

Theorem 3.51 *An associative, unital normed division algebra $\mathfrak{A}$ over $\mathbb{R}$ such that $||ab|| = ||a|| \, ||b||$, $a, b \in \mathfrak{A}$, is necessarily isometrically isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Such $\mathfrak{A}$ is therefore finite-dimensional, Banach and can be turned into a real C^*-algebra using the isomorphism. If one demands $||ab|| = ||a|| \, ||b||$ in the finite-dimensional case, as established by Hurwitz (and published posthumously in 1923), dropping associativity adds to the list only one other instance, the *Cayley numbers*, also known as *octonions*. The infinite-dimensional non-associative case is more complicated [UrWr60]. ■

To conclude the section let us see a very relevant definition in advanced formalisations of, for example, quantum field theories. We will come back to these notions later, especially in Chap. 14.

Definition 3.52 Let $\mathfrak{A}$ be a *-algebra (not necessarily unital, nor C^*) and H a Hilbert space. A *-homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ is called a **representation of $\mathfrak{A}$ on H**. (Note π preserves units if present.) Furthermore, one says that
(a) π is **faithful** if it is one-to-one;
(b) a subspace $\mathsf{M} \subset \mathsf{H}$ is **invariant** under π (or π-**invariant**) if $\pi(a)(\mathsf{M}) \subset \mathsf{M}$ for any $a \in \mathfrak{A}$.
(c) π is **irreducible** if there are no π-invariant closed subspaces other than $\{\mathbf{0}\}$ and H itself.
(d) If $\pi' : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}')$ is another representation of $\mathfrak{A}$ on H', π and π' are said to be **unitarily equivalent**:

$$\pi \simeq \pi'$$

if there exists a surjective isometry $U : \mathsf{H} \to \mathsf{H}'$ such that:

$$U\pi(a)U^{-1} = \pi'(a) \text{ for any } a \in \mathfrak{A}.$$

(e) A vector $\psi \in \mathsf{H}$ is called **cyclic** for π if $\overline{\{\pi(a)\psi \mid a \in \mathfrak{A}\}} = \mathsf{H}$.

Remarks 3.53 **(1)** One can also consider representations of *-algebras in terms of unbounded operators and operators defined on a common invariant domain of the Hilbert space.
(2) In case $\mathfrak{A}$ is a C^*-algebra with unit, every representation is automatically continuous with respect to the norm of $\mathfrak{A}$ on the domain and the operator norm on the codomain, as $||\pi(a)|| \leq ||a||$ for any $a \in \mathfrak{A}$. Then π is faithful iff isometric: $||\pi(a)|| = ||a||$ for any $a \in \mathfrak{A}$. All this will be proved in Theorem 8.22.
(3) With our definition of representation, the map $\mathfrak{A} \ni a \mapsto 0$ is a representation called the **zero representation** (also known as the **trivial representation**) in two cases only: either the *-algebra $\mathfrak{A}$ has *no* unit, or $\mathsf{H} = \{\mathbf{0}\}$. This is because our definition requires $\pi(\mathbb{I}) = I$, but $I \neq 0$ when $\mathsf{H} \neq \{\mathbf{0}\}$. If one drops the requirement $\pi(\mathbb{I}) = I$ in the definition of a representation, the zero representation makes sense also for unital *-algebras and $\mathsf{H} \neq \{\mathbf{0}\}$. ■

Here is an elementary yet important fact.

Proposition 3.54 *If $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ is an irreducible representation of a *-algebra $\mathfrak{A}$ with unit on $\mathsf{H} \neq \{\mathbf{0}\}$, then every non-zero vector in H is cyclic for π.*

Proof If $\psi \neq \mathbf{0}$, define $\mathsf{M}_\psi := \overline{\{\pi(a)\psi \mid a \in \mathfrak{A}\}}$. Since every $\pi(b)$ is bounded, necessarily $\pi(b)(\mathsf{M}_\psi) \subset \mathsf{M}_\psi$ for every $b \in \mathfrak{A}$, and therefore either $\mathsf{M}_\psi = \mathsf{H}$ or $\mathsf{M}_\psi = \{\mathbf{0}\}$ because π is irreducible. The second case is excluded, as it would imply $\pi(a) = 0$ for every $a \in \mathfrak{A}$ and in particular $\pi(\mathbb{I}) = 0 \neq I$ (because $\mathsf{H} \neq \{\mathbf{0}\}$) which is impossible by definition of representation of unital *-algebra. $\square$

Definition 3.52 explicitly requires that a representation map the unit of the algebra to the identity operator: $\pi(\mathbb{I}) = I$. The reader should pay attention to this fact, since some books do *not* impose $\pi(\mathbb{I}) = I$ when defining representations of *-algebras *with unit* (see Proposition 3.42 for the analogous issue regarding *-homomorphisms). The two inequivalent definitions are related by the following elementary result, whose proof requires the mathematical technology presented in Sect. 3.4 and which appears in the solution of Exercise 3.32.

Proposition 3.55 *Let $\mathfrak{A}$ be a *-algebra with unit, H a Hilbert space, and consider a linear map $\phi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ which preserves the products and the involutions. Then*
(a) $\mathsf{H}_\phi := Ran(\phi(\mathbb{I}))$ *and $\mathsf{H}_\phi^\perp$ are closed subspaces of H satisfying $\mathsf{H} = \mathsf{H}_\phi \oplus \mathsf{H}_\phi^\perp$, and each one is invariant under $\phi(a)$ for every $a \in \mathfrak{A}$.*
(b) $\phi(a)\restriction_{\mathsf{H}_\phi^\perp} = 0$ *for every $a \in \mathfrak{A}$.*
(c) *The restriction to the complement*

$$\pi_\phi : \mathfrak{A} \ni a \mapsto \phi(a)\restriction_{\mathsf{H}_\phi} \in \mathfrak{B}(\mathsf{H}_\phi)$$

is a representation of $\mathfrak{A}$ over H_ϕ according to Definition 3.52. It also satisfies

(i) *π_ϕ is faithful $\Leftrightarrow$ ϕ is injective;*
(ii) *π_ϕ is the zero representation $\Leftrightarrow$ $\phi(\mathbb{I}) = 0$ (in this case $\mathsf{H}_\phi = \{\mathbf{0}\}$);*
(iii) *$\pi_\phi = \phi$ if ϕ is surjective;*
(iv) *$\pi_\phi = \phi$ if ϕ is not the zero map and is irreducible (i.e. there are no closed subspaces $\{\mathbf{0}\} \subsetneq \mathsf{M} \subsetneq \mathsf{H}$ such that $\phi(a)(\mathsf{M}) \subset \mathsf{M}$ for every $a \in \mathfrak{A}$).*

Proof See the solution of Exercise 3.32. $\square$

3.3.3 Normal, Self-Adjoint, Isometric, Unitary and Positive Operators

Returning to the C^*-algebra $\mathfrak{B}(\mathsf{H})$ (or more generally to $\mathfrak{B}(\mathsf{H}, \mathsf{H}_1)$), we recall the most important types of operators we will encounter in subsequent chapters.

Definition 3.56 Let $(\mathsf{H}, (\,|\,))$, $(\mathsf{H}_1, (\,|\,)_1)$ be Hilbert spaces and I_H, I_{H_1} their respective identity operators.
(a) $T \in \mathfrak{B}(\mathsf{H})$ is said to be **normal** if $TT^* = T^*T$.

(b) $T \in \mathfrak{B}(\mathsf{H})$ is **self-adjoint** if $T = T^*$.
(c) $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ is **isometric** if bounded and $T^*T = I_{\mathsf{H}}$; equivalently, $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ is isometric if $(Tx|Ty)_1 = (x|y)$ for any pair $x, y \in \mathsf{H}$.
(d) $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ is **unitary** if bounded, $T^*T = I_{\mathsf{H}}$ and $TT^* = I_{\mathsf{H}_1}$; equivalently, $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ is unitary if it is isometric and onto, i.e. an isomorphism of Hilbert spaces.
(e) $T \in \mathfrak{L}(\mathsf{H})$ is **positive**, written $T \geq 0$, if $(u|Tu) \geq 0$ for any $u \in \mathsf{H}$.
(f) If $U \in \mathfrak{L}(\mathsf{H})$, we write $T \geq U$ in case $T - U \geq 0$.

Remarks 3.57 **(1)** Let us comment on the equivalence in (c): if $T \in \mathfrak{B}(\mathsf{H}, \mathsf{H}_1)$ and $T^*T = I_{\mathsf{H}}$, then $(Tx|Ty)_1 = (x|y)$ for any pair $x, y \in \mathsf{H}$, since $(x|y) = (x|T^*Ty) = (Tx|Ty)_1$. On the other hand, if $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ and $(Tx|Ty)_1 = (x|y)$ for any $x, y \in \mathsf{H}$, then T is bounded (set $y = x$), so T^* is well defined. At last $T^*T = I_{\mathsf{H}}$, because $(x|T^*Ty) = (Tx|Ty)_1 = (x|y)$ for any pair $x, y \in \mathsf{H}$, so in particular $(x|(T^*T - I)y) = 0$ with $x = (T^*T - I)y$.

As for the equivalence in (d), notice that any isometric operator T is obviously injective, for $Tu = \mathbf{0}$ implies $||u|| = 0$ and hence $u = \mathbf{0}$. Thus surjectivity is equivalent to the existence of a right inverse that coincides with the left inverse (the latter exists by injectivity, and equals T^*). From this it follows immediately that $T^*T = I_{\mathsf{H}}$ and $TT^* = I_{\mathsf{H}_1}$ are together equivalent to saying that $T \in \mathfrak{L}(\mathsf{H}, \mathsf{H}_1)$ is isometric (hence bounded) and surjective. Our definition of a unitary operator agrees with Definition 3.10.
(2) There exist isometric operators in $\mathfrak{B}(\mathsf{H})$ that are not unitary (this cannot happen if H has finite dimension). For instance, the operator on $\ell^2(\mathbb{N})$:

$$A : (z_0, z_1, z_2, \ldots) \mapsto (0, z_0, z_1, , \ldots) ,$$

for any $(z_0, z_1, z_2, \ldots) \in \ell^2(\mathbb{N})$.
(3) Unitary and self-adjoint operators in $\mathfrak{B}(\mathsf{H})$ are normal, but not conversely in general. ∎

To close the section we consider properties of normal, self-adjoint, unitary and positive operators on one Hilbert space. First, though, a definition that should be known from elementary courses.

Definition 3.58 Let X be a vector space over $\mathbb{K} = \mathbb{C}$, or $\mathbb{R}$, and take $T \in \mathfrak{L}(\mathsf{X})$. The number $\lambda \in \mathbb{K}$ is an **eigenvalue** of T if:

$$Tu = \lambda u \tag{3.44}$$

for some $u \in \mathsf{X} \setminus \{\mathbf{0}\}$ called an **eigenvector** of T relative (or associated) to λ. The subspace of X made of the null vector and all eigenvectors relative to a given eigenvalue λ is called the **eigenspace** of T with eigenvalue λ (of, associated to, relative to λ).

Notation 3.59 Only to avoid cumbersome and long sentences we shall also adopt the terms λ-**eigenvector** and λ-**eigenspace** for eigenvectors and eigenspaces relative to a given eigenvalue λ. ■

Now here is the proposition summarising the aforementioned properties.

Proposition 3.60 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space.*
(a) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is self-adjoint:*

$$||T|| = \sup\{|(x|Tx)| \mid x \in \mathsf{H}\,,\ \ ||x|| = 1\}\ . \tag{3.45}$$

More generally, if $T \in \mathfrak{L}(\mathsf{H})$ *satisfies* $(x|Tx) = (Tx|x)$ *for any* $x \in \mathsf{H}$ *and the right-hand side of (3.45) is finite, then* T *is bounded.*
(b) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is normal (in particular self-adjoint or unitary):*
(i) $\lambda \in \mathbb{C}$ *is an eigenvalue of* T *with eigenvector* $u \Leftrightarrow \overline{\lambda}$ *is an eigenvalue for* T^* *with the same eigenvector* u*;*
(ii) eigenspaces of T *relative to distinct eigenvalues are orthogonal;*
(iii) the relation:

$$||Tx|| = ||T^*x|| \ \ \textit{for any } x \in \mathsf{H} \tag{3.46}$$

holds, so $Ker(T) = Ker(T^*)$ *and* $\overline{Ran(T)} = \overline{Ran(T^*)}$*.*
(c) *Let* $T \in \mathfrak{L}(\mathsf{H})$*:*
(i) if T *is positive, its possible eigenvalues are real and non-negative;*
(ii) if T *is bounded and self-adjoint, its possible eigenvalues are real;*
(iii) if T *is isometric (in particular unitary), its possible eigenvalues are unit complex numbers.*
(d) *If* $T \in \mathfrak{L}(\mathsf{H})$ *satisfies* $(y|Tx) = (Ty|x)$ *for any pair* $x, y \in \mathsf{H}$*, then* T *is bounded and self-adjoint.*
(e) *If* $T \in \mathfrak{B}(\mathsf{H})$ *satisfies* $(x|Tx) = (Tx|x)$ *for any* $x \in \mathsf{H}$*,* T *is self-adjoint.*
(f) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is positive, it is self-adjoint.*
(g) *The relation* $\geq$ *is a partial order on* $\mathfrak{L}(\mathsf{H})$ *(hence on* $\mathfrak{B}(\mathsf{H})$*).*

Proof (a) Set $Q := \sup\{|(x|Tx)| \mid x \in \mathsf{H}\,,\ \ ||x|| = 1\}$. Since we take $||x|| = 1$

$$|(x|Tx)| \leq ||Tx||||x|| \leq ||Tx|| \leq ||T||\,,$$

hence $Q \leq ||T||$. To conclude it suffices to show $||T|| \leq Q$. The immediate identity

$$4(x|Ty) = (x+y|T(x+y)) - (x-y|T(x-y)) - i(x+iy|T(x+iy)) + i(x-iy|T(x-iy))\,,$$

together with the fact that $\overline{(z|Tz)} = (Tz|z) = (z|Tz)$, allow to rephrase $4Re(x|Ty) = 2(x|Ty) + 2\overline{(x|Ty)}$ as:

$$4Re(x|Ty) = (x+y|T(x+y)) - (x-y|T(x-y)) \leq Q||x+y||^2 + Q||x-y||^2$$

$$= 2Q||x||^2 + 2Q||y||^2\ .$$

Thus we proved:

$$4Re(x|Ty) \leq 2Q||x||^2 + 2Q||y||^2 \,.$$

Let $y \in \mathsf{H}$, $||y|| = 1$. If $Ty = \mathbf{0}$, it is clear that $||Ty|| \leq Q$; otherwise, define $x := Ty/||Ty||$ and we obtain the above inequality:

$$4||Ty|| = 4Re(x|Ty) \leq 2Q(||x||^2 + ||y||^2) = 2Q(1+1) = 4Q,$$

from which $||Ty|| \leq Q$ once again. Overall, $||Ty|| \leq Q$ if $||y|| = 1$, so

$$||T|| = \sup\{||Ty|| \mid y \in \mathsf{H}\,,\ \ ||y|| = 1\} \leq Q\,.$$

The more general statement follows from the second part of the above proof ($||T|| \leq Q$).
(b)(iii). The claim follows from the observation that $TT^* = T^*T$ implies $||Tx||^2 = (Tx|Tx) = (x|T^*Tx) = (x|TT^*x) = ||T^*x||^2$. The remaining identities are now obvious, in the light of Proposition 3.38(d). Let us prove (i). As $T - \lambda I$ is normal with adjoint $T^* - \overline{\lambda} I$, (iii) gives

$$||Tu - \lambda u|| = ||T^*u - \overline{\lambda} u||$$

and the claim is proved. (ii) Let u be a λ-eigenvector of T, v a μ-eigenvector of T. By (i), $\lambda(v|u) = (v|Tu) = (T^*v|u) = (\overline{\mu}v|u) = \mu(v|u)$, so $(\lambda - \mu)(v|u) = 0$. But $\lambda \neq \mu$, so $(v|u) = 0$.
(c) If $T \geq 0$ and $Tu = \lambda u$ with $u \neq \mathbf{0}$, then $0 \leq (u|Tu) = \lambda(u|u)$, and since $(u|u) > 0$, $\lambda \geq 0$. Let now $T = T^*$ and $Tu = \lambda u$ with $u \neq \mathbf{0}$. Then $\lambda(u|u) = (u|Tu) = (Tu|u) = \overline{\lambda}(u|u)$. From $(u|u) \neq 0$ we have $\lambda = \overline{\lambda}$, i.e. $\lambda \in \mathbb{R}$. If, instead, T is isometric, $(u|u) = (Tu|Tu) = |\lambda|^2(u|u)$, so $|\lambda| = 1$ as $u \neq \mathbf{0}$.
(d) It is enough to prove T is bounded. The adjoint's uniqueness implies that $T = T^*$ because $(y|Tx) = (Ty|x)$ for any pair $x, y \in \mathsf{H}$. By the closed graph Theorem 2.99, to prove T bounded we can just show it is closed. Let then $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ be a sequence converging to x and suppose the vectors Tx_n form a converging sequence: the claim is $Tx_n \to Tx$. Given $y \in \mathsf{H}$, our assumptions imply

$$(y|Tx_n) = (Ty|x_n) \to (Ty|x) = (y|Tx)\,.$$

The inner product is continuous, and by hypothesis $\lim_{n\to+\infty} Tx_n$ exists, so

$$\left(y \,\middle|\, \lim_{n\to+\infty} (Tx - Tx_n)\right) = 0\,.$$

Given that y is arbitrary, by choosing precisely $y := \lim_{n\to+\infty} (Tx - Tx_n)$ we obtain $\lim_{n\to+\infty} (Tx - Tx_n) = \mathbf{0}$.

(e)–(f) We have $((T^* - T)x|x) = 0$ for any $x \in \mathsf{H}$. By Exercise 3.21 $T^* - T = 0$ i.e. $T = T^*$. If $T \in \mathfrak{B}(\mathsf{H})$ is positive, then $(x|Tx)$ is real and coincides with its complex conjugate $(Tx|x)$ (by the properties of the inner product), so we fall back into the previous case.
(g) We have to prove three things. (i) $T \geq T$: this is obvious because it means $((T - T)x|x) \geq 0$ for any $x \in \mathsf{H}$. (ii) if $T \geq U$ and $U \geq S$ then $T \geq S$: this is immediate by noting $T - S = (T - U) + (U - S)$, so $((T - S)x|x) = ((T - U)x|x) + ((U - S)x|x) \geq 0$ for any $x \in \mathsf{H}$, since $T \geq U$ and $U \geq S$. (iii) if $T \geq U$ and $U \geq T$ then $T = U$. For this last one notice $(x|(T - U)x) = 0$ for any $x \in \mathsf{H}$. Exercise 3.21 forces $T - U = 0$, so $T = U$. $\square$

Remark 3.61 On real Hilbert spaces the relation $\geq$ is not a partial order, because $A \geq 0$ and $0 \geq A$ do not imply $A = 0$. For example consider a skew-symmetric matrix A acting on $\mathbb{R}^n$ (seen as real vector space with the ordinary inner product). Then $A \geq 0$ and also $0 \geq A$, since $(x|Ax) = 0$ for any $x \in \mathbb{R}^n$, but A can be very different from the null matrix. ■

3.4 Orthogonal Structures and Partial Isometries

In this section we introduce further mathematical structures related to the notions of isometry and orthogonality in a Hilbert space.

3.4.1 Orthogonal Projectors

The elementary notion we wish to introduce is that of *orthogonal projectors*, which will play a role to construct the formalism of QM.

Definition 3.62 (*Orthogonal projector*) If $(\mathsf{H}, (\ |\ ))$ is a Hilbert space, a projector $P \in \mathfrak{B}(\mathsf{H})$ (Definition 2.100) is called an **orthogonal projector** if $P^* = P$.

Remark 3.63 With this in place, orthogonal projectors are precisely the bounded operators $\mathsf{H} \to \mathsf{H}$ defined by $P = PP$ (P is idempotent) and $P = P^*$ (P is self-adjoint). An immediate consequence is their *positivity*: for any $x \in \mathsf{H}$

$$(u|Pu) = (u|PPu) = (P^*u|Pu) = (Pu|Pu) = ||Pu||^2 \geq 0\,. \qquad ■$$

The next couple of propositions characterise orthogonal projectors.

Proposition 3.64 *Let* H *be a Hilbert space and* $P \in \mathfrak{B}(\mathsf{H})$ *an orthogonal projector onto a subspace* M *(necessarily closed by Proposition 2.101(b)). Then*
(a) $Q := I - P$ *is an orthogonal projector.*

(b) $Q(\mathsf{H}) = \mathsf{M}^\perp$, *so the direct sum decomposition associated to* P *and* Q, *as of Proposition 2.101(b), is given by* M *and its orthogonal* $\mathsf{M}^\perp$:

$$\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp .$$

(c) *For any* $x \in \mathsf{H}$, $||x - P(x)|| = \min\{||x - y|| \mid y \in M\}$.
(d) *If* N *is a basis on* M, *then:*

$$Px = \sum_{u \in N} (u|x)u , \quad x \in \mathsf{H} .$$

(e) $I \geq P$; *moreover,* $||P|| = 1$ *if* P *is not the null operator (the projector onto* $\{\mathbf{0}\}$*).*

Proof (a) We know already that $Q := I - P$ is a projector (Proposition 2.4). By Proposition 3.38(c), since $I^* = I$, we have $Q^* = Q$, so Q is an orthogonal projector.
(b) By Proposition 2.101(b) it is enough to show $Q(\mathsf{H}) = \mathsf{M}^\perp$. For that notice that if $x \in Q(\mathsf{H})$ and $y \in \mathsf{M}$, then $(x|y) = (Qx|y) = (x|Qy) = (x|y - Py) = (x|y - y) = 0$, so $Q(\mathsf{H}) \subset \mathsf{M}^\perp$. We claim $\mathsf{M}^\perp \subset Q(\mathsf{H})$, hence $\mathsf{M}^\perp = Q(\mathsf{H})$. By Proposition 2.101, we have a direct sum:

$$\mathsf{H} = \mathsf{M} \oplus Q(\mathsf{H})$$

At the same time Theorem 3.13(d) gives the (orthogonal) decomposition:

$$\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp .$$

If $y \in \mathsf{M}^\perp$, the first decomposition induces $y = y_\mathsf{M} + z$ with $y_\mathsf{M} \in \mathsf{M}$ and $z \in Q(\mathsf{H})$. As we saw, $Q(\mathsf{H}) \subset \mathsf{M}^\perp$, so the uniqueness of the above splitting implies that $y = y_\mathsf{M} + z$ must also be the decomposition of y induced by $\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp$. Thus $y_\mathsf{M} \in \mathsf{M}$ and $z \in \mathsf{M}^\perp$. Then by assumption $y_\mathsf{M} = \mathbf{0}$, and $y = z \in Q(\mathsf{H})$. Since $y \in \mathsf{M}^\perp$ is arbitrary, we have proved $\mathsf{M}^\perp \subset Q(\mathsf{H})$.
(c) The statement is a straightforward consequence of Theorem 3.13(d) when $K := \mathsf{M}$, because the decomposition is unique.
(d) We may extend N to a basis of H by adding a basis N' of $\mathsf{M}^\perp$ (in fact $N \cup N'$ is an orthonormal system by construction; moreover, part (b) gives $\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp$, so any $x \in \mathsf{H}$ orthogonal to both N and N' must be null. Then, by definition, $N \cup N'$ is basis for H.) We can immediately verify that, varying $x \in \mathsf{H}$,

$$Rx = \sum_{u \in N} (u|x)u$$

and

$$R'x = \sum_{u \in N'} (u|x)u$$

define bounded operators (at most countably many products $(u|x)$ are non-zero for every fixed $x \in \mathsf{H}$), and they satisfy $RR = R$, $R(\mathsf{H}) = \mathsf{M}$, $R'R' = R'$, $R'(\mathsf{H}) = \mathsf{M}^\perp$ and also $R'R = RR' = 0$ and $R + R' = I$. By Proposition 2.101 R and R' are projectors associated to $\mathsf{M} \oplus \mathsf{M}^\perp$. By uniqueness of the decomposition of any vector we must have $R = P$ (and $R' = Q$).
(e) $Q = I - P$ is an orthogonal projector such that:

$$0 \leq (Qx|Qx) = (x|QQx) = (x|Qx) = (x|Ix) - (x|Px)\,,$$

for any $x \in \mathsf{H}$. This means $I \geq P$. What we have just seen also implies

$$||Px||^2 = (Px|Px) = (x|PPx) = (x|Px) \leq (x|x) = ||x||^2\,.$$

Therefore taking the supremum over all x with $||x|| = 1$ yields $||P|| \leq 1$. If $P \neq 0$, there will be a unit vector $x \in \mathsf{H}$ so that $Px = x$, hence $||Px|| = 1$. If so, $||P|| = 1$. □

Proposition 3.65 *Let* H *be a Hilbert space and* $\mathsf{M} \subset \mathsf{H}$ *a closed subspace. The projectors* P *and* Q *that decompose* $\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp$ *as in Proposition 2.102 (with* $\mathsf{N} := \mathsf{M}^\perp$*), and project onto* M *and* $\mathsf{M}^\perp$ *respectively, are orthogonal.*

Proof It is enough to prove $P = P^*$. That $Q = Q^*$ follows from $Q = I - P$.

If $x \in \mathsf{H}$ we have a unique decomposition $x = y + z$, $y = P(x) \in \mathsf{M}$ and $z = Q(x) \in \mathsf{M}^\perp$. Let $x' = y' + z'$ be the analogous splitting of $x' \in \mathsf{H}$. Then $(x'|Px) = (y' + z'|y) = (y'|y)$. We also have $(Px'|x) = (y'|y + z) = (y'|y)$, hence $(x'|Px) = (Px'|x)$ i.e. $((P^* - P)x'|x) = 0$ for any $x, x' \in \mathsf{H}$. By choosing $x = (P^* - P)x'$ we obtain $Px' = P^*x'$ for any x', so $P = P^*$. □

Our last result characterises commuting orthogonal projections.

Proposition 3.66 *Two orthogonal projectors* $P \neq 0$ *and* $P' \neq 0$ *on a Hilbert space* H **commute**,

$$PP' = P'P,$$

if and only if there exists a Hilbert basis N *of* H *such that, for every fixed* $x \in \mathsf{H}$*:*

$$Px = \sum_{u \in N_P} (u|x)u \quad \text{and, simultaneously,} \quad P'x = \sum_{u \in N_{P'}} (u|x)u,$$

for some pair of subsets $N_P, N_{P'} \subset N$.

Proof If $Px = \sum_{u \in N_P} (u|x)u$ and $P' = \sum_{u \in N_{P'}} (u|x)u$ for subsets $N_P, N_{P'} \subset N$, where N is a basis in H, then trivially $PP' = P'P$, as a direct computation, involving the orthogonality relations of $u \in N$, shows. Conversely, assume $PP' = P'P$. If $M := P(\mathsf{H})$, it is not hard to see $P'(M) \subset M$ and $P'(M^\perp) \subset M^\perp$. Moreover, $P'\restriction_M$ and $P'\restriction_{M^\perp}$ are orthogonal projectors on Hilbert spaces M and $M^\perp$ respectively, and in addition $P' = P'\upharpoonright_M \oplus P'\upharpoonright_{M^\perp}$, corresponding to $\mathsf{H} = M \oplus M^\perp$. By Proposition 3.64(d) (as any orthonormal system can be completed to a basis) we can fix bases N_M of M and $N_{M^\perp}$ of $M^\perp$ such that, for suitable subsets $N'_M \subset N_M$, $N'_{M^\perp} \subset N_{M^\perp}$, with obvious notation,

$$P'\upharpoonright_M = \sum_{u \in N'_M} u(u|\ )\ , \quad P'\upharpoonright_{M^\perp} = \sum_{v \in N'_{M^\perp}} v(v|\ )\ .$$

Therefore

$$P' = P'\upharpoonright_M \oplus P'\upharpoonright_{M^\perp} = \sum_{w \in N'_M \cup N'_{M^\perp}} w(w|\ )\ .$$

By construction, from $\mathsf{H} = M \oplus M^\perp$ we have that $N_M \cup N_{M^\perp}$ is a basis of H, and

$$Px = \sum_{w \in N_M} (w|x)w\ , \quad x \in \mathsf{H}$$

again from Proposition 3.64(d). The basis $N := N_M \cup N_{M^\perp}$ of H satisfies the requirements once we set $N_P := N_M$ and $N_{P'} := N'_M \cup N'_{M^\perp}$. $\square$

3.4.2 Hilbert Sum of Hilbert Spaces

The notion of *direct Hilbert sum of a family of Hilbert spaces* plays a relevant role in many technical constructions. We discuss here the most general case where the family's cardinality may be arbitrary.

Definition 3.67 (*Hilbert sum of Hilbert spaces*) Let $\{(\mathsf{H}_j, (\cdot|\cdot)_j\}_{j\in J}$ be a family of *arbitrary cardinality* of Hilbert spaces $\mathsf{H}_j \neq \{\mathbf{0}\}$ for every $j \in J$.

The **(direct orthogonal) Hilbert sum of the Hilbert spaces** H_j is the Hilbert space $\bigoplus_{j\in J} \mathsf{H}_j$ formed by families of vectors $\{\psi_j\}_{j\in J} \in \times_{j\in J}\mathsf{H}_j$ such that

$$\sum_{j\in J} ||\psi_j||_j^2 < +\infty \tag{3.47}$$

(in the sense of Definition 3.19). The linear structure is defined by

$$\alpha\{\psi_j\}_{j\in J} + \beta\{\phi_j\}_{j\in J} := \{\alpha\psi_j + \beta\phi_j\}_{j\in J} \quad \alpha, \beta \in \mathbb{C}\ ,$$

and the inner product:

$$\left(\{\psi_j\}_{j\in J} \middle| \{\phi_j\}_{j\in J}\right) := \sum_{j\in J} (\psi_a|\phi_a)_j\ . \tag{3.48}$$

An element $\{\psi_j\}_{j\in J} \in \oplus_{j\in J}\mathsf{H}_j$ is denoted by $\oplus_{j\in J}\psi_j$.

Remark 3.68 **(1)** We shall typically use the same symbol $\bigoplus_{\alpha\in A}\mathsf{H}_\alpha$ for both the ordinary direct sum and the Hilbert sum, since the nature of the structure in question will be clear from the context.
(2) In the finite case $J = \{1, \ldots, n\}$, the simpler notation

$$\mathsf{H}_1 \oplus \cdots \oplus \mathsf{H}_n := \bigoplus_{j\in J}\mathsf{H}_j$$

and

$$\psi_1 \oplus \cdots \oplus \psi_n := \oplus_{j\in J}\psi_j$$

are more common.
(3) When J is finite, condition (3.47) is automatically true, so $\mathsf{H}_1\oplus\cdots\oplus\mathsf{H}_n$ coincides with the product $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ equipped with the natural component-wise linear structure and the standard inner product

$$(v_1 \oplus \cdots \oplus v_n|u_1 \oplus \cdots \oplus u_n) := \sum_{i=1}^{n}(v_i|u_i)_i$$

which makes it a Hilbert space. It is easy to show that the topology of Hilbert space on $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ coincides with the product topology of the factors H_i. ■

Irrespective of the cardinality of J in Definition 3.67, we have

Proposition 3.69 *The Hilbert space $\bigoplus_{j\in J}\mathsf{H}_j$ is well defined.*

Proof First of all, if vectors $\oplus_{j\in J}\psi_j, \oplus_{j\in J}\phi_j$ satisfy (3.47), their inner product (3.48) is well defined (it complies with Definition 3.1). This is because there are at most countably many non-vanishing pairs (ψ_{j_n}, ψ_{j_n}), so the Cauchy–Schwarz inequality can be exploited to achieve

$$\sum_{j\in J}|(\psi_j|\phi_j)_j| = \sum_{n\in\mathbb{N}}|(\psi_{j_n}|\phi_{j_n})_{j_n}| \leq \sum_{n}||\psi_{j_n}||_{j_n}||\phi_{j_n}||_{j_n} \leq \sqrt{\sum_{j\in J}||\psi_j||_j^2\sum_{i\in J}||\phi_i||_i^2} < +\infty\,.$$

In particular, the sum (3.48), which is actually a series (or a finite sum), converges absolutely and can be rearranged.

The only nontrivial fact to check is the completeness of $\bigoplus_{j\in J}\mathsf{H}_j$, so let us prove it. Consider a Cauchy sequence $\{\oplus_{j\in J}\psi_{jn}\}_{n\in\mathbb{N}}$. Since $\sum_{j\in J}||\psi_{jn}||_j^2 < +\infty$ for every n, there is only a finite or countable set of indices j with $\psi_{jn} \neq \mathbf{0}$ for every fixed n. Hence the full set of $j \in J$ such that $\psi_{jn} \neq \mathbf{0}$ for some $n \in \mathbb{N}$ is at most countable, and we call them j_k with $k \in \mathbb{N}$ (the finite case being a trivial subcase). Thus we reduce to dealing with a countable family $\{\psi_{j_kn}\}_{n,k\in\mathbb{N}}$. The remaining vectors ψ_{jn} necessarily vanish. For every $\varepsilon > 0$ there exists N_ε with

$$|| \oplus_{j\in J} \psi_{jn} - \oplus_{j\in J}\psi_{jm}||^2 = \sum_{k\in\mathbb{N}} ||\psi_{j_k n} - \psi_{j_k m}||^2 < \varepsilon^2 \tag{3.49}$$

for $n, m > N_\varepsilon$. In particular, for a fixed k, we have $||\psi_{j_k n} - \psi_{j_k m}||^2 < \varepsilon^2$ if $n, m > N_\varepsilon$. Since H_{j_k} is complete, $\psi_{j_k n} \to \psi_{j_k} \in \mathsf{H}_{j_k}$ as $n \to +\infty$ and, from (3.49),

$$\sum_{k=0}^{N} ||\psi_{j_k n} - \psi_{j_k m}||^2 < \varepsilon^2 \tag{3.50}$$

Taking first the limit as $m \to \infty$ and then as $N \to +\infty$ in (3.50) we find

$$|| \oplus_{j\in J} \psi_{jn} - \oplus_{j\in J}\psi_{j}||^2 = \sum_{k\in\mathbb{N}} ||\psi_{j_k n} - \psi_{j_k}||^2 < \varepsilon^2 \tag{3.51}$$

for $n > N_\varepsilon$, where we defined $\oplus_{j\in J}\psi_j$ to have components ψ_{j_k}, or $\psi_j = \mathbf{0}$ when $j \notin \{j_k\}_{k\in\mathbb{N}}$. Condition (3.51) proves that $\oplus_{j\in J}\psi_{jn} \to \oplus_{j\in J}\psi_j$ as $n \to +\infty$ provided $\oplus_{j\in J}\psi_j \in \bigoplus_{j\in J} \mathsf{H}_j$. This is the case because:

$$\sum_{j\in J} ||\psi_j||^2 = \sum_{k\in\mathbb{N}} ||\psi_{j_k}||^2 \leq \sum_{k\in\mathbb{N}} (||\psi_{j_k n}|| + ||\psi_{j_k n} - \psi_{j_k}||)^2$$

$$\leq \sum_{k\in\mathbb{N}} ||\psi_{j_k n}||^2 + \sum_{k\in\mathbb{N}} ||\psi_{j_k n} - \psi_{j_k}||^2 + 2\sum_{k\in\mathbb{N}} ||\psi_{j_k n}||\, ||\psi_{j_k n} - \psi_{j_k}||$$

$$\leq || \oplus_{j\in J} \psi_{jn}||^2 + \varepsilon^2 + 2|| \oplus_{j\in J} \psi_{jn}|| \sqrt{\sum_{k\in\mathbb{N}} ||\psi_{j_k n} - \psi_{j_k}||^2}$$

$$= (|| \oplus_{j\in J} \psi_{jn}|| + \varepsilon)^2 < +\infty\,,$$

if $n > N_\varepsilon$. $\square$

The summands H_j in a Hilbert sum are naturally identified with (non-trivial) closed subspaces of $\bigoplus_{j\in J} \mathsf{H}_j$ itself, and $\mathsf{H}_j \perp \mathsf{H}_i$ if $i \neq j$. The converse is true too, for we have the following proposition.

Proposition 3.70 *Let H be a Hilbert space and $\{\mathsf{H}_\alpha\}_{\alpha\in A}$ a collection of arbitrary cardinality of closed subspaces such that*
(i) $\mathsf{H}_\alpha \neq \{\mathbf{0}\}$ for every $\alpha \in A$,
(ii) $\mathsf{H}_\alpha \perp \mathsf{H}_\beta$ if $\alpha, \beta \in A$ with $\alpha \neq \beta$.
Then

$$\bigoplus_{\alpha\in A} \mathsf{H}_\alpha \quad \textit{is isomorphic to} \quad \overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >} \tag{3.52}$$

under the map

$$L : \overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >} \ni \psi \;\mapsto\; \oplus_{\alpha\in A} P_\alpha \psi \in \bigoplus_{\alpha\in A} \mathsf{H}_\alpha \,, \tag{3.53}$$

where $P_\alpha : \mathsf{H} \to \mathsf{H}$ *is the orthogonal projector onto* H_α*, for every* $\alpha \in A$.

Proof Take a Hilbert basis $N_\alpha \subset \mathsf{H}_\alpha$, for every $\alpha \in A$. The orthonormal set $N := \cup_{\alpha\in A} N_\alpha$ is a Hilbert basis of the closed subspace $\overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >}$ because the span of N is dense therein. If we take $\psi \in \overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >}$, then

$$||\psi||^2 = \sum_{z\in N} |(z|\psi)|^2$$

where, as usual, only a finite, or countable, set of summands are strictly positive. Thus, interpreting the sum as an integral for the counting measure on $\mathbb{N}$, we may use the dominated convergence theorem and safely rearrange the sum as

$$||\psi||^2 = \sum_{\alpha\in A} \sum_{z\in N_\alpha} |(z|\psi)|^2 \,,$$

In other words,

$$||\psi||^2 = \sum_{\alpha\in A} ||P_\alpha\psi||^2 \,.$$

This identity proves at once that the linear map L in (3.53) is both well defined and isometric. To prove that L is a Hilbert space isomorphism it is enough to show it is surjective. If $\{\phi_\alpha\}_{\alpha\in A} \in \oplus_{\alpha\in A}\mathsf{H}_\alpha$, we have $\sum_{\alpha\in A} ||\phi_\alpha||^2 < \infty$ by hypothesis, so that $\phi := \sum_{\alpha\in A} \phi_\alpha$ exists in H in view of Lemma 3.25 (again, only a finite or countable number of ϕ_α do not vanish, and $\phi_\alpha \perp \phi_\beta$ for $\alpha \neq \beta$). Moreover $\phi \in \overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >}$, since ϕ is a limit of elements in the span of $\{\mathsf{H}_\alpha\}_{\alpha\in A}$. Evidently $L\phi = \{\phi_\alpha\}_{\alpha\in A}$. □

Notation 3.71 In the rest of the book, under the assumptions of Proposition 3.70, we shall often write $\bigoplus_{\alpha\in A} \mathsf{H}_\alpha$ in place of $\overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >}$, since the isomorphism between the two spaces is canonical. ■

From the proof of Proposition 3.70 we can extract the identity

$$||P\psi||^2 = \sum_{\alpha\in A} ||P_\alpha\psi||^2 \quad \text{for every } \psi \in \mathsf{H} \,, \tag{3.54}$$

where P is the orthogonal projector onto $\oplus_{\alpha\in A}\mathsf{H}_\alpha$ viewed as a subspace of H in accordance with Proposition 3.70. All norms refer to H, and $P_\alpha \neq 0$ is the orthogonal projector onto H_α for any $\alpha \in A$. Note that $P_\alpha \perp P_\beta$ if $\alpha \neq \beta$, forcing the map $A \ni \alpha \mapsto P_\alpha$ to be injective, as $P_\alpha \neq 0$. Furthermore, we also found the other useful relation

$$P\psi = \sum_{\alpha\in A} P_\alpha\psi \quad \text{for every } \psi \in \mathsf{H} \,, \tag{3.55}$$

where the series may be rearranged arbitrarily. The sum is a standard (infinite) series or a finite sum, since at most countably many terms $P_\alpha \psi$ do not vanish (Proposition 3.21(b)). If A is countable, as when H is separable and infinite-dimensional, (3.55) can be interpreted as a series in the strong operator topology

$$P = \text{s-}\sum_{a \in A} P_a \,. \tag{3.56}$$

3.4.3 Partial Isometries

We can pass to the useful notion of a *partial isometry*, a weaker version of the isometries seen earlier.

Definition 3.72 A bounded operator $U : \mathsf{H} \to \mathsf{H}$, with H a Hilbert space, is a **partial isometry** when:

$$||Ux|| = ||x|| \,, \quad \text{for } x \in [Ker(U)]^\perp .$$

If so, $[Ker(U)]^\perp$ is called the **initial space** of U and $Ran(U)$ the **final space**.

Any unitary operator $U : \mathsf{H} \to \mathsf{H}$ is a special partial isometry whose initial and final spaces coincide with the entire Hilbert space H. Observe also that if $U : \mathsf{H} \to \mathsf{H}$ is a partially isometric operator then H decomposes orthogonally into $Ker(U) \oplus [Ker(U)]^\perp$, and U restricts to an honest isometry on the second summand (with values in $Ran(U)$), while it is null on the first summand. This self-evident fact can be made stronger by proving that $Ran(U)$ is closed, hence showing $U\restriction_{[Ker(U)]^\perp}: [Ker(U)]^\perp \to Ran(U)$ is indeed a unitary operator between Hilbert spaces (closed subspaces in H). The second statement in the ensuing proposition shows U^* is a partial isometry if U is, and its initial and final spaces are those of U, but swapped.

Proposition 3.73 *Let $U : \mathsf{H} \to \mathsf{H}$ be a partial isometry on the Hilbert space. Then*
(a) *$Ran(U)$ is closed.*
(b) *$U^* : \mathsf{H} \to \mathsf{H}$ is a partial isometry with initial space $Ran(U)$ and final space $[Ker(U)]^\perp$.*

Proof (a) Let $y \in \overline{Ran(U)} \setminus \{\mathbf{0}\}$ (if $\mathsf{H} = \{\mathbf{0}\}$ the proof of the whole proposition is trivial). There is a sequence of vectors $x_n \in [Ker(U)]^\perp$ such that $Ux_n \to y$ as $n \to +\infty$. Since $||U(x_n - x_m)|| = ||x_n - x_m||$ by definition of partial isometry, the sequence x_n is Cauchy and converges to some $x \in \mathsf{H}$. By continuity $y = Ux$, so $y \in Ran(U)$. But $y = \mathbf{0}$ clearly belongs to $Ran(U)$, so we have proved $Ran(U)$ contains all its limits points, and as such it is closed.
(b) Begin by observing $Ker(U^*) = \overline{Ran(U)}$ and $\overline{Ran(U^*)} = [Ker(U)]^\perp$ by Proposition 3.38, so if we use part (a) there remains only to prove U^* is an isometry when restricted to $\overline{Ran(U)}$. Notice preliminarily that if $z, z' \in [Ker(U)]^\perp$, U being a partial isometry implies:

$$(Uz|Uz') = \frac{1}{4}\left[||U(z+z')||^2 - ||U(z-z')||^2 - i||U(z+iz')||^2 + i||U(z-iz')||^2)\right]$$

$$= \frac{1}{4}\left[||z+z'||^2 - ||z-z'||^2 - i||z+iz'||^2 + i||z-iz'||^2)\right] = (z|z') \, .$$

From what we have seen, suppose $y = Ux$ with $x \in [Ker(U)]^\perp$. Then

$$||U^*y||^2 = (U^*Ux|U^*Ux) = (Ux|U(U^*Ux)) = (x|U^*Ux) = (Ux|Ux) = ||y||^2.$$

In other terms U^* is isometric on $Ran(U)$, and it remains isometric also on the closure by continuity. This proves (b). □

At last, we present a relationship between partial isometries and orthogonal projectors.

Proposition 3.74 *Let $U : \mathsf{H} \to \mathsf{H}$ be a bounded linear operator on the Hilbert space H.*
(a) *U is a partial isometry if and only if U^*U is an orthogonal projector. In such a case UU^* is an orthogonal projector as well.*
(b) *If U is a partial isometry, U^*U projects onto the initial space of U, and UU^* projects onto the final space of U.*

Proof Suppose U is partially isometric, and let us show U^*U is an orthogonal projector. Since the latter is patently self-adjoint, it suffices to show it is idempotent. Decompose $\mathsf{H} \ni x = x_1 + x_2$ by $x_1 \in Ker(U)$ and $x_2 \in [Ker(U)]^\perp$. Then

$$(x|(U^*U)^2x) = (Ux|U^*UUx) = (Ux_2|U(U^*Ux_2)) = (x_2|U^*Ux_2) = (Ux_2|Ux_2)$$

$$= (Ux|Ux) \, .$$

That is to say, $(x|((U^*U)^2 - U^*U)x) = 0$ whichever $x \in \mathsf{H}$ is taken. Choose $x = y \pm iz$ and $x = y \pm y$, and then $(y|((U^*U)^2 - U^*U)z) = 0$ for any $y, z \in \mathsf{H}$. Therefore U^*U is idempotent, so an orthogonal projector. Conversely if U^*U is an orthogonal projector, let N be the closed subspace onto which it projects. If $U^*Ux = \mathbf{0}$ then $Ux \in Ker(U^*) = [Ran(U)]^\perp$. But $Ux \in Ran(U)$, hence $Ux = \mathbf{0}$. Therefore $U^*Ux = \mathbf{0}$ if and only if $x \in Ker(U)$, so $\mathsf{N}^\perp = Ker(U^*U) = Ker(U)$ and $\mathsf{N} = [Ker(U)]^\perp$. If additionally $x \in [Ker(U)]^\perp = \mathsf{N}$, then $U^*Ux = x$ and $||Ux||^2 = (U^*Ux|x) = ||x||^2$, proving U is a partial isometry. Throughout we also proved U^*U projects onto the initial space $\mathsf{N} = [Ker(U)]^\perp$. The remaining part follows easily from Proposition 3.73(b). In fact, if U is a partial isometry, U^* is partially isometric and so $UU^* = (U^*)^*U^*$ is an orthogonal projector. From the previous part it projects onto the closed subspace $[Ker(U^*)]^\perp = Ran(U)$. □

3.5 Polar Decomposition

This section is rather technical and contains useful notions in the theory of bounded operators on a Hilbert space. The central result is the so-called *polar decomposition theorem for bounded operators*, that generalises the polar form of a complex number whereby $z = |z|e^{i\, arg\, z}$ splits as product of the modulus times an exponential with purely imaginary logarithm. In the analogy z corresponds to a bounded operator, $|z|$ plays the role of a certain positive operator called *modulus*, and $e^{i\, arg\, z}$ is represented by a unitary operator when restricted to a subspace. The *modulus of an operator* is useful to generalise the "absolute convergence" of numerical series, and is built using operators and bases. We shall use these series to define *Hilbert–Schmidt operators* and *operators of trace class*, some of which represent states in QM. Part of the ensuing proofs are taken from [Mar82, KaAk82].

3.5.1 Square Roots of Bounded Positive Operators

Definition 3.75 Given a Hilbert space H and $A \in \mathfrak{B}(\mathsf{H})$, one says that $B \in \mathfrak{B}(\mathsf{H})$ is a **square root** of A if $B^2 = A$. If additionally $B \geq 0$, we call B a **positive square root**.

We will show in a moment that any bounded positive operator has one, and one only, positive square root. For this we need the preliminary result below, about sequences and series of orthogonal projectors in the strong topology, which is on its own a useful fact in spectral theory.

Proposition 3.76 *Let* H *be a Hilbert space and* $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}(\mathsf{H})$ *a non-decreasing (or non-increasing) sequence of self-adjoint operators. If* $\{A_n\}$ *is bounded from above (resp. below) by* $K \in \mathfrak{B}(\mathsf{H})$*, there exists a self-adjoint operator* $A \in \mathfrak{B}(\mathsf{H})$ *such that* $A \leq K$ *(*$A \geq K$*) and:*

$$A = s\text{-}\lim_{n\to+\infty} A_n\ . \tag{3.57}$$

Proof We prove it in the non-decreasing case, for the other case falls back to this situation if one considers $K - A_n$.

Set $B_n := A_n + ||A_0||I$. Then we can prove the following facts.

(i) The B_n form a *non-decreasing sequence of positive operators*. If $||x|| = 1$, in fact, $(x|A_n x) + ||A_0|| \geq (x|A_0 x) + ||A_0||$, but $-||A_0|| \leq (x|A_0 x) \leq ||A_0||$ by Proposition 3.60(a). Therefore $(x|A_n x) + ||A_0|| \geq 0$ for any unit vector x. That is to say $(y|A_n y) + ||A_0||(y|y) \geq 0$ for any $y \in \mathsf{H}$, i.e. $B_n = A_n + ||A_0||I \geq 0$.

(ii) $B_n \leq K + ||A_0||I =: K_1$, and K_1 is *positive* (K cannot be).

(iii) $(x|K_1 x) \geq (x|B_n x) - (x|B_m x) \geq 0$ for any $x \in \mathsf{H}$ if $n \geq m$. In fact, $(x|K_1 x) \geq (x|B_n x)$, and also $-(x|B_m x) \leq 0$ and $(x|B_n x) - (x|B_m x) \geq 0$.

Since any positive operator T defines a semi-inner product that satisfies Schwarz's inequality:

$$|(x|Ty)|^2 \leq (x|Tx)(y|Ty) , \tag{3.58}$$

we have, if $n \geq m$:

$$\begin{aligned}|(x|(B_n - B_m)y)|^2 &\leq (x|(B_n - B_m)x)(y|(B_n - B_m)y) \leq (x|K_1x)(y|K_1y)\\ &\leq ||K_1||^2||x||^2||y||^2 .\end{aligned}$$

Hence

$$|(x|(B_n - B_m)y)|^2 \leq ||K_1||^2||x||^2||y||^2 .$$

If we set $x = (B_n - B_m)y$ and take the supremum over unit vectors $y \in \mathsf{H}$, we find:

$$||B_n - B_m|| \leq ||K_1|| . \tag{3.59}$$

From (3.58), putting $y = (B_n - B_m)x$ and $T = B_n - B_m$, we obtain

$$||(B_n - B_m)x||^4 = ((B_n - B_m)x|(B_n - B_m)x)^2 \leq (x|(B_n - B_m)x)((B_n - B_m)x|(B_n - B_m)^2x)$$

for $x \in \mathsf{H}$, $n \geq m$. By (3.59), the last term is bounded by

$$(x|(B_n - B_m)x)||B_n - B_m||^3||x||^2 \leq ||K_1||^3||x||^2[(x|B_nx) - (x|B_mx)] ,$$

and so

$$||(B_n - B_m)x||^4 \leq ||K_1||^3||x||^2[(x|B_nx) - (x|B_mx)] .$$

The non-decreasing, bounded sequence of positive numbers $(x|B_kx)$ has to converge, so it is Cauchy. Therefore also the B_kx must form a Cauchy sequence, and as $k \to +\infty$ the limit exists. Define

$$B : \mathsf{H} \ni x \mapsto \lim_{n\to+\infty} B_nx .$$

By construction B is linear, and it satisfies

$$0 \leq (Bx|x) = (x|Bx) \leq (x|K_1x)$$

since $0 \leq (B_kx|x) = (x|B_kx) \leq (x|K_1x)$ for any $k \in \mathbb{N}$.

Now, K_1 is bounded and self-adjoint (it is positive), so Proposition 3.60(a) forces B to be bounded, since:

$$\sup\{|(x|Bx)| \mid x \in \mathsf{H} , ||x|| = 1\} \leq \sup\{|(x|K_1x)| \mid x \in \mathsf{H} , ||x|| = 1\} = ||K_1|| .$$

But B is also self-adjoint because of Proposition 3.60(e). Therefore $A := B - ||A_0||I$ is a bounded, self-adjoint operator and

$$Ax = \lim_{n\to+\infty}(B_n - ||A_0||I)x = \lim_{n\to+\infty} A_n x \ .$$

Eventually, $A \le K$ because for any $x \in \mathsf{H}$ we have $(x|A_n x) \le (x|Kx)$ by assumption, and this is still true when taking the limit as $n \to +\infty$. $\square$

The above result allows us to prove that bounded positive operators admit square roots.

Theorem 3.77 *Let* H *be a Hilbert space and* $A \in \mathfrak{B}(\mathsf{H})$ *a positive operator. Then there exists a unique positive square root, indicated by* $\sqrt{A}$*. Furthermore:*
(a) $\sqrt{A}$ *commutes with any* $B \in \mathfrak{B}(\mathsf{H})$ *that commutes with* A*:*

$$\textit{if } AB = BA \textit{ with } B \in \mathfrak{B}(\mathsf{H}), \textit{ then } \sqrt{A}B = B\sqrt{A}.$$

(b) *if* A *is bijective,* $\sqrt{A}$ *is bijective.*

Proof We may as well suppose $||A|| \le 1$ without any loss of generality, so let us set $A_0 := I - A$. We shall show $A_0 \ge 0$ and $||A_0|| \le 1$.

First of all $A_0 \ge 0$ because $(x|A_0 x) = (x|x) - (x|Ax) \ge ||x||^2 - ||A|| ||x||^2$, where we have used $A = A^*$, so (Proposition 3.60(a)) $||A|| = \sup\{|(z|Az)| \mid ||z|| = 1\}$, and $|(z|Az)| = (z|Az)$ by positivity. Since $(x, y) \mapsto (x|A_0 y)$ is a semi-inner product, from $A_0 \ge 0$, the Cauchy–Schwarz inequality:

$$|(x|A_0 y)|^2 \le (x|A_0 x)(y|A_0 y) \le ||x||^2 ||y||^2$$

holds, having used the positivity of $A = I - A_0$ and A_0 in the final step. Since $A = A^*$, using $y = A_0 x$ in the inequality gives

$$|(A_0 x|A_0 x)|^2 \le ||x||^2 ||A_0 x||^2 \ ,$$

hence $||A_0 x|| \le ||x||$, and so:

$$||A_0|| \le 1 \ . \tag{3.60}$$

Define a sequence of bounded operators $B_n : \mathsf{H} \to \mathsf{H}$, $n = 1, 2, \ldots$:

$$B_1 := 0 \ , \qquad B_{n+1} := \frac{1}{2}(A_0 + B_n^2) \ . \tag{3.61}$$

From (3.60), using the norm's properties,

$$||B_n|| \le 1 \quad \text{for any } n \in \mathbb{N} \ . \tag{3.62}$$

By induction, the operators B_n are polynomials in A_0 with non-negative coefficients. *Recall, here and in the sequel, that all operators* B_k *commute with one another and with* A_0*, by construction*. Equation (3.61) implies

$$B_{n+1} - B_n = \frac{1}{2}(A_0 + B_n^2) - \frac{1}{2}(A_0 + B_{n-1}^2) = \frac{1}{2}(B_n^2 - B_{n-1}^2)$$

i.e.

$$B_{n+1} - B_n = \frac{1}{2}(B_n + B_{n-1})(B_n - B_{n-1}) .$$

This identity implies, via induction, that $B_{n+1} - B_n$ are polynomials in A_0 with non-negative coefficients: every $B_n + B_{n-1}$ is a sum of polynomials with non-negative coefficients, and is itself a polynomial with non-negative coefficients; moreover, the product of two such is still of the same kind.

Since $A_0 \geq 0$, any polynomial in A_0 with non-negative coefficients is a positive operator: the polynomial is a sum of terms $a_{2n} A_0^{2n}$ (all positive, as $a_{2n} \geq 0$ and $A_0^{2n} = A_0^n A_0^n$ with A_0^n self-adjoint, so $a_{2n}(x|A_0^{2n}x) = a_{2n}(A_0^n x|A_0^n x) \geq 0$), and of terms $a_{2n+1} A_0^{2n+1}$ (also positive, for $a_{2n+1} \geq 0$ and $(x|A_0^{2n+1}x) = (x|A_0^n A A_0^n x) = (A_0^n x|A\ A_0^n x) \geq 0$).

We conclude the bounded operators B_n and $B_{n+1} - B_n$ are positive. So the sequence of positive, bounded (and self-adjoint) operators B_n is non-decreasing. The sequence is also bounded from above by I. In fact, $B_n^* = B_n \geq 0$ implies, due to Proposition 3.60(a), that $(x|B_n x) = |(x|B_n x)| \leq ||B_n|| ||x||^2$. From (3.62) follows $B_n \leq I$. So, we may apply Proposition 3.76 to obtain a positive bounded operator $B_0 \leq I$ such that

$$B_0 = \text{s-}\lim_{n\to+\infty} B_n .$$

By definition of strong topology, and because the continuous operators B_k commute,

$$B_0 B_m x = (\lim_{n\to+\infty} B_n) B_m x = \lim_{n\to+\infty} B_n B_m x = \lim_{n\to+\infty} B_m B_n x = B_m \lim_{n\to+\infty} B_n x = B_m B_0 x .$$

Thus B_0 commutes with every B_m,

$$B_0^2 - B_n^2 = (B_0 + B_n)(B_0 - B_n)$$

and so, as $n \to +\infty$:

$$||B_0^2 x - B_n^2 x|| \leq ||B_0 + B_n|| ||B_0 x - B_n x|| \leq (||B_0|| + ||B_n||) ||B_0 x - B_n x|| \leq 2||B_0 x - B_n x|| \to 0.$$

Rephrasing,

$$B_0^2 x = \lim_{n\to+\infty} B_n^2 x .$$

Taking the limit in

$$B_{n+1} x = \frac{1}{2}(A_0 x + B_n^2 x) ,$$

obtained from (3.61), we find

$$B_0 x = \frac{1}{2}(A_0 x + B_0^2 x) ,$$

for any $x \in \mathsf{H}$, i.e.

$$2B_0 = A_0 + B_0^2 .$$

To conclude, let us write the above identity in terms of $B := I - B_0$:

$$B^2 = I - A_0 ,$$

i.e.

$$B^2 = A .$$

Therefore B is a square root of A. Note that $B \geq 0$ because $B_0 \leq I$ and $B = I - B_0$, so B is a positive root of A. Moreover, if C is bounded and commutes with A, it commutes with A_0 and hence with any B_n. Therefore C commutes also with B_0 and $B = I - B_0$.

Let us now show the uniqueness of a positive square root V of A. The above positive root B commutes with all operators that commute with A. Since

$$AV = V^3 = VA ,$$

V and A commute, forcing B to commute with V. Fix an arbitrary $x \in \mathsf{H}$ and set $y := Bx - Vx$. Then:

$$||Bx-Vx||^2 = ([B-V]x|[B-V]x) = ([B-V]x|y) = (x|[B^*-V^*]y) = (x|[B-V]y) \quad (3.63)$$

We will show that $By = \mathbf{0}$ and $Vy = \mathbf{0}$ independently. This will end the proof, because then $||Bx - Vx|| = 0$ will imply $B = V$.

Now,

$$(y|By) + (y|Vy) = (y|[B+V][B-V]x) = (y|[B^2 - V^2]x) = (y|[A-A]x) = 0 .$$

Since $(y|Vy) \geq 0$ and $(y|By) \geq 0$,

$$(y|Vy) = (y|By) = 0 .$$

This means $Vy = By = \mathbf{0}$, for if W is a positive root of V, from

$$||Wy||^2 = (Wy|Wy) = (y|W^2 y) = (y|Vy) = 0$$

it follows that $Wy = \mathbf{0}$ and a fortiori $Vy = W(Wy) = \mathbf{0}$. Analogously, $By = \mathbf{0}$.

There remains to prove $\sqrt{A}$ is bijective if A is. If A is bijective, it commutes with A^{-1}, so $\sqrt{A}$ too commutes with A^{-1}. Then, immediately, $A^{-1}\sqrt{A} = \sqrt{A}A^{-1}$ is the left and right inverse of $\sqrt{A}$, which becomes bijective. $\square$

Corollary 3.78 *Let* H *be a Hilbert space. If* $A, B \in \mathfrak{B}(\mathsf{H})$ *are positive and commute, their product is a positive element of* $\mathfrak{B}(\mathsf{H})$.

Proof $\sqrt{B}$ commutes with A, hence

$$(x|ABx) = (x|A\sqrt{B}^2 x) = (x|\sqrt{B}A\sqrt{B}x) = (\sqrt{B}x|A\sqrt{B}x) \geq 0\,.$$

□

Remark 3.79 That the square root of $0 \leq A \in \mathfrak{B}(\mathsf{H})$ commutes with every operator of $\mathfrak{B}(\mathsf{H})$ that commutes with A can be expressed, equivalently, by saying $\sqrt{A}$ belongs to the *von Neumann algebra* generated by I and A in $\mathfrak{B}(\mathsf{H})$. ■

3.5.2 *Polar Decomposition of Bounded Operators*

To conclude the section we will show that any bounded operator A in a Hilbert space admits a decomposition $A = UP$ as a product of a uniquely-determined positive operator P with an isometric operator U, defined and unique on the image of P. The splitting is called *polar decomposition* and has a host of applications in mathematical physics. A preparatory definition is needed first.

Definition 3.80 Let H be a Hilbert space and $A \in \mathfrak{B}(\mathsf{H})$. The bounded, positive and hence self-adjoint operator

$$|A| := \sqrt{A^*A} \tag{3.64}$$

is called **modulus** of A.

Remark 3.81 For any $x \in \mathsf{H}$: $||\,|A|\,x||^2 = (x|\,|A|^2x) = (x|A^*Ax) = ||Ax||^2$, so:

$$||\,|A|\,x|| = ||Ax||\,, \tag{3.65}$$

whence:

$$Ker(|A|) = Ker(A) \tag{3.66}$$

and so $|A|$ is injective if and only if A is. Another useful property is

$$\overline{Ran(|A|)} = (Ker(A))^{\perp}\,, \tag{3.67}$$

consequence of

$$\overline{Ran(|A|)} = ((Ran(|A|))^{\perp})^{\perp} = (Ker(|A|^*))^{\perp} = (Ker(|A|))^{\perp} = (Ker(A))^{\perp}\,.$$

■

Now to the polar decomposition theorem. We present the version for bounded operators. Theorem 10.39 will give us a more general statement about a special class of unbounded operators.

Theorem 3.82 (Polar decomposition of bounded operators) *Let* H *be a Hilbert space and* $A \in \mathfrak{B}(\mathsf{H})$.
(a) *There exist unique operators* $P, U \in \mathfrak{B}(\mathsf{H})$ *such that:*
(1) the decomposition

$$A = UP \tag{3.68}$$

holds,
(2) P *is positive,*
(3) U *is isometric on* $Ran(P)$,
(4) U *is null on* $Ker(P)$.
(b) $P = |A|$, *so* $Ker(U) = Ker(A) = Ker(P) = [Ran(P)]^{\perp}$.
(c) *If* A *is bijective,* U *coincides with the unitary operator* $A|A|^{-1}$.

Proof (a) We start with the uniqueness. Suppose we have (3.68), $A = UP$ with $P \geq 0$ (beside bounded) and U bounded. Then $A^* = PU^*$, since P is self-adjoint as positive (Theorem 3.77(c)), and hence

$$A^*A = PU^*UP\ . \tag{3.69}$$

That U is isometric on $Ran(P)$ is expressed as $(UPx|UPy) = (Px|Py)$ for any $x, y \in \mathsf{H}$, or $(x|[PU^*UP - P^2]y) = 0$ for any $x, y \in \mathsf{H}$. Therefore $PU^*UP = P^2$. Substituting in (3.69) we have $P^2 = A^*A$. As P is positive and extracting the only positive root (Theorem 3.77) of both sides we get $P = |A|$. So if a decomposition as claimed exists, necessarily $P = |A|$. Let us prove U is unique as well. From $\mathsf{H} = Ker(P) \oplus (Ker(P))^{\perp}$, Proposition 3.38(d) and Theorem 3.13(e) imply $(Ker(P))^{\perp} = \overline{Ran(P^*)} = \overline{Ran(P)}$ because P is self-adjoint. Hence $\mathsf{H} = Ker(P) \oplus \overline{Ran(P)}$. To define an operator on H it suffices to have it on both summands: $U = 0$ on $Ker(P)$, while $UPx = Ax$ for any $x \in \mathsf{H}$ determines U on $Ran(P)$ uniquely. By assumption, on the other hand, U is bounded, and it remains bounded if restricted to $Ran(P)$. A bounded operator over a dense domain can be extended to a unique bounded operator on the closed domain (cf. Proposition 2.47). Therefore U is completely determined on $\overline{Ran(P)}$, hence on H. This concludes the proof of the uniqueness, so let us deal with the existence.

We must show that $UP = A$, $P = |A|$, or better: $U : |A|x \mapsto Ax$, for any $x \in \mathsf{H}$, actually defines an operator, say U_0, on $Ran(|A|)$. To make it well defined, it is necessary and sufficient that $|A|x = |A|y \Rightarrow Ax = Ay$, otherwise it would not be a function. By (3.65), if $|A|x = |A|y$, then $Ax = Ay$, so $U_0 : Ran(|A|) \ni |A|x \mapsto Ax$ is well defined (not multi-valued). That U_0 is linear is obvious by construction, as is the fact that it is an isometry, for U_0 preserves norms by (3.65) (cf. Exercise 3.8). Being an isometry on $Ran(|A|)$ implies, by continuity, that we can extend it uniquely to an isometry, still called U_0, on the closure of $Ran(|A|)$. Now define

$U : \mathsf{H} \to \mathsf{H}$ by setting $U\restriction_{Ker(|A|)} := 0$ and $U\restriction_{\overline{Ran(|A|)}} := U_0$, relative to the splitting $\mathsf{H} = Ker(|A|) \oplus \overline{Ran(|A|)}$. It is immediate to see that $U \in \mathfrak{B}(\mathsf{H})$ and U satisfies (3.68). Furthermore, by construction $Ker(U) \supset Ker(|A|)$. We claim the latter two are equal. Any u with $Uu = \mathbf{0}$ splits into $u_0 + x$, with $u_0 \in Ker(|A|)$ annihilated by U, and $x \in \overline{Ran(|A|)}$ such that $U_0 x = \mathbf{0}$. Since on $\overline{Ran(|A|)}$ U_0 is isometric, then $x = \mathbf{0}$ and so $u = u_0 \in Ker(|A|)$. Therefore $Ker(U) \subset Ker(|A|)$, so overall $Ker(U) = Ker(|A|) = Ker(A)$ by (3.66).
(b) was proved within part (a).
(c) If A is injective, using (b) we see $Ker(A) = Ker(U)$ is trivial and so U is injective. Directly from $A = UP$, though, we have $Ran(U) \supset Ran(A)$, so if A is onto also U is. Hence, if A is bijective then U must be as well. If so, U is a surjective isometry on $\overline{Ran(P)} = (Ker(P))^\perp = \{\mathbf{0}\}^\perp = \mathsf{H}$ by (b), hence unitary. At last, from $A = U|A|$ follows that $|A|$ is bijective because A and U are, whence we can write $U = A|A|^{-1}$. $\square$

Remarks 3.83 **(1)** The operator U showing up in (3.69) is a *partial isometry* (Definition 3.72) with *initial space*

$$[Ker(U)]^\perp = \overline{Ran(|A|)} = [Ker(A)]^\perp = \overline{Ran(A^*)}\,.$$

Bearing in mind Proposition 3.73(a) we see easily that the *final space* of U is

$$Ran(U) = \overline{Ran(A)}\,.$$

(2) Theorem 10.39 gives a polar decomposition under much weaker assumptions on A. We will also prove that the partial isometry U has the same initial and final spaces above, and is unitary precisely when A is injective and, simultaneously, $Ran(A)$ is dense in H. ■

Definition 3.84 Let H be a Hilbert space and $A \in \mathfrak{B}(\mathsf{H})$. The splitting

$$A = UP\,, \tag{3.70}$$

with $P \in \mathfrak{B}(\mathsf{H})$ positive, $U \in \mathfrak{B}(\mathsf{H})$ isometric on $Ran(P)$ and null on $Ker(P)$, is called **polar decomposition** of the operator A.

A corollary of the polar decomposition, useful in several applications, is this.

Corollary 3.85 (to Theorem *3.82*) *Under the assumptions of Theorem 3.82, if $U|A| = A$ is the polar decomposition of A, then:*

$$|A^*| = U|A|U^*\,. \tag{3.71}$$

Proof From $A = U|A|$ follows $A^* = |A|U^* = U^*\,U|A|U^*$, where we used $U^*U|A| = |A|$, since U is isometric on $Ran(|A|)$. Thus the self-adjoint operator AA^* satisfies

$$AA^* = U|A|U^* \, U|A|U^* .$$

As $U|A|U^*$ is clearly positive, by uniqueness of the root we have

$$|A^*| = \sqrt{(A^*)^*A^*} = \sqrt{AA^*} = U|A|U^* ,$$

proving the claim. □

We cite, in the form of the next theorem, yet another consequence of the polar decomposition theorem valid when $A \in \mathfrak{B}(\mathsf{H})$ is normal, i.e. commuting with A^*.

Theorem 3.86 (Polar decomposition of normal operators) *Let* $A \in \mathfrak{B}(\mathsf{H})$ *be a normal operator on the Hilbert space* H*, and* $W_0 : Ker(A) \to Ker(A)$ *a given unitary operator.*

There exists a unique pair $W, P \in \mathfrak{B}(\mathsf{H})$ *such that* $P \geq 0$*,* W *is unitary and:*

$$A = WP \quad \textit{with} \quad W\restriction_{Ker(A)} = W_0 .$$

Moreover, $P = |A|$*,* $W\restriction_{Ker(A)^\perp}$ *does not depend on* W_0*, and* W *commutes with* A*,* A^* *and* $|A|$*.*

Proof Under the assumptions made, $A = WP$ implies $A^*A = PW^*WP = P^2$, so $P = |P|$. Then consider the polar decomposition $A = U|A|$. As we know (Remark 3.83(1)) U is partially isometric, with initial space $Ker(A)^\perp$ and final space $\overline{Ran(A)}$. We have $\mathsf{H} = Ker(A) \oplus Ker(A)^\perp = Ker(A) \oplus \overline{Ran(A^*)}$, and since A is normal, by (iii) in Proposition 3.60(b) we can write $\mathsf{H} = Ker(A) \oplus \overline{Ran(A)}$. So U is unitary from $\overline{Ran(A)}$ to $\overline{Ran(A)}$, and is null from $Ker(A)$ to itself. Notice $Ker(|A|) = Ker(A)$, as seen earlier, so $\overline{Ran(|A|)} = \overline{Ran(A)}$. Now if there exists W unitary with $A = W|A|$, it must be isometric on $Ker(A)^\perp = \overline{Ran(A)}$, so it must coincide with U there by the polar decomposition theorem. Therefore the restriction of W to $\overline{Ran(A)}$ gives a unitary operator from $\overline{Ran(A)}$ to $\overline{Ran(A)}$. The condition $W\restriction_{Ker(A)} = W_0$ fixes W completely on the whole Hilbert space as a unitary operator, ending the proof of uniqueness. As far as existence is concerned, it is enough to verify that $W := W_0 \oplus U$, corresponding to $\mathsf{H} = Ker(A) \oplus \overline{Ran(A)}$, is an operator that commutes with A, A^*, and that $|A|$ fulfils $A = W|A|$. The latter request is true by the polar decomposition theorem, for $Ker(|A|) = Ker(A)$. Since $A^*(Ker(A^*)) \subset Ker(A^*)$, we have $A(Ker(A^*)^\perp) \subset Ker(A^*)^\perp$, i.e. $A(\overline{Ran(A)}) \subset \overline{Ran(A)}$. With respect to the usual orthogonal splitting of the Hilbert space, $A = 0 \oplus A\restriction_{\overline{Ran(A)}}$. Since we have $W = W_0 \oplus U\restriction_{\overline{Ran(U)}}$ and $A = 0 \oplus A\restriction_{\overline{Ran(A)}}$, the condition $AW = WA$ holds if $AU = UA$. So let us prove the latter. By polar decomposition, $A = U|A|$. Normality ($AA^* = A^*A$) can be rephrased as $U|A|^2U^* = |A|U^*U|A| = |A|^2$, since U is isometric on $Ran(|A|)$. Therefore $U^*U|A|^2U^* = U^*|A|^2$ i.e. $|A|^2U^* = U^*|A|^2$. Taking adjoints gives $U|A|^2 = |A|^2U$. Theorem 3.77(a) says the root of an operator commutes with anything that commutes with the given operator, so $U|A| = |A|U$.

Still by polar decomposition we infer $UA = AU$, so $WA = AW$, as required. Eventually, taking adjoints: $A^*W^* = W^*A^*$, and using W on both sides produces $WA^* = A^*W$. Consequently W commutes with $A^*A = |A|^2$, and hence also with its square root $|A|$. □

3.6 Introduction to von Neumann Algebras

Prominent examples of C^*-algebras, which are absolutely fundamental for the applications in Quantum Field Theory (but not only), are *von Neumann algebras*. Before we introduce them, let us first define the *commutant* of an operator algebra, and state an important theorem.

3.6.1 The Notion of Commutant

If $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ is a subset of bounded operators on a complex Hilbert space, the **commutant** of $\mathfrak{M}$ is:

$$\mathfrak{M}' := \{T \in \mathfrak{B}(\mathsf{H}) \mid TA - AT = 0 \quad \text{for any } A \in \mathfrak{M}\} .$$

Remark 3.87 If $\mathfrak{M}$ is closed under the Hermitian conjugation (i.e. $A^* \in \mathfrak{M}$ if $A \in \mathfrak{M}$) the commutant $\mathfrak{M}'$ is surely a *-algebra with unit. In general, $\mathfrak{M}_2 \subset \mathfrak{M}_1 \Rightarrow \mathfrak{M}_1' \subset \mathfrak{M}_2'$, and $\mathfrak{M} \subset (\mathfrak{M}')'$, which imply $\mathfrak{M}' = ((\mathfrak{M}')')'$. Hence we cannot reach beyond the second commutant by iterating the construction. ■

The continuity of the product of operators says that the commutant $\mathfrak{M}'$ is closed in the uniform topology, so if $\mathfrak{M}$ is closed under Hermitian conjugation, its commutant $\mathfrak{M}'$ is a C^*-subalgebra in $\mathfrak{B}(\mathsf{H})$.

$\mathfrak{M}'$ has other pivotal topological properties in this general setup. It is easy to prove $\mathfrak{M}'$ is both strongly and weakly closed. This holds, despite the product of operators is not continuous, because separate continuity in each variable is sufficient.

In the sequel we shall adopt the standard convention used for von Neumann algebras and write $\mathfrak{M}''$ instead of $(\mathfrak{M}')'$ *etc.*. The next crucial result is due to von Neumann [BrRo02].

Theorem 3.88 (Double commutant theorem) *If* H *is a complex Hilbert space and* $\mathfrak{A}$ *a unital* **-subalgebra of* $\mathfrak{B}(\mathsf{H})$*, the following facts are equivalent.*
(a) $\mathfrak{A} = \mathfrak{A}''$.
(b) $\mathfrak{A}$ *is weakly closed.*
(c) $\mathfrak{A}$ *is strongly closed.*
Furthermore, if $\mathfrak{B}$ *is a unital* **-subalgebra of* $\mathfrak{B}(\mathsf{H})$*, then*

$$\mathfrak{B}'' = \overline{\mathfrak{B}}^w = \overline{\mathfrak{B}}^s ,$$

where the bars denote the closure in the weak ($\overline{\cdot}^w$) or strong ($\overline{\cdot}^s$) topology.

Proof (a) implies (b) because the commutant S' of a set $S \subset \mathfrak{B}(\mathsf{H})$ is closed in the weak topology, as we can also show directly: if $A_n \in S'$ for any $n \in \mathbb{N}$ and $A_n \to A \in \mathfrak{B}(\mathsf{H})$ weakly, then $A_n^* \to A^*$ weakly (this is immediate), so for $B \in S$:

$$0 = (\psi|(A_nB - BA_n)\phi) = \overline{(B\phi|A_n^*\psi)} - (B^*\psi|A_n\phi) \to (A\psi|B\phi) - (B^*\psi|A\phi)$$

$$= (\psi|(AB - BA)\phi) ,$$

hence $(\psi|(AB - BA)\phi) = 0$. Being ψ, ϕ arbitrary in H, A commutes with B, and $A \in S'$ because $B \in S$ is generic.
(b) implies (c) because strong convergence implies weak convergence, and therefore a limit point in the strong topology remains a limit point in the weak one.

To finish the proof of the first part we have to show that (c) implies (a). Since $S \subset S''$ for any set, so that $\mathfrak{A} \subset \mathfrak{A}''$, we only need to prove that $\mathfrak{A}'' \subset \overline{\mathfrak{A}}^s$ since $\overline{\mathfrak{A}}^s = \mathfrak{A}$ by (c).

Lemma 3.89 *If $\mathfrak{A}$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$, then $\mathfrak{A}'' \subset \overline{\mathfrak{A}}^s$.*

Proof of Lemma 3.89. For $\psi \in \mathsf{H}$ define the closed subspace: $\mathsf{H}_\psi := \overline{\{A\psi \mid A \in \mathfrak{A}\}}$, and call P the orthogonal projector onto H_ψ. By construction, if $\phi \in \mathsf{H}_\psi$ then $B\phi \in \mathsf{H}_\psi$ for any $B \in \mathfrak{A}$ and so $PB\phi = B\phi$. That is to say $PBP = BP$ for $B \in \mathfrak{A}$. Taking adjoints gives $PB^* = B^*P$, since $P = P^*$ by definition of orthogonal projector. Since $B^* = A$ for some $A \in \mathfrak{A}$, and since B varies in the whole $\mathfrak{A}$ as $A \in \mathfrak{A}$, we conclude $P \in \mathfrak{A}'$. Therefore for any $X \in \mathfrak{A}''$ we have $PX = XP$. But $I \in \mathfrak{A}$, so $\psi \in \mathsf{H}_\psi$ and $X\psi \in \mathsf{H}_\psi$ (since $PX\psi = XP\psi = X\psi$). By definition of H_ψ, $X\psi \in \mathsf{H}_\psi$ implies that for any $\varepsilon > 0$ there exists $A \in \mathfrak{A}$ with $||A\psi - X\psi|| < \varepsilon$. Consider then a finite collection of vectors $\psi_1, \psi_2, \dots, \psi_n$ and the direct sum $\mathsf{H}_n := \mathsf{H} \oplus \cdots \oplus \mathsf{H}$ (n copies) with inner product $((x_1, \dots, x_n)|(y_1, \dots, y_n)) := \sum_{k=1}^n (x_k|y_k)$ (see Sect. 3.4.2 for a precise description of this structure, called the *n-fold Hilbert sum* of H). On this Hilbert space, take the algebra $\mathfrak{A}_n \subset \mathfrak{B}(\mathsf{H}_n)$ of operators of the form $A \oplus \cdots \oplus A : (v_1, \dots, v_n) \mapsto (Av_1, \dots, Av_n)$, with $v_k \in \mathsf{H}$ for $k = 1, \dots, n$, $A \in \mathfrak{A}$. It is immediate that $\mathfrak{A}_n$ is a *-subalgebra (with unit) in $\mathfrak{B}(\mathsf{H}_n)$. If $X \in \mathfrak{A}''$, then $X \oplus \cdots \oplus X \in \mathfrak{A}_n''$. With the same argument as before, for any $\varepsilon > 0$ there exists $A \in \mathfrak{A}$ such that $||A\psi_k - X\psi_k|| < \varepsilon$, for $k = 1, \dots, n$. By definition of strong topology, this implies that if $X \in \mathfrak{A}''$ then $X \in \overline{\mathfrak{A}}^s$. In other words $\mathfrak{A}'' \subset \overline{\mathfrak{A}}^s$, as wanted. □

Let us come to the last statement of Theorem 3.88 and end its proof. First observe that $\overline{\mathfrak{B}}^w$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$ (this is direct). Then (1) $\overline{\mathfrak{B}}^w \subset \mathfrak{B}''$, since $\mathfrak{B}'' = (\mathfrak{B}'')''$ and so $\mathfrak{B}''$ is a weakly closed set containing $\mathfrak{B}$ (by (b) with $\mathfrak{A} = \mathfrak{B}''$). On the other hand $\mathfrak{B} \subset \overline{\mathfrak{B}}^w$ trivially, so that (2) $\mathfrak{B}'' \subset \overline{\mathfrak{B}}^{w\prime\prime} = \overline{\mathfrak{B}}^w$ where, in the last passage, we have used (a) because $\overline{\mathfrak{B}}^w$ is a weakly closed unital *-subalgebra. So now (1) and (2) imply $\overline{\mathfrak{B}}^w = \mathfrak{B}''$. To conclude observe that $\mathfrak{B} \subset \overline{\mathfrak{B}}^s \subset \overline{\mathfrak{B}}^w = \mathfrak{B}''$,

where the second inclusion is just due to the definitions of strong and weak operator topologies. The only thing left is showing $\mathfrak{B}'' \subset \overline{\mathfrak{B}}^s$. But this is true by Lemma 3.89.
□

3.6.2 Von Neumann Algebras, Also Known as W*-Algebras

At this juncture we are ready to define von Neumann algebras.

Definition 3.90 (*von Neumann algebra a.k.a. W^*-algebra*) If $\mathsf{H} \neq \{\mathbf{0}\}$ is a complex Hilbert space, a **von Neumann algebra** over H, also known as W^***-algebra** over H, is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$ that satisfies one of the three equivalent properties (a), (b), (c) in von Neumann's double commutant Theorem 3.88.

The **centre** of a von Neumann algebra $\mathfrak{R}$ is the subset $\mathfrak{R} \cap \mathfrak{R}'$.

A **factor** is a von Neumann algebra $\mathfrak{R}$ whose centre is trivial: $\mathfrak{R} \cap \mathfrak{R}' = \{cI\}_{c\in\mathbb{C}}$.

Remarks 3.91 **(1)** Von Neumann algebras are also known as W^*-algebras, even if the latter were originally introduced to describe a certain abstract category of C^*-algebras: a W^*-algebra, officially, is a C^*-algebra arising as topological dual of a Banach space. In 1970 Sakai proved [Tak00] that abstract W^*-algebras are in one-to-one correspondence (up to isomorphism) to concrete von Neumann algebras of operators on Hilbert spaces. For this reason in the rest of the book we shall use the two terms as synonyms.

(2) A von Neumann algebra $\mathfrak{A}$ in $\mathfrak{B}(\mathsf{H})$ is a C^*-algebra with unit, or better, a C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$ with unit. This is because if $\mathfrak{A}$ is closed in the strong operator topology then it is also closed in the uniform topology.

(3) The commutant $\mathfrak{M}'$ is a von Neumann algebra provided $\mathfrak{M}$ is a *-closed subset of $\mathfrak{B}(\mathsf{H})$, because $(\mathfrak{M}')'' = \mathfrak{M}'$ as we saw above.

(4) A von Neumann algebra $\mathfrak{R}$ on a Hilbert space H is not just *closed* in the weak and strong operator topologies: it is also *complete* with respect to both. In other words, if $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{R}$ is such that $\{A_n\psi\}_{n\in\mathbb{N}}$ is Cauchy in H for every $\psi \in \mathsf{H}$ (or $\{(\phi|A_n\psi)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ for every $\phi, \psi \in \mathsf{H}$), then $A_n \to A$ for some $A \in \mathfrak{R}$ in the strong (or weak, respectively) operator topology. See Exercises 3.34 and 3.35.

(5) If $\{\mathfrak{R}_\alpha\}_{\alpha\in A}$ is a family of von Neumann algebras on the Hilbert space H, it is easy to prove that

$$\bigwedge_{\alpha\in A} \mathfrak{R}_\alpha := \bigcap_{\alpha\in A} \mathfrak{R}_\alpha$$

is a von Neumann algebra on H. It is the largest von Neumann algebra contained in each $\mathfrak{R}_\alpha$.

(6) The classification of factors (see Sect. 7.6.2), initiated by von Neumann and Murray, is one of the key chapters in the theory of operator algebras, and has enormous consequences in the algebraic theory of quantum fields. ■

If $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ (with $\mathsf{H} \neq \{\mathbf{0}\}$) is closed under Hermitian conjugation, $\mathfrak{M}''$ turns out to be the smallest (set-theoretically) von Neumann algebra containing $\mathfrak{M}$ as a subset. Indeed, if $\mathfrak{A}$ is a von Neumann algebra and $\mathfrak{M} \subset \mathfrak{A}$, then $\mathfrak{M}' \supset \mathfrak{A}'$ and $\mathfrak{M}'' \subset \mathfrak{A}'' = \mathfrak{A}$. This fact leads to the following definition.

Definition 3.92 Let $\mathsf{H} \neq \{\mathbf{0}\}$ be a complex Hilbert space and $\mathfrak{M} \subset \mathfrak{B}(\mathsf{H})$ a closed set under Hermitian conjugation. The von Neumann algebra $\mathfrak{M}''$ is called the **von Neumann algebra generated** by $\mathfrak{M}$.

If $\mathfrak{M}$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$, the von Neumann algebra $\mathfrak{M}''$ is nothing but the weak, and strong, closure of $\mathfrak{M}$, due to the double commutant theorem.

If $\{\mathfrak{R}_\alpha\}_{\alpha \in A}$ is a family of von Neumann algebras on the Hilbert space H, the following notation is used for the smallest von Neumann algebra containing every $\mathfrak{R}_\alpha$:

$$\bigvee_{\alpha \in A} \mathfrak{R}_\alpha := \left(\bigcup_{\alpha \in A} \mathfrak{R}_\alpha \right)'' .$$

An elementary fact regarding the simplest case $\mathfrak{R} = \mathfrak{B}(\mathsf{H})$ is the following.

Proposition 3.93 *Let* $\mathsf{H} \neq \{\mathbf{0}\}$ *be a complex Hilbert space. Then*
(a) $\mathfrak{B}(\mathsf{H})' = \{cI\}_{c \in \mathsf{H}}$, *so that* $\mathfrak{B}(\mathsf{H})'' = \mathfrak{B}(\mathsf{H})$*;*
(b) *if* $P \in \mathfrak{B}(\mathsf{H})'$ *is a projector, then either* $P = I$ *or* $P = 0$*;*
(c) *there are no non-trivial closed subspaces that are invariant under the action of every element of* $\mathfrak{B}(\mathsf{H})$.

Proof If H has dimension 1 all statements are true, trivially, so let us consider higher dimensions. (a) Suppose that $A \in \mathfrak{B}(\mathsf{H})'$. If $x \in \mathsf{H}$, the orthogonal projector onto the linear space generated by x commutes with A and so $Ax = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$. If $y \neq x$, similarly $Ay = \lambda_y y$ and $A(x+y) = \lambda_{x+y}(x+y)$. Linearity yields

$$\lambda_x x + \lambda_y y = \lambda_{x+y}(x+y)$$

so that

$$(\lambda_x - \lambda_{x+y})x = -(\lambda_y - \lambda_{x+y})y .$$

This immediately implies that both $(\lambda_x - \lambda_{x+y}) = 0$ and $(\lambda_y - \lambda_{x+y}) = 0$ must be valid whenever x and y are linearly independent (such y exists if $\dim(\mathsf{H}) \geq 2$). In particular $\lambda_x = \lambda_y$. To conclude the proof, consider a Hilbert basis B for H, so that z and z' are linearly independent if $z, z' \in B$ and $z \neq z'$. What we established above immediately implies that $Az = cz$ for some fixed $c \in \mathbb{C}$ and every $z \in B$. In view of the continuity of A,

$$Ax = A \sum_{z \in B} (z|x)z = \sum_{z \in B} (z|x)Az = c \sum_{z \in B} (z|x)z = cx ,$$

for every $x \in \mathsf{H}$. We have found that any $A \in \mathfrak{B}(\mathsf{H})'$ has the form $A = cI$ for some $c \in \mathbb{C}$. And then obviously $\mathfrak{B}(\mathsf{H})'' = \{cI\}'_{c \in \mathsf{H}} = \mathfrak{B}(\mathsf{H})$. The first part of (b) immediately follows from $PP = P$ since $P = cI$ for some $c \in \mathbb{C}$. Let us finally prove (c). If $\mathsf{M} \subset \mathsf{H}$ is a closed subspace and $A \in \mathfrak{B}(\mathsf{H})$, the inclusion $A(\mathsf{M}) \subset \mathsf{M}$ easily implies $PAP = AP$, where P is the orthogonal projector onto M. By assuming $A = A^*$ and taking the adjoint of both sides we also get $PAP = PA$, so $PA = AP$ for every self-adjoint $A \in \mathfrak{B}(\mathsf{H})$. Decomposing $B \in \mathfrak{B}(\mathsf{H})$ as $B = \frac{1}{2}(B + B^*) + \frac{1}{2i}i(B - B^*)$, and observing that both $(B + B^*)$ and $i(B - B^*)$ are self-adjoint, then $PB = BP$ holds for every element $B \in \mathfrak{B}(\mathsf{H})$. Therefore either $P = 0$ or $P = I$ by (b) and, correspondingly, $\mathsf{M} = \{\mathbf{0}\}$ or $\mathsf{M} = \mathsf{H}$. $\square$

3.6.3 Further Relevant Operator Topologies

In addition to the uniform, strong and weak topologies, there are at least two further operator topologies that are relevant when dealing with von Neumann algebras. Given a Hilbert space H, consider a sequence $X := \{x_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ such that $\sum_{n\in\mathbb{N}} ||x_n||^2 < +\infty$. Define the seminorm on $\mathfrak{B}(\mathsf{H})$

$$\sigma_X(A) := \sqrt{\sum_{n\in\mathbb{N}} ||Ax_n||^2}\,, \quad \text{for every } A \in \mathfrak{B}(\mathsf{H}). \tag{3.72}$$

Similarly, if $Y := \{y_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ is another sequence such that $\sum_{n\in\mathbb{N}} ||y_n||^2 < +\infty$, define the seminorm on $\mathfrak{B}(\mathsf{H})$

$$\sigma_{XY}(A) := \left|\sum_{n\in\mathbb{N}} (x_n|Ay_n)\right|\,, \quad \text{for every } A \in \mathfrak{B}(\mathsf{H}). \tag{3.73}$$

It easy to prove that the family of seminorms σ_X separates points, i.e., $\sigma_X(A-B) = 0$ for all X as above $\Rightarrow A = B$. The same property is true for the family of seminorms σ_{XY} with X, Y as above.

Definition 3.94 Given a Hilbert space H consider the families of seminorms defined by (3.72) and (3.73) for all sequences $X := \{x_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ and $Y := \{y_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ such that $\sum_{n\in\mathbb{N}} ||x_n||^2 < +\infty$ and $\sum_{n\in\mathbb{N}} ||y_n||^2 < +\infty$.

The topologies on $\mathfrak{B}(\mathsf{H})$ induced by these two families of seminorms, in accordance with Definition 2.68, are respectively called **σ-strong topology** (or **ultrastrong topology**) and **σ-weak topology** (or **ultraweak topology**).

It is easy to prove that, on $\mathfrak{B}(\mathsf{H})$,

(a) the uniform topology is finer than the σ-strong topology;
(b) the σ-strong topology is finer than the σ-weak topology;
(c) the σ-strong topology is finer than the strong topology;
(d) the σ-weak topology is finer than the weak topology.

Remark 3.95 **(1)** A von Neumann algebra in $\mathfrak{B}(\mathsf{H})$ is closed in the uniform topology (being a C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$), in the weak topology and in the strong topology (Theorem 3.88). Hence it is also closed for both the σ-weak and the σ-strong topologies, as a consequence of (c) and (d) above.
(2) *A surjective *-homomorphism α between two von Neumann algebras is necessarily continuous with respect to the σ-strong and σ-weak topologies* [BrRo02, Vol.1, Theorem 2.4.23].
(3) A special class of *-isomorphisms between von Neumann algebras $\mathfrak{R}_1 \subset \mathfrak{B}(\mathsf{H}_1)$ and $\mathfrak{R}_2 \subset \mathfrak{B}(\mathsf{H}_2)$ are the so-called **spatial isomorphisms**. These are maps of the form

$$\alpha : \mathfrak{R}_1 \ni A \mapsto V_\alpha A V_\alpha^* \in \mathfrak{R}_2 ,$$

where $V_\alpha : \mathsf{H}_1 \to \mathsf{H}_2$ is some fixed unitary operator (Definition 3.56(d)). Spatial isomorphisms always extend to the whole $\mathfrak{B}(\mathsf{H}_i)$, and to the commutants in particular:

$$\alpha(\mathfrak{R}_1') = \alpha(\mathfrak{R}_2') .$$

■

In the simplest case where $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{B}(\mathsf{H})$, it is possible to prove that a *-isomorphism (to be precise, a *-automorphism) $\alpha : \mathfrak{B}(\mathsf{H}) \to \mathfrak{B}(\mathsf{H})$ is always spatial.

Theorem 3.96 *If $\mathsf{H} \neq \{\mathbf{0}\}$ is a complex Hilbert space and $\alpha : \mathfrak{B}(\mathsf{H}) \to \mathfrak{B}(\mathsf{H})$ a *-automorphism, then α is* spatial*: there exists a unitary operator $U_\alpha : \mathsf{H} \to \mathsf{H}$ such that*

$$\alpha(A) = U_\alpha A U_\alpha^* \quad \textit{for every } A \in \mathfrak{B}(\mathsf{H}).$$

Every unitary operator defines a *-automorphism by the formula above, and a unitary operator U_1 satisfies the formula (in place of U) for the given α if and only if $U_1 = \chi U$ for some $\chi \in U(1)$.

Proof If $\dim(\mathsf{H}) = 1$, the only *-automorphism is the identity and so the thesis is obvious. Let us pass to $\dim(\mathsf{H}) > 1$. The last statement is trivial, in particular $U_1 = \chi U$ follows from Proposition 3.93 because $U_1 U^* \in \mathfrak{B}(\mathsf{H})'$, so we shall prove the first claim only. Let N be a Hilbert basis of H and define the orthogonal projector $P_x := (x|\cdot)x$ for every $x \in N$. Every operator $P_x' := \alpha(P_x)$ is an orthogonal projector as well, because $\alpha(P_x)\alpha(P_x) = \alpha(P_x P_x) = \alpha(P_x)$ and $\alpha(P_x)^* = \alpha(P_x^*) = \alpha(P_x)$. We intend to construct a second Hilbert basis N' using α. The projection space of P_x' is one-dimensional as is the projection space of P_x. Indeed, if there existed a pair of orthonormal vectors z_1, z_2 in $P_x'(\mathsf{H})$ and we set $Q_{z_i} := (z_i|\cdot)z_i$ for $i = 1, 2$, we would have $P_x' \neq Q_{z_i} P_x' = Q_{z_i} \neq 0$. Then applying α^{-1} would give $P_x \neq \alpha^{-1}(Q_{z_i})P_x = \alpha^{-1}(Q_{z_i}) \neq 0$. However, since P_x projects onto a one-dimensional subspace, no such orthogonal projector $\alpha^{-1}(Q_{z_i})$ can exist. Next, observe that $\oplus_{x\in N} P_x(\mathsf{H}) = \mathsf{H}$ because N is a Hilbert basis, but also $\oplus_{x\in N} P_x'(\mathsf{H}) = \mathsf{H}$. In fact, first of all $P_x' P_y' = \alpha^{-1}(P_x P_y) = 0$ if $x \neq y$, so that $P_x'(\mathsf{H}) \perp P_y'(\mathsf{H})$ when $x \neq y$. Furthermore, if there were $z \perp \oplus_{x\in N} P_x'(\mathsf{H})$ with unit norm, by defining $Q_z := (z|\cdot)z$ we would have $Q_z P'_x = 0$ for every $x \in N$ and hence $\alpha^{-1}(Q_z)\alpha^{-1}(P'_x) = 0$. That would mean $\alpha^{-1}(Q_z)P_x = 0$ for every $x \in N$ and, in turn, $\alpha^{-1}(Q_z) = 0$ so that $Q_z = 0$, against the hypothesis. If $x, y \in N$, the maps $T_{xy} := (y|\cdot)x$ restrict to Hilbert space isomorphisms between the one-dimensional, mutually orthogonal subspaces $P_y(\mathsf{H})$ and $P_x(\mathsf{H})$. If we define $T'_{xy} := \alpha\,((y|\cdot)x)$, we have $P'_z T'_{xy} = \alpha(P_z T_{xy}) = 0$ for $z \neq x$ or $\alpha(P_z T_{xy}) = T'_{xy}$ if $z = x$. Since $\oplus_{x\in N} P'_x(\mathsf{H}) = \mathsf{H}$ and $T'_{xy} \neq 0$, we conclude that the range of T'_{xy} is one-dimensional and coincides with $P'_x(\mathsf{H})$ itself. Using a similar argument and $T'^{*}_{xy}T'_{xy} = \alpha(T^*_{xy}T_{xy}) = \alpha(P_x) = P'_x$, we finally conclude that the maps T'_{xy} restrict to Hilbert space isomorphisms between the one-dimensional, mutually orthogonal subspaces $P'_y(\mathsf{H})$ and $P'_x(\mathsf{H})$. To construct the new Hilbert basis N', fix a unit vector $y' \in P'_y(\mathsf{H})$ and define the other unit vectors $x' := T'_{xy}y'$ for every $x \in N$. Evidently $N' := \{x'\}_{x\in N}$ is a Hilbert basis since each $P'_x(\mathsf{H})$ is one-dimensional and $\oplus_{x\in N} P'_x(\mathsf{H}) = \mathsf{H}$, as previously established. One can check the fundamental identity

$$\alpha(T_{xy}) = U_\alpha T_{xy} U^*_\alpha \quad \text{for every } x, y \in N, \tag{3.74}$$

directly, where $U_\alpha : \mathsf{H} \to \mathsf{H}$ is the unique linear and bounded operator such that $U_\alpha x := x'$ for every $x \in N$. This operator is unitary because the vectors x form a Hilbert basis, as do the x'. To conclude the proof, consider $A \in \mathfrak{B}(\mathsf{H})$ and the operators $T_{xx}AT_{yy}$ for $x, y \in N$. As $T_{xy} := (y|\cdot)x$, it is clear that $T_{yy}AT_{xx} = aT_{xy}$ where $a = (x|Ay)$, so that (3.74) produces

$$\alpha(T_{yy}AT_{xx}) = (x|Ay)U_\alpha T_{yx}U^*_\alpha$$

because α is linear. Since α preserves the products, the above reads

$$\alpha(T_{yy})\alpha(A)\alpha(T_{xx}) = (x|Ay)U_\alpha T_{yx}U^*_\alpha\ .$$

Exploiting (3.74) again,

$$U_\alpha T_{yy}U^*_\alpha \alpha(A)U_\alpha T_{xx}U^*_\alpha = (x|Ay)U_\alpha T_{yx}U^*_\alpha\ ,$$

that is

$$T_{yy}U^*_\alpha \alpha(A)U_\alpha T_{xx} = (x|Ay)T_{yx}\ ,$$

so that, applying both sides to x and taking the inner product with y,

$$(y|U^*_\alpha \alpha(A)U_\alpha x) = (x|Ay)\ .$$

Since x, y are generic elements of a Hilbert basis and A, $U^*_\alpha\alpha(A)U_\alpha$ are bounded, we have found that

$$U^*_\alpha \alpha(A)U_\alpha = A\ ,$$

which is equivalent to $\alpha(A) = U_\alpha A U^*_\alpha$, concluding the proof. □

3.6.4 Hilbert Sum of von Neumann Algebras

There exists a nice interplay between Hilbert sums of Hilbert spaces and direct sums of C^*-algebras, which specialises to the case of von Neumann algebras.

Proposition 3.97 *Consider a family of non-trivial Hilbert spaces* $\{\mathsf{H}_j\}_{j\in J}$ *and a family of* C^**-algebras of operators* $\{\mathfrak{A}_j\}_{j\in J}$*, with* $\mathfrak{A}_j \subset \mathfrak{B}(\mathsf{H}_j)$ *for every* $j \in J$.
(a) *If* $\oplus_{j\in J} A_j \in \bigoplus_{j\in J} \mathfrak{A}_j$ *(Proposition 3.45), the operator*

$$\hat{\oplus}_{j\in J} A_j : \bigoplus_{j\in J} \mathsf{H}_j \to \bigoplus_{j\in J} \mathsf{H}_j$$

given by

$$\hat{\oplus}_{j\in J} A_j : \oplus_j x_j \mapsto \oplus_j A_j x_j \tag{3.75}$$

is a well-defined element of $\mathfrak{B}(\bigoplus_{j\in J} \mathsf{H}_j)$*. Moreover*

$$||\hat{\oplus}_{j\in J} A_j|| = || \oplus_{j\in J} A_j|| ,$$

where the norm on the left is the uniform norm of $\mathfrak{B}(\bigoplus_{j\in J} \mathsf{H}_j)$*, whereas the one on the right is the* C^**-norm of* $\bigoplus_{j\in J} \mathfrak{A}_j$*. Therefore the map*

$$\bigoplus_{j\in J} \mathfrak{A}_j \ni \oplus_{j\in J} A_j \mapsto \hat{\oplus}_{j\in J} A_j \in \mathfrak{B}\left(\bigoplus_{j\in J} \mathsf{H}_j\right)$$

is an isometric **-homomorphism, making the set* $\hat{\bigoplus}_{j\in J} \mathfrak{A}_j$ *of operators* $\hat{\oplus}_{j\in J} A_j$ *a* C^**-subalgebra (not necessarily unital) of* $\mathfrak{B}(\bigoplus_{j\in J} \mathsf{H}_j)$*.*
(b) *We have*

$$\left(\hat{\bigoplus}_{j\in J} \mathfrak{A}_j\right)' = \hat{\bigoplus}_{j\in J} \mathfrak{A}'_j \, .$$

(c) $\hat{\bigoplus}_{j\in J} \mathfrak{A}_j$ *is a von Neumann algebra if each summand* $\mathfrak{A}_j$ *is a von Neumann algebra.*

Proof (a) First of all we must prove that $\hat{\oplus}_j A_j \in \mathfrak{B}(\bigoplus_j \mathfrak{A}_j)$. For every $\oplus_j x_j \in \bigoplus_j \mathsf{H}_j$ we have:

$$||\hat{\oplus}_j A_j(\oplus_j x_j)||^2 \leq \sup_j ||A_j||^2 \sum_j ||x_j||^2 = || \oplus_j A_j||^2 \, || \oplus_j x_j||^2 \, .$$

The result implies at once that $\hat{\oplus}_j A_j \in \mathfrak{B}(\bigoplus_j \mathfrak{A}_j)$ and $||\hat{\oplus}_j A_j|| \leq || \oplus_j A_j||$. To prove that $||\hat{\oplus}_j A_j|| = || \oplus_j A_j||$ it suffices to find a sequence $\{\oplus_j y_{j,n}\}_{n\in\mathbb{N}} \subset \bigoplus_j \mathsf{H}_j$ such that $|| \oplus_j y_{j,n}|| = 1$ and $||\hat{\oplus}_j A_j(\oplus_j y_{j,n})|| \to || \oplus_j A_j||$ as $n \to +\infty$.

If we look at $|| \oplus_j A_j|| := \sup_j ||A_j||$ there are two possibilities. Either

(1) $|| \oplus_j A_j|| = ||A_{j_0}||$ for some j_0

or

(2) there is a sequence $j_1, j_2, \ldots$ with $||A_{j_n}|| \to || \oplus_j A_j||$ as $n \to +\infty$.

In case (1), as we know that $||B|| = \sup_{||x||=1} ||Bx||$, there must be a sequence $x_n \in \mathsf{H}_{j_0}$ with $||x_n|| = 1$ and $||A_{j_0}x_n|| \to ||A_{j_0}||$ as $n \to +\infty$. In this situation, the required sequence $\{\oplus_j y_{j,n}\}_{n\in\mathbb{N}}$ is defined by $y_{j,n} = \mathbf{0}$ for $j \neq j_0$ and $y_{j_0,n} := x_n$.

In case (2), by exploiting again the general identity $||B|| = \sup_{||x||=1} ||Bx||$ on each operator A_{j_n} we conclude that there must exist a sequence of vectors $x_{j_n} \in \mathsf{H}_{j_n}$ with $||x_{j_n}|| = 1$ and $|\,||A_{j_n}x_{j_n}|| - ||A_{j_n}||\,| < 1/n$. The sequence $\{\oplus_j y_{j,n}\}_{n\in\mathbb{N}}$ we need is therefore defined by $y_{j,n} = \mathbf{0}$ for $j \neq j_n$ and $y_{j_n,n} := x_{j_n}$.

The fact that the map $\oplus_j A_j \mapsto \hat{\oplus}_j A_j$ is linear, preserves the product and unit elements is straightforward. Regarding Hermitian conjugation, observe that

$$\left(\oplus_j x_j|\hat{\oplus}_j A_j(\oplus_j y_j)\right) = \sum_j (x_j|A_j y_j)_j = \sum_j (A_j^* x_j|y_j)_j = \left(\hat{\oplus}_j A_j^*(\oplus_j x_j)| \oplus_j y_j\right)$$

$$= \left(\oplus_j x_j|(\hat{\oplus}_j A_j^*)^*(\oplus_j y_j)\right) .$$

Since $\oplus_j x_j, \oplus_j y_j \in \bigoplus_j \mathsf{H}_j$ are arbitrary, we have $\hat{\oplus}_j A_j = (\hat{\oplus}_j A_j^*)^*$ and hence $(\hat{\oplus}_j A_j)^* = \hat{\oplus}_j A_j^*$. We have obtained that $\oplus_{j\in J} A_j \mapsto \hat{\oplus}_{j\in J} A_j$ is an isometric *-homomorphism from the C^*-algebra $\bigoplus_j \mathfrak{A}_j$ into the C^*-algebra $\mathfrak{B}(\bigoplus_j \mathsf{H}_j)$, so the image $\hat{\bigoplus}_{j\in J}\mathfrak{A}_j$ is a C^*-subalgebra of $\mathfrak{B}(\bigoplus_j \mathsf{H}_j)$.

(b) Establishing that

$$\left(\hat{\bigoplus}_{j\in J}\mathfrak{A}_j\right)' \supset \hat{\bigoplus}_{j\in J}\mathfrak{A}_j'$$

is rather elementary, so let us prove the opposite inclusion. First of all notice that $\bigoplus_j \mathfrak{A}_j$ contains elements $\oplus_j \delta_{ij} I_j$, where $i \in J$ and $I_j : \mathsf{H}_j \to \mathsf{H}_j$ is the identity operator. It is easy to prove that $P_i := \hat{\oplus}_j \delta_{ij} I_j$ is the orthogonal projector onto H_i, viewed as a closed subspace of $\bigoplus_j \mathsf{H}_j$. If $B \in \left(\hat{\bigoplus}_{j\in J}\mathfrak{A}_j\right)'$ then $P_i B = B P_i$ for all $i \in J$ in particular. Obviously, if $\Psi \in \bigoplus_{i\in J} \psi$, we have $\Psi = \oplus_i \psi_i$ with $\psi_i = P_i \Psi$. Therefore $\Psi = \sum_{i\in I} P_i \Psi$ is a finite or countable sum, for at most countably many vectors ψ_{i_k}, $k \in \mathbb{N}$, do not vanish. Consequently, $P_i B = B P_i$ and $P_i P_i = P_i$ imply

$$B(\oplus_i \psi_i) = B \sum_{k\in\mathbb{N}} P_{i_k} \Psi = \sum_{k\in\mathbb{N}} P_{i_k} B P_{i_k} \Psi = \oplus_i (P_i B P_i)\psi_i .$$

Set $B_i := P_i B P_i \in \mathfrak{B}(\mathsf{H}_i)$. Then $\oplus_i B_i \in \bigoplus_i \mathfrak{B}(\mathsf{H}_i)$ is well defined because $\sup_i ||B_i|| = \sup_i ||P_i B|| \leq \sup_i ||P_i||\,||B|| \leq ||B|| < +\infty$. By construction

$$B(\oplus_i \psi_i) = \oplus_i B_i \psi_i = \hat{\oplus}_i B_i (\oplus_i \psi_i) .$$

Therefore imposing $(BA - AB)\Psi = \mathbf{0}$ for every $A = \hat{\oplus}_i A_i \in \hat{\bigoplus}_i \mathfrak{A}_i$ and every $\Psi \in \bigoplus_i \psi_i$ implies $(B_i A_i - A_i B_i)\psi_i = \mathbf{0}$ for every $i \in J$ and $\psi_i \in \mathsf{H}_i$. In other words $B_i \in \mathfrak{A}_i'$. In summary, $B \in [\bigoplus_j \mathfrak{A}_j]'$ implies $B = \hat{\oplus}_j B_j$ where $B_j \in \mathfrak{A}_j'$, so

$$\left(\hat{\bigoplus}_{j\in J}\mathfrak{A}_j\right)' \subset \hat{\bigoplus}_{j\in J}\mathfrak{A}_j' \;,$$

as we wanted.
(c) Suppose each $\mathfrak{A}_i$ is a von Neumann algebra. Then $\mathfrak{A}_i'' = \mathfrak{A}_i$ and

$$\left(\hat{\bigoplus}_{j\in J}\mathfrak{A}_j\right)'' = \left(\hat{\bigoplus}_{j\in J}\mathfrak{A}_j'\right)' = \hat{\bigoplus}_{j\in J}\mathfrak{A}_j'' = \hat{\bigoplus}_{j\in J}\mathfrak{A}_j \;,$$

and $\hat{\bigoplus}_{j\in J}\mathfrak{A}_j$ contains the identity operator (the sum of the identity operators of each H_j). We conclude that $\hat{\bigoplus}_{j\in J}\mathfrak{A}_j$ is a von Neumann algebra. $\square$

Definition 3.98 Consider a family of non-trivial Hilbert spaces $\{\mathsf{H}_j\}_{j\in J}$ and a family of von Neumann algebras $\{\mathfrak{R}_j\}_{j\in J}$ with $\mathfrak{R}_j \subset \mathfrak{B}(\mathsf{H}_j)$ for every $j \in J$.

The von Neumann algebra $\hat{\bigoplus}_{j\in J}\mathfrak{R}_j$ on $\bigoplus_{j\in J}\mathsf{H}_j$ defined in Proposition 3.97 is called the **direct sum of the family of von Neumann algebras** $\{\mathfrak{R}_j\}_{j\in J}$.

Notation 3.99 In the rest of the book, a direct sum of von Neumann algebras $\hat{\bigoplus}_{j\in J}\mathfrak{R}_j$ will be denoted by $\bigoplus_{j\in J}\mathfrak{R}_j$, despite the latter should – to be precise – indicate the isometric C^*-algebra associated to it.

Similarly, the operator $\hat{\oplus}_{j\in J}A_j$ of (3.75) will be indicated by the corresponding abstract element $\oplus_{j\in J}A_j$. ■

3.7 The Fourier–Plancherel Transform

In the last section of the chapter we introduce, rather concisely, the basics on Fourier and Fourier–Plancherel transforms, without any mention to Schwartz distributions [Rud91, ReSi80, Vla02].

Notation 3.100 From now on we will use the notations of Example 2.91, originally introduced for differential operators: in particular $x_k \in \mathbb{R}$ will denote the kth component of $x \in \mathbb{R}^n$, dx the ordinary Lebesgue measure on $\mathbb{R}^n$, and

$$M^\alpha(x) := x_1^{\alpha_1}\cdots x_n^{\alpha_n} \quad \text{for any multi-index}\, \alpha = (\alpha_1,\ldots,\alpha_n)\,.$$

By $\mathscr{D}(\mathbb{R}^n)$ we shall denote the space of smooth, complex-valued functions with compact support (in the literature this is also called $C_c^\infty(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R}^n)$), while $\mathscr{S}(\mathbb{R}^n)$ will indicate the **Schwartz space** on $\mathbb{R}^n$ (cf. Example 2.91). In these notations $\mathscr{S}(\mathbb{R}^n)$ is the $\mathbb{C}$-vector space of complex-valued maps in $C^\infty(\mathbb{R}^n)$ with this property:

for any $f \in \mathscr{S}(\mathbb{R}^n)$ and any multi-indices α, β, there exists $K < +\infty$ (depending on f, α, β) such that

$$|M^\alpha(x)\partial_x^\beta f(x)| \leq K, \quad \text{for any } x \in \mathbb{R}^n . \tag{3.76}$$

The norms $|| \ ||_1$, $|| \ ||_2$ and $|| \ ||_\infty$ will denote, throughout the section, the norms of $L^1(\mathbb{R}^n, dx)$, $L^2(\mathbb{R}^n, dx)$, $L^\infty(\mathbb{R}^n, dx)$ and the corresponding seminorms of $\mathscr{L}^1(\mathbb{R}^n, dx)$, $\mathscr{L}^2(\mathbb{R}^n, dx)$, $\mathscr{L}^\infty(\mathbb{R}^n, dx)$ (see Examples 2.29(6) and (8)). ■

Below we recall a number of known properties.

(1) The spaces $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ are invariant under $M^\alpha(x)$ (seen as multiplicative operator) and ∂_x^α. Put otherwise, functions stay in their respective spaces when acted upon by $M^\alpha(x)$ and ∂_x^α.

(2) Clearly $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{L}^p(\mathbb{R}, dx)$ as a subspace, for any $1 \leq p \leq \infty$, since compact sets in $\mathbb{R}^n$ have finite Lebesgue measure and any $f \in \mathscr{D}(\mathbb{R}^n)$ is continuous, hence bounded on compact sets.

(3) For any $1 \leq p \leq \infty$ we have $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{L}^p(\mathbb{R}, dx)$ as a subspace. In fact, if $C \subset \mathbb{R}^n$ is a compact set containing the origin, $f \in \mathscr{S}(\mathbb{R}^n)$ is bounded on C because continuous, while outside C we have $|f(x)| \leq C_m|x|^{-m}$ for any $n = 0, 1, 2, 3, \ldots$ as long as we choose $C_n \geq 0$ big enough. In summary, $|f|$ is bounded on $\mathbb{R}^n$, so it belongs to $\mathscr{L}^\infty$. But it is also bounded by some map in $\mathscr{L}^p$, for any $p \in [1, +\infty)$: the bounding function is constant on C, and equals $C_m/|x|^m$, $m > n/p$, outside C.

(4) Beside the obvious inclusion $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, recall a notorious fact (independent of this section) that we will use shortly [KiGv82]:

Proposition 3.101 *The spaces $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ are dense in $\mathscr{L}^p(\mathbb{R}, dx)$, for any $1 \leq p < \infty$.*

(5) The next important lemma, whose proof can be found in [Bre10, Corollary IV.24], is independent of this section's results.

Lemma 3.102 *Suppose $f \in \mathscr{L}^1(\mathbb{R}^n, dx)$ satisfies*

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = 0 \ \ \textit{for any } g \in \mathscr{D}(\mathbb{R}^n) .$$

Then $f(x) = 0$ almost everywhere for the Lebesgue measure dx on $\mathbb{R}^n$.

Let us introduce the first elementary definitions concerning the Fourier transform.

Definition 3.103 The linear maps $\mathscr{L}^1(\mathbb{R}^n, dx) \to \mathscr{L}^\infty(\mathbb{R}^n, dx)$ given by

$$(\mathscr{F} f)(k) := \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\,dx\,, \quad f \in \mathscr{L}^1(\mathbb{R}^n, dx), k \in \mathbb{R}^n\,, \tag{3.77}$$

$$(\mathscr{F}_- g)(x) := \int_{\mathbb{R}^n} \frac{e^{ik\cdot x}}{(2\pi)^{n/2}} g(k)\,dk\,, \quad g \in \mathscr{L}^1(\mathbb{R}^n, dk), x \in \mathbb{R}^n \tag{3.78}$$

are respectively called **Fourier transform** and **inverse Fourier transform**.

Remarks 3.104 (**1**) Above, dk always denotes the Lebesgue measure on $\mathbb{R}^n$. We have used a different name for the variable on $\mathbb{R}^n$ (k, not x) in the inverse Fourier formula, only to respect the traditional notation, and to simplify subsequent calculations.
(2) By the integral's properties it is obvious that

$$|(\mathscr{F} f)(k)| \leq \left|\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x) \frac{dx}{(2\pi)^{n/2}}\right| \leq \int_{\mathbb{R}^n} |e^{-ik\cdot x}|\,|f(x)| \frac{dx^n}{(2\pi)^{n/2}} = \int_{\mathbb{R}^n} |f(x)| \frac{dx}{(2\pi)^{n/2}}$$

$$= \frac{||f||_1}{(2\pi)^{n/2}},$$

and similarly $|(\mathscr{F}_- g)(x)| \leq ||g||_1/(2\pi^{n/2})$ for any $x, k \in \mathbb{R}^n$. Therefore it makes sense to define the Fourier and inverse Fourier transforms as operators with values in $\mathscr{L}^\infty(\mathbb{R}^n, dx)$. ■

In the sequel we will discuss features of the Fourier transform that most immediately relate to the Fourier–Plancherel transform. We shall inevitably overlook many results, like the continuity in the seminorm topology in the Schwartz space, for which we refer to any text on functional analysis or distributions [Rud91, ReSi80, Vla02] (see also Sect. 2.3.4).

Proposition 3.105 *The Fourier and inverse Fourier transforms enjoy the following properties.*
(a) *They are continuous in the natural norms of domain and codomain:*

$$||\mathscr{F} f||_\infty \leq \frac{||f||_1}{(2\pi)^{n/2}} \quad \textit{and} \quad ||\mathscr{F}_- g||_\infty \leq \frac{||g||_1}{(2\pi)^{n/2}}.$$

(b) *The Schwartz space is invariant under $\mathscr{F}$ and $\mathscr{F}_-$: $\mathscr{F}(\mathscr{S}(\mathbb{R}^n)) \subset \mathscr{S}(\mathbb{R}^n)$ and $\mathscr{F}_-(\mathscr{S}(\mathbb{R}^n)) \subset \mathscr{S}(\mathbb{R}^n)$.*
(c) *When restricted to the invariant space $\mathscr{S}(\mathbb{R}^n)$ they are one the inverse of the other: if $f \in \mathscr{S}(\mathbb{R}^n)$, then*

$$g(k) = \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\, dx$$

if and only if

$$f(x) = \int_{\mathbb{R}^n} \frac{e^{ik\cdot x}}{(2\pi)^{n/2}} g(k)\, dk.$$

(d) *When restricted to the invariant space $\mathscr{S}(\mathbb{R}^n)$ they are isometric for the semi-inner product of $\mathscr{L}^2(\mathbb{R}^n, dx^n)$: if $f_1, f_2, g_1, g_2 \in \mathscr{S}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \overline{(\mathscr{F} f_1)(k)}(\mathscr{F} f_2)(k)dk = \int_{\mathbb{R}^n} \overline{f_1(x)}\, f_2(x)dx$$

and

$$\int_{\mathbb{R}^n} \overline{(\mathscr{F}_- g_1)(x)}(\mathscr{F}_- g_2)(x)dx = \int_{\mathbb{R}^n} \overline{g_1(k)} g_2(k)dk \,.$$

(e) *They determine bounded maps from $L^1(\mathbb{R}^n, dx)$ to $C_0(\mathbb{R}^n)$ (continuous maps that vanish at infinity, cf. Example 2.29(4)), and so the* **Riemann–Lebesgue lemma** *holds: for any $f \in L^1(\mathbb{R}^n, dx)$*

$$(\mathscr{F} f)(k) \to 0 \quad as\ |k| \to +\infty$$

and analogously for $\mathscr{F}_-$.
(f) *They are injective if defined on $L^1(\mathbb{R}^n, dx)$.*

Remark 3.106 Concerning statement (f), more can be proved [Rud91], namely: if $f \in L^1(\mathbb{R}^n, dx)$ is such that $\mathscr{F} f \in L^1(\mathbb{R}^n, dk)$, then $\mathscr{F}_-(\mathscr{F} f) = f$. The same holds if we swap $\mathscr{F}$ and $\mathscr{F}_-$. ■

Proof of Proposition 3.105. Part (a) was proved in Remark 3.104(2). As for (b), let us prove the claim about $\mathscr{F}$, the one about $\mathscr{F}_-$ being similar. Set

$$g(k) := \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\, dx.$$

The right-hand side can be differentiated in k by passing the operator ∂_k^α inside the integral. In fact,

$$\left|\partial_k^\alpha e^{-ik\cdot x} f(x)\right| = \left|i^{|\alpha|} M^\alpha(x) e^{-ik\cdot x} f(x)\right| \leq |M^\alpha(x) f(x)| \,.$$

The function $x \mapsto |M^\alpha(x) f(x)|$ is in $\mathscr{L}^1$ because $f \in \mathscr{S}(\mathbb{R}^n)$. Since the absolute value of the derivative of the integrand is uniformly bounded by an integrable, positive map, known theorems on exchanging derivatives and integrals allow to say:

$$\partial_k^\alpha g(k) = (-i)^{|\alpha|} \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} M^\alpha(x) f(x)\, dx \,. \tag{3.79}$$

But f vanishes faster than any negative power of $|x|$, as $|x| \to +\infty$, so:

$$M^\beta(k) g(k) = \int_{\mathbb{R}^n} i^{|\beta|} \partial_x^\beta \left(\frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} \right) f(x)\, dx$$

and, integrating by parts,

$$M^\beta(k) g(k) = (-i)^{|\beta|} \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} \partial_x^\beta f(x)\, dx \,. \tag{3.80}$$

Writing $\partial_k^\alpha g$ instead of g in (3.80), and by (a), we have:

$$|M^\beta(k)\partial_k^\alpha g(k)| \leq \left|\left|\partial^\beta\left(M^\alpha f\right)\right|\right|_1 \ ,$$

for any $k \in \mathbb{R}^n$. The right-hand-side term is finite, since $f \in \mathscr{S}(\mathbb{R}^n)$; and because α and β are arbitrary, we conclude $g \in \mathscr{S}(\mathbb{R}^n)$.
(c) Identities (3.79) and (3.80) read:

$$\partial^\alpha \ \mathscr{F} = (-i)^{|\alpha|} \ \mathscr{F} M^\alpha \ , \tag{3.81}$$
$$M^\beta \ \mathscr{F} = (-i)^{|\beta|} \ \mathscr{F} \partial^\beta \ , \tag{3.82}$$

where $\mathscr{F}$ is the restriction of the Fourier transform to $\mathscr{S}(\mathbb{R}^n)$. Observing that

$$\overline{\mathscr{F} h} = \mathscr{F}_- \overline{h}$$

for any $h \in \mathscr{S}(\mathbb{R}^n)$, it is easy to obtain

$$\partial^\alpha \ \mathscr{F}_- = i^{|\alpha|} \ \mathscr{F}_- M^\alpha \ , \tag{3.83}$$
$$M^\beta \ \mathscr{F}_- = i^{|\beta|} \ \mathscr{F}_- \partial^\beta \ . \tag{3.84}$$

Then (3.81), (3.82), (3.83) and (3.84) imply in particular that:

$$\mathscr{F}\mathscr{F}_- \ M^\alpha = M^\alpha \mathscr{F}\mathscr{F}_- \ , \tag{3.85}$$
$$\mathscr{F}_-\mathscr{F} \ M^\alpha = M^\alpha \mathscr{F}_-\mathscr{F} \ , \tag{3.86}$$

where M^α is thought of as *multiplicative operator* $(M^\alpha f)(x) := M^\alpha(x) f(x)$, and

$$\mathscr{F}\mathscr{F}_- \ \partial^\alpha = \partial^\alpha \mathscr{F}\mathscr{F}_- \ , \tag{3.87}$$
$$\mathscr{F}_-\mathscr{F} \ \partial^\alpha = \partial^\alpha \mathscr{F}_-\mathscr{F} \ . \tag{3.88}$$

By virtue of those commuting relations, we claim $J := \mathscr{F}\mathscr{F}_-$ and $J_- := \mathscr{F}_-\mathscr{F}$ are the identity of $\mathscr{S}(\mathbb{R}^n)$. To begin with, we show, given $x_0 \in \mathbb{R}^n$ and $f \in \mathscr{S}(\mathbb{R}^n)$, that the value $(Jf)(x_0)$ depends only on $f(x_0)$. If $f \in \mathscr{S}(\mathbb{R}^n)$ we can write:

$$f(x) = f(x_0) + \int_0^1 \frac{df(x_0 + t(x - x_0))}{dt} dt = f(x_0) + \sum_{i=1}^n (x_i - x_{0i}) g_i(x) \ ,$$

where the g_i (in $C^\infty(\mathbb{R}^n)$, as is easy to see) are:

$$g_i(x) := \frac{\partial}{\partial x_i} \int_0^1 f(x_0 + t(x - x_0)) dt.$$

Hence if $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$ and $f_1(x_0) = f_2(x_0)$:

$$f_1(x) = f_2(x) + \sum_{i=1}^{n} (x_i - x_{0i}) h_i(x) , \tag{3.89}$$

where, subtracting, the map $x \mapsto \sum_{i=1}^{n}(x_i - x_{0i})h_i(x)$ and also the h_i belong to $\mathscr{S}(\mathbb{R}^n)$. Using J on both sides of (3.89) and recalling J commutes with polynomials in x by (3.85), we have:

$$(Jf_1)(x) = (Jf_2)(x) + \sum_{i=1}^{n} (x_i - x_{0i})(Jh_i)(x) .$$

Taking $x = x_0$ shows $(Jf_1)(x_0) = (Jf_2)(x_0)$ under the initial assumption $f_1(x_0) = f_2(x_0)$. Hence, as claimed, $(Jf)(x_0)$ is a map of $f(x_0)$ only. This map must be linear, as J is linear by construction. Consequently $(Jf)(x_0) = j(x_0)f(x_0)$ for some map $j : \mathbb{R}^n \to \mathbb{C}$. Given that x_0 was arbitrary, J acts as multiplication by a function j. The latter must be C^∞. To justify this, choose $f \in \mathscr{S}(\mathbb{R}^n)$ equal to 1 on a neighbourhood $I(x_0)$ of x_0. If $x \in I(x_0)$, then $(Jf)(x) = j(x)$. The left-hand side is C^∞ on $I(x_0)$, so also the right term is. That being valid around any point in $\mathbb{R}^n$, we have $j \in C^\infty(\mathbb{R}^n)$. Equation (3.87) implies

$$j(x)\frac{\partial}{\partial x^i} f(x) = \frac{\partial}{\partial x^i} j(x) f(x)$$

for any $f \in \mathscr{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. Choose as before f equal 1 on an open set, so the above identity forces all derivatives of j to vanish there. This holds around any point, and $\mathbb{R}^n$ is connected, so the continuous map j is constant on $\mathbb{R}^n$. The constant value clearly does not depend on the argument of J, and may be computed by evaluating J on an arbitrary function $\mathscr{S}(\mathbb{R}^n)$. Computing J on $x \mapsto e^{-x^2}$ is a useful exercise, and reveals the constant value is exactly 1. The argument for J_- is similar.
(d) Using (c) the claim is immediate. Let us carry out the proof for $\mathscr{F}$; the one for $\mathscr{F}_-$ is the same, essentially. Let $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$ and set, $i = 1, 2$:

$$g_i(k) := \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f_i(x)\, dx .$$

With the assumptions made, the theorem of Fubini–Tonelli gives:

$$\int_{\mathbb{R}^n} \overline{g_1(k)} g_2(k) dk = \int_{\mathbb{R}^n} \overline{g_1(k)} \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f_2(x) dx dk$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} \overline{g_1(k)} f_2(x) dx \otimes dk .$$

Now we rephrase the last integral and apply Fubini–Tonelli again:

$$\int_{\mathbb{R}^n} \overline{g_1(k)} g_2(k) dk$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \overline{\frac{e^{ik\cdot x}}{(2\pi)^{n/2}} g_1(k)} f_2(x) dx \otimes dk = \int_{\mathbb{R}^n} f_2(x) \int_{\mathbb{R}^n} \overline{\frac{e^{ik\cdot x}}{(2\pi)^{n/2}} g_1(k) dk} dx$$

$$= \int_{\mathbb{R}^n} \overline{f_1(x)} f_2(x) dx \; ,$$

where part (c) was used. This was what we wanted.
(e) We prove the statement for $\mathscr{F}$, and leave the similar assertion about $\mathscr{F}_-$ to the reader. Notice that both transformations are well defined on $L^1(\mathbb{R}^n, dx)$ since the integral does not change by altering the maps by sets of zero Lebesgue measure. The estimate $||\mathscr{F} f||_\infty \leq \frac{||f||_1}{(2\pi)^{n/2}}$ guarantees the linear map $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ is continuous when the domain has the L^1 norm and the codomain has $|| \cdot ||_\infty$. Now recall $\mathscr{S}(\mathbb{R}^n)$ is dense in L^1 in the given norm, and the codomain is complete in the second norm. Hence the Fourier transform, initially defined on $\mathscr{S}(\mathbb{R}^n)$, can be extended by continuity – in a unique way – to a bounded linear map $L^1(\mathbb{R}^n, dx) \to C_0(\mathbb{R}^n)$ that preserves the same norm by Proposition 2.47 (and coincides with the aforementioned linear transformation on $L^1(\mathbb{R}^n, dx)$). If $f \in L^1$, $\mathscr{F} f \in C_0(\mathbb{R}^n)$, then for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathbb{R}^n$ such that $|(\mathscr{F} f)(k)| < \varepsilon$ if $k \notin K_\varepsilon$. Choose, for any $\varepsilon > 0$, a ball at the origin of radius r_ε large enough to contain K_ε. Then there exists, for any $\varepsilon > 0$, a real number $r_\varepsilon > 0$ such that $|(\mathscr{F} f)(k)| < \varepsilon$ if $|k| > r_\varepsilon$.
(f) We prove the claim for $\mathscr{F}$, as the one for $\mathscr{F}_-$ is analogous. Since $\mathscr{F}$ is a linear operator, it suffices to show that if $\mathscr{F} f$ is the zero map then f is null almost everywhere. Therefore assume:

$$\int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\, dx = 0 \; , \quad \text{for any } k \in \mathbb{R}^n .$$

If $g \in \mathscr{S}(\mathbb{R}^n)$, the Fubini–Tonelli theorem gives

$$0 = \int_{\mathbb{R}^n} g(k) \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\, dx\, dk = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} g(k)\, dk \right) f(x)\, dx \; .$$

Since $\mathscr{F}$ is bijective on $\mathscr{S}(\mathbb{R}^n)$, what we have proved is equivalent to:

$$\int_{\mathbb{R}^n} \psi(x) f(x)\, dx = 0 \quad \text{for any } \psi \in \mathscr{S}(\mathbb{R}^n)$$

(note $\psi f \in \mathscr{L}^1(\mathbb{R}^n, dx)$ for any $\psi \in \mathscr{S}(\mathbb{R}^n)$, as ψ is bounded). As $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, Lemma 3.102 forces f to vanish almost everywhere. □
We move on to the *Fourier–Plancherel transform*. As $\mathscr{S}(\mathbb{R}^n)$ is dense in $\mathscr{L}^2(\mathbb{R}^n)$, by considering equivalence classes we can say $\mathscr{S}(\mathbb{R}^n)$ determines a dense subset,

still called $\mathscr{S}(\mathbb{R}^n)$, in the Hilbert space $L^2(\mathbb{R}^n)$. The operators $\mathscr{F}$ and $\mathscr{F}_-$ can be seen as defined on that dense subspace of $L^2(\mathbb{R}, dx)$. Proposition 3.105(d) says in particular that these operators are bounded with norm 1, since they are isometric. Then Proposition 2.47 tells us $\mathscr{F}$ and $\mathscr{F}_-$ determine unique bounded linear operators on $L^2(\mathbb{R}^n, dx)$. For instance, the operator extending $\mathscr{F}$ to $L^2(\mathbb{R}^n, dx)$ is defined as

$$\widehat{\mathscr{F}} f := \lim_{n\to+\infty} \mathscr{F} f_n ,$$

for $f \in L^2(\mathbb{R}^n, dx)$. Above, $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n)$ is an arbitrary sequence converging to f in the topology of $L^2(\mathbb{R}^n, dx)$. By the inner product's continuity, the extended operator $\widehat{\mathscr{F}}$ will preserve the inner product of $L^2(\mathbb{R}^2, dx)$, and as such $\widehat{\mathscr{F}}$ will be 1-1 on $L^2(\mathbb{R}^n, dx)$. The following elementary argument explains why $\widehat{\mathscr{F}}$ is surjective, too. Beside $\widehat{\mathscr{F}}$, we can construct the operator $\widehat{\mathscr{F}_-}$ that extends to $L^2(\mathbb{R}^n, dx)$ the inverse Fourier transform. On $\mathscr{S}(\mathbb{R}^n, dx)$

$$\mathscr{F}\mathscr{F}_- = I_{\mathscr{S}(\mathbb{R}^n)} .$$

Now pass to the L^2 extensions, by linearity and continuity, and recall that the unique linear extension of the identity from $\mathscr{S}(\mathbb{R}^n, dx)$ to $L^2(\mathbb{R}^n, dx)$ is the latter's identity operator I (constructed in the general way explained above). Then

$$\widehat{\mathscr{F}}\widehat{\mathscr{F}}_- = I ,$$

This condition implies $\widehat{\mathscr{F}}$ is onto.

Definition 3.107 The unique operator $\widehat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dx)$ that extends linearly and continuously the Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ is called **Fourier–Plancherel transform**.

Theorem 3.108 (Plancherel) *The Fourier–Plancherel transform:*

$$\widehat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dx)$$

is a bijective and isometric linear operator.

Proof The proof was given immediately before Definition 3.107. □

Remark 3.109 Define the bilinear map $L^2(\mathbb{R}^n, dx) \times L^2(\mathbb{R}^n, dx) \to \mathbb{C}$ that associates $(f, g) \in L^2(\mathbb{R}^n, dx) \times L^2(\mathbb{R}^n, dx)$ to

$$\langle f, g\rangle := \int_{\mathbb{R}^n} fg dx .$$

Exploiting Theorem 3.108, it is simple to prove that

$$\langle \widehat{\mathscr{F}} f, g\rangle = \langle f, \widehat{\mathscr{F}} g\rangle .$$

This result is the starting point to define *Fourier transforms of distributions*. ■

There is still one issue we have to deal with. If $f \in L^1(\mathbb{R}^n, dx) \cap L^2(\mathbb{R}^n, dx)$ (but $f \notin \mathscr{S}(\mathbb{R}^n)$), a priori $\mathscr{F} f$ and $\widehat{\mathscr{F}} f$ may be different, because to define $\widehat{\mathscr{F}}$ we did not extend $\mathscr{F}$ from $L^1(\mathbb{R}^n, dx)$, but rather from the subspace $\mathscr{S}(\mathbb{R}^n)$. This was the only possible choice because $L^1(\mathbb{R}^n, dx) \not\subset L^2(\mathbb{R}^n, dx)$.

The next proposition sheds light on the matter, and provides a practical method to compute the Fourier–Plancherel transform by means of limits of Fourier transforms.

Remark 3.110 Recall that if $K \subset \mathbb{R}^n$ is a finite-measure set, in particular compact, (compact sets have finite Lebesgue measure):

(1) $L^2(K, dx) \subset L^1(K, dx)$;

(2) if $\{f_n\}_{n\in\mathbb{N}} \subset L^2(K, dx)$ converges in norm $|| \,||_2$ to $f \in L^2(K, dx)$, it converges in norm $|| \,||_1$ to f;

(3) $L^\infty(K, dx) \subset L^p(K, dx)$, $1 \leq p < \infty$;

(4) if $\{f_n\}_{n\in\mathbb{N}} \subset L^\infty(K, dx)$ converges in norm $|| \,||_\infty$ to $f \in L^\infty(K, dx)$, it converges to f in norm $|| \,||_p$ as well.

These four statements are proved as follows: concerning the first two, recall the constant map 1 on a compact (of finite measure) set is integrable. Since

$$2|f(x)| \leq |f(x)|^2 + 1 \,,$$

the integral of the left is bounded by the integral of the right, so we have statement (1). As for the second claim, the Cauchy–Schwarz inequality

$$\left(\int_K |g(x)|\, 1\, dx\right)^2 \leq \left(\int_K |g(x)|^2 dx\right)\left(\int_K 1 dx\right)$$

with $f(x) - f_n(x)$ replacing $g(x)$ proves (2). To settle the other two, note that by definition of Lebesgue integral:

$$\int_K |g|^p dx \leq \operatorname*{ess\,sup}_K |g|^p \int_K dx = (||g||_\infty)^p \int_K dx$$

for any measurable map g on K. ■

Proposition 3.111 *The Fourier–Plancherel and Fourier transforms satisfy the following properties.*

(a) *If $f \in L^2(\mathbb{R}^n, dx) \cap L^1(\mathbb{R}^n, dx)$, the Fourier–Plancherel transform reduces to the Fourier transform $\mathscr{F} f$ computed by integral formula (3.77).*

(b) *If $f \in L^2(\mathbb{R}^n, dx)$, the Fourier–Plancherel transform can be computed as the limit (understood in $L^2(\mathbb{R}^n, dk)$)*

$$\hat{\mathscr{F}} f = \lim_{n\to+\infty} g_n$$

of

$$g_n(k) := \int_{K_n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)dx\ , \tag{3.90}$$

where $K_n \subset \mathbb{R}^n$ are compact, $K_{m+1} \supset K_m$, $m = 1, 2, \ldots$, and $\cup_{m=1}^{\infty} K_m = \mathbb{R}^n$.

Proof (a) Begin with proving the claim for $f \in L^2(\mathbb{R}^n, dx)$ different from 0 on a zero-measure set outside a compact set K_0. Such an f belongs to $L^1(\mathbb{R}^n, dx)$. Let then $\{s_n\}_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n)$ be a sequence converging to f in $L^2(\mathbb{R}^n, dx)$. If B, B' are open balls of finite radius with $B \supset \overline{B'} \supset B' \supset K_0$, we can construct a function $h \in \mathscr{D}(\mathbb{R}^n)$ equal to 1 on B' and null outside B. Obviously, letting $f_n := h \cdot s_n$, the sequence $\{f_n\}$ is in $\mathscr{D}(\mathbb{R}^n)$ and hence in $\mathscr{S}(\mathbb{R}^n)$. The supports lie in the compact set $K := \overline{B}$. Therefore every f_n belongs to $L^1(\mathbb{R}^n, dx)$ and the sequence $\{f_n\}$ tends to f in $L^2(\mathbb{R}^n, dx)$ and $L^1(\mathbb{R}^n, dx)$.

By definition, as $f_n \to f$ in norm $||\ ||_2$,

$$||\mathscr{F} f_n - \hat{\mathscr{F}} f||_2 \to 0 \tag{3.91}$$

as $n \to +\infty$. At the same time, since $f_n \to f$ in norm $||\ ||_1$, by Proposition 3.105(a) we have $||\mathscr{F} f_n - \mathscr{F} f||_\infty \to 0$ as $n \to +\infty$. But on finite-measure sets, convergence for $||\ ||_\infty$ implies convergence for $||\ ||_2$, so

$$||\mathscr{F} f_n - \mathscr{F} f||_2 \to 0 \tag{3.92}$$

and hence $\widehat{\mathscr{F}} f = \mathscr{F} f$ by (3.91) and by uniqueness of the limit.

Suppose now $f \in L^2(\mathbb{R}^n, dx) \cap L^1(\mathbb{R}^n, dx)$, and nothing more. Consider an exhaustion of $\mathbb{R}^n$ by compact sets $\{K_n\}$. Define maps $f_n := \chi_{K_n} \cdot f$, where χ_E is the characteristic function of E ($\chi_E(x) = 0$ if $x \notin E$ and $\chi_E(x) = 1$ if $x \in E$). It is clear that $f_n \to f$ pointwise, as $n \to +\infty$. Moreover $|f(x) - f_n(x)|^p \leq |f(x)|^p$, $p = 1, 2, \ldots$. By Lebesgue's dominated convergence theorem, $f_n \to f$ as $n \to +\infty$, both for $||\ ||_1$ and $||\ ||_2$. On the other hand what we have proved just above tells:

$$\mathscr{F} f_n = \widehat{\mathscr{F}} f_n\ .$$

Proposition 3.105(a) gives $||\mathscr{F} f - \mathscr{F} f_n||_\infty \to 0$ and at the same time $||\widehat{\mathscr{F}} f - \mathscr{F} f_n||_2 \to 0$. These facts hold also when restricting $\widehat{\mathscr{F}} f, \mathscr{F} f, \mathscr{F} f_n$ to any compact set K. For maps that are zero outside a compact set uniform convergence implies L^2 convergence, so if x belongs to a compact set, $(\mathscr{F} f)(x) = (\widehat{\mathscr{F}} f)(x)$ almost everywhere. But every point $x \in \mathbb{R}^n$ belongs to some compact set, so $\mathscr{F} f = \widehat{\mathscr{F}} f$ as elements in $L^2(\mathbb{R}^n, dx)$.
(b) This was proved in the final part of (a). $\square$

Examples 3.112 **(1)** There is an important property distinguishing $\mathscr{D}(\mathbb{R}^n)$ from $\mathscr{S}(\mathbb{R}^n)$: only the former is *not* invariant under the Fourier transform (and inverse Fourier transform). Since $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, in fact, it is clear that $\mathscr{F}(\mathscr{D}(\mathbb{R}^n)) \subset \mathscr{F}(\mathscr{S}(\mathbb{R}^n)) \subset \mathscr{S}(\mathbb{R}^n)$. This cannot be sharpened:

Proposition 3.113 *Take $f \in \mathscr{D}(\mathbb{R}^n)$. If $\mathscr{F} f \in \mathscr{D}(\mathbb{R}^n)$ then $f = 0$. The same holds for the inverse Fourier transform.*

Proof The proof is easy, and we show it only for $\mathscr{F}$, because the case $\mathscr{F}_-$ is similar. If

$$g(k) = \int_{\mathbb{R}^n} \frac{e^{-ik\cdot x}}{(2\pi)^{n/2}} f(x)\, dx \ ,$$

where f has compact support, the integral converges also for $k \in \mathbb{C}^n$. Using Lebesgue's dominated convergence, moreover, we can differentiate the components k_i of k inside the integral, and their real and imaginary parts. Since $k \mapsto e^{-ik\cdot x}$ is analytic (in each variable k_i separately), it solves the Cauchy–Riemann equations in each k_i. Consequently also g will solve those equations in each k_i, becoming analytic on $\mathbb{C}^n$. The restriction of g to $\mathbb{R}^n$ defines, via its real and imaginary parts, real analytic maps on $\mathbb{R}^n$. If g has compact support, there will be an open, non-empty set in $\mathbb{R}^n$ where $Re\ g$ and $Im\ g$ vanish. A known property of real-analytic maps (of one real variable) on open connected sets (here $\mathbb{R}^n$) is that they vanish everywhere if they vanish on an open non-empty set of the domain. Therefore if g has compact support it must be the zero map. Then also f is zero, since $\mathscr{F}$ is invertible on $\mathscr{S}(\mathbb{R}^n)$. $\square$

(2) Related to (1) is the known *Paley-Wiener theorem* (see for instance [KiGv82]):

Theorem 3.114 (Paley-Wiener) *Take $a > 0$ and consider $L^2([-a, a], dx)$ as subspace of $L^2(\mathbb{R}, dx)$. The space $\hat{\mathscr{F}}(L^2([-a, a], dx))$ consists of maps $g = g(k)$ that can be extended uniquely to analytic maps on the complex plane ($k \in \mathbb{C}$) such that*

$$|g(k)| \leq Ce^{2\pi a|Imk|}\ , \quad k \in \mathbb{C}$$

for some constant $C \geq 0$ depending on g.

Since $\hat{\mathscr{F}}(L^2([-a, a], dx)) \subset L^2(\mathbb{R}, dk)$ by Plancherel's theorem, the result of Paley-Wiener implies that analytic maps g bounded as above determine elements of $L^2(\mathbb{R}, dk)$ when k is real. ■

To conclude, consider the space $L^2((a, b), dx)$, where $-\infty \leq a < b \leq +\infty$ and dx is the usual Lebesgue measure on $\mathbb{R}$. The following extremely practical fact, used in Example 3.32(4) to build bases, descends from the Fourier–Plancherel theory.

Proposition 3.115 *Let $f : (a, b) \to \mathbb{C}$ be a measurable map such that:*

(1) the set $\{x \in (a, b) \mid f(x) = 0\}$ has zero measure,

(2) there exist $C, \delta > 0$ for which $|f(x)| < Ce^{-\delta|x|}$ for any $x \in (a, b)$.

Then the finite linear span of the maps $x \mapsto x^n f(x) =: f_n(x)$, $n = 0, 1, 2, \ldots$, is dense in $L^2((a, b), dx)$.

Proof Let $S := \{f_n\}_{n\in\mathbb{N}}$. It is enough to prove $S^\perp = \{\mathbf{0}\}$, because $S^\perp \oplus \overline{< S >} = L^2((a, b), dx)$ by Theorem 3.13. So take $h \in L^2((a, b), dx)$ such that

$$\int_a^b x^n f(x)\overline{h(x)} dx = 0$$

for any $n = 0, 1, 2, \ldots$. Extend h to the whole real line by setting it to zero outside (a, b), so now:

$$\int_{\mathbb{R}} x^n f(x)\overline{h(x)}dx = 0\,, \tag{3.93}$$

for any $n = 0, 1, 2, \ldots$. Moreover, the following three facts hold:
(i) $f \cdot \overline{h} \in L^1(\mathbb{R}, dx)$: both maps are in $L^2(\mathbb{R}, dx)$, so their product is in $L^1(\mathbb{R}, dx)$;
(ii) $f \cdot \overline{h} \in L^2(\mathbb{R}, dx)$, because $|f(x)|^2 < C^2 e^{-2\delta|x|} < C^2 < +\infty$ and $|h|^2$ is integrable by assumption;
(iii) the map sending $x \in \mathbb{R}$ to $e^{\delta'|x|} f(x)\overline{h}(x)$ is in $L^1(\mathbb{R}, dx)$ for any $\delta' < \delta$. In fact, since $x \mapsto |e^{\delta'|x|} f(x)| \le Ce^{-(\delta-\delta')|x|}$, the function $x \mapsto e^{\delta'|x|} f(x)$ is in $L^2(\mathbb{R}, dx)$, and $h \in L^2(\mathbb{R}, dx)$ by hypothesis, so the product belongs to $L^1(\mathbb{R}, dx)$.
Using (i) we compute the Fourier transform:

$$g(k) = \int_{\mathbb{R}} \frac{e^{-ik\cdot x}}{\sqrt{2\pi}} f(x)\overline{h}(x)\, dx\,.$$

This coincides with the Fourier–Plancherel transform of $f \cdot \overline{h}$ by (i), (ii) and Proposition 3.111(a). Using (iii), if k is complex and $|Imk| < \delta$, then $g = g(k)$ is well defined and analytic on the open strip $B \subset \mathbb{C}$ given by $Rek \in \mathbb{R}$, $|Imk| < \delta$; this is proved similarly to what we did in example (1). Lebesgue's dominated convergence and exchanging derivatives and integrals allow to see that

$$\frac{d^n g}{dk^n}|_{k=0} = \frac{(-i)^n}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n f(x)\overline{h(x)}dx$$

for any $n = 0, 1, \ldots$. All derivatives vanish by (3.93), and so the Taylor expansion of g at the origin is zero. This annihilates g on an open disc contained in B, so analyticity guarantees g is zero on the open connected set B, and in particular on the real line parametrised by k. Therefore the Fourier–Plancherel transform of $f \cdot \overline{h}$ is the null vector of $L^2(\mathbb{R}, dk)$. Since the transform is unitary we conclude $f \cdot \overline{h} = 0$ almost everywhere on $\mathbb{R}$: in particular on (a, b), where by assumption $f \neq 0$ almost everywhere. But then $h = 0$ almost everywhere on (a, b), which is to say each $h \in S^\perp$ coincides with the null element in $L^2((a, b), dx)$, ending the proof. □

Exercises

3.1 Let X be a real vector space and $\langle \cdot, \cdot \rangle : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ a bilinear map. Prove that the polarisation identity

$$\langle x, y \rangle = \frac{1}{4}\left(\langle x + y, x + y \rangle - \langle x - y, x - y \rangle\right)$$

holds for all $x, y \in \mathsf{X}$ if and only if $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathsf{X}$.

3.2 Let X be a complex vector space and $\langle\cdot,\cdot\rangle : \mathsf{X}\times\mathsf{X} \to \mathbb{C}$ a map which is linear in the right entry and antilinear in the left entry. Prove the polarisation identity

$$\langle x, y\rangle = \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle - i\langle x+iy, x+iy\rangle + i\langle x-iy, x-iy\rangle\right)$$

for every $x, y \in \mathsf{X}$. Next show that $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathsf{X}$ if and only if $\langle z, z\rangle \in \mathbb{R}$ for every $z \in \mathsf{X}$.

3.3 Definition 3.1 of a (semi-)inner product makes sense on real vector spaces as well, simply by replacing **H2** with $S(u, v) = S(v, u)$, and using real linear combinations in **H1**.

Show that with this definition Proposition 3.3 still holds, provided the polarisation formula is written as in (3.7).

3.4 (*Hard.*) Consider a real vector space and prove that if a (semi)norm p satisfies the parallelogram rule (3.3):

$$p(x+y)^2 + p(x-y)^2 = 2(p(x)^2 + p(y)^2)\,, \tag{3.94}$$

then there exists a unique (semi-)inner product S, defined in Exercise 3.3, inducing p via (3.2).

Solution. If S is a (semi-)inner product on the real vector space X, we have the polarisation formula (3.7):

$$S(x, y) = \frac{1}{4}\left(S(x+y, x+y) - S(x-y, x-y)\right)\,.$$

This implies S is unique, for $S(z, z) = p(z)^2$. For the existence from a given norm p set:

$$S(x, y) := \frac{1}{4}\left(p(x+y)^2 - p(x-y)^2\right)\,.$$

We shall prove S is a semi-inner product or an inner product according to whether p is a norm or a seminorm. If this is true and p is a norm, then substituting S to p above, on the right, gives $S(x, x) = 0$ and $x = \mathbf{0}$, making S an inner product.

To finish we need to prove, for any $x, y, z \in \mathsf{X}$:
(a) $S(\alpha x, y) = \alpha S(x, y)$ if $\alpha \in \mathbb{R}$,
(b) $S(x+y, z) = S(x, z) + S(y, z)$,
(c) $S(x, y) = S(y, x)$,
(d) $S(x, x) = p(x)^2$.

Properties (c) and (d) are straightforward from the definition of S. Let us prove (a) and (b). By (3.3) and the definition of S:

$$S(x, z) + S(y, z) = 4^{-1}\left(p(x+z)^2 - p(x-z)^2 + p(y+z)^2 - p(y-z)^2\right)$$

$$= 2^{-1}\left(p\left(\frac{x+y}{2}+z\right)^2 - p\left(\frac{x+y}{2}-z\right)^2\right) = 2S\left(\frac{x+y}{2}, z\right) .$$

Hence

$$S(x,z) + S(y,z) = 2S\left(\frac{x+y}{2}, z\right) . \tag{3.95}$$

Then (a) clearly implies (b), and we have to prove (a) only. Take $y = \mathbf{0}$ in (3.95) and recall $S(0, z) = 0$ by definition of S. Then

$$S(x,z) = 2S(x/2, z) .$$

Iterating this formula gives (a) for $\alpha = m/2^n, m, n = 0, 1, 2, \ldots$. These numbers are dense in $[0, +\infty)$. At the same time $\mathbb{R} \ni \alpha \mapsto p(\alpha x + z)$ and $\mathbb{R} \ni \alpha \mapsto p(\alpha x - z)$ are both continuous (in the topology induced by p), so

$$S(x, y) := \frac{1}{4}\left(p(x+y, x+y) - p(x-y, x-y)\right)$$

allows to conclude $\mathbb{R} \ni \alpha \mapsto S(\alpha x, y)$ is continuous in α. That is to say, (a) holds for any $\alpha \in [0, +\infty)$. Again by definition of S we have $S(-x, y) = -S(x, y)$, so the previous result is valid for any $\alpha \in \mathbb{R}$, ending the proof.

3.5 (*Hard.*) Suppose a (semi)norm p satisfies the parallelogram rule (3.3):

$$p(x+y)^2 + p(x-y)^2 = 2(p(x)^2 + p(y)^2) \tag{3.96}$$

on a $\mathbb{C}$-vector space. Show that there is a unique (semi-)inner product S inducing p by means of (3.2).

Solution. If S is a (semi-)inner product on the complex vector space X we have the polarisation formula (3.4):

$$4S(x, y) = S(x+y, x+y) - S(x-y, x-y) - iS(x+iy, x+iy) + iS(x-iy, x-iy).$$

Since $S(z, z) = p(z)^2$, as in the real case, that implies uniqueness of S for a given norm p on X. Existence: define, for a given (semi)norm p and $x, y \in \mathsf{X}$:

$$S_1(x, y) := 4^{-1}(p(x+y)^2 - p(x-y)^2) , \quad S(x, y) := S_1(x, y) - iS_1(x, iy) .$$

Notice $S(x, x) = p(x)^2$, and if p is a norm, by construction $S(x, x) = 0$ implies $x = \mathbf{0}$. There remains to show that the above S is a Hermitian (semi-)inner product. By Definition 3.1 we have to check:

(a) $S(\alpha x, y) = \alpha S(x, y)$ if $\alpha \in \mathbb{C}$,
(b) $S(x + y, z) = S(x, z) + S(y, z)$,
(c) $S(x, y) = \overline{S(y, x)}$,

(d) $S(x,x) = p(x)^2$.
The last one is true by construction. Proceeding as in the previous exercise, using S_1 instead of S, we can prove (b) for S_1, (a) for S_1 *with* $\alpha \in \mathbb{R}$, and also $S_1(x,y) = S_1(y,x)$. These, using the definition of S in terms of S_1, imply (a), (b) and (c).

3.6 Let X be a real vector space equipped with a real inner product $\langle\cdot,\cdot\rangle$ and associated norm $||\cdot||$. Prove that

$$||x_1 + \cdots + x_n|| \leq ||x_1|| + \cdots + ||x_n|| \,,$$

where

$$||x_1 + \cdots + x_n|| = ||x_1|| + \cdots + ||x_n||$$

if and only if $x_k = \lambda_k x$ for some $x \in \mathsf{X}$ and $\lambda_k \in [0, +\infty)$ for all $k = 1, \ldots, n$.

Hint. First show that $||x_1 + \cdots + x_p|| \leq ||x_1|| + \cdots + ||x_p||$. Then prove the second part of the thesis for $n = 2$, using the fact that $|\langle x, y\rangle| = ||x||\,||y|| \Leftrightarrow x$ and y are linearly independent, then extend the result using induction. Observe that $||x_1 + \cdots + x_p + x_{p+1}|| = ||x_1|| + \cdots + ||x_p|| + ||x_{p+1}||$ implies $||x_1 + \cdots + x_p|| = ||x_1|| + \cdots + ||x_p||$ since $||x_1 + \cdots + x_p + x_{p+1}|| \leq ||x_1 + \cdots + x_p|| + ||x_{p+1}||$.

3.7 Prove the claim in Remark 3.4(1) on a (semi-)inner product space (X, S): the (semi-)inner product $S : \mathsf{X} \times \mathsf{X} \to \mathbb{C}$ is continuous in the product topology of $\mathsf{X} \times \mathsf{X}$, having on X the topology induced by the (semi-)inner product itself. Consequently S is continuous in both arguments separately.

Hint. Suppose $\mathsf{X} \times \mathsf{X} \ni (x_n, y_n) \to (x, y) \in \mathsf{X} \times \mathsf{X}$ as $n \to +\infty$. Use the Cauchy–Schwarz inequality to show that if S is the (semi-)inner product associated to p, then:

$$|S(x,y) - S(x_n, y_n)| \leq p(x_n)p(y_n - y) + p(x_n - x)p(y) \,.$$

Recall that $p(x_n) \to p(x)$ and that the canonical projections are continuous in the product topology.

3.8 Prove Proposition 3.8: a linear operator $L : \mathsf{X} \to \mathsf{Y}$ between inner product spaces is an isometry, in the sense of Definition 3.6, if and only if

$$||Lx||_{\mathsf{Y}} = ||x||_{\mathsf{X}} \quad \text{for any } x \in \mathsf{X},$$

where norms are associated to the respective inner products.

Hint. Polarise.

3.9 Consider the Banach space $\ell^p(\mathbb{N})$, $p \geq 1$. Show that for $p \neq 2$ one cannot define any Hermitian inner product inducing the usual norm $||\ ||_p$. Conclude that $\ell^p(\mathbb{N})$ cannot be rendered a Hilbert space for $p \neq 2$.

Hint. Show there are pairs of vectors f, g violating the parallelogram rule. E.g. $f = (1, 1, 0, 0, \ldots)$ and $g = (1, -1, 0, 0, \ldots)$.

3.10 Prove that the Banach space $(C([0, \pi/2]), || \ ||_\infty)$ does not admit a Hermitian inner product inducing $|| \ ||_\infty$, i.e.: $(C([0, \pi/2]), || \ ||_\infty)$ cannot be made into a Hilbert space.

Hint. Show there are pairs of vectors f, g violating the parallelogram rule. Consider for example $f(x) = \cos x$ and $g(x) = \sin x$.

3.11 In the Hilbert space H consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ converging to $x \in \mathsf{H}$ weakly: $f(x_n) \to f(x)$, $n \to +\infty$, for any $f \in \mathsf{H}'$. Show that, in general, $x_n \not\to x$ in the topology of H. However, if we additionally assume $||x_n|| \to ||x||$, $n \to +\infty$, then $x_n \to x$, $n \to +\infty$, also in the topology of H.

Hint. Riesz's theorem implies that $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ weakly converges to $x \in \mathsf{H}$ iff $(z|x_n) \to (z|x)$, $n \to +\infty$, for any $z \in \mathsf{H}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis of H, thought of as separable. Then $x_n \to \mathbf{0}$ weakly but not in the topology of H. For the second claim, note $||x - x_n||^2 = ||x||^2 + ||x_n||^2 - 2Re(x|x_n)$.

3.12 Consider the basis of $L^2([-L/2, L/2], dx)$ formed by the functions (up to zero-measure sets):

$$e_n(x) := \frac{e^{i2\pi nx/L}}{\sqrt{L}} \quad n \in \mathbb{Z}\,.$$

Suppose, for $f \in L^2([-L/2, L/2], dx)$, that the series

$$\sum_{n\in\mathbb{Z}} (e_n|f)e_n(x)$$

converges to some g in norm $|| \ ||_\infty$. Prove $f(x) = g(x)$ a.e.

Hint. Compute the components $(e_n|g)$ using the fact that the integral of an absolutely convergent series on $[a, b]$ is the series of the integrated summands. Check that $(e_n|g) = (e_n|f)$ for any $n \in \mathbb{Z}$.

3.13 Consider the basis of $L^2([-L/2, L/2], dx)$ made by the functions e_n of Exercise 3.12. Suppose $f : [-L/2, L/2] \to \mathbb{C}$ is continuous, $f(-L/2) = f(L/2)$, and f is piecewise C^1 on $[-L/2, L/2]$ (i.e. $[-L/2, L/2] = [a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_{n-2}, a_{n-1}] \cup [a_{n-1}, a_n]$ and $f\restriction_{[a_i, a_{i+1}]} \in C^1([a_i, a_{i+1}])$ for any i, understanding boundary derivatives as left and right derivatives). Show:

$$f(x) = \sum_{n\in\mathbb{Z}} (e_n|f)e_n(x) \quad \text{for any } x \in [-L/2, L/2]$$

where

$$e_n(x) := \frac{e^{i2\pi nx/L}}{\sqrt{L}} \quad n \in \mathbb{Z}\,.$$

Prove the series converges uniformly.

Hint. Compute the components $(e_n|df/dx)$ by integration by parts: $|n(e_n|f)| = 2c|(e_n|df/dx)|$, where $c = L/(4\pi)$. Then

$$|(e_n|f)| = c2|(e_n|df/dx)||1/n| \leq c(|(e_n|df/dx)|^2 + 1/n^2), \quad n \neq 0\,.$$

Now, df/dx gives an L^2 map, the series with generic term $1/n^2$ converges, and $|e_n(x)| = 1$ for any x. Therefore the series

$$\sum_{n\in\mathbb{Z}} (e_n|f)e_n(x)$$

converges uniformly, i.e. in norm $||\ \ ||_\infty$. Apply Exercise 3.12.

3.14 Rephrase and prove Exercise 3.13, replacing the requirement that f be continuous and piecewise C^1 with the demand that f be *absolutely continuous* on $[-L/2, L/2]$ and either with *essentially bounded* derivative, or with derivative in $\mathscr{L}^2([-L/2, L/2], dx)$.

Hint. Remember Theorem 1.76(a).

3.15 Consider the basis $\{e_n\}$ of $L^2([-L/2, L/2], dx)$ of Exercise 3.12. Let $f : [-L/2, L/2] \to \mathbb{C}$ be of class C^N, suppose $d^k f/dx^k|_{-L/2} = d^k f/dx^k|_{L/2}$, $k = 0, 1, \ldots, N$ and that f is piecewise C^{N+1} on $[-L/2, L/2]$. Prove

$$\frac{d^k f(x)}{dx^k} = \sum_{n\in\mathbb{Z}} (e_n|f)\frac{d^k}{dx^k}e_n(x) \quad \text{for any } x \in [-L/2, L/2]$$

where

$$e_n(x) := \frac{e^{i2\pi nx/L}}{\sqrt{L}} \quad n \in \mathbb{Z}\,,$$

and the series' convergence is uniform, $k = 0, 1, 2, \ldots, N$

Hint. Iterate the procedure of Exercise 3.13, bearing in mind that we can swap derivatives and sum in a convergent series of C^1 maps whose series of derivatives converges uniformly.

3.16 Prove that the functions $[0, L] \ni x \mapsto s_n(x) := \sqrt{\frac{2}{L}}\sin\left(\frac{\pi nx}{L}\right)$, $n = 1, 2, 3, \ldots$, form an orthonormal system in $L^2([0, L], dx)$.

Sketch. A direct computation tells $||s_n|| = 1$. Then observe that $\Delta s_n = \left(\frac{\pi nx}{L}\right)^2 s_n$ where $\Delta := -\frac{d^2}{dx^2}$. Therefore if $(\ |\ )$ is the inner product in $L^2([0, L], dx)$:

$$(s_n|s_m) = \frac{1}{n}(\Delta s_n|s_m) = \frac{1}{n}(s_n|\Delta s_m) = \frac{m}{n}(s_n|s_m)$$

where, in the middle, we integrated twice by parts in order to shift Δ from left to right, and we used $s_k(0) = s_k(L) = 0$ to annihilate the boundary terms.

Therefore

$$\left(1 - \frac{m}{n}\right)(s_n|s_m) = 0,$$

implying $(s_n|s_m) = 0$ if $n \neq m$.

3.17 Prove that the maps $[0, L] \ni x \mapsto c_n(x) := \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right)$, $n = 0, 1, 2, \ldots$, form an orthonormal system in $L^2([0, L], dx)$.

Hint. Proceed exactly as in Exercise 3.16. If $\Delta := -\frac{d^2}{dx^2}$ we still have $\Delta c_n = \left(\frac{\pi n x}{L}\right)^2 c_n$, but now it is the derivatives of c_n that vanish on the boundary of $[0, L]$.

3.18 Recall that the space $\mathscr{D}((0, L))$ of smooth maps with compact support in $(0, L)$ is dense in $L^2([0, L], dx)$ in the latter's topology. Using Exercise 3.13 prove that the functions $[0, L] \ni x \mapsto s_n(x) := \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$, $n = 1, 2, 3, \ldots$, are a basis of $L^2([0, L], dx)$.

Outline. It suffices to prove that the space $< s_n >_{n=1,2,\ldots}$ of finite linear combinations of the s_n is dense in $\mathscr{D}((0, L))$ for $|| \; ||_\infty$, because this would imply, by elementary integral properties, that they are dense in the topology of $L^2([0, L], dx)$. Since $\mathscr{D}((0, L))$ is dense in $L^2([0, L], dx)$, we would have $< s_n >_{n=1,2,\ldots}$ dense in $L^2([0, L], dx)$. Because $\{s_n\}_{n=1,2,\ldots}$ is an orthonormal system (Exercise 3.16), this would in turn imply the claim, by Theorem 3.26. To show that $< s_n >_{n=1,2,\ldots}$ is dense in $\mathscr{D}((0, L))$ with respect to $|| \; ||_\infty$, fix $f \in \mathscr{D}((0, L))$ and extend it to F on $[-L, L]$ by imposing F be an odd map. By construction F is in $\mathscr{D}((-L, L))$ and satisfies $F(-L) = F(L)$, because it and its derivatives vanish around $x = 0$ and $x = \pm L$. A fortiori F is continuous and piecewise C^1 on $[-L, L]$. Applying Exercise 3.13 we conclude:

$$F(x) = \sum_{n\in\mathbb{N}} (F|e_n) \frac{e^{i\pi n x/L}}{\sqrt{2L}}$$

where now

$$e_n(x) := \frac{e^{i\pi n x/L}}{\sqrt{2L}},$$

and the series' convergence is in norm $|| \; ||_\infty$. Since F is odd:

$$F(x) = -F(-x) = -\sum_{n\in\mathbb{N}} (F|e_n) \frac{e^{-i\pi n x/L}}{\sqrt{2L}}$$

adding which to the previous expression of $F(x)$ gives:

$$2F(x) = \sum_{n\in\mathbb{N}} \frac{2i(F|e_n)}{\sqrt{2L}} \sin\left(\frac{\pi n x}{L}\right).$$

Restricting to $x \in [0, L]$:

$$f(x) = \sum_{n\in\mathbb{N}} \frac{i(F|e_n)}{\sqrt{2L}} \sin\left(\frac{\pi n x}{L}\right) .$$

Since the convergence is in norm $|| \; ||_\infty$, we have the claim.

3.19 Recall that the space $\mathscr{D}((0, L))$ of smooth maps with compact support in $(0, L)$ is dense in $L^2([0, L], dx)$ in the latter's topology. Using Exercise 3.13 prove that the functions $[0, L] \ni x \mapsto c_n(x) := \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right)$ are a basis of $L^2([0, L], dx)$.

Hint. Proceed as in Exercise 3.18, extending f to an even function on $[-L, L]$.

3.20 Let $C \subset \mathsf{H}$ be a closed subspace in the Hilbert space H. Prove C is weakly closed. Put otherwise, show that if $\{x_n\}_{n\in\mathbb{N}} \subset C$ converges weakly (cf. Exercise 3.11) to $x \in \mathsf{H}$, then $x \in C$.

Hint. If $P_C : \mathsf{H} \to C$ is the orthogonal projector onto C, show $P_C x_n \to P_C x$ weakly.

3.21 Let H be a Hilbert space and $T : D(T) \to \mathsf{H}$ a linear operator, where $D(T) \subset \mathsf{H}$ is a dense subspace in H ($D(T) = \mathsf{H}$, possibly). Prove that if $(u|Tu) = 0$ for any $u \in D(T)$ then $T = 0$, i.e. T is the null operator (sending everything to $\mathbf{0}$).

Solution. We have

$$0 = (u + v|T(u + v)) = (u|Tu) + (v|Tv) + (u|Tv) + (v|Tu) = (u|Tv) + (v|Tu)$$

and similarly

$$0 = i(u + iv|T(u + iv)) = i(u|Tu) + i(v|Tv) - (u|Tv) + (v|Tu) = -(u|Tv) + (v|Tu) .$$

Adding these two gives $(v|Tu) = 0$ for any $u, v \in D(T)$. Choose $\{v_n\}_{n\in\mathbb{N}} \subset D(T)$ such that $v_n \to Tu$, $n \to +\infty$. Then $||Tu||^2 = (Tu|Tu) = \lim_{n\to+\infty}(v_n|Tu) = 0$ for any $u \in D(T)$, i.e. $Tu = \mathbf{0}$ for any $u \in D(T)$, hence $T = 0$.

3.22 Consider $L^2([0, 1], m)$ where m is the Lebesgue measure, and take $f \in \mathscr{L}^2([0, 1], m)$. Let $T_f : L^2([0, 1], m) \ni g \mapsto f \cdot g$, where $\cdot$ is the standard pointwise product of functions. Prove T_f is well defined, bounded with norm $||T_f|| \leq ||f||$ and normal. Moreover, show T_f is self-adjoint iff f is real-valued up to a zero-measure set in $[0, 1]$.

3.23 Let $T \in \mathfrak{B}(\mathsf{H})$ be *self-adjoint*. Given $\lambda \in \mathbb{R}$, consider the series of operators

$$U(\lambda) := \sum_{n=0}^{\infty} (i\lambda)^n \frac{T^n}{n!} ,$$

where $T^0 := I$, $T^1 := T$, $T^2 := TT$ and so on, and the convergence is uniform. Prove the series converges to a unitary operator.

Hint. Proceed as when proving the properties of the exponential map using its definition as a series.

3.24 Referring to the previous exercise, show that $\lambda, \mu \in \mathbb{R}$ imply $U(\lambda)U(\mu) = U(\lambda + \mu)$.

3.25 Show that the series of Exercise 3.23 converges for any $\lambda \in \mathbb{C}$ to a bounded operator, and that $U(\lambda)$ is always normal.

3.26 Show that the operator $U(\lambda)$ of Exercise 3.23 is positive if $\lambda \in i\mathbb{R}$. Are there values $\lambda \in \mathbb{C}$ for which $U(\lambda)$ is a projector (not necessarily orthogonal)?

3.27 Compute explicitly $U(\lambda)$ in Exercise 3.23 if T is the operator T_f of Exercise 3.22 and $f = \overline{f}$.

3.28 In $\ell^2(\mathbb{N})$ consider the operator $T : \{x_n\} \mapsto \{x_{n+1}/n\}$. Prove T is bounded and compute T^*.

3.29 Consider the Volterra operator $T : L^2([0, 1], dx) \to L^2([0, 1], dx)$:

$$(Tf)(x) = \int_0^x f(t)dt \; .$$

Prove it is well defined, bounded and its adjoint satisfies:

$$(T^* f)(x) = \int_x^1 f(t)dt \quad \text{for any } f \in L^2([0, 1], dx).$$

Hint. Since $[0, 1]$ has finite Lebesgue measure, $L^2([0, 1], dx) \subset L^1([0, 1], dx)$. Then use Theorem 1.76.

3.30 Let $U : \mathsf{H} \to \mathsf{H}$ be a bounded operator over a Hilbert space H. Prove that if $(x|y) = 0$ implies $(Ux|Uy) = 0$ for $x, y \in \mathsf{H}$ and $U \neq 0$, then there is $a > 0$ such that $V := aU$ is isometric.

Solution. The first part of the hypothesis can be rephrased as $y \perp x$ implies $y \perp U^*Ux$. As a consequence $U^*Ux \in \{\{x\}^\perp\}^\perp$ which is the linear span of x. In other words, if $x \in \mathsf{H}$ then $U^*Ux = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$. Let us prove that λ_x does not depend on x. To this end, consider a couple of vectors $x \perp y$ with $x, y \neq \mathbf{0}$. Using the argument above we have $U^*Ux = \lambda_x x$, $U^*Uy = \lambda_y y$, $U^*U(x+y) = \lambda_{x+y}(x+y)$. Linearity of U^*U applied to the last identity leads to $U^*Ux + U^*Uy = \lambda_{x+y}x + \lambda_{x+y}y$, namely $U^*Ux - \lambda_{x+y}x = -(U^*Uy - \lambda_{x+y}y)$. Exploiting $U^*Ux = \lambda_x x$ and $U^*Uy = \lambda_y y$ we get $(\lambda_x - \lambda_{x+y})x = -(\lambda_y - \lambda_{x+y})y$. Since $x \perp y$ and $x, y \neq \{\mathbf{0}\}$, the only possibility is that $\lambda_x = \lambda_{x+y} = \lambda_y$. In summary, a couple of orthogonal non-vanishing vectors x, y satisfies $\lambda_x = \lambda_y$. Next consider a Hilbert basis $\{x_n\}_{n\in J} \subset \mathsf{H}$ so that, if $z \in \mathsf{H}$, $z = \sum_{n\in J} c_n x_n$. for complex numbers c_n. Since U^*U is bounded,

$$U^*Uz = U^*U\sum_{n\in J} c_n x_n = \sum_{n\in J} c_n U^*Ux_n = \sum_{n} c_n \lambda_{x_n} x_n$$

From the previous argument that $\lambda_{x_n} = \lambda_{x_m}$ so that, indicating with c the common value of the λ_{x_n}, we have

$$U^*Uz = \sum_{n\in J} c_n c x_n = c\sum_{n\in J} c_n x_n = cz \ .$$

Since $z \in \mathsf{H}$ is arbitrary, we have found that

$$U^*U = cI \ .$$

Finally, if $x \in \mathsf{H}$, it must hold $0 \leq (Ux|Ux) = (x|U^*Ux) = c(x|x)$ so that $c \geq 0$ and $c \neq 0$ because $U \neq 0$. $V := c^{-1/2}U$ satisfies $V^*V = I$ and therefore is isometric.

3.31 Let $\mathfrak{A}$ be a C^*-algebra without unit. Consider the direct sum $\mathfrak{A} \oplus \mathbb{C}$ and define the product:

$$(x, c) \cdot (y, c') := (x \circ y + cy + c'x, cc'), \quad (x', c'), (x, c) \in \mathfrak{A} \oplus \mathbb{C} \ ,$$

where $\circ$ is the product on $\mathfrak{A}$. Define the norm:

$$||(x, c)|| := \sup\{||cy + xy|| \mid y \in \mathfrak{A}, ||y|| = 1\}$$

and the involution: $(x, c)^* = (x^*, \overline{c})$, where $\overline{c}$ is the complex conjugate of c and the involution on the right is the one of $\mathfrak{A}$. Prove that the vector space $\mathfrak{A} \oplus \mathbb{C}$ with the above structure is a C^*-algebra with unit $(0, 1)$.

Hint. The triangle inequality is easy. The proof that $||(x, c)|| = 0$ implies $c = 0$ and $x = 0$ goes as follows. If $c = 0$, $||(x, 0)|| = 0$ means $||x|| = 0$, so $x = 0$. If $c \neq 0$, we can simply look at $c = 1$. In that case $||y + xy|| \leq ||y|| \, ||(x, 1)||$, so $||(x, 1)|| = 0$ implies $y = xy$ for any $y \in \mathfrak{A}$. Using the involution we have $y = yx^*$ for any $y \in \mathfrak{B}$. In particular $x^* = xx^* = x$, and then $y = xy = yx$ for any $y \in \mathfrak{A}$. Therefore x is the unit of $\mathfrak{A}$, a contradiction. Hence $c = 0$ is the only possibility, and we fall back to the previous case. Let us see to the C^* properties of the norm. By definition of norm: $||(c, x)||^2 = \sup\{||cy + xy||^2 \mid y \in \mathfrak{A}, ||y|| = 1\} = \sup\{||y^*(\overline{c}cy + \overline{c}xy + cx^*y + x^*xy)|| \mid y \in \mathfrak{A}, ||y|| = 1\}$. Hence $||(c, x)||^2 \leq ||(c, x)^*(c, x)|| \leq ||(c, x)^*|| \, ||(c, x)||$. In particular $||(c, x)|| \leq ||(c, x)^*||$, and replacing (c, x) with $(c, x)^*$ gives $||(c, x)^*|| = ||(c, x)||$. The inequality $||(c, x)||^2 \leq ||(c, x)^*(c, x)|| \leq ||(c, x)^*|| \, ||(c, x)||$ implies $||(c, x)||^2 \leq ||(c, x)^*(c, x)|| \leq ||(c, x)||^2$, and so $||(c, x)||^2 = ||(c, x)^*(c, x)||$.

3.32 Prove Proposition 3.55:

Proposition. *Let* $\mathfrak{A}$ *be a* **-algebra with unit,* H *a Hilbert space, and consider a linear map* $\phi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ *which preserves the products and the involutions. The following facts hold.*

(a) $\mathsf{H}_\phi := Ran(\phi(\mathbb{I}))$ *and* $\mathsf{H}_\phi^\perp$ *are closed subspaces of* H *satisfying* $\mathsf{H} = \mathsf{H}_\phi \oplus \mathsf{H}_\phi^\perp$, *and each subspace is invariant under* $\phi(a)$, *for every* $a \in \mathfrak{A}$.

(b) $\phi(a)\restriction_{\mathsf{H}_\phi^\perp} = 0$ *for every* $a \in \mathfrak{A}$.

(c) *The restriction to the complement*

$$\pi_\phi : \mathfrak{A} \ni a \mapsto \phi(a)\restriction_{\mathsf{H}_\phi} \in \mathfrak{B}(\mathsf{H}_\phi)$$

is a representation of $\mathfrak{A}$ over H_ϕ *according to Definition* 3.52. *It also satisfies*

(i) π_ϕ *is faithful* $\Leftrightarrow$ ϕ *is injective*;

(ii) $\pi_\phi(a)$ *is the zero representation* $\Leftrightarrow$ $\phi(\mathbb{I}) = 0$ *(in this case* $\mathsf{H}_\phi = \{\mathbf{0}\}$*)*;

(iii) $\pi_\phi = \phi$ *if* ϕ *is surjective*;

(iv) $\pi_\phi = \phi$ *if* ϕ *is not the zero map and irreducible (i.e. there are no closed subspaces* M *of* H *with* $\{\mathbf{0}\} \subsetneq \mathsf{M} \subsetneq \mathsf{H}$ *and such that* $\phi(a)(\mathsf{M}) \subset \mathsf{M}$ *for every* $a \in \mathfrak{A}$*)*.

Solution. (a) First of all we notice that $P := \phi(\mathbb{I})$ is an orthogonal projector because $PP = \phi(\mathbb{I})\phi(\mathbb{I}) = \phi(\mathbb{I}\mathbb{I}) = \phi(\mathbb{I}) = P$ and $P^* = \phi(\mathbb{I})^* = \phi(\mathbb{I}^*) = \phi(\mathbb{I}) = \phi$, where we have used Proposition 3.44(c). Therefore $\mathsf{H}_\phi := P(\mathsf{H})$ is a closed subspace by Proposition 2.101(b). Moreover $I - P$ projects orthogonally onto $\mathsf{H}_\phi^\perp$, and $\mathsf{H} = \mathsf{H}_\phi \oplus \mathsf{H}_\phi^\perp$ by Proposition 3.64(a, b). Now observe that $P\phi(a) = \phi(\mathbb{I})\phi(a) = \phi(\mathbb{I}a) = \phi(a) = \phi(a\mathbb{I}) = \phi(a)P$, which entails that H_ϕ is invariant under every $\phi(a)$. The same fact holds for $\mathsf{H}_\phi^\perp$ when we replace P by $I - P$ in the previous argument. (b) If $x \in \mathsf{H}_\phi^\perp$, $\phi(a)x = \phi(a)(I - P)x = \phi(a)x - \phi(a\mathbb{I})x = \phi(a)x - \phi(a)x = \mathbf{0}$. (c) Since P restricts to the identity map I on H_ϕ, we have $\phi(a)\restriction_{\mathsf{H}_\phi}(\mathbb{I}) = I$. Then the restriction $\mathfrak{A} \ni a \mapsto \pi(a) := \phi(a)\restriction_{\mathsf{H}_\phi}$ is a representation on the Hilbert space H_ϕ in the sense of Definition 3.52, because it is linear and it preserves the product and the involution, as the reader immediately proves using the same properties of ϕ. (i) It is obvious that $\pi_\phi(a) = 0 \Leftrightarrow \phi(a) = 0$, since $\phi(a)\restriction_{\mathsf{H}_\phi^\perp} = 0$ and this is the same as saying that π_ϕ is injective (faithful) if and only if ϕ is injective. (ii) If π_ϕ is the zero representation then $\pi_\phi(\mathbb{I}) = 0$ in particular, and hence $\phi(\mathbb{I}) = 0$ as well, by construction. If $\phi(\mathbb{I}) = 0$ then $\pi_\phi(\mathbb{I}) = 0$ and so $\pi_\phi(a) = \pi_\phi(\mathbb{I}a) = \pi_\phi(\mathbb{I})\pi_\phi(a) = 0\pi_\phi(a) = 0$ for every $a \in \mathfrak{A}$. (If $\phi(\mathbb{I}) = 0$, we also have $\mathsf{H}_\phi = P(\mathsf{H}) = \{\mathbf{0}\}$ since $P = \phi(\mathbb{I})$.) (iii) If ϕ is surjective, there exists $a_1 \in \mathfrak{A}$ with $\phi(a_1) = I$ so $I = \phi(a_1) = \phi(\mathbb{I}a_1) = \phi(\mathbb{I})\phi(a_1) = \phi(\mathbb{I})I = \phi(\mathbb{I}) = P$. Therefore $\mathsf{H}_\phi = \mathsf{H}$, which implies $\phi = \pi_\phi$. (iv) $\mathsf{M} := P(\mathsf{H})$ satisfies $\phi(a)(\mathsf{M}) \subset \mathsf{M}$ for every $a \in \mathfrak{A}$, so either $\mathsf{M} = \mathsf{H}$, and thus $\phi = \pi_\phi$ as we wanted, or $\mathsf{M} = \{\mathbf{0}\}$. The latter is forbidden by hypothesis, because it would imply $\phi(a) = 0$ for every $a \in \mathfrak{A}$.

3.33 If $\{\mathfrak{R}_\alpha\}_{\alpha \in A}$ is a family of von Neumann algebras on a Hilbert space H, prove that

$$\left(\bigvee_{\alpha \in A} \mathfrak{R}_\alpha\right)' = \bigwedge_{\alpha \in A} \mathfrak{R}'_\alpha \quad \text{and} \quad \left(\bigwedge_{\alpha \in A} \mathfrak{R}_\alpha\right)' = \bigvee_{\alpha \in A} \mathfrak{R}'_\alpha \,.$$

3.34 Prove that a von Neumann algebra $\mathfrak{R}$ on a Hilbert space H is complete in the strong operator topology. In other words, given $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{R}$ such that $\{A_n x\}_{n\in\mathbb{N}}$ is Cauchy in H for every fixed $x \in \mathsf{H}$, there exists $A \in \mathfrak{R}$ such that $A_n \to A$ strongly (the converse being trivially true).

Solution. As $\mathfrak{R}$ is strongly closed in $\mathfrak{B}(\mathsf{H})$, it is enough to prove that $\mathfrak{B}(\mathsf{H})$ is strongly complete. Fix $x \in \mathsf{H}$. As $\{A_n x\}_{n\in\mathbb{N}}$ is Cauchy, $Ax := \lim_{n\to+\infty} A_n x$ does exist. Linearity of each A_n implies linearity of $A : \mathsf{H} \ni x \mapsto Ax$. There remains to prove that A is bounded. As $A_n x$ admits limit, we have $||A_n x|| < +\infty$ for every $x \in \mathsf{H}$ and every $n \in \mathbb{N}$. The Banach–Steinhaus Theorem 2.62 implies that $||A_n|| < S < +\infty$ for every $n \in \mathbb{N}$. Moreover, if $x \in \mathsf{H}$, for every $\varepsilon_x > 0$ there is $n_{\varepsilon_x} \in \mathbb{N}$ such that $||(A - A_n)x|| < \varepsilon_x$ if $n > n_{\varepsilon_x}$. Summing up,

$$||Ax|| \leq ||Ax - A_n x|| + ||A_n x|| < \varepsilon_x + S||x|| \quad \text{for } n > n_{\varepsilon_x}.$$

Since $\varepsilon_x > 0$ is arbitrary,

$$||Ax|| \leq S||x|| \quad \forall x \in \mathsf{H} \,,$$

which entails $||A|| \leq S < +\infty$ and therefore $A \in \mathfrak{B}(\mathsf{H})$.

3.35 Prove that a von Neumann algebra $\mathfrak{R}$ on a Hilbert space H is complete in the weak operator topology. In other words, if $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{R}$ is such that $\{(y|A_n x)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ for every fixed $x, y \in \mathsf{H}$, then there exists $A \in \mathfrak{R}$ such that $A_n \to A$ weakly (the converse being trivially true).

Solution. As $\mathfrak{R}$ is weakly closed in $\mathfrak{B}(\mathsf{H})$, it suffices to prove that $\mathfrak{B}(\mathsf{H})$ is weakly complete. Fix $x, y \in \mathsf{H}$. As $\{(y|A_n x)\}_{n\in\mathbb{N}}$ is Cauchy, $A_{yx} := \lim_{n\to+\infty}(y|A_n x) \in \mathbb{C}$ exists. Each A_n is linear and the inner product is Hermitian, so $\mathsf{H} \ni x \mapsto A_{yx}$ is linear and $\mathsf{H} \ni y \mapsto A_{yx}$ antilinear. We claim there exist a linear operator $A : \mathsf{H} \to \mathsf{H}$ such that $A_{yx} = (y|Ax)$ for every $x, y \in \mathsf{H}$. For $x, y \in \mathsf{H}$ fixed, $|(y|A_n x)| < +\infty$ for every $n \in \mathbb{N}$. Therefore, for fixed x, applying the Banach–Steinhaus Theorem 2.62 to the family of linear functionals $\mathsf{H} \ni y \mapsto \overline{(y|A_n x)}$, we conclude that there is a positive number $S_x < +\infty$ satisfying $||\overline{(\cdot|A_n x)}|| < S_x$ uniformly in n. Moreover, if $y \in \mathsf{H}$, for every $\varepsilon_y > 0$ there is $n_{\varepsilon_y} \in \mathbb{N}$ such that $|\overline{A_{yx}} - \overline{(y|A_n x)}| < \varepsilon_y$ if $n > n_{\varepsilon_y}$. Summing up,

$$|\overline{A_{yx}}| \leq |\overline{A_{yx}} - \overline{(y|A_n x)}| + |\overline{(y|A_n x)}| < \varepsilon_y + S_x||y|| \quad \text{for } n > n_{\varepsilon_y}.$$

Since $\varepsilon_y > 0$ is arbitrary,

$$|\overline{A_{yx}}| \leq S_x||y|| \quad \forall y \in \mathsf{H} \,,$$

which entails $||\overline{A_{\cdot x}}|| \leq S_x < +\infty$. Therefore Riesz's lemma proves that there exists $A_x \in \mathsf{H}$ with

$$A_{yx} = (y|A_x) \quad \forall y \in \mathsf{H}\,.$$

It easy to prove that $\mathsf{H} \ni x \mapsto A_x =: Ax$ is linear. So now we have a linear operator $A : \mathsf{H} \to \mathsf{H}$ such that $(y|A_n x) \to (y|Ax)$ for every $x, y \in \mathsf{H}$. To finish it suffices to prove that A is bounded. Since $(A_n^* y|x) = (y|A_n x)$ we know the sequence $\{(A_n^* y|x)\}_{n\in\mathbb{N}}$ is Cauchy for every choice of $x, y \in \mathsf{H}$. Repeating the procedure above, there exists an operator $B : \mathsf{H} \to \mathsf{H}$ such that $(A_n^* y|x) \to (By|x)$ for every $x, y \in \mathsf{H}$, so that $(y|Ax) = (By|x)$ for every $x, y \in \mathsf{H}$. This identity easily implies that B is closed (Definition 2.98). Applying the closed graph Theorem 2.99, we conclude that $B \in \mathfrak{B}(\mathsf{H})$. The identity $(y|Ax) = (By|x)$, $x, y \in \mathsf{H}$, rewritten as $(Ax|y) = (x|By) = (B^*x|y)$, implies $A = B^* \in \mathfrak{B}(\mathsf{H})$.

3.36 Let H, K be complex Hilbert spaces, $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ and $\mathfrak{B} \subset \mathfrak{B}(\mathsf{K})$ unital C^*-algebras of operators, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a continuous linear map relatively to the strong operator topology (or weak operator topology) of both $\mathfrak{A}$ and $\mathfrak{B}$. Prove that ϕ extends uniquely to a continuous linear map $\Phi : \mathfrak{A}'' \to \mathfrak{B}''$ relatively to the strong operator topology (resp. weak operator topology) both on $\mathfrak{A}''$ and $\mathfrak{B}''$.

Solution. Let us treat the case of strong topologies. Since $\mathfrak{A}$ is strongly dense in $\mathfrak{A}''$, any strongly continuous extension Φ must be unique. Let us prove that Φ exists by using Exercise 3.34. Since $\mathfrak{A}$ is dense in $\mathfrak{A}''$, if $A \in \mathfrak{A}''$ there is a sequence $\{A_n\}_{n\in\mathbb{N}} \in \mathfrak{A}$ converging to A strongly. Define $\Phi(A) := s\text{-}\lim_{n\to+\infty} \phi(A_n)$. First of all we prove that $\{\phi(A_n)\}_{n\in\mathbb{N}} \subset \mathfrak{B}''$ is Cauchy (with respect to the strong operator topology) and hence its limit belongs to $\mathfrak{B}''$, which is complete in that topology as established in Exercise 3.34. Fix a seminorm $p_y(\cdot) := ||\cdot y||$ in $\mathfrak{B}(\mathsf{K})$ associated to the vector $y \in \mathsf{K}$. Since ϕ is strongly continuous, for every $\varepsilon > 0$ there is an open set $O_{y,\varepsilon}$ containing $0 \in \mathfrak{B}(\mathsf{H})$, given by intersecting a finite number of balls defined by seminorms $p_{x_1^{(y)}}, \ldots p_{x_{n_y}^{(y)}}$. The vectors $x_k^{(y)}$ and the number n_y also depend on ε, with $p_y(\phi(A)) < \varepsilon$ if $A \in O_{y,\varepsilon} \cap \mathfrak{A}$. In other words $||\phi(A)y|| < \varepsilon$ if $\sum_{k=1}^{n_y} ||Ax_k^{(y)}|| < \delta$ for some $\delta > 0$ depending on y and ε. If A does not satisfy the last condition, $A' := \frac{\delta A}{\sum_{k=1}^{n_y} ||Ax_k^{(y)}||}$ certainly will. Therefore, for every $y \in \mathsf{K}$ there is a constant $C_y \geq 0$ $(= \varepsilon/\delta)$ and there are vectors $x_1^{(y)}, \ldots, x_{n_y}^{(y)} \in \mathsf{H}$, such that

$$||\phi(A)y|| \leq C_y \sum_{k=1}^{n_y} ||Ax_k^{(y)}|| \quad \forall A \in \mathfrak{A}\,.$$

This inequality implies that the sequence $\{\phi(A_n)\}_{n\in\mathbb{N}}$ is Cauchy in the strong operator topology if $\{A_n\}_{n\in\mathbb{N}}$ is. We conclude that $\Phi(A) := s\text{-}\lim_{n\to+\infty} \phi(A_n)$ is well defined. Notice that a different sequence $\{A'_n\}_{n\in\mathbb{N}}$ tending to A (strongly) would produce the same limit, because

$$||\phi(A_n)y - \phi(A'_n)y|| \leq C_y \sum_{k=1}^{n_y} ||(A_n - A'_n)x_k^{(y)}||$$

and $||(A_n - A'_n)x_k^{(y)}|| \to 0$ (both sequences converge to A strongly). It is therefore clear that Φ extends ϕ, because for any $A \in \mathfrak{A}$ we may define $\Phi(A) := s\text{-}\lim_{n\to+\infty} \phi(A_n)$, using a constant sequence $A_n = A$. The limiting process used to define Φ immediately proves that Φ is linear as well, since ϕ is. Regarding the strong continuity of Φ, observe that, by construction

$$||\Phi(A)y|| = \left|\left|\lim_{n\to+\infty} \phi(A_n)y\right|\right| = \lim_{n\to+\infty} ||\phi(A_n)y|| \,.$$

Since $A_n x_k^{(y)} \to Ax_k^{(y)}$ and

$$||\phi(A_n)y|| \leq C_y \sum_{k=1}^{n_y} ||A_n x_k^{(y)}|| \,,$$

the limit above produces

$$||\Phi(A)y|| \leq C_y \sum_{k=1}^{n_y} ||Ax_k^{(y)}|| \,, \quad \text{if } A \in \mathfrak{A}'',$$

valid for every $y \in \mathsf{K}$, for the corresponding $C_y \geq 0$ and for $x_1^{(y)}, \ldots, x_{n_y}^{(y)} \in \mathsf{H}$. This inequality is equivalent to the continuity of Φ in the strong topology of the two von Neumann algebras $\mathfrak{A}''$ and $\mathfrak{B}''$.

The case of weak topologies is completely analogous if we replace the seminorms $p_x(\cdot) = ||\cdot x||$ with the seminorms $p_{x,y}(\cdot) = |(x|\cdot y)|$. Then Φ is the unique map satisfying $(x|\Phi(A)y) = \lim_{n\to+\infty}(x|\phi(A_n)y)$ for $x, y \in \mathsf{K}$, $A \in \mathfrak{A}''$ and $\mathfrak{A} \ni A_n \to A$ weakly.

3.37 Let H, K be complex Hilbert spaces, $\mathfrak{A} \subset \mathfrak{B}(\mathsf{H})$ and $\mathfrak{B} \subset \mathfrak{B}(\mathsf{K})$ unital C^*-algebras of operators. Prove that a continuous *-homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ (relatively to the weak operator topology of $\mathfrak{A}$ and $\mathfrak{B}$) extends to a unique continuous *-homomorphism of von Neumann algebras $\Phi : \mathfrak{A}'' \to \mathfrak{B}''$ (in the weak topology).

Solution. As ϕ is linear it defines a unique weakly continuous linear extension $\Phi : \mathfrak{A}'' \to \mathfrak{B}''$, as established in Exercise 3.36. We shall exploit the fact that $\mathfrak{A}$ is weakly dense in $\mathfrak{A}''$, by the double commutant theorem. The claim is that $\Phi(I_\mathsf{H}) = I_\mathsf{K}$, $\Phi(A^{*_\mathsf{H}}) = \Phi(A)^{*_\mathsf{K}}$ and $\Phi(A \circ_\mathsf{H} B) = \Phi(A) \circ_\mathsf{K} \Phi(B)$ for $A, B \in \mathfrak{A}''$. (We henceforth omit the subscripts $_{\mathsf{H},\mathsf{K}}$ for simplicity.) The first requirement automatically holds because it is valid for ϕ. Let us prove $\Phi(A^*) = \Phi(A)^*$. If $\mathfrak{A} \ni A_n \to A$ weakly, then $\mathfrak{A} \ni A_n^* \to A^*$ weakly. From now on all limits will be meant in the weak sense. We have

$$\Phi(A^*) = \lim_{n\to+\infty} \phi(A_n^*) = \lim_{n\to+\infty} \phi(A_n)^* = \Phi(A)^*$$

because Φ is weakly continuous and extends the *-homomorphism ϕ. To conclude, we need to establish $\Phi(AB) = \Phi(A)\Phi(B)$ for $A, B \in \mathfrak{A}''$. Let us start by assuming $A \in \mathfrak{A}''$ but $B \in \mathfrak{A}$. Then $\mathfrak{A} \ni A_n \to A$ weakly implies $\mathfrak{A} \ni A_n B \to AB$ weakly. Therefore, by the very definition of Φ:

$$\Phi(AB) = \lim_{n\to+\infty} \Phi(A_n B) = \lim_{n\to+\infty} \phi(A_n B) = \lim_{n\to+\infty} \phi(A_n)\phi(B)$$

$$= \lim_{n\to+\infty} \phi(A_n)\Phi(B) = \Phi(A)\Phi(B)\,,$$

where we have used the fact that $\phi(A_n) \to \Phi(A)$ weakly implies $\phi(A_n)\Phi(B) \to \Phi(A)\Phi(B)$ weakly. (With a similar proof, $\Phi(AB) = \Phi(A)\Phi(B)$ holds true also if $A \in \mathfrak{A}$ but $B \in \mathfrak{A}''$.) Let us finally focus on the general case $A, B \in \mathfrak{A}''$. Since Φ is weakly continuous, and $\mathfrak{A} \ni B_n \to B \in \mathfrak{A}''$ weakly implies $AB_n \to AB$, we have:

$$\Phi(AB) = \lim_{n\to+\infty} \Phi(AB_n) = \lim_{n\to+\infty} \Phi(A)\Phi(B_n) = \Phi(A)\Phi(B)\,.$$

3.38 Let H be a complex Hilbert space and suppose $h : \mathfrak{B}(\mathsf{H}) \to \mathfrak{B}(\mathsf{H})$ is an antilinear map preserving the product, the operation *, and fixing the identity: $h(I) = I$. Prove that if h is bijective, then there exists an isometric and surjective antilinear map $U : \mathsf{H} \to \mathsf{H}$ (called an *anti-unitary operator*) such that $h(A) = UAU^{-1}$.

Hint. If $N \subset \mathsf{H}$ is a Hilbert basis, define the map $C : \mathsf{H} \to \mathsf{H}$ by

$$C : \sum_{x\in N} (x|z)x \mapsto \sum_{x\in N} \overline{(x|z)}x$$

for every $z \in \mathsf{H}$. Next, set $\alpha(A) := Ch(A)C^{-1}$ for every $A \in \mathfrak{B}(\mathsf{H})$, so that the map α becomes a *-automorphism of $\mathfrak{B}(\mathsf{H})$. Finally, exploit Theorem 3.96.

Chapter 4
Families of Compact Operators on Hilbert Spaces and Fundamental Properties

Measure what can be measured, and make measurable what can't be.

Galilei

The aim of this chapter, from the point of view of quantum mechanical applications, is to introduce certain types of operators used to define *quantum states* in the standard formulation of the theory. These operators, known in the literature as *operators of trace class*, or *nuclear operators*, are bounded operators on a Hilbert space that admit a trace. In order to introduce them it is necessary to define first *compact operators*, also known as *completely continuous* operators, which play an important role in several branches of mathematics and physical applications irrespective of quantum theories.

The first section introduces *compact operators* on normed spaces, then briefly discusses general properties in normed and Banach spaces. We will prove the classical result on the non-compactness of the infinite-dimensional unit ball.

In section two we specialise to Hilbert spaces, with an eye to L^2 spaces where compact operators (such as Hilbert–Schmidt operators) admit an integral representation. We will show that compact operators determine a closed *-ideal in the C^*-algebra of bounded operators on a Hilbert space, hence, a fortiori, a C^*-subalgebra. We will prove Hilbert's celebrated theorem on the spectral expansion of compact operators, to be considered as a precursor of all spectral decomposition results of subsequent chapters.

The *-ideal of *Hilbert–Schmidt operators* and their elementary properties are the subject of section four. We will show that Hilbert–Schmidt operators form a Hilbert space.

The penultimate section is concerned with the *-ideal of operators of trace class, and the basic (and most useful in physics) properties. In particular we shall prove

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that *the trace of a product of operators is invariant under cyclic permutations of the factors*.

The final section is devoted to a short introduction to *Fredholm's alternative theorem* for Fredholm's integral equations of the second kind.

4.1 Compact Operators on Normed and Banach Spaces

This section deals with compact operators on normed spaces. We start by recalling general results about compact subsets in normed spaces, especially infinite-dimensional ones. In the next section we shall discuss the theory in Hilbert spaces.

4.1.1 Compact Sets in (Infinite-Dimensional) Normed Spaces

Compactness in a completely general topological space X was addressed in Definition 1.19, which we recall below.

Definition Let $(\mathsf{X}, \mathscr{T})$ be a topological space and $K \subset \mathsf{X}$ a subset. One says that
(a) K is **compact** if any open covering of it admits a finite sub-covering: if $\{A_i\}_{i\in I} \subset \mathscr{T}$, $\cup_{i\in I} A_i \supset K$, em then $\cup_{i\in J} A_i \supset K$ for some finite subset $J \subset I$;
(b) K *is* **relatively compact** if $\overline{K}$ is compact;
(c) X is **locally compact** if every point admits a relatively compact open neighbourhood.

Related to this is the notion of *sequential compactness*.

Definition 4.1 A subset K in a topological space is **sequentially compact** if any sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ has a subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ that converges in K.

Remark 4.2 Let us list below a few general features of compact sets that should be known from basic topology courses (Ser94II). We shall make use of them later.
(1) Compactness is hereditary, in the sense that it is passed on to induced topologies (cf. Remark 1.22(2)).
(2) Closed subsets in compact sets are compact, and in Hausdorff spaces (like normed vector spaces or Hilbert spaces), compact sets are closed.
(3) In metrisable spaces (in particular normed vector spaces, Hilbert spaces), compactness is *equivalent* to sequential compactness. ■

The next, useful, property is also valid in metric spaces.

Proposition 4.3 *Let* $(\mathsf{X}, ||\ ||)$ *be a normed space and* $A \subset \mathsf{X}$. *If any sequence in* A *admits a converging subsequence (not in* A *necessarily), then* A *is relatively compact.*

Proof The only thing to prove is that $\{y_k\}_{k\in\mathbb{N}} \subset \overline{A}$ admits a subsequence that converges (in $\overline{A}$, $\overline{A}$ being closed). Given $\{y_k\}_{k\in\mathbb{N}} \subset \overline{A}$, there will be sequences $\{x_n^{(k)}\}_{n\in\mathbb{N}} \subset A$, one for each k, with $x_n^{(k)} \to y_k$ as $n \to +\infty$. Fix k and pick a number n_k large enough to construct, term by term, a new sequence $\{z_k := x_{n_k}^{(k)}\}_{k\in\mathbb{N}} \subset A$ such that $||y_k - z_k|| < 1/k$. Under the assumptions made on A there will be a subsequence $\{z_{k_p}\}_{p\in\mathbb{N}}$ of $\{z_k\}_{k\in\mathbb{N}}$ converging to some $y \in \overline{A}$. Then

$$||y_{k_p} - y|| \leq ||y_{k_p} - z_{k_p}|| + ||z_{k_p} - y|| \, .$$

Since $1/k_p \to 0$ when $p \to +\infty$, for any given $\varepsilon > 0$ there will be P such that, if $p > P$, $||z_{k_p} - y|| < \varepsilon/2$ and $1/k_p < \varepsilon/2$, so $||y_{k_p} - y|| < \varepsilon$. In other words $y_{k_p} \to y$ as $p \to +\infty$. □

Remark 4.4 This proposition holds on metric spaces too, and the proof is the same with minor modifications. ■

Examples of compact sets in an infinite-dimensional normed space $(\mathsf{X}, ||\ ||)$ are easily obtained from finite-dimensional subspaces. As we know from Sect. 2.5, any finite-dimensional space S is homeomorphic to $\mathbb{C}^n$ (or $\mathbb{R}^n$ for real vector spaces). Therefore any closed and bounded set $K \subset \mathsf{S}$ (e.g., the closure of an open ball of finite radius) is compact in S by the Heine–Borel theorem. Since compactness is a hereditary property (cf. Remark 1.22(2)) K is compact also in the topology of $(\mathsf{X}, ||\ ||)$.

The following is an important result, that discriminates between finite- and infinite-dimensional normed spaces. We leave to the reader the proof of the fact that the closure of an open ball in a normed space is nothing but the corresponding closed ball with the same centre and radius:

$$\overline{\{x \in \mathsf{X} \mid ||x - x_0|| < r\}} = \{x \in \mathsf{X} \mid ||x - x_0|| \leq r\} \, .$$

Proposition 4.5 *Let $(\mathsf{X}, ||\ ||)$ be a normed space of infinite dimension. The closure of the open unit ball $\{x \in \mathsf{X} \mid ||x|| < 1\}$ (that is, the closed unit ball $\{x \in \mathsf{X} \mid ||x|| \leq 1\}$) cannot be compact.*
The same is true for any ball, with arbitrary centre and positive, finite radius.

Proof Let us indicate by B the open unit ball centred at the origin, and suppose by contradiction that $\overline{B}$ is compact. Then we can cover $\overline{B}$, hence B, with $N > 0$ open balls B_k of radius $1/2$ centred at certain points x_k, $k = 1, \ldots, N$. Consider a subspace X_n in X, of finite dimension n, containing the vectors x_k. Since $\dim \mathsf{X}$ is infinite, we may choose $n > N$ as large as we want. Define further "balls" $P := B \cap \mathsf{X}_n$ of radius 1 and $P_k := B_k \cap X_n$, $k = 1, \ldots, N$, all of radius $1/2$. Let us identify X_n with $\mathbb{R}^{2n}$ (or $\mathbb{R}^n$ if the base field is $\mathbb{R}$) by choosing a basis of X_n, say $\{z_k\}_{k=1,\ldots,n}$. Notice that a "ball" P_k does not necessarily have the round shape of a Euclidean ball. If we normalise the Lebesgue measure m on $\mathbb{R}^{2n}$ by dividing by the volume of P (non-zero since P is open and non-empty by Proposition 2.107), then $m(P) = 1$. We claim $m(P_k) = (1/2)^n$. The Lebesgue measure is translation-invariant, so we may only consider balls $B(r)$

centred at the origin of radius r. Since every norm is a homogeneous function, $B(\lambda r) = \{\lambda u \mid u \in B(r)\} =: \lambda B(r)$ for any $\lambda > 0$. The Lebesgue measure on $\mathbb{R}^{2n}$ satisfies $m(\lambda E) = \lambda^{2n} m(E)$, hence $m(P_k) = m((1/2)P) = (1/2)^n m(P) = (1/2)^n$. Eventually, as $B \subset \cup_{k=1}^N B_k$ and $P \subset \cup_{k=1}^N P_k$, we have $m(P) \leq \sum_{k=1}^N m(P_k)$ by subadditivity, i.e. $1 \leq N(1/2)^{2n}$. But this is impossible if n is large enough (N is fixed). The argument is completely analogous for open balls of finite radius centred at any other point in the space. □

The next result explains, once more, how compact sets acquire 'counter-intuitive' properties when passing to infinitely many dimensions. In the standard spaces $\mathbb{C}^n$ or $\mathbb{R}^n$ there exist compact sets with non-empty interior: it is enough to close a bounded open set, for the Heine–Borel theorem warrants that the closure (still bounded) is compact and clearly has points in its interior.

The complete space $\mathbb{C}$ can be viewed as the countable union of compact subsets: take all open discs with rational centres and rational radii. In the infinite-dimensional case, instead, the picture changes dramatically.

Corollary 4.6 *Let* X *be a normed space of infinite dimension.*
(a) *If* $K \subset \mathsf{X}$ *is compact, the interior of* K *is empty.*
(b) *If* X *is also complete (i.e. a Banach space),* X *cannot be obtained as a countable union of compact subsets.*

Proof (a) If the interior of K were not empty, it would contain an open ball B, since open balls form a basis for the topology. Compact subsets are closed because normed spaces are Hausdorff, so $\overline{B} \subset \overline{K} = K$, and closed subsets in compact sets are compact, hence $\overline{B}$ would be compact, contradicting the previous proposition.
(b) The claim follows from (a) and the last statement in Baire's Theorem 2.93, where X is our complete normed space. □

4.1.2 Compact Operators on Normed Spaces

We are ready to introduce *compact operators*.

Recall that a subset M in a normed space $(\mathsf{X}, || \, ||)$ is **bounded** (relative to $|| \, ||$) if there is an open ball $B_\delta(x_0)$, of finite radius $\delta > 0$ and centred at some $x_0 \in \mathsf{X}$, such that $M \subset B_\delta(x_0)$.

Clearly, M is bounded if and only if there is a metric ball of finite radius $\delta > 0$ and *centred at the origin* of X, containing M (just choose $\delta + ||x_0||$ as radius).

Definition 4.7 Let X, Y be normed spaces over $\mathbb{R}$ or $\mathbb{C}$. The operator $T \in \mathfrak{L}(\mathsf{X}, \mathsf{Y})$ is called **compact** (or **completely continuous**) when either of the following equivalent conditions holds:

(a) for any bounded subset $M \subset \mathsf{X}$, $T(M)$ is relatively compact in Y;

(b) if $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ is bounded, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ converges in Y.

The set of compact operators from X to Y is indicated by $\mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$, and $\mathfrak{B}_\infty(\mathsf{X}) := \mathfrak{B}_\infty(\mathsf{X}, \mathsf{X})$ is the set of compact operators on X.

Remark 4.8 **(1)** Clearly (a) $\Rightarrow$ (b). The opposite implication (b) $\Rightarrow$ (a) is an immediate consequence of Proposition 4.3.
(2) Any compact operator is certainly bounded, for it maps the unit closed ball centred at the origin to a set (containing the origin) with compact closure K. The latter can be covered by N open balls of radius $r > 0$, say $B_r(y_i)$. Then $K \subset \cup_{i=1}^N B_r(y_i) \subset B_{R+r}(\mathbf{0})$, where R is the largest distance between the centres y_i and the origin. In particular $||T(x)|| \leq (R + r)$ for $||x|| = 1$, so $||T|| \leq r + R < +\infty$. ■

The sets $\mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$ and $\mathfrak{B}_\infty(\mathsf{X})$ are actually vector spaces under the usual linear combinations of operators, hence subspaces of $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$ and $\mathfrak{B}(\mathsf{X})$ respectively. But there is more:

Proposition 4.9 *If* X, Y *are normed spaces, then*
(a) $\mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$ *is a vector subspace of* $\mathfrak{B}(\mathsf{X}, \mathsf{Y})$.
(b) *if* Z *is a normed space and* $A \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$*:*
(i) $B \in \mathfrak{B}(\mathsf{Z}, \mathsf{X})$ *implies* $AB \in \mathfrak{B}_\infty(\mathsf{Z}, \mathsf{Y})$*,*
(ii) $B \in \mathfrak{B}(\mathsf{Y}, \mathsf{Z})$ *implies* $BA \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Z})$*;*
(c) *If, additionally,* Y *is a Banach space and* $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$ *converges uniformly to* $A \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$*, then* $A \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$. *In other words* $\mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$ *is a closed subspace in the Banach space* $(\mathfrak{B}(\mathsf{X}, \mathsf{Y}), ||\ ||)$*, where* $||\ ||$ *is the operator norm.*

Proof (a) Consider the operator $\alpha A + \beta B$, $\alpha, \beta \in \mathbb{C}$, $A, B \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$. Let us prove it is compact by showing that any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ has a subsequence $\{x_{n_r}\}_{r\in\mathbb{N}} \subset \mathsf{X}$ whose image $\{(\alpha A + \beta B)(x_{n_r})\}_{r\in\mathbb{N}} \subset \mathsf{Y}$ converges.

Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ be a bounded sequence. There is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ for which $Ax_{n_k} \subset \mathsf{Y}$ converges, as A is compact. The subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ is also bounded by assumption, so there is a sub-subsequence $\{x_{n_{k_m}}\}_{m\in\mathbb{N}}$ such that $Bx_{n_{k_m}} \in \mathsf{Y}$ converges. Now, by construction, $\{x_{n_{k_m}}\}_{m\in\mathbb{N}}$ is a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$ for which $\alpha Ax_{n_{k_m}} + \beta Bx_{n_{k_m}} \subset \mathsf{Y}$ converges.
(b) In case (i), if $\{z_k\}_{k\in\mathbb{N}} \subset \mathsf{Z}$ is bounded by $M > 0$, the set $\{Bz_k\}_{k\in\mathbb{N}}$ is bounded by $||B||M$, as B is bounded. But A is compact, so there is a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ for which ABz_{n_k} converges. Thus AB is compact. In case (ii), as A is compact, if $\{x_k\}_{k\in\mathbb{N}} \subset \mathsf{X}$ is bounded there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that the Ax_{n_k} converge. Since B is continuous, also the BAx_{n_k} converge, and so BA is compact.
(c) Let $\mathfrak{B}(\mathsf{X}, \mathsf{Y}) \ni A = \lim_{i\to+\infty} A_i$ with $A_i \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$. Take a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in X: $||x_n|| \leq C$ for any n. We want to prove the existence of a convergent subsequence of $\{Ax_n\}$. Using a hopefully-clear notation, we build recursively a family of subsequences:

$$\{x_n\} \supset \{x_n^{(1)}\} \supset \{x_n^{(2)}\} \supset \cdots \tag{4.1}$$

such that, for any $i = 1, 2, \ldots$, $\{x_n^{(i+1)}\}$ is a subsequence of $\{x_n^{(i)}\}$ with $\{A_{i+1}x_n^{(i+1)}\}$ convergent. This is always possible, because any $\{x_n^{(i)}\}$ is bounded by C, being a subsequence of $\{x_n\}$, and A_{i+1} is compact by assumption. We claim that $\{Ax_i^{(i)}\}$ is the subsequence of $\{Ax_n\}$ that will converge. From the triangle inequality

$$||Ax_i^{(i)} - Ax_k^{(k)}|| \le ||Ax_i^{(i)} - A_n x_i^{(i)}|| + ||A_n x_i^{(i)} - A_n x_k^{(k)}|| + ||A_n x_k^{(k)} - Ax_k^{(k)}|| .$$

With this estimate,

$$||Ax_i^{(i)} - Ax_k^{(k)}|| \le ||A - A_n||(||x_i^{(i)}|| + ||x_k^{(k)}||) + ||A_n x_i^{(i)} - A_n x_k^{(k)}||$$

$$\le 2C||A - A_n|| + ||A_n x_i^{(i)} - A_n x_k^{(k)}|| .$$

Given $\varepsilon > 0$, if n is large enough then $2C||A - A_n|| \le \varepsilon/2$, since $A_n \to A$. Fix n and take $r \ge n$. Then $\{A_n(x_p^{(r)})\}_p$ is a subsequence of the converging sequence $\{A_n(x_p^{(n)})\}_p$. Consider the sequence $\{A_n(x_p^{(p)})\}_p$, for $p \ge n$: it picks up the "diagonal" terms of all those subsequences, *each of which is a subsequence of the preceding one by (4.1)*; moreover, it is still a subsequence of the convergent sequence $\{A_n(x_p^{(n)})\}_p$, so it, too, converges (to the same limit). We conclude that if $i, k \ge n$ are large enough, then $||A_n x_i^{(i)} - A_n x_k^{(k)}|| \le \varepsilon/2$. Hence if i, k are large enough then $||Ax_i^{(i)} - Ax_k^{(k)}|| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. This finishes the proof, for we have produced a Cauchy subsequence in the Banach space Y, which must converge in the space. □

Keeping in mind Proposition 2.74, a remarkable property of compact operators is spelt out by the next fact.

Proposition 4.10 *Let* X, Y *be normed spaces. If* $\mathsf{X} \ni x_n \to x \in \mathsf{X}$ *weakly and* $T \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$*, then* $||T(x_n) - T(x)||_Y \to 0$ *as* $n \to +\infty$*. In words, compact operators map weakly convergent sequences to sequences that converge in norm.*

Proof See Exercise 4.5. □

The last general property of compact operators on normed spaces concerns eigenvalues. We will give a proof before moving on to compact operators on Hilbert spaces.

Theorem 4.11 (On the eigenvalues of compact operators on normed spaces) *Let* X *be a normed space and* $T \in \mathfrak{B}_\infty(\mathsf{X})$.
(a) *For any* $\delta > 0$ *there exist finitely many* λ*-eigenspaces of* T *such that* $|\lambda| > \delta$.
(b) *If* $\lambda \neq 0$ *is an eigenvalue of* T*, the* λ*-eigenspace has finite dimension.*
(c) *The eigenvalues of* T*, in general complex numbers, form a bounded, at most countable, set; they can be ordered by decreasing modulus:*

$$|\lambda_1| \ge |\lambda_2| \ge \cdots 0,$$

with 0 *as unique possible limit point.*

Proof We shall need a lemma in order to prove parts (a) and (b).

Lemma 4.12 (Banach) *Let $x_1, x_2, \ldots$ be a (finite or infinite) sequence of linearly independent vectors in a normed space X, and set $\mathsf{X}_n :=< \{x_1, x_2, \ldots, x_n\} >$. Then there exists a corresponding sequence $y_1, y_2, \ldots \subset \mathsf{X}$ such that:*
(i) $||y_n|| = 1$,
(ii) $y_n \in \mathsf{X}_n$,
(iii) $d(y_n, \mathsf{X}_{n-1}) > 1/2$,
for any $n = 1, 2, \ldots$, where

$$d(y_n, \mathsf{X}_{n-1}) = \inf_{x \in \mathsf{X}_{n-1}} ||x - y_n||$$

is the distance of y_n to X_{n-1}.

Proof of lemma 4.12. Observe $d(y_n, \mathsf{X}_{n-1})$ exists and is finite, since it is the infimum of a non-empty set of real numbers bounded from below by 0. Choose $y_1 := x_1/||x_1||$ and build the sequence $\{y_n\}$ inductively, as follows. The vectors $x_1, x_2, \ldots$ are linearly independent, so $x_n \notin \mathsf{X}_{n-1}$ and $d(x_n, \mathsf{X}_{n-1}) = \alpha > 0$. So let $x' \in \mathsf{X}_{n-1}$ be such that $\alpha < ||x_n - x'|| < 2\alpha$. As $\alpha = d(x_n, \mathsf{X}_{n-1}) = d(x_n - x', \mathsf{X}_{n-1})$, the vector

$$y_n := \frac{x_n - x'}{||x_n - x'||}$$

satisfies (i), (ii), (iii). □

Let us resume the proof of (a) and (b). If X is finite-dimensional the claims hold because eigenvectors with distinct eigenvalues are linearly independent. So consider X infinite-dimensional, where there can be infinitely many eigenvalues and eigenvectors. The proof of both (a) and (b) follows simultaneously from the existence, for any $\delta > 0$, of a finite number of linearly independent λ-eigenvectors with $|\lambda| > \delta$. Let us prove that, then. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of eigenvalues of T, possibly repeated, such that $|\lambda_n| > \delta$. Assume, *by contradiction*, there is an infinite sequence $x_1, x_2, \ldots$, of corresponding linearly independent eigenvectors. We are claiming, by refuting the theorem, that there are *infinitely many* linearly independent λ-eigenvectors such that $|\lambda| > \delta$. Using Banach's lemma, construct the sequence $y_1, y_2, \ldots$ fulfilling (i), (ii) and (iii), where X_n is spanned by $x_1, x_2, \ldots, x_n$. Since $|\lambda_n| > \delta$, the sequence $\{\frac{y_n}{\lambda_n}\}_{n=1,2,\ldots}$ is bounded. We will show that we cannot extract a convergent subsequence from the images $\{T\frac{y_n}{\lambda_n}\}_{n=1,2,\ldots}$. By construction, in fact,

$$y_n := \sum_{k=1}^{n} \beta_k x_k \ ,$$

so

$$T\frac{y_n}{\lambda_n} = \sum_{k=1}^{n-1} \frac{\beta_k \lambda_k}{\lambda_n} x_k + \beta_n x_n = y_n + z_n \ ,$$

where

$$z_n := \sum_{k=1}^{n-1} \beta_k \left(\frac{\lambda_k}{\lambda_n} - 1\right) x_k \in \mathsf{X}_{n-1} \,.$$

For any $i > j$, then:

$$\left\| T\left(\frac{y_i}{\lambda_i}\right) - T\left(\frac{y_j}{\lambda_j}\right)\right\| = ||y_i + z_i - (y_j + z_j)||$$

$$= ||y_i - (y_j + z_j - z_i)|| > 1/2$$

as $y_j + z_j - z_i \in \mathsf{X}_{i-1}$. This is clearly incompatible with the compactness of T. Therefore we have to conclude that an infinite sequence of linearly independent eigenvectors $x_1, x_2, \ldots$ cannot exist. This ends the proof of (a) and (b).
(c) This part follows from (a) by picking a sequence of numbers $\delta > 0$ of the form $\delta_n = 1/n, n = 1, 2, 3, \ldots$. □

Remark 4.13 **(1)** One final property, that we shall not prove, states that *in the Banach setting* the adjoint operator (cf. Definition 2.45) to a compact operator is compact. We will prove it for Hermitian adjoints to compact operators on Hilbert spaces.
(2) From Lemma 4.12 descends an alternative proof that the closed unit ball in an infinite-dimensional normed space is not compact (see Exercise 4.2). ■

4.2 Compact Operators on Hilbert Spaces

From now on we will consider compact operators on Hilbert spaces, even if certain properties are valid in less structured spaces, like normed or Banach spaces.

4.2.1 General Properties and Examples

In the first theorem about compact operators, the space's completeness is necessary only for the last statement.
Before, though, we need a preparatory proposition.

Proposition 4.14 *Let* H *be a Hilbert space. Then* $A \in \mathfrak{B}(\mathsf{H})$ *is compact if and only if* $|A|$ *is compact (see Definition 3.80).*

Proof Assume A is compact. Let $\{x_k\}_{k\in\mathbb{N}}$ be a bounded sequence in H and $\{Ax_{k_n}\}_{n\in\mathbb{N}}$ a subsequence of $\{Ax_k\}_{k\in\mathbb{N}}$ that converges, by virtue of compactness. Since the latter is a Cauchy subsequence by (3.65), the subsequence $\{|A|x_{k_n}\}_{n\in\mathbb{N}}$ of $\{|A|x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and converges. Hence $|A|$ is compact. For the reverse implication just swap A and $|A|$ and repeat the argument. □

Theorem 4.15 *Let* H *be a Hilbert space. The set of compact operators* $\mathfrak{B}_\infty(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H})$ *is*
(a) *a linear subspace;*
(b) *a two-sided* **-ideal:* TK, KT, $T^* \in \mathfrak{B}_\infty(\mathsf{H})$ *for any* $T \in \mathfrak{B}_\infty(\mathsf{H})$, $K \in \mathfrak{B}(\mathsf{H})$*;*
(c) *a* C^**-subalgebra (without unit if* $\dim \mathsf{H} = \infty$*), as closed in the uniform topology.*

Proof (a) We proved this, for general normed spaces, in Proposition 4.9(a).
(b) Proposition 4.9(b) shows that, in normed spaces, the left and right multiplications by a bounded operator preserve $\mathfrak{B}_\infty(\mathsf{H})$. To show closure under Hermitian conjugation, observe that $|T|$ is compact iff T is compact, by Proposition 4.14. From the polar decomposition $T = U|T|$ of Theorem 3.82, we have $T^* = |T|U^*$, where we used $|T| \geq 0$, so $|T|$ is self-adjoint. The boundedness of U^* together with the compactness of $|T|$ force the product $T^* = |T|U^*$ to be compact.
(c) This part follows directly from Proposition 4.9(c) and the definition of C^*-algebra (recall H has infinite dimension, so the identity I is not compact, for otherwise the closed ball would be compact, and we know this cannot be). □

Remark 4.16 The statement of the previous theorem can be reversed if H is separable, as we have the following result due to Calkin (Cal41).

Theorem 4.17 *If the Hilbert space* H *is separable,* $\mathfrak{B}_\infty(\mathsf{H})$ *is the unique non-trivial two-sided* **-ideal of* $\mathfrak{B}(\mathsf{H})$ *which is closed with respect to the uniform topology.*

■

Examples 4.18 **(1)** If X, Y are normed spaces and $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ has finite-dimensional final space $Ran(T)$, then T must be compact. Let us prove it. If $V \subset \mathsf{X}$ is bounded, i.e. $V \subset B_r(\mathbf{0})$ for some number $r > 0$, then $||T(V)|| \leq r||T|| < +\infty$, whence $T(V)$ is bounded. Therefore $\overline{T(V)}$ is closed and bounded in a finite-dimensional normed space homeomorphic to $\mathbb{C}^n$ (Proposition 2.107). By the Heine–Borel theorem $\overline{T(V)}$ is compact in the topology induced on the range of T. Hence T is compact, because compactness in the induced topology is the same as compactness in the ambient space.

In particular, suppose H is a Hilbert space and consider the operator $T_x \in \mathfrak{L}(\mathsf{H})$:

$$T_x : u \mapsto (x|u)y ,$$

where $x, y \in \mathsf{H}$ are given vectors (possibly equal). As it has finite-dimensional range, T_x is compact.
(2) If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are orthogonal sets in H, and if $T = \sum_{n\in\mathbb{N}}(x_n|\ )y_n$ is a bounded operator (interpreting the series in the uniform topology), then T is compact by Theorem 4.15(a, c).
(3) Consider the operator $A : \{x_n\} \mapsto \{x_{n+1}/n\}$ on $\ell^2(\mathbb{N})$. It is compact because is arises as uniform limit of:

$$A_m : \{x_n\} \mapsto \{x_2/1, x_3/2, \ldots, x_{m+1}/m, 0, 0, \ldots\}$$

for $n = 1, 2 \dots$ In fact, it is easy to prove (exercise)

$$||A - A_n|| \leq 1/(n+1) .$$

(4) Consider a measure μ on the σ-algebra Σ of X, and suppose μ is σ-finite, so to define the product measure $\mu \otimes \mu$. We will use the simpler notation $L^2(\mu) := L^2(\mathsf{X}, \mu)$ and $L^2(\mu \otimes \mu) := L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$. Given $K \in L^2(\mu \otimes \mu)$, we shall prove that

$$T_K : L^2(\mu) \ni f \mapsto \int_{\mathsf{X}} K(x, y) f(y) d\mu(y)$$

defines a compact operator $T_K \in \mathfrak{B}(L^2(\mu))$ in case μ is separable (cf. Examples 3.32(3)). First of all, irrespective of separability, if $f \in L^2(\mu)$:

$$\int_{\mathsf{X}} K(\cdot, y) f(y) d\mu(y) \in L^2(\mu)$$

and

$$\left|\left|\int_{\mathsf{X}} K(\cdot, y) f(y) d\mu(y)\right|\right|_{L^2(\mu)} \leq ||K||_{L^2(\mu\otimes\mu)} ||f||_{L^2(\mu)} ,$$

which is to say:

$$||T_K|| \leq ||K||_{L^2(\mu\otimes\mu)} . \tag{4.2}$$

The proof of this is entirely based on the Fubini–Tonelli theorem: if $K \in L^2(\mu \otimes \mu)$, by Fubini–Tonelli we have:

(1) $|K(x, \cdot)|^2 \in L^1(\mu)$, a.e. in μ,

(2) $\int_{\mathsf{X}} |K(\cdot, y)|^2 d\mu(y) \in L^1(\mu)$.

From (1) $K(x, \cdot) \in L^2(\mu)$ a.e., so $K(x, \cdot) f \in L^1(\mu)$ a.e. By the Cauchy–Schwarz inequality:

(3) $\int_{\mathsf{X}} |K(x, y)||f(y)| d\mu(y) \leq ||K(x, \cdot)||_{L^2} ||f||_{L^2}$.

Setting $F(x) := \int_{\mathsf{X}} K(x, y) f(y) d\mu(y)$, F is measurable, and by (3):

(4) $|F(x)|^2 \leq ||f||^2_{L^2} \int_{\mathsf{X}} |K(x, y)|^2 d\mu(y)$.

From (2) we have $|F|^2 \in L^2(\mu)$, so it is true that

$$\int_{\mathsf{X}} K(\cdot, y) f(y) d\mu(y) \in L^2(\mu) .$$

By (4) and Fubini–Tonelli, finally, we obtain

$$\left|\left|\int_{\mathsf{X}} K(\cdot, y) f(y) d\mu(y)\right|\right|_{L^2(\mu)} \leq ||K||_{L^2(\mu\otimes\mu)} ||f||_{L^2(\mu)} ,$$

hence (4.2).

In order to show T_K is compact, let us further assume μ is separable, so to make $L^2(\mathsf{X}, \mu)$ separable (see Proposition 3.33). For instance, X could be an interval in $\mathbb{R}$ (or a Borel set) and μ the Lebesgue measure on $\mathbb{R}$. If $\{u_n\}_{n\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mu)$, $\{u_n \cdot \overline{u_m}\}_{n,m\in\mathbb{N}}$ is a Hilbert basis of $L^2(\mu \otimes \mu)$ ($\cdot$ is the ordinary pointwise product of functions). Then, in the topology of $L^2(\mu \otimes \mu)$, we have

$$K = \sum_{n,m} k_{nm} u_n \cdot \overline{u_m} ,$$

where the numbers $k_{nm} \in \mathbb{C}$ depend on K. So, setting

$$K_p := \sum_{n,m\leq p} k_{nm} u_n \cdot \overline{u_m}$$

we have $K_p \to K$ as $p \to +\infty$ in $L^2(\mu \otimes \mu)$. Applying (4.2) to $T_{K_p - K} = T_{K_p} - T_K$, where T_{K_p} is induced by the integral kernel K_p, we have

$$||T_K - T_{K_p}|| = \left|\left|\sum_{n,m>p} k_{nm} u_n \cdot \overline{u_m}\right|\right|_{L^2(\mu\otimes\mu)} \to 0 ,$$

as $p \to +\infty$. Then T_K is compact, because the operators T_{K_p} are compact, being finite sums of operators with finite-dimensional ranges, like those of example (1) above.

Even without assuming μ separable, and demanding instead $K \in L^2(\mu \otimes \mu)$ and μ σ-finite, it is easy to see that

$$T_{\overline{K}} = T_K^* , \tag{4.3}$$

where $\overline{K}(x, y) := \overline{K(x, y)}$ for any $x, y \in \mathsf{X}$, and the bar is complex conjugation. The proof follows from Proposition 3.36 and Fubini–Tonelli. ∎

4.2.2 *Spectral Decomposition of Compact Operators on Hilbert Spaces*

Compact operators on Hilbert spaces enjoy remarkable properties: concerning eigenvectors, eigenvalues and eigenspaces, in particular, the features of compact and self-adjoint operators generalise to infinite dimensions the properties of Hermitian matrices. The first two results clarify this matter.

Theorem 4.19 (Hilbert) *Let* H *be a Hilbert space,* $T \in \mathfrak{B}_\infty(\mathsf{H})$ *an operator satisfying* $T = T^*$.
(a) *Every* λ*-eigenspace of* T*, for* $\lambda \neq 0$*, has finite dimension.*
(b) *The set* $\sigma_p(T)$ *of eigenvalues of* T *is:*
(1) non-empty,
(2) real,
(3) at most countable;
(4) it has one limit point at most, and this is 0*;*
(5) it satisfies:

$$\sup\{|\lambda| \mid \lambda \in \sigma_p(T)\} = ||T|| \,.$$

More precisely, the least upper bound is $\Lambda \in \sigma_p(T)$*, where*

$$\Lambda = ||T|| \quad \text{if} \quad \sup_{||x||=1}(x|Tx) = ||T|| \,, \tag{4.4}$$

or

$$\Lambda = -||T|| \quad \text{if} \quad \inf_{||x||=1}(x|Tx) = -||T|| \,. \tag{4.5}$$

(6) T *coincides with the null operator if and only if* 0 *is the only eigenvalue.*

Partial proof.
(a) Let H_λ be the λ-eigenspace of T, with $\lambda \neq 0$. If $B \subset \mathsf{H}_\lambda$ is the open unit ball at the origin, we can write $B = T(\lambda^{-1}B)$, and $\lambda^{-1}B$ is bounded by construction. Since T is compact, $\overline{B}$ is compact too. Hence in the Hilbert space H_λ the closure of the open unit ball is compact, and so $\dim \mathsf{H}_\lambda < +\infty$ by Proposition 4.5. An alternate proof immediately arises from Theorem 4.11(b).
(b) We will prove all items but (3) and (4), which will be part of the next theorem. If $\sigma_p(T)$ is not empty it must consist of real numbers by (ii) in Proposition 3.60(c), T being self-adjoint. Proposition 3.60(a) says that $-||T|| \leq (x|Tx) \leq ||T||$ for any unit vector x, so only one of two possibilities can occur: either $\sup_{||x||=1}(x|Tx) = ||T||$ or $\inf_{||x||=1}(x|Tx) = -||T||$. Suppose the former is true, the other case being analogous by flipping the sign of T. Assume $||T|| > 0$, otherwise the theorem is trivial. For any eigenvalue λ choose a unit λ-eigenvector x, so $||T|| \geq |(x|Tx)| = |\lambda|(x|x) = |\lambda|$, and then $\sup|\sigma_p(T)| \leq ||T||$. To prove (5) it suffices to exhibit an eigenvector with eigenvalue $\Lambda = ||T||$. This also proves $\sigma_p(T) \neq \varnothing$ by the way. By assumption there is a sequence of unit vectors x_n such that $(x_n|Tx_n) \to ||T|| =: \Lambda > 0$. Using $||Tx_n|| \leq ||T||||x_n|| = ||T||$, we have

$$||Tx_n - \Lambda x_n||^2 = ||Tx_n||^2 - 2\Lambda(x_n|Tx_n) + \Lambda^2 \leq ||T||^2 + \Lambda^2 - 2\Lambda(x_n|Tx_n) \,.$$

As $||T|| = \Lambda$, taking the limit as $n \to +\infty$ in the inequality gives

$$Tx_n - \Lambda x_n \to \mathbf{0} \,. \tag{4.6}$$

To conclude it would be enough to show either that $\{x_n\}_{n\in\mathbb{N}}$ converges, or that a subsequence does. As $||x_n|| = 1$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. But T is compact, so we may extract from $\{Tx_n\}_{n\in\mathbb{N}}$ a convergent subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$.
Then (4.6) implies that

$$x_{n_k} = \frac{1}{\Lambda}[Tx_{n_k} - (Tx_{n_k} - \Lambda x_{n_k})]$$

converges to some $x \in \mathsf{H}$, as $k \to +\infty$, for it is a *linear combination of converging sequences*. Since T is continuous and $x_{n_k} \to x$, formula (4.6) forces

$$Tx = \Lambda x\ .$$

Observe that $x \neq \mathbf{0}$ because $||x|| = \lim_{k\to+\infty} ||x_{n_k}|| = 1$. So we have shown x is a Λ-eigenvector.
(6) is an immediate consequence of (5). □

Let us move on to the celebrated theorem of Hilbert on the expansion of self-adjoint compact operators in terms of eigenvector bases.

Theorem 4.20 (Hilbert decomposition of compact operators) *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space and assume* $T \in \mathfrak{B}_\infty(X)$*, with eigenvalue set* $\sigma_p(T)$*, satisfies* $T = T^*$.
(a) *If* P_λ *is the orthogonal projector onto the* λ*-eigenspace, then*

$$T = \sum_{\lambda\in\sigma_p(T)} \lambda P_\lambda\ . \tag{4.7}$$

If $\sigma_p(T)$ *is infinite, series (4.7) is understood in uniform topology, and the eigenvalues:* $\lambda_0, \lambda_1, \dots$ $(\lambda_i \neq \lambda_j,\ i \neq j)$ *are ordered so that* $|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \dots$.
(b) *If* B_λ *is a Hilbert basis of the* λ*-eigenspace, then* $\cup_{\lambda\in\sigma_p(T)} B_\lambda$ *is a Hilbert basis of* H*. Equivalently,* H *admits a basis made of eigenvectors of* T*.*

Remark 4.21 Notice that there can only be at most two distinct non-trivial eigenvalues with equal absolute value (the eigenvalues are real). As

$$|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \dots,$$

the ambiguity in arranging the terms of the series regards pairs λ, λ' with $|\lambda| = |\lambda'|$. As we shall see after the proof, the sum of the series does not depend on this choice. ■

Proof (including parts (3), (4) of Theorem 4.19). (a) Let λ be an eigenvalue with eigenspace H_λ. Call P_λ the orthogonal projector onto H_λ and $Q_\lambda := I - P_\lambda$ the orthogonal projector onto $\mathsf{H}_\lambda^\perp$. Then

$$TP_\lambda = P_\lambda T = \lambda P_\lambda\ . \tag{4.8}$$

In fact, if $x \in \mathsf{H}$, $P_\lambda x \in \mathsf{H}_\lambda$ then $TP_\lambda x = \lambda P_\lambda x$, so $TP_\lambda = \lambda P_\lambda$. Recalling that $\lambda \in \mathbb{R}$, $T = T^*$, $P_\lambda = P_\lambda^*$, by taking adjoints we find $P_\lambda T = \lambda P_\lambda = TP_\lambda$. A further consequence is, directly by definition of Q_λ and the above, that

$$Q_\lambda T = T Q_\lambda \ . \tag{4.9}$$

Observe that from $I = P_\lambda + Q_\lambda$ we infer $T = P_\lambda T + Q_\lambda T$, i.e.

$$T = \lambda P_\lambda + Q_\lambda T \ . \tag{4.10}$$

The operator $Q_\lambda T$:
(i) is self-adjoint, for $(Q_\lambda T)^* = T^* Q_\lambda^* = T Q_\lambda = Q_\lambda T$,
(ii) is compact by Theorem 4.15(b),
(iii) satisfies, by construction, $P_\lambda(Q_\lambda T) = (Q_\lambda T)P_\lambda = 0$ since $P_\lambda Q_\lambda = Q_\lambda P_\lambda = 0$.

In the rest of the proof these identities will be used without further mention, and we shall write P_n, Q_n, H_n instead of P_{λ_n}, Q_{λ_n}, H_{λ_n}.

Let us begin by choosing an eigenvalue $\lambda = \lambda_0$ with largest absolute value: there are, at most, two such eigenvalues differing by a sign, and we pick either one. If $T_1 := Q_0 T$ then

$$T = \lambda_0 P_0 + T_1$$

where T_1 satisfies the above (i), (ii) and (iii). If $T_1 = 0$ the proof ends; if not, we know T_1 is self-adjoint and compact, so we can iterate the procedure using T_1 in place of T and find

$$T = \lambda_0 P_0 + \lambda_1 P_1 + T_2$$

where $T_2 := Q_1 T_1$. Now λ_1 is a non-null eigenvalue of T_1 of largest absolute value (if the largest eigenvalue were zero, then $T_1 = 0$ by (6) in Theorem 4.19(b)). Furthermore, P_1 is the orthogonal projector onto the λ_1-eigenspace of T_1.
Observe that any eigenvalue λ_1 of T_1 is also an eigenvalue of T, because, if $T_1 u_1 = \lambda_1 u_1$,

$$T u_1 = (\lambda_0 P_0 + T_1)u_1 = \lambda_0 P_0 T_1 \frac{1}{\lambda_1} u_1 + T_1 u_1 = \lambda_0 P_0 Q_0 T \frac{1}{\lambda_1} u_1 + T_1 u_1$$

$$= \lambda_0 \cdot 0 \cdot T \frac{1}{\lambda_1} u_1 + \lambda_1 u_1 = \lambda_1 u_1 \ .$$

What is more, $\lambda_1 \neq \lambda_0$ since $u_1 \in Ran T_1 = Ran(Q_0 T) \subset \mathsf{H}_0^\perp$. At last, every λ_1-eigenvector of T is a λ_1-eigenvector of T_1. In fact, using $T_1 = Q_0 T = (I - P_0)T$ we have, with $Tu = \lambda_1 u$,

$$T_1 u = \lambda_1 u - \lambda_1 P_0 u = \lambda_1 u + \mathbf{0} = \lambda_1 u \ ,$$

also using $P_0 u = \mathbf{0}$ (because eigenspaces with distinct eigenvalues are orthogonal for self-adjoint operators, like T). Overall, the λ_1-eigenspace $\mathsf{H}_1^{(T_1)}$ coincides with the

λ_1-eigenspace H_1 of T. Therefore P_1 is the orthogonal projector onto this common eigenspace.

Since $|\lambda_0|$ is the maximum,

$$|\lambda_1| \leq |\lambda_0|.$$

There is an important consequence to this. Since $||T|| = |\lambda_0|$ and $||T_1|| = \lambda_1$ by the previous theorem,

$$||T_1|| \leq ||T|| \, .$$

If $T_2 = 0$ the proof ends, otherwise we proceed as before, finding

$$T - \sum_{k=0}^{n} \lambda_k P_k = T_n \, , \tag{4.11}$$

where

$$|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_k| \geq \ldots$$

and

$$||T_k|| = |\lambda_k| \tag{4.12}$$

If none of the T_k is null, the process never stops. In such a case we claim that the decreasing sequence of positive numbers $|\lambda_k|$ must tend to 0 (there cannot be a positive limit point). Suppose $|\lambda_0| \geq |\lambda_1| \geq \cdots \geq |\lambda_k| \geq \ldots \geq \varepsilon > 0$ and pick a unit vector $x_n \in \mathsf{H}_n$ for any n. The sequence of the x_n is bounded, so the sequence of Tx_n, or a subsequence of it, must converge as T is compact. But this is impossible: x_n and x_m are perpendicular (their eigenvalues are distinct and the operator is self-adjoint, cf. Proposition 3.60(b), part (ii)), so

$$||Tx_n - Tx_m||^2 = ||\lambda_n x_n - \lambda_m x_m||^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon \, ,$$

for any n, m. Therefore neither the Tx_n, nor any subsequence, can converge, for they are not Cauchy sequences. This is a contradiction, so the sequence of the λ_n (if there really are infinitely many thereof) converges to 0. By (4.11) and (4.12), what we have proved implies

$$T = \sum_{k=0}^{+\infty} \lambda_k P_k \tag{4.13}$$

in the uniform topology. By construction, this result does not depend on how we decide to order pairs of eigenvalues with equal absolute value. Now we claim that (4.13) coincides with (4.7), because the sequence of eigenvalues $\{\lambda_k\}$ exhausts $\sigma_p(T)$ except, possibly, for 0 (which at any rate interferes neither with (4.13) nor with (4.7)).

Let $\lambda \neq \lambda_n$ for any n be an eigenvalue of T, and P_λ its orthogonal projector. Given that $P_n P_\lambda = 0$ for any n (again, by (ii) in Proposition 3.60(b)), (4.13) implies

$$TP_\lambda = \sum_{k=0}^{+\infty} \lambda_k P_k P_\lambda = 0 \,,$$

whence, if $u \in \mathsf{H}_\lambda$,

$$Tu = TP_\lambda u = 0 \,.$$

This means $\lambda = 0$.

The proof of part (a) ends here, and in due course we have also justified the leftovers of Theorem 4.19.

(b) The bases B_λ always exist by Theorem 3.27, eigenspaces being closed (exercise) in H and hence Hilbert spaces themselves. Call $B := \cup_{\lambda \in \sigma_p(T)} B_\lambda$ the union. We assert that if $u \in B^\perp$ then $u = \mathbf{0}$; since B is orthonormal, by Definition 3.22 the proof would end. Take then $u \in B^\perp$, so $u \perp B_\lambda$ for any $\lambda \in \sigma_p(T)$, and hence $P_\lambda u = \mathbf{0}$ for any $\lambda \in \sigma_p(T)$. Using decomposition (4.7) for T we find $Tu = \mathbf{0}$. Hence u belongs to the 0-eigenspace H_0. But u is orthogonal to *every* eigenspace of T by construction, so $u \in \mathsf{H}_0^\perp$. This forces $u = \mathbf{0}$, as claimed. $\square$

Hilbert's theorem, together with the polar decomposition Theorem 3.82, allows to generalise formula (4.7) to compact operators that are not self-adjoint. First, let us see a definition useful for the sequel.

Definition 4.22 Let H be a Hilbert space and $A \in \mathfrak{B}_\infty(\mathsf{H})$. Non-zero eigenvalues λ of $|A|$ are called **singular values** of A, and their set is denoted by $sing(A)$.
The (finite) dimension m_λ of the eigenspace of $\lambda \in sing(A)$ is called **multiplicity** of λ.

Theorem 4.23 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space and* $A \in \mathfrak{B}_\infty(\mathsf{H})$, $A \neq 0$. *Suppose* $sing(A)$ *is non-empty and ordered as* $\lambda_0 > \lambda_1 > \lambda_2 > \ldots > 0$. *Then*

$$A = \sum_{\lambda \in sing(A)} \sum_{i=1}^{m_\lambda} \lambda \, (u_{\lambda,i}| \quad) \, v_{\lambda,i} \,, \tag{4.14}$$

$$A^* = \sum_{\lambda \in sing(A)} \sum_{i=1}^{m_\lambda} \lambda \, (v_{\lambda,i}| \quad) \, u_{\lambda,i} \,. \tag{4.15}$$

The sums, if infinite, are meant in the uniform topology, and for any $\lambda \in sing(A)$ *the set of* $u_{\lambda,1}, \ldots, u_{\lambda,m_\lambda}$ *is an orthonormal basis for the* λ*-eigenspace of* $|A|$. *Moreover, for any* $\lambda \in sing(A)$, $i = 1, 2, \ldots, m_\lambda$, *the vectors*

$$v_{\lambda,i} := \frac{1}{\lambda} A u_{\lambda,i} \tag{4.16}$$

form an orthonormal system, and

$$v_{\lambda,i} = U u_{\lambda,i} , \tag{4.17}$$

where U is defined by the polar decomposition $A = U|A|$.

Proof The operator $|A|$ is self-adjoint, positive and compact. Its eigenvalues are real, positive and satisfy conditions (a) and (b) in Theorem 4.19. We examine the case where the eigenvalue set is infinite (countable), leaving the finite case to the reader. Theorem 4.20 allows to expand $|A|$:

$$|A| = \sum_{\lambda \in \sigma_p(|A|)} \lambda P_\lambda ,$$

where the convergence is uniform. It is clear we can reduce to non-zero eigenvalues since 0 does contribute to the series, so

$$|A| = \sum_{\lambda \in sing(A)} \lambda P_\lambda .$$

If U is bounded and $\mathfrak{B}(\mathsf{H}) \ni T_n \to T \in \mathfrak{B}(\mathsf{H})$ uniformly, then $UT_n \to UT$ in the uniform topology. Since U (from $A = U|A|$) is bounded, in the uniform topology we have:

$$A = U|A| = \sum_{\lambda \in sing(A)} \lambda U P_\lambda . \tag{4.18}$$

Theorem 4.19(a) says the closed projection space of each P_λ ($\lambda > 0$) has finite dimension m_λ, so we may find an orthonormal basis $\{u_{\lambda,i}\}_{i=1,\ldots,m_\lambda}$ for it. Note $(u_{\lambda,i}|u_{\lambda',j}) = \delta_{\lambda\lambda'}\delta_{ij}$ by construction, as eigenvectors with distinct eigenvalues are orthogonal ($|A|$ is normal because positive) by virtue of (ii) in Proposition 3.60(b). From $u_{\lambda,i} = |A|(u_{\lambda,i}/\lambda)$ we have $u_{\lambda,i} \in Ran(|A|)$. Therefore U acts on $u_{\lambda,i}$ isometrically, and the vectors on the right in (4.17) are still orthonormal. Equation (4.17) is equivalent to (4.16) by polar decomposition:

$$Au_{\lambda,i} = U|A|u_{\lambda,i} = U\lambda u_{\lambda,i} = \lambda v_{\lambda,i} .$$

It is an easy exercise to show that the orthogonal projector P_λ ($\lambda > 0$) can be written as

$$P_\lambda = \sum_{i=1}^{m_\lambda} (u_{\lambda,i}| \quad) \, u_{\lambda,i} .$$

Consequently,

$$UP_\lambda = \sum_{i=1}^{m_\lambda} (u_{\lambda,i}| \quad) \, Uu_{\lambda,i} = \sum_{i=1}^{m_\lambda} (u_{\lambda,i}| \quad) \, v_{\lambda,i} .$$

Substituting in (4.18) gives (4.14). Equation (4.15) arises from (4.14) if we consider the following two facts: (i) the conjugation $A \mapsto A^*$ is antilinear (it transforms linear combinations of operators into linear combinations of the adjoints with conjugated coefficients); (ii) conjugation is continuous in the uniform topology of $\mathfrak{B}(\mathsf{H})$ because Proposition 3.38(a) implies $||A^*|| = ||A||$.

From these two facts, (4.14) gives (recall $\lambda \in \mathbb{R}$):

$$A^* = \sum_{\lambda \in sing(A)} \sum_{i=1}^{m_\lambda} \lambda \,[(u_{\lambda,i}| \quad) \, v_{\lambda,i}]^* \,,$$

where the series converges in the uniform topology. An easy exercise shows that

$$[(u_{\lambda,i}| \quad) \, v_{\lambda,i}]^* = (v_{\lambda,i}| \quad) \, u_{\lambda,i} \,,$$

from which (4.15) is immediate. □

The theorem just proved allows us to introduce Hilbert–Schmidt operators and operators of trace class, which we will describe in the ensuing sections.

4.3 Hilbert–Schmidt Operators

A particular class of compact operators is that of Hilbert–Schmidt operators. They have several applications in wide-ranging areas of mathematical physics, besides QM. This section is devoted to their study.

4.3.1 Main Properties and Examples

Warning. In this section, and sometimes also elsewhere, the operator norm $|| \quad ||_2$ will denote the *Hilbert–Schmidt* norm (see later) and not the usual L^2 norm. This should not be cause of ambiguity, for the correct meaning will be clear from the context. ■

Definition 4.24 Let $(\mathsf{H}, (\,|\,))$ be a Hilbert space and $|| \quad ||$ the inner product norm. An operator $A \in \mathfrak{B}(\mathsf{H})$ is a **Hilbert–Schmidt operator** (HS) if there exists a basis U such that $\sum_{u\in U} ||Au||^2 < +\infty$ in the sense of Definition 3.19.
The class of Hilbert–Schmidt operators on H will be indicated with $\mathfrak{B}_2(\mathsf{H})$. If $A \in \mathfrak{B}_2(\mathsf{H})$,

$$||A||_2 := \sqrt{\sum_{u\in U} ||Au||^2} \,, \tag{4.19}$$

where U is the chosen Hilbert basis.

As first thing let us prove that the choice of a particular Hilbert basis is not important, and that $||A||_2$ does not depend on it.

Proposition 4.25 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space with norm* $||\ ||$ *induced by the inner product,* U *and* V *bases (possibly coinciding) and* $A \in \mathfrak{B}(\mathsf{H})$*. Then:*
(a) $\{||Au||^2\}_{u\in U}$ *has finite sum* $\Leftrightarrow$ $\{||Av||^2\}_{v\in V}$ *has finite sum. In this case:*

$$\sum_{u\in U} ||Au||^2 = \sum_{v\in V} ||Av||^2 \,. \tag{4.20}$$

(b) $\{||Au||^2\}_{u\in U}$ *has finite sum* $\Leftrightarrow$ $\{||A^*v||^2\}_{v\in V}$ *has finite sum. If so:*

$$\sum_{u\in U} ||Au||^2 = \sum_{v\in V} ||A^*v||^2 \,. \tag{4.21}$$

Proof In the light of Theorem 3.26(d),

$$||Au||^2 = \sum_{v\in V} |(v|Au)|^2 < +\infty \,,$$

so, given u, only a countable number of coefficients $|(v|Au)|$, at most, is non-zero by Proposition 3.21(b). This gives at most a countable set $V(u) \subset V$ such that

$$\sum_{u\in U} ||Au||^2 = \sum_{u\in U} \sum_{v\in V(u)} |(v|Au)|^2 < +\infty \,. \tag{4.22}$$

In particular, using Proposition 3.21(b) again, it means that a countable (at most) set of $u \in U$ gives non-zero sum $\sum_{v\in V(u)} |(v|Au)|^2$. Therefore the coefficients $(v|Au)$ do not vanish only for a countable (at most) set Z of pairs $(u, v) \in U \times V$. Define sets (at most countable):

$$U_0 := \{u \in U \mid \text{ there exists } v \in V \text{ with } (v|Au) \neq 0\} \,,$$

$$V_0 := \{v \in V \mid \text{ there exists } u \in U \text{ with } (v|Au) \neq 0\} \,.$$

Thus $Z \subset U_0 \times V_0$. Endow U_0 and V_0 with counting measures μ and ν, and write the above series using integrals and these measures (Proposition 3.21(c)). In particular (4.22) becomes:

$$\sum_{u\in U} ||Au||^2 = \sum_{u\in U} \sum_{v\in V(u)} |(v|Au)|^2 = \int_{U_0} d\mu(u) \int_{V_0} d\nu(v) |(v|Au)|^2 < +\infty \,. \tag{4.23}$$

As U_0 and V_0 are at most countable, μ and ν are σ-finite, so we can define the product $\mu \otimes \nu$ and use the Fubini–Tonelli theorem. Concerning the last part of (4.23), this theorem ensures that $(v, u) \mapsto |(v|Au)|^2$ is integrable in the product measure and we can swap integrals:

$$\sum_{u \in U} ||Au||^2 = \int_{U_0 \times V_0} |(v|Au)|^2 d\mu(u) \otimes d\nu(v) = \int_{V_0} d\nu(v) \int_{U_0} d\mu(u) |(v|Au)|^2 < +\infty \,.$$

Note $(v|Au) = (A^*v|u)$, so just countably many, at most, products $(A^*v|u)$ (with $(u, v) \in U \times V$) will be different from zero, and in particular:

$$\sum_{v \in V} \sum_{u \in U} |(A^*v|u)|^2 = \int_{V_0} d\nu(v) \int_{U_0} d\mu(u) |(A^*v|u)|^2 = \sum_{u \in U} ||Au||^2 < +\infty \,.$$

But the left-hand side is precisely $\sum_{v \in V} ||A^*v||^2$. Therefore we have proved the following part of assertion (b): *if* $\{||Au||^2\}_{u \in U}$ *has finite sum, so does* $\{||A^*v||^2\}_{v \in V}$*, and the sums coincide.* Recalling that $(A^*)^* = A$ for bounded operators, we can now use the same proof, just exchanging the bases and starting from A^*, to prove the remaining part of (b): *if* $\{||A^*v||^2\}_{v \in V}$ *has finite sum, then also* $\{||Au||^2\}_{u \in U}$ *does, and then (4.21) holds.*

The proof of (a) is straightforward from (b) because the bases used are arbitrary. □

With that settled we can discuss some of the many and interesting mathematical properties of HS operators. The most fascinating from a mathematical viewpoint is stated as (b) in the next theorem: HS operators A form a Hilbert space whose inner product induces precisely the norm we called $||A||_2$. Another important fact is that HS operators are compact, and their space is a *-closed ideal inside bounded operators.

Theorem 4.26 *Hilbert–Schmidt operators on a Hilbert space* H *enjoy the following properties.*
(a) $\mathfrak{B}_2(\mathsf{H})$ *is a subspace in* $\mathfrak{B}(\mathsf{H})$ *and, actually, a two-sided* **-ideal in* $\mathfrak{B}(\mathsf{H})$*; moreover:*

(i) $||A||_2 = ||A^*||_2$ *for any* $A \in \mathfrak{B}_2(\mathsf{H})$*;*

(ii) $||AB||_2 \leq ||B||\,||A||_2$ *and* $||BA||_2 \leq ||B||\,||A||_2$ *for any* $A \in \mathfrak{B}_2(\mathsf{H})$*,* $B \in \mathfrak{B}(\mathsf{H})$*;*

(iii) $||A|| \leq ||A||_2$ *for any* $A \in \mathfrak{B}_2(\mathsf{H})$.

(b) *If* $A, B \in \mathfrak{B}_2(\mathsf{H})$ *and if* N *is a basis in* H*, define:*

$$(A|B)_2 := \sum_{x \in N} (Ax|Bx) \,. \tag{4.24}$$

The map

$$(\ |\ )_2 : \mathfrak{B}_2(\mathsf{H}) \times \mathfrak{B}_2(\mathsf{H}) \to \mathbb{C}$$

is well defined (the sum always reduces to an absolutely convergent series and does not depend on the basis) and determines an inner product on $\mathfrak{B}_2(\mathsf{H})$ *such that:*

$$(A|A)_2 = ||A||_2^2 \tag{4.25}$$

for any $A \in \mathfrak{B}_2(\mathsf{H})$.
(c) $(\mathfrak{B}_2(\mathsf{H}), (\ |\ )_2)$ *is a Hilbert space.*
(d) $\mathfrak{B}_2(\mathsf{H}) \subset \mathfrak{B}_\infty(\mathsf{H})$. *More precisely,* $A \in \mathfrak{B}_2(\mathsf{H})$ *if and only if* A *is compact and the set of positive numbers* $\{m_\lambda \lambda^2\}_{\lambda \in sing(A)}$ *(*m_λ *is the multiplicity of* λ*) has finite sum. In this case:*

$$||A||_2 = \sqrt{\sum_{\lambda \in sing(A)} m_\lambda \lambda^2}\,. \tag{4.26}$$

Proof (a) Obviously $\mathfrak{B}_2(\mathsf{H})$ is closed under multiplication by a scalar. Let us show it is closed under sums. If $A, B \in \mathfrak{B}_2(\mathsf{H})$ and N is any Hilbert basis:

$$\sum_{u\in N} ||(A+B)u||^2 \leq \sum_{u\in N} (||Au|| + ||Bu||)^2 \leq 2\left[\sum_{u\in N} ||Au||^2 + \sum_{u\in N} ||Bu||^2\right].$$

Thus $\mathfrak{B}_2(\mathsf{H})$ is a subspace in $\mathfrak{B}(\mathsf{H})$. A consequence of Proposition 4.25(b) is the closure under Hermitian conjugation, which proves (i). We prove (ii) and at the same time that $\mathfrak{B}_2(\mathsf{H})$ is closed under left and right composition with bounded operators. If $A \in \mathfrak{B}_2(\mathsf{H})$ and $B \in \mathfrak{B}(\mathsf{H})$:

$$\sum_{u\in N} ||BAu||^2 \leq \sum_{u\in N} ||B||^2 ||Au||^2 = ||B||^2 \sum_{u\in N} ||Au||^2\,.$$

This shows $\mathfrak{B}_2(\mathsf{H})$ is closed under composition on the left, and also the second inequality in (ii). Closure under right composition follows from closure under Hermitian conjugation and left composites: $B^*A^* \in \mathfrak{B}_2(\mathsf{H})$ if $A \in \mathfrak{B}_2(\mathsf{H})$ and $B \in \mathfrak{B}(\mathsf{H})$, so $(B^*A^*)^* \in \mathfrak{B}_2(\mathsf{H})$, i.e. $AB \in \mathfrak{B}_2(\mathsf{H})$. From (i) we find that if $A \in \mathfrak{B}_2(\mathsf{H})$ and $B \in \mathfrak{B}(\mathsf{H})$, then $||AB||_2 = ||(AB)^*||_2 = ||B^*A^*||_2 \leq ||B^*||\,||A^*||_2 = ||B||\,||A||_2$, finishing part (ii). As for (iii) observe:

$$||A|| = \sup_{||x||=1} ||Ax|| = \sup_{||x||=1} (||Ax||^2)^{1/2} = \sup_{||x||=1} \left(\sum_{u\in N} |(u|Ax)|^2\right)^{1/2}$$

$$= \sup_{||x||=1} \left(\sum_{u\in N} |(A^*u|x)|^2\right)^{1/2},$$

where Theorem 3.26(d) was used for the Hilbert basis N. Using Cauchy–Schwarz on the last term above gives:

$$||A|| \le \sup_{||x||=1} \left(\sum_{u\in N} ||A^*u||^2 ||x||^2 \right)^{1/2} = \left(\sum_{u\in N} ||A^*u||^2 \right)^{1/2} = ||A^*||_2 = ||A||_2 .$$

(b) If $A, B \in \mathfrak{B}_2(\mathsf{H})$ and N is a Hilbert basis, the number of non-zero terms Au and Bu, for $u \in N$, is at most countable by Definition 4.24 and Proposition 3.21(b). Since

$$|(Au|Bu)| \le ||Au||\,||Bu|| \le \frac{1}{2}(||Au||^2 + ||Bu||^2) ,$$

the number of non-vanishing products $(Au|Bu)$, $u \in N$, is also countable at most, and the series of non-null terms $(Au|Bu)$ is absolutely convergent, so the term ordering in (4.24) is irrelevant. In a moment we will show that the choice of Hilbert basis is not important. First, though, notice that (4.25) holds trivially and $(\ |\ )_2$ satisfies the axioms of a semi-inner product, as is easy to check. Positive definiteness (axiom **H3**) follows directly from (iii), so $(\,|\,)_2$ is an inner product inducing $||\ ||_2$. Therefore we have polarisation, for this formula holds for any inner product:

$$4(A|B)_2 = ||A+B||_2^2 + ||A-B||_2^2 - i||A+iB||_2^2 + i||A-iB||_2^2 .$$

Since, by Proposition 4.25, the number on the right does not depend on any Hilbert basis, neither will the left-hand side.
(c) We need only prove the space is complete. Take a Hilbert basis N of H and $\{A_n\}_{n\in\mathbb{N}}$ a Cauchy sequence of HS operators with respect to $||\ ||_2$. From part (iii) in (a), that is a Cauchy sequence also in the uniform topology, and since $\mathfrak{B}(\mathsf{H})$ is complete by Theorem 2.44, there will be $A \in \mathfrak{B}(\mathsf{H})$ with $||A - A_n|| \to 0$ as $n \to +\infty$. Cauchy's property asserts that however we take $\varepsilon > 0$ there is N_ε such that $||A_n - A_m||_2^2 \le \varepsilon^2$ if $n, m > N_\varepsilon$. By definition of $||\ ||_2$, for the same ε we will also have that for any finite subset $I \subset N$:

$$\sum_{u\in I} ||(A_n - A_m)u||^2 \le ||A_n - A_m||_2^2 \le \varepsilon^2$$

whenever $n, m > N_\varepsilon$. Passing to the limit as $m \to +\infty$, we find

$$\sum_{u\in I} ||(A_n - A)u||^2 \le \varepsilon^2 ,$$

for any finite $I \subset N$ if $n > N_\varepsilon$. Overall, given that I is arbitrary,

$$||A - A_n||_2 \le \varepsilon \qquad \text{if } n > N_\varepsilon . \tag{4.27}$$

In particular, then, $A - A_n \in \mathfrak{B}_2(\mathsf{H})$, and so:

$$A = A_n + (A - A_n) \in \mathfrak{B}_2(\mathsf{H}) .$$

Furthermore, also $\varepsilon > 0$ was arbitrary in (4.27), so A_n tends to A in norm $|| \, ||_2$. Therefore every Cauchy sequence for $|| \, ||_2$ converges inside $\mathfrak{B}_2(\mathsf{H})$, making the latter complete.

(d) Let $A \in \mathfrak{B}_2(\mathsf{H})$: we claim it is compact and fulfils (4.26). Take a Hilbert basis N. Then $\sum_{u\in N} ||Au||^2 < +\infty$, where at most countably many summands do not vanish, and the sum can be written as series, or finite sum, by taking only the u_n for which $||Au_n||^2 > 0$. Therefore, for any $\varepsilon > 0$ there exists N_ε such that

$$\sum_{n=N_\varepsilon}^{+\infty} ||Au_n||^2 \leq \varepsilon^2 \, .$$

The same property can be expressed in terms of N: for any $\varepsilon > 0$ there is a finite subset $I_\varepsilon \subset N$ such that

$$\sum_{u\in N\setminus I_\varepsilon} ||Au||^2 \leq \varepsilon^2 \, .$$

Let then A_{I_ε} be the operator defined by $A_{I_\varepsilon} u := Au$ if $u \in I_\varepsilon$ and $A_{I_\varepsilon} u := \mathbf{0}$ if $u \in N \setminus I_\varepsilon$. The range of A_{I_ε} is spanned by the Au with $u \in I_\varepsilon$, because these are finite in number, and A_{I_ε} is bounded and compact (Example 4.18(1)). By construction $||A - A_{I_\varepsilon}||_2$ exists, and:

$$||A - A_{I_\varepsilon}||_2^2 = \sum_{u\in N} ||(A - A_{I_\varepsilon})u||^2 = \sum_{u\in N\setminus I_\varepsilon} ||Au||^2 \, .$$

By part (iii) in (a), therefore,

$$||A - A_{I_\varepsilon}|| \leq ||A - A_{I_\varepsilon}||_2 = \left(\sum_{u\in N\setminus I_\varepsilon} ||Au||^2\right)^{1/2} \leq \varepsilon \, .$$

Hence A is a limit point in the space of compact operators in the uniform topology. As the ideal of compact operators is uniformly closed (Theorem 4.15(c)), A is compact. Now let us prove (4.26). Consider the positive compact operator $|A| = \sqrt{A^*A}$ and let $\{u_{\lambda,i}\}_{\lambda\in sing(A), i=\{1,2,\ldots,m_\lambda\}}$ be a Hilbert basis of $Ker(|A|)^\perp$, built as in Theorem 4.23. We may complete it to a basis of the Hilbert space by adding a basis for $Ker(|A|)$, and $Ker(|A|) = Ker(A)$ by Remark 3.81. (Using the orthogonal splitting $\mathsf{H} = Ker(|A|) \oplus Ker(|A|)^\perp$, if $\{u_i\}$ is a basis for the closed subspace $Ker(|A|)$ and $\{v_j\}$ a basis for the closed $Ker(|A|)^\perp$, the orthonormal system $N := \{u_i\} \cup \{v_j\}$ is a basis of H, since $x \in \mathsf{H}$ orthogonal to N implies $x = \mathbf{0}$.) Using that basis to write $||A||_2$:

$$||A||_2^2 = \sum_{\lambda\in sing(A)} \sum_{i=1}^{m_\lambda} (Au_{\lambda,i}|Au_{\lambda,i}) = \sum_{\lambda\in sing(A)} \sum_{i=1}^{m_\lambda} (A^*Au_{\lambda,i}|u_{\lambda,i}) = \sum_{\lambda\in sing(A)} m_\lambda \lambda^2 \, , \tag{4.28}$$

where the basis of $Ker(A)$, by construction, does not contribute, $|A|u_{\lambda,i} = \sqrt{A^*A}u_{\lambda,i} = \lambda u_{\lambda,i}$ and $(u_{\lambda,i}|u_{\lambda',j}) = \delta_{\lambda\lambda'}\delta_{ij}$.

If, conversely, A is compact and $\{m_\lambda \lambda^2\}_{\lambda\in sing(A)}$ has finite sum, then (4.28) implies $||A||_2 < +\infty$, so $A \in \mathfrak{B}_2(\mathsf{H})$. □

Examples 4.27 **(1)** Let us go back to example (4) in (4.18). Consider the operators:

$$T_K : L^2(\mathsf{X}, \mu) \to L^2(\mathsf{X}, \mu)$$

induced by integral kernels $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$

$$(T_K f)(x) := \int_{\mathsf{X}} K(x, y) f(y) d\mu(y) \quad \text{for any } f \in L^2(\mathsf{X}, \mu)$$

(μ is a σ-finite separable measure). We did prove earlier that these operators are compact. Now we show they are Hilbert–Schmidt operators.

Using the same definition of the example mentioned above, if $f \in L^2(\mathsf{X}, \mu)$ we saw (cf. part (3) in Example 4.18(4)) that for any $x \in \mathsf{X}$

$$F(x) = \int_{\mathsf{X}} |K(x, y)|\, |f(y)| d\mu(y) < +\infty\,.$$

Since $F \in L^2(\mathsf{X}, \mu)$, for any $g \in L^2(\mathsf{X}, \mu)$ the map $x \mapsto g(x)F(x)$ is integrable (so we can define the inner product of g and F). The Fubini–Tonelli theorem guarantees the map $(x, y) \mapsto g(x)K(x, y) f(y)$ is in $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ and that

$$\int_{\mathsf{X}\times\mathsf{X}} \overline{g(x)} K(x, y) f(y)\, d\mu(x) \otimes d\mu(y) = \int_{\mathsf{X}} d\mu(x)\, \overline{g(x)} \int_{\mathsf{X}} K(x, y) f(y) d\mu(y) = (g|T_k f)\,. \tag{4.29}$$

So let us consider a basis of $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ of type $\{u_i \cdot \overline{u_j}\}_{i,j}$, where $\{u_k\}_k$ is a basis for $L^2(\mathsf{X}, \mu)$ (and so is $\{\overline{u_k}\}_k$, as is easy to prove). As $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$, we have an expansion:

$$K = \sum_{i,j} \alpha_{ij} u_i \cdot \overline{u_j}\,, \tag{4.30}$$

and then

$$||K||^2_{L^2} = \sum_{i,j} |\alpha_{ij}|^2 < +\infty\,. \tag{4.31}$$

On the other hand, (4.29) and (4.30) imply

$$(u_i|T_K u_j) = \int_{\mathsf{X}\times\mathsf{X}} \overline{u_i(x)} u_j(y) K(x, y)\, d\mu(x) \otimes d\mu(y) = (u_i \cdot \overline{u_j}|K) = \alpha_{ij}\,,$$

hence (4.31) rephrases as:

$$||K||_{L^2}^2 = \sum_{i,j} |(u_i|T_K u_j)|^2 < +\infty .$$

By definition T_K is therefore a Hilbert–Schmidt operator, and

$$||T_K||_2 = ||K||_{L^2} \tag{4.32}$$

(2) It is not so hard to prove that if $\mathsf{H} = L^2(\mathsf{X}, \mu)$ with μ separable and σ-finite, $\mathfrak{B}_2(\mathsf{H})$ consists precisely of the operators T_K with $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$, so that the map identifying K with T_K is a Hilbert space isomorphism between $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ and $\mathfrak{B}_2(\mathsf{H})$. To see this, we take $T \in \mathfrak{B}_2(\mathsf{H})$ and shall exhibit $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ for which $T = T_K$. Given any basis $\{u_n\}_{n\in\mathbb{N}}$ of $L^2(\mathsf{X}, \mu)$ we have $\sum_{n\in\mathbb{N}} ||Tu_n||^2 < +\infty$. Consequently, by expanding Tu_n in $\{u_n\}_{n\in\mathbb{N}}$ we obtain:

$$+\infty > \sum_{n\in\mathbb{N}} ||Tu_n||^2 = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} |(u_m|Tu_n)|^2 .$$

Interpreting the series as integrals on $\{u_n\}_{n\in\mathbb{N}}$, and applying Fubini–Tonelli, we conclude $\sum_{(n,m)\in\mathbb{N}^2} |(u_m|Tu_n)|^2 < +\infty$, so the HS operator T_K with integral kernel:

$$K := \sum_{(n,m)\in\mathbb{N}^2} (u_m|Tu_n)u_m \cdot \overline{u_n} \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$$

is well defined. At the same time, the results of the previous example tell that, by construction:

$$(u_m|T_K u_n) = \int_{\mathsf{X}} d\mu(x)\overline{u_m(x)} \int_{\mathsf{X}} d\mu(y) K(x, y) u_n(y) = (u_m|Tu_n)$$

and so $T_K u_n = Tu_n$ for any $n \in \mathbb{N}$. By continuity $T_K = T$ follows immediately.

(3) Consider the **Volterra equation** in the unknown function $f \in L^2([0, 1], dx)$:

$$f(x) = \rho \int_0^x f(y)dy + g(x) , \quad \text{with given } g \in L^2([0, 1], dx) \text{ and } \rho \in \mathbb{C} \setminus \{0\}. \tag{4.33}$$

Above, dx is the Lebesgue measure, and the integral exists because, for any given $x \in [0, 1]$, we can view it as the inner product of f and the map $[0, 1] \ni y \mapsto \theta(x - y)$:

$$\int_0^x f(y)dy = \int_0^1 \theta(x - y) f(y)dy ,$$

where $\theta(u) = 1$ if $u \geq 0$ and $\theta(u) = 0$ if $u < 0$. Clearly $(x, y) \mapsto \theta(x - y)$ is also in $L^2([0, 1]^2, dx \otimes dy)$, so the equation can be rephrased using a Hilbert–Schmidt operator T:

$$f = \rho T f + g\,, \quad \text{with given } g \in L^2([0, 1], dx) \text{ and } \rho \in \mathbb{C} \setminus \{0\} \tag{4.34}$$

where we defined the **Volterra operator**:

$$(Tg)(x) := \int_0^x g(y)dy \quad g \in L^2([0, 1], dx)\,. \tag{4.35}$$

Volterra operators, and the associated equations, are more generally defined as:

$$(T_V f)(x) := \int_0^x V(x, y)g(y)dy\,,$$

for some suitably regular $V : [0, 1]^2 \to \mathbb{R}$. We will study the simplest situation, given by (4.35). If the operator $(I - \rho T)$ is invertible, the solution to (4.34) reads:

$$f = (I - \rho T)^{-1} g\,. \tag{4.36}$$

Formally, using the geometric series we see that the (left and right) inverse to $I - \rho T$ is:

$$(I - \rho T)^{-1} = I + \sum_{n=0}^{+\infty} \rho^{n+1} T^{n+1}\,, \tag{4.37}$$

where the convergence is in the uniform topology. A sufficient condition for convergence is $||\rho T|| < 1$, proved in analogy to the geometric series. Yet we will look for a finer estimate. Use the norm of $\mathfrak{B}_2(L^2([0, 1], dx))$ and recall part (iii) in Theorem 4.26(a). Moreover, if $||A_n|| \leq a_n \geq 0$ for any $A_n \in \mathfrak{B}(L^2([0, 1], dx))$ where $\sum_{n=0}^{+\infty} a_n$ converges, then also $\sum_{n=0}^{+\infty} A_n$ converges in $\mathfrak{B}(L^2([0, 1], dx))$. The proof of the latter fact is similar to that of Weierstrass's 'M-test' in elementary calculus. A direct computation with (4.35) shows that if $n \geq 1$:

$$(T^{n+1} g)(x) = \int_0^x \frac{(x - y)^n}{n!} g(y)dy\,,$$

so $T^n \in \mathfrak{B}_2(L^2([0, 1], dx))$ and:

$$||T^n|| \leq ||T^n||_2 = \sqrt{\int_{[0,1]^2} |\theta(x - y)|^2 \left|\frac{(x - y)^{n-1}}{(n - 1)!}\right|^2 dx \otimes dy} \leq \frac{2^{n-1}}{(n - 1)!}\,.$$

Since the series with general term $\frac{\rho^n 2^n}{n!}$ converges, for any $\rho \neq 0$ the operator $(I - \rho T)^{-1}$ exists in $\mathfrak{B}(L^2([0,1], dx))$ and is given by sum (4.37). Therefore (4.36) solves the initial Volterra equation. It is possible to make $(I - \rho T)^{-1}$ explicit

$$\left((I - \rho T)^{-1} g\right)(x) = g(x) + \sum_{n=0}^{+\infty} \rho^{n+1} \left(T^{n+1} g\right)(x) = g(x) + \rho \sum_{n=0}^{+\infty} \int_0^x \frac{(\rho(x-y))^n}{n!} g(y) dy \,.$$

The theorem of dominated convergence warrants we may swap sum and integral, so that Volterra's solution reads:

$$f(x) = \left((I - \rho T)^{-1} g\right)(x) = g(x) + \rho \int_0^x e^{\rho(x-y)} g(y) dy \,.$$

For these operations we used a notion of pointwise convergence that is different from the uniform operator convergence. That the above expression is indeed the explicit inverse to $I - \rho T$ can be checked by a direct computation, using (4.35) and integrating by parts with $g \in C([0,1])$. The result extends to $L^2([0,1], dx)$ because the operator with integral kernel $\theta(x-y)e^{\rho(x-y)}$ is bounded (HS), and $C([0,1])$ is dense in $L^2([0,1], dx)$. The inverse's uniqueness concludes the proof.

(4) Take $L^2(\mathsf{X}, \mu)$ with μ separable. An integral operator $T_K : L^2(\mathsf{X}, \mu) \to L^2(\mathsf{X}, \mu)$ given by the kernel:

$$K(x, y) = \sum_{k=1}^{N} p_k(x) q_k(y) \,,$$

where $p_k, q_k \in L^2(\mathsf{X}, \mu)$, $k = 1, 2, 3, \ldots, N$, are arbitrary maps and $N \in \mathbb{N}$ is chosen at random, is called **degenerate operator**. Degenerate operators form a two-sided *-ideal $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ in $\mathfrak{B}(L^2(\mathsf{X}, \mu))$ that is a subspace of both $\mathfrak{B}_\infty(L^2(\mathsf{X}, \mu))$ and $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$. It is easy to show that $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ is dense in $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ in the latter's norm topology. ■

4.3.2 Integral Kernels and Mercer's Theorem

The content of Examples 4.27(1) and (2) can be subsumed in a theorem. The final assertion is easy and left as exercise.

Theorem 4.28 *If μ is a positive, σ-additive and separable measure on X, the space $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ consists of the operators T_K:*

$$(T_K f)(x) := \int_{\mathsf{X}} K(x, y) f(y) dy \,, \quad \textit{for any } f \in L^2(\mathsf{X}, \mu), \tag{4.38}$$

where $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$. *Moreover:*

$$||T_K||_2 = ||K||_{L^2(\mathsf{X}\times\mathsf{X},\mu\otimes\mu)} .$$

In particular, if $T \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ *and* U *is a Hilbert basis of* $L^2(\mathsf{X}, \mu)$*, then* $T = T_K$ *for the kernel*

$$K = \sum_{u,v\in U\times U} (u|Tv)u \cdot \overline{v} \tag{4.39}$$

and the convergence is in $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$.

The map $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu) \ni K \mapsto T_K \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ *is an isomorphism of Hilbert spaces.*

Mercer's theorem (RiNa53) can be useful when we deal with a compact space $\mathsf{X} \subset \mathbb{R}^n$ with Lebesgue's measure μ. In such a case, if K is continuous and T_K is positive, the convergence of (4.39) is in $||\ ||_\infty$, provided one uses a basis of eigenvectors for T_K. We state and prove the theorem in a slightly more general setup, so to cover Lebesgue measures on compact sets in $\mathbb{R}^n$.

Theorem 4.29 (Mercer) *Let* μ *be a positive, separable Borel measure on a compact Hausdorff space* X *such that* $\mu(\mathsf{X}) < +\infty$ *and* $\mu(A) > 0$ *for any open set* $A \neq \varnothing$. *Assume* $K : \mathsf{X} \times \mathsf{X} \to \mathbb{C}$ *is a continuous function.*
If T_K *in (4.38) is positive, i.e.* $(f|T_K f) \geq 0$ *for* $f \in L^2(\mathsf{X}, \mu)$*, then*

$$K(x, y) = \sum_{\lambda\in\sigma(T_K)} \sum_{i=1}^{m_\lambda} \lambda u_{\lambda,i}(x)\overline{u_{\lambda,i}(y)} , \tag{4.40}$$

where the series converges on $\mathsf{X} \times \mathsf{X}$ *in norm* $||\ ||_\infty$. *The number* m_λ *indicates the dimension (finite if* $\lambda \neq 0$*) of the* λ*-eigenspace of* T_K*, and* $\{u_{\lambda,i}\}_{\lambda\in\sigma_p(T_k),i=1,\ldots,m_\lambda}$ *is a Hilbert basis of eigenvectors (continuous if* $\lambda \neq 0$*) of* T_K.

Proof For simplicity let us relabel eigenvectors as u_j with $j \in \mathbb{N}$, and call λ_j the corresponding eigenvalues (it may happen that $\lambda_j = \lambda_k$ if $j \neq k$). To begin with, notice the eigenvectors with $\lambda \neq 0$ are continuous, by the Cauchy–Schwarz inequality:

$$|u_j(x) - u_j(x')|^2 \leq \int_{\mathsf{X}} |K(x, y) - K(x', y)|^2 d\mu(y) \int_{\mathsf{X}} |u_j(y)|^2 d\mu(y) \to 0 \quad \text{as } x \to x'.$$

We used dominated convergence for the first integral on the right, since K is integrable on X, as $\mu(\mathsf{X})$ is finite, and also continuous on the compact set $\mathsf{X} \times \mathsf{X}$ and hence $|K(x, \cdot) - K(x', \cdot)|^2$ is bounded, uniformly in x, x', by some constant map $C \geq 0$. So take the continuous maps

$$K_n(x, y) := K(x, y) - \sum_{j=0}^{n} \lambda_j u_j(x)\overline{u_j(y)} = \sum_{j=n+1}^{+\infty} \lambda_j u_j(x)\overline{u_j(y)} ,$$

where the last series converges in $L^2(\mathsf{X} \times \mathsf{X}, \mu(x) \otimes \mu(y))$ by Theorem 4.28. Note $\lambda_j \geq 0$, because $0 \leq (u_j | T_K u_j) = \lambda_j$. In the topology of $L^2(\mathsf{X} \times \mathsf{X}, \mu(x) \otimes \mu(y))$ we have:

$$K_n(x, y) = \sum_{j=n+1}^{+\infty} \lambda_j u_j(x) \overline{u_j(y)} \,,$$

so if $f \in L^2(\mathsf{X}, \mu)$

$$\int_{\mathsf{X}} \int_{\mathsf{X}} K_n(x, y) f(y) \overline{f(x)} \mu(x) \mu(y) = \sum_{j=n+1}^{+\infty} \lambda_j (u_j | f)(f | u_j) \geq 0$$

We claim that this fact implies $K_n(x, x) \geq 0$. If there were $x_0 \in \mathsf{X}$ with $K_n(x_0, x_0) < 0$, as K_n is continuous we would be able to find an open neighbourhood (x_0, x_0) where $K_n(x, y) \leq K_n(x_0, x_0) + \varepsilon < 0$. Since $\mathsf{X} \times \mathsf{X}$ has the product topology, we could choose the neighbourhood to be $B_{x_0} \times B_{x_0}$ where B_{x_0} is an open neighbourhood of x_0. By Urysohn's lemma (Theorem 1.24) we could find a continuous map f with support in B_{x_0} such that $0 \leq f \leq 1$ and $f(x_0) = 1$ ($\{x_0\}$ is compact because closed inside a compact space). Then a contradiction would ensue, since $\mu(B_{x_0}) > 0$ by assumption:

$$(f | T_K f) = \int_{\mathsf{X}} \int_{\mathsf{X}} K_n(x, y) f(y) \overline{f(x)} d\mu(x) d\mu(y) = \int_{B_{x_0}} \int_{B_{x_0}} K_n(x, y) f(y) \overline{f(x)} d\mu(x) d\mu(y)$$

$$\leq \left(\int_{B_{x_0}} f(x) d\mu(x) \right)^2 (K_n(x_0, x_0) + \varepsilon) \leq \left(\int_{B_{x_0}} 1 d\mu(x) \right)^2 (K_n(x_0, x_0) + \varepsilon)$$

$$= \mu(B_{x_0})(K_n(x_0, x_0) + \varepsilon) < 0 \,.$$

Therefore if $n = 0, 1, 2, \dots$

$$0 \leq K_n(x, x) = K(x, y) - \sum_{j=0}^{n} \lambda_j u_j(x) \overline{u_j(x)} \,,$$

and the positive-term series $\sum_{j=0}^{+\infty} \lambda_j u_j(x) \overline{u_j(x)}$ converges, with sum bounded by $K(x, x)$. Hence the series $\sum_{j=0}^{+\infty} \lambda_j u_j(x) \overline{u_j(y)}$ converges for any x, uniformly in y. In fact if $M = \max_{x \in \mathsf{X}} K(x, x)$, from Cauchy–Schwarz:

$$\left| \sum_{j=m}^{n} \lambda_j u_j(x) \overline{u_j(y)} \right|^2 \leq \sum_{j=m}^{n} \lambda_j |u_j(x)|^2 \sum_{j=m}^{n} \lambda_j |u_j(y)|^2 \leq M \sum_{j=m}^{n} \lambda_j |u_j(x)|^2 \,.$$

Therefore $B(x, y) := \sum_{j=0}^{+\infty} \lambda_j u_j(x)\overline{u_j(y)}$ is continuous in y for any given x. By dominated convergence, for any continuous $f : \mathsf{X} \to \mathbb{C}$ and any given x

$$\int_{\mathsf{X}} B(x, y) f(y) d\mu(y) = \sum_{j=0}^{+\infty} \lambda_j u_j(x) \int_{\mathsf{X}} \overline{u_j(y)} f(y) d\mu(y) \tag{4.41}$$

by virtue of the above series' uniform convergence on X (of finite measure). But we also know the series on the right converges to $T_K f$ in $L^2(\mathsf{X}, d\mu(x))$: we claim it converges to $(T_K f)(x)$ *also pointwise* at x. Since K is continuous on X (compact and of finite measure), an obvious consequence of dominated convergence shows $\mathsf{X} \ni x \mapsto \int_{\mathsf{X}} |K(x, y)|^2 d\mu(y)$ is continuous, so a constant C exists such that:

$$\int_{\mathsf{X}} |K(x, y)|^2 d\mu(y) < C^2 \quad \text{for any } x \in \mathsf{X}\,.$$

Hence if $f_n \to f$ in $L^2(\mathsf{X}, \mu)$, then $T_K f_n \to T_K f$ in norm $|| \;\; ||_\infty$:

$$|(Tf)(x) - (Tf_n)(x)|^2 = \left| \int_{\mathsf{X}} K(x, y)(f(y) - f_n(y)) d\mu(y) \right|^2$$

$$\leq \int_{\mathsf{X}} |K(x, y)|^2 d\mu(y) \int_{\mathsf{X}} |f(y) - f_n(y)|^2 d\mu(y) \leq C^2 \mu(\mathsf{X}) ||f - f_n||_\infty^2\,.$$

Now if we decompose f in (4.41) using the eigenvector basis of T_K,

$$f(x) = \sum_{j=0}^{+\infty} u_j(x) \int_{\mathsf{X}} \overline{u_j(y)} f(y) d\mu(y),$$

we obtain a convergent expansion in $L^2(\mathsf{X}, d\mu(x))$. Applying T_K must give a uniformly converging series at $x \in \mathsf{X}$. Therefore the last series in (4.41) converges pointwise (and uniformly) to $(T_K f)(x)$ for any $x \in \mathsf{X}$. Comparing with the left-hand side of (4.41) and recalling that $(T_K f)(x) = \int K(x, y) f(y) d\mu(y)$ we finally get

$$\int_{\mathsf{X}} (B(x, y) - K(x, y))\, f(y) d\mu(y) = 0\,.$$

Choosing $f(y) := \overline{B(x, y) - K(x, y)}$, for any given $x \in \mathsf{X}$, allows to conclude $B(x, y) = K(x, y)$ by using Proposition 1.71 suitably. So

$$K(x, x) = B(x, x) = \sum_{j=0}^{+\infty} \lambda_j |u_j(x)|^2\,.$$

The terms are continuous, non-negative and the sum is a continuous map, so Dini's Theorem 2.20 forces uniform convergence. From Cauchy–Schwarz we deduce that $\sum_{j=0}^{+\infty} \lambda_j u_j(x)\overline{u_j(y)}$ converges uniformly, jointly in x, y, to some $K'(x, y)$. But $\mathsf{X} \times \mathsf{X}$ has finite measure, so the convergence is in $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$. Since we know the series converges to $K(x, y)$ in that topology, then it converges uniformly to $K(x, y)$ and the proof ends. □

Remark 4.30 The theorem is still valid if T_K has a finite number of negative eigenvalues, as is easy to prove. ■

4.4 Trace-Class (or Nuclear) Operators

In this part we introduce operators of *trace class*, also known as *nuclear operators*. We shall follow the approach of (Mar82) essentially (a different perspective is found in (Pru81)).

4.4.1 General Properties

Proposition 4.31 *Let* H *be a Hilbert space and* $A \in \mathfrak{B}(\mathsf{H})$. *The following three facts are equivalent.*
(a) *There exists a Hilbert basis* N *of* H *such that* $\{(u||A|u)\}_{u\in N}$ *has finite sum:*

$$\sum_{u\in N}(u||A|u) < +\infty\ .$$

(a)' $\sqrt{|A|}$ *is a Hilbert–Schmidt operator.*
(b) *A is compact and the indexed set* $\{m_\lambda \lambda\}_{\lambda\in sing(A)}$, *where* m_λ *is the multiplicity of* λ, *has finite sum.*

Proof Statement (a)' is a mere translation of (a), for $\sqrt{|A|}\sqrt{|A|} = |A|$, so (a) and (a)' are equivalent.

We will show (a) $\Rightarrow$ (b). Recall that any HS operator, in particular $\sqrt{|A|}$, is compact (Theorem 4.26(d)); secondly, the product of compact operators, e.g. $|A| = (\sqrt{|A|})^2$, is compact (Theorem 4.15(b)); at last, $|A| = (\sqrt{|A|})^2$ is compact iff A is compact (Proposition 4.14). As a consequence of all this, A is compact. Let us take a basis of H made of eigenvectors of $|A|$: $u_{\lambda,i}$, $i = 1, \ldots, m_\lambda$ ($m_\lambda = +\infty$ possibly, only for $\lambda = 0$) and $|A|u_{\lambda,i} = \lambda u_{\lambda,i}$. In such basis:

$$\left|\left|\sqrt{|A|}\right|\right|_2^2 = \sum_{\lambda,i}\left(\sqrt{|A|}u_{\lambda,i}\,\middle|\,\sqrt{|A|}u_{\lambda,i}\right) = \sum_{\lambda,i}\left(u_{\lambda,i}\,\middle|(\sqrt{|A|})^2 u_{\lambda,i}\right) = \sum_{\lambda,i}(u_{\lambda,i}||A|u_{\lambda,i})$$

$$= \sum_{\lambda} m_\lambda \lambda\ .$$

So $\{m_\lambda \lambda\}_{\lambda \in sing(A)}$ has finite sum because $\left|\left|\sqrt{|A|}\right|\right|_2^2 < +\infty$ by assumption. Conversely, it is obvious that (b) $\Rightarrow$ (a)' by proceeding backwards in the argument and computing $\left|\left|\sqrt{|A|}\right|\right|_2^2$ in a basis of eigenvectors of $|A|$. □

Definition 4.32 Let H be a Hilbert space. The operator $A \in \mathfrak{B}(\mathsf{H})$ is said to be of **trace class**, or a **nuclear operator**, if it satisfies either of (a), (a)' or (b) in Proposition 4.31. The set of trace-class operators on H will be denoted by $\mathfrak{B}_1(\mathsf{H})$. In the notation of Proposition 4.31, if $A \in \mathfrak{B}_1(\mathsf{H})$,

$$||A||_1 := \left|\left|\sqrt{|A|}\right|\right|_2^2 = \sum_{\lambda \in sing(A)} m_\lambda \lambda \,. \tag{4.42}$$

Remark 4.33 **(1)** The name "trace class" has its origin in the following observation. For an operator A of trace class, the real number $||A||_1$ generalises to infinite dimensions the notion of trace of the matrix corresponding to $|A|$ (not A). As a matter of fact, the analogies do not end here, as we shall soon see.
(2) The following inclusions hold:

$$\mathfrak{B}_1(\mathsf{H}) \subset \mathfrak{B}_2(\mathsf{H}) \subset \mathfrak{B}_\infty(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H}) \,.$$

The only relation we have not yet proved is the first one. To this end, if $A \in \mathfrak{B}_1(\mathsf{H})$, by definition $\sqrt{|A|} \in \mathfrak{B}_2(\mathsf{H})$, so $|A| = \sqrt{|A|}\sqrt{|A|}$ is HS by Theorem 4.26(a). From the polar decomposition $A = U|A|$, $U \in \mathfrak{B}(\mathsf{H})$, we have $A \in \mathfrak{B}_2(\mathsf{H})$ by Theorem 4.26(a).
(3) Each of the above sets is a subspace in the vector space of bounded operators, and also a two-sided *-ideal (for trace-class operators we will prove it in a moment). Furthermore, each has a natural Hilbert or Banach structure: compact operators are closed in $\mathfrak{B}(\mathsf{H})$ in the uniform topology, so they form a Banach space for the operator norm; HS operators form a Hilbert space with the Hilbert–Schmidt inner product; trace-class operators form a Banach space with norm $|| \;\; ||_1$, as we will explain later. ■

Before we extend the notion of trace to the infinite-dimensional case, let us review the key features of nuclear operators.

Theorem 4.34 *Let* H *be a Hilbert space. Nuclear operators on* H *enjoy the following properties.*
(a) *If* $A \in \mathfrak{B}_1(\mathsf{H})$ *there exist two operators* $B, C \in \mathfrak{B}_2(\mathsf{H})$ *such that* $A = BC$. *Conversely, if* $B, C \in \mathfrak{B}_2(\mathsf{H})$ *then* $BC \in \mathfrak{B}_1(\mathsf{H})$ *and:*

$$||BC||_1 \le ||B||_2 \, ||C||_2 \,. \tag{4.43}$$

(b) $\mathfrak{B}_1(\mathsf{H})$ *is a subspace of* $\mathfrak{B}(\mathsf{H})$*, and actually a two-sided* **-ideal. Moreover:*
(i) $||AB||_1 \leq ||B||\,||A||_1$ *and* $||BA||_1 \leq ||B||\,||A||_1$ *for any* $A \in \mathfrak{B}_1(\mathsf{H})$ *and* $B \in \mathfrak{B}(\mathsf{H})$*;*
(ii) $||A||_1 = ||A^*||_1$ *for any* $A \in \mathfrak{B}_1(\mathsf{H})$*;*
(c) $||\ \ ||_1$ *is a norm on* $\mathfrak{B}_1(\mathsf{H})$.

Remark 4.35 It can be proved $(\mathfrak{B}_1(\mathsf{H}), ||\ \ ||_1)$ is a Banach space (Sch60, BiSo87).■

Proof of Theorem 4.34 *(part (b)(ii) is deferred to Proposition* 4.38*)*. (a) If A is of trace class, the polar decomposition $A = U|A|$ tells B and C can be taken to be $B := U\sqrt{|A|}$ and $C := \sqrt{|A|}$. By definition of trace-class operator $\sqrt{|A|}$ is a Hilbert–Schmidt operator, so C is, too. Also B is HS, for $U \in \mathfrak{B}(\mathsf{H})$ and $\mathfrak{B}_2(\mathsf{H})$ is a two-sided ideal in $\mathfrak{B}(\mathsf{H})$, by Theorem 4.26. Let now B, C be HS operators, and let us show $A := BC$ is of trace class. By Theorems 4.26(d) and 4.15(b) A is compact. Hence we need only show $\sum_{\lambda \in sing(A)} m_\lambda \lambda < +\infty$. If $BC = 0$, the claim is obvious. Assume $BC \neq 0$ and expand the compact operator BC after Theorem 4.23:

$$A = BC = \sum_{\lambda \in sing(A)} \sum_{i=1}^{m_\lambda} \lambda (u_{\lambda,i}|\ \ ) v_{\lambda,i}\ .$$

Lest the notation become too heavy, set

$$\Gamma := \{(\lambda, i) | \lambda \in sing(A),\ \ i = 1, 2, \ldots, m_\lambda\}$$

and suppose λ_j is the first element in the pair $j = (\lambda, i)$. Then, clearly,

$$\sum_{j \in \Gamma} \lambda_j = \sum_{\lambda \in sing A} m_\lambda \lambda\ .$$

From the polar decomposition theorem $A = U|A|$, with $U^*U = I$ on the range of $|A|$. Since $v_j = Uu_j$ implies $U^* v_j = u_j$, we have:

$$(v_j|BCu_j) = (v_j|Au_j) = (v_j|U|A|u_j) = \lambda_j (v_j|Uu_j) = \lambda_j (U^* v_j|u_j) = \lambda_j (u_j|u_j) = \lambda_j\ .$$

If $S \subset \Gamma$ is finite:

$$\sum_{j \in S} \lambda_j = \sum_{j \in S} (v_j|BCu_j) = \sum_{j \in S} (B^* v_j|Cu_j)$$

$$\leq \sum_{j \in S} ||B^* v_j||\,||Cu_j|| \leq \sqrt{\sum_{j \in S} ||B^* v_j||^2} \sqrt{\sum_{j \in S} ||Cu_j||^2}\ .$$

As the orthonormal systems $u_j = u_{\lambda,i}$ and $v_j = v_{\lambda,i}$ can be both completed to give bases of H, the final term in the chain of inequalities above is smaller than

$$||B^*||_2 \, ||C||_2 = ||B||_2 \, ||C||_2 \,.$$

Taking the supremum over all finite sets S we conclude

$$||BC||_1 = \sum_{\lambda \in sing(A)} m_\lambda \lambda \leq ||B||_2 \, ||C||_2$$

and in particular $A = BC \in \mathfrak{B}_1(\mathsf{H})$.

(b)–(c). The closure of $\mathfrak{B}_1(\mathsf{H})$ under inner product is immediate from the definition itself. Let us show $\mathfrak{B}_1(\mathsf{H})$ is closed under sums. Take $A, B \in \mathfrak{B}_1(\mathsf{H})$. If $A + B = 0$, $A + B$ is clearly nuclear. So assume $A + B \neq 0$ (compact anyway) and write polar decompositions: $A = U|A|$, $B = V|B|$, $A + B = W|A + B|$. Use the familiar sum over singular values, like we did in (a), to find:

$$A + B = \sum_{\beta \in sing(A+B)} \sum_{i=1}^{m_\beta} \beta (u_{\beta,i} | \;) v_{\beta,i} \,.$$

Set $\Gamma := \{(\beta, i) | \beta \in sing(A + B), \;\; i = 1, 2, \dots, m_\beta\}$, take $S \subset \Gamma$ finite and let β_j be the first element in the pair $j \in \Gamma$. Then:

$$\sum_{j \in S} \beta_j = \sum_{j \in S} (v_j | (A + B) u_j) = \sum_{j \in S} (v_j | A u_j) + \sum_{j \in S} (v_j | B u_j) \,.$$

We can rewrite this as follows:

$$\sum_{j \in S} \beta_j = \sum_{j \in S} (\sqrt{|A|} U^* v_j | \sqrt{|A|} u_j) + \sum_{j \in S} (\sqrt{|B|} V^* v_j | \sqrt{|B|} u_j) \,.$$

Proceeding as in (a) gives:

$$\sum_{j \in S} \beta_j \leq ||\sqrt{|A|} U^*||_2 \, ||\sqrt{|A|}||_2 + ||\sqrt{|B|} V^*||_2 \, ||\sqrt{|B|}||_2 \leq ||\sqrt{|A|}||_2^2 + ||\sqrt{|B|}||_2^2$$

(in the final passage we use the inequality $||\sqrt{|A|} U^*||_2 \leq ||\sqrt{|A|}||_2 \, ||U^*||$ (part (ii) in Theorem 4.26(a)), since $\sqrt{|A|}$ is Hilbert–Schmidt. Furthermore, it is easy to see $||U^*|| \leq 1$, because U^* is isometric on $Ker(|A|)^\perp$ and vanishes on $Ker(|A|)$).

Eventually note:

$$||\sqrt{|A|}||_2^2 + ||\sqrt{|B|}||_2^2 = ||A||_1 + ||B||_1 \,.$$

So we have proved $A + B \in \mathfrak{B}_1(\mathsf{H})$ and that the triangle inequality

$$||A + B||_1 \leq ||A||_1 + ||B||_1$$

holds on $\mathfrak{B}_1(\mathsf{H})$. This turns $|| \;\; ||_1$ into a seminorm. It is indeed a norm, for $||A||_1 = 0$ implies the eigenvalues of $|A|$ are all zero. By compactness $|A| = 0$, from (b) in Theorem 4.19(6). The polar decomposition of $A = U|A|$ forces $A = 0$. At this stage we have proved $\mathfrak{B}_1(\mathsf{H})$ is a subspace of $\mathfrak{B}(\mathsf{H})$ and $|| \;\; ||_1$ is a norm. Let us show that $\mathfrak{B}_1(\mathsf{H})$ is closed under composition with bounded operators on either side. Take $A \in \mathfrak{B}_1(\mathsf{H})$, $B \in \mathfrak{B}(\mathsf{H})$ and write $A = U|A|$. Then $BA = (BU\sqrt{|A|})\,\sqrt{|A|}$, where the two factors are HS operators, so $BA \in \mathfrak{B}_1(\mathsf{H})$ by part (a). Using Theorem 4.26(a)(ii), Eq. (4.42) and part (a):

$$||BA||_1 \le ||BU\sqrt{|A|}||_2\, ||\sqrt{|A|}||_2 \le ||BU||\, ||\sqrt{A}||_2\, ||\sqrt{A}||_2 \le ||B||\, ||\sqrt{A}||_2^2 = ||B||\, ||A||_1 \,.$$

Moreover $AB = (U\sqrt{|A|})\,\sqrt{|A|}B \in \mathfrak{B}_1(\mathsf{H})$ because both factors are HS and (a) holds. In a manner similar to part (a) one proves $||AB||_1 \le ||B||\, ||A||_1$. Statement (ii) in part (b) will be justified in the proof of Proposition 4.38. □

4.4.2 The Notion of Trace

To conclude we introduce the notion of trace of a nuclear operator, and we show how it has the same formal properties of the trace of a matrix.

Proposition 4.36 *If* $(\mathsf{H}, (\,|\,))$ *is a Hilbert space,* $A \in \mathfrak{B}_1(\mathsf{H})$ *and* N *is a basis of* H*, then*

$$tr A := \sum_{u\in N} (u|Au) \tag{4.44}$$

is well defined, since the series on the right is finite or absolutely convergent. Moreover:

(a) $tr A$ *does not depend on the chosen Hilbert basis;*

(b) *for any pair* (B, C) *of Hilbert–Schmidt operators such that* $A = BC$*:*

$$tr A = (B^*|C)_2 \,; \tag{4.45}$$

(c) $|A| \in \mathfrak{B}_1(\mathsf{H})$ *and:*

$$||A||_1 = tr|A| \,; \tag{4.46}$$

(d) $|tr A| \le tr|A|$*, so that* $tr : \mathfrak{B}_1(\mathsf{H}) \to \mathbb{C}$ *is continuous with respect to* $|| \;\; ||_1$.

Proof (a)–(b). Any trace-class operator can be decomposed in the product of two HS operators as we saw in Theorem 4.34(a). We begin by noticing that if $A = BC$, with B, C Hilbert–Schmidt, then

$$(B^*|C)_2 = \sum_{u\in N} (B^*u|Cu) = \sum_{u\in N} (u|BCu) = \sum_{u\in N} (u|Au) = tr A \,.$$

This justifies (4.45) but also explains that $tr A$ is well defined, being a Hilbert–Schmidt inner product. Moreover, it says that in the infinite sum (4.44) only countably many summands, at most, are non-zero, and that the sum reduces to a finite sum or to an absolutely convergent series, since $\sum_{u\in N} |(B^*u|Cu)| < +\infty$ by definition of HS inner product. (It also shows $(B^*|C)_2 = (B'^*|C')_2$ if $BC = B'C'$, for B, B', C, C' are HS operators.) The result eventually proves the invariance of $tr A$ under changes of basis, because $(\ |\ )_2$ does not depend on the chosen Hilbert basis.
(c) Firstly, by uniqueness of the square root, $|\,(|A|)\,| = |A|$. In fact $|\,(|A|)\,|$ is the only positive bounded operator whose square is $|A|^*|A| = |A|^2$, and $|A|$ is bounded, positive and squaring to $|A|^2$. As A is of trace class:

$$+\infty > \sum_{u\in N}(u||A|u) = \sum_{u\in N}(u||\,(|A|)\,|u)\ ,$$

so Definition 4.32 implies $|A|$ itself is of trace class. Choosing a basis $\{u_{\lambda,i}\}$ of eigenvectors for $|A|$ we have:

$$tr|A| = \sum_{\lambda\in sing(A)}\sum_{i=1}^{m_\lambda}(u_{\lambda,i}|\ |A|u_{\lambda,i}) = \sum_{\lambda\in sing(A)}\sum_{i=1}^{m_\lambda}\lambda = \sum_{\lambda\in sing(A)} m_\lambda\lambda = ||A||_1\ .$$

Eventually, (d) is easy: to compute $tr A$ we use again a basis N of eigenvectors u of $|A|$ and exploit the polar decomposition of $A = U|A|$.

$$|tr A| \le \sum_{\lambda\in sing(A)}\sum_{i=1}^{m_\lambda}|(u_{\lambda,i}|\ U|A|u_{\lambda,i})| = \sum_{\lambda\in sing(A)}\sum_{i=1}^{m_\lambda}\lambda|(u_{\lambda,i}|Uu_{\lambda,i})| \le \sum_{\lambda\in sing(A)}\sum_{i=1}^{m_\lambda}\lambda = tr|A|\ .$$

Hence the proof ends. □

Definition 4.37 Let H be a Hilbert space and $A \in \mathfrak{B}_1(\mathsf{H})$. The number $tr A \in \mathbb{C}$ is called the **trace** of the operator A.

The next proposition summarises other useful properties of nuclear operators on Hilbert spaces: in particular – and precisely as in the finite-dimensional case – the trace is invariant under cyclic permutations. We remark that the operators of the statements below need not necessarily be all of trace class (an important fact in physical applications).

Proposition 4.38 *Let H be a Hilbert space. The trace enjoys the following properties.*
(a) *If $A, B \in \mathfrak{B}_1(\mathsf{H})$ and $\alpha, \beta \in \mathbb{C}$, then:*

$$tr A^* = \overline{tr A}\ , \tag{4.47}$$
$$tr(\alpha A + \beta B) = \alpha\, tr A\ +\ \beta\, tr B\ . \tag{4.48}$$

(b) *If A is of trace class and $B \in \mathfrak{B}(\mathsf{H})$, or A and B are both HS operators, then*

$$tr\,AB = tr\,BA\,. \tag{4.49}$$

(c) *Let $A_1, A_2, \dots A_n$ be in $\mathfrak{B}(\mathsf{H})$. If one is of trace class, or two are HS operators, then the trace is* **invariant under cyclic permutations***:*

$$tr\,(A_1 A_2 \cdots A_n) = tr\,(A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)})\,, \tag{4.50}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a cyclic permutation of $(1, 2, \dots, n)$.
(d) *If $A \in \mathfrak{B}_1(\mathsf{H})$ then:*

$$||A|| \le ||A||_2 = \sqrt{tr(|A|^2)} \le tr|A| = ||A||_1\,.$$

Proof of proposition 4.38 and of part (ii) in Theorem 4.34(b).
(a) Immediate by definition of trace.
(b) Let us begin by proving the statement if A and B are both HS operators. By Theorem 4.26(b), Eq. (4.49) is equivalent to

$$(A^*|B)_2 = (B^*|A)_2\,. \tag{4.51}$$

The proof of (4.51) is straightforward using the polarisation formula (valid for any inner product and the induced norm)

$$4(X|Y) = ||X+Y||^2 + ||X-Y||^2 - i||X+iY||^2 + i||X-iY||^2\,,$$

and recalling that, for HS norms, $||Z||_2 = ||Z^*||_2$ ((i) in Theorem 4.26(a)).

Now suppose A is of trace class and $B \in \mathfrak{B}(\mathsf{H})$. Then $A = CD$, with C and D Hilbert–Schmidt operators by Theorem 4.34(a). In addition, DB and BC are Hilbert–Schmidt, for $\mathfrak{B}_2(\mathsf{H})$ is a two-sided ideal in $\mathfrak{B}(\mathsf{H})$. By swapping two HS operators at a time:

$$tr\,AB = tr((CD)B) = tr(C(DB)) = tr((DB)C) = tr(D(BC)) = tr((BC)D)$$

$$= tr(B(CD)) = tr\,BA\,.$$

(c) Since $\mathfrak{B}_1(\mathsf{H})$ is a two-sided ideal in $\mathfrak{B}(\mathsf{H})$, one operator of trace class among $A_1, \dots, A_n$ is enough to render their product of trace class. In particular, using Theorem 4.34(a) and the fact that $\mathfrak{B}_2(\mathsf{H})$ is a two-sided ideal of $\mathfrak{B}(\mathsf{H})$ we see clearly that if two among $A_1, \dots, A_n$ are HS, their product is of trace class. Then (4.50) is equivalent to:

$$tr\,(A_1 A_2 \cdots A_n) = tr\,(A_2 A_3 \cdots A_n A_1)\,. \tag{4.52}$$

In fact, one transposition at a time, we obtain any cyclic permutation. Let us prove (4.52). Consider first the case of two HS operators A_i, A_j, $i < j$. If $i = 1$, the claim follows from (b) with $A = A_1$ and $B = A_2 \cdots A_n$. If $i > 1$, the four operators (i) $A_1 \cdots A_i$, (ii) $A_{i+1} \cdots A_n$, (iii) $A_{i+1} \cdots A_n A_1$, (iv) $A_2 \cdots A_i$ are necessarily HS, for they involve either A_i or A_j as factor (never both). Hence:

$$tr(A_1 \cdots A_n) = tr(A_1 \cdots A_i \; A_{i+1} \cdots A_n) = tr(A_{i+1} \cdots A_n \; A_1 A_2 \cdots A_i)$$

$$= tr(A_2 \cdots A_i \; A_{i+1} \cdots A_n A_1) ,$$

which is what we wanted.

Let us prove invariance under permutations, assuming A_i is of trace class. If $i = 1$ the claim follows from part (b) by taking $A = A_1$ and $B = A_2 \cdots A_n$. So suppose $i > 1$. Then $A_1 \cdots A_i$ and $A_2 \cdots A_i$ are of trace class because both contain A_i, and then:

$$tr(A_1 \cdots A_n) = tr(A_1 \cdots A_i \; A_{i+1} \cdots A_n) = tr(A_{i+1} \cdots A_n \; A_1 A_2 \cdots A_i)$$

$$= tr(A_2 \cdots A_i A_{i+1} \cdots A_n A_1) ,$$

recalling part (b). Invariance under permutations allows to prove part (ii) in Theorem 4.34(b). Using (4.46) we have to prove $tr|A| = tr|A^*|$. By the corollary to the polar decomposition theorem (Theorem 3.82) we deduce $|A^*| = U|A|U^*$, where $U|A| = A$ is the polar decomposition of A. Hence

$$||A^*||_1 = tr|A^*| = tr(U|A|U^*) = tr(U^*U|A|) = tr|A| = ||A||_1 ,$$

where we used $U^*U|A| = |A|$, for U is isometric on $Ran(|A|)$ (Theorem 3.82).

(d) Taking (iii) in Theorem 4.26(a) into account, observing that $|A|^2 = A^*A$, and applying the various definitions, the only thing to be proved is $\sqrt{tr(|A|^2)} \le tr(|A|)$, that is $tr(|A|^2) \le (tr(|A|))^2$. This inequality is trivially true if we write traces using a basis of eigenvectors of $|A|$. □

Remark 4.39 **(1)** If $A \in \mathfrak{B}_1(\mathsf{H})$ and $A = A^*$, computing the trace of A through a basis of eigenvectors of A itself (this exists by Theorem 4.20), we conclude $tr(A) = \sum_{\lambda \in \sigma_p(A)} m_\lambda \lambda$, where $\sigma_p(A)$ is, as always, the set of eigenvalues of A and m_λ the dimension of the λ-eigenspace. As for finite dimensions, the trace of a *self-adjoint* operator of trace class coincides with the sum of the eigenvalues. This is true even if A is *not* self-adjoint, provided we clarify the meaning of eigenvalue multiplicity.

Theorem 4.40 (Lidskii) *If* H *is a complex Hilbert space and* $T \in \mathfrak{B}_1(\mathsf{H})$*, then* $tr(T) = \sum_{\lambda \in \sigma_p(T)} \mu_\lambda \lambda$*, where* $\sigma_p(T)$ *is the eigenvalue set of* T*,* μ_λ *is the* algebraic multiplicity *of the eigenvalue* λ*, and the series converges absolutely.*

The result is far from obvious, and a proof can be found in (GGK00, BiSo87). The algebraic multiplicity is discussed in (BiSo87, p.77), so we just mention that $0 < m_\lambda \leq \mu_\lambda$.

(2) Taking the inclusions $\mathfrak{B}_1(\mathsf{H}) \subset \mathfrak{B}_2(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H})$ into account, Proposition 4.38(d) now has the following consequence. If $\{x_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}_1(\mathsf{H})$ converges to $x \in \mathfrak{B}_1(\mathsf{H})$ in the topology of $||\ \ ||_1$, then it converges in the topology of $|| \cdot ||_2$. The same result holds replacing $(\mathfrak{B}_1(\mathsf{H}), ||\ \ ||_1)$ by $(\mathfrak{B}_2(\mathsf{H}), ||\ \ ||_2)$ and $(\mathfrak{B}_2(\mathsf{H}), ||\ \ ||_2)$ by $(\mathfrak{B}(\mathsf{H}), ||\ \ ||)$. In other words the topology of $\mathfrak{B}_1(\mathsf{H})$ is finer than the topology of $\mathfrak{B}_2(\mathsf{H})$, and this is in turn finer than the topology of $\mathfrak{B}(\mathsf{H})$. ■

To conclude we present alternative definitions of trace-class operators (stated in the form of propositions), which can be found in textbooks.

We will be able to justify the first one only after the spectral theorem for self-adjoint operators:

Proposition 4.41 *If H is a complex Hilbert space, $T \in \mathfrak{B}(\mathsf{H})$ is of trace class if and only if $\sum_{u\in N} |(u|Tu)| < +\infty$ for every Hilbert basis $N \subset \mathsf{H}$.*

Proof See the solution to Exercise 8.20. □

If the Hilbert spaces is infinite-dimensional and separable, there is a third characterisation:

Proposition 4.42 *If H is an infinite-dimensional, separable complex Hilbert space, $T \in \mathfrak{B}(\mathsf{H})$ is of trace class if and only if its trace is well defined, i.e. $\sum_{n=0}^{+\infty}(u_n|Tu_n)$ converges to a unique $s \in \mathbb{C}$ for every Hilbert basis $\{u_n\}_{n\in\mathbb{N}}$.*

Proof If T is of trace class, $\sum_{n=0}^{+\infty}(u_n|Tu_n)$ converges for any basis $\{u_n\}_{n\in\mathbb{N}}$, and the sum does not depend on the basis because it is nothing but trT. If, conversely, $\sum_{n=0}^{+\infty}(u_n|Tu_n) = s \in \mathbb{C}$ for every basis $\{u_n\}_{n\in\mathbb{N}}$, then $\sum_{n=0}^{+\infty}(u'_n|Tu'_n) = s$ where $u'_n := u_{f(n)}$ for any bijection $f : \mathbb{N} \to \mathbb{N}$. In other words the sum does not depend of the summation order and may be rearranged. In view of Theorem 1.83, the series of complex numbers $\sum_{n=0}^{+\infty}(u_n|Tu_n)$ converges absolutely, i.e., $\sum_{n=0}^{+\infty}|(u_n|Tu_n)| < +\infty$. Since this result is valid for every basis $\{u_n\}_{n\in\mathbb{N}}$ of H, Proposition 4.41 proves that T is of trace class. □

Example 4.43 Here some familiarity with Riemannian geometry is required.

A important trace-class operator in physics arises (see, e.g. (Mor99)) when studying the *Laplace-Beltrami operator* (or *Laplacian*) on a *Riemannian manifold* (M, g). In local coordinates $x_1, \ldots, x_n$ on the n-manifold M, the Laplacian is the differential operator:

$$\Delta = \sum_{i=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} g^{ij}(x)\sqrt{g}\frac{\partial}{\partial x_j} \ ,$$

where g is the determinant of the matrix $(g_{ij})_{i,j=1,\ldots,n}$ that describes the metric tensor in the given coordinates, and g^{ij} are the coefficients of the inverse matrix. If $V : M \to (K, +\infty)$, for a certain $K > 0$, is an arbitrary smooth map, we consider the operator $A = -\Delta + V$ defined on the space $\mathscr{D}(M)$ of smooth complex-valued maps on M. We may view $\mathscr{D}(M)$ as a (dense) subspace in $L^2(M, \mu_g)$, where μ_g, the natural Borel measure associated to the metric, reads $\sqrt{g}dx_1 \cdots dx_n$ in local coordinates. The operator A is positive, not bounded, and admits a unique inverse (also positive): $A^{-1} : L^2(M, \mu_g) \to \mathscr{D}(M)$. Thinking of A^{-1} as an $L^2(M, \mu_g)$-valued operator, it turns out that $A^{-1} \in \mathfrak{B}(L^2(M, \mu_g))$. The first interesting fact is that $A^{-1} \in \mathfrak{B}_\infty(L^2(M, \mu_g))$ actually. But there is more to the story. According to a theorem of Weyl the eigenvalues $\lambda_j \in \sigma_p(A)$ of A (where j labels eigenvectors ϕ_j and not eigenvalues, so that $\phi_i \neq \phi_k$ if $k \neq i$ but $0 < K \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$) satisfy the asymptotic formula:

$$\lim_{j \to +\infty} j^{-1} \lambda_j^{n/2} = \frac{k_n}{vol(M)} , \tag{4.53}$$

where $vol(M)$ is the manifold's volume (finite by compactness) and k_n a universal constant that depends only on the dimension n. Furthermore, the eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ form a basis of $L^2(M, \mu_g)$, which implies that the eigenvalue set of A^{-k} is $\sigma_p(A^{-k}) = \{\lambda_j^{-k}\}_{j\in\mathbb{N}}$. Computing the norms $|| \ ||_1$ and $|| \ ||_2$ of $A^{-k} = |A|^{-k}$, and using the basis of eigenvectors of A, we have:

$$||A^{-k}||_1 = \sum_{j=0}^{+\infty} \lambda_j^{-k} \quad \text{and} \quad ||A^{-k}||_2^2 = \sum_{j=0}^{+\infty} \lambda_j^{-2k} .$$

Weyl's law (4.53) implies that $A^{-k} \in \mathfrak{B}_1(L^2(M, \mu_g))$ if $k > n/2$, and $A^{-k} \in \mathfrak{B}_2(L^2(M, \mu_g))$ if $k > n/4$. ■

4.5 Introduction to the Fredholm Theory of Integral Equations

Integral equations are a central branch of functional analysis, especially with regard to applications in physics (for instance in the theory of quantum scattering (Pru81) and the study of inverse problems) and other sciences. In the sequel we shall present general results, due to Fredholm for the most part, and we will regularly take an abstract viewpoint, whereby integral operators are seen as particular compact operators on Hilbert spaces (even though several results can be extended to Banach spaces). We shall essentially follow (KoFo99).

To fix ideas, let us consider a measure space $(\mathsf{X}, \Sigma, \mu)$, where $\mu : \Sigma \to [0, +\infty]$ is a positive (σ-additive) measure that is σ-finite and separable, and take a map $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ with no further properties. Define $T_K \in \mathfrak{B}_2(\mathsf{H})$ to be the usual integral operator (cf. Examples 4.18(3), (4) and 4.27(1), (2)) on $\mathsf{H} = L^2(\mathsf{X}, \mu)$:

$$(T_K \psi)(x) := \int_{\mathsf{X}} K(x, y)\psi(y)d\mu(y) . \tag{4.54}$$

We wish to study, in broad terms, the following integral equation in the unknown $\varphi \in \mathsf{H}$:

$$T_K\varphi - \lambda\varphi = f \tag{4.55}$$

where $f \in \mathsf{H}$ is given and $\lambda \in \mathbb{C}$ is a constant.

To begin with, consider the case $\lambda = 0$. This is the so-called **Fredholm equation of the first kind** on the Hilbert space H.

From the abstract point of view we have to solve for $\varphi \in \mathsf{H}$ the equation:

$$A\varphi = f ,$$

where $A : \mathsf{H} \to \mathsf{H}$ is a compact operator (in the concrete case A is T_K, a Hilbert–Schmidt operator) and $f \in \mathsf{H}$ a given element.

An important general result, valid also with an infinite-dimensional Banach space B replacing H, which assumes A compact, is that the equation has *no* solution for certain $f \in \mathsf{H}$, irrespective of $A \in \mathfrak{B}_\infty(\mathsf{H})$. This follows from the next proposition.

Proposition 4.44 *If B is a Banach space of infinite dimension and $A \in \mathfrak{B}_\infty(\mathsf{B})$, then $Ran(A) \neq \mathsf{B}$.*

Proof We can write $\mathsf{B} = \cup_{n\in\mathbb{N}} B_n$, where B_n is the open ball of radius n at the origin, so:

$$Ran(A) = \cup_{n\in\mathbb{N}} A(B_n) .$$

If $Ran(A)$ were equal to B we could write:

$$\mathsf{B} = \cup_{n\in\mathbb{N}} A(B_n) \subset \cup_{n\in\mathbb{N}} \overline{A(B_n)} \subset \mathsf{B},$$

hence

$$\mathsf{B} = \cup_{n\in\mathbb{N}} \overline{A(B_n)} ,$$

where any $\overline{A(B_n)}$ is compact because A is compact and B_n bounded. Therefore B would become a countable union of compact sets, which is impossible by Corollary 4.6. □

The next proposition raises a second issue concerning Fredholm equations of the first kind.

Proposition 4.45 *Let X be a normed space. Every left inverse to a compact injective operator $A \in \mathfrak{B}_\infty(\mathsf{X})$ cannot be bounded if X is infinite-dimensional.*

Proof The proof is in Exercise 4.1. □

Because of this result, the solutions to $A\varphi = f_1$ and $A\varphi = f_2$ may be very different, even if f_1 and f_2 are close in norm. Fredholm equations of the first kind, in other terms, are *ill posed* problems *à la* Hadamard. This does not entail, obviously, that Fredholm equations of the 1st kind are mathematically uninteresting, nor that they are useless in applied sciences. What it means is that their study is hard and requires advanced and specialised topics, that reach well beyond the present elementary treatise.

Equation (4.55), when $\lambda \neq 0$, is called **Fredholm equation of the second kind**. For a short moment we assume that T_K admits a *Hermitian kernel*. In other terms we consider:

$$T_K\varphi - \lambda\varphi = f \ , \tag{4.56}$$

where T_K has the form (4.54) with $\lambda \neq 0$ fixed, and $K(x, y) = \overline{K(y, x)}$, so that, by (4.3), $T_K = T_K^*$. In such a case we can state a more general theorem.

Theorem 4.46 (Fredholm equations of the 2nd kind with Hermitian kernels) *Let $\mathsf{H} = L^2(\mathsf{X}, \mu)$ be a Hilbert space with σ-finite and separable, positive, σ-additive measure μ. Given $f \in \mathsf{H}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, consider Eq. (4.56) in $\varphi \in \mathsf{H}$ with*

$$(T_K\varphi)(x) = \int_{\mathsf{X}} K(x, y)\varphi(y)d\mu(y) \ ,$$

where $K \in L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu)$ and $K(x, y) = \overline{K(y, x)}$. Then the following hold.
(a) *If λ is not an eigenvalue of T_K, Eq. (4.56) has always a unique solution, whichever $f \in \mathsf{H}$.*
(b) *If λ is an eigenvalue of T_K, (4.56) has solutions if and only if f is orthogonal to the λ-eigenspace. In such case there exist infinitely many solutions.*

Proof Multiplying (4.56) by λ^{-1} allows to study only $\lambda = 1$ (redefining $\lambda^{-1}K$ as K and $\lambda^{-1}f$ as f). Hence we prove it in this case only. We know T_K is compact by Example 4.18(4) and self-adjoint up to a possible and inessential factor $1/\lambda$. So we shall refer to Theorems 4.19 and 4.20. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a basis of $Ker(T_K)^\perp$ of eigenvectors of T_K. We can decompose, uniquely, any $\varphi \in \mathsf{H}$ as

$$\varphi = \sum_{n=1}^{+\infty} a_n\psi_n + \varphi' \ , \tag{4.57}$$

where $\varphi' \in Ker(T_K)$ and the $a_n \in \mathbb{C}$ are uniquely determined by φ. In particular

$$f = \sum_{n=1}^{+\infty} b_n\psi_n + f' \ .$$

Let us seek a solution to (4.56):

$$\varphi = T_K\varphi - f$$

in the form (4.57). We must find the numbers a_n and the map φ' once T_K and f are given. Substituting (4.57) and the expression of f in (4.56), we easily find

$$\sum_n a_n \psi_n + \varphi' = \sum_n a_n \lambda_n \psi_n - \sum_n b_n \psi_n - f' ,$$

where λ_n are the *non-null* eigenvalues of T_K corresponding to eigenvectors ψ_n (in general it may happen that $\lambda_n = \lambda_{n'}$, for we have labelled eigenvectors and not eigenvalues). That is to say:

$$\sum_n a_n(1 - \lambda_n + b_n)\psi_n = -f' - \varphi'_n .$$

The two sides are orthogonal by construction, and so are the vectors ψ_n, pairwise. Therefore the identity is equivalent to:

$\varphi' = -f'$,
$a_n(1 - \lambda_n) = -b_n, n = 1, 2, \ldots$.

In any case φ' is always determined, for it coincides with f'. The existence of solutions to (4.56) amounts to:

$\varphi' = -f'$,
$a_n = \frac{b_n}{\lambda_n - 1}$ for $\lambda_n \neq 1$
$b_0 = 0$ for $\lambda_m = 1$.

If $\lambda_n \neq 1$ for every n, the coefficients a_n are uniquely determined by the b_n. If $\lambda_m = 1$ for some m and $b_m \neq 0$, the last condition is false, and Eq. (4.56) has no solution. Instead, if $b_m = 0$ for any m such that $\lambda_m = 1$ (i.e. if f is normal to the 1-eigenspace of T_K), the coefficients a_m can be chosen at will, whereas the remaining a_n are determined. In this case there exist infinitely many solutions to (4.56). □

To conclude we move to the general case and drop the assumption on Hermitian kernels. In order to stay general we shall study the abstract *Fredholm equation of the 2nd kind* in the Hilbert space H:

$$A\varphi - \lambda\varphi = f , \tag{4.58}$$

where $f \in \mathsf{H}, \lambda \in \mathbb{C} \setminus \{0\}$ and $A \in \mathfrak{B}_\infty(\mathsf{H})$ are given on H and $\varphi \in \mathsf{H}$ is the problem's unknown. Nothing more is assumed on A, apart from compactness. In particular, we do not suppose A is a Hilbert–Schmidt operator. Let us prove the following general theorem, due to Fredholm, which can be stated also for $A \in \mathfrak{B}_\infty(\mathsf{B})$ on a Banach space B.

Theorem 4.47 (Fredholm) *On the Hilbert space* H *consider the abstract* **Fredholm equation of the second kind**

$$A\varphi - \lambda\varphi = f \tag{4.59}$$

in the unknown $\varphi \in \mathsf{H}$, *with* $f \in \mathsf{H}$, $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $A \in \mathfrak{B}_\infty(\mathsf{H})$ *given. Consider as well the corresponding* **homogeneous equation**, *the* **adjoint equation** *and the* **homogeneous adjoint equation***:*

$$A\varphi_0 \ -\lambda\varphi_0 = \mathbf{0}\ , \tag{4.60}$$

$$A^*\psi \ -\overline{\lambda}\psi = g\ , \tag{4.61}$$

$$A^*\psi_0 \ -\overline{\lambda}\psi_0 = \mathbf{0}\ , \tag{4.62}$$

respectively, with $g \in \mathsf{H}$ *given and* $\varphi_0, \psi, \psi_0 \in \mathsf{H}$ *unknown. Then*

(a) *Equation (4.59) admits solutions* $\Leftrightarrow$ f *is orthogonal to each solution* ψ_0 *to (4.62);*

(b) *either (4.59) admits exactly one solution for any* $f \in \mathsf{H}$, *or (4.60) has a non-zero solution;*

(c) *Equation (4.60), (4.62) admit the same, finite, number of linearly independent solutions.*

Remark 4.48 **(1)** Statement (b) is the acclaimed *Fredholm alternative*.
(2) The above theorem holds in particular when A is self-adjoint, and becomes Theorem 4.46. ■

Proof of theorem 4.47. Here, too, dividing the initial equation by $\lambda \neq 0$ permits to reduce to $\lambda = 1$ (after redefining $\lambda^{-1}A$ as A, $\lambda^{-1}f$ as f and $\overline{\lambda}^{-1}g$ as g). Henceforth, then, $\lambda = 1$. Observe $T := A - I$ is bounded but not compact, for I is not compact. The theorem relies on three lemmas. Let us first notice that $Ker(T)$ is always closed in H if T is continuous, as in the present situation, whereas $Ran(T)$ may not. The first lemma shows that $Ran(T)$ is closed as well, provided $T := A - I$ with $A \in \mathfrak{B}_\infty(\mathsf{H})$.

Lemma 4.49 *Under the assumptions made on* T, $Ran(T)$ *is closed.*

Proof Let $y_n \in Ran(T)$, $n \in \mathbb{N}$, and suppose $y_n \to y$ as $n \to +\infty$. We need to prove $y \in Ran(T)$. By assumption:

$$y_n = Tx_n = Ax_n - x_n \tag{4.63}$$

for some sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathsf{H}$. With no loss of generality we may assume $x_n \in Ker(T)^\perp$, possibly eliminating from the sequence what projects onto $Ker(T)$. The claim is proven if we can show the sequence $\{x_n\}$ is bounded: in fact, A being compact, there will exist a subsequence x_{n_k} such that $Ax_{n_k} \to y' \in \mathsf{H}$ as $k \to \infty$. Substituting in (4.63) we conclude $x_{n_k} \to x$ for some $x \in \mathsf{H}$ as $k \to +\infty$. By the continuity of A, $Tx = Ax - x = y$, so $y \in Ran(T)$.

We will proceed by contradiction, and assume $\{x_n\}_{n\in\mathbb{N}} \subset Ker(T)^\perp$ is bounded. If not, there would be a subsequence x_{n_m} with $0 < ||x_{n_m}|| \to +\infty$ as $m \to +\infty$. Since the y_n form a convergent, hence bounded, sequence, dividing by $||x_{n_m}||$ in (4.63) gives:

$$T\frac{x_{n_m}}{||x_{n_m}||} = A\frac{x_{n_m}}{||x_{n_m}||} - \frac{x_{n_m}}{||x_{n_m}||} = \frac{y_{n_m}}{||x_{n_m}||} \to \mathbf{0}\ . \tag{4.64}$$

But A is compact and the $\frac{x_{n_m}}{||x_{n_m}||}$ are bounded, so we can extract a further subsequence $x_{n_{m_k}}/||x_{n_{m_k}}||$ such that:

$$\frac{x_{n_{m_k}}}{||x_{n_{m_k}}||} \to x' \in \mathsf{H} \quad \text{and} \quad T\frac{x_{n_{m_k}}}{||x_{n_{m_k}}||} \to Tx' \quad \text{as } k \to +\infty.$$

By (4.64) we infer $x' \in Ker(T)$. By assumption $\frac{x_{n_{m_k}}}{||x_{n_{m_k}}||} \in Ker(T)^\perp$, and $Ker(T)^\perp$ is closed, so $x' \in Ker(T)^\perp$. Consequently $x' \in Ker(T) \cap Ker(T)^\perp = \{\mathbf{0}\}$, in contradiction to

$$||x'|| = \lim_{k\to+\infty} \frac{||x_{n_{m_k}}||}{||x_{n_{m_k}}||} = 1 .$$

This ends the proof. □

The second lemma claims the following.

Lemma 4.50 *Under the same assumptions on T we have orthogonal decompositions:*

$$\mathsf{H} = Ker(T) \oplus Ran(T^*) = Ker(T^*) \oplus Ran(T) . \tag{4.65}$$

Proof Since $T, T^* \in \mathfrak{B}(\mathsf{H})$, Theorem 3.13(e, d) and Proposition 3.38(d) imply $Ker(T) = (Ran(T^*)^\perp)^\perp = \overline{Ran(T^*)}$, $Ker(T^*) = (Ran(T)^\perp)^\perp = \overline{Ran(T)}$, and:

$$H = Ker(T) \oplus \overline{Ran(T^*)} = Ker(T^*) \oplus \overline{Ran(T)} .$$

Lemma 4.49 also holds if we replace T with T^*, for $(A - I)^* = A^* - I$ where A^* is compact if A is. □

Theorem 4.47(a) now follows from Lemma 4.50 (we still had to finish the proof for $\lambda = 1$). In fact the lemma implies $f \perp Ker(T^*) \Leftrightarrow f \in Ran(T) \Leftrightarrow T\varphi = f$ for some $\varphi \in \mathsf{H}$.

To continue with the proof of part (b) in the main theorem we define the subspaces $\mathsf{H}^k := Ran(T^k)$, $k = 1, 2, \ldots$ (all closed by Lemma 4.49), so that:

$$\mathsf{H} \supset \mathsf{H}^1 \supset \mathsf{H}^2 \supset \mathsf{H}^3 \supset \cdots$$

By construction $T(\mathsf{H}^k) = \mathsf{H}^{k+1}$. And now we have the third lemma.

Lemma 4.51 *With T as above and $\mathsf{H}^k = Ran(T^k)$, $k = 1, 2, \ldots$, there exists $j \in \mathbb{N}$ such that:*

$$\mathsf{H}^{k+1} = \mathsf{H}^k \quad \text{if } k \geq j.$$

Proof Assume, by contradiction, that such an index j does not exist. Then $\mathsf{H}^k \neq \mathsf{H}^h$ if $k \neq h$, and we can manufacture a sequence of orthonormal vectors $x_k \in \mathsf{H}^k$ such that $x_k \perp H^{k+1}$, $k = 1, 2, \ldots$. If $l > k$

$$Ax_l - Ax_k = -x_k + (x_l + Tx_l - Tx_k),$$

so $||Ax_l - Ax_k||^2 \geq 1$ because $x_l + Tx_l - Tx_k \in \mathsf{H}^{k+1}$. But now we cannot extract any convergent subsequence from $\{Ax_k\}$, contradicting the compactness of A. □

Next we prove two lemmas that, combined, will eventually yield the proof of Theorem 4.47(b) (for $\lambda = 1$).

Lemma 4.52 *Under the previous assumptions on T, $Ker(T) = \{\mathbf{0}\} \Rightarrow Ran(T) = \mathsf{H}$.*

Proof Assume $Ker(T) = \{\mathbf{0}\}$, making T one-to-one, but by contradiction suppose $Ran(T) \neq \mathsf{H}$. Then the H^k, $k = 1, 2, 3, \ldots$, would be all distinct, violating lemma 4.51. □

Lemma 4.53 *Under the assumptions made on T, $Ran(T) = \mathsf{H} \Rightarrow Ker(T) = \{\mathbf{0}\}$.*

Proof If $Ran(T) = \mathsf{H}$, by Lemma 4.50 we have $Ker(T^*) = \{\mathbf{0}\}$. Then the previous lemma (with T^* instead of T) guarantees $Ran(T^*) = \mathsf{H}$. Now Lemma 4.50 again forces $Ker(T) = \{\mathbf{0}\}$. □

Now it is patent that Lemmas 4.52 and 4.53 together prove statement (b) in Theorem 4.47.

We finish by proving part (c) (always for $\lambda = 1$).

Suppose $\dim Ker(T) = +\infty$, rebutting (c). Then there is an infinite orthonormal system $\{x_n\}_{n\in\mathbb{N}} \subset Ker(T)$. By construction $Ax_n = x_n$ and so $||Ax_k - Ax_h||^2 = 2$. But this cannot be, for it would run up against the existence of a subsequence in the bounded set $\{x_n\}_{n\in\mathbb{N}}$ such that $\{Ax_{n_k}\}_{k\in\mathbb{N}}$ converges, which is granted by compactness. Hence $\dim Ker(T) = m < +\infty$. Similarly $\dim Ker(T^*) = n < +\infty$. Assume, contradicting the statement, that $m \neq n$. In particular we may suppose $m < n$. Let $\{\varphi_j\}_{j=1,\ldots,n}$ and $\{\psi_j\}_{j=1,\ldots,m}$ be orthonormal bases for $Ker(T)$ and $Ker(T^*)$ respectively. Define $S \in \mathfrak{B}(\mathsf{H})$ by:

$$Sx := Tx + \sum_{j=1}^{m} (\varphi_j|x)\psi_j \; .$$

As $S = A' - I$, with A' compact (obtained from the compact operator A by adding a compact operator, since the range is finite-dimensional), the results found above for $T = A - I$ hold for S, too.

We claim $Sx = \mathbf{0} \Rightarrow x = \mathbf{0}$. Write, explicitly,

$$Tx + \sum_{j=1}^{m} (\varphi_j|x)\psi_j = \mathbf{0} \; . \tag{4.66}$$

By virtue of Lemma 4.50, all vectors ψ_j are orthogonal to any one of the form Tx, so (4.66) implies $Tx = \mathbf{0}$. Moreover, $(\varphi_j|x) = 0$ if $1 \leq j \leq m$, because x is a linear combination of the φ_j and it must simultaneously be orthogonal to them, so $x = \mathbf{0}$. Hence $Sx = \mathbf{0}$ implies $x = \mathbf{0}$. From part (b), then, there exists $y \in \mathsf{H}$ such that

$$Ty + \sum_{j=1}^{m} (\varphi_j|y)\psi_j = \psi_{m+1} \; .$$

Taking the inner product with ψ_{m+1} gives a contradiction: 1 on the right equals 0 on the left, because $Ty \in Ran(T)$ and $Ran(T) \perp Ker(T^*)$. This shows we cannot assume $m < n$. Had we instead supposed $n < m$ earlier on, we would have run into a similar contradiction by arguing exactly as above with the roles of T and T^* swapped. This eventually ends the proof of (c). □

Examples 4.54 An interesting instance of a Fredholm equation of the 2nd kind is **Volterra's equation of the second kind**:

$$\varphi(x) = \int_a^x K(x,t)\varphi(t)dt + f(x)\,, \tag{4.67}$$

where $\varphi \in L^2([a,b], dx)$ is the unknown function, $f \in L^2([a,b], dx)$ is given and the integral kernel satisfies $|K(x,t)| < M < +\infty$ for any $x, t \in [a,b]$, $t \le x$. (Any multiplicative factor $\rho \in \mathbb{C} \setminus \{0\}$ is absorbed in K.)

This equation befits the theory of Fredholm's theorem if we rewrite the integral as an integral over all $[a,b]$ and assume $K(x,t) = 0$ if $t \ge x$. For this type of equation, though, there is a better result based on contraction maps (cf. Sect. 2.6). It turns out, namely, that a certain high power of $T_K : L^2([a,b], dx) \to L^2([a,b], dx)$ is a contraction, where T_K is the integral operator in (4.67)

$$(T_K\varphi)(x) = \int_a^x K(x,t)\varphi(t)dt\,.$$

Consequently the homogeneous equation $T_K\varphi = \mathbf{0}$ has one, and one only, solution by Theorem 2.112, and the solution must be $\varphi = \mathbf{0}$. The proof that T_K^n is a contraction if n is large enough is similar to what we saw in Example 2.113(1), where the Banach space $(C([a,b]), ||\ ||_\infty)$ is replaced by $(L^2([a,b], dx), ||\ ||_2)$ (cf. Exercise 4.19). That said, parts (a) and (b) in Fredholm's theorem imply that Eq. (4.67) has *exactly* one solution, for any choice of the source term $f \in L^2([a,b], dx)$. ■

Exercises

4.1 Prove that if X is a normed space and $T : \mathsf{X} \to \mathsf{X}$ is compact and injective, then any linear operator $S : Ran(T) \to \mathsf{X}$ that inverts T on the left cannot be bounded if $\dim \mathsf{X} = \infty$.

Solution. If S were bounded, Proposition 2.47 would allow to extend it to a bounded operator $\tilde{S} : \mathsf{Y} \to \mathsf{X}$, where $\mathsf{Y} := \overline{Ran(T)}$, so that $\tilde{S}T = I$. Precisely as in the proof of Proposition 4.9(b), we can prove that $\tilde{S}T$ is compact if $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ is compact and $\tilde{S} \in \mathfrak{B}(\mathsf{Y}, \mathsf{X})$. Then $I : \mathsf{X} \to \mathsf{X}$ would be compact, and thus the unit ball in X would have compact closure, breaching Proposition 4.5.

4.2 Using Banach's Lemma 4.12 prove that in an infinite-dimensional normed space the closed unit ball is not compact.

Outline. Let $x_1, x_2, \ldots$ be an infinite sequence of linearly independent vectors with $||x_n|| = 1$ (hence all belonging to the closure of the unit ball). Banach's lemma constructs a sequence of vectors $y_1, y_2, \ldots$ such that $||y_n|| = 1$ and $||y_{n-1} - y_n|| > 1/2$. This sequence cannot contain converging subsequences.

4.3 Prove that if $A^* = A \in \mathfrak{B}_\infty(\mathsf{H})$ on a Hilbert space H, then

$$\sigma_p(|A|) = \{|\lambda| \ |\lambda \in \sigma_p(A)\} .$$

Conclude that if $A^* = A \in \mathfrak{B}_\infty(\mathsf{H})$,

$$sing(A) = \{|\lambda| \ |\lambda \in \sigma_p(A) \setminus \{0\}\} .$$

Solution. Expand the compact, self-adjoint operators A and $|A|$ according to Theorem 4.20:

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda \quad \text{and} \quad |A| = \sum_{\mu \in \sigma_p(|A|)} \mu P'_\mu ,$$

with the obvious notation. By squaring A and $|A|$ and using their continuity (this allows to consider all series as finite sums), using the idempotency and orthogonality of projectors relative to distinct eigenvectors, and recalling $|A|^2 = A^*A = A^2$, we have

$$\sum_{\lambda \in \sigma_p(A)} \lambda^2 P_\lambda = \sum_{\mu \in \sigma_p(|A|)} \mu^2 P'_\mu . \tag{4.68}$$

Now keep in mind $P_\lambda P_{\lambda_0} = 0$ if $\lambda \neq \lambda_0$ and $P_\lambda P_{\lambda_0} = P_{\lambda_0}$ otherwise, and the same holds for the projectors in the decomposition of $|A|$. Composing with P_{λ_0} on the right in (4.68), taking adjoints and eventually right-composing with P'_{μ_0} produces $\lambda_0^2 P_{\lambda_0} P'_{\mu_0} = \mu_0^2 P_{\lambda_0} P'_{\mu_0}$, i.e.

$$(\lambda_0^2 - \mu_0^2) P_{\lambda_0} P'_{\mu_0} = 0 , \tag{4.69}$$

for any $\lambda_0 \in \sigma(A)$ and $\mu_0 \in \sigma_p(|A|)$. The fact that A admits a basis of eigenvectors (Theorem 4.20) is known to be equivalent to

$$I = s\text{-}\sum_{\lambda_0 \in \sigma(A)} P_{\lambda_0} .$$

Fix $\mu_0 \in \sigma_p(|A|)$. If $P_{\lambda_0} P'_{\mu_0} = 0$ for any $\lambda_0 \in \sigma(A)$, from the above identity we would have $P'_{\mu_0} = 0$, absurd by definition of eigenspace. Therefore, (4.69) withstanding, there must exist $\lambda_0 \in \sigma(A)$ such that $\lambda_0^2 = \mu_0^2$, i.e. $\mu_0 = |\lambda_0|$. If $\lambda_0 \in \sigma_p(A)$, swapping A and $|A|$ and using a similar argument would produce $\mu_0 \in \sigma_p(|A|)$ such that $\lambda_0^2 = \mu_0^2$, i.e. $\mu_0 = |\lambda_0|$. The first assertion is thus proved. The second one is evident by the definition of singular value.

4.4 Consider a separable Hilbert space H with basis $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ and the sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ where $|s_n| \geq |s_{n+1}|$ and $|s_n| \to 0$ as $n \to +\infty$. Using the uniform topology prove that

$$T := \sum_{n=0}^{+\infty} s_n f_n(f_n| \ )$$

is well defined and $T \in \mathfrak{B}_\infty(\mathsf{H})$. Show that if every s_n is real, T is further self-adjoint and every s_n is an eigenvalue of T.

Hint. Under the assumptions made, the operator $T_N := \sum_{n=0}^{N} s_n f_n(f_n| \ )$ satisfies:

$$||T_N x - T_M x||^2 \leq |s_M|^2 \sum_{n=M}^{N-1} |(f_n|x)|^2 \leq |s_M|^2 ||x||^2$$

for $N \geq M$. Taking the least upper bound over unit vectors $x \in \mathsf{H}$ gives:

$$||T_N - T_M|| \leq |s_M|^2 \to 0 \quad \text{as } N, M \to +\infty,$$

whence the first part. The rest follows by direct inspection.

4.5 Prove that if $T \in \mathfrak{B}_\infty(\mathsf{H})$ and if $\mathsf{H} \ni x_n \to x \in \mathsf{H}$ *weakly*, i.e.

$$(g|x_n) \to (g|x) \quad \text{as } n \to +\infty, \text{ for any given } g \in \mathsf{H},$$

then $||T(x_n) - T(x)|| \to 0$ as $n \to +\infty$. Put otherwise, compact operators map weakly convergent sequences to sequences converging in norm. Extend the result to the case $T \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$, X and Y normed.

Solution. Suppose $x_n \to x$ weakly. If we bear in mind Riesz's theorem, the set $\{x_n\}_{n\in\mathbb{N}}$ is immediately weakly bounded in the sense of Corollary 2.64. According to this corollary, $||x_n|| \leq K$ for any $n \in \mathbb{N}$ and for some $K > 0$. So define $y_n := Tx_n$, $y := Tx$ and note that for any $h \in \mathsf{H}$,

$$(h|y_n) - (h|y) = (T^*h|x_n) - (T^*h|x) \to 0 \quad \text{as } n \to +\infty,$$

hence also the y_n converge weakly to y. Suppose, by contradiction, $||y_n - y|| \not\to 0$ as $n \to +\infty$. Then there exist $\varepsilon > 0$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ with $||y - y_{n_k}|| \geq \varepsilon$ for any $k \in \mathbb{N}$. Since $\{x_{n_k}\}_{k\in\mathbb{N}}$ is bounded by K, and T is compact, there must be a subsequence $\{y_{n_{k_r}}\}_{r\in\mathbb{N}}$ converging to some $y' \neq y$. This subsequence $\{y_{n_{k_r}}\}_{r\in\mathbb{N}}$ has to converge to y' also weakly. But this cannot be, for $\{y_n\}_{n\in\mathbb{N}}$ converges weakly to $y \neq y'$. Therefore $y_n \to y$ in the norm of H. The argument works in the more general case where $T \in \mathfrak{B}_\infty(\mathsf{X}, \mathsf{Y})$, X and Y normed spaces, by interpreting $\mathsf{X} \ni x_n \to x \in \mathsf{X}$ in *weak* sense:

$$g(x_n) \to g(x) \quad \text{as } n \to +\infty, \text{ for any given } g \in \mathsf{X}',$$

because Corollary 2.64 still holds. In the proof one uses the fact that $h \in \mathsf{Y}' \Rightarrow h \circ T \in \mathsf{X}'$ ($h \circ T$ is a composite of continuous linear mappings).

4.6 Referring to Example 4.27, take $T_K, T_{K'} \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ (μ is assumed separable) with integral kernels K, K'. Prove that the HS operator $aT_K + bT_{K'}$, $a, b \in \mathbb{C}$, has kernel $aK + bK'$.

4.7 Given $T_K \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ with integral kernel K, prove the Hilbert–Schmidt operator T_K^* has integral kernel $K^*(x, y) = \overline{K(y, x)}$.

4.8 With the same hypotheses as Exercise 4.6, show that the integral kernel of $T_K T_{K'}$ is

$$K''(x, z) := \int_{\mathsf{X}} K(x, y) K'(y, z)\, d\mu(y) .$$

4.9 Let $L^2(\mathsf{X}, \mu)$ be separable. Prove that the mapping $L^2(\mathsf{X} \times \mathsf{X}, \mu \otimes \mu) \ni K \mapsto T_K \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ is an isomorphism of Hilbert spaces. Discuss whether one can view this map as an isometry of normed spaces, taking $\mathfrak{B}(L^2(\mathsf{X}, \mu))$ as codomain. Discuss whether it is continuous if viewed as a homeomorphisms only.

4.10 With reference to Exercise 4.27(3), prove that if $g \in C_0([0, 1])$ then

$$\left((I - \rho T)^{-1} g\right)(x) = g(x) + \rho \int_0^x e^{\rho(x-y)} g(y) dy .$$

Hint. Use the operator $I - \rho T$, recalling the integral expression of T and noticing $\rho e^{\rho(x-y)} = \frac{\partial}{\partial x} e^{\rho(x-y)}$.

4.11 Let $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ be the set of *degenerate operators* on $L^2(\mathsf{X}, \mu)$ (cf. Example 4.27(4)), with μ separable. Prove the following are equivalent statements.

(a) $T \in \mathfrak{B}_D(L^2(\mathsf{X}, \mu))$.

(b) $Ran(T)$ has finite dimension.

(c) $T \in \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ (hence T is an integral operator) with kernel $K(x, y) = \sum_{k=1}^N p_k(x) q_k(y)$, where $p_1, \ldots, p_N \in L^2(\mathsf{X}, \mu)$, $q_1, \ldots, q_N \in L^2(\mathsf{X}, \mu)$ are linearly independent.

4.12 Take the set $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ of degenerate operators (cf. Example 4.27(4)) on $L^2(\mathsf{X}, \mu)$, with μ separable. Show $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ is a two-sided *-ideal in $\mathfrak{B}(L^2(\mathsf{X}, \mu))$ and a subspace in $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$. In other words, prove that $\mathfrak{B}_D(L^2(\mathsf{X}, \mu) \subset \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$, that it is a closed subspace under Hermitian conjugation, and that $AD, DA \in \mathfrak{B}_D(L^2(\mathsf{X}, \mu)$ if $A \in \mathfrak{B}(L^2(\mathsf{X}, \mu))$ and $D \in \mathfrak{B}_D(L^2(\mathsf{X}, \mu))$.

4.13 Consider $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ (cf. Example 4.27(4)) with μ separable. Does the closure of $\mathfrak{B}_D(L^2(\mathsf{X}, \mu))$ in $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ in the norm topology of $\mathfrak{B}(L^2(\mathsf{X}, \mu))$ coincide with $\mathfrak{B}_2(L^2(\mathsf{X}, \mu))$?

Hint. Consider the operator

$$T := \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n}} T_{K_n} ,$$

where $K_n(x, y) = \phi_n(x)\phi_n(y)$, $\{\phi_n\}_{n\in\mathbb{N}}$ is a basis of $L^2(\mathsf{X}, \mu)$ and the convergence is uniform. Prove $T \in \mathfrak{B}(L^2(\mathsf{X}, \mu))$ is well defined, but $T \notin \mathfrak{B}_2(L^2(\mathsf{X}, \mu))$ since $||T\phi_n||^2 = 1/n$.

4.14 Under the assumptions of Mercer's Theorem 4.29, prove that if $T_K \in \mathfrak{B}_1(L^2(\mathsf{X}, d\mu))$ then $tr(T_K) = \int_{\mathsf{X}} K(x, x)d\mu(x)$.

Hint. Expand the trace in the basis of eigenvectors given by the continuous maps in Mercer's statement. Since the series that defines K converges uniformly on the compact set X of finite measure, a clever use of dominated convergence allows to show

$$\int_{\mathsf{X}} K(x, x)d\mu(x) = \int_X \sum_{\lambda,i} \lambda u_{\lambda,i}(x)\overline{u_{\lambda,i}(x)}d\mu(x) = \sum_{\lambda,i} \lambda \int_X u_{\lambda,i}(x)\overline{u_{\lambda,i}(x)}d\mu(x)$$

$$= \sum_{\lambda,i} \lambda = tr(T_K) .$$

4.15 Consider an integral operator T_K on $L^2([0, 2\pi], dx)$ with integral kernel:

$$K(x, y) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} e^{in(x-y)} .$$

Prove T_K is a compact Hilbert–Schmidt operator of trace class.

4.16 Consider the operator T_K of Exercise 4.15 and the differential operator

$$A := -\frac{d^2}{dx^2} ,$$

defined on smooth maps over $[0, 2\pi]$ that satisfy periodicity conditions (together with all derivatives). What is $T_K A$?

Hint. Let $\mathbf{1}$ be the constant map 1 on $[0, 2\pi]$, and $P_0 : f \mapsto (\frac{1}{2\pi} \int_0^{2\pi} f(x)dx)\mathbf{1}$ the orthogonal projector onto the space of constant maps in $L^2([0, 2\pi], dx)$. Then $T_K A = I - P_0$.

4.17 Consider an integral operator T_s on $L^2([0, 2\pi], dx)$ with kernel:

$$K_s(x, y) = \frac{1}{2\pi} \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n^{2s}} e^{in(x-y)} .$$

Prove that if $Re\, s$ is sufficiently large (how large?) the following identity makes sense:

$$tr(T_s) = \zeta_R(2s)\ ,$$

where $\zeta_R = \zeta_R(s)$ is *Riemann's zeta function*.

4.18 Take $B \in \mathfrak{B}(\mathsf{H})$ on a Hilbert space H, and a basis N such that $\sum_{u\in N} ||Bu|| < +\infty$. Prove $B \in \mathfrak{B}_1(\mathsf{H})$.

Hint. Observe $|||B|\psi|| = ||B\psi||$ and $|(\psi||B|\psi)| \leq ||\psi||\, |||B|\psi||$.

4.19 Consider the integral operator $T_K : L^2([a,b], dx) \to L^2([a,b], dx)$

$$(T_K\varphi)(x) = \int_a^x K(x,t)\varphi(t)dt$$

where K is a measurable map such that $|K(x,t)| \leq M$ for some $M \in \mathbb{R}$, for all $x, t \in [a,b]$, $t \leq x$. Prove

$$||T_K^n|| \leq \frac{M^n(b-a)^n}{\sqrt{(n+1)!}}$$

and conclude that there exists a positive integer n rendering T_K^n a contraction.

Solution. In the ensuing computations $\varphi \in L^2([a,b], dx)$ implies $\varphi \in L^1([a,b], dx)$ by the Cauchy–Schwarz inequality, because the constant map 1 is in $L^2([a,b], dx)$. Define $\theta(z) = 1$ for $z \geq 0$ and $\theta(z) = 0$ otherwise. By construction,

$$|(T_K^n\varphi)(x)| = \int_a^b dx_1 \int_a^b dx_2 \cdots \int_a^b dx_n \theta(x - x_1)\theta(x_1 - x_2)\cdots\theta(x_{n-1} - x_n)$$
$$\times K(x, x_1)K(x_1 - x_2)\cdots K(x_{n-1}, x_n)\varphi(x_n)\ .$$

Hence

$$|(T_K^n\varphi)(x)| \leq M^n \int_{[a,b]^n} dx_1 \cdots dx_n |\theta(x - x_1)\cdots\theta(x_{n-1} - x_n)|\, |\varphi(x_n)|\ .$$

Using Cauchy–Schwarz on $L^2([a,b]^n, dx_1 \cdots dx_n)$, and $\theta(z)^2 = \theta(z) = |\theta(z)|$, we have:

$$|(T_K^n\varphi)(x)| \leq M^n \sqrt{\int_{[a,b]^n} dx_1 \cdots dx_n \theta(x-x_1)\cdots\theta(x_{n-1}-x_n)}\,[b-a]^{(n-1)/2}||\varphi||_2\,,$$

i.e.

$$|(T_K^n\varphi)(x)| \leq M^n \frac{(x-a)^{n/2}}{\sqrt{n!}}[b-a]^{(n-1)/2}||\varphi||_2\,.$$

Consequently

$$||T_K^n\varphi||_2 \leq \frac{M^n(b-a)^n}{\sqrt{(n+1)!}}||\varphi||_2\,,$$

and so

$$||T_K^n|| \leq \frac{M^n(b-a)^n}{\sqrt{(n+1)!}}\,.$$

But since:

$$\lim_{n\to+\infty} \frac{M^n(b-a)^n}{\sqrt{(n+1)!}} \to 0 \quad \text{as } n \to +\infty,$$

for n large enough there must exist $0 < \lambda < 1$ such that

$$||T_K^n\varphi - T_K^n\varphi'||_2 \leq \lambda||\varphi-\varphi'||_2\,,$$

making T_K^n a contraction operator.

4.20 Denote by $\mathfrak{B}_D(\mathsf{H})$ the family of degenerate operators on a general Hilbert space H. This is the collection of operators $A \in \mathfrak{B}(\mathsf{H})$ such that $Ran(A)$ is finite-dimensional. Prove that:

$$\mathfrak{B}_D(\mathsf{H}) \subset \mathfrak{B}_1(\mathsf{H}) \subset \mathfrak{B}_2(\mathsf{H}) \subset \mathfrak{B}_\infty(\mathsf{H}) \subset \mathfrak{B}(\mathsf{H})\,,$$

where $\mathfrak{B}_D(\mathsf{H})$ is dense in $\mathfrak{B}_i(\mathsf{H})$ with respect to $||\ \ ||_i$, for $i = 1, 2, \infty$.

Chapter 5
Densely-Defined Unbounded Operators on Hilbert Spaces

Von Neumann had just about ended his lecture when a student stood up and in a vaguely abashed tone said he hadn't understood the final argument. Von Neumann answered:
"Young man, in mathematics you don't understand things. You just get used to them."

David Wells

This chapter will extend the theory seen so far to unbounded operators that are not necessarily defined on the entire space.

In section one we will define, in particular, the *standard domain* of an operator built by composing operators with non-maximal domains. We will introduce *closed* and *closable operators*. Then we shall study *adjoint operators* to unbounded and densely-defined operators, thus generalising the similar notion for bounded operators defined on the whole Hilbert space.

The second section deals with generalisations of *self-adjoint* operators to the unbounded case. For this we will introduce *Hermitian*, *symmetric*, *essentially self-adjoint* and *self-adjoint* operators, and discuss their main properties. In particular we will define the *core* of an operator and the *deficiency index*.

Section three is entirely devoted to two examples of self-adjoint operators of the foremost importance in Quantum Mechanics, namely the operators *position* and *momentum* on the Hilbert space $L^2(\mathbb{R}^n, dx)$. We will study their mathematical properties and present several equivalent definitions.

In the final section we shall discuss more advanced criteria to establish whether a symmetric operator admits self-adjoint extensions. We will present *von Neumann's criterion* and *Nelson's criterion*. The technical instruments needed for this study are the *Cayley transform* and *analytic vectors*: the latter, defined by Nelson, turned out to be crucial in the applications of operator theory to QM.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_5

5.1 Unbounded Operators with Non-maximal Domains

Let us introduce the theory of unbounded operators with non-maximal domains. *The domains under exam will always be vector subspaces of some ambient space*, and we will often consider dense subspaces. Despite the operators of concern will not be bounded, all definitions will reduce in the bounded case to the ones seen in earlier chapters.

5.1.1 *Unbounded Operators with Non-maximal Domains in Normed Spaces*

The first definitions are completely general and do not require any Hilbert structure. A notion of graph was already given in Definition 2.98 for operators with maximal domain. The following definition extends Definition 2.98 slightly.

Definition 5.1 Let X be a vector space. We shall call a linear mapping

$$T : D(T) \to \mathsf{X}\,,$$

an **operator on** X, where $D(T) \subset \mathsf{X}$ is a subspace called the **domain** of T. The **graph** of the operator T is the subspace of $\mathsf{X} \oplus \mathsf{X}$ (see Definition 2.97(3))

$$G(T) := \{(x, Tx) \in \mathsf{X} \oplus \mathsf{X} \mid x \in D(T)\}\,.$$

If $\alpha \in \mathbb{C}$, and A, B are operators on X with domains $D(A)$, $D(B)$, we define the following operators on H:

(a) AB, given by $ABf := A(Bf)$ on the **standard domain**:

$$D(AB) := \{f \in D(B) \,|\, Bf \in D(A)\}$$

(b) $A + B$, given by $(A + B)f := Af + Bf$ on the **standard domain**:

$$D(A + B) := D(A) \cap D(B)$$

(c) αA, given by $\alpha Af := \alpha(Af)$ on the **standard domain**: $D(\alpha A) = D(A)$ if $\alpha \neq 0$, and $D(0A) := \mathsf{X}$.

Remark 5.2 By taking standard domains the usual *associative properties* of the sum and product of operators hold. If A, B, C are operators on X:

$$A + (B + C) = (A + B) + C\,, \quad (AB)C = A(BC)\,.$$

Distributive properties are weaker than expected (see Definition 5.3 below for the meaning of $\supset$):

$$(A+B)C = AC + BC\,, \quad A(B+C) \supset AB + AC\,,$$

for it may happen that $(B+C)x \in D(A)$ even if Bx or Cx do not belong in $D(A)$. ■

The above notion of graph evidently coincides with the familiar graph of $T \in \mathfrak{L}(\mathsf{X})$, where the latter is nothing else than an operator on X with $D(T) = \mathsf{X}$.

Extensions of *closed* operators play a central role in the sequel. The first notion is straightforward.

Definition 5.3 If A is an operator on the vector space X, an operator B on X is called an **extension** of A, written $A \subset B$, or $B \supset A$, if $G(A) \subset G(B)$.

Remark 5.4 With the above definitions of standard domains, it is easy to prove that for any operators A, B, C on X,

(i) $A \subset B$ and $B \subset C \Rightarrow A \subset C$;

(ii) $A \subset B$ and $B \subset A \Leftrightarrow A = B$;

(iii) $A \subset B \Rightarrow AC \subset BC$ and $CA \subset CB$;

(iv) if $D(A) = \mathsf{X}$, then $AB \subset BA \Rightarrow A(D(B)) \subset D(B)$;

(v) if $D(A) = \mathsf{X}$, then $AB = BA \Rightarrow A(D(B)) \subset D(B)$ and $D(B) = A^{-1}(D(B))$ (so that $A(D(B)) = D(B)$ if A is surjective).

■

5.1.2 Closed and Closable Operators

We shall extend the reach of Definition 2.98, allowing for domains smaller than the whole space. This will accommodate new concepts, such as *closed operators*.

We remind that if X is normed, the *product topology* on the Cartesian product $\mathsf{X}\times\mathsf{X}$ is the one whose open sets are $\varnothing$ and unions of products of open balls $B_\delta(x) \times B_{\delta_1}(x_1)$ centred at $x, x_1 \in \mathsf{X}$ with any radii $\delta, \delta_1 > 0$.

Definition 5.5 Let A be an operator on the normed space X.
(a) A is called **closed** if its graph is closed in the product topology of $\mathsf{X} \times \mathsf{X}$.
Consequently A is closed if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ such that:

(i) $x_n \to x \in \mathsf{X}$ as $n \to +\infty$ and
(ii) $Ax_n \to y \in \mathsf{X}$ as $n \to +\infty$,

it follows $x \in D(A)$ and $y = Ax$.
(b) A is **closable** if the closure $\overline{G(A)}$ of its graph is the graph of a (necessarily closed) operator. The latter is denoted $\overline{A}$ and is called the **closure** of A.

The next proposition characterises closable operators.

Proposition 5.6 *Let A be an operator on the normed space X. The following facts are equivalent:*

(i) A is closable,
(ii) $\overline{G(A)}$ does not contain elements of type $(\mathbf{0}, z)$, $z \neq \mathbf{0}$,
(iii) A admits closed extensions.

Proof (i) $\Leftrightarrow$ (ii). A is not closable iff there exist sequences in $D(A)$, say $\{x_n\}_{n\in\mathbb{N}}$ and $\{x'_n\}_{n\in\mathbb{N}}$, such that $x_n \to x \leftarrow x'_n$, and $Ax_n \to y \neq y' \leftarrow Ax'_n$. By linearity this is the same as saying there is a sequence $x''_n = x_n - x'_n \to \mathbf{0}$ such that $Ax''_n \to y - y' = z \neq \mathbf{0}$. In turn, this amounts to $\overline{G(A)}$ containing points $(\mathbf{0}, z) \neq (\mathbf{0}, \mathbf{0})$.
(i) $\Leftrightarrow$ (iii). If A is closable, $\overline{A}$ is a closed extension of A. Conversely, if there is a closed extension B of A, there cannot be in $\overline{G(A)}$ elements of the kind $(\mathbf{0}, z) \neq (\mathbf{0}, \mathbf{0})$, for otherwise $G(B) = \overline{G(B)} \supset \overline{G(A)} \ni (\mathbf{0}, z)$, since $A \subset B$ and B is closed. Therefore B would not be linear as $B(\mathbf{0}) \neq \mathbf{0}$. □

Here is a useful general property of closable operators on Banach (hence Hilbert) spaces.

Proposition 5.7 *Let X, Y be Banach spaces, $T \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$ and $A : D(A) \to \mathsf{Y}$ an operator on Y (in general unbounded, and with $D(A)$ properly contained in Y). If*

(i) A is closable,
(ii) $Ran(T) \subset D(A)$,
then $AT \in \mathfrak{B}(\mathsf{X}, \mathsf{Y})$.

Proof As the closure of A extends A, $AT = \overline{A}T$ is well defined. Now it suffices to show $\overline{A}T : \mathsf{X} \to \mathsf{Y}$ is closed and invoke the closed graph theorem (Theorem 2.99) to conclude. To prove $\overline{A}T$ is closed, assume $\mathsf{X} \ni x_n \to x \in \mathsf{X}$ and $(\overline{A}T)(x_n) \to y \in \mathsf{Y}$ as $n \to +\infty$. Then $Tx_n \to z \in \mathsf{Y}$, for T is continuous. As $\overline{A}$ is closed and $\overline{A}(Tx_n) \to y$, then $z \in D(\overline{A})$ and $\overline{A}z = y$. That is to say, $(\overline{A}T)(x) = y$. Therefore $\overline{A}T$ is closed by definition. □

5.1.3 *The Case of Hilbert Spaces: The Structure of* $\mathsf{H} \oplus \mathsf{H}$ *and the Operator* τ

Let us look at the situation in which $\mathsf{X} = \mathsf{H}$ is a Hilbert space with inner product $(\ |\ )$. We know that there is a convenient way to define a Hilbert space structure on the direct sum $\mathsf{H} \oplus \mathsf{H}$, yielding the *Hilbert sum* of H with itself. This structure was presented in Definition 3.67, referring to a much more general situation. Here we wish to make further comments on the elementary case at hand. From Definition 3.67, we know that the inner product:

$$((x, x')|(y, y'))_{\mathsf{H}\oplus\mathsf{H}} := (x|y) + (x'|y') \quad \text{if } (x, x'), (y, y') \in \mathsf{H} \oplus \mathsf{H} \tag{5.1}$$

makes the two summands of $\mathsf{H}\oplus\mathsf{H}$ mutually orthogonal, so the sum is not only direct, but orthogonal. Furthermore, it turns $\mathsf{H}\oplus\mathsf{H}$ into a Hilbert space, for the induced norm $|| \; ||_{\mathsf{H}\oplus\mathsf{H}}$ satisfies:

$$||(z, z')||^2_{\mathsf{H}\oplus\mathsf{H}} = ||z||^2 + ||z'||^2 \quad \text{for any } (z, z') \in \mathsf{H} \oplus \mathsf{H}. \tag{5.2}$$

Let us see how. Any Cauchy sequence $\{(x_n, x'_n)\}_{n\in\mathbb{N}} \subset \mathsf{H} \oplus \mathsf{H}$ for the norm $|| \; ||_{\mathsf{H}\oplus\mathsf{H}}$ determines Cauchy sequences in H: $\{x_n\}_{n\in\mathbb{N}}$ and $\{x'_n\}_{n\in\mathbb{N}}$. The latter converge to x and x' in H respectively. It is therefore immediate to see $(x_n, x'_n) \to (x, x')$ as $n \to +\infty$ in norm $|| \; ||_{\mathsf{H}\oplus\mathsf{H}}$, by (5.2). Therefore $(\mathsf{H} \oplus \mathsf{H}, || \; ||_{\mathsf{H}\oplus\mathsf{H}})$ is complete. Topologically speaking, $\mathsf{H} \oplus \mathsf{H}$ can be endowed with the product topology of $\mathsf{H} \times \mathsf{H}$, the same used to define the closure of an operator. But this is exactly the topology induced by the inner product. To see this directly, in analogy to the discussion of Sect. 2.3.6, it suffices to recall the inclusions of open balls

$$B_{\delta/2}(x) \times B_{\delta/2}(y) \subset B^{(\mathsf{H}\oplus\mathsf{H})}_\delta((x, y)) \subset B_\delta(x) \times B_\delta(y),$$

where $B^{(\mathsf{H}\oplus\mathsf{H})}_\delta((x, y)) \subset \mathsf{H} \oplus \mathsf{H}$ has centre $(x, y) \in \mathsf{H} \oplus \mathsf{H}$ and radius $\delta > 0$, while $B_\varepsilon(z) \subset \mathsf{H}$ has centre z and radius $\varepsilon > 0$.

A useful tool to prove results quickly is the bounded operator

$$\tau : \mathsf{H} \oplus \mathsf{H} \ni (x, y) \mapsto (-y, x) \in \mathsf{H} \oplus \mathsf{H}\,. \tag{5.3}$$

If we refer * and $^\perp$ to the Hilbert space $\mathsf{H} \oplus \mathsf{H}$, then:

$$\tau^* = \tau^{-1} = -\tau\,, \tag{5.4}$$

so, in particular, τ is *unitary* on $\mathsf{H} \oplus \mathsf{H}$. Moreover, a direct computation shows τ and $^\perp$ *commute*:

$$\tau(F^\perp) = (\tau(F))^\perp \tag{5.5}$$

for any $F \subset \mathsf{H} \oplus \mathsf{H}$.

5.1.4 General Properties of the Hermitian Adjoint Operator

We now pass to define the *Hermitian adjoint* of an (unbounded) operator T whose domain $D(T)$ is *dense* in a Hilbert space H.

We cannot rely on Riesz's theorem and must proceed differently. First of all let us define the *domain* $D(T^*)$ of the adjoint, bearing in mind we are aiming at obtaining $(T^*x|y) = (x|Ty)$ with $x \in D(T^*)$ and $y \in D(T)$. To this end we put

$$D(T^*) := \left\{x \in \mathsf{H} \mid \text{ there exists } z_{T,x} \in \mathsf{H} \text{ with } (x|Ty) = (z_{T,x}|y) \text{ for any } y \in D(T)\right\} , \tag{5.6}$$

and later we will set $T^*x := z_{T,x},\ x \in D(T^*)$.

At any rate, let us show definition (5.6) is well posed, first, and that it determines: (a) a *subspace* $D(T^*) \subset \mathsf{H}$, and (b) an *operator* $T^* : D(T^*) \ni x \mapsto z_{T,x}$.

(a) $D(T^*) \neq \varnothing$ for $\mathbf{0} \in D(T^*)$ if we define $z_{T,\mathbf{0}} := \mathbf{0}$. Moreover, by linearity of the inner product and of T, if $x, x' \in D(T^*)$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha x + \beta x' \in D(T^*)$. That happens because $(\alpha x + \beta x'|Ty) = (z_{T,\alpha x+\beta x'}|y)$ if $z_{T,\alpha x+\beta x'} := \alpha z_{T,x} + \beta z_{T,x'}$. Hence $D(T^*)$ is a subspace.

(b) The assignment $D(T^*) \ni x \mapsto z_{T,x} =: T^*x$ will define a function, linear by construction as we saw above, only if any $x \in D(T^*)$ determines a *unique* element $z_{T,x}$. We claim that this is the case when $D(T^*)$ is dense, as we have assumed. If $(z'_{T,x}|y) = (x|Ty) = (z_{T,x}|y)$ for any $y \in D(T)$, then $0 = (x|Ty) - (x|Ty) = (z_{T,x} - z'_{T,x}|y)$. Since $\overline{D(T)} = \mathsf{H}$, there exists $\{y_n\}_{n\in\mathbb{N}} \subset D(T)$ with $y_n \to z_{T,x} - z'_{T,x}$. The inner product is continuous, so $(z_{T,x} - z'_{T,x}|y) = 0$ implies $||z_{T,x} - z'_{T,x}||^2 = 0$ and then $z_{T,x} = z'_{T,x}$.

Definition 5.8 (*Adjoint operator*) If T is an operator on the Hilbert space H with $\overline{D(T)} = \mathsf{H}$, the **(Hermitian) adjoint operator** to T, denoted T^*, is the operator on H with domain

$$D(T^*) := \left\{x \in \mathsf{H} \mid \text{ there exists } z_{T,x} \in \mathsf{H} \text{ with } (x|Ty) = (z_{T,x}|y) \text{ for any } y \in D(T\right\}$$

defined by $T^* : x \mapsto z_{T,x}$.

Remark 5.9 **(1)** It is clear that by construction

$$(T^*x|y) = (x|Ty)\,, \quad \text{for any pair } (x, y) \in D(T^*) \times D(T)$$

as we wanted.

(2) If $T \in \mathfrak{B}(\mathsf{H})$, Definition 5.8 implies immediately $D(T^*) = \mathsf{H}$ by Riesz's Theorem 3.16. Hence:

Definitions 5.8 and 3.37 *coincide for adjoints to operators in* $\mathfrak{B}(\mathsf{H})$.

(3) If T is a densely-defined operator on the Hilbert space H, $D(T^*)$ is not automatically dense in H. Therefore $(T^*)^*$ will *not* exist, in general.

(4) If A, B are densely-defined operators on the Hilbert space H:

$$A \subset B \quad \Rightarrow \quad A^* \supset B^* . \tag{5.7}$$

The proof is straightforward from Definition 5.8.

(5) If A, B are operators on the Hilbert space H with dense domains, and $D(AB)$ is dense, then

$$B^*A^* \subset (AB)^* \ .$$

Furthermore

$$B^*A^* = (AB)^*$$

if $A \in \mathfrak{B}(\mathsf{H})$.

Similarly, if $D(A+B)$ is dense,

$$A^* + B^* \subset (A+B)^* \ .$$

Furthermore

$$A^* + B^* = (A+B)^*$$

if $A \in \mathfrak{B}(\mathsf{H})$.
The proofs are deferred to the exercise section at the end of this chapter. ■

Theorem 5.10 *Let A be an operator on the Hilbert space H with $\overline{D(A)} = \mathsf{H}$. Then*

(a) *A^* is closed and*

$$G(A^*) = \tau(G(A))^{\perp} \ . \tag{5.8}$$

(b) *A is closable $\Leftrightarrow$ $D(A^*)$ is dense, in which case*

$$A \subset \overline{A} = (A^*)^* \ .$$

(c) *$Ker(A^*) = [Ran(A)]^{\perp}$ and $Ker(A) \subset [Ran(A^*)]^{\perp}$, with equality if $D(A^*)$ is dense in H and A is closed.*

(d) *If A is closed then $\mathsf{H} \oplus \mathsf{H}$ splits orthogonally:*

$$\mathsf{H} \oplus \mathsf{H} = \tau(G(A)) \oplus G(A^*) \ . \tag{5.9}$$

Proof (a) Write $\tau(G(A))^\perp$ explicitly, using (5.5):

$$\tau(G(A))^\perp = \{(x, y) \in \mathsf{H} \oplus \mathsf{H} \mid -(x|Az) + (y|z) = 0 \quad \text{for any } z \in D(A)\}\,.$$

That is to say, $\tau(G(A))^\perp$ is the graph of A^* (so long as the operator is defined!) and (5.8) holds. By construction $\tau(G(A))^\perp$ is closed, being an orthogonal complement (Theorem 3.13(a)), so A^* is closed.

(b) Consider the closure of the graph of A. Then we have $\overline{G(A)} = (G(A)^\perp)^\perp$ by Theorem 3.13. Since $\tau\tau = -I$, $S^\perp = -S^\perp$ for any set S, and because (5.5), (5.8) hold, we have:

$$\overline{G(A)} = -\tau(\,\tau(G(A))^\perp\,)^\perp = -\tau(G(A^*))^\perp = \tau(G(A^*))^\perp\,. \tag{5.10}$$

By Proposition 5.6, $\overline{G(A)}$ is the graph of an operator (the closure of A) iff $\overline{G(A)}$ does not contain elements $(\mathbf{0}, z)$, $z \neq \mathbf{0}$. I.e., $\overline{G(A)}$ is *not* the graph of an operator iff there exists $z \neq \mathbf{0}$ such that $(\mathbf{0}, z) \in \tau(G(A^*))^\perp$. More explicitly

$$\text{there exists } z \neq \mathbf{0} \text{ such that} \quad 0 = ((\mathbf{0}, z)|(-A^*x, x))\,, \quad \text{for any } x \in D(A^*)\,.$$

Put equivalently, $\overline{G(A)}$ is *not* the graph of an operator iff $D(A^*)^\perp \neq \{\mathbf{0}\}$, which happens precisely when $D(A^*)$ is not dense in H. To sum up: $\overline{G(A)}$ is a graph $\Leftrightarrow$ $\overline{D(A^*)} = \mathsf{H}$.

If $D(A^*)$ is dense in H, then $(A^*)^*$ exists, and by (5.10), (5.8) we have

$$\overline{G(A)} = \tau(G(A^*))^\perp = G((A^*)^*)\,.$$

Eventually, by definition of closure, $\overline{G(A)} = G(\overline{A})$. Substituting above:

$$G(\overline{A}) = G((A^*)^*)\,,$$

so $\overline{A} = (A^*)^*$.

(c) The claims descend directly from

$$(A^*x|y) = (x|Ay)\,, \quad \text{for any pair} (x, y) \in D(A^*) \times D(A)$$

by the density of $D(A)$, and from (b) when A is closed.

(d) Since A is closed, $G(A)$ is closed and so $\tau(G(A))$ is closed, because $\tau : \mathsf{H} \oplus \mathsf{H} \to \mathsf{H} \oplus \mathsf{H}$ is unitary. From (5.8) and Theorem 3.13 (b, d) we have immediately (5.9). This ends the proof. □

Remark 5.11 The density of $D(A)$ implies $(A - \lambda I)^* = A^* - \overline{\lambda} I$ for $\lambda \in \mathbb{C}$, so the first equation in (c) has the immediate consequence:

$$Ker(A^* - \overline{\lambda} I) = [Ran(A - \lambda I)]^\perp\,,$$

while the second equation yields:

$$Ker(A - \lambda I) \subset [Ran(A^* - \overline{\lambda} I)]^\perp .$$

In the rest of the book these relations will be used repeatedly. ■

5.2 Hermitian, Symmetric, Self-adjoint and Essentially Self-adjoint Operators

We are now in a position to define in full generality *self-adjoint* operators and related objects.

Definition 5.12 Let $(\mathsf{H}, (\,|\,))$ be a Hilbert space. An operator $A : D(A) \to \mathsf{H}$ on H is called

(a) Hermitian if $(Ax||y) = (x|Ay)$ for any $x, y \in D(A)$;
(b) symmetric if:

(i) A is Hermitian and
(ii) $D(A)$ is dense;

therefore A is symmetric if and only if:

(i)' $\overline{D(A)} = \mathsf{H}$ and
(ii)' $A \subset A^*$;

(c) self-adjoint if:

(i) $D(A)$ is dense and
(ii) $A = A^*$;

(d) essentially self-adjoint if:

(i) $D(A)$ is dense,
(ii) $D(A^*)$ is dense and
(iii) $A^* = (A^*)^*$ (the adjoint is self-adjoint).

Equivalently (by Theorem 5.10(b)), A is essentially self-adjoint if:

(i)' $D(A)$ is dense,
(ii)' A is closable and
(iii)' $A^* = \overline{A}$;

(e) normal if $A^*A = AA^*$, where either side is defined on its standard domain.

Remark 5.13 **(1)** A comment on (c) in Definition 5.12: by Theorem 5.10(a), every self-adjoint operator is automatically closed.

(2) It is worth noting that:

(i) *the definitions of Hermitian, symmetric, self-adjoint and essentially self-adjoint operator coincide when the operator's domain is the whole Hilbert space;*
(ii) the following important result holds.

Theorem 5.14 (Hellinger-Toeplitz) *A Hermitian operator with the entire Hilbert space as domain is necessarily bounded (and self-adjoint in the sense of definition 3.56).*

Proof Boundedness follows from Proposition 3.60(d). The operator is therefore self-adjoint, according to Definition 3.9. □

(iii) *Bounded self-adjoint operators for Definition* 3.56 *are precisely the self-adjoint operators of Definition* 5.12 *with domain the whole space*.
(3) Essential self-adjointness is by far the most important property of the four for applications to QM, on the following grounds. As we will explain soon, an essentially self-adjoint operator admits a unique self-adjoint extension, so it retains the information of a self-adjoint operator, essentially. For reasons we shall see later in the book, paramount operators in QM are self-adjoint. At the same time it is a fact that differential operators are the easiest to handle in QM. It often turns out that QM's differential operators become essentially self-adjoint if defined on suitable domains. Thus self-adjoint differential operators are on one hand easy to employ, on the other they carry, in essence, the information of self-adjoint operators useful in QM. Because of this we will indulge on certain features related to essential self-adjointness.
(4) Given an operator $A : D(A) \to \mathsf{H}$ on the Hilbert space H, $B \in \mathfrak{B}(\mathsf{H})$ **commutes** with A when:

$$BA \subset AB\,.$$

If the domain of A is dense and so A^* exists, it is easy to check that if $B \in \mathfrak{B}(\mathsf{H})$ commutes with A then B^* commutes with A^* (prove it as an exercise). Denote by $\{A\}'$ the **commutant** of $A : D(A) \to \mathsf{H}$:

$$\{A\}' := \{B \in \mathfrak{B}(\mathsf{H}) \mid BA \subset AB\}$$

If $A = A^*$ then $\{A\}'$ is a unital *-subalgebra of $\mathfrak{B}(\mathsf{H})$ that is closed in the strong topology (prove it as an exercise). Therefore it is a *von Neumann algebra* (see Sect. 3.3.2). The double commutant $\{A\}'' := \{\{A\}'\}'$ is still a von Neumann algebra, called the **von Neumann algebra** generated by A.

■

The following important, yet elementary, proposition will be frequently used without explicit mention. Its easy proof is left to the reader.

Proposition 5.15 *Let* $\mathsf{H}_1, \mathsf{H}_2$ *be Hilbert spaces and* $U : \mathsf{H}_1 \to \mathsf{H}_2$ *a unitary operator. If* $A : D(A) \to \mathsf{H}_1$ *is an operator on* H_1*, consider the operator on* H_2

$$A_2 : D(A_2) \to \mathsf{H}_2 \quad \textit{with } A_2 := UA_1U^{-1} \textit{and } D(A_2) := UD(A_1).$$

Then A_2 *is closable, or closed, Hermitian, symmetric, essentially self-adjoint, self-adjoint, or normal if and only if* A_1 *is alike.*

Notation 5.16 From now on we shall also write $A^{**\cdots*}$ instead of $(((A^*)^*)\cdots)^*$. ■

Proposition 5.17 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space and* A *an operator on* H.

(a) *If* $D(A)$ *and* $D(A^*)$*, are dense,*

$$A^* = \overline{A}^* = \overline{A^*} = A^{***}\,. \tag{5.11}$$

In particular the identities are true when A *is symmetric.*

(b) *A is essentially self-adjoint* $\Leftrightarrow$ $\overline{A}$ *is self-adjoint.*

(c) *If A is self-adjoint, it is maximal symmetric, i.e. it has no proper symmetric extensions.*

(d) *If A is essentially self-adjoint, A admits only one self-adjoint extension:* $\overline{A}$ *(coinciding with* A^**).*

Proof (a) If $D(A)$ and $D(A^*)$ are dense, the operators A^*, A^{**} and A^{***} exist (in particular $D(A^{**}) \supset D(A)$ is dense). Moreover

$$\overline{A}^* = (A^{**})^* = A^{***} = (A^*)^{**} = \overline{A^*}$$

by Theorem 5.10(b). Since A^* is closed (by Theorem 5.10(a)) we have $\overline{A^*} = A^*$. If A is symmetric, it has dense domain, so that $A^* \supset A$ has dense domain as well.

(b) If A is essentially self-adjoint, $\overline{A} = A^*$, and in particular $D(\overline{A}) = D(A^*)$ is dense. Compute the adjoint of $\overline{A}$ and recall Theorem 5.10(b): $\overline{A}^* = (A^*)^* = \overline{A}$, i.e. $\overline{A}$ is self-adjoint.

Vice versa, if $\overline{A}$ is self-adjoint, i.e. there exists $\overline{A}^* = \overline{A}$, then $D(A), D(A^*), D(A^{**})$ are dense and by part (a): $A^* = \overline{A^*} = \overline{A}^*$. Hence $A^* = \overline{A}$, and A is essentially self-adjoint.

(c) Let A be self-adjoint and $A \subset B$, B symmetric. Taking adjoints gives $A^* \supset B^*$. But $B^* \supset B$ by symmetry, so

$$A \subset B \subset B^* \subset A^* = A\,,$$

and then $A = B = B^*$.

(d) Let $A^* = A^{**}$, $A \subset B$ with $B = B^*$. Taking the adjoint of $A \subset B$ we see that $B = B^* \subset A^*$. Taking the adjoint twice yields $A^{**} \subset B$, but then

$$B = B^* \subset A^* = A^{**} \subset B\,,$$

hence $B = A^{**}$. The latter coincides with $\overline{A}$ by Theorem 5.10(b). □

Now we discuss two crucial features that characterise self-adjoint and essentially self-adjoint operators.

Theorem 5.18 *Let A be a symmetric operator on the Hilbert space* H. *The following are equivalent:*

(a) *A is self-adjoint;*

(b) *A is closed and* $Ker(A^* \pm iI) = \{\mathbf{0}\}$*;*

(c) $Ran(A \pm iI) = \mathsf{H}$.

Proof (a) ⇒ (b). If $A = A^*$, A is closed because A^* is. If $x \in Ker(A^* + iI)$, then $Ax = -ix$, so

$$i(x|x) = (Ax|x) = (x|Ax) = (x| - ix) = -i(x|x)\,,$$

whence $(x|x) = 0$ and $x = \mathbf{0}$.

The proof that $Ker(A^* - iI) = \{\mathbf{0}\}$ is analogous.

(b) ⇒ (c). By definition of adjoint we have (see Remark 5.11) $[Ran(A - iI)]^{\perp} = Ker(A^* + iI)$. Hence part (b) implies $Ran(A - iI)$ is dense in H. Now we shall use the closure of A to show that $Ran(A - iI) = \mathsf{H}$. Fix $y \in \mathsf{H}$ arbitrarily and choose $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ so that $(A - iI)x_n \to y \in \mathsf{H}$. For $z \in D(A)$,

$$||(A - iI)z||^2 = ||Az||^2 + ||z||^2 \geq ||z||^2\,,$$

whence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and $x = \lim_{n\to+\infty} x_n$ exists. The closure of A forces $A - iI$ to be closed, so $(A - iI)x = y$ and then $Ran(A - iI) = \overline{Ran(A - iI)} = \mathsf{H}$. The proof of $Ker(A^* - iI) = \{\mathbf{0}\}$ is similar.

(c) ⇒ (a). Since $A \subset A^*$ by symmetry, it is enough to show $D(A^*) \subset D(A)$. Take $y \in D(A^*)$. Given that $Ran(A - iI) = \mathsf{H}$, there is a vector $x_- \in D(A)$ such that

$$(A - iI)x_- = (A^* - iI)y\,.$$

On $D(A)$ the operator A^* coincides with A and therefore, by the previous identity,

$$(A^* - iI)(y - x_-) = \mathbf{0}\,.$$

But $Ker(A^* - iI) = Ran(A + iI)^{\perp} = \{\mathbf{0}\}$, so $y = x_-$ and $y \in D(A)$. The argument for $Ran(A + iI)$ is completely analogous. □

Theorem 5.19 *Let A be a symmetric operator on the Hilbert space* H. *The following are equivalent:*

(a) *A is essentially self-adjoint;*

(b) $Ker(A^* \pm iI) = \{\mathbf{0}\}$*;*

(c) $\overline{Ran(A \pm iI)} = \mathsf{H}$.

Proof (a) $\Rightarrow$ (b). If A is essentially self-adjoint, then $A^* = A^{**}$ and A^* is self-adjoint (and closed). Applying Theorem 5.18 gives $Ker(A^{**} \pm iI) = \{\mathbf{0}\}$ and so (b) holds, for $A^{**} = A^*$.
(b) $\Rightarrow$ (a). $A \subset A^*$ by assumption, and because $D(A)$ is dense so is $D(A^*)$. Consequently, Theorem 5.10(b) implies A is closable and $A \subset \overline{A} = A^{**}$ (in particular $D(A^{**}) = D(\overline{A}) \supset D(A)$ is dense). Therefore $A \subset A^*$ implies $\overline{A} \subset A^*$, and Proposition 5.17(a) tells $A^* = \overline{A}^*$. Overall, $\overline{A} \subset \overline{A}^*$, i.e. $\overline{A}$ is symmetric. Then we may apply Theorem 5.18 to $\overline{A}$, for this operator satisfies (b) in the theorem. We conclude $\overline{A}$ is self-adjoint. From Proposition 5.17(b) it follows A is essentially self-adjoint.
(b) $\Leftrightarrow$ (c). Since $Ran(A \pm iI)^{\perp} = Ker(A^* \mp iI)$ and $\overline{Ran(A \pm iI)} \oplus Ran(A \pm iI)^{\perp} = \mathsf{H}$, (b) and (c) are equivalent. □

To finish we present a useful notion for the applications: the *core* of an operator.

Definition 5.20 Let A be a closed, densely-defined operator on the Hilbert space H. A dense subspace $\mathsf{S} \subset D(A)$ is a **core** of A if

$$\overline{A \restriction_{\mathsf{S}}} = A\,.$$

The next proposition is obvious, yet important.

Proposition 5.21 *If A is a self-adjoint operator on the Hilbert space* H, *a subspace* $\mathsf{S} \subset D(A)$ *is a core for A if and only if $A \restriction_{\mathsf{S}}$ is essentially self-adjoint.*

Proof If $A \restriction_S$ is essentially self-adjoint, it admits a unique self-adjoint extension, which coincides with its closure by Proposition 5.17(d). In our case the extension necessarily coincides with A, which is self-adjoint. Hence $A \restriction_S$ is a core.
Conversely, if $A \restriction_S$ is a core, the closure of $A \restriction_S$ is self-adjoint because it coincides with the self-adjoint A. By Proposition 5.17(b) $A \restriction_S$ is essentially self-adjoint. □

5.3 Two Major Applications: The Position Operator and the Momentum Operator

To exemplify the formalism described so far we study the features of two self-adjoint operators of the foremost relevance in QM, called *position operator* and *momentum operator*. Their physical meaning will be clarified in the second part of the book.

In the sequel we shall adopt the conventions and notations of Sect. 3.7, and $x = (x_1, \ldots, x_n)$ will be a generic point in $\mathbb{R}^n$.

5.3.1 The Position Operator

Definition 5.22 (*Position operator*) Consider $\mathsf{H} := L^2(\mathbb{R}^n, dx)$, where dx is the Lebesgue measure on $\mathbb{R}^n$. If $i \in \{1, 2, \ldots, n\}$ is given, the operator on H:

$$(X_i f)(x) = x_i f(x)\,, \tag{5.12}$$

with domain:

$$D(X_i) := \left\{ f \in L^2(\mathbb{R}^n, dx) \;\middle|\; \int_{\mathbb{R}^n} |x_i f(x)|^2\, dx < +\infty \right\}, \tag{5.13}$$

is called ith **position operator**.

Proposition 5.23 *The operator X_i of Definition 5.22 satisfies these properties.*

(a) *X_i is self-adjoint.*

(b) *$\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ are cores: $X_i = \overline{X_i \restriction_{\mathscr{D}(\mathbb{R}^n)}} = \overline{X_i \restriction_{\mathscr{S}(\mathbb{R}^n)}}$.*

Proof (a) The domain of X_i is certainly dense in H for it contains the space $\mathscr{D}(\mathbb{R}^n)$ of smooth maps with compact support, and also the space of Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ (see Notation 3.100), both of which are dense in $L^2(\mathbb{R}^n, dx)$. Therefore X_i admits an adjoint. By definition we have $(g|X_i f) = (X_i g|f)$ if $f, g \in D(X_i)$. Consequently X_i is Hermitian and symmetric. We claim it is self-adjoint, too. By symmetry $X_i \subset X_i^*$, so it suffices to show $D(X_i^*) \subset D(X_i)$. Let us define the adjoint to X_i directly: $f \in D(X_i^*)$ if and only if there exists $h \in L^2(\mathbb{R}^n, dx)$ ($h = X_i^* f$ by definition) such that

$$\int_{\mathbb{R}^n} \overline{f(x)} x_i g(x) dx = \int_{\mathbb{R}^n} \overline{h(x)} g(x) dx \quad \text{for any} g \in D(X_i).$$

Since $D(X_i)$ is dense and

$$\int_{\mathbb{R}^n} [\overline{x_i f(x) - h(x)}] g(x) dx = 0 \quad \text{for any } g \in D(X_i),$$

we can also say $f \in L^2(\mathbb{R}^n, dx)$ *belongs to* $D(X_i^*) \;\Leftrightarrow\; x_i f(x) = h(x)$ *almost everywhere, with* $h \in L^2(\mathbb{R}^n, dx)$.

Hence $D(X_i^*)$ consists precisely of maps $f \in L^2(\mathbb{R}^n, dx)$ for which

$$\int_{\mathbb{R}^n} |x_i f(x)|^2\, dx < +\infty\,,$$

and so $D(X_i^*) = D(X_i)$ and X_i is self-adjoint.

(b) If we define X_i as above, apart from restricting the domain to $\mathscr{D}(\mathbb{R}^n)$ or $\mathscr{S}(\mathbb{R}^n)$, the operator thus obtained is no longer self-adjoint, but stays symmetric. The adjoints to $X_i \restriction_{\mathscr{D}(\mathbb{R}^n)}$ and $X_i \restriction_{\mathscr{S}(\mathbb{R}^n)}$ both coincide with the above X_i^*, for in the construction we only used that X_i is the operator that multiplies by x_i on a *dense* domain: whether this is the $D(X_i)$ of (5.13), or a dense subspace, does not alter the result. If we define X_i through (5.12) and (5.13), the adjoint X_i^* must satisfy $Ker(X_i^* \pm iI) = \{\mathbf{0}\}$ by Theorem 5.18(b). But as X_i^* is the same as we get by restricting the domain of X_i to $\mathscr{D}(\mathbb{R}^n)$ or $\mathscr{S}(\mathbb{R}^n)$ by Theorem 5.19(b), the restricted X_i is essentially self-adjoint. Part (b) is now an immediate consequence of Proposition 5.21. □

5.3.2 The Momentum Operator

Let us introduce the *momentum operator*. Henceforth we adopt the definitions and conventions taken from Example 2.91, and retain Notation 3.100. First, though, we need a few definitions.

We say $f : \mathbb{R}^n \to \mathbb{C}$ is a **locally integrable** function on $\mathbb{R}^n$ if $f \cdot g \in L^1(\mathbb{R}^n, dx)$ for any map $g \in \mathscr{D}(\mathbb{R}^n)$.

Definition 5.24 Let f be locally integrable and α a multi-index. A map $h : \mathbb{R}^n \to \mathbb{C}$ is the α**th weak derivative** of f, written w-$\partial^\alpha f = h$, if h is locally integrable and:

$$\int_{\mathbb{R}^n} h(x)g(x)\,dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\partial_x^\alpha g(x)\,dx \tag{5.14}$$

for any map $g \in \mathscr{D}(\mathbb{R}^n)$.

Remark 5.25 **(1)** If it exists, a weak derivative is uniquely determined up to sets of zero measure: if h and h' are locally integrable (the following is trivial if they are in $L^2(\mathbb{R}^n, dx)$) and satisfy (5.14), then:

$$\int_{\mathbb{R}^n} (h(x) - h'(x))g(x)\,dx = 0 \quad \text{for any}\, g \in \mathscr{D}(\mathbb{R}^n). \tag{5.15}$$

This implies $h(x) - h'(x) = 0$ almost everywhere by the *Du Bois–Reymond lemma* [Vla02]:

Lemma 5.26 (Du Bois–Reymond) *Suppose ϕ is a locally integrable map on $\mathbb{R}^n$. Then: ϕ is zero almost everywhere $\Leftrightarrow$ $\int_{\mathbb{R}^n} \phi(x)f(x)\,dx = 0$ for any $f \in \mathscr{D}(\mathbb{R}^n)$.*

(2) In case $f \in C^{|\alpha|}(\mathbb{R}^n)$, the αth weak derivative of f exists and coincides with the usual derivative (up to a zero-measure set). However, there are situations in which the ordinary derivative *does not exist*, whereas the weak derivative is defined.

(3) Maps in $L^2(\mathbb{R}^n, dx)$ are locally integrable, for $\mathscr{D}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, dx)$ and $f \cdot g \in L^1$ if $f, g \in L^2$.

■

In order to define the *momentum operator* let us construct the operator A_j on $\mathsf{H} := L^2(\mathbb{R}^n, dx)$:

$$(A_j f)(x) = -i\hbar \frac{\partial}{\partial x_j} f(x) \quad \text{with } D(A_j) := \mathscr{D}(\mathbb{R}^n), \tag{5.16}$$

where $\hbar$ is a positive constant (*Planck's constant*), whose precise value is irrelevant at present. By definition we have $(g|A_j f) = (A_j g|f)$ if $f, g \in D(A_j)$. Thus A_j is symmetric because $\overline{D(A_j)} = \mathsf{H}$. Let us find the adjoint to A_j, denoted $P_j := A_j^*$, directly from the definition. Given $f \in D(A_j^*) = D(P_j)$ there must be $\phi \in L^2(\mathbb{R}^n, dx)$ (coinciding with $P_j f$ by definition) such that:

$$\int_{\mathbb{R}^n} \overline{\phi(x)} g(x) dx = -i\hbar \int_{\mathbb{R}^n} \overline{f(x)} \frac{\partial}{\partial x_j} g(x) dx\,, \quad \text{for any } g \in \mathscr{D}(\mathbb{R}^n)\,. \tag{5.17}$$

Conjugating the equation we may rephrase (5.17) as follows: $f \in L^2(\mathbb{R}^n, dx)$ *belongs in* $D(P_j)$ *if and only if it admits weak derivative* $\phi \in L^2(\mathbb{R}^n, dx)$.

Definition 5.27 (*Momentum operator*) Let $\mathsf{H} := L^2(\mathbb{R}^n, dx)$, dx being the Lebesgue measure on $\mathbb{R}^n$. Given $j \in \{1, 2, \ldots, n\}$, the operator on H:

$$(P_j f)(x) = -i\hbar \text{w-}\frac{\partial}{\partial x_j} f(x)\,, \tag{5.18}$$

with domain:

$$D(P_j) := \left\{ f \in L^2(\mathbb{R}^n, dx) \;\middle|\; \text{there exists w-}\frac{\partial}{\partial x_j} f \in L^2(\mathbb{R}^n, dx) \right\}, \tag{5.19}$$

is called the jth **momentum operator**.

Remark 5.28 If $n = 1$, $D(P_j)$ is identified with the *Sobolev space* $\mathsf{H}^1(\mathbb{R}, dx)$. ■

Proposition 5.29 *Let P_j be the momentum operator of Definition 5.27. Then*
(a) *P_j is self-adjoint;*
(b) *$\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ are cores of P_j. Therefore:*

$$(A_j f)(x) = -i\hbar \frac{\partial}{\partial x_j} f(x) \quad \textit{with } f \in D(A_j) := \mathscr{D}(\mathbb{R}^n)\,, \tag{5.20}$$

$$(A'_j f)(x) = -i\hbar \frac{\partial}{\partial x_j} f(x) \quad \textit{with } f \in D(A'_j) := \mathscr{S}(\mathbb{R}^n)\,, \tag{5.21}$$

are essentially self-adjoint and $\overline{A_j} = \overline{A'_j} = P_j$.

Proof To simplify notations, in the sequel we will set $\hbar = 1$ (absorbing the constant $\hbar^{-1}$ in the unit of measure of the coordinate x_j), and denote by ∂_j the jth derivative and by w-∂_j the weak derivative. We want to prove $Ker(A_j^* \pm iI) = \{\mathbf{0}\}$. This would imply, owing to Theorem 5.19, that A_j is essentially self-adjoint, i.e. $P_j = A_j^*$ is self-adjoint. The space $Ker(A_j^* \pm iI)$ consists of maps $f \in L^2(\mathbb{R}^n, dx)$ admitting weak derivative and such that $i(\text{w-}\partial_j f \pm f) = 0$. Let us consider the equation:

$$\text{w-}\partial_j f \pm f = 0\,, \tag{5.22}$$

with $f \in L^2(\mathbb{R}^n, dx)$. Multiplying by an exponential gives:

$$\text{w-}\partial_j \left(e^{\pm x_j} f\right) = 0\,, \tag{5.23}$$

So we can reduce to proving the following.

Lemma 5.30 *If $h : \mathbb{R}^n \to \mathbb{C}$ is locally integrable and*

$$w\text{-}\partial_j h = 0\,, \tag{5.24}$$

h coincides almost everywhere with a constant function in x_j.

Proof of lemma 5.30. Without loss of generality we can suppose $j = 1$. We indicate by (x, y) the coordinates of $\mathbb{R}^n$, where x is x_1 and y subsumes the remaining $n - 1$ components. Take h locally integrable satisfying (5.24). Explicitly:

$$\int_{\mathbb{R}^n} h(x, y) \frac{\partial}{\partial x} g(x, y) dx \otimes dy = 0\,, \quad \text{for any } g \in \mathscr{D}(\mathbb{R}^n)\,. \tag{5.25}$$

Pick $f \in \mathscr{D}(\mathbb{R}^n)$, and choose $a > 0$ large, so to have $supp(f) \subset [-a, a] \times [-a, a]^{n-1}$. Define a map $\chi \in \mathscr{D}(\mathbb{R})$ with $supp(\chi) = [-a, a]$ and $\int_{\mathbb{R}} \chi(x)dx = 1$. Then there is a map $g \in \mathscr{D}(\mathbb{R}^n)$ such that

$$\frac{\partial}{\partial x} g(x, y) = f(x, y) - \chi(x) \int_{\mathbb{R}} f(u, y)du\,.$$

In fact, it is enough to consider

$$g(x, y) := \int_{-\infty}^{x} f(u, y)du - \int_{-\infty}^{x} \chi(v)dv \int_{\mathbb{R}} f(u, y)du\,. \tag{5.26}$$

This map is smooth by construction, and its x-derivative coincides with:

$$f(x, y) - \chi(x) \int_{\mathbb{R}} f(u, y)du\,.$$

Moreover the support of g is bounded: if some coordinate satisfies $|y_k| > a$, then $f(u, y) = 0$ whichever u we have, so $g(x, y) = 0$ for any x. If $x < -a$ the first integral in (5.26) vanishes, and also the second one, for χ is supported in $[-a, a]$. Conversely, if $x > a$

$$g(x, y) := \int_{-\infty}^{+\infty} f(u, y)du - 1 \int_{\mathbb{R}} f(u, y)du = 0\,,$$

where we used $supp\chi = [-a, a]$ and $\int_{\mathbb{R}} \chi(x)dx = 1$. Altogether g vanishes outside $[-a, a] \times [-a, a]^{n-1}$. Inserting g in (5.25) and using the theorem of Fubini–Tonelli gives

$$\int_{\mathbb{R}^n} h(x, y)f(x, y)\, dx \otimes dy - \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} h(x, y)\chi(x)dx\right) f(u, y)\, du \otimes dy = 0\,.$$

Relabelling variables:

$$\int_{\mathbb{R}^n} \left\{h(x, y) - \left(\int_{\mathbb{R}} h(u, y)\chi(u)du\right)\right\} f(x, y)\, dx \otimes dy = 0\,, \tag{5.27}$$

f being arbitrary in $\mathscr{D}(\mathbb{R}^n)$. Notice that

$$(x, y) \mapsto k(y) := \int_{\mathbb{R}} h(u, y)\chi(u)du$$

is locally integrable on $\mathbb{R}^n$, because

$$(x, y, u) \mapsto f(x, y)h(u, y)\chi(u)$$

is integrabile on $\mathbb{R}^{n+1}$ for any $f \in \mathscr{D}(\mathbb{R}^n)$ (it is enough to observe $|f(x, y)| \leq |f_1(x)||f_2(y)|$ for suitable f_1 in $\mathscr{D}(\mathbb{R})$ and f_2 in $\mathscr{D}(\mathbb{R}^{n-1})$). Equation (5.27), valid for any $f \in \mathscr{D}(\mathbb{R}^n)$, implies immediately

$$h(x, y) - \int_{\mathbb{R}} h(u, y)\chi(u)du = 0$$

almost everywhere on $\mathbb{R}^n$ by the Du Bois–Reymond Lemma 5.26. That is to say

$$h(x, y) = k(y)$$

almost everywhere on $\mathbb{R}^n$. $\square$

In the case under scrutiny the result implies that every solution to (5.22) must have the form $f(x) = e^{\pm x_j}h(x)$, where h does *not* depend on x_j. The theorem of Fubini–Tonelli then tells $\int_{\mathbb{R}^n} |f(x)|^2 dx = ||h||^2_{L^2(\mathbb{R}^{n-1})} \int_{\mathbb{R}} e^{\pm 2x_j} dx_j$. Hence h must be null almost everywhere if, as required, $f \in L^2(\mathbb{R}^n, dx)$. Therefore $Ker(A_j^* \pm iI) = \{\mathbf{0}\}$ and so $P_j = A_j^*$ is self-adjoint (A_j is essentially self-adjoint).

Because $\mathscr{S}(\mathbb{R}^n) \supset \mathscr{D}(\mathbb{R}^n)$ it is easy to see that A'_j is symmetric, that f admits generalised derivative if $f \in D(A'^*_j)$, and that

$$A'^*_j f = -i\text{w-}\frac{\partial}{\partial x_j} f \,.$$

Using the same procedure, if $f \in Ker(A'^*_j \pm I)$ then $f = 0$, so A'_j is essentially self-adjoint, too. Since $A_j \subset A'_j$ and A_j is essentially self-adjoint, then $A'^{**}_j = \overline{A'_j} = A_j^{**} = \overline{A_j} = P_j$ by Proposition 5.17(d). $\square$

There is another way to introduce the operator P_j, using the Fourier–Plancherel transform $\hat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dk)$ seen in Sect. 3.7.

We define on $L^2(\mathbb{R}^n, dk)$ the analogue to X_j, which we call K_j (conventionally, the target space $\mathbb{R}^n$ of the Fourier–Plancherel transform has coordinates $(k_1, \dots, k_n)$). Since $\hat{\mathscr{F}}$ is unitary, the operator $\hat{\mathscr{F}}^{-1} K_j \hat{\mathscr{F}}$ is self-adjoint if defined on the domain $\hat{\mathscr{F}}^{-1} D(K_j)$.

Proposition 5.31 *Let K_j be the* jth *position operator on the target space of the Fourier–Plancherel transform $\hat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dk)$. Then*

$$P_j = \hbar \hat{\mathscr{F}}^{-1} K_j \hat{\mathscr{F}} \,.$$

Proof It suffices to show the operators coincide on a domain where they are both essentially self-adjoint. To this end consider $\mathscr{S}(\mathbb{R}^n)$. From Sect. 3.7 we know the Fourier–Plancherel transform is the Fourier transform on this space, and $\hat{\mathscr{F}}(\mathscr{S}(\mathbb{R}^n)) = \mathscr{S}(\mathbb{R}^n)$. Moreover, the properties of the Fourier transform imply

$$-i\hbar \frac{\partial}{\partial x_j} f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ik\cdot x}\, \hbar k_j g(k)\, dk$$

provided $g \in \mathscr{S}(\mathbb{R}^n)$ and

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ik\cdot x}\, g(k)\, dk.$$

Therefore

$$P_j \restriction_{\mathscr{S}(\mathbb{R}^n)} = \hbar \hat{\mathscr{F}}^{-1} K_j \restriction_{\mathscr{S}(\mathbb{R}^n)} \hat{\mathscr{F}} \,.$$

Notice K_j is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^n)$ by Proposition 5.23, so also the operator $\hbar \hat{\mathscr{F}}^{-1} K_j \restriction_{\mathscr{S}(\mathbb{R}^n)} \hat{\mathscr{F}}$ is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^n)$, because $\hat{\mathscr{F}}$ is unitary. As $P_j \restriction_{\mathscr{S}(\mathbb{R}^n)} = A'_j$ is essentially self-adjoint as well (Proposition 5.29), and since self-adjoint extensions of essentially self-adjoint operators are unique and coincide with the closure (Proposition 5.17(d)), we conclude

$$P_j = \overline{\hbar \hat{\mathscr{F}}^{-1} K_j \restriction_{\mathscr{S}(\mathbb{R}^n)} \hat{\mathscr{F}}} = \hbar \hat{\mathscr{F}}^{-1} \overline{K_j \restriction_{\mathscr{S}(\mathbb{R}^n)}} \hat{\mathscr{F}} = \hbar \hat{\mathscr{F}}^{-1} K_j \hat{\mathscr{F}} .$$

□

5.4 Existence and Uniqueness Criteria for Self-adjoint Extensions

In this remaining part of the chapter we examine a few criteria to determine whether an operator admits self-adjoint extensions, and how many. A deeper analysis on these topics can be found in [ReSi80, vol. II], also in terms of quadratic forms, or in [Wie80, Rud91, Tes09, Schm12].

5.4.1 The Cayley Transform and Deficiency Indices

One crucial technical tool is the *Cayley transform*, introduced below. Before that, we generalise the notion of isometry (Definition 3.6) to operators with non-maximal domain.

Definition 5.32 An operator $U : D(U) \to \mathsf{H}$, on the Hilbert space H, is an **isometry** if

$$(Ux|Uy) = (x|y) \quad \text{for any } x, y \in D(U).$$

Remark 5.33 **(1)** Clearly, if $D(U) = \mathsf{H}$ the above definition pins down isometric operators in the sense of Definition 3.56.

(2) By Proposition 3.8, the above condition is the same as demanding $||Ux|| = ||x||$, for any $x \in D(U)$. ■

The transformation $\mathbb{R} \ni t \mapsto (t-i)(t+i)^{-1} \in \mathbb{C}$ is a well-known bijection between the real line $\mathbb{R}$ and the unit circle in $\mathbb{C}$ minus the point 1. There is a similar correspondence that maps isometric operators to symmetric operators, called the *Cayley transform*.

Theorem 5.34 *Let* H *be a Hilbert space.*
(a) *If* A *is a symmetric operator on* H*:*

(i) $A + iI$ *is injective,*

(ii) the **Cayley transform of** *A:*

$$V := (A - iI)(A + iI)^{-1} : Ran(A + iI) \to \mathsf{H}, \tag{5.28}$$

is a well-defined operator,

(iii) V is an isometry with $Ran(V) = Ran(A - iI)$.

(b) *If (5.28) holds for some operator $A : D(A) \to \mathsf{H}$ with $(A + iI)$ injective, then:*

(i) $I - V$ is injective,

(ii) $Ran(I - V) = D(A)$ and

$$A := i(I + V)(I - V)^{-1}. \tag{5.29}$$

(c) *If A is symmetric on H, the following facts are equivalent:*

(i) A is self-adjoint,

(ii) its Cayley transform V is unitary on H.

(d) *If $V : \mathsf{H} \to \mathsf{H}$ is unitary and $I - V$ injective, then V is the Cayley transform of some self-adjoint operator on H.*

Proof (a) A direct computation using the symmetry of A and the linearity of inner products proves that

$$||(A \pm iI)f||^2 = ||Af||^2 + ||f||^2 \tag{5.30}$$

if $f \in D(A)$. Therefore if $(A + iI)f = \mathbf{0}$ or $(A - iI)f = \mathbf{0}$ then $f = \mathbf{0}$. The operators $A \pm iI$ are therefore injective on $D(A)$, turning V into an operator $D(V) := Ran(A + iI) \to \mathsf{H}$. From (5.30)

$$||(A - iI)g|| = ||(A + iI)g||$$

for any $g \in D(A)$. Set $g = (A + iI)^{-1}h$, with $h \in Ran(A + iI)$. Then

$$||Vh|| = ||(A - iI)(A + iI)^{-1}h|| = ||h||,$$

so V is an isometry with domain $D(V) = Ran(A + iI)$ and range $Ran(V) = Ran(A - iI)$.

(b) The domain $D(V)$ consists of vectors $g = (A + iI)f$ with $f \in D(A)$. Applying V to g gives $Vg = (A - iI)f$. Adding and subtracting $g = (A + iI)f$ produces

$$(I + V)g = 2Af, \tag{5.31}$$

$$(I - V)g = 2if. \tag{5.32}$$

Now, (5.32) tells $(I - V)$ is injective, for if $(I - V)g = \mathbf{0}$ then $f = \mathbf{0}$ and so $g = (A + iI)f = \mathbf{0}$. Therefore if $f \in D(A)$ we can write

$$g = 2i(I - V)^{-1} f . \tag{5.33}$$

Furthermore, $Ran(I-V) = D(A)$ follows immediately from (5.32). Applying $(I+V)$ to Eq. (5.33) and using (5.31):

$$Af = i(I + V)(I - V)^{-1} f \quad \text{for any } f \in D(A).$$

(c) Suppose $A = A^*$. By Theorem 5.18 $Ran(A + iI) = Ran(A - iI) = \mathsf{H}$. Then part (a) implies V is an isometry from $Ran(A + iI) = \mathsf{H}$ onto $\mathsf{H} = Ran(A - iI)$. Hence V is a surjective isometry, i.e. a unitary operator.

Suppose now $V : \mathsf{H} \to \mathsf{H}$ is the unitary Cayley transform of a symmetric operator A on H. By part (a) $Ran(A+iI) = Ran(A-iI) = \mathsf{H}$. This means $A = A^*$ by Theorem 5.18.

(d) It is enough to prove V is the Cayley transform of a symmetric operator. By part (c) this symmetric operator is self-adjoint. By assumption there is a bijective map $z \mapsto x$, from $D(V) = \mathsf{H}$ to $Ran(I - V)$, given by $x := z - Vz$. Define $A : Ran(I - V) \to \mathsf{H}$ as

$$Ax := i(z + Vz) , \quad \text{if} x = z - Vz. \tag{5.34}$$

By taking $x, y \in D(A) = Ran(I - V)$ we have $x = z - Vz$ and $y = u - Vu$ for some $z, u \in D(V)$. But V is an isometry, so

$$(Ax|y) = i(z + Vz|u - Vu) = i(Vz|u) - i(z|Vu) = (z - Vz|iu - iVu) = (x|Ay) ,$$

and A is Hermitian. To show it is symmetric, note $D(A) = Ran(I - V)$ is dense. In fact $[Ran(I - V)]^\perp = Ker(I - V^*)$. If $Ker(I - V^*)$ were not $\{\mathbf{0}\}$, there would exist a non-zero vector $u \in \mathsf{H}$ such that $V^*u = u$, and then applying V would give $u = Vu$. But that is not possible, for $I - V$ is injective by assumption.

To finish, we prove V is the Cayley transform of A. Equation (5.34) reads:

$$2iVz = Ax - ix , \quad 2iz = Ax + ix , \quad \text{if } z \in \mathsf{H}. \tag{5.35}$$

Hence $V(Ax + ix) = Ax - ix$ for $x \in D(A)$ and $\mathsf{H} = D(V) = Ran(A + iI)$. But then V is the Cayley transform of A because $V(A + iI) = A - iI$, and so

$$V = (A - iI)(A + iI)^{-1} .$$

This ends the proof. □

Remark 5.35 From the statement and proof we infer that $Ker(A \pm iI) = \{\mathbf{0}\}$ if A is symmetric. The further condition $Ker(A^* \pm iI) = \{\mathbf{0}\}$ is very restrictive, equivalent to the essential self-adjointness of A (if A is symmetric) by Theorem 5.19. ■

Before we pass to the consequences of Theorem 5.34 concerning the existence of self-adjoint extensions of a symmetric operator, let us make a general remark.

Other textbooks that discuss these issues suppose the symmetric operator A be also *closed*. We shall not make that hypothesis, because the (differential) operators typically handled in practical computations of QM are symmetric but not closed. Besides, imposing closure from the start is not that essential, in view of the following elementary result.

Proposition 5.36 *If A is symmetric, then $\overline{A}$ is symmetric, and B is a self-adjoint extension of A if and only if it is a self-adjoint extension of $\overline{A}$.*

Proof By direct inspection, one sees that A symmetric $\Rightarrow \overline{A}$ symmetric (see the solutions of Exercises 5.7 and 5.8). A self-adjoint operator is closed because an adjoint operator is always closed. If $B = B^*$ extends A, then B is a closed extension of A so that $\overline{A} \subset B$, since $\overline{A}$ is the *smallest* closed extension of A. The converse is trivial: if B is self-adjoint and $B \supset \overline{A}$ then $B \supset \overline{A} \supset A$. □

The first result we introduce is about *deficiency indices*. If A is a symmetric operator on the Hilbert space H, we call

$$d_{\pm}(A) := \dim\, Ker(A^* \pm iI)$$

the **deficiency indices** of A. They can be defined equivalently as:

$$d_{\pm}(A) := \dim\,[Ran(A \mp iI)]^{\perp}\,,$$

because $Ker(A^* \pm iI) = [Ran(A \mp iI)]^{\perp}$. Furthermore, the deficiency indices of the symmetric operator A and those of $\overline{A}$ coincide, and more strongly we have

$$Ker(A^* \pm iI) = Ker(\overline{A}^* \pm iI)\,.$$

In fact, since $\overline{A} = A^{**}$ and $A^* = A^{***}$ due to Proposition 5.17(a), and because $(B + aI)^* = B^* + \overline{a}I$ and $\overline{B + aI} = \overline{B} + \overline{a}I$ for densely-defined, closable operators B (as one immediately proves), then A and $\overline{A}$ have the same adjoint.

Theorem 5.37 *If A is a symmetric operator on the Hilbert space H, the following facts hold.*
(a) *A admits self-adjoint extensions $\Leftrightarrow$ $d_+(A) = d_-(A)$.*
(b) *If $d_+(A) = d_-(A)$, there is a 1-1 correspondence*

$$U_0 : Ker(A^* - iI) \to Ker(A^* + iI)$$

between self-adjoint extensions of A and surjective isometries.

In particular, A admits more than one self-adjoint extension whenever $d_+(A) = d_-(A) > 0$.
As a matter of fact, the domain of the self-adjoint extension A_{U_0} is

$$D(A_{U_0}) = D(\overline{A}) + (I - U_0)(Ker(A^* - iI))\,, \tag{5.36}$$

where the sum is direct (but not orthogonal in general). Furthermore, A_{U_0} *acts by*

$$A_{U_0}z = \overline{A}x + i(I + U_0)y \,, \tag{5.37}$$

for $z = x + (I + U_0)y$ *with* $x \in D(\overline{A})$ *and* $y \in Ker(A^* - iI)$*,*

Proof (a) Consider the Cayley transform V of A. Suppose A has a self-adjoint extension B and let $U : \mathsf{H} \to \mathsf{H}$ be the Cayley transform of B. It is straightforward to see U is an extension of V using (5.28), recalling $(B + iI)^{-1}$ extends $(A + iI)^{-1}$ and $B - iI$ extends $A - iI$. Hence U maps $Ran(A + iI)$ to $Ran(A - iI)$. As U is unitary, $y \perp Ran(A + iI) \Leftrightarrow Uy \perp U(Ran(A + iI))$, that is to say $U([Ran(A + iI)]^{\perp}) = [Ran(A - iI)]^{\perp}$. By Theorem 5.10(c) this means $U(Ker(A^* + iI)) = Ker(A^* - iI)$. Since U is an isometry, dim $Ker(A^* + iI) =$ dim $Ker(A^* - iI)$, i.e. $d_+(A) = d_-(A)$.

(b) Let us show, conversely, that if $d_+(A) = d_-(A)$ then A has a self-adjoint extension, not unique in case $d_+(A) = d_-(A) > 0$. The Cayley transform V of A is bounded, so Proposition 2.47 says we can extend it, uniquely, to an isometric operator $U : \overline{Ran(A + iI)} \to \overline{Ran(A - iI)}$. The same we can do for V^{-1}, extending it to a unique isometry $\overline{Ran(A - iI)} \to \overline{Ran(A + iI)}$. By continuity this operator is $U^{-1} : \overline{Ran(A - iI)} \to \overline{Ran(A + iI)}$. Now recall $\overline{Ran(A \pm iI)}^{\perp} = [Ran(A \pm iI)]^{\perp} = Ker(A^* \mp iI)$. Having assumed $d_+(A) = d_-(A)$, we can define a unitary operator

$$U_0 : Ker(A^* - iI) \to Ker(A^* + iI) \,.$$

Since

$$\mathsf{H} = \overline{Ran(A + iI)} \oplus Ker(A^* - iI) = \overline{Ran(A - iI)} \oplus Ker(A^* + iI)$$

is an orthogonal decomposition by closed spaces,

$$W := U \oplus U_0 : (x, y) \mapsto (Ux, U_0y) \,, \quad \text{with } x \in \overline{Ran(A + iI)} \text{ and } y \in Ker(A^* - iI) \,,$$

is a unitary operator on H. Moreover $I - W$ is injective. In fact, $Ker(I - W)$ consists of pairs $(x, y) \neq (\mathbf{0}, \mathbf{0})$ with $Ux = x$ and $U_0y = y$: the first condition has only the solution $x = \mathbf{0}$ because U is an isometry, and the second one implies $y \in Ker(A^* + iI) \cap Ker(A^* - iI)$, giving $y = \mathbf{0}$. Therefore Theorem 5.34(d) applies, and W is the Cayley transform of a self-adjoint operator B. As W extends U, B is a self-adjoint extension of A.

We now claim that the correspondence between self-adjoint extensions of A and surjective isometries U_0 is bijective. First of all we must prove that every self-adjoint extension is constructed out of a surjective isometry U_0. If, as above, $U : \overline{Ran(A + iI)} \to \overline{Ran(A - iI)}$ denotes the unique unitary extension of the Cayley transform V of A, each self-adjoint extension B of A has a unitary Cayley transform W extending U to a unitary operator on the whole Hilbert space H. Since

$$\mathsf{H} = \overline{Ran(A+iI)} \oplus Ker(A^* - iI) = \overline{Ran(A-iI)} \oplus Ker(A^* + iI)\,,$$

and W extends U, the only possibility is that W determines a surjective isometry $U_0 : Ker(A^* - iI) \to Ker(A^* + iI)$. Therefore B is determined by U_0. Let us prove that the correspondence $B \mapsto U_0$ is one-to-one. Two distinct self-adjoint extensions B, B' give distinct operators U_0, U_0', otherwise the Cayley transforms W, W' would coincide. But Theorem 5.34(a, b) tells that two symmetric operators are distinct iff their Cayley transforms differ. The map is also onto by what we said above, because the choice of U_0 determines a self-adjoint extension of A, i.e., the one with Cayley transform $W := U \oplus U_0$.

There are many choices for U_0 if $d_+(A) = d_-(A) > 0$, and each one produces a different self-adjoint extension of A.

We conclude the proof by showing (5.36) and (5.37). First assume that A is symmetric and *closed*. In this case $Ran(A \pm iI) = \overline{Ran(A \pm iI)}$ (since $A \pm iI$ is also closed, $(A \pm iI)^{-1}$ is bounded and by the definition of closed operator in terms of sequences), so $V = U$. Denote by A_{U_0} the self-adjoint extension constructed out of U_0, and by $W_{U_0} = V \oplus U_0$ its Cayley transform. Using the splitting $\mathsf{H} = \overline{Ran(A+iI)} \oplus Ker(A^* - iI)$, from (ii) in Theorem 5.34(b) we have

$$D(A_{U_0}) = Ran(I - W_{U_0}) = Ran(I + V) + (I - U_0)Ker(A^* - iI)\,,$$

that is

$$D(A_{U_0}) = D(A) + (I - U_0)(Ker(A^* - iI))\,.$$

We claim that this is a direct sum. Suppose $x + (I - U_0)y = \mathbf{0}$ with $x \in D(A)$ and $y \in Ker(A^* - iI)$, so that $U_0 y \in Ker(A^* + iI)$. We want to prove that $x = (I - U_0)y = \mathbf{0}$. Indeed, applying A^* to $x + y - U_0 y = \mathbf{0}$ and noticing that $A^* \supset A$, we have

$$\mathbf{0} = A^*x + A^*y - A^*U_0 y = Ax + (A^* - iI)y + (A^* + iI)U_0 y + iy - iU_0 = Ax + iy - iU_0 y\,.$$

Comparing the result with $x + y - U_0 y = \mathbf{0}$ we find $(A - iI)x = \mathbf{0}$ which, in turn, implies $x = \mathbf{0}$ because $A - iI$ is injective since A is symmetric. As a consequence $(I - U_0)y = \mathbf{0}$, as required.

Let us prove (5.37) to conclude the proof for A symmetric and closed. In accordance with (5.36), take $D(A_{U_0}) \ni z = x + (I - U_0)y$ with $x \in D(A)$ and $y \in Ker(A^* - iI)$. Then

$$A_{U_0} z = i(I + W_{U_0})(I - W_{U_0})^{-1} z = i(I + V)(I - V)^{-1} x + i(I + U_0)(I - U_0)^{-1}(I - U_0)y\,,$$

that is

$$A_{U_0} z = Ax + i(I + U_0)y\,,$$

which is (5.37).

Eventually, observe that (5.36), (5.37) are also valid for A symmetric with $A \subsetneq \overline{A}$, since the self-adjoint extensions of A and $\overline{A}$ coincide and $Ker(A^* \pm iI) = Ker(\overline{A}^* \pm iI)$. □

Next comes the first important corollary to Theorem 5.37.

Theorem 5.38 *A symmetric operator A on the Hilbert space H is essentially self-adjoint if and only if it admits a unique self-adjoint extension.*

Proof If A is essentially self-adjoint it has a unique self-adjoint extension by Proposition 5.17(d). Theorem 5.37 implies that a symmetric operator A has self-adjoint extensions only if $d_+ = d_-$. In particular, if the extension is unique $d_+ = d_- = 0$. But then Theorem 5.19(b) forces A to be essentially self-adjoint. □

5.4.2 Von Neumann's Criterion

Another consequence of Theorem 5.34, proved by von Neumann, establishes sufficient conditions for a symmetric operator to admit self-adjoint extensions. First we need two definitions.

Definition 5.39 Let X and X' be $\mathbb{C}$-vector spaces with Hermitian inner products $(\,|\,)_{\mathsf{X}}$ and $(\,|\,)_{\mathsf{X}'}$ respectively. A surjective map $V : \mathsf{X} \to \mathsf{X}'$ is an **anti-unitary operator** if:

(a) V is antilinear: $V(\alpha x + \beta y) = \overline{\alpha} Vx + \overline{\beta} Vy$ for any $x, y \in \mathsf{X}$, $\alpha, \beta \in \mathbb{C}$;

(b) V is anti-isometric: $(Vx|Vy)_{\mathsf{X}'} = \overline{(x|y)_{\mathsf{X}}}$ for any $x, y \in \mathsf{X}$.

Remark 5.40 Despite the complex conjugation in (b), note that $||Vz||_{\mathsf{X}'} = ||z||_{\mathsf{X}}$ for any $z \in \mathsf{X}$. Moreover, V is bijective. ■

Definition 5.41 If $(\mathsf{H}, (\,|\,))$ is a Hilbert space, an anti-unitary operator $C : \mathsf{H} \to \mathsf{H}$ is called a **conjugation operator**, or just **conjugation**, if it is *involutive*, i.e. $CC = I$.

Remark 5.42 Conjugations are defined on Hermitian inner product spaces. In general they *differ* from involutions in the sense of Definition 3.40, as the latter are defined on algebras. ■

Theorem 5.43 (Von Neumann's criterion) *Let A be a symmetric operator on the Hilbert space H. If there exists a conjugation $C : \mathsf{H} \to \mathsf{H}$ such that*

$$CA \subset AC\,,$$

then A admits self-adjoint extensions.

Proof To begin with, let us show $C(D(A^*)) \subset D(A^*)$ and $CA^* \subset A^*C$. By definition of adjoint $(A^*f|Cg) = (f|ACg)$ for any $f \in D(A^*)$ and $g \in D(A)$. As C is anti-unitary, $(CCg|CA^*f) = (CACg|Cf)$. As C commutes with A and $CC = I$, we have $(g|CA^*f) = (Ag|Cf))$, i.e. $(CA^*f|g) = (Cf|Ag)$ for any $f \in D(A^*)$ and $g \in D(A)$. By definition of adjoint, this means $Cf \in D(A^*)$ if $f \in D(A^*)$ and $CA^*f = A^*Cf$.

Let us pass to the existence, using Theorem 5.37. According to what we have just proved, if $A^*f = if$, applying C and using that C is antilinear and commutes with A^*, we obtain $A^*Cf = -iCf$. Thus C is a map (injective because it preserves norms) from $Ker(A^* - iI)$ to $Ker(A^* + iI)$. It is also onto, for if $A^*g = -ig$, picking $f := Cg$ we have $A^*f = +if$. Applying C to f again (recall $CC = I$) gives $Cf = g$. Therefore C is a bijection $Ker(A^* - iI) \to Ker(A^* + iI)$. That it is also anti-isometric, i.e. it preserves orthonormal vectors, implies it must map bases to bases. In particular it preserves their cardinality, so $d_+(A) = d_-(A)$. The claim now follows from Theorem 5.37. □

5.4.3 Nelson's Criterion

We present, in conclusion, *Nelson's criterion*, which provides sufficient conditions for a symmetric operator to be essentially self-adjoint. Although we will be able to appreciate the theorem in full only after delving into spectral theory (Chaps. 8 and 9), we believe it is better to present the result at this juncture. The reader might want to postpone the proof until he becomes familiar with the material of those chapters. First, though, a few preliminaries are in order.

Definition 5.44 Let A be an operator on the Hilbert space H.

(a) A vector $\psi \in D(A)$ such that $A^n\psi \in D(A)$ for any $n \in \mathbb{N}$ ($A^0 := I$) is called a C^∞ **vector for** A, and we denote by $C^\infty(A)$ the subspace of C^∞ vectors for A.

(b) A vector $\psi \in C^\infty(A)$ is an **analytic vector for** A if:

$$\sum_{n=0}^{+\infty} \frac{||A^n\psi||}{n!} t^n < +\infty \quad \text{for some } t > 0.$$

(d) A vector $\psi \in C^\infty(A)$ is a **vector of uniqueness for** A if $A\restriction_{D_\psi}$ is an essentially self-adjoint operator on the Hilbert space $\mathsf{H}_\psi := \overline{D_\psi}$, where $D_\psi \subset \mathsf{H}$ is the span of $A^n\psi, n = 0, 1, 2 \ldots$

We shall come back to analytic vectors more extensively in Sect. 9.2. Here we will just introduce some results towards Nelson's criterion. If ψ is an analytic vector for A, the series:

$$\sum_{n=0}^{+\infty} \frac{||A^n\psi||}{n!} t^n$$

converges for some $t > 0$. Known results on convergence of power series guarantee the complex series

$$\sum_{n=0}^{+\infty} \frac{||A^n \psi||}{n!} z^n$$

converges absolutely for any $z \in \mathbb{C}$, $|z| < t$ and uniformly on $\{z \in \mathbb{C} \mid |z| < r\}$ for every positive $r < t$. Furthermore, for $|z| < t$, also the series of derivatives of any order

$$\sum_{n=0}^{+\infty} \frac{||A^{n+p} \psi||}{n!} z^n$$

converges, for any given $p = 1, 2, 3, \ldots$. The last fact has an important consequence, easily proved, that comes from using the triangle inequality and the norm's homogeneity repeatedly (the details will appear in the proof of Proposition 9.25(f)).

Proposition 5.45 *If ψ is an analytic vector for an operator A on the Hilbert space H, every vector in D_ψ is analytic for A. More precisely, if the series*

$$\sum_{n=0}^{+\infty} \frac{||A^n \psi||}{n!} t^n ,$$

converges for $t > 0$ and $\phi \in D_\psi$, then

$$\sum_{n=0}^{+\infty} \frac{||A^n \phi||}{n!} s^n ,$$

converges for any $s \in \mathbb{C}$ with $|s| < t$.

We have a proposition, called *Nussbaum lemma*.

Proposition 5.46 ("Nussbaum lemma") *Let A be a symmetric operator on the Hilbert space H. If $D(A)$ contains a set of vectors of uniqueness whose linear span is dense in H, A is essentially self-adjoint.*

Proof By Theorem 5.19 it is enough to prove the spaces $Ran(A \pm iI)$ are dense. With our assumptions, given $\phi \in \mathsf{H}$ and $\varepsilon > 0$, there is a finite linear combination of vectors of uniqueness ψ_i with $||\phi - \sum_{i=1}^N \alpha_i \psi_i|| < \varepsilon/2$. Since $\psi_i \in \mathsf{H}_\psi$ and $A \restriction_{D_\psi}$ is essentially self-adjoint on this Hilbert space, Theorem 5.19(c) implies there exist vectors $\eta_i \in \mathsf{H}_\psi$ with $||(A \restriction_{D_\psi} + iI)\eta_i - \psi_i|| \leq \varepsilon/2 \left(\sum_{j=1}^N |\alpha_j|\right)^{-1}$. Setting $\eta := \sum_{i=1}^N \alpha_i \eta_i$ and $\psi := \sum_{i=1}^N \alpha_i \psi_i$, we have $\eta \in D(A)$ and

$$||(A + iI)\eta - \phi|| \leq ||(A \restriction_{D_\psi} + iI)\eta - \psi|| + ||\phi - \psi|| < \varepsilon .$$

But $\varepsilon > 0$ is arbitrary, so $Ran(A + iI)$ is dense. The claim about $Ran(A - iI)$ is similar. So, A is essentially self-adjoint by Theorem 5.19(c). □

The above result prepares the ground for the proof of Nelson's 'analytic-vector theorem'. As we mentioned, the proof needs the spectral theory of unbounded self-adjoint operators (which is logically independent from the criterion, albeit presented in Chaps. 8 and 9).

Theorem 5.47 (Nelson's criterion) *Let A be a symmetric operator on the Hilbert space H. If $D(A)$ contains a set of analytic vectors for A whose span is dense in H, A is essentially self-adjoint.*

Proof By Proposition 5.46 it suffices to show that an analytic vector ψ_0 for A is a vector of uniqueness for A. Note $A\restriction_{D_{\psi_0}}$ is surely a symmetric operator on $\mathsf{H}_{\psi_0} := \overline{D_{\psi_0}}$, because it is Hermitian and its domain is dense in H_{ψ_0}. Suppose $A\restriction_{D_{\psi_0}}$ has a self-adjoint extension B in H_{ψ_0}. (NB: we are talking about self-adjoint extensions of $A\restriction_{D_{\psi_0}}$ on the Hilbert space H_{ψ_0}, and *not* on H!) Let μ_ψ be the spectral measure of $\psi \in D_{\psi_0}$ for the PVM of the spectral expansion of B (cf. Theorems 8.52 (c) and 9.13), defined by $\mu_\psi(E) := (\psi|P^{(B)}_E\psi)$ for any Borel set $E \subset \sigma(B) \subset \mathbb{R}$, where $P^{(B)}_E$ is the PVM associated to the self-adjoint operator B. As ψ_0 is analytic

$$\sum_{n=0}^{+\infty} \frac{||A^n\psi_0||}{n!} t_0^n < +\infty \quad \text{for some } t_0 > 0.$$

If $z \in \mathbb{C}$ and $0 < |z| < t_0$,

$$\sum_{n=0}^{+\infty} \int_{\sigma(B)} \left|\frac{z^n}{n!}x^n\right| d\mu_\psi(x) = \sum_{n=0}^{+\infty} \left|\frac{z^n}{n!}\right| \int_{\sigma(B)} 1 \cdot |x^n| d\mu_\psi(x)$$

$$\leq \sum_{n=0}^{+\infty} \frac{t_0^n}{n!} \left(\int_{\sigma(B)} d\mu_\psi(x)\right)^{1/2} \left(\int_{\sigma(B)} x^{2n} d\mu_\psi(x)\right)^{1/2}$$

$$= \sum_{n=0}^{+\infty} \frac{t_0^n}{n!} ||\psi||\, ||B^n\psi|| = ||\psi|| \sum_{n=0}^{+\infty} \frac{t_0^n}{n!} ||A^n\psi|| < +\infty\,,$$

where we used Theorem 9.4(c) for the spectral measure $P^{(B)}$ of the expansion of B (spectral Theorem 9.13). The theorem of Fubini–Tonelli implies, for $0 < |z| < t_0$, that we can swap series and integral

$$\sum_{n=0}^{+\infty} \int_{\sigma(B)} \frac{z^n}{n!} x^n d\mu_\psi(x) = \int_{\sigma(B)} \sum_{n=0}^{+\infty} \frac{z^n}{n!} x^n d\mu_\psi(x)\,.$$

Hence if $0 \leq |z| < t_0$ and if $\psi \in D_{\psi_0}$ also belongs to the domain of e^{zB} (cf. Definition 9.14),

$$(\psi|e^{zB}\psi) = \int_{\sigma(B)} e^{zx} d\mu_\psi(x) = \int_{\sigma(B)} \sum_{n=0}^{+\infty} \frac{z^n}{n!} x^n d\mu_\psi(x) = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\sigma(B)} x^n d\mu_\psi(x)$$

$$= \sum_{n=0}^{+\infty} \frac{z^n}{n!} (\psi|A^n\psi)\,.$$

In particular, if $\psi \in D_{\psi_0}$, this happens if $z = it$ (with $|t| < t_0$) because the domain of e^{itB} is the entire Hilbert space, by Corollary 9.5

$$(\psi|e^{itB}\psi) = \sum_{n=0}^{+\infty} \frac{(it)^n}{n!} (\psi|A^n\psi)\,. \tag{5.38}$$

(Note the power series on the right converges on an open disc of radius t_0, i.e. it defines an analytic extension of the function on the left when it is replaced by z in the disc, even if ψ does not belong to the domain of e^{zB}.) Now consider another self-adjoint extension of $A_{D_{\psi_0}}$, say B'. Arguing as before, for $|t| < t_0$ we have

$$(\psi|e^{itB'}\psi) = \sum_{n=0}^{+\infty} \frac{(it)^n}{n!} (\psi|A^n\psi)\,. \tag{5.39}$$

Then (5.38) and (5.39) imply, for any $|t| < t_0$ and any $\psi \in D_{\psi_0}$,

$$(\psi|(e^{itB} - e^{itB'})\psi) = 0\,.$$

But D_{ψ_0} is dense in H_{ψ_0}, so (cf. Exercise 3.21) for any $|t| < t_0$:

$$e^{itB} = e^{itB'}\,.$$

Compute the strong derivatives at $t = 0$ and invoke Stone's theorem (Theorem 9.33), to the effect that

$$B = B'\,.$$

Therefore all possible self-adjoint extensions of $A \upharpoonright_{D_\psi}$ are the same. We claim there exists at least one. Define $C : D_{\psi_0} \to \mathsf{H}_{\psi_0}$ by

$$C : \sum_{n=0}^{N} a_n A^n \psi_0 \mapsto \sum_{n=0}^{N} \overline{a_n} A^n \psi_0\,.$$

Easily C extends to a unique conjugation operator on H_{ψ_0}, which we still call C (see Exercise 5.17). What is more, by construction $CA\restriction_{D_{\psi_0}} = A\restriction_{D_{\psi_0}} C$, so $A\restriction_{D_{\psi_0}}$ admits self-adjoint extensions by Theorem 5.43.

Altogether, for any analytic vector ψ_0, $A\restriction_{D_{\psi_0}}$ must be essentially self-adjoint on H_{ψ_0} by Theorem 5.38, because it is symmetric and it admits precisely one self-adjoint extension. We have thus proved that any analytic vector ψ_0 is a vector of uniqueness, ending the proof. $\square$

Examples 5.48 **(1)** A standard example to which von Neumann's criterion applies is an operator of chief importance in QM, namely $H := -\Delta + V$, where Δ is the usual Laplacian on $\mathbb{R}^n$

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,,$$

and V is a locally integrable real-valued function.
By setting the domain of H to be $\mathscr{D}(\mathbb{R}^n)$, H becomes immediately a symmetric operator on $L^2(\mathbb{R}^n, dx)$. Define C as the anti-unitary operator mapping $f \in L^2(\mathbb{R}^n, dx)$ to its pointwise-conjugate function. Clearly $CH = HC$, so H admits self-adjoint extensions. By choosing a specific V it is possible to prove H is essentially self-adjoint, as we will see at the end of Chap. 10.

(2) We know the operator $A_i := -i\frac{\partial}{\partial x_i}$ on $\mathscr{D}(\mathbb{R}^n)$ (see Proposition 5.29) is essentially self-adjoint, and as such it admits self-adjoint extensions. Is there a conjugation C that commutes with A_i? (The issue is moot, as such a C it might not exist). The conjugation operator of (1) does not commute with A_i despite its invariant subspace is the domain. Another possibility is $C : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dx)$, $(Cf)(x) := \overline{f(-x)}$ (almost everywhere) for any $f \in L^2(\mathbb{R}^n, dx)$. It is not hard to see $C(\mathscr{D}(\mathbb{R}^n)) \subset \mathscr{D}(\mathbb{R}^n)$ and $CA_i = A_iC$.

(3) Consider the Hilbert space $\mathsf{H} := L^2([0, 1], dx)$ with Lebesgue measure dx. Take the operator $A := i\frac{d}{dx}$ acting on functions in $C^1([0, 1])$ (i.e. maps $f \in C^1((0, 1))$ that are continuous on $[0, 1]$ and whose first derivative has finite limit at 0, 1) that further vanish at 0 and 1. The operator is Hermitian, as can be seen integrating by parts and because the maps annihilate boundary terms, so they vanish at the endpoints of the integral. One can also verify the domain of A is dense, making A symmetric. Let us show A is *not* essentially self-adjoint. The condition that $g \in D(A^*)$ satisfies $A^*g = ig$ (resp. $A^*g = -ig$) reads:

$$\int_0^1 \overline{g(x)}\left[f'(x) + f(x)\right] dx = 0$$

(resp. $\int_0^1 \overline{g(x)}\left[f'(x) - f(x)\right] dx = 0$) for any $f \in D(A)$. Integrating by parts shows that the L^2 map $g(x) = e^x$ ($g(x) = e^{-x}$) solves the above equation for any f in $C^1([0, 1])$ that vanishes at 0, 1. This latter fact is crucial when integrating by parts, because the exponential does *not* vanish at 0 and 1. Theorem 5.19 says A cannot be essentially self-adjoint.

Theorem 5.43 warrants, nonetheless, the existence of self-adjoint extensions. The antilinear transformation $C : L^2([0,1], dx) \to L^2([0,1], dx)$, $(Cf)(x) := \overline{f(1-x)}$ maps the space of C^1 functions on $[0,1]$ vanishing at the endpoints to itself. In addition

$$\left(Ci\frac{d}{dx}f\right)(x) = -i\frac{d}{d(1-x)}\overline{f(1-x)} = i\frac{d}{dx}\overline{f(1-x)} = i\frac{d}{dx}(Cf)(x)\,,$$

whence $CA = AC$. There must be more than one such extension, otherwise A would be essentially self-adjoint by Theorem 5.37, a contradiction.

The argument does not change if one takes domains akin to the above, in particular the space of C^∞ maps on $[0,1]$ that vanish at 0 and 1, or smooth maps on $[0,1]$ with compact support in $(0,1)$.

(4) Take $\mathsf{H} := L^2([0,1], dx)$ with the usual Lebesgue measure dx, and consider $A := -i\frac{d}{dx}$ defined on smooth periodic maps on $[0,1]$ with periodic derivatives of any order (of period 1). Integration by parts reveals that A is Hermitian. The exponential maps $e_n(x) := e^{i2\pi nx}$, $x \in [0,1]$, $n \in \mathbb{Z}$, form a basis of H, as shown in Exercise 3.32(1). They are all defined on $D(A)$, and their span is dense in H, so $D(A)$ is dense in H and A is symmetric.

Any $f \in \mathsf{H}$ corresponds 1-1 to the sequence of Fourier coefficients $\{f_n\}_{n\in\mathbb{Z}} \subset \ell^2(\mathbb{Z})$ of the expansion

$$f = \sum_{n\in\mathbb{Z}} f_n e_n\,.$$

This defines a unitary operator $U : \mathsf{H} \to \ell^2(\mathbb{Z})$, $f \mapsto \{f_n\}_{n\in\mathbb{Z}}$ (see Theorem 3.28). The elementary theory of Fourier series tells that $UD(A)U^{-1} =: D(A')$ is the space of sequences $\{f_n\}$ in $\ell^2(\mathbb{Z})$ such that $n^N|f_n| \to 0, n \to +\infty$, for any $N \in \mathbb{N}$. Moreover, if $A' := UAU^{-1}$ and $\{f_n\}_{n\in\mathbb{Z}} \in D(A')$, then

$$A' : \{f_n\}_{n\in\mathbb{Z}} \mapsto \{2\pi n f_n\}_{n\in\mathbb{Z}}\,.$$

Replicating the argument used for X_i in the proof of Proposition 5.23 allows to arrive at

$$D(A'^*) = \left\{ \{g_n\}_{n\in\mathbb{Z}} \subset \ell^2(\mathbb{Z}) \;\middle|\; \sum_{n\in\mathbb{Z}} |2\pi n g_n|^2 < +\infty \right\}.$$

On this domain

$$A'^* : \{f_n\} \mapsto \{2\pi n f_n\}\,.$$

As in Proposition 5.23, we can verify without problems that the adjoint to A'^* is A'^* itself. Hence A'^* is self-adjoint and A' essentially self-adjoint. As U is unitary, also A is essentially self-adjoint and the unique self-adjoint extension $\overline{A}$ satisfies $\overline{A} = U\overline{A'}U^{-1}$. (Fill in all details as exercise.)

(5) Example (4) can be settled in a much quicker way using Nelson's criterion. The domain of A contains the functions e_n, whose span is dense in $\mathsf{H} := L^2([0, 1], dx)$. Moreover $Ae_n = 2\pi n e_n$. Then

$$\sum_{k=0}^{+\infty} \frac{||A^k e_n||}{k!} t^k = \sum_{k=0}^{+\infty} \frac{(2\pi |n|)^k}{k!} (t)^k = e^{2\pi |n| t} < +\infty \,,$$

for any $t > 0$. As a consequence, A is essentially self-adjoint.

(6) Let us go back again to example (4), this time using Theorem 5.37 (for more details see [ReSi80, vol. II] and [Tes09]). For the sake of computational simplicity we replace $[0, 1]$ by $[0, 2\pi]$. Referring to the Hilbert space $L^2([0, 2\pi], dx)$, consider the operator $A := -i\frac{d}{dx}$ with domain

$$D(A) = \left\{\psi : [0, 2\pi] \to \mathbb{C} \;\middle|\; \psi \in C^1([0, 2\pi]) \;\;, \psi(0) = \psi(1) = 0\right\}$$

where $\psi \in C^1([0, 2\pi])$ means that both ψ and $\frac{d\psi}{dx}$ (with the derivatives at the boundary of $[0, 2\pi]$ defined by limits) are continuous. Observe that $D(A)$ is dense in $L^2([0, 2\pi], dx)$ (because it contains the dense space of smooth, compactly supported functions) and, by direct inspection, A is symmetric. Then $D(A^*)$ turns out to be the subspace of $L^2([0, 2\pi], dx)$

$$D(A^*) = \left\{\psi \in C([0, 2\pi]) \;\middle|\; \exists \text{w-}\frac{d\psi}{dx} \in L^2([0, 2\pi], dx)\right\} .$$

Equivalently, $\psi \in D(A^*)$ if and only if ψ is *absolutely continuous* (Definition 1.74 and Theorem 1.76) and its derivative (defined a.e.) belongs to $L^2([0, 2\pi], dx)$ (see [Tes09]). The derivative coincides with the *weak derivative* w-$\frac{d\psi}{dx}$ and, obviously, $A^*\psi = -i\text{w-}\frac{d\psi}{dx}$. The analysis of $Ker(A^* \pm iI)$ proves that, as expected, these spaces are one-dimensional and $Ker(A^* \pm iI)$ is the span of $e^{\pm x}$. So on the one hand A is not essentially self-adjoint, and hence it does not determine (precisely) one self-adjoint operator. On the other hand it admits an infinite family of self-adjoint extensions because $d_+(A) = d_-(A) = 1$.

The unitary operators $Ker(A^* - iI) \to Ker(A^* + iI)$ are completely and faithfully labelled by the parameter $\theta \in [0, 2\pi)$ and act in this way

$$U_\theta : e^{2\pi} e^{-x} \mapsto e^{i\theta} e^x$$

(the factor $e^{2\pi}$ makes both sides have the same L^2-norm $\sqrt{2^{-1}(e^{4\pi} - 1)}$, as due). Denoting by A_θ the self-adjoint extension of A associated with U_θ, according to (5.36), $f_\theta \in D(A_\theta)$ iff

$$f_\theta(x) = f(x) + \alpha(e^{2\pi - x} - e^{i\theta} e^x) \tag{5.40}$$

for some $\alpha \in \mathbb{C}$ and $f \in D(\overline{A})$, where it is possible to prove that

$$D(\overline{A}) = \left\{ \psi \in C([0, 2\pi]) \;\middle|\; \psi(2\pi) = \psi(0) = 0\,,\ \text{w-}\frac{d\psi}{dx} \in L^2([0, 2\pi], dx) \right\}\,.$$

Equivalently, $\psi \in D(\overline{A}) \Leftrightarrow \psi$ is absolutely continuous, its derivative (defined a.e.) belongs to $L^2([0, 2\pi], dx)$ and ψ vanishes at the boundary of $[0, 2\pi]$. Since f in (5.40) vanishes at 0 and 2π, the value attained by the generic $f_\theta \in D(A_\theta)$ at $0, 2\pi$ is determined by the second summand in (5.40):

$$f_\theta(2\pi) = e^{i\theta'} f_\theta(0)\,, \quad e^{i\theta'} = \frac{1 - e^{i\theta} e^{2\pi}}{e^{2\pi} - e^{i\theta}}\,. \tag{5.41}$$

It easy to see that θ' ranges over the whole $[0, 2\pi)$ when $\theta \in [0, 2\pi)$. In other words, the choice of a self-adjoint extension of A relaxes the boundary conditions of the functions on which it acts, as a glance at

$$D(A_\theta) = \left\{ \psi \in C([0, 2\pi]) \;\middle|\; \psi(2\pi) = e^{i\theta'} \psi(0)\,,\ \text{w-}\frac{d\psi}{dx} \in L^2([0, 2\pi], dx) \right\}$$

and $D(A)$ (or $D(\overline{A})$) confirms. We may also phrase the condition $\psi \in D(A_\theta)$ by demanding that ψ is absolutely continuous, its derivative (defined a.e.) belongs to $L^2([0, 2\pi], dx)$ and it satisfies the almost-periodic boundary conditions written above. A nice physical analysis of the self-adjoint extensions of $-i\frac{d}{dx}$ on the interval $[a, b]$ appears in [ReSi80, vol. II]. ■

Exercises

5.1 Let B be a closable operator on the Hilbert space H with dense domain $D(B)$. Prove that $\overline{Ran(B)} = \overline{Ran(\overline{B})}$ and therefore $Ker(B^*) = Ker(\overline{B}^*)$.

Solution. If $y \in Ran(\overline{B})$, there is a sequence of elements $x_n \in D(B)$ such that $x_n \to x \in \mathsf{H}$ and $Bx_n \to y$, so $y \in \overline{Ran(B)}$. Hence $Ran(\overline{B}) \subset \overline{Ran(B)}$. Since $B \subset \overline{B}$ we finally conclude that: $Ran(B) \subset Ran(\overline{B}) \subset \overline{Ran(B)}$. Taking again the closure: $\overline{Ran(B)} \subset \overline{Ran(\overline{B})} \subset \overline{Ran(B)}$, so that $\overline{Ran(B)} = \overline{Ran(\overline{B})}$. Eventually: $Ker(B^*) = [Ran(B)]^\perp = [\overline{Ran(B)}]^\perp = [\overline{Ran(\overline{B})}]^\perp = [Ran(\overline{B})]^\perp = Ker(\overline{B}^*)$.

5.2 Let A be an operator on the Hilbert space H with dense domain $D(A)$. Take $\alpha, \beta \in \mathbb{C}$ and consider the standard domain $D(\alpha A + \beta I) := D(A)$. Prove that
(i) $\alpha A + \beta I : D(\alpha A + \beta I) \to \mathsf{H}$ admits an adjoint and

$$(\alpha A + \beta I)^* = \overline{\alpha} A^* + \overline{\beta} I\,;$$

(ii) assuming $\alpha, \beta \in \mathbb{R}$, $\alpha A + \beta I$ is (respectively) Hermitian, symmetric, self-adjoint, or essentially self-adjoint $\Leftrightarrow$ A is Hermitian, symmetric, self-adjoint, or essentially self-adjoint;
(iii) $\alpha A + \beta I$ is closable $\Leftrightarrow$ A is closable; in that case

$$\overline{\alpha A + \beta I} = \alpha \overline{A} + \beta I\,.$$

Hint. Apply directly the definitions.

5.3 Let A and B be densely-defined operators on the Hilbert space H. If $A + B : D(A) \cap D(B) \to \mathsf{H}$ is densely defined, prove

$$A^* + B^* \subset (A + B)^*\,.$$

5.4 Let A and B be densely-defined operators on the Hilbert space H. If the standard domain $D(AB)$ is densely defined, show $AB : D(AB) \to \mathsf{H}$ admits an adjoint and

$$B^* A^* \subset (AB)^*\,.$$

5.5 Let A be a densely-defined operator on the Hilbert space H and $L : \mathsf{H} \to \mathsf{H}$ a bounded operator. Using the definition of adjoint prove that

$$(LA)^* = A^* L^*\,.$$

Then show

$$(L + A)^* = L^* + A^*\,.$$

5.6 Let $A : D(A) \to \mathsf{H}$ be a symmetric operator on the Hilbert space H. Prove that A bijective $\Rightarrow$ A self-adjoint. (Bear in mind that the inverse to a self-adjoint operator, if it exists, is self-adjoint. This falls out of the spectral theorem for unbounded self-adjoint operators, that we shall see later.)

Solution. If A is symmetric so is $A^{-1} : \mathsf{H} \to D(A)$. The latter is defined on the whole Hilbert space, so it is self-adjoint. Its inverse will, in turn, be self-adjoint.

5.7 Let $A : D(A) \to \mathsf{H}$ be a symmetric operator on the Hilbert space H. Prove that the closure $\overline{A}$ is symmetric, using the properties of * in Theorem 5.18.

Solution. $A \subset A^*$ by hypothesis, then $A^* \supset A^{**} = \overline{A}$, and finally $A^{**} \subset \overline{A}^*$, that is $\overline{A} \subset \overline{A}^*$.

5.8 Let $A : D(A) \to \mathsf{H}$ be a symmetric operator on the Hilbert space H. Prove that the closure $\overline{A}$ is symmetric, using the continuity of the inner product and the definition of closure of an operator in terms of sequences.

Solution. If $f, g \in D(\overline{A})$, we have $D(A) \ni f_n \to f$ for some sequence such that $Af_n \to \overline{A}f$ and $D(A) \ni g_n \to g$ for some sequence such that $Ag_n \to \overline{A}g$. Then

$$(f|\overline{A}g) = \lim_{m\to+\infty}(f|Ag_m) = \lim_{m\to+\infty}\lim_{n\to+\infty}(f_n|Ag_m) = \lim_{n\to+\infty}\lim_{m\to+\infty}(f_n|Ag_m)$$

$$\lim_{n\to+\infty}\lim_{m\to+\infty}(Af_n|g_m) = \lim_{n\to+\infty}(Af_n|g) = (\overline{A}f|g)\,.$$

(The two limits can be swapped, because the inner product is jointly continuous.)

5.9 In the sequel the commutant $\{A\}'$ of an operator A on H indicates the set of operators B in $\mathfrak{B}(\mathsf{H})$ such that $BA \subset AB$. Let $A : D(A) \to \mathsf{H}$ be an operator on the Hilbert space H. If $D(A)$ is dense and A closed, prove that $\{A\}' \cap \{A^*\}'$ is a strongly closed *-subalgebra in $\mathfrak{B}(\mathsf{H})$ with unit.

5.10 Prove Proposition 5.15.

5.11 Discuss whether and where the operator $-d^2/dx^2$ is Hermitian, symmetric, or essentially self-adjoint on the Hilbert space $\mathsf{H} = L^2([0,1], dx)$. Take as domain: (i) periodic maps in $C^\infty([0,1])$, and then (ii) maps in $C^\infty([0,1])$ that vanish at the endpoints.

5.12 Prove that

$$H := -\frac{d^2}{dx^2} + x^2$$

is essentially self-adjoint on $L^2(\mathbb{R}, dx)$ if $D(H) := \mathscr{S}(\mathbb{R})$.

Hint. Seek a basis of $L^2(\mathbb{R}, dx)$ made of eigenvectors of H.

5.13 Consider the Laplace operator on $\mathbb{R}^n$ seen in Example 5.48(1):

$$\Delta := \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\,.$$

Prove explicitly Δ is essentially self-adjoint on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ inside $L^2(\mathbb{R}^n, dx)$, and as such it admits one self-adjoint extension $\overline{\Delta}$.
Then show that if $\widehat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dk)$ is the Fourier–Plancherel transform (Sect. 3.7),

$$\left(\widehat{\mathscr{F}}\overline{\Delta}\widehat{\mathscr{F}}^{-1}f\right)(k) := -k^2 f(k)\,,$$

where $k^2 = k_1^2 + k_2^2 + \ldots + k_n^2$, on the standard domain:

$$\left\{f \in L^2(\mathbb{R}^n, dk) \;\middle|\; \int_{\mathbb{R}^n} k^4 |f(k)|^2 dk < +\infty\right\}\,.$$

Hint. The operator Δ is symmetric on $\mathscr{S}(\mathbb{R}^n)$, so we can use Theorem 5.19, verifying condition (b). Since the Schwartz space is invariant under the action of the unitary operator $\widehat{\mathscr{F}}$ given by the Fourier–Plancherel transform, as seen in Sect. 3.7, we may consider Theorem 5.19(b) for $\widehat{\Delta} := \widehat{\mathscr{F}}\Delta\widehat{\mathscr{F}}^{-1}$. This operator is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$ iff Δ is defined on $\mathscr{S}(\mathbb{R}^3)$. Now, $\widehat{\Delta}$ acts on $\mathscr{S}(\mathbb{R}^n)$ by multiplication by $-k^2 = -(k_1^2 + k_2^2 + \ldots + k_n^2)$, giving a self-adjoint operator on the aforementioned standard domain. Condition (b) can then be verified easily for $\widehat{\Delta}^*$, by using the definition of adjoint plus the fact that $\mathscr{S}(\mathbb{R}^n) \supset \mathscr{D}(\mathbb{R}^n)$. The uniqueness of self-adjoint extensions for essentially self-adjoint operators proves the last part, because $\widehat{\mathscr{F}}$ is unitary.

5.14 Recall $\mathscr{D}(\mathbb{R}^n)$ denotes the space of smooth complex-valued functions with compact support in $\mathbb{R}^n$. Referring to the previous exercise let $\overline{\Delta}$ be the unique self-adjoint extension of $\Delta : \mathscr{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n, dx)$. Prove $\mathscr{D}(\mathbb{R}^n)$ is a core for $\overline{\Delta}$. In other words show $\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)}$ is essentially self-adjoint and $\overline{\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)}} = \overline{\Delta}$.

Hint. It suffices to show $(\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^* = \overline{\Delta}$ (because that implies, by taking adjoints, $\overline{\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)}} = ((\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^*)^* = \overline{\Delta}^* = \overline{\Delta}$). For this identity note that if $\psi \in D((\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^*)$ then $(\Delta\varphi|\psi) = (\varphi|\psi')$, with $\psi' = (\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^*\psi \in L^2(\mathbb{R}^n, dx)$, for any $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Applying the Fourier–Plancherel transform immediately gives $\widehat{\mathscr{F}}\psi' = -k^2\widehat{\mathscr{F}}\psi$, since $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^n))$ is dense in $L^2(\mathbb{R}^n, dk)$. Therefore we obtained $\psi \in D(\overline{\Delta})$ and $\psi' = \overline{\Delta}\psi$, and so $(\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^* \subset \overline{\Delta}$. Now suppose, conversely, $\psi \in D(\overline{\Delta})$. Using the Fourier–Plancherel transform gives $-k^2\widehat{\mathscr{F}}\psi \in L^2(\mathbb{R}^n, dk)$, and for any $\varphi \in \mathscr{D}(\mathbb{R}^n)$ we may write $(\Delta\varphi|\psi) = -\int dk k^2\overline{(\widehat{\mathscr{F}}\varphi)}\widehat{\mathscr{F}}\psi = -\int dk\overline{(\widehat{\mathscr{F}}\varphi)}k^2\widehat{\mathscr{F}}\psi = (\varphi|\overline{\Delta}\psi)$. By definition of adjoint we found $\psi \in D((\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^*)$ and $(\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^*\psi = \overline{\Delta}\psi$. Hence we have the other inclusion, $(\Delta\restriction_{\mathscr{D}(\mathbb{R}^n)})^* \supset \overline{\Delta}$.

5.15 Let $A : D(A) \to \mathsf{H}$ be self-adjoint and T its Cayley transform. Prove that the von Neumann algebra $(\{A\}')'$ generated by A coincides with the von Neumann algebra $(\{T\}')'$ generated by $\{T\}$ (cf. Sect. 3.3.2).

5.16 Prove Proposition 5.45.

5.17 Take a symmetric operator $A : D(A) \to \mathsf{H}$ on the Hilbert space H and suppose $\psi \in C^\infty(A)$ is such that the finite linear span of $A^n\psi$, $n \in \mathbb{N}$, is dense in H. Prove that for any chosen $N = 0, 1, 2, \ldots$ and $a_n \in \mathbb{C}$,

$$C : \sum_{n=0}^{N} a_n A^n\psi \mapsto \sum_{n=0}^{N} \overline{a_n} A^n\psi$$

determines a conjugation operator $C : \mathsf{H} \to \mathsf{H}$ (Definition 5.39).

Outline. The first thing to prove is C is well defined as a map, i.e. if $\sum_{n=0}^{N} a_n A^n \psi = \sum_{n=0}^{N_1} a'_n A^n \psi$ then $\sum_{n=0}^{N} \overline{a_n} A^n \psi = \sum_{n=0}^{N_1} \overline{a'_n} A^n \psi$. For that it is enough to observe that if $\Psi = \sum_{m=0}^{M} b_m A^m \psi$, then

$$\left(\psi \left| \sum_{n=0}^{N} a_n A^n \Psi \right.\right) = \left(\psi \left| \sum_{n=0}^{N_1} a'_n A^n \Psi \right.\right) \text{ so that } \left(\left. \sum_{n=0}^{N} \overline{a_n} A^n \psi \right| \Psi \right) = \left(\left. \sum_{n=0}^{N_1} \overline{a'_n} A^n \psi \right| \Psi \right) .$$

Since the vectors Ψ are dense, $\sum_{n=0}^{N} \overline{a_n} A^n \psi = \sum_{n=0}^{N_1} \overline{a'_n} A^n \psi$, as required. By construction one verifies that if Ψ and Ψ' are as above then $(C\Psi|C\Psi') = \overline{(\Psi|\Psi')}$. Since the Ψ are dense in H and $||C\Psi|| = ||\Psi||$, it is straightforward to see C extends to H by continuity and antilinearity. The antilinear extension C satisfies $(C\Psi|C\Psi') = \overline{(\Psi|\Psi')}$ on H and is onto, as one obtains by extending the relation $CC\Psi = I\Psi$ by continuity.

Chapter 6
Phenomenology of Quantum Systems and Wave Mechanics: An Overview

Two are the possible outcomes: if the result confirms the hypotheses, you only took a measurement. But if the result contradicts the assumptions, then you made a discovery.
Enrico Fermi

In this chapter we shall present a circle of ideas aiming at understanding the meaning of *quantum systems* and *quantum phenomenology*. The more mathematically-oriented reader, perhaps not so interested in the genesis of QM notions in physics, may skip the sections following the first. Starting from sections two, in fact, we address a number of experimental facts, and briefly review the theoretical "proto-quantum" methods that led to the formulation of *Wave Mechanics* first, and then to proper QM. Many of the physics details can be found in [Mes99, CCP82]. We shall eschew discussing important steps in this historical development, e.g. atomic spectroscopy, the models of the atom (Rutherford's, Bohr's, Bohr–Sommerfeld's), the Franck–Hertz experiment, for which we recommend physics textbooks (e.g. [Mes99, CCP82]). This overview is meant to shed light on the basic theoretical model behind QM, which will be fully developed in ensuing chapters.

Notation 6.1 As is customary in physics texts, in this chapter, and possibly others too, we will denote vectors in three-space (identified with $\mathbb{R}^3$ once Cartesian coordinates have been fixed in a frame system), by boldface letters, e.g. $\mathbf{x}$. In the same way, Lebesgue's measure on $\mathbb{R}^3$ will be written $d^3\mathbf{x}$. ■

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_6

6.1 General Principles of Quantum Systems

We use the term *physical system* loosely, as a manner of speaking. It is quite hard to define, from a physical point of view, what a *quantum system* actually is. We can start by saying that rather than talking of a *physical quantum state* it may be more suitable to discuss a physical system with *quantum behaviour*, thus distinguishing these systems more by their phenomenological/experimental aspects than by theoretical ones. Within the theoretical formulation of QM there is no clear borderline separating classical systems from quantum systems. The divide is forced artificially; demarcation issues are very debated, today more than in the past, and the object of intense theoretical and experimental research work.

Generically speaking we can talk of quantum nature for *microphysical* systems, i.e. *molecules*, *atoms*, *nuclei* and *subatomic particles* when taken singularly or in small numbers. Physical systems made of several copies of those subsystems (like *crystals*) can display a quantum behaviour. Certain macroscopic systems behave in a typical quantum fashion only under specific circumstances that are hard to achieve (e.g. *Bose-Einstein condensates*, or *L.A.S.E.R.*). There is a way to refine slightly the rough distinction between the above micro- and macrosystems. We may say that when any physical system behaves in a quantum manner, the system's characteristic action, i.e. the number of physical dimensions of *energy* $\times$ *time* (equivalently, *momentum* $\times$ *length* or *angular momentum*), obtained by combining suitably the characteristic physical dimensions (mass, speed, length,...) in the processes examined, is of order smaller than **Planck's constant**:

$$h = 6.6262 \cdot 10^{-34}\,\text{Js}.$$

Planck's constant, and the word *quantum* stamped on Quantum Mechanics, were first introduced by Planck in a 1900 work on the *black-body theory*: this dealt with the issue of the theoretically-infinite total energy of a physical system consisting of the electromagnetic radiation in thermodynamical equilibrium with the walls of an enclosure at fixed temperature. Planck's theoretical prediction, later proved to be correct, was that the radiation could exchange with the walls quantities of energy proportional to the frequencies of the atomic oscillators inside the walls, whose universal factor is the *Planck constant*. These packets of energy were called by the Latin name *quanta*. But let us return to the criterion for distinguishing quantum from classical systems using h, and look for instance at an electron orbiting around a hydrogen nucleus. A characteristic action of the electron is, for example, the product of its mass ($\sim 9 \cdot 10^{-31}$ Kg), the estimated orbiting speed ($\sim 10^6$ m/s) and the value of Bohr's radius for the hydrogen atom ($\sim 5 \cdot 10^{-11}$ m). This gives $4.5 \cdot 10^{-35}$ Js, smaller than Planck's constant. One would therefore expect the hydrogen electron behaved in a quantum manner, and this is indeed the case. A similar computation can be carried out for macroscopic systems like a pendulum, of mass a few grams and length one centimetre, swinging under gravity's pull. A characteristic action for this can be the maximum kinetic energy times the period of oscillation, which is several orders of magnitude bigger than h.

Remark 6.2 The set of values taken by physical quantities, like energy, that characterise a quantum systems's state is called the *spectrum*, in the jargon. One of the peculiarities of quantum systems is that their spectrum is usually quite dissimilar to the spectrum measured on comparable macroscopic systems. Sometimes the difference is astonishing, for one passes from a *continuous* spectrum of possible values in the classical case, to a *discrete* spectrum in quantum situations. It is important to point out that in QM it is *not* essential for a given physical quantity to have a discrete spectrum: there are quantum quantities in QM with a continuous spectrum. This misunderstanding is the cause – or the consequence, at times – of a recurrent oversimplified interpretation of the word *quantum* in QM. ■

6.2 Particle Aspects of Electromagnetic Waves

Under special experimental circumstances electromagnetic waves, hence light as well, reveal a behaviour that is typical of *collections of particles*. The mathematical description of these anomalies (from a classical viewpoint) involves Planck's constant. In this respect we can cite two examples of classically deviant behaviour: the *photoelectric effect* and *Compton's effect*. In the infant stages of the development of QM these played a fundamental role in the construction of the proto-quantum models meant to explain them.

6.2.1 The Photoelectric Effect

The *photoelectric effect* is the emission of electrons (a current) by a metal irradiated with an electromagnetic wave, a phenomenon known since the first half of the XIX century. Some of its features remained with no explanation within the classical theory of interactions between matter and electromagnetic waves for a long time [Mes99, CCP82]. One conundrum, in particular, was to make sense of a threshold beaming frequency below which no emission could be measured. This minimum value depends on the metal employed. At the time it did not seem possible to explain why the emission started instantaneously once that particular frequency was exceeded.

According to the classical theory, an electronic emission should be detected independently of the frequency used, as long as enough time lapses for the metal's electrons to absorb sufficient energy to bond with atoms.

In 1905 A. Einstein proposed a very daring model to account for the strange properties of the photoelectric effect.[1] Compared to the experimental data, the precision

[1]Einstein was awarded the Nobel Prize in Physics for this work.

of his predictions was outstanding. Following Planck, Einstein's point was that a *monochromatic* electromagnetic wave, i.e. one with fixed frequency ν, was in reality made of particles of matter, called *light quanta*, each having energy prescribed by Planck's radiation formula:

$$E = h\nu \,. \tag{6.1}$$

The total energy of the electromagnetic wave in this model would then be the sum of the energies of the single quanta of light "associated" to the wave.

All this was, and still is, in contrast with classical electromagnetism, according to which an electromagnetic wave is a continuous system whose energy is proportional to the wave's *amplitude* rather than its *frequency*. What happened in the photoelectric effect, Einstein said, was that by irradiating the metal with a monochromatic wave each energy packet associated to the wave was absorbed by an electron in the metal, and transformed into kinetic energy. To justify the experimental evidence Einstein postulated, more precisely, that the packet could be absorbed either *completely* or *not at all*, without intermediate possibilities. If, and only if, the energy of the quantum was equal to, or bigger than, the electron's bonding energy E_0 to the metal (which depends on the metal, and can be measured irrespective of the photoelectric effect), would the electron be *instantaneously* emitted, transforming the energetic excess of the absorbed quantum into kinetic energy. The frequency $\nu_0 := E_0/h$ would thus detect the threshold observed experimentally. This conjecture turned out to match the experimental data *perfectly*.

6.2.2 The Compton Effect

The first observation and study of the *Compton effect* dates back 1923. It concerns the scattering of monochromatic electromagnetic waves of extremely high frequency – x-rays ($> 10^{17}$ Hz) and γ rays ($> 10^{18}$ Hz) – caused by matter (gases, fluids and solids). It is useful to remind that monochromatic electromagnetic waves have both a fixed frequency ν and a fixed wavelength λ, whose product is constant and equal to the speed of light, $\nu\lambda = c$, irrespective of the kind of wave. Hence in the sequel we will talk about the wavelength of monochromatic waves. Simplifying as much as possible, the Compton effect consists in the following phenomenon. Suppose we irradiate a substance (the *obstacle*) with a plane monochromatic electromagnetic wave that moves along the direction z with given wavelength λ. Then we observe a wave scattered by the obstacle and decomposed into several components (i.e. several wavelengths or frequencies). One component is scattered in every direction and has the same wavelength of the incoming wave. Every other component has a wavelength $\lambda(\theta)$, depending on the angle θ of observation, that is slightly bigger than λ. If we define θ to be the angle between the wave's incoming direction z and the outgoing direction (after hitting the obstacle, wavelength $\lambda(\theta)$), we have the equation

$$\lambda(\theta) = \lambda + f\,(1 - \cos\theta)\,, \tag{6.2}$$

where the constant f has the dimension of a length and comes from the experimental data. Its value[2] is $f = 0.024(\pm 0.001)$ Å. The region around the z-axis is isotropic.

Classical electromagnetic theory was, and still is, inadequate to explain this phenomenon. However, as Compton proved, the effect could be clarified by making three assumptions, all incompatible with the classical theory but in agreement with Planck and Einstein's speculations about light quanta.

(a) The electromagnetic wave is made of particles that carry energy according to Planck's equation (6.1), exactly as Einstein predicted.

(b) Each quantum of light possesses a momentum

$$\mathbf{p} := \hbar\mathbf{k}\,, \tag{6.3}$$

where $\mathbf{k}$ is the *wavevector* of the wave associated to the quantum (see below). Here and henceforth, following the standard notation of physicists:

$$\hbar := \frac{h}{2\pi}\,.$$

(c) Quanta interact, through collisions (in relativistic regime, in general), with the outer-most atoms and electrons of the obstacle, obeying the conservation laws of momentum and energy.

Remark 6.3 Concerning assumption (c), let us stress that the wavevector $\mathbf{k}$ associated to a plane monochromatic wave has, by definition, the same direction and orientation of the travelling wave, and its modulus is $2\pi/\lambda$, where λ is the wavelength. Equivalently, if ν denotes the frequency,

$$|\mathbf{k}| = 2\pi/\lambda = 2\pi\nu/c\,, \tag{6.4}$$

where we have used the well-known relationship

$$\nu\lambda = c \tag{6.5}$$

for monochromatic electromagnetic waves and $c = 2.99792 \cdot 10^8$ m/s is the speed of light in vacuum. ■

The interested reader will find below a few more details. Under the assumptions made above, the energy and momentum conservation laws to be used in a relativistic regime, i.e. when (certain) speeds approach c, read as follow:

[2] Recall 1 Å$= 10^{-10}m$.

$$m_e c^2 + h\nu = \frac{m_e c^2}{\sqrt{1 - v^2/c^2}} + h\nu(\theta) \,, \tag{6.6}$$

$$\hbar \mathbf{k} = \frac{m_e \mathbf{v}}{\sqrt{1 - v^2/c^2}} + \hbar \mathbf{k}(\theta) \,. \tag{6.7}$$

On the left we have the quantities before the collision, on the right those after the interaction. The constant $m_e = 9.1096 \cdot 10^{-31}$ Kg is the mass of the electron. The electron is thought of as at rest prior to the collision with the quantum of light. After the collision the quantum is scattered at an angle θ, while the electron travels at velocity $\mathbf{v}$. The wavevector before the collision, $\mathbf{k}$, is parallel to z (this direction is arbitrary, but fixed), while the wavevector of the quantum after the collision, $\mathbf{k}(\theta)$, forms an angle θ with z.

By (6.7) and by definition of wavevector,

$$\frac{m_e^2 c^2}{1 - v^2/c^2} = \frac{h^2 \nu^2}{c^2} + \frac{h^2 \nu(\theta)^2}{c^2} - 2\frac{h\nu}{c}\frac{h\nu(\theta)}{c}\cos\theta \,.$$

Eliminating v from this and (6.6) gives

$$\nu(\theta) = \nu - \frac{h\nu\nu(\theta)}{m_e c^2}\,(1 - \cos\theta) \,. \tag{6.8}$$

Because of (6.1) and $\nu = c/\lambda$, we easily find Eq. (6.2) written in the form

$$\lambda(\theta) = \lambda + \frac{h}{m_e c}\,(1 - \cos\theta), \tag{6.9}$$

so that $f = h/(m_e c)$. The actual numerical value coincides with the experimental one when one substitutes the values for h, m_e, c. By taking the formal limit as $m_e \to +\infty$, Eq. (6.9) gives $\lambda(\theta) \to \lambda$. This explains the isotropic component of the scattered wave with identical wavelength (to the incoming one), as if this component were due to quanta of light interacting with particles of much bigger mass than the electron's (an atom of the substance, or the entire obstacle).

Remark 6.4 The models of Einstein and Compton explain the photoelectric effect and formula (6.2) perfectly, both in quantitative and qualitative terms. Yet they must be considered *ad hoc* models, unrelated and actually conflicting with the physics knowledge of their times. The idea that electromagnetic waves, hence also light, have a particle structure cannot account for the wavelike phenomena – such as *interference* and *diffraction* – known since Newton and Huygens. Somehow, the wave and corpuscular nature of light (or an electromagnetic wave) must *co-exist* in the real world. This is forbidden in the paradigm of classical physics, but possible in the totally-relativistic formulation of QM by the introduction of *photons*, massless particles which we shall not examine thoroughly. ■

6.3 An Overview of Wave Mechanics

In this text we will not discuss the quantum properties of light, which would require a deeper study of the formalism of Quantum Mechanics. In a reversal of viewpoint, the ideas about the early attempts to describe light from a quantum perspective, introduced previously, proved very practical to formulate *Wave Mechanics*, which has rights to be considered the first step towards a formulation of QM.

Wave Mechanics is among the first rudimentary versions[3] of QM *for particles with mass*. We will spend only a little time on spelling out the foundational ideas of Wave Mechanics that shed light on the cornerstones of the QM formalism. In particular, we will forego result that are historically related to Schrödinger's stationary equation and the ensuing description of the energy spectrum of the hydrogen atom. We will return to these issues after the formalism has been set up.

6.3.1 De Broglie Waves

A quantum of light, according to Einstein and Compton, is associated to a monochromatic plane electromagnetic wave with wavenumber $\mathbf{k} = \mathbf{p}/\hbar$ and *angular frequency* $\omega = E/\hbar$, which is just the 2π-multiple of the frequency ν. Each component of the plane wave (one along each of the three orthonormal directions of the electric or magnetic field vibrating perpendicularly to $\mathbf{k}$) has the form of a scalar wave:

$$\psi(t, \mathbf{x}) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-t\omega)} . \tag{6.10}$$

Actually only the *real part* of the above has any physical meaning, but it is much more convenient to use complex-valued waves, for a number of reasons. For instance, complex waves appear in Fourier's decomposition (see Sect. 3.7) of a general solution to the electromagnetic field equations (Maxwell's equations) or, more generally, d'Alembert's equation. In terms of *momentum* and *energy* of the quantum of light, the same wave may be written as

$$\psi(t, \mathbf{x}) = Ae^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-tE)} . \tag{6.11}$$

Note how only the momentum and the energy of the quantum of light appear in the expression. In 1924 de Broglie put forward a truly revolutionary conjecture: *just like particles (photons) are associated to electromagnetic waves in certain experimental contexts, so one should be able to relate some sort of wave to a particle of matter.* According to de Broglie, this 'wave of matter' should be of the form (6.11), where now $\mathbf{p}$ and E are understood as the momentum and (kinetic) energy of a *particle*. The wavelength associated to a particle of momentum $\mathbf{p}$,

[3] Another version, developed in parallel by Heisenberg, consists in so-called *Matrix Mechanics*, which we will not treat.

$$\lambda = h/|\mathbf{p}| , \tag{6.12}$$

is called *de Broglie wavelength of the particle*.

It was not at all clear what the nature of these alleged waves could be until 1927, when *experimental evidence* was gained of waves associated to electronic behaviour, through two experiments carried out by Davisson and Germer, and independently G.P. Thompson. Without going into details, let us just say the following. It is a known fact that when a (sound, electromagnetic,...) wave hits an obstacle with an inner structure of dimensions *comparable to, or larger* than the wavelength, the scattered wave undergoes so-called *diffraction*. The various internal parts of the obstacle interact with the wave creating constructive and destructive interference. The resulting wave creates a pattern made of areas of alternating intensity on a screen placed behind the obstacle (*darker* and *brighter* areas in the case of light beams). These figures are called *diffraction patterns*. If the obstacle is a crystal, the analysis of the diffraction pattern allows to determine the inner structure of the crystal itself. Davisson, Germer and G.P. Thompson shot beams of electrons through a crystalline structure of mesh 1 Å, thus generating patterns made of *clusters of pinhead dots: the traces left by the electrons* that had hit the screen. The incredible fact, endorsing de Broglie's thesis, was that the diffraction patterns would appear only if de Broglie's wavelength was comparable to or smaller than the mesh, exactly as in electromagnetic diffraction.

Remark 6.5 It is important to underline that the diffraction phenomenon strictly depends on the wavelike nature of waves (it is due to something that oscillates and to the superposition principle). Diffraction patterns *cannot* be generated by particles that obey the usual laws of classical mechanics, whatever the obstacle. ■

6.3.2 Schrödinger's Wavefunction and Born's Probabilistic Interpretation

In 1926 Schrödinger took de Broglie's ideas seriously and in two famous and extraordinary papers made a more detailed hypothesis: he associated to a particle not a single plane wave like (6.11), but rather a *wave packet* made by the superposition of de Broglie waves (in the sense of the Fourier transform). For *free* particles, whose energy is purely kinetic, Schrödinger's wave reads:

$$\psi(t, \mathbf{x}) = \int_{\mathbb{R}^3} \frac{e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - tE(\mathbf{p}))}}{(2\pi\hbar^2)^{3/2}} \hat{\psi}(\mathbf{p})\, d^3\mathbf{p} , \tag{6.13}$$

where $E(\mathbf{p}) := \mathbf{p}^2/(2m)$, and m is the particle's mass. Schrödinger observed that ray optics relies on a relation, called *eikonal equation* [GPS01, CCP82], that bears a strong formal resemblance to the *Hamilton-Jacobi* equation [FaMa06, GPS01, CCP82] of classical mechanics. He was looking for a fundamental equation for matter in a *wave mechanics* of sorts, hoping it would stand to Hamilton-Jacobi's equation in a similar way *d'Alembert's equation* approximates the eikonal equation [GPS01, CCP82]. In a nutshell, wave mechanics should stand to classical mechanics as wave optics stands to ray optics. The celebrated *Schrödinger equation* was born. Only after we have constructed the formalism will we recover Schrödinger's equation. For a particle subject to a force with potential U, say $\mathbf{f}(t, \mathbf{x}) = -\nabla U(t, \mathbf{x})$, the equation reads:

$$i\hbar \frac{\partial \psi(t, \mathbf{x})}{\partial t} = \left[-\frac{\hbar^2}{2m} \Delta + U(t, \mathbf{x}) \right] \psi(t, \mathbf{x}) \tag{6.14}$$

where $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^3$.

The de Broglie-Schrödinger wave ψ is a *complex*-valued function, and was called the *wavefunction* of the particle to which it is attached. The physical interpretation of the wavefunction ψ – at least in the standard interpretation ("Copenhagen's interpretation") of the QM formalism – came from Born in 1926:

$$\rho(t, \mathbf{x}) := \frac{|\psi(t, \mathbf{x})|^2}{\int_{\mathbb{R}^3} |\psi(t, \mathbf{y})|^2 d^3\mathbf{y}}$$

is the probability density of detecting the particle at the point $\mathbf{x}$ *and at time* t *measured during an experiment meant to determine the particle's position.*

Born's interpretation turned out to agree with later experience, but essentially was already in line with the experimental evidence found by Davisson, Germer and G.P. Thompson, by which the traces left by particles on the screen clustered in regions where $\rho(t, \mathbf{x}) > 0$ and were absent where $\rho(t, \mathbf{x}) = 0$, thus giving rise to the diffraction pattern.

Remark 6.6
(1) From a mathematical point of view Born's hypothesis only requires *square-integrable* wavefunctions that are *not almost everywhere zero*. Put equivalently, *non-zero* elements in $L^2(\mathbb{R}^3, d^3\mathbf{x})$ make physical sense, and a *Hilbert space* makes its appearance for the very first time in the construction of QM. (It is physically irrelevant that de Broglie's plane waves have no straightforward meaning in the light of Born's interpretation, for they do not belong in $L^2(\mathbb{R}^3, d^3\mathbf{x})$. Plane monochromatic waves, used to understand experimental results *à la* de Broglie, can be approximated arbitrarily well by elements of $L^2(\mathbb{R}^3, d^3\mathbf{x})$ by using distributions $\hat{\psi}(\mathbf{p})$ close to a value $\mathbf{p}_0$, which in turn determines with the desired accuracy the wavelength $\lambda_0 = |\mathbf{p}_0|/h$ of de Broglie.)

(2) Assuming Born's interpretation, and in absence of experiments to determine its position, the particle with wavefunction ψ cannot evolve in time by the laws of classical mechanics. For if it followed a regular trajectory, as classically prescribed, the function $|\psi|^2$ would have to vanish almost everywhere away from the trajectory. But any regular curve in $\mathbb{R}^3$ has zero measure, so $|\psi|^2$ would be null almost everywhere in $\mathbb{R}^3$, a contradiction. In other terms when *no* experiment is made to detect a particle's position, the particle *cannot* be thought of as a classical object, for its time evolution is governed by the evolution of the wave ψ (solution to Schrödinger's equation).

(3) If we accept, as in the Copenhagen interpretation, that the wavefunction ψ describes *in full* the physical state of the particle, then the particle's position must be *physically* indefinite before an experiment is conducted to pin it down. Furthermore, it must be attached indissolubly to the *experiment* in a probabilistic way. It is wrong to think that the probabilistic description is meant to cover for our ignorance about the system's state ("the position is well determined, but we do not know it"). In the Copenhagen interpretation the position does *not* exist until we make an experiment to determine it and until the particle's state (the maximum amount of information about its physical properties in time) is described by ψ. In wave mechanics a quantum has, thus, a dual *wave-particle* essence, but the two *never clash because they never manifest themselves simultaneously*. ■

6.4 Heisenberg's Uncertainty Principle

When one tries to evaluate experimentally an arbitrary quantity in a physical system, the state of the system may be altered by interacting with it. In principle, the classical description would allow to make this perturbation negligible. In 1927 Heisenberg realised that the combined hypotheses of Planck, Einstein, Compton and de Broglie had a momentous (and epistemologically relevant) consequence. In quantitative terms Heisenberg's principle asserts that if we consider quantum systems and particular quantities to be measured, it is not always possible to disregard (as infinitesimal) the variation in the state of the system generated by a measurement: that is because Planck's constant bounds from below the product of certain quantities. Having made thought experiments involving some of the hypotheses in the Planck, Einstein, Compton and de Broglie models, Heisenberg concluded that:

in trying to determine the position or the momentum of a particle moving along a given axis x, we alter the momentum or the position, respectively, along the same axis, in such a way that the product of the two smallest variations Δx, Δp (of the final values of position and momentum) obeys

$$\Delta x \Delta p \gtrsim h \, . \tag{6.15}$$

Instead, if position and momentum are measured along orthogonal axes the above product can be made arbitrarily small.

Equation (6.15) is *Heisenberg's uncertainty principle* for position and momentum. An analogous relationship holds for the uncertainty ΔE of a particle's energy E and the uncertainty Δt of the instant t of measurement of the energy[4]:

$$\Delta E \Delta t \gtrsim h \, . \tag{6.16}$$

To illustrate the matter let us consider the thought experiment whereby one seeks to determine the position X of an electron, with known initial momentum P, by hitting it with a monochromatic beam of light of wavelength λ that propagates in the direction x. Let us imagine we can read the position off a screen parallel to the axis x using a lens placed between the axis and the screen. A quantum of light that has interacted with the electron will go through the lens and hit the screen, thus producing an image X'. Since the lens' aperture is finite, the outgoing direction of the quantum of light generating X' cannot be pinned down with absolute precision. Wave optics predicts at X' a diffraction pattern by which we may measure the coordinate X with a bounded precision

$$\Delta X \gtrsim \frac{\lambda}{\sin \alpha} \, ,$$

where α is half the angle under which we see the lens from X. To the quantum of light there corresponds a momentum h/λ, so the uncertainty in the component P_x of the outgoing quantum will approximately be $h(\sin \alpha)/\lambda$. The total momentum of the system 'particle + quantum of light + microscope' will remain constant, hence the uncertainty in the x-component of the particle's exit momentum must equal the corresponding uncertainty in the quantum itself:

$$\Delta P_x \gtrsim \frac{h}{\lambda} \sin \alpha \, .$$

The product of the variations along the axis x is then at least

$$\Delta X \Delta P_x \gtrsim h \, .$$

Remark 6.7 Heisenberg's principle, at this level, bears the same logical (in)consistency of the proto-quantum models of Planck, Einstein, Compton *et al*. It should be viewed more like a *working assumption* towards a novel notion of particle, for which the classical terms position and momentum make sense only within the boundaries fixed by the principle itself: *a quantum particle is allowed only in physical states in which momentum and position are neither defined, nor definable, simultaneously.*

It is worth stressing, as we will see, that Heisenberg's *principle* is a *theorem* in the final formulation of QM. ∎

[4]This second uncertainty relationship has a controversial status and its interpretation is a much thornier issue than the former's. We will not enter this territory, and refer to classical textbooks as [Mes99] in this respect.

6.5 Compatible and Incompatible Quantities

Quantum phenomenology, irrespective of the uncertainty principle, shows that there are pairs of quantities A and B that are *incompatible*. This means that, in principle, arbitrarily accurate and simultaneous measurements of A, B cannot be carried out. More explicitly, suppose we first measure quantity A, obtaining a as result, and immediately after we measure B obtaining b. Any further reading of A, infinitesimally close to B (so not to blame the time lapse), will give a value a_1 that is typically different from a, even by far. The same happens if we swap the roles of A and B.

For instance, position and momentum along a fixed direction are incompatible pairs, and comply with Heisenberg's principle. The allegiance to Heisenberg and incompatibility have to do with each other, but the precise relationship can be explained properly only after the formalism has been set up completely. In general, incompatible quantities do not satisfy the uncertainty principle.

It turns out that incompatible quantum quantities never depend on one another, nor do there exist devices capable of measuring them simultaneously.

There is a point to call the attention to: quantum phenomenology shows that *compatible* quantities A' and B' do exist. This entails that if we measure first A' on the system and read a', and immediately afterwards measure B' reading b', the next measurement of A' – as close to B' as we want (so that time evolution does not interfere with measurements) – gives the same result a'. The same happens swapping A' and B'. In particular, any physical quantity A is compatible with itself and with any function depending on A (like the position of a particle along a line and its position squared.)

An example of an incompatible pair on which Heisenberg's principle does not pronounce itself is that of two distinct components of a particle's **spin**. The spin of the electron (and of all nuclear and subnuclear particles) was introduced by Goudsmit and Uhlenbeck in 1925 [Mes99, CCP82] in order to make sense of some "bizarre" properties, the so-called *anomalous Zeeman effect*, of atomic energy spectra (spectral lines) in alkali metals. In a semi-classical sense the spin represents the intrinsic angular momentum of the electron, which may be considered, from a certain point of view, a consequence of the nonstop rotation of the electron around its centre of mass. This explanation, however, is misleading and cannot be taken verbatim. The deeper meaning of the spin emerges in Wigner's framework, whereby an elementary quantum particle is defined as an elementary system that is invariant under the action of the Poincaré group.

Associated to the spin is an intrinsic magnetic momentum responsible for the anomalous Zeeman effect. At any rate, the spin is a vector-valued quantity with characteristic quantum features, which distinguish it from a classical angular momentum and hence make it a quantum angular momentum. The first difference is in the range of the spin's modulus. In the unit $\hbar$ these values are always of the type $\sqrt{s(s+1)}$, where s is an integer fixed by the kind of particle; for instance, $s = 1/2$ for the electron. Each of the three components of the spin, in a positive orthonormal frame system, can take any of the $2s + 1$ discrete values $-s, -s + 1, \ldots, s - 1, s$. The spin's three components are pairwise *incompatible* quantities, for measuring in rapid sequence two of them gives distinct readings, as explained above. It is important to say that the components of a particle's orbital momentum and total angular momentum are incompatible exactly in the same way.

Compatible quantities for a quantum particle are, for instance, the x-component of the momentum and the y-component of the position vector.

Chapter 7
The First 4 Axioms of QM: Propositions, Quantum States and Observables

Some historians claim that it is very difficult, nowadays, to find the line separating – and at the same time joining – the experimental level from the so-called theoretical one. But their view already includes several arbitrary elements, the so-called approximations.

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In this chapter we will describe the overall mathematical structure of Quantum Mechanics. The strategy essentially goes back to von Neumann, but we shall present a more contemporary account based on *Gleason's theorem*. It will entail extending classical (Hamiltonian) mechanics and keeping track of the experimental evidence about the nature of quantum systems seen in the previous chapter.

The first section summarises the results of Chap. 6, emphasising aspects that will be fundamental later.

In section two we will re-examine a few facets of Hamilton's formulation of mechanics from a set-theoretical and formal/logical perspective. We shall present the interpretation of the theory's foundations, whereby elementary propositions on the physical system are described by a σ-algebra, while states can be described by Borel probability measures (possibly, Dirac measures) on the σ-algebra.

Section three will show how the classical structure may be modified to comprise quantum phenomenology. Now the σ-algebra is replaced by the lattice of projectors on a suitable Hilbert space, and a generalised σ-additive measure on the projectors' lattice takes the place of states. Similar approaches have been explored in depth by [Mac63, Jau73, Pir76, Var07].

We shall enter the heart of the matter in sections four. With the aid of *Gleason's theorem* we shall explain that the aforementioned generalised measures are nothing but positive trace-class operators with trace one. This will allow us to introduce the convex space of quantum states, in which *pure states* (or *rays*) are identified with extreme points.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_7

Section five is devoted to the heuristic definition of *observables* as collections of elementary propositions giving *projector-valued measures* (PVM) on the Hilbert space of the system. The construction will also motivate the *spectral theorem*, proved later in the book. We will also describe formally the notion of compatible propositions.

A number of advanced issues, both foundational and technical, will be addressed in the penultimate section. One example for all is the famous *Solèr theorem*, that tells how to recover the Hilbert space form the lattice of elementary propositions of a quantum system. This part also includes a survey of the features of von Neumann algebras relevant in quantum theories. A general critical discussion appears in [BeCa81], while more recent results can be found in [EGL09].

The last section introduces superselection rule at the level of the lattice of elementary propositions.

7.1 The Pillars of the Standard Interpretation of Quantum Phenomenology

Let us begin by summarising a few cardinal features of the behaviour of quantum systems, which were briefly described in Chap. 6.

QM1. (i) On a quantum system whose state has been fixed, measurements have a probabilistic outcome. Hence it not possible to foresee the measurement's outcome, but only its probability.

(ii) However, if a quantity has been measured and gives a certain reading, repeating the measurement immediately after (so that the system does not evolve in the meantime) will give the same result.

QM2. (i) There exist **incompatible** physical quantities, in the following sense. Call A, B two such quantities. If we first measure A on the system (in a given state) and read a as outcome, and immediately after we measure B obtaining b, a subsequent measuring of A – as close as we want to the measurement of B to avoid ascribing the result to the evolution of the state – produces a reading $a_1 \neq a$, in general. The same happens swapping A and B.
Consequently: (a) incompatible quantum quantities never depend on one another, and (b) there are no instruments capable of simultaneous measurements.

(ii) There exist **compatible** physical quantities in the following sense. Call A', B' two such quantities. If we first measure A' on the system (in a given state) and obtain a as result, and immediately after we measure B obtaining b, a subsequent measuring of A – as close as we want to the measurement of B to avoid attributing the result to the evolution of the state – produces the same reading a. The same happens swapping A and B.
Consequently: (a) every physical quantity is compatible with itself, and (b) if two quantities are functionally dependent (e.g. the energy and its square), then they are compatible.

Remark 7.1 **(1) QM1** and **QM2** refer to physical quantities that do not characterise a physical system. By this we intend quantities whose range does *not* depend on the state and therefore allow to distinguish a system from another. On the contrary, the remaining quantities mentioned by QM1 and QM2 take values that *depend on the state of the system*.

The physical quantities that **QM1** and **QM2** refer to, in relation to whether the outcome of successive experiments can or not be reproduced, are of course quantities that attain discrete values. As far as continuous quantities are concerned the matter is much more delicate, and we will not examine it [BGL95]. Irrespective of the type of quantities (continuous vs. discrete) what we can say, in general, is: *two quantities are compatible if and only if there exists a device capable of simultaneous measurements*.

Furthermore, **QM1** and **QM2** refer to extremely idealised measuring processes, in particular those during which the microscopic physical system is not destroyed by the measurement itself. The measuring procedures employed in the experimental practice are rather diversified.

(2) It is clear we cannot be absolutely certain that quantum systems satisfy (i) in QM1. We could be tempted to think that the stochastic outcome of measurements is really due to the lack of full knowledge scientists have of the system's state, and that by knowing it in toto they would be able to predict the outcome of measurements. In this sense quantum probability would merely have an *epistemic* nature. In the standard interpretation of QM, the so-called **Copenhagen interpretation**, the stochastic outcome of a measurement is assumed as a *primitive* feature of quantum systems. There are nonetheless interesting attempts to interpret quantum phenomenology based on alternative formalisms (the so-called formulations by *hidden variables*) [Bon97]. There, the stochastic feature is explained as it were due to partial human knowledge about the system's true state, which is described by more variables (and in different fashion) than those needed in the standard formulation. None of these attempts is considered nowadays completely satisfactory, and does not threaten the standard interpretation and formulation of QM when one also considers relativistic quantum theories, and relativistic QFT in particular (despite some are indeed deep, like *Bohm's theory*).

But we must stress that one *cannot* build a *completely classical* physical theory (that counts non-quantum relativistic theories among classical ones) that is capable of explaining the experimental phenomenology of a quantum system in its entirety. Hidden variables, in order to agree with known evidence, must at any rate satisfy a rather unusual *contextuality* property for classical theories. Furthermore, any theory that wishes to explain the quantum phenomenology, QM included, must be *nonlocal* [Bon97]. As we shall see in Sect. 13.4.3, actually, subsequent to the work of Einstein, Podolsky and Rosen first, and then Bell, experiments have proved the existence of correlations among measurements made in different regions of space and at lapses so short that transmission of information between events is out of the question, by whichever physical mean moving slower than the speed of light in vacuum.

(3) It is implicit in **QM1** and **QM2** that the physical systems of interest, both in quantum theory and quantum phenomenology, are divided in two large categories: *measuring instruments* and *quantum systems*. The Copenhagen formulation assumes that measuring devices are systems obeying the laws of classical physics. These hypotheses match the data coming from experiments, and although quite crude, theoretically-speaking, they lie at the heart of the interpretation's formalism. Therefore not much can be said about them within the standard formulation. At the moment, for instance, it is not clear where to draw the line between classical and quantum systems, nor how this boundary may be described inside the formalism, and not even whether the compound system 'instrument + quantum system' can be itself considered a larger quantum system, and as such treated by the formalism. In closing, the interaction between an instrument and a quantum system, which produces the actual measurement, is not described from within the standard quantum formalism as a dynamical process. For a deeper discussion on these stimulating and involved issues we refer to [Bon97, Des99], and also to the superb section dedicated to foundational aspects of quantum theories in the *Stanford Encyclopedia of Philosophy*.[1] ■

7.2 Classical Systems: Elementary Propositions and States

Let us see how (Borel) probability measures can be employed to represent the physical states of classical systems. A generalisation will be used later to describe the states of a quantum system mathematically.

7.2.1 *States as Probability Measures*

The modern formal treatment of probability theory, due to Kolmogorov, translates into the study of *probability measures*. We recap below a few definitions taken from Sect. 1.4.

Definition A positive, σ-additive measure μ on the measure space (X, Σ) is called a **probability measure** if $\mu(\mathsf{X}) = 1$.

The simplest case of a probability measure on (X, Σ) is certainly the **Dirac measure** δ_x **concentrated at** $x \in \mathsf{X}$:

$$\delta_x(E) = 0 \text{ if } x \notin E, \qquad \delta_x(E) = 1 \text{ if } x \in E, \quad \text{for any } E \in \Sigma.$$

We shall work with *Borel measures*, so we recall the following notions from Sect. 1.4, which we have already used and will be useful in the rest of the book.

[1] http://plato.stanford.edu/.

Definition Let X be a topological space.

(a) The **Borel σ-algebra** of X, $\mathscr{B}(\mathsf{X})$, is the smallest (under intersections) σ-algebra containing the open sets of X.

(b) The elements of $\mathscr{B}(\mathsf{X})$ are called **Borel sets** of X.

(c) A map $f : \mathsf{X} \to \mathbb{C}$ is (**Borel**) **measurable** if it is measurable with respect to $\mathscr{B}(\mathsf{X})$ and $\mathscr{B}(\mathbb{C})$, i.e. $f^{-1}(E) \in \mathscr{B}(\mathsf{X})$ for any $E \in \mathscr{B}(\mathbb{C})$.

Obviously, in (c), $\mathscr{B}(\mathbb{C})$ refers to the standard topology of $\mathbb{C}$, and the definition can be stated to comprise $\mathbb{R}$-valued maps and $\mathscr{B}(\mathbb{R})$.

Definition If X is a *locally compact Hausdorff* space, a **Borel measure** on X is a positive, σ-additive measure on $\mathscr{B}(\mathsf{X})$.

Consider a classical physical system with n spatial freedom degrees, so $2n$ degrees overall, including kinetic degrees ("velocities"). The *Hamiltonian formulation* [GPS01, FaMa06] of the system's dynamics, very briefly, goes as follows.

(i) The ambient space is the **phase spacetime** $\mathscr{H}_{n+1}$. This is a smooth manifold of real dimension $2n+1$ formed by the disjoint union[2] of $2n$-dimensional submanifolds $\mathscr{F}_t$, all diffeomorphic and smoothly depending on $t \in \mathbb{R}$:

$$\mathscr{H}_{n+1} = \bigsqcup_{t\in\mathbb{R}} \mathscr{F}_t \ .$$

(ii) The coordinate $t \in \mathbb{R}$ is **time**, every $\mathscr{F}_t$ is the **phase space** at time t and any point in $\mathscr{F}_t$ represents a **state** of the system **at time** t.

(iii) $\mathscr{H}_{n+1}$ admits an atlas with local coordinates: $t, q^1, \ldots, q^n, p_1, \ldots, p_n$ (where $q^1, \ldots, q^n, p_1, \ldots, p_n$ are **symplectic coordinates** on $\mathscr{F}_t$) in which the system's evolution is governed by **Hamilton's equations**:

$$\frac{dq^k}{dt} = \frac{\partial H(t, q(t), p(t))}{\partial p_k} \quad k = 1, 2, \ldots, n \ , \tag{7.1}$$

$$\frac{dp_k}{dt} = -\frac{\partial H(t, q(t), p(t))}{\partial q^k} \quad k = 1, 2, \ldots, n \ , \tag{7.2}$$

where H, the **Hamiltonian (function)** of the system in local coordinates, is known once the system is known.

With this representation the system's evolution in time is described by the integral curves of Hamilton's differential equations. If $s(t) \in \mathscr{F}_t$ is the state of the system at time t, each integral curve determines, at any given time $t \in \mathbb{R}$, a point $(t, s(t)) \in \mathscr{H}_{n+1}$ where the curve meets $\mathscr{F}_t$.

We remark that (in absence of constraints) the choice of a frame system $\mathscr{I}$ allows to decompose locally $\mathscr{H}_{n+1}$ as a the Cartesian product $\mathbb{R} \times \mathscr{F}$, where $\mathbb{R}$ is the

[2] $\mathscr{H}_{n+1}$ is the total space of a *fibre bundle* with base $\mathbb{R}$ (the *time axis*) and fibres $\mathscr{F}_t$ given by $2n$-dimensional *symplectic manifolds*. There is an atlas on $\mathscr{H}_{n+1}$ whose local charts have coordinates $t, q^1, \ldots, q^n, p_1, \ldots, p_n$, where t is the natural parameter on the base $\mathbb{R}$ while the remaining $2n$ coordinates define a local symplectic frame on each $\mathscr{F}_t$.

time axis (once the origin has been fixed) and $\mathscr{F}$ is identified with phase space at time $t = 0$. Other choices of the framing give different identifications. Similarly, the Hamiltonian H, identified in certain circumstances with the total *mechanical energy* of the system, depends on the frame system. However, Hamilton's equations of motion are independent of any frame: their solutions do not depend on choices, but are the same on $\mathscr{H}_{n+1}$ irrespective of the coordinates.

In certain, fundamental, contexts, like *Statistical Mechanics* or *Thermodynamical Statistics*, the system's state is not known with absolute precision, so neither is the evolution of the system. In these cases one uses statistical ensembles [Hua87, FaMa06]: rather than considering a single system, one takes a statistical ensemble of identical and independent copies of the system, whose states are distributed in the various $\mathscr{F}_t$ with a certain probability density given locally by a C^1 map $\rho = \rho(t, q, p)$. The density evolves in time in accordance to **Liouville's equation**:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \frac{\partial \rho}{\partial p_i} \right) = 0 \,. \tag{7.3}$$

The function $\rho(t, s)$, with $s \in \mathscr{F}_t$, represents the probability density that the system is in the state s at time t. The interpretation of ρ requires, for any t:

$$\rho(t, s) \geq 0 \quad \text{and} \quad \int_{\mathscr{F}_t} \rho \, d\mu_t = 1 \,. \tag{7.4}$$

The measure μ_t on $\mathscr{B}(\mathscr{F}_t)$ is the Lebesgue measure $dq^1 \cdots dq^n dp_1 \cdots dp_n$ on every local symplectic chart of $\mathscr{F}_t$ (and extended to $\mathscr{F}_t$ using a partition of unity). The known **Liouville theorem** states that with this choice μ_t on every phase space, the integral in (7.4) does not depend on $t \in \mathbb{R}$ provided ρ solves (7.3) [Hua87, FaMa06, CCP82].

In case one works with statistical ensembles, the density ρ_t is still thought of as the *system's state at time* t, even if this notion of state is more general. We shall abide by this convention, and distinguish **sharp states**, given by points $r(t) \in \mathscr{F}_t$, from **probabilistic states**, determined by a Liouville density ρ_t on $\mathscr{F}_t$. In either case the state at time t can be viewed as a *Borel probability measure* $\{\nu_t\}_{t \in \mathbb{R}}$ defined on $\mathscr{F}_t$. More precisely:

(i) for a probabilistic state[3] $\nu_t(E) := \int_E \rho(t, s) d\mu_t$ if $E \in \mathscr{B}(\mathscr{F}_t)$;

(ii) for a sharp state $\nu_t := \delta_{r(t)}$.

Remark 7.2 In order to represent the system's states at time t in a completely general way, thereby foregoing the evolution problem and also abandoning the standard Hamiltonian formulation, one could use topological manifolds $\mathscr{F}_t$ rather than smooth ones. States (at time t) could be represented in terms of probability measures for the Borel σ-algebra. The existence of a topology on $\mathscr{F}_t$ is intrinsically tied to the exis-

[3] $\mathscr{F}_t$ is a smooth manifold hence a locally compact Hausdorff space (since locally homeomorphic to $\mathbb{R}^n$). As μ_t is defined on $\mathscr{B}(\mathscr{F}_t)$ and ρ_t is continuous, ν_t is well defined on $\mathscr{B}(\mathscr{F}_t)$.

tence of "neighbourhoods" for its points, arising from experimental errors that may be infinitesimal but not negligible. More precisely, the possibility of distinguishing points in $\mathscr{F}_t$, despite the measuring errors, is expressed mathematically by requesting a Hausdorff topology on $\mathscr{F}_t$ (as happens on a smooth manifold). ■

7.2.2 Propositions as Sets, States as Measures on Them

If we assume that the Hamiltonian description of our system retains all physical properties, then it must be possible to describe, in phase space $\mathscr{F}_t$ at time t, all statements about the system that at time t are true, false, or true with a certain probability, in some way or another. Moreover, it should be possible to recover the truth value of those propositions, i.e. the probability they are true, from the state ν_t of the system. Here is a natural way to do this.

Observe first that every proposition P determines a subset in $\mathscr{F}_t$ that contains the points (thought of as sharp states) that render P true (at time t). We indicate this set by the same symbol $P \subset \mathscr{F}_t$. Next, suppose we work with a sharp state, so that ν_t is a Dirac measure. Then proposition P is true at time t if and only if the point $r(t)$ describing the system at time t belongs to the set P. Now assign the conventional value 0 to a false proposition at time t, and 1 to a true one at t, in relation to state $\nu_t = \delta_{r(t)}$. The crucial observation is that the truth value of P is $\nu_t(P)$, when the state is ν_t, thought of as *the measure of $P \subset \mathscr{F}_t$ with respect to the (Dirac) measure ν_t*.

This fact clarifies the concrete meaning of the Dirac measure ν_t viewed as system's state at time t. Furthermore, the same interpretation can be employed when the state is probabilistic: *$\nu_t(P)$ represents the probability that proposition $P \subset \mathscr{F}_t$ is true at time t when the state ν_t is probabilistic.*

Remark 7.3 **(1)** Everything we said makes sense if the set P belongs to the σ-algebra on which the measures ν_t are defined. This is the Borel σ-algebra, and hence it is reasonably large.
(2) One proposition may be formulated in different yet equivalent ways. When we identify propositions with sets in $\mathscr{F}_t$ we are explicitly assuming that *if two propositions determine the same subset in $\mathscr{F}_t$, they must be considered identical.* ■

7.2.3 Set-Theoretical Interpretation of the Logical Connectives

Given two propositions P, Q, we can compose them using *logical connectives* to obtain other propositions. In particular, we can form the propositions $P \mathscr{O} Q$ and $P \mathscr{E} Q$ using the binary connectives called *disjunction* (*'inclusive or'*), and *conjunction* (*'and'*). Negifying one proposition produces its *negation* $\dashv P$.

We can interpret these propositions in terms of sets in the Borel σ-algebra on $\mathscr{F}_t$:

(i) $P \mathscr{O} Q$ corresponds to $P \cup Q$;
(ii) $P \mathscr{E} Q$ corresponds to $P \cap Q$;
(iii) $\dashv P$ corresponds to $\mathscr{F}_t \setminus P$.
There is a *partial order relation* on subsets of $\mathscr{F}_t$ given by the inclusion: $P \leq Q$ if and only if $P \subset Q$.

At the level of propositions, the most natural interpretation of $P \subset Q$ *is to say that* P *implies* Q*, i.e.* $P \Rightarrow Q$. Equivalently: each time the system is in a sharp state satisfying P, the state satisfies Q as well. For non-sharp states, the probability that Q is true is not smaller than the probability that P is true.

Remark 7.4 **(1)** The truth probability of composite propositions can be computed from the starting propositions using the measure ν_t, because a σ-algebra is closed under the set-theoretical operations corresponding to $\mathscr{O}$, $\mathscr{E}$, $\dashv$.
(2) It is easy to see that if ν_t is a Dirac measure, the truth probability (in this case either 0 or 1) assigned to each expression (i), (ii), (iii), coincides with the value found on the truth tables of the connective used. For instance, $P \mathscr{O} Q$ is true ($\nu_t(P \cup Q) = 1$) if and only if at least one of its constituent propositions is true ($\nu_t(P) = 1$ or $\nu_t(Q) = 1$); in fact the point x at which the Dirac measure $\delta_x = \nu_t$ concentrates lies in $P \cup Q$ iff x lies in P or in Q. ■

7.2.4 "Infinite" Propositions and Physical Quantities

Propositional calculus normally disregards propositions made by infinitely many propositions and connectives like $P_1 \mathscr{O} P_2 \mathscr{O} \ldots$ Interpreting propositions and connectives in terms of points and operations on a σ-algebra, though, allows to "handle" infinitely-long propositions.

We can relate (at least) some of these propositions to measurable *physical quantities* on the system. Generally speaking, we may consider the *physical quantities* defined on our Hamiltonian system as a collection of functions, regular to some degree, defined on phase spacetime and real-valued: $f : \mathscr{H}_{n+1} \to \mathbb{R}$. A fairly broad choice for *regularity* is to take the class of maps that restrict to *Borel measurable* maps on each fibre $\mathscr{F}_t$. Less radical options are continuous maps, C^1 maps, or even C^∞ maps. From the point of view of physics it may seem natural to require physical quantities be described by functions that are at least continuous, because measurements are always affected by experimental errors when finding the point in $\mathscr{F}_t$ representing the state at time t: if the maps were not continuous, small errors would cause enormous variations in a quantity's values. Nevertheless we must also remember there might be quantities with discrete range, for which the above issue is meaningless (discrete values can be distinguished using instruments with sufficient – finite – precision). As we are interested in the passage to the quantum case rather than in analysing the classical case, we shall not go deep into this kind of problem. We shall limit ourselves to working at instants t for which the quantities of concern will be measurable functions $f : \mathscr{F}_t \to \mathbb{R}$. If $f : \mathscr{F}_t \to \mathbb{R}$ is a quantity that can be measured on the system (at time t), using it we can construct statements of this kin:

$$P_E^{(f)} =$$

"The value that f assumes on the system's state belongs to the Borel subset $E \subset \mathbb{R}$",

Considering Borel sets E, and not just open intervals or singlets for example, allows to treat quantities with both continuous and discrete ranges in the same way, and also keep track of the fact that the measurement made by an instrument is a set, not just a point, owing to the finite precision of the instrument. As a matter of fact $\mathscr{B}(\mathbb{R})$ contains closed sets, finite sets, countable sets and so on. In set-theoretical terms $P_E^{(f)}$ is associated to the Borel set

$$P_E^{(f)} = f^{-1}(E) \subset \mathscr{F}_t \, ,$$

that we continue to denote by the same symbol. (As explained above, by this convention the probability that $P_E^{(f)}$ is true for the system at time t is $\nu_t(P_E^{(f)})$, once the state ν_t is known.)

Consider an interval $[a, b)$, $b \leq +\infty$. Decompose it in the disjoint union of infinitely many subintervals: $[a, b) = \cup_{i=1}^{\infty}[a_i, a_{i+1})$, where $a_1 := a$, $a_i < a_{i+1}$ and $a_i \to b$ as $i \to \infty$. Then the proposition

$$P_{[a,b)}^{(f)} =$$

"The value of f on the state of the system falls in the Borel set $[a, b)$"

can be clearly written as an *infinite disjunction*

$$P_{[a,b)}^{(f)} = \mathscr{O}_{i=1}^{+\infty} P_{[a_i,a_{i+1})}^{(f)}$$

of statements of the form:

$$P_{[a_i,a_{i+1})}^{(f)} =$$

"The value of f on the state of the system falls in the Borel set $[a_i, a_{i+1})$".

This corresponds to writing the set $P_{[a,b)}^{(f)}$ as the disjoint union:

$$P_{[a,b)}^{(f)} = \cup_{i=1}^{+\infty} P_{[a_i,a_{i+1})}^{(f)} \, .$$

Therefore it makes physical sense to assume the existence of statements built by infinitely many connectives and propositions.

Since negating $\mathscr{O}$ gives $\mathscr{E}$, if we assume the set of admissible propositions is closed under $\dashv$, then we must also accept propositions involving infinitely many $\mathscr{E}$s as physically meaningful.

The possibility of representing propositions as sets in a σ-algebra, and thus compute the probability they are true on a state using the corresponding measure, suggests to allow for propositions with countably many connectives $\mathscr{O}$ or $\mathscr{E}$, because the corresponding sets still belong in the σ-algebra, which is closed under countable unions and intersections.

To obtain a structure "isomorphic" to the σ-algebra built from atomic formulas and $\mathscr{O}, \mathscr{E}, \dashv$, we need to add two more propositions, playing the role of the sets $\varnothing$ and $\mathscr{F}_t$. These are the contradiction (whose truth probability is 0, whichever the state), denoted $\mathbf{0}$, and the tautology $\mathbf{1}$ (of truth probability equal to 1, whichever the state).

Once propositions and sets are identified, the σ-algebra structure enables us to declare that *the set of elementary propositions P relative to the physical system of concern, equipped with the logical connectives $\mathscr{O}$, $\mathscr{E}$, $\dashv$, is "isomorphic" to a σ-algebra. The truth value of a proposition P, for sharp states, or its truth probability, for probabilistic states, on a given state at time t equals $\nu_t(P)$, where now $P \subset \mathscr{F}_t$ is the set corresponding to the proposition.*

Remark 7.5 **(1)** We may ask whether the σ-algebra of all propositions on the system corresponds to the full Borel σ-algebra of $\mathscr{F}_t$, or if it is smaller. If we assume every bounded measurable real map on $\mathscr{F}_t$ is a physical quantity, then the answer is clearly yes, because among those maps are the characteristic functions of measurable subsets of $\mathscr{F}_t$.
(2) As earlier remarked, once we fix a frame system $\mathscr{I}$ (in absence of constraints, as in the cases at hand) the phase spacetime $\mathscr{H}_{n+1}$ of the system is locally diffeomorphic to the Cartesian product $\mathbb{R} \times \mathscr{F}$, where $\mathscr{F}$ is the phase space at time 0 and $\mathbb{R}$ the time line (with given origin). Thus we may regard propositions at any given instant t as Borel subsets of $\mathscr{F}$, and any state at time t as a probability measure on $\mathscr{F}$. Henceforth, especially when generalising to the quantum case, we will harness this simplification of the formalism that results from a choice of frame. ∎

7.2.5 Basics on Lattice Theory

In physical systems we can identify propositions and sets, and think of states as measures on sets. In order to pass to quantum systems, where there is no notion of phase space, it is important to raise the following question. Do there exist mathematical structures that are not σ-algebras of sets but sort of isomorphic to one? The answer is yes and comes from the theory of lattices.

In the sequel we will use some basic notions about posets. We shall assume they are known to the reader; if not they can be found in Sect. A.1.

Definition 7.6 A partially ordered set $(\mathsf{X}, \geq)$ is a **lattice** when, for any $a, b \in \mathsf{X}$,
(a) $\sup\{a, b\}$ exists, denoted $a \vee b$ (sometimes called 'join' of a, b);
(b) $\inf\{a, b\}$ exists, written $a \wedge b$ (sometimes 'meet' of a, b).
(The poset is not required to be totally ordered.)

Remark 7.7 **(1)** If $(\mathsf{X}, \geq)$ is partially ordered, as usual $a \leq b$ means $b \geq a$, for any $a, b \in \mathsf{X}$. The following equivalences are easy: $a \wedge b = a \;\Leftrightarrow\; a \vee b = b \;\Leftrightarrow\; a \leq$

b.
(2) On any lattice X, by definition of inf, sup we have the following properties, for any $a, b, c \in \mathsf{X}$:
associativity: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.
idempotency: $a \vee a = a$ and $a \wedge a = a$.
By the associative property we can write $a \vee b \vee c \vee d$ and $a \wedge b \wedge c \wedge d$ without ambiguity.
(3) The above properties characterise lattices: a set X equipped with binary operations $\wedge : \mathsf{X} \times \mathsf{X} \to \mathsf{X}, \vee : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ that satisfy the properties of (2) is partially ordered by the relation $a \geq b \Leftrightarrow b = b \wedge a$. In that case $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$. ■

Various types of lattices exist, and the next definition describes some of them.

Definition 7.8 (*Lattice*) A lattice $(\mathsf{X}, \geq)$ is called:
(a) distributive if $\vee$ and $\wedge$ distribute over one another: for any $a, b, c \in \mathsf{X}$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\ , \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ ; \tag{7.5}$$

(b) bounded if it admits a minimum $\mathbf{0}$ and a maximum $\mathbf{1}$ (sometimes called 'bottom' and 'top');
(c) orthocomplemented if it is bounded and admits a mapping $\mathsf{X} \ni a \mapsto \neg a$, where $\neg a$ is the **orthogonal complement** of a, such that:
(i) $a \vee \neg a = \mathbf{1}$ for any $a \in \mathsf{X}$,
(ii) $a \wedge \neg a = \mathbf{0}$ for any $a \in \mathsf{X}$,
(iii) $\neg(\neg a) = a$ for any $a \in \mathsf{X}$,
(iv) $a \geq b$ implies $\neg b \geq \neg a$ for any $a, b \in \mathsf{X}$;
two elements a, b are
orthogonal, written $a \perp b$, if $\neg a \geq b$ (or equivalently $\neg b \geq a$);
commuting, if $a = c_1 \vee c_3$ and $b = c_2 \vee c_3$ with $c_i \perp c_j$ if $i \neq j$;
the **centre** of the lattice is the set of elements commuting with every element of the lattice and the lattice is **irreducible** if its centre is $\{\mathbf{0}, \mathbf{1}\}$;
(d) modular, if $q \geq p$ implies $(p \vee r) \wedge q = p \vee (r \wedge q)$, $\forall p, q, r \in \mathsf{X}$;
(e) orthomodular, if orthocomplemented and $q \geq p$ implies $q = p \vee ((\neg p) \wedge q)$, $\forall p, q \in \mathsf{X}$;
(f) σ-complete, if every countable set $\{a_n\}_{n\in\mathbb{N}} \subset \mathsf{X}$ admits least upper bound (indicated by $\vee_{n\in\mathbb{N}} a_n$) and greatest lower bound (indicated by $\wedge_{n\in\mathbb{N}} a_n$).
(f)' complete, if every set $A \subset \mathsf{X}$ admits both least upper bound and greatest lower bound;
(g) Boolean algebra, if has properties (a), (b) and (c) (hence (d) and (e));
(g)' Boolean σ-algebra if is a Boolean algebra satisfying (f).

A (**distributive, bounded, orthocomplemented, σ-complete, complete**) **sublattice** is a subset in X admitting a lattice structure (distributive, bounded, orthocomplemented, σ-complete, complete, respectively) for the restrictions of $\geq$ and $\neg$.

Remark 7.9 **(1)** It is immediate to prove that arbitrary intersections of orthocomplemented sublattices are orthocomplemented sublattices (with the same minimum, maximum and orthogonal complement of X).
(2) If X is an orthocomplemented lattice and $p, q \in \mathsf{X}$ belong to a Boolean subalgebra of X, then p and q commute (remark (6) below). The converse also holds [BeCa81].

Proposition 7.10 *Let X be an orthocomplemented lattice. Then $p, q \in \mathsf{X}$ commute if and only if the orthocomplemented sublattice generated by $\{p, q\}$ (the intersection of all bounded orthocomplemented sublattices containing $\{p, q\}$) is Boolean.*

Proof See Exercise 7.7. □

(3) In a general orthocomplemented lattice:

$$\text{if } a \perp b \text{ then } a \wedge b = \mathbf{0}.$$

(4) It is easy to see that orthocomplemented lattices X satisfy **De Morgan's laws**: for any $a, b \in \mathsf{X}$,

$$\neg(a \vee b) = \neg a \wedge \neg b\,, \quad \neg(a \wedge b) = \neg a \vee \neg b\,. \tag{7.6}$$

These relations can be generalized. The proof of the following statement is immediate, simply by using the definition of inf and sup (§A.1).

Proposition 7.11 *Let X be an orthocomplemented lattice. Then for any subset $A \subset \mathsf{X}$*
(a) *if A is* finite*, then*

$$\neg \sup_{a\in A} a = \inf_{a\in A} \neg a\,, \quad \neg \inf_{a\in A} a = \sup_{a\in A} \neg a\,; \tag{7.7}$$

(b) *if A is infinite, then $\sup_{a\in A} a$ exists $\Leftrightarrow$ $\inf_{a\in A} \neg a$ exists, and similarly, $\inf_{a\in A} a$ exists $\Leftrightarrow$ $\sup_{a\in A} \neg a$ exists. In either case, the corresponding relation in (7.7) is valid.*
(c) *(7.7) holds for* every *(countable) subset $A \subset \mathsf{X}$ if X is complete (σ-complete).*

(5) We have the following implications:

$$\textbf{distributivity} \Rightarrow \textbf{modularity} \Rightarrow \textbf{orthomodularity}$$

Therefore the orthomodular condition (the only one satisfied by quantum lattices, as we shall see shortly) is a weaker form of distributivity and modularity.

(6) A Boolean algebra X is distributive, modular, orthomodular and every pair

$a, b \in \mathsf{X}$ commutes: using the distributive law, in particular, $a = (a \wedge \neg b) \vee (a \wedge b)$ and $b = (b \wedge \neg a) \vee (a \wedge b)$. ■

We leave the following elementary proposition as an exercise for the reader.

Proposition 7.12 *The centre of an orthomodular lattice $\mathscr{L}$ is a Boolean subalgebra of $\mathscr{L}$.*

Definition 7.13 If X, Y are lattices, a map $h : \mathsf{X} \to \mathsf{Y}$ is a **(lattice) homomorphism** when

$$h(a \vee_{\mathsf{X}} b) = h(a) \vee_{\mathsf{Y}} h(b)\ , \quad h(a \wedge_{\mathsf{X}} b) = h(a) \wedge_{\mathsf{Y}} h(b)\ , \quad a, b \in \mathsf{X}$$

(with the obvious notations.) We further require that a homomorphism h satisfies: if X, Y are bounded

$$h(\mathbf{0}_{\mathsf{X}}) = \mathbf{0}_{\mathsf{Y}}\ , \quad h(\mathbf{1}_{\mathsf{X}}) = \mathbf{1}_{\mathsf{Y}}\ ;$$

if X, Y are orthocomplemented

$$h(\neg_{\mathsf{X}} a) = \neg_{\mathsf{Y}} h(x)\ ;$$

if X, Y are σ-complete

$$h(\vee_{n\in\mathbb{N}} a_n) = \vee_{n\in\mathbb{N}} h(a_n) \quad \textit{and} \quad h(\wedge_{n\in\mathbb{N}} a_n) = \wedge_{n\in\mathbb{N}} h(a_n) \quad \textit{if}\ \{a_n\}_{n\in\mathbb{N}} \subset \mathsf{X}\ ;$$

and if X, Y are complete

$$h(\sup A) = \sup h(A) \quad \textit{and} \quad h(\inf A) = \inf h(A) \quad \textit{if}\ A \subset X\ .$$

In every case (bounded, orthocomplemented, (σ-)complete lattices, Boolean (σ-)algebras) if h is bijective it is called **isomorphism** of the corresponding structure; in particular, it is called **automorphism** if $\mathsf{X} = \mathsf{Y}$.

Remark 7.14 **(1)** Since $b \geq a \ \Leftrightarrow\ b \wedge a = a$, the following facts hold. If $h : \mathsf{X} \to \mathsf{Y}$ is a homomorphism then for any $a, b \in \mathsf{X}$, $a \geq_{\mathsf{X}} b$ implies $h(a) \geq_{\mathsf{Y}} h(b)$, with the obvious notation.
(2) It is immediate to see that the inverse $h^{-1} : \mathsf{Y} \to \mathsf{X}$ of an isomorphism $h : \mathsf{X} \to \mathsf{Y}$ (of lattices or Boolean (σ-)algebras) is an isomorphism.

Concerning isomorphisms of lattices:

Proposition 7.15 *Let $h : \mathsf{X} \to \mathsf{Y}$ be an orthocomplemented lattice isomorphism. If both X and Y are (σ-)complete, then h is a (σ-)complete lattice isomorphism.*

Proof See Exercise 7.4. □

(3) Given an abstract Boolean σ-algebra X, does there exist a concrete σ-algebra of sets that is isomorphic to X? The answer is contained a general result known as **Loomis–Sikorski theorem** [Sik48]. This guarantees that every Boolean σ-algebra is isomorphic to a quotient Boolean σ-algebra $\Sigma/\mathscr{N}$, where Σ is a concrete σ-algebra of sets over a measurable space and $\mathscr{N} \subset \Sigma$ is closed under countable unions; moreover, $\varnothing \in \mathscr{N}$ and for any $A \in \Sigma$ with $A \subset N \in \mathscr{N}$, then $A \in \mathscr{N}$. The equivalence relation is $A \sim B \Leftrightarrow A \cup B \setminus (A \cap B) \in \mathscr{N}$, for any $A, B \in \Sigma$. It is easy to see the coset space $\Sigma/\mathscr{N}$ inherits the structure of Boolean σ-algebra from Σ with respect to the (well-defined) partial order $[A] \geq [B]$ if $A \supset B$, $A, B \in \Sigma$.

This is the sharpest result in the general case. Consider, for instance, the σ-algebra $\mathscr{B}([0,1])$ of Borel sets in $[0, 1]$. Take the quotient $\mathscr{B}^*([0,1]) := \mathscr{B}([0,1])/\mathscr{N}$, where $\mathscr{N}$ consists of subsets in $[0, 1]$ of zero (Lebesgue) measure. It can be proved that $\mathscr{B}^*([0,1])$ is isomorphic to no σ-algebra of subsets on any measurable space.

But if one restricts to Boolean algebras only, the known **Stone representation theorem** [Sto36] asserts that an abstract Boolean algebra is always isomorphic to some concrete algebra of sets, without the need of a quotient. ■

Notation 7.16 In case $\{a_j\}_J$ is a collection of elements of a lattice X, and J has any cardinality, we shall also use the natural notation

$$\vee_{j\in J}\, a_j := \sup_{j\in J} a_j\,, \quad \wedge_{j\in J} a_j := \inf_{j\in J} a_j \tag{7.8}$$

henceforth. ■

7.2.6 The Boolean Lattice of Elementary Propositions for Classical Systems

We can revert to σ-algebras of sets, and with the definitions given above the following assertions are trivial, so their proof is left as exercise.

Proposition 7.17 *Every σ-algebra on X is a Boolean σ-algebra where:*
(i) the partial order is the inclusion (hence $\vee$ corresponds to $\cup$ and $\wedge$ to $\cap$),
(ii) the maximum and minimum in the Boolean algebra are X and $\varnothing$,
(iii) orthocomplements correspond to set-theoretical complements with respect to X.

Proposition 7.18 *Let Σ, Σ' be σ-algebras on X and X' respectively, and $f : \mathsf{X} \to \mathsf{X}'$ a measurable function.*
(a) *The sets $P^{(f)}_E := f^{-1}(E)$, $E \in \Sigma'$, define a Boolean σ-subalgebra of the Boolean σ-algebra of Proposition 7.17.*
(b) *The mapping $\Sigma' \ni E \mapsto P^{(f)}_E$ is a homomorphism of Boolean σ-algebras.*

The same assertions hold for the set of propositions relative to a physical system.

Proposition 7.19 *Propositions relative to a classical physical system form a distributive, bounded, orthocomplemented and σ-complete lattice, i.e. a Boolean σ-algebra, where:*

(i) the order relation is the logical implication, the conjunction is the intersection and the disjunction is the union;

(ii) the maximum and minimum are the tautology **1** *and the contradiction* **0***;*

(iii) orthocomplementation corresponds to negation.

If a measurable function $f : \mathscr{F} \to \mathbb{R}$ represents a physical quantity, then:
(a) *as E varies in the Borel σ-algebra of $\mathbb{R}$, the propositions*

$$P_E^{(f)} =$$

"The value that f takes on the state of the system belongs in the Borel set $E \subset \mathbb{R}$",

define a Boolean σ-algebra;
(b) *the map that sends a Borel set $E \subset \mathbb{R}$ to the proposition $P_E^{(f)}$ is a homomorphism of Boolean σ-algebras.*

7.3 Quantum Systems: Elementary Propositions

We can now move on to quantum systems. We shall begin by examining the structure of elementary propositions, and later we shall discuss the notion of quantum state.

7.3.1 Quantum Lattices and Related Structures in Hilbert Spaces

In trying to follow an approach that is closest to the classical case, we first aim at finding a mathematical model for the class of elementary propositions relative to a quantum system. Then we will evaluate at time t by conducting experiments with the aid of suitable instruments, whose results will be merely 0 (= *the proposition is false*) or 1 (= *the proposition is true*). At this stage we still do not know how to describe the system, but we do know that the quantum quantities that are measurable have to satisfy **QM1** and **QM2**.

For the moment let us concentrate on **QM2**. As there exist incompatible *quantities*, necessarily there must be incompatible *propositions*. If A and B are incompatible, then

$$P_J^{(A)} =$$

"*The value of A on the state of the system belongs to the Borel set* $J \subset \mathbb{R}$",

$$P_K^{(B)} =$$

"*The value of B on the state of the system belongs to the Borel set* $K \subset \mathbb{R}$",

are, in general, incompatible propositions. Their truth values interfere with each other when we measure them within infinitesimally short lapses (so that the system's state is not responsible for the time evolution). We also know no instrument exists that is capable of evaluating simultaneously two incompatible quantities. Hence it is physically meaningless, in this context, to say that the above propositions, which are associated to incompatible quantities, can assume on the system a given truth value *simultaneously*. The propositions $P_J^{(A)}$ and $P_K^{(B)}$, in this sense, are called **incompatible**.

Important remark. The propositions we are considering must be understood as statements about physical systems to which we assign a truth value, 0 or 1, *as a consequence of a corresponding experimental measuring process*. In this light the *incompatibility* of two propositions does not prevent them from being both false, so that their conjunction is always false, for example. The meaning is much deeper: 'incompatible' points to the fact that *it makes no (physical) sense to give them, simultaneously, any truth value whatsoever*. Nor should we try to make sense of propositions like $P_J^{(A)}\ \mathscr{O}\ P_K^{(B)}$ or $P_J^{(A)}\ \mathscr{E}\ P_K^{(B)}$ in this case, *because there is no experiment that can evaluate the truth of such propositions.*

By this remark we cannot take, as model for the set of elementary propositions to be tested on our quantum system, a σ-algebra of sets *where* $\cap$ *and* $\cup$ *are interpreted as* $\mathscr{E}$ *and* $\mathscr{O}$ *respectively*. If we were to do so, we would then have to impose constraints on the model, for instance veto symbolic combinations built by incompatible propositions. An alternative idea of von Neumann turns out to be successful: elementary propositions are modelled using *orthogonal projectors* of a complex Hilbert space. As we will see, the set of projectors is a lattice. Although the structure is not that of a Boolean σ-algebra, it will allow us to distinguish among compatible and incompatible propositions, and to interpret $\mathscr{E}$ and $\mathscr{O}$ as the standard $\wedge$ and $\vee$ provided the former are used on compatible propositions.

7.3.2 The Non-Boolean (Non-Distributive) Lattice of Projectors on a Hilbert Space

The set of orthogonal projectors on a Hilbert space enjoys properties that closely resemble those of Boolean lattices. There are, however, important differences that will enable us to model the incompatible propositions of a quantum system. First of all we shall deal with a number of technical features of commuting projectors.

Proposition 7.20 *Let* $(\mathsf{H}, (\,|\,))$ *be a Hilbert space and* $\mathscr{L}(\mathsf{H})$ *the set of orthogonal projectors on* H.

The following properties hold for any $P, Q \in \mathscr{L}(\mathsf{H})$.

(a) *The following facts are equivalent:*

(i) $P \leq Q$*;*

(ii) $P(\mathsf{H})$ *is a subspace of* $Q(\mathsf{H})$*;*

(iii) $PQ = P$

(iv) $QP = P$.

(b) *The following facts are equivalent:*

(i) $PQ = 0$*;*

(ii) $QP = 0$*;*

(iii) $P(\mathsf{H})$ *and* $Q(\mathsf{H})$ *are orthogonal;*

(iv) $Q \leq I - P$*;*

(v) $P \leq I - Q$.

If (i)–(v) hold, $P + Q$ *is an orthogonal projector and projects onto* $P(\mathsf{H}) \oplus Q(\mathsf{H})$.

(c) *If* $PQ = QP$ *then* PQ *is an orthogonal projector and projects onto* $P(\mathsf{H}) \cap Q(\mathsf{H})$.

(d) *If* $PQ = QP$ *then* $P + Q - PQ$ *is an orthogonal projector and projects onto the closed space* $\overline{< P(\mathsf{H}), Q(\mathsf{H}) >}$.

(e) $PQ = QP$ *if and only if there exist* $R_1, R_2, R_3 \in \mathscr{L}(\mathsf{H})$ *such that:*

$$P = R_1 + R_3\,, \quad Q = R_1 + R_2 \quad \textit{with } R_i(\mathsf{H}) \perp R_j(\mathsf{H}) \textit{ for } i \neq j\,.$$

Proof (a) First, notice that if P is a projector onto M, then $Pu = 0 \Leftrightarrow u \in \mathsf{M}^\perp$, by the orthogonal decomposition $\mathsf{H} = \mathsf{M} \oplus \mathsf{M}^\perp$ (Theorem 3.13(d)) and because the component of u on M is precisely Pu.

(i) $\Rightarrow$ (ii). If $P \leq Q$ then $(u|Qu) \geq (u|Pu)$. Since projectors are idempotent and self-adjoint, the latter is equivalent to $(Qu|Qu) \geq (Pu|Pu)$, i.e. $||Qu|| \geq ||Pu||$. In particular $Qu = 0$ implies $Pu = 0$, so $Q(\mathsf{H})^\perp \subset P(\mathsf{H})^\perp$. Using Theorem 3.13(e) and noting that $Q(\mathsf{H})$ and $P(\mathsf{H})$ are closed, we find $P(\mathsf{H}) \subset Q(\mathsf{H})$.

(ii) $\Rightarrow$ (iii). If S is a basis for $P(\mathsf{H})$, complete it to a basis of $Q(\mathsf{H})$ by adding the orthogonal set S' to S. By Proposition 3.64(d), $P = s\text{-}\sum_{u\in S} u(u|\;)$ and $Q = s\text{-}\sum_{u\in S\cup S'} u(u|\;)$. Since S and S' are orthogonal and orthonormal systems, and because the inner product is continuous, the claim follows.

(iii) $\Leftrightarrow$ (iv). The statements are implied by one another by taking adjoints.

(iii) + (iv) $\Rightarrow$ (i). If $u \in \mathsf{H}$, $(u|Qu) = ((P + P^\perp)u|Q(P + P^\perp)u)$ where $P^\perp = I - P$. Notice P and $P^\perp$ commute with Q by (iii) and (iv), and moreover $PP^\perp = P^\perp P = 0$. Expanding the right side of $(u|Qu) = (u|(P + P^\perp)Q(P + P^\perp)u)$, and neglecting terms that are null by the above considerations, gives

$$(u|Qu) = (u|PQPu) + (u|P^\perp QP^\perp u)\,.$$

On the other hand by (iii) and (iv): $(u|PQPu) = (u|PPu) = (u|Pu)$. Therefore

$$(u|Qu) = (u|Pu) + (u|P^\perp QP^\perp u)\,,$$

so $(u|Qu) \geq (u|Pu)$.
(b) Assuming $PQ = 0$ and taking adjoints gives $QP = 0$, hence $P(\mathsf{H})$ and $Q(\mathsf{H})$ are orthogonal, for $PQ = QP = 0$. If $P(\mathsf{H})$ and $Q(\mathsf{H})$ are orthogonal we fix on each a basis, say N and N' respectively. By writing P and Q as prescribed by Proposition 3.64(d): $P = \sum_{u \in N}(u|\ )u$, $Q = \sum_{u \in N'}(u|\ )u$ we have immediately $PQ = QP = 0$. At last, $Q \leq I - P$ $(P \leq I - Q)$ iff Q (resp. P) projects on a subspace in the orthogonal space to $P(\mathsf{H})$ (resp. $Q(\mathsf{H})$) by part (a), i.e. $P(\mathsf{H}) \perp Q(\mathsf{H})$. Using the above expressions for P, Q, recalling $N \cup N'$ is a basis of $P(\mathsf{H}) \oplus Q(\mathsf{H})$ and using again Proposition 3.64(d) implies $P + Q$ is the orthogonal projector onto $P(\mathsf{H}) \oplus Q(\mathsf{H})$.
(c) That PQ is an orthogonal projector (self-adjoint and idempotent) if $PQ = QP$, with P, Q orthogonal projectors, is straightforward. If $u \in \mathsf{H}$, then $PQu \in P(\mathsf{H})$ but also $PQu = QPu \in Q(\mathsf{H})$, so $PQu \in P(\mathsf{H}) \cap Q(\mathsf{H})$. We have shown $PQ(\mathsf{H}) \subset P(\mathsf{H}) \cap Q(\mathsf{H})$, so to conclude it suffices to show $P(\mathsf{H}) \cap Q(\mathsf{H}) \subset PQ(\mathsf{H})$. If $u \in P(\mathsf{H}) \cap Q(\mathsf{H})$ then $Pu = u$, $Qu = u$, so also $Pu = PQu = u$, i.e. $u \in PQ(\mathsf{H})$. This means $P(\mathsf{H}) \cap Q(\mathsf{H}) \subset PQ(\mathsf{H})$.
(d) That $R := P + Q - PQ$ is an orthogonal projector is straightforward. Consider the space $\overline{< P(\mathsf{H}), Q(\mathsf{H}) >}$. We can build a basis as follows. Begin with a basis N for the closed subspace $P(\mathsf{H}) \cap Q(\mathsf{H})$. Then add a basis N' for the rest, i.e. the closed orthogonal complement $P(\mathsf{H}) \cap (P(\mathsf{H}) \cap Q(\mathsf{H}))^{\perp}$. With the same criterion build a third basis N'' for $Q(\mathsf{H}) \cap (P(\mathsf{H}) \cap Q(\mathsf{H}))^{\perp}$. The three bases thus obtained are pairwise orthogonal and together give a basis of $\overline{< P(\mathsf{H}), Q(\mathsf{H}) >}$. All this shows that

$$\overline{< P(\mathsf{H}), Q(\mathsf{H}) >} =$$

$$(P(\mathsf{H}) \cap Q(\mathsf{H})) \oplus (P(\mathsf{H}) \cap (P(\mathsf{H}) \cap Q(\mathsf{H}))^{\perp}) \oplus (Q(\mathsf{H}) \cap (P(\mathsf{H}) \cap Q(\mathsf{H}))^{\perp})$$

is an orthogonal sum. With our assumptions the projector onto the first summand is PQ by (c). Hence the projector onto $(P(\mathsf{H}) \cap Q(\mathsf{H}))^{\perp}$ is $I - PQ$. Again by (c) the orthogonal projector onto the second summand is $P(I - PQ) = P - PQ$, and similarly the third projector is

$$Q(I - PQ) = Q - PQ\,.$$

By part (b) the projector onto the whole sum $\overline{< P(\mathsf{H}), Q(\mathsf{H}) >}$ is

$$PQ + (P - PQ) + (Q - PQ) = P + Q - PQ\,.$$

Statement (e) is another way to phrase Proposition 3.66. $\square$

Based on what we have proved, consider orthogonal projectors $P, Q \in \mathscr{L}(\mathsf{H})$ that commute, and suppose they are associated to statements about the physical system (i.e. propositions, denoted by the same letters). Under the correspondence

$$\begin{aligned} P \mathscr{E} Q &\longleftrightarrow PQ\,, \\ P \mathscr{O} Q &\longleftrightarrow P + Q - PQ\,, \\ \dashv P &\longleftrightarrow I - P\,, \end{aligned}$$

the right-hand sides are orthogonal projectors. The latter, moreover, satisfy properties that are formally identical to those of propositional calculus. For example, $\dashv (P \mathscr{E} Q) = \dashv P \mathscr{O} \dashv Q$. In fact,

$$\dashv P \mathscr{O} \dashv Q \longleftrightarrow (I-P)+(I-Q)-(I-P)(I-Q) = 2I - P - Q - I + PQ + P + Q$$

$$= I - PQ \longleftrightarrow \dashv (P \mathscr{E} Q)\,.$$

and in the same way one may check *every* relation written previously, provided the projectors commute. Note, further, that if P, Q commute and $P \leq Q$ then $PQ = QP = P$ and $P + Q - PQ = Q$. If we interpret the latter by their truth value, the above correspondence will tell that $P \leq Q$ corresponds to Q being a logical consequence of P.

The real difference between orthogonal projectors and the propositions of a classical system is the following. If the projectors P, Q *do not commute*, PQ and $P + Q - PQ$ are not even projectors in general, so *the above correspondence breaks down*.

All this seems very interesting in order to find a model for the propositions of quantum system, under axiom **QM2**. The idea is that
the propositions of quantum systems are in 1–1 *correspondence with the orthogonal projectors of a Hilbert space. The correspondence is such that:*

(i) the logical implication $P \Rightarrow Q$ between propositions P and Q corresponds to the relation $P \leq Q$ of the corresponding projectors;

(ii) two propositions are compatible if and only if the respective projectors commute.

Remark 7.21 Before going any further let us shed some light on the nature of commuting orthogonal projectors. One might be led to suspect that P and Q commute only when: (a) projection spaces are one contained in the other, or (b) projection spaces are orthogonal. With the following explicit example we show that there are other possibilities. Consider $L^2(\mathbb{R}^2, dx \otimes dy)$, dx, dy being Lebesgue measures on the real line, and take the sets $A = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ in the plane, with $a < b$, $c < d$ given. If $G \subset \mathbb{R}^2$ is a measurable set, define the linear operator

$$P_G : L^2(\mathbb{R}^2, dx \otimes dy) \to L^2(\mathbb{R}^2, dx \otimes dy)$$

by $P_G f = \chi_G \cdot f$ for any $f \in L^2(\mathbb{R}^2, dx \otimes dy)$, where χ_G is, as always, the characteristic function of G and $\cdot$ is the pointwise product of two maps. The operator P_G is an orthogonal projector, and moreover

$$P_G(L^2(\mathbb{R}^2, dx \otimes dy)) = \{f \in L^2(\mathbb{R}^2, dx \otimes dy) \mid ess\ supp\ f \subset G\} .$$

Then it is immediate to prove $P_A P_B = P_B P_A = P_{A\cap B}$, whilst:

(a) none of $P_A(L^2(\mathbb{R}^2, dx \otimes dy))$, $P_B(L^2(\mathbb{R}^2, dx \otimes dy))$ is included in the other,

(b) nor are they orthogonal. ■

If the speculative correspondence between propositions about quantum systems and orthogonal projectors on a suitable Hilbert space is to be meaningful, the structural analogies of orthogonal projectors and the σ-complete Boolean algebra of propositions must reach farther than the case of two propositions. We expect, in particular, to be able to determine the structure of a Boolean (σ-)algebra on some set of projectors representing pairwise-compatible properties. The following fact asserts that the space of *all* orthogonal projectors is a non-distributive lattice, and establishes some of its peculiarities. We shall see later that the lack of distributivity is salvaged by a weaker form of it, the orthomodularity property.

Referring to part (c) let us remark that if $A \subset \mathscr{L}(\mathsf{H})$ is a set of commuting orthogonal projectors, by Zorn's lemma there exists a maximal commutative subset $\mathscr{L}_0(\mathsf{H}) \subset \mathscr{L}(\mathsf{H})$ with $A \subset \mathscr{L}_0(\mathsf{H})$: every projector in $\mathscr{L}(\mathsf{H})$ commuting with any element in $\mathscr{L}_0(\mathsf{H})$ belongs to $\mathscr{L}_0(\mathsf{H})$.

Theorem 7.22 *Let H be a (complex) Hilbert space.*
(a) *The collection $\mathscr{L}(\mathsf{H})$ of orthogonal projectors on H is an orthocomplemented, complete (in particular σ-complete) lattice, typically non-distributive. More precisely:*

(i) $\geq$ is the order relation between projectors. If $\{P_i\}_{i\in I}$ is family of orthogonal projectors of arbitrary cardinality:
(a) $\vee_{i\in I} P_i := \sup_{i\in I} P_i$ is the projector onto $\overline{< \{P_i(\mathsf{H})\}_{i\in I} >}$,
(b) $\wedge_{i\in I} P_i := \inf_{i\in I} P_i$ is the projector onto $\cap_{i\in I} P_i(\mathsf{H})$;

(ii) the maximum and minimum elements in $\mathscr{L}(\mathsf{H})$ are: I (identity operator) and 0 (null operator) respectively;

(iii) the orthocomplement to the projector P corresponds to

$$\neg P = I - P \tag{7.9}$$

and furthermore

$$\neg\left(\wedge_{\alpha\in A} P_\alpha\right) = \vee_{\alpha\in A} \neg P_\alpha\ , \quad \neg\left(\vee_{\alpha\in A} P_\alpha\right) = \wedge_{\alpha\in A} \neg P_\alpha \tag{7.10}$$

for every family $\{P_\alpha\}_{\alpha\in A} \subset \mathscr{L}(\mathsf{H})$;

(iv) the projection spaces of $P, Q \in \mathscr{L}(\mathsf{H})$ are orthogonal iff P, Q are orthogonal elements in $\mathscr{L}(\mathsf{H})$;

(v) the projectors $P, Q \in \mathscr{L}(\mathsf{H})$ commute iff they commute as lattice elements;

(vi) $\mathscr{L}(\mathsf{H})$ is not distributive if $dim\ \mathsf{H} \geq 2$.

(b) *In $\mathscr{L}(\mathsf{H})$ the following hold:*
(i) if $P, Q \in \mathscr{L}(\mathsf{H})$ commute:

$$P \wedge Q = PQ\,, \tag{7.11}$$
$$P \vee Q = P + Q - PQ\,, \tag{7.12}$$

(ii) if $\{Q_n\}_{n\in\mathbb{N}} \subset \mathscr{L}(\mathsf{H})$ consists of commuting elements:

$$\vee_{n\in\mathbb{N}}\, Q_n = s\text{-}\lim_{n\to+\infty} Q_0 \vee \cdots \vee Q_n\,, \tag{7.13}$$
$$\wedge_{n\in\mathbb{N}} Q_n = s\text{-}\lim_{n\to+\infty} Q_0 \wedge \cdots \wedge Q_n\,, \tag{7.14}$$

independently of the labelling of the Q_n.

(c) *If $\mathscr{L}_0(\mathsf{H}) \subset \mathscr{L}(\mathsf{H})$ is a maximal commutative set of orthogonal projectors, then $\mathscr{L}_0(\mathsf{H})$ is a Boolean σ-subalgebra. In particular $0, I \in \mathscr{L}_0(\mathsf{H})$, $\mathscr{L}_0(\mathsf{H})$ is closed under orthocomplementation, the inf and sup of a countable subset in $\mathscr{L}_0(\mathsf{H})$ exist in $\mathscr{L}_0(\mathsf{H})$ and coincide with the inf and sup on $\mathscr{L}(\mathsf{H})$.*

Proof (a) Recall $\geq$ is a partial order on $\mathscr{L}_0(\mathsf{H})$ by Proposition 3.60(f). By Proposition 7.20(a):

$$P \leq Q \quad \Leftrightarrow \quad P(\mathsf{H}) \subset Q(\mathsf{H})\,. \tag{7.15}$$

This partial ordering of orthogonal projectors corresponds one-to-one to the partial order of projection spaces. The class of closed subspaces in H is a lattice: we claim that if M, N are closed, their least upper bound is $\mathsf{M} \vee \mathsf{N} = \overline{<\mathsf{M}, \mathsf{N}>}$ and the greatest lower bound $\mathsf{M} \wedge \mathsf{N} = \mathsf{M} \cap \mathsf{N}$. Now, $\overline{<\mathsf{M}, \mathsf{N}>}$ is closed and contains M, N; moreover, any closed space L containing M, N must contain $\overline{<\mathsf{M}, \mathsf{N}>}$ as well, so $\mathsf{M} \vee \mathsf{N} = \overline{<\mathsf{M}, \mathsf{N}>}$. By construction $\mathsf{M} \cap \mathsf{N}$ closed in M and N, and if L is another such space, it must be contained in $\mathsf{M} \cap \mathsf{N}$, whence $\mathsf{M} \wedge \mathsf{N} = \mathsf{M} \cap \mathsf{N}$. Passing to projectors and using (7.15), we have that for $P, Q \in \mathscr{L}(\mathsf{H})$, $P \vee Q$ is the orthogonal projector onto $\overline{<P(\mathsf{H}), Q(\mathsf{H})>}$, while $P \wedge Q$ the projector onto $P(\mathsf{H}) \cap Q(\mathsf{H})$. The same argument applies to a family of orthogonal projectors $\{P_i\}_{i\in I}$ of arbitrary cardinality. In that case $\vee_{i\in I} P_i := \sup_{i\in I}\{P_i\}$ is the projector onto $\overline{<\{P_i(\mathsf{H})\}_{i\in I}>}$ and $\wedge_{i\in I} P_i := \inf_{i\in I}\{P_i\}$ the projector onto $\cap_{i\in I} P_i(\mathsf{H})$, so the lattices of orthogonal projectors and closed subspaces are both complete (and in particular σ-complete). In the lattice of closed subspaces the min and max are clearly $\{0\}$ and H. Passing to orthogonal projectors via (7.15), the minimum and maximum are the orthogonal projectors onto $\{0\}$ and H, i.e. the null operator and the identity. Orthocomplementation of projectors, $\neg P := I - P$, corresponds to complementation of closed subspaces $\neg P(\mathsf{M}) := P(\mathsf{M})^{\perp}$, by Proposition 3.64(b). Identities (7.10) immediately descend from Proposition 7.11 by exploiting the completeness of the lattice. Part (iv) in (a) follows directly from Proposition 7.20(b), whilst (v) in (a) descends from Proposition 7.20(e). To prove (vi), we shall exhibit a counterexample to distributivity.

Take a two-dimensional subspace S in a (complex) Hilbert space H of dimension ≥ 2, and identify S with $\mathbb{C}^2$ by fixing an orthonormal basis $\{e_1, e_2\}$. Now consider the subspaces: $\mathsf{H}_1 :=< e_1 >$, $\mathsf{H}_2 :=< e_2 >$ and $\mathsf{H}_3 :=< e_1 + e_2 >$.

From $\mathsf{H}_1 \wedge (\mathsf{H}_2 \vee \mathsf{H}_3) = \mathsf{H}_1 \wedge \mathsf{S} = \mathsf{H}_1$ and $(\mathsf{H}_1 \wedge \mathsf{H}_2) \vee (\mathsf{H}_1 \wedge \mathsf{H}_3) = \{0\} \vee \{0\} = \{0\}$ follows

$$\mathsf{H}_1 \wedge (\mathsf{H}_2 \vee \mathsf{H}_3) \neq (\mathsf{H}_1 \wedge \mathsf{H}_2) \vee (\mathsf{H}_1 \wedge \mathsf{H}_3) .$$

Let us prove (b) and (c) together. If the projectors P and Q commute, or if the Q_n pairwise commute, by Zorn's lemma there is a maximal commuting chain $\mathscr{L}_0(\mathsf{H})$ containing P and Q, or $\{Q_n\}_{n\in\mathbb{N}}$ respectively. Let us first prove that $\mathscr{L}_0(\mathsf{H})$ is a Boolean subalgebra and (i) holds. Clearly 0 and I belong to $\mathscr{L}_0(\mathsf{H})$ because they commute with everything in $\mathscr{L}_0(\mathsf{H})$. The same happens for $\neg P = I - P$ if $P \in \mathscr{L}_0(\mathsf{H})$. We have to prove, for any $P, Q \in \mathscr{L}_0(\mathsf{H})$, the existence of the sup and the inf of $\{P, Q\}$ inside $\mathscr{L}_0(\mathsf{H})$, that they are computed as prescribed in part (b), and that these projectors actually coincide with the sup and inf of $\{P, Q\}$ inside $\mathscr{L}(\mathsf{H})$. The distributive laws of $\vee$ and $\wedge$ follow easily from (7.12) and (7.11), from the projectors' commutation and from the idempotency of any projector, $PP = P$.

By Proposition 7.20(c), the projector onto $M \cap N$, corresponding to $P \wedge Q$ in $\mathscr{L}(\mathsf{H})$, is exactly PQ, and this belongs to $\mathscr{L}_0(\mathsf{H})$ because by construction it commutes with any element of the maximal space $\mathscr{L}_0(\mathsf{H})$. Therefore

$$P \wedge Q := \inf_{\mathscr{L}_0(\mathsf{H})} \{P, Q\} = \inf_{\mathscr{L}(\mathsf{H})} \{P, Q\} = PQ .$$

As P, Q commute, the projector onto $\overline{< \mathsf{M}, \mathsf{N} >}$, corresponding to $P \vee Q$ in $\mathscr{L}(\mathsf{H})$, is $P + Q - PQ$ by Proposition 7.18(d). The latter lives in $\mathscr{L}_0(\mathsf{H})$ for it commutes with $\mathscr{L}_0(\mathsf{H})$. As before,

$$P \vee Q := \sup_{\mathscr{L}_0(\mathsf{H})} \{P, Q\} = \sup_{\mathscr{L}(\mathsf{H})} \{P, Q\} = P + Q - PQ .$$

This makes $\mathscr{L}_0(\mathsf{H})$ a Boolean subalgebra and proves (b)(i).

To conclude we need to show $\mathscr{L}_0(\mathsf{H})$ is σ-complete by proving (b)(ii). Consider a countable family of projectors $\{Q_n\}_{n\in\mathbb{N}}$ and associate to each the projector P_n defined recursively by: $P_0 := Q_0$, and for $n = 1, 2, \ldots$:

$$P_n := Q_n(I - P_1 - \ldots - P_{n-1}) .$$

By induction we can prove with ease:
(i) $P_n P_m = 0$ if $n \neq m$;
(ii) $Q_1 \vee \cdots \vee Q_n = P_1 \vee \cdots \vee P_n = P_1 + \cdots + P_n$, $n = 0, 1, \ldots$.
If we introduce bounded operators

$$A_n := P_1 + \cdots + P_n ,$$

then:
(iii) $A_n = A_n^*$ and $A_n A_n = A_n$ for any $n = 0, 1, \ldots$, i.e. the A_n are orthogonal projectors, so $A_n \leq I$, for any $n = 0, 1, \ldots$ by Proposition 3.64(e).
(iv) $A_n \leq A_{n+1}$ for any $n = 0, 1, \ldots$
By virtue of Proposition 3.76 there exists a bounded self-adjoint operator A defined by the strong limit:

$$A = \text{s-}\lim_{n\to+\infty} P_n = \text{s-}\lim_{n\to+\infty} Q_0 \vee \cdots \vee Q_n \ .$$

Immediately, then, $AA = A$, making A an orthogonal projector in $\mathscr{L}_0(\mathsf{H})$ because (strong) limit of operators commuting with $\mathscr{L}_0(\mathsf{H})$. Still by Proposition 3.76, $A_n \leq A$ and in particular $Q_n \leq Q_1 \vee \cdots \vee Q_n \leq A$ for any $n \in \mathbb{N}$. We claim A is the least upper bound of the Q_n, in $\mathscr{L}(\mathsf{H})$ and in $\mathscr{L}_0(\mathsf{H})$. Suppose an orthogonal projector $K \in \mathscr{L}(\mathsf{H})$ satisfies $K \geq Q_n$ for any $n \in \mathbb{N}$. Then $KQ_n = Q_n$ by Proposition 7.20(a). By definition of the P_n, $KP_n = P_n$ and hence $KA_n = A_n$, so also $K \geq A_n$ for any natural number n, by Proposition 7.20(a). Hence Proposition 3.76 warrants $K \geq A$. In other words $A \in \mathscr{L}_0(\mathsf{H})$ bounds the Q_n from above, and any other upper bound $K \in \mathscr{L}(\mathsf{H})$ is bigger than A. By definition of supremum, $A = \sup_{\mathscr{L}(\mathsf{H})}\{Q_n\}_{n\in\mathbb{N}} =: \vee_{n\in\mathbb{N}} Q_n$. As $A \in \mathscr{L}_0(\mathsf{H})$, A is also the sup in $\mathscr{L}_0(\mathsf{H})$. In the above identity

$$\vee_{n\in\mathbb{N}} Q_n = \text{s-}\lim_{n\to+\infty} Q_0 \vee \cdots \vee Q_n$$

the indexing order of the Q_n is not relevant, given that the left-hand side, i.e. the supremum of $\{Q_n\}_{n\in\mathbb{N}}$, does not depend on arrangements. Formula (7.14) is easy using $\neg$ and (7.13). $\square$

7.4 Propositions and States on Quantum Systems

In this section we set out to discuss the first two axioms of the general formulation of QM, and describe propositions and states of quantum systems by using a suitable Hilbert space. An important theorem due to Gleason characterises those states. We will also show that quantum states form a convex set, and can be obtained as linear combinations of extreme states. The latter, called *pure states*, are in one-to-one correspondence with elements (*rays*) of the projective space associated to the physical system's Hilbert space.

7.4.1 *Axioms* A1 *and* A2*: Propositions, States of a Quantum System and Gleason's Theorem*

Based on what we saw in the previous section, we shall assume the following QM axiom. Propositions and projectors are denoted by the same symbol, as customary.

A1. *Let S be a quantum system described in a frame system $\mathscr{I}$. Then the set of testable propositions on S at any given time corresponds one-to-one to a subset of the lattice $\mathscr{L}(\mathsf{H}_S)$ of orthogonal projectors of S on the separable complex Hilbert space $\mathsf{H}_S \neq \{\mathbf{0}\}$. The space $\mathscr{L}(\mathsf{H}_S)$ is called the* **logic of elementary propositions** *of the system, and H_S is the* **Hilbert space associated to** *S.*

Moreover:

(1) *compatible propositions correspond to commuting orthogonal projectors;*
(2) *the logical implication of compatible propositions $P \Rightarrow Q$ corresponds to the relation $P \leq Q$ of the associated projectors;*
(3) *I (identity operator) and 0 (null operator) correspond to the tautology and the contradiction;*
(4) *the negation $\neg P$ of a proposition P corresponds to the projector $\neg P = I - P$;*
(5) *only when the propositions P, Q are compatible, the propositions $P \mathscr{O} Q$, $P \mathscr{E} Q$ make physical sense and correspond to the projectors $P \vee Q$, $P \wedge Q$;*
(6) *if $\{Q_n\}_{n\in\mathbb{N}}$ is a countable set of pairwise-compatible propositions, the propositions corresponding to $\vee_{n\in\mathbb{N}} Q_n$, $\wedge_{n\in\mathbb{N}} Q_n$* make sense.

Remark 7.23 **(1)** A proper explanation of why H_S should be *separable* will be given later, when we will consider concrete quantum system and give an explicit representation of H_S. The hypothesis is also necessary in some theoretical results in this book.
(2) From now on we shall assume that the subset of $\mathscr{L}(\mathsf{H}_S)$ is the entire logic of elementary propositions $\mathscr{L}(\mathsf{H}_S)$, leaving out for the moment *superselection rules*. As we will see in Sect. 7.6.2 the matter is quite subtle. A weaker assumption would be to have elementary propositions described by the sublattice of orthogonal projectors of a *von Neumann algebra* $\mathfrak{R}_S \subset \mathfrak{B}(\mathsf{H}_S)$. Self-adjoint elements $\mathfrak{R}_S$ identify with bounded observables on S, as we will have time to explain, especially in Chap. 11.
(3) As we shall discuss better in Sect. 7.6, the set $\mathscr{L}(\mathsf{H}_S)$ contains the subset of so-called **atomic propositions**. These correspond (see Theorem 7.56) to the *atoms* of the lattice. Atomic propositions are defined as follows: $P \neq 0$ is atomic if there is no $P' \in \mathscr{L}(\mathsf{H}_S)$ such that $P' \Rightarrow P$ apart from $P' = 0$, $P' = P$. It turns out that atomic propositions P, Q commute ($PQ = QP$) if and only if they are either mutually exclusive $PQ = 0$ or they coincide $P = Q$. In $\mathscr{L}(\mathsf{H}_S)$ atomic propositions are the orthogonal projectors onto *one-dimensional* subspaces, which implies that $R, S \in \mathscr{L}(\mathsf{H}_S)$ are compatible if and only if they can be written, separately, as disjunctions (at most countable) of sets N_R, N_S of atomic propositions, so that the atomic propositions of the union $N_R \cup N_S$ are pairwise compatible. The proof, by the above argument, follows immediately by Proposition 3.66. That atomic propositions exist in classical systems, by the way, is not at all obvious (see [Jau73]).

The existence of a subset of atoms is physically remarkable. If we restrict the family of physically admissible elementary propositions to some proper sublattice of $\mathscr{L}(\mathsf{H})$, atoms may still exist but they might not necessarily be one-dimensional projectors.
(4) Apart from S, also the Hilbert space H_S depends on the frame system of choice. Picking a different (inertial) frame system boils down to having a new, yet isomorphic Hilbert space, as we will see in Chaps. 12 and 13.

Another formulation, alternative to ours, to build the quantum formalism is the following. Given a quantum system S, one assigns to any instant $t \in \mathbb{R}$ a Hilbert space $\mathsf{H}_S(t)$ that *does not* depend on any frame system. This reminds of *absolute space at time t* in classical physics, a notion that is independent of frames. The way the various $\mathsf{H}_S(t)$ are related depends on the frame system and the time evolution, the latter described by isomorphisms between spaces at different times t; it does not depend upon the chosen frame, in contrast to what we will obtain in Chap. 13 (albeit the formalism will be equivalent). If we chose to use frame-independent (but time-dependent) Hilbert spaces $\{\mathsf{H}_S(t)\}_{t\in\mathbb{R}}$, we would not be able to describe the evolution by a one-parameter group of unitary operators on the same Hilbert space. This is precisely what happens after having fixed a frame and the Hilbert space once and for all, as we will see in Chap. 13.[4] ■

Let us pass to the second axiom about quantum states. The crux of the matter is that a quantum state at time t gives the "probability" that every proposition of the system is true. Hence the idea is to generalise the notion of σ-additive probability measure. Instead of defining the measure on a σ-algebra, we must think of it as living on the set of associated projectors. We know every maximal set of compatible propositions defines a σ-finite Boolean algebra, itself an extension of a σ-algebra where measures are defined. So here is the natural principle.

A2 (measure-theory version). *A state μ at time t on a quantum system S is a* **σ-additive probability measure** *on $\mathscr{L}(\mathsf{H}_S)$, i.e., a map $\mu : \mathscr{L}(\mathsf{H}_S) \to [0, 1]$ such that:*
(1) $\mu(I) = 1$;
(2) if $\{P_i\}_{i\in\mathbb{N}} \subset \mathscr{L}(\mathsf{H}_S)$ satisfy $P_i P_j = 0, i \neq j$, then

$$\mu\left(\text{s-}\sum_{i=0}^{+\infty} P_i\right) = \sum_{i=0}^{+\infty} \mu(P_i)\ .$$

Remark 7.24 **(1)** Demand (1) just says the tautology is true on every state.
(2) Demand (2) clearly holds for finitely many propositions: it is enough that $P_i = 0$ from a certain index i onwards.
(3) We may rephrase (2) as:

$$\mu\left(\vee_{i\in\mathbb{N}} P_i\right) = \sum_{i=1}^{+\infty} \mu(P_i)\ ,$$

for any collection $\{P_i\}_{i\in\mathbb{N}} \subset \mathscr{L}(\mathsf{H}_S)$ of compatible, mutually-exclusive propositions, so that $\sum_{i=0}^{+\infty} P_i = \vee_{i\in\mathbb{N}} P_i$ exists by Theorem 7.22.

[4] Although we will not do so, one could also use two-parameter groupoids of unitary transformations between different Hilbert spaces.

The proof of the existence of $\sum_{i=0}^{+\infty} P_i$ is spelt out next, at any rate. Under the assumptions, partial sums give self-adjoint idempotent operators, hence orthogonal projectors. Therefore $\sum_{i=0}^{N} P_i \leq I$ by Proposition 3.64(e). Moreover $\sum_{i=0}^{N+1} P_i \geq \sum_{i=0}^{N} P_i$, as is easy to see. So by Proposition 3.76 the sequence admits a strong limit. Immediately, this limit is idempotent and self-adjoint, hence a projector.
(4) Every state μ clearly determines the equivalent of a positive σ-additive probability measure on any maximal commutative set of projectors $\mathscr{L}_0(\mathsf{H}_s)$, which, as seen before, generalises a σ-algebra. Thus we have extended the notion of probability measure, as suggested by the term *σ-additive probability measure* itself.
(5) The reader should however be careful when identifying the *probability measure* μ on the non-Boolean lattice $\mathscr{L}(\mathsf{H}_S)$ with an honest probability measure on a σ-algebra: *the fact that we now consider quantum incompatible propositions alters drastically the rules of conditional probability*. The probability that "P is true when Q holds" abides by a different set of rules from the classical theory if P and Q are incompatible in quantum sense.
(6) If H_S is separable, a σ-additive probability measure μ on $\mathscr{L}(\mathsf{H}_S)$ in the sense of **A2** is completely determined by its range over *atomic propositions* (see Remark 7.23(3)), i.e. over orthogonal projectors onto subspaces of dimension 1 in H. The proof follows directly property (2) in **A2**, to which μ is subjected. ■

Important remark. When we assign a state there will be propositions with probability 1 of being true, and propositions with probability less than 1, if the system undergoes a measurement. We may view the first class as properties that the system really possesses in the state considered.

Under the standard interpretation of QM, where probability *has no epistemic meaning*, we are forced to conclude that the properties relative to the second class of propositions *are not defined* for the state examined.

An important example for physics is this. Consider a system formed by a quantum particle on the real line and take propositions P_E of the form: "the particle's position is in the Borel set $E \in \mathscr{B}(\mathbb{R})$". If the state μ assigns to each P_E, E bounded, a probability less than 1 (such states come by easily, as we shall see with Heisenberg's uncertainty principle/theorem) then we must conclude that *the particle's position, in state μ, is not defined.*

From this point of view, the spatial description of particles as points in a manifold – here $\mathbb{R}$, representing the "physical space at rest" of a frame system – does not play a central role anymore, unlike in classical physics. In some sense all the properties of a system (which may vary with the state) are put on an equal footing, and the "space" in which system and states are described is a Hilbert space. ■

From the mathematical perspective the first question to raise is whether maps such as μ in **A2** exist at all.

Given a Hilbert space H we will show they do exist. Recall that $\mathfrak{B}_1(\mathsf{H})$ denotes *trace-class operators* on H (Chap. 4).

Proposition 7.25 *Let* H *be a Hilbert space and* $T \in \mathfrak{B}_1(\mathsf{H})$ *a positive (hence self-adjoint) operator with trace* 1. *Define* $\mu_T : \mathscr{L}(\mathsf{H}) \to \mathbb{R}$ *by* $\mu_T(P) := tr(TP)$ *for any* $P \in \mathscr{L}(\mathsf{H})$. *Then:*
(a) $\mu_T(P) \in [0, 1]$ *for any* $P \in \mathscr{L}(\mathsf{H})$,
(b) $\mu_T(I) = 1$,
(c) *if* $\{P_i\}_{i\in\mathbb{N}} \subset \mathscr{L}(\mathsf{H})$ *satisfies* $P_i P_j = 0$, $i \neq j$, *then*

$$\mu_T\left(s\text{-}\sum_{i=0}^{+\infty} P_i\right) = \sum_{i=1}^{+\infty} \mu_T(P_i)\ .$$

Proof (a) The operator TP is of trace class for any $P \in \mathscr{L}(\mathsf{H})$ by Theorem 4.34(b), for P is bounded, hence we can compute $tr(TP)$. The positivity of T ensures the eigenvalues of T are non-negative (Proposition 3.60(c)). We claim they all belong to $[0, 1]$. First, T is compact and self-adjoint (as positive). Using the decomposition of Theorem 4.23, since $|A| = A$ ($A \geq 0$) and so in $A = U|A|$ we have $U = I$,

$$T = \sum_{\lambda\in\sigma_p(A)} \sum_{i=1}^{m_\lambda} \lambda\, (u_{\lambda,i}|\ )\, u_{\lambda,i}\ . \tag{7.16}$$

The finite (or countable) set $\sigma_p(A)$ consists of the eigenvalues of A, and if $\lambda > 0$, $\{u_{\lambda,i}\}_{i=1,\ldots,m_\lambda}$ is a basis of the λ-eigenspace. The convergence is in the uniform topology. Let us write the above expansion as

$$T = \sum_j \lambda_j (u_j|\ )u_j\ . \tag{7.17}$$

We labelled over $\mathbb{N}$ (or a finite subset thereof, if $\dim(\mathsf{H}) < +\infty$) the set of eigenvectors $u_j = u_{\lambda_j,i}$, $\lambda_j > 0$, where λ_j is the eigenvalue of u_j. Moreover, the set of eigenvectors was completed to a basis of H by adding a, generally uncountable, basis for the kernel of T.

Computing the trace of T with respect to the u_j gives

$$1 = tr(T) = \sum_j \lambda_j\ ,$$

so $\lambda_j \in [0, 1]$. Note that the above equation proves part (b) as well, for $TI = I$. Take now $P \in \mathscr{L}(\mathsf{H})$ and compute the trace of TP in said basis:

$$tr(TP) = \sum_j \lambda_j (u_j|Pu_j)\ .$$

As $(u_j|Pu_j) = (Pu_j|Pu_j)$, we have $0 \leq (u_j|Pu_j) \leq ||P||^2||u_j||^2 \leq 1$, where we used $||u_j|| = 1$ and $||P|| \leq 1$ (Proposition 3.64(e)). Therefore

$$0 \leq \sum_j \lambda_j (u_j|Pu_j) \leq \sum_j \lambda_j = 1$$

and (a) holds.
(b) is trivially true as seen above. Let us prove (c). Referring to the previous eigenvector basis, we have:

$$\mu_T(P) = \sum_j \lambda_j \left(u_j \middle| \text{s-}\sum_i P_i \; u_j \right) = \sum_j \lambda_j \sum_i (u_j|P_i u_j) \,. \tag{7.18}$$

Both i and j range over a finite or countable set (only the eigenvalues $\lambda_j > 0$ and the corresponding finite or countable set of eigenvectors appear in (7.18)). We assume that the set of indices is $\mathbb{N}$ in either case, and the remaining possibilities can be treated similarly. We may interpret the above double sum as an iterated integral with respect to the counting measure μ on $\mathbb{N}$:

$$\mu_T(P) = \int_{\mathbb{N}} \left(\int_{\mathbb{N}} \lambda_j (u_j|P_i u_j) d\mu(i) \right) d\mu(j) \,.$$

Since $\mathbb{N}$ is σ-finite we can define the product measure $\mu \otimes \mu$ and we are allowed to swap the integrals, provided the function $\mathbb{N} \times \mathbb{N} \ni (i, j) \mapsto |\lambda_j (u_j|P_i u_j)|$ is integrable with respect to the product measure, in view of the Fubini–Tonelli theorem. The function is, again by Fubini–Tonelli, $\mu \otimes \mu$-integrable if

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} |\lambda_j (u_j|P_i u_j)| d\mu(i) \right) d\mu(j) < +\infty \,.$$

However, $\lambda_j (u_j|P_i u_j) = \lambda_j (P_j u_j|P_i u_j) \geq 0$ because $T \geq 0$, so we can replace $|\lambda_j (u_j|P_i u_j)|$ by $\lambda_j (u_j|P_i u_j)$ above. The required condition is, indeed, fulfilled since:

$$\int_{\mathbb{N}} \int_{\mathbb{N}} \lambda_j (u_j|P_i u_j) d\mu(i) d\mu(j) = \sum_j \lambda_j \sum_i (u_j|P_i u_j) \leq \sum_j \lambda_j (u_j|u_j) = \sum_j \lambda_j = 1 \,,$$

where once again we have exploited the bound

$$\sum_i (u_j|P_i u_j) = \left(u_j \middle| \text{s-}\sum_i P_i \; u_j \right) \leq \left|\left| \text{s-}\sum_i P_i \right|\right|^2 ||u_j||^2 \leq ||u_j||^2 = 1 \,.$$

Swapping the two summation signs in (7.18), we have

$$\mu_T(P) = \sum_i \sum_j \lambda_j (u_j | P_i u_j) = \sum_i tr(T P_i) = \sum_i \mu_T(P_i) ,$$

i.e. statement (c). $\square$

The next result, due to Gleason [Gle57, Dvu93], is truly paramount, in that it provides a complete characterisation of the functions that satisfy axiom **A2**.

Theorem 7.26 (Gleason) *Let H be a Hilbert space of finite dimension $\neq 2$, or infinite-dimensional and separable.*

For any map $\mu : \mathscr{L}(\mathsf{H}) \to [0, +\infty]$ with $\mu(I) < +\infty$ satisfying statement (2) in **A2***, there exists a positive operator $T \in \mathfrak{B}_1(\mathsf{H})$ such that*

$$\mu(P) = tr(T P) \quad \textit{for any } P \in \mathscr{L}(\mathsf{H}).$$

Sketch of proof. Take a Hilbert space H, either separable and infinite-dimensional, or just finite-dimensional. If $\dim \mathsf{H} = 1$ the thesis is obvious as H is isomorphic to $\mathbb{C}$. The convex set of positive trace-class operators is made by multiplicative operators $S_m : \mathbb{C} \ni z \to mz \in \mathbb{C}$ where $tr(S_m) = m \geq 0$. On the other hand $\mathscr{L}(\mathbb{C}) = \{0, 1\}$, viewed as multiplicative operators. There is only one map $\mu : \mathscr{L}(\mathbb{C}) \to [0, +\infty]$ for every fixed value $0 \leq m < +\infty$ satisfying statement (2) in **A2**, *viz.* the map $\mu(0) := 0$, $\mu(1) = m$. For each such μ, $T := S_\mu$ satisfies Gleason's thesis.

Let us consider the case $\dim \mathsf{H} > 2$. Define a non-negative *frame function* on H to be a mapping $f : \mathbb{S}_\mathsf{H} \to [0, +\infty)$, $\mathbb{S}_\mathsf{H} := \{x \in \mathsf{H} \mid ||x|| = 1\}$, for which there exists $W \in [0, +\infty)$ such that

$$\sum_{i \in K} f(x_i) = W$$

for any Hilbert basis $\{x_i\}_{i \in K} \subset \mathsf{H}$. A lengthy argument relying on results of von Neumann (cf. Gleason, *op. cit.*) proves the following lemma.

Lemma 7.27 *On any Hilbert space, either separable or of finite dimension > 2, for any non-negative frame function f there exists a bounded, self-adjoint operator T such that $f(x) = (x|Tx)$, for every $x \in \mathbb{S}_\mathsf{H}$.*

Consider the projectors $P_x := (x| \ ) x$, $x \in \mathbb{S}_\mathsf{H}$. With the assumption made on μ it is straightforward that $f(x) := \mu(P_x)$ is a non-negative frame function, since $\mu \geq 0$ and

$$\sum_{i \in K} f(x_i) = \sum_{i \in K} \mu(P_{x_i}) = \mu\left(\sum_{i \in K} P_{x_i}\right) = \mu\,(I) < +\infty\,.$$

By the lemma there is a self-adjoint operator T such that $\mu(P_x) = (x|Tx)$ for any $x \in \mathbb{S}_\mathsf{H}$. Since $(x|Tx) \geq 0$ for any x, then T is positive, so $T = |T|$ (in fact:

$|T|^2 = T^*T$ by polar decomposition, but $T^*T = T^2$ because positive roots are unique (Theorem 3.77)). If $\{x_i\}_{i\in K} \subset \mathsf{H}$ is a Hilbert basis,

$$+\infty > \mu(I) = \sum_{i\in K} f(x_i) = \sum_{i\in K} (x_i|Tx_i) = \sum_{i\in K} (x_i|\,|T|x_i)\;.$$

By Definition 4.32 then $T = |T|$ is of trace class. Take now $P \in \mathscr{L}(\mathsf{H})$, pick a Hilbert basis $\{x_i\}_{i\in J}$ of $P(\mathsf{H})$ and complete it by adding a Hilbert basis $\{x_i\}_{i\in J'}$ of $P(\mathsf{H})^\perp$. Then J is countable (or finite) by Theorem 3.30, plus:

$$P = \text{s-}\sum_{i\in J} P_{x_i}$$

by Proposition 3.64(d). Eventually,

$$P_{x_i} P_{x_j} = 0$$

if $i \neq j$ are in J. Since $Px_i = x_i$ if $i \in J$, and $Px_i = 0$ if $i \in J'$, we have

$$\mu(P) = \sum_{i\in J} \mu(P_{x_i}) = \sum_{i\in J} (x_i|Tx_i) = \sum_{i\in J\cup J'} (x_i|TPx_i) = tr(TP)\;.$$

The sketch of the proof ends here. $\square$

Remark 7.28 **(1)** Gleason's proof works for separable real and quaternionic Hilbert spaces, too [Var07].
(2) The operator T has trace 1 if $\mu(I) = 1$, as in the case of **A2**.
(3) If the Hilbert space is complex, as in **A2** and always in this text, the operator T associated to μ is unique: any other T' of trace class such that $\mu(P) = tr(T'P)$ for any $P \in \mathscr{L}(\mathsf{H})$ must also satisfy $(x|(T - T')x) = 0$ for any $x \in \mathsf{H}$. If $x = \mathbf{0}$ this is clear, while if $x \neq \mathbf{0}$ we may complete the vector $x/||x||$ to a basis, in which $tr((T - T')P_x) = 0$ reads $||x||^{-2}(x|(T - T')x) = 0$, where P_x is the projector onto $< x >$. By Exercise 3.21 we obtain $T - T' = 0$.
(4) Imposing $\dim \mathsf{H} \neq 2$ is mandatory, as the next example shows. On $\mathbb{C}^2$ the orthogonal projectors are 0, I and any matrix of the form

$$P_n := \frac{1}{2}\left(I + \sum_{i=1}^{3} n_i\sigma_i\right)\,, \quad \text{with } n = (n_1, n_2, n_3) \in \mathbb{R}^3 \text{ such that } |n| = 1,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the *Pauli matrices*:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,,\quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\,,\quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,. \tag{7.19}$$

There is a one-to-one correspondence between projectors P_n and points $n \in \mathbb{S}^2$ on the unit two-sphere. The functions μ of Gleason's theorem can be thought of as maps on $\mathbb{S}^2 \cup \{0, I\}$. Gleason's assumptions boil down to $\mu(0) = 0$, $\mu(I) = 1$ and $\mu(n) = 1 - \mu(-n)$. Positive trace-class operators with unit trace are precisely those of the form:

$$\rho_u = \frac{1}{2}\left(I + \sum_{i=1}^{3} u_i \sigma_i\right) \quad \text{with } u \in \mathbb{R}^3 \text{ such that } |u| \leq 1 \,. \tag{7.20}$$

If $\cdot$ is the standard dot product on $\mathbb{R}^3$, a direct computation using Pauli matrices gives

$$tr(\rho_u P_n) = \frac{1}{2}(1 + u \cdot n) \ .$$

The function μ defined by $\mu(0) = 0$, $\mu(I) = 1$ and

$$\mu(P_n) = \frac{1}{2}\left(1 + (v \cdot n)^3\right) \ ,$$

for any $n \in \mathbb{S}^2$ and a fixed $v \in \mathbb{S}^2$, satisfies the hypotheses of Gleason's theorem. It is easy to prove, however, that there are no operators ρ_u like (7.20) such that $\mu(P_n) := tr(\rho_u P_n)$ for any projector P_u; that is to say, there are no $u \in \mathbb{R}^3$, $|u| \leq 1$, such that

$$(1 + u \cdot n) = \left(1 + (v \cdot n)^3\right) \quad \text{for any } n \in \mathbb{S}^2.$$

■

Gleason's theorem, the fact that the trace-class operator representing a state is unique as discussed Remark 7.28(3), plus the fact that, in absence of superselection rules, every known quantum system has a Hilbert space satisfying Gleason's assumptions,[5] lead to a reformulation of axiom **A2** involving a technically more useful notion of state.

A2. *A* **state** *ρ at time t, on a quantum system S with associated Hilbert space H_S, is a positive trace-class operator on H_S with unit trace.*

The probability that the proposition $P \in \mathscr{L}(\mathsf{H}_S)$ is true on state ρ equals $tr(\rho P)$.

In conclusion, and more generally, we can say the following.

Definition 7.29 (*State*) Let H be a Hilbert space (not necessarily separable nor finite-dimensional). A positive trace-class operator with trace 1 is called a **state** on H. The set of states on H is denoted by $\mathfrak{S}(\mathsf{H})$.

---

[5] Particles with spin 1/2 admit a Hilbert space – in which the observable spin is defined – of dimension 2. The same occurs to the Hilbert space in which the polarisation of light is described (cf. helicity of photons). When these systems are described in full, however, for instance by including freedom degrees relative to position or momentum, they are representable on a separable Hilbert space of infinite dimension.

We have the following almost obvious result (valid also for H non-separable or $\dim \mathsf{H} = 2$) which proves that there is no redundancy in our definition of states and elementary propositions.

Proposition 7.30 *If* H *is a Hilbert space, the following facts hold.*
(a) $\mathfrak{S}(\mathsf{H})$ *separates* $\mathscr{L}(\mathsf{H})$*: if* $P, P' \in \mathscr{L}(\mathsf{H})$ *satisfy* $tr(\rho P) = tr(\rho P')$ *for all* $\rho \in \mathfrak{S}(\mathsf{H})$*, then* $P = P'$.
(b) $\mathscr{L}(\mathsf{H})$ *separates* $\mathfrak{S}(\mathsf{H})$*: if* $\rho, \rho' \in \mathscr{L}(\mathsf{H})$ *satisfy* $tr(\rho P) = tr(\rho' P)$ *for all* $P \in \mathscr{L}(\mathsf{H})$*, then* $\rho = \rho'$.

Proof Both statements immediately arise form the property that if $A, A' \in \mathfrak{B}(\mathsf{H})$ satisfy $(\psi|(A - A')\psi) = 0$ for every $\psi \in \mathsf{H}$, then $A = A'$ (Exercise 3.21). To prove (a) it suffices to use states $\rho = \psi(\psi|\cdot)$. For (b) it is enough to exploit projectors $P = \psi(\psi|\cdot)$ with $\psi \in \mathsf{H}$ and $||\psi|| = 1$.

Remark 7.31 For H separable, the statement of the theorem is still valid when we replace $\mathfrak{S}(\mathsf{H})$ with the space of σ-additive probability measures over $\mathscr{L}(\mathsf{H})$: part (b) is nothing but the definition of measure; part (a) is a consequence of Proposition 7.30(a), because positive trace-class operators with trace 1 are σ-additive probability measures even when $\dim \mathsf{H} = 2$ (a case not covered by Gleason's theorem). ■

7.4.2 *The Kochen–Specker Theorem*

Gleason's theorem has a momentous consequence in physics, which distinguishes the states of classical systems from quantum ones. Classical systems admit *completely deterministic* states, described by what we have called sharp states: Dirac measures with support at a point in phase space at the time considered. Each such measure maps sets either to 0 or to 1. These are states on which every statement is either true or false, and there is no intermediate option. *States of this kind do not occur in quantum systems because of the following important fact* [KoSp67].

Theorem 7.32 (Kochen–Specker) *If* H *is a Hilbert space, separable or of finite dimension* > 2*, there is no function* $\mu : \mathscr{L}(\mathsf{H}) \to [0, 1]$ *fulfilling (1) and (2) in axiom* **A2** *(measure-theory version) and taking values in* $\{0, 1\}$.

Proof If x belongs to $\mathbb{S}_\mathsf{H}$ (unit length) and P_x is the orthogonal projector $(x|\cdot)x$, any such μ gives by Gleason's theorem (the dimension is > 2) a map $\mathbb{S}_\mathsf{H} \ni x \mapsto \mu(P_x) = (x|Tx)$, where μ determines a unique $T \in \mathfrak{B}_1(\mathsf{H})$ with $T \geq 0$, $trT = 1$. This map is patently continuous for the topology of $\mathbb{S}_\mathsf{H}$ induced by the ambient H. We claim $\mathbb{S}_\mathsf{H}$ is path-connected, i.e., for any $x, y \in \mathbb{S}_\mathsf{H}$ there is a continuous path $\gamma : [a, b] \to \mathbb{S}_\mathsf{H}$ starting at $\gamma(a) = x$ and ending at $\gamma(b) = y$. If so, since $\mathbb{S}_\mathsf{H} \ni x \mapsto \mu(P_x) = (x|Tx)$ is continuous, its image is clearly path-connected (as composite of paths in $\mathbb{S}_\mathsf{H}$ with μ itself). As this image belongs in $\{0, 1\}$, the possibilities are that it is $\{0, 1\}$, or $\{0\}$, or

$\{1\}$. But there is no path joining 0 and 1 *contained in* $\{0, 1\}$, so necessarily $\mu(P_x) = 0$ for any $x \in \mathbb{S}_{\mathsf{H}}$, or $\mu(P_x) = 1$ for any $x \in \mathbb{S}_{\mathsf{H}}$. In the former case $(x|Tx) = 0$ for any x, hence $tr(T) = 0$, violating $tr(T) = 1$. In the latter case $(x|Tx) = 1$ for any x, again contradicting $tr(T) = 1$ by dimensional reasons.

To conclude we must then show $\mathbb{S}_{\mathsf{H}}$ is indeed path-connected. Taking $x, y \in \mathbb{S}_{\mathsf{H}}$ we have two options. The first is that $x = e^{i\alpha_0} y$ for some $\alpha_0 > 0$, so x is joined to y by the curve $[0, \alpha_0] \ni \alpha \mapsto e^{i\alpha} y$. The curve is continuous in the Hilbert topology and totally contained in S. The second option is that x is a linear combination of y and some $y' \in \mathbb{S}_{\mathsf{H}}$ orthogonal to y, obtained from completing y to an orthonormal basis for the span of y, x. Since $||x|| = ||y|| = ||y'|| = 1$ and $y \perp y'$, then $x = e^{i\alpha}(\cos\beta)y + e^{i\delta}(\sin\beta)y'$ for three real numbers α, β, δ. But then x is joined to y by the continuous curve, all contained in $\mathbb{S}_{\mathsf{H}}$, defined by varying each of the three parameters on suitable adjacent intervals. $\square$

In particular, if H is separable or of finite dimension > 2, it is impossible to define a state such that in each maximal set of compatible elementary propositions $\mathscr{L}_0(\mathsf{H})$, every proposition $P \in \mathscr{L}_0(\mathsf{H})$ is certainly either true or false. This is because every $P \in \mathscr{L}(\mathsf{H})$ belongs to a maximal set $\mathscr{L}_0(\mathsf{H})$ and the theorem above holds. This no-go result is relevant when one tries to construct classical models of QM by introducing "hidden variables" of classical type, essentially, because these severely restrict the models. The result by Kochen and Specker (obtained independently from Gleason's theorem, with a much more complicated and involved proof) implies that it is possible to embed the set of quantum-mechanical observables in a set of classical quantities only in presence of constraints on the simultaneous values attained by the variables. Hidden variables must satisfy a constraint known as *contextuality* and thus cannot be completely classic. For a general discussion on the use of hidden variables and the obstruction due to the lack of dispersion-free states, i.e. sharp states (also in more general contexts than the formulation of QM in Hilbert spaces) we recommend [Jau73, Chap. 7] and [BeCa81, Chap. 25].

7.4.3 Pure States, Mixed States, Transition Amplitudes

Let us now study the set of states $\mathfrak{S}(\mathsf{H}_S)$ when H_S is the Hilbert space associated to the quantum system S. A few reminders will be useful.

Given a vector space X, a finite linear combination $\sum_{i\in F} \alpha_i x_i$ is called **convex** if $\alpha_i \in [0, 1]$, $i \in F$, and $\sum_{i\in F} \alpha_i = 1$.

Moreover (Definition 2.65) $C \subset \mathsf{X}$ is called **convex** if for any pair $x, y \in C$, $\lambda x + (1-\lambda)y \in C$ for all $\lambda \in [0, 1]$ (and thus every convex combination of elements in C belongs to C).

If C is convex, $e \in C$ is called **extreme** if it cannot be written as $e = \lambda x + (1-\lambda)y$, with $\lambda \in (0, 1)$, $x, y \in C \setminus \{e\}$.

Definition 7.33 (*Ray of a projective space*) Let X be a vector space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Consider the equivalence relation:

$$u \sim v \;\Leftrightarrow\; v = \alpha u \text{ for some } \alpha \in \mathbb{K} \setminus \{\mathbf{0}\}.$$

The quotient space $\mathsf{X}/\!\sim$ is the **projective space** over X. We call elements of $\mathsf{X}/\!\sim$ other than $[\{\mathbf{0}\}]$ (the equivalence class of the null vector) **rays** of X.

Proposition 7.34 *Let* $(\mathsf{H}, (\ |\ ))$ *be a Hilbert space.*
(a) $\mathfrak{S}(\mathsf{H})$ *is a convex closed subset in* $\mathfrak{B}_1(\mathsf{H})$.
(b) *The extreme points in* $\mathfrak{S}(\mathsf{H})$ *are those of the form:*

$$\rho_\psi := (\psi|\ )\psi\ , \quad \textit{for every vector } \psi \in \mathsf{H} \textit{ with } ||\psi|| = 1.$$

This sets up a bijection between extreme states and rays of H, *which maps the extreme state* $(\psi|\ )\psi$ *to the ray* $[\psi]$.
(c) *Any state* $\rho \in \mathfrak{S}(\mathsf{H})$ *satisfies*

$$\rho \geq \rho\rho,$$

and is extreme if and only if

$$\rho\rho = \rho\ .$$

(d) *Any state* $\rho \in \mathfrak{S}(\mathsf{H})$ *is a linear combination of extreme states, including infinite combinations in the strong operator topology (or the uniform one if we rearrange the sum in accordance with the decomposition of Theorem 4.23). In particular there is always a decomposition*

$$\rho = \sum_{\phi \in N} p_\phi (\phi|\ )\phi,$$

where N *is an orthonormal eigenvector basis for* ρ, $p_\phi \in [0, 1]$ *is the eigenvalue of* $\phi \in N$, *every* p_ϕ*-eigenspace* ($p_\phi \neq 0$) *is finite-dimensional, and*

$$\sum_{\phi \in N} p_\phi = 1\ .$$

Proof (a) Take two states ρ, ρ'. It is clear $\lambda\rho + (1-\lambda)\rho'$ is of trace class because trace-class operators form a subspace in $\mathfrak{B}(\mathsf{H})$ (Theorem 4.34). By the trace's linearity (Proposition 4.36):

$$tr[\lambda\rho + (1-\lambda)\rho'] = \lambda tr\rho + (1-\lambda)tr\rho' = \lambda 1 + (1-\lambda)1 = 1\ .$$

If $f \in \mathsf{H}$ and $\lambda \in [0, 1]$, since ρ and ρ' are positive:

$$(f|(\lambda\rho + (1-\lambda)\rho')f) = \lambda(f|\rho f) + (1-\lambda)(f|\rho' f) \geq 0\ .$$

Hence $\lambda\rho + (1-\lambda)\rho'$ is a state if ρ, ρ' are states and $\lambda \in [0, 1]$. To conclude, we prove that $\mathfrak{S}(\mathsf{H})$ is closed in the natural topology of $\mathfrak{B}_1(\mathsf{H})$. Let ρ be an operator in $\mathfrak{B}_1(\mathsf{H})$ such that $\rho_n \to \rho$ as $n \to +\infty$ for some sequence of elements $\rho_n \in \mathfrak{S}(\mathsf{H})$. Consequently $\rho_n \to \rho$ also holds in the natural topology of $\mathfrak{B}(\mathsf{H})$ ((Proposition 4.38(d)) and hence in the weak topology. Therefore, as $\rho_n \geq 0$ for every $n \in \mathbb{N}$, we finally have that $\rho \geq 0$. The fact that $tr(\rho) = 1$ immediately follows from $tr(\rho_n) = 1$ for every n, since the trace is continuous with respect to $||\cdot||_1$. We conclude that $\rho \in \mathfrak{S}(\mathsf{H})$. But ρ was an arbitrary limit point of $\mathfrak{S}(\mathsf{H})$, so $\mathfrak{S}(\mathsf{H})$ is closed in $\mathfrak{B}_1(\mathsf{H})$.

(b)–(d) Consider $\rho \in \mathfrak{S}(\mathsf{H})$. The operator ρ is compact and self-adjoint (as positive). Using the decomposition of Theorem 4.23, and since $|\rho| = \rho$ ($\rho \geq 0$), so $U = I$ in the polar decomposition of $\rho = U|\rho|$, we find:

$$\rho = \sum_{\lambda \in \sigma_p(\rho)} \sum_{i=1}^{m_\lambda} \lambda \,(u_{\lambda,i}|\ )\, u_{\lambda,i} \,. \tag{7.21}$$

Above, $\sigma_p(\rho)$ is the set of eigenvectors of ρ, and if $\lambda > 0$, $\{u_{\lambda,i}\}_{i=1,\dots,m_\lambda}$ is a basis of the λ-eigenspace. At last, convergence is understood in the uniform topology if the eigenspaces are ordered in accordance with Theorem 4.23. If not, convergence is in the strong operator topology (we shall see in Example 8.60(1) that this is customary for spectral expansions). We have proved (d).

Completing $\cup_{\lambda>0}\{u_{\lambda,i}\}_{i=1,\dots,m_\lambda}$ by adding a basis for $Ker\rho$, by Proposition 4.36 we obtain:

$$1 = tr(\rho) = \sum_{\lambda \in \sigma_p(\rho)} m_\lambda \lambda \,. \tag{7.22}$$

Suppose now $\rho_\psi := (\psi|\ )\psi$, $||\psi|| = 1$. Immediately, $\rho_\psi \in \mathfrak{S}(\mathsf{H})$. We want to prove ρ_ψ is extreme in $\mathfrak{S}(\mathsf{H})$. So assume there are $\rho, \rho' \in \mathfrak{S}(\mathsf{H})$ and $\lambda \in (0, 1)$ such that

$$\rho_\psi = \lambda\rho + (1-\lambda)\rho' \,.$$

We claim $\rho = \rho' = \rho_\psi$.

Consider the orthogonal projector $P_\psi = (\psi|\ )\psi$. It is clear (completing ψ to a basis) that $tr(\rho_\psi P_\psi) = 1$, so

$$1 = \lambda tr(\rho P_\psi) + (1-\lambda) tr(\rho' P_\psi) \,.$$

As $\lambda \in (0, 1)$ and $0 \leq tr(\rho P_\psi) \leq 1$, we have $0 \leq tr(\rho' P_\psi) \leq 1$, which is possible only if $tr(\rho P_\psi) = tr(\rho' P_\psi) = 1$. So let us prove that $tr(\rho P_\psi) = 1$ and $tr(\rho' P_\psi) = 1$ imply $\rho = \rho' = \rho_\psi$.

Decomposing ρ as in (7.21), $tr(\rho P_\psi) = 1$ becomes

$$\sum_j \lambda_j |(u_j|\psi)|^2 = 1 \,, \tag{7.23}$$

where: the eigenvectors $u_j = u_{\lambda,i}$, $\lambda > 0$, were labelled by $\mathbb{N}$ or a finite subset, λ_j is the eigenvalue of u_j and we added to these eigenvectors a basis for the null space of ρ. The overall basis may be uncountable, but the values j for which $\lambda_j \neq 0$ form a finite or countable set, because ρ is compact. Also the indices j such that $|(u_j|\psi)| \neq 0$ are countable at most. For the rest of the proof only the union of these two subsets of indices is relevant, so we shall assume that $j \in \mathbb{N}$ (the finite case being trivial, by the way). By assumption we have

$$\sum_j \lambda_j = 1\,, \tag{7.24}$$

$$\sum_j |(u_j|\psi)|^2 = 1\,. \tag{7.25}$$

Since $\lambda_j \in [0, 1]$ and $|(u_j|\psi)|^2 \in [0, 1]$ for any $j \in \mathbb{N}$, we obtain

$$\sum_j \lambda_j^2 \leq 1\,, \tag{7.26}$$

$$\sum_j |(u_j|\psi)|^4 \leq 1\,. \tag{7.27}$$

Therefore the sequences of the λ_j and $|(u_j|\psi)|^2$ belong to $\ell^2(\mathbb{N})$. Identity (7.23), plus (7.26), (7.27) and the Cauchy–Schwarz inequality in $\ell^2(\mathbb{N})$, give

$$\sum_j \lambda_j^2 = 1\,, \tag{7.28}$$

$$\sum_j |(u_j|\psi)|^4 = 1\,. \tag{7.29}$$

Since $\lambda_j \in [0, 1]$ for any $j \in \mathbb{N}$, (7.24) and (7.28) are consistent only if all λ_i vanish except one, say $\lambda_p = 1$. Likewise, since $|(u_j|\psi)|^2 \in [0, 1]$ for any $j \in \mathbb{N}$, (7.25) and (7.29) are consistent only if all $|(u_j|\psi)|$ are zero except for $|(u_k|\psi)| = 1$. As the u_i are a basis and $||\psi|| = 1$, necessarily $\psi = \alpha u_k$, with $|\alpha| = 1$. Clearly, then, $k = p$, for otherwise $tr(\rho P_\psi) = 0$. But

$$\rho = \sum_j \lambda_j (u_j|\ )u_j\,,$$

so eventually

$$\rho = \lambda_k (u_k|\ )u_k = 1 \cdot (u_k|\ )u_k = \overline{\alpha^{-1}}\alpha^{-1}(\psi|\ )\psi = |\alpha|^{-1}(\psi|\ )\psi = (\psi|\ )\psi = \rho_\psi\,.$$

In the same way we can prove $\rho' = \rho_\psi$.

If a state ρ is not of the type $(\psi|\ )\psi$, we can still decompose it orthogonally as

$$\rho = \sum_j \lambda_j (u_j|\ )u_j \ ,$$

where at least two vectors u_p, u_q do not vanish and are perpendicular. Hence in particular $\lambda_p, 1-\lambda_p \in (0,1)$. Then we can write ρ as

$$\rho = \lambda_p(u_p|\ )u_p \ + (1-\lambda_p)\sum_{j\neq p} \frac{\lambda_j}{(1-\lambda_p)}(u_j|\ )u_j \ .$$

As already said $(u_p|\ )u_p$ is a state, and easily, we also have

$$\rho' := \sum_{j\neq p} \frac{\lambda_j}{(1-\lambda_p)}(u_j|\ )u_j$$

is a state of $\mathfrak{S}(\mathsf{H})$ (obviously $\rho' \neq (u_p|\ )u_p$ by construction, as $u_q \not\sim u_p$). So we have proved ρ is not extreme.

The function f mapping the extreme state $(\psi|\ )\psi$ to the ray $[\psi]$ is well defined. In fact, let us first notice that $||\psi|| = 1$ by definition of extreme state, so $\psi \neq \mathbf{0}$ and $[\psi]$ is a ray. Extremes states may be expressed in different ways: namely (as is immediate to see from $||\phi|| = 1$) $(\psi|\ )\psi = (\phi|\ )\phi$ iff $\psi = e^{i\alpha}\phi$ for some $\alpha \in \mathbb{R}$. But then by definition of ray $[\psi] = [\phi]$. We claim f is one-to-one: if ϕ, ψ are unit vectors and $[\psi] = [\phi]$, then $\psi = e^{i\alpha}\phi$ for some $\alpha \in \mathbb{R}$, so $(\psi|\ )\psi = (\phi|\ )\phi$. The function is also onto, because if $[\phi]$ is a ray, $||\phi|| \neq 0$ so there exists $\psi \in [\phi]$ with $||\psi|| = 1$. Then $f((\psi|\ )\psi) = [\phi]$ since $\psi = \alpha\phi$ for some non-zero $\alpha \in \mathbb{C}$.
(c) Begin with the second claim. If ρ is extreme, $\rho\rho = \rho$ using the form in part (b) for extreme points. Decomposing a state ρ as:

$$\rho = \sum_j \lambda_j (u_j|\ )u_j$$

(see the meaning above) gives

$$\rho\rho = \sum_j \lambda_j^2 (u_j|\ )u_j \ .$$

If $\rho\rho = \rho$, passing to traces gives

$$\sum_j \lambda_j^2 = \sum_j \lambda_j = 1$$

with $\lambda_j \in [0, 1]$. This is possible only if all λ_j are zero but one, $\lambda_k = 1$. Then

$$\rho = \sum_j \lambda_j (u_j|\ )u_j = 1 \cdot (u_k|\ )u_k \ ,$$

which is an extreme state by (b).

Now to the first claim. Let $x = \sum_j \alpha_j u_j$ be a point in H (the u_j are a basis of H). Since $\lambda_j \in [0, 1]$,

$$(x|\rho\rho x) = \sum_j \lambda_j^2 (x|u_j)(u_j|x) = \sum_j \lambda_j^2 |\alpha_j|^2 \leq \sum_j \lambda_j |\alpha_j|^2 = \sum_j \lambda_j (x|u_j)(u_j|x) = (x|\rho x) \ .$$

Therefore $\rho\rho \leq \rho$. □

Remark 7.35 Let $||\ ||_i$ refer to the spaces of compact operators $\mathfrak{B}_i(\mathsf{H})$, for $i = 1, 2$. The following four facts are equivalent:

(i) $\rho \in \mathfrak{S}(\mathsf{H})$ is extreme,
(ii) $||\rho|| = ||\rho||_1$,
(iii) $||\rho||_2 = ||\rho||_1$,
(iv) $||\rho|| = ||\rho||_2$,
(v) $||\rho||_2 = 1$.

The elementary proof is left to the reader. ■

Now we have a definition.

Definition 7.36 Let $(\mathsf{H}, (\ |\ ))$ be a Hilbert space.
(a) Extreme elements in $\mathfrak{S}(\mathsf{H})$ are called **pure states**, and their set is indicated by $\mathfrak{S}_p(\mathsf{H})$. Non-extreme states are **mixed states**, **mixtures** or **nonpure states**.
(b) Suppose:

$$\psi = \sum_{i \in I} \alpha_i \phi_i \ ,$$

where I is finite, or countable (and the series converges), the vectors $\phi_i \in \mathsf{H}$ are non-null and $0 \neq \alpha_i \in \mathbb{C}$ for any $i \in I$. One says the state $(\psi|\ )\psi/||\psi||^2$ is a **coherent superposition** of the states $(\phi_i|\ )\phi_i/||\phi_i||^2$.
(c) If $\rho \in \mathfrak{S}(\mathsf{H})$ satisfies:

$$\rho = \sum_{i \in I} p_i \rho_i$$

with I finite, $\rho_i \in \mathfrak{S}(\mathsf{H})$, $0 \neq p_i \in [0, 1]$ for any $i \in I$, and $\sum_i p_i = 1$, the state ρ is called **incoherent superposition** of the ρ_i (possibly pure).
(d) If $\psi, \phi \in \mathsf{H}$ satisfy $||\psi|| = ||\phi|| = 1$:

(i) the complex number $(\psi|\phi)$ is the **transition amplitude** or **probability amplitude** of state $(\phi|\ )\phi$ on state $(\psi|\ )\psi$;

(ii) the non-negative real number $|(\psi|\phi)|^2$ is the **transition probability** of state $(\phi|\ )\phi$ on state $(\psi|\ )\psi$.

Remark 7.37 **(1)** The vectors of the Hilbert space of a quantum system associated to pure states are often referred to, in physics, as **wavefunctions**. The reason for the name is due to the earliest formulation of QM in terms of Wave Mechanics (see Chap. 6). Similarly, operators in $\mathfrak{S}(\mathsf{H})$ are very often called **statistical operators** or **density matrices** in physics' literature. We will use this language sometimes.
(2) The possibility of creating pure states by non-trivial combinations of vectors associated to other pure states is called, in the jargon of QM, **superposition principle of (pure) states**.
(3) In (c), in case $\rho_i = \psi_i(\psi_i|\ )$, we do *not* require $(\psi_i|\psi_j) = 0$ if $i \neq j$. However it is immediate to see that if I is finite, if ρ_i is a mixed or pure state and if $p_i \in [0, 1]$ for any $i \in I$, $\sum_i p_i = 1$, then:

$$\rho = \sum_{i \in I} p_i \rho_i$$

is of trace class (obvious: trace-class operators are a vector space and every ρ_i is of trace class), positive (as positive linear combination of positive operators), and it has unit trace: this because by the trace's linearity (Proposition 4.36) we have

$$tr\rho = tr\left(\sum_{i \in I} p_i \rho_i\right) = \sum_{i \in I} p_i tr\rho_i = \sum_{i \in I} p_i \cdot 1 = 1 .$$

The decomposition of ρ over an eigenvector basis can be considered a limiting case of the above: when I is countable, in fact, $\rho_i = \psi_i(\psi_i|\ )$ and $(\psi_i|\psi_j) = \delta_{ij}$.

It is important to remark that *in general, a given mixed state admits several incoherent decompositions by pure and nonpure states.*
(4) Consider a pure state $\rho_\psi \in \mathfrak{S}_p(\mathsf{H})$, written $\rho_\psi = (\psi|\ )\psi$ for some $\psi \in \mathsf{H}$ with $||\psi|| = 1$. What we want to emphasise is that this pure state is also an orthogonal projector $P_\psi := (\psi|\ )\psi$, so it must correspond to a proposition about the system. *The naïve and natural interpretation*[6] *of the proposition is: "the system's state is the pure state given by the vector ψ".*

This interpretation is due, if $\rho \in \mathfrak{S}(\mathsf{H})$, to the fact that $tr(\rho P_\psi) = 1 \Leftrightarrow \rho = (\psi|\ )\psi$. In fact, if $\rho = (\psi|\ )\psi$, by completing ψ to a basis and taking the trace, we have $tr(\rho P_\psi) = 1$. Conversely, suppose $tr(\rho P_\psi) = 1$ for the state ρ. Then $\rho = (\psi|\ )\psi$ from the proof of Proposition 7.34.
(5) Part (4) allows to interpret the squared modulus of the transition amplitude $(\phi|\psi)$. If $||\phi|| = ||\psi|| = 1$, as the definition of transition amplitude imposes, $tr(\rho_\psi P_\phi) = |(\phi|\psi)|^2$, where $\rho_\psi := (\psi|\ )\psi$ and $P_\phi = (\phi|\ )\phi$. Using (4) we conclude:
$|(\phi|\psi)|^2$ *is the probability that the state, given (at time t) by the vector ψ, following a measurement (at time t) on the system becomes determined by ϕ.*

[6] We cannot but notice how this interpretation muddles the semantic and syntactic levels. Although this could be problematic in a formulation within formal logic, the use physicists make of the interpretation eschews the issue.

Notice $|(\phi|\psi)|^2 = |(\psi|\phi)|^2$, so the probability transition of the state determined by ψ on the state determined by ϕ coincides with the analogous probability where the vectors are swapped. This fact is, a priori, highly non-evident in physics. ■

To conclude we state and prove an elementary but useful technical result concerning states.

Proposition 7.38 *Let* H *be a Hilbert space, take* $\rho \in \mathfrak{S}(\mathsf{H})$, $P \in \mathscr{L}(\mathsf{H})$ *and consider the eigenvectors decomposition*

$$\rho = \sum_{\phi \in N} p_\phi (\phi|\ )\phi$$

of Proposition 7.34(d). The following facts are equivalent.
(a) $tr(P\rho) = 1$.
(b) $\phi \in P(\mathsf{H})$ *for every eigenvector* ϕ *of* ρ *in the basis* N.
(c) $P\rho P = \rho$.

Proof Evidently (b) implies (a). Indeed, if (b) holds true, we have

$$tr(P\rho) = \sum_{\phi \in N} p_\phi ||P\phi||^2 = \sum_{\phi \in N} p_\phi ||\phi||^2 = \sum_{\phi \in N} p_\phi = 1 \, .$$

Let us prove the converse implication. First of all, observe that, for $P \in \mathscr{L}(\mathsf{H})$ and $\psi \in \mathsf{H}$, we immediately have $||P\psi - \psi||^2 = ||P\psi||^2 + ||\psi||^2 - (P\psi|\psi) - (\psi|P\psi) = ||P\psi||^2 + ||\psi||^2 - (P\psi|P\psi) - (P\psi|P\psi) = ||P\psi||^2 + ||\psi||^2 - 2\,||P\psi||^2$. We have found that $||P\psi - \psi||^2 = ||\psi||^2 - ||P\psi||^2$. We conclude that $\psi \in P(\mathsf{H})$ is equivalent to $||P\psi|| = ||\psi||$. Now assume that (a) is valid. It can be rephrased as

$$1 = tr(P\rho) = \sum_{\phi \in N} p_\phi ||P\phi||^2 \, .$$

Since $||P\phi||^2 \leq ||\phi||^2 = 1$ and $p_\phi \geq 0$ with $\sum_{\phi \in N} p_\phi = 1$, the identity above can be fulfilled only if $||P\phi||^2 = 1$ for every $\phi \in N$. In other words $||P\phi|| = ||\phi||$ and thus $P\phi \in P(\mathsf{H})$ for every $\phi \in N$. To conclude, notice that (b) implies (c) immediately by the eigenvector decomposition of ρ. Moreover, if (c) is valid, $tr(P\rho) = tr(PP\rho) = tr(P\rho P) = tr\rho = 1$, so that (a) is also true. □

7.4.4 *Axiom* A3*: Post-Measurement States and Preparation of States*

The standard formulation of QM assumes a third axiom, introduced by von Neumann and generalised by Lüders [BeCa81], about what occurs to a system S, in state

$\rho \in \mathfrak{S}(\mathsf{H}_S)$ at time t, when we measure a proposition $P \in \mathscr{L}(\mathsf{H}_S)$ that is true (so in particular $tr(\rho P) > 0$, prior to the measurement).

A3. *If the quantum system S is in state $\rho \in \mathfrak{S}(\mathsf{H}_S)$ at time t and a proposition $P \in \mathscr{L}(\mathsf{H}_S)$ is true after a measurement at time t, the system's state immediately afterwards is:*

$$\rho_P := \frac{P\rho P}{tr(\rho P)} . \tag{7.30}$$

In particular, if ρ is pure and determined by the unit vector ψ, immediately after measurement the state is still pure, and determined by:

$$\psi_P = \frac{P\psi}{||P\psi||} .$$

Obviously, in either case ρ_P and ψ_P define states. In the former, in fact, ρ_P is positive of trace class, with unit trace, while in the latter $||\psi_P|| = 1$.

Remark 7.39 **(1)** As already highlighted, measuring a property of a physical quantity goes through the interaction between the system and an instrument (supposed macroscopic and obeying the laws of classical physics). Quantum Mechanics, in its standard formulation, does not establish what a measuring instrument is; it only says they exist. Nor is it capable of describing the interaction of instrument and quantum system beyond the framework set in **A3**. Several viewpoints and conjectures exist on how to complete the physical description of the measuring process. These are called, in the slang of QM, **collapse**, or **reduction**, **of the state** or **of the wavefunction**. For various reasons, though, none of the current proposals is entirely satisfactory [Des99, Bon97, Ghi07, Alb94]. A very interesting proposal was put forward in 1985 by G.C. Girardi, T. Rimini and A. Weber (*Physical Review* D34, 1985 p. 470), who described in a dynamically nonlinear way the measuring process and assumed it is due to a self-localisation process, *rather than* to an instrument. This idea, alas, still has several weak points: in particular it does not allow – at least not in an obvious manner – for a relativistic description.

(2) Axiom **A3** refers to *non-destructive* testing, also known as *indirect measurement* or *measurement of the first kind* [BrKh95], where the physical system examined (typically a particle) is not absorbed/annihilated by the instrument. They are idealised versions of the actual processes used in labs, and only in part can they be modelled in such a way.

(3) Measuring instruments are commonly employed to *prepare a system in a certain pure state*. Theoretically-speaking the preparation of a *pure* state is carried out like this: a finite collection of *compatible* propositions $P_1, \ldots, P_n$ is chosen so that the projection subspace of $P_1 \wedge \cdots \wedge P_n = P_1 \cdots P_n$ is *one-dimensional*. In other words $P_1 \cdots P_n = (\psi|\ )\psi$ for some vector with $||\psi|| = 1$. The existence of such propositions is seen in practically all quantum systems used in experiments. (From a theoretical point of view these are *atomic* propositions in the sense of Remark 7.23(3), and must exist because of the Hilbert space.) The propositions P_i are then simultaneously measured on several identical copies of the physical system of concern (e.g., electrons), whose initial states, though, are unknown. If the measurements of all propositions are successful for one system, the post-measurement state is determined by the vector ψ, and the system was **prepared** in that particular pure state.

Normally each projector P_i is associated to a measurable quantity A_i on the system, and P_i defines the proposition "the quantity A_i belongs to the set E_i". So in practice, in order to prepare a system (available in arbitrarily many identical copies) in the pure state ψ, one measures a number of *compatible* quantities A_i simultaneously, and selects the systems for which the readings of the A_i belong to the given sets E_i.

(4) Let us explain how to obtain nonpure states from pure ones. Consider q_1 identical copies of system S prepared in the pure state associated to ψ_1, q_2 copies of S prepared in a *different* pure state associated to ψ_2 and so on, up to ψ_n. If we mix these states each system will be in the *necessarily nonpure* state (see Exercise 7.18):

$$\rho = \sum_{i=1}^{n} p_i (\psi_i|\ \ )\psi_i \ ,$$

where $p_i := q_i / \sum_{i=1}^{n} q_i$. In general, $(\psi_i|\psi_j)$ is not zero if $i \neq j$, so the above expression for ρ is not the decomposition in an eigenvector basis for ρ. This procedure hints at the existence of two different types of probability: one intrinsic and due to the quantum nature of state ψ_i; the other epistemic, and encoded in the probability p_i. But this is not true: once a nonpure state has been created, as above, there is no way, within QM, to distinguish the states forming the mixture. For example, the same ρ could have been obtained mixing other pure states than those determined by the ψ_i. In particular, one could have used those in the eigenvector decomposition. For physics, no kind of measurement (under the axioms of QM available thus far) would distinguish the two mixtures.

(5) Axiom **A3** permits us to give an interesting and natural interpretation to quantities like $||PQ\psi||^2 = tr(\rho_\psi PQ)$ where P, Q are *incompatible* propositions and ψ a normalised vector representing a pure state. If P and Q were compatible $||PQ\psi||^2$ would simply be the probability that $P \wedge Q$ is true in the pure state defined by ψ. This interpretation is now untenable. Taking the second case in **A3** into account, we can easily interpret the right-hand side of

$$||PQ\psi||^2 = \left|\left|\frac{PQ\psi}{||Q\psi||}\right|\right|^2 ||Q\psi||^2 \ .$$

It has the natural meaning of *the probability of measuring first Q true and next P true in a consecutive measurement of Q and P*. The asymmetry $||PQ\psi||^2 \neq ||QP\psi||^2$ is in agreement with this interpretation. It is worth noticing that the same interpretation can be given if P and Q are compatible. In that case, however, we will find the same result as by a simultaneous measurement of P and Q. ■

Axiom **A3** can equivalently be stated using σ-additive probability measures on $\mathscr{L}(\mathsf{H}_S)$ as, for dim $H_S \neq 2$, these are in one-to-one correspondence with the elements of $\mathfrak{S}(\mathsf{H}_S)$ by Gleason's theorem. This equivalent formulation provides a natural interpretation of the axiom itself in terms of conditional probability:

A3 (measure-theory version). *If $\mu : \mathscr{L}(\mathsf{H}_S) \to [0,1]$ is the σ-additive probability measure representing the state of the system, and proposition $P \in \mathscr{L}(\mathsf{H}_S)$ is true after a measurement at time t, the system's state immediately afterwards is represented by the σ-additive probability measure μ_P given by*

$$\mu_P(Q) := \frac{\mu(PQP)}{\mu(P)} \quad \text{for every } Q \in \mathscr{L}(\mathsf{H}_S)\,. \tag{7.31}$$

This formula is nothing but the translation of **A3** in terms of associated measures. The formulation of the post-measurement postulate in terms of measures, in principle, is valid also for $\dim \mathsf{H}_S = 2$.

The measure μ_P actually enjoys a natural conditional-probability property which completely fixes it, and therefore it can be used as a justification of axiom **A3**. The idea is that as soon as a proposition P has been proved to be true for a state (probability measure) μ, another proposition $Q \leq P$ must have probability $\mu(Q)/\mu(P)$ to be true in the post-measurement state μ_P. This constraint is completely natural when the probability measures are defined on Boolean lattices (σ-algebras), and is the basic idea of conditional probability. In those Boolean cases, the requirement completely fixes the new probability measure μ_P over the whole lattice and not only over the sublattice of the events $P \leq Q$. The reader can prove this easily, by observing that a probability measure μ_P satisfying $\mu_P(Q) = \mu(Q)/\mu(P)$ for $Q \leq P$ must have support contained in P since $\mu_P(P) = \mu(P)/\mu(P) = 1$. Actually, the result is true even if $\mathscr{L}(\mathsf{H}_S)$ is not Boolean.

Proposition 7.40 *Suppose* $\dim(\mathsf{H}_S) \neq 2$ *in axiom* **A3**. *The post-measurement probability measure μ_P (relative to the probability measure μ with value 1 on $P \in \mathscr{L}(\mathsf{H}_S)$) is the unique probability measure on $\mathscr{L}(\mathsf{H}_S)$ such that*

$$\mu_P(Q) = \frac{\mu(Q)}{\mu(P)} \quad \textit{for every } Q \in \mathscr{L}(\mathsf{H}_S) \textit{ with } Q \leq P\,.$$

Proof If $Q \leq P$ then $PQP = Q$ so that $\mu_P(Q) = \frac{\mu(PQP)}{\mu(P)} = \frac{\mu(Q)}{\mu(P)}$. Conversely, let ν be a σ-additive probability measure on the whole space $\mathscr{L}(\mathsf{H}_S)$ such that $\nu(Q) = \frac{\mu(Q)}{\mu(P)}$ if $Q \leq P$, and indicate by T_ν the trace-class operator associated with ν and by T_{μ_P} the one associated with μ_P. For $Q = P$ we find $tr(T_\nu P) = \mu(P)/\mu(P) = 1 = tr(T_{\mu_P} P)$. Now, Proposition 7.38 implies $T_\nu = PT_\nu P$ and $T_{\mu_P} = PT_{\mu_P}P$. Hence both T_ν and T_{μ_P} keep $\mathsf{M} = P(\mathsf{H}_S)$ invariant and vanish on $\mathsf{M}^\perp$. As a consequence it is enough to prove that $T_{\mu_P}\restriction_{\mathsf{M}} = T_\nu\restriction_{\mathsf{M}}$ to conclude that $T_{\mu_P} = T_\nu$, i.e., $\nu = \mu_P$. As a matter of fact we already know that $tr(T_\nu\restriction_{\mathsf{M}} Q) = \nu(Q) = \mu_P(Q) = tr(T_{\mu_P}\restriction_{\mathsf{M}} Q)$

for $Q \leq P$. If we choose $Q = (\psi|\cdot)\psi$ with $\psi \in \mathsf{M}$ of unit norm then the above identity reads $(\psi|(T_\nu\restriction_{\mathsf{M}} - T_{\mu_P}\restriction_{\mathsf{M}})\psi) = 0$, which implies $T_\nu\restriction_{\mathsf{M}} - T_{\mu_P}\restriction_{\mathsf{M}} = 0$, as we wanted. $\square$

7.4.5 Quantum Logics

The discussion in Sect. 7.3.2 explains that it makes sense to describe propositions about quantum system in terms of the (non-Boolean) projector lattice of a Hilbert space, and incompatible propositions in terms of non-commuting projectors. Moreover, it makes sense to assign the usual meaning to $\wedge$, $\vee$ in terms of the connectives $\mathscr{E}$, $\mathscr{O}$, provided the former are employed with projectors describing compatible propositions.

In the general case $R := P \wedge Q$ denotes simply the projector onto the intersection of the targets of P, Q. This R may be a meaningful statement about the system, but as we noted earlier it *does not correspond to the proposition $P \mathscr{E} Q$ when P, Q relate to incompatible propositions*. Conversely, the approach of Birkhoff and von Neumann, that befits the so-called *Standard Quantum Logic*, uses $\vee$ and $\wedge$ as proper connectives (yielding an algebra different from the usual one), even if they operate between projectors of incompatible propositions (i.e. for which no instrument can evaluate the truth of P, Q simultaneously). This is the reason why the point of view of *Quantum Logic* has been criticised by physicists (cf. [Bon97, Chap. 5] for a thorough discussion). In the past years, alongside the modern development of Birkhoff's and von Neumann's approach [EGL09], many authors have introduced new formal strategies that differ from *Quantum Logic à la* Birkhoff–von Neumann, in particular by means of *topos theory* [DoIs08, HLS09].

A difficult issue is the operational meaning of $P \wedge Q$ and $P \vee Q$ when P and Q are incompatible. We know that $P \wedge Q$ and $P \vee Q$ are orthogonal projectors and correspond to some elementary proposition in their own right, which can be experimentally tested by some procedure. However, just because P and Q cannot be simultaneously tested, this procedure does not have an evident meaning in terms of the outcome of the measurements of P and Q. Even if we will shall not enter into the details of this ongoing debate (see also Sect. 7.6.1), let us observe that as soon as one assumes that elementary propositions are described by orthogonal projectors on the Hilbert space H_S, a proposition $P \in \mathscr{L}(\mathsf{H}_S)$ is completely determined by the class of states $\rho \in \mathfrak{S}(\mathsf{H}_S)$ for which P is *always true*: $tr(\rho P) = 1$. This is an immediate consequence of Proposition 7.38. We therefore may identify a proposition with the class of states for which the proposition is always true. This approach permits us to partially grasp the operational meaning of $P \wedge Q$ even when P and Q are incompatible. A state ρ makes $P \wedge Q$ always true ($tr(\rho P \wedge Q) = 1$) if and only if the outcome of *separate* measurements of P, Q on that state (either on the same system or on different copies, all prepared in the same state ρ) is always 1: $tr(\rho P) = tr(\rho Q) = 1$. The proof of this fact is trivial: a measurement of P or Q does not change the state ψ, in agreement with **A3**, because $tr(\rho P \wedge Q) = 1$ implies that each eigenvector in the decomposition of ρ belongs to the intersection of the projection spaces of P and Q, by Proposition 7.38. Irrespective of whether P and Q are compatible, therefore, $P \wedge Q$ *is the elementary proposition which is always true on* ρ *if and only if* P *and* Q *are always true on* ρ *when measured separately (independently from the order, when the measurements are performed on a single system).* This interpretation of the Boolean operation 'meet' can be taken to the level of the abstract elementary propositions $\mathscr{L}$ [BeCa81, Sect. 16.5], see Sect. 7.6.1. A similar proposal for $P \vee Q$ would use the identity $P \vee Q = \neg(\neg P \wedge \neg Q)$, valid in orthocomplemented lattices. But the ensuing interpretation is much more difficult to accept, for it involves a counterfactual conditional statement.

The observation above on the practical meaning of $P \wedge Q$ in case of incompatible propositions also leads to a suggestive operational interpretation of $P \wedge Q$ due to Jauch [Jau73], based on a result of von Neumann (Theorem 13.7 in [Neu50]).

Theorem 7.41 (von Neumann's theorem on iterated projectors) *Let* H *be a complex Hilbert space and* $P, Q : \mathsf{H} \to \mathsf{H}$ *orthogonal projectors, in general not commuting. Calling, as usual,* $P \wedge Q$ *the orthogonal projector onto* $P(\mathsf{H}) \cap Q(\mathsf{H})$*, we have:*

$$(P \wedge Q)x = \lim_{n \to +\infty} (PQ)^n x \quad \textit{for any } x \in \mathsf{H}. \tag{7.32}$$

Proof First, $Q(PQ)^n = (QPQ)^n = (QP)^n Q$. The sequence $A_n = (QPQ)^n \in \mathfrak{B}(\mathsf{H})$ satisfies $A_n \geq A_{n+1} \geq 0$: $||\sqrt{QPQ}||^2 = ||QPQ|| \leq ||Q||^2||P|| \leq 1$ and $0 \leq (x|A_{n+1}x) = ||\sqrt{QPQ}(QPQ)^{n/2}x||^2 \leq ||\sqrt{QPQ}||^2||(QPQ)^{n/2}x||^2 = ||\sqrt{QPQ}||^2 (x|A_n x)$. By Proposition 3.76, $\text{s-}\lim_{n\to+\infty}(QPQ)^n = R \in \mathfrak{B}(\mathsf{H})$. Immediately, $RR = R$ and $(Rx|y) = \lim_n (x|(Q(PQ)^n)^* y) = \lim_n (x|(QP)^n Qy) = (x|Ry)$, so $R = R^*$. By construction $PR = \text{s-}\lim_n PQ(PQ)^n = R$. Therefore $(PQ)^n \to R \in \mathscr{L}(\mathsf{H})$ in the strong topology. Analogously $(QP)^n \to R' \in \mathscr{L}(\mathsf{H})$ in the same topology. However, $(x|(PQ)^n y) \to (x|Ry)$ is equivalent to $((QP)^n x|y) \to (x|Ry)$, i.e. $(R'x|y) = (x|Ry)$. Since $R' = R'^*$ we have $R' = R$. Clearly $RP = R = RQ$, so $R(\mathsf{H}) \supset P(\mathsf{H}) \cap Q(\mathsf{H})$. The orthogonal to the latter space is generated (by De Morgan's laws) by $(\neg P)(\mathsf{H})$ and $(\neg Q)(\mathsf{H})$. As $R(\neg P) = R(\neg Q) = 0$, we conclude $R(\mathsf{H}) = P(\mathsf{H}) \cap Q(\mathsf{H})$. $\square$

There is an interesting physical point of view that interprets the right-hand side of (7.32) as the *consecutive and alternated* measurement of an *infinite* sequence of propositions P, Q. From this perspective the *proposition* $P \wedge Q$ *is always true for a pure state represented by the unit vector* ψ *of a quantum system* ($||P \wedge Q\psi||^2 = 1$) *only if all propositions in the sequence turn out to be true (see (5) in Remark 7.39) when performed on the system in the initial pure state* ψ.[7]

The extension to mixed states is easy.

[7]If $||P \wedge Q\psi||^2 = 1$, the state does not change after each single measurement of P or Q, in accordance with **A3**, because of Proposition 7.38, as already observed for a general ρ.

7.5 Observables as Projector-Valued Measures on $\mathbb{R}$

At this point we want to introduce *observables* by means of *projector-valued measures* (PVM). This notion lies at the heart of the mathematical formulation of standard QM. In ensuing chapters this notion will be generalised and made more precise, but it will be only in the spectral decomposition of unbounded self-adjoint operators that PVMs will reveal their full potential.

7.5.1 Axiom **A4***: The Notion of Observable*

In Quantum Mechanics, a physical quantity that is testable on a physical system and whose behaviour is described by **QM1** and **QM2** is called an **observable**.

As seen in Sect. 7.2.4, it is reasonable to label measurement readings by Borel subsets of $\mathbb{R}$. From the physical point of view it is natural to assume that if the orthogonal projectors $P_E^{(A)}$ associated to the observable A commute with each other then $E \in \mathscr{B}(\mathbb{R})$ (the Borel σ-algebra of $\mathbb{R}$), since we expect, for $E \in \mathscr{B}(\mathbb{R})$, propositions like

$$P_E^{(A)} :=$$

"*The value of A on the state of the system belongs to the Borel set $E \subset \mathbb{R}$*"

to be all compatible. If it were not so, we would not have an observable, but distinct incompatible quantities. Since the outcome belongs to both E and E' if and only if it belongs to $E \cap E'$, we take $P_E^{(A)} \wedge P_{E'}^{(A)} = P_{E\cap E'}^{(A)}$. Assume also $P_{\mathbb{R}}^{(A)} = I$, because the result certainly belongs to $\mathbb{R}$, so $P_{\mathbb{R}}^{(A)}$ is a tautology, independent of the state on which the measurement is done. Eventually, for physically self-evident reasons and because of the *logical* meaning of $\vee$, it is reasonable to suppose

$$\vee_{n\in\mathbb{N}} P_{E_n}^{(A)} = P_{\cup_{n\in\mathbb{N}} E_n}^{(A)}$$

for any finite or countable collection $\{E_n\}_{n\in\mathbb{N}}$ of Borel sets of $\mathbb{R}$. Although one could also take sets of arbitrary cardinality, we will stop at countable, as we did in the classical case.

Definition 7.42 If H is a Hilbert space, a function A mapping $E \in \mathscr{B}(\mathbb{R})$ to an orthogonal projector $P_E^{(A)} \in \mathscr{L}(\mathsf{H})$ is called an **observable** if:
(a) $P_E^{(A)} P_{E'}^{(A)} = P_{E'}^{(A)} P_E^{(A)}$ for any $E, E' \in \mathscr{B}(\mathbb{R})$;
(b) $P_E^{(A)} \wedge P_{E'}^{(A)} = P_{E\cap E'}^{(A)}$ for any $E, E' \in \mathscr{B}(\mathbb{R})$;
(c) $P_{\mathbb{R}}^{(A)} = I$;
(d) for any family $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{R})$:

$$\vee_{n\in\mathbb{N}} P_{E_n}^{(A)} = P_{\cup_{n\in\mathbb{N}} E_n}^{(A)} .$$

Remark 7.43 **(1)** It is straightforward to see that $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$ is a Boolean σ-algebra for the usual partial order relation $\leq$ of projectors. In general $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$ is not maximal commutative.
(2) Bearing in mind Definition 7.13 it is easy to prove that an observable is nothing but a *homomorphism of Boolean σ-algebras*, mapping the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ to the Boolean σ-algebra of projectors $\{P_E^{(A)}\}_{E\subset\mathscr{B}(\mathbb{R})}$. It can be proved ([Jau73, Chaps. 5–6]) that if H is a separable Hilbert space, any subset of projectors in $\mathscr{L}(\mathsf{H})$ forming a Boolean σ-algebra is automatically an observable, i.e. of the form $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$, and satisfies Definition 7.42. ■

Observables may be redefined in an equivalent way, but mathematically simpler, as the next proposition shows.

Proposition 7.44 *Let H be a Hilbert space. A map $P : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ is an observable if and only if the following hold.*
(a) *$P(B) \geq 0$ for any $B \in \mathscr{B}(\mathbb{R})$;*
(b) *$P(B)P(B') = P(B \cap B')$ for any $B, B' \in \mathscr{B}(\mathbb{R})$;*
(c) *$P(\mathbb{R}) = I$;*
(d) *for any family $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{R})$ with $B_n \cap B_m = \varnothing$ if $n \neq m$:*

$$s\text{-}\sum_{n=0}^{+\infty} P(B_n) = P(\cup_{n\in\mathbb{N}} B_n) \,.$$

Proof If $P : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ is an observable properties (a), (b), (c), (d) are trivially true. So we have to prove that any $P : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ satisfying them is an observable.
Let us collect all operators $P(B)$ with $B \in \mathscr{B}(\mathbb{R})$ in one maximal set of commuting projectors $\mathscr{L}_0(\mathsf{H})$ (which exists by Zorn's lemma), and from now on we shall work in it without loss of generality.
(a) says that every operator $P(B)$ is self-adjoint by Proposition 3.60(f), so (b) implies $P(B)P(B) = P(B \cap B) = P(B)$, whence every $P(B)$ is an orthogonal projector. Moreover (b) implies, if $P(B)P(B') = P(B \cap B') = P(B' \cap B) = P(B')P(B)$, that all projectors commute with one another. Using the first identity in (i) of Theorem 7.22(b), condition (b) above reads $P(B) \wedge P(B') = P(B \cap B')$. To finish we need to show property (d) of Definition 7.42. Consider countably many sets $\{E_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{R})$, in general not disjoint. We want to find $\vee_{n\in\mathbb{N}} P(E_n)$ and prove that

$$\vee_{n\in\mathbb{N}} P(E_n) = P(\cup_{n\in\mathbb{N}} E_n) \,.$$

To do so define a collection $\{B_n\}_{n\in\mathbb{N}}$ of pairwise disjoint Borel sets: $B_0 := E_0$ and

$$B_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$$

for $n > 0$. Then

$$\cup_{n=0}^{p} E_n = \cup_{n=0}^{p} B_n \quad \text{for any } p \in \mathbb{N} \cup \{+\infty\}.$$

From this, using $I - P(B) = P(\mathbb{R} \setminus B)$ and the second identity in (i) of Theorem 7.22(b) recursively, we find

$$\vee_{n=0}^{p} P(E_n) = \vee_{n=0}^{p} P(B_n) \quad \text{for any } n \in \mathbb{N}.$$

As part (d) of the present proposition implies

$$\vee_{n=0}^{p} P(B_n) = \sum_{n=0}^{p} P(B_n)$$

for finitely many disjoint B_n (this collection may be made countable by adding infinitely many empty sets), we have

$$\vee_{n=0}^{p} P(E_n) = \sum_{n=0}^{p} P(B_n) \,. \tag{7.33}$$

To conclude we take the strong limit as $p \to +\infty$ in (7.33). This exists by Theorem 7.22(b), and we also have

$$\vee_{n\in\mathbb{N}} P(E_n) = \text{s-}\lim_{p\to+\infty} \sum_{n=0}^{p} P(B_n) = P(\cup_{n\in\mathbb{N}} B_n) = P(\cup_{n\in\mathbb{N}} E_n) \,.$$

□

Remark 7.45 **(1)** Notice that (c) and (d) alone imply $I = P(I \cup \varnothing) = I + P(\varnothing)$, so

$$P(\varnothing) = 0 \,.$$

(2) If $B \in \mathscr{B}(\mathbb{R})$ then $\mathbb{R} \setminus B \in \mathscr{B}(\mathbb{R})$ and $\mathbb{R} = B \cup (\mathbb{R} \setminus B)$. By (d), taking $B_0 = B$, $B_1 = \mathbb{R} \setminus B$ and all remaining $B_k = \varnothing$, we obtain $I = P(B) + P(\mathbb{R} \setminus B)$. Therefore

$$\neg P(B) = P(\mathbb{R} \setminus B) \,.$$

■

The above proposition sets up a 1-1 correspondence between observables and well-known objects in mathematics, namely *projector-valued measures* on $\mathbb{R}$. The latter will be generalised in the next chapter.

Definition 7.46 A map $P : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$, H a Hilbert space, satisfying (a), (b), (c) and (d) in Proposition 7.44 is called a **projector-valued measure (PVM) on** $\mathbb{R}$ or **spectral measure on** $\mathbb{R}$.

We are finally in the position to disclose the fourth axiom of the general mathematical formulation of Quantum Mechanics.

A4. *Every observable A on a quantum system S is described by a projector-valued measure $P^{(A)}$ on $\mathbb{R}$ in the Hilbert space H_S of the system. If E is a Borel set in $\mathbb{R}$, the projector $P^{(A)}(E)$ corresponds to the proposition "the reading of a measurement of A falls in the Borel set E".*

Remark 7.47 **(1)** Let us suppose, owing to a superselection rule (see Sect. 7.7), that the Hilbert space splits into coherent sectors $\mathsf{H}_S = \oplus_{k\in K}\mathsf{H}_{Sk}$. Call P_k the orthogonal projector onto H_{Sk}. From Sect. 7.7.1, every projector $P_E^{(A)}$ of an observable A satisfies $P_k P_E^{(A)} = P_E^{(A)} P_k$ for any $k \in K$ and any Borel set $E \subset \mathbb{R}$.

(2) We say that **the observable** B **is a function of the observable** A, written $B = f(A)$, when there is a measurable map $f : \mathbb{R} \to \mathbb{R}$ such that $P^{(B)}(E) = P^{(A)}(f^{-1}(E))$ for any Borel set $E \subset \mathbb{R}$. This is totally natural: if "$B = f(A)$" then to measure B we can measure A and use f on the reading. In this sense the outcome of measuring B belongs to E iff the outcome of A belongs in $f^{-1}(E)$. In particular, the elementary propositions (orthogonal projectors) $P_E^{(B)}$ and $P_F^{(A)}$ are always compatible, and $\{P_E^{(B)}\}_{E\in\mathscr{B}(\mathbb{R})} \subset \{P_F^{(A)}\}_{F\in\mathscr{B}(\mathbb{R})}$. It is possible to prove [Jau73] that for given observables A, B in a separable Hilbert space, the previous inclusion is equivalent to the existence of a measurable map f such that $B = f(A)$. More important is a result of von Neumann and Varadarajan [Jau73, Chaps. 6–7] (valid for any orthocomplemented, σ-complete, separable lattice, not necessarily the projector lattice of a Hilbert space):

Theorem 7.48 *If $\{A_j\}_{j\in J}$ is a family of pairwise-compatible observables (that is, $P^{(j)}(E)P^{(i)}(F) = P^{(i)}(F)P^{(j)}(E)$ if $P^{(j)}(E) \in A_j$, $P^{(i)}(F) \in A_i$) on a separable Hilbert space, there exists an observable A and a corresponding family of measurable maps $f_j : \mathbb{R} \to \mathbb{R}$, $j \in J$ such that $A_j = f_j(A)$ for any $j \in J$.*

■

7.5.2 *Self-adjoint Operators Associated to Observables: Physical Motivation and Basic Examples*

This section contains the idea underpinning the correspondence between self-adjoint operators and observables. We will, in other words, provide the physical motivation for the spectral theorems of Chaps. 8 and 9.

For classical systems, at time t on phase space $\mathscr{F}$, we know that observables correspond to what have been called physical quantities, i.e. measurable maps

$f : \mathscr{F} \to \mathbb{R}$. To any physical quantity f we can associate the collection of all propositions/Borel sets of the form:

$$P_E^{(f)} :=$$

"The value of f on the system's state belongs to the Borel set $E \subset \mathbb{R}$",

or, set-theoretically,

$$P_E^{(f)} := f^{-1}(E) \in \mathscr{B}(\mathscr{F}) .$$

Propositions 7.18 and 7.19 told us that $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$ is a (Boolean) σ-algebra and the map $\mathscr{B}(\mathbb{R}) \ni E \mapsto P_E^{(f)} \in \mathscr{B}(\mathscr{F})$ a Boolean σ-algebra homomorphism. The picture is the same in the quantum case when we look at the class $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$ of propositions/projectors associated to an observable A: they form a Boolean σ-algebra and $\mathscr{B}(\mathbb{R}) \ni E \mapsto P_E^{(A)} \in \mathscr{L}(\mathsf{H})$ is a homomorphism of Boolean σ-algebras. If we compare $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$ and $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$ the situation is analogous. In the classical case, though, *there exists a function* f consenting to *build* the collection $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$: this map retains, alone, all possible information about the propositions $P_E^{(f)}$. This is no surprise since we defined propositions/sets starting from f! In the quantum case, when an observable $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$ is given, we have nothing, at least at present, that may correspond to a function f "generating" the PVM $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$. *So is there a quantum analogue to f?*

In order to answer the question we must dig deeper into the relationship between f and the associated family $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$. We know how to get the latter out of the former, but now we are interested in recovering the map from the family, because in the quantum formulation one starts from the analogue of $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$. As a matter of fact the σ-algebra $\{P_E^{(f)}\}_{E\in\mathscr{B}(\mathbb{R})}$ allows to *reconstruct* f by means of a certain limiting process reminiscent of *integration*.

To explain this point we need a technical result. Recall that if (X, Σ) is a measure space, a Σ-measurable map $s : \mathsf{X} \to \mathbb{C}$ is **simple** if its range is finite.

Proposition 7.49 *Let (X, Σ) be a measure space, $S(\mathsf{X})$ the space of complex-valued simple functions with respect to Σ, $M(\mathsf{X})$ the space of $\mathbb{C}$-valued, Σ-measurable maps, and $M_b(\mathsf{X}) \subset M(\mathsf{X})$ the subspace of bounded maps. Then*
(a) *$S(\mathsf{X})$ is dense in $M(\mathsf{X})$ pointwise.*
(b) *$S(\mathsf{X})$ is dense in $M_b(\mathsf{X})$ in norm $|| \; ||_\infty$.*
(c) *If $f \in M(\mathsf{X})$ ranges over non-negative reals, there is a sequence $\{s_n\}_{n\in\mathbb{N}} \subset S(\mathsf{X})$ with:*

$$0 \le s_0 \le s_1 \le \cdots \le s_n(x) \to f(x) \quad \textit{as } n \to +\infty, \textit{for any } x \in \mathsf{X}.$$

Furthermore, the convergence is in norm $|| \; ||_\infty$ as well, provided $f \in M_b(\mathsf{X})$.

Proof It is enough to prove the claim for real-valued maps, for the complex case is a consequence of decomposing complex functions into real and imaginary parts. Define $f_+(x) := \sup\{0, f(x)\}$ and $f_-(x) := \inf\{0, f(x)\}$, $x \in \mathsf{X}$; then $f = f_+ +$

f_-, where $f_+ \geq 0$, $f_- \leq 0$ are known to be measurable since f is. Now we construct a sequence of simple maps converging to f_+ (whence part (c) is proven, as $f = f_+$ if $f \geq 0$). For given $0 < n \in \mathbb{N}$ let us partition the real semi-axis $[0, +\infty)$ into Borel sets:

$$E_{n,i} := \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) , \qquad E_n := [n, +\infty) ,$$

$1 \leq i \leq n2^n$. For each n let

$$P^{(f)}_{n,i} := f^{-1}\left(E_{n,i}\right) , \qquad P^{(f)}_n := f^{-1}(E_n)$$

be subsets in Σ. Then define $s_0(x) := 0$ if $x \in \mathsf{X}$, and

$$s_n := \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{P^{(f)}_{n,i}} + n\chi_{P^{(f)}_n} . \tag{7.34}$$

for any $n \in \mathbb{N} \setminus \{0\}$. By construction $0 \leq s_n \leq s_{n+1} \leq f, n = 1, 2, \ldots$. Furthermore, for any given x we have $|f_+(x) - s_n(x)| \leq 1/2^n$ definitely. Evidently, then, $s_n \to f_+$ pointwise if $n \to +\infty$. The estimate $|f_+(x) - s_n(x)| \leq 1/2^n$ is uniform in x if f_+ is bounded (take $n > \sup f_+$), and then $s_n \to f_+$ uniformly, too. Similarly, by partitioning the negative semi-axis we may construct another simple sequence $\{s_n^{(-)} \leq 0\}$ tending to f_- pointwise. Overall, the simple sequence $s_n^{(-)} + s_n$ converges to f pointwise, and uniformly if f is additionally bounded. □

Remark 7.50 If f is non-negative, part (a) still holds even when $f : \mathsf{X} \to [0, +\infty]$, by taking simple maps that attain the value $+\infty$. ■

It is clear thus that a given classical quantity $f : \mathscr{F} \to \mathbb{R}$ (measurable) can be recovered using a sequence of maps that are constant non-zero only on sets in $\{P^{(f)}_E\}_{E\in\mathscr{B}(\mathbb{R})}$. Without loss of generality we focus on the situation $f : \mathscr{F} \to \mathbb{R}_+$ and further suppose f bounded. This entitles us to forget, in (7.34) and for n large enough: (i) *all intervals* E_n and (ii) the $E_{n,i}$ with left endpoint $((i-1)2^{-n})$ larger, say, than $(\sup f) + 1/2^n$, for the pre-image of these sets under f is empty. If we do so the sum in (7.34) can be truncated:

$$f = \lim_{n\to+\infty} \sum_{i=1}^{2+2^n \sup f} \frac{i-1}{2^n} \chi_{P^{(f)}_{n,i}} . \tag{7.35}$$

This limit may be understood as an *integration* of sorts with respect to a "measure with values on characteristic functions":

$$\nu^{(f)} : \mathscr{B}(\mathbb{R}) \ni E \mapsto \chi_{f^{-1}(E)} \in S(\mathsf{X}),$$

associating to a Borel subset $E \subset \mathbb{R}$ (in the range of the map) a characteristic function $\chi_{f^{-1}(E)} : \mathsf{X} \to \mathbb{C}$. Observe, in fact, that $\frac{i-1}{2^n}$ is approximately the value f assumes at

$P^{(f)}_{n,i}$ – the estimate becomes more accurate as n increases – and the right-hand side in (7.35) is just a "Cauchy sum". Equation (7.35) might be formally written as:

$$f = \int_{\mathbb{R}} \lambda d\nu^{(f)}(\lambda) \,. \tag{7.36}$$

But as we are concerned with the quantum setting, we will not push the analogy further, even though doing that would give a rigorous meaning to the above integral. In such case the similar formula to (7.36) is:

$$A = \int_{\mathbb{R}} \lambda dP^{(A)}(\lambda)$$

where the characteristic functions $\chi_{f^{-1}(E)}$ have been replaced by the orthogonal projectors $P^{(A)}_E$ of the observable A. This relation defines a *self-adjoint operator A* associated to an observable, that was called A and that corresponds to the classical quantity f. From such an operator the observable $\{P^{(A)}_E\}_{E\in\mathscr{B}(\mathbb{R})}$ can be recovered, a posteriori, in a similar manner to what we do to get $\{P^{(f)}_E\}_{E\in\mathscr{B}(\mathbb{R})}$ out of f. We will see all this in full generality, and rigorously, in the sequel (Chaps. 8 and 9). At this juncture we shall describe an elementary example of observable and show how to associate to it a self-adjoint operator.

Examples 7.51 **(1)** Consider a quantum system described on a Hilbert space H, and take a quantity ranging, from the point of view of physics, over a discrete and finite set of *distinct* values $\{a_n\}_{n=1,\cdots,N} \subset \mathbb{R}$. We first show how to find an observable A given by a family of orthogonal projectors $P^{(A)}_E$, $E \in \mathscr{B}(\mathbb{R})$. We posit that there are *non-null* orthogonal projectors labelled by a_n, $\{P_{a_n}\}_{n=1,\cdots,N}$, such that $P_{a_n}P_{a_m} = 0$ if $n \neq m$ (i.e., taking adjoints, $P_{a_m}P_{a_n} = 0$ if $n \neq m$), and moreover:

$$\sum_{n=1}^{N} P_{a_n} = I \,. \tag{7.37}$$

The meaning of P_{a_n}, clearly, is:
"the value of A, read by a measurement on the system, is precisely a_n".
Obviously the equations $P_{a_n}P_{a_m} = P_{a_m}P_{a_n} = 0$, i.e. $P_{a_n} \wedge P_{a_m} = 0$ for $n \neq m$ correspond to two physical requirements: (a) the propositions P_{a_n}, P_{a_m} are physically compatible, but (b) the observable's measurement cannot produce distinct values a_n and a_m simultaneously (the proposition associated to the null projector is contradictory). Demanding $\sum_{n=1}^{N} P_{a_n} = I$, i.e. $P_{a_1} \vee \cdots \vee P_{a_n} = I$, amounts to asking that at least one proposition P_{a_n} is true when measuring A. The observable $A : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathsf{H})$ is built as follows: for any Borel set $E \subset \mathbb{R}$

$$P_E^{(A)} := \sum_{a_n \in E} P_{a_n} , \qquad \text{with } P_{\varnothing}^{(A)} := 0 . \tag{7.38}$$

Properties (a), (b), (c) and (d) in Proposition 7.44 are immediate.
(2) Referring to example (1), to the observable A we can associate an operator still called A:

$$A := \sum_{n=0}^{N} a_n P_{a_n} . \tag{7.39}$$

This A is bounded and self-adjoint by construction, being a real linear combination of self-adjoint operators. It has another interesting property: *the eigenvalue set $\sigma_p(A)$ of A coincides with the values the observable A can assume*.

The proof is direct: if $0 \neq u \in P_{a_n}(\mathsf{H})$ then $P_{a_m} u = P_{a_m} P_{a_n} u = u$ if $n = m$ or 0 if $n \neq m$. Inserting this in (7.39) gives $Au = a_n u$, so $a_n \in \sigma_p(A)$. Conversely, if $u \neq 0$ is a λ-eigenvector of A (λ real since $A = A^*$), then (7.39) implies

$$\lambda u = \sum_{n=0}^{N} a_n P_{a_n} u .$$

On the other hand, since $\sum_{a_n} P_{a_n} = I$ we obtain

$$\sum_{n=0}^{N} \lambda P_{a_n} u = \sum_{n=0}^{N} a_n P_{a_n} u ,$$

hence

$$\sum_{n=0}^{N} (\lambda - a_n) P_{a_n} u = \mathbf{0} . \tag{7.40}$$

Now apply P_m and recall $P_m P_n = \delta_{m,n} P_n$, resulting in N identities:

$$(\lambda - a_m) P_{a_m} u = \mathbf{0} .$$

If all of them were solved by $P_m u = \mathbf{0}$ for any m, we would obtain a contradiction, because

$$\mathbf{0} \neq u = Iu = \sum_{n=0}^{N} P_{a_n} u .$$

Therefore there must be some n in (7.40) for which $\lambda = a_n$. This can happen for one value n only, since by assumption the a_n are distinct. Overall the eigenvalue λ of A must be one particular a_n. So we proved that the eigenvalue set of A coincides with the values A can assume. The self-adjoint operator A here plays a role similar to that of f for the classical quantity $\{P_E^{(f)}\}_{E \in \mathscr{T}(\mathbb{R})}$.

(3) Suppose A is the operator of an observable in the sense of (7.39), and $g : \mathbb{R} \to \mathbb{R}$ is a map. We can define a new observable, as if it were a function of the previous one, determined entirely by the self-adjoint operator

$$C := g(A) := \sum_{n=0}^{N} g(a_n) P_{a_n} . \tag{7.41}$$

By construction the possible values of the new observable are the images $g(a_n)$, that in turn determine the eigenvalues of C. ■

In the next chapters we will develop a procedure for associating to each observable A (i.e. a PVM on $\mathbb{R}$) a unique self-adjoint operator (typically unbounded) denoted by the same letter A, thereby generalising the previous examples. The values the observable can take will be elements in the *spectrum* $\sigma(A)$, which is normally larger than the set $\sigma_p(A)$ of eigenvalues. The major tool will be the integration with respect to a projector-valued measure, corresponding to a generalisation of

$$\sum_{\lambda \in \sigma_p(A)} h(\lambda) P_\lambda =: \int_{\sigma(A)} h(\lambda)\, dP^{(A)}(\lambda)$$

to the case when the λ can be infinite. In particular

$$A = \int_{\sigma(A)} \lambda\, dP^{(A)}(\lambda) , \quad I = \int_{\sigma(A)} 1\, dP^{(A)}(\lambda),$$

whose interpretation befits the theory of spectral measures.

7.5.3 Probability Measures Associated to Couples State/Observable

Here is yet another remarkable property about PVMs on $\mathbb{R}$, with important consequences in physics.

Proposition 7.52 *Let* H *be a Hilbert space and* $A = \{P_E\}_{E \in \mathscr{B}(\mathbb{R})}$ *a projector-valued measure on* $\mathbb{R}$. *If* $\rho \in \mathfrak{S}(\mathsf{H})$ *is a state, the map* $\mu_\rho^{(A)} : E \mapsto tr(\rho P_E)$ *is a Borel probability measure on* $\mathbb{R}$.

Proof The proof is elementary. It suffices to show $\mu_\rho^{(A)}$ is positive, σ-additive and $\mu_\rho^{(A)}(\mathbb{R}) = 1$. As $\mathbb{R}$ is Hausdorff and locally compact, every positive σ-additive measure on the Borel algebra is a Borel measure. Decompose ρ in the usual way with an eigenvector basis:

$$\rho = \sum_{j \in \mathbb{N}} p_j (\psi_j |\ )\psi_j ,$$

where the p_j are non-negative and their sum is 1. Then $\mu_\rho^{(A)}(E) = tr(\rho P_E) \geq 0$ because orthogonal projectors are positive, $P_j \geq 0$ and $tr(\rho P_E) = \sum_{j\in\mathbb{N}} p_j(\psi_j | P_E \psi_j)$. Moreover $\mu_\rho^{(A)}(\mathbb{R}) = 1$, since $P_{\mathbb{R}} = I$ implies

$$\sum_{j\in\mathbb{N}} p_j(\psi_j|I\psi_j) = tr\rho = 1 .$$

Let us show σ-additivity. If $\{E_n\}_{n\in\mathbb{N}}$ are pairwise-disjoint Borel sets and $E := \cup_{n\in\mathbb{N}} E_n$, by Proposition 7.44(d):

$$+\infty > tr(\rho P_E) = \sum_{j=0}^{+\infty} p_j \left(\psi_j \left| \sum_{i=0}^{+\infty} P_{E_i}\psi_j \right.\right) = \sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} p_j(\psi_j|P_{E_i}\psi_j) .$$

Since $p_j \geq 0$ and $(\psi_j|P_{E_i}\psi_j) \geq 0$, Fubini's theorem allows to swap the series:

$$tr(\rho P_E) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_j(\psi_j|P_{E_i}\psi_j) = \sum_{i=0}^{+\infty} tr(\rho P_{E_i}) .$$

That is to say, if $\{E_n\}_{n\in\mathbb{N}}$ are pairwise-disjoint Borel sets then

$$\mu_\rho^{(A)}(\cup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{+\infty} \mu_\rho^{(A)}(E_n) ,$$

ending the proof. $\square$

Examples 7.53 **(1)** The observable A of (1) and (2) in Example 7.51 assumes a finite number N of discrete values a_n. Let A (cf. 7.39) also denote the self-adjoint operator of the observable. Fix a state $\rho \in \mathfrak{S}(\mathsf{H})$ and consider its probability measure relative to the observable $\{P_E^{(A)}\}_{E\in\mathscr{B}(\mathbb{R})}$. By construction, if $E \in \mathscr{B}(\mathbb{R})$:

$$\mu_\rho^{(A)}(E) := tr(\rho P_E^{(A)}) = \sum_{a_n\in E} tr(\rho P_{a_n}) = \sum_{a_n} p_n \delta_{a_n}(E)$$

with

$$p_n := tr(\rho P_{a_n}) .$$

Hence

$$\mu_\rho^{(A)} = \sum_{a_n} p_n \delta_{a_n} , \tag{7.42}$$

where δ_{a_n} are Dirac measures centred at a_n: $\delta_a(E) = 1$ if $a \in E$, $\delta_a(E) = 0$ if $a \notin E$. Note $0 \leq p_n \leq 1$ and $\sum_n p_n = 1$ by construction. Thus the probability measure associated to the state ρ and relative to A is a convex combination of Dirac measures.

(2) The mean value $\langle A\rangle_\rho$, of A and its standard deviation ΔA^2_ρ, on state ρ, can be written succinctly using the associated operator A of (7.39). By definition of mean value

$$\langle A\rangle_\rho = \int_{\mathbb{R}} a\, d\mu^{(A)}_\rho(a)\,.$$

On the other hand, by (7.42) we have

$$\int_{\mathbb{R}} a\, d\mu^{(A)}_\rho(a) = \sum_n p_n a_n = \sum_n a_n tr(\rho P_{a_n})\,.$$

Using (7.39) and the linearity of the trace, we conclude

$$\langle A\rangle_\rho = tr(A\rho)\,. \tag{7.43}$$

In case ρ is pure, i.e. $\rho = \psi(\psi|\cdot)$, $||\psi|| = 1$, (7.43) implies

$$\langle A\rangle_\psi = (\psi|A\psi)\,, \tag{7.44}$$

where $\langle A\rangle_\psi$ indicates the mean value of A on the state of the vector ψ. By definition the deviation equals

$$\Delta A^2_\rho = \int_{\mathbb{R}} a^2\, d\mu^{(A)}_\rho(a) - \langle A\rangle^2_\rho\,.$$

Proceeding as before,

$$\int_{\mathbb{R}} a^2\, d\mu^{(A)}_\rho(a) = \sum_n p_n a_n^2 = \sum_n a_n^2 tr(\rho P_{a_n}) = tr\left(\rho \sum_n a_n^2 P_{a_n}\right)\,.$$

Now observe

$$A^2 = \sum_n a_n P_{a_n} \sum_m a_m P_{a_m} = \sum_{n,m} a_n a_m P_{a_n} P_{a_m} = \sum_n a_n^2 P_{a_n}\,,$$

where we used $P_{a_n} P_{a_m} = \delta_{n,m} P_n$. Therefore

$$\Delta A^2_\rho = tr\left(\rho A^2\right) - \left(tr\left(\rho A\right)\right)^2\,. \tag{7.45}$$

If ρ is a pure state, i.e. $\rho = \psi(\psi|\cdot)$, $||\psi|| = 1$, we have, from (7.45),

$$\Delta A^2_\psi = \left(\psi|A^2\psi\right) - (\psi|A\psi)^2 = \left(\psi|\left(A^2 - \langle A\rangle^2_\psi\right)\psi\right)\,, \tag{7.46}$$

where ΔA^2_ψ is the standard deviation of A on the state determined by the vector ψ. ∎

The formulas above are actually valid, under suitable technical assumptions, in a broader context. This will be proved in Proposition 11.27, after we show in full generality the procedure for associating self-adjoint operators to observables.

7.6 More Advanced, Foundational and Technical Issues

In this section we shall mainly focus on a number of foundational issues concerning the nature of the quantum lattice of elementary propositions and on historically related technical result, such as the direct decomposition into von Neumann algebra of definite type. The last subsection will characterise the space of states $\mathfrak{S}(\mathsf{H})$ and will have a purely mathematical flavour.

Foundational studies on the role of the projector lattice, in relationship to the logical formulation of QM, are found in [Mac63, Jau73, Pir76, BeCa81, Red98, DCGi02, Var07, EGL09] besides [Bon97]. The reader can read about a different approach in [Emc72]: this book is based on *Jordan algebras* and prepares for the *algebraic formulation* following ideas of Segal.

7.6.1 Recovering the Hilbert Space from the Lattice: The Theorems of Piron and Solèr

A reasonable question to ask is whether there are more cogent reasons for choosing to describe quantum systems via a projector lattice, other than the kill-all argument "it works". To tackle the problem we shall need certain special properties of the lattice of projectors. We start with some abstract definitions.

Definition 7.54 In a lattice $(\mathsf{X}, \geq)$, we say that $a \in \mathsf{X}$ **covers** $b \in \mathsf{X}$ if $a \geq b$, $a \neq b$, and $a \geq c \geq b$ implies either $c = a$ or $c = b$.

In a bounded lattice $(\mathsf{X}, \geq)$, an element $a \in \mathsf{X} \setminus \{\mathbf{0}\}$ is called an **atom** if $\mathbf{0} \leq r \leq a \Rightarrow r = \mathbf{0}$ or $r = a$.

An orthocomplemented lattice $(\mathsf{X}, \geq)$ is said:
(a) separable if any collection $\{r_j\}_{j\in A} \subset \mathsf{X}$ of orthogonal elements, $r_i \perp r_j$, $i \neq j$, is countable at most;
(b) atomic if for any $r \in \mathsf{X} \setminus \{\mathbf{0}\}$ there exists an atom $a \leq r$;
(b)' atomistic if it is atomic and every $r \in \mathsf{X} \setminus \{\mathbf{0}\}$ is the join of the atoms $a \leq r$;
(c) to satisfy the covering property if, for any $p \in \mathsf{X}$ and any atom a, then $a \wedge p = \mathbf{0} \Rightarrow a \vee p$ covers p;
(d) irreducible if the centre of X only contains $\mathbf{0}$ and $\mathbf{1}$.

Remark 7.55 It is easy to prove two atoms a, b in an orthocomplemented lattice commute if and only if either $a \perp b$ or $a = b$. ■

The following result holds for the projection lattice $\mathscr{L}(\mathsf{H})$.

Theorem 7.56 *The orthocomplemented, complete lattice $\mathscr{L}(\mathsf{H})$ of Theorem 7.22 is*

(i) separable $\Leftrightarrow$ H *is separable,*

(ii) atomic *and* atomistic*,*

(iii) orthomodular*;*

(iv) it satisfies the covering property*, and is*

(v) irreducible.

The atoms in $\mathscr{L}(\mathsf{H})$ are the orthogonal projectors onto one-dimensional subspaces.

Proof (i) If H is separable, every set of orthogonal projectors $\{P_j\}_{j\in J}$ with $P_j \perp P_h$, i.e. $P_j P_h = 0$, if $h \neq j$ must be finite or countable. Having an uncountable set of pairwise-orthogonal vectors (one could choose one in each $P_j(\mathsf{H})$) is forbidden by Theorem 3.30, for maximal sets of orthonormal vectors in a separable Hilbert space are finite or countable. If $\mathscr{L}(\mathsf{H})$ is separable, the set N of orthogonal projectors P_u associated to the elements of a Hilbert basis $U \ni u$ of H satisfy $P_u \perp P_v$ if $u \neq v$, and so N must be finite or countable. Then H is separable by Theorem 3.30.
(ii) It is easy to prove that the atoms are the orthogonal projectors onto one-dimensional subspaces just by applying the definition. Hence $\mathscr{L}(\mathsf{H})$ is atomic. Let us prove that it is atomistic. If $P \in \mathscr{L}(\mathsf{H})$ projects onto M and U is a (Hilbert) basis of M, evidently $\sup_{u\in U} P_u = P$, where P_u is the orthogonal projector onto the space generated by u. If A_M is the set of atoms $Q \leq P$, then $\sup_{Q\in A_\mathsf{M}} Q \geq \sup_{u\in U} P_u$, just because $P_u \in A_\mathsf{M}$. On the other hand, (a) in Theorem 7.22(i) implies that $\sup_{Q\in A_\mathsf{M}} Q$ projects onto $\overline{< \{Q(\mathsf{H})\}_{Q\in A_\mathsf{M}} >}$, which is contained in M because closure of finite combinations of elements in M, and M is closed. Hence $\sup_{Q\in A_\mathsf{M}} Q \leq \sup_{u\in U} P_u$. Summing up, $\sup_{Q\in A_\mathsf{M}} Q = \sup_{u\in U} P_u = P$, which means $\mathscr{L}(\mathsf{H})$ is atomistic.
(iii) We must prove that $P \leq Q \Rightarrow Q = P \vee ((I - P) \wedge Q)$ for any orthogonal projectors P, Q. To this end, by Proposition 7.20 $P \leq Q \Rightarrow PQ = QP = P$. Therefore, again by Proposition 7.20 the identity to prove is $Q = P + (I - P)Q - P(I - P)Q$. But that is trivial, by direct inspection.
(iv) If $A, P \in \mathfrak{B}(\mathsf{H})$ and A is an atom, asking $A \wedge P = 0$ is equivalent to saying that $A(\mathsf{H})$ is a one-dimensional subspace not contained in the closed subspace $\mathsf{M} := P(\mathsf{H})$. Hence $A \vee P$ projects onto a subspace of dimension one more than $P(\mathsf{H})$. We can write $A \vee P(\mathsf{H}) = \mathsf{M} \oplus < n >$ for some unit vector $n \perp \mathsf{M}$. The covering property is then equivalent to the obvious fact that if $\mathbf{0} \neq n \perp \mathsf{M}$, there are no closed subspaces N with $\mathsf{M} \subsetneq \mathsf{N} \subsetneq \mathsf{M} \oplus < n >$.
(v) Irreducibility can be proved using the same proof of Proposition 3.93(a): $A \in \mathfrak{B}(\mathsf{H})'$ can be replaced by $A \in \mathscr{L}(\mathsf{H})'$ to give the same final result, namely that $A = cI$ for some $c \in \mathbb{C}$. Since at present $A \in \mathscr{L}(\mathsf{H})$, so that $A = A^*$ and $AA = A$, we must have either $A = 0$ or $A = I$. In other words, the centre of $\mathscr{L}(\mathsf{H})$ is $\{0, I\}$ and $\mathscr{L}(\mathsf{H})$ is irreducible. $\square$

Remark 7.57 **(1)** *Atomicity* and *atomisticity* necessarily coalesce under orthomodularity:

Proposition 7.58 *An orthomodular lattice is atomic if and only if it is atomistic.*

Proof See Exercise 7.6 □

(2) An orthocomplemented, atomistic lattice with the covering property is sometimes called an **AC lattice**. In an atomic orthomodular lattice X, the covering property is equivalent to each of the following statements [BeCa81].
(i) For every $a \in \mathsf{X}$ and for every pair of atoms $p, q \in \mathsf{X}$ with $p \not\leq a$ and $q \not\leq a$, the relation $p \leq a \vee q$ implies $q \leq a \vee p$.
(ii) For every $a \in \mathsf{X}$ and every atom $p \in \mathsf{X}$ with $p \not\leq a^{\perp}$, the element (called **Sasaki projection of** p **onto** a) $a \wedge (p \vee a^{\perp})$ is an atom. ■

All the properties of Theorem 7.56 admit an operational interpretation (e.g. see [BeCa81]). Based on the experimental evidence of quantum systems, therefore, we might try to prove, in absence of an Hilbert space, that the elementary propositions about a quantum system with experimental outcome in $\{0, 1\}$ form a poset. For that we would first need a bounded, orthocomplemented, complete lattice that verifies conditions (i)–(v) above. Then we should prove that this lattice is described by the orthogonal projectors of a Hilbert space. The second step is an entire, and challenging, programme, known as the *coordinatisation problem* [BeCa81]: it was initiated by several researchers in the 1960s and completed by M.P. Solèr in 1995, as we shall see below.

Although the partial order of elementary propositions can be defined in various ways, it will always correspond to the logical implication, in some way or another. Starting from [Mac63] a number of approaches (either of physical nature, essentially, or formal) have been developed toward this end: among them are those making use of the notion of (quantum) *state*. The objects are now [Mac63] pairs $(\mathscr{O}, \mathscr{S})$, where $\mathscr{O}$ is the class of observables and $\mathscr{S}$ are the states. Elementary propositions form a subclass $\mathscr{L}$ of $\mathscr{O}$ equipped with a natural poset structure $(\mathscr{L}, \leq)$ (and satisfying weakened versions of conditions (i)–(v)). A state $s \in \mathscr{S}$, in particular, defines the probability $m_s(P)$ that P is true for every $P \in \mathscr{L}$ [Mac63]. As a matter of fact, if $P, Q \in \mathscr{L}$, $P \leq Q$ is defined to mean $m_s(P) \leq m_s(Q)$ for every state $s \in \mathscr{S}$. A different, and apparently weaker, interpretation due to Jauch appears in [Jau68]: there $P \leq Q$ means that if P is true in a state s, also Q is true in s. More difficult is to justify that the poset thus obtained is a lattice, i.e. that it admits greatest lower bound $P \vee Q$ and least upper bound $P \wedge Q$ for any P, Q. Nowadays there exist several proposals, very different in character, to justify the lattice structure. See Aerts in [EGL09] for a recent operational viewpoint, and [BeCa81, Sect. 21.1] for a summary on several possible ways to introduce the lattice structure on the poset of abstract elementary propositions $\mathscr{L}$. If we accept the lattice structure on elementary propositions of a quantum system, then we may define orthocomplementation by the familiar logical/physical negation. Compatible propositions can then be defined in terms of commuting propositions as of Definition 7.8 (by (v) in Theorem 7.22(a) this notion of compatibility is the usual one when propositions are interpreted via projectors). Now fully-fledged with an orthocomplemented lattice of elementary propositions and the notion of compatible propositions, we can attach a physical meaning (an interpretation backed by experimental evidence) to the requests that the lattice be orthocomplemented, complete, atomic, irreducible, and to the covering property [BeCa81]. Under these hypotheses, and supposing there exist at least 4 pairwise-orthogonal atoms, Piron ([Pir64, JaPi69, Pir76], [BeCa81, Chap. 21], Aerts in [EGL09]) used projective geometry techniques to show that the lattice of quantum propositions can be identified canonically with a certain class of subspaces in a generalised vector space of sorts. We shall merge Piron's theorem with a result by Maeda–Maeda [MaMa70] (part (b) below). There are other versions of this result where the lattice is atomic, rather than atomistic (this is only apparently weaker, for the lattice is orthomodular).

Theorem 7.59 (Piron–Maeda–Maeda) *Let $\mathscr{L}$ be an orthocomplemented, complete, irreducible and atomistic lattice satisfying the covering property. Suppose $\mathscr{L}$ contains at least* 4 *pairwise orthogonal atoms.*[8] *Then, there exist*

(i) a (generally noncommutative) division ring $\mathbb{B}$ *with unit* 1 *and zero* 0*,*

(ii) an involutive anti-automorphism *(i.e., a map $\mathbb{B} \ni \lambda \mapsto \overline{\lambda} \in \mathbb{B}$ such that $\overline{\mu + \nu} = \overline{\mu} + \overline{\nu}$, $\overline{\mu\nu} = \overline{\nu}\,\overline{\mu}$, and $\overline{\overline{\mu}} = \mu$),*

(iii) a 'generalised' vector space E over $\mathbb{B}$*,*

(iv) a Hermitian form $\langle\cdot|\cdot\rangle : \mathsf{E} \times \mathsf{E} \to \mathbb{B}$ *(i.e. linear in the second slot and conjugate-symmetric) which is* non-singular *($\langle x|x\rangle = 0$ implies $x = \mathbf{0}$),*
satisfying the following properties.

(a) *$\mathscr{L}$ is isomorphic to the orthocomplemented lattice of subspaces $\mathsf{M} \subset \mathsf{E}$ satisfying $(\mathsf{M}^\perp)^\perp = \mathsf{M}$ ($^\perp$ in reference to the Hermitian product), ordered by inclusion.*

(b) *$\mathscr{L}$ is* orthomodular *if and only if $\mathsf{M} + \mathsf{M}^\perp = \mathsf{E}$ for any $\mathsf{M} \subset \mathsf{E}$ with $\mathsf{M} = (\mathsf{M}^\perp)^\perp$.*

And this is not the whole story. Many people conjecture (see [BeCa81]) that if the lattice is also orthomodular, the division ring $\mathbb{B}$ becomes a real division algebra and can only be picked among $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternion algebra). More recently Solèr [Sol95], Holland [Hol95] and Aerts-van Steirteghem have found sufficient hypotheses for this to happen. We state below, in a joint proposition, the results of Solèr and Holland. The notation is the same as in Theorem 7.59.

Theorem 7.60 (Solèr–Holland) *Consider a lattice $\mathscr{L}$ that is orthocomplemented, complete, irreducible and atomistic, satisfies the covering property and contains at least four pairwise-orthogonal atoms. Assume $\mathscr{L}$ is orthomodular. Suppose that one of the following (equivalent) conditions holds:*

(a) *(Solèr) E contains an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}} \subset \mathsf{E}$, i.e. $\langle e_n|e_n\rangle = 1$ for every $n \in \mathbb{N}$ (1 is the unit of the division ring $\mathbb{B}$).*

(b) *(Holland) E contains an infinite orthogonal sequence $\{e_n\}_{n\in\mathbb{N}} \subset \mathsf{E}$ with $\langle e_n|e_n\rangle = a$ for every $n \in \mathbb{N}$ and some fixed $a \in \mathbb{B} \setminus \{0\}$.*

[8] With our hypotheses, this is equivalent to supposing $\mathscr{L}$ contains 4 distinct elements $\mathbf{0} \neq p_1 < p_2 < p_3 < p_4$, as in other formulations of this result.

(c) *(Holland) E does not have finite (algebraic) dimension and, for every orthogonal atoms* $p, q \in \mathscr{L}$ *there exists a linear bijective map* $U : \mathsf{E} \to \mathsf{E}$ *such that* $\langle x|y\rangle = \langle Ux|Uy\rangle$ *for all* $x, y \in \mathsf{E}$ *and* $U(p) = q$.
Then the following hold:
(i) $\mathbb{B}$ *is either* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$ *with the respective conjugation as anti-automorphism,*[9]
(ii) either $\langle\cdot|\cdot\rangle$ *or* $-\langle\cdot|\cdot\rangle$ *is positive,*
(iii) E *is complete for the norm induced by* $\langle\cdot|\cdot\rangle$*, hence a real or complex Hilbert space or a generalised structure (e.g., see [GMP13]) if* $\mathbb{B} = \mathbb{H}$.
The Hilbert space E *is separable if and only if* $\mathscr{L}$ *is separable.*

Irreducibility is not really essential. If we remove it the lattice can be split into irreducible sublattices [Jau73, BeCa81] and the argument goes through on each summand. Physically speaking this situation is natural in presence of *superselection rules*, of which more soon.

It is worth stressing that the covering property, on the contrary, is a crucial hypothesis for Theorem 7.60. Indeed there are other relevant lattices in physics that verify all remaining properties. Remarkably, the family of so-called *causally closed subsets* in some physically meaningful spacetime satisfies all properties but the covering law [Cas02, CFJ17]. This obstruction prevents one to endow a spacetime with the structure of a (generalised) Hilbert space. Having said that, it might suggest a way towards a formulation of quantum gravity.

Recently, M. Oppio and the author of this book [MoOp17, MoOp18] have explained why one can rule out real and quaternionic Hilbert spaces as possible tools to describe elementary physical systems acted upon by the Poincaré group (elementary particles), when some hypotheses in the Solèr–Piron theorem are relaxed (such as the covering property).

7.6.2 The Projector Lattice of von Neumann Algebras and the Classification of von Neumann Algebras and Factors

A historically important point for developing von Neumann's theory was that the lattice of elementary propositions on a quantum system should satisfy the *modularity* condition (Definition 7.8(d)). We will not go into explaining the manifold reasons for this (see Rédei in [EGL09]). It will be enough to remark, as von Neumann himself proved, that $\mathscr{L}(\mathsf{H})$ is not modular if H does not have finite dimension. The way out proposed by von Neumann and Murray is to reduce the number of elementary observables on the quantum system, so to guarantee modularity also in the infinite-dimensional case. The ensuing theoretical results have become of the utmost importance in quantum theories, independently from the initial problem of von Neumann, so we shall spend a few words about them. Later in the book these facts we be needed, especially when we deal with superselection rules.

[9]In $\mathbb{R}$ this is the identity map, in $\mathbb{C}$ the standard conjugation, and in $\mathbb{H}$ it is $\overline{a + bi + cj + dk} := a - bi - cj - dk$ for every $a, b, c, d \in \mathbb{R}$.

The point is to start from a *von Neumann algebra* $\mathfrak{R}$ (also called *W^*-algebra*, Definition 3.90) on a (not necessarily separable) Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$, as opposed to the complete lattice $\mathscr{L}(\mathsf{H})$. Then consider the **logic of the von Neumann algebra** $\mathfrak{R}$:

$$\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) := \mathfrak{R} \cap \mathscr{L}(\mathsf{H}) ,$$

i.e., the set of orthogonal projectors belonging to $\mathfrak{R}$. Here $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ represents the actual set of elementary propositions associated to the physical system, and self-adjoint elements in $\mathfrak{R}$ are interpreted as bounded observables (this will be clearer in Chap. 11). There is a intimate relationship between $\mathfrak{R}$ and $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ as the latter generates the former as a von Neumann algebra. Moreover the poset structure of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is very similar to the one on the whole $\mathscr{L}(\mathsf{H})$, and allows for analogous physical interpretations. Everything is stated in the following proposition.

Proposition 7.61 *Let $\mathfrak{R}$ be a von Neumann algebra on the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ and $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ the set of orthogonal projectors $P \in \mathfrak{R}$. The following facts hold.*

(a) *$\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is an orthomodular (hence bounded and orthocomplemented) complete lattice, with structure inherited from $\mathscr{L}(\mathsf{H})$.*

(b) *The von Neumann algebra generated by $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is $\mathfrak{R}$ itself:*

$$\mathfrak{R} = \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'' .$$

(c) *The centre of the lattice generates the centre of the algebra,*

$$\left(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'\right)'' = \mathfrak{R} \cap \mathfrak{R}' .$$

(d) *$\mathfrak{R}$ is a factor (Definition 3.90) if and only if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is irreducible.*

(e) *$\mathfrak{R} = \mathfrak{B}(\mathsf{H})$ if and only if $\mathscr{L}_{\mathfrak{R}'}(\mathsf{H}) = \{0, I\}$.*

Proof (a) $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ trivially inherits a poset structure from $\mathscr{L}(\mathsf{H})$, including a minimum and a maximum because $0, I \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$. It also inherits the orthocomplemented structure from $\mathscr{L}(\mathsf{H})$. In fact, if $P, Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$, $P \wedge Q$ in $\mathscr{L}(\mathsf{H})$ can be computed by (7.32). Since $\mathfrak{R}$ is closed in the strong operator topology, $P \wedge Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ and, in particular, the $\wedge$ (inf) of $\mathscr{L}(\mathsf{H})$ must coincide with the meet of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ since $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \subset \mathscr{L}(\mathsf{H})$. Next observe that $\neg P = I - P \in \mathfrak{R}$ if $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$, since $\mathfrak{R}$ is an algebra. Thus $\neg P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ if $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$. Consequently, $P \vee Q = \neg(\neg P \wedge \neg Q) \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ if $P, Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$, and the $\vee$ (sup) of $\mathscr{L}(\mathsf{H})$ must coincide with the join of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ since $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \subset \mathscr{L}(\mathsf{H})$. Very easily, orthomodularity passes from $\mathscr{L}(\mathsf{H})$ to $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$. At this stage $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is just an orthocomplemented, orthomodular lattice with the induced operations. More difficult is to establish completeness. From (iii) in Theorem 7.22(a)

$$\wedge_{\alpha \in A} P_\alpha = \neg\left(\vee_{\alpha \in A} \neg P_\alpha\right) ,$$

if $\{P_\alpha\}_{\alpha\in A} \subset \mathscr{L}(\mathsf{H})$. Therefore, since the logic $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is closed under $\neg$, its completeness is equivalent to

$$\vee_{\alpha\in A} P_\alpha \in \mathfrak{R} \quad \text{if} \quad \{P_\alpha\}_{\alpha\in A} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H}), \tag{7.47}$$

where $\vee$ *is the join of* $\mathscr{L}(\mathsf{H})$. Let us prove (7.47). Define $P := \vee_{\alpha\in A} P_\alpha$ (which does exist in $\mathscr{L}(\mathsf{H})$). Take $X \in \mathfrak{R}'$ and an element P_α in $\{P_\alpha\}_{\alpha\in A} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$. Since $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \subset \mathfrak{R}$, we have $P_\alpha X - XP_\alpha = 0$. On the other hand $PP_\alpha = P_\alpha$ (since $P \geq P_\alpha$) so that

$$(PX - XP)P_\alpha = PXP_\alpha - XPP_\alpha = PP_\alpha X - XPP_\alpha = P_\alpha X - XP_\alpha = 0\,.$$

In other words $P_\alpha(\mathsf{H}) \subset Ker(PX - XP)$. If K denotes the orthogonal projector onto $Ker(PX - XP)$, we therefore have $K \geq P_\alpha$. That inequality is valid for every P_α, so it must also hold for the supremum of the P_α. Hence $K \geq P$, so that $(PX - XP)P = 0$, i.e., $PXP = XP$. Since $X \in \mathfrak{R}'$ is arbitrary and $\mathfrak{R}'$ is closed under Hermitian conjugation, starting with X^* in place of X gives $PX^*P = X^*P$ and the adjoint relation $PXP = PX$. This and the earlier, analogous condition prove $XP = PX$. We have established that $P \in (\mathfrak{R}')' = \mathfrak{R}$, and therefore (7.47) holds.
(b) Here we need a few (independent) results from Chap. 8. As $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \subset \mathfrak{R}$, we have $\mathfrak{R}' \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'$ and finally $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})'' \subset \mathfrak{R}'' = \mathfrak{R}$. Let us prove that $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})'' \supset \mathfrak{R}$ to conclude. Since the von Neumann algebra $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})''$ is closed in the strong operator topology (Theorem 3.88), it is sufficient to prove that if $A \in \mathfrak{R}$, there is a sequence of elements $A_n \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})''$ such that $A_n \to A$ strongly. As $A = \frac{1}{2}(A + A^*) + i\frac{1}{2i}(A - A^*)$ and both $\frac{1}{2}(A + A^*)$ and $\frac{1}{2i}(A - A^*)$ are self-adjoint elements of $\mathfrak{R}$, we may prove our claim for self-adjoint elements $A \in \mathfrak{R}$. Since self-adjoint operators of $\mathfrak{B}(\mathsf{H})$ are normal, we can apply the spectral theorem for bounded normal operators Theorem 8.56, and write $A = \int_K x dP^{(A)}(x)$ where $\{P^{(A)}(E)\}_{E\in\mathscr{B}(\mathbb{R})}$ is the PVM associated to A, and $K \subset \mathbb{R}$ is a sufficiently large compact set containing the spectrum of A. By Theorem 8.56(c) $P^{(A)}(E) = \int_K \chi_E(x) dP^{(A)}(x)$ commutes with every bounded operator commuting with A (and $A^* = A$) where χ_E is the characteristic function of the Borel set $E \subset \mathbb{R}$. As $A \in \mathfrak{R} = (\mathfrak{R}')'$, $P^{(A)}(E)$ commutes with every element of $\mathfrak{R}'$ and thus $P^{(A)}(E) \in \mathfrak{R}'' = \mathfrak{R}$. In particular, $P^{(A)}(E) \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ because it is an orthogonal projector. Finally, consider a bounded sequence of simple functions s_n tending pointwise to $id : K \ni x \mapsto x \in \mathbb{R}$ (such a sequence exists as a consequence of Proposition 1.49). Then Theorem 8.54(c) yields $\int_K s_n(x) dP^{(A)}(x) \to A$ strongly as $n \to +\infty$. On the other hand

$$A_n = \int_K s_n(x) dP^{(A)}(x) = \sum_{k=1}^{N_n} s_{kn} P^{(A)}(E_{kn})$$

commutes with every operator which commutes with the elements of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ because we saw $P^{(A)}(E_{kn}) \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$. We conclude that $A_n \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})''$. Summarising, we have found that $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})'' \ni A_n \to A$ strongly, as we wanted.

(c) From (b), $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})' = \mathscr{L}_{\mathfrak{R}}(\mathsf{H})''' = \mathfrak{R}'$ so that $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})' \subset \mathfrak{R} \cap \mathfrak{R}'$. As the right-hand side is a von Neumann algebra, it contains the smallest von Neumann algebra containing the left-hand side: $(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})')'' \subset \mathfrak{R} \cap \mathfrak{R}'$. Let us establish the converse inclusion. We will prove that if $A \in \mathfrak{R} \cap \mathfrak{R}'$, there is a sequence $< \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})' >=< \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathfrak{R}' >\ni A_n \to A$ strongly, where $< \mathscr{A} >$ is the space of finite, complex linear combinations of elements of $\mathscr{A} \subset \mathfrak{B}(\mathsf{H})$. As $(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})')'' \supset< \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})' >$ and $(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})')''$ is strongly closed, this concludes the proof. It suffices to restrict to self-adjoint operators A in view of the same argument used to prove (b). If $A^* = A \in \mathfrak{R} \cap \mathfrak{R}'$, the spectral measure $P^{(A)}$ of A belongs to $\mathfrak{R}$ and commutes with the same operators of $\mathfrak{B}(\mathsf{H})$ commuting with A (Theorem 8.56). Therefore $P^{(A)}(E) \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathfrak{R}' = \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'$ for every Borel set $E \subset \mathbb{R}$. In analogy to part (b) we can construct $A_n \in< \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})' >$ with

$$A_n = \sum_{k=1}^{N_n} s_{kn} P^{(A)}(E_{kn}) \to A \quad \text{strongly as } n \to +\infty,$$

for some real numbers s_{kn} and Borel sets $E_{kn} \subset \mathbb{R}$. This concludes the proof of (c).
(d) From (c), if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is irreducible, $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})' = \{0, I\}$ so that $(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})')' = \mathfrak{B}(\mathsf{H})$ and thus $\mathfrak{R}$ is a factor because

$$\mathfrak{R} \cap \mathfrak{R}' = \left(\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'\right)'' = \{cI\}_{c\in\mathbb{C}} .$$

Conversely, if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is not irreducible, there is a non-trivial orthogonal projector P in $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})' \cap \mathfrak{R}$. In particular P commutes with every spectral measure of every self-adjoint operator $A \in \mathfrak{R}$, hence it commutes with A itself by the spectral decomposition of A (Theorem 8.56). Since every element of $\mathfrak{R}$ is a linear combination of two self-adjoint elements of $\mathfrak{R}$, $P \in \mathfrak{R}' \cap \mathfrak{R}$ and hence $\mathfrak{R}$ cannot be a factor.
(e) $\mathfrak{R} = \mathfrak{B}(\mathsf{H})$ is equivalent to $\mathfrak{R}' = \{cI\}_{c\in\mathbb{C}}$ (Proposition 3.93). If $\mathfrak{R}' = \{cI\}_{c\in\mathbb{C}}$ then $\mathscr{L}_{\mathfrak{R}'}(\mathsf{H}) = \{0, I\}$. On the other hand, if $\mathscr{L}_{\mathfrak{R}'}(\mathsf{H}) = \{0, I\}$, then $\mathfrak{R}' = \mathscr{L}_{\mathfrak{R}'}(\mathsf{H})'' = \{0, I\}'' = \mathfrak{B}(\mathsf{H})' = \{cI\}_{c\in\mathbb{C}}$. $\square$

Ultimately it is possible to choose $\mathfrak{R}$ so that $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ becomes modular. This happens for certain *factors* (Definition 3.90), i.e. von Neumann algebras $\mathfrak{R}$ with trivial centre, $\mathfrak{R}' \cap \mathfrak{R} = \{cI\}_{c\in\mathbb{C}}$. The classification of factors of Murray and von Neumann proves, among other things, that only the logic $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ of so-called *factors of type* II_1 is modular if H is not finite-dimensional. Although modularity is nowadays no longer deemed fundamental, and some of von Neumann motivations have proved indefensible,[10] *von Neumann algebras*, *factors*, and the classification and study of factors have been decisive for the development of the mathematical formulation of quantum theories, including algebraic Quantum Field Theories [Haa96].

[10] See Rédei in [EGL09] and it was established by Piron that modularity is incompatible with localisability of elementary particles [Pir64, p. 452].

Whereas in algebraic Quantum Field Theory all types of factors are relevant, the factors that play a role in standard QM are the so-called *factors of type I* (special *-algebras isomorphic to some $\mathfrak{B}(\mathsf{K})$, see below). This fact easily implies that the bounded, orthomodular, complete, irreducible lattices of projectors onto type-I factors are also *atomic*, hence *atomistic* (Remark 7.57(1)), and satisfy the *covering property*. The projector lattice of factors of other types, though bounded, orthomodular, complete and irreducible, does not contain atoms.

Following Murray and von Neumann (for a complete account, see [Tau61]), factors are classified in the following way (see [Red98, Pan93] for a quick report, and [Tak00, vol. I], [KaRi97, vol. II] for a complete and detailed discussion on the subject). If $\mathfrak{R}$ is a von Neumann algebra in the complex (non-separable, in general) Hilbert space H we define an equivalence relation on $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$,

$P \sim Q$ for $P, Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ means that there is a partial isometry $U \in \mathfrak{R}$ such that $P = U^*U$ and $Q = UU^*$.

(Refer to Definition 3.72 and Proposition 3.74.) In other words, P is equivalent to Q if and only if there is a Hilbert-space isomorphism $P(\mathsf{H}) \to Q(\mathsf{H})$, and the map U that extends it on $P(\mathsf{H})^\perp$, where $U = 0$, belongs to $\mathfrak{R}$.

This equivalence relation has the property that, if $P_i \sim Q_i$ for $i = 1, 2$ and $P_1 \perp P_2$ and $Q_1 \perp Q_2$, we also have $P_1 + P_2 \sim Q_1 + Q_2$.

We next introduce an order relation on $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ by saying that $P \preceq Q$ when $P \sim P' \leq Q$ for some $P' \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$.

Definition 7.62 If $\mathfrak{R}$ is a von Neumann algebra over the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$, an element $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is said to be

(a) finite if $P \sim Q \leq P$ for some $Q \in \mathfrak{R} \Rightarrow P = Q$ (P is not equivalent to any proper subprojector);

(b) infinite if it is not finite;

(c) properly infinite if $P \neq 0$ is infinite and QP is either 0 or infinite for every $Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ that is **central** (in the centre of $\mathfrak{R}$).

It is possible to prove that $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})/\sim$, with the order relation induced by $\preceq$, is *totally* ordered if $\mathfrak{R}$ is a *factor*. Moreover, the following crucial result holds (see for instance [Red98] for a short exposition).

Proposition 7.63 *If $\mathfrak{R}$ is a factor for the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$, there exists a map (unique up to a positive multiplicative constant) $d : \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \to [0, +\infty]$ such that*

(i) $d(P) = 0 \Leftrightarrow P = 0$,

(ii) $d(P) = d(Q) \Leftrightarrow P \sim Q$,

(iii) $d(P) \leq d(Q) \Leftrightarrow P \preceq Q$,

(iv) $d(P) < +\infty \Leftrightarrow P$ is a finite projection,

(v) $d(P) + d(Q) = d(P \wedge Q) + d(P \vee Q)$,

(vi) $d(P + Q) = d(P) + d(Q)$ if $P \perp Q$,

for any $P, Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$.

The map d, called the **dimension function** on $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$, has necessarily one of the following ranges – after suitable normalisation, depending on the nature of the factor $\mathfrak{R}$:
1. **Type** I_n: $d(\mathscr{L}_{\mathfrak{R}}(\mathsf{H})) = \{0, 1, \ldots, n\}$ where $n = 1, 2, \ldots, \infty$.
2. **Type** II_1: $d(\mathscr{L}_{\mathfrak{R}}(\mathsf{H})) = [0, 1]$.
3. **Type** II_∞: $d(\mathscr{L}_{\mathfrak{R}}(\mathsf{H})) = [0, \infty) \cup \{\infty\}$.
4. **Type** III: $d(\mathscr{L}_{\mathfrak{R}}(\mathsf{H})) = \{0, \infty\}$, where $d(P) = 0$ only if $P = 0$.
The factor $\mathfrak{R}$ is called **of type** I_n, II_1, II_∞, III in accordance with the above table for d. Factors of type I_∞ can be further subdivided in more refined types, still denoted by I_n, *where now n indicates an infinite cardinal number*, whose meaning will be discussed shortly. With this refinement the classification is exhaustive: *a factor is necessarily of one, and only one, type among I_n, II_1, II_∞ and III, where n can be any non-zero cardinal (finite or infinite).*

Henceforth, by a **type-I** factor we shall mean a factor of type I_n with $n > 0$ a finite or infinite cardinal, while **type-I_∞** will indicate a factor for type I_n with n infinite. Similarly, a **type-II** factor may be of type II_1 or II_∞. The next result can be found as a subcase of Theorem 9.1.3 and Example 9.1.5 in [KaRi97, vol. II]. It is technically important for it relates a factor $\mathfrak{R}$ with $\mathfrak{R}'$ (the commutant is a factor since $\mathfrak{R}$ and $\mathfrak{R}'$ have the same centre).

Proposition 7.64 *A factor $\mathfrak{R}$ on a complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ is of type I_n, II or III if and only if its commutant $\mathfrak{R}'$ is of type I_m, II or III respectively, where n, m are arbitrary cardinals.*

Type-I factors enjoy very nice properties [KaRi97, vol. II]:

(a) among the factors of type I_n, with $n > 0$ finite or infinite, are the **minimal projectors**, i.e. the atoms of the projector lattice;

(b) type-I_n factors are *-isomorphic to $\mathfrak{B}(\mathsf{H}_n)$, where $\dim(\mathsf{H}_n) = n$ may be infinite. As a matter of fact, there exist Hilbert spaces $\mathsf{H}_n, \mathsf{H}_m$ of dimensions n, m and a unitary operator $U : \mathsf{H} \to \mathsf{H}_n \otimes \mathsf{H}_m$ such that $U\mathfrak{R}U^{-1} = \mathfrak{B}(\mathsf{H}_n) \otimes \mathbb{C}I_m$, and $U\mathfrak{R}'U^{-1} = \mathbb{C}I_n \otimes \mathfrak{B}(\mathsf{H}_m)$, where I_l is the identity on H_l (the notion of tensor product will be introduced in Sect. 10.2). This property can actually be taken as a definition of type-I_n factors;

(c) only factors of type I_n may be irreducible ($\mathfrak{R}$ is irreducible iff $\mathfrak{R}' = \{cI\}_{c\in\mathbb{C}}$. Since $\{cI\}_{c\in\mathbb{C}}$ is of type I_1, $\mathfrak{R} = \{cI\}'_{c\in\mathbb{C}}$ must be of type I by Proposition 7.64.)
Other general features of the lattices of orthogonal projectors for the various factors are listed below (see [Tak00, KaRi97] and [Red98] for a summary).

The projector lattice of type-I_n factors, with finite n, is non-distributive for $n \geq 2$, modular and, as already said atomic (thus atomistic because orthomodular); moreover, $d(P) = \dim P(\mathsf{H}_n)$.

The projector lattice of type-I_∞ factors is orthomodular, non-modular, atomic (thus atomistic because orthomodular).

The projector lattice of type-II_1 factors is modular but non-atomic.

The projector lattice of type-II_∞ and type-III factors is neither modular nor atomic.

Von Neumann algebras $\mathfrak{R}$ that are not factors can be classified, similarly, in *disjoint* types: I_n, II_1, II_∞, III. In contrast to the previous sorting, however, this classification does *not* exhaust all instances, for there are W^*-algebras that do not belong to any one class above. But every von Neumann algebra is a direct sum of the above types. This classification of von Neumann algebras of definite type requires a pair of definitions.

Definition 7.65 Let $\mathfrak{R}$ be a von Neumann algebra over the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$.
(a) A *finite* projector $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is called **Abelian** if $P\mathfrak{R}P := \{PAP \mid A \in \mathfrak{R}\}$ is an Abelian W^*-algebra.
(b) If $A \in \mathfrak{R}$, its **central carrier** is the orthogonal projector $C_A := I - P_A$ where $P_A := \vee\{P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathfrak{R}' \mid PA = 0\}$.

We are in a position to state the involved classification of von Neumann algebras of definite type (see [Tak00, vol. I], [KaRi97, vol. II] and the discussion in [BEH07, Chap. 6]).

Definition 7.66 A von Neumann algebra $\mathfrak{R}$ over the Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ is said to be

(a) of **type** I if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ contains an Abelian projector with central carrier I; it is, further, of **type** I_n if I is the orthogonal sum of n equivalent Abelian projectors (n can be any finite or infinite cardinal);

(b) of **type** II if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ does not contain any non-zero Abelian projectors, but contains a central projector with central carrier I; it is of **type** II_1 if I is finite, and of **type** II_∞ when I is properly infinite;

(c) of **type** III if $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ has no non-zero finite projectors.

All these types are pairwise *distinct*, so in particular I_n and I_m are different if $n \neq m$. The fundamental link between the classification of factors and the classification of von Neumann algebras of definite type is the following: *A factor is of type I_n, II_1, II_∞ or III if and only if it is of type I_n, II_1, II_∞ or III respectively, as a von Neumann algebra.*

Conversely, *in a separable Hilbert space*, (see the paragraph below Remark 7.69) a von Neumann algebra is of type I_n, II_1, II_∞, III if and only if it is a *direct integral* of factors of type, respectively, I_n, II_1, II_∞, III.

Remark 7.67 The von Neumann algebras of quantum field operators associated to bounded regions in Minkowski spacetime are usually of type III, while the whole algebra may be a factor of type I or III depending on the state used to construct the representation: type I is typical for ground states and type III for extended thermodynamical (KMS) states. These types also appear in the theory when one describes extended thermodynamical systems. After the breakthrough brought by the *Tomita–Takesaki modular theory* [Haa96], and its implications in thermodynamical Quantum Field Theory, at the end of the 1960s, it became possible for Connes to further refine the classification of type-III algebras in an uncountable family of algebras of type III_λ, $\lambda \in [0, 1]$ [KaRi97]. Type III_1 is of particular interest in physics. A nice and rapid discussion on the occurrence of these types of von Neumann algebras and factors for describing quantum physical systems, with all the relevant references, can be found in [ReSu07]. ■

7.6.3 Direct Decomposition into Factors and Definite-Type von Neumann Algebras and Factors

A number of theoretical results later mentioned in the book will require that we know a crucial result concerning the decomposition of a von Neumann algebra into definite-type algebras (a reformulation of Theorem 6.5.2 in [KaRi97]).

Theorem 7.68 (Type decomposition) *Let $\mathfrak{R}$ be a von Neumann algebra on the Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$. There is a unique family $\{Q_j\}_{j\in J} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'$ with*
(i) $Q_j \neq 0$,
(ii) $Q_j \perp Q_i$ if $i \neq j$,
(iii) $\vee_{j\in J} Q_j = I$,
such that

$$\mathsf{H} = \bigoplus_{j\in J} \mathsf{H}^{(j)} \quad \textit{and} \quad \mathfrak{R} = \bigoplus_{j\in J} \mathfrak{R}^{(j)}\,, \tag{7.48}$$

where $\mathsf{H}^{(j)} := Q_j(\mathsf{H})$, $\mathfrak{R}^{(j)} := Q_j\mathfrak{R}\restriction_{\mathsf{H}^{(j)}} \subset \mathfrak{B}(\mathsf{H}^{(j)})$ and the direct sums refer to Definition 3.98. Moreover, each von Neumann algebra $\mathfrak{R}^{(j)}$ is of definite type I_n (for any possible finite or infinite cardinal $0 < n \leq \dim(\mathsf{H})$), II_1, II_∞, and III, and each such type appears at most once in the sums.

Remark 7.69 **(1)** A *von Neumann algebra* is said to be of type I_∞ if it is a sum (as in (7.48)) of von Neumann algebras of type I_n where every n is infinite.
(2) It follows from the theorem ([KaRi97, vol. II], p. 422) that, if $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \setminus \{0\}$ is central and the von Neumann algebra $\mathfrak{R}$ is of type I_n, II_1, II_∞ or III, then the von Neumann algebra $P\mathfrak{R} := \{PA \mid A \in \mathfrak{R}\}$ has the same definite type.
(3) Proposition 7.64 is valid for general von Neumann algebras $\mathfrak{R}$ [KaRi97, vol. II, Theorem 9.1.3], if one replaces I_n and I_m by I. ■

Von Neumann proved a related result, that in a sense [BrRo02] is finer than Theorem 7.68: if a Hilbert space H is separable, von Neumann algebras on H can always be written as *direct integrals* of factors which, as we know, are of defined type. Let us show how this decomposition arises and how it is related to Theorem 7.68. We shall consider a simplified situation where the centre of the associated projector lattice is *atomic* and the direct integral becomes a Hilbert sum. This is the only case that is truly relevant for this book. Besides, H does not even need to be separable.

Given a von Neumann algebra $\mathfrak{R}$ on a Hilbert space H, by Zorn's lemma the centre $\mathfrak{R} \cap \mathfrak{R}'$ (i.e. the centre $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'$) contains a maximal set of non-vanishing orthogonal projectors $\{P_k\}_{k\in K}$ such that $P_k \perp P_h$ if $k \neq h$. At this point we make the non-trivial assumption that there exists a maximal set of central pairwise-orthogonal projectors such that, for every fixed $k \in K$, there is no projector $Q \in \mathfrak{R} \cap \mathfrak{R}'$ satisfying $0 \leq Q \leq P_k$ and $Q \neq 0, P_k$. In other words *the central projectors P_k are atoms of $\mathscr{L}_{\mathfrak{R}\cap\mathfrak{R}'}(\mathsf{H})$*. This condition determines uniquely this maximal family, and the following result holds.

Proposition 7.70 *Let $\mathfrak{R}$ be a von Neumann algebra on the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ and $\{P_k\}_{k\in K} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ a family of orthogonal projectors such that:*

(i) every P_k is non-zero and central*: $0 \neq P_k \in \mathfrak{R} \cap \mathfrak{R}'$,*

(ii) $P_k \perp P_h$ if $k \neq h$,

(iii) the family is maximal (among all families satisfying (i)-(ii)),

(iv) each P_k is an atom of $\mathfrak{R} \cap \mathfrak{R}'$.

Then the following facts hold.

(a) *Irrespective of (iv), the set of conditions (i), (ii), (iii) is equivalent to the set (i), (ii), (iii)' where*

(iii)' $\vee_{k\in K} P_k = I$.

(b) *The family $\{P_k\}_{h\in K} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ satisfying (i)-(iv) is unique up to term relabelling.*

(c) *Every closed subspace $\mathsf{H}_k := P_k(\mathsf{H})$ is invariant under $\mathfrak{R}$, i.e.,*

$$A(\mathsf{H}_k) \subset \mathsf{H}_k \quad \textit{if } A \in \mathfrak{R}\,.$$

(d) *Each map*

$$\mathfrak{R} \ni A \mapsto \pi_k(A) := A\restriction_{\mathsf{H}_k} : \mathsf{H}_k \to \mathsf{H}_k$$

is a *-algebra representation of $\mathfrak{R}$.

(e) *Each $\mathfrak{R}_k := \pi_k(\mathfrak{R})$ is a factor on the Hilbert space H_k.*

(f) *We have splittings*

$$\mathsf{H} = \bigoplus_{k\in K} \mathsf{H}_k\,, \qquad \mathfrak{R} = \bigoplus_{k\in K} \mathfrak{R}_k\,, \tag{7.49}$$

called **direct decompositions into factors**.

(g) *There is a partition $K = \sqcup_{j\in J} K_j$ such that*

$$\mathsf{H}^{(j)} = \bigoplus_{k\in K_j} \mathsf{H}_k \quad \textit{and} \quad \mathfrak{R}^{(j)} = \bigoplus_{k\in K_j} \mathfrak{R}_k\,, \tag{7.50}$$

where $\mathsf{H}^{(j)}$ and $\mathfrak{R}^{(j)}$ are the closed subspaces and definite-type W^*-algebras of Theorem 7.68. In particular, each factor $\mathfrak{R}_k$ with $k \in K_j$ has the same type as the corresponding $\mathfrak{R}^{(j)}$.

Proof (a) Assume that (i)–(iii) are valid. The element $\sum_{k\in K} P_k = \vee_{k\in K} P_k$ (the sum is in the strong operator topology, and we use this notation everywhere in the proof, considering families of pairwise-orthogonal projectors), exists in $\mathfrak{R} \cap \mathfrak{R}'$ (a von Neumann algebra, hence strongly closed). Therefore (iii)' follows, for otherwise $I - \sum_{k\in K} P_k \neq 0$ would be central and orthogonal to every P_k, contradicting maximality. If, conversely, (i),(ii),(iii)' hold, take a central projector Q

satisfying $Q \perp P_k$ for all $k \in K$. We therefore have $P_k Q = Q P_k = 0$ and thus $Qx = Q \sum_{k\in K} P_k x = \sum_{k\in K} Q P x = \mathbf{0}$ for every $x \in \mathsf{H}$ so that $Q = 0$ and (i) is false. This means $\{P_k\}_{k\in K}$ is maximal ((iii) holds) if we require (i),(ii), (iii)'.
(b) Let us pass to the uniqueness property. If $\{Q_h\}_{h\in H}$ is another maximal set of non-vanishing, pairwise-orthogonal, atomic and central orthogonal projectors, we have $P_k Q_h = Q_h P_k$ and so $P_k Q_h$ is an orthogonal projector with $P_k Q_h \le P_k$. Under our hypotheses either $P_k Q_h = P_k$ or $P_k Q_h = 0$. Since $Q_h = \sum_{k\in K} P_k Q_h$ we also have $Q_h = \sum_{k\in K_h} P_k$ for some subset $K_h \subset K$. If K_h is contained more than one index, Q_h would not be an atom. We conclude that $Q_h = P_{k_h}$ for a unique k_h depending on h. The condition $\sum_h Q_h = I$ implies that k_h ranges over the whole K when $h \in H$, since $\sum_{k\in K'} P_k < I$ if K' is properly included in K. Therefore $\{Q_h\}_{h\in H}$ and $\{P_k\}_{h\in K}$ denote the same set.
(c)–(d). We observe that any element of $\mathfrak{R}$ commutes with every central projector P_k, and therefore the closed subspace $\mathsf{H}_k := P_k(\mathsf{H})$ is invariant under the whole algebra $\mathfrak{R}$. Moreover

$$\mathfrak{R} \ni A \mapsto \pi_k(A) := A\restriction_{\mathsf{H}_k} : \mathsf{H}_k \to \mathsf{H}_k$$

is a *-algebra representation of $\mathfrak{R}$, the proof being straightforward.
(e) The space $\mathfrak{R}_k := \pi_k(\mathfrak{R}) \subset \mathfrak{B}(\mathsf{H}_k)$ contains I and is *-closed. We claim that $\mathfrak{R}_k$ is strongly closed, and hence a von Neumann algebra (Theorem 3.88). Suppose $\mathfrak{R}_k \ni B_n \to B \in \mathfrak{B}(\mathsf{H}_k)$ strongly. We know that $B_n = A_n\restriction_{\mathsf{H}_k}$ for some $A_n \in \mathfrak{R}$, so that $A_n P_k \phi \to B'\phi$ for every $\phi \in \mathsf{H}$ where B' extends B as 0 on $\mathsf{H}_k^\perp$. Then $B' \in \mathfrak{R}$, for it is the strong limit of $A_n P_k \in \mathfrak{R}$ and $\mathfrak{R}$ is a von Neumann algebra. By construction, if $\psi \in \mathsf{H}_k$, $B'\psi = B'\restriction_{\mathsf{H}_k} \psi$ and so $B_n\psi = A_n P_k \psi \to B'\psi = B'\restriction_{\mathsf{H}_k} \psi = \pi_k(B')\psi$. Consequently $\mathfrak{R}_k$ is strongly closed, as required. Now, $\mathfrak{R}_k'$ is a factor because, if Q is an orthogonal projector in the centre of $\mathfrak{R}_k'$, by extending it to the zero operator on $\mathsf{H}_k^\perp$ we obtain an orthogonal projector Q' in the centre of $\mathfrak{R}$ which satisfies $Q' \le P_k$. Under our hypotheses either $Q' = 0$ or $Q' = P_k$, *viz.* $Q = 0$ or $Q = I_k$ (the identity in $\mathfrak{R}_k$) and hence $\mathfrak{R}_k$ is a factor due to Proposition 7.61(d).
(f) In terms of subspaces, $I = \vee_{k\in K} P_k$ means $\mathsf{H} = \overline{< \mathsf{H}_k >_{k\in K}}$. Since $P_k \neq 0$ and $P_k \perp P_h$ if $k \neq h$, just by definition of Hilbert sum we find $\mathsf{H} = \oplus_{k\in K}\mathsf{H}_k$. With this decomposition of H, applying Definition 3.98 immediately gives $\mathfrak{R} = \bigoplus_{k\in K} \mathfrak{R}_k$ where $\mathfrak{R}_k = \pi_k(\mathfrak{R})$.
(g) To prove the last statement, let P_k be a central orthogonal projector onto H_k and Q_j a central projector in the type-decomposition of Theorem 7.68. The following facts hold: (1) if $P_k \le Q_j$ then Q_j is uniquely determined by P_k. (Since $Q_j \perp Q_i$ for $i \neq j$, if $P_k \le Q_i$, i.e., $P_k = P_k Q_i$, we also have $P_k = P_k Q_j = P_k Q_i Q_j = 0$, a contradiction). (2) For every P_k there exists Q_{j_k} such that $P_k \le Q_{j_k}$. ($P_k Q_j \le P_k$ and, as P_k is an atom of the centre, either $P_k Q_j = 0$ or $P_k Q_j = P_k$. If $P_k Q_j = 0$ for every $j \in J$, since $\sum_{j\in J} Q_j = I$, we would have $P_k = \sum_{j\in J} P_k Q_j = 0$, impossible.) (3) Every Q_j satisfies $P_{k_j} \le Q_j$ for some $k_j \in K$. (Since $Q_j = \sum_{k\in K} Q_j P_k$ and $Q_j P_k \le P_k$ we have either $Q_j P_k = 0$ or $Q_j P_k = P_k$ because P_k is an atom of the centre. Hence $Q_j = \sum_{k\in K_j} P_k$ and $P_{k_j} \le Q_j$ if $k_j \in K_j$.) Summing up, K can be decomposed in non-empty, pairwise-disjoint subsets $K_j = \{k \in K \mid P_k \le Q_j\}$ where $j \in J$. Furthermore, rephrasing (1), (2) and (3) in terms of $\mathfrak{R}_k = P_k\mathfrak{R}$ and

$\mathfrak{R}^{(j)} = Q_j\mathfrak{R}$, we find that each factor $\mathfrak{R}_k$ is contained in a unique von Neumann algebra $\mathfrak{R}^{(j_k)}$, so that $\mathfrak{R}_k$ has the same type of $\mathfrak{R}^{(j_k)}$ (Remark 7.69(2)). Moreover, every von Neumann algebra $\mathfrak{R}^{(j)}$ contains a factor $\mathfrak{R}_k$, necessarily of the same type. Finally, since $Q_j = \sum_{k\in K_j} P_k$ (where $P_k \perp P_h$ if $k \neq h$), (7.50) holds true. □

It is mathematically relevant that the four requirements (i), (ii), (iii), (iv) (or (i), (ii), (iii)', (iv)) are together equivalent to a unique demand on the centre of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$.

Proposition 7.71 *Let $\mathfrak{R}$ be a von Neumann algebra on the complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$. Then $\mathfrak{R}$ contains a (unique) family $\{P_k\}_{k\in K} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ satisfying (i)-(iv) of Proposition 7.70 if and only if the centre of the lattice $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is atomic (Definition 7.54).*

Proof If the centre of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is an atomic lattice, there are sets of pairwise orthogonal atoms. Zorn's lemma immediately proves that there are maximal such sets. A maximal set of pairwise orthogonal atoms $\{P_k\}_{k\in K}$ satisfies (i)-(iv) evidently. Suppose, conversely, that the family of central projectors $\{P_k\}_{k\in K}$ satisfies (i)–(iv) and therefore (iii)' of Proposition 7.70 in particular. Let $Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \cap \mathscr{L}_{\mathfrak{R}}(\mathsf{H})'$. We have $Q = \vee_{k\in K} QP_k$ where the projectors QP_k and QP_h are orthogonal if $h \neq k$. If $Q \neq 0$, there must be $l \in K$ with $QP_l \neq 0$. As $QP_l \leq P_l$, and since P_l is an atom, we also have $P_l = QP_l \leq Q$. The arbitrariness of Q implies that the centre of $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is atomic. □

We have seen that $\mathfrak{R}$ can be decomposed into a direct sum of factors. However, to achieve this nice result, we made the overall quite strong assumption that the centre of the lattice $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ is atomic or, equivalently, that it contains a maximal set of pairwise orthogonal atoms $\{P_k\}_{k\in K}$. Similar central maximal sets always exist by Zorn's lemma, but there is no guarantee that they are made of atoms of the centre. In the general case, if H is separable, a finer but analogous decomposition of a von Neumann algebra can be constructed in terms of a *direct integral* of factors, taking the place of the Hilbert sum. This generalised decomposition reduces to (7.49) as soon as the centre is atomic. In the general case, the direct integral decomposition into factors is essentially unique.

7.6.4 Gleason's Theorem for Lattices of von Neumann Algebras

There is a version of Gleason's theorem that holds for measures on lattices of von Neumann algebras. It necessitates certain conditions [Mae89] on the definite-type decomposition of the von Neumann algebra.

Theorem 7.72 (General Gleason theorem) *Consider a von Neumann algebra $\mathfrak{R}$ on a complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ whose definite-type decomposition (Theorem 7.68) does not include a type-I_2 algebra. Denote, as before, by $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ the (orthomodular, complete) lattice of orthogonal projectors in $\mathfrak{R}$.*

Suppose $\mu : \mathscr{L}_{\mathfrak{R}}(\mathsf{H}) \to [0, +\infty]$ *satisfies* $0 < \mu(I) < +\infty$ *and is* σ**-additive***:*

$$\mu\left(s\text{-}\sum_{i=0}^{+\infty} P_i\right) = \sum_{i=0}^{+\infty} \mu(P_i) \quad \textit{for}\ \{P_i\}_{i\in\mathbb{N}} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H}_S)\ \textit{with}\ P_i \perp P_j\ \textit{if}\ i \neq j. \tag{7.51}$$

Each of the following three conditions is equivalent to the existence of a positive, trace-class operator $T \in \mathfrak{B}_1(\mathsf{H})$ *such that* $\mu(P) = tr(TP)$ *for every* $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$.

(i) μ *is* **completely additive***:*

$$\mu\left(\vee_{j\in J} P_j\right) = \sum_{j\in J} \mu(P_j)\,, \tag{7.52}$$

for every family $\{P_j\}_{j\in J} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ *of pairwise orthogonal elements. (The projector* $\vee_{j\in J} P_j$ *exists since* $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}_S)$ *is a complete lattice (Proposition 7.61), and the right-hand side of (7.52) is well-defined because its terms are non-negative).*

(ii) μ *admits a* **support***, namely an element* $P \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ *such that* $\mu(Q) = 0$ *for* $Q \in \mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ *if and only if* $Q \perp P$.

(iii) $\frac{1}{\mu(I)}\mu$ *is the restriction of a* normal algebraic state *on the* C^**-algebra* $\mathfrak{R}$ *(Definition 14.10).*

Remark 7.73 **(1)** If H is *separable* the theorem immediately implies that μ is automatically represented by an operator of trace class. This is because every family of pairwise orthogonal projectors is at most countable if H is separable, so μ is automatically completely additive since σ-additive. In contrast to the case $\mathfrak{R} = \mathfrak{B}(\mathsf{H})$, the trace-class operator representing μ is not unique in general.
(2) A probability measure μ_T over a von Neumann algebra $\mathfrak{R}$ on H (possibly with type-I_2 summands) that is induced by a positive trace-class operator T of trace one is called a **normal state** of $\mathfrak{R}$. Such a measure is nothing but the restriction to $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ of the measure defined by Proposition 7.25 on $\mathscr{L}(\mathsf{H})$. A *normal state* as understood in (iii) is more appropriately the functional $\omega_T := \mathfrak{R} \ni A \mapsto tr(TA)$. However, it is obvious that $\omega_T \restriction_{\mathscr{L}_{\mathfrak{R}}(\mathsf{H})} = \mu_T$. The general notion of normal (algebraic) state will be introduced and discussed in Chap. 14. ■

7.6.5 Algebraic Characterisation of a State as a Noncommutative Riesz Theorem

This section is purely mathematical, and focuses on a characterisation of the space of states $\mathfrak{S}(\mathsf{H})$. The description has a certain interest for the *algebraic formulation* of QM [Str05a] and quantum theories in general, which we will briefly see in Sect. 14.1. Its mathematical relevance resides in that it implies a *noncommutative* version of Riesz's Theorem 1.58 on (finite) positive Borel measures. The word *noncommutative* refers to measures on the projector lattice $\mathscr{L}(\mathsf{H})$ in the sense of axiom **A2** (measure-theory version), rather than on a σ-algebra.

First of all observe that a positive trace-class operator T determines a linear functional of the C^*-algebra of compact operators $\mathfrak{B}_\infty(\mathsf{H})$, given by $\omega_T : \mathfrak{B}_\infty(\mathsf{H}) \to \mathbb{C}$, $\omega_T(A) = tr(TA)$. This is **positive**:

$$\omega_T(A^*A) \geq 0 \quad \text{for any } A \in \mathfrak{B}_\infty(\mathsf{H}). \tag{7.53}$$

In fact, $tr(A^*TA) = tr(A^*T^{1/2}T^{1/2}A) = tr((T^{1/2}A)^*T^{1/2}A) \geq 0$. The last inequality comes from expanding the trace in some basis of H. Viewing ω_T as linear operator on the Banach space $\mathfrak{B}_\infty(\mathsf{H})$ (with the norm of $\mathfrak{B}(\mathsf{H})$), we have

$$||\omega_T|| = trT\ . \tag{7.54}$$

In fact, if $A \in \mathfrak{B}_\infty(\mathsf{H})$ and $||A|| \leq 1$, taking the trace in a basis $\{\psi_j\}_{j\in J}$ of eigenvectors of $T = T^*$ gives

$$|\omega_T(A)| \leq \sum_{j\in J} |p_j(\psi_j|A\psi_j)| \leq \sum_{j\in J} p_j||A\psi_j|| \leq \sum_{j\in J} p_j = trT\ .$$

Eventually $\omega_T(A_N) \to trT$, as $N \to +\infty$, if $A_N := \sum_{0\leq p_j<N} \psi_j(\psi_j|\ \ )$, where $||A_N|| \leq 1$ and $A_N \in \mathfrak{B}_\infty(\mathsf{H})$ because the latter's range is finite-dimensional (see Example 4.18(1)).

Definition 7.74 Let H be a complex Hilbert space. Positive and linear functionals $\omega : \mathfrak{B}_\infty(\mathsf{H}) \to \mathbb{C}$ of unit norm are called **algebraic states** on the C^*-algebra $\mathfrak{B}_\infty(\mathsf{H})$. Their set will be denoted $\mathfrak{C}(\mathfrak{B}_\infty(\mathsf{H}))$.

Therefore every state $T \in \mathfrak{S}(\mathsf{H})$ determines an algebraic state $\omega_T \in \mathfrak{C}(\mathfrak{B}_\infty(\mathsf{H}))$.

This can be accompanied by the following characterisation.

Theorem 7.75 *If* H *is a complex Hilbert space, the mapping* $\mathfrak{S}(\mathsf{H}) \ni T \mapsto \omega_T \in \mathfrak{C}(\mathfrak{B}_\infty(\mathsf{H}))$*, with* $\omega_T(A) = tr(TA)$*,* $A \in \mathfrak{B}_\infty(\mathsf{H})$*, is well defined and bijective. Equivalently: states in* $\mathfrak{S}(\mathsf{H})$ *are in one-to-one correspondence with algebraic states in* $\mathfrak{C}(\mathfrak{B}_\infty(\mathsf{H}))$.

Proof The map $T \mapsto \omega_T$ is well defined by the above argument, and also one-to-one: if $\omega_T = \omega_{T'}$ in fact, then $tr((T - T')A) = 0$ for any compact operator A. By decomposing the self-adjoint, trace-class operator $T - T'$ over an eigenvector basis $\{\phi_i\}_{i\in I}$ and choosing $A = \phi_i(\phi_i|\ \ )$ for any $i \in I$, we conclude the eigenvalues of $T - T'$ must all vanish, so $T - T' = 0$ by (6) in Theorem 4.19(b).

Let us prove the surjectivity of $T \mapsto \omega_T$. Considering $\omega \in \mathfrak{C}(\mathfrak{B}_\infty(\mathsf{H}))$ we try to find $T \in \mathfrak{S}(\mathsf{H})$ such that $\omega = \omega_T$. If $\psi, \phi \in \mathsf{H}$, define $A_{\psi,\phi} := \psi(\phi|\ \ ) \in \mathfrak{B}_\infty(\mathsf{H})$. By definition of norm $||A_{\psi,\phi}|| = ||\psi||\,||\phi||$. The coefficients $\omega(\phi, \psi) := \omega(A_{\psi,\phi})$ define a function $\mathsf{H} \times \mathsf{H} \to \mathbb{C}$, linear in the right argument and antilinear in the left one. Further, $|\omega(\phi, \psi)| = |\omega(A_{\psi,\phi})| \leq 1||A_{\psi,\phi}|| = ||\psi||\,||\phi||$. Then Riesz's representation Theorem 3.16 implies there is a linear map $T' : \mathsf{H} \to \mathsf{H}$ such that $\omega(A_{\psi,\phi}) = (T'\psi|\phi)$, for any $\psi, \phi \in \mathsf{H}$. As $||T'\psi||^2 = |(T'\psi|T'\psi)| = |\omega(A_{\psi,T'\psi})| \leq ||\psi||\,||T'\psi||$, we conclude $||T'|| \leq 1$. Setting $T := T'^*$, we have an operator $T \in \mathfrak{B}(\mathsf{H})$ with $||T|| \leq 1$ and $\omega(A_{\psi,\phi}) = (\psi|T\phi)$ for any $\psi, \phi \in \mathsf{H}$. As ω is positive, taking $\psi = \phi$ implies $T \geq 0$, so in particular $T = T^*$ and $|T| = T$. Now take a Hilbert basis N of H. If $F \subset N$ is finite define $L_F := \sum_{z\in F} z(z|\ )$. By construction $L_F \in \mathfrak{B}_\infty(\mathsf{H})$ and $||L_F|| \leq 1$ (orthogonal projector). Therefore

$$0 \leq \sum_{z\in F}(z||T|z) = \sum_{z\in F}(z|Tz) = \sum_{z\in F}\omega(A_{z,z}) = \omega\left(L_F\right) = |\omega\left(L_F\right)| \leq ||\omega||\,.$$

But F is arbitrary, so $\sum_{z\in N}(z||T|z) \leq 1 < +\infty$, and by definition of trace class, $T \in \mathfrak{B}_1(\mathsf{H})$. Splitting T over an eigenvector basis, $T = \sum_{i\in I} p_i\psi_i(\psi_i|\ )$ (by construction $p_i \geq 0$, $trT = \sum_i p_i \leq ||\omega||$), and taking the trace, by linearity we have

$$|\omega(A)| = |tr(TA)| \leq \sum_{i\in I} p_i|(\psi_i|A\psi_i)| \leq (trT)||A||$$

if $A \in \mathfrak{B}_\infty(\mathsf{H})$ is a finite combination of the $A_{\psi,\phi}$. Since the above operators A are dense in $\mathfrak{B}_\infty(\mathsf{H})$ in the uniform topology (Theorem 4.23), by continuity and linearity $\omega(A) = tr(TA)$ and $|\omega(A)| \leq trT\ ||A||$ for any $A \in \mathfrak{B}_\infty(\mathsf{H})$. The latter tells $||\omega|| \leq tr(T)$; but since we know $trT \leq ||\omega||$, then $trT = ||\omega||$. In particular $trT = 1$, for $||\omega|| = 1$ by assumption. Hence we have $\omega = \omega_T$ for some $T \in \mathfrak{S}(\mathsf{H})$, rendering the map onto. □

Remark 7.76 One fact becomes evident from the proof: we may drop the hypothesis that ω has unit norm, and demand, more weakly, that the norm be finite. Then the positive operator $T_\omega \in \mathfrak{B}_1(\mathsf{H})$ corresponding to ω will satisfy $tr(T_\omega) = ||\omega||$. ■

We wish to interpret the result in the light of the theory of the probability measure ρ on the lattice $\mathscr{L}(\mathsf{H})$, in the sense of axiom **A2** (measure-theory version). To this end recall *Riesz's theorem on positive Borel measures* Theorem 1.58 on the locally compact Hausdorff space X. Consider, slightly modifying the theorem's hypotheses, positive linear functionals $\Lambda : C_0(\mathsf{X}) \to \mathbb{C}$, where $C_0(\mathsf{X})$ is the space of continuous complex functions on X that vanish at infinity with norm $||\ \ ||_\infty$ (Example 2.29(4)).

Proposition 7.77 *If X is locally compact, Hausdorff and $\Lambda : C_0(\mathsf{X}) \to \mathbb{C}$ is a bounded positive linear functional (with norm $||\ \ ||_\infty$ on the domain), there exists a unique positive and σ-additive regular measure μ_Λ on $\mathscr{B}(\mathsf{X})$, such that $\Lambda(f) = \int_{\mathsf{X}} f d\mu_\Lambda$ for any $f \in C_0(\mathsf{X})$. Moreover μ_Λ is finite and $||\Lambda|| = \mu_\Lambda(\mathsf{X})$.*

Proof The restriction $\Lambda\restriction_{C_c(\mathsf{X})}$ gives a positive functional as in Riesz's Theorem 1.58. Applying the theorem produces a measure $\mu_\Lambda : \mathscr{B}(\mathsf{X}) \to [0, +\infty]$ mapping compact sets to a finite measure, uniquely determined by $\Lambda(f) = \int_{\mathsf{X}} f d\mu_\Lambda$, $f \in C_c(\mathsf{X})$, if we impose μ_Λ is regular. So we assume regularity from now on. If Λ is bounded, easily $\mu_\Lambda(\mathsf{X}) = ||\Lambda||$, so $\mu_\Lambda(\mathsf{X})$ is finite. (For any $f \in C_0(\mathsf{X})$ we have

$|\Lambda(f)| \leq \int_X |f| d\mu_\lambda \leq ||f||_\infty \mu_\Lambda(\mathsf{X})$, hence $||\Lambda|| \leq ||\mu_\Lambda(X)||$. For any compact set $K \subset \mathsf{X}$, by local compactness and Hausdorff's property, *Urysohn's lemma* (Theorem 1.24) gives $f_K \in C_c(\mathsf{X})$ such that $f : \mathsf{X} \to [0, 1]$ with $f_K(K) = \{1\}$, so $\mu_\Lambda(K) \leq \int_\mathsf{X} f_K d\mu_\Lambda \leq ||\Lambda|| ||f_K||_\infty = ||\Lambda||$ and then $\mu_\Lambda(\mathsf{X}) \leq ||\Lambda||$, because $\mu_\Lambda(\mathsf{X}) = \sup\{\mu_\Lambda(K) \mid K \subset \mathsf{X}\,, \text{compact}\}$ by inner regularity of μ_Λ.) That μ_Λ is finite implies that any map of $C_0(\mathsf{X})$ can be integrated. Then the above constraint on the integral in $d\mu_\Lambda$, fixing the regular measure μ_Λ on $\mathscr{B}(\mathsf{X})$, becomes $\Lambda(f) = \int_\mathsf{X} f d\mu_\Lambda$ for any $f \in C_0(\mathsf{X})$. $\square$

We know that any positive operator $T \in \mathfrak{B}_1(\mathsf{H})$ gives a generalised measure on $\mathscr{L}(\mathsf{H})$ (a probability measure if $trT = 1$) in the sense of Proposition 7.25. Then Theorem 7.75 implies a *noncommutative version* of Riesz's representation theorem for *finite measures*, stated in Proposition 7.77. This comes about as follows: think of the projector lattice $\mathscr{L}(\mathsf{H})$ as the noncommutative variant of the Borel σ-algebra $\mathscr{B}(\mathsf{X})$, and the C^*-algebra of compact operators $\mathfrak{B}_\infty(\mathsf{H})$ as the noncommutative correspondent to the commutative C^*-algebra $C_0(\mathsf{X})$. (Both algebras are without unit if H is infinite-dimensional and X non-compact, respectively.) In that case the bounded positive functional Λ on $C_0(\mathsf{X})$ becomes the bounded positive functional ω on $\mathfrak{B}_\infty(\mathsf{H})$. In either case the existence of positive functionals ω, Λ implies the existence of corresponding finite measures on $\mathscr{L}(\mathsf{H})$, $\mathscr{B}(\mathsf{X})$ respectively. The latter is what we denoted μ_Λ above, whilst the former is simply defined as $\rho_\omega(P) := tr(T_\omega P)$ for any $P \in \mathscr{L}(\mathsf{H})$, where $T_\omega \in \mathfrak{B}_1(\mathsf{H})$ and ω correspond to one another as in Theorem 7.75 (there T_ω was called T and ω was ω_T). The requests fixing μ_Λ (assumed regular) and ρ_ω are

$$\Lambda(f) = \int_\mathsf{X} f(x)\, d\mu(x) \quad \forall f \in C_0(\mathsf{X}) \quad \text{and} \quad \omega(A) = tr(T_\omega A) \quad \forall A \in \mathfrak{B}_\infty(\mathsf{H})$$

respectively. The identity $||\Lambda|| = \mu_\Lambda(\mathsf{X})$ now is $||\omega|| = trT_\omega$.

Remark 7.78 This discussion serves to explain that the generalisation of the integral of maps in $C_0(\mathsf{X})$ with respect to μ_Λ should be viewed, in the noncommutative setting, as the trace $tr(T_\omega \cdot)$ acting on $\mathfrak{B}_\infty(\mathsf{H})$. Hence if $T_\rho \in \mathfrak{B}_1(\mathsf{H})$ is the operator associated (by Remark 7.28(3) only) to a probability measure $\rho : \mathscr{L}(\mathsf{H}) \to [0, 1]$ (fulfilling (1) and (2) of axiom **A2** (measure-theory version)) by Gleason's theorem, we will use the writing

$$\int_{\mathscr{L}(\mathsf{H})} A d\rho := tr(T_\rho A)\,. \tag{7.55}$$

$\blacksquare$

Now we can prove the noncommutative version of Proposition 7.77.

Theorem 7.79 *If* H *is a complex Hilbert space, separable or of finite dimension* $\neq 2$, *and* $\omega : \mathfrak{B}_\infty(\mathsf{H}) \to \mathbb{C}$ *is a bounded positive linear functional with unit norm, there exists a unique probability measure* $\rho_\omega : \mathscr{L}(\mathsf{H}) \to [0, 1]$ *(satisfying (1), (2) in axiom* **A2** *(measure-theory version)), such that:*

$$\omega(A) = \int_{\mathscr{L}(\mathsf{H})} A d\rho_\omega \quad \forall A \in \mathfrak{B}_\infty(\mathsf{H}) .$$

Suppose further $||\omega||$ *is finite, but not necessarily one. Then* $\rho : \mathscr{L}(\mathsf{H}) \to [0, ||\omega||]$ *and* $\rho_\omega(I) = ||\omega||$ *instead of* $\rho(I) = 1$.

Proof Define $\rho_\omega(P) := tr(T_\omega P)$ for any $P \in \mathscr{L}(\mathsf{H})$, where ω and $T_\omega \in \mathfrak{B}_1(\mathsf{H})$ correspond bijectively as in Theorem 7.75. Then $\omega(A) = tr(T_\omega A) =: \int_{\mathscr{L}(\mathsf{H})} A d\rho_\omega$ for any $A \in \mathfrak{B}_\infty(\mathsf{H})$, because of (7.55) and T_ω is by construction associated to the measure ρ_ω by Gleason's theorem. (Proposition 7.25 ensures ρ_ω fulfils (1), (2) in axiom **A2** (measure-theory version)). Let us prove uniqueness. By Gleason's Theorem 7.26 and Remark 7.28(3), every probability measure ρ on $\mathscr{L}(\mathsf{H})$ satisfies $\rho(P) = tr(T_\rho P)$ for a unique positive operator T_ρ of trace class with unit trace and any $P \in \mathscr{L}(\mathsf{H})$. If $\omega(A) = \int_{\mathscr{L}(\mathsf{H})} A d\rho := tr(T_\rho A)$ for any compact operator A, since we saw $\omega(A) = tr(T_\omega A)$, choosing $A = \psi(\psi|\ \ )$ will give $(\psi|(T_\omega - T_\rho)\psi) = 0$ for any $\psi \in \mathsf{H}$. Hence $T_\rho = T_\omega$, and consequently $\rho(P) = tr(T_\rho P) = tr(T_\omega P) = \rho_\omega(P)$ for any $P \in \mathscr{L}(\mathsf{H})$. All this extends to the case $0 < ||\omega|| \neq 1$, by using the functional $\omega' := ||\omega||^{-1}\omega$. If $||\omega|| = 0$ then $\omega = 0$. Therefore a possible measure ρ compatible with $0 = \omega(A) = Tr(T_\rho A)$ for any $A \in \mathfrak{B}_\infty(\mathsf{H})$ is the null measure. It is unique by the same argument. $\square$

7.7 Introduction to Superselection Rules

For known quantum systems, not all normalised ψ determine states that are physically admissible for describing the quantum system. There are various theoretical reasons (which we shall return to in the sequel) that force the existence of so-called *superselection rules*. This section is an introduction to this notion. A more advanced presentation, relying on spectral theory technicalities that we will discuss in ensuing chapters, appears in Sect. 11.2.

7.7.1 Coherent Sectors, Admissible States and Admissible Elementary Propositions

For some quantum systems affected by so-called *superselection rules*, the system's (separable) Hilbert space H is a Hilbert sum of preferred closed subspaces called *coherent sectors*:

$$\mathsf{H} = \bigoplus_{k \in K} \mathsf{H}_k , \tag{7.56}$$

where $\mathsf{H}_k \neq \{\mathbf{0}\}$ and $\mathsf{H}_k \perp \mathsf{H}_h$ for every $k \neq h$. (We wrote H instead of H_S for the sake of notational simplicity. This will be the convention in this section.) The relevance of these subspaces comes from the fact that physically admissible states in $\mathfrak{S}_p(\mathsf{H})$ are only those represented by vectors in one of the H_k. States given by linear combinations over distinct coherent sectors are not physically permitted. Coherent sectors are associated to collections of mutually exclusive propositions – i.e. orthogonal projectors P_k onto the corresponding coherent sectors, with $\sum_{k\in K} P_k = I$. The sum, if infinite, is meant in strong sense since K is at most countable (otherwise $\sum_{k\in K} P_k := \vee_{k\in K} P_k$ when H is not separable, but this is not our situation). The proposition associated to P_k corresponds to the assertion that the quantity determining the superselection rule has a certain value. More generally, the quantity will not be required to take a specific value on each subspace, but only to range over a certain set specified by the proposition. Let us explain this with two examples. A further example will be given in Sect. 11.2.2 and a fourth one, regarding Bargmann's superselection rule, in Sect. 12.3.4.

Example 7.80 **(1)** The first instance is the **superselection rule of the electric charge** for a charged quantum system. It prescribes that each vector ψ, determining a system's state, satisfy a proposition P_Q of the type: "the system's charge equals Q" for some value Q. Mathematically, then, $tr(P_Q(\psi|\ )\psi) = 1$ for some Q, which amounts to saying $P_Q\psi = \psi$ for some Q (Proposition 7.38). In other words: *states that are determined by single vectors and whose charge is not a definite value are not admissible*. This demand is obvious in classical physics, but not in QM, where an electrically charged system could, a priori, admit states with indefinite charge. Asking the Hilbert space to be separable requires that the number of values Q of the charge, i.e. the coherent sectors with given charge, be at most countable, so that the electric charge cannot vary with continuity.[11]

(2) Another superselection rule concerns the angular momentum of any physical system. From QM we know that the squared modulus J^2 of the angular momentum, when in a definite state, can only take values $j(j+1)$ with j integer or semi-integer (in $\hbar = \frac{h}{2\pi}$ units, where h is the usual Planck constant). The Hilbert space of the system decomposes in two closed orthogonal subspaces, one associated to integer-valued j, the other associated to semi-integer j. The **superselection rule of the angular momentum** dictates that vectors representing states are not linear combinations over both subspaces. It is important to remark that a pure state can have an *undefined* angular momentum, since the state/associated vector is a linear combination of pure states/vectors with distinct angular momenta; by superselection, however, these values must be either all integer or all semi-integer. We will return to this point in Sect. 12.3.2. ■

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[11] Here we are using notions that will be introduced in Chaps. 8 and 9. If the charge is taken to be continuous and H_q is the subspace where it takes the value $q \in \mathbb{R}$, i.e. the q-eigenspace of a self-adjoint operator Q, then the Hilbert space (non-separable) is still a sum $\oplus_{q\in\mathbb{R}}\mathsf{H}_q$, and $\mathbb{R}$ is the *point* spectrum of Q. Some authors, instead, prefer to think of the Hilbert space as a *direct integral*, thereby preserving its separability, and in this case $\mathbb{R} = \sigma_c(Q)$.

In presence of superselection rules associated to the coherent decomposition (7.56) we can define spaces of states $\mathfrak{S}(\mathsf{H}_k)$ and pure states $\mathfrak{S}_p(\mathsf{H}_k)$ of each sector. These can be identified with subsets in $\mathfrak{S}(\mathsf{H})$ and $\mathfrak{S}_p(\mathsf{H})$ respectively, by the following obvious argument: if M is a closed subspace in H, $A \in \mathfrak{B}(\mathsf{M})$ is identified with an operator of $\mathfrak{B}(\mathsf{H})$ simply by extending it as the null operator on $\mathsf{M}^\perp$. If A is positive and of trace class, the extension is positive, of trace class, and the trace is preserved. If A is of the form $(\psi|\ )\psi$, $\psi \in \mathsf{M}$, $||\psi|| = 1$, the extension is alike. In the case considered we identify every $\mathfrak{S}(\mathsf{H}_k)$ and $\mathfrak{S}_p(\mathsf{H}_k)$ with a subset in $\mathfrak{S}(\mathsf{H})$ and $\mathfrak{S}_p(\mathsf{H})$ respectively, extending each state ρ, nonpure or pure, to the zero operator on $\mathsf{H}_k^\perp$. Hence $\mathfrak{S}(\mathsf{H}_k) \cap \mathfrak{S}(\mathsf{H}_j) = \varnothing$ and $\mathfrak{S}_p(\mathsf{H}_k) \cap \mathfrak{S}_p(\mathsf{H}_j) = \varnothing$ if $k \neq j$. Physically admissible pure states for the system described on H are precisely those in:

$$\mathfrak{S}_p(\mathsf{H})_{adm} := \bigsqcup_{k \in K} \mathfrak{S}_p(\mathsf{H}_k)\ . \tag{7.57}$$

We call the family above the set of *admissible pure states* in presence of superselection rules. Physically-admissible nonpure states for the system described on H are then those that can be built as mixtures of admissible pure states. Hence physically-admissible mixed states will be convex combinations of elements of

$$\bigsqcup_{k \in K} \mathfrak{S}(\mathsf{H}_k)\ . \tag{7.58}$$

Certain physically-admissible mixed states will be finite sums

$$\rho = \sum_{i=1}^{n} q_i \rho_i$$

where $q_i \in (0, 1]$, $\sum_i q_i = 1$ and $\rho_i \in \mathsf{H}_{k_i}$ for some $k_i \in K$. The subtle point is that one should also comprise infinite convex combinations. For the moment we shall stick to the finite case. Each state ρ_i decomposes along a basis of eigenvectors, which belong to the corresponding coherent sector H_{k_i}, in accordance with Theorem 4.20:

$$\rho_i = \sum_{j=1}^{+\infty} p_{ij} \phi_{ij} (\phi_{ij}|\cdot)$$

where $p_{ij} \in [0, 1]$ with $\sum_j p_{ij} = 1$, and these numbers are arranged to guarantee the convergence of the series in the uniform topology. Collecting the decompositions of all states we obtain the overall expression for ρ:

$$\rho = \sum_{j=1}^{+\infty} \sum_{i=1}^{n_i} q_i p_{ij} \phi_{ij} (\phi_{ij}|\cdot)\ .$$

By using the strong operator topology, which does not care about the order we follow to sum terms, we may rearrange:

$$\rho = \text{s-}\sum_{h \in H} p_h \psi_h (\psi_h|\cdot) \quad \text{with } \psi_h \in \mathsf{H}_{k_h} \text{ for } k_h \in K \tag{7.59}$$

where $p_h \in [0, 1]$ and $\sum_h p_h = 1$. States $\rho \in \mathfrak{S}(\mathsf{H})$ of form (7.59) are supposed to be *all* admitted by the superselection rule (the elements of $\mathfrak{S}_p(\mathsf{H})_{adm}$ are obviously among them). We stress that, here, we can assume the indexing set H is countable. It is clear that a statistical operator of the form (7.59) satisfies

$$\rho P_k = P_k \rho \quad \text{for any } k \in K, \tag{7.60}$$

where P_k is the orthogonal projector onto the coherent sector H_k. Using the spectral theory developed in Chap. 8, it is easy to establish that if $\rho \in \mathfrak{S}(\mathsf{H})$ satisfies (7.60) then it is of the form (7.59). Hence equation (7.60) characterises the set of physically admissible states completely.

Based on the discussion above we can state the first axiom for quantum systems with superselection rules. *This, and the second axiom below, are to be understood as constraints on* **A1**, **A2**, **A3**.

Ss1. *In presence of superselection rules in a physical system described on a separable Hilbert space* $\mathsf{H} \neq \{\mathbf{0}\}$, *there is a preferred, at most countable family of orthogonal projectors* $\{P_k\}_{k \in K}$ *that is assumed to describe elementary propositions and satisfies*

(i) $P_k \neq 0$,

(ii) $P_k \perp P_h$ if $h \neq k$,

(iii) s-$\sum_k P_k = I$.

The pairwise-orthogonal, closed subspaces $\mathsf{H}_k := P_k(\mathsf{H})$ *are called* **coherent sectors** *or* **superselection sectors**.

The only possible states of the system, called **admissible states**, *are the elements in:*

$$\mathfrak{S}(\mathsf{H})_{adm} := \{\rho \in \mathfrak{S}(\mathsf{H}) \mid \rho P_k = P_k \rho \quad \text{for any } k \in K\} . \tag{7.61}$$

The **admissible pure states** *are therefore those in the subset* $\mathfrak{S}_p(\mathsf{H})_{adm}$ *of (7.57).*

As one might expect,

Proposition 7.81 *The subset* $\mathfrak{S}(\mathsf{H})_{adm}$ *is convex in* $\mathfrak{S}(\mathsf{H})$, *and* $\mathfrak{S}_p(\mathsf{H})_{adm}$ *is the set of extreme elements of* $\mathfrak{S}(\mathsf{H})_{adm}$.

Proof Evidently $\mathfrak{S}(\mathsf{H})_{adm}$ is convex, due to (7.61). The extreme elements of $\mathfrak{S}(\mathsf{H})$ are those in $\mathfrak{S}_p(\mathsf{H})$ (Proposition 7.34), so $\mathfrak{S}_p(\mathsf{H}) \cap \mathfrak{S}(\mathsf{H})_{adm} = \mathfrak{S}_p(\mathsf{H})_{adm}$ is made of extreme elements of $\mathfrak{S}(\mathsf{H})_{adm}$. If $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ does not belong to $\mathfrak{S}_p(\mathsf{H})_{adm}$, it is an incoherent superposition of elements of $\mathfrak{S}_p(\mathsf{H})_{adm}$ and therefore it cannot be extreme. $\square$

Let us now look at the lattice of elementary propositions in presence of superselection rules. Axiom **A3** about post-measurement states has some remarkable implications as soon as states are bound by **Ss1**. Suppose $P \in \mathscr{L}(\mathsf{H})$ satisfies $PP_k = P_k P$ for every $k \in K$, and that a measurement says it is true when the state is $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$. Then the post-measurement state $\rho_P := \frac{P\rho P}{tr(P\rho)}$ still satisfies $P_k \rho_P = \rho_P P_k$, hence it is admissible. As it turns out, *only* these propositions can be admitted, as established by the following proposition.

Proposition 7.82 *Assume that* **Ss1** *holds. In view of axiom* **A3** *on post-measurement states, only certain orthogonal projectors of* $\mathscr{L}(\mathsf{H})$ *can be admitted as elementary propositions on the physical system. These are the projectors* P *satisfying* $P_k P = P P_k$ *for every* k *labelling a coherent sector* H_k.

Proof If $\psi \in \mathsf{H}_k \setminus \{0\}$ is a pure state and we measure the pair of mutually exclusive, compatible elementary observables P, $P' := I - P$, one of them must be true. The post-measurement state, up to renormalisation, is given by vectors $P\psi$ or $P'\psi$. The corresponding states must still belong to $\mathfrak{S}_p(\mathsf{H})_{adm}$ in view of **Ss1**. This requirement implies that $P\psi \in \mathsf{H}_r$ and $P\psi \in \mathsf{H}_s$ for some r, s. If $P\psi = \mathbf{0}$, we automatically think $P\psi$ in H_k. But $\mathsf{H}_k \ni \psi = P\psi + P'\psi \in \mathsf{H}_r \oplus \mathsf{H}_s$. As $\mathsf{H}_k, \mathsf{H}_r, \mathsf{H}_s$ are pairwise orthogonal if k, r, s are distinct, $k = r$ cannot be if $P\psi \neq \mathbf{0}$. In conclusion $P\psi \in \mathsf{H}_k$ if $\psi \in \mathsf{H}_k$ (when $\psi = \mathbf{0}$ this is trivial). If $\phi \in \mathsf{H}$, we can say $PP_k\phi \in \mathsf{H}_k$, and so $P_h P P_k \phi = \delta_{hk} P P_k \phi = P_k P P_h \phi$. Using $I =$ s-$\sum_h P_h$ and the continuity of $P_k P$, then $P_h P P_k \phi = P_k P P_h \phi$ implies $PP_k = P_k P$. $\square$

Example 7.83 Referring to Example 7.80(1), all elementary propositions of a physical system carrying an electric charge must commute with every projector P_Q corresponding to the statement: "the system's charge equals Q" for some value Q. $\blacksquare$

By direct inspection one easily proves that the subset $\mathscr{L}(\mathsf{H})_{adm} \subset \mathscr{L}(\mathsf{H})$ of elements P commuting with the projectors P_k (pairwise orthogonal and satisfying s-$\sum_k P_k = I$) is a complete orthomodular sublattice of $\mathscr{L}(\mathsf{H})$. The P_k are clearly central in $\mathscr{L}(\mathsf{H})_{adm}$, so $\mathscr{L}(\mathsf{H})_{adm}$ cannot coincide with $\mathscr{L}(\mathsf{H})$, whose centre is just $\{0, I\}$.

If the family of projectors P_k describes all superselection rules of the system, in each H_k all possible coherent combinations of vectors must be physically admissible. Therefore H_k cannot admit finer coherent decompositions: there is no element P in the centre of $\mathscr{L}(\mathsf{H})_{adm}$ such that $0 < P < P_k$, making P_k an *atom* of the centre of $\mathscr{L}(\mathsf{H})_{adm}$. The second axiom incorporates everything we have just seen.

Ss2. *The set* $\mathscr{L}(\mathsf{H})_{adm}$ *of elementary propositions permitted by superselection rules, called the* **logic of admissible elementary propositions** *of the system, is a complete, orthomodular sublattice of* $\mathscr{L}(\mathsf{H})$ *satisfying the following properties.*

(i) *The centre of* $\mathscr{L}(\mathsf{H})_{adm}$ *contains the family* $\{P_k\}_{k\in K}$.

(ii) *Each* P_k *is an atom of the centre of* $\mathscr{L}(\mathsf{H})_{adm}$ *(when* $\{P_k\}_{k\in K}$ *describes all superselection rules of the physical system, as we assumed).*

Remark 7.84 **(1)** The meaning of the propositions P_k is provided by physics. Some example have been illustrated previously. A broader physical discussion on the relationship between the P_k and relevant observables defining superselection rules is undertaken in Sects. 11.2.2 and 14.1.7 specifically in comparison with the so-called algebraic approach.
(2) The description above is appropriate for the so-called *discrete* superselection rules. There is another type, called continuous superselection rules, to which we will come back in Chap. 11. ■

Here is an elementary but important point. The states in $\mathfrak{S}(\mathsf{H})_{adm}$, even if fewer than those in $\mathfrak{S}(\mathsf{H})$, are however sufficient to separate admissible elementary propositions.

Proposition 7.85 *Under* **Ss1** *and* **Ss2***, the space of admissible states separates the lattice of admissible elementary propositions: if* $P, P' \in \mathscr{L}(\mathsf{H})_{adm}$ *and* $tr(\rho P) = tr(\rho P')$ *for every* $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$*, then* $P = P'$.

Proof If $P(\mathsf{H}) \subset \mathsf{H}_k$ and $P'(\mathsf{H}) \subset \mathsf{H}_h$ with $h \neq k$, then $tr(\rho P) = tr(\rho P')$ for every $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ is impossible, because $\rho = \psi(\psi|\cdot)$ with $\psi \in P(\mathsf{H})$ and $||\psi|| = 1$ satisfies $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ and gives $tr(\rho P) = 1$ but $tr(\rho P') = 0$. If P, P' project onto the same sector H_k, the claim is true in view of Proposition 7.30(a). □

7.7.2 An Alternate Formulation of the Theory of Superselection Rules

At the beginning of the description of superselection rules we assumed that admissible states are the ones in $\mathfrak{S}(\mathsf{H})_{adm}$, which is strictly contained in $\mathfrak{S}(\mathsf{H})$. Irrespective of our previous choices of description, it is theoretically reasonable to think of the states of the system as *σ-additive probability measures* on the lattice $\mathscr{L}(\mathsf{H})_{adm}$. Without superselection rules, and if $\dim \mathsf{H} \neq 2$, these measures correspond one-to-one, by Gleason's theorem, to positive trace-class operators with unit trace, namely elements of $\mathfrak{S}(\mathsf{H})$. In that case the description in terms of states (trace-class operators) is equivalent to the description in terms of measures.

Now that the focus is on situations where superselection rules are present, we wish to propose an analogous alternative description of the space of states. It is based on *two axioms, understood as constraints on* **A1**, **A2 (measure-theory formulation)** *and* **A3 (measure-theory version)**. The Hilbert space H is implicitly always separable and non-trivial. Since we start by looking at observables instead of states, as opposed to what we did in **Ss1** and **Ss2**, now the first axiom regards observables and the second one states.

Ss1 (measure-theory formulation). *The set* $\mathscr{L}(\mathsf{H})_{adm}$ *of elementary propositions permitted by superselection rules, called* **logic of admissible elementary propositions***, is a complete, orthomodular sublattice of* $\mathscr{L}(\mathsf{H})$ *satisfying the following properties.*

(i) *The centre of* $\mathscr{L}(\mathsf{H})_{adm}$ *contains the countable (at most) family* $\{P_k\}_{k\in K}$ *of orthogonal projectors satisfying* $P_k \neq 0$, $P_k \perp P_h$ *if* $h \neq k$, *and* $s\text{-}\sum_k P_k = I$.

(ii) *Each* P_k *is an atom of the centre of* $\mathscr{L}(\mathsf{H})_{adm}$ *(since* $\{P_k\}_{k\in K}$ *is assumed to describe all superselection rules of the physical system).*

By the same argument that gave Proposition 7.70(b) we have that the family $\{P_k\}_{k\in K}$, if it exists, is uniquely determined by those properties.

Ss2 (measure-theory formulation). *In the presence of superselection rules, the quantum states of the system are described by* σ**-additive probability measures** *on* $\mathscr{L}(\mathsf{H})_{adm}$. *These are maps*

$$\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1] \quad \textit{with} \quad \mu(I) = 1 \tag{7.62}$$

such that

$$\mu\left(\text{s-}\sum_{i=0}^{+\infty} Q_i\right) = \sum_{i=0}^{+\infty} \mu(Q_i) \quad \textit{for } \{Q_i\}_{i\in\mathbb{N}} \subset \mathscr{L}_{\mathfrak{R}}(\mathsf{H})_{adm} \textit{ with } Q_i \perp Q_j \textit{ if } i \neq j. \tag{7.63}$$

As we saw about admissible states, it is easy to see (but important to note) that the σ-additive probability measures on $\mathscr{L}(\mathsf{H})_{adm}$ are sufficient to separate admissible elementary propositions. By contrast to states, however, measures representing quantum states also satisfy the reciprocal statement.

Proposition 7.86 *Under* **Ss1 (measure-theory formulation)** *and* **Ss2 (measure-theory formulation)***:*

(a) σ*-additive probability measures on* $\mathscr{L}(\mathsf{H})_{adm}$ *separate admissible elementary propositions: if* $P, P' \in \mathscr{L}(\mathsf{H})_{adm}$ *and* $\mu(P) = \mu(P')$ *for every* σ*-additive probability measure, then* $P = P'$*;*

(b) *the lattice* $\mathscr{L}(\mathsf{H})_{adm}$ *of admissible elementary propositions separates the set of* σ*-additive probability measures on* $\mathscr{L}(\mathsf{H})_{adm}$*: if* $\mu'(P) = \mu(P)$ *for every* $P \in \mathscr{L}(\mathsf{H})_{adm}$, *then* $\mu = \mu'$.

Proof The proof of (a) descends immediately from Proposition 7.85 and the fact that states define σ-additive probability measures on $\mathscr{L}(\mathsf{H})$ (Proposition 7.25) and hence on $\mathscr{L}(\mathsf{H})_{adm}$ by restriction. Part (b) is true just by definition of measure. □

Theoretically speaking, the formulation relying on the notion of state in terms of a measure is preferable to the one where states are trace-class operators. That is because the physical objects we can really handle are just the probabilities that an elementary proposition P is found true after measurement. The class of all those probabilities $\mu(P)$, $P \in \mathscr{L}(\mathsf{H})_{adm}$, univocally fixes a (generalised) probability measure $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1]$. Conversely, there could, in principle, be different states $\rho \neq \rho' \in \mathfrak{S}(\mathsf{H})_{adm}$ with $tr(\rho P) = tr(\rho' P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$, so our definition of state may be affected by redundancies. Obviously this surplus is impossible in the absence of superselection rules, since $\mathscr{L}(\mathsf{H})$ contains all orthogonal projectors onto one-dimensional subspaces and these are enough to separate the elements of

$\mathfrak{S}(\mathsf{H})$ (Proposition 7.30). The next result (which uses Proposition 7.70) proves that, apart from this possible excess, the use of measures or states is essentially equivalent when the von Neumann algebra $\mathscr{L}(\mathsf{H})''_{adm}$ has certain properties.

Proposition 7.87 *Let $\mathsf{H} \neq \{\mathbf{0}\}$ be a separable Hilbert space. Suppose that*

(i) $\mathfrak{S}(\mathsf{H})_{adm} \subset \mathfrak{S}(\mathsf{H})$ is defined by a family, at most countable, $\{P_k\}_{k\in K} \subset \mathscr{L}(\mathsf{H})$, with $P_k \neq 0$, $P_h \perp P_k$ if $h \neq k$ and s-$\sum_{k\in K} P_k = I$, as in (7.61);

(ii) $\mathscr{L}(\mathsf{H})_{adm} \subset \mathscr{L}(\mathsf{H})$ is a complete orthomodular sublattice whose centre contains $\{P_k\}_{k\in K}$ as atoms.

Then the following hold.

(a) *If $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$, there exists a unique σ-additive probability measure $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0,1]$ such that $tr(\rho P) = \mu(P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$.*

(b) *If $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0,1]$ is a σ-additive probability measure, and $\mathscr{L}(\mathsf{H})''_{adm}$ does not contain algebras of type I_2 as summands (Theorem 7.68), then there exists $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ such that $tr(\rho P) = \mu(P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$.*

(c) *Suppose that either (a) or (b) hold, and fix $P \in \mathscr{L}(\mathsf{H})_{adm}$. Then the associated post-measurement state and measure, in accordance with* **A3**, *(7.30) and* **A3 (measure-theory version)**, *(7.31), satisfy*

$$tr(\rho_P Q) = \mu_P(Q)$$

for every $Q \in \mathscr{L}(\mathsf{H})_{adm}$.

Proof (a) Proposition 7.25 establishes that $\mu' : \mathscr{L}(\mathsf{H}) \ni P \mapsto tr(P\rho)$ is a σ-additive probability measure on $\mathscr{L}(\mathsf{H})$. The map $\mu := \mu'\restriction_{\mathscr{L}(\mathsf{H})_{adm}}$ is therefore a σ-additive probability measure on $\mathscr{L}(\mathsf{H})_{adm}$, the latter being a complete sublattice of $\mathscr{L}(\mathsf{H})$. By construction, $tr(\rho P) = \mu(P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$.
(b) Theorem 7.72 and Remark 7.73 entail that there is a state $\rho_0 \in \mathfrak{S}(\mathsf{H})$ with $tr(\rho_0 P) = \mu(P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$. We claim that there also exists $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ with the same property. If $\psi \in \mathsf{H}$, define $\rho\psi := \sum_{k\in K} P_k\rho_0 P_k\psi$. The series converges as $\{P_k\rho_0 P_k\psi\}_{k\in K}$ is a system of orthogonal vectors and (Lemma 3.25)

$$\sum_{k\in K} ||P_k\rho_0 P_k\psi||^2 \leq \sum_{k\in K} ||P_k||^2\, ||\rho_0||^2\, ||P_k\psi||^2 \leq ||\rho_0||^2 \sum_{k\in K} ||P_k\psi||^2 \leq ||\rho_0||^2\, ||\psi||^2 \,.$$

This argument implies furthermore that $\rho \in \mathfrak{B}(\mathsf{H})$ with $||\rho|| \leq ||\rho_0||$. It is clear that $\rho \geq 0$ because $\rho_0 \geq 0$ and

$$(\psi|\rho\psi) = \left(\psi \left| \sum_{k\in K} P_k\rho_0 P_k\psi \right.\right) = \sum_{k\in K} (\psi|P_k\rho_0 P_k\psi) = \sum_{k\in} (P_k\psi|\rho_0 P_k\psi) \geq 0\,,$$

so that $\rho = |\rho|$. Moreover $P_k\rho = \rho P_k$ for every $k \in K$ just by construction. Let us prove that $\rho \in \mathfrak{B}_1(\mathsf{H})$ and that its trace is 1. If $\{\psi^{(k)}_{j_k}\}_{j_k\in J_k}$ is a basis of $\mathsf{H}_k := P_k(\mathsf{H})$, then $\{\psi^{(k)}_{j_k}\}_{j_k\in J_k, k\in K}$ is a basis of H, because $\mathsf{H} = \oplus_{k\in K}\mathsf{H}_k$ implies that it is a maximal orthonormal system. By construction, $\rho\psi_{j_k}^{(k)} = P_k\rho_0 P_k\psi_{j_k}^{(k)} = P_k\rho_0\psi_{j_k}^{(k)}$ and so

$$\sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}||\rho|\psi_{j_k}^{(k)}) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|\rho\psi_{j_k}^{(k)}) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|P_k\rho_0\psi_{j_k}^{(k)})$$

$$= \sum_{j_k \in J_k, k \in K} (P_k\psi_{j_k}^{(k)}|\rho_0\psi_{j_k}^{(k)}) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|\rho_0\psi_{j_k}^{(k)}) = tr\rho_0 = 1\,.$$

Thus far we have established $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$. To conclude, observe that, if $P \in \mathscr{L}(\mathsf{H})_{adm}$ then $P\rho\psi_{j_k}^{(k)} = PP_k\rho_0 P_k\psi_{j_k}^{(k)} = P_k P\rho_0\psi_{j_k}^{(k)}$, and consequently

$$tr(P\rho) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|P\rho\psi_{j_k}^{(k)}) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|P_k P\rho_0\psi_{j_k}^{(k)})$$

$$= \sum_{j_k \in J_k, k \in K} (P_k\psi_{j_k}^{(k)}|P\rho_0\psi_{j_k}^{(k)}) = \sum_{j_k \in J_k, k \in K} (\psi_{j_k}^{(k)}|P\rho_0\psi_{j_k}^{(k)}) = tr(P\rho_0) = \mu(P)\,,$$

as wanted.
(c) This claim immediately follows from the definitions of μ_P and ρ_P, using $PQP \in \mathscr{L}(\mathsf{H})_{adm}$. □

There remains the issue about whether there exist pairs of distinct elements $\rho, \rho' \in \mathfrak{S}(\mathsf{H})_{adm}$ determining the same σ-additive probability measure $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1]$, in the sense that $tr(\rho P) = tr(\rho' P) =: \mu(P)$ for every $P \in \mathscr{L}(\mathsf{H})_{adm}$. Equivalently, whether or not $\mathscr{L}(\mathsf{H})_{adm}$ *separates* $\mathfrak{S}(\mathsf{H})_{adm}$.

The following statement provides sufficient conditions.

Proposition 7.88 *Assume the hypotheses (i) and (ii) of Proposition 7.87. Suppose, further, that*

$$\mathscr{L}(\mathsf{H})_{adm} \supset \mathscr{L}(\mathsf{H}_k) \quad \textit{with } \mathsf{H}_k := P_k(\mathsf{H}) \textit{ for every } k \in K, \tag{7.64}$$

where elements in $\mathscr{L}(\mathsf{H}_k)$ are viewed as elements in $\mathscr{L}(\mathsf{H})$ by extending them trivially (as zero) on $\mathsf{H}_k^\perp$. The following facts hold.
(a) *$\mathscr{L}(\mathsf{H})_{adm}$ separates the elements of $\mathfrak{S}(\mathsf{H})_{adm}$: if $\rho, \rho' \in \mathfrak{S}(\mathsf{H})_{adm}$ satisfy*

$$tr(P\rho) = tr(P\rho') \quad \textit{for every } P \in \mathscr{L}(\mathsf{H})_{adm}$$

then $\rho = \rho'$.
(b) *Under the hypotheses of Proposition 7.87(b), for every σ-additive probability measure $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1]$ there is exactly one $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ such that $\mu(P) = tr(\rho P)$ for all $P \in \mathscr{L}(\mathsf{H})_{adm}$.*

Proof Obviously (a) implies (b), so it is sufficient to prove (a). Suppose $\rho, \rho' \in \mathfrak{S}(\mathsf{H})_{adm}$ satisfy $tr(\rho Q) = tr(\rho' Q)$ for every $Q \in \mathscr{L}(\mathsf{H})_{adm}$. Since Q can be decomposed as $Q = \sum_k Q_k$ in the strong sense, where $Q_k := QP_k$, using bases in H_k gives

$$0 = tr((\rho - \rho')Q) = \sum_{k\in K} tr((\rho_k - \rho'_k)Q_k) ,$$

where $\rho_k := \rho\restriction_{\mathsf{H}_k} : \mathsf{H}_k \to \mathsf{H}_k$. If $\mathscr{L}(\mathsf{H})_{adm} \supset \mathscr{L}(\mathsf{H}_k)$ for every $h \in K$, we can choose $Q_h = \psi_h(\psi_h|\cdot)$ for any unit vector $\psi_h \in \mathsf{H}_h$ so that

$$0 = \sum_{k\in K} tr((\rho_k - \rho'_k)Q_k) = tr((\rho_h - \rho'_h)Q_h) = (\psi_h|(\rho_h - \rho'_h)\psi_h) = 0 .$$

The arbitrariness of ψ_h implies $(\rho_h - \rho'_h) = 0$. As the latter is valid for any $h \in K$, then $\rho - \rho' = \oplus_h \rho_h - \rho'_h = 0$. $\square$

Remark 7.89 **(1)** Concerning *von Neumann algebras of observables* (Proposition 7.71), in
Sect. 11.2.2 we shall discuss sufficient conditions to guarantee the validity of (7.64). However, in general (7.64) fails and there exist many states $\rho \in \mathfrak{S}(\mathsf{H})_{adm}$ corresponding to a given measure $\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1]$. The logic of admissible elementary propositions is not able to separate states, as opposed to probability measures. This fact suggests that the notion of quantum state, in the presence of superselection rules, is better described by σ-additive measures on the lattice of admissible elementary propositions rather than trace-class operators and vectors. Besides, we have already hinted at operators being in excess to describe the quantum features of a quantum system.
(2) Sometimes, in presence of superselection rules, no constraint the states is assumed and every $\rho \in \mathfrak{S}(\mathsf{H})$ is supposed admissible, though the lattice of elementary propositions is restricted to $\mathscr{L}(\mathsf{H})_{adm}$. With this third formulation the redundancy becomes huge, due to the profusion of states that determine the same probability measure on $\mathscr{L}(\mathsf{H})_{adm}$, and hence carry exactly the same physical information. The following instructive example elucidates that the difference between coherent and incoherent superposition becomes unsustainable within this picture. Take

$$\rho := \sum_{k\in K} p_k \psi_k(\psi_k|\cdot) \in \mathfrak{S}(\mathsf{H})_{adm} ,$$

where we selected one vector $\psi_k \in \mathsf{H}_k$ for every $k \in K$, and fixed the $p_k \in (0, 1]$ with $\sum_{k\in K} p_k = 1$. Next consider a unit vector of the form

$$\Psi := \sum_{k\in K} e^{ic_k}\sqrt{p_k}\psi_k$$

for any choice of $c_k \in \mathbb{R}$. We claim ρ and Ψ give rise to the same σ-additive probability measure on $\mathscr{L}(\mathsf{H})_{adm}$. Indeed, if $Q \in \mathscr{L}(\mathsf{H})_{adm}$, we define $\rho' := \Psi(\Psi|\cdot)$ and complete both Ψ and $\{\psi_k\}_{k\in K}$ to separate Hilbert bases of H. Then

$$tr(\rho' Q) = (\Psi|Q\Psi) = \sum_{k\in K}\sum_{h\in K} \sqrt{p_k p_h}(\psi_k|Q\psi_h) = \sum_{k\in K}\sum_{h\in K} \sqrt{p_k p_h}\delta_{hk}(\psi_k|Q\psi_h)$$

$$= \sum_{k \in K} p_k (\psi_k | Q \psi_k) = tr(\rho Q) ,$$

where, since $P_k Q = Q P_k$, we have used the identity

$$(\psi_k | Q \psi_h) = (\psi_k | Q P_h \psi_h) = (\psi_k | P_h Q \psi_h) = (P_h \psi_k | Q \psi_h) = \delta_{hk} (\psi_k | Q \psi_h) .$$

In general $\Psi(\Psi|\cdot)$ will not belong in $\mathfrak{S}(\mathsf{H})_{adm}$, because it is a coherent superposition of pure states of different sectors as soon as $p_k > 0$ for at least two values of k. The picture can be made even more general by allowing for more than one vector in each coherent sector H_k. This is yet another bit of suggesting evidence that – in the presence of superselection rules – the notion of quantum state is better portrayed by σ-additive measures on admissible elementary propositions than by trace-class operators and vectors. ■

Exercises

7.1 Prove that in a Boolean algebra X, for any $a \in \mathsf{X}$ there exists a unique element, written $\neg a$, that satisfies properties (i), (ii) in Definition 7.8(c).

7.2 Prove that every orthocomplemented lattice X satisfies De Morgan's laws (7.6). Then prove Proposition 7.11:

Proposition. *Let* X *be an orthocomplemented lattice. The following facts hold for any subset* $A \subset \mathsf{X}$.
(a) *If* A *is finite, then* $\neg \sup_{a \in A} a = \inf_{a \in A} \neg a$, $\neg \inf_{a \in A} a = \sup_{a \in A} \neg a$.
(b) *If* A *is infinite, then* $\sup_{a \in A} a$ ($\inf_{a \in A} a$) *exists if and only if* $\inf_{a \in A} \neg a$ ($\sup_{a \in A} \neg a$) *exists. If so, the former (latter) relation in (7.7) holds.*
(c) *If* X *is complete (*σ*-complete), then (7.7) holds for every (countable) subset* $A \subset \mathsf{X}$.

Hint. Just use the definition of sup, inf and requirement (iv) ($a \geq b \Leftrightarrow \neg b \geq \neg a$) in the definition of orthocomplemented lattice.

7.3 Show that an orthocomplemented lattice is complete (σ-complete) iff every set (resp. countable set) $A \subset \mathsf{X}$ admits greatest lower bound.

Hint. Exploit Exercise 7.2.

7.4 Prove Proposition 7.15:

Proposition. *Let* $h : \mathsf{X} \to \mathsf{Y}$ *be an isomorphism of orthocomplemented lattices. If both* X *and* Y *are (*σ*-)complete, then* h *is an isomorphism of (*σ*-)complete lattices.*

Solution. We only consider complete lattices, for the other case is similar. We have to prove that if $A \subset \mathsf{X}$ then $h(\sup A) = \sup h(A)$ and $h(\inf A) = \inf h(A)$. Remark 7.9(4) permits us to prove the former relation only. Since $a \leq \sup A$ for every $a \in A$ and h preserves the ordering (Remark 7.14(1)), then $h(a) \leq h(\sup A)$ so that (i) $\sup h(A) \leq h(\sup A)$. Since $h(a) \leq \sup h(A)$ for $a \in A$, applying the isomorphism of orthocomplemented lattices h^{-1} (it preserves the order), we get $a \leq h^{-1}(\sup h(A))$ so that $\sup A \leq h^{-1}(\sup h(A))$. Applying h, we conclude (ii) $h(\sup A) \leq \sup h(A)$. Now (i) and (ii) together imply $h(\sup A) = \sup h(A)$.

7.5 Let $(\mathsf{X}, \geq)$ be atomic and orthomodular. Prove that if $p \in \mathsf{X}$ is not an atom, then $A_p := \{a \leq p \mid a \text{is an atom}\}$ contains more than one atom.

Solution. Since $(\mathsf{X}, \geq)$ is atomic, there exists $a \in A_p$. Then observe that orthomodularity implies $p = a \vee (\neg a \wedge p)$ and hence either $\neg a \wedge p = \mathbf{0}$ (and $p = a$) or $\neg a \wedge p \neq \mathbf{0}$. The second case is only possible when $p \neq a$. Next, atomicity implies that $\neg a \wedge p \geq a'$ for some atom a'. Moreover, $a' \neq a$ because otherwise $a' \leq \neg a$ and $a' \leq \neg a'$ would imply $a' = \mathbf{0}$, and atoms cannot be zero. Since then $a' \leq (\neg a \wedge p \leq)\ p$, we conclude that $A_p \ni a' \neq a$.

7.6 Prove that an orthomodular lattice $(\mathsf{X}, \geq)$ is atomistic iff atomic.

Solution. The only thing that needs to be proved is that 'atomicity $\Rightarrow$ atomisticity'. Consider $p \in \mathsf{X}$ and $A_p := \{a \leq p \mid a \text{is an atom}\}$. Suppose that there exists $q \neq p$ such that $a \leq q \leq p$ if $a \in A_p$. Orthomodularity requires $p = q \vee (\neg q \wedge p)$, and so $\neg q \wedge p \neq \mathbf{0}$ because $p \neq q$. Since X is atomic, there is an atom $a' \leq \neg q \wedge p$, so in particular $a' \leq \neg q$ and also $a' \leq p$. Hence $a' \in A_p$ and, in turn, $a' \leq q$. Therefore $a' \leq q \wedge \neg q = \mathbf{0}$, which is forbidden because a' is an atom. In summary, the element q cannot exist, and p is the smallest element satisfying $a \leq p$ if $a \in A_p$, namely $p = \sup A_p$. This shows X is atomistic.

7.7 Prove Proposition 7.10:

Proposition. *If X is an orthocomplemented lattice, then $p, q \in \mathsf{X}$ commute if and only if the orthocomplemented lattice generated by $\{p, q\}$ (the intersection of all bounded orthocomplemented sublattices containing $\{p, q\}$) is Boolean.*

Solution. Suppose the intersection of all (orthocomplemented) sublattices containing $\{p, q\}$ is Boolean. Then $r_1 := p \vee \neg q$, $r_2 := q \vee \neg p$, $r_3 := p \wedge q$ are contained in that Boolean lattice and hence satisfy $r_i \perp r_j$ if $i \neq j$, $p = r_1 \vee r_3$, and $q = r_2 \vee r_3$. Assume, conversely, p_1 and p_2 commute so that $p = r_1 \vee r_2$ and $q = r_2 \vee r_3$ with $r_i \perp r_j$ if $i \neq j$. A sublattice X_0 containing p and q is made of the following elements: $\mathbf{0}$, r_1, r_2, r_3, $r_4 := \neg(r_1 \vee r_2 \vee r_3)$, all possible two-fold, three-fold and four-fold joins of them ($\mathbf{1}$ is one of these). It is easy to prove that X_0 is Boolean. Every sublattice containing p and q necessarily contains r_1, r_2, r_3, and so it also contains X_0. In summary, X_0 is the intersection of all (orthocomplemented) sublattices containing $\{p, q\}$.

7.8 Prove Propositions 7.17 and 7.18.

7.9 Let $\mathscr{L}$ be a complete lattice. A map $\mathscr{L} \ni a \mapsto a^g \in \mathscr{L}$ is called a **Galois connection** if $a \le b$ entails $b^g \le a^g$ and $a \le (a^g)^g$ for every $a, b \in \mathscr{L}$. The map

$$\mathscr{L} \ni a \mapsto a^c := (a^g)^g \in \mathscr{L}$$

is called the **closure operator associated to** g.
Prove that in a complete lattice $\mathscr{L}$ equipped with a closure operator ^{c}, for every $a, b \in \mathscr{L}$,
(i) $a^c \le a$,
(ii) $a^c = (a^c)^c$,
(iii) $a \le b$ implies $a^c \le b^c$.

7.10 If U is any set, consider the bounded Boolean lattice of the subsets of U ordered by inclusion, and let $R \subset U \times U$ be a reflexive and symmetric relation. Prove that

$$A^g := \{x \in U \mid (x, y) \notin R \quad \forall y \in A\}\,, \quad A \subset U$$

defines a Galois connection.

7.11 Referring to Exercise 7.9, suppose that $\mathscr{L}$ is a complete bounded lattice equipped with a Galois connection ^{g} and induced closure ^{c}.
Assume further that $a^g \wedge a = \mathbf{0}$ for all $a \in \mathscr{L}$ and $\mathbf{0}^g = \mathbf{1}$, and set
$a^\perp := a^g$ if $a \in \mathscr{L}^c$.
Prove that $\mathscr{L}^c := \{a \in \mathscr{L} \mid a = a^c\}$ is a complete orthocomplemented lattice with the same partial order as $\mathscr{L}$ and the same top and bottom elements.
(With obvious notation) show that
(a) $a \wedge_{\mathscr{L}^c} b = a \wedge_{\mathscr{L}} b$,
(b) $a \vee_{\mathscr{L}^c} b = (a \vee_{\mathscr{L}} b)^c$
for every $a, b \in \mathscr{L}^c$.

7.12 Study the relationship between Exercise 3.33 and Exercises 7.9 and 7.11.

7.13 Consider two self-adjoint operators

$$A = \sum_{n=1}^{N} a_n P_{a_n} \quad \text{and} \quad B = \sum_{m=1}^{M} b_m Q_{b_m}$$

that represent, as in Examples 7.51, observables in the d-dimensional Hilbert space H (d finite). Show that A, B commute iff the orthogonal projectors P_{a_n}, Q_{b_m} commute, irrespective of how we choose the eigenvalues a_n, b_m.

Hint. Identify H with $\mathbb{C}^d$ and diagonalise simultaneously the matrices representing A and B.

7.14 Consider two self-adjoint operators A, B representing, as in the previous exercise, observables in a Hilbert space H of finite dimension d. Prove that if A and B commute, there exists a third observable (self-adjoint operator) C such that: $A = f(C)$

and $B = g(C)$ in the sense of (7.41), for some real-valued maps on $\mathbb{R}$. Show that C, f and g can be chosen in infinitely many ways.

Hint. If $\{\psi_n\}_{n=1,\ldots,d}$ is an orthonormal basis of H of eigenvectors for both A and B, define $C := \sum_{k=1}^d k\psi_k(\psi_k|\cdot)$. We must find f, g such that $A = \sum_{k=1}^d f(k)\psi_k(\psi_k|\cdot)$ and $B = \sum_{k=1}^d g(k)\psi_k(\psi_k|\cdot)$. At this point the choice for f, g should be patent.

7.15 Prove that two mixed states ρ_1, ρ_2 on the Hilbert space H satisfy $\overline{Ran(\rho_1)} \perp \overline{Ran(\rho_2)}$ iff there exists an orthogonal projector $P \in \mathscr{L}(\mathsf{H})$ with $tr(\rho_1 P) = 1$, $tr(\rho_2 P) = 0$.

Solution. If $\overline{Ran(\rho_1)} \perp \overline{Ran(\rho_2)}$, the orthogonal projector onto $\overline{Ran(\rho_1)}$ solves the problem. Conversely, if $tr(\rho_1 P) = 1$ and $tr(\rho_2 P) = 0$ for some $P \in \mathscr{L}(\mathsf{H})$, let $P' := I - P$. Then $1 = tr(\rho_1) = tr(P\rho_1 P) + tr(P'\rho_1 P') + tr(P'\rho_1 P) + tr(P\rho_1 P')$. But $tr(P\rho_1 P) = tr(\rho_1 P) = 1$, $tr(P'\rho_1 P) = tr(\rho_1 PP') = 0$, $tr(P\rho_1 P') = tr(\rho_1 P'P) = 0$, and therefore $tr(P'\rho_1 P') = 0$. Since $P'\rho_1 P'$ is positive, self-adjoint and of trace class, and the trace equals the sum of the eigenvalues, the latter all vanish. By the spectral decomposition Theorem 4.20 we have $P'\rho_1 P' = 0$, so $\rho_1 = P\rho_1 P + P'\rho_1 P + P\rho_1 P'$. From this identity we easily see that $P\rho_1 P' \neq 0$ implies $(x + ay|\rho_1(x + ay)) < 0$ for some $x \in P(\mathsf{H})$, $y \in P'(\mathsf{H})$, with $a \in \mathbb{R}$ or $a \in i\mathbb{R}$ of sufficiently large modulus. Hence $P\rho_1 P' = P'\rho_1 P = 0$ and $\rho_1 = P\rho_1(P + P') = P\rho_1$, and then $Ran(\rho_1) \subset P(\mathsf{H})$. A similar reasoning gives $P'\rho_2 P = P\rho_2 P' = 0$, whence $\rho_2 = P'\rho_2(P + P') = P'\rho_2$. This implies $Ran(\rho_2) \subset P'(\mathsf{H})$, and therefore $\overline{Ran(\rho_1)} \perp \overline{Ran(\rho_2)}$.

7.16 Prove that a state $\rho \in \mathfrak{S}(\mathsf{H})$ is pure if and only if $tr(\rho^2) = (tr(\rho))^2$.

Hint. Decompose ρ over a basis of eigenvectors and exploit the fact that the eigenvalues are non-negative.

7.17 Prove the statement in Remark 7.35.

Hint. Use the result in Exercise 7.16 and the fact that $\rho \geq 0$ and $||\rho|| = \sup\{|\lambda| \mid \lambda \in \sigma_p(\rho)\}$.

7.18 Consider $N \geq 1$ vectors $\phi_1, \ldots, \phi_N \in \mathsf{H}$, with H a complex Hilbert space, and suppose that $||\phi_i|| = 1$ for $i = 1, \ldots, N$ (notice that we do not require that $(\phi_i|\phi_j) = 0$ for $i \neq j$). Define

$$\rho := \sum_{i=1}^N p_i \phi_i(\phi_i|\cdot)$$

where $p_i \in (0, 1)$ for every $i = 1, \ldots, N$ and $\sum_{i=1}^N p_i = 1$. Prove that ρ defines a pure state if and only if $N = 1$ or $N > 1$ but $\phi_i(\phi_i|\cdot) = \phi_j(\phi_j|\cdot)$ for every $i, j = 1, \ldots, N$.

Hint. If $\rho := \sum_{i=1}^{N} p_i \phi_i(\phi_i|\cdot)$ is a pure state, then $||\rho||_2 = 1$, namely

$$\left|\left|\sum_{i=1}^{N} p_i \phi_i(\phi_i|\cdot)\right|\right|_2 = 1 = \sum_{i=1}^{N} p_i 1 = \sum_{i=1}^{N} p_i ||\phi_i(\phi_i|\cdot)||_2 = \sum_{i=1}^{N} ||p_i \phi_i(\phi_i|\cdot)||_2 \, .$$

Hence

$$\left|\left|\sum_{i=1}^{N} p_i \phi_i(\phi_i|\cdot)\right|\right|_2 = \sum_{i=1}^{N} ||p_i \phi_i(\phi_i|\cdot)||_2 \, .$$

This is the limiting case of the triangle inequality for a norm of the *real* inner product space of self-adjoint elements of $\mathfrak{B}_2(\mathsf{H})$. In a real vector space with real inner product, $||x_1 + \cdots + x_N|| = ||x_1|| + \cdots + ||x_N||$ only if there exist a vector x and non-negative scalars a_i with $x = a_i x_i$ for $i = 1, \ldots, N$ (see Exercise 3.6). Therefore there must exist $T = T^* \in \mathfrak{B}_2(\mathsf{H})$ such that $p_i \phi_i(\phi_i|\cdot) = a_i T$ for real coefficients a_i, $i = 1, \ldots, N$.

Chapter 8
Spectral Theory I: Generalities, Abstract C^*-Algebras and Operators in $\mathfrak{B}(\mathsf{H})$

A mathematician plays a game and invents the rules. A physicist plays a game whose rules are dictated by Nature. As time goes by it is more and more evident that the rules the mathematician finds appealing are precisely the ones Nature has chosen.

P.A.M. Dirac

In this purely mathematically-flavoured chapter we introduce the basic spectral theory on normed spaces, leading up to spectral measures and the spectral decomposition theorem for normal operators in $\mathfrak{B}(\mathsf{H})$, with H a Hilbert space. (The spectral theorem for *unbounded* self-adjoint operators will be discussed in the next chapter.) Here we present a number of general results about abstract C^*-algebras and *-homomorphisms.

The first part is devoted to the *resolvent set* and *spectrum* of an operator, or an element in a Banach algebra with unit. Given a normal element in a unital C^*-algebra, possibly a normal operator in a concrete algebra of bounded operators on a Hilbert space, we shall prove there exists a *-homomorphism mapping continuous functions defined on a compact subset in $\mathbb{C}$ (the spectrum of the element) to algebra elements, i.e. operators. In case we are dealing with operators we will show that this *-homomorphism extends to the C^*-algebra of bounded measurable functions defined on the compact set.

The spectrum of an operator is a collection of complex numbers that generalise eigenvalues. The spectral theorem, proved afterwards, decomposes any operator – in this chapter always bounded and normal – by integrating the spectrum with respect to a suitable "projector-valued" measure. Altogether, the spectral theorem may be viewed as a generalisation to Hilbert spaces of the diagonalisation of complex-valued normal matrices. The tools necessary to establish the spectral theorem are useful also for other reasons. Through them, namely, we will be able to define "operators depending on operators", a notion with several applications in mathematical physics.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_8

The relationship between spectral theory and Quantum Mechanics lies in the fact that projector-valued measures are nothing but the observables defined in the previous chapter. Via the spectral theorem, observables are in one-to-one correspondence with self-adjoint operators (typically unbounded), and the latter's spectra are the sets of possible measurements of observables. The correspondence observables/self-adjoint operators will allow us to formulate Quantum Theory in tight connection to Classical Mechanics, where the observables are the physical quantities represented by real functions. Let us present the contents in detail.

In section one we will define the *spectrum*, the *resolvent set* and the *resolvent operator*, establish their main properties and discuss the formula for the *spectral radius*. All this will be generalised to abstract Banach algebras or C^*-algebras, and include the proof of the *Gelfand–Mazur theorem* and a brief overview of the major features of C^*-algebra representations. We shall state the important *Gelfand-Najmark theorem*, whereby any unital C^*-algebra is a concrete C^*-algebra of operators on a Hilbert space. The proof will be given in Chap. 14, after the GNS theorem.

In section two we shall construct continuous *-homomorphisms of C^*-algebras of functions, induced either by normal elements in an abstract C^*-algebra or by bounded self-adjoint operators on Hilbert spaces. These homomorphisms represent the primary tool towards the spectral theorem. We will discuss the general properties of *-homomorphisms of unital C^*-algebras and positive elements of C^*-algebras. Then we will introduce the *Gelfand transform* to study commutative C^*-algebras with unit, and prove the *commutative Gelfand-Najmark theorem*.

In the third section we shall introduce *spectral measures*, also known as *projector-valued measures* (PVMs), and define the integral of a bounded function with respect to a PVM.

The spectral theorem for normal bounded operators (in particular self-adjoint or unitary) and further technical facts will be dealt with in Sect. 8.4.

The final section is devoted to *Fuglede's theorem* and some consequences.

8.1 Spectrum, Resolvent Set and Resolvent Operator

In this section we study the structural concepts and results of spectral theory in normed, Banach and Hilbert spaces, but also in the more general context of Banach and C^*-algebras.

We shall make use of *analytic functions* defined on domains in $\mathbb{C}$ with values in a complex Banach space [Rud86], rather than in $\mathbb{C}$.

Definition 8.1 Let $(\mathsf{X}, ||\ ||)$ be a Banach space over $\mathbb{C}$ and $\Omega \subset \mathbb{C}$ a non-empty open set. A function $f : \Omega \to \mathsf{X}$ is called **analytic** if for any $z_0 \in \Omega$ there exists $\delta > 0$ such that

$$f(z) = \sum_{n=0}^{+\infty} (z - z_0)^n a_n \quad \text{for any } z \in B_\delta(z_0),$$

where $B_\delta(z_0) \subset \Omega$, $a_n \in \mathsf{X}$ for any $n \in \mathbb{N}$ and the series converges in norm $||\ ||$.

The theory of analytic functions in Banach spaces is essentially the same as that of complex-valued analytic functions, which we take for granted; the only difference is that on the range the Banach norm replaces the modulus of complex numbers. With this proviso, all definitions, theorems and proofs are the same as in the holomorphic case.

8.1.1 Basic Notions in Normed Spaces

We begin with operators on normed spaces, and recall that if X is a vector space, the sentence "A is a operator on X" (Definition 5.1) means $A : D(A) \to \mathsf{X}$, where the domain $D(A) \subset \mathsf{X}$ is a subspace, usually not closed, in X.

Definition 8.2 Let A be an operator on the complex normed space X.
(a) One calls **resolvent set** of A the set $\rho(A)$ of numbers $\lambda \in \mathbb{C}$ such that:
(i) $\overline{Ran(A - \lambda I)} = \mathsf{X}$;
(ii) $(A - \lambda I) : D(A) \to \mathsf{X}$ is injective;
(iii) $(A - \lambda I)^{-1} : Ran(A - \lambda I) \to \mathsf{X}$ is bounded.
(b) If $\lambda \in \rho(A)$, the **resolvent** of A is the operator

$$R_\lambda(A) := (A - \lambda I)^{-1} : Ran(A - \lambda I) \to D(A) .$$

(c) The **spectrum** of A is the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$.
The spectrum of A is the disjoint union of the three subsets below:
(i) the **point spectrum** $\sigma_p(A)$, made by the $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not injective;
(ii) the **continuous spectrum** $\sigma_c(A)$, made by the $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is injective and $\overline{Ran(A - \lambda I)} = \mathsf{X}$, but $(A - \lambda I)^{-1}$ is not bounded;
(iii) the **residual spectrum** $\sigma_r(A)$, made by the $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is injective, but $\overline{Ran(A - \lambda I)} \neq \mathsf{X}$.

Remarks 8.3 **(1)** It is clear that $\sigma_p(A)$ consists precisely of the eigenvalues of A (see Definition 3.58). In case $\mathsf{X} = \mathsf{H}$ is a Hilbert space and the eigenvectors of A form a basis in H one says A has **purely point spectrum**. This does not mean, generally speaking, that $\sigma_p(T) = \sigma(T)$. For example compact self-adjoint operators have purely point spectrum, but 0 may belong to the continuous spectrum.
(2) There exist other decompositions of the spectrum in the case that $\mathsf{X} = \mathsf{H}$ is a Hilbert space and A is normal in $\mathfrak{B}(\mathsf{H})$, or self-adjoint on H. We shall consider alternative splittings in the following chapter, after the spectral theorem for unbounded self-adjoint operators. An in-depth analysis of these classifications, with reference to important operators in QM, can be found in [ReSi80, BiSo87, AbCi97, BEH07, Schm12]. ■

We start making a few precise assumptions, like taking X Banach and working with *closed* operators. In particular, the next result holds if $T \in \mathfrak{B}(\mathsf{X})$ or, on a Hilbert space $\mathsf{X} = \mathsf{H}$, if $T : D(T) \to \mathsf{H}$ is self-adjoint or an adjoint operator on H, since both are automatically closed.

Theorem 8.4 *Let T be a closed operator on a Banach space $\mathsf{X} \neq \{\mathbf{0}\}$. Then*
(a) $\lambda \in \rho(T) \Leftrightarrow T - \lambda I : D(T) \to \mathsf{X}$ *is a bijection.*
(b) *(i) $\rho(T)$ is open,*
(ii) $\sigma(T)$ is closed,
(iii) if $\rho(T) \neq \emptyset$, the map $\rho(T) \ni \lambda \mapsto R_\lambda(T) \in \mathfrak{B}(\mathsf{X})$ is analytic.
(c) *If $D(T) = \mathsf{X}$ (hence $T \in \mathfrak{B}(\mathsf{X})$):*
(i) $\rho(T) \neq \emptyset$,
(ii) $\sigma(T)$ is non-empty and compact,
(iii) $|\lambda| \leq ||T||$ for any $\lambda \in \sigma(T)$.
(d) *For any $\lambda, \mu \in \rho(T)$ the* **resolvent identity** *holds:*

$$R_\lambda(T) - R_\mu(T) = (\lambda - \mu) R_\lambda(T) R_\mu(T) .$$

Remarks 8.5 **(1)** A comment on (c): if X is a Banach space and $D(T) = \mathsf{X}$, then $T : D(T) \to \mathsf{X}$ is closed iff $T \in \mathfrak{B}(\mathsf{X})$, by the closed graph Theorem 2.99.
(2) Part (a) is rather useful for deciding whether λ belongs in $\rho(T)$. It is not necessary to consider the topology, i.e. the density of $Ran(T - \lambda I)$ and the boundedness of $(T - \lambda I)^{-1}$. As a matter of fact, it is enough to check $T - \lambda I : D(T) \to \mathsf{X}$ is bijective, a set-theoretical property. ■

Proof of Theorem 8.4. (a) If $\lambda \in \rho(T)$, it suffices to show $Ran(T - \lambda I) = \mathsf{X}$. Since $(T - \lambda I)^{-1}$ is continuous, there exists $K \geq 0$ such that $||(T - \lambda I)^{-1} x|| \leq K||x||$ for any $x = (T - \lambda I)y \in Ran(T - \lambda I)$. Consequently, for any $y \in D(T)$:

$$||y|| \leq K||(T - \lambda I)y|| . \tag{8.1}$$

Because $\overline{Ran(T - \lambda I)} = \mathsf{X}$, if $x \in \mathsf{X}$ there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset D(T)$ for which $(T - \lambda I)y_n \to x$, as $n \to +\infty$. From (8.1) we conclude $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and so it admits a limit $y \in \mathsf{X}$. As T is a closed operator, $y \in D(T)$ and $(T - \lambda I)y = x$, hence $x \in Ran(T - \lambda I)$. Therefore $Ran(T - \lambda I) = \mathsf{X}$, as claimed.

Suppose now $T - \lambda I$ is a bijection from $D(T)$ to X; to prove the claim we need to show $(T - \lambda I)^{-1}$ is continuous. Since T is closed, then also $T - \lambda I$ is closed, i.e. its graph is closed. But $T - \lambda I$ is a bijection, so $(T - \lambda I)^{-1}$ has a closed graph and is therefore closed. Being $(T - \lambda I)^{-1}$ defined on X by assumption, Theorem 2.99 implies $(T - \lambda I)^{-1}$ is bounded.
(b) If $\mu \in \rho(T)$, the series

$$S(\lambda) := \sum_{n=0}^{+\infty} (\lambda - \mu)^n R_\mu(T)^{n+1}$$

converges absolutely in operator norm (hence in the uniform topology) provided

$$|\lambda - \mu| < 1/||R_\mu(T)|| \,. \tag{8.2}$$

In fact,

$$\sum_{n=0}^{+\infty} |\lambda - \mu|^n ||R_\mu(T)^{n+1}|| \le \sum_{n=0}^{+\infty} |\lambda - \mu|^n ||R_\mu(T)||^{n+1} = ||R_\mu(T)|| \sum_{n=0}^{+\infty} |\,(\lambda - \mu)\; ||R_\mu(T)||\,|^n \,.$$

The last series is geometric, of reason $|\,(\lambda - \mu)\;||R_\mu(T)||\,|$, and converges because $|\,(\lambda - \mu)\;||R_\mu(T)||\,| < 1$ by (8.2).

If λ satisfies the above condition, applying $T - \lambda I = (T - \mu I) + (\mu - \lambda)I$ first to the left, and then to right of $S(\lambda)$ gives (again using the definition $R_\mu(T)^0 := I$):

$$(T - \lambda I)S(\lambda) = I_{\mathsf{X}}$$

while:

$$S(\lambda)(T - \lambda I) = I_{D(T)} \,.$$

Hence if $\mu \in \rho(T)$ there exists an open neighbourhood of μ such that, for any λ in that neighbourhood, the left and right inverses of $T - \lambda I$, from X to $D(T)$, exist and are finite. By (a) then, the neighbourhood is contained in $\rho(T)$, and so $\rho(T)$ is open and $\sigma(T) = \mathbb{C} \setminus \rho(T)$ closed. Moreover $R_\lambda(T)$ has a Taylor series around any point of $\rho(T)$ in the uniform topology, so by definition $\rho(T) \ni \lambda \mapsto R_\lambda(T)$ is analytic and maps $\rho(T)$ to the Banach space $\mathfrak{B}(\mathsf{X})$.

(c) In case $D(T) = \mathsf{X}$, since T is closed and X Banach, the closed graph theorem makes T bounded. If $\lambda \in \mathbb{C}$ satisfies $|\lambda| > ||T||$, the series

$$S(\lambda) = \sum_{n=0}^{+\infty} (-\lambda)^{-(n+1)} T^n$$

($T^0 := I$), converges absolutely in operator norm. A direct computation, as before, gives the identities

$$(T - \lambda I)S(\lambda) = I$$

and

$$S(\lambda)(T - \lambda I) = I \,,$$

hence $S(\lambda) = R_\lambda(T)$ by (a). Again (a) implies that every $\lambda \in \mathbb{C}$ with $|\lambda| > ||T||$ belongs to $\rho(T)$, which is thus non-empty. Furthermore, if $\lambda \in \sigma(T)$, $|\lambda| \le ||T||$, and $\sigma(T)$ will be compact if it is non-empty, being closed and bounded. Let us show $\sigma(T) \neq \emptyset$. Assume the contrary and argue by contradiction. Then $\lambda \mapsto R_\lambda(T)$ is defined on $\mathbb{C}$. Fix $f \in \mathsf{X}'$ (dual to X) and $x \in \mathsf{X}$, and consider the complex-valued function $\rho(T) \ni \lambda \mapsto g(\lambda) := f(R_\lambda(T)x)$. It is certainly analytic on $\mathbb{C}$, because if

$\mu \in \rho(T)$, on a neighbourhood of μ contained in $\rho(T)$ we have a Taylor expansion

$$f(R_\lambda(T)x) := \sum_{n=0}^{+\infty}(\lambda - \mu)^n f(R_\mu(T)^{n+1}x) .$$

We have used the continuity of the linear functional f, and the fact the series converges uniformly (and so weakly). Hence assuming $\sigma(T) = \emptyset$, g is analytic on $\mathbb{C}$. We notice that for $|\lambda| > ||T||$ we have

$$g(\lambda) := f(R_\lambda(T)x) = \sum_{n=0}^{+\infty}(-\lambda)^{-(n+1)} f(T^n x) .$$

This series converges absolutely (by Abel's theorem on power series), so we can write, for $|\lambda| \geq 1 + ||T||$:

$$|g(\lambda)| \leq \frac{1}{|\lambda|}\sum_{n=0}^{+\infty}\frac{|f(T^n x)|}{|\lambda|^n} \leq \frac{||f||||x||}{|\lambda|}\sum_{n=0}^{+\infty}\left(\frac{||T||}{|\lambda|}\right)^n = \frac{||f||||x||}{|\lambda|}\frac{|\lambda|}{|\lambda| - ||T||} \leq \frac{K}{|\lambda|}$$

with $K > 0$. Thus $|g|$, everywhere continuous and bounded from above by $K|\lambda^{-1}|$, when $|\lambda| \geq \Lambda$ for some constant Λ, must be bounded on the *entire complex plane*. Being analytic on $\mathbb{C}$, g is constant by Liouville's theorem. As $|g(\lambda)|$ vanishes at infinity, g is the null map. Then $f(R_\lambda(T)x) = 0$. But the result holds for any $f \in \mathsf{X}'$, so Corollary 2.56 to Hahn–Banach (where $\mathsf{X} \neq \{\mathbf{0}\}$), implies $||R_\lambda(T)x|| = 0$. As $x \in \mathsf{X} \neq \{\mathbf{0}\}$ was arbitrary, we have to conclude $R_\lambda(T) = 0$ for any $\lambda \in \rho(T)$. Therefore $R_\lambda(T)$ cannot invert $T - \lambda I$, and the contradiction disproves the assumption $\sigma(T) = \emptyset$.

(d) The resolvent identity is proved as follows. First, we have

$$(T - \lambda I)R_\lambda(T) = I \quad \text{and} \quad (T - \mu I)R_\mu(T) = I .$$

Consider $TR_\lambda(T) - \lambda R_\lambda(T) = I_{\mathsf{X}}$ and $TR_\mu(T) - \mu R_\mu(T) = I_{\mathsf{X}}$, multiply the first by $R_\mu(T)$ on the left and the second by $R_\lambda(T)$ on the right, and then subtract them. Recalling $R_\mu(T)R_\lambda(T) = R_\lambda(T)R_\mu(T)$ and $R_\mu(T)TR_\lambda(T) = R_\lambda(T)TR_\mu(T)$, we obtain the resolvent equation. The first commutation relation used above follows from the evident fact

$$(T - \mu I)(T - \lambda I) = (T - \lambda I)(T - \mu I),$$

which also gives a similar equation for inverses. The other relation is explained as follows:

$$R_\mu(T)TR_\lambda(T) = R_\mu(T)(T-\lambda I)R_\lambda(T) + R_\mu(T)\lambda I R_\lambda(T) = R_\mu(T)I + \lambda R_\mu(T)R_\lambda(T)$$

$$= R_\mu(T) + \lambda R_\lambda(T)R_\mu(T) = (I + \lambda R_\lambda(T))R_\mu(T) = (R_\lambda(T)(T-\lambda I) + \lambda R_\lambda(T))R_\mu(T)$$

$$= R_\lambda(T)TR_\mu(T) .$$

This ends the proof. □

A useful corollary is worth citing that descends immediately from the resolvent identity and (a), (b) in Proposition 4.9.

Corollary 8.6 *Let $T : D(T) \to \mathsf{X}$ be a closed operator on the Banach space X. If for one $\mu \in \rho(T)$ the resolvent $R_\mu(T)$ is compact, then $R_\lambda(T)$ is compact for any $\lambda \in \rho(T)$.*

8.1.2 The Spectrum of Special Classes of Normal Operators on Hilbert Spaces

Let us focus on unitary operators and self-adjoint operators on Hilbert spaces, and discuss the structure of their spectrum. Using Definition 8.2 we shall work in full generality and consider unbounded operators with non-maximal domains.

Proposition 8.7 *Let H be a Hilbert space.*
(a) *If A is self-adjoint on H (but not necessarily bounded, nor defined on the whole H in general):*
  (i) $\sigma(A) \subset \mathbb{R}$,
  (ii) $\sigma_r(A) = \emptyset$,
  (iii) the eigenspaces of A with distinct eigenvalues (points in $\sigma_p(A)$) are orthogonal.[1]
(b) *If $U \in \mathfrak{B}(\mathsf{H})$ is unitary:*
  (i) $\sigma(U)$ is a non-empty compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$,
  (ii) $\sigma_r(U) = \emptyset$.
(c) *If $T \in \mathfrak{B}(\mathsf{H})$ is normal:*
  (i) $\sigma_r(T) = \sigma_r(T^*) = \emptyset$,
  (ii) $\sigma_p(T^*) = \overline{\sigma_p(T)}$,
  (iii) $\sigma_c(T^*) = \overline{\sigma_c(T)}$, where the bar denotes complex conjugation.

Proof (a) Let us begin with (i). Suppose $\lambda = \mu + i\nu$, $\nu \neq 0$ and let us prove $\lambda \in \rho(A)$. If $x \in D(A)$,

$$((A-\lambda I)x|(A-\lambda I)x) = ((A-\mu I)x|(A-\mu I)x) + \nu^2(x|x) + i\nu[(Ax|x) - (x|Ax)] .$$

[1] The analogous property for normal operators (hence unitary or self-adjoint too) in $\mathfrak{B}(\mathsf{H})$ is contained in Proposition 3.60(b).

The last summand vanishes for A is self-adjoint. Hence

$$||(A - \lambda I)x|| \geq |\nu| \, ||x|| \, .$$

With a similar argument we obtain

$$||(A - \overline{\lambda} I)x|| \geq |\nu| \, ||x|| \, .$$

The operators $A - \lambda I$ and $A - \overline{\lambda} I$ are then one-to-one, and $||(A - \lambda I)^{-1}|| \leq |\nu|^{-1}$, where $(A - \lambda I)^{-1} : Ran(A - \lambda I) \to D(A)$. Notice

$$\overline{Ran(A - \lambda I)}^{\perp} = [Ran(A - \lambda I)]^{\perp} = Ker(A^* - \overline{\lambda} I) = Ker(A - \overline{\lambda} I) = \{\mathbf{0}\} \, ,$$

where the last equality makes use of the injectivity of $A - \overline{\lambda} I$. Summarising: $A - \lambda I$ in injective, $(A - \lambda I)^{-1}$ bounded and $\overline{Ran(A - \lambda I)}^{\perp} = \{\mathbf{0}\}$, i.e. $Ran(A - \lambda I)$ is dense in H. Therefore $\lambda \in \rho(A)$, by definition of resolvent set.

Now to (ii). Suppose $\lambda \in \sigma(A)$, but $\lambda \notin \sigma_p(A)$. Then $A - \lambda I$ must be one-to-one and $Ker(A - \lambda I) = \{\mathbf{0}\}$. Since $A = A^*$ and $\lambda \in \mathbb{R}$ by (i), we have $Ker(A^* - \overline{\lambda} I) = \{\mathbf{0}\}$, so $[Ran(A - \lambda I)]^{\perp} = Ker(A^* - \overline{\lambda} I) = \{0\}$ and $\overline{Ran(A - \lambda I)} = \mathsf{H}$. Consequently $\lambda \in \sigma_c(A)$.

Proving (iii) is easy: if $\lambda \neq \mu$ and $Au = \lambda u$, $Av = \mu v$, then

$$(\lambda - \mu)(u|v) = (Au|v) - (u|Av) = (u|Av) - (u|Av) = 0 \, ;$$

from $\lambda, \mu \in \mathbb{R}$ and $A = A^*$. But $\lambda - \mu \neq 0$, so $(u|v) = 0$.
(b) (i) The closure of $\sigma(U)$ is a consequence of Theorem 8.4(b), because any unitary operator is defined on H, bounded and so closed. As $||U|| = 1$, part (c) of that theorem implies $\sigma(U)$ is a compact non-empty subset in $\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. To finish, consider the series

$$S(\lambda) = \sum_{n=0}^{+\infty} \lambda^n (U^*)^{n+1}$$

with $|\lambda| < 1$. Since $||U|| = ||U^*|| = 1$, the series converges absolutely in operator norm, so it defines an operator in $\mathfrak{B}(\mathsf{H})$. Because $U^*U = UU^* = I$,

$$(U - \lambda I)S(\lambda) = S(\lambda)(U - \lambda I) = I \, .$$

By Theorem 8.4(a) $\lambda \in \rho(U)$. To sum up: $\sigma(U)$ is compact and non-empty inside $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.
(ii) This follows from part (i) of (c), because every unitary operator is normal.
(c) Recall that $T \in \mathfrak{B}(\mathsf{H})$ normal implies $\lambda \in \mathbb{C}$ is an eigenvalue iff $\overline{\lambda}$ is an eigenvalue of T^* ((i) in Proposition 3.60(b)). This is enough to give (ii). The three parts of the spectrum are disjoint, and $\sigma(T) = \overline{\sigma(T^*)}$ (by Proposition 8.14(b), whose proof is independent from this theorem), so to prove (iii) it is enough to show (i). Assume

$\lambda \in \sigma(T)$, but $\lambda \notin \sigma_p(T)$. Since $\sigma(T) = \overline{\sigma(T^*)}$ and $\sigma_p(T) = \overline{\sigma_p(T^*)}$, the hypothesis is equivalent to $\overline{\lambda} \in \sigma(T^*)$, but $\overline{\lambda} \notin \sigma_p(T^*)$. Then $T^* - \overline{\lambda} I$ must be one-to-one and $Ker(T^* - \overline{\lambda} I) = \{\mathbf{0}\}$. Now Proposition 3.38(d) tells $[Ran(T - \lambda I)]^\perp = Ker(T^* - \overline{\lambda} I) = \{0\}$, hence $\overline{Ran(T - \lambda I)} = \mathsf{H}$ (here the bar denotes the closure). Therefore $\lambda \in \sigma_c(T)$, i.e. $\sigma_r(T) = \emptyset$. The proof for T^* is the same, because $(T^*)^* = T$ (Proposition 3.38(b)). $\square$

8.1.3 Abstract C*-Algebras: Gelfand–Mazur Theorem, Spectral Radius, Gelfand's Formula, Gelfand–Najmark Theorem

Now we consider, more abstractly, unital Banach algebras and unital C^*-algebras (Definitions 2.24 and 3.40). Recall that $\mathfrak{B}(\mathsf{X})$ is a unital Banach algebra if X is normed, by (i) in Theorem 2.44(c). If H is a Hilbert space, $\mathfrak{B}(\mathsf{H})$ is a unital C^*-algebra whose involution is the Hermitian conjugation, by Theorem 3.49.

First of all we generalise the notions of resolvent set and spectrum to an abstract setting. We shall use $\mathfrak{B}(\mathsf{X})$ as model, with X Banach, so to have Theorem 8.4 at work. Recall that in an algebra $\mathfrak{A}$ with unit $\mathbb{I}$ the inverse a^{-1} to $a \in \mathfrak{A}$ is the unique element, if present, such that $a^{-1}a = aa^{-1} = \mathbb{I}$.

Definition 8.8 Let $\mathfrak{A}$ be a Banach algebra with unit $\mathbb{I}$ and take $a \in \mathfrak{A}$.
(a) The **resolvent set** of a is the set:

$$\rho(a) := \{\lambda \in \mathbb{C} \mid \exists (a - \lambda\mathbb{I})^{-1} \in \mathfrak{A}\} .$$

(b) The **spectrum** of a is the complement $\sigma(a) := \mathbb{C} \setminus \rho(a)$.

The following fact generalises the assertion in Theorem 8.4 about operators of $\mathfrak{B}(\mathsf{X})$.

Theorem 8.9 *Let $\mathfrak{A} \neq \{0\}$ be a Banach algebra with unit $\mathbb{I}$, $a \in \mathfrak{A}$ an arbitrary element.*
(a) *$\rho(a) \neq \emptyset$ is open, $\sigma(a) \neq \emptyset$ is compact and:*

$$|\lambda| \leq ||a|| , \quad \textit{for any } \lambda \in \sigma(a).$$

(b) *The map $\rho(a) \ni \lambda \mapsto R_\lambda(a) := (a - \lambda\mathbb{I})^{-1} \in \mathfrak{A}$ is analytic.*
(c) *If $\lambda, \mu \in \rho(a)$ the* **resolvent identity** *holds:*

$$R_\lambda(a) - R_\mu(a) = (\lambda - \mu) R_\lambda(a) R_\mu(a) .$$

Proof The argument is the same as for properties (b), (c), (d) of Theorem 8.4, because of Remark 8.5(1): just replace, in the proof of (c), $f(R_\lambda(T)x)$ by $f(R_\lambda(a))$, where $f \in \mathfrak{A}'$ (Banach dual to $\mathfrak{A}$). $\square$

A straightforward, yet important corollary is the *Gelfand–Mazur theorem*, whereby every complex, normed division algebra is isomorphic to $\mathbb{C}$, so in particular is commutative.

Theorem 8.10 (Gelfand–Mazur) *A complex Banach algebra $\mathfrak{B} \neq \{0\}$ with unit, in which every non-zero element is invertible, is naturally isomorphic to $\mathbb{C}$. (In particular $\mathfrak{B}$ is commutative.)*

Proof Take $x \in \mathfrak{B}$, so $\sigma(x) \neq \emptyset$ by part (a) in the previous theorem. Then $x - c\mathbb{I}$ is not invertible for some $c \in \mathbb{C}$ by definition of spectrum. In our case $x - c\mathbb{I} = 0$, so $x = c\mathbb{I}$. But c is completely determined by x, for $c\mathbb{I} \neq c'\mathbb{I}$ if $c \neq c'$. The map $\mathfrak{B} \ni x \mapsto c \in \mathbb{C}$ is a Banach algebra isomorphism, as is easy to see. □

Remark 8.11 The assumption that the field is $\mathbb{C}$ is crucial. There exist Banach division algebras that are not commutative, like the algebra $\mathbb{H}$ of *quaternions* introduced in Example 3.48(3). The latter, though, is a real algebra. ■

According to Theorem 8.9(a), the spectrum of $a \in \mathfrak{A}$ is contained in the disc of radius $||a||$ centred at the origin of $\mathbb{C}$. Yet there might be a disc of smaller radius at the origin enclosing $\sigma(a)$. In this respect we have the next definition.

Definition 8.12 Let $\mathfrak{A}$ be a Banach algebra with unit. The **spectral radius** of $a \in \mathfrak{A}$ is the non-negative real number

$$r(a) := \sup\{|\lambda| \mid \lambda \in \sigma(a)\} .$$

This applies in particular when $\mathfrak{A} = \mathfrak{B}(\mathsf{X})$, X a Banach space.

Remark 8.13 Any element a in a unital Banach algebra $\mathfrak{A}$ satisfies the elementary (yet fundamental) property:

$$0 \le r(a) \le ||a|| , \tag{8.3}$$

immediately ensuing from Theorem 8.9(a). ■

There exists an explicit expression for the spectral radius, due to the mathematician I.Gelfand. We shall recover Gelfand's formula using a property of the spectrum of polynomials over $\mathfrak{A}$.

Proposition 8.14 *Let $\mathfrak{A}$ be a Banach algebra with unit $\mathbb{I}$, $a \in \mathfrak{A}$ and $p = p(z)$ a complex-valued polynomial in the variable $z \in \mathbb{C}$.*
(a) *Let $p(a)$ be the element in $\mathfrak{A}$ obtained by formally substituting the element a to z in $p(z)$ and interpreting powers a^n in the obvious way ($a^0 := \mathbb{I}$); then*

$$\sigma(p(a)) = p(\sigma(a)) := \{p(\lambda) \mid \lambda \in \sigma(a)\} . \tag{8.4}$$

(This holds in particular for $\mathfrak{A} = \mathfrak{B}(\mathsf{X})$, X Banach.)

(b) *If $\mathfrak{A}$ is additionally a *-algebra, the spectrum of a^* satisfies*

$$\sigma(a^*) = \overline{\sigma(a)} := \{\overline{\lambda} \mid \lambda \in \sigma(a)\} . \tag{8.5}$$

(This holds in particular for $\mathfrak{A} = \mathfrak{B}(\mathsf{H})$ with H a Hilbert space.)

Proof (a) If $\alpha_1, \ldots, \alpha_n$ denote the roots of a polynomial q (not necessarily distinct), $q(z) = c\prod_{i=1}^n (z-\alpha_i)$ for some complex number c. Hence $q(a) = c\prod_{i=1}^n (a-\alpha_i\mathbb{I})$. Let $\lambda \in \sigma(a)$, so $(a - \lambda\mathbb{I})$ *is not invertible* by definition. Set $\mu := p(\lambda)$. Consider now the polynomial $q := p-\mu$. As $q(\lambda) = 0$, one factor in the above decomposition of q will be $(z - \lambda)$, and so choosing the root order appropriately, and recalling that the $a - \alpha_i\mathbb{I}$ commute, we have:

$$p(a) - \mu\mathbb{I} = c\left[\prod_{i=1}^{n-1}(a - \alpha_i\mathbb{I})\right](a - \lambda\mathbb{I}) = c(a - \lambda\mathbb{I})\prod_{i=1}^{n-1}(a - \alpha_i\mathbb{I}) .$$

Thus $p(a) - \mu\mathbb{I}$ cannot be invertible, for $a - \lambda\mathbb{I}$ is not. (If $p(a) - \mu\mathbb{I}$ were invertible, we would have

$$\mathbb{I} = (a - \lambda\mathbb{I})\left[\left(\prod_{i=1}^{n-1} c(a - \alpha_i\mathbb{I})\right)(p(a) - \mu\mathbb{I})^{-1}\right] ,$$

$$\mathbb{I} = \left((p(a) - \mu\mathbb{I})^{-1}\prod_{i=1}^{n-1} c(a - \alpha_i\mathbb{I})\right)(a - \lambda\mathbb{I}) ,$$

implying $(a - \lambda\mathbb{I})$ invertible. Applying the big bracket to the first equation would say that the right and left inverses of $(a - \lambda\mathbb{I})$ coincide:

$$(p(a) - \mu\mathbb{I})^{-1}\prod_{i=1}^{n-1} c(a - \alpha_i\mathbb{I}) = \left(\prod_{i=1}^{n-1} c(a - \alpha_i\mathbb{I})\right)(p(a) - \mu\mathbb{I})^{-1} ,$$

as it should be.) By definition we must have $\mu \in \sigma(p(a))$, hence we proved $p(\sigma(a)) \subset \sigma(p(a))$. Now we go for the other inclusion. Let $\mu \in \sigma(p(a))$, set $q = p - \mu$ and factor $q(z) = c\prod_{i=1}^n (z - \alpha_i)$. Therefore

$$p(a) - \mu\mathbb{I} = c\prod_{i=1}^{n}(a - \alpha_i\mathbb{I})$$

as before. If all roots α_i belonged to $\rho(a)$, every $(a - \alpha_i\mathbb{I}) : \mathsf{X} \to \mathsf{X}$ would be invertible, so $p(a) - \mu\mathbb{I}$ would become invertible, which is excluded by assumption. Therefore there is a root α_k such that $(a - \alpha_k\mathbb{I})$ is not invertible, so $\alpha_k \in \sigma(a)$. But then $p(\alpha_k) - \mu = 0$, so $\mu \in p(\sigma(a))$, and hence $p(\sigma(a)) \supset \sigma(p(a))$.

(b) $(a - \lambda\mathbb{I})$ is invertible if and only if $(a - \lambda\mathbb{I})^* = a^* - \overline{\lambda}\mathbb{I}$ by Proposition 3.44(c), hence the claim. □

Theorem 8.15 *Let $\mathfrak{A}$ be a Banach algebra with unit and $a \in \mathfrak{A}$.*
(a) *The spectral radius of a can be computed by* **Gelfand's formula***:*

$$r(a) = \lim_{n\to+\infty} ||a^n||^{1/n} ,$$

where the limit always exists. (This holds in particular when $\mathfrak{A} = \mathfrak{B}(\mathsf{X})$ with X Banach.)
(b) *If $\mathfrak{A}$ is a C^*-algebra with unit and a is normal ($a^*a = aa^*$), then*

$$r(a) = ||a|| , \tag{8.6}$$

and consequently:

$$||a|| = r(a^*a)^{1/2} \quad \text{for any } a \in \mathfrak{A}. \tag{8.7}$$

(Valid in particular for $\mathfrak{A} = \mathfrak{B}(\mathsf{H})$, H Hilbert.)

Proof (a) By Proposition 8.14(a) $(\sigma(a))^n = \sigma(a^n)$, so $r(a)^n = r(a^n) \leq ||a^n||$, and then

$$r(a) \leq \liminf_n ||a^n||^{1/n} . \tag{8.8}$$

(In contrast to the limit infimum, which always exists, the limit might not.) If $|\lambda| > r(a)$,

$$R_\lambda(a) = \sum_{n=0}^{+\infty} (-\lambda)^{-(n+1)} a^n , \tag{8.9}$$

because a *theorem of Hadamard* guarantees that the convergence disc of the Laurent series of an analytic function touches the singularity closest to the point at infinity. In our case all singularities belong to the spectrum $\sigma(a)$, so the boundary consists of points $\lambda \in \mathbb{C}$ with $|\lambda| > r(a)$. Therefore the above series converges for any $\lambda \in \mathbb{C}$ such that $|\lambda| > r(a)$, hence it converges absolutely on any disc, centred at infinity, passing through such λ. In particular

$$|\lambda|^{-(n+1)} ||a^n|| \to 0 ,$$

as $n \to +\infty$, for any $\lambda \in \mathbb{C}$ with $|\lambda| > r(a)$. Hence for any $\varepsilon > 0$

$$||a^n||^{1/n} < \varepsilon^{1/n} |\lambda|^{(n+1)/n} = (\varepsilon|\lambda|)^{1/n} |\lambda|$$

definitely. Since $(\varepsilon|\lambda|)^{1/n} \to 1$ for $n \to +\infty$, we have $\limsup_n ||a^n||^{1/n} \leq |\lambda|$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > r(a)$. We can get as close as we want to $r(a)$ with $|\lambda|$, so $\limsup_n ||a^n||^{1/n} \leq r(a)$. Finally, by (8.8),

$$r(a) \leq \liminf_n ||a^n||^{1/n} \leq \limsup_n ||a^n||^{1/n} \leq r(a) .$$

This shows the limit of $||a^n||^{1/n}$ exists as $n \to +\infty$, and it coincides with $r(a)$.
(b) By Proposition 3.44(a) we have $||a^n|| = ||a||^n$ if a is normal. Gelfand's formula gives

$$r(a) = \lim_{n\to+\infty} ||a^n||^{1/n} = \lim_{n\to+\infty} (||a||^n)^{1/n} = ||a|| .$$

Equation (8.7) follows from a property of C^*-algebras, i.e. $||a^*a|| = ||a||^2$ for any a, because a^*a is self-adjoint hence normal. □

Identity (8.7) explains that the norm of a C^*-algebra is *uniquely determined* by algebraic properties, because the spectral radius is obtainable from the spectrum, and this in turn is built by algebraic means entirely.

Corollary 8.16 *A unital *-algebra $\mathfrak{A}$ admits one norm at most that makes it a unital C^*-algebra.*

Notation 8.17 Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be C^*-algebras with unit and take $a \in \mathfrak{A}_1 \cap \mathfrak{A}$. *A priori*, the element a could have two different spectra if thought of as element of $\mathfrak{A}_1$ or of $\mathfrak{A}$. That is why here, and in other similar situations where confusion might arise, we will label spectra: $\sigma_{\mathfrak{A}}(a)$ or $\sigma_{\mathfrak{A}_1}(a)$. ■

There is another important consequence of (8.7) concerning algebra homomorphisms, to which we will return later with a general theorem. Remarkably enough, *-homomorphisms mapping unital Banach *-algebras to unital C^*-algebras are continuous. Later we will see something stronger when domain and target are unital C^*-algebras: if injective, namely, *-homomorphisms are automatically isometric.

Corollary 8.18 *Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a *-homomorphism between a unital Banach *-algebra $\mathfrak{A}$ and a unital C^*-algebra $\mathfrak{B}$. Then*
(a) *ϕ is continuous, for $||\phi(a)||_{\mathfrak{B}} \leq ||a||_{\mathfrak{A}}$ for any $a \in \mathfrak{A}$;*
(b) *for every $a \in \mathfrak{A}$, $\sigma_{\mathfrak{B}}(\phi(a)) \subset \sigma_{\mathfrak{A}}(a)$;*
(c) *if $\mathfrak{A}$ is a unital C^*-algebra and ϕ is additionally a *-isomorphism, it is also isometric: $||\phi(a)||_{\mathfrak{B}} = ||a||_{\mathfrak{A}}$ for any $a \in \mathfrak{A}$, and $\sigma_{\mathfrak{B}}(\phi(a)) = \sigma_{\mathfrak{A}}(a)$ for any $a \in \mathfrak{A}$.*

Proof We start with the proof of (b). If a' exists such that $(a - \lambda\mathbb{I}_{\mathfrak{A}})a' = a'(a - \lambda\mathbb{I}_{\mathfrak{A}}) = \mathbb{I}_{\mathfrak{A}}$, applying the *-homomorphism ϕ we conclude $(\phi(a) - \lambda\mathbb{I}_{\mathfrak{B}})\phi(a') = \phi(a')(\phi(a) - \lambda\mathbb{I}_{\mathfrak{B}}) = \mathbb{I}_{\mathfrak{B}}$, so $\rho(\phi(a)) \supset \rho(a)$ and the claim follows.
Let us pass to (a). By part (b), $\sigma(\phi(a)) \subset \sigma(a)$, and $r(\phi(a)) \leq r(a)$. Equation (8.7) implies $||\phi(a)||^2_{\mathfrak{B}} = r_{\mathfrak{B}}(\phi(a)^*\phi(a)) = r_{\mathfrak{B}}(\phi(a^*a)) \leq r_{\mathfrak{A}}(a^*a) \leq ||a||^2_{\mathfrak{A}}$, where we have also used (8.3).
(c) is obvious from (a) and (b), replicating the argument for ϕ^{-1}. □

To end the abstract considerations we are making, let us present the next result on C^*-algebras in relationship to the classification (i)-(iv) of Definition 3.40.

Proposition 8.19 *Let $\mathfrak{A}$ be a unital C^*-algebra (in particular $\mathfrak{B}(\mathsf{H})$, H Hilbert space).*
(a) *If $a \in \mathfrak{A}$ admits a left inverse, then $\sigma(a^{-1}) = \sigma(a)^{-1} := \{\lambda^{-1} \mid \lambda \in \sigma(A)\}$.*
(b) *If $a \in \mathfrak{A}$ is isometric, i.e. $a^*a = \mathbb{I}$, then $r(a) = 1$.*
(c) *If $a \in \mathfrak{A}$ is unitary, i.e. $a^*a = aa^* = \mathbb{I}$, then $\sigma(a) \subset \mathbb{S}^1 \subset \mathbb{C}$.*
(d) *If $a \in \mathfrak{A}$ is self-adjoint, i.e. $a = a^*$, then $\sigma(a) \subset \mathbb{R}$. More precisely, $\sigma(a) \subset [-||a||, ||a||]$, and $\sigma(a^2) \subset [0, ||a||^2]$.*
(e) *If $a, b \in \mathfrak{A}$ then $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.*

Proof (a) If a is left-invertible, $0 \notin \sigma(a) \cup \sigma(a^{-1})$. For $\lambda \neq 0$, then,

$$\lambda\mathbb{I} - a = \lambda a(a^{-1} - \lambda^{-1}\mathbb{I}) \quad \text{and} \quad \lambda^{-1}\mathbb{I} - a^{-1} = \lambda^{-1}a^{-1}(a - \lambda\mathbb{I}) .$$

Thus $a - \lambda\mathbb{I}$ is invertible iff $a^{-1} - \lambda^{-1}\mathbb{I}$ is.
(b) If $a^*a = \mathbb{I}$ then $||a^n||^2 = ||(a^n)^*a^n|| = ||(a^*)^n a^n|| = ||\mathbb{I}|| = 1$. Gelfand's formula implies $r(a) = 1$.
(c) By (b) and the definition of spectral radius we infer $\sigma(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. On the other hand we know from Proposition 8.14 that $\sigma(a) = \overline{\sigma(a^*)}$. As $a^* = a^{-1}$ and using part (a) we have $\sigma(a) = \overline{\sigma(a)}^{-1}$. Hence any element $\lambda \in \sigma(a)$ satisfies $|\lambda| \leq 1$ and can be written as $\lambda = \overline{\mu}^{-1}$, $|\mu| \leq 1$. This implies $|\lambda| = 1$.
(d) First of all we prove $\sigma(a) \subset \mathbb{R}$. Fix $\lambda \in \mathbb{R}$, $\lambda^{-1} > ||a||$, so that $| - i\lambda^{-1}| = \lambda^{-1} > r(a)$ and consequently $\mathbb{I} + i\lambda a = i\lambda(-i\lambda^{-1}\mathbb{I} + a)$ is invertible. Define $b := (\mathbb{I} - i\lambda a)(\mathbb{I} + i\lambda a)^{-1}$. Then $b^* = (\mathbb{I} - i\lambda a)^{-1}(\mathbb{I} + i\lambda a)$, and since the terms in brackets trivially commute,

$$b^*b = (\mathbb{I} - i\lambda a)^{-1}(\mathbb{I} + i\lambda a)(\mathbb{I} - i\lambda a)(\mathbb{I} + i\lambda a)^{-1} = \mathbb{I} .$$

A similar computation gives $bb^* = \mathbb{I}$, making b unitary. We may then invoke part (c), so that $\sigma(b) \subset \mathbb{S}^1$. Directly, $|(1 - i\lambda\mu)(1 + i\lambda\mu)^{-1}| = 1$ iff $\mu \in \mathbb{R}$. Therefore

$$z := (1 - i\lambda\mu)(1 + i\lambda\mu)^{-1}\mathbb{I} - b$$

is invertible when $\mu \in \mathbb{C} \setminus \mathbb{R}$. Solving the expression of b for a gives

$$z = 2i\lambda(\mathbb{I} + i\lambda\mu)^{-1}(a - \mu\mathbb{I})(\mathbb{I} + i\lambda a)^{-1},$$

hence $a - \mu\mathbb{I}$ is invertible for any $\mu \in \mathbb{C} \setminus \mathbb{R}$. It follows $\sigma(a) \subset \mathbb{R}$. But $r(a) = ||a||$, so $\sigma(a) \subset [-||a||, ||a||]$ is immediate by definition of spectral radius.
(d) follows from Proposition 8.14(a, b).
(e) If c is the inverse of $\mathbb{I} - ab$, then $(\mathbb{I} + bca)(\mathbb{I} - ba) = \mathbb{I} - ba + bc(\mathbb{I} - ab)a = \mathbb{I}$ and $(\mathbb{I} - ba)(\mathbb{I} + bca) = \mathbb{I} - ba + b(\mathbb{I} - ab)ca = \mathbb{I}$. Hence $\mathbb{I} + bca$ inverts $\mathbb{I} - ba$, implying (e). $\square$

We might ask ourselves whether there exist C^*-algebras that cannot be realised as algebras of operators on Hilbert spaces. The answer is no, even if the identification between the C^*-algebra and a C^*-algebra of operators is not fixed uniquely. In fact, the following truly paramount result holds, which we shall prove as Theorem 14.29 after the *GNS theorem* (Chap. 14).

Theorem (Gelfand–Najmark). *If $\mathfrak{A}$ is a C^*-algebra with unit, there exist a Hilbert space* H *and an isometric* **-isomorphism* $\phi : \mathfrak{A} \to \mathfrak{B}$, *where* $\mathfrak{B} \subset \mathfrak{B}(\mathsf{H})$ *is a unital* C^**-subalgebra of* $\mathfrak{B}(\mathsf{H})$.

8.2 Functional Calculus: Representations of Commutative C^*-Algebras of Bounded Maps

This section aims to show how to represent an algebra of bounded measurable functions f by an algebra of *functions* $f(T, T^*)$ *of a bounded normal operator* T. We shall construct a continuous map

$$\widehat{\Phi}_T : \mathsf{M}_b(K) \ni f \mapsto f(T, T^*) \in \mathfrak{B}(\mathsf{H}) ,$$

preserving the structure of unital, commutative C^*-algebra, from bounded measurable functions defined on a compact set K, to bounded operators on a Hilbert space H (see Examples 2.29(4) and 3.48(1)). This will be a *representation* (Definition 3.52) of the unital, commutative C^*-algebra $\mathsf{M}_b(K)$ on H. It will be "generated" by a normal operator $T \in \mathfrak{B}(\mathsf{H})$, and $K = \sigma(T)$. The idea of viewing $\widehat{\Phi}_T(f)$ as $f(T)$, when $T = T^*$, arises also from the physical interpretation related to the notion of observable, as we shall see. This theory goes under the name of *functional calculus*. In a subsequent section we will show how the operator $f(T, T^*)$ can be understood as an integral of f with respect to an operator-valued measure. For the time being we shall construct $f(T, T^*)$ with no mention to spectral measures.

The first part of the construction involves only continuous maps f, and one speaks about *continuous functional calculus*. Continuous functional calculus overlooks the concrete C^*-algebra of bounded operators, and is valid more abstractly if we replace T by a normal element a in a given C^*-algebra. Therefore we shall work first in an abstract setting, and build a continuous functional calculus for self-adjoint elements. Afterwards we will consider normal elements in a general C^*-algebra with unit, using the *Gelfand transform*. Eventually, when dealing with measurable functions, we will return to operator algebras. By the way, continuous functional calculus touches upon *-homomorphisms of C^*-algebras, and allows to characterise positive elements of a C^*-algebra, as we will explain in a moment.

8.2.1 Abstract C^*-Algebras: Functional Calculus for Continuous Maps and Self-adjoint Elements

Let us put ourselves in a general case where $\mathfrak{A}$ is a C^*-algebra with unit $\mathbb{I}$. We may think rather concretely that $\mathfrak{A} = \mathfrak{B}(\mathsf{H})$ for some Hilbert space H, although the following considerations transcend this case.

The first step to build the aforementioned *-homomorphisms is to study *polynomial functions* of a *self-adjoint element* $a^* = a \in \mathfrak{A}$.

Define the function ϕ_a that maps a polynomial with complex coefficients $p = p(x)$, $x \in \mathbb{R}$, to the normal element $p(a)$ of $\mathfrak{A}$, in the obvious way: i.e., evaluating at a and interpreting the product in the algebra. Set also $a^0 := \mathbb{I}$.
This map ϕ_a has interesting features, of immediate proof:

(a) it is linear: $\phi_a(\alpha p + \beta p') = \alpha\phi_a(p) + \beta\phi_a(p')$ for any $\alpha, \beta \in \mathbb{C}$;

(b) it transforms a product of polynomials into a composite in the algebra: $\phi_a(p \cdot p') = \phi_a(p)\phi_a(p')$;

(c) it maps the constant polynomial 1 to the neutral element: $\phi_a(1) = \mathbb{I}$.
By Definition 2.24 these properties make ϕ_a a *homomorphism of algebras with unit*, from the unital commutative *-algebra of complex polynomials to the unital C^*-algebra $\mathfrak{A}$.

Here are other properties:

(d) ϕ_a maps the polynomial $\mathbb{R} \ni x \mapsto x$ (denoted x, inappropriately) to a, i.e. $\phi_a(x) = a$;

(e) if $\overline{p}$ is the conjugate polynomial to p ($\overline{p}(x) = \overline{p(x)}$, $x \in \mathbb{R}$), then $\phi_a(p)^* = \phi_a(\overline{p})$;

(f) if $ba = ab$ for some $b \in \mathfrak{A}$, then $b\phi_a(p) = \phi_a(p)b$ for any polynomial p.
Property (e) establishes that ϕ_a is a *-homomorphism (Definition 3.40) from the unital *-algebra of polynomials to the unital C^*-algebra $\mathfrak{A}$.

There is a further property if we deal with self-adjoint elements. Since $a = a^*$, $\phi_a(p) = p(a)$ is normal. By virtue of Theorem 8.15(b)

$$||p(a)|| = r(p(a)) = \sup\{|\mu| \mid \mu \in \sigma(p(a))\} .$$

The fact that $\sigma(p(a)) = p(\sigma(a))$ (Proposition 8.14(a)) implies

$$||\phi_a(p)|| = \sup\{|p(x)| \mid x \in \sigma(a)\} . \tag{8.10}$$

That is to say: if the algebra of polynomials on $\sigma(a)$ is endowed with norm $||\ ||_\infty$, ϕ_a is an *isometry*. As we shall see, this fact can be generalised beyond polynomials.

Remark 8.20 Assuming $\sigma(a)$ is *not* a *finite* set, with a minor reinterpretation of the symbols we denote, henceforth, by ϕ_a the map sending a function $p\restriction_{\sigma(a)}$ to $p(a) \in \mathfrak{A}$, where p is a polynomial. Thus $||p||_\infty$ will for instance indicate the least upper bound of the absolute value of p over the compact set $\sigma(a)$. Properties (a)-(f) still hold, because a polynomial's restriction to an infinite set determines the polynomial: the difference of two polynomials (in $\mathbb{R}$ or $\mathbb{C}$, with complex coefficients) is a polynomial, and if it has infinitely many zeroes it must be the null polynomial.

In the case $\sigma(a)$ is *finite*, the matter is more delicate, because the restriction $q\restriction_{\sigma(a)}$ of a polynomial q does not determine the polynomial completely. However,

$$||q(a)|| = \sup\{|q(x)| \mid x \in \sigma(a)\}$$

implies immediately that if $q\restriction_{\sigma(a)} = q'\restriction_{\sigma(a)}$ then $q(a) = q'(a)$. Therefore everything we say will work for $\sigma(a)$ finite as well, even though we will not distinguish much the two situations. ■

Recall that the space $C(\mathsf{X})$ of complex-valued, continuous maps on a compact space X (cf. Example 2.29(4), 3.48(1) in §2, 3), is a commutative C^*-algebra with unit: the norm is $|| \ ||_\infty$, sum and product are the standard pointwise operations, the involution is the complex conjugation and the unit is the constant map 1.

Theorem 8.21 (Functional calculus for continuous maps and self-adjoint elements) *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{I}$ and $a \in \mathfrak{A}$ a self-adjoint element.*
(a) *There exists a unique *-homomorphism defined on the unital, commutative C^*-algebra $C(\sigma(a))$:*

$$\Phi_a : C(\sigma(a)) \ni f \mapsto f(a) \in \mathfrak{A}$$

such that

$$\Phi_a(x) = a, \tag{8.11}$$

x being the map $\sigma(a) \ni x \mapsto x$.
(b) *The following properties hold:*
(i) Φ_a is isometric: for any $f \in C(\sigma(a))$, $||\Phi_a(f)|| = ||f||_\infty$;
(ii) if $ba = ab$ with $b \in \mathfrak{A}$, then $bf(a) = f(a)b$ for any $f \in C(\sigma(a))$;
(iii) Φ_a preserves involutions: $\Phi_a(\overline{f}) = \Phi_a(f)^*$ for any $f \in C(\sigma(a))$.
(c) *$\sigma(f(a)) = f(\sigma(a))$ for any $f \in C(\sigma(a))$.*
(d) *If $\mathfrak{B}$ is a C^*-algebra with unit and $\pi : \mathfrak{A} \to \mathfrak{B}$ a *-homomorphism:*

$$\pi(f(a)) = f(\pi(a)) \quad \textit{for any } f \in C(\sigma(a)).$$

Proof In the sequel we assume the spectrum of a is infinite; the finite case must be treated separately by keeping in account the previous remark. We leave this easy task to the reader.
(a) Let us show existence. The spectrum $\sigma(a) \subset \mathbb{C}$ is compact by Theorem 8.4(c), and $C(\sigma(a))$ is Hausdorff because $\mathbb{C}$ is, so we can use Stone-Weierstrass (Theorem 2.30). The space $P(\sigma(a))$ of polynomials $p = p(x)$ restricted to $x \in \sigma(a)$ and with complex coefficients is a subalgebra in $C(\sigma(a))$ that contains the unit (the function 1), separates points in $\sigma(a)$ and is closed under complex conjugation. Hence Theorem 2.30 guarantees it is dense in $C(\sigma(a))$. Consider the map

$$\phi_a : P(\sigma(a)) \ni p \mapsto p(a) \in \mathfrak{A}\,,$$

to properties (a)–(f). We know ϕ_a is linear and $||\phi_a(p)|| = ||p||_\infty$ by (8.10), which implies continuity. By Proposition 2.47 there is a unique bounded linear operator $\Phi_a : C(\sigma(a)) \to \mathfrak{A}$ extending ϕ_a to $C(\sigma(a))$ and preserving the norm. This must be a homomorphism of unital algebras because: (a) it is linear, (b) $\Phi_a(f \cdot g) = \Phi_a(f)\Phi_a(g)$ by continuity (it is true on the subalgebra of polynomials, by definition of ϕ_a), (c) it maps the constant function $1 \in P(\sigma(a))$ to the identity $\mathbb{I} \in \mathfrak{A}$, by definition of ϕ_a. Equation (8.11) holds trivially by property (d). That Φ_a is a *-homomorphism is due to this argument: if $\{p_n\}$ are polynomials converging uniformly on $\sigma(a)$ to the continuous map f, $\{\overline{p_n}\}$ tends uniformly on $\sigma(a)$ to the continuous map $\overline{f}$; as seen above, though (cf. property (e)), $\Phi_a(\overline{p_n}) = \phi_a(\overline{p_n}) = \phi_a(p_n)^* = \Phi_a(p_n)^*$ and Hermitian conjugation is continuous in the uniform topology. By continuity of Φ_a, $\Phi_a(\overline{f}) = \Phi_a(f)^*$.

Now to uniqueness. Any *-homomorphism χ_a of C^*-algebras with unit, fulfilling (8.11), must agree with Φ_a on integer powers of x, hence on any polynomial by definition of *-homomorphism. Moreover χ_a must be continuous by Corollary 8.18(a). Since χ_a and Φ_a are linear, by Proposition 2.47 χ_a coincides with Φ_a.
(b) Property (iii) was proved above. (i) and (ii) are immediate for polynomials, so they extend by continuity to $C(\sigma(a))$.
(c) Observe first that the set of non-invertible elements in $\mathfrak{A}$ is closed under the norm because its complement is open (Proposition 2.27). Consider a polynomial sequence $\{p_n\}$ converging to some $f \in C(\sigma(a))$ uniformly on $\sigma(a)$. Then $p_n(\lambda) \in \sigma(p_n(a))$ by Proposition 8.14(a), i.e. $p_n(a) - p_n(\lambda)\mathbb{I}$ is not invertible. The set of non-invertible elements is closed in $\mathfrak{A}$, so we can take the limit and obtain that $f(a) - f(\lambda)\mathbb{I}$ is not invertible. Hence $f(\lambda) \in \sigma(f(a))$ and then $f(\sigma(a)) \subset \sigma(f(a))$. Conversely, if $\mu \notin f(\sigma(a))$, then $g : \sigma(a) \ni \lambda \mapsto (f(\lambda) - \mu)^{-1}$ is in $C(\sigma(a))$. That is because f is continuous and $f(\sigma(a))$ closed (continuous image in $\mathbb{C}$ of a compact set). By construction $g(a)(f(a) - \mu\mathbb{I}) = (f(a) - \mu\mathbb{I})g(a) = \mathbb{I}$, so $f(a) - \mu\mathbb{I}$ is invertible, hence $\mu \notin \sigma(f(a))$.
(d) The statement is true if f is a polynomial. By the continuity of π (Corollary 8.18(a)) it stays true when passing to continuous maps. □

What we would like to do now is generalise the above theorem to normal elements, not necessarily self-adjoint, in a unital C^*-algebra $\mathfrak{A}$. We want to define an element $f(a, a^*) \in \mathfrak{A}$ for an arbitrary continuous map f defined on the spectrum $\sigma(a) \subset \mathbb{C}$ of a, so that its norm is $||f||_\infty$.

One possibility is to do as follows:

(1) start from polynomials $p(z, \overline{z})$ (dense in $C(\sigma(a))$ by the Stone-Weierstrass theorem) defined on the spectrum of a;

(2) associate to each $p(z, \overline{z})$ the polynomial operator $p(a, a^*) \in \mathfrak{A}$;

(3) show the above correspondence is a continuous *-homomorphism of unital *-algebras.

Yet there is a problem when we pass from polynomials to continuous maps by a limiting procedure. We should prove $||p(a, a^*)|| \le ||p||_\infty$. In case a is self-adjoint the equality was proved using 'spectral invariance' (Proposition 8.14(a), i.e. $\sigma(p(a)) = p(\sigma(a))$) and Theorem 8.15(b), for which $||p(a)|| = r(p(a)) = \sup\{|\mu| \in \mathbb{C} \mid \mu \in \sigma(p(a))\}$. In the case at stake there is nothing guaranteeing $\sigma(p(a, a^*)) = p(\sigma(a, a^*))$. The failure of the fundamental theorem of algebra for complex polynomials in the variables z and $\overline{z}$ is the main cause of the lack of a direct proof of the above fact, and the reason why we have to look for an alternative, albeit very interesting, way.

8.2.2 *Key Properties of *-Homomorphisms of C*-Algebras, Spectra and Positive Elements*

This section is devoted to a series of technical corollaries to Theorem 8.21, essential to extend continuous functional calculus to normal, not self-adjoint, elements. A number of results are nonetheless interesting on their own.

Corollary 8.18(c) tells that a *-homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ between C^*-algebras with unit is isometric if one-to-one and onto. But surjectivity is not necessary, for a consequence of the previous theorem is that injectivity is equivalent to the norm being preserved. We encapsulate in the next statement also Corollary 8.18(a), which we proved earlier.

Theorem 8.22 (On *-homomorphisms of unital C^*-algebras) *A *-homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ of unital C^*-algebras is continuous, for*

$$||\pi(a)||_{\mathfrak{B}} \leq ||a||_{\mathfrak{A}} \quad \textit{for any } a \in \mathfrak{A}.$$

Furthermore
(a) π *is one-to-one iff isometric, i.e.* $||\pi(a)|| = ||a||$ *for any* $a \in \mathfrak{A}$.
(b) $\pi(\mathfrak{A})$ *is a unital C^*-subalgebra inside* $\mathfrak{B}$.

Proof As said, the first statement is Corollary 8.18(a).
(a) If π is isometric it is obviously injective, so we prove the converse. We have $||\pi(a)|| \leq ||a||$ by Corollary 8.18, so it suffices to prove that injectivity forces $||\pi(a)|| \geq ||a||$. If that is true for self-adjoint elements in a C^*-algebra with unit, it holds for any element:

$$||\pi(a)||^2 = ||\pi(a)^*\pi(a)||^2 = ||\pi(a^*a)|| \geq ||a^*a|| = ||a||^2 .$$

So assume there is a self-adjoint element $a \in \mathfrak{A}$ with $||\pi(a)|| < ||a||$. Then Proposition 8.19 says $\sigma_{\mathfrak{A}}(a) \subset [-||a||, ||a||]$ and $r(a) = ||a||$, so $||a|| \in \sigma_{\mathfrak{A}}(a)$ or $-||a|| \in \sigma_{\mathfrak{A}}(a)$. Similarly $\sigma_{\mathfrak{B}}(\pi(a)) \subset [-||\pi(a)||, ||\pi(a)||]$. Choose a continuous map $f : [-||a||, ||a||] \to \mathbb{R}$ that vanishes on $[-||\pi(a)||, ||\pi(a)||]$ and such that $f(-||a||) = f(||a||) = 1$. Theorem 8.21(d) implies $\pi(f(a)) = f(\pi(a)) = 0$, for $f\restriction_{\sigma_{\mathfrak{B}}(\pi(a))} = 0$ and $||f(a)|| = ||f||_{\infty, C(\sigma_{\mathfrak{A}}(a))} \geq 1$. Then $f(a) \neq 0$, contradicting the injectivity of π.
(b) Consider the set $K := Ker(\pi) := \{a \in \mathfrak{A} \mid \pi(a) = 0\}$. One easily proves that K is a closed two-sided *-ideal of $\mathfrak{A}$. The closure arises from $||\pi(a)|| \leq ||a||$. In view of the fact that K is a two-sided *-ideal of $\mathfrak{A}$, the quotient vector space $\mathfrak{A}_1 := \mathfrak{A}/K$ has a natural *-algebra structure induced from that of $\mathfrak{A}$. Moreover, it is known that $||[a]|| := \inf\{||a + b|| \mid b \in K\}$ is a C^*-norm on $\mathfrak{A}_1$ (see Theorem 3.1.4 in [Mur90]). By construction $\pi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\pi_1([a]) := \pi(a)$ for $a \in \mathfrak{A}$, is a well-defined *-homomorphism. However π_1 is injective – and therefore it is isometric by (a) – and, by construction $\pi_1(\mathfrak{A}_1) = \pi(\mathfrak{A})$. Now claim (b) is immediate because π_1 is isometric

and by definition of unital C^*-subalgebra. In particular, π_1 isometric guarantees that $\pi(\mathfrak{A}) = \pi_1(\mathfrak{A}_1)$ is closed in $\mathfrak{B}$, hence complete as a normed space. $\square$

A second result shows that the spectrum does not change by restricting to C^*-subalgebras or taking *-isomorphic images.

Theorem 8.23 (Invariance of spectrum) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with unit.*
(a) *If $\mathfrak{A}$ is a unital C^*-subalgebra in $\mathfrak{B}$,*

$$\sigma_{\mathfrak{A}}(a) = \sigma_{\mathfrak{B}}(a) \quad \textit{for any } a \in \mathfrak{A}.$$

(b) *If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism,*

$$\sigma_{\mathfrak{B}}(\pi(a)) = \sigma_{\pi(\mathfrak{A})}(\pi(a)) \subset \sigma_{\mathfrak{A}}(a) \quad \textit{for any } a \in \mathfrak{A}.$$

The last inclusion is an equality if π is one-to-one.

Proof (a) Let us observe, preliminarily, that the unit $\mathbb{I}$ is the same in $\mathfrak{A}$ and $\mathfrak{B}$. If $a \in \mathfrak{A}$ moreover, also a^* is the same in $\mathfrak{A}$ and $\mathfrak{B}$. It is clear that $\rho_{\mathfrak{A}}(a) \subset \rho_{\mathfrak{B}}(a)$, or equivalently $\sigma_{\mathfrak{B}}(a) \subset \sigma_{\mathfrak{A}}(a)$. Therefore it is enough to prove, for any $a \in \mathfrak{A}$, that $(a - \lambda\mathbb{I})$ has inverse $(a - \lambda\mathbb{I})^{-1} \in \mathfrak{B}$ belonging to $\mathfrak{A}$. This is the same as demanding that the possible inverse $a^{-1} \in \mathfrak{B}$ to $a \in \mathfrak{A}$ is in $\mathfrak{A}$. Let us consider the subcase where $a = a^*$ is invertible in $\mathfrak{B}$. Then $\sigma_{\mathfrak{B}}(a) \subset \mathbb{R}$, and since $\rho_{\mathfrak{B}}(a)$ is open and $0 \in \rho_{\mathfrak{B}}(a)$ there is a disc $D \subset \mathbb{C}$ of radius $r > 0$ at the origin that does not intersect $\sigma_{\mathfrak{B}}(a)$. Hence $f : x \mapsto 1/x$ is continuous and bounded on $\sigma_{\mathfrak{B}}(a)$, and we can define $f(a) = \Phi_a(f)$ using Theorem 8.21 on $a = a^* \in \mathfrak{B}$. By construction $af(a) = f(a)a = \mathbb{I}$, i.e. $f(a) = a^{-1}$ in $\mathfrak{B}$. If $f(a) \in \mathfrak{A}$ the proof ends here. By construction of the one-to-one *-homomorphism Φ_a, we have $f(a) = \lim_{n\to+\infty} p_n(a)$, where the p_n are polynomials and the limit is understood in $\mathfrak{B}$. But $p_n(a) \in \mathfrak{A}$ by definition, for $\mathfrak{A}$ is closed under algebraic operations. Since $\mathfrak{A}$ has the induced topology of $\mathfrak{B}$ and is closed, $f(a) \in \mathfrak{A}$ as required, hence $a^{-1} \in \mathfrak{A}$.

Now consider the case $a \in \mathfrak{A}$ not self-adjoint, such that $a^{-1} \in \mathfrak{B}$. Then also $(a^*)^{-1} = (a^{-1})^* \in \mathfrak{B}$ and we can write $a^{-1} = (a^*a)^{-1}a^*$. Notice $a^*a \in \mathfrak{A}$ is self-adjoint, so $(a^*a)^{-1} \in \mathfrak{A}$ by the previous argument. Trivially $a^* \in \mathfrak{A}$, so $a^{-1} = (a^*a)^{-1}a^* \in \mathfrak{A}$, thus ending part (a).
(b) The inclusion $\sigma_{\pi(\mathfrak{A})}(\pi(a)) \subset \sigma_{\mathfrak{A}}(a)$ was proved in Corollary 8.18(b). The equality $\sigma_{\mathfrak{B}}(\pi(a)) = \sigma_{\pi(\mathfrak{A})}(\pi(a))$ follows from part (a) and the fact $\pi(\mathfrak{A})$ is a unital C^*-subalgebra in $\mathfrak{B}$ by Theorem 8.22(b). If π is one-to-one, $\sigma_{\pi(\mathfrak{A})}(\pi(a)) = \sigma_{\mathfrak{A}}(a)$ follows from (a) and the fact that $\pi : \mathfrak{A} \to \pi(\mathfrak{A})$ is a *-isomorphism of unital C^*-algebras by Theorem 8.22(a). $\square$

Theorem 8.21 also permits to give a reasonable meaning to *positive elements* in a C^*-algebra with unit. The definition and characterisation that follow will play an important role in advanced formulations of quantum fields.

Definition 8.24 An element a in a unital C^*-algebra $\mathfrak{A}$ is **positive** if $a = a^*$ and $\sigma(a) \subset [0, +\infty)$. The set of positive elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}^+$.

We have arrived at the characterisation of positive elements, together with other properties, given in the next result.

Theorem 8.25 (On positive elements in a C^*-algebra with unit) *Let $\mathfrak{A}$ be a C^*-algebra with unit.*
(a) *If $\alpha_1, \dots \alpha_n \in [0, +\infty)$ and $a_1, \dots, a_n \in \mathfrak{A}$ are positive, then $\sum_{j=1}^n \alpha_j a_j$ is positive, so $\mathfrak{A}^+$ is a closed convex cone in $\mathfrak{A}$.*
(b) *The following assertions, for any $a \in \mathfrak{A}$, are equivalent.*
 (i) *a is positive.*
 (ii) *$a = a^*$ and $a = c^*c$ for some $c \in \mathfrak{A}$.*
 (iii) *$a = a^*$ and $a = b^2$ for some self-adjoint $b \in \mathfrak{A}$.*
(c) *If $\mathfrak{A}_0 \subset \mathfrak{A}$ is a unital C^*-subalgebra, then $\mathfrak{A}_0^+ = \mathfrak{A}_0 \cap \mathfrak{A}^+$, and $\mathfrak{A}_0 =< \mathfrak{A}_0^+ >$.*
(d) *If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism of unital C^*-algebras, and $a \in \mathfrak{A}$ is positive, then $\pi(a)$ is positive.*

Proof (a) The claim is clearly true if $n = 1$, for $\sigma(\alpha_1 a_1) = \alpha_1 \sigma(a_1)$, and $\alpha_1 a$ is self-adjoint iff a_1 is and $\alpha_1 \geq 0$. So we will just prove the claim for $n = 2$ with α_1 and α_2 both non-zero. We will use the fact that d is positive iff self-adjoint, plus

$$\left|\left|\mathbb{I} - ||d||^{-1} d\right|\right| \leq 1 \, .$$

The above condition implies $\sigma\left(\mathbb{I} - ||d||^{-1} d\right) \subset [-1, 1]$ i.e. $1 - ||d||^{-1}\sigma(d) \subset [-1, 1]$, by the properties of the spectral radius. This implies $\sigma(d) \subset [0, 2||d||]$, so d is positive. Conversely, if d is positive then $\sigma(d) \subset [0, ||d||]$, so as before $||\mathbb{I} - ||d||^{-1}d|| \leq 1$. If $d = d^*$ and

$$||\mathbb{I} - d|| \leq 1$$

then d is positive with $||d|| \leq 2$. The proof is the same as the previous one. All these facts in turn imply, if a_1 and a_2 are self-adjoint, positive, with $||a_1|| = ||a_2|| = 1$ and $\alpha_1, \alpha_2 \in (0, 1)$, $\alpha_1 + \alpha_2 = 1$, that the self-adjoint element $\alpha_1 a_1 + \alpha_2 a_2$ is positive. In fact,

$$||\mathbb{I} - \alpha_1 a_1 + \alpha_2 a_2|| \leq \alpha_1 ||\mathbb{I} - a_1|| + \alpha_2 ||\mathbb{I} - a_2|| \leq \alpha_1 + \alpha_2 = 1$$

so $\alpha_1 a_1 + \alpha_2 a_2$ is positive. Multiply by $\lambda > 0$, so (renaming constants) $\lambda\mu a_1 + \lambda(1 - \mu)a_2$ is positive whichever $\mu \in (0, 1)$ and $\lambda \in (0, +\infty)$ are chosen. If we now take $\alpha_1, \alpha_2 > 0$ without further conditions, $\lambda = \alpha_1 + \alpha_2 \in (0, +\infty)$ and $\mu = \alpha_1/(\alpha_1 + \alpha_2) \in (0, 1)$ immediately, and

$$\lambda\mu a_1 + \lambda(1 - \mu)a_2 = \alpha_1 a_1 + \alpha_2 a_2 \, .$$

But now $\alpha_1 a_1 + \alpha_2 a_2$ is positive for arbitrary $\alpha_1, \alpha_2 > 0$, so the claim is proved (note that the constraint $||a_1|| = ||a_2|| = 1$ has disappeared). Let us show $\mathfrak{A}^+$ is closed. If $\mathfrak{A}^+ \ni a_n \to a \in \mathfrak{A}$ then $||a_n - a|| \to 0$, so $||a_n|| - ||a|| \to 0$. That $a_n \in \mathfrak{A}^+$, by the

properties of spectrum and spectral radius, implies $|| \, ||a_n||\mathbb{I} - a_n|| \leq ||a_n||$. In the limit $n \to +\infty$ we find $|| \, ||a_n||\mathbb{I} - a|| \leq ||a||$, hence $a \in \mathfrak{A}^+$.
(b) If (iii) holds, Proposition 8.19(d) gives $\sigma(a) = \sigma(b^2) = \{\lambda^2 \mid \lambda \in \sigma(a)\} \subset [0, +\infty)$, so (iii) implies (i). Now the converse. Using continuous functional calculus, and recalling $a = a^*$, the real continuous map $\sqrt{\cdot} : \sigma(a) \ni x \mapsto \sqrt{x}$ allows to define $\sqrt{a} := \Phi_a(\sqrt{\cdot})$. Set $b := \sqrt{a}$, so $b = b^*$ and $b^2 = a$, because Φ_a is a *-homomorphism. Hence (i) and (iii) are equivalent. That (iii) implies (ii) is obvious. So there remains to show (ii) $\Rightarrow$ (i). Let $a = a^*$, $a = c^*c$, and we claim $\sigma(a) \subset [0, +\infty)$. By contradiction assume $\sigma(-a) \subset (0, +\infty)$. Then Proposition 8.19(e) tells $\sigma(-cc^*) \setminus \{0\} = \sigma(-c^*c) \setminus \{0\} \subset (0, +\infty)$. By decomposing $c := c_1 + ic_2$, with c_1, c_2 self-adjoint, we have

$$c^*c + cc^* = 2c_1^2 + 2c_2^2 \,.$$

But c_1^2 and c_2^2 are positive by (iii), and $-cc^*$ is positive by assumption. Hence the linear combination with positive coefficients $2c_1^2 + 2c_2^2 - cc^* = c^*c$ is a positive operator by (a). Therefore $\sigma(c^*c) \subset [0, +\infty)$, but since $\sigma(-c^*c) \setminus \{0\} \subset (0, +\infty)$ as well, we have $\sigma(c^*c) = \{0\}$ i.e. $\sigma(a) = \sigma(-a) = \{0\}$, a contradiction. Hence $\sigma(-a) \subset (-\infty, 0]$, i.e. $\sigma(a) \subset [0, +\infty)$, so (ii) implies (i).
(c) If $a \in \mathfrak{A}_0$ is positive in $\mathfrak{A}$, it is positive in $\mathfrak{A}_0$ and conversely, for $\sigma_{\mathfrak{A}}(a) = \sigma_{\mathfrak{A}_0}(a)$ by Theorem 8.23(a), and also $a = a^*$ is invariant. Hence $\mathfrak{A}_0^+ = \mathfrak{A}_0 \cap \mathfrak{A}^+$. If $a \in \mathfrak{A}_0$, write $a = a_1 + ia_2$, with $a_1 := (a + a^*)/2$ and $a_2 := (a - a^*)/(2i)$ self-adjoint. If b is self-adjoint, we can define $b_+ := (|b| + b)/2$ and $b_- := (|b| - b)/2$, where $|b| = \Phi_b(|\cdot|)$ and $|\cdot| : \mathbb{C} \to [0, +\infty)$ is the modulus. Since Φ_b is a *-homomorphism and $|\cdot|$ is real-valued, b_+ and b_- are self-adjoint because b is (in particular $\sigma(b) \subset \mathbb{R}$). Property (c) in Theorem 8.21 says b_+ and b_- are positive, as $|x| \pm x \geq 0$ for any $x \in \sigma(b) \subset \mathbb{R}$. In conclusion, every $a \in \mathfrak{A}_0$ is the complex linear combination of 4 positive elements in $\mathfrak{A}_0$, so $\mathfrak{A}_0 =< \mathfrak{A}_0^+ >$.
(d) This follows immediately from (b) bearing in mind π is a *-homomorphism. $\square$

8.2.3 Commutative Banach Algebras and the Gelfand Transform

In order to generalise the isometric *-homomorphism $\Phi_a : C(\sigma(a)) \to \mathfrak{A}$ (defined for $a^* = a \in \mathfrak{A}$ in Sect. 8.2.1) to subsume a normal, not self-adjoint, we shall introduce some technical results in the theory of commutative Banach (C^*-)algebras, due to Gelfand and interesting by their own means. We will prove a characterisation called *commutative Gelfand–Najmark theorem*, according to which any commutative C^*-algebra with unit is canonically a C^*-algebra $C(\mathsf{X})$ of functions with norm $|| \, ||_\infty$ over the compact Hausdorff space X given by the algebra itself.

We need a technical result that explains the relationship between *maximal ideals* in Banach algebras and *multiplicative linear functionals*, after the mandatory definitions. In the sequel every Banach algebra will be complex.

Definition 8.26 If $\mathfrak{A}$ is a Banach algebra with unit, a subset $I \subset \mathfrak{A}$ is a **maximal ideal** if
(i) I is a linear subspace in $\mathfrak{A}$,
(ii) $ba, ab \in I$ for any $a \in I, b \in \mathfrak{A}$,
(iii) $I \neq \mathfrak{A}$;
(iv) if $I \subset J$, with J as in (i), (ii), then either $J = I$ or $J = \mathfrak{A}$.

Remark 8.27 Conditions (i) and (ii) say I is an ideal, whereas (iii) prescribes the ideal must be *proper*. *Maximality* is expressed by (iv). ■

Definition 8.28 If $\mathfrak{A}$ is a Banach algebra with unit, a multiplicative linear functional $\phi : \mathfrak{A} \to \mathbb{C}$ is called a **character** of $\mathfrak{A}$ (where multiplicative means $\phi(ab) = \phi(a)\phi(b)$).
If $\mathfrak{A}$ is also commutative, the set of non-trivial characters is denoted by $\sigma(\mathfrak{A})$ and called the **spectrum of the algebra**.

Now we can state and prove the advertised proposition.

Proposition 8.29 *Let $\mathfrak{A}$ be a Banach algebra with unit $\mathbb{I}$.*
(a) *A character χ of $\mathfrak{A}$ is non-zero iff $\chi(\mathbb{I}) = 1$.*
(b) *A maximal ideal $I \subset \mathfrak{A}$ is closed.*
(c) *If $\mathfrak{A}$ is commutative, the map $\mathscr{I} : \sigma(\mathfrak{A}) \ni \chi \mapsto Ker(\chi) \subset \mathfrak{A}$ is a bijection onto the set of maximal ideals.*
(d) *If $\mathfrak{A}$ is commutative, then characters are continuous.*

Proof Observe preliminarily that the existence of the unit $\mathbb{I}$ in $\mathfrak{A}$, with $||\mathbb{I}|| = 1$, implies $\mathfrak{A} \neq \{0\}$.
(a) If χ is a character $\chi(a) = \chi(\mathbb{I}a) = \chi(\mathbb{I})\chi(a)$. If $\chi \neq 0$ then $\chi(a) \neq 0$ for some $a \in \mathfrak{A}$. Then $\chi(\mathbb{I}) = 1$. If $\chi(\mathbb{I}) \neq 0$, clearly $\chi \neq 0$.
(b) By assumption $I \neq \mathfrak{A}$, so $\mathbb{I} \notin I$ (otherwise $a = a\mathbb{I} \in I$ for any $a \in \mathfrak{A}$). Hence $\mathbb{I} \notin \overline{I}$. In fact, if $\mathbb{I} \in \overline{I}$, since the set of invertible elements is open (Proposition 2.27), there would be an open neighbourhood B of $\mathbb{I}$ of invertible elements intersecting I. For any $a \in B \cap I$, then, $\mathbb{I} = a^{-1}a \in I$, which cannot be. Therefore $\overline{I} \neq \mathfrak{A}$, $\mathbb{I}$ being excluded. Since $\overline{I} \supset I$ and $\overline{I}$ satisfies (i), (ii), (iii) in Definition 8.26, we have $I = \overline{I}$ by Definition 8.26(iv).
(c) If $\chi \in \sigma(\mathfrak{A})$, $Ker(\chi)$ is a maximal ideal: (i),(ii) in Definition 8.26 are true as χ is linear and multiplicative, and (iii) holds for $\chi \neq 0$. Notice $\mathfrak{A} = Ker(\chi) \oplus V$, where $\dim(V) = 1$, for this must be the dimension of the target space $\mathbb{C}$ of χ. Hence any subspace $J \subset \mathfrak{A}$ containing $Ker(\chi)$ properly must be $\mathfrak{A}$ itself, so $Ker(\chi)$ is a maximal ideal. Therefore the map $\mathscr{I}$ sends characters to maximal ideals. Let us show it is one-to-one. If $\chi, \chi' \in \sigma(\mathfrak{A})$ and $Ker(\chi) = Ker(\chi') = N$, from $\mathfrak{A} = N \oplus V$ we have $\chi(a) = \chi(v_a)$ and $\chi'(a) = \chi'(v_a)$, where $n_a \in N$ and $v_a \in V$ are the projections of a on N and V. If e is a basis of V (1-dimensional), $v_a = c_a e$

for some complex number c_a determined by a. Hence $Ker(\chi) = Ker(\chi')$ implies $\chi(a) = a_v\chi(e)$ and $\chi'(a) = a_v\chi'(e)$. By (a) $\chi(\mathbb{I}) = \chi'(\mathbb{I}) = 1$, so $\chi(e) = \chi'(e)$ and $\chi = \chi'$, proving injectivity. Now surjectivity. If I is a maximal ideal, it is closed by (b). It is easy to show that the quotient space $\mathfrak{A}/I$ of cosets $[a]$ ($a \sim a' \Leftrightarrow a-a' \in I$) inherits a natural Banach space structure and a commutative Banach algebra structure with unit $[\mathbb{I}]$ from $\mathfrak{A}$. By construction $\mathfrak{A}/I$ does not contain ideals other than $\mathfrak{A}/I$ itself. So any non-null element $[a] \in \mathfrak{A}/I$ is invertible, otherwise $[a]\mathfrak{A}/I$ would be a proper ideal in $\mathfrak{A}/I$. The Gelfand–Mazur Theorem (8.10) guarantees the existence of a Banach space isomorphism $\psi : \mathfrak{A}/I \to \mathbb{C}$. If $\pi : \mathfrak{A} \ni a \to [a] \in \mathfrak{A}/I$ denotes the canonical projection (continuous by construction), $\chi := \psi \circ \pi : \mathfrak{A} \to \mathbb{C}$ is an element of $\sigma(\mathfrak{A})$ with trivial null space $Ker(\chi) = I$.
(d) As $\chi \mapsto Ker(\chi)$ is a bijection, the last argument also tells that any character χ must look like $\psi \circ \pi$, for $Ker(\chi) = I$. Hence χ is continuous, because ψ and π are. $\square$

Now it is time for the first theorem of Gelfand on commutative Banach algebras with unit. We shall refer to the *-*weak topology* on the dual $\mathfrak{A}'$ of $\mathfrak{A}$ (seen as Banach space) introduced by Definition 2.72. More precisely, viewing $\sigma(\mathfrak{A})$ as subset of $\mathfrak{A}'$ with the induced topology, we consider the unital algebra $C(\sigma(\mathfrak{A}))$ of continuous maps $\sigma(\mathfrak{A}) \to \mathbb{C}$ with norm $|| \, ||_\infty$. One part of the theorem establishes that $\sigma(\mathfrak{A})$ is a compact Hausdorff space. As we saw in Chaps. 2 and 3 (Examples 2.29(4), 3.48(1)), in fact, $C(\sigma(\mathfrak{A}))$ is a Banach algebra with unit (and also a C^*-algebra).

Theorem 8.30 *Take a commutative Banach algebra $\mathfrak{A}$ with unit $\mathbb{I}$ and let*

$$\mathscr{G} : \mathfrak{A} \ni x \mapsto \widehat{x} : \sigma(\mathfrak{A}) \to \mathbb{C} \tag{8.12}$$

denote the **Gelfand transform**

$$\widehat{x}(\chi) := \chi(x)\,, \quad x \in \mathfrak{A},\ \chi \in \sigma(\mathfrak{A}), \tag{8.13}$$

Then
(a) *$\sigma(\mathfrak{A})$ is a *-weakly compact Hausdorff space, and $||\chi|| \leq 1$ if $\chi \in \sigma(\mathfrak{A})$ ($||.||$ is the strong norm on $\mathfrak{A}'$).*
(b) *If $x \in \mathfrak{A}$:*

$$\sigma(x) = \{\widehat{x}(\chi) \mid \chi \in \sigma(\mathfrak{A})\}\,.$$

(c) *$\widehat{\mathfrak{A}} \subset C(\sigma(\mathfrak{A}))$, and $\mathscr{G} : \mathfrak{A} \to C(\sigma(\mathfrak{A}))$ is a homomorphism of unital Banach algebras.*
(d) *$\mathscr{G} : \mathfrak{A} \to C(\sigma(\mathfrak{A}))$ is continuous, $||\widehat{x}||_\infty \leq ||x||$ for any $x \in \mathfrak{A}$.*

Proof (a) Consider $\chi \in \sigma(\mathfrak{A})$ and the maximal ideal $I = Ker(\chi)$ associated to it under Proposition 8.29(c). If $x \in \mathfrak{A}$, $\chi(x - \chi(x)\mathbb{I}) = 0$, so $x - \chi(x)\mathbb{I} \in I$ cannot be invertible (cf. Proposition 8.29(b)). Then $\chi(x) \in \sigma(x)$, so $|\chi(x)| \leq ||x||$ by elementary properties of the spectral radius. Consequently $||\chi|| \leq 1$, where the norm defines the strong topology. Therefore $\sigma(\mathfrak{A})$ is contained in the unit ball of the

dual $\mathfrak{A}'$. We know this set is *-weakly compact by Theorem 2.80 (Banach–Alaoglu). Since the *-weak topology is Hausdorff, to finish it suffices to show $\sigma(\mathfrak{A})$ is *-weakly closed. Saying $\sigma(\mathfrak{A}) \ni \chi_n \to \chi \in \mathfrak{A}'$ in that topology means $\chi_n(x) \to \chi$ for any $x \in \mathfrak{A}$. By continuity χ is a character if all χ_n are. Thus $\sigma(\mathfrak{A})$ is closed in the *-weak topology.
(b) Above we proved $\chi(x) \in \sigma(x)$, so $\{\widehat{x}(\chi) \mid \chi \in \sigma(\mathfrak{A})\} \subset \sigma(x)$. Let us prove the converse inclusion. If $\lambda \in \sigma(x)$ then $x - \lambda\mathbb{I}$ is not invertible, so $x\mathfrak{A} := \{(x-\lambda\mathbb{I})y \,|\, y \in \mathfrak{A}\}$ is a proper ideal. Zorn's lemma gives us a maximal ideal I containing $x\mathfrak{A}$. Let $\chi_I \in \sigma(\mathfrak{A})$ be the associated character by Proposition 8.29(c). Then $\widehat{x}(\chi_I) = \chi_I(x) = \lambda$ and so $\{\widehat{x}(\chi) \mid \chi \in \sigma(\mathfrak{A})\} \supset \sigma(x)$, as required.
(c)–(d). That $\mathscr{G}$ is a homomorphism of algebras with unit is straightforward, because $\widehat{x}$ acts on characters χ (linear and multiplicative, plus $\widehat{\mathbb{I}}(\chi) := \chi(\mathbb{I}) = 1$). Moreover, from (b) and the definition of spectral radius we have $||\widehat{x}||_\infty = r(x)$. On the other hand $r(x) \leq ||x||$ by elementary properties of the spectral radius. $\square$

Example 8.31 Let $\ell^1(\mathbb{Z})$ be the Banach space of maps $f : \mathbb{Z} \to \mathbb{C}$ such that

$$||f||_1 := \sum_{n\in\mathbb{Z}} |f(n)| < +\infty \,.$$

Equip $\ell^1(\mathbb{Z})$ with the structure of a unital Banach algebra by defining the product using the *convolution*:

$$(f * g)(m) := \sum_{n\in\mathbb{Z}} f(m-n)g(n), \quad f, g \in \ell^1(\mathbb{Z}).$$

This product is well defined and satisfies $||f * g||_1 \leq ||f||_1||g||_1$, because:

$$\sum_{n\in\mathbb{Z}} |(f * g)(n)| = \sum_{n\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} f(n-m)g(m) \right| \leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |f(n-m)|\,|g(m)|$$

$$= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |f(n-m)|\,|g(m)| \leq \sum_{m\in\mathbb{Z}} \left(|g(m)| \sum_{n\in\mathbb{Z}} |f(n-m)| \right) = \sum_{m\in\mathbb{Z}} |g(m)|\,||f||_1$$

$$= ||f||_1\,||g||_1 \,.$$

There is a unit $\mathbb{I}$, namely the map $\mathbb{I}(n) = 1$ if $n = 0$ and $\mathbb{I}(n) = 0$ if $n \neq 0$. Since $f * g = g * f$, as is easy to see, $\ell^1(\mathbb{Z})$ becomes a commutative Banach algebra with unit, and we can apply Gelfand's theory.

Set $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ and define characters χ_z associated to $z \in \mathbb{S}^1$:

$$\chi_z(f) := \sum_{n\in\mathbb{Z}} f(n)z^n \,.$$

Trivially, these are well-defined characters. Hence we have a function $\Gamma : \mathbb{S}^1 \ni z \mapsto \chi_z \in \sigma(\ell^1(\mathbb{Z}))$ easily seen to be invertible. Actually, it is a homeomorphism, for we shall prove it is a continuous bijection between compact Hausdorff spaces (Proposition 1.23). Continuity, using the *-weak topology on $\sigma(\ell^1(\mathbb{Z}))$, amounts to continuity of $\mathbb{S}^1 \ni z \mapsto \chi_z(f) \in \mathbb{C}$ with $f \in \ell^1(\mathbb{Z})$ fixed, because $z \mapsto \chi_z(f)$ is the uniform limit of the continuous map $g_m(z) := \sum_{|n|<m} f(n)z^n$, for $\sum_{n\in\mathbb{Z}} |f(n)z^n| = ||f||_1 < +\infty$ with $|z| = 1$.

Therefore we may identify the spectrum $\sigma(\ell^1(\mathbb{Z}))$ with $\mathbb{S}^1$ under the homeomorphism Γ. The Gelfand transform $\widehat{f}$ of $f \in \ell^1(\mathbb{Z})$ is therefore continuous on $\mathbb{S}^1$, and defined by

$$\widehat{f}(z) := \sum_{n\in\mathbb{Z}} f(n)z^n .$$

■

The elementary theory of Fourier series forces $f(n)$ to be the Fourier coefficient

$$f(n) = \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(e^{i\theta})e^{-in\theta} d\theta .$$

Therefore $\mathscr{G}(\ell^1(\mathbb{Z}))$ is the subset, in the unital Banach algebra $(C(\mathbb{S}^1), || \ ||_\infty)$, of maps with absolutely convergent Fourier series. Gelfand observed that there is an interesting consequence to this fact, which corresponds to a classical statement due to Wiener (but proved by different means):

Proposition 8.32 *If $h \in C(\mathbb{S}^1)$ has absolutely convergent Fourier series and no zeroes, the map $\mathbb{S}^1 \ni z \mapsto 1/h(z)$ (belonging in $C(\mathbb{S}^1)$) has absolutely convergent Fourier series.*

Proof First, $h = \widehat{f}$ for some $f \in \ell^1(\mathbb{Z})$. Since $\widehat{f}(z) \neq 0$, then $0 \notin \sigma(f)$ by Theorem 8.30(b). Hence f has inverse $g \in \ell^1(\mathbb{Z})$ and $\widehat{g} = 1/h$. We conclude that the Fourier series of $1/h$ must converge absolutely. □

To conclude we consider the more rigid case in which $\mathfrak{A}$ is a commutative C^*-algebra with unit. Then the Gelfand transform defines an honest *-isomorphism of C^*-algebras with unit, and must be isometric by Theorem 8.22(a). In fact we have the following commutative version of the Gelfand–Najmark theorem.

Theorem 8.33 (Commutative Gelfand–Najmark theorem) *Let $\mathfrak{A}$ be a commutative C^*-algebra with unit. Then every character of $\mathfrak{A}$ is a *-homomorphism. Moreover, if we think of $C(\sigma(\mathfrak{A}))$ as a commutative C^*-algebra with unit (for the norm $|| \ ||_\infty$), the Gelfand transform*

$$\mathscr{G} : \mathfrak{A} \ni x \mapsto \widehat{x} \in C(\sigma(\mathfrak{A})) \quad \textit{where} \quad \widehat{x}(\chi) := \chi(x) \quad , \quad x \in \mathfrak{A},\ \chi \in \sigma(\mathfrak{A}),$$

is an isometric *-isomorphism.

Proof The only thing to prove is that the Gelfand transform defines a *-isomorphism, because the rest follows from Theorem 8.22(a). Knowing the Gelfand transform is an algebra homomorphism, though, requires only that we prove surjectivity and the involution property. The first lemma is that $\widehat{x}$ is real if $x^* = x \in \mathfrak{A}$. If so, with $t \in \mathbb{R}$ we define

$$u_t := e^{itx} := \sum_{n=0}^{+\infty} \frac{(it)^n}{n!} x^n$$

with respect to the norm of $\mathfrak{A}$. Since $\mathfrak{A}$ is commutative, and working as we were in $\mathbb{C}$, we have $u_t^* = u_t$ and $u_t^* u_t = u_0 = \mathbb{I}$. Taking norms gives $||u_t|| = ||u_{-t}|| = 1$. If now χ is a character (continuous, linear and multiplicative), we see $\chi(u_t) = e^{it\chi(x)}$ and $\chi(u_{-t}) = e^{-it\chi(x)}$. So by Theorem 8.30(d):

$$|\chi(u_{\pm t})| = |\widehat{u_{\pm t}}(\chi)| \leq ||\widehat{u_{\pm t}}||_\infty \leq ||u_{\pm t}|| \leq 1\,.$$

That is to say $|e^{\pm it\chi(x)}| \leq 1$, implying $\chi(x) \in \mathbb{R}$. Now if $x \in \mathfrak{A}$ we can decompose $x = a + ib$, $a = a^*$, $b = b^*$ so that $\overline{\chi(x)} = \chi(a) - i\chi(b) = \chi(x^*)$. This proves that characters of commutative C^*-algebras are *-homomorphisms and not only homomorphisms. Hence, coming back to the Gelfand transform,

$$\widehat{x}^*(\chi) = \chi(x^*) = \chi(a - ib) = \chi(a) - i\chi(b) = \overline{\chi(a) + i\chi(b)} = \overline{\chi(x)} = \overline{\widehat{x}(\chi)}\,.$$

Therefore the Gelfand transform preserves the involution.

To conclude we settle surjectivity, showing $\{\widehat{x} \mid x \in \mathfrak{A}\} = C(\sigma(\mathfrak{A}))$. The set on the left is closed as compact (continuous image of a compact set, Theorem 8.30) in a Hausdorff space. By construction, this set is a closed *-subalgebra of $C(\sigma(\mathfrak{A}))$ containing the identity ($\widehat{\mathbb{I}} = 1$, identity map). The elements of that algebra separate points of $\sigma(\mathfrak{A})$: if $\chi_1 \neq \chi_1$ then $\chi_1(x) \neq \chi_2(x)$ for some $x \in \mathfrak{A}$, so $\widehat{x}(\chi_1) \neq \widehat{x}(\chi_2)$. The Stone-Weierstrass theorem implies $\{\widehat{x} \mid x \in \mathfrak{A}\} = C(\sigma(\mathfrak{A}))$. $\square$

Remarks 8.34 **(1)** The commutative Gelfand–Najmark theorem proves that every commutative C^*-algebra $\mathfrak{A}$ with unit is canonically a C^*-algebra $C(\mathsf{X})$ of functions with norm $||\ ||_\infty$ on a compact set $\mathsf{X} = \sigma(\mathfrak{A})$. The "points" of X are the characters of the C^*-algebra. Put equivalently, commutative C^*-algebras with unit are C^*-algebras of functions built in a canonical manner via the algebra's spectrum $\sigma(\mathfrak{A})$.
(2) If we start from a concrete C^*-algebra $C(\mathsf{X})$ of functions on a compact Hausdorff space X, Gelfand's procedure recovers exactly this algebraic construction, because characters, in the present case, are nothing but points in X. In fact, any $x \in \mathsf{X}$ can be mapped one-to-one to the corresponding character $\chi_x : C(\mathsf{X}) \to \mathbb{C}$, $\chi_x(f) := f(x)$ for any $f \in C(\mathsf{X})$. It can be proved that every character has this form by showing it is positive (by multiplicativity), and that it must be a positive Borel measure by the theorem of Riesz. Since the only multiplicative Borel measures are Dirac measures δ_x, we have $\chi(f) = \int_\mathsf{X} f d\delta_x = f(x)$ for some $x \in \mathsf{X}$ determined by χ. Observe that

the topology on X coincides with the *-weak topology if we interpret points $x \in \mathsf{X}$ as characters χ_x, as is immediate to verify.

Naïvely speaking, a compact Hausdorff space can be fully described by the commutative C^*-algebra of its continuous complex functions. This remark can be taken, and indeed was by A. Connes, as a starting point to develop *noncommutative geometry*: instead of using a commutative C^*-algebra with unit one takes a noncommutative algebra, and the associated "space" is defined in terms of continuous linear functionals on the algebra. ■

8.2.4 Abstract C^*-Algebras: Functional Calculus for Continuous Maps and Normal Elements

We wish to extend Sect. 8.2.1 to *normal elements* $a \in \mathfrak{A}$ ($a^*a = aa^*$) in a C^*-algebra $\mathfrak{A}$ with unit $\mathbb{I}$. We want to make sense of the function $f(a, a^*) \in \mathfrak{A}$ of a, a^* when f is a continuous complex-valued map defined on the spectrum of a.

A few preliminary remarks and notational issues must be seen to, before we proceed to define $f(a, a^*)$.

We can always decompose a and a^* into linear combinations of two commuting *self-adjoint* elements x, y:

$$a = x_a + iy_a \ , \quad a^* = x_a - iy_a \ , \tag{8.14}$$

where by definition

$$x_a := \frac{a + a^*}{2} \ , \quad y_a := \frac{a - a^*}{2i} \ . \tag{8.15}$$

x_a and y_a are clearly self-adjoint. That they commute is also obvious, for a and a^* commute.

Decomposition (8.14) reminds of the analogue splitting of a complex number into real and imaginary parts

$$z = x + iy \ , \quad \overline{z} = x - iy \ , \tag{8.16}$$

where

$$x := \frac{z + \overline{z}}{2} \ , \quad y := \frac{z - \overline{z}}{2i} \ . \tag{8.17}$$

Remark 8.35 The maps $f : \sigma(a) \to \mathbb{C}$ we shall deal with are to be thought of as functions in x and y, imagining $\sigma(a)$ as subset of $\mathbb{R}^2$ rather than $\mathbb{C}$. Equivalently, the variables may be taken to be z and $\overline{z}$, considered *independent*. They are bijectively determined by x, y, so maps in x, y are in one-to-one correspondence to maps in z, $\overline{z}$: to $f = f(z, \overline{z})$ we may associate

$$g = g(x, y) := f(x + iy, x - iy)$$

and conversely, to $g = g(x, y)$ is associated

$$f_1 = f_1(z, \overline{z}) := g((z + \overline{z})/2, (z - \overline{z})/2i)\ .$$

Clearly, $f_1 = f$. This fact will be used often without further notice. ■

Now we are ready for the continuous functional calculus for normal elements. The proof will be substantially different from Theorem 8.21, in that it will involve the Gelfand transform of the previous section. We shall still use the name Φ_a for the *-isomorphism, because it generalises the morphism of Theorem 8.21.

Theorem 8.36 (Functional calculus for continuous maps and normal elements) *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{I}$ and $a \in \mathfrak{A}$ a normal element. View f as a function of the independent variables z and $\overline{z}$.*
(a) *There exists a unique *-homomorphism on the unital commutative C^*-algebra $C(\sigma(a))$:*

$$\Phi_a : C(\sigma(a)) \ni f \mapsto f(a, a^*) \in \mathfrak{A}\ ,$$

such that

$$\Phi_a(z) = a \tag{8.18}$$

with z being the polynomial $\sigma(a) \ni (z, \overline{z}) \mapsto z$.
(b) *The following properties hold:*
(i) Φ_a is isometric: for any $f \in C(\sigma(a))$, $||\Phi_a(f)|| = ||f||_\infty$;
(ii) if $ba = ab$ and $ba^* = a^*b$ for some $b \in \mathfrak{A}$, then $bf(a, a^*) = f(a, a^*)b$ for any $f \in C(\sigma(a))$;
(iii) Φ_a preserves involutions: $\Phi_a(\overline{f}) = \Phi_a(f)^*$ for any $f \in C(\sigma(a))$.
(c) *$\sigma(f(a, a^*)) = f(\sigma(a), \overline{\sigma(a)})$ for any $f \in C(\sigma(a))$.*
(d) *If $\mathfrak{B}$ is a C^*-algebra with unit and $\pi : \mathfrak{A} \to \mathfrak{B}$ a *-homomorphism,*

$$\pi(f(a, a^*)) = f(\pi(a), \pi(a^*)) \quad \textit{for any } f \in C(\sigma_{\mathfrak{A}}(a)).$$

(e) *If $a = a^*$ the *-homomorphism Φ_a coincides with its analogue of Theorem 8.21.*

Proof (a)–(b)–(e). Uniqueness is evident because if two *-homomorphisms $\Phi_a : C(\sigma(a)) \to \mathfrak{A}$ and $\Phi'_a : C(\sigma(a)) \to \mathfrak{A}$ satisfy $\Phi_a(z) = \Phi'_a(z) = a$, by definition they coincide on the polynomial algebra in z and $\overline{z}$, which is dense in $C(\sigma(a))$ in norm $||\,||_\infty$ by Stone-Weierstrass ($\sigma(a)$ is compact and Hausdorff). As *-homomorphisms are continuous (Theorem 8.22), $\Phi_a(f) = \Phi'_a(f)$ for any $f \in C(\sigma(a))$. The same argument proves, in the case $a = a^*$, that the *-homomorphism Φ_a coincides with its cousin in Theorem 8.21. Likewise, if Φ_a is defined, then (ii) in (b) holds, because if b commutes with a and a^* it commutes with every polynomial in a, a^*, and by continuity with any $\Phi_a(f)$.

Let us show Φ_a exists and satisfies the remaining requests in (a) and (b). Consider the unital commutative C^*-(sub)algebra $\mathfrak{A}_a \subset \mathfrak{A}$ spanned by $\mathbb{I}$, a and a^*. It is the closure, for the norm of $\mathfrak{A}$, of the set of polynomials $p(a, a^*)$ with complex coefficients. The idea is to define $\Phi_a(a)$ by $\mathscr{G}^{-1}(f)$, because the inverse Gelfand transform $\mathscr{G}^{-1} : C(\sigma(\mathscr{A})) \to \mathfrak{A}_a$ is an isometric *-isomorphism by Theorem 8.33. The problem is that now f is defined on $\sigma(\mathfrak{A}_a)$, not on $\sigma(a)$. So let us prove $\sigma(\mathfrak{A}_a)$ and $\sigma(a)$ are homeomorphic under $F : \sigma(\mathfrak{A}_a) \ni \chi \mapsto \chi(a) \in \sigma(a)$. That $\chi(a) \in \sigma(a)$ follows from Theorem 8.30(b). The function is continuous because characters are continuous, by Proposition 8.29(d), and it acts between compact Hausdorff spaces. Hence it is enough to show it is bijective to have a homeomorphism (Proposition 1.23). If $F(\chi) = F(\chi')$ then $\chi(a) = \chi'(a)$, $\overline{\chi(a)} = \overline{\chi'(a)}$ (see the proof of Theorem 8.33) and $\chi(a^*) = \chi'(a^*)$. On the other hand $\chi(\mathbb{I}) = \chi'(\mathbb{I}) = 1$ by Proposition 8.29(a). Since χ preserves sums and products, by continuity $\chi(b) = \chi'(b)$ if $b \in \mathfrak{A}_a$, and F is injective. F is onto by Theorem 8.30(b). Define

$$\Phi_a(f) := \mathscr{G}^{-1}(f \circ F)$$

for $f \in C(\sigma(a))$. By construction Φ_a is an isometric *-isomorphism from $C(\sigma(a))$ to $\mathfrak{A}_a$ such that $\Phi_a^{-1}(a) = z$, where z is $\sigma(a) \ni (z, \overline{z}) \mapsto z$. In fact, $\Phi_a^{-1}(a) = z$ means $\mathscr{G}(a) = z \circ F$ i.e. $\chi(a) = z(\chi)$ for any character $\chi \in \sigma(\mathfrak{A}_a)$. But the latter is true by definition of F. Hence (a), (b) are valid by redefining Φ_a as valued in the larger algebra $\mathfrak{A}$.

(c) By Theorem 8.23, first of all, $\sigma_{\mathfrak{A}}(f(a, a^*)) = \sigma_{\mathfrak{A}_a}(f(a, a^*))$, so we look at the spectrum of $f(a, a^*)$ in $\mathfrak{A}_a$. Then $\Phi_a : C(\sigma(a)) \to \mathfrak{A}_a$ defines an isometric *-isomorphism. The abstract function $f(a, a^*) - \lambda\mathbb{I}$ corresponds to the concrete map $\sigma(s) \ni (z, \overline{z}) \mapsto f(z, \overline{z}) - \lambda$. Therefore $f(a, a^*) - \lambda\mathbb{I}$ is invertible iff $\sigma(s) \ni (z, \overline{z}) \mapsto (f(z, \overline{z}) - \lambda)^{-1}$ is in $C(\sigma(a))$. Since the range of f is compact (continuous image of a compact set), the assertion is equivalent to $\lambda \notin f(\sigma(a), \overline{\sigma(a)})$. Now (c) is immediate.

(d) We prove the equivalent fact $\pi(\Phi_a(f)) = \Phi_{\pi(a)}(f)$. By construction $C(\sigma(a)) \ni f \to \pi(\Phi_a(f)) \in \pi(\mathfrak{A})$ and $C(\sigma(a)) \ni f \to \Phi_{\pi(a)}(f) \in \pi(\mathfrak{A})$ are continuous *-homomorphisms. Trivially, $\pi(\Phi_a(z)) = \pi(a) = \Phi_{\pi(a)}(z)$, $\pi(\Phi_a(\overline{z})) = \pi(a)^* = \Phi_{\pi(a)}(\overline{z})$ and $\pi(\Phi_a(1)) = \mathbb{I} = \Phi_{\pi(a)}(1)$. Therefore $\pi(\Phi_a(p)) = \Phi_{\pi(a)}(p)$ on polynomials $p = p(z, \overline{z})$, and by continuity they coincide on any $f \in \sigma(a)$. $\square$

8.2.5 C^*-Algebras of Operators in $\mathfrak{B}(\mathsf{H})$: Functional Calculus for Bounded Measurable Functions

Let us return to functional calculus for operators and specialise Sect. 8.2.4 to $\mathfrak{A} = \mathfrak{B}(\mathsf{H})$, H a Hilbert space. Instead of a normal element $a \in \mathfrak{A}$ consider a normal operator $T \in \mathfrak{B}(\mathsf{H})$. Then the *-homomorphism Φ_T is a *representation* of $C(\sigma(T))$ on H (definition 3.52). Here as well it is convenient to decompose T into self-adjoint operators $X, Y \in \mathfrak{B}(\mathsf{H})$:

$$T = X + iY\,, \quad T^* \;=\; X - iY\,, \tag{8.19}$$

where

$$X := \frac{T + T^*}{2}\,, \quad Y \;:=\; \frac{T - T^*}{2i}\,. \tag{8.20}$$

The operators X and Y are patently self-adjoint by construction, and commute since T is normal and commutes with T^*.

As previously remarked, decomposition (8.19) is akin to the real/imaginary decomposition of a complex number

$$z = x + iy\,, \quad \overline{z} \;=\; x - iy\,, \tag{8.21}$$

where

$$x := \frac{z + \overline{z}}{2}\,, \quad y \;:=\; \frac{z - \overline{z}}{2i}\,. \tag{8.22}$$

As before, we may view $f(z, \overline{z})$ as a complex function in x and y. Theorem 8.36 specialises, with identical proof, as follows. We refer to Definition 3.52 for the notion of representation of a C^*-algebra.

Proposition 8.37 *Let H be a Hilbert space and $T \in \mathfrak{B}(\mathsf{H})$ a normal operator.*
(a) *There exists a unique representation of the unital commutative C^*-algebra $C(\sigma(T))$ on H:*

$$\Phi_T : C(\sigma(T)) \ni f \mapsto f(T, T^*) \in \mathfrak{B}(\mathsf{H})\,,$$

such that

$$\Phi_T(z) = T \tag{8.23}$$

if z is the polynomial $\sigma(T) \ni (z, \overline{z}) \mapsto z$.
(b) *Moreover:*

(i) Φ_T is faithful, as isometric: for any $f \in C(\sigma(T))$, $||\Phi_T(f)|| = ||f||_\infty$;

(ii) if, for $A \in \mathfrak{B}(\mathsf{H})$, $AT = TA$ and $AT^* = T^*A$, then $A\Phi_T(f) = \Phi_T(f)A$ for any $f \in C(\sigma(T))$;

(iii) Φ_T preserves involutions: $\Phi_T(\overline{f}) = \Phi_T(f)^*$ for any $f \in C(\sigma(T))$.
(c) *$\sigma(\Phi_T(f)) = f(\sigma(T), \overline{\sigma(T)})$, for any $f \in C(\sigma(T))$.*

One consequence is worth making explicit.

Corollary 8.38 *Let H be a Hilbert space and $T \in \mathfrak{B}(\mathsf{H})$ a normal operator. Consider the isometric *-homomorphism $\Phi_T : C(\sigma(T)) \to \mathfrak{B}(\mathsf{H})$ defined in Proposition 8.37. Then the set $\Phi_T(C(\sigma(T)))$ of continuous functions in the variables T, T^* (defined on $\sigma(T)$) is the smallest C^*-subalgebra with unit in $\mathfrak{B}(\mathsf{H})$ containing I and T.*

Proof Every unital C^*-subalgebra $\mathfrak{A}$ of $\mathfrak{B}(\mathsf{H})$ containing I and T must contain polynomials in T, T^* (restricted to $\sigma(T)$). The construction that led to Φ_T shows $\mathfrak{A}$ contains all continuous maps in T and T^*, i.e. $\Phi_T(C(\sigma(T)))$. The latter, being the image of a C^*-algebra with unit under an injective *-homomorphism, is a C^*-subalgebra with unit of $\mathfrak{B}(\mathsf{H})$ (Theorem 8.22(b)). □

The fact that we are now working with a concrete C^*-algebra of operators allows to make one step forward in functional calculus. We can generalise the above theorem by defining $f(T, T^*)$ when f is a *bounded and measurable, not necessarily continuous,* map. In order to do so, in the absence of a Stone-Weierstrass-type theorem for bounded measurable functions ($C(\mathsf{X})$ is not dense in $M_b(\mathsf{X})$ if X is compact with non-empty interior in $\mathbb{R}^n$, cf. Remark 2.29(4)), we shall use heavily Riesz's representation results (for Hilbert spaces and Borel measures).

Recall that on a topological space X, $\mathscr{B}(\mathsf{X})$ is the Borel σ-algebra on X. The C^*-algebra of bounded measurable maps $f : \mathsf{X} \to \mathbb{C}$ is indicated by $M_b(\mathsf{X})$ (Examples 2.29(3) and 3.48(1)).

Proposition 8.37 can be generalised to prove the existence and uniqueness of a *-homomorphism of unital C^*-algebras $M_b(\sigma(T)) \to \mathfrak{B}(\mathsf{H})$ (the topology on $\sigma(T)$ is induced by $\mathbb{C} \supset \sigma(T)$). The consequences of the next theorem are legion. It will, in particular, be a crucial ingredient to prove the existence of spectral measures, in Theorem 8.56. Statement (iii) in (b) will be completed by Theorem 9.11.

Theorem 8.39 (Functional calculus for bounded measurable functions of normal operators) *Let* H *be a Hilbert space and* $T \in \mathfrak{B}(\mathsf{H})$ *a normal operator.*
(a) *There is a unique representation of the unital commutative* C^**-algebra* $M_b(\sigma(T))$ *(with respect to* $|| \ ||_\infty$*) on* H*:*

$$\widehat{\Phi}_T : M_b(\sigma(T)) \ni f \mapsto f(T, T^*) \in \mathfrak{B}(\mathsf{H})$$

such that:

(i) if z *is the polynomial* $\sigma(a) \ni (z, \overline{z}) \mapsto z$*,*

$$\widehat{\Phi}_T(z) = T \ ; \tag{8.24}$$

(ii) if $\{f_n\}_{n\in\mathbb{N}} \subset M_b(\sigma(T))$ *is bounded and converges pointwise to* $f : \sigma(T) \to \mathbb{C}$*, then*

$$\widehat{\Phi}_T(f) = w\text{-}\lim_{n\to+\infty} \widehat{\Phi}_T(f_n) \ .$$

(b) *The mapping* $\widehat{\Phi}_T$ *enjoys these properties:*

(i) the restriction of $\widehat{\Phi}_T$ *to* $C(\sigma(T))$ *is the* **-homomorphism* Φ_T *of Proposition 8.37;*

(ii) for any $f \in M_b(\sigma(T))$, $||\widehat{\Phi}_T(f)|| \leq ||f||_\infty$*;*

(iii) with $A \in \mathfrak{B}(\mathsf{H})$*, if* $AT = TA$ *and* $AT^* = T^*A$ *then* $A\widehat{\Phi}_T(f) = \widehat{\Phi}_T(f)A$ *for any* $f \in M_b(\sigma(T))$*;*

(iv) $\widehat{\Phi}_T$ *preserves involutions:* $\widehat{\Phi}_T(\overline{f}) = \widehat{\Phi}_T(f)^*$ *for any* $f \in M_b(\sigma(T))$*;*

(v) if $\{f_n\}_{n\in\mathbb{N}} \subset M_b(\sigma(T))$ *is bounded and converges pointwise to* $f : \sigma(T) \to \mathbb{C}$, *then*

$$\widehat{\Phi}_T(f) = s\text{-}\lim_{n\to+\infty} \widehat{\Phi}_T(f_n) ;$$

(vi) if $f \in M_b(\sigma(T))$ *takes only real values and* $f \geq 0$, *then* $\widehat{\Phi}_T(f) \geq 0$.

Proof (a). Fix $x, y \in \mathsf{H}$. The map

$$L_{x,y} : C(\sigma(T)) \ni f \mapsto (x|\Phi_T(f)y) \in \mathbb{C}$$

is linear and $||L_{x,y}||$ is given by:

$$\sup\{|L_{x,y}(f)| \mid f \in C(\sigma(T)) , \quad ||f||_\infty = 1\}$$

$$\leq ||x||\,||y|| \sup\{||\Phi_T(f)|| \mid f \in C(\sigma(T)) , \quad ||f||_\infty = 1\}$$

(Cauchy–Schwarz was used). Since Φ_T is isometric we find

$$||L_{x,y}|| \leq ||x||\,||y|| ,$$

so $L_{x,y}$ is bounded.

By Theorem 2.52 (Riesz's representation theorem for complex measures) there exists a unique complex measure $\mu_{x,y}$ (Definition 1.81) on the compact set $\sigma(T) \subset \mathbb{C}$, such that for any $f \in C(\sigma(T))$:

$$L_{x,y}(f) = (x|\Phi_T(f)y) = \int_{\sigma(T)} f(\lambda)\, d\mu_{x,y}(\lambda) . \tag{8.25}$$

Moreover, $|\mu_{x,y}|(\sigma(T)) = ||L_{x,y}|| \leq ||x||\,||y||$. Aside, note that $x = y$ forces $\mu_{x,x}$ to be a real, positive, finite measure: in fact, if $f \in C(\sigma(T))$ is real-valued $\Phi_T(f) = \Phi_T(f)^*$ by part (iii) of Proposition 8.37(b), so

$$\int_{\sigma(T)} f(\lambda)\, \overline{h(\lambda)} d|\mu_{x,x}(\lambda)| = \overline{\int_{\sigma(T)} f(\lambda)\, h(\lambda) d|\mu_{x,x}(\lambda)|} = \overline{(x|\Phi_T(f)x)} = (\Phi_T(f)x|x)$$

$$= (x|\Phi_T(f)x) = \int_{\sigma(T)} f(\lambda)\, h(\lambda) d|\mu_{x,x}(\lambda)| ,$$

where we have written $d\mu_{x,x}$ as $hd|\mu_{x,x}|$, h being a measurable map of unit norm determined, almost everywhere, by $\mu_{x,x}$ (Theorem 1.87), and $|\mu_{x,x}|$ being the *total variation* of $\mu_{x,x}$ (Remark 1.82(2)). By linearity

$$\int_{\sigma(T)} f(\lambda)\, \overline{h(\lambda)} d|\mu_{x,x}(\lambda)| = \int_{\sigma(T)} f(\lambda)\, h(\lambda) d|\mu_{x,x}(\lambda)|$$

must hold when $f \in C(\sigma(T))$ is complex-valued. Riesz's Theorem 2.52 on complex measures guarantees $hd|\mu_{x,x}| = \overline{h}d|\mu_{x,x}|$, so $\overline{h(\lambda)} = h(\lambda)$ almost everywhere; but $|h(\lambda)| = 1$, so $h(\lambda) = 1$ almost everywhere, and hence $\mu_{x,x}$ is a real, positive and finite measure (so is $|\mu_{x,x}|$).

Use (8.25) to generalise $L_{x,y}(f)$ to the case $f \in M_b(\sigma(T))$, since the right-hand side is well defined anyway: if $g \in M_b(\sigma(T))$,

$$L_{x,y}(g) := \int_{\sigma(T)} g(\lambda)\, d\mu_{x,y}(\lambda)\,. \tag{8.26}$$

By general properties of complex measures (cf. Example 2.48(1)):

$$|L_{x,y}(g)| \leq ||g||_\infty |\mu_{x,y}|(\sigma(T)) \leq ||g||_\infty ||x||\, ||y||\,. \tag{8.27}$$

By construction, given $g \in C(\sigma(T))$, $(x, y) \mapsto L_{x,y}(g)$ is antilinear in x and linear in y. One can prove this is still valid for $g \in M_b(\sigma(T))$. Let us for instance show linearity in y, the other part being similar. Given $x, y, z \in \mathsf{H}$ and $g \in M_b(\sigma(T))$, if $\alpha, \beta \in \mathbb{C}$ then

$$\alpha \int_{\sigma(T)} g(\lambda)\, d\mu_{x,y}(\lambda) + \beta \int_{\sigma(T)} g(\lambda)\, d\mu_{x,z}(\lambda) = \int_{\sigma(T)} g(\lambda)\, d\nu(\lambda)\,, \tag{8.28}$$

where ν is the complex measure $\nu(E) := \alpha\mu_{x,y}(E) + \beta\mu_{x,z}(E)$ for any Borel set $E \subset \sigma(T)$. Remembering how we defined the $\mu_{x,y}$ (cf. (8.25)) and using the inner product's linearity on the right, we immediately see that for any $f \in C(\sigma(T))$ replacing g in (8.28):

$$\int_{\sigma(T)} f(\lambda)\, d\mu_{x,\alpha y+\beta z}(\lambda) = \int_{\sigma(T)} f(\lambda)\, d\nu(\lambda)\,.$$

Riesz's theorem now tells $\mu_{x,\alpha y+\beta z} = \nu$. Therefore (8.28) reads, *for any* $g \in M_b(\sigma(T))$:

$$\alpha \int_{\sigma(T)} g(\lambda)\, d\mu_{x,y}(\lambda) + \beta \int_{\sigma(T)} g(\lambda)\, d\mu_{x,z}(\lambda) = \int_{\sigma(T)} g(\lambda)\, \mu_{x,\alpha y+\beta z}(\lambda)\,.$$

We proved $L_{x,y}(g)$ is linear in y for any given $x \in \mathsf{H}$ and any $g \in M_b(\sigma(T))$.

Equation (8.27) implies the linear operator $y \mapsto L_{x,y}(g)$ is bounded, so by Theorem 3.16 (Riesz once again), given $g \in M_b(\sigma(T))$ and $x \in \mathsf{H}$, there exists a unique $v_x \in \mathsf{H}$ such that $L_{x,y}(g) = (v_x|y)$ for any $y \in \mathsf{H}$. Since v_x is linear in x ($L_{x,y}(g)$ is antilinear in x and the inner product $(v_x|y)$ is antilinear in v_x), there is also a unique operator $g(T, T^*)' \in \mathfrak{L}(\mathsf{H})$ such that $v_x = g(T, T^*)'x$ for any $x \in \mathsf{H}$. Hence $L_{x,y}(g) = (g(T, T^*)'x|y)$. Condition (8.27) implies $g(T, T^*)'$ is bounded, for:

$$||g(T, T^*)'x||^2 = |(g(T, T^*)'x|g(T, T^*)'x)| = |L_{x,g(T,T^*)'x}(g)| \leq ||g||_\infty\, ||x||\, ||g(T, T^*)'x||\,,$$

hence

$$\frac{||g(T, T^*)'x||}{||x||} \leq ||g||_\infty$$

and then $||g(T, T^*)'|| \leq ||g||_\infty$.

Setting $g(T, T^*) := g(T, T^*)'^*$, we proved that for $g \in M_b(\sigma(T))$ there is a unique operator $g(T, T^*) \in \mathfrak{B}(\mathsf{H})$ such that

$$L_{x,y}(g) = (x|g(T, T^*)y)$$

for any $x, y \in \mathsf{H}$. The linear mapping

$$\widehat{\Phi}_T : M_b(\sigma(T)) \ni f \mapsto f(T, T^*) \in \mathfrak{B}(\mathsf{H}) ,$$

where, for any $x, y \in \mathsf{H}$,

$$L_{x,y}(f) = (x|f(T, T^*)y) := \int_{\sigma(T)} f(\lambda)\, d\mu_{x,y}(\lambda) ,$$

is, by construction, an extension of Φ_T: in particular (8.24) holds and $\widehat{\Phi}_T(1) = \Phi_T(1) = I$. The extension is continuous because $||\widehat{\Phi}_T(f)|| \leq ||f||_\infty$ for any $f \in M_b(\sigma(T))$, in fact:

$$||\widehat{\Phi}_T(f)|| = ||f(T, T^*)|| = ||f(T, T^*)'^*|| = ||f(T, T^*)'|| \leq ||f||_\infty .$$

As $\widehat{\Phi}_T$ extends the algebra homomorphism Φ_T, to prove $\widehat{\Phi}_T$ is an algebra homomorphism it suffices to show $\widehat{\Phi}_T(f \cdot g) = \widehat{\Phi}_T(f)\widehat{\Phi}_T(g)$ when $f, g \in M_b(\sigma(T))$. If the two maps belong in $C(\sigma(T))$, the claim is true by Proposition 8.37 above. Suppose $f, g \in C(\sigma(T))$. Then

$$\int_{\sigma(T)} f \cdot g\, d\mu_{x,y} = (x|\widehat{\Phi}_T(f \cdot g)y) = (x|\widehat{\Phi}_T(f)\widehat{\Phi}_T(g)y) = \int_{\sigma(T)} f\, d\mu_{x,\widehat{\Phi}_T(g)y} .$$

The aforementioned theorem of Riesz on complex measures implies that $d\mu_{x,\widehat{\Phi}_T(g)y}$ coincides with $g\, d\mu_{x,y}$. Hence, if $f \in M_b(\sigma(T))$,

$$\int_{\sigma(T)} f \cdot g\, d\mu_{x,y} = \int_{\sigma(T)} f\, d\mu_{x,\widehat{\Phi}_T(g)y}.$$

From this follows, for any $x, y \in \mathsf{H}$, $f \in M_b(\sigma(T))$ and $g \in C(\sigma(T))$:

$$\int_{\sigma(T)} f \cdot g\, d\mu_{x,y} = \int_{\sigma(T)} f\, d\mu_{x,\widehat{\Phi}_T(g)y} = (x|\widehat{\Phi}_T(f)\widehat{\Phi}_T(g)y) = (\widehat{\Phi}_T(f)^*x|\widehat{\Phi}_T(g)y)$$

$$= \int_{\sigma(T)} g\, d\mu_{\widehat{\Phi}_T(f)^*x,y} .$$

Arguing as before, and by Riesz's theorem, the equality

$$\int_{\sigma(T)} f \cdot g \, d\mu_{x,y} = \int_{\sigma(T)} g \, d\mu_{\widehat{\Phi}_T(f)^* x, y} , \tag{8.29}$$

valid for any $g \in C(\sigma(T))$, forces $f d\mu_{x,y} = d\mu_{\widehat{\Phi}_T(f)^* x, y}$, so (8.29) must hold for any $x, y \in \mathsf{H}$, and any $f, g \in M_b(\sigma(T))$. Therefore

$$(x|\widehat{\Phi}_T(f \cdot g)y) = \int_{\sigma(T)} f \cdot g \, d\mu_{x,y} = \int_{\sigma(T)} g \, d\mu_{\widehat{\Phi}_T(f)^* x, y}$$

$$= (\widehat{\Phi}_T(f)^* x|\widehat{\Phi}_T(g)y) = (x|\widehat{\Phi}_T(f)\widehat{\Phi}_T(g)y) ,$$

and consequently

$$\left(x \,\middle|\, (\widehat{\Phi}_T(f \cdot g) - \widehat{\Phi}_T(f)\widehat{\Phi}_T(g))y\right) = 0 .$$

Choosing x as the second argument in the inner product gives

$$\widehat{\Phi}_T(f \cdot g)y = \widehat{\Phi}_T(f)\widehat{\Phi}_T(g)y$$

for any $y \in \mathsf{H}$, $f, g \in M_b(\sigma(T))$, whence

$$\widehat{\Phi}_T(f \cdot g) = \widehat{\Phi}_T(f)\widehat{\Phi}_T(g) .$$

To show we have indeed a *-homomorphism we need to prove property (iv). Let $x \in \mathsf{H}$ and $g \in M_b(\sigma(T))$. Since $\mu_{x,x}$ is real, we have (beware that complex conjugation does not act on $\sigma(T)$, here thought of as subset in $\mathbb{R}^2$):

$$(x|\widehat{\Phi}_T(\overline{g})x) = \int_{\sigma(T)} \overline{g} \, d\mu_{x,x} = \overline{\int_{\sigma(T)} g \, d\mu_{x,x}} = (\widehat{\Phi}_T(g)x|x) = (x|\widehat{\Phi}_T(g)^* x) .$$

Hence $(x|(\widehat{\Phi}_T(\overline{g}) - \widehat{\Phi}_T(g)^*)x) = 0$ for any $x \in \mathsf{H}$. From Exercise 3.21 we have $\widehat{\Phi}_T(\overline{g}) = \widehat{\Phi}_T(g)^*$.

Property (ii) of (a) follows from (v) in (b), which we will prove below independently.

To finish (a), we show $\widehat{\Phi}_T$ is unique under (a). Let $\Psi : M_b(\sigma(T)) \to \mathfrak{B}(\mathsf{H})$ satisfy (a). It must coincide with $\widehat{\Phi}_T$ on polynomials, so by continuity (it is continuous being a *-homomorphism of C^*-algebras with unit, and Theorem 8.22 holds) it coincides with $\widehat{\Phi}_T$ on $C(\sigma(T))$. Given $x, y \in \mathsf{H}$, the map

$$\nu_{x,y} : E \mapsto (x|\Psi(\chi_E)y) ,$$

where E is an arbitrary Borel set in $\sigma(T)$ and χ_E its characteristic function, is a complex measure on $\sigma(T)$. In fact $\nu_{x,y}(\emptyset) = (x|\Psi(0)y) = 0$; moreover, if $\{S_k\}_{k\in\mathbb{N}}$ is a family of pairwise disjoint Borel sets,

$$\nu_{x,y}(\cup_k S_k) = (x|\Psi(\chi_{\cup_k S_k})y) = \left(x \,\middle|\, \lim_{n\to+\infty} \Psi\left(\sum_{k=0}^{n} \chi_{S_k}\right) y\right) = \lim_{n\to+\infty} \sum_{k=0}^{n} (x|\Psi(\chi_{S_k})y)$$

$$= \sum_{k=0}^{+\infty} \nu_{x,y}(S_k) \ ,$$

where the left-hand side is always finite, we used (ii) in (a) and also that, pointwise:

$$\chi_{\cup_k S_k} = \sum_{k=0}^{+\infty} \chi_{S_k} \ . \tag{8.30}$$

Observe that (8.30) does not depend on the order of the S_k, for the series has positive terms. Consequently

$$\nu_{x,y}(\cup_k S_k) = \sum_{k=0}^{+\infty} \nu_{x,y}(S_k)$$

holds irrespective of the arrangement of the terms, and the series converges absolutely (Theorem 1.83). This means $\nu_{x,y}$ is a complex measure.

Bearing in mind the linearity of both Ψ and the inner product, plus the definition of integral of a simple map, we easily see

$$\int_{\sigma(T)} s \, d\nu_{x,y} = (x|\Psi(s)y)$$

for any simple map $s \in S(\sigma(T))$. If $f \in M_b(\sigma(T))$ and $\{s_n\} \subset S(\sigma(T))$ converges uniformly to f (the sequence exists by Proposition 7.49(b)), then the continuity of Ψ in norm $|| \ ||_\infty$ and dominated convergence relative to $|\nu_{x,y}S|$ imply

$$(x|\Psi(f)y) = \int_{\sigma(T)} f \, d\nu_{x,y} \tag{8.31}$$

for any $f \in M_b(\sigma(T))$. In particular, this must hold for $f \in C(\sigma(T))$, on which Ψ coincides with $\widehat{\Phi}_T$. Therefore, Riesz's Theorem 2.52 implies that $\nu_{x,y}$ coincides with the complex measure $\mu_{x,y}$ of the beginning, using which we defined $\widehat{\Phi}_T$ by

$$(x|\widehat{\Phi}_T(f)y) = \int_{\sigma(T)} f \, d\mu_{x,y} \ ,$$

for $x, y \in \mathsf{H}$ and $f \in M_b(\sigma(T))$. But then (8.31) implies $\Psi(f) = \widehat{\Phi}_T(f)$ for any $f \in M_b(\sigma(T))$, for $\nu_{x,y} = \mu_{x,y}$.
(b). We only need to prove (iii), (v) and (vi), because the rest were shown in part (a). Property (iii) holds when $f \in C(\sigma(T))$, as we know from Proposition 8.37(b). If $AT = TA$ and $AT^* = T^*A$,

$$\int_{\sigma(T)} f\, d\mu_{x,Ay} = (x|\widehat{\Phi}_T(f)Ay) = (x|A\widehat{\Phi}_T(f)y) = (A^*x|\widehat{\Phi}_T(f)y) = \int_{\sigma(T)} f\, d\mu_{A^*x,y}\,,$$

for any vectors $x, y \in \mathsf{H}$ and any $f \in C(\sigma(T))$. Riesz's Theorem 2.52 on the representation of complex measures on Borel sets ensures $\mu_{A^*x,y} = \mu_{x,Ay}$, hence

$$(x|\widehat{\Phi}_T(f)Ay) = \int_{\sigma(T)} f\, d\mu_{x,Ay} = \int_{\sigma(T)} f\, d\mu_{A^*x,y} = (A^*x|\widehat{\Phi}_T(f)y) = (x|A\widehat{\Phi}_T(f)y)$$

for any $x, y \in \mathsf{H}$, $f \in M_b(\sigma(T))$. As the vectors x, y are arbitrary, $\widehat{\Phi}_T(f)A = A\widehat{\Phi}_T(f)$ if $f \in M_b(\sigma(T))$.

Let us prove (v). Take a sequence $\{f_n\}_{n\in\mathbb{N}} \subset M_b(\sigma(T))$ that is bounded (in absolute value) by $K > 0$ and that converges to $f : \sigma(T) \to \mathbb{C}$. Therefore $||f||_\infty \leq K$ and f is measurable, forcing $f \in M_b(\sigma(T))$. Given $x, y \in \mathsf{H}$ and using (iv) in (b),

$$||(\widehat{\Phi}_T(f_n) - \widehat{\Phi}_T(f))x||^2 = ((\widehat{\Phi}_T(f_n) - \widehat{\Phi}_T(f))x|(\widehat{\Phi}_T(f_n) - \widehat{\Phi}_T(f))x)$$
$$= (x|(\widehat{\Phi}_T(f_n - f)^*\widehat{\Phi}_T(f_n - f)x) = (x|\widehat{\Phi}_T(|f - f_n|^2)x)\,.$$

The last terms can be written as

$$\int_{\sigma(T)} |f - f_n|^2\, d\mu_{x,x} = \int_{\sigma(T)} |f - f_n|^2\, hd|\mu_{x,x}|\,,$$

where $|\mu_{x,x}|$ is the positive measure (the total variation of Remark 1.82(2)) associated the real (signed) measure $\mu_{x,x}$, and h is a measurable function of constant modulus 1 (Theorem 1.87). (Actually, we saw in part (a) that $\mu_{x,x}$ is a positive real measure, so $|\mu_{x,x}| = \mu_{x,x}$ and $h = 1$.) Because

$$|\mu_{x,x}|(\sigma(T)) < +\infty\,,$$

the dominated convergence theorem implies $|h||f - f_n|^2$ converges to 0 in $L^1(\sigma(T), |\mu_{x,x}|)$. Hence as $n \to +\infty$, $||(\widehat{\Phi}_T(f_n) - \widehat{\Phi}_T(f))x||^2 \to 0$ for any $x \in \mathsf{H}$.

Eventually, let us prove (vi). The proof is easy and follows from Theorem 8.25(iii), but here is an alternative argument. If $M_b(\sigma(T)) \ni f \geq 0$, then $f = g^2$ where $0 \leq g \in M_b(\sigma(T))$. By (a), $\widehat{\Phi}_T(f) = \widehat{\Phi}_T(g \cdot g) = \widehat{\Phi}_T(g)\widehat{\Phi}_T(g)$. Moreover, $\widehat{\Phi}_T(g)^* = \widehat{\Phi}_T(\overline{g}) = \widehat{\Phi}_T(g)$ (by (iv)), so $\widehat{\Phi}_T(g \cdot g) = \widehat{\Phi}_T(g)^*\widehat{\Phi}_T(g)$. The right-hand side is patently positive. $\square$

Remark 8.40 The spectral decomposition theorem, proved later, is in some sense a way to interpret the operator $f(T, T^*)$ in terms of an integral of f with respect to an operator-valued measure: integrating bounded measurable functions produces, instead of numbers, operators. The version of the spectral decomposition theorem presented in this chapter states that there is always such a measure, for any bounded normal operator. ■

8.3 Projector-Valued Measures (PVMs)

In this section we introduce *projector-valued measures* (PVMs), also called *spectral measures*. They are the central tool to state spectral theorems, and represent a generalisation of the notion of measure on a measurable space $(\mathsf{X}, \Sigma(\mathsf{X}))$. Now the measure's range is no longer contained in $\mathbb{R}$. Rather, it is a subset of orthogonal projectors $\mathscr{L}(\mathsf{H})$ in a Hilbert space H:

$$\Sigma(\mathsf{X}) \ni E \mapsto P(E) \in \mathscr{L}(\mathsf{H}) ,$$

using which we will be able to *integrate functions to obtain operators*. We will see, in particular, that the homomorphism $\widehat{\Phi}_T$ associated to a bounded normal operator T, studied in the previous section, is nothing but an integral with respect to a PVM generated by T:

$$\widehat{\Phi}_T(f) = \int_{\sigma(T)} f(x) dP^{(T)}(x) ,$$

where $\mathsf{X} := \sigma(T) \subset \mathbb{C}$ is viewed as a second-countable topological space equipped with the topology induced from $\mathbb{R}^2$, and $\Sigma(\mathsf{X})$ is the associated Borel σ-algebra $\mathscr{B}(\sigma(T))$.

Projector-valued measures made their appearance already in Chap. 7 (Definition 7.46), in the special situation where the σ-algebra of the PVM was the Borel σ-algebra $\mathscr{B}(\mathbb{R})$. A quantum *observable*, in the sense of the previous chapter, is a special spectral measure, by virtue of Proposition 7.44. In that case the operator to which such a PVM is attached is not just normal, but self-adjoint as well.

8.3.1 Spectral Measures, or PVMs

We remind that for $T, U \in \mathfrak{B}(\mathsf{H})$ on a Hilbert space H, the writing $T \geq U$ means $(x|Tx) \geq (x|Ux)$ for any $x \in \mathsf{H}$ (see Definition 3.56(f) and the ensuing comments).

Definition 8.41 (*Spectral measure*) Let H be a Hilbert space and $(\mathsf{X}, \Sigma(\mathsf{X}))$ a measurable space. One calls $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ a **spectral measure** on X, or **projector-valued measure on** X (PVM), if the following requisites are satisfied.

(a) $P(B) \geq 0$ for any $B \in \Sigma(\mathsf{X})$;
(b) $P(B)P(B') = P(B \cap B')$ for any $B, B' \in \Sigma(\mathsf{X})$;
(c) $P(\mathsf{X}) = I$;
(d) if $\{B_n\}_{n\in\mathbb{N}} \subset \Sigma(\mathsf{X})$, with $B_n \cap B_m = \emptyset$, $n \neq m$:

$$\text{s-}\sum_{n=0}^{+\infty} P(B_n) = P(\cup_{n\in\mathbb{N}} B_n) \, .$$

Remark 8.42 Notice that (c) and (d) imply $P(\emptyset) = 0$. ■

A relevant definition is that of *support of a PVM* in the special case where X is a topological space and $\Sigma(\mathsf{X})$ is the Borel σ-algebra $\mathscr{B}(\mathsf{X})$.

Definition 8.43 Specialising Definition 8.41 to the case of a PVM $P : \mathscr{B}(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ defined on the Borel σ-algebra $\mathscr{B}(\mathsf{X})$ of a topological space $(\mathsf{X}, \mathscr{T})$, the **support of** P is the closed set

$$supp(P) := \mathsf{X} \setminus \bigcup_{A\in\mathscr{T}, P(A)=0} A \, .$$

When $\mathsf{X} = \mathbb{R}^n$ or $\mathbb{C}^n$ with the standard topology, such P is called **bounded** if $supp(P)$ is bounded.

The next proposition treats the basic properties of PVMs. In particular, as the name PVM itself suggests, every $P(E)$ is an orthogonal projector onto the Hilbert space H.

Proposition 8.44 *Retaining Definition 8.41, the following facts hold.*
(a) *$P(B)$ is an orthogonal projector for any $B \in \Sigma(\mathsf{X})$.*
Keeping (c) and (d), condition (a) and (b) in Definition 8.41 may be replaced by the equivalent requirement that $P(B)$ is an orthogonal projector if $B \in \Sigma(\mathsf{X})$.
(b) *P is* **monotone***: $C \subset B \Rightarrow P(C) \leq P(B)$, for all $B, C \in \Sigma(\mathsf{X})$.*
(c) *P is* **sub-additive***: if $B_n \in \Sigma(\mathsf{X})$, $n \in \mathbb{N}$, then*

$$(x\,|P\,(\cup_{n\in\mathbb{N}} B_n)\,x) \leq \sum_{n\in\mathbb{N}} (x|P(B_n)x) \quad \textit{for any } x \in \mathsf{H}.$$

(d) *Assume $\Sigma(\mathsf{X}) := \mathscr{B}(\mathsf{X})$ for some topological space $(\mathsf{X}, \mathscr{T})$, and that at least one of the following conditions is true:*

1. *$(\mathsf{X}, \mathscr{T})$ is second-countable;*
2. *$(\mathsf{X}, \mathscr{T})$ is Hausdorff, locally compact, and the positive Borel measure $\mathscr{B}(\mathsf{X}) \ni E \mapsto (x|P(E)x)$ is* inner regular *for every $x \in \mathsf{H}$;*

then P is **concentrated** *on $supp(P)$, i.e.*

$$P(B) = P(B \cap supp(P)) \quad \textit{for } B \in \mathscr{B}(\mathsf{X}).$$

In particular $P(supp(P)) = I$.

Proof (a) The operators $P(B)$ are idempotent, $P(B)P(B) = P(B \cap B) = P(B)$, by Definition 8.41(b), and self-adjoint because bounded and positive (by Definition 8.41(a)), so they are orthogonal projectors. We claim that if (c) and (d) in definition 8.41 hold, and every $P(B)$ is an orthogonal projector, then also (a) and (b) hold in Definition 8.41. Part (a) is trivial, for any $P(E)$ is an orthogonal projector, hence positive. By part (d), if $E_1, E_2 \in \Sigma(\mathsf{X})$ and $E_1 \cap E_2 = \emptyset$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$. Multiplying by $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, and recalling we are using (idempotent) projectors, gives $P(E_1)P(E_2) + P(E_2)P(E_1) = 0$ and so $P(E_1)P(E_2) = -P(E_2)P(E_1)$. Applying now $P(E_1)$ and recalling that $P(E_1)P(E_2) = -P(E_2)P(E_1)$, we also find $P(E_1)P(E_2) = P(E_2)P(E_1)$. Therefore $P(E_1)P(E_2) = 0$ if $E_1 \cap E_2 = \emptyset$. Now set $C = B \cap B'$, $E_1 = B \setminus C$, $E_2 = B' \setminus C$ for $B, B' \in \Sigma(\mathsf{X})$. Remember $E_1 \cap E_2 = E_1 \cap C = E_2 \cap C = \emptyset$. Then

$$P(B)P(B') = (P(E_1) + P(C))(P(E_2) + P(C)) = P(C)P(C) = P(C) = P(B \cap B')$$

i.e. property (b) in 8.41.
(b) $B = C \cup (B \setminus C)$ and $C \cap (B \setminus C) = \emptyset$ so by Definition 8.41(d) it follows that $P(B) = P(C) + P(B \setminus C)$. But $P(B \setminus C) \geq 0$, so $P(C) \leq P(B)$.
(c) Define $B := \cup_{n \in \mathbb{N}} B_n$ and the sequence $\{C_n\}_{n \in \mathbb{N}}$, with $C_0 := B_0$, $C_1 := B_1 \setminus B_0$, $C_2 := B_2 \setminus (B_0 \cup B_1)$ and so on. Clearly $C_k \cap C_h = \emptyset$ if $h \neq k$ and $B = \cup_{n \in \mathbb{N}} C_n$. By Definition 8.41(d), then, $P(B)x = \sum_{n=0}^{+\infty} P(C_k)x$ and so $(x|P(B)x) = \sum_{n=0}^{+\infty}(x|P(C_k)x)$. Since $C_k \subset B_k$ for any $k \in \mathbb{N}$, by monotonicity $(x|P(C_k)x) \leq (x|P(B_k)x)$, i.e.

$$(x|P(B)x) \leq \sum_{n=0}^{+\infty}(x|P(B_k)x) \; .$$

(d) $P(supp(P)) = I$ is obviously equivalent to $P(A) = 0$, where $A := \mathsf{X} \setminus supp(P)$. To prove $P(A) = 0$ under hypothesis (1), notice that by definition A is the union of open sets with null spectral measure. As X is second-countable, Lindelöf's lemma (Theorem 1.8) says we can extract a countable subcovering $A = \cup_{n \in \mathbb{N}} A_n$, with $P(A_n) = 0$ for any $n \in \mathbb{N}$. Using sub-additivity, for any $x \in \mathsf{H}$,

$$0 \leq ||P(A)x||^2 = (P(A)x|P(A)x) = (x|P(A)x) \leq \sum_{n \in \mathbb{N}}(x|P(A_n)x) = 0 \; ,$$

hence $P(A) = 0$. If, instead, hypothesis (2) holds, consider the inner-regular Borel measure $\mathscr{B}(\mathsf{X}) \ni E \mapsto \mu_x(E) := (x|P(E)x)$. If $O \subset \mathsf{X}$ is open and $P(O) = 0$, then $(x|P(O)x) = ||P(O)x||^2 = 0$ for every $x \in \mathsf{H}$. Hence the union $A := \mathsf{X} \setminus supp(P)$ of all such O satisfies $||P(A)x||^2 = \mu_x(A) = 0$ by Proposition 1.45(ii), so $P\,(\mathsf{X} \setminus supp(P)) = 0$ since x is arbitrary.

With A defined as above, decomposing $B = (B \cap supp(P)) \cup (B \cap A)$, Definition 8.41(d) in fact gives $P(B) = P(B \cap supp(P)) + P(B \cap A)$, and monotonicity $0 \leq (x|P(B \cap A)x) \leq (x|P(A)x) = 0$. In summary, since x is arbitrary, $P(B) = P(B \cap supp(P))$, as required. □

Another related and useful definition is the following.

Definition 8.45 If $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ is a PVM, a measurable function $f : \mathsf{X} \to \mathbb{C}$ is called **essentially bounded for** P when

$$P(\{x \in \mathsf{X} \mid |f(x)| \geq M\}) = 0 \quad \text{for some } M < +\infty. \tag{8.32}$$

If f is essentially bounded, the greatest lower bound $||f||^{(P)}_\infty$ on the set of numbers $M \geq 0$ satisfying (8.32) is called **essential (semi)norm** of f in P.

Remarks 8.46 **(1)** Let $f : \mathsf{X} \to \mathbb{C}$ be measurable. If Proposition 8.44(d) holds, then P is concentrated on $supp(P)$. Hence the first inequality below is easy to prove:

$$||f||^{(P)}_\infty \leq ||f\restriction_{supp(P)}||_\infty \leq ||f||_\infty \,. \tag{8.33}$$

(the second one is obvious). Equation (8.33) holds trivially when one among $||f||^{(P)}_\infty$, $||f\restriction_{supp(P)}||_\infty$, $||f||_\infty$ is $+\infty$.
(2) The set of measurable functions that are essentially bounded for P is a vector space, and $||\ \ ||^{(P)}_\infty$ is a *seminorm* on it. ■

8.3.2 Integrating Bounded Measurable Functions in a PVM

We pass now to define a procedure to integrate bounded measurable functions *with respect to a PVM*.

Recall that given a space X with a σ-algebra Σ, a (complex-valued) map $s : \mathsf{X} \to \mathbb{C}$, measurable for Σ, is called **simple** when its range is finite.

Notation 8.47 If X is a measurable space, $S(\mathsf{X})$ denotes the vector space of complex-valued simple functions on X, relative to the σ-algebra $\Sigma(\mathsf{X})$. ■

Let a PVM be given on X, with values in $\mathfrak{B}(\mathsf{H})$ for some Hilbert space H. Consider a map $s \in S(\mathsf{X})$. We can always write it, for suitable $c_i \in \mathbb{C}$ and I finite, as follows:

$$s = \sum_{i \in I} c_i \chi_{E_i} \,. \tag{8.34}$$

As, by definition, the range of a simple function consists of finitely many distinct values, the expression above is uniquely determined by s once we require the measurable sets E_i to be pairwise disjoint, and that the complex numbers c_i are distinct. We define the **integral of** s **with respect to** P (or **in** P) as the operator in $\mathfrak{B}(\mathsf{H})$:

$$\int_{\mathsf{X}} s(x)\, dP(x) := \sum_{i \in I} c_i P(E_i) \,. \tag{8.35}$$

Remark 8.48 If we do not insist the above c_i be distinct, there are several ways to write s as a linear combination of characteristic functions of disjoint measurable sets. Using the same argument as for an ordinary measure it is easy to prove, however, that the integral of s does not depend on the particular representation of s chosen. ■

The mapping

$$\mathfrak{I} : S(\mathsf{X}) \ni s \mapsto \int_{\mathsf{X}} s(x)\, dP(x) \in \mathfrak{B}(\mathsf{X}) , \tag{8.36}$$

is linear, i.e. $\mathfrak{I} \in \mathfrak{L}(S(\mathsf{X}), \mathfrak{B}(\mathsf{H}))$, as the previous remark easily implies. Since $S(\mathsf{X})$ and $\mathfrak{B}(\mathsf{H})$ are normed spaces, $\mathfrak{L}(S(\mathsf{X}), \mathfrak{B}(\mathsf{H}))$ is equipped with the operator norm. Then $\mathfrak{I}$ turns out to be a *bounded* operator for this norm. Let us prove this fact, and consider $s \in S(\mathsf{X})$ of the form (8.34). As the E_k are pairwise disjoint, $P(E_j)P(E_i) = P(E_j \cap E_i) = 0$ if $i \neq j$ or $P(E_j)P(E_i) = P(E_i)$ if $i = j$. If $x \in \mathsf{H}$

$$||\mathfrak{I}(s)x||^2 = (\mathfrak{I}(s)x|\mathfrak{I}(s)x) = \left(\sum_{i\in I} c_i P(E_i)x \middle| \sum_{j\in I} c_j P(E_j)x\right) = \sum_{i,j\in I} \left(c_i P(E_j)^* P(E_i)x \middle| c_j x\right)$$

$$= \sum_{i,j\in I} \left(c_i P(E_j)P(E_i)x \middle| c_j x\right) = \sum_{i\in I} |c_i|^2 (x|P(E_i)x) \leq \sup_{i\in I'} |c_i|^2 \sum_{i\in I'} (x|P(E_i)x) ,$$

where $I' \subset I$ is made by indices for which $P(E_i) \neq 0$. By additivity and monotonicity

$$\sum_{i\in I'} (x|P(E_i)x) \leq (x|P(\cup_{i\in I'} E_i)x) \leq (x|P(\mathsf{X})x) = (x|x) = ||x||^2 .$$

But I' is finite, so trivially $||s||_\infty^P = \sup_{i\in I'} |c_i|$, and hence $||\mathfrak{I}(s)x||^2 \leq ||x||^2(||s||_\infty^P)^2$. Taking the least upper bound over unit vectors $x \in \mathsf{H}$:

$$||\mathfrak{I}(s)|| \leq ||s||_\infty^{(P)} .$$

But $||s||_\infty^{(P)}$ coincides with one of the values of $|s|$, say $|c_k|$ if we choose $x \in P(E_k)(\mathsf{H})$ ($\neq \{\mathbf{0}\}$ by construction). Hence $x = P(E_k)x$ implies

$$\mathfrak{I}(s)x = \sum_{i\in I'} c_i P(E_i)x = \sum_{i\in I'} c_i P(E_i)P(E_k)x = c_k P(E_k)x = c_k x .$$

So choosing x with $||x|| = 1$ we obtain $||\mathfrak{I}(s)x|| = ||s||_\infty^{(P)}$. Therefore $\mathfrak{I}$ is certainly continuous on $S(\mathsf{X}) \subset M_b(\mathsf{X})$ in norm $||\ ||_\infty$, by what we have just proved and by (8.33):

$$||\mathfrak{I}(s)|| = ||s||_\infty^{(P)} \leq ||s||_\infty \tag{8.37}$$

This settled, we can define integrals of bounded measurable functions, by prolonging $\mathfrak{I}$ by linearity and continuity to the whole Banach space $M_b(\mathsf{X})$ of bounded measurable maps $f : \mathsf{X} \to \mathbb{C}$. The space $M_b(\mathsf{X})$ contains $S(\mathsf{X})$ as dense subspace for the norm $|| \ ||_\infty$, by Proposition 7.49(b). The operator $\mathfrak{I} : S(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ is continuous. By Proposition 2.47 there exists one, and only one, bounded operator $M_b(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ extending $\mathfrak{I}$.

Definition 8.49 Let $(\mathsf{X}, \Sigma(\mathsf{X}))$ a measurable space, H a Hilbert space and $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ a PVM.
(a) The unique bounded extension $\widehat{\mathfrak{I}} : M_b(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ of the operator $\mathfrak{I} : S(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ (cf. (8.35)–(8.36)) is called **integral operator** in P.
(b) For any $f \in M_b(\mathsf{X})$:

$$\int_{\mathsf{X}} f(x)\, dP(x) := \widehat{\mathfrak{I}}(f)$$

is the **integral of** f with respect to the projector-valued measure P.
(c) Let $f : \mathsf{X} \to \mathbb{C}$ be measurable, not necessarily bounded. If $f\restriction_E \in M_b(E)$ with $E \subset \Sigma(\mathsf{X})$, we define:

$$\int_E f(x)\, dP(x) := \int_{\mathsf{X}} \chi_E(x) f(x)\, dP(x)\,.$$

If $g \in M_b(E)$, with $E \subset \Sigma(\mathsf{X})$, we set:

$$\int_E g(x)\, dP(x) := \int_{\mathsf{X}} g_0(x)\, dP(x)\,,$$

where $g_0(x) := g(x)$ if $x \in E$, or $g_0(x) := 0$ if $x \notin E$.

Remark 8.50 Let us concentrate on the topological case: take X a topological space, $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ and suppose hypotheses (1) or (2) in Proposition 8.44(d) are valid. If P is a spectral measure on X and $supp(P) \neq \mathsf{X}$, we can *restrict* P to a spectral measure $P\restriction_{supp(P)}$ on $supp(P)$ (with induced topology), by defining $P\restriction_{supp(P)}(E) := P(E)$ for any measurable set $E \subset \mathscr{B}(supp(P))$. The fact that $P\restriction_{supp(P)}$ is a PVM is immediate using Proposition 8.44, especially part (d). From (d) we have, for any $s \in S(\mathsf{X})$,

$$\int_{\mathsf{X}} s\, dP = \int_{supp(P)} s\, dP = \int_{supp(P)} s\restriction_{supp(P)} dP\restriction_{supp(P)}\,,$$

where the second integral is understood in the sense of Definition 8.49(c). If $S(\mathsf{X}) \ni s_n \to f$ in norm $|| \ ||_\infty$, then $S(\mathsf{X}) \ni s_n\restriction_{supp(P)} \to f\restriction_{supp(P)}$ in the same norm. Therefore the definition of integral of $f \in M_b(\mathsf{X})$ with respect to P tells that

$$\int_{\mathsf{X}} f\, dP = \int_{supp(P)} f\, dP = \int_{supp(P)} f\restriction_{supp(P)} dP\restriction_{supp(P)} \quad \text{for any } f \in M_b(\mathsf{X}). \tag{8.38}$$

■

Examples 8.51
(1) Let us see a concrete example lest the procedure seem too abstract. The (generalisation of this) example actually covers all possibilities, as we shall explain.

Consider the Hilbert space $\mathsf{H} = L^2(\mathsf{X}, \mu)$, where X is a topological space and μ a positive σ-additive measure on the Borel σ-algebra of X. For any $\psi \in L^2(\mathsf{X}, \mu)$ and $E \in \mathscr{B}(\mathsf{X})$, set

$$(P(E)\psi)(x) := \chi_E(x)\psi(x)\,, \quad \text{for almost every } x \in \mathsf{X}\,. \tag{8.39}$$

The map $\mathscr{B}(\mathsf{X}) \ni E \mapsto P(E)$ easily defines a spectral measure on $L^2(\mathsf{X}, \mu)$. We want to understand what the operators $\int_{\mathsf{X}} f(x)\, dP(x)$ look like, for any map of $M_b(\mathsf{X})$.

If $\psi \in L^2(\mathsf{X}, \mu)$ and $f \in M_b(\mathsf{X})$, then $f \cdot \psi \in L^2(\mathsf{X}, \mu)$, where $\cdot$ is the pointwise product of maps, for:

$$\int_{\mathsf{X}} |f(x)\psi(x)|^2\, d\mu(x) \leq ||f||_\infty^2 \int_{\mathsf{X}} |\psi(x)|^2\, d\mu(x) < +\infty\,.$$

In particular, we proved

$$||f \cdot \psi|| \leq ||f||_\infty ||\psi||$$

if $f \in M_b(\mathsf{X})$ and $\psi \in L^2(\mathsf{X}, \mu)$. Consequently:
if $\{f_n\}_{n\in\mathbb{N}} \subset M_b(\mathsf{X})$ *and* $f_n \to f \in M_b(\mathsf{X})$ *in norm* $||\ ||_\infty$, *as* $n \to +\infty$, *then also* $f_n \cdot \psi \to f \cdot \psi$ in $L^2(\mathsf{X}, \mu)$.

Moreover, if $s \in S(\mathsf{X})$, the operator $\int_{\mathsf{X}} s(x)\, dP(x)$ can be made explicit using (8.39) and (8.35): for any $\psi \in L^2(\mathsf{X})$, in fact,

$$\left(\int_{\mathsf{X}} s(y)\, dP(y)\psi\right)(x) = s(x)\psi(x)\,.$$

Hence if $\{s_n\} \subset S(\mathsf{X})$ converges uniformly to $f \in M_b(\mathsf{X})$ (by Proposition 7.49(b) such a sequence exists for any $f \in M_b(\mathsf{X})$), we have

$$s_n \cdot \psi = \int_{\mathsf{X}} s_n(x)\, dP(x)\psi \to \int_{\mathsf{X}} f(x)\, dP(x)\psi$$

as $n \to +\infty$, by the definition of integral via the continuous prolongation $\widehat{\mathfrak{I}}$ of $\mathfrak{I}$. On the other hand we saw at the beginning that under our assumptions (with $f_n := s_n$) we have $s_n \cdot \psi \to f \cdot \psi$ in $L^2(\mathsf{X})$, as $n \to +\infty$, so

$$\left(\int_{\mathsf{X}} f(y)\, dP(y)\psi\right)(x) = f(x)\psi(x) \quad \text{for almost every } x \in \mathsf{X}, \tag{8.40}$$

for any $f \in M_b(\mathsf{X})$, $\psi \in L^2(\mathsf{X}, \mu)$. Equation (8.40) gives the explicit form of the integral operator of f with respect to the PVM of (8.39).

Regarding the supports, we have $supp(P) = supp(\mu)$ ($P(E) = 0$ is equivalent to $\int_E |\psi|^2 d\mu = 0$ for every $\psi \in L^2(\mathsf{X}, \mu)$ so that, taking $\psi = \chi_E$, $P(E) = 0$ iff $\mu(E) = 0$). If X is second countable then P is concentrated on $supp(P)$, because hypothesis (1) in Proposition 8.44(d) holds. Even if X is not second countable, but Hausdorff and locally compact, and μ is inner continuous, then both P and μ are concentrated on their respective supports, as follows from Proposition 1.45(ii).

(2) In the second example we consider a Hilbert basis N of a (generally non-separable) Hilbert space H, and define $\Sigma(N)$ to be the power set of N. Actually this σ-algebra can be viewed as a Borel σ-algebra if we endow N with the discrete topology of the power set of N, for which singlets are open and the associated Borel σ-algebra is the topology itself. This topology is Hausdorff and locally compact, and furthermore second countable if H is separable. If $E \subset N$, consider the closed subspace $\mathsf{H}_E := \overline{< \{z\}_{z\in E} >}$. The orthogonal projector onto it is (cf. Proposition 3.64(d))

$$P(E)x := \sum_{z\in E}(z|x)z \,, \quad x \in \mathsf{H}$$

E being a basis of H_E. It is easy to check $P : \mathscr{B}(N) \ni E \mapsto P(E)$ is a PVM. One can also prove, for any $f : N \to \mathbb{C}$ bounded and $x \in \mathsf{H}$

$$\int_N f(z)\, dP(z)x = \sum_{z\in N} f(z)\, (z|x)\, z \,. \tag{8.41}$$

The proof can be obtained using example (1), because (Theorem 3.28) H and $L^2(N,\mu)$ are isomorphic Hilbert spaces under the surjective isometry $U : \mathsf{H} \to L^2(N,\mu)$ sending $x \in \mathsf{H}$ to the map $z \mapsto \psi_x(z) := (z|x)$, where μ is the counting measure of N. Indeed, $Q(E) := UP(E)U^{-1}$ is the operator in $L^2(N,\mu)$ that multiplies by the characteristic function of E: we obtain thus a spectral measure $Q : \mathscr{B}(N) \ni E \mapsto Q(E)$ of the kind of example (1). Using the integral of a map $f \in M_b(\mathsf{X})$ defined by simple integrals, for which

$$\int_N s(z)\, dQ(z) = \sum_i c_i Q(E_i) = U \sum_i c_i P(E_i) U^{-1} = U \int_N s(z)\, dP(z) U^{-1} \,,$$

we obtain

$$\int_N f(z)\, dQ(z) = U \int_N f(z)\, dP(z) U^{-1} \,, \tag{8.42}$$

by continuity of the composite in $\mathfrak{B}(\mathsf{H})$. Equation (8.40) implies

$$\int_N f(z)\, dQ(z)\psi = f \cdot \psi \,. \tag{8.43}$$

From (8.42) and (8.43), then

$$\int_N f(z)\, dP(z)\phi = U^{-1} f \cdot U\phi = \sum_{z\in N} f(z)\,(z|\phi)\,z\,,$$

where we used the definition of U (cf. Theorem 3.28):

$$U : \mathsf{H} \ni \phi \mapsto \{(z|\phi)\}_{z\in N} \in L^2(N, \mu)$$

and the inverse:

$$U^{-1} : L^2(N, \mu) \ni \{\alpha_z\}_{z\in N} \mapsto \sum_{z\in N} \alpha_z z \in \mathsf{H}\,.$$

Altogether, we proved that

$$\int_N f(z)\, dP(z)x = \sum_{z\in N} f(z)\,(z|x)\,z$$

for $x \in \mathsf{H}$ as required.

Concerning $supp(P)$, it is clear that it coincides with N itself, since every subset $E \subset N$ satisfies $P(E) \neq 0$, unless $E = \emptyset$.

(3) The third example generalises the previous one. Consider a set X equipped with a σ-algebra Σ in which every singlet $\{x\}$, $x \in \mathsf{X}$, belongs to $\Sigma(\mathsf{X})$. Let us define a family of orthogonal projectors $0 \neq P_\lambda : \mathsf{H} \to \mathsf{H}$ on the Hilbert space H, for any $\lambda \in \mathsf{X}$. In order to have a PVM on $\mathscr{B}(\mathsf{X})$ we impose three conditions:

(a) $P_\lambda P_\mu = 0$, for $\lambda, \mu \in \mathsf{X}$, $\lambda \neq \mu$;

(b) $\sum_{\lambda\in\mathsf{X}} ||P_\lambda\psi||^2 < +\infty$, for any $\psi \in \mathsf{H}$;

(c) $\sum_{\lambda\in\mathsf{X}} P_\lambda\psi = \psi$, for any $\psi \in \mathsf{H}$.

Condition (b) implies that only countably many (at most, see Proposition 3.21) elements $P_\lambda\psi$ are non-zero, even if X is not countable; by (a), the vectors $P_\lambda\psi$ and $P_\mu\psi$ are orthogonal if $\lambda \neq \mu$, so Lemma 3.25 guarantees that the sum of (c) is well defined and may be rearranged at will.

That (a), (b), (c) hold is proved by exhibiting a family that satisfies them. The simplest case is given by the projectors $P(\{z\})$, $z \in N$, of example (2) when $\mathsf{X} = N$ is a Hilbert basis. An instance where X is not a basis is the following. Take a self-adjoint compact operator T, set $\mathsf{X} = \sigma_p(T) \subset \mathbb{R}$ (with induced topology and the associated Borel σ-algebra) and define P_λ, $\lambda \in \sigma_p(T)$, to be the orthogonal projector onto the λ-eigenspace. By Theorems 4.19 and 4.20 conditions (a), (b) and (c) follow. With these assumptions, $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$, where

$$P(E)\psi = \sum_{\lambda\in E} P_\lambda\psi\,, \tag{8.44}$$

for any $E \subset \Sigma(\mathsf{X})$ and any $\psi \in \mathsf{H}$, is a PVM on H. The sum $\sum_{\lambda \in E} P_\lambda \psi$ always exists in H, for any $\psi \in \mathsf{H}$, and does not depend on the ordering: this fact is a consequence of condition (b), because of lemma 3.25. Now we wish to prove

$$\int_{\mathsf{X}} f(x)\, dP(x)\psi = \sum_{x \in \mathsf{X}} f(x) P_x \psi \tag{8.45}$$

for any $f \in M_b(\mathsf{X})$ and $\psi \in \mathsf{H}$. The right-hand side is well defined and can be rearranged by Lemma 3.25, because for any $\psi \in \mathsf{H}$:

$$\sum_{x \in \mathsf{X}} ||f(x)P_x\psi||^2 \leq ||f||_\infty^2 \sum_{x \in \mathsf{X}} ||P_x\psi||^2 = ||f||_\infty^2 \sum_{x \in \mathsf{X}} (P_x\psi|P_x\psi) = ||f||_\infty^2 \sum_{x \in \mathsf{X}} (\psi|P_x^2\psi)$$

$$= ||f||_\infty^2 \sum_{x \in \mathsf{X}} (\psi|P_x\psi) = ||f||_\infty^2 \left(\psi \,\middle|\, \sum_{x \in \mathsf{X}} P_x\psi \right) = ||f||_\infty^2 (\psi|\psi) = ||f||_\infty^2 ||\psi||^2 ,$$

the last equality coming from (c). If $s \in S(\mathsf{X})$ is simple, using (8.44) and the definition of integral, we have

$$\int_{\mathsf{X}} s(x)\, dP(x)\psi = \sum_i c_i P(E_i)\psi = \sum_i \sum_{x \in E_i} s(x) P_x \psi = \sum_{x \in \mathsf{X}} s(x) P_x \psi \ , \tag{8.46}$$

for any $\psi \in \mathsf{H}$. Note that in the second equality we used that $s(x) = \sum_i c_i \chi_{E_i}$ implies $c_i = s(x)$ for all $x \in E_i$.
If $\{s_n\} \subset S(\mathsf{X})$ and $s_n \to f \in M_b(\mathsf{X})$ uniformly, then for any $\psi \in \mathsf{H}$:

$$\int_{\mathsf{X}} f(x)\, dP(x)\psi - \int_{\mathsf{X}} s_n(x)\, dP(x)\psi \to 0 \, , \tag{8.47}$$

as $n \to +\infty$, by definition of integral of bounded measurable maps. At the same time, (8.46) and condition (a) give

$$\left|\left| \sum_{x \in \mathsf{X}} f(x)\, P_x\psi - \int_{\mathsf{X}} s_n(x)\, dP(x)\psi \right|\right|^2 = \sum_{x \in \mathsf{X}} |f(x) - s_n(x)|^2 ||P_x\psi||^2 \leq ||f - s_n||_\infty^2 ||\psi||^2 \, .$$

The last term goes to zero as $n \to +\infty$. By (8.47) and uniqueness of limits in H,

$$\sum_{x \in \mathsf{X}} f(x)\, P_x\psi = \int_{\mathsf{X}} f(x)\, dP(x)\psi \, ,$$

for any $\psi \in \mathsf{H}$, so (8.45) holds. ∎

8.3.3 Properties of Operators Obtained Integrating Bounded Maps with Respect to PVMs

In this section we examine the properties of the integral operator, separating them in two groups. The first theorem establishes basic features.

Theorem 8.52 *Let* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *a measurable space,* $(\mathsf{H}, (\ |\ ))$ *a Hilbert space and* $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ *a PVM.*
(a) *For any* $f \in M_b(\mathsf{X})$,

$$\left|\left|\int_{\mathsf{X}} f(x)\, dP(x)\right|\right| = ||f||_\infty^{(P)} . \tag{8.48}$$

(b) *The integral operator with respect to* P *is* **positive**:

$$\int_{\mathsf{X}} f(x)\, dP(x) \geq 0 \quad \text{if}\ 0 \leq f \in M_b(\mathsf{X}) .$$

(c) *For any* $\psi, \phi \in \mathsf{H}$, *the map*

$$\mu_{\psi,\phi} : \mathscr{B}(\mathsf{X}) \ni E \mapsto \left(\psi \left| \int_{\mathsf{X}} \chi_E\, dP(x)\phi \right.\right)$$

satisfies the following properties:

(i) $\mu_{\psi,\phi}$ *is a complex measure on* X, *called* **complex spectral measure associated to** ψ *and* ϕ,

(ii) if $\psi = \phi$, *then* $\mu_\psi := \mu_{\psi,\psi}$ *is a finite positive measure on* X, *called* **(positive) spectral measure associated to** ψ,

(iii) $\mu_{\psi,\phi}(\mathsf{X}) = (\psi|\phi)$, *and in particular* $\mu_\psi(\mathsf{X}) = ||\psi||^2$,

(iv) for any $f \in M_b(\mathsf{X})$:

$$\left(\psi \left| \int_{\mathsf{X}} f(x)\, dP(x)\phi \right.\right) = \int_{\mathsf{X}} f(x)\, d\mu_{\psi,\phi}(x) , \tag{8.49}$$

(v) if X *is a topological space,* $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ *and condition (1) or (2) of Proposition 8.44(d) is valid, then*

$$supp(\mu_{\psi,\phi}) \subset supp(P) \quad and \quad supp(\mu_\psi) \subset supp(P) ,$$

and furthermore

$$supp(P) = \overline{\bigcup_{\psi \in \mathsf{H}} supp(\mu_\psi)} . \tag{8.50}$$

(d) *If* $f \in M_b(\mathsf{X})$, $\int_{\mathsf{X}} f(x) dP(x)$ *commutes with every operator* $B \in \mathfrak{B}(\mathsf{H})$ *such that* $P(E)B = BP(E)$ *for any* $E \in \Sigma(\mathsf{X})$.

Proof (a) Consider a sequence of simple functions s_n converging to f in norm $||\ ||_\infty$. Then $||s_n - f||_\infty^{(P)} \leq ||s_n - f||_\infty \to 0$, so $|\,||s_n||_\infty^{(P)} - ||f||_\infty^{(P)}| \leq ||s_n - f||_\infty^{(P)}$ implies $||s_n||_\infty^{(P)} \to ||f||_\infty^{(P)}$. We also know $||\int_\mathsf{X} s_n dP|| = ||s_n||_\infty^{(P)}$ by (8.37). From the definition of integral of bounded maps $||\int_\mathsf{X} s_n dP|| \to ||\int_\mathsf{X} f dP||$, hence $||s_n||_\infty^{(P)} \to ||f||_\infty^{(P)} = ||\int_\mathsf{X} f dP||$, proving (8.48).
(b) Using Proposition 7.49(c) if $0 \leq f \in M_b(\mathsf{X})$ there is a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$, $0 \leq s_n \leq s_{n+1} \leq f$ for any n, that converges uniformly to f. Keeping in mind the definition of integral in P, and that uniform convergence implies weak convergence, we have $(\psi|\int_\mathsf{X} s_n dP\psi) \to (\psi|\int_\mathsf{X} f dP\psi)$, as $n \to +\infty$, for any $\psi \in \mathsf{H}$. For the positivity of $\int_\mathsf{X} f dP$ it suffices to show $(\psi|\int_\mathsf{X} s_n dP\psi) \geq 0$ for any n. Directly from (8.35) we find

$$\left(\psi \left| \int_\mathsf{X} s_n dP \ \psi \right.\right) = \sum_{i\in I_n} c_i^{(n)} \left(\psi \left| P(E_i^{(n)}) \ \psi \right.\right) \geq 0\,,$$

because every orthogonal projector is positive and the numbers $c_i^{(n)}$ are non-negative for $s_n \geq 0$.
(c) By (8.35),

$$\mu_{\psi,\phi}(E) = \left(\psi \left| \int_\mathsf{X} \chi_E(x)\, dP(x)\phi \right.\right) = (\psi|1 \cdot P(E)\phi) = (\psi|P(E)\phi)\,, \quad (8.51)$$

and $(\psi|P(E)\psi) \geq 0$. Then Definition 8.41(d) and the inner product's continuity imply $\mu_{\psi,\phi}$ is a complex measure on $\Sigma(\mathsf{X})$. Moreover, parts (d) and (a) in Definition 8.41 say that if $\psi = \phi$, μ_ψ is a positive, σ-additive, finite measure on $\Sigma(\mathsf{X})$. At last Definition 8.41(c) forces $\mu_{\psi,\phi}(\mathsf{X}) = (\psi|\phi)$, in particular $\mu_\psi(\mathsf{X}) = (\psi|\psi) = ||\psi||^2$. As μ_ψ and $|\mu_{\psi,\phi}|$ are finite measures, their integral is continuous in norm $||\ ||_\infty$ on $M_b(\mathsf{X})$. (In fact, for any $f \in M_b(\mathsf{X})$,

$$\left|\int_\mathsf{X} f(x)\, d\mu_{\psi,\phi}(x)\right| \leq \int_\mathsf{X} |f(x)|\, d|\mu_{\psi,\phi}(x)| \leq ||f||_\infty |\mu_{\psi,\phi}|(\mathsf{X})\,,$$

whence the integral's continuity in sup norm.)
If $s_n \in S(\mathsf{X})$, using (8.51) and (8.35) we immediately see

$$\left(\psi \left| \int_\mathsf{X} s_n(x)\, dP(x)\phi \right.\right) = \int_\mathsf{X} s_n(x)\, d\mu_{\psi,\phi}(x)\,.$$

If now $f \in M_b(\mathsf{X})$ and $\{s_n\}_{n\in\mathbb{N}} \subset S(\mathsf{X})$ converges to f uniformly, as $n \to +\infty$ (cf. Proposition 7.49(b)), we can use the continuity of the inner product and of the integral associated to $\mu_{\psi,\phi}$ (uniform convergence) to obtain

$$\left(\psi \left| \int_\mathsf{X} f(x)\, dP(x)\phi \right.\right) = \left(\psi \left| \lim_{n\to+\infty} \int_\mathsf{X} s_n(x)\, dP(x)\phi \right.\right)$$

$$= \lim_{n\to+\infty}\left(\psi\left|\int_{\mathsf{X}} s_n(x)\,dP(x)\phi\right.\right) = \lim_{n\to+\infty}\int_{\mathsf{X}} s_n(x)\,d\mu_{\psi,\phi}(x) = \int_{\mathsf{X}} f(x)\,d\mu_{\psi,\phi}(x)\,.$$

Let us prove (v), or equivalently, $\mathsf{X}\setminus supp(\mu_{\psi,\phi}) \supset \mathsf{X}\setminus supp(P)$. Take $x \in \mathsf{X}\setminus supp(P)$, so that there is an open set $A \subset \mathsf{X}$ with $x \in A$ and $P(A) = 0$. By monotonicity $P(B) = 0$ if $\mathscr{B}(\mathsf{X}) \ni B \subset A$, and therefore

$$\mu_{\psi,\phi}(B) = \int_{\mathsf{X}} \chi_B(x)\,d\mu_{\psi,\phi}(x) = \left(\psi\left|\int_{\mathsf{X}} \chi_B(x)\,dP(x)\phi\right.\right) = (\psi|P(B)\phi) = 0\,.$$

By the definition of total variation (Remark 1.82(2)) $|\mu_{\psi,\phi}|(A) = 0$, so $x \in \mathsf{X}\setminus supp(\mu_{\psi,\phi})$. The case μ_ψ is analogous.

Let us prove (8.50). If $x \in \mathsf{X}\setminus supp(P)$, there exists an open set $O \ni x$ with $P(O) = 0$ and therefore $\mu_\psi(O) = ||P(O)\psi||^2 = 0$ for every $\psi \in \mathsf{H}$. As a consequence $\mathsf{X}\setminus supp(P) \subset \cap_{\psi\in\mathsf{H}}\mathsf{X}\setminus supp(\mu_\psi)$, namely $supp(P) \supset \cup_{\psi\in\mathsf{H}} supp(\mu_\psi)$ and also, since $supp(P)$ is closed, $supp(P) \supset \overline{\cup_{\psi\in\mathsf{H}} supp(\mu_\psi)}$. Let us prove the converse inclusion. If $x \in supp(P)$ and $O \ni x$ is open, then $P(O) \neq 0$ (otherwise $x \in \mathsf{X}\setminus supp(P)$). Therefore, there must exist $\psi_O \in \mathsf{H}$ such that $\mu_{\psi_O}(O) = ||P(O)\psi_O||^2 \neq 0$. Consequently $O \cap supp(\mu_{\psi_O}) \neq \emptyset$ due to Proposition 1.45. Taking $x_O \in O \cap supp(\mu_{\psi_O})$ we also have $x_O \in O \cap \cup_{\psi\in\mathsf{H}} supp(\mu_\psi)$. In summary, if $x \in supp(P)$, for every open set $O \ni x$ there is $x_O \in O$ such that $x_O \in \cup_{\psi\in\mathsf{H}} supp(\mu_\psi)$. This is equivalent to saying $x \in \overline{\cup_{\psi\in\mathsf{H}} supp(\mu_\psi)}$, proving $supp(P) \subset \overline{\cup_{\psi\in\mathsf{H}} supp(\mu_\psi)}$, as we wanted.

(d) The claim is obvious when f is simple, and extends by continuity to any f. □

Remarks 8.53 **(1)** It must be said that if we want the positive measures μ_ψ, defined on $\Sigma(\mathsf{X}) := \mathscr{B}(\mathsf{X})$ when X is a topological space, to be proper *Borel measures*, then we should also demand X be Hausdorff and locally compact (Definition 1.42(iv)). In concrete situations, like when we use PVMs for the spectral expansion of an operator, X is always (a subset of) $\mathbb{R}$ or $\mathbb{R}^2$, so the extra assumptions hold. In such case the measures μ_ψ are also regular, see Remark 8.46(3), so that P is concentrated on $supp(P)$ by Proposition 8.44(d)(2). The same result follows from Proposition 8.44(d)(1), since the standard topology of $\mathbb{R}^n$ is second countable.

(2) A useful remark is that the complex measure $\mu_{\psi,\phi}$ decomposes as complex linear combination of 4 positive finite measures μ_χ. Since $\mu_{\psi,\phi}(E) = (\psi|P(E)\phi) = (P(E)\psi|P(E)\phi)$, by the polarisation formula (3.4) we obtain:

$$\mu_{\psi,\phi}(E) = \mu_{\psi+\phi}(E) - \mu_{\psi-\phi}(E) - i\mu_{\psi+i\phi}(E) + i\mu_{\psi-i\phi}(E) \quad \text{for any } E \in \Sigma(\mathsf{X}).$$

■

The next theorem establishes the primary feature of a PVM: it gives rise to an isometric *-homomorphism of C^*-algebras $M_b(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$. In the topological case, when $\mathsf{X} = \mathbb{R}^2$ and $\Sigma(\mathsf{X}) = \mathscr{B}(\mathbb{R}^2)$, this *-homomorphism extends the representation $\widehat{\Phi}_T$ of Theorem 8.39, provided we define the normal operator T suitably.

This will be a crucial point in the spectral theorem, proved immediately after.

Theorem 8.54 *Let* H *be a Hilbert space,* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *a measurable space and* $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ *a projector-valued measure.*
(a) *The integral operator:*

$$\widehat{\mathfrak{I}} : M_b(\mathsf{X}) \ni f \mapsto \int_{\mathsf{X}} f(x)dP(x) \in \mathfrak{B}(\mathsf{H})$$

is a (continuous) representation on H *of the unital* C^**-algebra* $M_b(\mathsf{X})$*. Equivalently: beside (8.48) the following hold:*
(i) if 1 *is the constant map on* X,

$$\int_{\mathsf{X}} 1\, dP(x) = I\ ,$$

(ii) for any $f, g \in M_b(\mathsf{X})$, $\alpha, \beta \in \mathbb{C}$,

$$\int_{\mathsf{X}} (\alpha f(x) + \beta g(x))\, dP(x) = \alpha \int_{\mathsf{X}} f(x)dP(x) + \beta \int_{\mathsf{X}} g(x)dP(x)\ ,$$

(iii) for any $f, g \in M_b(\mathsf{X})$,

$$\int_{\mathsf{X}} f(x)\, dP(x) \int_{\mathsf{X}} g(x)\, dP(x) = \int_{\mathsf{X}} f(x)g(x)\, dP(x)\ ,$$

(iv) for any $f \in M_b(\mathsf{X})$,

$$\int_{\mathsf{X}} \overline{f(x)}\, dP(x) = \left(\int_{\mathsf{X}} f(x)dP(x)\right)^*\ .$$

(b) *If* $\psi \in \mathsf{H}$ *and* $f \in M_b(\mathsf{X})$ *then*

$$\left|\left|\int_{\mathsf{X}} f(x)\, dP(x)\psi\right|\right|^2 = \int_{\mathsf{X}} |f(x)|^2\, d\mu_\psi(x)\ .$$

(c) *If* $\{f_n\}_{n\in\mathbb{N}} \subset M_b(\mathsf{X})$ *is bounded and converges to* $f : \mathsf{X} \to \mathbb{C}$ *pointwise, the integral of* f *in the spectral measure* P *exists and equals:*

$$\int_{\mathsf{X}} f(x)\, dP(x) = s\text{-}\lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x)\, dP(x)\ .$$

(d) *If* $\mathsf{X} = \mathbb{R}^2$ *with the Euclidean topology,* $\Sigma(\mathsf{X}) := \mathscr{B}(\mathbb{R}^2)$*, and* $supp(P)$ *is bounded, then*

$$supp(P) = \sigma(T)\ ,$$

where $\sigma(T)$ is viewed as subset in $\mathbb{R}^2$, and we defined the normal operator:

$$T := \int_{supp(P)} z \, dP(x, y) \, ,$$

with z denoting the map $\mathbb{R}^2 \ni (x, y) \mapsto z := x + iy$.

In this case let us identify $M_b(\sigma(T))$ with the closed subspace of $M_b(\mathbb{R}^2)$ of maps vanishing outside the compact set $\sigma(T)$. Then the restriction

$$\widehat{\mathfrak{I}}\restriction_{M_b(\sigma(T))} : M_b(\sigma(T)) \to \mathfrak{B}(\mathsf{H})$$

coincides with the representation $\widehat{\Phi}_T$ of the normal operator T of Theorem 8.39, and we may write

$$f(T, T^*) = \int_{\sigma(T)} f(x, y) dP(x, y) \, , \quad f \in M_b(\sigma(T)),$$

where $f(T, T^*) := \widehat{\Phi}_T(f)$.

Proof of Theorem 8.54. (a) The only facts that are not entirely trivial are (iii) and (iv), so let us begin with the former. Choose two sequences of simple functions $\{s_n\}$ and $\{t_m\}$ that converge uniformly to f and g respectively. A direct computations shows

$$\int_{\mathsf{X}} s_n(x) \, dP(x) \int_{\mathsf{X}} t_m(x) \, dP(x) = \int_{\mathsf{X}} s_n(x) t_m(x) \, dP(x) \, .$$

Given m, $s_n \cdot t_m$ tends to $f \cdot t_m$ uniformly, as $n \to +\infty$, because t_m is bounded. By continuity (in the sense of Theorem 8.52(a)) and linearity of the integral, the limit as $n \to +\infty$ gives

$$\int_{\mathsf{X}} f(x) \, dP(x) \int_{\mathsf{X}} t_m(x) \, dP(x) = \int_{\mathsf{X}} f(x) t_m(x) \, dP(x) \, ,$$

where we used the fact that the composite of bounded operators is continuous in its arguments. Similarly, since $f \cdot t_m$ tends to $f \cdot g$ uniformly as $m \to +\infty$, we obtain (iii). Property (iv) is proven by choosing a sequence of simple functions $\{s_n\}$ converging to f uniformly. Take $\psi, \phi \in \mathsf{H}$. Directly by definition of integral of a simple function (orthogonal projectors are self-adjoint), we have

$$\left(\int_{\mathsf{X}} \overline{s_n(x)} \, dP(x) \psi \middle| \phi \right) = \left(\psi \middle| \int_{\mathsf{X}} s_n(x) \, dP(x) \phi \right) \, .$$

Notice $\overline{s_n} \to \overline{f}$ uniformly, as $n \to +\infty$. Hence by continuity and linearity of the integral (in the sense of Theorem 8.52(a)), plus the continuity of the inner product, when we take the limit as $n \to +\infty$, the above identity gives

$$\left(\int_{\mathsf{X}} \overline{f(x)}\, dP(x)\psi \middle| \phi \right) = \left(\psi \middle| \int_{\mathsf{X}} f(x)\, dP(x)\phi \right) ,$$

so:

$$\left(\left[\int_{\mathsf{X}} \overline{f(x)}\, dP(x) - \left(\int_{\mathsf{X}} f(x)\, dP(x) \right)^* \right] \psi \middle| \phi \right) = 0 .$$

As $\psi, \phi \in \mathsf{H}$ are arbitrary, (iv) holds.
(b) If $\psi \in \mathsf{H}$, using (iii) and (iv) of (a), we find

$$\left|\left| \int_{\mathsf{X}} f(x)\, dP(x)\psi \right|\right|^2 = \left(\psi \middle| \int_{\mathsf{X}} |f(x)|^2\, dP(x)\psi \right) = \int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) ,$$

where the last equality uses Theorem 8.52(c).
(c) As first thing let us observe $f \in M_b(\mathsf{X})$, because f is measurable, as limit of measurable functions, and bounded by the constant that bounds the sequence f_n. If $\psi \in \mathsf{H}$ the integral's linearity and (b) imply

$$\left|\left| \left(\int_{\mathsf{X}} f(x)\, dP(x) - \int_{\mathsf{X}} f_n(x)\, dP(x) \right) \psi \right|\right|^2 = \int_{\mathsf{X}} |f(x) - f_n(x)|^2\, d\mu_\psi(x) .$$

The measure μ_ψ is finite, so constant maps are integrable. By assumption $|f_n| < K < +\infty$ for any $n \in \mathbb{N}$, so $|f| \leq K$ and then $|f_n - f|^2 \leq (|f_n| + |f|)^2 < 4K^2$. Since $|f_n - f|^2 \to 0$ pointwise, we can invoke the dominated convergence theorem to obtain, as $n \to +\infty$,

$$\left|\left| \int_{\mathsf{X}} f(x)\, dP(x)\psi - \int_{\mathsf{X}} f_n(x)\, dP(x)\psi \right|\right| = \sqrt{\int_{\mathsf{X}} |f(x) - f_n(x)|^2\, d\mu_\psi(x)} \to 0 .$$

Given the freedom in choosing $\psi \in \mathsf{H}$, (c) is proved.
(d) If $supp(P)$ is bounded, it is compact (as closed by definition) and every continuous map on it is bounded. The mapping $\mathbb{R}^2 \ni (x, y) \mapsto z\chi_{supp(P)}(x, y) \in \mathbb{C}$ is thus bounded, so $T := \int_{supp(P)} z\, dP(x, y) := \int_{\mathbb{R}^2} z\chi_{supp(P)}(x, y)\, dP(x, y)$ is well defined and a normal operator ((ii) and (iv) in (a)) in $\mathfrak{B}(\mathsf{H})$. Its residual spectrum is in particular empty, by Proposition 8.7(c).

By definition of resolvent set, the claim is the same as asking:
$\mathbb{C} \ni \lambda \notin supp(P) \Leftrightarrow \lambda \in \rho(T)$.

Let us prove $\lambda \notin supp(P) \Rightarrow \lambda \in \rho(T)$. Since $\mathbb{R}^2 \ni (x, y) \mapsto z = x + iy$ is bounded on $supp(P)$, suppose $\lambda \notin supp(P)$. If so, there is an open subset in $\mathbb{R}^2$, $A \ni (x_0, y_0)$ with $x_0 + iy_0 = \lambda$, such that $P(A) = 0$. It follows that $(x, y) \mapsto (z - \lambda)^{-1}$ is bounded on the closed set $supp(P)$. Then we have the operator

$$\int_{supp(P)} \frac{1}{z - \lambda}\, dP(x, y)$$

of $\mathfrak{B}(\mathsf{H})$. By virtue of (iii) and (i) in (a),

$$\int_{supp(P)} \frac{1}{z-\lambda}\, dP(x,y) \int_{supp(P)} (z-\lambda)\, dP(x,y)$$

$$= \int_{supp(P)} (z-\lambda)\, dP(x,y) \int_{supp(P)} \frac{1}{z-\lambda}\, dP(x,y)$$

$$= \int_{supp(P)} 1\, dP(x,y) = \int_{\mathbb{R}^2} 1\, dP(x,y) = I\,,$$

which we may write, by (i) and (ii) of (a),

$$\int_{supp(P)} \frac{1}{z-\lambda}\, dP(x,y)(T-\lambda I) = (T-\lambda I)\int_{supp(P)} \frac{1}{z-\lambda}\, dP(x,y) = I\,.$$

Put differently, $T - \lambda I$ is a bijection of H so that $\lambda \in \rho(T)$. By Theorem 8.4(a) $\lambda \in \rho(T)$.

Now we show $\lambda \in \rho(T) \Rightarrow \lambda \notin supp(P)$, and actually we shall prove the equivalent statement: $\lambda \in supp(P) \Rightarrow \lambda \in \sigma(T) = \sigma_p(T) \cup \sigma_c(T)$.
If $\lambda \in supp(P)$, is may happen that $T - \lambda I : \mathsf{H} \to \mathsf{H}$ is not one-to-one, in which case $\lambda \in \sigma_p(T)$ and the proof ends. If, on the contrary, $T - \lambda I : \mathsf{H} \to \mathsf{H}$ is injective we can prove the inverse $(T - \lambda I)^{-1} : Ran(T - \lambda I) \to \mathsf{H}$ cannot be bounded, so $\lambda \in \sigma_c(T)$. For that it is enough to show, for $\lambda \in supp(P)$, $n = 1, 2, \ldots$, that there exists $\psi_n \in \mathsf{H}$, $\psi_n \neq 0$, such that

$$||(T-\lambda I)\psi_n||/||\psi_n|| \leq 1/n\,.$$

(under our assumptions $\psi_n = (T - \lambda I)^{-1}\phi_n$, for any $n = 1, 2, \ldots$, with $\phi_n \neq 0$ so that $\psi_n \neq 0$. Then

$$1/n \geq ||(T-\lambda I)\psi_n||/||\psi_n|| = ||(T-\lambda I)(T-\lambda I)^{-1}\phi_n||/||(T-\lambda I)^{-1}\phi_n||\,.$$

In other terms, for $n = 1, 2, \ldots$, there is $\phi_n \in \mathsf{H}$, $\phi_n \neq 0$, such that

$$\frac{||(T-\lambda I)^{-1}\phi_n||}{||\phi_n||} \geq n\,.$$

Then $(T - \lambda I)^{-1}$ cannot be bounded, and hence $\lambda \in \sigma_c(T)$.)

If $\lambda \in supp(P)$, any open set $A \ni \lambda$ must satisfy $P(A) \neq 0$. Set $x_0 + iy_0 := \lambda$ and consider the family of open discs $D_n \subset \mathbb{R}^2$, centred at (x_0, y_0) and of radii $1/n$, $n = 1, 2, \ldots$. As $P(D_n) \neq 0$, there exists $\psi_n \neq 0$ with $\psi_n \in P(D_n)(\mathsf{H})$. In such a case

$$(T - \lambda I)\psi_n = \int_{supp(P)} (z - \lambda)\, dP(x, y)\psi_n$$

$$= \int_{supp(P)} (z - \lambda)\, dP(x, y) \int_{supp(P)} \chi_{D_n}(z)\, dP(x, y)\psi_n\ ,$$

where we used $P(D_n) = \int_{\mathbb{R}^2} \chi_{D_n}(z)\, dP(x, y)$ and $P(D_n)\psi_n = \psi_n$. By part (iii) in (a) we find

$$(T - \lambda I)\psi_n = \int_{\mathbb{R}^2} \chi_{D_n}(z)(z - \lambda) dP(x, y)\ .$$

Hence property (b) yields

$$||(T - \lambda I)\psi_n||^2 = \int_{\mathbb{R}^2} |\chi_{D_n}(z)|^2 |z - \lambda|^2 d\mu_{\psi_n}(x, y) \le \int_{\mathbb{R}^2} 1 \cdot n^{-2}\, d\mu_{\psi_n}(x, y)$$

$$= n^{-2} \int_{\mathbb{R}^2} 1\, d\mu_{\psi_n}(x, y) = n^{-2} ||\psi_n||^2\ ,$$

where the last equality made use of $\mu_{\psi_n}(\mathbb{R}^2) = ||\psi_n||^2$, by (iii) in Theorem 8.52(c). Taking the square root of both sides produces

$$\frac{||(T - \lambda I)\psi_n||}{||\psi_n||} \le 1/n\ ,$$

and concludes the proof.

The final statement is an easy consequence of the uniqueness of $\widehat{\Phi}_T$, because $\widehat{\mathfrak{I}}$ restricted to $M_b(\sigma(T)) = M_b(supp(P))$ trivially satisfies all the conditions (see Theorem 8.39). $\square$

It is worth to state explicitly an important elementary consequence of the theorem (a consequence that was actually used in the proof already).

Corollary 8.55 *With reference to Theorem 8.54(a), the commutation relationship*

$$\int_{\mathsf{X}} f(x)\, dP(x) \int_{\mathsf{X}} g(x)\, dP(x) = \int_{\mathsf{X}} g(x)\, dP(x) \int_{\mathsf{X}} f(x)\, dP(x)$$

holds for $f, g \in M_b(\mathsf{X})$. In particular, the operator $\int_{\mathsf{X}} f(x)\, dP(x)$ is normal, for any $f \in M_b(\mathsf{X})$.

Proof Commutativity comes from (a)(iii), since $f \cdot g = g \cdot f$, and (iv) implies normality. $\square$

8.4 Spectral Theorem for Normal Operators in $\mathfrak{B}(\mathsf{H})$

At this juncture enough material has been gathered to allow to state the first important *spectral decomposition theorem* for normal operators in $\mathfrak{B}(\mathsf{H})$. Later in this section we will prove another version of the theorem that concerns a useful *spectral representation* for bounded normal operators T, in terms of multiplicative operators on certain L^2 spaces built on the spectrum of T.

8.4.1 Spectral Decomposition of Normal Operators in $\mathfrak{B}(\mathsf{H})$

The spectral decomposition theorem, reversing (d) in Theorem 8.54, explains how any normal operator in $\mathfrak{B}(\mathsf{H})$ can be constructed by integrating a certain PVM, whose support is the operator's spectrum and which is completely determined by the operator. In view of the applications it is important to point out that the theorem holds in particular for self-adjoint operators in $\mathfrak{B}(\mathsf{H})$ and unitary operators, since both are subclasses of normal operators.

Theorem 8.56 (Spectral decomposition of normal operators in $\mathfrak{B}(\mathsf{H})$) *Let H be a Hilbert space and $T \in \mathfrak{B}(\mathsf{H})$ a normal operator.*
(a) *There exists a unique bounded projector-valued measure $P^{(T)} : \mathscr{B}(\mathbb{R}^2) \to \mathfrak{B}(\mathsf{H})$ ($\mathbb{R}^2$ has the standard topology) such that:*

$$T = \int_{supp(P^{(T)})} z \, dP^{(T)}(x, y) \,, \tag{8.52}$$

where z is the map $\mathbb{R}^2 \ni (x, y) \mapsto z := x + iy \in \mathbb{C}$.
(a)' *If T is self-adjoint, or unitary, statement (a) can be refined by replacing $\mathbb{R}^2$ with:*

$$\mathbb{R} \quad or \quad \mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \; respectively.$$

(b) *$P^{(T)}$ is concentrated on its support and*

$$supp(P^{(T)}) = \sigma(T) \,.$$

In particular, for $\lambda = x + iy \in \mathbb{C}$ ($\lambda = x \in \mathbb{R}$, or $\lambda = x + iy \in \mathbb{S}^1$ respectively):

(i) $\lambda \in \sigma_p(T) \Leftrightarrow P^{(T)}(\{\lambda\}) \neq 0$. In this case $P^{(T)}(\{\lambda\})$ is the orthogonal projector onto the λ-eigenspace of T;

(ii) $\lambda \in \sigma_c(T) \Leftrightarrow P^{(T)}(\{\lambda\}) = 0$, and $P^{(T)}(A_\lambda) \neq 0$ for any open set $A_\lambda \subset \mathbb{R}^2$ ($\mathbb{R}$ or $\mathbb{S}^1$ respectively), $A_\lambda \ni \lambda$;

(iii) if $\lambda \in \sigma(T)$ is isolated, then $\lambda \in \sigma_p(T)$;

(iv) if $\lambda \in \sigma_c(T)$, then for any $\varepsilon > 0$ there exists $\phi_\varepsilon \in \mathsf{H}$ with $||\phi_\varepsilon|| = 1$ and

$$0 < ||T\phi_\varepsilon - \lambda\phi_\varepsilon|| \leq \varepsilon \,.$$

(c) *If $f \in M_b(\sigma(T))$, the operator $\int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y)$ commutes with every operator in $\mathfrak{B}(\mathsf{H})$ that commutes with T and T^*.*

Remarks 8.57 **(1)** In practice, property (iv) of (b) says that when $\lambda \in \sigma_c(T)$, despite T has no λ-eigenvectors (the continuous and discrete spectra are disjoint), we can still construct non-zero vectors of constant norm that solve the characteristic equation with arbitrary approximation.
(2) We may rephrase part (c) as follows: *the* **-subalgebra of* $\mathfrak{B}(\mathsf{H})$ *of operators* $f(T, T^*)$, *for* $f \in M_b(\sigma(T))$, *is contained in the von Neumann algebra generated by* T, T^* *in* $\mathfrak{B}(\mathsf{H})$. In Theorem 9.11 we will prove that this inclusion is actually an equality, provided H is separable. ■

Proof of Theorem 8.56
(a)–(a)'–(c). *Uniqueness*. We begin with the spectral measure's uniqueness. First, observe that if $P^{(T)}$ exists it must be concentrated on its support, since the standard topology of $\mathbb{R}^2$ is second countable and hence (1) in Proposition 8.44(d) applies. Next, note that if a spectral measure P satisfies (8.52) it must have bounded support, since the map z is not bounded on unbounded sets and the right-hand side in (8.52) is understood as in Definition 8.49(c). So let P, P' be PVMs over $\mathscr{B}(\mathbb{R}^2)$ with bounded support (so compact, for $supp(P)$ is closed in $\mathbb{R}^2$ by definition) and such that:

$$T = \int_{supp(P)} z\, dP(x, y) = \int_{supp(P')} z\, dP'(x, y)\,. \tag{8.53}$$

Using (i)–(iv) in Theorem 8.54(a), this gives, for any polynomial $p = p(z, \overline{z})$,

$$p(T, T^*) = \int_{supp(P)} p(x + iy, x - iy)\, dP(x, y) = \int_{supp(P')} p(x + iy, x - iy)\, dP'(x, y)\,,$$

where the polynomial $p(T, T^*)$ is defined in the most obvious manner, i.e. reading multiplication as composition of operators and setting $T^0 := I$, $(T^*)^0 := I$. If $u, v \in \mathsf{H}$ are arbitrary, for any complex polynomial $p = p(z, \overline{z})$ on $\mathbb{R}^2$,

$$\int_{supp(\mu_{u,v})} p(z, \overline{z})\, d\mu_{u,v}(x, y) = \left(u \middle| \int_{supp(P)} p(z, \overline{z})\, dP(x, y) v\right)$$

$$= \left(u \middle| \int_{supp(P')} p(z, \overline{z})\, dP'(x, y) v\right) = \int_{supp(\mu'_{u,v})} p(z, \overline{z})\, d\mu'_{u,v}(x, y)\,.$$

The two complex measures $\mu_{u,v}$ and $\mu'_{u,v}$ are those of Theorem 8.52(c) (where u, v were called ψ, ϕ). Moreover $supp(\mu_{u,v})$, $supp(\mu'_{u,v})$ are compact subsets of $\mathbb{R}^2$ (by (v) Theorem 8.52(c)), so there exists a compact set $K \subset \mathbb{R}^2$ containing both. Let us extend in the obvious way the measures to K, maintaining the supports intact, by defining the measure of a Borel set E in K by $\mu_{u,v}(E \cap supp(\mu_{u,v}))$, and similarly for $\mu'_{u,v}$.

Since polynomials in $z, \overline{z}$ with complex coefficients correspond bijectively to complex polynomials $q(x, y)$ in the real variables x, y (under the usual identification $z := x + iy$ and $\overline{z} := x - iy$, so $p(x + iy, x - iy) = q(x, y)$), we can rewrite the above identities in terms of polynomials with complex coefficients in $(x, y) \in K$:

$$\int_K p(x + iy, x - iy)\, d\mu_{u,v}(x, y) = \int_K p(x + iy, x - iy)\, d\mu'_{u,v}(x, y)\,.$$

By the Stone–Weierstrass Theorem (2.30), the algebra of complex polynomials $q(x, y)$ is dense inside $C(K)$ for the sup norm. Therefore the algebra of complex polynomials $p(x + iy, x - iy)$ restricted to K is dense in $C(K)$ for the sup norm. Since integrals of complex measures are continuous functionals in sup norm,

$$\int_K f(x, y)\, d\mu_{u,v}(x, y) = \int_K f(x, y)\, d\mu'_{u,v}(x, y) \quad \text{for any } f \in C(K)\,.$$

Riesz's Theorem 2.52 for complex measures ensures the two extended measures must coincide. Consequently the yet-to-be-extended measures must have the same support and coincide. Now by (iv) in Theorem 8.52(c), for any pair of vectors $u, v \in \mathsf{H}$ and any bounded measurable g on $\mathbb{R}^2$ we have

$$\left(u \left| \int_{supp(P)} g(x, y)\, dP(x, y) v \right.\right) = \left(u \left| \int_{supp(P')} g(x, y)\, dP'(x, y) v \right.\right)\,,$$

i.e.

$$\left(u \left| \int_{\mathbb{R}^2} g(x, y)\, dP(x, y) v \right.\right) = \left(u \left| \int_{\mathbb{R}^2} g(x, y)\, dP'(x, y) v \right.\right)\,.$$

Therefore

$$\int_{\mathbb{R}^2} g(x, y)\, dP(x, y) = \int_{\mathbb{R}^2} g(x, y)\, dP'(x, y)$$

because u and v are arbitrary. If E is an arbitrary Borel set of $\mathbb{R}^2$ and we pick $g = \chi_E$, the above equation implies

$$P(E) = \int_{\mathbb{R}^2} \chi_E(x, y)\, dP(x, y) = \int_{\mathbb{R}^2} \chi_E(x, y)\, dP'(x, y) = P'(E)\,,$$

proving $P = P'$.

Observe, furthermore, that (8.53) and Theorem 8.54(d) give $supp(P^{(T)}) = \sigma(T)$. Uniqueness for the cases of (a)' follows by what we have just proved, because $supp(P^{(T)}) = \sigma(T)$ and by (i) in Proposition 8.7(a, b).

Existence. Let us see to the existence of the spectral measure $P^{(T)}$.

Consider the *-homomorphism $\widehat{\Phi}_T$ associated to the normal operator $T \in \mathfrak{B}(\mathsf{H})$, as of Theorem 8.39. If E is a Borel set in $\mathbb{R}^2$, define $E' := E \cap \sigma(T)$ whence $P^{(T)}(E) := \widehat{\Phi}_T(\chi_{E'})$. The operator $P^{(T)}(E)$ is patently idempotent, for $\widehat{\Phi}_T$ is a homomorphism and $\chi_{E'} \cdot \chi_{E'} = \chi_{E'}$. By property (vi) of Theorem 8.39(b) and the positivity of characteristic functions, $P^{(T)}(E) \geq 0$, so $P^{(T)}(E)$ is self-adjoint. Therefore every $P^{(T)}(E)$ is an orthogonal projector. It is easy to check $\mathscr{B}(\mathbb{R}) \ni E \mapsto P^{(T)}(E)$ is a PVM: $P^{(T)}(E) \geq 0$ we saw above. Concerning Definition 8.41: (b) follows from $\chi_{E'} \cdot \chi_{F'} = \chi_{E' \cap F'}$ and because $\widehat{\Phi}_T$ is a homomorphism; (c) descends from $\widehat{\Phi}_T(\chi_{\sigma(T)}) = I$, which holds by definition of algebra homomorphism; eventually, (d) follows from (v) in Theorem 8.41(b), because, pointwise, $\lim_{N\to+\infty} \sum_{n=0}^{N} \chi_{E'_n} = \chi_{\cup_{n\in\mathbb{N}} E'_n}$ when the E'_n are pairwise disjoint. By construction, $supp(P^{(T)})$ is bounded because $supp(P^{(T)}) \subset \sigma(T)$, the latter being compact by Theorem 8.4(c).

To continue with the proof, let us remark the following fact. By the above argument, and because the integral operator of $P^{(T)}$ and $\widehat{\Phi}_T$ are both linear,

$$\widehat{\Phi}_T(s \restriction_{\sigma(T)}) = \int_{supp(P^{(T)})} s(x, y)\, dP^{(T)}(x, y)\,,$$

for any simple function $s : \mathbb{R}^2 \to \mathbb{C}$. Either functional is continuous in the sup topology ((ii) in Theorem 8.39(b) and (a)), so Proposition 7.49 gives

$$\widehat{\Phi}_T(f \restriction_{\sigma(T)}) = \int_{supp(P^{(T)})} f(x, y)\, dP^{(T)}(x, y)\,, \tag{8.54}$$

for any $f : \mathbb{R}^2 \to \mathbb{C}$ measurable and bounded. In particular, by (i) Theorem 8.39(a)

$$T = \int_{supp(P^{(T)})} z\, dP^{(T)}(x, y)\,.$$

As far as the proof of (c) is concerned, notice that (8.54) implies that $A \in \mathfrak{B}(\mathsf{H})$ commutes with $\int_{supp(P^{(T)})} f(x, y)\, dP^{(T)}(x, y)$, when A commutes with T, T^*; that is because A commutes with $\widehat{\Phi}_T(f \restriction_{\sigma(T)})$ in consequence of (iii) in Theorem 8.39(b).
(b) Let us prove the claim for the general case where T is not necessarily self-adjoint nor unitary; these special cases are easily proved with this argument. As already said above, $P^{(T)}$ must be concentrated on its support. The fact that $supp(P^{(T)}) = \sigma(T)$ is a straightforward consequence of Theorem 8.54(d). Let us prove the equivalence $\lambda \in \sigma_c(T) \Leftrightarrow P^{(T)}(\{\lambda\}) = 0$ in (i). We shall write P instead of $P^{(T)}$ to simplify the notation. Let $\lambda := x_0 + iy_0$ be a complex number. By (iii) of Theorem 8.54(a),

$$TP(\{(x_0, y_0)\}) = \int_{\sigma(T)} (x + iy)\chi_{\{(x_0,y_0)\}}(x, y)\, dP(x, y)$$

$$= \int_{\sigma(T)} (x_0 + iy_0)\chi_{\{(x_0,y_0)\}}(x, y)\, dP(x, y) = \lambda \int_{\sigma(T)} \chi_{\{(x_0,y_0)\}}(x, y)\, dP(x, y)\,.$$

Hence $TP(\{(x_0, y_0)\}) = \lambda P(\{(x_0, y_0)\})$. We conclude that $P(\{(x_0, y_0)\}) \neq 0$ implies $\lambda := x_0 + iy_0$ is an eigenvalue of T and hence $\lambda \in \sigma_p(T)$, because any vector $u \neq 0$ in the target subspace of $P(\{(x_0, y_0)\})$ is a λ-eigenvector.

Suppose conversely $\lambda \in \sigma_p(T)$ so that $Tu = \lambda u$, $u \neq 0$ and $\lambda := x_0 + iy_0$. Then (cf. (b) of (i) in Proposition 3.60) $T^*u = \overline{\lambda}u$, $T^n(T^*)^m u = \lambda^n \overline{\lambda}^m u$, and by linearity

$$p(T, T^*)u = \int_{supp(P)} p(x + iy, x - iy)\, dP(x, y)u = p(\lambda, \overline{\lambda})u \tag{8.55}$$

for any polynomial $p = p(x+iy, x-iy)$, because the integral defines a *-homomorphism. Every polynomial $p = p(x + iy, x - iy)$ is also a complex polynomial $q = q(x, y)$ in the real variables x, y by setting $q(x, y) := p(x+iy, x-iy)$ pointwise; the correspondence is bijective. Since the $q(x, y)$ approximate continuous maps $f(x, y)$ in sup norm, the second equality of (8.55) holds when $p(x + iy, x - iy) = q(x, y)$ is replaced by the continuous map $f = f(x, y)$. If $\lambda = x_0 + iy_0$, it is not hard to see $\chi_{\{(x_0,y_0)\}}$ is the pointwise limit of a bounded sequence of continuous maps f_n. Using Theorem 8.52(c) and dominated convergence (μ_u is finite), we have:

$$(u|P_{\{(x_0,y_0)\}}u) = \left(u \left| \int_{supp(P)} \chi_{\{(x_0,y_0)\}}(x, y)dP(x, y)\, u \right.\right) = \int_{supp(P)} \chi_{\{(x_0,y_0)\}}(x, y)d\mu_u(x, y)$$

$$= \lim_{n\to+\infty} \int_{supp(P)} f_n(x, y)d\mu_u(x, y) = \lim_{n\to+\infty} \left(u \left| \int_{supp(P)} f_n(x, y)dP(x, y)\, u \right.\right)$$

$$= \lim_{n\to+\infty} (u|f_n(x_0, y_0)u) = \chi_{\{(x_0,y_0)\}}(x_0, y_0)(u|u)\,.$$

Since orthogonal projectors are idempotent and self-adjoint, and since $\chi_{\{(x_0,y_0)\}}(x_0, y_0) = 1$ by definition,

$$(P_{\{(x_0,y_0)\}}u|P_{\{(x_0,y_0)\}}u) = (u|u) \neq 0\,.$$

Hence $P_{\{(x_0,y_0)\}} \neq 0$.

We conclude the proof of (i) by proving that $P_{\{\lambda\}}$ is the orthogonal projector onto the λ-eigenspace H_λ of T, if $\lambda \in \sigma_p(T)$. We proved above that

$$TP(\{\lambda\}) = \lambda P(\{\lambda\})$$

for $\lambda \in \sigma_p(T)$, so that $P(\{\lambda\})(\mathsf{H}) \subset \mathsf{H}_\lambda$. Let us show the opposite inclusion. Suppose that $u \in \mathsf{H}_\lambda$, so that $(T - \lambda I)u = 0$. We therefore have

$$0 = ||(T - \lambda I)u||^2 = \int_{\mathbb{R}^2} |z - \lambda|^2 d\mu_u(z)$$

in view of (b) in Theorem 8.54. Define a partition of $\mathbb{R}^2$ made of sets

$$D_n := \left\{ z \in \mathbb{R}^2 \left| \frac{1}{n+1} < |z-\lambda| \leq \frac{1}{n} \right.\right\} , \quad n = 1, 2, \ldots ,$$

$$D_0 := \left\{ z \in \mathbb{R}^2 \;\middle|\; 1 < |z-\lambda| \right\} ,$$

and

$$D_\infty = \{\lambda\}$$

With this partition we can decompose the above integral, for instance using the dominated convergence theorem, into:

$$0 = ||(T-\lambda I)u||^2 = \int_{D_\infty} |z-\lambda|^2 d\mu_u(z) + \sum_{n=0}^{+\infty} \int_{D_n} |z-\lambda|^2 d\mu_u(z)$$

$$= 0 + \sum_{n=0}^{+\infty} \int_{D_n} |z-\lambda|^2 d\mu_{P_{D_n}u}(z) . \tag{8.56}$$

In the last passage we have exploited the fact that $|z-\lambda|^2 = 0$ on D_∞ and

$$\int_{D_n} f(z) d\mu_u(z) = \int_{\mathbb{R}^2} \chi_{D_n}(z) f(z) d\mu_u(z) = \left(u \left| \int_{\mathbb{R}^2} \chi_{D_n}(z) f(z) dP(z) u \right.\right)$$

$$= \left(u \left| \int_{\mathbb{R}^2} \chi_{D_n}(z)\chi_{D_n}(z)\chi_{D_n}(z) f(z) dP(z) u \right.\right) = \left(u \left| P_{D_n} \int_{\mathbb{R}^2} \chi_{D_n}(z) f(z) dP(z) P_{D_n} u \right.\right)$$

$$\left(P_{D_n} u \left| \int_{\mathbb{R}^2} \chi_{D_n}(z) f(z) dP(z) P_{D_n} u \right.\right) = \int_{D_n} \chi_{D_n}(z) f(z) d\mu_{P_{D_n}u}(z) .$$

Finally (8.56) yields:

$$0 \geq \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} \int_{D_n} 1 d\mu_{P_{D_n}u}(z) = \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} ||P_{D_n} x||^2 .$$

We conclude that $P_{D_n} u = 0$ for $n = 0, 1, \ldots$. On the other hand, exploiting an argument similar to the one leading to (8.56), using here the operator I in place of $T - \lambda I$, we obtain

$$||u||^2 = ||P_{\{\lambda\}} u||^2 + \sum_{n=0}^{+\infty} ||P_{D_n} u||^2 .$$

That, by elementary properties of Hilbert spaces, is equivalent to

$$u = P_{\{\lambda\}} u + \sum_{n=0}^{+\infty} P_{D_n} u ,$$

because the vectors in the right-hand side are pairwise orthogonal as the D_n are pairwise disjoint. Since $P_{D_n}u = 0$, we conclude that $u = P_{\{\lambda\}}u$, that is $\mathsf{H}_\lambda \subset P_{\{\lambda\}}(\mathsf{H})$ as wanted.

Let us pass to (ii). As $\sigma_c(T) \cup \sigma_p(T) = \sigma(T)$ (by (i) Proposition 8.7(c)) and $\sigma_c(T) \cap \sigma_p(T) = \emptyset$ by definition, we must have $\lambda \in \sigma_c(T)$ if and only if $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_p(T)$. But $supp(P^{(T)}) = \sigma(T)$, so $\lambda \in \sigma(T)$ is the same as saying, for any open set A in $\mathbb{R}^2$ containing (x_0, y_0), $x_0 + iy_0 = \lambda$, that $P(A) \neq 0$. On the other hand, by (i), $\lambda \notin \sigma_p(T)$ means $P^{(T)}(\{(x_0, y_0)\}) = 0$.

Now (iii). If $\lambda = x_0 + iy_0 \in \mathbb{C}$ is an isolated point in $\sigma(T)$, by definition there is an open set $A \ni \{(x_0, y_0)\}$ disjoint from the remaining part of $\sigma(T)$. If $P(\{(x_0, y_0)\})$ were 0, then λ would belong to $supp(P^{(T)})$, for in that case $P(A) = 0$. Necessarily, then, $P^{(T)}(\{(x_0, y_0)\}) \neq 0$. By (i) we have $\lambda \in \sigma_p(T)$.

The proof of (iv) is contained in Theorem 8.54(d), where we proved, among other things, that if $\lambda \in \sigma_c(T)$, for any natural number $n > 0$ there exists $\psi_n \in \mathsf{H}$ with $||\psi_n|| \neq 0$ and $0 < ||T\psi_n - \lambda\psi_n||/||\psi_n|| \leq 1/n$. To have (iv) it suffices to define $\phi_n := \psi_n/||\psi_n||$ with n such that $1 \leq \varepsilon n$ for the given ε. $\square$

8.4.2 Spectral Representation of Normal Operators in $\mathfrak{B}(\mathsf{H})$

The next important result provides a *spectral representation* for any normal operator $T \in \mathfrak{B}(\mathsf{H})$: it is shown that every bounded normal operator can be viewed as a multiplicative operator, on a suitable L^2 space, basically built on the spectrum of T.

Theorem 8.58 (Spectral representation of normal operators in $\mathfrak{B}(\mathsf{H})$) *Let H be a Hilbert space, $T \in \mathfrak{B}(\mathsf{H})$ a normal operator, $P^{(T)}$ the spectral measure of T over the Borel sets of $\sigma(T)$, as of Theorem 8.56(a).*

(a) *H splits as a Hilbert sum $\mathsf{H} = \oplus_{\alpha \in A}\mathsf{H}_\alpha$ (A at most countable if H is separable), where the subspaces H_α are closed and mutually orthogonal, such that:*

(i) for any $\alpha \in A$, $T\mathsf{H}_\alpha \subset \mathsf{H}_\alpha$ and $T^*\mathsf{H}_\alpha \subset \mathsf{H}_\alpha$;

(ii) for any $\alpha \in A$ there exist a positive, finite Borel measure μ_α on the spectrum $\sigma(T) \subset \mathbb{R}^2$, and a surjective isometry $U_\alpha : \mathsf{H}_\alpha \to L^2(\sigma(T), \mu_\alpha)$, such that

$$U_\alpha \left(\int_{\sigma(T)} f(x, y) dP^{(T)}(x, y) \right) \restriction_{\mathsf{H}_\alpha} U_\alpha^{-1} = f \cdot ,$$

for $f \in M_b(\sigma(T))$. In particular:

$$U_\alpha T \restriction_{\mathsf{H}_\alpha} U_\alpha^{-1} = (x + iy) \cdot , \quad U_\alpha T^* \restriction_{\mathsf{H}_\alpha} U_\alpha^{-1} = (x - iy) \cdot$$

where $f\cdot$ is the multiplication by f in $L^2(\sigma(T), \mu_\alpha)$:

$$(f \cdot g)(x, y) = f(x, y)g(x, y) \quad \text{almost everywhere on } \sigma(T) \text{ if } g \in L^2(\sigma(T), \mu_\alpha) ;$$

(ii)' if T is self-adjoint or unitary, there exist, for any $\alpha \in A$, a positive finite Borel measure, on Borel sets of $\sigma(T) \subset \mathbb{R}$ or $\sigma(T) \subset \mathbb{S}^1$ (respectively), and a surjective isometry $U_\alpha : \mathsf{H}_\alpha \to L^2(\sigma(T), \mu_\alpha)$, such that

$$U_\alpha \left(\int_{\sigma(T)} f(x) dP^{(T)}(x) \right) \restriction_{\mathsf{H}_\alpha} U_\alpha^{-1} = f\cdot \,,$$

for $f \in M_b(\sigma(T))$. In particular,

$$U_\alpha T \restriction_{\mathsf{H}_\alpha} U_\alpha^{-1} = x\cdot \,,$$

where $f\cdot$ is the multiplication by f on $L^2(\sigma(T), \mu_\alpha)$:

$$(f \cdot g)(x) = f(x)g(x) \quad \textit{almost everywhere on } \sigma(T)$$

for any $g \in L^2(\sigma(T), \mu_\alpha)$.
(b) *We have*

$$\sigma(T) = \overline{\bigcup_{\alpha \in A} supp(\mu_\alpha)}\,.$$

(c) *If H is separable, there exist a measure space $(\mathsf{M}_T, \Sigma_T, \mu_T)$, with $\mu_T(\mathsf{M}_T) < +\infty$, a bounded map $F_T : M_T \to \mathbb{C}$ ($M_T \to \mathbb{R}$ if T is self-adjoint, or $M_T \to \mathbb{S}^1$ if T is unitary), and a unitary operator $U_T : \mathsf{H} \to L^2(\mathsf{M}_T, \mu_T)$, satisfying*

$$\left(U_T T U_T^{-1} f\right)(m) = F_T(m) f(m)\,, \quad \left(U_T T^* U_T^{-1} f\right)(m) = \overline{F_T(m)} f(m) \quad \textit{for any } f \in \mathsf{H}. \tag{8.57}$$

Proof (a) We prove (i), (ii) and (iii). The proof of (ii)' is similar to (ii).

To begin with, assume there is a vector $\psi \in \mathsf{H}$ whose subspace H_ψ, containing vectors of type $\int_{\sigma(T)} g(x, y)\, dP^{(T)}(x, y)\psi$, $g \in M_b(\sigma(T))$, is dense in H. If μ_ψ is the spectral measure associated to ψ (finite since

$$\int_{supp(P^{(T)})} 1 \, d\mu_\psi = ||\psi||^2 \,)$$

we have $supp(\mu_\psi) \subset supp(P^{(T)})$ by (iv) in Theorem 8.52(c). Consider the Hilbert space $L^2(\sigma(T), \mu_\psi)$ and the linear surjective operator

$$V_\psi : M_b(\sigma(T)) \ni g \mapsto \int_{\sigma(T)} g(x, y)\, dP^{(T)}(x, y)\psi \in H_\psi \,.$$

As μ_ψ is finite, $M_b(\sigma(T)) \subset L^2(\sigma(T), \mu_\psi)$ as subspace. Hence for any $g_1, g_2 \in M_b(\sigma(T))$,

$$\int_{\sigma(T)} \overline{g_1(x,y)} g_2(x,y)\, d\mu_\psi(x,y)$$

$$= \left(\int_{\sigma(T)} g_1(x,y)\, dP^{(T)}(x,y)\psi \,\middle|\, \int_{\sigma(T)} g_2(x,y)\, dP^{(T)}(x,y)\psi \right) , \tag{8.58}$$

or equivalently,

$$\int_{\sigma(T)} \overline{g_1(x,y)} g_2(x,y)\, d\mu_\psi(x,y) = \left(V_\psi g_1 | V_\psi g_2 \right) . \tag{8.59}$$

The proof of (8.58) descends by the following observation. If $E, E' \subset \sigma(T)$ are Borel sets, using (iv) in Theorem 8.52(c), (iii) in Theorem 8.54(a) and (iv) in Theorem 8.54(a),

$$\int_{\sigma(T)} \overline{\chi_E} \chi_{E'}\, d\mu_\psi = \int_{\sigma(T)} \chi_{E\cap E'}\, d\mu_\psi = \left(\psi \,\middle|\, \int_{\sigma(T)} \chi_{E\cap E'}\, dP^{(T)}\, \psi \right) =$$

$$\left(\psi \,\middle|\, \int_{\sigma(T)} \chi_E \chi_{E'}\, dP^{(T)}\, \psi \right) = \left(\psi \,\middle|\, \int_{\sigma(T)} \chi_E\, dP^{(T)} \int_{\sigma(T)} \chi_{E'}\, dP^{(T)}\, \psi \right)$$

$$= \left(\int_{\sigma(T)} \chi_E\, dP^{(T)}\psi \,\middle|\, \int_{\sigma(T)} \chi_{E'}\, dP^{(T)}\, \psi \right) .$$

By linearity of the inner product, if s and t are simple,

$$\int_{\sigma(T)} \overline{s} t\, d\mu_\psi = \left(\int_{\sigma(T)} s\, dP^{(T)}\psi \,\middle|\, \int_{\sigma(T)} t\, dP^{(T)}\, \psi \right) .$$

By Proposition 7.49, using the definition of integral of a measurable bounded map for a spectral measure, by dominated convergence with respect to the finite measure μ_ψ and by the inner product's continuity, the above identity implies (8.58). Thus we have proved V_ψ is a surjective isometry mapping $M_b(\sigma(T))$ to H_ψ. Notice that $M_b(\sigma(T))$ is dense in $L^2(\sigma(T), \mu_\psi)$, because if $g \in L^2(\sigma(T), \mu_\psi)$, the maps $g_n := \chi_{E_n} \cdot g$,

$$E_n := \{(x,y) \in \sigma(T) \mid |g(x,y)| < n\} ,$$

belong in $M_b(\sigma(T))$, and $g_n \to g$ in $L^2(\sigma(T), \mu_\psi)$ by dominated convergence (pointwise $|g_n - g|^2 \to 0$, as $n \to +\infty$, and $|g_n - g|^2 \le 2|g|^2 \in L^1(\sigma(T), \mu_\psi)$). So we can extend V_ψ to a unique surjective isometry $\widehat{V_\psi} : L^2(\sigma(T), \mu_\psi) \to \overline{H_\psi}$, whose inverse will be denoted by U_ψ. Then $\overline{H_\psi} = \mathsf{H}$.

If $f \in M_b(\sigma(T))$, from (8.58) and using (iii) in Theorem 8.54(a) we see that

$$\int_{\sigma(T)} \overline{g_1(x,y)} f(x,y) g_2(x,y)\, d\mu_\psi(x,y)$$

$$= \left(\int_{\sigma(T)} g_1(x, y)\, dP^{(T)}(x, y)\psi \,\middle|\, \int_{\sigma(T)} f(x, y) g_2(x, y)\, dP^{(T)}(x, y)\psi \right)$$

$$= \left(\int_{\sigma(T)} g_1(x, y)\, dP^{(T)}(x, y)\psi \,\middle|\, \int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y) \int_{\sigma(T)} g_2(x, y)\, dP^{(T)}(x, y)\psi \right)$$

$$= \left(V_\psi g_1 \,\middle|\, \int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y) V_\psi g_2 \right) .$$

We have proved that for any triple $g_1, g_2, f \in M_b(\sigma(T))$

$$\int_{\sigma(T)} \overline{g_1(x, y)} f(x, y) g_2(x, y)\, d\mu_\psi(x, y) = \left(V_\psi g_1 \,\middle|\, \int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y) V_\psi g_2 \right) .$$

The operator $f\cdot : L^2(\sigma(T), \mu_\psi) \to L^2(\sigma(T), \mu_\psi)$, i.e. the multiplication by $f \in M_b(\sigma(T))$, is easily bounded. Since $M_b(\sigma(T))$ is dense in $L^2(\sigma(T), \mu_\psi)$, by definition of U_ψ, because

$$\int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y)$$

is bounded and, eventually, by continuity of the inner product, we have

$$\int_{\sigma(T)} \overline{g_1(x, y)} f(x, y) g_2(x, y)\, d\mu_\psi(x, y) = \left(U_\psi^{-1} g_1 \,\middle|\, \int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y) U_\psi^{-1} g_2 \right) ,$$

for any $g_1, g_2 \in L^2(\sigma(T), \mu_\psi)$. That is to say

$$U_\psi \int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y) U_\psi^{-1} = f \cdot \ . \tag{8.60}$$

Now we pass to the case in which there is no ψ with $\overline{H_\psi} = \mathsf{H}$.

If so, let ψ be an arbitrary vector in H, and indicate by H_ψ the space of vectors $\int_{\sigma(T)} f(x, y)\, dP^{(T)}(x, y)\psi$, $f \in M_b(\sigma(T))$. We have $T(H_\psi) \subset H_\psi$ and $T^*(H_\psi) \subset H_\psi$, because for any $f \in M_b(\sigma(T))$

$$T \int_{\sigma(T)} f(x, y)\, dP^{(T)}\psi = \int_{\sigma(T)} (x + iy)\, dP^{(T)} \int_{\sigma(T)} f(x, y)\, dP^{(T)}\psi$$

$$= \int_{\sigma(T)} (x + iy) f(x, y)\, dP^{(T)}\psi ,$$

(Theorem 8.56(a) and (iii) in Theorem 8.54(a)). Hence $T \int_{\sigma(T)} f(x, y)\, dP^{(T)}\psi \in H_\psi$, for $(x, y) \mapsto (x + iy) f(x, y)$ is an element of $M_b(\sigma(T))$). The proof for T^* is analogous, using

$$T^* = \int_{\sigma(T)} (x - iy)\, dP^{(T)} .$$

By continuity $T(\overline{\mathsf{H}_\psi}) \subset \overline{\mathsf{H}_\psi}$ and $T^*(\overline{H_\psi}) \subset \overline{\mathsf{H}_\psi}$. Defining U_ψ as before we have (8.60).

Now let us show how to build another closed subspace $\overline{H_{\psi'}}$, orthogonal to $\overline{H_\psi}$, invariant under T, T^* and satisfying (8.60) for a corresponding surjective isometry $U_{\psi'} : \overline{H_{\psi'}} \to L^2(\sigma(T), \mu_{\psi'})$. If $0 \neq \psi' \perp \mathsf{H}_\psi$ then

$$\left(\psi' \middle| \int_{\sigma(T)} f(x,y)\, dP^{(T)}(x,y)\psi\right) = 0 ,$$

for any $f \in M_b(\sigma(T))$. But then the properties of the integral with respect to spectral measures ((iii)-(iv) in Theorem 8.54(a)) imply, for any $g, f \in M_b(\sigma(T))$:

$$\left(\int_{\sigma(T)} g\, dP^{(T)}\psi' \middle| \int_{\sigma(T)} f\, dP^{(T)}\psi\right) = \left(\psi' \middle| \int_{\sigma(T)} \overline{g}\, dP^{(T)} \int_{\sigma(T)} f\, dP^{(T)}\psi\right)$$

$$= \left(\psi' \middle| \overline{g(x,y)} f(x,y)\, dP^{(T)}(x,y)\psi\right) = 0 ,$$

where we used $\overline{g} \cdot f \in M_b(\sigma(T))$. Overall, if $\psi' \perp H_\psi$ then $H_{\psi'}$ is orthogonal to H_ψ, so the same holds for the closures by continuity of the inner product. The space $\overline{H_{\psi'}}$ is invariant under T and T^* (the proof is the same as for $\overline{\mathsf{H}_\psi}$), and (8.60) holds for the surjective isometry $U_{\psi'} : \overline{H_{\psi'}} \to L^2(\sigma(T), \mu_{\psi'})$ (see the proof at the beginning). Thus, choosing $\{\psi_\alpha\}$ suitably, we can construct closed subspaces $\mathsf{H}_\alpha = \overline{H_{\psi_\alpha}}$, each with a surjective isometry $U_\alpha : \overline{H_\alpha} \to L^2(\sigma(T), \mu_{\psi'})$, so that: (a) the spaces are pairwise orthogonal; (b) each one is T-invariant and T^*-invariant; (c) they satisfy

$$U_\alpha \int_{\sigma(T)} f(x,y)\, dP^{(T)}(x,y) \upharpoonright_{\mathsf{H}_\alpha} U_\alpha^{-1} = f \cdot \ . \tag{8.61}$$

for any $f \in M_b(\sigma(T))$. Call $\mathfrak{C}$ the collection of these subspaces. We can order $\mathfrak{C}$ (partially) by inclusion. Then every ordered subset in $\mathfrak{C}$ is upper bounded, and Zorn's lemma gives us a maximal element $\{\mathsf{H}_\alpha\}_{\alpha\in A}$ in $\mathfrak{C}$. We claim $\mathsf{H} = \oplus_{\alpha\in A}\mathsf{H}_\alpha$. It suffices to show that if ψ belongs to the orthogonal complement of every H_a, then $\psi = \mathbf{0}$. If there existed $\psi \in \mathsf{H}$ with $\psi \perp \mathsf{H}_\alpha$ for any $\alpha \in A$ and $\psi \neq \mathbf{0}$, we would be able to construct $\overline{H_\psi}$, distinct from every H_α but satisfying (a), (b), (c). Consequently $\{\mathsf{H}_\alpha\}_{\alpha\in A} \cup \{\overline{H_\psi}\}$ would contain $\{\mathsf{H}_\alpha\}_{\alpha\in A}$, producing a contradiction. We conclude that if ψ is orthogonal to every H_α, it must vanish $\psi = \mathbf{0}$. Put equivalently, $\overline{< \{\mathsf{H}_\alpha\}_{\alpha\in A} >} = \mathsf{H}$, so $\mathsf{H} = \oplus_{\alpha\in A}\mathsf{H}_\alpha$ for the spaces are mutually orthogonal.

Now we prove (b) when T is normal (T self-adjoint or unitary is obtained specialising the same proof). We shall prove that $\lambda \notin \overline{\cup_\alpha supp(\mu_\alpha)} \Leftrightarrow \lambda \in \rho(T)$, which is equivalent to the claim. In turn, $\lambda \notin \overline{\cup_\alpha supp(\mu_\alpha)}$ is equivalent to

$\lambda \in \mathbb{C} \setminus \overline{\cup_\alpha supp(\mu_\alpha)}$, that is $\lambda \in Int(\mathbb{C} \setminus \cup_\alpha supp(\mu_\alpha))$. The latter means $\lambda \in Int(\cap_\alpha(\mathbb{C} \setminus supp(\mu_\alpha))$.
($\Rightarrow$) If $\lambda \in Int(\cap_\alpha(\mathbb{C} \setminus supp(\mu_\alpha))$, take an open disc D_R of radius $R > 0$ centred at λ, with $\mu_\alpha(D_R) = 0$ for any $\alpha \in A$; such a disc exists by the assumptions. On every space $L^2(\sigma(T), \mu_\alpha)$ the multiplication by $(x, y) \mapsto (x + iy - \lambda)^{-1}$ is bounded, with norm not exceeding $1/R$ (independent from α), and inverts (on the left and the right) the multiplication by $(x + iy - \lambda)$. Let $R_\lambda(\alpha) : \mathsf{H}_\alpha \to \mathsf{H}_\alpha$ be the operator $U_\alpha^{-1}(x + iy - \lambda)^{-1} \cdot U_\alpha$. It has the same norm of the operator $(x + iy - \lambda)^{-1}\cdot$, since U_a is a surjective isometry, so $||R_\lambda(\alpha)|| \leq 1/R$. Define $R_\lambda : \mathsf{H} \to \mathsf{H}$ so that

$$R_\lambda : \sum_{\alpha\in A} P_\alpha \psi \mapsto \sum_{\alpha\in A} R_\lambda(\alpha) P_\alpha \psi \ ,$$

for any $\psi \in \mathsf{H}$. Remembering the H_α are invariant under T and R_λ (i.e. $R_\lambda(\alpha)$ on each H_α), we easily see that $||R_\lambda|| \leq 1/R$ and $R_\lambda(T - \lambda I) = (T - \lambda I)R_\lambda = I$. In fact, $Ran R_\lambda(\alpha) = \mathsf{H}_\alpha$ implies

$$||R_\lambda\psi||^2 = ||\sum_{\alpha\in A} R_\lambda(\alpha)P_\alpha\psi||^2 = ||\sum_{\alpha\in A} P_a R_\lambda(\alpha)P_\alpha\psi||^2 = \sum_{\alpha\in A} ||P_a R_\lambda(\alpha)P_\alpha\psi||^2$$

$$= \sum_{\alpha\in A} ||R_\lambda(\alpha)P_\alpha\psi||^2 \leq R^{-2}\sum_{\alpha\in A} ||P_\alpha\psi||^2 = R^{-2}||\psi||^2 \ .$$

Moreover

$$(T - \lambda I)R_\lambda\psi = (T - \lambda I)R_\lambda \sum_{\alpha\in A} P_\alpha\psi$$

$$\sum_{\alpha\in A}(T - \lambda I)R_\lambda P_\alpha\psi = \sum_{\alpha\in A}(T - \lambda I)\restriction_{H_\alpha} R_\lambda(a)P_\alpha\psi = \sum_{\alpha\in A} IP_\alpha\psi = \psi \ ,$$

hence $(T - \lambda I)R_\lambda = I$. Similarly

$$R_\lambda(T - \lambda I)\psi = R_\lambda(T - \lambda I)\sum_{\alpha\in A} P_\alpha\psi$$

$$\sum_{\alpha\in A} R_\lambda(T - \lambda I)P_\alpha\psi = \sum_{\alpha\in A} R_\lambda(a)(T - \lambda I)\restriction_{H_\alpha} P_\alpha\psi = \sum_{\alpha\in A} IP_\alpha\psi = \psi \ ,$$

so $R_\lambda(T - \lambda I) = I$. By Theorem 8.4(a) $\lambda \in \rho(T)$.
($\Leftarrow$) Suppose now $\lambda \in \rho(T)$, so $(T - \lambda I)^{-1} : \mathsf{H} \to \mathsf{H}$ is the closed inverse to $T - \lambda I$. Pick $\varepsilon > 0$ so that $||(T - \lambda I)^{-1}|| =: 1/\varepsilon$. We claim $\mu_\alpha(D_\varepsilon) = 0$ for any $\alpha \in A$, D_ε being the open disc of radius ε centred at λ, so that $\lambda \in Int(\cap_\alpha(\mathbb{C} \setminus supp(\mu_\alpha))$. We proceed by contradiction. Suppose the last assertion is false, so there exists $\beta \in A$ such that $\mu_\beta(D_\varepsilon) > 0$. Consider an element $\psi \in \mathsf{H} \setminus \{0\}$ defined by $P_\alpha\psi = \mathbf{0}$ if $\alpha \neq \beta$sps and $U_\beta\psi = f$, such that $f = 0$ outside D_ε and $\int_{D_\varepsilon} |f|^2 d\mu_\beta > 0$. We can always redefine ψ so that $||\psi|| = 1$. Then,

$$||(T - \lambda I)\psi||^2 = \int_{D_\varepsilon} |(x + iy) - \lambda|^2 |f(x, y)|^2 \, d\mu_\beta(x, y) < \varepsilon^2 \int_{D_\varepsilon} |f(x, y)|^2 \, d\mu_\beta(x, y) = \varepsilon^2 \,.$$

Therefore

$$||(T - \lambda I)\psi|| < \varepsilon \,.$$

On the other hand, by definition of norm,

$$||(T - \lambda I)^{-1}|| \geq \frac{||(T - \lambda I)^{-1}\phi||}{||\phi||}$$

for any $\phi \in \mathsf{H} \setminus \{\mathbf{0}\}$. Setting $(T - \lambda I)^{-1}\phi = \psi$, we have

$$1/\varepsilon = ||(T - \lambda I)^{-1}|| \geq \frac{||\psi||}{||(T - \lambda I)\psi||} \,,$$

so, as $||\psi|| = 1$,

$$1/\varepsilon \geq \frac{1}{||(T - \lambda I)\psi||} > 1/\varepsilon \,.$$

But that is a contradiction.

We finish the proof by showing (c). If H is separable, consider the collection of orthogonal non-zero vectors $\{\psi_n\}_{n\in\mathbb{N}}$ built as the $\{\psi_\alpha\}_{\alpha\in A}$ above, where now the index α is called $n \in \mathbb{N}$ (if there are finitely many vectors ψ_α the argument is similar). We are free to choose them so that $||\psi_n||^2 = 2^{-n}$. Define $M_T := \bigsqcup_{n=1}^{+\infty} \sigma(T)$ to be the disjoint union of infinitely many copies of $\sigma(T)$. Now call μ_T the measure that restricts to μ_n on the nth factor $\sigma(T)$. It is clear that, in this way, $\mu_T(M_T) = \sum_{n=0}^{+\infty} ||\psi_n||^2 < +\infty$. The map F_T clearly restricts to $(x + iy)\cdot$ on each component $\sigma(T)$. Hence F_T is bounded, because every copy of $\sigma(T)$ is bounded. The operator U_T is built in the obvious manner using the U_n. $\square$

One can rearrange canonically the decomposition of H into spaces isomorphic to L^2. In particular [DuSc88, vol.I] the following fact holds. In the statement we use the symbol $A \triangle B := (A \cup B) \setminus (A \cap B)$ for the symmetric difference of two sets.

Theorem 8.59 *Let $T \in \mathfrak{B}(\mathsf{H})$ be a normal operator on the separable Hilbert space* H.
(a) *There exists a pair $(\mu_T, \{E_{Tn}\}_{n\in\mathbb{N}})$, where μ_T is a positive, finite Borel measure on $\mathbb{C}$ and $\{E_{Tn}\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathbb{C})$ satisfies $\mathbb{C} = E_{T1} \supset E_{T2} \supset \cdots$, so that Theorem 8.58(a) holds (replacing $\sigma(T)$ by $\mathbb{C}$ everywhere) with $A = \mathbb{N}$, $\mu_\alpha(F) := \mu_T(F \cap E_{T\alpha})$ for any $\alpha \in \mathbb{N}$, $F \in \mathscr{B}(\mathbb{C})$.*
(b) *If $(\mu'_T, \{E'_{Tn}\}_{n\in\mathbb{N}})$ satisfies part (a) then $\mu_T \prec \mu'_T \prec \mu_T$ and $\mu_T(E_{Tn} \triangle E'_{Tn}) = \mu'(E_{Tn} \triangle E'_{Tn}) = 0$ for any $n \in \mathbb{N}$.*
(c) *Let $S \in \mathfrak{B}(\mathsf{H})$ be a normal operator and suppose $(\mu_S, \{E_{Sn}\}_{n\in\mathbb{N}})$ satisfies (a) together with S. Then there exists a unitary operator $U : \mathsf{H} \to \mathsf{H}$ with $USU^{-1} = T$ if and only if $\mu_T \prec \mu_S \prec \mu_T$ and $\mu_T(E_{Tn} \triangle E_{Sn}) = \mu_S(E_{Tn} \triangle E_{Sn}) = 0$ for any $n \in \mathbb{N}$.*

Evidently $\sigma(T) = supp(\mu_T)$. In case $\mu_T(E_{Tn}) \neq 0$ but $\mu_T(E_{Tn+1}) = 0$, one says T has **spectral multiplicity** n (including $n = +\infty$ if $\mu_T(E_{Tn}) \neq 0$ for any n). The definition is clearly independent of the pair $(\mu_T, \{E_{Tn}\}_{n\in\mathbb{N}})$ of (a). If $\mu_T(E_{Tk}) = 0$ for some k then $\mu'_T(E_{Tk}) = 0$, for $\mu_T \prec \mu'_T \prec \mu_T$. Since $\mu'_T(E_{Tk}\triangle E'_{Tk}) = 0$ we have $\mu'_T(E'_{Tk}) = 0$. By symmetry $\mu'_T(E'_{Tk}) = 0$ implies $\mu_T(E_{Tk}) = 0$. The unabridged theory of the spectral multiplicity can be found in [Hal51].

Examples 8.60
(1) Consider a compact self-adjoint operator $T \in \mathfrak{B}(\mathsf{H})$ on the Hilbert space H. By Theorem 4.19, $\sigma_p(T)$ is discrete in $\mathbb{R}$, with possible unique limit point 0. Consequently $\sigma(T) = \sigma_p(T)$, except in case $\sigma_p(T)$ accumulates at 0, but $0 \notin \sigma_p(T)$. In that case ($\sigma(T)$ is closed by Theorem 8.4) $\sigma(T) = \sigma_p(T) \cup \{0\}$ and 0 is the *only point in* $\sigma_c(T)$ (for $\sigma_r(T) = \emptyset$ by Proposition 8.7). Following Example 8.51(3), we can define a PVM on $\mathbb{R}$ that vanishes outside $\sigma(T)$:

$$P_E x := \sum_{\lambda\in E} P_\lambda x \quad x \in \mathsf{H}$$

where $E \subset \sigma(T)$, while P_λ is either the null projector $P_\lambda = 0$, or an orthogonal projector onto the λ-eigenspace. The former case can occur only when $\lambda = 0$ is not an eigenvalue. Mimicking Example 8.51(3), we see

$$\int_{\sigma(T)} \lambda P(\lambda)\psi = \sum_{\lambda\in\sigma(T)} \lambda P_\lambda \psi \ ,$$

for any $\psi \in \mathsf{H}$. On the other hand Theorem 4.20 gives

$$\sum_{\lambda\in\sigma(T)} \lambda P_\lambda = T \ ,$$

where $P_0 = 0$ if $0 \in \sigma_c(T)$.

The statement of Theorem 4.20 explains that the decomposition is valid in the uniform topology provided we label eigenvalues properly. Using such an ordering, for any $\psi \in \mathsf{H}$

$$\sum_{\lambda\in\sigma(T)} \lambda P_\lambda \psi = T\psi \ .$$

We may interpret the sum as an integral in the PVM on $\sigma(T)$ defined above. This also proves that the series on the left can be rearranged (when projectors are applied to some $\psi \in \mathsf{H}$). By construction, $supp(P) = \sigma(T)$. In conclusion: the above measure on $\sigma(T)$ is the spectral measure of T, uniquely associated to T by the spectral theorem. Moreover, the spectral decomposition of T is precisely the eigenspace decomposition with respect to the strong topology:

$$T = \text{s-}\sum_{\lambda\in\sigma_p(T)} \lambda P_\lambda \ .$$

The point $0 \in \sigma_c(T)$, if present, brings no contribution to the integral.

(2) Consider the operator T on $\mathsf{H} := L^2([0,1] \times [0,1], dx \otimes dy)$ defined by

$$(Tf)(x, y) = xf(x, y)$$

almost everywhere on $\mathsf{X} := [0,1] \times [0,1]$, for any $f \in \mathsf{H}$. It is not hard to show T is bounded, self-adjoint and its spectrum is $\sigma(T) = \sigma_c(T) = [0,1]$.

A spectral measure on $\mathbb{R}$, with bounded support, that reproduces T as integral operator is given by orthogonal projectors $P_E^{(T)}$ that multiply by characteristic functions $\chi_{E'}$, $E' := (E \cap [0,1]) \times [0,1]$, for any Borel set $E \subset \mathbb{R}$. Proceeding as in Example 8.51(1), and choosing appropriate domains, allows to see that

$$\left(\int_{[0,1]} g(\lambda)\, dP(\lambda) f\right)(x, y) = g(x) f(x, y)\ , \quad \text{almost everywhere on } \mathsf{X}$$

for any $g \in M_b(\mathsf{X})$. In particular

$$\left(\int_{[0,1]} \lambda\, dP(\lambda) f\right)(x, y) = xf(x, y)\ , \quad \text{almost everywhere on } \mathsf{X}\ ,$$

so

$$T = \int_{[0,1]} \lambda\, dP(\lambda)\ ,$$

as required. This spectral measure is therefore the unique measure on $\mathbb{R}$ satisfying condition (a) in the spectral representation theorem.

Let us concentrate on part (c) in the spectral representation theorem. A decomposition of H of the kind prescribed in (c) can be obtained as follows. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Hilbert basis of $L^2([0,1], dy)$. Consider subspaces of $\mathsf{H} := L^2([0,1] \times [0,1], dx \otimes dy)$ given, for any $n \in \mathbb{N}$, by

$$\mathsf{H}_n := \{f \cdot u_n \mid f \in L^2([0,1], dx)\}\ .$$

It is easy to see that these subspaces, with respect to T, fulfil every request of item (c). In particular, H_n is by construction isomorphic to $L^2([0,1], dx)$ under the surjective isometry $f \cdot u_n \mapsto f$, so $\mu_n = dx$. ■

8.5 Fuglede's Theorem and Consequences

In the final section we state and prove a well-known result. *Fuglede's theorem* establishes that an operator $B \in \mathfrak{B}(\mathsf{H})$ commutes with A^*, for $A \in \mathfrak{B}(\mathsf{H})$ normal, provided it commutes with A. The result is far from obvious, and has immediate consequences in the light of the previous section. Due to the spectral decomposition Theorem 8.56(c), for example, one corollary is that if B commutes with A then it commutes with every operator $\int_{\sigma(T)} f(x, y) dP^{(A)}(x, y)$, for any measurable bounded map $f : \sigma(T) \to \mathbb{C}$.

8.5.1 Fuglede's Theorem

Theorem 8.61 (Fuglede) *Let* H *be a Hilbert space. If* $A \in \mathfrak{B}(\mathsf{H})$ *is normal and* $B \in \mathfrak{B}(\mathsf{H})$ *commutes with* A*, then* B *commutes with* A^* *as well.*

Proof For $s \in \mathbb{C}$ consider the function $Z(s) = e^{-sA^*} B e^{sA^*}$, where the exponentials are defined spectrally by integrals of $\mathbb{C} \ni x + iy \mapsto e^{\mp s(x-iy)}$ with respect to the spectral measure $P^{(A)}$ of A. As usual $z = x + iy$ and $\overline{z} = x - iy$. Now observe $e^{\mp s(x-iy)} = \sum_{n=0}^{+\infty} \frac{(\mp s(x-iy))^n}{n!}$, and for given s, the convergence is uniform in (x, y) on every compact set, like $\sigma(A)$. In particular this means the sequence of partial sums is bounded in norm $|| \ ||_\infty$. Using again the PVM associated spectrally to A, by Theorem 8.54(c) we have

$$e^{\mp sA^*} = \text{s-}\sum_{n=0}^{+\infty} \frac{(\mp sA^*)^n}{n!} . \tag{8.62}$$

Expanding $Z(s)A\psi$ and $AZ(s)\psi$ as above, and recalling A^* and B commute with A, we see $A^n Z(s)\psi = Z(s)A^n\psi$ for any $\psi \in \mathsf{H}$. Hence the exponential expansion gives

$$e^{\mp\overline{s}A} Z(s)\psi = Z(s)e^{\mp\overline{s}A}\psi \quad \text{for any } \psi \in \mathsf{H}.$$

Therefore

$$Z(s) = Z(s)e^{+\overline{s}A}e^{-\overline{s}A} = e^{+\overline{s}A}Z(s)e^{-\overline{s}A} = e^{-sA^*}e^{+\overline{s}A}Be^{sA^*}e^{-\overline{s}A} = e^{-sA^*+\overline{s}A}Be^{sA^*-\overline{s}A} .$$

To obtain this we need the identities $e^{-sA^*}e^{+\overline{s}A} = e^{-sA^*+\overline{s}A}$ and $e^{+\overline{s}A}e^{-\overline{s}A} = I$, which are proved exactly as in $\mathbb{C}$, i.e. using (8.62) and the commutation of A and A^*. With the same technique one proves $U_s := e^{-sA^*+\overline{s}A} = \left(e^{sA^*-\overline{s}A}\right)^*$ and $U_s^* = U_s^{-1}$. Therefore U_s is unitary and $||Z(s)|| = ||U_s B U_s^*|| \leq ||U_s|| \, ||B|| \, ||U_s^*|| = 1||B||1 = ||B||$. The map $\mathbb{C} \ni s \mapsto (\psi|Z(s)\phi)$ is then bounded on the entire complex plane. If this function were entire (i.e. analytic on $\mathbb{C}$), Liouville's theorem would force it to be constant. So let us assume the map is entire, hence constant. Consequently, since ψ, ϕ are arbitrary, $Z(s) = Z(0)$ for any $s \in \mathbb{C}$. This identity reads $e^{-sA^*}Be^{sA^*} = B$, i.e. $Be^{sA^*} = e^{sA^*}B$. Therefore $(\psi|Be^{sA^*}\phi) = (\psi|e^{sA^*}B\phi)$ for any $\psi, \phi \in \mathsf{H}$. This equation can be written $(B^*\psi|e^{sA^*}\phi) = (\psi|e^{sA^*}B\phi)$, and by the properties of PVMs and the spectral theorem:

$$\int_K e^{s\bar{z}} d\mu_{B^*\psi,\phi} = \int_K e^{s\bar{z}} d\mu_{\psi,B\phi} \ ,$$

where $K \subset \mathbb{R}^2 \equiv \mathbb{C}$ is a compact set large enough to contain the supports of the measures of the integrals. Let us differentiate in s, and evaluate at $s = 0$, by swapping derivative and integral (the derivatives of the integrands are continuous in $(s, (x, y))$, hence bounded on the compact set $C \times K$, where C is some compact subset containing $s = 0$; hence Theorem 1.88 applies). The outcome is

$$\int_K \bar{z} d\mu_{B^*\psi,\phi} = \int_K \bar{z} d\mu_{\psi,B\phi} \ ,$$

which we can write $(\psi|BA^*\phi) = (\psi|A^*B\phi)$. Varying ψ and ϕ, we obtain that B commutes with A^*: $BA^* = A^*B$. There remains to prove $\mathbb{C} \ni s \mapsto (\psi|Z(s)\phi)$ is an analytic function. Expansion (8.62) and the inner product's continuity imply

$$(\psi|Z(s)\phi) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} \frac{(-s)^{n+m}}{n!m!}(\psi|(A^*)^n B(A^*)^m\phi) \ . \tag{8.63}$$

We may interpret the double series as an iterated integral *for the counting measure of* $\mathbb{N}$; we shall denote the latter by $d\mu(n)$. By the Schwarz inequality and known norm properties:

$$\left|\frac{(-s)^{n+m}}{n!m!}(\psi|(A^*)^n B(A^*)^m\phi)\right| \le \frac{(|s|\,||A||)^n}{n!}\frac{(|s|\,||A||)^m}{m!}||B||\,||\psi||\,||\phi|| \ .$$

The positive function on $\mathbb{N} \times \mathbb{N}$ of the right-hand side is integrable in the product measure by Fubini-Tonelli (the integral is clearly $e^{|s|\,||A||}e^{|s|\,||A||}||B||\,||\psi||\,||\phi||$), so $(n, m) \mapsto \frac{(-s)^{n+m}}{n!m!}(\psi|(A^*)^n B(A^*)^m\phi) =: f_s(n, m)$ is L^1 for the product measure, and (8.63) reads:

$$(\psi|Z(s)\phi) = \int_{\mathbb{N}\times\mathbb{N}} f_s(n, m) d\mu(n) \otimes d\mu(m) \ . \tag{8.64}$$

By dominated convergence we have

$$\int_{\mathbb{N}\times\mathbb{N}} f_s(n, m) d\mu(n) \otimes d\mu(m) = \lim_{N\to+\infty} \int_{\mathbb{N}\times\mathbb{N}} \chi_{\{(n,m)\in\mathbb{N}\times\mathbb{N}\,|\,n+m\le N\}} f_s(n, m) d\mu(n) \otimes d\mu(m) \ .$$

Writing the right side using sums:

$$(\psi|Z(s)\phi) = \lim_{N\to+\infty} \sum_{M=0}^{N} \sum_{n+m=M} \frac{(-s)^{n+m}}{n!m!}(\psi|(A^*)^n B(A^*)^m\phi) \ ,$$

i.e.

$$(\psi|Z(s)\phi) = \sum_{N=0}^{+\infty} C_N s^N \quad \forall s \in \mathbb{C}\,, \tag{8.65}$$

where

$$C_N = (-1)^N \sum_{n+m=N} \frac{(\psi|(A^*)^n B(A^*)^m \phi)}{n!m!}\,.$$

Now, (8.65) says we may express $(\psi|Z(s)\phi)$ as a power series in s, with s roaming the whole complex plane. Hence $\mathbb{C} \ni s \mapsto (\psi|Z(s)\phi)$ is an entire function, as claimed. □

The theorem generalises, if we drop the boundedness of A (but keeping that of B). This was Fuglede's original statement [Fug50], whose proof requires the spectral theory of unbounded normal operators that we will not develop.

8.5.2 Consequences to Fuglede's Theorem

Corollary 8.62 *Let H be a Hilbert space. If $M, N \in \mathfrak{B}(\mathsf{H})$ are normal and satisfy $NM = MN$, then NM is normal.*

Proof First, $MN(MN)^* = MNM^*N^*$. By Fuglede's theorem the right-hand side is $MM^*NN^* = M^*MN^*N = M^*N^*MN = (NM)^*MN$. But N, M commute, so $(NM)^*MN = (MN)^*MN$. Hence we have proved $MN(MN)^* = (MN)^*MN$, i.e. the claim. □

Corollary 8.63 (Fuglede–Putnam–Rosenblum) *Let H be a Hilbert space and $T, M, N \in \mathfrak{B}(\mathsf{H})$. If M, N are normal and $MT = TN$ then $M^*T = TN^*$.*

Proof Consider the Hilbert space $\mathsf{H} \oplus \mathsf{H}$ with standard inner product $((u, v)|(u', v')) := (u|u')(v|v')$, and the operators of $\mathfrak{B}(\mathsf{H} \oplus \mathsf{H})$:

$$T' : (u, v) \mapsto (0, Tu)\,, \quad N' : (u, v) \mapsto (Nu, Mv)\,.$$

By direct computation $N'N'^* = N'^*N'$, i.e. N' is normal, and $N'T' = T'N'$ by the fact that $MT = TN$. We can apply Fuglede's theorem to get $N'^*T' = T'N'^*$. Since $N'^* : (u, v) \mapsto (N^*u, M^*v)$, taking the components of the identity $N'^*T'(u, v) = T'N'^*(u, v)$ gives $M^*Tu = TN^*u$ for any $u \in \mathsf{H}$, i.e. $M^*T = TN^*$. □

Corollary 8.64 *Let $M, N \in \mathfrak{B}(\mathsf{H})$ be normal operators on the Hilbert space H. If there is a bijection $S \in \mathfrak{B}(\mathsf{H})$ such that*

$$MS = SN\,,$$

then there is also a unitary operator $U \in \mathfrak{B}(\mathsf{H})$ *such that*

$$UMU^{-1} = N .$$

Proof Observe preliminarily $S^{-1} \in \mathfrak{B}(\mathsf{H})$ by Theorem 2.96. By polar decomposition $S = U|S|$, with U unitary, and therefore $MU|S| = U|S|N$. In our case $|S|^{-1}$ exists and equals $|S^{-1}|$, as is easy to see. The proof finishes if we can show that $|S|N = N|S|$, for then we can left-apply $|S|^{-1}$ on $MU|S| = UN|S|$. Let us prove that. By by Fuglede–Putnam–Rosenblum theorem $MS = SN$ implies $M^*S = SN^*$. Taking adjoints, $S^*M = NS^*$. Using $MS = SN$ again, we get $S^*MS = S^*SN = NS^*S$, i.e. $|S|^2N = N|S|^2$. By Theorem 3.77(a), $|S|N = N|S|$. □

Exercises

8.1 Take $\mathsf{H} = \ell^2(\mathbb{N})$ and consider the operator

$$T : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots) .$$

Determine the spectrum of T.

8.2 Let H be a Hilbert space and $T = T^* \in \mathfrak{B}(\mathsf{H})$ compact. Show that if $\dim(RanT)$ is not finite, then $\sigma_c(T) \neq \emptyset$ and consists of one point.

Hint. Decompose T as in Theorem 4.20, use Theorem 4.19 and the fact that $\sigma(T)$ is closed by Theorem 8.4.

8.3 If T is self-adjoint on the Hilbert space H and $\lambda \in \rho(T)$, show $R_\lambda(T)$ is a normal operator of $\mathfrak{B}(\mathsf{H})$ such that

$$R_\lambda(T)^* = R_{\overline{\lambda}}(T) .$$

8.4 Prove that the residual spectrum of a unitary operator is empty, without using the fact that 'unitary ⇒ normal'.

Solution. If $\lambda \in \sigma_r(U)$, $Ran(U - \lambda I)$ is not dense, so there exists $f \neq 0$ orthogonal to $Ran(U - \lambda I)$. For any $g \in \mathsf{H}$, $(f|\lambda g) = (f|Ug)$, so $(\overline{\lambda} f|g) = (U^* f|g)$ for any $g \in \mathsf{H}$. Hence $U^* f = \overline{\lambda} f$. Letting U act on this relation gives $f = \overline{\lambda} Uf$, and then $Uf = \lambda f$, because $1/\lambda = \overline{\lambda}$ by $|\lambda| = 1$. Consequently $\lambda \in \sigma_p(U)$. But this is absurd, for the point and residual spectra are disjoint, and hence $\sigma_r(U) = \emptyset$.

8.5 Assume $U : \mathsf{H} \to \mathsf{H}$ is an isometry on a Hilbert space H that is *not surjective*. Prove $\sigma_r(U) \neq \emptyset$.

Solution. $0 \in \sigma_r(U)$ and $U - 0I$ is one-to-one, but $Ran(U - 0I) = RanU$ is not dense. Let us prove that by contradiction. If it were dense, for any $f \in \mathsf{H}$ there would exist $\{f_n\}_{n\in\mathbb{N}} \subset \mathsf{H}$ with $Uf_n \to f$. Since $||f_n - f_m|| = Uf_n - Uf_m$, then $\{f_n\}$ would be a Cauchy sequence, and $f_n \to g \in \mathsf{H}$. Hence $Ug = f$ for any $f \in \mathsf{H}$, which cannot be, for U is not surjective.

8.6 Build a self-adjoint operator with point spectrum dense in, but not coinciding with, [0, 1].

Hint. Take the Hilbert space $\mathsf{H} = \ell^2(\mathbb{N})$, and label rationals in [0, 1] arbitrarily: $r_0, r_1, \ldots$ Define

$$T : (x_0, x_1, x_2, \ldots) \mapsto (r_0x_0, r_1x_1, r_2x_2, \ldots)$$

with domain $D(T)$ given by sequences $(x_0, x_1, x_2, \ldots) \in \ell^2(\mathbb{N})$ such that

$$\sum_{n=0}^{+\infty} |r_n x_n|^2 < +\infty .$$

8.7 Define a bounded normal operator $T : \mathsf{H} \to \mathsf{H}$, for some Hilbert space H, such that $\sigma(T) = \sigma_p(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. Can H be separable?

Hint. Define $\mathsf{H} := L^2(D, \mu)$, where μ is the counting measure and $D := \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$. Then set $(Tf)(z) := zf(z)$, $f \in \mathsf{H}$.

8.8 If $P : \mathsf{X} \to \mathfrak{B}(\mathsf{H})$ is a PVM, prove: (1) the set of P-essentially bounded, measurable maps $f : \mathsf{X} \to \mathbb{C}$ is a vector space, and (2) $|| \; ||_\infty^{(P)}$ is a seminorm on that space.

8.9 Let A be an operator on the Hilbert space H with domain $D(A)$, and let $U : \mathsf{H}' \to \mathsf{H}$ be an isometry onto H. If $A' := U^{-1}AU : D(A') \to \mathsf{H}'$, $D(A') = U^{-1}D(A)$, prove $\sigma_c(A) = \sigma_c(A')$, $\sigma_p(A) = \sigma_p(A')$, $\sigma_r(A) = \sigma_r(A')$.

Hint. Just apply the definition of the various parts of the spectrum and use the fact that the isomorphisms U is bijective and norm-preserving.

8.10 Consider the position operator X_i introduced in definition 5.22. Show $\sigma(X_i) = \sigma_c(X_i) = \mathbb{R}$.

8.11 Consider the momentum operator P_i introduced in definition 5.27. Show $\sigma(P_i) = \sigma_c(P_i) = \mathbb{R}$.

Hint. Use Proposition 5.31.

8.12 Find two operators A and B on a Hilbert space such that $\sigma(A) = \sigma(B)$, but $\sigma_p(A) \neq \sigma_p(B)$.

Hint. Consider the operator of Exercise 8.6 and the operator that multiplies by the coordinate x in $L^2([0,1], dx)$, where dx is the Lebesgue measure on $\mathbb{R}$.

8.13 Take Volterra's operator $A : L^2([0,1], dx) \to L^2([0,1], dx)$:

$$(Af)(x) = \int_0^x f(t)dt \; .$$

Study its spectrum and prove $\sigma(A) = \sigma_c(A) = \{0\}$. Conclude, without computations, that A cannot be normal.

Solution. Since $[0,1]$ has finite Lebesgue measure, then $L^2([0,1], dx) \subset L^1([0,1], dx)$, and we can view Lebesgue's integral as a function of the upper limit of integration (in particular, Theorem 1.76 holds). Notice the spectrum of A cannot be empty by Theorem 8.4, since A is bounded and hence closed. If $\lambda \neq 0$, then $(\lambda^{-1}A)^n$ is a contraction operator for n large enough, as we saw in Exercise 4.19. By the fixed-point theorem $\lambda \neq 0$ cannot be an eigenvalue, since the unique solution ψ to the characteristic equation $\lambda^{-1}A\psi = \psi$ is $\psi = \mathbf{0}$, which is not an eigenvector. As A is compact, Lemma 4.52 guarantees that if $0 \neq \lambda$ (hence $\lambda \notin \sigma_p(A)$), then $Ran(A - \lambda I) = \mathsf{H}$ (i.e. the Hilbert space $L^2([0,1], dx)$). Moreover, since $A - \lambda I$ is bijective, $(A - \lambda I)^{-1} : \mathsf{H} \to \mathsf{H}$ is bounded by the inverse-operator theorem. Therefore $\lambda \notin \sigma(A)$ if $\lambda \neq 0$. So the unique point in the spectrum is $\lambda = 0$. By Theorem 1.76(b) there are no non-zero solutions to $A\psi = \mathbf{0}$, and we conclude $0 \in \sigma_r(A) \cup \sigma_c(A)$. If 0 were in $\sigma_r(A)$, $Ran(A)$ would not be dense in $L^2([0,1], dx)$, i.e. $Ker(A^*) \neq \{0\}$ because $\mathsf{H} = \overline{Ran(A)} \oplus Ker(A^*)$. This is not possible, because $(A^* f)(x) = \int_x^1 f(t)dt$ (see Exercise 3.29), so applying Theorem 1.76(b) would give a contradiction.

If A were normal, its boundedness would imply $||A|| = r(A)$. But $r(A) = 0$, for $\sigma(A) = \{0\}$. Therefore A would be forced to be null.

8.14 Consider the bounded, self-adjoint operator T on $\mathsf{H} := L^2([0,1], dx)$ that multiplies functions by x^2:

$$(Tf)(x) := x^2 f(x) \; .$$

Find its spectral measure.

Hint. Find a unitary transformation $\mathsf{H} \to L^2([0,1], dy)$ that maps the multiplication by x^2 to the multiplication by y.

8.15 Consider the bounded, self-adjoint operator T on $\mathsf{H} := L^2([-1,1], dx)$ that multiplies by x^2:

$$(Tf)(x) := x^2 f(x) \; .$$

Determine its spectral measure.

Hint. Argue as in Exercise 8.14, after decomposing

$$L^2([-1,1], dx) = L^2([-1,0], dx) \oplus L^2([0,1], dx) \; .$$

8.16 Let $T \in \mathfrak{B}(\mathsf{H})$ be a normal operator on a Hilbert space H. Prove, for any $\alpha \in \mathbb{C}$, that

$$e^{\alpha T} = \int_{\sigma(T)} e^{\alpha(x+iy)}\, dP^{(T)}(x, y)\,,$$

where the term on the left is defined, in the *uniform topology*, as

$$e^{\alpha T} := \sum_{n=0}^{+\infty} \frac{\alpha^n T^n}{n!}\,.$$

Hint. The series $\sum_{n=0}^{+\infty} \frac{\alpha^n z^n}{n!}$ converges absolutely and uniformly on any closed disc of finite radius and centred at the origin of $\mathbb{C}$. Moreover, for any polynomial $p(z)$ $(z = x + iy)$,

$$p(T) = \int_{\sigma(T)} p(x + iy)\, dP^{(T)}(x, y)\,.$$

Now use the first part of Theorem 8.52.

8.17 For any given Hilbert space H, build a compact self-adjoint operator $T : \mathsf{H} \to \mathsf{H}$ such that $T \notin \mathfrak{B}_1(\mathsf{H})$, $T \notin \mathfrak{B}_2(\mathsf{H})$.

Hint. It suffices to show $\sum_{\lambda\in\sigma_p(T)} |\lambda| = +\infty$ and $\sum_{\lambda\in\sigma_p(T)} |\lambda|^2 = +\infty$, see Exercise 4.4.

8.18 Take $T \in \mathfrak{B}(\mathsf{H})$ with $T \geq 0$ and H a Hilbert space. Prove that if T is compact then

$$T^\alpha := \int_{\sigma(T)} \lambda^\alpha dP^{(T)}(\lambda)$$

is compact for any real $\alpha > 0$.

Outline of solution. If $\sigma(T)$ is finite the claim is obvious by the spectral theorem and because operators with finite-dimensional range are compact. Consider the other case. Expand T spectrally: $T = \sum_j \lambda_j(\psi_j|\ \ )\psi_j$, where $||T|| \geq \lambda_j \geq \lambda_{j+1} \to 0_+$ by compactness, and for any given j, $\lambda_{j+k} = \lambda_j$ only for a finite number of k (if the eigenvalue is non-null). Recall that for compact operators the expansion converges in the uniform topology, too. If $\alpha \geq 1$ and m, n are large enough, then $\sum_{j=n}^m \lambda_j^\alpha(\psi_j|\ \ )\psi_j \leq \sum_{j=n}^m \lambda_j(\psi_j|\ \ )\psi_j$, hence (positive operators are self-adjoint) $||\sum_{j=n}^m \lambda_j^\alpha(\psi_j|)\psi_j|| \leq ||\sum_{j=n}^m \lambda_j(\psi_j|)\psi_j||$. Bearing in mind the spectral decomposition theorem for the self-adjoint operator T^α, conclude that T^α, $\alpha > 1$, is compact as uniform limit of compact operators (ranges are finite-dimensional). When $\alpha < 1$, observe $||T^{1/2}x - T^{1/2}y||^2 = ((x - y)|T(x - y)) \leq ||x - y||||Tx - Ty||$, then conclude that $T^{1/2}$ is compact if T is compact, by using the definition of compact operator. When $\alpha \in [1/2, 1)$, $T^\alpha = (T^{1/2})^\beta$ for some $\beta \in [1, 2)$. Relying on the previous proof recover that T^α is compact if T is when $\alpha \in [1/2, 1)$. Iterating the procedure obtain that $T^{1/4}$ and T^α, $\alpha \in [1/4, 1/2)$, are compact if T is, and so on; hence reach any T^α, with $\alpha \in (0, 1)$, because in that case $\alpha \in [1/2^{k+1}, 1/2^k)$ for some $k = 0, 1, 2, \ldots$.

8.19 Prove that $T = T^* \in \mathfrak{B}(\mathsf{H})$ is positive if and only if $\sigma(T) \subset [0, +\infty)$

Hint. Use $(x|Tx) = \int_{\sigma(T)} \lambda d\mu_x(\lambda)$. If $\sigma(T) \ni \lambda_0 < 0$ then either $\lambda_0 \in \sigma_p(T)$ so $P^{(T)}(\{\lambda_0\}) \neq 0$, or $\lambda_0 \in \sigma_c(T)$ and then $P^{(T)}((\lambda_0 - \delta, \lambda_0 + \delta)) \neq 0$. In both cases find $x \in \mathsf{H}$ such that $\int_{\sigma(T)} \lambda d\mu_x(\lambda) < 0$.

8.20 If H is a Hilbert space, prove $T \in \mathfrak{B}(\mathsf{H})$ is of trace class ($T \in \mathfrak{B}_1(\mathsf{H})$) $\Leftrightarrow$ $\sum_{u \in N} |(u|Tu)| < +\infty$ for every Hilbert basis $N \subset \mathsf{H}$.

Solution. Suppose $\sum_{u \in N} |(u|Tu)| < +\infty$ for every Hilbert basis N. Assume, first, $T = T^*$. By the spectral theorem $T = \int_{\mathbb{R}} \lambda dP^{(T)}(\lambda)$. Define $T_- := \int_{(-\infty,0)} \lambda dP^{(T)}(\lambda)$ and $T_+ := \int_{[0,+\infty)} \lambda dP^{(T)}(\lambda)$. Clearly $T_\pm \in \mathfrak{B}(\mathsf{H})$ ($|\lambda \chi_{(-\infty,0)}| \leq |\lambda|$, $|\lambda \chi_{[0,+\infty)}| \leq |\lambda|$ and λ is bounded on the support of $P^{(T)}$ by the spectral theorem for self-adjoint bounded operators). Moreover $\pm T_\pm \geq 0$ by exercise 8.19. Furthermore, $\mathsf{H} = \mathsf{H}_- \oplus \mathsf{H}_+$ is a closed orthogonal sum where $\mathsf{H}_- := P^{(T)}((-\infty, 0))\mathsf{H}$, $\mathsf{H}_+ := P^{(T)}([0, +\infty))\mathsf{H}$. Let $N_- \subset \mathsf{H}_-$, $N_+ \subset \mathsf{H}_+$ be Hilbert bases, so $N := N_- \cup N_+$ is a Hilbert basis of H. As $-T_-, T_+ \geq 0$ and $T_\pm u = 0$ if $u \in \mathsf{H}_\mp$, we have

$$+\infty > \sum_{u \in N} |(u|Tu)| = \sum_{u \in N_-} |(u|T_- u)| + \sum_{u \in N_+} |(u|T_+ u)| = \sum_{u \in N_-} -(u|T_- u) + \sum_{u \in N_-} (u|T_+ u)$$

$$= \sum_{u \in N_-} (u||T_-|u) + \sum_{u \in N_-} (u||T_+|u) = \sum_{u \in N} (u||T_-|u) + \sum_{u \in N} (u||T_+|u)\,.$$

Therefore $T_\pm \in \mathfrak{B}_1(\mathsf{H})$ by Definition 4.32, so $T = T_+ + T_- \in \mathfrak{B}_1(\mathsf{H})$ by Theorem 4.34(b). In case T is not self-adjoint, we can decompose $T = A + iB$, with $A := \frac{1}{2}(T + T^*)$, $B := \frac{1}{2i}(T - T^*)$, A, B self-adjoint. For any given basis $N \subset \mathsf{H}$, $|(u|Tu)| = |(u|Au) + i(u|Bu)| = \sqrt{|(u|Au)|^2 + |(u|Bu)|^2} \geq |(u|Au)|\,, |(u|Bu)|$, with $u \in N$. Applying the result proved above for self-adjoint operators gives $A, B \in \mathfrak{B}_1(\mathsf{H})$, so $T \in \mathfrak{B}_1(\mathsf{H})$. If, instead, $T \in \mathfrak{B}_1(\mathsf{H})$, for any basis $\sum_{u \in N} |(u|Tu)| < +\infty$ by Proposition 4.36.

Chapter 9
Spectral Theory II: Unbounded Operators on Hilbert Spaces

The language of mathematics turns out to be unreasonably effective in natural sciences, a wonderful gift that we don't understand nor deserve.

Eugene Paul Wigner

In this second chapter on spectral theory we shall examine a number of mathematical issues concerning typically unbounded self-adjoint operators.

The first section is devoted to extending the spectral decomposition theorem of the previous chapter to unbounded self-adjoint operators. The proof relies on a generalisation of the integration procedure for spectral measures to *unbounded* functions, and on the Cayley transform. The resulting technique will also enable us to prove, in passing, an important characterisation of the von Neumann algebra generated by a bounded normal operator and its adjoint. Then we will describe two physically-relevant examples of unbounded self-adjoint operators and their spectral decomposition: the Hamiltonian of the harmonic oscillator, and the position and momentum operators. Finally we will state a spectral representation theorem for unbounded self-adjoint operators and introduce *joint spectral measures*.

The second, very short, section is dedicated to exponentiating unbounded operators, in relationship to earlier-defined analytic vectors.

In section three we will focus on the theory of strongly continuous one-parameter groups of unitary operators. First we will establish that the various notions of continuity are equivalent. Next we will show *von Neumann's theorem* on the continuity of measurable one-parameter groups of unitary operators, and then go on to prove *Stone's theorem* and a few important corollaries. In particular, we will discuss a useful criterion, based on one-parameter unitary groups generated by self-adjoint operators, to establish when the spectral measures of two self-adjoint operators commute. We recommend [Schm12] for a recent and quite complete treatise on the whole subject.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_9

9.1 Spectral Theorem for Unbounded Self-adjoint Operators

We now set out to generalise some of the material of Chap. 8. In particular we want to prove the spectral decomposition theorem in the case of unbounded self-adjoint operators. The physical relevance lies in that most self-adjoint operators representing interesting observables in Quantum Mechanics are *unbounded*. The paradigmatic case is the position operator of Chap. 5.

9.1.1 *Integrating Unbounded Functions with Respect to Spectral Measures*

We will often use the following natural definition.

Definition 9.1 Let X be a complex vector space, T an operator on X with domain $D(T)$ and $p(x) = \sum_{k=0}^{m} a_k x^k$ a polynomial of degree m with complex coefficients.
(a) The operator $p(T)$ on X is defined by writing T in place of the variable x, with $T^0 := I$, $T^1 := T$, $T^2 := TT$, and so on.
(b) The domain of $p(T)$ is

$$D(p(T)) := \bigcap_{k=0}^{m} D(a_k T^k) , \tag{9.1}$$

with $D(a_k T \cdots T)$ given in Definition 5.1.

Extending the previous chapter's results to unbounded operators requires first a definition for the integral of *unbounded* functions with respect to a PVM. If P is a spectral measure on the measurable space $(\mathsf{X}, \Sigma(\mathsf{X}))$, in the sense of Definition 8.41, and if $f : \mathsf{X} \to \mathbb{C}$ is a measurable function (for the σ-algebra $\Sigma(\mathsf{X})$), but not necessarily bounded, based on what we know $\int_{\mathsf{X}} f(x) dP(x)$ is meaningless. The point is to make sense of this integral.

Consider a vector ψ, in a Hilbert space H, of the projector-valued measure P such that

$$\int_{\mathsf{X}} |f(x)|^2 \, d\mu_\psi(x) < +\infty , \tag{9.2}$$

where the spectral measure μ_ψ for ψ was defined in Theorem 8.52(c). We can find a sequence of bounded measurable maps f_n such that, for every fixed $\psi \in \mathsf{H}$, we have $f_n \to f$ as $n \to +\infty$ in $L^2(\mathsf{X}, \mu_\psi)$. For example, using Lebesgue's dominated convergence it suffices to consider $f_n := \chi_{F_n} \cdot f$, where $\{F_n\}_{n\in\mathbb{N}}$ is any family of Borel subsets of X such that

$$\cup_{m\in\mathbb{N}} F_m = \mathsf{X}\,, \quad F_{n+1} \supset F_n\,, \quad |f(x)| < C_n \quad \forall x \in F_n, \text{for some } C_n \in \mathbb{R}, \forall n \in \mathbb{N}. \tag{9.3}$$

A simple choice is

$$F_n := \{x \in \mathsf{X} \mid |f(x)| < n\}\,. \tag{9.4}$$

Using (ii) in (a) and (b) of Theorem 8.54 we immediately find

$$\left|\left|\int_{\mathsf{X}} f_n(x)dP(x)\psi - \int_{\mathsf{X}} f_m(x)dP(x)\psi\right|\right|^2 = \int_{\mathsf{X}} |f_n(x) - f_m(x)|^2 d\mu_\psi(x)\,. \tag{9.5}$$

Therefore the sequence of vectors $\int_{\mathsf{X}} f_n(x)dP(x)\psi$ converges to some $\int_{\mathsf{X}} f(x)dP(x)\psi$:

$$\int_{\mathsf{X}} f(x)dP(x)\psi := \lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x)dP(x)\psi\,. \tag{9.6}$$

We may use (9.6) as *definition* of the integral in P of the unbounded measurable function f. This is *well defined* since $\int_{\mathsf{X}} f(x)dP(x)\psi$ *does not* depend on the sequence $\{f_n\}_{n\in\mathbb{N}}$. In fact if $\{g_n\}_{n\in\mathbb{N}}$ is another sequence of bounded measurable maps converging to f in $L^2(\mathsf{X}, \mu_\psi)$, proceeding as before we obtain

$$\left|\left|\int_{\mathsf{X}} f_n(x)dP(x)\psi - \int_{\mathsf{X}} g_n(x)dP(x)\psi\right|\right|^2 = \int_{\mathsf{X}} |f_n(x) - g_n(x)|^2 d\mu_\psi(x)\,,$$

so

$$\lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x)dP(x)\psi = \lim_{n\to+\infty} \int_{\mathsf{X}} g_n(x)dP(x)\psi\,.$$

If we use (9.6) to define the integral of an unbounded function we must remember that this operator is not defined on the whole Hilbert space, but only on vectors satisfying (9.2). Consequently we have to check that these vectors form a subspace in H. To show this, and much more, we need to a lemma that relates the spectral measure μ_ψ to $\mu_{\phi,\psi}$ via (9.2), for $\psi \in \mathsf{H}$.

Lemma 9.2 *Let $(\mathsf{X}, \Sigma(\mathsf{X}))$ be a measurable space, H a Hilbert space, $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ a PVM, and $f : \mathsf{X} \to \mathbb{C}$ a measurable function.*

Given $\phi, \psi \in \mathsf{H}$, if the measures μ_ψ and $\mu_{\phi,\psi}$ are defined as in Theorem 8.52(c) and

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) < +\infty\,,$$

then $f \in L^1(\mathsf{X}, |\mu_{\phi,\psi}|)$ and

$$\int_{\mathsf{X}} |f(x)|d|\mu_{\phi,\psi}(x)| \leq ||\phi|| \sqrt{\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x)}\,. \tag{9.7}$$

Proof If f is bounded, by (iv) in Theorem 8.52(c):

$$\left(\phi \left| \int_{\mathsf{X}} |f(x)| dP(x)\psi \right.\right) = \int_{\mathsf{X}} |f(x)| d\mu_{\phi,\psi}(x)$$

By Theorem 1.87 there exists a map $h : \mathsf{X} \to \mathbb{C}$, $|h(x)| = 1$, such that $d\mu_{\phi,\psi} = hd|\mu_{\phi,\psi}|$, and so

$$\int_{\mathsf{X}} |f(x)| d|\mu_{\phi,\psi}(x)| = \int_{\mathsf{X}} |f(x)| h^{-1}(x) d\mu_{\phi,\psi}(x) = \left(\phi \left| \int_{\mathsf{X}} |f(x)| h^{-1}(x) dP(x)\psi \right.\right) .$$

Using Theorem 8.54(b) and noting $||f(x)|h^{-1}(x)|^2 = |f(x)|^2$, we have

$$\int_{\mathsf{X}} |f(x)| d|\mu_{\phi,\psi}(x)| \leq ||\phi|| \left|\left| \int_{\mathsf{X}} |f(x)| h^{-1}(x) dP(x)\psi \right|\right| = ||\phi|| \sqrt{\int_{\mathsf{X}} |f(x)|^2 d\mu_{\psi}(x)} .$$

Let now f be unbounded. Define bounded maps $f_n := \chi_{F_n} \cdot f$ as above, so that $0 \leq |f_n(x)| \leq |f_{n+1}(x)| \to |f(x)|$ as $n \to +\infty$. By monotone convergence, since $f \in L^2(\mathsf{X}, d\mu_\psi)$, we obtain

$$\int_{\mathsf{X}} |f(x)| d|\mu_{\phi,\psi}(x)| = \lim_{n\to+\infty} \int_{\mathsf{X}} |f_n(x)| d|\mu_{\phi,\psi}(x)| \leq ||\phi|| \lim_{n\to+\infty} \sqrt{\int_{\mathsf{X}} |f_n(x)|^2 d\mu_{\psi}(x)}$$

$$= ||\phi|| \sqrt{\int_{\mathsf{X}} |f(x)|^2 d\mu_{\psi}(x)} < +\infty .$$

This proves the general case. $\square$

The next theorem gathers several technical facts seen above, and establishes the first general properties of integrals of unbounded maps with respect to a spectral measure.

Theorem 9.3 *Let* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *be a measurable space,* H *a Hilbert space and* $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ *a PVM.*

For any measurable $f : \mathsf{X} \to \mathbb{C}$ *define*

$$\Delta_f := \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\mathsf{X}} |f(x)|^2 \, d\mu_\psi(x) < +\infty \right\} . \tag{9.8}$$

(a) Δ_f *is a dense subspace in* H.
(b) *If* $f \in L^2(\mathsf{X}, \mu_\psi)$*, the mapping*

$$\int_{\mathsf{X}} f(x) dP(x) : \; \Delta_f \ni \psi \mapsto \int_{\mathsf{X}} f(x) dP(x)\psi \tag{9.9}$$

(with right-hand-side term as in (9.6), and using any sequence of bounded measurable maps $\{f_n\}_{n\in\mathbb{N}}$ converging to f in $L^2(\mathsf{X}, \mu_\psi)$):

(i) is a linear operator;

(ii) extends the integral operator of measurable bounded functions $f \in M_b(\mathsf{X})$ of Definition 8.49(b).

(c) *Assume that X is a topological space, $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ and at least one of the hypotheses (1) and (2) in Proposition 8.44(d) is valid, so that P is concentrated on $supp(P)$. If $f\restriction_{supp(P)}$ is bounded, then:*

$$\Delta_f = \mathsf{H} \quad and \quad \int_{\mathsf{X}} f(x)dP(x) = \int_{supp(P)} f(x)dP(x) \in \mathfrak{B}(\mathsf{H}) \,,$$

where the right side is the operator of Definition 8.49(c).

Proof (a)–(b). As first thing we have to prove, for any given measurable $f : \mathsf{X} \to \mathbb{C}$, that $\phi + \psi \in \Delta_f$ and $c\phi \in \Delta_f$ for any $c \in \mathbb{C}$ if $\phi, \psi \in \Delta_f$. Note Δ_f contains the null vector of H, so it is non-empty.

If $\phi, \psi \in \Delta_f$, $E \in \Sigma(\mathsf{X})$:

$$||P_E(\phi + \psi)||^2 \le (||P_E\phi|| + ||P_E\psi||)^2 \le 2||P_E\phi||^2 + 2||P_E\psi||^2 \,;$$

since $\mu_\chi(E) = (\chi|P_E\chi) = (\chi|P_E P_E\chi) = (P_E\chi|P_E\chi) = ||P_E\chi||^2$:

$$\mu_{\phi+\psi}(E) \le 2(\mu_\phi(E) + \mu_\psi(E)) \,.$$

This implies, for $L^2(\mathsf{X}, \mu_\phi) \ni f$ and $L^2(\mathsf{X}, \mu_\psi) \ni f$, that $L^2(\mathsf{X}, \mu_{\phi+\psi}) \ni f$. That is to say, $\phi, \psi \in \Delta_f \Rightarrow \phi + \psi \in \Delta_f$. On the other hand $\mu_{c\phi}(E) = |c|^2(P_E\phi|\phi) = |c|^2\mu_\phi(E)$, so $f \in L^2(\mathsf{X}, \mu_{c\phi})$ for $f \in L^2(\mathsf{X}, \mu_\phi)$ and $c \in \mathbb{C}$. That is to say, $\phi \in \Delta_f \Rightarrow c\phi \in \Delta_f$, so Δ_f is a subspace. That $\int_{\mathsf{X}} f(x)dP(x) : \Delta_f \ni \psi \mapsto \int_{\mathsf{X}} f(x)dP(x)\psi$ is linear is a consequence of the definition of $\int_{\mathsf{X}} f(x)dP(x)\psi$ and of the linearity of the integral of a bounded map in a PVM.

Now we show Δ_f is dense in H. Given f as in the statement, let:

$$E_n := \{x \in \mathsf{X} \mid n - 1 \le |f(x)| < n\} \,, \quad \text{for any } n \in \mathbb{N}, n \ge 1.$$

Note $E_n \cap E_m = \varnothing$ if $n \ne m$ and $\cup_n E_n = \mathsf{X}$. By Definition 8.41 the closed subspaces $\mathsf{H}_n := P(E_n)\mathsf{H}$ are mutually orthogonal, and by propriety (d) of the same definition finite combinations over the H_n form a dense space inside H. We claim Δ_f contains this subspace. By monotone convergence, if $\psi \in \mathsf{H}$:

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) = \lim_{k\to+\infty} \sum_{n=1}^{k} \int_{\mathsf{X}} |\chi_{E_n}(x) f(x)|^2 d\mu_\psi(x) \le +\infty \,. \tag{9.10}$$

The integral inside the sum can be written as follows, using Theorem 8.54(b):

$$\left(\int_{\mathsf{X}} \chi_{E_n}(x) f(x) dP(x)\psi \,\middle|\, \int_{\mathsf{X}} \chi_{E_n}(x) f(x) dP(x)\psi \right) .$$

But since $x \mapsto \chi_{E_n}(x) f(x)$ is bounded and $\chi_{E_n} = \chi_{E_n} \cdot \chi_{E_n}$, using (iii) in Theorem 8.54(a) gives

$$\int_{\mathsf{X}} \chi_{E_n}(x) f(x) dP(x)\psi = \int_{\mathsf{X}} \chi_{E_n}(x) f(x) dP(x) \int_{\mathsf{X}} \chi_{E_n}(x) dP(x)\psi$$

$$= \int_{\mathsf{X}} \chi_{E_n}(x) f(x) dP(x) \circ P(E_n)\psi .$$

If $\psi \in \mathsf{H}_n$, then, as projectors $P(E_m)$ are orthogonal,

$$\int_{\mathsf{X}} \chi_{E_k}(x) f(x) dP(x)\psi = 0, \quad \text{for } k \neq n .$$

Under the assumptions on ψ, therefore, the series of (9.10) becomes

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) = \int_{\mathsf{X}} |\chi_{E_n}(x) f(x)|^2 d\mu_\psi(x) \leq \int_{\mathsf{X}} n^2 d\mu_\psi(x) = n^2 ||\psi||^2 < +\infty .$$

We conclude $\mathsf{H}_n \subset \Delta_f$, for any $n = 1, 2, \ldots$. But Δ_f is a subspace so it contains also the dense space of finite combinations of the H_n. Hence Δ_f itself is dense.

The new definition of integral extends the old one for functions in $M_b(\mathsf{X})$ of Definition 8.49(b). For if the sequence $M_b(\mathsf{X}) \ni f_n \to f \in M_b(\mathsf{X})$ in (9.6) is chosen as $f_n = \chi_{F_n} \cdot f$ with F_n defined in (9.4), then this sequence is bounded and converges pointwise to f. Hence, Theorem 8.54(c) implies that the limit (9.6) defining the new notion of integral coincides with that of Definition 8.49(b).

(c) Given $f : \mathsf{X} \to \mathbb{C}$, let F_n be defined as in (9.4). Suppose $f \restriction_{supp(P)}$ is bounded. Define *bounded* measurable maps $f_n := \chi_{supp(P)} \cdot f + g_n$ where $g_n = \chi_{F_n} \cdot \chi_{\mathsf{X} \setminus supp(P)} \cdot f$. Since $supp(\mu_\psi) \subset supp(P)$ by Theorem 8.52(v), for any $\psi \in \mathsf{H}$ we have $f \in L^2(\mathsf{X}, \mu_\psi)$, hence $\Delta_f = \mathsf{H}$ because μ_f is finite, and:

$$\int_{\mathsf{X}} |f_n(x) - f(x)|^2 d\mu_\psi(x) = \int_{supp(P)} |f(x) - f(x)|^2 d\mu_\psi(x) = 0 .$$

Consequently $f_n \to f$ in $L^2(\mathsf{X}, \mu_\psi)$ for any $\psi \in \mathsf{H}$, so:

$$\int_{\mathsf{X}} f(x) dP(x)\psi := \lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x) dP(x)\psi = \lim_{n\to+\infty} \int_{\mathsf{X}} \chi_{supp(P)} f_n(x) dP(x)\psi$$

$$= \lim_{n\to+\infty} \int_{\mathsf{X}} \chi_{supp(P)} f(x) dP(x)\psi = \int_{\mathsf{X}} \chi_{supp(P)} f(x) dP(x)\psi =: \int_{supp(P)} f(x) dP(x)\psi ,$$

where the last integral is meant as in Definition 8.49(c) (first case), so the operator $\int_{supp(P)} f(x)dP(x)$ belongs in $\mathfrak{B}(\mathsf{H})$. $\square$

Now a result that deals, in particular, with composites of integrals of spectral measures, and characterises in a *very precise* way the corresponding domains.

Theorem 9.4 *Let* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *be a measurable space,* H *a Hilbert space and* $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ *a PVM. For any measurable* $f : \mathsf{X} \to \mathbb{C}$*, in the same notation of Theorem 9.3:*
(a) $\int_{\mathsf{X}} f(x)dP(x) : \Delta_f \to \mathsf{H}$ *is a closed operator;*
(b) $\int_{\mathsf{X}} f(x)dP(x)$ *is self-adjoint if* f *is real, and more generally:*

$$\left(\int_{\mathsf{X}} f(x)dP(x)\right)^* = \int_{\mathsf{X}} \overline{f(x)}dP(x) \quad and \quad \Delta_{\overline{f}} = \Delta_f \,. \tag{9.11}$$

(c) *Suppose* $f : \mathsf{X} \to \mathbb{C}$*,* $g : \mathsf{X} \to \mathbb{C}$ *are measurable,* D *is the standard domain (Definition 5.1) and* $f \cdot g$ *denotes the pointwise product. Then*

$$\int_{\mathsf{X}} f(x)dP(x) + \int_{\mathsf{X}} g(x)dP(x) \subset \int_{\mathsf{X}} (f+g)(x)dP(x) \tag{9.12}$$

$$D\left(\int_{\mathsf{X}} f(x)dP(x) + \int_{\mathsf{X}} g(x)dP(x)\right) = \Delta_f \cap \Delta_g \tag{9.13}$$

with equality in (9.12) $\Leftrightarrow$ $\Delta_{f+g} = \Delta_f \cap \Delta_g$*;*

$$\int_{\mathsf{X}} f(x)dP(x) \int_{\mathsf{X}} g(x)dP(x) \subset \int_{\mathsf{X}} (f \cdot g)(x)dP(x) \tag{9.14}$$

$$D\left(\int_{\mathsf{X}} f(x)dP(x) \int_{\mathsf{X}} g(x)dP(x)\right) = \Delta_{f\cdot g} \cap \Delta_g \tag{9.15}$$

with equality in (9.14) $\Leftrightarrow$ $\Delta_{f\cdot g} \subset \Delta_g$*. In particular:*

$$\int_{\mathsf{X}} \overline{f(x)}dP(x) \int_{\mathsf{X}} f(x)dP(x) = \int_{\mathsf{X}} |f(x)|^2 dP(x) \tag{9.16}$$

$$D\left(\int_{\mathsf{X}} \overline{f(x)}dP(x) \int_{\mathsf{X}} f(x)dP(x)\right) = \Delta_{|f|^2} \,. \tag{9.17}$$

Moreover

$$\left(\int_{\mathsf{X}} f(x)dP(x) \int_{\mathsf{X}} g(x)dP(x)\right) \restriction_{\Delta_f \cap \Delta_g \cap \Delta_{f\cdot g}} = \left(\int_{\mathsf{X}} g(x)dP(x) \int_{\mathsf{X}} f(x)dP(x)\right) \restriction_{\Delta_f \cap \Delta_g \cap \Delta_{f\cdot g}} . \tag{9.18}$$

Eventually, if f is bounded on $E \in \Sigma(\mathsf{X})$, then $\Delta_{\chi_E \cdot f} = \mathsf{H}$ and

$$\int_{\mathsf{X}} \chi_{E(x)} dP(x) \int_{\mathsf{X}} f(x) dP(x) \subset \int_{\mathsf{X}} f(x) dP(x) \int_{\mathsf{X}} \chi_{E(x)} dP(x) = \int_{\mathsf{X}} (\chi_E \cdot f)(x) dP(x) \in \mathfrak{B}(\mathsf{H}). \tag{9.19}$$

(d) *If $\mathsf{X} = \mathbb{R}$, $\Sigma(\mathbb{R}) := \mathscr{B}(\mathbb{R})$, $p : \mathbb{R} \to \mathbb{C}$ is a polynomial of degree $m \in \mathbb{N}$, and $T := \int_{\mathbb{R}} x dP(x)$, then*

$$p(T) = \int_{\mathbb{R}} p(x) dP(x) \quad \text{and} \quad D(p(T)) = D(T^m) = \Delta_p \,. \tag{9.20}$$

(e) *Defining $\mu_{\phi,\psi}$ as in Theorem 8.52(c), $\int_{\mathsf{X}} f(x) dP(x)$ is the unique operator on H with domain Δ_f such that, for any $\psi \in \Delta_f$, $\phi \in \mathsf{H}$:*

$$\left(\phi \middle| \int_{\mathsf{X}} f(x) dP(x) \psi \right) = \int_{\mathsf{X}} f(x) d\mu_{\phi,\psi}(x) \,. \tag{9.21}$$

(f) *For any $\psi \in \Delta_f$:*

$$\left|\left| \int_{\mathsf{X}} f(x) dP(x) \psi \right|\right|^2 = \int_{\mathsf{X}} |f(x)|^2 d\mu_{\psi}(x) \,. \tag{9.22}$$

(g) *Every operator $\int_{\mathsf{X}} f(x) dP(x)$ is positive when f is positive:*

$$\left(\psi \middle| \int_{\mathsf{X}} f(x) dP(x) \psi \right) \geq 0 \quad \text{for any } \psi \in \Delta_f, \text{ if } f(x) \geq 0, x \in \mathsf{X}. \tag{9.23}$$

(h) *If $(\mathsf{X}', \Sigma'(\mathsf{X}'))$ is a measurable space and $\phi : \mathsf{X} \to \mathsf{X}'$ is measurable (i.e. $\phi^{-1}(E') \in \Sigma(\mathsf{X})$ for $E' \in \Sigma'(\mathsf{X}')$), then*

$$\Sigma'(\mathsf{X}') \ni E' \mapsto P'(E') := P(\phi^{-1}(E'))$$

is a PVM on X', and for any measurable map $f : \mathsf{X}' \to \mathbb{C}$:

$$\int_{\mathsf{X}'} f(x') dP'(x') = \int_{\mathsf{X}} (f \circ \phi)(x) dP(x) \quad \text{and} \quad \Delta'_f = \Delta_{f \circ \phi} \,, \tag{9.24}$$

where Δ'_f is the domain of the operator on the left.

Proof (a) As a preliminary result, we observe that, defining the sets F_k as in (9.3), we have

$$P(F_n)\psi \to \psi \quad \text{as } n \to +\infty, \text{ for every } \psi \in \mathsf{H}. \tag{9.25}$$

To prove it, define $E_0 = F_0$ and $E_n := F_n \setminus F_{n-1}$ so that: (i) $E_k \cap E_h = \varnothing$ for $h \neq k$, (ii) $F_n = \cup_{j=0}^n E_j$, and (iii) $\cup_{j=0}^\infty E_n = \mathsf{X}$. Then, the general properties of a PVM yield:

$$P(F_n)\psi = \sum_{j=0}^{n} P(E_j)\psi \to P\left(\cup_{j\in\mathbb{N}} E_j\right)\psi = \psi \quad \text{as } n \to +\infty, \text{ for every } \psi \in \mathsf{H}. \tag{9.26}$$

We claim $T := \int_{\mathsf{X}} f(x)dP(x)$, defined on Δ_f, is closed. Notice, first, that the bounded operators

$$T_k := \int_{\mathsf{X}} \chi_{F_k}(x) f(x) dP(x) , \tag{9.27}$$

for $\psi \in \Delta_f$, are such that: (1) $TP_{F_k}\psi = P_{F_k}T\psi = T_k\psi$ and (2) $T_k\psi \to T\psi$, $k \to +\infty$. The proof of (1) is similar to Theorem 9.3(c), whilst (2) follows from the argument preceding Lemma 9.2. So let $\{\psi_n\}_{n\in\mathbb{N}} \subset \Delta_f$ be such that $\psi_n \to \psi \in \mathsf{H}$ and $T\psi_n \to \phi$, $n \to +\infty$. We claim $\psi \in \Delta_f$ and $T\psi = \phi$, implying the closure of T. By (1) and because $P_{F_k} \to I$ strongly as $k \to +\infty$ for (9.25):

$$T_k\psi = \lim_{n\to+\infty} T_k\psi_n = \lim_{n\to+\infty} P_{F_k}T\psi_n = P_{F_k}\phi \to \phi \quad \text{in } \mathsf{H} \text{ as } k \to +\infty.$$

Define $\phi_k := T_k\psi$; then

$$\int_{\mathsf{X}} \chi_{F_k}(x) f(x) dP(x)\psi = \phi_k \to \phi \quad \text{in } \mathsf{H} \text{ as } k \to +\infty. \tag{9.28}$$

By Theorem 8.54(b):

$$\int_{\mathsf{X}} \chi_{F_k}(x)|f(x)|^2 d\mu_\psi(x) = ||\phi_k||^2 \to ||\phi||^2 < +\infty \quad \text{as } n \to +\infty.$$

Monotone convergence ensures $f \in L^2(\mathsf{X}, \mu_\psi)$, i.e. $\psi \in \Delta_f$. Rewriting (9.28) as $T_k\psi = \phi_k$, and taking the limit as $k \to +\infty$ using (2), gives $T\psi = \phi$, as required.
(b) $\Delta_f = \Delta_{\overline{f}}$ is an obvious consequence of the definition of Δ_f and $|f| = |\overline{f}|$. We will show $\int_{\mathsf{X}} \overline{f(x)}dP(x) \subset \left(\int_{\mathsf{X}} f(x)dP(x)\right)^*$. If $\psi \in \Delta_{\overline{f}}$, $\phi \in \Delta_f$ and $f_n \to f$ in $L^2(\mathsf{X}, \mu_\phi)$ so $\overline{f_n} \to \overline{f}$ in $L^2(\mathsf{X}, \mu_\psi)$, where f_n are bounded, we have:

$$\left(\psi \middle| \int_{\mathsf{X}} f(x)dP(x)\phi\right) = \lim_{n\to+\infty} \left(\psi \middle| \int_{\mathsf{X}} f_n(x)dP(x)\phi\right)$$

$$= \lim_{n\to+\infty} \left(\int_{\mathsf{X}} \overline{f_n(x)}dP(x)\psi \middle| \phi\right) = \left(\int_{\mathsf{X}} \overline{f(x)}dP(x)\psi \middle| \phi\right)$$

where we used the definition of integral of f and $\overline{f}$ in P, plus property (iv) in Theorem 8.54(a). This means $\int_{\mathsf{X}} \overline{f(x)}dP(x) \subset \left(\int_{\mathsf{X}} f(x)dP(x)\right)^*$. We will prove $\int_{\mathsf{X}} \overline{f(x)}dP(x) \supset \left(\int_{\mathsf{X}} f(x)dP(x)\right)^*$ by showing $D\left(\left(\int_{\mathsf{X}} f(x)dP(x)\right)^*\right) \subset \Delta_{\overline{f}}$. Let

$T := \int_{\mathsf{X}} \overline{f(x)} dP(x)$ and take the bounded operators T_k of (9.27). Fix $\psi \in D(T^*)$. Then there exists $h \in \mathsf{H}$ such that, for any $\phi \in \Delta_f$, $(\psi|T\phi) = (h|\phi)$. Choosing $\phi = T_k^*\psi$ we obtain $(\psi|T_k T_k^*\psi) = (h|T_k^*\psi)$, where we used $TT_k^* = T_k T_k^*$ because $T_k^* = P_{F_k} T_k^*$ from $TP_{F_k} = T_k$. Therefore $||T_k^*\psi||^2 = (h|T_k^*\psi)$, so $||T_k^*\psi||^2 \leq ||T_k^*\psi||\, ||h||$, i.e. $||T_k^*\psi|| \leq ||h||$. Consequently, by Theorem 8.54(b):

$$\int_{\mathsf{X}} \chi_{F_k}(x) |\overline{f(x)}|^2 d\mu_\psi(x) \leq ||h||^2 \quad \text{for any } k \in \mathbb{N},$$

which implies $\psi \in \Delta_{\overline{f}}$ by monotone convergence. So we have $D(T^*) \subset \Delta_{\overline{f}}$.
(c) Formulas (9.12), (9.13) and the ensuing remark are trivial consequences of the given definitions and of standard domains. Let us prove (9.14), (9.15). Assume f is bounded so that $\Delta_{f\cdot g} \subset \Delta_g$, and $\psi \in \Delta_g$. Take a sequence $\{g_n\}_{n\in\mathbb{N}}$ of bounded measurable maps converging to g in $L^2(\mathsf{X}, d\mu_g)$. Then $f \cdot g_n \to f \cdot g$ in $L^2(\mathsf{X}, d\mu_g)$, and because the integrals of f, g_n, $f \cdot g_n$ are as in Definition 8.49, plus (iii) in Theorem 8.54(a), we immediately have, for $n \to +\infty$:

$$\int_{\mathsf{X}} f(x) dP(x) \int_{\mathsf{X}} g_n(x) dP(x)\psi = \int_{\mathsf{X}} (f \cdot g_n)(x) dP(x)\psi \to \int_{\mathsf{X}} (f \cdot g)(x) dP(x)\psi.$$

As $\int_{\mathsf{X}} f dP$ is continuous, we will prove that f bounded and $\psi \in \Delta_g$ imply

$$\int_{\mathsf{X}} f(x) dP(x) \int_{\mathsf{X}} g(x) dP(x)\psi = \int_{\mathsf{X}} (f \cdot g)(x) dP(x)\psi\,. \tag{9.29}$$

Let now $\phi := \int_{\mathsf{X}} g dP \psi$. By (f) the identity shows

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\phi(x) = \int_{\mathsf{X}} |(f\cdot g)(x)|^2 d\mu_\psi(x) \quad \text{if } f \text{ is bounded and } \psi \in \Delta_g. \tag{9.30}$$

Take now f just measurable, possibly unbounded. As (9.30) holds for bounded maps, it holds for unbounded ones too. Since

$$D\left(\int_{\mathsf{X}} f(x) dP(x) \int_{\mathsf{X}} g(x) dP(x)\right)$$

contains all $\psi \in \Delta_g$ such that $\phi \in \Delta_f$, which happens by (9.30) precisely when $\psi \in \Delta_{f\cdot g}$, we conclude:

$$D\left(\int_{\mathsf{X}} f(x) dP(x) \int_{\mathsf{X}} g dP(x)\right) = \Delta_g \cap \Delta_{f\cdot g}\,.$$

Suppose $\phi \in \Delta_g \cap \Delta_{f\cdot g}$. When $\psi = \int_{\mathsf{X}} g(x) dP(x)\phi$ and $f_n := \chi_{F_n} \cdot f$ (F_n as previously), then $f_n \to f$ in $L^2(\mathsf{X}, \mu_\psi)$, $f_n \cdot g \to f \cdot g$ in $L^2(\mathsf{X}, \mu_\phi)$ and (9.29) and part (f) (f_n replacing f) imply:

$$\int_{\mathsf{X}} f(x)\, dP(x) \int_{\mathsf{X}} g(x)\, dP(x)\phi = \int_{\mathsf{X}} f(x)\, dP(x)\psi = \lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x)\, dP(x)\psi =$$

$$= \lim_{n\to+\infty} \int_{\mathsf{X}} (f_n \cdot g)(x)\, dP(x)\phi = \int_{\mathsf{X}} (f \cdot g)(x)\, dP(x)\phi \,.$$

This ends the proof of (9.14) and (9.15).

Inclusion (9.14) plus the equality in case $\Delta_g \supset \Delta_{f\cdot g}$ easily imply (9.18) and (9.19). Concerning (9.18), we have $\Delta_f \supset \Delta_{\overline{f}\cdot f} = \Delta_{|f|^2}$ for the following reason: as μ_ψ is finite, if $\psi \in \Delta_{|f(x)|^2}$

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) = \int_{\mathsf{X}} |f(x)|^2 \cdot 1 d\mu_\psi(x) \leq \sqrt{\int_{\mathsf{X}} |f(x)|^4 d\mu_\psi(x)} \sqrt{\int_{\mathsf{X}} 1^2 d\mu_\psi(x)} < +\infty.$$

(d) By (9.14) and (9.12) we have

$$p(T) \subset \int_{\mathsf{X}} p(x)\, dP(x) \,.$$

Hence the claim is true when $D(p(T)) = \Delta_p$. To prove this we shall start from showing by induction

$$D(T^n) = \Delta_{x^n} \quad \text{for } n \in \mathbb{N}. \tag{9.31}$$

When $n = 0, 1$, the identity is true: $D(T^0) = \Delta_1 = \mathsf{H}$, $D(T) = \Delta_x$. Assume it true for a given n and let us prove it for $n+1$: $D(T^{n+1}) = \Delta_{x^{n+1}}$. We have to show $D(TT^n) = \Delta_{x\circ x^n}$. Using the special property stated after (9.15), we know the claim is equivalent to $\Delta_{x\circ x^n} \subset \Delta_{x^n}$. The latter holds because μ_ψ is positive and finite, and $|x^{n+1}| > |x^n|$ outside a compact set $J \subset \mathbb{R}$, so $\psi \in \Delta_{x^{n+1}}$ implies

$$\int_{\mathbb{R}} |x|^{2n} d\mu_\psi(x) = \int_{\mathbb{R}\setminus J} |x|^{2n} d\mu_\psi(x) + \int_J |x|^{2n} d\mu_\psi(x)$$

$$\leq \int_{\mathbb{R}\setminus J} |x|^{2n+2} d\mu_\psi(x) + \sup_J |x|^{2n} \int_J 1 d\mu_\psi(x)$$

$$\leq \int_{\mathbb{R}} |x|^{2n+2} d\mu_\psi(x) + \sup_J |x|^{2n} \int_{\mathbb{R}} 1 d\mu_\psi(x) < +\infty \,.$$

We remark, for later, that we have also obtained

$$D(T^{n+1}) = \Delta_{x^{n+1}} \subset \Delta_{x^n} = D(T^n) \,.$$

To finish the proof of $D(p(T)) = \Delta_p$ we compute separately the two sides. Take the leading coefficient $a_m \neq 0$ in p. As $D(T^{n+1}) \subset D(T^n)$, and in general $D(A+B) = D(A) \cap D(B)$, we have

$$D(p(T)) = D(T^m) . \tag{9.32}$$

Let us compute Δ_p. Since $\Delta_{x^{n+1}} \subset \Delta_{x^n}$, we find $\Delta_{x^m} \subset \Delta_p$. Let us prove the opposite inclusion. From $|p(x)|/|x|^m \to |a_m|, |x| \to +\infty$, follows $|p(x)|/|x|^m \leq |a_m|+\varepsilon > 0$ for any $\varepsilon > 0$, provided x does not belong in too large a compact set $J_\varepsilon \subset \mathbb{R}$. Hence if $\psi \in \Delta_p$:

$$\int_{\mathbb{R}} |x|^{2m} d\mu_\psi$$

$$\leq \int_{\mathbb{R}\setminus J_\varepsilon} |x|^{2m} d\mu_\psi + \int_{J_\varepsilon} |x|^{2m} d\mu_\psi \leq \int_{\mathbb{R}\setminus J_\varepsilon} \frac{|p(x)|^2}{(|a_m|+\varepsilon)^2} d\mu_\psi + \sup_{J_\varepsilon} |x|^{2m} \int_{\mathbb{R}} d\mu_\psi$$

$$\leq \frac{1}{(|a_m|+\varepsilon)^2} \int_{\mathbb{R}} |p(x)|^2 d\mu_\psi + \sup_{J_\varepsilon} |x|^{2m} \int_{\mathbb{R}} d\mu_\psi < +\infty ,$$

and so $\psi \in \Delta_{x^m}$. Therefore $\Delta_p \subset \Delta_{x^m}$ and $\Delta_p = \Delta_{x^m}$. From (9.31) and (9.32) we have $\Delta_p = \Delta_{x^m} = D(T^m) = D(p(T))$, ending this part.

(e) Define the usual bounded maps $f_n := \chi_{F_n} \cdot f$ tending to f in $L^2(\mathsf{X}, \mu_\psi)$. By definition of integral, and by (iv) in Theorem 8.52(c):

$$\left(\phi \middle| \int_{\mathsf{X}} f(x) dP(x)\, \psi\right) = \lim_{n\to+\infty} \left(\phi \middle| \int_{\mathsf{X}} f_n(x) dP(x)\, \psi\right) = \lim_{n\to+\infty} \int_{\mathsf{X}} f_n(x) d\mu_{\phi,\psi}(x) .$$

Now we show

$$\lim_{n\to+\infty} \int_{\mathsf{X}} (f_n(x) - f(x)) d\mu_{\phi,\psi}(x) = 0 .$$

By Lemma 9.2 (same notations):

$$\left|\int_{\mathsf{X}} (f_n(x) - f(x)) d\mu_{\phi,\psi}(x)\right| = \left|\int_{\mathsf{X}} (f_n(x) - f(x)) h(x) d|\mu_{\phi,\psi}(x)|\right|$$

$$\leq \int_{\mathsf{X}} |f_n(x) - f(x)| d|\mu_{\phi,\psi}(x)| \leq ||\phi|| \sqrt{\int_{\mathsf{X}} |f_n(x) - f(x)|^2 d\mu_\psi(x)} \to 0$$

as $n \to +\infty$, by definition of $\int_{\mathsf{X}} f(x) dP(x)\psi$. Uniqueness now follows. If $T : \Delta_f \to \mathsf{H}$ satisfies the same property of $\int_X f(x) dP(x)$, then $T' := T - \int_X f(x) dP(x)$ solves $(\phi|T'\psi) = 0$ for any $\phi \in \mathsf{H}$, irrespective of $\psi \in \Delta_f$. Choosing $\phi = T'\psi$ gives $||T'\psi|| = 0$ and so $T = \int_X f(x) dP(x)$.

(f) This statement follows, by continuity, from the similar property of bounded maps, seen in Theorem 8.54(b), when we use our definition of integral of unbounded maps.

(g) Likewise, Theorem 8.52(b) implies (g). In fact if $f \geq 0$, $\psi \in \Delta_f$, the sequence of maps $\chi_{F_n} \cdot f_n \geq 0$ (F_n as in (9.3)) tends to f in $L^2(\mathsf{X}, \mu_\psi)$, so

$$0 \leq \left(\psi \middle| \int_{\mathsf{X}} (\chi_{F_n} \cdot f(x)) dP(x) \psi\right) \to \left(\psi \middle| \int_{\mathsf{X}} f(x) dP(x) \psi\right), \quad \text{as } n \to +\infty,$$

and $\left(\psi \middle| \int_{\mathsf{X}} f(x) dP(x) \psi\right) \geq 0$.
(h) We shall outline the proof as it is elementary, if tedious. By direct inspection P' is a PVM. If f is simple, assertion (9.24) is trivial. Using Definition 8.49 one generalises (9.24) to bounded measurable maps, so (9.24) extends by virtue of the definition of integral for unbounded f. $\square$

Corollary 9.5 *Under the assumptions of Theorem 9.3, if $f : \mathsf{X} \to \mathbb{C}$ is measurable the following facts are equivalent.*
(a) $\Delta_f = \mathsf{H}$.
(b) *f is essentially bounded with respect to the projector-valued measure P (Definition 8.45).*
(c) *$\int_{\mathsf{X}} f(x) dP(x)$ is bounded.*
(d) *$\int_{\mathsf{X}} f(x) dP(x) \in \mathfrak{B}(\mathsf{H})$.*
Under either of (a), (b), (c), (d), the estimate

$$||f||_{\infty}^{(P)} \leq \left|\left|\int_{\mathsf{X}} f dP\right|\right| . \tag{9.33}$$

holds.

Suppose X is a topological space, $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ and at least one of the hypotheses (1) and (2) in Proposition 8.44(d) is valid, so that P is concentrated on $supp(P)$. Then we can redefine f on a zero-measure set for P obtaining $||f||_{\infty} < +\infty$, and without altering $\int_{\mathsf{X}} f dP$: that latter can be computed with Definition 8.49 and yields the same result.

Proof Let us prove (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (9.33). Regarding (d), we will show (a) + (c) $\Rightarrow$ (d), while (d) $\Rightarrow$ (c) is trivial.
(a) $\Rightarrow$ (c) by the closed graph theorem (2.99), for $\int_{\mathsf{X}} f dP$ is closed by Theorem 9.4(a). On the other hand, (c) $\Rightarrow$ (a) because $\int_{\mathsf{X}} f dP$ is closed with dense domain by Theorem 9.3(a). Indeed, if $x \in \mathsf{H}$, there is a sequence $D(\int_{\mathsf{X}} f dP) \ni x_n \to x$ and $\int_{\mathsf{X}} f dP x_n$ converges to some point in H because the operator is bounded assuming (c). Next, closure implies that $x \in D(\int_{\mathsf{X}} f dP) = \Delta_f$, so that (a) holds.
(b) $\Rightarrow$ (c). Define F_n as in (9.4). If $f_n := \chi_{F_n} \cdot f$, then $f_n \to f$ pointwise as $n \to +\infty$. If f is essentially bounded, for n large enough $f - f_n$ is not 0 on a set $G_n \in \Sigma(\mathsf{X})$ with $P(G_n) = 0$. Hence

$$\int_{\mathsf{X}} f + (-f_n) dP = \int_{\mathsf{X}} \chi_{G_n}(f - f_n) dP = \int_{\mathsf{X}} f - f_n dP \int_{\mathsf{X}} \chi_{G_n} dP$$

$$= \left(\int_{\mathsf{X}} f - f_n dP\right) P(G_n) = 0 \, .$$

By Theorem 9.4(c), $\int_X f(x)dP(x) = -\int_X(-f_n(x))dP(x)$ belongs to $\mathfrak{B}(\mathsf{H})$ ($-f_n$ being bounded by Theorem 9.3(c)).
(c) ⇒ (b) + (9.33). Consider $f : \mathsf{X} \to \mathbb{C}$ measurable, with no boundedness assumption, and assume (c) (i.e. (a)). Take the usual sequence $f_n \in M_b(\mathsf{X})$. By (8.48):

$$||f_n||_\infty^{(P)} = \left|\left|\int_{\mathsf{X}} f\chi_n dP\right|\right| = \left|\left|\int_{\mathsf{X}} fdP\int_{\mathsf{X}}\chi_{F_n}dP\right|\right| \leq \left|\left|\int_{\mathsf{X}} fdP\right|\right| \left|\left|\int_{\mathsf{X}}\chi_{F_n}dP\right|\right|$$

$$\leq \left|\left|\int_{\mathsf{X}} fdP\right|\right| =: M < +\infty\,.$$

By construction $\{x \in \mathsf{X}\,|\,|f(x)| \geq M\} \subset \cup_{n\in\mathbb{N}}\{x \in \mathsf{X}\,|\,|f_n(x)| \geq M\}$. Monotonicity and sub-additivity imply

$$(\psi|P(\{x \in \mathsf{X}\,|\,|f(x)| \geq M\}\psi) \leq \sum_{n\in\mathbb{N}}(\psi|P(\{x \in \mathsf{X}\,|\,|f_n(x)| \geq M\})\psi) = 0\,,$$

which means $||f||_\infty^{(P)} \leq M < +\infty$, as required. The final assertion is clear if we redefine f null on the zero-measure set $|f(x)| > N$ for some finite $N > ||f||_\infty^{(P)}$. □

***** Inclusions (9.12) and (9.14) can be turned into equalities by taking the closures of the left-hand sides, as shown in the next proposition. We need a preliminary technical lemma, whose proof was essentially contained in Theorem 9.4(a), but is interesting in its own right.

Lemma 9.6 *Under the assumptions of Theorem 9.3, take a measurable $f : \mathsf{X} \to \mathbb{C}$ and a family $\{F_n\}_{n\in\mathbb{N}} \subset \Sigma(\mathsf{X})$ as in (9.3).*

The subspace $< P_{F_n}(\mathsf{H}) >_{n\in\mathbb{N}}$ containing finite linear combinations of the $P_{F_n}(\mathsf{H})$ satisfies:
(a) $< P_{F_n}(\mathsf{H}) >_{n\in\mathbb{N}} \subset \Delta_f$, *and it is dense in* H,
(b) $\int_{\mathsf{X}} fdP(< P(F_n)(\mathsf{H}) >_{n\in\mathbb{N}}) \subset < P(F_n)(\mathsf{H}) >_{n\in\mathbb{N}}$,
(c) $< P_{F_n}(\mathsf{H}) >_{n\in\mathbb{N}}$ *is a core for* $\int_{\mathsf{X}} fdP$.

Proof (a) Define $E_0 = F_0$ and $E_n := F_n \setminus F_{n-1}$ as in the proof of Theorem 9.4(a), so that: (i) $E_k \cap E_h = \varnothing$ for $h \neq k$, (ii) $F_n = \cup_{j=0}^n E_j$, (iii) $\cup_{j=0}^\infty E_n = \mathsf{X}$, and (9.26) holds true. Next (i), (ii) and the basic properties of a PVM imply

$$D_0 :=< P(F_n)(\mathsf{H}) >_{n\in\mathbb{N}} =< P(E_n)(\mathsf{H}) >_{n\in\mathbb{N}}$$

and (9.26) entails that D_0 is dense in H. Furthermore $D_0 \subset \Delta_f$. Indeed, we know that $\psi \in D_0 \Leftrightarrow \psi = \sum_{j=0}^m P(E_j)\psi$ for some finite m. Hence (i) and elementary properties of a PVM imply

$$\mu_\psi(L) = \left(\sum_{j=1}^m P(E_j)\psi \,\middle|\, P(L)\sum_{k=1}^m P(E_k)\psi\right) = \sum_{j=0}^m \mu_\psi(L \cap E_j)$$

for every $L \in \Sigma(\mathsf{X})$, which yields

$$\int_{\mathsf{X}} |f(x)|^2 d\mu_\psi(x) = \sum_{j=0}^{m} \int_{E_j} |f(x)|^2 d\mu_\psi(x) < +\infty \,,$$

since f is bounded on each $E_j \subset F_j$ by definition of F_j.
(b) If $\psi \in D_0$, from (9.19) we have

$$\int_{\mathsf{X}} f dP\psi = \sum_{j=0}^{m} \int_{\mathsf{X}} f dP \; P(E_j)\psi = \sum_{j=0}^{m} P(E_j) \int_{\mathsf{X}} f dP\psi$$

which means that $\int_{\mathsf{X}} f dP(D_0) \subset D_0$.
(c) We start by observing that $\int_{\mathsf{X}} f dP$ is closed by Theorem 9.4(a), so $\int_{\mathsf{X}} f dP \restriction_{D_0}$ is closable and $\overline{\int_{\mathsf{X}} f dP \restriction_{D_0}} \subset \int_{\mathsf{X}} f dP$. To obtain the converse inclusion, fix a point $(\psi, \int_{\mathsf{X}} f dP\psi)$ in the graph of $\int_{\mathsf{X}} f dP$ and consider the sequence of points of the graph of $\int_{\mathsf{X}} f(x)dP(x) \restriction_{D_0}$:

$$\left(P(F_n)\psi, \int_{\mathsf{X}} f dP \restriction_{D_0} P(F_n)\psi\right) \quad n \in \mathbb{N} \,,$$

where the second entry is well defined as $P(F_n)\psi \in D_0$. We have:

$$\left(P(F_n)\psi, \int_{\mathsf{X}} f dP \restriction_{D_0} P(F_n)\psi\right) = \left(P(F_n)\psi, \int_{\mathsf{X}} f dP P(F_n)\psi\right) = \left(P(F_n)\psi, \int_{\mathsf{X}} f_n dP\psi\right)$$

where $f_n := \chi_{F_n} \cdot f$, and we have exploited (9.19) with $E = F_n$. Just in view of (9.25) and the very definition (9.9) of $\int_{\mathsf{X}} f(x)dP(x)$ we have that, as $n \to +\infty$, in the topology of the graph

$$\left(P(F_n)\psi, \int_{\mathsf{X}} f_n dP\psi\right) \to \left(\psi, \int_{\mathsf{X}} f dP\psi\right) .$$

This result can be rephrased as $\overline{\int_{\mathsf{X}} f dP(x) \restriction_{D_0}} \supset \int_{\mathsf{X}} f dP$, concluding the proof. □

Proposition 9.7 *Under the assumptions of Theorem 9.3, if $f, g : \mathsf{X} \to \mathbb{C}$ are measurable functions,*

$$\overline{\int_{\mathsf{X}} f(x)dP(x) + \int_{\mathsf{X}} g(x)dP(x)} = \int_{\mathsf{X}} (f+g)(x)dP(x) \,, \tag{9.34}$$

and

$$\overline{\int_{\mathsf{X}} f(x)dP(x) \int_{\mathsf{X}} g(x)dP(x)} = \int_{\mathsf{X}} (f \cdot g)(x)dP(x) \,. \tag{9.35}$$

Proof Define Borel sets $F_n := \{x \in \mathsf{X} \mid |f(x)| + |g(x)| + |f(x)g(x)| < n\}$, when $n \in \mathbb{N}$. This family satisfies (9.3) for f, g, $f+g$ and $f \cdot g$ simultaneously, so in particular $< P(F_k) >_{k\in\mathbb{N}}$ is contained in $\Delta_f, \Delta_g, \Delta_{f+g}, \Delta_{f\cdot g}$ by Lemma 9.6(a), and it is a core for the corresponding integral operators by Lemma 9.6(c).

Let us first focus on (9.34). From (9.12) we have

$$\left(\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP\right)\restriction_{<P(F_k)>_{k\in\mathbb{N}}} \subset \int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP \subset \int_{\mathsf{X}} f + g\, dP$$

so that, observing that the last integral is a closed operator,

$$\overline{\left(\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP\right)\restriction_{<P(F_k)>_{k\in\mathbb{N}}}} \subset \overline{\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP} \subset \int_{\mathsf{X}} f + g\, dP\ .$$

To conclude the proof it suffices to establish

$$\overline{\left(\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP\right)\restriction_{<P(F_k)>_{k\in\mathbb{N}}}} = \int_{\mathsf{X}} f + g\, dP\ . \tag{9.36}$$

To prove (9.36), define the pairwise disjoint sets E_k associated to the F_k as in the proof of Lemma 9.6, so that $\psi \in < P(F_k) >_{k\in\mathbb{N}} \Leftrightarrow \psi = \sum_{k=0}^{n} P(E_k)\psi$ for some finite n. Moreover

$$\left(\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP\right)\psi = \sum_{k=0}^{n} \int_{\mathsf{X}} f dP\ P(E_k)\psi + \int_{\mathsf{X}} g dP\ P(E_k)\psi$$

$$= \sum_{k=0}^{n} \int_{\mathsf{X}} f_n dP\psi + \int_{\mathsf{X}} g_n dP\psi$$

where we made use of (9.19) and where $f_n := \chi_{E_n} \cdot f$ and $g_n := \chi_{E_n} \cdot g$. Since these functions are bounded, from (ii) in Theorem 8.54(a), we have

$$\sum_{k=0}^{n} \int_{\mathsf{X}} f_n dP\psi + \int_{\mathsf{X}} g_n dP\psi = \sum_{k=0}^{n} \int_{\mathsf{X}} f_n + g_n dP\psi = \sum_{k=0}^{n} \int_{\mathsf{X}} f + g\, dP\ P(E_k)\psi$$

$$= \int_{\mathsf{X}} f + g\, dP\psi\ .$$

Summing up:

$$\left(\int_{\mathsf{X}} f dP + \int_{\mathsf{X}} g dP\right)\psi = \int_{\mathsf{X}} f + g\, dP\psi \quad \forall \psi \in < P(F_k) >_{k\in\mathbb{N}}\ .$$

In other words

$$\left(\int_X f dP + \int_X g dP\right) \restriction_{<P(F_k)>_{k\in\mathbb{N}}} = \int_X f + g \, dP \restriction_{<P(F_k)>_{k\in\mathbb{N}}} .$$

Since $< P(F_k) >_{k\in\mathbb{N}}$ is a core of $\int_X f + g \, dP$ (Lemma 9.6(c)), taking closures we obtain (9.36), proving (9.34).

Let us pass to (9.35). As in the previous case, from (9.14) we have

$$\overline{\left(\int_X f dP \int_X g dP\right) \restriction_{<P(F_k)>_{k\in\mathbb{N}}}} \subset \overline{\int_X f dP \int_X g dP} \subset \int_X f \cdot g \, dP ,$$

where the first integral is well defined because of Lemma 9.6(b). The proofs ends as soon as one proves that

$$\overline{\left(\int_X f dP \int_X g dP\right) \restriction_{<P(F_k)>_{k\in\mathbb{N}}}} = \int_X f \cdot g \, dP . \tag{9.37}$$

With a reasoning similar to the previous case, taking Lemma 9.6(b) into account, one sees that

$$\left(\int_X f dP \int_X g dP\right) \restriction_{<P(F_k)>_{k\in\mathbb{N}}} = \int_X f \cdot g \, dP \restriction_{<P(F_k)>_{k\in\mathbb{N}}} .$$

Since $< P(F_k) >_{k\in\mathbb{N}}$ is a core for $\int_X f \cdot g \, dP$, taking closures yields (9.37), concluding the proof. $\square$

The next definition is based on (9.9). We can also make use of Theorem 9.4(e) to obtain a more elegant, equivalent definition.

Definition 9.8 Let $(\mathsf{X}, \Sigma(\mathsf{X}))$ be a measurable space, H a Hilbert space and $P : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ a PVM.
(a) If $f : \mathsf{X} \to \mathbb{C}$ is measurable with Δ_f as in (9.8), the operator

$$\int_X f(x) dP(x) : \Delta_f \to \mathsf{H}$$

of (9.9) is called **integral of** f with respect to the projector-valued measure P.

Equivalently, $\int_X f(x) dP(x)$ is the unique operator $S : \Delta_f \to \mathsf{H}$ such that

$$(\phi | S\psi) = \int_X f(x) d\mu_{\phi,\psi}(x) , \quad \text{for any } \phi \in H, \psi \in \Delta_f ,$$

where the complex spectral measure $\mu_{\phi,\psi}$ is defined in Theorem 8.52(c).
(b) For every $E \in \Sigma(\mathsf{X})$, $f : \mathsf{X} \to \mathbb{C}$ and $g : E \to \mathbb{C}$ measurable, the integrals

$$\int_E f(x)\, dP(x) := \int_{\mathsf{X}} \chi_{E(x)} f(x)\, dP(x) \;\text{ and }\; \int_E g(x)\, dP(x) := \int_{\mathsf{X}} g_0(x)\, dP(x)\,,$$

with $g_0(x) := g(x)$ if $x \in E$ and $g_0(x) := 0$ if $x \notin E$, are respectively called **integral of** f **on** E and **integral of** g **on** E (in the projector-valued measure P).

Remarks 9.9
(1) By Theorem 9.3(c), the above extends Definition 8.49 for bounded maps.
(2) Assume that X is a topological space, $\Sigma(\mathsf{X}) = \mathscr{B}(\mathsf{X})$ and at least one of the hypotheses (1) and (2) in Proposition 8.44(d) is valid, so that P is concentrated on $supp(P)$. For any $f : \mathsf{X} \to \mathbb{C}$ measurable,

$$\int_{\mathsf{X}} f(x)dP(x) = \int_{supp(P)} f(x)dP(x) \quad \text{and so} \quad \int_{\mathsf{X}\setminus supp(P)} f(x)dP(x) = 0\,. \tag{9.38}$$

The proof is straightforward: $\chi_{supp(P)}$ is bounded, by definition its integral is ($\chi_{supp(P)}$ is simple):

$$\int_{supp(P)} 1dP := \int_{\mathsf{X}} \chi_{supp(P)}dP = P(supp(P)) = I\,,$$

where Proposition 8.44(d) was used in the last equality. Now the second identity in (9.19) gives

$$\int_{\mathsf{X}} f(x)dP(x) = \int_{\mathsf{X}} f(x)dP(x) \int_{\mathsf{X}} \chi_{supp(P)}(x)dP(x) = \int_{supp(P)} \chi_{supp(P)}(x) f(x)dP(x)$$

$$=: \int_{supp(P)} f(x)dP(x)\,.$$

The rest of (9.38) follows, similarly, by using $P(\mathsf{X} \setminus supp(P)) = 0$. ■

Example 9.10 **(1)** Consider the spectral measure:

$$P : \mathscr{B}(N) \ni E \mapsto P_E = \sum_{z\in E} z(z|\ \ )$$

of Example 8.51(2) on a basis N of a separable Hilbert space H, and equip N with the second-countable topology of power sets. We are interested in writing an explicit formula for the integral of an unbounded map $f : N \to \mathbb{C}$ relying on definition (9.6). In the case under exam $\int_N |f(z)|^2 d\mu_\psi(z) < +\infty$ becomes $\sum_{z\in N} |f(z)|^2 |(z|\psi)|^2 < +\infty$. We aim to show

$$\int_N f(z)dP(z) = \text{s-}\sum_{z\in N} f(z)z(z|\ \ )$$

for f unbounded. This formula was proved in Example 8.51(2) for f bounded. Suppose $\{N_n\}_{n\in\mathbb{N}}$ are finite subsets in N, $N_{n+1} \supset N_n$ and $\cup_{n\in\mathbb{N}} N_n = N$. The sequence of bounded maps $f_n := \chi_{N_n} \cdot f$ converges in $L^2(N, \mu_\psi)$, for any $\psi \in \mathsf{H}$ such that $\sum_z |f(z)|^2|(z|\psi)|^2 < +\infty$, simply by Lebesgue's dominated convergence. By definition (9.6) we have, if $\sum_{z\in N} |f(z)|^2|(z|\psi)|^2 < +\infty$:

$$\int_N f(z)dP(z)\psi := \lim_{n\to+\infty} \int_N f_n(z)dP(z)\psi \tag{9.39}$$

But f_n is bounded, so Example 8.51(2) guarantees

$$\int_N f_n(z)dP(z)\psi = \text{s-}\sum_{z\in N} f_n(z)(z|\psi) = \sum_{z\in N_n} f(z)z(z|\psi) ,$$

where the sum is finite for N_n contains a finite number of points. Definition (9.39) reduces to

$$\int_N f(z)dP(z)\psi = \lim_{n\to+\infty} \sum_{z\in N_n} f(z)z(z|\psi) ,$$

i.e.

$$\int_N f(z)dP(z) = \text{s-}\sum_{z\in N} f(z)z(z|\quad) . \tag{9.40}$$

Later we will see a concrete example of an unbounded self-adjoint operator built with this type of spectral measure.

(2) Consider the spectral measure of Example 8.51. Take the Hilbert space $\mathsf{H} = L^2(\mathsf{X}, \mu)$, with X second countable and μ positive, σ-additive on the Borel σ-algebra of X. The spectral measure on H we wish to consider is the following. For any $\psi \in L^2(\mathsf{X}, \mu)$, $E \in \mathscr{B}(\mathsf{X})$, let

$$(P(E)\psi)(x) := \chi_{E(x)}\psi(x) , \quad \text{for almost every } x \in \mathsf{X} . \tag{9.41}$$

With $\psi \in \mathsf{H}$, the measure μ_ψ is

$$\mu_\psi(E) = (\psi|P(E)\psi) = \int_E |\psi(x)|^2 d\mu(x) , \quad \text{for any } E \in \mathscr{B}(\mathsf{X}) .$$

Consequently if $g : \mathsf{X} \to \mathbb{C}$ is measurable:

$$\int_{\mathsf{X}} g(x)d\mu_\psi(x) = \int_{\mathsf{X}} g(x)|\psi(x)|^2 d\mu(x) .$$

In Example 8.51(1) we saw that if $f : \mathsf{X} \to \mathbb{C}$ is measurable and bounded:

$$\left(\int_{\mathsf{X}} f(y)\, dP(y)\psi\right)(x) = f(x)\psi(x) \quad \text{for every } \psi \in L^2(\mathsf{X}, \mu) \text{ and almost every } x \in \mathsf{X}. \tag{9.42}$$

This is valid for unbounded measurable maps, too, as long as $\psi \in \Delta_f$. If $f : \mathsf{X} \to \mathbb{C}$ is unbounded and measurable take a sequence of bounded measurable maps f_n such that $f_n \to f$, $n \to +\infty$, in $L^2(\mathsf{X}, \mu_\psi)$, with $\psi \in \Delta_f$. In other words, by the above expression for μ_ψ we take

$$\int_{\mathsf{X}} |f_n(x) - f(x)|^2 |\psi(x)|^2 d\mu(x) \to 0 \quad \text{as } n \to +\infty.$$

By (9.42):

$$||f \cdot \psi - f_n \cdot \psi||^2_{\mathsf{H}} = \int_{\mathsf{X}} |f(x) - f_n(x)|^2 |\psi(x)|^2 d\mu(x) \to 0 \quad \text{as } n \to +\infty.$$

Therefore the definition of integral in P implies that for any $\psi \in \Delta_f$, with $f : \mathsf{X} \to \mathbb{C}$ measurable and possibly unbounded:

$$\left(\int_{\mathsf{X}} f(x) dP(x)\psi\right)(y) = f(y)\psi(y) \quad \text{for almost every } y \in \mathsf{X}. \tag{9.43}$$

■

9.1.2 Von Neumann Algebra of a Bounded Normal Operator

Corollary 9.5 has an important consequence for the von Neumann algebra generated by a bounded normal operator and the adjoint.

Theorem 9.11 (Von Neumann algebra generated by a bounded normal operator and its adjoint) *Take a normal operator $T \in \mathfrak{B}(\mathsf{H})$ with H separable. The von Neumann algebra generated by T and T^* (the subspace in $\mathfrak{B}(\mathsf{H})$ that commutes with every operator commuting with T and T^*) consists precisely of the operators $f(T, T^*)$ of Theorem 8.39, for $f \in M_b(\sigma(T))$.*

Proof Indicate by $\mathfrak{M}$ the von Neumann algebra generated by T, T^*. We know that any $f(T, T^*)$, with $f : \sigma(T) \to \mathbb{C}$ measurable and bounded, belongs to $\mathfrak{M}$ by (iii) in Theorem 8.39(b) (in the sequel we will need Theorem 8.54, the spectral Theorems 8.56 and 8.58). Let us show the converse. Clearly $\mathfrak{M}$ coincides with the von Neumann algebra generated by the unital *-algebra of complex polynomials in T, T^* (restricted to $\sigma(T)$, from now on always assumed). By the double commutant theorem (3.88) $\mathfrak{M}$ is the strong closure of complex polynomials in T, T^*. That is to say, if $B \in \mathfrak{M}$ there is a sequence of bounded measurable f_n (better: restrictions of polynomials to $\sigma(T)$) such that $f_n(T, T^*)x \to Bx$,

$n \to +\infty$, for any $x \in \mathsf{H}$. We claim $B = f(T, T^*)$ for some bounded measurable map f defined on $\sigma(T)$. As $g(T, T^*) = \int_{\sigma(T)} g dP^{(T)}$ by Theorem 8.54(d) for g bounded and measurable, let $\{\psi_\alpha\}_{\alpha\in\mathbb{N}} \subset \mathsf{H}$ be an orthonormal system (countable since H is separable – the finite case is alike) as the one of the proof of Theorem 8.58. As in the mentioned proof, build corresponding orthogonal spaces $\overline{\mathsf{H}_{\psi_\alpha}} \subset \mathsf{H}$ whose orthogonal sum is H. Each $\overline{\mathsf{H}_{\psi_\alpha}}$ is invariant under any $g(T, T^*)$, and isomorphic to $L^2(\sigma(T), \mu_\alpha)$, where $\mu_\alpha(E) := (\psi_\alpha|P^{(T)}(E)\psi_\alpha)$ are the usual positive probability measures, $P^{(T)}$ is the PVM of T and $E \subset \sigma(T)$ a Borel set. The vector ψ_α is described in $L^2(\sigma(T), \mu_\alpha)$ by the constant map 1. The operator $f_n(T, T^*)$ is described in each $L^2(\sigma(T), \mu_\alpha)$ by the multiplication by f_n. Now look at $x := \oplus_{\alpha\in\mathbb{N}} 2^{-\alpha/2}\psi_\alpha \in \mathsf{H}$. The sequence $f_n(T, T^*)x$ is a Cauchy sequence by assumption. Expanding $\mathsf{H} = \oplus_{\alpha\in\mathbb{N}}\overline{\mathsf{H}_{\psi_\alpha}}$, the inequality $||f_n(T, T^*)x - f_m(T, T^*)x||^2 < \varepsilon$, $n, m > N_\varepsilon$, is equivalent to $\sum_{\alpha\in\mathbb{N}} \int_{\sigma(T)} |f_n - f_m|^2 2^{-\alpha} d\mu_\alpha < \varepsilon$, $n, m > N_\varepsilon$, by Theorem 8.58(a). Let $\mu(E) := \sum_{\alpha\in\mathbb{N}} 2^{-\alpha}\mu_\alpha(E)$ be a bounded positive Borel measure. Then the previous condition says $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\sigma(T), \mu)$ (and in every $L^2(\sigma(T), \mu_\alpha)$), so there is a subsequence (of the same name, for simplicity) converging μ-almost everywhere to a measurable map $f \in L^2(\sigma(T), \mu)$, possibly unbounded. Since zero-measure sets for μ are so also for each μ_α, convergence holds almost everywhere for $P^{(T)}$ as well, by Theorem 8.58. (In fact for any $z = \oplus_\alpha \phi_\alpha \in \mathsf{H}$ with $\phi_\alpha \in L^2(\sigma_\alpha(T), \mu_\alpha)$, we have $(z|P^{(T)}(E)z) = \sum_{\alpha\in\mathbb{N}} \int_{\sigma(T)} \chi_E |\phi_\alpha|^2 d\mu_\alpha = 0$ if $\mu_\alpha(E) = 0$ for any $\alpha \in \mathbb{N}$.) So, according to Definition 9.8, we may define a generally unbounded, closed operator $A := \int_{\sigma(T)} f dP^{(T)}$, with dense domain Δ_f. Call $\mathscr{D} \subset \Delta_f$ the linear space, dense in H, of elements $\oplus_{\alpha\in F}\phi_\alpha$ where $\phi_\alpha \in M_b(\sigma(T)) \subset L^2(\sigma(T), \mu_\alpha)$ and F is finite but arbitrary. By linearity $A\restriction_{\mathscr{D}} = B\restriction_{\mathscr{D}}$ as $||A\phi_\alpha - B\phi_\alpha|| \leq ||(A - f_n(T, T^*))\phi_\alpha|| + ||f_n(T, T^*)\phi_\alpha - B\phi_\alpha||$, where the last term is infinitesimal as $n \to +\infty$ by construction, whereas the penultimate term squared is smaller than $||\phi_\alpha||^2_\infty \int |f - f_n|^2 d\mu_\alpha$, and $\int |f - f_n|^2 d\mu_\alpha \to 0$ as $n \to +\infty$, as we know. Since B is bounded and $\mathscr{D}$ dense, closing $A\restriction_{\mathscr{D}} = B\restriction_{\mathscr{D}}$ gives $\overline{A\restriction_{\mathscr{D}}} = B$. But $A = \overline{A} \supset \overline{A\restriction_{\mathscr{D}}}$, and B is defined on H, whence $A = B$. Recalling $A = \int_{\sigma(T)} f dP^{(T)}$, by Corollary 9.5 $||f||^{(P)}_\infty < +\infty$. We may redefine f on a zero-measure set for $P^{(T)}$ without changing $A = \int_{\sigma(T)} f dP^{(T)}$. Since f is now bounded, we can define $f(T, T^*)$ as in Theorem 8.39. Thus $f(T, T^*) = \int_{\sigma(T)} f dP^{(T)} = B$.
□

Remark 9.12 The above result may be somehow generalised by looking at unbounded PVMs, as proven in Exercises 9.5 and 9.6. ■

9.1.3 Spectral Decomposition of Unbounded Self-adjoint Operators

The time is right to prove the *spectral decomposition theorem for unbounded self-adjoint operators*, exploiting the properties of the Cayley transform. A similar statement is valid also for unbounded (closed) normal operators. The classical proof of this general proposition can be found in [Rud91], which uses the self-adjoint case, and thus relies again on the Cayley transform as a crucial tool. A general proof for (closed) normal unbounded operators, which does not rely on the Cayley transform, appears in [Schm12]. Extensions of the spectral decomposition theorem of (generally unbounded) normal operators to quaternionic Hilbert spaces appear in [GMP13] and [GMP17].

Theorem 9.13 (Spectral decomposition of unbounded self-adjoint operators) *Let T be a self-adjoint (possibly unbounded) operator on the Hilbert space H.*
(a) *There exists a unique projector-valued measure $P^{(T)} : \mathscr{B}(\mathbb{R}) \to \mathfrak{B}(\mathsf{H})$ ($\mathbb{R}$ equipped with the standard topology) such that*

$$T = \int_{\mathbb{R}} \lambda \, dP^{(T)}(\lambda) \,. \tag{9.44}$$

(b) *$P^{(T)}$ is concentrated on its support and*

$$supp(P^{(T)}) = \sigma(T) \,. \tag{9.45}$$

In particular:

(i) $\lambda \in \sigma_p(T) \;\Leftrightarrow\; P^{(T)}(\{\lambda\}) \neq 0$. In this case $P^{(T)}(\{\lambda\})$ is the orthogonal projector onto the λ-eigenspace of T;

(ii) $\lambda \in \sigma_c(T) \;\Leftrightarrow\; P^{(T)}(\{\lambda\}) = 0$, and any open set $A_\lambda \subset \mathbb{R}$ containing λ satisfies $P^{(T)}(A_\lambda) \neq 0$;

(iii) if $\lambda \in \sigma(T)$ is isolated, then $\lambda \in \sigma_p(T)$;

(iv) if $\lambda \in \sigma_c(T)$, then for any $\varepsilon > 0$ there exists $\phi_\varepsilon \in D(T)$, $||\phi_\varepsilon|| = 1$ with

$$0 < ||T\phi_\varepsilon - \lambda\phi_\varepsilon|| \leq \varepsilon \,.$$

Proof (a) Let V be the Cayley transform of T, a unitary operator by Theorem 5.34 because T is self-adjoint. Setting $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, define $\mathsf{X} := \mathbb{S}^1 \setminus \{(1, 0)\}$ and write $z = x + iy$. Put on X the topology induced by $\mathbb{R}^2$ (or $\mathbb{S}^1$, which is the same) and consider its Borel σ-algebra $\mathscr{B}(\mathsf{X}) \subset \mathscr{B}(\mathbb{S}^1)$. Let also $P_0^{(V)}$ be the spectral measure of V on $\mathbb{S}^1$, stemming from the spectral decomposition Theorem 8.56(a)'. Then

$$V = \int_{\mathbb{S}^1} z dP_0^{(V)}(x, y) \,. \tag{9.46}$$

The operator $I - V$ is one-to-one by (i) in Theorem 5.34(b), so $1 = 1 + i0 \notin \sigma_p(V)$. This in turn implies $P_0^{(V)}(\{(1, 0)\}) = 0$ by (i) in Theorem 8.56(b). Consider orthogonal projectors

$$P^{(V)} : \mathscr{B}(\mathsf{X}) \ni E \mapsto P_0^{(V)}(E) \in \mathscr{L}(\mathsf{H}) \,,$$

where $\mathscr{B}(\mathsf{X}) \subset \mathscr{B}(\mathbb{S}^1)$. By construction $P^{(V)}$ is a PVM on X (cf. Definition 8.41); note that $P^{(V)}(\mathsf{X}) = I$ because $P_0^{(V)}(\{(1, 0)\}) = 0$:

$$P^{(V)}(\mathsf{X}) := P_0^{(V)}(\mathsf{X}) = P_0^{(V)}(\mathbb{S}^1\setminus\{(1,0)\}) = P_0^{(V)}(\mathbb{S}^1) - P_0^{(V)}(\{(1,0)\}) = I - 0 = I.$$

For the same reason the integral of a simple map s on $\mathbb{S}^1$ in $P_0^{(V)}$ coincides trivially with the integral of $s\restriction_{\mathsf{X}}$ in $P^{(V)}$. From the construction of the integral of bounded maps, taking $f \in M_b(\mathbb{S}^1)$, hence $f\restriction_{\mathsf{X}} \in M_b(\mathsf{X})$, it follows that $\int_{\mathsf{X}} f\restriction_{\mathsf{X}} dP^{(V)} = \int_{\mathsf{X}} f dP_0^{(V)}$. However we choose $\phi, \psi \in \mathsf{H}$, $E \subset \mathscr{B}(\mathbb{S}^1)$:

$$\mu_{\phi,\psi}^{(P^{(V)})}(E \setminus \{(1,0)\}) = (\phi|P^{(V)}(E \setminus \{(1,0)\})\psi) = (\phi|P_0^{(V)}(E \setminus \{(1,0)\})\psi)$$

$$= (\phi|P_0^{(V)}(E)\psi) = \mu_{\phi,\psi}^{(P_0^{(V)})}(E)\,,$$

(in the obvious notation). Then using the definition of integral of measurable maps we find $\int_{\mathsf{X}} f\restriction_{\mathsf{X}} dP^{(V)} = \int_{\mathsf{X}} f dP_0^{(V)}$ for any $f : \mathbb{S}^1 \to \mathbb{C}$ measurable. In particular, from (9.46) and dropping $\restriction_{\mathsf{X}}$, we obtain:

$$V = \int_{\mathsf{X}} z dP^{(V)}(x,y)\,. \tag{9.47}$$

Now define the real-valued, measurable unbounded map on X:

$$f(z) := i\frac{1+z}{1-z} \quad z \in \mathsf{X}\,, \tag{9.48}$$

and integrate it in $P^{(V)}$ on X, to get the operator (unbounded, in general):

$$T' := \int_{\mathsf{X}} f(z) dP^{(V)}(x,y)\,. \tag{9.49}$$

As f ranges in the reals ($(x,y) \in \mathsf{X}$), T' must be self-adjoint by Theorem 9.4(b). The equation $f(z)(1-z) = i(1+z)$, by virtue of Theorem 9.4(c), implies:

$$T'(I-V) = i(I+V) \tag{9.50}$$

(it is easy to see that there is '$=$' in (9.14)). In particular (9.50) implies $Ran(I-V) \subset \Delta_f =: D(T')$. From Theorem 5.34 we know

$$T(I-V) = i(I+V) \quad \text{and } D(T) = Ran(I-V) \subset \Delta_f.$$

Comparing with (9.50) allows to conclude T' is a self-adjoint extension of T. As $T = T^*$ has no proper self-adjoint extension (Proposition 5.17(c)), then $T = T'$. Hence

$$T = \int_{\mathsf{X}} f(z) dP^{(V)}(x,y)\,. \tag{9.51}$$

The function $f : \mathsf{X} \to \mathbb{R}$ is actually bijective and so its range is $\mathbb{R}$. From Theorem 9.4(h) $\mathscr{B}(\mathbb{R}) \ni E \mapsto P^{(T)}(E) := P^{(V)}(f^{-1}(E))$ is a PVM on $\mathbb{R}$ and (9.51) may be written as in (9.44):

$$T = \int_{\mathbb{R}} \lambda dP^{(T)}(\lambda) \, .$$

But this is exactly the spectral expansion we wanted. So let us pass to the uniqueness of the measure solving (9.44). Let P' be a PVM on $\mathbb{R}$ with

$$T = \int_{\mathbb{R}} \lambda dP'(\lambda) \, .$$

The Cayley transform, by Theorem 9.4(c), reads

$$V = (T - iI)(T + iI)^{-1} = \int_{\mathbb{R}} \frac{\lambda - i}{\lambda + i} dP'(\lambda) \, .$$

Using statement (h) in the same theorem, with the same measurable $f : \mathsf{X} \to \mathbb{R}$ with measurable inverse (9.48), we find

$$V = \int_{\mathsf{X}} z dP'(f(x, y)) \, ,$$

where $\mathscr{B}(\mathsf{X}) \ni F \mapsto Q(F) := P'(f(F))$ is a PVM on X which we can extend to a PVM on $\mathbb{S}^1$ by $Q_0(F) := Q(F \setminus \{(1, 0)\})$, $F \in \mathscr{B}(\mathbb{S}^1)$. Thus

$$V = \int_{\mathbb{S}^1} z dQ_0(x, y) \, .$$

By (9.46), then, as the spectral measure associated to a bounded normal operator is unique by Theorem 8.56, necessarily $Q_0(F) = P_0^{(V)}(F)$ for any Borel set in $\mathbb{S}^1$. Hence $Q(F) = P^{(V)}(F)$ for any Borel set of X. Therefore, for any $E \in \mathscr{B}(\mathbb{R})$, $Q(f^{-1}(E)) = P^{(V)}(f^{-1}(E))$, i.e. $P'(E) = P^{(T)}(E)$, as required.

(b) First, observe that $P^{(T)}$ is concentrated on its support since the standard topology of $\mathbb{R}$ is second countable and so (1) in Proposition 8.44(d) applies. Now let us show $\sigma(T) = supp(P^{(T)})$, or equivalently, $\lambda_0 \in \rho(T) \Leftrightarrow \lambda_0 \notin supp(P^{(T)})$. First of all we shall prove $\lambda_0 \notin supp(P^{(T)}) \Rightarrow \lambda_0 \in \rho(T)$. In fact, there exists an open interval $(a, b) \subset \mathbb{R} \setminus supp(P^{(T)})$, $\lambda_0 \in (a, b)$. Hence $I = \int_{\mathbb{R}} \chi_{\mathbb{R}\setminus(a,b)} dP$, and from the last result in Theorem 9.4(c)

$$\int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0} dP^{(T)}(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0} dP^{(T)}(\lambda) \int_{\mathbb{R}} \chi_{\mathbb{R}\setminus(a,b)} dP =$$

$$\int_{\mathbb{R}} \chi_{\mathbb{R}\setminus(a,b)}(\lambda) \frac{1}{\lambda - \lambda_0} dP^{(T)}(\lambda) \, .$$

By Theorem 9.4(c), as the last integrand is bounded,

$$R_{\lambda_0}(T) := \int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0}\, dP^{(T)}(\lambda) \in \mathfrak{B}(\mathsf{H}) .$$

Always by Theorem 9.4(c) (and keeping an eye on the domains of the products):

$$R_{\lambda_0}(T)(T - \lambda_0 I) = I \restriction_{D(T)} , \quad (T - \lambda_0 I)R_{\lambda_0}(T) = I .$$

The second is true everywhere on H, so $Ran(T - \lambda_0 I) = \mathsf{H}$. The operator $R_{\lambda_0}(T)$ is therefore the resolvent of T associated to λ_0 by Theorem 8.4(a), as the name suggests. By definition, then, $\lambda_0 \in \rho(T)$.

Conversely, let us prove $\lambda_0 \in \rho(T) \Rightarrow \lambda_0 \notin supp(P^{(T)})$. Under the assumptions on λ_0, $P^{(T)}(\{\lambda_0\}) = 0$, otherwise there would exist $\psi \in P^{(T)}_{\{\lambda_0\}}(\mathsf{H}) \setminus \{0\}$ such that (by Theorem 9.4(c)):

$$T\psi = \int_{\mathbb{R}} \lambda dP^{(T)}(\lambda) P^{(T)}_{\{\lambda_0\}}\psi = \int_{\mathbb{R}} \lambda dP^{(T)}(\lambda) \int_{\mathbb{R}} \chi_{\{\lambda_0\}}(\lambda) dP^{(T)}(\lambda)\psi$$

$$= \int_{\mathbb{R}} \lambda \chi_{\{\lambda_0\}}(\lambda) dP^{(T)}(\lambda)\psi = \int_{\mathbb{R}} \lambda_0 \chi_{\{\lambda_0\}}(\lambda) dP^{(T)}(\lambda)\psi = \lambda_0 P^{(T)}_{\{\lambda_0\}}\psi = \lambda_0\psi$$

and then $\psi \in \sigma_p(T)$, contradicting $\lambda_0 \in \rho(T)$. Furthermore, the resolvent exists (as T is closed and by Theorem 8.4(a, b)). This is the operator $R_{\lambda_0}(T) \in \mathfrak{B}(\mathsf{H})$ that satisfies

$$(T - \lambda_0 I)R_{\lambda_0}(T) = I \quad \text{and} \quad R_{\lambda_0}(T)(T - \lambda_0 I) = I \restriction_{D(T)} .$$

On the other hand Theorem 9.4(c) and $P^{(T)}(\{\lambda_0\}) = 0$ imply

$$\left(\int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0}\, dP^{(T)}(\lambda)\right)(T - \lambda_0 I) = I \restriction_{D(T)} , \quad (T - \lambda_0 I)\int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0}\, dP^{(T)}(\lambda) = I$$

(again, beware of domains). From the first we also see that the domain of $\int_{\mathbb{R}} \frac{1}{\lambda-\lambda_0}\, dP^{(T)}(\lambda)$ is $D(T - \lambda_0 I)$, i.e. H. It does not really matter how one defines $\lambda \mapsto \frac{1}{\lambda-\lambda_0}$ at $\lambda = \lambda_0$, because $P^{(T)}(\{\lambda_0\}) = 0$. By uniqueness of the inverse, then,

$$\int_{\mathbb{R}} \frac{1}{\lambda - \lambda_0}\, dP^{(T)}(\lambda) = R_{\lambda_0}(T) ,$$

and the operator on the left is bounded. Now suppose by contradiction that $\lambda_0 \in supp(P^{(T)})$. Then any open set containing λ_0, in particular any interval $I_n := (\lambda_0 - 1/n, \lambda_0 + 1/n)$, must satisfy $P^{(T)}(I_n) \neq 0$. Take $\psi_n \in P^{(T)}_{I_n}(\mathsf{H}) \setminus \{0\}$ for any $n = 1, 2, \ldots$. Without loss of generality assume $||\psi_n|| = 1$. Using Theorem 9.4(f) we obtain

$$||R_{\lambda_0}(T)\psi_n||^2 = \left|\left|\int_{\mathbb{R}} \frac{1}{\lambda-\lambda_0}\, dP^{(T)}(\lambda)\psi_n\right|\right|^2 = \int_{I_n} \frac{1}{|\lambda-\lambda_0|^2} d\mu_{\psi_n}(\lambda)$$

$$\geq \inf_{I_n} \frac{1}{|\lambda-\lambda_0|^2} \int_{I_n} d\mu_{\psi_n}(\lambda) \geq \inf_{I_n} \frac{1}{|\lambda-\lambda_0|^2} = n^2 \to +\infty \quad \text{as}\, n \to +\infty.$$

We have reached the absurd that $R_{\lambda_0}(T)$ cannot be bounded. Therefore $\lambda_0 \notin supp(P^{(T)})$.

Let us prove the first part of (i); the second part can be proved as we did for the corresponding part of (i) in Theorem 8.56(b), using (9.22). As above, if $P^{(T)}(\{x\}) \neq 0$ then $x \in \sigma_p(T)$. Suppose $x \in \sigma_p(T)$. By definition of Cayley transform, $((x-i)/(x+i)) \in \sigma_p(V)$. We may apply (i) in Theorem 8.56(b) to the normal (unitary) operator V replacing T. Then $P^{(V)}(\{\frac{x-i}{x+i}\}) \neq 0$. Looking at the way the PVM associated to T was obtained from the PVM of V, we see $P^{(T)}(x) = P^{(V)}(\{\frac{x-i}{x+i}\}) \neq 0$.

Now to (ii). By Proposition 8.7(a), $x \in \sigma_c(T)$ means: (1) $x \in \sigma(T)$ but (2) $x \notin \sigma_p(T)$. Assertion (1) implies $x \in supp(P^{(T)})$, so any open set A_x containing x must satisfy $P^{(T)}(A_x) \neq 0$. Number (2) is equivalent to $P^{(T)}(\{x\}) = 0$ (otherwise (i) would give a contradiction).

The proof of (iii) is immediate: if $x \in supp(P^{(T)})$ is an isolated point, then $P^{(T)}(\{x\}) \neq 0$, otherwise x could not belong to $supp(P^{(T)})$, and using (i) the claim follows.

At last, let us prove (iv). If $x \in \sigma_c(T)$, using (ii) on the intervals $I_n := (x - 1/n, x + 1/n)$, $n = 1, 2, \ldots$, we have $P^{(T)}(I_n) \neq 0$. So choose $\psi_n \in P^{(T)}_{I_n}(\mathsf{H})$ with $||\psi_n|| = 1$ for any n. Then

$$||T\psi_n - x\psi_n||^2 = \left(\int_{\mathbb{R}} (\lambda - x) dP^{(T)}(\lambda)\psi_n \middle| \int_{\mathbb{R}} (\lambda - x) dP^{(T)}(\lambda)\psi_n\right) =$$

$$\left(\int_{\mathbb{R}} (\lambda - x) dP^{(T)}(\lambda) P^{(T)}_{I_n}\psi_n \middle| \int_{\mathbb{R}} (\lambda - x) dP^{(T)}(\lambda)\psi_n\right)$$

Using Theorem 9.4(c) the last inner product is

$$\int_{\mathbb{R}} \chi_{I_n}(x)(\lambda - x)^2 d\mu_{\psi_n}(\lambda) \leq \int_{I_n} \sup_{I_n} (\lambda - x)^2 d\mu_{\psi_n}(\lambda) =$$

$$= n^{-2} \int_{I_n} d\mu_{\psi_n}(\lambda) = n^{-2} \int_{\mathbb{R}} d\mu_{\psi_n}(\lambda) = n^{-2} ||\psi_n||^2 \,.$$

So for any $n = 1, 2, \ldots$ there exists a unit vector ψ_n with $||T\psi_n - x\psi_n|| \leq 1/n$. The claim follows, since $x \notin \sigma_p(T)$ by assumption, and $0 < ||T\psi_n - x\psi_n||$. $\square$

Having eventually settled the spectral theorem we can pass to a definition useful for the applications to QM.

Definition 9.14 Consider a self-adjoint operator T on the Hilbert space H and a Borel-measurable map $f : \sigma(T) \to \mathbb{C}$. The operator:

$$f(T) := \int_{\sigma(T)} f(x) dP^{(T)}(x) , \tag{9.52}$$

with domain

$$D(f(T)) = \Delta_f := \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\sigma(T)} |f(x)|^2 d\mu_\psi^{(T)}(x) < +\infty \right\} ,$$

where $\mu_\psi^{(T)}(E) := (\psi | P^{(T)}(E)\psi)$ for any $E \in \mathscr{B}(\sigma(T))$, is called **function of the operator** T.

Since $supp(P^{(T)}) = \sigma(T)$, the PVM $P^{(T)}$ associated to T can be thought of as being defined either on $\sigma(T)$ or on $\mathbb{R}$. Even when defined on $\sigma(T)$ only (precisely, on the Borel σ-algebra $\mathscr{B}(\sigma(T))$) we still have $supp(P^{(T)}) = \sigma(T)$ by the definition of support in the subspace $\sigma(T)$ of $\mathbb{R}$ with induced topology. Therefore we can view the right integral in (9.52) as living on $\mathbb{R}$, by extending f trivially (as zero) outside $\sigma(T)$ or directly taking a measurable $f : \mathbb{R} \to \mathbb{C}$:

$$f(T) := \int_{\mathbb{R}} f(x) dP^{(T)}(x) ,$$

with

$$D(f(T)) = \Delta_f := \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\mathbb{R}} |f(x)|^2 d\mu_\psi^{(T)}(x) < +\infty \right\} .$$

In the sequel we shall use the most convenient viewpoint without further explanations. We leave to the reader the obvious check that the definition of $f(T)$ coincides with the known one when $T \in \mathfrak{B}(\mathsf{H})$, $f \in M_b(\sigma(T))$ (relying on the functional calculus for bounded measurable functions, cf. Chap. 8).

Remarks 9.15
(1) The spectral theorem allows for a second decomposition of the spectrum of a self-adjoint operator $T : D(T) \to \mathsf{H}$. Its constituents are the **discrete spectrum**

$$\sigma_d(T) := \left\{ \lambda \in \sigma(T) \;\middle|\; \dim\left(P^{(T)}_{(\lambda-\varepsilon,\lambda+\varepsilon)}(\mathsf{H})\right) \text{ is finite for some } \varepsilon > 0 \right\} ,$$

and the **essential spectrum** $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_d(T)$.
It is not hard to see that $\lambda \in \sigma_d(T) \;\Leftrightarrow\; \lambda$ is an isolated point in $\sigma(T)$, and as such, λ is an eigenvalue for T with finite-dimensional eigenspace. By Theorem 9.13 $\sigma_d(T) \subset \sigma_p(T)$. In general, though, the opposite inclusion fails, for instance because there may be non-isolated points in $\sigma_p(T)$.
(2) A third spectral decomposition for a self-adjoint operator $T : D(T) \to \mathsf{H}$ arises by splitting the Hilbert space into the closed span H_p of the eigenvectors and its

orthogonal complement: $\mathsf{H} = \mathsf{H}_p \oplus \mathsf{H}_p^\perp$. Both $\mathsf{H}_p \cap D(T)$ and $\mathsf{H}_p^\perp \cap D(T)$ are easily T-invariant. With the obvious symbols:

$$T = T\restriction_{\mathsf{H}_p} \oplus T\restriction_{\mathsf{H}_p^\perp} \ .$$

One calls **purely continuous spectrum** the set $\sigma_{pc}(T) := \sigma(T\restriction_{\mathsf{H}_p^\perp})$, where for simplicity $T\restriction_{\mathsf{H}_p^\perp}$ stands for $T\restriction_{D(T)\cap\mathsf{H}_p^\perp}$ here and in the sequel. Then $\sigma(T) = \overline{\sigma_p(T)} \cup \sigma_{pc}(T)$. The latter is not necessarily a disjoint union, and in general $\sigma_{pc}(T) \neq \sigma_c(T)$.
(3) A fourth spectral decomposition of $T : D(T) \to \mathsf{H}$ on the Hilbert space H (and even on a normed space), is that into **approximate point spectrum**

$$\sigma_{ap}(T)$$

$$:= \left\{\lambda \in \sigma(T) \mid (T - \lambda I)^{-1} : Ran(T - \lambda I) \to D(T) \text{ does not exist or is not bounded}\right\}$$

and **purely residual spectrum** $\sigma_{pr}(T) := \sigma(T) \setminus \sigma_{ap}(T)$. The unboundedness of $(T - \lambda I)^{-1}$ is equivalent to the existence of $\delta > 0$ with $||(T - \lambda I)\psi|| \geq \delta||\psi||$ for any $\psi \in D(T)$, so we immediately see how the next result comes about, thereby justifying the names: $\lambda \in \sigma_{ap}(T) \Leftrightarrow$ there exists a unit vector $\psi_\varepsilon \in D(T)$ such that

$$||T\psi - \lambda\psi|| \leq \varepsilon$$

for any $\varepsilon > 0$. For self-adjoint operators the above holds for any $\lambda \in \sigma_c(T)$ due to Theorem 8.56(b), but clearly also for $\lambda \in \sigma_p(T)$; since $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$ in this case, we conclude $\sigma_{ap}(T) = \sigma(T)$ and $\sigma_{pr}(T) = \varnothing$ for every self-adjoint operator.
(4) The last partial spectral classification for self-adjoint operators (cf. [ReSi80, vol. I] and [Gra04]) descends from Lebesgue's Theorem 1.77 on the decomposition of Borel measures on $\mathbb{R}$. If T is self-adjoint on the Hilbert space H and μ_ψ is the spectral measure of the vector ψ (Theorem 8.52(c)), we define the sets (all closed spaces):
$\mathsf{H}_{ac} := \{\psi \in \mathsf{H} \mid \mu_\psi$ is absolutely continuous for the Lebesgue measure$\}$,
$\mathsf{H}_{sing} := \{\psi \in \mathsf{H} \mid \mu_\psi$ is continuous and singular for the Lebesgue measure$\}$,
$\mathsf{H}_{pa} := \{\psi \in \mathsf{H} \mid \mu_\psi$ is purely atomic (hence singular for the Lebesgue measure)$\}$.
Then set $\sigma_{ac}(T) := \sigma(T\restriction_{\mathsf{H}_{ac}})$, $\sigma_{sing}(T) := \sigma(T\restriction_{\mathsf{H}_{sing}})$, respectively called **absolutely continuous spectrum of** T and **singular spectrum of** T. It turns out that $\sigma_{ac}(T) \cup \sigma_{sing}(T) = \sigma_{pc}(T)$ and $\overline{\sigma_p(T)} = \sigma(T\restriction_{\mathsf{H}_{pa}})$.
(5) As $supp(P^{(T)}) = \sigma(T)$, definition (9.52) reads:

$$f(T) := \int_{\sigma(T)} f(x)dP^{(T)}(x) \ . \tag{9.53}$$

Likewise, the domain of $f(T)$ is

$$D(f(T)) = \left\{ \psi \in \mathsf{H} \;\middle|\; \int_{\sigma(T)} |f(x)|^2 d\mu_\psi^{(T)}(x) \; < +\infty \right\} ,$$

as $supp(\mu_{\phi,\psi}) \subset supp(P^{(T)})$. Eventually, if $f : \mathbb{R} \to \mathbb{R}$ is measurable, decomposing $f(T)$ (self-adjoint by Theorem 9.4(b)) under the spectral Theorem 9.13 produces

$$\int_{\sigma(f(T))} \lambda dP^{(f(T))}(\lambda) = \int_{\sigma(T)} f(\lambda) dP^{(T)}(\lambda) = \int_{\sigma(f(T))} \lambda dP^{(T)}(f^{-1}(\lambda)) \,. \quad (9.54)$$

The last identity follows from Theorem 9.4(h). By uniqueness of the PVM associated to $f(T)$ we have

$$P^{f(T)}(E) = P^{(T)}(f^{-1}(E)) \quad \text{for any } E \in \mathscr{B}(\sigma(f(T))). \quad (9.55)$$

(6) Theorem 9.3(d) implies that for any self-adjoint T, the standard domain of a polynomial $p(T)$ coincides with the domain of $p(T)$ thought of as function of T according to Definition 9.14. By definition of standard domain we also have, for any self-adjoint T:

$$D(T^m) \subset D(T^n) \,, \qquad \text{for any } 0 \leq n \leq m \text{ in } \mathbb{N}. \quad (9.56)$$

■

Functions of an operator enjoy properties that descend directly from Theorems 9.3 and 9.4. The next proposition specifies more features of the spectrum of $f(T)$. In order to stay general, we shall state the first result for spectral measures that do not necessarily come from self-adjoint operators. But first a definition.

Definition 9.16 If $P : \mathscr{B}(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ is a PVM on the topological space X and $f : \mathsf{X} \to \mathbb{C}$ is Borel measurable, the **essential rank of f with respect to P**

$$ess\ ran_P(f) \subset \mathbb{C}$$

is the closed complement of the union of all open sets $A \subset \mathbb{C}$ such that $P(f^{-1}(A)) = 0$. I.e., $z \in ess\ ran_P(f) \;\Leftrightarrow\; P(f^{-1}(A)) \neq 0$ if $A \subset \mathbb{C}$ open and $z \in A$.

(Note that $f^{-1}(A) \in \mathscr{B}(\mathsf{X})$ since f is Borel measurable and A is open.) If V is the union of said sets A then $P(f^{-1}(V)) = 0$, because V is the union of countably many sets A by Lindelöf's lemma, and PVMs are sub-additive.

Proposition 9.17 *Let $P : \mathscr{B}(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ be a PVM on the topological space X and $f : \mathsf{X} \to \mathbb{C}$ a Borel measurable map. If $E_z := f^{-1}(\{z\})$, $z \in \mathbb{C}$ then:*
(a)

$$\sigma\left(\int_{\mathsf{X}} f dP\right) = ess\ ran_P(f) \,,$$

and in particular, for $z \in ess\ ran_P(f)$:

(i) $P(E_z) \neq 0 \Rightarrow z \in \sigma_p\left(\int_X f dP\right)$,
(ii) $P(E_z) = 0 \Rightarrow z \in \sigma_c\left(\int_X f dP\right)$,
(hence $\sigma_r\left(\int_X f dP\right) = \varnothing$ *even if* f *is not essentially bounded).*
Specialising to the PVM $P^{(T)} : \sigma(T) \to \mathfrak{B}(\mathsf{H})$ *of a normal or self-adjoint operator* T*, the following results hold for* $f : \sigma(T) \to \mathbb{C}$ *(Borel measurable).*
(b) *If* $f : \sigma(T) \to \mathbb{C}$ *is continuous and* T *self-adjoint,* $\sigma(f(T)) = \overline{f(\sigma(T))}$*, with bar denoting closure.*
(c) *If* $f : \sigma(T) \to \mathbb{C}$ *is continuous and* $T \in \mathfrak{B}(\mathsf{H})$ *normal,* $\sigma(f(T)) = f(\sigma(T))$.
(d) *If* $f : \sigma(T) \to \mathbb{C}$ *is measurable and* T *as in (b) or (c), then* $\sigma_p(f(T)) \supset f(\sigma_p(T))$ *(not an equality, in general).*

Proof (a) In the sequel $\Psi(f) := \int_X f dP$ and we assume $z = 0$ without loss of generality. Let us prove (i). If $P(E_0) \neq 0$, there exists $x \in P(E_0)(\mathsf{H})$ with $||x|| = 1$. Call $\chi := \chi_{E_0}$, so $f\chi = 0$ and $\Psi(f)\Psi(\chi) = \mathbf{0}$ by Theorem 9.4(c). As $\Psi(\chi) = P(E_0)$, $\Psi(f)x = \Psi(f)P(E_0)x = \Psi(f)\Psi(\chi)x = \mathbf{0}$, proving (i).

Now to (ii). By assumption $P(E_0) = 0$, but $P(F_n) \neq 0$ if $F_n := \{s \in X \,|\, |f(s)| < 1/n\}$, $n = 1, 2, \ldots$, because $z \in ess\, ran_P(f)$. Choose $x_n \in P(F_n)(\mathsf{H})$, $||x_n|| = 1$ and let $\chi_n := \chi_{F_n}$. As before $||\Psi(f)x_n|| = ||\Psi(f\chi_n)x_n|| \leq ||\Psi(f\chi_n)|| = ||f\chi_n||_\infty \leq 1/n$. Therefore $\Psi(f)x_n \to \mathbf{0}$, notwithstanding $||x_n|| = 1$. This shows that $\Psi(f)^{-1}$ (and, similarly, $(\Psi(f) - zI)^{-1}$) cannot be bounded if it exists. To show 0 (z in general) is in the continuous spectrum we need prove $Ker(\Psi(f)) = \{\mathbf{0}\}$ and $\overline{Ran(\Psi(f))} = \mathsf{H}$. Suppose $\Psi(f)x = \mathbf{0}$ for some $x \in \Delta_f$. Then

$$\int_X |f| d\mu_x = ||\Psi(f)x||^2 = 0\,. \tag{9.57}$$

As $P(f^{-1}(0)) = 0$ and hence $(x|P(f^{-1}(0))x) = 0$, we have $|f| > 0$ almost everywhere for μ_x, so that (9.57) entails $0 = \mu_x(X) = ||x||^2$. That is, $Ker(\Psi(f)) = \{\mathbf{0}\}$. To finish part (ii) we prove $\overline{Ran(\Psi(f))} = \mathsf{H}$. Since $\Psi(f)^* = \Psi(\overline{f})$, the same argument used above tells $Ker(\Psi(f)^*) = \{\mathbf{0}\}$ and $\overline{Ran(\Psi(f))} = (Ker(\Psi(f)^*)^\perp)^\perp = \{\mathbf{0}\}^\perp = \mathsf{H}$.

There remains the first assertion in (a). By (i)–(ii) we have $ess\, ran_P(f) \subset \sigma(\Psi(f))$. For the opposite inclusion assume $z = 0 \notin ess\, ran_P(f)$ ($z \neq 0$ is analogous). Then $f' := 1/f$ is bounded and $ff' = 1$, so $\psi(f)\Psi(f') = I$ and $Ran(\Psi(f)) = \mathsf{H}$. Since $|f| > 0$, $\Psi(f)$ is one-to-one as in case (ii). Therefore $\Psi(f)^{-1} \in \mathfrak{B}(\mathsf{H})$ by the closed graph theorem. This ends the proof because Theorem 8.4(a) implies $0 \notin \sigma(\Psi(f))$.
(b) Recall $supp(P^{(T)}) = \sigma(T)$ (viewing $P^{(T)}$ on $\mathbb{C}$, or only on $\sigma(T)$ if we use the induced topology). Certainly $f(supp(P^{(T)})) \subset \operatorname{ran\,ess}_P(f)$ (if $z \in f(supp(P^{(T)}))$, for any open set $A \subset \mathbb{C}$, $A \ni z$, the open set $f^{-1}(A)$ includes $x \in supp(P^{(T)})$ with $f(x) = z$ and so $P^{(T)}(f^{-1}(A)) \neq 0$ otherwise $x \notin supp(P^{(T)})$). Because $\operatorname{ran\,ess}_{P^{(T)}}(f)$ is closed, we have $\overline{f(supp(P^{(T)}))} \subset \operatorname{ran\,ess}_{P^{(T)}}(f)$. If $z \in \operatorname{ran\,ess}_{P^{(T)}}(f)$ but $z \notin \overline{f(supp(P^{(T)}))}$, there would be an open set $A \ni z$ not intersecting $f(supp(P^{(T)}))$. Thus $P^{(T)}(f^{-1}(A)) = P^{(T)}(\varnothing) = 0$, which cannot be by definition of $\operatorname{ran\,ess}_{P^{(T)}}(f)$.

(c) The statement is straightforward from (b): if $T \in \mathfrak{B}(\mathsf{H})$, then the spectrum $\sigma(T)$ is compact and its continuous image in $\mathbb{C}$ under f is compact, so closed and $\overline{f(\sigma(T))} = f(\sigma(T))$.
(d) If $\lambda \in \sigma_p(T)$ then $P^{(T)}(\{\lambda\}) \neq 0$. If $x \in P^{(T)}(\{\lambda\})(\mathsf{H}) \setminus \{\mathbf{0}\}$, using Theorem 9.4(c) we get $f(T)x = f(T)\chi_{\{\lambda\}}(T)x = f(\lambda)x$, hence $f(\lambda) \in \sigma_p(f(T))$.
Here is an example where $\sigma_p(f(T)) \supsetneq f(\sigma_p(T))$ for $T = T^*$. Take $(a, b) \subset \sigma_c(T)$ so that $P^{(T)}((a, b)) \neq 0$. If f is measurable, it equals a constant $c > 0$ on (a, b) and $f(\lambda) = 0$ outside the interval, so $c \in \sigma_p(f(T))$ by (i) in (a). But $c \notin f(\sigma_p(T))$ (which at most includes $0 < c$) and hence $\sigma_p(f(T)) \supsetneq f(\sigma_p(T))$. $\square$

9.1.4 *Example of Operator with Point Spectrum: The Hamiltonian of the Harmonic Oscillator*

On the complex Hilbert space $L^2(\mathbb{R}, dx)$ (dx is the Lebesgue measure on $\mathbb{R}$) consider the operator

$$H_0 := \frac{1}{2m}\left(P\restriction_{\mathscr{S}(\mathbb{R})}\right)^2 + \frac{m\omega^2}{2}\left(X\restriction_{\mathscr{S}(\mathbb{R})}\right)^2 ,$$

where X, P are the position and momentum operators for a particle moving on the real line, seen in Chap. 5. In other terms

$$H_0 := -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 ,$$

where x^2 stands for the multiplication by $\mathbb{R} \ni x \mapsto x^2$ and $\hbar$, ω, m are positive constants. Define $D(H_0) := \mathscr{S}(\mathbb{R})$, where $\mathscr{S}(\mathbb{R})$ is the Schwartz space of $\mathbb{R}$, i.e. the space of smooth complex functions that vanish at infinity, together with any derivative, faster than any negative power of x (see Example 2.91).

The numbers $\hbar$, ω, m have no mathematical relevance (and could be set to 1 in the sequel), yet it is their physical meaning that is important. The operator H_0 is called the **Hamiltonian of the one-dimensional harmonic oscillator** with characteristic frequency $\omega/(2\pi)$ for a particle of mass m, and $h := 2\pi\hbar$ is Planck's constant. Note that H_0 is not an observable as it is not self-adjoint. However, its closure $\overline{H_0}$ is self-adjoint: it is the *energy observable* of the system under exam. In this section we shall not be concerned with the physical background, and just study the operator from a mathematical perspective, leaving any comment about the physics to Chaps. 12 and 13.

Evidently H_0 is symmetric, as Hermitian and because $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R}, dx)$. Moreover, it admits self-adjoint extensions by von Neumann's criterion (Theorem 5.43), for it commutes with the (anti-unitary) complex conjugation of $L^2(\mathbb{R}, dx)$. We will show H_0 is essentially self-adjoint, provide an explicit expression for it in terms of the spectral expansion of its unique self-adjoint extension $\overline{H_0}$, and also describe the spectrum.

Let us introduce three operators, called **creation operator**, **annihilation operator** and **number operator**:

$$A^\star := \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right), \quad A := \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right), \quad \mathscr{N} := A^\star A\,. \tag{9.58}$$

In this case, as well, we assume the operators are densely defined on $D(A) = D(A^\star) = D(\mathscr{N}) := \mathscr{S}(\mathbb{R})$. It should be clear that $A^\star \subset A^*$, justifying the notation, and $\mathscr{N}$ is further symmetric. Notice $\mathscr{S}(\mathbb{R})$ is dense and invariant under H_0, A, $A^\star$. Using A, $A^\star$ we will build eigenvectors for $\mathscr{N}$ and H_0 that form a basis in $L^2(\mathbb{R}, dx)$. As eigenvectors are obviously analytic vectors, by Nelson's criterion (Theorem 5.47) H_0 and $\mathscr{N}$ are essentially self-adjoint on their domain $\mathscr{S}(\mathbb{R})$.

We start by observing that, by definition, the commutation relation

$$[A, A^\star] = I\,, \tag{9.59}$$

holds, *where either side acts on the dense invariant space* $\mathscr{S}(\mathbb{R})$.[1] The proof is immediate. What is more, still by definition,

$$H_0 = \hbar\omega\left(A^\star A + \frac{1}{2}I\right) = \hbar\omega\left(\mathscr{N} + \frac{1}{2}I\right)\,. \tag{9.60}$$

Consider the equation in $\mathscr{S}(\mathbb{R})$:

$$A\psi_0 = 0\,, \tag{9.61}$$

A solution is, easily,

$$\psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{s}}\, e^{-\frac{x^2}{2s^2}}\,, \quad s := \sqrt{\frac{\hbar}{m\omega}}\,,$$

where the factor was chosen so to normalise $||\psi_0|| = 1$. The function ψ_0 is the first Hermite function introduced in Example 3.32(4), provided we use the variable $x' = x/s$ and consider the factor $1/\sqrt{s}$ not to destroy the normalisation. Now define vectors:

$$\psi_n := \frac{(A^\star)^n}{\sqrt{n!}}\psi_0 \tag{9.62}$$

for $n = 1, 2, \ldots$. Only using (9.61), (9.59) it is easy to prove by induction that

$$A\psi_n = \sqrt{n}\psi_{n-1}\,, \quad A^\star\psi_n = \sqrt{n+1}\psi_{n+1}\,, \quad (\psi_n|\psi_m) = \delta_{nm}\,, \tag{9.63}$$

$n, m \in \mathbb{N}$. The second identity actually follows from the definition of the ψ_n, whilst the first is proved like this:

[1] More appropriately, identity (9.59) should be written $[A, A^\star] \subset I$.

$$A\psi_n = \frac{1}{\sqrt{n!}}A(A^\star)^n\psi_0 = \frac{1}{\sqrt{n!}}[A,(A^\star)^n]\psi_0 + \frac{1}{\sqrt{n!}}(A^\star)^n A\psi_0 = \frac{1}{\sqrt{n!}}[A,(A^\star)^n]\psi_0 + 0\,;$$

but (9.59) implies $[A,(A^\star)^n] = n(A^\star)^{n-1}$, substituting which above gives what needed. Here is the proof of the third identity (for $n \geq m$, the other case is similar):

$$(\psi_m|\psi_n) = \frac{1}{\sqrt{n!m}}(\psi_{m-1}|A(A^\star)^n\psi_0) = \frac{1}{\sqrt{n!m}}(\psi_{m-1}|[A,(A^\star)^n]\psi_0)$$

$$= \frac{n}{\sqrt{n!m}}(\psi_{m-1}|(A^\star)^{n-1}\psi_0) = \sqrt{\frac{n}{m}}(\psi_{m-1}|\psi_{n-1}) = \cdots = \sqrt{\frac{n!}{m!(n-m)!}}(\psi_0|\psi_{n-m})\,.$$

If $n = m$ the result is 1, otherwise 0, for

$$(\psi_0|\psi_{n-m}) = (n-m)^{-1/2}(\psi_0|A^\star\psi_{n-m-1}) = (n-m)^{-1/2}(A\psi_0|\psi_{n-m-1}) = 0\,.$$

The second equation in (9.63) (the normalisation is preserved when using $x' = x/s$ because of $1/\sqrt{s}$) is the recurrence relationship of Hermite functions mentioned in Example 3.32(4). Hence the ψ_n are (up to a multiplicative constant and a change of variable) Hermite functions, and so they are a basis of $L^2(\mathbb{R}, dx)$. The last equation in (9.63) implies $\{\psi_n\}_{n\in\mathbb{N}}$ is, as it should, an orthonormal system in $L^2(\mathbb{R}, dx)$; the first two tell

$$\mathscr{N}\psi_n = n\psi_n\,, \tag{9.64}$$

so by (9.60) the ψ_n are a Hilbert basis of eigenvectors of H_0, as:

$$H_0\psi_n = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n\,. \tag{9.65}$$

By the way this proves H_0 (but also $\mathscr{N}$) is unbounded, since the set $\{||H_0\psi|| \mid \psi \in D(H_0)\,,\, ||\psi|| = 1\}$ contains all numbers $\hbar\omega(n + 1/2)$, $n \in \mathbb{N}$. By Nelson's criterion (Theorem 5.47) the symmetric operators $\mathscr{N}$, H_0 are both essentially self-adjoint, since their domains contain a set $\{\psi_n\}_{n\in\mathbb{N}}$ of analytic vectors spanning a dense subset in $L^2(\mathbb{R}, dx)$.

To obtain the spectral decomposition of $\overline{H_0}$, construct a spectral measure on $\mathbb{R}$ with support on the naturals $n \in \mathbb{N}$:

$$P_F := \text{s-}\sum_{n\in F\cap\mathbb{N}} \psi_n(\psi_n|\ \ ) \quad \text{for } F \in \mathscr{B}(\mathbb{R}).$$

The PVM we have found can be reinterpreted as a PVM defined on $\mathbb{N}$, identified with $N := \{\psi_n\}_{n\in\mathbb{N}}$. Following in the footsteps of Example 9.10(1), for any measurable map $f : \mathbb{R} \to \mathbb{C}$ we have

$$\int_{\mathbb{R}} f(x)dP(x) = \int_{\mathbb{N}} f(\phi(z))dP(z) = \text{s-}\sum_{n\in\mathbb{N}} f\left(\hbar\omega\left(n+\frac{1}{2}\right)\right)\psi_n(\psi_n|\ \ )\,,$$

where the last equality is (9.40). Taking f to be $\mathbb{R} \ni x \mapsto x$, we obtain the self-adjoint operator (for Theorem 9.4(b))

$$H := \int_{\mathbb{R}} x dP(x) = \text{s-}\sum_{n\in\mathbb{N}} \hbar\omega\left(n+\frac{1}{2}\right)\psi_n(\psi_n|\ \ )\,. \tag{9.66}$$

We claim $H = \overline{H_0}$. Let $< N >$ be the dense space spanned by finite combinations of the ψ_n. By Nelson's criterion $H_0\restriction_{<N>}$ is still essentially self-adjoint. Therefore $\overline{H_0} = \overline{H_0\restriction_{<N>}}$, i.e. H_0 and $H_0\restriction_{<N>}$ have the same (unique) self-adjoint extension (their closure). On the other hand H is certainly a self-adjoint extension of $H_0\restriction_{<N>}$, because (9.66) implies

$$H\psi_n = \omega\left(n+\frac{1}{2}\right)\psi_n = H_0\psi_n$$

for any n, and so $H\restriction_{<N>} = H_0\restriction_{<N>}$. Therefore H must be the unique self-adjoint extension of $H_0\restriction_{<N>}$, hence of H_0. This means $H = \overline{H_0}$, as was claimed. Under the spectral decomposition Theorem 9.13 the spectral measure associated to $\overline{H_0}$ is $\mathscr{B}(\mathbb{R}) \ni F \mapsto P_F$, and we also have the spectral decomposition of $\overline{H_0}$ into

$$\overline{H_0} = \text{s-}\sum_{n\in\mathbb{N}} \hbar\omega\left(n+\frac{1}{2}\right)\psi_n(\psi_n|\ \ )\,.$$

Eventually, using Theorem 9.13(b), from the latter we obtain

$$\sigma(\overline{H_0}) = \sigma_p(\overline{H_0}) = \left\{\hbar\omega\left(n+\frac{1}{2}\right)\,\middle|\, n\in\mathbb{N}\right\}\,.$$

We must remark that the spectrum of $\overline{H_0}$ is a point spectrum and eigenspaces are all finite-dimensional, even though the operator itself is not compact (it is unbounded). Yet the first and second inverse powers of $\overline{H_0}$ are compact, for they are a Hilbert–Schmidt operator and a trace-class operator respectively (exercise).

The numbers in $\sigma_p(\overline{H_0})$ are, physically, the levels of total mechanical energy that a quantum oscillator may assume for given ω, m, in contrast to the classical case where the energy varies with continuity.

9.1.5 *Examples with Continuous Spectrum: The Operators Position and Momentum*

We return to the operators position (5.12)–(5.13) and momentum (5.18)–(5.19) on the Hilbert space $\mathsf{H} = L^2(\mathbb{R}^3, dx)$ with Lebesgue measure. In the sequel we shall set $x = (x_1, x_2, x_3)$. We saw that X_i and P_i, $i = 1, 2, 3$, are self-adjoint, and we set out to determine their spectra and spectral expansion.

Start by the position operator X_1. The findings will work for X_2 and X_3 by swapping names. A PVM on $\mathbb{R}$ with values in $\mathfrak{B}(\mathsf{H}) = \mathfrak{B}(L^2(\mathbb{R}^3, dx))$ is

$$(P(E)\psi)(x_1, x_2, x_3) = \chi_E(x_1)\psi(x_1, x_2, x_3) \quad \text{for any } E \in \mathscr{B}(\mathbb{R}), \psi \in L^2(\mathbb{R}^3, dx). \tag{9.67}$$

If $\psi \in L^2(\mathbb{R}^3, dx)$, it is easy to see the measure μ_ψ on $\mathscr{B}(\mathbb{R})$ is defined by:

$$\mu_\psi(E) = \int_{E\times\mathbb{R}^2} |\psi(x_1, x_2, x_3)|^2 dx\ , \quad \mathrm{E} \in \mathscr{B}(\mathbb{R}),$$

so

$$\int_{\mathbb{R}} g(y) d\mu_\psi(y) = \int_{E\times\mathbb{R}^2} f(x_1)\psi(x_1, x_2, x_3) dx \tag{9.68}$$

for $g : \mathbb{R} \to \mathbb{C}$ Borel measurable. In analogy to Example 9.10(2) it is easy to check, for $f : \mathbb{R} \to \mathbb{C}$ Borel measurable and $\psi \in \Delta_f$ (i.e. $\int_{\mathbb{R}} |f(x_1)\psi(x_1, x_2, x_3)|^2 dx < +\infty$), that

$$\left(\int_{\mathbb{R}} f(y) dP(y)\psi\right)(x_1, x_2, x_3) = f(x_1)\psi(x_1, x_2, x_3) \quad \text{a.e. for } (x_1, x_2, x_3) \in \mathbb{R}^3. \tag{9.69}$$

We can then introduce the operator X_1' associated, in (9.69), to the map $f := f_1 : \mathbb{R} \ni y \mapsto y$. It is self-adjoint by Theorem 9.4(b), as the map is real. By comparison with (5.13) we infer $\Delta_{f_1} = D(X_1)$, and from (9.69) we get

$$X_1'\psi = X_1\psi \quad \text{for any } \psi \in D(X_1).$$

The spectral decomposition Theorem 9.13 warrants uniqueness of the spectral measure, whence (9.67) is the spectral measure associated to X_1. The spectral expansion of X_i, $i = 1, 2, 3$, must therefore be

$$\left(\int_{\mathbb{R}} y dP^{(X_i)}(y)\psi\right)(x_1, x_2, x_3) = (X_i\psi)(x_1, x_2, x_3) \quad \text{a.e. for } (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{9.70}$$

where

$$(P^{(X_i)}(E)\psi)(x_1, x_2, x_3) = \chi_E(x_i)\psi(x_1, x_2, x_3) \quad \forall \mathrm{E} \in \mathscr{B}(\mathbb{R}), \psi \in L^2(\mathbb{R}^3, dx). \tag{9.71}$$

This spectral measure allows to find the spectrum of X_i, $i = 1, 2, 3$. Applying (ii) in Theorem 9.13(b) immediately gives

$$\sigma(X_i) = \sigma_c(X_i) = \mathbb{R} \,. \tag{9.72}$$

Now to momenta. The argument is rather straightforward because of Proposition 5.31, since the Fourier-Plancherel transform is unitary. As such, it preserves spectra (Exercise 8.9), so if K_i are the position operators (as in Proposition 5.31):

$$\sigma(P_i) = \sigma(\hbar\hat{\mathscr{F}}^{-1} K_i \hat{\mathscr{F}}) = \hbar\mathbb{R} = \mathbb{R} \,,$$

i.e.

$$\sigma(P_i) = \sigma_c(P_i) = \mathbb{R} \,. \tag{9.73}$$

The spectral measure of P_i must be supported on the whole $\mathbb{R}$. The reader may prove easily, using Proposition 5.31 and Exercises 9.1–9.4, that the PVM associated to the momentum P_i is just

$$P^{(P_i)}(E) = \hat{\mathscr{F}}^{-1} P^{(K_i)} \hat{\mathscr{F}} \,, \quad E \in \mathscr{B}(\mathbb{R}). \tag{9.74}$$

where $P^{(K_i)}$ is the spectral measure of K_i.

9.1.6 *Spectral Representation of Unbounded Self-adjoint Operators*

The next *spectral representation* generalises Theorem 8.58 to self-adjoint unbounded operators. The details are left as exercise, as they essentially replicate the proof of Theorem 8.58.

Theorem 9.18 (Spectral representation of unbounded self-adjoint operators) *Let* H *be a Hilbert space,* $T : D(T) \to \mathsf{H}$ *a self-adjoint operator on* H*,* $P^{(T)}$ *the PVM of* T *over Borel sets of* $\sigma(T)$ *according to Theorem 9.13.*
(a) H *may be decomposed as a Hilbert sum* $\mathsf{H} = \oplus_{\alpha\in A} H_\alpha$ *(*A *countable, at most, if* H *is separable), whose summands* H_α *are closed and orthogonal. Moreover:*
(i) for any $\alpha \in A$ *and any measurable map* $f : \sigma(T) \to \mathbb{C}$*,*

$$f(T)(\mathsf{H}_\alpha \cap D(f(T))) \subset \mathsf{H}_\alpha \,,$$

in particular

$$T(\mathsf{H}_\alpha \cap D(T)) \subset \mathsf{H}_\alpha \,,$$

(ii) for any $\alpha \in A$ *there exist a unique finite positive Borel measure* μ_α *on* $\sigma(T) \subset \mathbb{R}$*, and a surjective isometric operator* $U_\alpha : \mathsf{H}_\alpha \to L^2(\sigma(T), \mu_\alpha)$*, such that:*

$$U_\alpha \left(\int_{\sigma(T)} f(x) dP^{(T)}(x) \right) \restriction_{\mathsf{H}_\alpha \cap D(f(T))} U_\alpha^{-1} = f \cdot$$

for any measurable $f : \sigma(T) \to \mathbb{C}$*. In particular*

$$U_\alpha T \restriction_{\mathsf{H}_\alpha \cap D(T)} U_\alpha^{-1} = x \cdot \, ,$$

where $f\cdot$ *is the multiplication by* f *on* $L^2(\sigma(T), \mu_\alpha)$*:*

$$(f \cdot g)(x) = f(x)g(x) \quad \textit{a.e. on}\, \sigma(T) \textit{if}\, g \in D(f\cdot),$$

and

$$D(f\cdot) := \{g \in L^2(\sigma(T), \mu_\alpha) \mid f \cdot g \in L^2(\sigma(T), \mu_\alpha)\} \, .$$

(b) *We have*

$$\sigma(T) = \overline{\bigcup_{\alpha \in A} supp(\mu_\alpha)} \, .$$

(c) *If* H *is separable, there exist a measure space* $(\mathsf{M}_T, \Sigma_T, \mu_T)$*,* $\mu_T(\mathsf{M}_T) < +\infty$*, a map* $F_T : M_T \to \mathbb{R}$ *and a unitary operator* $U_T : \mathsf{H} \to L^2(\mathsf{M}_T, \mu_T)$ *such that:*

$$\left(U_T T U_T^{-1} f\right)(m) = F_T(m) f(m) \, , \quad f \in L^2(\mathsf{M}_T, \mu_T), U_T^{-1} f \in D(T). \tag{9.75}$$

Proof The proof mimics Theorem 8.58 for T self-adjoint and any H_ψ. Apart from the obvious adaptations, it suffices to replace bounded measurable maps $M_b(\sigma(T))$ with the space $L^2(\sigma(T), \mu_\psi)$, paying attention to domains. □

9.1.7 Joint Spectral Measures

The final notion of this section is the *joint spectral measure* of self-adjoint operators with commuting spectral measures.

Theorem 9.19 (Joint spectral measure) *Let* $\mathbf{A} := \{A_1, A_2, \ldots, A_n\}$ *be a set of self-adjoint operators (even unbounded) on the separable Hilbert space* H*, and suppose the associated spectral measures* $P^{(A_k)}$ *commute:*

$$P^{(A_k)}(E) P^{(A_h)}(E') = P^{(A_h)}(E') P^{(A_k)}(E) \, , \quad E, E' \in \mathscr{B}(\mathbb{R}), h, k \in \{1, 2, \ldots, n\}.$$

Then there exists a unique PVM $P^{(\mathbf{A})} : \mathscr{B}(\mathbb{R}^n) \to \mathfrak{B}(\mathsf{H})$ *such that*

$$P^{(\mathbf{A})}(E^{(1)}\times\cdots\times E^{(n)}) = P^{(A_1)}(E^{(1)})\cdots P^{(A_n)}(E^{(n)}), \quad E^{(k)} \in \mathscr{B}(\mathbb{R}), k = 1, \ldots, n. \tag{9.76}$$

This PVM $P^{(\mathbf{A})}$ *is called the* **joint spectral measure** *of* $A_1, A_2, \ldots, A_n$ *and* $supp(P^{(\mathbf{A})})$ *is the* **joint spectrum** *of* $\mathbf{A}$.

For any measurable $f : \mathbb{R} \to \mathbb{C}$*:*

$$\int_{\mathbb{R}^n} f(x_k(x))dP^{(\mathbf{A})}(x) = \int_{\mathbb{R}} f(x_k)dP^{(A_k)}(x_k) = f(A_k)\,, \quad k = 1, 2, \ldots, n, \tag{9.77}$$

where $x_k(x)$ *is the kth component of* $x = (x_1, x_2, \ldots, x_k \ldots, x_n) \in \mathbb{R}^n$.

Proof We need a couple of technical lemmas.

Lemma 9.20 *Let* H *be a Hilbert space,* $\{P_\alpha\}_{\alpha\in A} \subset \mathscr{L}(\mathsf{H})$ *an infinite family of orthogonal projectors such that* $P_\alpha P_{\alpha'} = P_{\alpha'} P_\alpha = P_\beta$ *for any* $\alpha, \alpha' \in A$ *and some* $\beta \in A$ *depending on* α, α'. *Define* $\mathsf{M}_a := P_\alpha(\mathsf{H})$, $\mathsf{M} := \cap_{\alpha\in A}\mathsf{M}_\alpha$ *and let* P_M *be the orthogonal projector onto* M.
(a) *If* H *is separable, there exists a countable subfamily* $\{\mathsf{M}_{\alpha_m}\}_{m\in\mathbb{N}}$ *such that* $\cap_{m\in\mathbb{N}}\mathsf{M}_{\alpha_m} = \mathsf{M}$.
(b) $(\psi|P_\mathsf{M}\psi) = \inf_{\alpha\in A}(\psi|P_\alpha\psi)$ *for any* $\psi \in \mathsf{H}$.

Proof (a) We have $\mathsf{H}\setminus\mathsf{M} = \cup_{\alpha\in A}(\mathsf{H}\setminus\mathsf{M}_\alpha)$, where the $\mathsf{H}\setminus\mathsf{M}_\alpha$ form an open covering of $\mathsf{H}\setminus\mathsf{M}$. As H is separable, it is second countable (see Remark 2.86(2) because the topology of H is induced by the norm distance). By Theorem 1.8 we can take a countable subcovering $\mathsf{H}\setminus\mathsf{M} = \cup_{m\in\mathbb{N}}(\mathsf{H}\setminus\mathsf{M}_{\alpha_m})$. Now we take complements in H, and obtain (a).
(b) Noting that $P_1 \geq P_2$ for orthogonal projectors means $(x|P_1x) \geq (x|P_2x)$ for all $x \in \mathsf{H}$, the claim is nothing but item (i) in Theorem 7.22(a). □

Lemma 9.21 *Let* $\mathscr{A}$ *be an algebra (Definition 1.30) or a* σ*-algebra of subsets in* X. *If* $P : \mathscr{A} \to \mathscr{L}(\mathsf{H})$, *with* H *a Hilbert space, satisfies (c) and (d) in definition 8.41 (the latter if* $\cup_n E_n \in \mathscr{A}$*), then it also satisfies (a) and (b) of that definition.*

Proof The proof is the same as for Proposition 8.44(a). □

It is easy to see that finite unions of products $E^{(1)} \times \cdots \times E^{(n)}$, with $E^{(k)} \in \mathscr{B}(\mathbb{R})$, form an *algebra of sets*, denoted $\mathscr{B}_0(\mathbb{R}^n)$; the same can be obtained by taking *disjoint* finite unions of products (just decompose further in case of non-empty intersections). The σ-algebra generated by $\mathscr{B}_0(\mathbb{R}^n)$ contains countable unions of products of open balls in $\mathbb{R}$: as $\mathbb{R}^n$ is second countable, the σ-algebra includes all open sets in $\mathbb{R}^n$, so a fortiori the Borel σ-algebra $\mathscr{B}(\mathbb{R}^n)$, and then it must coincide with the latter.
If $S = \cup_{j=1}^N E_j^{(1)}\times\cdots\times E_j^{(n)} \in \mathscr{B}_0(\mathbb{R}^n)$ with $(E_j^{(1)}\times\cdots\times E_i^{(n)})\cap(E_i^{(1)}\times\cdots\times E_j^{(n)}) = \varnothing$, $i \neq j$, define:

$$Q(S) := \sum_{j=1}^{N} P^{(A_1)}(E_j^{(1)}) \cdots P^{(A_n)}(E_j^{(n)}) .$$

Since $P^{(A_k)}(E_j^{(k)})$ are commuting orthogonal projectors, every $Q(S)$ is an orthogonal projector that commutes with every other $Q(S')$. It is not hard to prove $\mathscr{B}_0(\mathbb{R}^n) \ni S \mapsto Q(S)$ satisfies $Q(\varnothing) = 0$, $Q(\mathbb{R}^n) = I$, and $s\text{-}\sum_{n\in\mathbb{S}} Q(S_n) \in \mathcal{P}(\mathsf{H})$ exists when $S_k \cap S_h = \varnothing$, $h \neq k$. Moreover the 'strong' sum equals $Q(\cup_{k\in\mathbb{N}} S_k)$ if $\cup_{k\in\mathbb{N}} S_k \in \mathscr{B}_0(\mathbb{R}^n)$. Applying Lemma 9.21 gives $Q(S_1)Q(S_2) = Q(S_1 \cap S_2)$ if $S_1, S_2 \in \mathscr{B}_0(\mathbb{R}^n)$.

If $R \in \mathscr{B}(\mathbb{R}^n)$ let $P^{(\mathbf{A})}(R)$ be the projector onto the intersection of all projection spaces of $\sum_k Q(S_k)$, for any family $\{S_k\}_{k\in\mathbb{N}} \subset \mathscr{B}_0(\mathbb{R}^n)$ such that $S_k \cap S_h = \varnothing$ for $h \neq k$, $\cup_{n\in\mathbb{N}} S_k \supset R$. By construction $P^{(\mathbf{A})}(\mathbb{R}^n) = I$: if $\cup_{k\in\mathbb{N}} S_k = \mathbb{R}^n$, for $\mathbb{R} \in \mathscr{B}_0(\mathbb{R}^n)$, σ-additivity implies $\sum_k Q(S_k) = Q(\mathbb{R}^n) = I$. The latter projects onto H, so $P(\mathbb{R}^n) = I$. Using Lemma 9.20, with $\psi \in \mathsf{H}$:

$$(\psi | P^{(\mathbf{A})}(R)\psi)$$

$$= \inf \left\{ \left(\psi \left| \sum_{k\in\mathbb{N}} Q(S_k)\psi \right. \right) \left| \bigcup_{k\in\mathbb{N}} S_k \supset R \, , \{S_k\}_{k\in\mathbb{N}} \subset \mathscr{B}_0(\mathbb{R}^n) \, , S_k \cap S_h = \varnothing \text{ for } k \neq h \right. \right\} .$$

As consequence of Theorem 1.41, for $\psi \in \mathsf{H}$, $\mathscr{B}(\mathbb{R}^n) \ni E \mapsto (\psi | P^{(\mathbf{A})}(R)\psi)$ defines a positive σ-additive finite measure on $\mathscr{B}(\mathbb{R}^n)$, namely the unique extension of $\mathscr{B}_0(\mathbb{R}^n) \ni S \mapsto (\psi | Q(S)\psi)$. In other words, it is the only positive σ-additive measure ν_ψ on $\mathscr{B}(\mathbb{R}^n)$ such that $\nu_\psi(E_j^{(1)} \times \cdots \times E_j^{(n)}) = (\psi | P^{(A_1)}(E_j^{(1)}) \cdots P^{(A_n)}(E_j^{(n)})\psi)$, for any $E^{(k)} \in \mathscr{B}(\mathbb{R})$. Using the polarisation formula, $\mathscr{B}(\mathbb{R}^n) \ni E \mapsto (\psi | P(R)\phi)$ is, for $\psi, \phi \in \mathsf{H}$, a complex measure on $\mathscr{B}(\mathbb{R}^n)$. Therefore $\mathscr{B}(\mathbb{R}^n) \ni E \mapsto P(R)$ satisfies (a), (b), (c), (d) in Definition 8.41: (a) holds because $P^{(\mathbf{A})}(R)$ is a projector, (c) by construction and (d) by σ-additivity of $\mathscr{B}(\mathbb{R}^n) \ni E \mapsto (\psi | P^{(\mathbf{A})}(R)\phi)$. Eventually, (b) follows from Lemma 9.21. The identity $P^{(\mathbf{A})}(E^{(1)} \times \cdots \times E^{(n)}) = P^{(A_1)}(E^{(1)}) \cdots P^{(A_n)}(E^{(n)})$ implies $P^{(\mathbf{A})}(\Pi_k^{-1}(E^{(k)})) = P^{(A_k)}(E^{(k)})$ for any $E^{(k)} \in \mathscr{B}(\mathbb{R})$, where $\Pi_k : \mathbb{R}^n \to \mathbb{R}$ is the kth canonical projection of $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$. Using Theorem 9.4(h) with $\phi := \Pi_k$ and the spectral Theorem 9.13 for A_k gives

$$\int_{\mathbb{R}^n} f(\Pi_k(x)) dP^{(\mathbf{A})}(x) = \int_{\mathbb{R}} f(x_k) dP^{(A_k)}(x_k) = f(A_k) \, , \quad k = 1, 2, \ldots, n.$$

Now, every PVM $P' : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathsf{H})$ satisfying

$$P'(E^{(1)} \times \cdots \times E^{(n)}) = P^{(A_1)}(E^{(1)}) \cdots P^{(A_n)}(E^{(n)})$$

for any $E^{(k)} \in \mathscr{B}(\mathbb{R})$ must also satisfy

$$(\psi | P'(E^{(1)} \times \cdots \times E^{(n)})\psi) = (\psi | P^{(A_1)}(E^{(1)}) \cdots P^{(A_n)}(E^{(n)})\psi) .$$

As positive measures doing just that are unique, we have $(\psi|P'(R)\psi) = (\psi|P^{(\mathbf{A})}(R)\psi)$ for any $R \in \mathscr{B}(\mathbb{R}^n)$ and any $\psi \in \mathsf{H}$. Therefore $P^{(\mathbf{A})} = P'$, since the previous relation, by polarisation, implies $(\psi|P'(R)\phi) = (\psi|P^{(\mathbf{A})}(R)\phi)$ for $\psi, \phi \in \mathsf{H}$. $\square$

An exhaustive discussion on joint spectral measures, their integrals, and the meaning in QM can be found in [Pru81] and [BeCa81]. In analogy to Theorem 9.11 we could prove what follows (see [BeCa81], and Exercise 9.6 for $n = 1$). An introductory definition is necessary.

Definition 9.22 Let $\mathbf{A} = \{A_1, \ldots, A_n\}$ be a collection of self-adjoint operators on the Hilbert space H. The **commutant of A**, denoted by $\mathbf{A}'$, is the von Neumann algebra made of all elements of $\mathfrak{B}(\mathsf{H})$ that commute with the spectral measures of each $A_k \in \mathbf{A}$.

We leave to the reader to prove that this definition reduces to the standard one if $\mathbf{A} \subset \mathfrak{B}(\mathsf{H})$, and to the definition in Remark 5.13(4) when $n = 1$.

Proposition 9.23 *Let* $\mathbf{A} = \{A_1, \ldots, A_n\}$ *be a collection of self-adjoint operators on the separable Hilbert space* H *whose spectral measures commute. The von Neumann algebra* $\mathbf{A}''$ *(the set of operators in* $\mathfrak{B}(\mathsf{H})$ *commuting with operators in* $\mathfrak{B}(\mathsf{H})$ *that commute with all spectral measures) coincides with the collection of operators* $f(A_1, \ldots, A_n) := \int_{supp(P^{(\mathbf{A})})} f(x_1, \ldots, x_n) dP^{(\mathbf{A})}$ *with* $f : supp(P^{(\mathbf{A})}) \to \mathbb{C}$ *measurable and bounded.*

If f is real-valued, $f(A_1, \ldots, A_n)$ is self-adjoint: interpreted as an observable, it is a function of the observables $A_1, \ldots, A_n$ of the quantum system. This corresponds to the notion of Remark 7.47(2).

9.2 Exponential of Unbounded Operators: Analytic Vectors

This section is short and technical. We go back to *analytic vectors*, introduced at the end of Chap. 5, and uncover other properties in the light of the theory developed since. The results will be used at various places in the rest of the book.

Here is an interesting general problem. If A is a self-adjoint operator on the Hilbert space H, the exponential e^{zA} can be defined as a function of A (Definition 9.14). We expect, in some cases, to be able to employ the Taylor expansion:

$$e^{zA} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} A^n \ ,$$

using Definition 9.14 for the left-hand side. If $A \in \mathfrak{B}(\mathsf{H})$ the above identity does hold, provided we understand the expansion in the uniform topology, as is easy to see (Exercise 8.16). If A is not bounded, the issue is subtler and the above equation makes no sense in the uniform topology. As Nelson clarified, it has a meaning in the strong topology and over a dense subspace in the Hilbert space, which is a core for A: we shall prove in Proposition 9.25 that this dense core is the space of analytic vectors for A.

Let A be an operator with domain $D(A)$ on the Hilbert space H. Recall (Definition 5.44) that a vector $\psi \in D(A)$ such that $A^n\psi \in D(A)$ for any $n \in \mathbb{N}$ ($A^0 := I$) is called a C^∞ **vector for** A. The subspace of C^∞ vectors for A is written $C^\infty(A)$. Furthermore, $\psi \in C^\infty(A)$ is an **analytic vector for** A if

$$\sum_{n=0}^{+\infty} \frac{||A^n\psi||}{n!} t^n < +\infty \,, \quad \text{for some } t > 0. \tag{9.78}$$

Recall also Nelson's Theorem 5.47, for which a symmetric operator on a Hilbert space is essentially self-adjoint if its domain contains analytic vectors whose finite combinations are dense.

Notation 9.24 If A is an operator on H with domain $D(A)$, we shall indicate by $\mathscr{A}(A)$ the subset in $C^\infty(A)$ of elements satisfying (9.78). ■

The next proposition discusses useful properties of analytic vectors, in particular the exponential of (self-adjoint) unbounded operators.

Proposition 9.25 *Let A be an operator on the Hilbert space* H.
(a) *$\mathscr{A}(A)$ is a vector space.*
(b) *If A is closable:*

$$\mathscr{A}(A) \subset \mathscr{A}(\overline{A}) \,.$$

(c) *(i) For any $c \in \mathbb{C}$, defining $A + cI$ on its standard domain:*

$$\mathscr{A}(A + cI) = \mathscr{A}(A) \,.$$

(ii) For any $c \in \mathbb{C} \setminus \{0\}$, defining cA on its standard domain:

$$\mathscr{A}(cA) = \mathscr{A}(A) \,.$$

(iii) If A is Hermitian, defining A^2 on its standard domain:

$$\mathscr{A}(A^2) \subset \mathscr{A}(A) \,.$$

(d) *If A is self-adjoint and $\psi \in \mathscr{A}(A) \cap D(e^{zA})$, viewing e^{zA} as in Definition 9.14:*

$$e^{zA}\psi = \sum_{n=0}^{+\infty} \frac{z^n}{n!} A^n\psi \quad \textit{for any } z \in \mathbb{C},\, |z| \leq t, \textit{ and } t \textit{ satisfying } (9.78) \textit{ for } \psi \,. \tag{9.79}$$

If $Re z = 0$, equation (9.79) holds for $\psi \in \mathscr{A}(A)$, provided $|z| \leq t$ and t solves (9.78) for the given ψ.

(e) *If A is self-adjoint, viewing e^{zA} as in Definition 9.14:*

$$e^{isA}(\mathscr{A}(A)) \subset \mathscr{A}(A)\,, \quad s \in \mathbb{R}.$$

(f) *If A is self-adjoint, $D(A)$ contains a dense subset made of analytic vectors for $p(A)$ for any $t > 0$ in (9.78) (where A is replaced by $p(A)$) and for any complex polynomial $p(A)$ of A.*

Proof (a) The claim follows from the estimate

$$||A^n(a\psi + b\phi)|| \leq |a|\,||A^n\psi|| + |b|\,||A^n\phi||\,,$$

$\psi, \phi \in \mathscr{A}(A)$, by choosing $t > 0$ small enough to satisfy (9.78) for ψ and ϕ.
(b) This is a direct consequence of the definitions, for $\overline{A}$ is an extension of A and so $\overline{A}^n$ extends A^n.
(c) To prove (i), note that if $t > 0$ satisfies (9.78) for ψ, then:

$$+\infty > M \geq e^{|tc|}\sum_{k=0}^{+\infty}\frac{t^k\left|\left|A^k\psi\right|\right|}{k!} = \sum_{p=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{|tc|^p}{p!}\frac{\left|\left|t^kA^k\psi\right|\right|}{k!}\,.$$

The Fubini–Tonelli theorem on the counting product measure of $\mathbb{N}$ allows us to compute the product of the series (integral in the product measure) as a double integral in:

$$M \geq \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{|tc|^{n-k}}{(n-k)!}\frac{\left|\left|t^kA^k\psi\right|\right|}{k!} \geq \sum_{n=0}^{+\infty}\frac{t^n}{n!}\left|\left|\sum_{k=0}^{n}\frac{n!c^{n-k}A^k\psi}{k!(n-k)!}\right|\right| = \sum_{n=0}^{+\infty}\frac{t^n}{n!}||(A+cI)^n\psi||$$

Therefore $\mathscr{A}(A + cI) \supset \mathscr{A}(A)$. Now define $A' := A + cI$, so $A = A' + c'I$ and $c' = -c$. It follows that $\mathscr{A}(A' + c'I) \supset \mathscr{A}(A')$, which is equivalent to $\mathscr{A}(A) \supset \mathscr{A}(A + cI)$, so $\mathscr{A}(A) = \mathscr{A}(A + cI)$. Property (ii) is obvious by definition, so let us see to (iii). By construction $C^\infty(A) = C^\infty(A^2)$. Since A is Hermitian and $\sqrt{x} \leq 1 + x$ for $x \geq 0$, in $C^\infty(A)$ we have:

$$||A^n\psi|| = \sqrt{(\psi|A^{2n}\psi)} \leq \sqrt{||\psi||}\sqrt{||A^{2n}\psi||} \leq \sqrt{||\psi||}(1 + ||(A^2)^n\psi||)\,.$$

The claim is therefore true, since for $t > 0$:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}||A^n\psi|| \leq \sqrt{||\psi||}\sum_{n=0}^{+\infty}\frac{t^n}{n!}||(A^2)^n\psi|| + \sqrt{||\psi||}\sum_{n=0}^{+\infty}\frac{t^n}{n!}$$

$$= \sqrt{||\psi||}\left(\sum_{n=0}^{+\infty}\frac{t^n}{n!}||(A^2)^n\psi|| + e^t\right).$$

(d) For some $\phi \in \mathsf{H}$, $\mu_{\phi,\psi}$ is the complex measure $\mu_{\phi,\psi}(E) := \left(\phi \left| P^{(A)}(E)\, \psi\right.\right)$, and for any $\chi \in \mathsf{H}$, $\mu_\chi(E) := \left(\chi \left| P^{(A)}(E)\, \chi\right.\right)$ is the usual positive finite spectral measure. If $z \in \mathbb{C}$ and $|z| \leq t$ then, using Lemma 9.2:

$$\sum_{n=0}^{+\infty} \int_{\sigma(A)} \left|\frac{z^n}{n!} x^n\right| d|\mu_{\phi,\psi}(x)| \leq \sum_{n=0}^{+\infty} \int_{\sigma(A)} \left|\frac{z^n}{n!} x^n\right| d|\mu_{\phi,\psi}(x)| = \sum_{n=0}^{+\infty} \left|\frac{z^n}{n!}\right| \int_{\sigma(A)} |x^n| d|\mu_{\phi,\psi}(x)|$$

$$\leq \sum_{n=0}^{+\infty} \frac{t^n}{n!} ||\phi|| \left(\int_{\sigma(A)} x^{2n} d\mu_\psi(x)\right)^{1/2} = \sum_{n=0}^{+\infty} ||\phi|| \frac{t^n}{n!} \, ||A^n \psi|| < +\infty \,,$$

where (9.78) is needed in the last passage. Then Theorem 1.87 (for that $|h| = 1$ a.e.) and Fubini–Tonelli imply, for $|z| \leq t$, that we may swap sum and integral:

$$\sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\sigma(A)} x^n d\mu_{\phi,\psi}(x) = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\sigma(A)} x^n h d|\mu_{\phi,\psi}(x)|$$

$$= \int_{\sigma(A)} \sum_{n=0}^{+\infty} \frac{z^n}{n!} x^n h d|\mu_{\phi,\psi}(x)| = \int_{\sigma(A)} \sum_{n=0}^{+\infty} \frac{z^n}{n!} x^n d\mu_{\phi,\psi}(x) \,.$$

Hence for $|z| \leq t$, if ψ belongs to the domain of e^{zA} (cf. Definition 9.14) and by virtue of Theorem 9.4(e):

$$(\phi|e^{zA}\psi) = \int_{\sigma(A)} e^{zx} d\mu_{\phi,\psi} = \int_{\sigma(A)} \sum_{n=0}^{+\infty} \frac{z^n}{n!} x^n d\mu_{\phi,\psi} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{\sigma(A)} x^n d\mu_{\phi,\psi} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} (\phi|A^n\psi).$$

By (9.78) the series

$$\sum_{n=0}^{+\infty} \frac{z^n}{n!} A^n \psi$$

converges in H, and the inner product is continuous, so the above identity reads

$$(\phi|e^{zA}\psi) = \left(\phi \left| \sum_{n=0}^{+\infty} \frac{z^n}{n!} A^n \psi \right.\right).$$

As ϕ is arbitrary, we have (9.79). In case $Rez = 0$, i.e. $z = is$, $s \in \mathbb{R}$, the map $\mathbb{R} \ni x \mapsto e^{isx}$ is clearly bounded, so $e^{isA} \in \mathfrak{B}(\mathsf{H})$ (the domain is H) by Corollary 9.5.
(e) If A is self-adjoint, by Theorem 9.4(c) $e^{isA}(e^{isA})^* = (e^{isA})^* e^{isA} = I$, so e^{isA} is unitary. Using Theorem 9.4(c) with $\psi \in \mathscr{A}(A) \subset C^\infty(A)$ produces $A^n e^{isA}\psi = e^{isA} A^n \psi$, but e^{isA} is unitary and $||e^{isA} A^n \psi|| = ||A^n \psi||$, whence the claim follows.
(f) Consider the spectral decomposition $A = \int_{\mathbb{R}} x dP^{(A)}(x)$, partition the real line $\mathbb{R} = \cup_{n\in\mathbb{Z}} (n, n+1]$ and take its closed, pairwise-orthogonal subspaces $\mathsf{H}_n = P_n(\mathsf{H})$, where we define projectors $P_n := \int_{(n,n+1]} 1 dP^{(A)}(x)$. Choosing a basis $\{\psi_k^{(n)}\}_{k\in K_n} \subset$

H_n for any n, the union of all bases is a basis of H. Notice $supp(\mu_{\psi_k^{(n)}}) \subset (n, n+1]$ by definition of μ_ϕ (Theorem 8.52). From Theorem 9.4(e) every $\psi_k^{(n)}$ belongs in $D(A)$, since $\int_{\mathbb{R}} |x|^2 d\mu_{\psi_k^{(n)}}(x) = \int_{(n,n+1]} |x|^2 d\mu_{\psi_k^{(n)}}(x) \leq |n+1|^2$, Moreover (9.78) holds for any $t > 0$, as $||A^m \psi_k^{(n)}||^2 = \int_{(n,n+1]} |x|^{2m} d\mu_{\psi_k^{(n)}}(x) \leq |n+1|^{2m} ||\psi_k^{(n)}||^2$. Finite linear combinations are, by construction, a dense subspace in H, and analytic for A (for any $t > 0$) by (a).

Now take a complex polynomial $p_N(x) = \sum_{k=0}^N x^n$ of degree N, and define $p_N(A)$ on the domain $D(p_N(A)) = D(A^N)$ (Theorem 9.4(d)). We will check every $\psi_k^{(n)}$ is analytic for the closed (self-adjoint if p_N is real) $p_N(A)$ by Theorem 9.4. Choose one of them of unit norm, say ψ, and suppose its spectral measure μ_ψ has support in some interval $(-L, L]$. Then $||A^k\psi|| \leq L^k ||\psi|| = L^k$. Therefore

$$||p_N(A)\psi|| = \left|\left|\sum_{k=0}^N a_k A^k \psi\right|\right| \leq \sum_{k=0}^N |a_k| ||A^k\psi|| = \sum_{k=0}^N |a_k| L^k .$$

In a similar manner:

$$||p_N(A)^n\psi|| = \left|\left|\sum_{k_1,\ldots,k_n=0}^N a_{k_1}\cdots a_{k_n} A^{k_1+\cdots+k_n}\psi\right|\right| \leq \sum_{k_1,\ldots,k_n=0}^N |a_{k_1}|\cdots|a_{k_n}| ||A^{k_1+\cdots+k_n}\psi||$$

$$\leq \sum_{k_1,\ldots,k_n=0}^N |a_{k_1}|\cdots|a_{k_n}| L^{k_1+\cdots+k_n} .$$

We conclude that if $M_L := \sum_{k=0}^N |a_k| L^k$, then

$$||p_N(A)^n\psi|| \leq M_L^n \quad \text{and} \quad \sum_{n=0}^{+\infty} \frac{t^n}{n!} ||p_N(A)^n\psi|| \leq e^{tM_L}$$

and so ψ (by (a), any combination of such vectors) is analytic for $p_N(A)$, for any $t > 0$. $\square$

9.3 Strongly Continuous One-Parameter Unitary Groups

The goal of this section is to prove *Stone's theorem*, one of the most important results in view of the applications to QM (and not only that). To state it we will present some preliminary results about one-parameter groups of unitary operators, and in particular an important theorem due to von Neumann.

9.3.1 Strongly Continuous One-Parameter Unitary Groups, von Neumann's Theorem

Definition 9.26 *(One-parameter group of operators)* Let H be a Hilbert space. A collection $\{U_t\}_{t\in\mathbb{R}} \subset \mathfrak{B}(\mathsf{H})$ is called a **one-parameter group (of operators)** on H if

$$U_0 = I \quad \text{and} \quad U_t U_s = U_{t+s} \text{for any } t, s \in \mathbb{R} \tag{9.80}$$

A one-parameter group $\{U_t\}_{t\in\mathbb{R}} \subset \mathfrak{B}(\mathsf{H})$ is said:
(a) one-parameter unitary group if U_t is unitary for any $t \in \mathbb{R}$,
(b) weakly continuous at $t_0 \in \mathbb{R}$, or **strongly continuous at** $t_0 \in \mathbb{R}$, if the mapping $t \mapsto U_t$ is continuous at t_0 in the weak, resp. strong, topology (and $\mathbb{R}$ is standard);
(c) weakly continuous or **strongly continuous** if it is weakly, or strongly, continuous at each point of $\mathbb{R}$.

By (9.80), if the U_t are unitary:

$$(U_t)^* = U_t^{-1} = U_{-t} \,, \qquad \text{for any } t \in \mathbb{R} \,. \tag{9.81}$$

Proposition 9.27 *Let $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter unitary group on the Hilbert space* $(\mathsf{H}, (\cdot|\cdot))$. *The following assertions are equivalent.*
(a) $(\psi|U_t\psi) \to (\psi|\psi)$ *as* $t \to 0$ *for any* $\psi \in \mathsf{H}$.
(b) $\{U_t\}_{t\in\mathbb{R}}$ *is weakly continuous at* $t = 0$.
(c) $\{U_t\}_{t\in\mathbb{R}}$ *is weakly continuous.*
(d) $\{U_t\}_{t\in\mathbb{R}}$ *is strongly continuous at* $t = 0$.
(e) $\{U_t\}_{t\in\mathbb{R}}$ *is strongly continuous.*

Proof First, let us number the properties.
(1) $\{U_t\}_{t\in\mathbb{R}}$ is weakly continuous at $t = 0$.
(2) $(\psi|U_t\psi) \to (\psi|\psi)$ as $t \to 0$ for any $\psi \in \mathsf{H}$.
(3) $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous at $t = 0$.
(4) $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous.
(5) $\{U_t\}_{t\in\mathbb{R}}$ is weakly continuous.
We will show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Weak continuity at $t = 0$ implies, when $t \to 0$, that $(\psi|U_t\psi) \to (\psi|U_0\psi) = (\psi|\psi)$ and $(U_t\psi|\psi) \to (U_0\psi|\psi) = (\psi|\psi)$ by conjugation.

(2) $\Rightarrow$ (3). Strong continuity at $t = 0$ amounts to saying, for any $\psi \in \mathsf{H}$,

$$||U_t\psi - U_0\psi|| \to 0$$

as $t \to 0$. Since $U_0 = I$, squaring and writing norms via inner products transforms the above into

$$(U_t\psi|U_t\psi) - (\psi|U_t\psi) - (U_t\psi|\psi) + (\psi|\psi) \to 0 \,.$$

But U_t unitary implies $(U_t\psi|U_t\psi) = (\psi, \psi)$, so the identity reads

$$(\psi|\psi) - (\psi|U_t\psi) - (U_t\psi|\psi) + (\psi, \psi) \to 0\,, \quad \text{as}\, t \to 0\,.$$

As we said at the beginning, the latter holds if (2) does.

(3) $\Rightarrow$ (4). If $\psi \in \mathsf{H}$:

$$U_t\psi - U_{t_0}\psi = U_t(\psi - U_t^{-1}U_{t_0}\psi) = U_t(\psi - U_{t_0-t}\psi)\,,$$

where (9.81) was used. As U_t is unitary, for any $\psi \in \mathsf{H}$:

$$||U_s\psi - U_{t_0}\psi|| = ||U_s(\psi - U_{t_0-s}\psi)|| = ||\psi - U_{t_0-s}\psi||\,.$$

Under strong continuity at $t = 0$, since $t_0 - s \to 0$ for $s \to t_0$, we find $||U_s\psi - U_{t_0}\psi|| \to 0$. Hence strong continuity at $t = 0$ forces strong continuity at any $t_0 \in \mathbb{R}$.

(4) $\Rightarrow$ (5). Obvious because strong convergence implies weak convergence.

(5) $\Rightarrow$ (1). True by definition. $\square$

Here is another property of unitary groups.

Proposition 9.28 *Let $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter unitary group on the Hilbert space $(\mathsf{H}, (\cdot|\cdot))$, and $\mathscr{H} \subset \mathsf{H}$ a subset such that:*
(a) *the finitely-generated span $< \mathscr{H} >$ is dense in H,*
(b) $(\psi|U_t\psi) \to (\psi|\psi)$*, as $t \to 0$, for any $\psi \in \mathscr{H}$.*
Then $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group.

Proof The same argument used in Proposition 9.27 gives that $(\phi_0|U_t\phi_0) \to (\phi_0|\phi_0)$, as $t \to 0$, for $\phi_0 \in \mathscr{H}$ implies $||U_t\phi_0 - \phi_0|| \to 0$, $t \to 0$. If, more generally, $\phi \in< \mathscr{H} >$ then $\phi = \sum_{i\in I} c_i\phi_{0i}$ where I is finite and $\phi_{0i} \in \mathscr{H}$. Hence as $t \to 0$

$$||U_t\phi - \phi|| = \left|\left|U_t\left(\sum_i c_i\phi_{0i}\right) - \sum_i c_i\phi_{0i}\right|\right| = \left|\left|\sum_i c_i(U_t\phi_{0i} - \phi_{0i})\right|\right|$$

$$\leq \sum_i |c_i|||U_t\phi_{0i} - \phi_{0i}|| \to 0\,.$$

By Proposition 9.27 it now suffices to extend this to H. That is to say, $||U_t\phi-\phi|| \to 0$, as $t \to 0$, for any $\phi \in< \mathscr{H} >$ implies $||U_t\psi - \psi|| \to 0$, $t \to 0$, for any $\psi \in \mathsf{H}$. As $< \mathscr{H} >$ is dense, for any given $\psi \in \mathsf{H}$ there is a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset< \mathscr{H} >$ with $\psi_n \to \phi$, $n \to +\infty$. If $\{t_m\}_{m\in\mathbb{N}}$ is a real infinitesimal sequence, by the triangle inequality

$$||U_{t_m}\psi - \psi|| \leq ||U_{t_m}\psi - U_{t_m}\phi_n|| + ||U_{t_m}\phi_n - \phi_n|| + ||\phi_n - \psi||$$

for any given $n \in \mathbb{N}$. Since the U_{t_m} are unitary and the norm non-negative, that means

$$0 \leq ||U_{t_m}\psi - \psi|| \leq ||U_{t_m}\phi_n - \phi_n|| + 2||\phi_n - \psi|| . \tag{9.82}$$

For fixed n, $||U_{t_m}\phi_n - \phi_n|| \to 0$, $m \to +\infty$, by assumption, so:

$$\limsup_m ||U_{t_m}\phi_n - \phi_n|| = \liminf_m ||U_{t_m}\phi_n - \phi_n|| = \lim_{m\to+\infty} ||U_{t_m}\phi_n - \phi_n|| = 0 .$$

By (9.82), for any $n \in \mathbb{N}$:

$$0 \leq \limsup_m ||U_{t_m}\psi - \psi|| \leq 2||\phi_n - \psi|| , \quad 0 \leq \liminf_m ||U_{t_m}\psi - \psi|| \leq 2||\phi_n - \psi|| .$$

On the other hand for n large enough we can make $||\phi_n - \psi||$ infinitesimal, so:

$$\limsup_m ||U_{t_m}\psi - \psi|| = \liminf_m ||U_{t_m}\psi - \psi|| = 0 .$$

Therefore

$$\lim_{m\to+\infty} ||U_{t_m}\psi - \psi|| = 0 .$$

As $\psi \in \mathsf{H}$ and the $\{t_m\}_{m\in\mathbb{N}}$ are arbitrary, for any $\psi \in \mathsf{H}$ we have:

$$\lim_{t\to 0} ||U_t\psi - \psi|| = 0 ,$$

ending the proof. $\square$

The theory developed thus far puts us in the position to prove an important result due to von Neumann, which shows how the strong continuity of one-parameter unitary groups is, actually, not such restrictive a fact in separable Hilbert spaces.

Theorem 9.29 (Von Neumann) *Let $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter unitary group on the Hilbert space $(\mathsf{H}, (\cdot|\cdot))$. If H is separable, $\{U_t\}_{t\in\mathbb{R}}$ is strongly continuous if and only if the map $\mathbb{R} \ni t \mapsto (U_t\psi|\phi)$ is Borel measurable for any $\psi, \phi \in \mathsf{H}$.*

Proof Obviously if the group is strongly continuous then any $\mathbb{R} \ni t \mapsto (U_t\psi|\phi)$ is Borel measurable, being continuous. We show the converse. Suppose every such map is Borel measurable, hence Lebesgue measurable. Schwarz's inequality and $||U_t|| = 1$ imply that these maps are bounded. Given $a \in \mathbb{R}$, $\psi \in \mathsf{H}$,

$$\mathsf{H} \ni \phi \mapsto \int_0^a (U_t\psi|\phi)dt$$

is a bounded linear functional with norm not exceeding $|a|\ ||\psi||$ by Schwarz and $||U_t|| = 1$. Riesz's Theorem 3.16 provides $\psi_a \in \mathsf{H}$ such that

$$(\psi_a|\phi) = \int_0^a (U_t\psi|\phi)dt , \quad \text{for any } \phi \in \mathsf{H}.$$

So

$$(U_b\psi_a|\phi) = (\psi_a|U_{-b}\phi) = \int_0^a (U_t\psi|U_{-b}\phi)dt = \int_0^a (U_{t+b}\psi|\phi)dt = \int_b^{a+b} (U_t\psi|\phi)dt \; .$$

Splitting the integral in the obvious manner:

$$|(U_b\psi_a|\phi) - (\psi_a|\phi)| = \left|\int_b^{a+b} (U_t\psi|\phi)dt - \int_0^a (U_t\psi|\phi)dt\right|$$

$$\leq \left|\int_b^0 (U_t\psi|\phi)dt\right| + \left|\int_a^{a+b} (U_t\psi|\phi)dt\right| \leq 2b||\phi||\;||\psi||.$$

Then $(U_b\psi_a|\phi) \to (\psi_a|\phi)$, as $b \to 0$, and so by conjugation:

$$\lim_{t\to 0}(\phi \mid U_b\psi_a) \to (\phi \mid \psi_a) \; .$$

We are done if we can prove that the set $\{\psi_a \mid \psi \in \mathsf{H},\ a \in \mathbb{R}\}$ finitely generates a dense space in H, by the previous proposition and choosing $\phi = \psi_a$. Take $\phi \in \{\psi_a \mid \psi \in \mathsf{H},\ a \in \mathbb{R}\}^\perp$ and let $\{\psi^{(n)}\}_{n\in\mathbb{N}}$ be a countable Hilbert basis for H, which we have by separability (the finite-dimensional case is the same). For any $n \in \mathbb{N}$:

$$0 = (\psi_a^{(n)}|\phi) = \int_0^a (U_t\psi^{(n)}|\phi)dt \; , \quad a \in \mathbb{R} \; ,$$

implying (Theorem 1.76(b)) $\mathbb{R} \ni t \mapsto (U_t\psi^{(n)}|\phi)$ is null almost everywhere. Call $S_n \subset \mathbb{R}$ the set where the map *does not* vanish, and fix $t_0 \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} S_n$. The latter exists for $\bigcup_{n\in\mathbb{N}} S_n$ cannot coincide $\mathbb{R}$: the former, in fact, has zero measure as countable union of zero-measure sets. (This is the point where we need the basis to be countable, i.e. separability.) Then $(U_{t_0}\psi^{(n)}|\phi) = 0$ for any n, forcing $\phi = 0$ because U_{t_0} is unitary and $\{U_{t_0}\psi^{(n)}\}_{n\in\mathbb{N}}$ is a basis. Since $\{\psi_a \mid \psi \in \mathsf{H},\ a \in \mathbb{R}\}^\perp = \{0\}$, the span of $\{\psi_a \mid \psi \in \mathsf{H},\ a \in \mathbb{R}\}$ is dense, as required, and the theorem is proved. □

Remark 9.30 In the statement we may substitute Borel measurability with measurability for the Lebesgue σ-algebra. If Lebesgue measurability holds, in fact, the proof does not change and so the group is strongly continuous. Under strong continuity Borel measurability is granted, so also Lebesgue measurability. ■

9.3.2 One-Parameter Unitary Groups Generated by Self-adjoint Operators and Stone's Theorem

This section contains the celebrated *Stone's theorem*, that describes strongly continuous one-parameter unitary groups obtained by exponentiating self-adjoint operators.

Later we will use these groups to provide a necessary and sufficient condition for the spectral measures of self-adjoint operators to commute.

Before all this we need a technical result, which we state separately given its usefulness in many contexts. As usual, dx is the Lebesgue measure on $\mathbb{R}^n$ and $\chi_{[a,b]}$ the characteristic function of $[a, b]$.

Proposition 9.31 *Let* H *be a complex Hilbert space and* $\{V_t\}_{t\in\mathbb{R}^n} \subset \mathfrak{B}(\mathsf{H})$ *a family of operators satisfying the two following conditions:*

(i) $s\text{-}\lim_{t\to t_0} V_t = V_{t_0}$*, for any* $t_0 \in \mathbb{R}^n$*,*

(ii) there exists $C \geq 0$ *such that* $||V_t|| \leq C$ *for any* $t \in \mathbb{R}^n$*.*

Then for any $f \in L^1(\mathbb{R}^n, dx)$ *there is a unique operator on* $\mathfrak{B}(\mathsf{H})$*, denoted* $\int_{\mathbb{R}^n} f(t)V_t\, dt$*, such that:*

$$\left(\phi \middle| \int_{\mathbb{R}^n} f(t)V_t\, dt\; \psi\right) = \int_{\mathbb{R}^n} f(t)\, (\phi|V_t\psi)\; dt\;, \quad \phi, \psi \in \mathsf{H}. \tag{9.83}$$

If $f \in L^1(\mathbb{R}^n, dx)$ *has compact (essential) support, condition (i) is enough to guarantee the existence of* $\int_{\mathbb{R}^n} f(t)V_t dt$*.*

The latter satisfies:

(a) *for any* $\psi \in \mathsf{H}$*:*

$$\left|\left|\int_{\mathbb{R}^n} f(t)V_t\, dt\psi\right|\right| \leq \int_{\mathbb{R}^n} |f(t)|\, ||V_t\psi||\, dt\;. \tag{9.84}$$

(b) *If* $A \in \mathfrak{B}(\mathsf{H})$*:*

$$A\int_{\mathbb{R}^n} f(t)V_t dt = \int_{\mathbb{R}^n} f(t)AV_t dt \quad \textit{and} \quad \int_{\mathbb{R}^n} f(t)V_t dt A = \int_{\mathbb{R}^n} f(t)V_t A dt\;. \tag{9.85}$$

(c) *Let, for* $n = 1$*,* $\int_t^s f(\tau)V_\tau d\tau := \int_{\mathbb{R}} g(\tau) f(\tau)V_\tau d\tau$ *where* $g = \chi_{[t,s]}$ *if* $s \geq t$ *and* $g = -\chi_{[s,t]}$ *if* $s \leq t$*. Then*

$$(i) \quad \mathbb{R}^2 \ni (s,t) \mapsto \int_s^t f(\tau)V_\tau d\tau \quad \textit{is continuous in the uniform topology;}$$

$$(ii) \quad f \textit{ continuous implies} \quad s - \frac{d}{dt}\int_s^t f(\tau)V_\tau d\tau = f(t)V_t \quad \forall s,t \in \mathbb{R}\,. \tag{9.86}$$

Proof Take $\psi, \phi \in \mathsf{H}$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a map in $L^1(\mathbb{R}^n, dx)$. Consider the integral

$$I(\phi,\psi) := \int_{\mathbb{R}^n} f(t)\, (\phi|V_t\psi)\; dt\;.$$

It is well defined as $\mathbb{R}^n \ni t \mapsto (\phi|V_t\psi)$ is continuous, since $\{V_t\}_{t\in\mathbb{R}^n}$ is weakly continuous, and bounded by (ii) from Schwarz's inequality. Hence

$$|I(\phi, \psi)| \leq ||f||_1 C||\psi||||\phi|| \,.$$

Since $\mathsf{H} \ni \psi \mapsto I(\phi, \psi)$ is linear and we have the above inequality, Riesz's theorem gives, for any $\phi \in \mathsf{H}$, a unique $\Phi_\phi \in \mathsf{H}$ such that

$$I(\phi, \psi) = (\Phi_\phi|\psi)\,, \quad \text{for any } \psi \in \mathsf{H}.$$

The map $\mathsf{H} \ni \phi \mapsto T\phi := \Phi_\phi$ is linear, and by construction

$$|(\psi|T\phi)| = |(T\phi|\psi)| = |(\Phi_\phi|\psi)| = |I(\phi, \psi)| \leq ||f||_1 C||\psi||||\phi||\,, \quad \text{with } \phi, \psi \in \mathsf{H}.$$

Choosing $\psi = T\phi$ shows that T, and hence its adjoint $\int_{\mathbb{R}^n} f(t)V_t\, dt$, are bounded. By construction (9.83) holds, and the argument ensures uniqueness. From (9.83) follows

$$\left|\left(\phi\left|\int_{\mathbb{R}^n} f(t)V_t\, dt\ \psi\right.\right)\right| \leq \int_{\mathbb{R}^n} |f(t)|\,|(\phi|V_t\psi)|\, dt \leq \int_{\mathbb{R}^n} |f(t)|\,||V_t\psi||\, dt\ ||\phi||,$$

and taking $\phi = \int_{\mathbb{R}^n} f(t)V_t\, dt\psi$ leads to (9.84). Identity (9.85) follows from (9.83). In case the essential support of f is in a compact set K we can equivalently define $I(\psi, \phi)$ by integrating on it and then proceeding as before. In such a case the constant C of (ii) ($t \in K$) automatically exists. By continuity, in fact, whichever $\psi \in \mathsf{H}$ we take there is $C_\psi \geq 0$ such that $||V_t\psi|| \leq C_\psi$ if $t \in K$. By Banach–Steinhaus this implies that $C \geq 0$ exists with $||V_t|| \leq C$ if $t \in K$. So let us prove (c). Choose $[a, b]$ so that $[a, b] \times [a, b]$ contains an open neighbourhood of (t, s), to which (t', s') belongs. From (a) we have

$$\left|\left|\int_t^s f(\tau)V(\tau)d\tau\psi - \int_{t'}^{s'} f(\tau)V(\tau)d\tau\psi\right|\right| \leq (|t-t'| + |s-s'|) \sup_{\tau\in[a,b]} |f(\tau)| \sup_{\tau\in[a,b]} ||V_\tau\psi||\,,$$

where we used $\int_t^s - \int_{t'}^{s'} = \int_t^{t'} + \int_{t'}^{s} - \int_{t'}^{s'} = \int_t^{t'} + \int_s^{s'}$. Since $||V_\tau|| \leq C < +\infty$ for $\tau \in [a, b]$, taking the least upper bound over $||\psi|| = 1$ produces

$$\left|\left|\int_t^s f(\tau)V(\tau)d\tau - \int_{t'}^{s'} f(\tau)V(\tau)d\tau\right|\right| \leq (|t-t'| + |s-s'|) \sup_{\tau\in[a,b]} |f(\tau)|C\,,$$

whence continuity in uniform topology. As for the second property, by strong continuity of $t \mapsto f(t)V_t$, as $h \to 0$, we have

$$\left|\left|\frac{1}{h}\int_\tau^{\tau+h} f(t)V_t dt\ \psi - f(\tau)V_\tau\psi\right|\right| = \left|\left|\frac{1}{h}\left[\int_\tau^{\tau+h}(f(t)V_t - f(\tau)V_\tau)dt\right]\psi\right|\right|$$

$$\leq \frac{\left|\int_\tau^{\tau+h} dt\right|}{|h|}\sup_{|t'-\tau|\leq h}||f(t')V_{t'}\psi - f(\tau)V_\tau\psi|| = \sup_{|t'-\tau|\leq h}||f(t')V_{t'}\psi - f(\tau)V_\tau\psi|| \to 0\,.$$

□

Remark 9.32 As exercise the reader might prove **Stone's formula**, valid for $b > a$ and a self-adjoint operator $T : D(T) \to \mathsf{H}$ with spectral measure $P^{(T)}$:

$$\frac{1}{2}(P^{(T)}(\{a\})+P^{(T)}(\{b\}))+P^{(T)}((a,b)) = s-\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_a^b \frac{1}{T-\lambda-i\varepsilon}-\frac{1}{T-\lambda+i\varepsilon}d\lambda.$$

The integral is understood in the sense of Proposition 9.31 and

$$\frac{1}{T-\lambda\pm i\varepsilon} := (T-\lambda\pm i\varepsilon)^{-1} = R_{\lambda\mp i\varepsilon}(T)$$

is the resolvent of T. ■

It is time to pass to *Stone's theorem*. This name actually refers to assertion (b), the only non-elementary statement.

Theorem 9.33 (Stone) *Let* H *be a Hilbert space.*
(a) *If* $A : D(A) \to \mathsf{H}$*, with* $D(A)$ *dense in* H*, is a self-adjoint operator and* $P^{(A)}$ *is its spectral measure, then the operators*

$$U_t = e^{itA} := \int_{\sigma(A)} e^{i\lambda t}\, dP^{(A)}(\lambda)\,,\quad t\in\mathbb{R}\,,$$

form a strongly continuous one-parameter unitary group. Moreover:
(i) *the limit*

$$s\text{-}\left.\frac{dU_t}{dt}\right|_{t=0}\psi := \lim_{t\to 0}\frac{U_t\psi-\psi}{t} \tag{9.87}$$

exists in H *if and only if* $\psi \in D(A)$*;*
(ii) *if* $\psi \in D(A)$*:*

$$s\text{-}\left.\frac{dU_t}{dt}\right|_{t=0}\psi = iA\psi\,. \tag{9.88}$$

(b) *If* $\{U_t\}_{t\in\mathbb{R}}$ *is a strongly continuous one-parameter unitary group on* H*, there exists a unique self-adjoint operator* $A : D(A) \to \mathsf{H}$ *(with* $D(A)$ *dense in* H*) such that*

$$e^{itA} = U_t\,,\quad \textit{for any}\ t\in\mathbb{R}. \tag{9.89}$$

Proof (a) If $t\in\mathbb{R}$, $\mathbb{R}\ni\lambda\mapsto e^{it\lambda}$ is trivially bounded, so $e^{itA}\in\mathfrak{B}(\mathsf{H})$ by Corollary 9.5. Theorem 9.4(c) implies ($t\in\mathbb{R}$) $e^{itA}(e^{itA})^* = (e^{itA})^*e^{itA} = I$, making e^{itA}

unitary. To prove strong continuity it is enough to check $(\psi|U_t\psi) \to (\psi|\psi)$ for any $\psi \in \mathsf{H}$ as $t \to 0$, by Proposition 9.27. This is true, by Theorem 9.4(f) and since the domain of e^{itA} is all of H, because:

$$(\psi|U_t\psi) = \int_{\sigma(A)} e^{it\lambda} d\mu_\psi(\lambda) \to \int_{\sigma(A)} 1 d\mu_\psi(\lambda) = (\psi|\psi) \quad \text{as } t \to 0\,.$$

We used that $e^{it\lambda} \to 1$ and so Lebesgue's dominated convergence applies, as $|e^{it\lambda}| = 1$ for any t and the constant 1 is integrable as μ_ψ is finite.

Let us prove (i)–(ii). If $\psi \in D(A)$, from Theorem 9.4(c) we compute

$$\left|\left|\frac{U_t - I}{t}\psi - iA\psi\right|\right|^2 = \int_{\sigma(A)} \left|\frac{e^{i\lambda t} - 1}{t} - i\lambda\right|^2 d\mu_\psi(\lambda)\,. \tag{9.90}$$

On the other hand $|e^{i\lambda t} - 1| = 2|\sin(\lambda t/2)| \le |\lambda t|$, so

$$\left|\frac{e^{i\lambda t} - 1}{t} - i\lambda\right|^2 \le 4|\lambda|^2\,.$$

The map $\mathbb{R} \ni \lambda \mapsto |\lambda|^2$ is integrable in μ_ψ by definition of $D(A) \ni \psi$. At last,

$$\left|\frac{e^{i\lambda t} - 1}{t} - i\lambda\right|^2 \to 0 \quad \text{as } t \to 0, \text{ for any } \lambda \in \mathbb{R}.$$

The dominated convergence theorem on the right side of (9.90) gives

$$\left|\left|\frac{U_t - I}{t}\psi - iA\psi\right|\right| \to 0 \quad \text{as } t \to 0, \text{ for any } \psi \in D(A).$$

To finish we show that $\frac{U_t\psi - \psi}{t} \to \phi_\psi \in \mathsf{H}$, $t \to 0$, implies $\psi \in D(A)$. The set of $\psi \in \mathsf{H}$ for which the limit exists is a subspace $D(B)$ in H containing $D(A)$, and as such is dense. The mapping $\psi \mapsto iB\psi := \phi_\psi$ defines an operator with dense domain $D(B)$. If $\psi, \psi' \in D(B)$, using $U_t^* = U_{-t}$:

$$\left(\psi|B\psi'\right) = \left(\psi\left|-i\lim_{t\to 0}\frac{U_t\psi' - \psi'}{t}\right.\right) = -i\lim_{t\to 0}\left(\psi\left|\frac{U_t\psi' - \psi'}{t}\right.\right)$$

$$= -i\lim_{t\to 0}\left(\left.\frac{U_{-t}\psi - \psi}{t}\right|\psi'\right) = \left(\left.-i\lim_{t\to 0}\frac{U_{-t}\psi - \psi}{-t}\right|\psi'\right) = \left(B\psi|\psi'\right)\,.$$

Hence B is a symmetric extension of A. But A is self-adjoint, so $B = A$ by Proposition 5.17(c). Thus any vector ψ for which the limit of $\frac{U_t\psi - \psi}{t}$ exists as $t \to 0$ lives in $D(A)$. This concludes part (a).

(b) The uniqueness of A is immediate. If there were two self-adjoint operators A, A' with $e^{itA} = U_t = e^{itA'}$ for any $t \in \mathbb{R}$, (i)–(ii) in (a) would force $A = A'$. Let us manufacture a self-adjoint operator A satisfying $U_t = e^{itA}$ for a given strongly continuous one-parameter unitary group. Specialise Proposition 9.31 to a strongly continuous one-parameter unitary group $V_t = U_t$. Call $\mathscr{D}$ the space of vectors of the form $\int_{\mathbb{R}} f(t)U_t\, dt\phi$, $\phi \in \mathsf{H}$, with arbitrary $f \in \mathscr{D}(\mathbb{R})$ (smooth complex functions on $\mathbb{R}$ with compact support). This vector space $\mathscr{D}$ is called **Gårding space**. Equation (9.83) easily implies its invariance: $U_s\mathscr{D} \subset \mathscr{D}$ for any $s \in \mathbb{R}$, i.e.

$$U_s \int_{\mathbb{R}} f(t)U_t\, dt\psi = \int_{\mathbb{R}} f(t)U_{t+s}\, dt\psi = \int_{\mathbb{R}} f(t-s)U_t dt\psi \quad \text{for any } \psi \in \mathsf{H}. \tag{9.91}$$

Let us show, if $\psi \in \mathscr{D}$, that $\frac{U_t\psi-\psi}{t} \to \psi_0 \in \mathsf{H}$ as $t \to 0$. Suppose $\psi = \int_{\mathbb{R}} f(s)U_s\, ds\phi$. A few computations involving (9.91) and the definition of $\int_{\mathbb{R}} f(s)U_s\, ds\phi$, yield

$$\left|\left|\frac{U_t\psi-\psi}{t} - \int_{\mathbb{R}} f'(s)U_s\, ds\phi\right|\right|^2$$

$$= \left(\int_{\mathbb{R}}\left(\frac{f(s-t)-f(s)}{t} - f'(s)\right)U_s\, ds\phi \,\middle|\, \int_{\mathbb{R}}\left(\frac{f(r-t)-f(r)}{t} - f'(r)\right)U_r\, dr\phi\right)$$

$$= \int_{\mathbb{R}} ds \int_{\mathbb{R}} dr\overline{h_t(s)}h_t(r)\,(\phi|U_{r-s}\phi)\ ,$$

where

$$h_t(s) := \frac{f(s-t)-f(s)}{t} - f'(s)\ .$$

For any $t \in \mathbb{R}$, the function $s \mapsto h_t(s)$ has support contained in a compact set and is C^∞ (hence bounded). As $(r,s) \mapsto (\phi|U_{r-s}\phi)$ is also bounded, we may interpret the integral using the product Lebesgue measure:

$$\left|\left|\frac{U_t\psi-\psi}{t} - \int_{\mathbb{R}} f'(t)U_t\, dt\phi\right|\right|^2 = \int_{\mathbb{R}\times\mathbb{R}} dsdr\overline{h_t(s)}h_t(r)\,(\phi|U_{r-s}\phi)\ . \tag{9.92}$$

Now: the integrand is pointwise infinitesimal as $t \to 0$, the maps

$$(s,r) \mapsto \overline{h_t(s)}h_t(r)\,(\phi|U_{r-s}\phi)$$

all have support in one large-enough compact set if t varies in a bounded interval around 0, and they are, there, uniformly bounded by some constant not depending on t (as $(t,s,r) \mapsto \overline{h_t(s)}h_t(r)\,(\phi|U_{r-s}\phi)$ is jointly continuous in its variables). Because of all this, we apply dominated convergence and obtain that both sides in (9.92) vanish as $t \to 0$. Therefore, for $\psi \in \mathscr{D}$ we have proven $\frac{U_t\psi-\psi}{t} \to \psi_0 \in \mathsf{H}$

as $t \to 0$. The map $\psi \mapsto iS\psi := \psi_0$ is clearly linear. Continuing as in part (a) one can see S is Hermitian. As a matter of fact S is symmetric since $\mathscr{D}$ is dense, which is what we prove next. Given $\phi \in \mathsf{H}$ consider the sequence of $\int_{\mathbb{R}} f_n(t)U_t\, dt\phi$, where $f_n \in \mathscr{D}(\mathbb{R})$ satisfy $f_n \geq 0$, supp $f_n \subset [-1/n, 1/n]$ and $\int_{\mathbb{R}} f_n(s)ds = 1$. Then

$$\left|\left|\int_{\mathbb{R}} f_n U_t dt\psi - \psi\right|\right| = \left|\left|\int_{\mathbb{R}} f_n U_t dt\psi - \int_{\mathbb{R}} f_n dt\psi\right|\right| = \left|\left|\int_{\mathbb{R}} f_n (U_t - I) dt\psi\right|\right|$$

$$\leq \int_{\mathbb{R}} |f_n(t)|\, ||(U_t - I)\psi||\, dt$$

where we used (9.84) on $V_t = U_t - I$. Since

$$\int_{\mathbb{R}} |f_n(t)|\, ||(U_t - I)\psi||\, dt \leq \int_{-1/n}^{1/n} |f_n(t)|\ dt \sup_{t\in[-1/n,1/n]} ||(U_t - I)\psi||$$

$$= \sup_{t\in[-1/n,1/n]} ||(U_t - I)\psi||$$

and $\sup_{t\in[-1/n,1/n]} ||(U_t - I)\psi|| \to 0$ as $n \to \infty$, the U_t being strongly continuous, we conclude

$$\mathscr{D} \ni \int_{\mathbb{R}} f_n(t)U_t\, dt\phi \to \phi \in \mathsf{H}\,, \quad n \to \infty.$$

Hence $\mathscr{D}$ is dense in H and S is symmetric. Now we prove it is essentially self-adjoint on $\mathscr{D}$. If $\psi_\pm \in Ran(S \pm iI)^\perp$, then for any $\chi \in \mathscr{D}$ (recall $U_t\mathscr{D} \subset \mathscr{D}$):

$$\frac{d}{dt}(\psi_\pm|U_t\chi) = \lim_{h\to 0}\left(\psi_\pm \left| \frac{U_h U_t\chi - U_t\chi}{h}\right.\right) = (\psi_\pm|iSU_t\chi)$$

$$= i\,(\psi_\pm|(S \pm iI)U_t\chi) \pm (\psi_\pm|U_t\chi) = \pm(\psi_\pm|U_t\chi)$$

and $F_\pm(t) := (\psi_\pm|U_t\chi)$ is of the form $F_\pm(0)e^{\pm t}$. If we want it bounded ($||U_t|| = 1$ for any $t \in \mathbb{R}$), necessarily $F_\pm(0) = 0$ and $\psi_\pm = 0$, in turn implying $\overline{Ran(S \pm iI)} = \mathsf{H}$. By Theorem 5.19 that means $S : \mathscr{D} \to \mathsf{H}$ is essentially self-adjoint. Now let $\overline{S}$ be the self-adjoint extension of S. To finish observe that if $V_t := e^{it\overline{S}}$, for any $\psi, \phi \in \mathscr{D}$:

$$\frac{d}{dt}\left(\psi\left|(V_t)^* U_t\ \phi\right.\right) = \frac{d}{dt}(V_t\psi|U_t\phi) = (iSV_t\psi|U_t\phi) + (V_t\psi|iSU_t\phi)$$

$$= -(V_t\psi|iSU_t\phi) + (V_t\psi|iSU_t\phi) = 0\,.$$

Thus $(\psi|(V_t)^*U_t\phi) = (\psi|I\phi)$. As $\mathscr{D}$ is dense, $(V_t)^*U_t = I$, i.e. $U_t = e^{it\overline{S}}$ for any $t \in \mathbb{R}$. $\square$

Corollary 9.34 *If A is self-adjoint on the Hilbert space H and $\mathscr{D}_0 \subset D(A)$ is dense and such that $e^{itA}\mathscr{D}_0 \subset \mathscr{D}_0$ for any $t \in \mathbb{R}$, then $A\restriction_{\mathscr{D}_0}$ is essentially self-adjoint, i.e. $\mathscr{D}_0$ is a core of A.*

Proof Take $\psi \in \mathscr{D}_0 \subset D(A)$. Then $U_t\psi = e^{itA}\psi$ is differentiable and its derivative is $iAU_t\psi$. Going through the final part of Stone's proof and replacing the Gårding space $\mathscr{D}$ with $\mathscr{D}_0$ proves the claim. $\square$

Now comes a related technical, and useful, elementary result.

Proposition 9.35 *Let A be self-adjoint on the Hilbert space H, and define $U_t := e^{itA}$, $t \in \mathbb{R}$. For any measurable $f : \sigma(A) \to \mathbb{C}$:*

$$U_t f(A) = f(A)U_t\ , \quad \forall t \in \mathbb{R}, \tag{9.93}$$

and, consequently

$$U_t(D(f(A))) = D(f(A))\ , \quad \forall t \in \mathbb{R}. \tag{9.94}$$

Proof On one hand $\psi \in D(f(A)) \Leftrightarrow \int_{\sigma(A)} |f(\lambda)|^2 d\mu_\psi(\lambda) < +\infty$. On the other, the measures μ_ψ and $\mu_{U_t\psi}$ are the same, since

$$(U_t\psi|P^{(A)}(E)U_t\psi) = (\psi|U_t^* P^{(A)}(E)U_t\psi)\ ,$$

but $U_t^* P^{(A)}(E)U_t = P^{(A)}(E)$ from (9.14)–(9.15) in Theorem 9.4(c) (recall all integrals refer to bounded maps so the operators are defined on the entire space). In conclusion $\psi \in D(f(A)) \Leftrightarrow U_t\psi \in D(f(A))$. Conversely, $f(A)\psi \in D(U_t) = \mathsf{H}$ holds trivially, since U_t is unitary. With this, using (9.14)–(9.15) in Theorem 9.4(c), we get $U_t f(A)\psi = f(A)U_t\psi$ for any $\psi \in D(f(A))$, i.e. (9.93). Summing up, we have proved that $U_t f(A) \subset f(A)U_t$. Applying U_{-t} to both sides (see Remark 5.4) we also have $f(A)U_{-t} \subset U_{-t} f(A)$. Since $t \in \mathbb{R}$ is arbitrary, we can recast this identity as $f(A)U_t \subset U_t f(A)$. Since $U_t f(A) \subset f(A)U_t$ we finally obtain $U_t f(A) = f(A)U_t$. Remark 5.4(v) eventually proves $U_t(D(f(A))) = D(f(A))$. $\square$

The next definition will be fundamental for physical applications, as we will see in Chaps. 12 and 13.

Definition 9.36 (*Self-adjoint generator*) Let H be a Hilbert space, $\{U_t\}_{t\in\mathbb{R}} \subset \mathfrak{B}(\mathsf{H})$ a strongly continuous one-parameter unitary group. The unique self-adjoint operator A on H fulfilling (9.89) is called **(self-adjoint) generator of** $\{U_t\}_{t\in\mathbb{R}} \subset \mathfrak{B}(\mathsf{H})$.

The self-adjoint generator A is typically unbounded. It is bounded – and so defined on all H – precisely when $\{U_t\}_{t\in\mathbb{R}}$ is continuous at $t = 0$ (and hence everywhere) in the uniform topology. See Exercise 9.7.

Stone's theorem has a host of useful corollaries, and here is one.

Corollary 9.37 *If $A : D(A) \to \mathsf{H}$ has dense domain in the Hilbert space H and is self-adjoint (in general unbounded), and $U : \mathsf{H} \to \mathsf{H}_1$ is an isomorphism (surjective isometry), then*

$$Ue^{isA}U^{-1} = e^{isUAU^{-1}}, \quad s \in \mathbb{R}.$$

The same holds, in particular, when $\mathsf{H} = \mathsf{H}_1$ *and* U *is unitary.*

Proof The operator UAU^{-1} is clearly self-adjoint on $UD(A)$ by definition. Hence the strongly continuous one-parameter unitary group $\{e^{isUAU^{-1}}\}_{s\in\mathbb{R}}$ is well defined. As U is an isomorphism, $\{Ue^{isA}U^{-1}\}_{s\in\mathbb{R}}$ too is a strongly continuous one-parameter unitary group on H_1. Furthermore, if $\psi = U^{-1}\phi \in U^{-1}D(A)$ then

$$\lim_{s\to 0} \frac{Ue^{isA}U^{-1}\psi - \psi}{s} = \lim_{s\to 0} \frac{Ue^{isA}\phi - U\phi}{s} = U \lim_{s\to 0} \frac{e^{isA}\phi - \phi}{s} = iUA\phi = UAU^{-1}\psi.$$

By Stone's theorem the generator of $\{Ue^{isA}U^{-1}\}_{s\in\mathbb{R}}$ is a self-adjoint extension of UAU^{-1}; but the latter is already self-adjoint, so the generator of $\{Ue^{isA}U^{-1}\}_{s\in\mathbb{R}}$ is UAU^{-1} itself, and

$$Ue^{isA}U^{-1} = e^{isUAU^{-1}}, \quad s \in \mathbb{R}.$$

□

Remarks 9.38
(1) In a sense Stone's theorem is a special case in the larger picture created by the *Hille–Yosida theorem* [Rud91]. This has had a momentous impact in mathematical physics, esp. concerning the applications of the theory of *semigroups*. Let us remind that, in a Banach space $(\mathsf{X}, ||\ ||)$, a **strongly continuous semigroup of operators** $\{Q_t\}_{t\in[0,+\infty)}$ is a collection of operators $Q_t \in \mathfrak{B}(\mathsf{X})$ such that: (a) $Q(0) = I$, (b) $Q_{t+s} = Q_t Q_s$ for $s, t \in [0, +\infty)$, and (c) $||Q_t\psi - \psi|| \to 0$ as $t \to 0$ for any $\psi \in \mathsf{X}$. A **generator** is an operator A on X such that

$$\frac{d}{dt} Q_t\psi = -AQ_t\psi = -Q_t A\psi, \quad \psi \in D(A).$$

This condition determines A completely. The derivative is computed in the norm of X.

If we look at the subcase of normal (bounded) operators $\{Q_t\}_{t\in[0,+\infty)}$ on a Hilbert space $\mathsf{X} = \mathsf{H}$, then [Rud91]: (1) every semigroup has a densely defined generator A, (2) A is normal (unbounded, in general), and (3)

$$Q_t = e^{-tA},$$

where the right-hand side is defined by the integral of

$$\sigma(A) \ni \lambda \mapsto e^{-t\lambda}$$

in the PVM of the spectral decomposition of A (extending Theorem 9.13 to unbounded normal operators [Rud91]). At last: (4) the real part of the spectrum of A is lower bounded, i.e. there exists $\gamma \in \mathbb{R}$ such that $\gamma < Re(\lambda)$ for any $\lambda \in \sigma(A)$.

The following fundamental general result is however valid in Banach spaces [Rud91].

Theorem 9.39 (Hille–Yosida) *Let A be a closed linear operator defined on a linear subspace $D(A)$ of a Banach space X, and fix r, $M \in \mathbb{R}$ with $M > 0$.*

Then A is the generator of a strongly-continuous semigroup $\{Q_t\}_{t\in[0,+\infty)} \subset \mathfrak{B}(\mathsf{X})$ that satisfies

$$||Q_t|| \leq Me^{rt} \quad t \in [0, +\infty)$$

if and only if $D(A)$ is dense in X, every $\lambda > r$ belongs to $\rho(A)$ and for such λ and every positive $n \in \mathbb{N}$ the bound

$$||(A - \lambda I)^{-n}|| \leq \frac{M}{(\lambda - r)^n}$$

holds.

(2) A useful and important result in operator theory is the *Trotter formula*, which has a big impact in the rigorous theory of path integrals [AH-KM08]. We present the statement in the unitary case [Che74].

Theorem 9.40 *Let A, B be self-adjoint operators on the Hilbert space H and suppose that $A + B$, defined on $D(A) \cap D(B)$, is essentially self-adjoint. Then the corresponding strongly continuous unitary groups satisfy the* **Trotter formula**

$$e^{it\overline{A+B}} = s\text{-}\lim_{n\to+\infty} \left(e^{\frac{itA}{n}} e^{\frac{itB}{n}}\right)^n , \quad t \in \mathbb{R} . \tag{9.95}$$

9.3.3 Commuting Operators and Spectral Measures

To finish the chapter we prove a bunch of technical results about commuting spectral measures of self-adjoint operators, which rely on the one-parameter groups they generate. For *bounded* self-adjoint operators the spectral measures commute if and only if the operators themselves commute, an easy consequence of the spectral theorem (see also Corollary 9.42). For *unbounded* operators, instead, there are domain-related issues and the criterion cannot be used. Using unitary groups is a simple way to overcome this problem. The next result is widely applied in QM.

Theorem 9.41 *Let A, B be (in general unbounded) operators on the Hilbert space H, with A further self-adjoint.*

(i) *Suppose B is self-adjoint and call $P^{(A)}$, $P^{(B)}$ the respective spectral measures. Then the following statements are equivalent.*

(a) *For any Borel sets E, $E' \subset \mathbb{R}$:*

$$P^{(A)}(E)P^{(B)}(E') = P^{(B)}(E')P^{(A)}(E) .$$

(b) *For any Borel set $E \subset \mathbb{R}$ and any $s \in \mathbb{R}$:*

$$P^{(A)}(E)e^{isB} = e^{isB}P^{(A)}(E)\,.$$

(c) *For any $t, s \in \mathbb{R}$:*

$$e^{itA}e^{isB} = e^{isB}e^{itA}\,.$$

(d) *For any $t \in \mathbb{R}$:*

$$e^{itA}B \subset Be^{itA}\,,$$

(d)' *For any $t \in \mathbb{R}$:*

$$e^{itA}B = Be^{itA}\,,$$

If these conditions hold, then

$$e^{itA}(D(B)) = D(B) \quad \textit{for all}\, t \in \mathbb{R}.$$

(ii) *Under either of the above five conditions:*

$$\begin{aligned} AB\psi &= BA\psi && \textit{if}\, \psi \in D(AB) \cap D(BA) \\ (A\varphi|B\psi) - (B\varphi|A\psi) &= 0 && \textit{if}\, \psi, \varphi \in D(A) \cap D(B). \end{aligned}$$

(iii) *If $B \in \mathfrak{B}(\mathsf{H})$ (not necessarily self-adjoint) and $P^{(A)}$ is the PVM of A, the following are equivalent.*

(e) $BA \subset AB$.

(f) $Bf(A) \subset f(A)B$ *for any measurable $f : \sigma(A) \to \mathbb{R}$.*

(g) $BP^{(A)}(E) = P^{(A)}(E)B$ *for any Borel set $E \subset \mathbb{R}$.*

(h) $Be^{itA} = e^{itA}B$ *for every $t \in \mathbb{R}$.*

Proof (i) First of all we notice that (d)' is equivalent to (d): the former implies the latter, and the latter entails, applying e^{-itA}, $Be^{-itA} \subset e^{-itA}B$. Since $t \in \mathbb{R}$ is arbitrary, we have $Be^{itA} \subset e^{itA}B$ which, in turn, implies (d)'. (d') immediately implies the last statement of (i).

Let us pass to the remaining part of (i). Using Definition 9.14 the identity in (b) reads

$$\int_{\mathbb{R}} e^{it\lambda} dP^{(A)}_\lambda \int_{\mathbb{R}} e^{is\mu} dP^{(B)}_\mu = \int_{\mathbb{R}} e^{is\mu} dP^{(B)}_\mu \int_{\mathbb{R}} e^{it\lambda} dP^{(A)}_\lambda\,, \quad t, s \in \mathbb{R}, \tag{9.96}$$

where the standard definition of integral of a *bounded* measurable map in a spectral measure was employed, by Theorem 9.4(a). That (a) implies (c) is immediate by definition of integral of a bounded map in a spectral measure (Chap. 8) working in the strong topology. Let us prove (c) $\Rightarrow$ (b) $\Rightarrow$ (a). For the first implication, from (9.96), given $U_s := e^{isB}$, $\psi, \phi \in \mathsf{H}$ and $s \in \mathbb{R}$, we have $\left(\psi \,\middle|\, \int_{\mathbb{R}} e^{it\lambda} dP^{(A)}_\lambda U_s\phi\right) =$

$\left(U_s^*\psi \,\middle|\, \int_{\mathbb{R}} e^{it\lambda} dP_\lambda^{(A)}\phi\right)$ for any $t \in \mathbb{R}$, i.e.

$$\int_{\mathbb{R}} e^{it\lambda} d\mu^{(A)}_{\psi,U_s\phi}(\lambda) = \int_{\mathbb{R}} e^{it\lambda} d\mu^{(A)}_{U_s^*\psi,\phi}(\lambda)\,, \tag{9.97}$$

where we introduced complex measures as in Theorem 8.52(c). The above integrals can be transformed in integrals for *finite* positive measures by Theorem 1.87. Next, using Fubini–Tonelli in (9.97) we can say that if f is (up to the sign of t) the Fourier transform of a map in the Schwartz space $\mathcal{S}(\mathbb{R})$ (see Chap. 3):

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)\, e^{it\lambda} dt\right) d\mu^{(A)}_{\psi,U_s\phi}(\lambda) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t)\, e^{it\lambda} dt\right) d\mu^{(A)}_{U_s^*\psi,\phi}(\lambda)\,.$$

As the Fourier transform maps $\mathcal{S}(\mathbb{R})$ to itself bijectively, the identity becomes

$$\int_{\mathbb{R}} g(\lambda)\, d\mu^{(A)}_{\psi,U_s\phi}(\lambda) = \int_{\mathbb{R}} g(\lambda)\, d\mu^{(A)}_{U_s^*\psi,\phi}(\lambda)\,, \quad g \in \mathcal{S}(\mathbb{R}). \tag{9.98}$$

If $h \in C_c(\mathbb{R})$ (continuous with compact support), the sequence

$$g_n(x) := \sqrt{\frac{n}{4\pi}} \int_{\mathbb{R}} e^{-n(x-y)^2/4} h(y)dy$$

satisfies $g_n \in \mathscr{D}(\mathbb{R})$ and converges uniformly to h as $n \to +\infty$. As $g_n \in \mathscr{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and $g_n \to h \in C_c(\mathbb{R})$ in sup norm, and measures are finite, (9.98) implies

$$\int_{\mathbb{R}} h(\lambda)\, d\mu^{(A)}_{\psi,U_s\phi}(\lambda) = \int_{\mathbb{R}} h(\lambda)\, d\mu^{(A)}_{U_s^*\psi,\phi}(\lambda)\,, \quad h \in C_c(\mathbb{R}). \tag{9.99}$$

Riesz's Theorem 2.52 for complex measures ensures the measures involved in the integrals above coincide. By their explicit expression (Theorem 8.52(c)):

$$\left(\psi \,\middle|\, P^{(A)}(E)U_s\phi\right) = \left(U_s^*\psi \,\middle|\, P^{(A)}(E)\phi\right) \quad \text{for any Borel set } E \subset \mathbb{R} \text{and any } s \in \mathbb{R}. \tag{9.100}$$

As ψ, ϕ are arbitrary, obvious manipulations give (b):

$$P^{(A)}(E)e^{isB} = e^{isB}P^{(A)}(E) \quad \text{for any Borel set } E \subset \mathbb{R}, \text{ any } s \in \mathbb{R}. \tag{9.101}$$

Now we can prove (b) $\Rightarrow$ (a). Iterating the procedure that leads to (b) knowing (c), where now e^{isB} replaces e^{itA} and the unitary U_s is replaced by the projector $P^{(A)}(E)$, we obtain that (9.101) implies (a): $P_E^{(A)}P_{E'}^{(B)} = P_{E'}^{(B)}P_E^{(A)}$ for any pair of Borel sets $E, E' \subset \mathbb{R}$.

To finish (i) there remains to show that (d) is equivalent to one of the preceding statements. If (c) holds, by Stone's theorem and the continuity of e^{itA}, (d) follows immediately. On the other hand (d) $\Rightarrow$ (c), let us see why. First of all (d) amounts to $e^{itA}Be^{-itA} = B$, so exponentiating gives $e^{is(e^{itA}Be^{-itA})} = e^{isB}$. For any given $s \in \mathbb{R}$ the strongly continuous one-parameter unitary groups $t \mapsto e^{is(e^{itA}Be^{-itA})}$ and $t \mapsto e^{itA}e^{isB}e^{-itA}$ have the same generator, so they coincide by Stone's theorem. Hence $e^{itA}e^{isB}e^{-itA} = e^{isB}$, i.e. (c).

Let us prove (ii). For the first assertion, take $\psi \in D(AB) \cap D(BA)$ and look at (c) in (i): $e^{itA}e^{isB}\psi = e^{isB}e^{itA}\psi$. Differentiating in t at the origin, Stone's theorem gives $Ae^{isB}\psi = e^{isB}A\psi$. Now we differentiate in s at the origin. The right side gives $iBA\psi$ by Stone. On the left we can move the derivative past A, as $A = A^*$ is closed and because the limit exists. Hence $iAB\psi = iBA\psi$, as we wanted. Now we prove the second assertion, assuming again (c). If $\psi \in D(A)$ and $\varphi \in D(B)$, $(e^{itA}\psi|e^{isB}\varphi) = (e^{-isB}\psi|e^{-itA}\varphi)$. Differentiating in t and s at $t = s = 0$ proves the claim by Stone's theorem.

Assertion (iii) goes like this. It is obvious that (f) implies (e) and (g) (choose $f = \chi_E$ for (g)). So we show (e) $\Rightarrow$ (h) $\Rightarrow$ (f). First we prove that (e) forces B to commute with e^{itA} for any $t \in \mathbb{R}$, that is, (h). For this we shall use Proposition 9.25(d, f). Let ψ be analytic for A and for every power in the dense set of Proposition 9.25(f). As B is bounded, using Proposition 9.25(d):

$$Be^{itA}\psi = \sum_{n=0}^{+\infty} \frac{(it)^n}{n!} BA^n\psi = \sum_{n=0}^{+\infty} \frac{(it)^n}{n!} A^n B\psi = e^{-itA}B\psi \ .$$

In the last two equalities we used $BA\psi = AB\psi$ repeatedly, plus $||A^n B\psi|| = ||BA^n\psi|| \leq ||B||||A^n\psi||$, so $B\psi$ is analytic for A. But ψ moves in a dense set and the operators B, e^{itA} are continuous, so $Be^{itA} = e^{itA}B$. If B is bounded and commutes with every e^{itA}, B commutes with the spectral measure of A, and this incidentally proves that (h) $\Rightarrow$ (g). The proof is similar to the proof that, in (i), (c) implies (b): we just have to replace U_s by B. Hence by definition of $g(A)$, *if* g *is bounded* (and so is $g(A)$) then $Bg(A) = g(A)B$. At this point notice

$$\mu^{(A)}_{B\psi}(E) = (B\psi|P^{(A)}(E)B\psi) = (P^{(A)}B\psi|P^{(A)}(E)B\psi)$$
$$= (BP^{(A)}\psi|BP^{(A)}(E)\psi) \leq ||B||^2\mu^{(A)}_\psi(E) \tag{9.102}$$

so $\psi \in D(f(A))$ implies $B\psi \in D(f(A))$. Applying the definition of $f(A)$ for f measurable unbounded, and taking a sequence of bounded measurable maps f_n converging to f in $L^2(\sigma(A), \mu_\psi)$, we obtain (f), by taking the limit as $n \to +\infty$ of $Bf_n(A)\psi = f_n(A)B\psi$, for any $n \in \mathbb{N}$, since B is continuous (the equality holds for f_n is bounded). At last, (g) implies (9.102), and (e) follows from the previous argument with $f(x) = x$. $\square$

Corollary 9.42 *Consider two self-adjoint operators* $A : D(A) \to \mathsf{H}$, $B \in \mathfrak{B}(\mathsf{H})$ *on the Hilbert space* H. *They commute, i.e.*

$$BA \subset AB$$

if and only if their spectral measures commute.

Proof If the operators commute, (g) holds in (iii) above. Apply (iii) giving B the role of A and $P^{(A)}(E)$ the role of B. Then $P^{(B)}(F)P^{(A)}(E) = P^{(A)}(E)P^{(B)}(F)$ for any Borel sets $E, F \subset \mathbb{R}$. Conversely, if the spectral measures commute, by definition of integral of a bounded PVM it follows that $BP^{(A)}(E) = P^{(A)}(E)B$ for any Borel set $E \subset \mathbb{R}$. Now (iii) implies $AB\psi = BA\psi$ for any $\psi \in D(A)$, which is another way to write $BA \subset AB$. $\square$

Here is another useful technical consequence.

Corollary 9.43 *Let A be self-adjoint on the Hilbert space* H *and* $B_0 : D(B_0) \to \mathsf{H}$ *essentially self-adjoint. If*

$$e^{itA}B_0 \subset B_0 e^{itA}\,, \quad \forall t \in \mathbb{R}\,,$$

then A and $B := \overline{B_0}$ satisfy (a), (b), (c), (d) in Theorem 9.41(i).

Proof It suffices to note that by definition of closure, using the continuity of e^{itA}, the self-adjoint operator $B : D(B) \to \mathsf{H}$ satisfies

$$e^{itA}D(B) \subset D(B)\,, \quad e^{itA}B\phi = Be^{itA}\phi\,, \quad \forall t \in \mathbb{R}\,, \forall \phi \in D(B)\,.$$

Then part (i) in Theorem 9.41 produces the claim. $\square$

Exercises

9.1 Consider a spectral measure $P : \Sigma(\mathsf{X}) \ni E \mapsto P(E) \in \mathfrak{B}(\mathsf{H})$ and a unitary operator (isometric and onto) $V : \mathsf{H} \to \mathsf{H}'$, where H is a complex Hilbert space. Prove

$$P' : \Sigma(\mathsf{X}) \ni E \mapsto P'(E) := VP(E)V^{-1} \in \mathfrak{B}(\mathsf{H}')$$

is a PVM.

9.2 In relationship to Exercise 9.1, prove the following facts.
(i) If $f : \mathsf{X} \to \mathbb{C}$ is measurable then $\psi \in \Delta_f \Leftrightarrow V\psi \in \Delta'_f$, where Δ'_f is the domain of the integral of f in P'.
(ii) $V \int_{\mathsf{X}} f(x)dP(x)V^{-1} = \int_{\mathsf{X}} f(x)dP'(x)$.

9.3 Prove that (iv) in Theorem 9.13(b) can be strengthened as follows: *let $T : D(T) \to \mathsf{H}$ be self-adjoint on the Hilbert space H. Then $\lambda \in \sigma_c(T)$ is equivalent to asking that $0 < ||T\phi - \lambda\phi||$, $\forall \phi \in D(T)$ with $||\phi|| = 1$, and that for any $\varepsilon > 0$ there exists $\phi_\varepsilon \in D(T)$, $||\phi_\varepsilon|| = 1$, such that*

$$||T\phi_\varepsilon - \lambda\phi_\varepsilon|| \leq \varepsilon\,.$$

Hint. The second condition amounts to saying λ does not belong to $\sigma_p(T)$, so $(T-\lambda I)^{-1} : Ran(T-\lambda I) \to D(T)$ exists. Then $\lambda \in \sigma(T) = \sigma_p(T) \cup \sigma_c(T)$. Can $(T-\lambda I)^{-1}$ be bounded?

9.4 Consider the space $L^2(\mathsf{X}, \mu)$ with μ positive and finite on the Borel σ-algebra of a space X. Let $f : \mathsf{X} \to \mathbb{R}$ be an arbitrary real, measurable, and locally L^2 map (i.e. $f \cdot g \in L^2(\mathsf{X}, \mu)$ for any $g \in C_c(\mathsf{X})$). Consider the operator on $L^2(\mathsf{X}, \mu)$:

$$T_f : h \mapsto f \cdot h$$

where $D(T_f) := \{h \in L^2(\mathsf{X}, \mu) \,|\, f \cdot h \in L^2(\mathsf{X}, \mu)\}$. Prove T_f is self-adjoint. Without using Proposition 9.17 explicitly, show

$$\sigma(T_f) = \text{ess ran}(f)\ .$$

For $f : \mathsf{X} \to \mathbb{R}$, ess ran$(f)$ is the **essential rank** of the measurable map f, defined by $\mathbb{R} \ni v \in \text{ran ess}(f) \ \Leftrightarrow\ \mu\left(f^{-1}(v-\varepsilon, v+\varepsilon)\right) > 0$ for any $\varepsilon > 0$.

Hint. The domain of T_f is dense because f is locally L^2, and the self-adjointness comes from computing $T_f^* = T_f$. The second part goes along these lines. Observe that $\lambda \in \rho(T_f) \ \Leftrightarrow$ the resolvent $R_\lambda(T_f)$ exists on $L^2(\mathsf{X}, \mu)$ and is bounded, i.e. there is $M > 0$ such that $||R_\lambda(T_f)h|| \leq M$ for any unit map $h \in L^2(\mathsf{X}, \mu)$. That is to say, $\lambda \in \rho(T_f)$ if and only if:

$$\int_{\mathsf{X}} \frac{|h(x)|^2}{|f(x)-\lambda|^2} d\mu(x) < M \quad \text{for any unit}\, h \in L^2(\mathsf{X}, \mu)\ .$$

If $\lambda \notin ess\ ran(f)$, by definition of essential rank and $\mu(\mathsf{X}) < +\infty$ we see that the condition holds, so $\lambda \notin ess\ ran(f) \Rightarrow \lambda \notin \sigma(T_f)$. If $\lambda \in ess\ ran(f)$, still by definition of essential rank we may build a sequence of unit vectors h_n such that $\int_{\mathsf{X}} \frac{|h(x)|^2}{|f(x)-\lambda|^2} d\mu(x) > 1/n^2$ for any $n = 1, 2, \ldots$. Hence $\lambda \in ess\ ran(f) \Rightarrow \lambda \in \sigma(T_f)$.

9.5 Consider a PVM $P : \mathscr{B}(\mathbb{C}) \to \mathsf{H}$ with H separable. Prove $A \in \mathfrak{B}(\mathsf{H})$ has the form $A = \int_{\mathbb{C}} f dP$ for some $f \in M_b(\mathbb{C})$ if and only if it commutes with every $B \in \mathfrak{B}(\mathsf{H})$ satisfying $BP(E) = P(E)B$ for any $E \in \mathscr{B}(\mathbb{C})$.

Solution. The sufficient implication is known, so we just prove the necessary part of the equivalence. Divide $supp(P)$ in a disjoint collection, at most countable, of bounded sets E_n, and H in the corresponding orthogonal sum $\mathsf{H} = \oplus_n \mathsf{H}_n$, $\mathsf{H}_n := P(E_n)(\mathsf{H})$. Every H_n is A-invariant, since $AP(E_n) = P(E_n)A$ by assumption. If $A_n := A\restriction_{\mathsf{H}_n} : \mathsf{H}_n \to \mathsf{H}_n$, then $A\psi = \sum_n A_n\psi$ for any $\psi \in \mathsf{H}$. Moreover (see Corollary 9.42) A_n commutes with any operator in $\mathfrak{B}(\mathsf{H}_n)$ that commutes with the bounded normal operator $T_n := \int_{E_n} z dP(z)$ and its adjoint. By Theorem 9.11, $A_n = \int_{E_n} f_n dP$ for some $f_n \in M_b(E_n)$. Define $f(z) := f_n(z)$ on $z \in E_n$, for any $z \in \mathbb{C}$.

Then $f_n \to f$ (the f_n are null outside E_n) in $L^2(\mathbb{C}, \mu_\psi)$ by dominated convergence if $\psi \in \Delta_f$. Therefore $A\psi = \int_{\mathbb{C}} f dP\psi$ for $\psi \in \Delta_f$, by definition of $\int_{\mathbb{C}} f dP$. As A is bounded, Corollary 9.5 implies f must be bounded, $\Delta_f = \mathsf{H}$ and $A = \int_{\mathbb{C}} f dP$.

9.6 Let H be separable and $T : \mathscr{D}(T) \to \mathsf{H}$ self-adjoint on H (not necessarily bounded). Prove that $A \in \mathfrak{B}(\mathsf{H})$ has the form $A = f(T)$, for some $f : \mathbb{R} \to \mathbb{C}$ measurable and bounded, if and only if A commutes with every $B \in \mathfrak{B}(\mathsf{H})$ such that $BT\psi = TB\psi$ for any $\psi \in \mathscr{D}(T)$.

Solution. If $P^{(T)}$ is the PVM of T, $BP^{(T)}(E) = P^{(T)}(E)B \Leftrightarrow BT\psi = TB\psi$ for any $\psi \in \mathscr{D}(T)$. The claim boils down to proving $A = \int f dP^{(T)}$, f bounded, iff A commutes with any $B \in \mathfrak{B}(\mathsf{H})$ commuting with $P^{(T)}$. Exercise 9.5 does exactly that.

9.7 If A is the self-adjoint generator of a strongly continuous one-parameter unitary group $U_t = e^{itA}$, prove that A is bounded (and hence it is defined on the whole Hilbert space) if and only if $||U_t - I|| \to 0$, as $t \to 0$.

Hint. Passing to the spectral representation of A, we have $||U_t - I|| = ||f_t||_\infty$ where $f_t(\lambda) = |e^{it\lambda} - 1|$. Since $(a, b) \ni \lambda \mapsto f_t(\lambda)$ tends to 0 uniformly in λ, as $t \to 0$, if and only if a, b are finite, the claim follows.

9.8 Consider the operators A, $A^\star$ of Sect. 9.1.4. Prove they are closable, and

$$\sigma_p(\overline{A}) = \mathbb{C}$$

so that $\sigma(\overline{A}) = \mathbb{C}$ while $\sigma_c(\overline{A}) = \sigma_r(\overline{A}) = \varnothing$.

Outline of solution. The operators are closable because they admit closed extensions, for $A \subset (A^\star)^*$ and $A^\star \subset A^*$. Using the Hilbert basis $\{\psi_n\}_{n\in\mathbb{N}}$ of Sect. 9.1.4, construct explicitly an eigenvector $\psi \in \mathsf{H} \setminus \{\mathbf{0}\}$ of $\overline{A}$ (i.e. $\overline{A}\psi = \lambda\psi$) for every $\lambda \in \mathbb{C} \setminus \{0\}$. Supposing $\psi = \sum_{n\in\mathbb{N}} c_n\psi_n$ we may heuristically assume that $\overline{A}\psi = \lambda \sum_{n\in\mathbb{N}} c_n\sqrt{n}\psi_{n-1}$, so that

$$c_{n+1} = \frac{c_n}{\lambda\sqrt{n+1}} \, .$$

and thus the candidate eigenvector reads

$$\psi = \sum_{n\in\mathbb{N}} \frac{c_0\lambda^{-n}}{\sqrt{(n+1)!}}\psi_n \, .$$

It is easy to prove that, for $c_0 \neq 0$, the series converges to a non-zero element of H which belongs to $D(\overline{A})$ and satisfies $\overline{A}\psi = \lambda\psi$. We already know that ψ_0 satisfies $\overline{A}\psi_0 = A\psi_0 = \mathbf{0}$, so $0 \in \sigma_p(\overline{A})$.

9.9 Consider the operators A, $A^\star$ and the Hilbert basis $\{\psi_n\}_{n\in\mathbb{N}}$ of Sect. 9.1.4. Prove that $A^* = \overline{A}^* = \overline{A^\star}$, that

$$\overline{A}\left(\sum_{n=0}^{+\infty} c_n\psi_n\right) = \sum_{n=0}^{+\infty} \sqrt{n+1}c_{n+1}\psi_n$$

and that

$$A^*\left(\sum_{n=0}^{+\infty} c_n\psi_n\right) = \sum_{n=1}^{+\infty} \sqrt{n}c_{n-1}\psi_n \ ,$$

where

$$D(\overline{A}) = \left\{\psi \in \mathsf{H} \;\middle|\; \sum_{n=0}^{+\infty}(n+1)|(\psi|\psi_{n+1})|^2 < +\infty\right\}$$

and

$$D(A^*) = \left\{\psi \in \mathsf{H} \;\middle|\; \sum_{n=1}^{+\infty} n|(\psi|\psi_{n-1})|^2 < +\infty\right\} .$$

Conclude that $D(\overline{A}) = D(A^*)$.

Outline of solution. The solution mostly relies on Proposition 5.17 and on the very definition of adjoint. Apply the definition of adjoint of A and prove that $D(A^*)$ and A^* take the form written above. Next observe that $\overline{A} = (A^*)^*$. Then, applying the definition of adjoint, prove that $D(\overline{A})$ and $\overline{A}$ have the form claimed. Finally, again exploiting the definition of adjoint, demonstrate that $(A^\star)^* = \overline{A}$ and conclude that $\overline{A^\star} = ((A^\star)^*)^* = \overline{A}^* = A^*$. The last statement is quite evident if one simply rearranges the expressions of $D(\overline{A})$ and $D(A^*)$ and uses $\psi \in H$.

9.10 Consider the operators A, $A^\star$ of Sect. 9.1.4. Prove that

$$N := A^*\overline{A} = \overline{A^\star A}$$

is the unique self-adjoint extension of the symmetric operator N defined on the span of the vectors ψ_n satisfying $N\psi_n = n\psi_n$ for $n \in \mathbb{N}$. The Hilbert basis $\{\psi_n\}_{n\in\mathbb{N}}$ is the one in Sect. 9.1.4.

9.11 Consider the operators A, $A^\star$ of Sect. 9.1.4, prove that $A + A^\star$ and $i(A - A^\star)$ are essentially self-adjoint on $\mathscr{S}(\mathbb{R})$. Next, study the relation of the closures of those operators and the self-adjoint position and momentum operators.

9.12 Consider the operators A and $A^\star$ of Sect. 9.1.4. Compute $e^{\overline{\alpha A+\overline{\alpha}A^\star}}\psi_n$ with $\alpha \in \mathbb{C}$ given.

9.13 Prove Stone's formula, valid for a self-adjoint operator $T : D(T) \to \mathsf{H}$ with spectral measure $P^{(T)}$ and $b > a$. Use the weak operator topology:

$$\frac{1}{2}(P^{(T)}(\{a\})+P^{(T)}(\{b\}))+P^{(T)}((a,b)) = w-\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_a^b \frac{1}{T-\lambda-i\varepsilon}-\frac{1}{T-\lambda+i\varepsilon}d\lambda.$$

The integral is understood in the sense of Proposition 9.31. Is the identity still valid for $a = b$?

Outline of solution. Define $S(\varepsilon) := \frac{1}{2\pi i}\int_a^b \frac{1}{T-\lambda-i\varepsilon} - \frac{1}{T-\lambda+i\varepsilon}d\lambda$. Next take $\psi, \phi \in \mathsf{H}$ and prove that

$$(\psi|S(\varepsilon)\phi) = \frac{1}{2\pi i}\int_{\mathbb{R}}\left[\int_a^b \frac{2i\varepsilon}{(\ell-\lambda)^2+\varepsilon^2}d\lambda\right]d\mu_{\psi,\phi}(\ell)$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}\left[\tan^{-1}((b-\ell)/\varepsilon) - \tan^{-1}((a-\ell)/\varepsilon)\right]d\mu_{\psi,\phi}(\ell)\,.$$

Prove that, taking the limit when $\varepsilon \to 0^+$ one obtains

$$\lim_{\varepsilon\to 0^+}(\psi|S(\varepsilon)\phi) = \int_{\mathbb{R}}\left[\chi_{(a,b)}(\ell) + \frac{1}{2}(\chi_{\{b\}}(\ell)+\chi_{\{a\}}(\ell))\right]d\mu_{\psi,\phi}(\ell)$$

and conclude. The identity is generally *not* valid for $a = b$: the right-hand side always vanishes while the left-hand side may not.

9.14 Prove that the result of Exercise 9.13 is valid if we use the strong operator topology:

$$\frac{1}{2}(P^{(T)}(\{a\})+P^{(T)}(\{b\}))+P^{(T)}((a,b)) = s-\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_a^b \frac{1}{T-\lambda-i\varepsilon}-\frac{1}{T-\lambda+i\varepsilon}d\lambda.$$

The integral is understood in the sense of Proposition 9.31.

Outline of solution. Since we already know that the convergence is weak, it suffices to show that, if $\phi \in \mathsf{H}$,

$$\lim_{\varepsilon\to 0^+}||S(\varepsilon)\phi||^2 = \left|\left|\frac{1}{2}(P^{(T)}(\{a\})\phi + P^{(T)}(\{b\}))\phi + P^{(T)}((a,b))\phi\right|\right|^2\,.$$

The left-hand side can be written as

$$\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\left|\tan^{-1}((b-\ell)/\varepsilon) - \tan^{-1}((a-\ell)/\varepsilon)\right|^2 d\mu_\phi(\ell)\,,$$

and the limit produces the result we want.

9.15 Consider the operator H in formula (9.66), example Sect. 9.1.4. Show that $\rho_\beta = e^{-\beta H}$ is a well-defined trace-class operator for every $\beta \in \mathbb{C}$, $Re(\beta) > 0$. Compute $tr\rho_\beta$ for these values of β. For $A \in \mathfrak{B}(\mathsf{H})$, define

$$\alpha_z(A) := e^{izH} A e^{-izH} \ , \quad z \in \mathbb{C}$$

$$\langle A \rangle_\beta := tr(e^{-\beta H} A) \ .$$

and finally

$$F_{AB}^{(\beta)}(z) := \langle B \alpha_z(A) \rangle_\beta \ , \quad G_{AB}^{(\beta)}(z) := \langle \alpha_z(A) B \rangle_\beta \ .$$

Prove that (i) $F_{AB}^{(\beta)}(z)$ is an analytic function on the strip $0 < Im(z) < \beta$ and $G_{AB}^{(\beta)}(z)$ is analytic on the strip $-\beta < Im(z) < 0$; (ii) $F_{AB}^{(\beta)}$ and $G_{AB}^{(\beta)}$ are bounded, continuous functions, and they can be extended continuously to the boundaries of their strips; (iii) along the boundaries the *KMS condition*

$$G_{AB}^{(\beta)}(t) = F_{AB}^{(\beta)}(t + i\beta)$$

holds.

Chapter 10
Spectral Theory III: Applications

Particles are solutions to differential equations.

Werner Karl Heisenberg

In this chapter we examine applications of the theory of unbounded operators in Hilbert spaces, where spectral theory, as developed in Chaps. 8 and 9, plays a paramount technical role during the proofs. The final part of the chapter presents a series of classical results about certain operators of interest in Quantum Mechanics, in particular regarding self-adjointness and spectral lower bounds. We recommend [Tes09] for a quite complete, recent treatise on spectral theory applied to QM and Schrödinger operators, in addition to the classical book [ReSi80].

Section one is devoted to the study of abstract differential equations in Hilbert spaces.

The second section pertains the notion of *Hilbert tensor product* of Hilbert spaces and of operators (typically unbounded), plus their spectral properties. We apply this to one example, the orbital angular momentum of a quantum particle.

We extend the polar decomposition theorem to closed, densely-defined unbounded operators in the third section. The properties of operators of the form A^*A, with A densely defined and closed, are examined, together with square roots of unbounded positive self-adjoint operators.

Section four contains the statement, the proof and a few direct applications of the *Kato–Rellich theorem*, which gives criteria for a self-adjoint operator, perturbed by a symmetric operator, to remain self-adjoint, and establishes lower bounds for the perturbed spectrum.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_10

10.1 Abstract Differential Equations in Hilbert Spaces

Looking at spectral theory from the right angle allows to tackle the issue of existence and uniqueness of solutions to certain PDEs that are important in physics. Recall [Sal08] that a second-order linear differential equation in $u \in C^2(\Omega; \mathbb{R})$, for given open set $\Omega \subset \mathbb{R}^n$ and continuous real maps a_{ij}, b_i, c, has the form:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0 .$$

Equations of this kin are classified, *pointwise at each* $x \in \Omega$, into (a) *elliptic*, (b) *parabolic* or (c) *hyperbolic* type according to the spectrum of the symmetric matrix $a_{ij}(x)$. He have type (a) when the eigenvalues have the same sign $\pm$, (b) when there is a null eigenvalue, (c) when there are eigenvalues with opposite sign but none vanishes. An equation is called *elliptic*, *parabolic* or *hyperbolic* if it is such at each point $x \in \Omega$.

By a smart coordinate choice around each point in Ω, the equation can be written as:

$$a(t,y)\frac{\partial^2}{\partial t^2}u(t,y) + \sum_{i,j=1}^{n-1} a'_{ij}(t,y)\frac{\partial^2 u}{\partial y_i \partial y_j} + b(t,y)\frac{\partial u}{\partial t}$$

$$+\sum_{i=1}^{n-1} b'_i(t,y)\frac{\partial u}{\partial y_i} + c(t,y)u(t,y) = 0 .$$

For elliptic equations (e.g. *Poisson's equation*) $a(t,y)$ is never zero and has the same sign of the eigenvalues (all non-zero) of the symmetric matrix $a'_{ij}(t,y)$. Parabolic equations (e.g. the *heat equation* where $b(t,y) \neq 0$) have $a(t,y) = 0$. For hyperbolic equations (e.g. *d'Alembert's equation*) $a(t,y)$ has opposite sign to some eigenvalues (none zero) of the symmetric matrix $a'_{ij}(t,y)$.

We shall suppose all functions we consider are *complex-valued*, and study the theory of these PDEs from a different point of view. The above will be considered "abstract differential equations" in Hilbert spaces equipped with suitable topologies. The variable t will be regarded as a parameter upon which the solutions depends: this will give a curve in the Hilbert space. The differential operators determined by the matrix a'_{ij} and the vector b'_i will become operators acting on a subspace in the Hilbert space $L^2(\Omega, dy)$ containing the support of the solution curve. One can even use a completely abstract Hilbert space H, without mentioning coordinates, whence solutions become H-valued functions $t \mapsto u_t \in \mathsf{H}$. This generalisation will allow us to treat equations that do not befit the classical trichotomy (like the *Schrödinger equation*), equations of degree higher than the second, and equations that cannot be classified within the above scheme, like those where the differential operator given by the matrix a'_{ij} is formally replaced by a square root. For instance

$$a\frac{\partial u}{\partial t} + b\sqrt{-\frac{\partial^2}{\partial x^2}}\, u = 0\,.$$

Notation 10.1 If $J \subset \mathbb{R}$ is an interval, H a Hilbert space, $S \subset \mathsf{H}$ a subspace (closed or not) and $k = 0, 1, 2, \ldots$ is fixed, we let

$$C^k(J;S) := \left\{ J \ni t \mapsto u(t) \in S \,\middle|\, J \ni t \mapsto \frac{d^j u}{dt^j} \text{ exists and is continuous for } j = 0, 1 \ldots, k \right\},$$

where derivative and continuity refer to the topology of H.

We shall also write $C(J;S) := C^0(J;S)$.

Remark 10.2

(1) Of course $C(J;S)$ is a complex vector space.

(2) It is easy to prove, by inner product/norm continuity and Schwarz's inequality, that:

$$\frac{d}{dt}(u(t)|v(t)) = \left(\frac{du}{dt}\middle| v(t)\right) + \left(u(t)\middle|\frac{dv}{dt}\right) \tag{10.1}$$

for every $t \in J$, when $u, v : J \to \mathsf{H}$ are differentiable everywhere on J (in particular continuous on J).

(3) If $\mathsf{H} = L^2(\Omega, dx)$, with $\Omega \subset \mathbb{R}^n$ open, and we take a family of maps $u_t \in \mathscr{L}^2(\Omega, dx)$, $(t, x) \in J \times \Omega$ for a given open interval $J \subset \mathbb{R}$, the existence of the derivative at t forces the existence in $L^2(\Omega, dx)$, under rather weak hypotheses. For example

Proposition 10.3 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $J \subset \mathbb{R}$ an open interval and $\{u_t\}_{t\in J} \subset \mathscr{L}^2(\Omega, dx)$ a family defined on Ω.*

If the maps $u = u_t(x)$ are differentiable in t for every $x \in \Omega$ and $|\frac{\partial u_t}{\partial t}| \leq M$ in Ω for some $M \in \mathbb{R}$ and any $t \in J$, then (viewing $\{u_t\}_{t\in J} \subset L^2(\Omega, dx)$ for the derivative):

$$\exists\, \frac{du_t}{dt} \text{ for every } t \in J \text{ and } \frac{du_t}{dt} = \frac{\partial u_t}{\partial t} \text{ a.e. at } x \text{ for any } t \in J,$$

where the derivative is computed as usual.

(This generalises to higher derivatives in the obvious way.)

Proof Note $\Omega \ni x \mapsto \frac{\partial u_t}{\partial t}$ is measurable for any $t \in J$ as pointwise limit of measurable functions. For any given $t \in J$, the mean value theorem says that for every $x \in \Omega$ and some $x'(x, t, h) \in [t-h, t+h]$:

$$\int_\Omega \left| \frac{u_{t+h}(x) - u_t(x)}{h} - \frac{\partial u_t}{\partial t}(x) \right|^2 dx = \int_\Omega \left| \frac{\partial u_t}{\partial t}(x'(x,t,h)) - \frac{\partial u_t}{\partial t}(x) \right|^2 dx\,.$$

The right integrand is smaller, uniformly with respect to h, than the constant M in $L^2(\Omega, dx)$, since Ω has finite Lebesgue measure. As the integrand is pointwise infinitesimal when $h \to 0$, dominated convergence proves the claim. $\square$

Having Ω bounded can be dropped in favour of a uniform estimate in t, of the sort $|\frac{\partial u_t}{\partial t}(x)| \leq |g_{t_0}(x)|$ with $g_{t_0} \in \mathscr{L}^2(\Omega, dx)$, holding around every given $t_0 \in J$. $\blacksquare$

10.1.1 The Abstract Schrödinger Equation (With Source)

The first equation we study is Schrödinger's equation, for which we allow a source term to be present. The equation will be considered abstractly, in a Hilbert space, and without referring to physics. We shall return to it in chapter 13, when the physical meaning of the sourceless case will be discussed. For the standard theory of PDEs the Schrödinger equation has the following structure (numerical coefficients apart, whose great relevance is neglected for the time being):

$$-i\frac{\partial}{\partial t}u_t(x) + (A_0 u_t)(x) = S(t, x) \tag{10.2}$$

where $J \subset \mathbb{R}$ is a fixed open interval, $\Omega \subset \mathbb{R}^n$ a given open set,

$$A_0 := -\Delta_x + V(x) : D(A_0) \to L^2(\Omega, dx) \tag{10.3}$$

is defined on some domain $D(A_0) \subset C^2(\Omega)$, $V : \Omega \to \mathbb{R}$ and $S : J \times \Omega \to \mathbb{C}$ are given maps, say continuous, and finally Δ_x is the usual Laplacian on $\mathbb{R}^n$.

A function $u = u(t, x)$ is called **classical solution** to (10.2) if it is defined for $(t, x) \in J \times \Omega$, of class C^1 in t and C^2 in $x_1, \ldots, x_n$, and of course if it solves (10.2) on its domain.

If the functions in $D(A_0)$ decay quickly outside compact sets in Ω and the first derivatives are bounded, the operator A_0 is certainly Hermitian, as is clear by integrating by parts. At least for $\Omega := \mathbb{R}^n$, we expect that choosing $D(A_0)$ properly will make A_0 essentially self-adjoint in $L^2(\Omega, dx)$. We already know that for $\Omega := \mathbb{R}^n$ and $V := 0$, the operator A_0 of (10.3) is essentially self-adjoint on the domain $D(A_0) := \mathscr{S}(\mathbb{R}^n)$ (Exercises 5.13 and 5.14); as we shall see, the same holds on $D(A_0) := \mathscr{D}(\mathbb{R}^n)$. We will discuss more general cases, with $V \neq 0$, in Sect. 10.4.

Assuming $A := \overline{A_0}$ is self-adjoint leads to a different interpretation of Eq. (10.2), where A_0 is replaced by any self-adjoint operator and differentiation in t is defined with reference to the topology of the Hilbert space.

Let us fix an open interval $J \subset \mathbb{R}$, $J \ni 0$. If $A : D(A) \to \mathsf{H}$ is a self-adjoint operator on the Hilbert space H and $J \ni t \mapsto S_t \in \mathsf{H}$ a given map in $C(J; \mathsf{H})$, the abstract **Schrödinger equation** with source is:

$$-i\frac{d}{dt}u_t + Au_t = S_t \tag{10.4}$$

where $u \in C^1(J; D(A))$ is the unknown. As we said, the derivative is computed in the topology of H. The *source* is the function $S = S_t$. If $S_t = 0$ for any $t \in J$, Eq. (10.4) is as usual called *homogeneous*.

The **Cauchy problem for the Schrödinger equation**, whether with source or homogeneous, is the problem of finding a function $u \in C^1(J; \mathsf{H})$ solving (10.4), with or without source, together with the **initial condition**:

$$u_0 = v \in D(A). \tag{10.5}$$

Remark 10.4 If A_0 is of the form (10.3) and essentially self-adjoint, we can consider a classical solution $u = u(t, x)$ to (10.2), for which $u(t, \cdot) \in D(A_0)$ for any $t \in J$. Under assumptions of the kind of Proposition 10.3, u also solves the abstract equation (10.4), as $D(A) = D(\overline{A_0}) \supset D(A_0)$. Therefore classical solutions are abstract solutions, under mild assumptions. ■

The first result establishes the uniqueness of the solution to the abstract Schrödinger equation with any initial condition.

Proposition 10.5 *If $u = u_t$ solves the homogeneous equation (10.4):*

$$||u_t|| = ||u_0|| \quad \textit{for any } t \in J. \tag{10.6}$$

Hence if a solution to the Cauchy problem (10.4)–(10.5) exists, with $S_t \neq 0$ in general, it is unique.

Proof From (10.1) and (10.4), for $S_t = 0$:

$$\frac{d}{dt}||u_t||^2 = \frac{d}{dt}(u_t|u_t) = i(Au_t|u_t) - i(u_t|Au_t) = 0$$

because A is self-adjoint. So $||u_t|| = ||u_0||$. Uniqueness follows immediately because if u, u' both solve the Cauchy problem (S_t is the same), then $J \ni t \mapsto u_t - u'_t$ solves (10.4) with $S_t = 0$ and initial condition $u_0 = 0$, so $u_t - u'_t = 0$ for every $t \in J$. □

We are interested in existence now. Actually, we already have everything we need, because Stone's theorem (Theorem 9.33) implies existence in the homogeneous case:

Proposition 10.6 *A solution to the homogeneous Cauchy problem (10.4)–(10.5) has the form:*

$$u_t = e^{-itA}v\,, \quad t \in J\,,$$

where the exponential is understood in spectral sense.

Proof Immediate consequence of Theorem 9.33. □

Remark 10.7 If $v \notin D(A)$ we can still define $u_t := e^{-itA}v$, because the domain of the unitary operator e^{-itA} is H. The map $J \ni t \mapsto u_t$ *does not* solve the homogeneous Schrödinger equation. But trivially

$$\frac{d}{dt}(z|u_t) + i(Az|u_t) = 0 \quad \text{for any } z \in D(A), t \in J, \tag{10.7}$$

by Stone's theorem, because the inner product is continuous and e^{-itA} is unitary, implying $(z|e^{-itA}v) = (e^{itA}z|v)$. Due to (10.7) the map $J \ni t \mapsto u_t$ is called a **weak solution** to the homogeneous Schrödinger equation. ■

It should be clear that the solution set to the Schrödinger equation *with source* (10.4) – if non-empty – consists of functions

$$J \ni t \mapsto u_t^{(0)} + s_t \ ,$$

where: s is an arbitrary, but fixed, solution to the non-homogeneous equation (10.4), and $u^{(0)}$ is free in the vector space of *homogeneous* solutions. A solution to the equation with source satisfying the zero initial condition can be written as:

$$s_t = e^{-tiA} \int_0^t e^{\tau i A} S_\tau d\tau \ ,$$

assuming something on $S \in C(J; \mathsf{H})$. We can prove the next theorem.

Theorem 10.8 *Let $A : D(A) \to \mathsf{H}$ be a self-adjoint operator on the Hilbert space H, $J \subset \mathbb{R}$ an open interval with $0 \in J$. If:*

(i) $v \in D(A)$,
(ii) $J \ni t \mapsto S_t$ *is continuous in the topology of* H,
(iii) $S_t \in D(A)$ *for any* $t \in J$,
(iv) $J \ni t \mapsto AS_t$ *is continuous in the topology of* H,

there exists a unique solution $J \ni t \mapsto u_t \in C^1(J; D(A))$ to the Cauchy problem

$$\begin{cases} \dfrac{du_t}{dt} + Au_t = S_t \ , \\ u_0 = v \ . \end{cases} \tag{10.8}$$

The solution has the form:

$$u_t = e^{-itA}v + e^{-tiA} \int_0^t e^{\tau i A} S_\tau d\tau \ , \quad t \in J \ . \tag{10.9}$$

The integral above has to be interpreted as follows. If $J \ni t \mapsto L_t \in \mathfrak{B}(\mathsf{H})$ is continuous in the strong topology and $J \ni t \mapsto \psi_t \in \mathsf{H}$ is continuous, the vector $\int_a^b L_\tau \psi_\tau d\tau \in D(A)$, $a, b \in J$, is by definition the unique element in H satisfying:

$$\left(u \left| \int_a^b L_\tau \psi_\tau d\tau \right.\right) = \int_a^b (u\,|L_\tau \psi_\tau\,)\,d\tau\;. \tag{10.10}$$

Proof By continuity of $t \mapsto \psi_t$, of the inner product and Schwarz's inequality on the right-hand side of (10.10):

$$\left|\left|\left(u \left| \int_a^b L_\tau \psi_\tau d\tau \right.\right)\right|\right| \leq K_{a,b}||u|| \quad \text{for any } u \in \mathsf{H}$$

for some constant $K_{a,b} \geq 0$. By Riesz's representation Theorem 3.16 the vector $\int_a^b L_\tau \psi_\tau d\tau \in \mathsf{H}$ is well defined. Schwarz's inequality implies

$$\left|\left|\int_a^b L_\tau \psi_\tau d\tau\right|\right| \leq \int_a^b ||L_\tau \psi_\tau||d\tau\;, \tag{10.11}$$

for:

$$\left|\left|\int_a^b L_\tau \psi_\tau d\tau\right|\right|^2 = \int_a^b \int_a^b (L_\tau \psi_\tau\,|L_s \psi_s\,)\,ds d\tau = \left|\int_a^b \int_a^b (L_\tau \psi_\tau\,|L_s \psi_s\,)\,ds d\tau\right|$$

$$\leq \int_a^b \int_a^b |(L_\tau \psi_\tau\,|L_s \psi_s\,)|\,ds d\tau \leq \int_a^b \int_a^b ||L_\tau \psi_\tau||||L_s \psi_s||ds d\tau = \left(\int_a^b ||L\tau \psi_\tau||d\tau\right)^2.$$

Proposition 10.5 grants uniqueness, so we just need existence. We will show the right side of (10.8) solves the Cauchy problem (10.9). By definition $\int_0^t e^{\tau i A} S_\tau d\tau$ is the null vector if $t = 0$, so the right-hand side of (10.9) satisfies $u_0 = v$. We claim $u_t \in D(A)$. We know $e^{itA}v \in D(A)$ by Proposition 10.6. In reality u_t belongs in $D(A) = D(A^*)$, since

$$\left(Ax \left| \int_0^t e^{\tau i A} S_\tau d\tau \right.\right) = \int_0^t \left(Ax \left| e^{\tau i A} S_\tau \right.\right) = \int_0^t \left(x \left| e^{\tau i A} A S_\tau \right.\right) = \left(x \left| \int_0^t e^{\tau i A} A S_\tau d\tau \right.\right)\,,$$

by definition of adjoint, $A = A^*$, $S_t \in D(A)$ and with $x \in D(A)$. We have also proved that if $S_\tau \in D(A)$:

$$A \int_0^t e^{\tau i A} S_\tau d\tau = \int_0^t e^{\tau i A} A S_\tau d\tau\;. \tag{10.12}$$

Summing up we have $u_t \in D(A)$ since $e^{itA}v \in D(A)$, $\int_0^t e^{\tau i A} S_\tau d\tau \in D(A)$ and so $e^{-itA} \int_0^t e^{\tau i A} S_\tau d\tau \in D(A)$ since $e^{-itA}(D(A)) \subset D(A)$ for one-parameter unitary groups generated by self-adjoint operators.

Now we show u_t is a solution. The first term on the right in (10.9) admits derivative $-iAe^{-itA}v$ by Stone's theorem. We want to prove the derivative of the second term equals $-iAe^{-tiA}\int_0^t e^{\tau i A} S_\tau d\tau + S_t$. If so, u_t solves the problem. Define $t' := t + h$ and:

$$\Phi_t := \int_0^t e^{\tau i A} S_\tau d\tau.$$

The derivative of the second term on the right in (10.9) is the limit, as $h \to 0$, of:

$$h^{-1}\left(e^{it'A}\Phi_{t'} - e^{itA}\Phi_t\right) = h^{-1}e^{it'A}\left(\Phi_{t'} - \Phi_t\right) + h^{-1}\left(e^{itA} - e^{it'A}\right)\Phi_t \ .$$

The last term converges to $-iAe^{-tiA}\Phi_t$ by Stone's theorem, since $\Phi_t \in D(A)$. As for the first term:

$$h^{-1}e^{it'A}\left(\Phi_{t'} - \Phi_t\right) = h^{-1}e^{itA}\left(\Phi_{t'} - \Phi_t\right) + h^{-1}\left(e^{it'A} - e^{itA}\right)\left(\Phi_{t'} - \Phi_t\right) \ .$$

By the continuity of e^{itA} the first terms converges to

$$e^{itA}\left(\lim_{h\to 0}\frac{1}{h}\int_t^{t'} e^{-i\tau A}S_\tau d\tau\right) = e^{itA}e^{-itA}S_t = S_t \ ,$$

where, by (10.11), we used:

$$h^{-1}\left|\left|\int_t^{t'} e^{-i\tau A}S_\tau d\tau - e^{-itA}S_t\right|\right| = h^{-1}\left|\left|\int_t^{t'}\left(e^{-i\tau A}S_\tau - e^{-itA}S_t\right)d\tau\right|\right|$$

$$\leq h^{-1}\int_t^{t'}\left|\left|e^{-i\tau A}S_\tau - e^{-itA}S_t\right|\right| d\tau \leq \sup_{\tau\in[t,t']}\left|\left|e^{-i\tau A}S_\tau - e^{-itA}S_t\right|\right| \to 0$$

as $h \to 0$, since $\tau \mapsto e^{-i\tau A}S_\tau$ is continuous from

$$\left|\left|e^{-i\tau A}S_\tau - e^{-itA}S_t\right|\right|^2 = ||S_\tau||^2 + ||S_t||^2 - 2Re\left(S_\tau\left|e^{i\tau A}e^{-itA}\ S_t\right.\right) \ .$$

The last thing to prove is

$$R_h := h^{-1}\left(e^{it'A} - e^{itA}\right)\left(\Phi_{t'} - \Phi_t\right) \to 0\,, \quad h \to 0\,.$$

Set $\Psi_{t'} := \Phi_{t'} - \Phi_t$:

$$||R_h|| = \left|\left|\frac{e^{-ihA} - I}{h}\Psi_{t'}\right|\right| \leq \left|\left|\frac{e^{-ihA} - I}{h}\Psi_{t'} + iA\Psi_{t'}\right|\right| + ||iA\Psi_{t'}|| \ . \qquad (10.13)$$

The last term tends to zero as $h \to 0$ ($t' \to t$), since $\tau \mapsto ||AS_\tau||$ is continuous:

$$||A\Psi_{t'}|| = \left|\left|A\int_t^{t'} e^{\tau iA} S_\tau d\tau\right|\right| \le \int_t^{t'} ||e^{\tau iA} AS_\tau||d\tau = \int_t^{t'} ||AS_\tau||d\tau \to 0 \text{ as } t' \to t.$$

The first term on the right in (10.13), using the spectral measure of Ψ_t, reads:

$$\sqrt{\int_{\mathbb{R}} \lambda^2 \left|\frac{e^{-ih\lambda}-1}{h\lambda} + i\right|^2 d\mu_{\Psi_{t'}}(\lambda)}.$$

Since:

$$\left|\frac{e^{-ih\lambda}-1}{h\lambda} + i\right|^2 = \left(1 + 2\frac{1-\cos h\lambda}{h\lambda} - 2\frac{\sin h\lambda}{h\lambda}\right) < 5\,,$$

we have

$$\left|\left|\frac{e^{-ihA}-I}{h}\Psi_{t'} + iA\Psi_{t'}\right|\right| \le \sqrt{5}\sqrt{\int_{\mathbb{R}} \lambda^2 d\mu_{\Psi_{t'}}(\lambda)} = \sqrt{5}||A\Psi_{t'}|| \to 0\,, \quad t' \to t$$

as seen above.

So we proved u_t is a solution. Eventually we need to show it belongs to $C^1(J; D(A))$. By the equation and the assumptions on S_t, that means $t \mapsto Au_t$ is continuous. By definition of u_t, known properties of integrals in a PVM and (10.12) it follows:

$$Au_t = e^{-itA}Av + e^{-itA}\int_0^t e^{i\tau A}AS_\tau d\tau\ .$$

The map $t \mapsto e^{-itA}(Av)$ is clearly continuous, while

$$\left|\left|e^{-it'A}\int_0^{t'} e^{i\tau A}AS_\tau d\tau - e^{-itA}\int_0^t e^{i\tau A}AS_\tau d\tau\right|\right|$$

$$\le \left|\left|\int_t^{t'} e^{i\tau A}AS_\tau d\tau\right|\right| + \left|\left|\left(e^{-it'A} - e^{-itA}\right)\int_0^t e^{i\tau A}AS_\tau d\tau\right|\right|$$

$$\le \int_t^{t'} ||AS_\tau||d\tau + \left|\left|\left(e^{-it'A} - e^{-itA}\right)\int_0^t e^{i\tau A}AS_\tau d\tau\right|\right| \to 0$$

as $t' \to t$, since $t \mapsto ||AS_t||$ is continuous by assumption and $s \mapsto e^{isA}$ is strongly continuous. $\square$

Example 10.9 Under the hypotheses of the previous theorem, take $S_t := e^{\alpha t}\psi$, with $\psi \in D(A)$ and $\alpha \in \mathbb{R} \setminus \{0\}$ a given constant. The Cauchy problem (10.8) is solved by:

$$u_t = e^{-itA}v + i(A - i\alpha I)^{-1}(e^{-itA} - e^{\alpha t} I)\psi .$$

The resolvent $(A - i\alpha I)^{-1}$ is a well-defined operator in $\mathfrak{B}(\mathsf{H})$ because $\sigma(A) \subset \mathbb{R}$. To arrive at

$$\int_0^t e^{i\tau A} e^{\alpha\tau} \psi d\tau = i(iA - i\alpha I)^{-1}(I - e^{itA+\alpha I})\psi \ ,$$

implying the formula, notice that by definition

$$\left(\phi \left| \int_0^t e^{i\tau A} e^{\alpha\tau} \psi d\tau \right.\right) = \int_0^t \left(\phi \left| e^{i\tau A} e^{\alpha\tau} \psi \right.\right) d\tau = \int_0^t \int_{\mathbb{R}} e^{i\tau(\lambda+\alpha)} e^{\alpha\tau} d\mu_{\phi,\psi} d\tau \ .$$

As the complex measure $\mu_{\phi,\psi}$ is finite, $[0, t]$ has finite measure and $[0, t] \times \mathbb{R} \ni (\tau, \lambda) \mapsto e^{i\tau(\lambda+\alpha)}$ is bounded, we can swap the two integrals by the Fubini–Tonelli theorem (after decomposing $\mu_{\phi,\psi} = h|\mu_{\phi,\psi}|$, $|h| = 1$). Therefore by Theorem 9.4:

$$\left(\phi \left| \int_0^t e^{i\tau A} e^{\alpha\tau} \psi d\tau \right.\right) = \int_0^t \left(\phi \left| e^{i\tau A} e^{\alpha\tau} \psi \right.\right) d\tau = \int_{\mathbb{R}} \int_0^t e^{i\tau(\lambda-i\alpha)} d\tau d\mu_{\phi,\psi}$$

$$= \int_{\mathbb{R}} i(\lambda - i\alpha)^{-1}(1 - e^{i\tau(\lambda-i\alpha)}) d\mu_{\phi,\psi} = \left(\phi \left| i(A - i\alpha I)^{-1}(I - e^{itA+\alpha I})\, \psi \right.\right) \ ,$$

whence the claim, as ϕ is arbitrary. ∎

10.1.2 The Abstract Klein–Gordon/d'Alembert Equation (With Source and Dissipative Term)

The second equation we shall analyse is the *Klein–Gordon* equation. Once a again we will assume there is a source term, and now also a *dissipative* term proportional to the time derivative by a positive coefficient. We shall not return to it at a later stage, so the study begins and ends here. Yet it has to be remembered that the equation has great importance in Quantum Field Theory. Assuming a certain parameter (the mass, in physics) vanishes and in absence of dissipation, the equation goes under the name of *D'Alembert* equation and describes small deformations of (any kind of) waves in linear media. Under the lens of standard PDE theory, the Klein–Gordon equation (with dissipative term and source as well) is *hyperbolic* in nature. Its structure (ignoring the important physical meaning of the coefficients) is the following: given an open interval $J \subset \mathbb{R}$ and an open set $\Omega \subset \mathbb{R}^n$ the equation reads

$$\frac{\partial^2}{\partial t^2} u_t(x) + 2\gamma \frac{\partial}{\partial t} u_t(x) + (A_0 u_t)(x) = S(t, x) \qquad (10.14)$$

where, on some domain $D(A_0) \subset C^2(\Omega)$,

$$A_0 = -\Delta_x + m^2 : D(A_0) \to L^2(\Omega, dx) , \tag{10.15}$$

$m > 0$, $\gamma \geq 0$ are constants, $V : \Omega \to \mathbb{R}$ and $S : J \times \Omega \to \mathbb{C}$ are given functions, for instance continuous, and Δ_x is the Laplace operator on $\mathbb{R}^n$. The D'Alembert equation arises from setting $m = 0$, $\gamma = 0$ in (10.14)–(10.15). One can consider equations where m and γ are functions, too.

A map $u = u(t, x)$ is a **classical solution** to (10.14) if it is defined on $(t, x) \in J \times \Omega$, it is twice differentiable with continuity in every variable, and it solves the equation on its domain.

We can make the same comments of the previous section about A_0. Supposing $A := \overline{A_0}$ is self-adjoint, we can reinterpret Eq. (10.14), where now A_0 is replaced by any self-adjoint operator, here positive definite, and the t-derivative is in the topology of the Hilbert space.

Fix an open interval $J \subset \mathbb{R}$ with $J \ni 0$. Let $A : D(A) \to \mathsf{H}$ be self-adjoint on the Hilbert space H, $J \ni t \mapsto S_t \in \mathsf{H}$ a given map in $C(J; \mathsf{H})$, $\gamma > 0$ a constant. The abstract **Klein–Gordon equation** with source and dissipative term reads:

$$\frac{d^2}{dt^2} u_t + 2\gamma \frac{d}{dt} u_t + A u_t = S_t \tag{10.16}$$

where $u \in C^2(J; D(A))$ is the unknown function. Derivatives are defined with respect to H. The *source* is the function $S = S_t$ and the *dissipative term* is the one multiplied by $\gamma \geq 0$. If $S_t = 0$ for any $t \in J$, Eq. (10.16) is *homogeneous*.

The **Cauchy problem for the Klein–Gordon equation** with dissipative term, whether with source or homogeneous, is the problem that seeks a solution $u \in C^2(J; D(A))$ to (10.16), respectively with source or homogeneous, subject to the **initial conditions**:

$$u_0 = v \in D(A) , \quad \frac{du_t}{dt}|_{t=0} = v' \in \mathsf{H} . \tag{10.17}$$

Remark 10.10

(1) If A_0 is of type (10.15) and essentially self-adjoint, we may take a classical solution $u = u(t, x)$ to (10.14), for which $u(t, \cdot) \in D(A_0)$ for any $t \in J$. Under assumptions of the kind of Proposition 10.3 for the first and second time-derivatives, the solution also satisfies the abstract equation (10.16), as $D(A) = D(\overline{A_0}) \supset D(A_0)$. Therefore, under not so strong assumptions, classical solutions are solutions in the abstract sense.

(2) The abstract approach presented allows for operators A *different* from self-adjoint extensions of Laplacians. The abstract equation befits important situations in physics, like waves created by small deformations of elastic media with inner tensions at rest (a violin's sound board): A is a self-adjoint extension of the *squared* Laplacian Δ^2, which is a *fourth-order* differential operator. Allowing for dissipative term and source, the classical equation governing the deformation $u = u(t, x)$ is:

$$a\frac{\partial^2 u}{\partial t^2} + b\Delta_x^2 u + c\frac{\partial u}{\partial t} = S(t, \mathbf{x})$$

for $a, b > 0$, $c \geq 0$. ■

Our first result establishes uniqueness for the abstract Klein–Gordon with any given initial condition, provided A, apart from being positive, does not have zero as an eigenvalue. These assumptions are automatic for operators like (10.15), and if one works on reasonable domains such as $\mathscr{D}(\mathbb{R}^n)$. Note that closing the operator might cause 0 to appear as an eigenvalue.

Proposition 10.11 *Suppose $u = u_t$ solves the homogeneous equation (10.16), with $\gamma \geq 0$, and A is self-adjoint, $A \geq 0$ and $Ker(A) = \{0\}$. Then the* **energy estimate**

$$\frac{dE[u_t]}{dt} \leq -4\gamma \left|\left|\frac{du_t}{dt}\right|\right|^2 \tag{10.18}$$

holds, where the "energy of the solution at time t" is:

$$E[u_t] := \left|\left|\frac{du_t}{dt}\right|\right|^2 + (u_t|Au_t)\,. \tag{10.19}$$

Hence, if a solution $J \ni t \mapsto u_t$ to (10.16) exists ($S_t \neq 0$ in general), it is uniquely determined, for $t \in [0, +\infty) \cap J$, by u_0 and $du_t/dt|_{t=0}$. If $\gamma = 0$ the solution is unique everywhere on J.

Proof By continuity of the inner product:

$$\frac{d}{dt}E_t[u_t] = \left(\frac{d^2u_t}{dt^2}\middle|\frac{du_t}{dt}\right) + \left(\frac{du_t}{dt}\middle|\frac{d^2u_t}{dt^2}\right) + \frac{d}{dt}\left(u_t\,|Au_t\right)\,.$$

The last derivative is the limit, as $h \to 0$, of

$$\frac{1}{h}\left(\left(u_{t+h}\,|Au_{t+h}\right) - \left(u_t\,|Au_t\right)\right)$$

$$= \frac{1}{h}\left(\left(u_{t+h}\,|Au_{t+h}\right) - \left(u_{t+h}\,|Au_t\right)\right) - \frac{1}{h}\left(\left(u_{t+h}\,|Au_t\right) - \left(u_t\,|Au_t\right)\right)\,.$$

The last term, by inner product continuity, tends to

$$\left(\frac{du_t}{dt}\middle| Au_t\right)\,.$$

Furthermore, by (10.16):

$$\frac{1}{h}\left((u_{t+h}|Au_{t+h}) - (u_{t+h}|Au_t)\right) = \frac{1}{h}\left((Au_{t+h}|u_{t+h}) - (Au_{t+h}|u_t)\right)$$

$$= \left(Au_{t+h}\left|\frac{u_{t+h}-u_t}{h}\right.\right) = -\left(\left(\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}\right)u_{t+h}\left|\frac{u_{t+h}-u_t}{h}\right.\right).$$

As $t \mapsto u_t$ is in $C^2(J; D(A))$ and the inner product is continuous,

$$\frac{1}{h}\left((u_{t+h}|Au_{t+h}) - (u_{t+h}|Au_t)\right) \to -\left(\left(\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}\right)u_t\left|\frac{du_t}{dt}\right.\right)$$

as $h \to 0$. Therefore we obtain (10.18):

$$\frac{d}{dt}E[u_t] = \left(\frac{d^2u_t}{dt^2}\left|\frac{du_t}{dt}\right.\right) + \left(\frac{du_t}{dt}\left|\frac{d^2u_t}{dt^2}\right.\right) - \left(\left(\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}\right)u_t\left|\frac{du_t}{dt}\right.\right)$$

$$-\left(\frac{du_t}{dt}\left|\left(\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}\right)u_t\right.\right) = -4\gamma\left(\frac{du_t}{dt}\left|\frac{du_t}{dt}\right.\right) \leq 0.$$

Consider now two solutions to Eq. (10.16) with source, and suppose they have the same initial data. The difference of the solutions, $u = u_t$, solves the homogeneous equation, hence also (10.18). By construction $u_0 = 0$, $du_t/dt|_{t=0} = 0$, and the function on the right in (10.18) is continuous. Therefore, for any $t \geq 0$:

$$E[u_t] \leq E[u_0] = 0,$$

where we used $u_0 = 0$ and $du_t/dt|_{t=0} = 0$. As $E[u_t] \geq 0$ by definition (10.19), we conclude that

$$E[u_t] = 0 \quad \text{if } t \geq 0.$$

Definition (10.19) implies $(u_t|Au_t) = 0$, so by Theorem (9.4) $(\sqrt{A}u_t|\sqrt{A}u_t) = 0$, i.e. $u_t \in Ker(\sqrt{A})$, if $t \geq 0$ (recall $D(A) \subset D(\sqrt{A})$ for any self-adjoint operator $A \geq 0$, by definition of $D(f(A))$). If we had $\sqrt{A}u_t = 0$, then $\sqrt{A}\sqrt{A}u_t = 0$ i.e. $Au_t = 0$, which is impossible unless $u_t = 0$. So $u_t = 0$ when $t \geq 0$, and the two solutions coincide for $t \geq 0$. If $\gamma = 0$ the argument works for $t < 0$ as well, by flipping the sign of t to $-t$. □

We are interested in having global existence on J. We will establish a result in the homogeneous case with "small" dissipative term, when $\sigma(A)$ is bounded from below by a positive constant and restricting the initial condition v'.

Proposition 10.12 *Let $\gamma \geq 0$ be given, and assume $A - \gamma^2 I \geq \varepsilon I$ for some $\varepsilon > 0$. Given initial conditions (10.17) with $v \in D(A)$, $v' \in D(\sqrt{A})$, the homogeneous Cauchy problem (10.16)–(10.17) admits a solution, for $t \in J$:*

$$u_t = \frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} + e^{-it\sqrt{A-\gamma^2 I}}\right) v$$

$$- i\frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} - e^{-it\sqrt{A-\gamma^2 I}}\right)(A-\gamma^2 I)^{-\frac{1}{2}}(v' + \gamma v) \qquad (10.20)$$

where the exponential and the root are meant in spectral sense.

Proof A direct computation shows the right-hand side of (10.20) solves (10.16): for this we need Theorem 9.33, the fact that $(A-\gamma^2 I)^{-\frac{1}{2}}$ is bounded and defined on the whole Hilbert space, and $D(A) = D(A-\gamma^2 I) \subset D(\sqrt{A-\gamma^2 I}) = D(\sqrt{A})$ by the assumptions made. By Proposition 10.11 the solution found is unique, because $A \geq 0$ and $Ker(A) = \{0\}$ from the lower bound $\gamma^2 + \varepsilon > 0$ of $\sigma(A)$.

That u_t is C^1 (as it should) descends from a computation of the derivative, which needs Stone's theorem, and the boundedness of $(A-\gamma^2 I)^{-1/2}$ (it has bounded spectrum). Therefore

$$\frac{du_t}{dt} = -\gamma u_t + i\frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} - e^{-it\sqrt{A-\gamma^2 I}}\right)\sqrt{A-\gamma^2 I}v$$

$$i\frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} + e^{-it\sqrt{A-\gamma^2 I}}\right)(v' + \gamma v)\,,$$

is continuous since: u_t is continuous as differentiable, and the remaining part of du_t/dt is the action of strongly continuous one-parameter groups on given vectors (plus an extra continuous factor $e^{-\gamma t}$).

That u_t is in $C^2(J; \mathsf{H})$ goes as follows: write d^2u_t/dt^2 as combination of u_t, du_t/dt, Au_t and S_t using the differential equation, and recall u_t, du_t/dt and S_t are continuous together with:

$$Au_t = \frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} + e^{-it\sqrt{A-\gamma^2 I}}\right) Av$$

$$-i\frac{e^{-\gamma t}}{2}\left(e^{it\sqrt{A-\gamma^2 I}} - e^{-it\sqrt{A-\gamma^2 I}}\right) A(A-\gamma^2 I)^{-\frac{1}{2}}(v' + \gamma v)\,.$$

As before, in fact, the above is the action of strongly continuous one-parameter groups on fixed vectors. Eventually, from the expression of u_t and du_t/dt we see the initial conditions are satisfied. □

Remark 10.13 Here is a more suggestive way to write (10.20), which is legitimate if we recall the notion of a function of an operator A:

$$u_t = e^{-\gamma t}\cos\left(t\sqrt{A-\gamma^2 I}\right) v + e^{-\gamma t}\sin\left(t\sqrt{A-\gamma^2 I}\right)(A-\gamma^2 I)^{-1/2}(v' + \gamma v)\,. \qquad (10.21)$$

■

Example 10.14 A frequent situation in classical applications is that in which the self-adjoint operator A satisfies $A \geq \varepsilon I$ for some $\varepsilon > 0$ and has *compact resolvent* (by Corollary 8.6 it suffices for this to happen at one point of the resolvent set). The resolvent's spectrum computed at $\varepsilon/2$ (so that to have a self-adjoint operator) is made by eigenvalues only, possibly with 0 as point of the continuous spectrum, and every eigenspace is finite-dimensional by Theorem 4.19. Proposition 9.17 implies $\sigma(A) = \sigma_p(A)$, since $\sigma(A) = \overline{\{\mu^{-1} + \varepsilon/2 \mid \mu \in \sigma(R_{\varepsilon/2}(A))\}}$ and every eigenvector of the resolvent $R_{\varepsilon/2}(A)$ is an eigenvector for A. Each eigenspace of A has finite dimension, as it corresponds to an eigenspace (with non-zero eigenvalue) of the compact resolvent $R_{\varepsilon/2}(A)$.

For example, this is the case when $-A$ is the closure of the Laplacian on the relatively compact open set $\Omega \subset \mathbb{R}^n$, with $D(\Delta)$ containing maps $\psi \in C^2(\Omega)$ vanishing at the boundary and whose derivatives up to order two are finite on $\partial\Omega$. Then the Laplacian is essentially self-adjoint and the closure's resolvent is compact. If $c > 0$ is constant (the travelling speed of waves in the medium), the equation

$$\frac{d^2 u_t}{dt^2} - c^2 \overline{\Delta} u_t = 0$$

presides over the evolution of the vertical deformation $u_t(x)$ of a flat horizontal elastic membrane represented by the region $\Omega \subset \mathbb{R}^2$, assumed to be fixed at the rim.

Let $A := -c^2\overline{\Delta}$, and call $\{\phi_n\}_{n\in\mathbb{N}}$ an eigenvector basis for A with corresponding eigenvalues $0 < \varepsilon \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. Decompose the initial conditions v, v':

$$v = \sum_{n\in\mathbb{N}} c_n \phi_n \,, \quad v' = \sum_{n\in\mathbb{N}} c'_n \phi_n \,.$$

Using (10.21), with $\gamma = 0$, produces the explicit solution:

$$u_t = \sum_{n\in\mathbb{N}} \left(c_n \cos\left(\sqrt{\lambda_n} t\right) + c'_n \frac{\sin\left(\sqrt{\lambda_n} t\right)}{\sqrt{\lambda_n}} \right) \phi_n \tag{10.22}$$

The solution clearly oscillates by the system's **natural frequencies** (**eigenfrequencies**), i.e. the numbers $\sqrt{\lambda_n}$, for $\lambda_n \in \sigma_p(A)$. ■

It should be clear that the solution set to the Klein–Gordon equation *with source* and dissipative term (10.16) – if not empty – consists of maps

$$J \ni t \mapsto u_t^{(0)} + s_t \,,$$

where s is an arbitrary, but fixed, solution to (10.4), while $u^{(0)}$ varies in the vector space of solutions to the *homogeneous* equation (possibly with dissipative term). This solution exists if the source is regular enough. In fact the following analogue to Theorem 10.8 holds.

Theorem 10.15 *Let $A : D(A) \to \mathsf{H}$ be a self-adjoint operator on H, $\gamma \geq 0$ a fixed number, $J \subset \mathbb{R}$ an open interval with $0 \in J$. If*

(i) $A - \gamma^2 I \geq \varepsilon I$ *for some* $\varepsilon > 0$,
(ii) $v \in D(A)$, $v' \in D(\sqrt{A})$,
(iii) $J \ni t \mapsto S_t$ *is continuous in* H,
(iv) $S_t \in D(A)$ *for any* $t \in J$,
(v) $J \ni t \mapsto AS_t$ *is continuous in* H,

then there exists a solution $J \ni t \mapsto u_t \in C^2(J; D(A))$ *to the Cauchy problem*

$$\begin{cases} \dfrac{d^2u_t}{dt^2} + 2\gamma \dfrac{du_t}{dt} + Au_t = S_t \,, \\ u_0 = v \,, \quad \dfrac{du_t}{dt}|_{t=0} = v' \,, \end{cases} \tag{10.23}$$

of the form:

$$u_t = e^{-\gamma t} \cos\left(t\sqrt{A - \gamma^2 I}\right) v + e^{-\gamma t} \sin\left(t\sqrt{A - \gamma^2 I}\right) (A - \gamma^2 I)^{-1/2}(v' + \gamma v)$$

$$+ e^{-\gamma t - it\sqrt{A-\gamma^2 I}} \int_0^t d\tau e^{2i\tau\sqrt{A-\gamma^2 I}} \int_0^\tau dx e^{\gamma x - ix\sqrt{A-\gamma^2 I}} S_x \,. \tag{10.24}$$

The latter is unique on $[0, +\infty) \cap J$, and even on J if $\gamma = 0$.
Integrals in (10.24) are defined using (10.10) repeatedly.

Sketch of proof. Uniqueness was proven earlier, so we have to show

$$u'_t := e^{-\gamma t - it\sqrt{A-\gamma^2 I}} \int_0^t d\tau e^{2i\tau\sqrt{A-\gamma^2 I}} \int_0^\tau dx e^{\gamma x - ix\sqrt{A-\gamma^2 I}} S_x$$

is in $C^2(J; D(A))$ and solves the differential equation with zero initial data. The initial conditions are satisfied by direct computation. The rest is proved applying Theorem 10.8 twice and bearing in mind the following. Since $D(A) = D(A - \gamma^2 I) \subset D(\sqrt{A - \gamma^2 I}) = D(\sqrt{A})$, by Theorem 9.4

$$\frac{d^2u_t}{dt^2} + 2\gamma \frac{du_t}{dt} + Au_t = \left[\frac{d}{dt} - \left(-\gamma I + i\sqrt{A - \gamma^2 I}\right)\right]\left[\frac{d}{dt} - \left(-\gamma I - i\sqrt{A - \gamma^2 I}\right)\right] u_t$$

if $u \in C^2(J; D(A))$. Then the PDE reads:

$$\left[\frac{d}{dt} - \left(-\gamma I + i\sqrt{A - \gamma^2 I}\right)\right]\left[\frac{d}{dt} - \left(-\gamma I - i\sqrt{A - \gamma^2 I}\right)\right] u_t = S_t \,.$$

Theorem 10.8 generalises easily to an operator $A + iaI$ with $a \in \mathbb{R}$, A self-adjoint. The equation thus becomes

$$\left[\frac{d}{dt} - \left(-\gamma I - i\sqrt{A - \gamma^2 I}\right)\right] u_t = e^{-t\gamma I + it\sqrt{A-\gamma^2 I}} \int_0^t e^{\tau\gamma I - i\tau\sqrt{A-\gamma^2 I}} S_\tau d\tau + u_t^{(0)}\,, \tag{10.25}$$

where $u^{(0)}$ denotes the generic homogeneous solution to:

$$\left[\frac{d}{dt} - \left(-\gamma I + i\sqrt{A - \gamma^2 I}\right)\right] u_t^{(0)} = 0\,.$$

Fixing $u^{(0)} = 0$ and iterating for the remaining term on the left in (10.25)

$$\left[\frac{d}{dt} - \left(-\gamma I - i\sqrt{A - \gamma^2 I}\right)\right],$$

produces the solution in the needed form. What is still missing is to check the assumptions granting we can invoke Theorem 10.8: this is left as exercise. □

Example 10.16

(1) In the hypotheses of Theorem 10.15 let us consider a physical system described by the Klein–Gordon equation with dissipative term, and periodic source

$$S_t = e^{i\omega t}\psi$$

where $\omega \in \mathbb{R}$ is a given constant and $\psi \in D(A)$. Under Theorem 10.15, but explicitly with $\gamma > 0$, we want to study the solution $u = u_t$ of the Cauchy problem with initial conditions v, v' in the "far future", meaning $t \gg 1$. *Provided* $\gamma > 0$, a direct computation (see Exercise 10.1) following from (10.24) yields:

$$u_t = e^{-\gamma t}\left[\cos\left(t\sqrt{A-\gamma^2 I}\right) v + \sin\left(t\sqrt{A-\gamma^2 I}\right)(A-\gamma^2 I)^{-1/2}(v' + \gamma v)\right]$$
$$+ e^{-\gamma t} C_{\omega,t}\psi + e^{i\omega t}(A - \omega^2 I + 2i\gamma\omega I)^{-1}\psi\,.$$

for $C_{\omega,t} \in \mathfrak{B}(\mathsf{H})$, $||C_{\omega,t}|| \leq K_\omega$, some constant $K_\omega \geq 0$ and any $t \geq 0$. Assuming $\gamma > 0$, the resolvent of A at $\omega^2 - 2i\gamma$, i.e. $(A - \omega^2 I + 2i\gamma I\omega)^{-1}$, is well defined and in $\mathfrak{B}(\mathsf{H})$, as $\sigma(A) \subset (0, +\infty)$ by assumption. For large $t > 0$ only the last summand in u_t above survives. The term $e^{-\gamma t}C_{\omega,t}\psi$ tends to zero in the norm of H and the part of solution depending on the initial conditions also goes to zero, because:

$$\left|\left|\cos\left(t\sqrt{A-\gamma^2 I}\right)\right|\right| \leq 1 \text{ and } \left|\left|\sin\left(t\sqrt{A-\gamma^2 I}\right)(A-\gamma^2 I)^{-1/2}\right|\right| \leq K\,,$$

for some constant $K \geq 0$. Therefore at large times the solution oscillates at the same frequency of the source, and the information provided by the initial data gets lost:

$$||u_t - u_t^{(\infty,\psi,\omega)}|| \to 0 \quad \text{as } t \to +\infty,$$

where we call
$$u_t^{(\infty,\psi,\omega)} := e^{i\omega t}(A - \omega^2 I + 2i\gamma\omega I)^{-1}\psi$$

the **long-time solution**.

(2) Referring to example (1), we shall explain the phenomenon called *resonance*.If the source oscillates at a frequency ω that is (up to sign) in the spectrum of A, then the smaller the damping term γ is, the larger the long-time solution $u_t^{(\infty,\psi,\omega)}$ can be rendered by choosing a suitable ψ. In fact let $P^{(A)}$ be the PVM of A and set $I_\omega^\delta := [\omega^2 - \delta, \omega^2 + \delta]$, for $\delta > 0$ finite. If the source is given by the unit vector $\psi \in P^{(A)}_{I_\omega^\delta}(\mathsf{H})$, with $\delta > 0$ small enough we have:

$$\left|\left|e^{i\omega}(A - \omega^2 I + 2i\gamma I\omega)^{-1}\psi\right|\right|^2 = \int_{I_\omega^\delta} \frac{d\mu_\psi(\lambda)}{|\lambda - \omega^2|^2 + 4\gamma^2\omega^2} \geq \inf_{\lambda \in I_\omega^\delta} \frac{1}{|\lambda - \omega^2|^2 + 4\gamma^2\omega^2}$$

and so:

$$\left|\left|u_t^{(\infty,\psi,\omega)}\right|\right| \geq \frac{1}{\sqrt{\delta^2 + 4\gamma^2\omega^2}} .$$

This is all the more evident if the resolvent of A is compact (see Example 10.14), in which case $\sigma(A) = \sigma_p(A)$. If so, picking $\omega \in \sigma_p(A)$ and a corresponding unit eigenvector ψ, the previous estimate strengthens to:

$$\left|\left|u_t^{(\infty,\psi,\omega)}\right|\right| \geq \frac{1}{2\gamma|\omega|} .$$

Continuing with a compact resolvent for A (self-adjoint with strictly positive spectrum), so $\sigma(A) = \sigma_p(A)$, let us take:

$$S_t = \sum_{j\in J} e^{i\omega_j t}\psi_j ,$$

$\omega_j \in \mathbb{R}$ and $\psi_j \neq 0$, J finite. By linearity the long-time solution will be the superposition:
$$u_t^{(\infty)} = \sum_{j\in J} e^{i\omega_j t}(A - \omega^2 I + 2i\gamma I\omega)^{-1}\psi_j .$$

We can decompose every ψ_j using an eigenvector basis $\{\phi_n\}_{n\in\mathbb{N}}$ for A:

$$\psi_j = \sum_{n\in\mathbb{N}} c_{n,j}\phi_n .$$

As $(A - \omega^2 I + 2i\gamma I\omega)^{-1}$ is continuous and J finite:

$$u_t^{(\infty)} = \sum_{j\in J}\sum_{n\in\mathbb{N}} \frac{c_{n,j}e^{i\omega_j t}}{\lambda_n - \omega_j^2 + 2i\gamma\omega_j}\phi_n \tag{10.26}$$

In contrast to solution (10.22), which arises in absence of source and dissipation, the long-time solution, besides having lost track of the initial conditions, no longer oscillates by the natural frequencies $\sqrt{\lambda_n}$ of the system described by A as in (10.22); rather, the oscillations are forced by the frequencies of the source ω_j. However, the system's eigenfrequencies leave traces in the denominator on the right of (10.26), thus generating resonance. That is why the sound of a violin essentially depends on the *harmonic*[1] frequencies of the strings ω_j despite being produced by the sound board, whose *non-harmonic* resonance frequencies are $\lambda_j \neq \omega_j$. ■

10.1.3 The Abstract Heat Equation

In the standard theory of PDEs the heat equation is *parabolic*. Coefficients apart, whose meaning – albeit relevant – we shall ignore, the heat equation over a given open set $\Omega \subset \mathbb{R}^n$ reads:

$$\frac{\partial}{\partial t}u_t(x) + (A_0 u_t)(x) = S(t,x)\,. \tag{10.27}$$

Above

$$A_0 := -\Delta_x : D(A_0) \to L^2(\Omega, dx) \tag{10.28}$$

for some domain $D(A_0) \subset C^2(\Omega)$, with Δ_x being the Laplace operator on $\mathbb{R}^n$.

A map $u = u(t,x)$ is a **classical solution** to (10.27) if it is defined for $(t,x) \in [0,b) \times \Omega$ with $b \in (0,+\infty]$ given, continuous on $[0,b)\times\Omega$, differentiable with continuity in t and twice in $x_1,\dots,x_n$ on $(0,b)\times\Omega$, and clearly if it solves (10.27) on $(0,b)\times\Omega$.

Assuming, as before, that $A := \overline{A_0}$ is self-adjoint leads to a generalised interpretation of (10.27), where A_0 is replaced by any self-adjoint operator and t-differentiation is meant in the Hilbert space.

Fix $b \in (0,+\infty]$. If $A : D(A) \to \mathsf{H}$ is self-adjoint on the Hilbert space H and $[0,b) \ni t \mapsto S_t \in \mathsf{H}$ is continuous and fixed in $C((0,b);\mathsf{H})$, the abstract **heat equation** with source is

$$\frac{d}{dt}u_t + Au_t = S_t\,. \tag{10.29}$$

[1] *Harmonic* here means that these frequencies are integer multiples of a fundamental frequency. The presence of a fundamental frequency makes the distinction between sound and noise.

The unknown is the continuous map $u : [0, b) \to D(A)$ with $u \in C^1((0, b); D(A))$. Continuity and derivative are meant in H. The *source* is the map $S = S_t$. As usual, if $S_t = 0$ for any $t \in [0, b)$ the Eq. (10.29) is *homogeneous*.

The **Cauchy problem for the heat equation**, with source or homogeneous, seeks a C^1 map $u : [0, b) \to D(A)$ solving (10.29), respectively with source or homogeneous, together with the **initial condition**:

$$u_0 = v \in D(A). \tag{10.30}$$

Remark 10.17 If A_0 is of the form (10.28) and essentially self-adjoint, we may consider a classical solution $u = u(t, x)$ to (10.27), for which $u(t, \cdot) \in D_0(A_0)$ for any $t \in [0, b)$. Under assumptions of the kind of Proposition 10.3, this solution will solve the abstract equation (10.29) too, since $D(A) = D(\overline{A_0}) \supset D(A_0)$. Therefore under relatively mild hypotheses classical solutions are abstract solutions. ■

The next result guarantees the abstract heat equation has a unique solution for any initial condition, provided the operator A is positive.

Proposition 10.18 *If A is self-adjoint and $A \geq 0$, and $u = u_t$ solves the homogeneous equation (10.29), then*

$$||u_t|| \leq ||u_0|| \quad \textit{for any } t \in [0, b). \tag{10.31}$$

Hence, for $A \geq 0$, the Cauchy problem (10.29)–(10.30), with $S_t \neq 0$ in general, has at most one solution.

Proof By (10.1) and (10.29), for $S_t = 0$:

$$\frac{d}{dt}||u_t||^2 = \frac{d}{dt}(u_t|u_t) = -(Au_t|u_t) - (u_t|Au_t) \leq 0$$

as A is self-adjoint. Hence $||u_t|| \leq ||u_{t_0}||$ if $t \geq t_0 \in (0, b)$. By continuity the estimate holds at $t_0 = 0$. Uniqueness is proved as follows. If u, u' solve the Cauchy problem (with the same S_t), then $J \ni t \mapsto u_t - u'_t$ solves the Cauchy problem (10.29) with $S_t = 0$ and initial condition $u_0 = 0$. Then $0 \leq ||u_t - u'_t|| \leq ||0|| = 0$ for any $t \in J$ and hence $u_t = u'_t$ for any $t \geq 0$. □

Now we go for an existence result.

Proposition 10.19 *If A is self-adjoint and $A \geq 0$, a (hence, the) solution to the Cauchy problem (10.29)–(10.30) reads:*

$$u_t = e^{-tA}v\,, \quad t \in [0, b)\,,$$

where the exponential is meant in spectral sense.

Proof Take $\psi \in \mathsf{H}$ and $t, t' \in [0, b)$. Observe, as $\sigma(A) \subset [0, +\infty)$, that Theorem 9.4 implies $e^{-tA} \in \mathfrak{B}(\mathsf{H})$ if $t \geq 0$, $e^{-0A} = I$ and also:

$$||e^{-tA}\psi - e^{-t'A}\psi||^2 = \int_{\sigma(A)} |e^{-\lambda t} - e^{-\lambda t'}|^2 d\mu_\psi(\lambda) = \int_{[0,+\infty)} |e^{-\lambda t} - e^{-\lambda t'}|^2 d\mu_\psi(\lambda).$$

Since μ_ψ is finite, if we bound the integrand uniformly in t, t' in a given neighbourhood by some constant, dominated convergence forces $||e^{-tA}\psi - e^{-t'A}\psi||^2 \to 0$ as $t \to t'$. Consequently $u_t := e^{-tA}v$ is continuous on $[0, +\infty)$, and in particular $u_0 = v$.

Suppose $v \in D(A)$, so

$$\int_{[0,+\infty)} |\lambda|^2 d\mu_v(\lambda) < +\infty.$$

Assume as well

$$\int_{[0,+\infty)} \lambda^2 e^{-2\lambda t} d\mu_v(\lambda) < +\infty\,,$$

justified by $\lambda \geq 0$ and $t \geq 0$, so $0 \leq e^{-\lambda t} \leq 1$. Then for $t \in (0, b)$:

$$\int_{[0,+\infty)} \left| \frac{e^{-\lambda(t+h)} - e^{-\lambda t}}{h} - \lambda \right|^2 d\mu_v(\lambda) \to 0 \quad \text{as } h \to 0,$$

because the integrand tends pointwise to 0 as $h \to 0$ and is bounded, uniformly around $h = 0$, by the map $[0, +\infty)\lambda \mapsto C\lambda^2 e^{-2\lambda t}$ (integrable if $v \in D(A)$). Thus we proved

$$\left|\left| \frac{1}{h}(e^{-(t+h)A} - e^{-tA})v - Av \right|\right|^2 \to 0 \quad \text{as } h \to 0,$$

if $t \in (0, +\infty)$. That means $u_t := e^{-tA}v$ solves (10.29) for $t \in (0, b)$. Recalling Theorem 9.4(f):

$$\left|\left| \frac{du_t}{dt}(t) - \frac{du_t}{dt}(t') \right|\right|^2 = \int_{[0,+\infty)} |\lambda|^2 |e^{-t\lambda} - e^{-t'\lambda}|^2 d\mu_v(\lambda)\,.$$

So with a similar argument involving Lebesgue's dominated convergence and using $\lambda e^{-t\lambda} \leq 1/t$, we immediately find $\frac{du_t}{dt}(t)$ is continuous in $(0, +\infty)$. Consequently $u \in C^1((0, b); D(A))$, as needed. □

Remark 10.20

(1) If $v \notin D(A)$, we may still define $u_t := e^{-tA}v$, since the domain of the operator e^{-tA} is H. The map $[0, b) \ni t \mapsto u_t$ *does not* solve the homogeneous heat equation. Since $(e^{-tA})^* = e^{-tA}$, though:

$$\frac{d}{dt}(z|u_t) + (Az|u_t) = 0 \quad \text{for any } z \in D(A), t \in (0, b). \tag{10.32}$$

The map $[0, b) \ni t \mapsto u_t$ is called a **weak solution** to the homogeneous heat equation.

(2) For $A \geq 0$ self-adjoint, the set of operators $T_t := e^{-tA}$, $t \geq 0$, is a **strongly continuous semigroup of operators** (see Remark 9.38) generated by the self-adjoint operator A. Put otherwise the functions $[0, +\infty) \ni t \mapsto T_t$, beside being strongly continuous (cf. above), satisfy $T_0 = I$ and $T_t T_s = T_{t+s}$, $t, s \in [0, +\infty)$.

From $\lambda^n e^{-\lambda t} \leq C_n t^{-n}$, with $C_n := n^n e^{-n}$, $n \geq 0$, $t, \lambda > 0$, we obtain, for a self-adjoint operator $A \geq 0$ on H:

$$||A^n T_t \psi||^2 = \int_{[0,+\infty)} |\lambda^n e^{-\lambda t}|^2 d\mu_\psi(\lambda) \leq C_n^2 t^{-2n}$$

for any unit vector $\psi \in \mathsf{H}$ (bearing in mind Theorem 9.4(c)). Therefore:

$$||A^n T_t|| \leq C_n t^{-n} \ .$$

In particular:

$$Ran(T_t \psi) \subset D(A^n) \quad \text{for any } n = 0, 1, 2, \ldots, \psi \in \mathsf{H} \text{ and } t > 0.$$

When A is, say, the closure of $-\Delta$ on $\mathscr{S}(\mathbb{R}^n)$, then

$$\psi_t := e^{-t\overline{\Delta}}\psi \in D(\overline{\Delta}^n) \quad \text{for any } n = 0, 1, 2, \ldots, \psi \in L^2(\mathbb{R}^n, dx) \text{ and } t > 0.$$

Using the *Fourier–Plancherel transform* (see Sect. 3.7), we obtain easily that ψ_t admits *weak derivatives* (Definition 5.24) of any order, and the latter belong to $L^2(\mathbb{R}^n, dx)$. Well-known results of Sobolev (cf. [Rud91], always with $t > 0$) imply ψ_t is in $C^\infty(\mathbb{R}^n)$ up to a zero-measure set; on the other hand $\psi_t \to \psi$ as $t \to 0^+$ in $L^2(\mathbb{R}^n, dx)$. In this sense semigroups generated by elliptic operators like $-\Delta$ are employed to *regularise* functions.

(3) It should be clear, once again, that the solutions to the heat equation *with source* (10.29) – if they exist – have the form

$$J \ni t \mapsto u_t^{(0)} + s_t$$

where s is any fixed solution to (10.29) and $u^{(0)}$ roams the vector space of *homogeneous* solutions. ■

10.2 Hilbert Tensor Products

We shall see in Chap. 13 that composite quantum systems in elementary QM are described on *tensor products* of the Hilbert spaces of the component subsystems. We will explain in the sequel what exactly a tensor product in the category of Hilbert spaces is, and assume the reader has a familiarity with (standard) tensor products of finite-dimensional vector spaces [Lan10] for general motivations and the notations used. For the infinite-dimensional case we shall follow the approach of [ReSi80].

10.2.1 Tensor Product of Hilbert Spaces and Spectral Properties

Consider n (complex) Hilbert spaces $(\mathsf{H}_i, (\cdot|\cdot)_i)$, $i = 1, 2, \ldots, n$, and choose a vector v_i in each H_i. Mimicking the finite-dimensional situation one can define the *tensor product* $v_1 \otimes \cdots \otimes v_n$ of the v_i as the *multilinear functional*:

$$v_1 \otimes \cdots \otimes v_n : \mathsf{H}'_1 \times \cdots \times \mathsf{H}'_n \ni (f_1, \ldots, f_n) \mapsto f_1(v_1) \cdots f_n(v_n) \in \mathbb{C}\,,$$

where H'_i is the topological dual to H_i and the dots $\cdot$, on the far right, denote the product of complex numbers. *Equivalently*, by Riesz's theorem, we may define the action of $v_1 \otimes \cdots \otimes v_n$ on n-tuples in $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ rather than on $\mathsf{H}'_1 \times \cdots \times \mathsf{H}'_n$. This keeps track of the anti-isomorphism (built from the inner product) that identifies dual Hilbert spaces. In this manner $v_1 \otimes \cdots \otimes v_n$ acts on n-tuples $(u_1, \ldots, u_n) \in \mathsf{H}_1 \times \cdots \times \mathsf{H}_n$ via the inner products, and induces an *antilinear* functional in each variable. This latter, more viable, definition will be our choice.

Definition 10.21 Consider n (complex) Hilbert spaces $(\mathsf{H}_i, (\cdot|\cdot)_i)$, $i = 1, 2, \ldots, n$, and pick one vector v_i in each H_i. The **tensor product** $v_1 \otimes \cdots \otimes v_n$ of $v_1, \ldots, v_n$ is the mapping:

$$v_1 \otimes \cdots \otimes v_n : \mathsf{H}_1 \times \cdots \times \mathsf{H}_n \ni (u_1, \ldots, u_n) \mapsto (u_1|v_1)_1 \cdots (u_n|v_n)_n \in \mathbb{C}\,. \tag{10.33}$$

With $T_{i=1}^n \mathsf{H}_i$ we denote the collection of maps $\{v_1 \otimes \cdots \otimes v_n \,|\, v_i \in \mathsf{H}_i,\, i = 1, 2, \ldots, n\}$ while $\widetilde{\bigotimes}_{i=1}^n \mathsf{H}_i$ is the $\mathbb{C}$-vector space of finite linear combinations of tensor products $v_1 \otimes \cdots \otimes v_n \in T_{i=1}^n \mathsf{H}_i$.

Remark 10.22 From this definition it is evident that the mapping $v_1 \otimes \cdots \otimes v_n$: $\mathsf{H}_1 \times \cdots \times \mathsf{H}_n \to \mathbb{C}$ is *conjugate*-multilinear, that is antilinear in each variable $u_i \in \mathsf{H}_i$ separately, as we see from (10.33), since the inner product is conjugate-linear on the left. This notwithstanding, $(v_1, \ldots v_n) \mapsto v_1 \otimes \cdots \otimes v_n$ is multilinear, as one proves immediately. ■

Let us show how we may define an inner product $(\cdot|\cdot)$ on $\widetilde{\bigotimes_{i=1}^n \mathsf{H}_i}$. Consider the map $S : T_{i=1}^n \mathsf{H}_i \times T_{i=1}^n \mathsf{H}_i \to \mathbb{C}$,

$$S(v_1 \otimes \cdots \otimes v_n, v_1' \otimes \cdots \otimes v_n') := (v_1|v_1') \cdots (v_n|v_n'). \tag{10.34}$$

The following result holds.

Proposition 10.23 *The mapping $S : T_{i=1}^n \mathsf{H}_i \times T_{i=1}^n \mathsf{H}_i \to \mathbb{C}$ extends, by linearity in the right slot and antilinearity in the left, to a unique Hermitian inner product on the complex space $\widetilde{\bigotimes_{i=1}^n \mathsf{H}_i}$:*

$$(\Psi|\Phi) := \sum_i \sum_j \overline{\alpha_i}\beta_j S(v_{1i} \otimes \cdots \otimes v_{ni}, u_{1j} \otimes \cdots \otimes u_{nj})$$

for $\Psi = \sum_i \alpha_i v_{1i} \otimes \cdots \otimes v_{ni}$ and $\Phi = \sum_j \beta_j u_{1j} \otimes \cdots \otimes u_{nj}$ (both sums being finite).

Proof Just to preserve readability let us reduce to the case $n = 2$. If $n > 2$ the proof is conceptually identical, only more tedious to write. Take $\Psi, \Phi \in \mathsf{H}_1 \widetilde{\otimes} \mathsf{H}_2$. First we have to show that *distinct* decompositions of the *same* vectors

$$\Psi = \sum_j \alpha_j v_j \otimes v_j' = \sum_h \beta_h u_h \otimes u_h', \quad \Phi = \sum_k \gamma_k w_k \otimes w_k' = \sum_s \delta_s z_s \otimes z_s',$$

force

$$\sum_j \sum_k \overline{\alpha_j}\gamma_k S(v_j \otimes v_j', w_k \otimes w_k') = \sum_j \sum_s \overline{\alpha_j}\delta_s S(v_j \otimes v_j', z_s \otimes z_s') \tag{10.35}$$

and:

$$\sum_h \sum_k \overline{\beta_h}\gamma_k S(u_h \otimes u_h', w_k \otimes w_k') = \sum_h \sum_s \overline{\beta_h}\delta_s S(u_h \otimes u_h', z_s \otimes z_s'). \tag{10.36}$$

This would prove that the extension of S to $\mathsf{H}_1 \widetilde{\otimes} \mathsf{H}_2$ is *well defined*, for it does not depend on the particular representatives S acts on. So let us prove this fact just for the right variable (10.35), because for (10.36) the argument is similar. The left-hand side in (10.35) may be written:

$$\sum_j \sum_k \overline{\alpha_j}\gamma_k S(v_j \otimes v_j', w_k \otimes w_k') = \sum_j \left(\sum_k \gamma_k w_k \otimes w_k'\right)(\alpha_j v_j, v_j') = \sum_j \Phi(\alpha_j v_j, v_j')$$

and, analogously, the right side of (10.35) reads

$$\sum_j \sum_s \overline{\alpha_j} \delta_s S(v_j \otimes v'_j, z_s \otimes z'_s) = \sum_j \left(\sum_s \delta_s z_s \otimes z'_s \right) (\alpha_j v_j, v'_j) = \sum_j \Phi(\alpha_j v_j, v'_j) \, ,$$

where we used $\Phi = \sum_k \gamma_k w_k \otimes w'_k = \sum_s \delta_s z_s \otimes z'_s$. Therefore S extends uniquely to a map, linear in the second argument and antilinear in the first, $(\cdot|\cdot) : \mathsf{H}_1 \widetilde{\otimes} \mathsf{H}_2 \to \mathbb{C}$. By definition of S:

$$(\Psi|\Phi) = \overline{(\Phi|\Psi)} \, .$$

To prove that $(\cdot|\cdot)$ is indeed a Hermitian inner product we just have to show positive definiteness. That is easy. If $\Psi = \sum_{j=1}^n \alpha_j v_j \otimes v'_j$, where $n < +\infty$ by assumption, consider the (finite!) *orthonormal* basis $u_1, \ldots, u_m$ $(m \le n)$ in the span of $v_1, \ldots, v_n$, and a similar basis $u'_1, \ldots, u'_l$, $(l \le n)$ in the span of $v'_1, \ldots, v'_n$. Using the bilinearity of $\otimes$, we can write $\Psi = \sum_{j=1}^m \sum_{k=1}^l b_{jk} u_j \otimes u'_k$, for suitable coefficients b_{jk}. By definition of S and since the bases are orthonormal, we obtain

$$(\Psi|\Psi) = \left(\sum_{j=1}^m \sum_{k=1}^l b_{jk} u_j \otimes u'_k \,\middle|\, \sum_{i=1}^m \sum_{s=1}^l b_{is} u_i \otimes u'_s \right) = \sum_{j=1}^n \sum_{k=1}^l |b_{jk}|^2 \, .$$

Now it is patent that $(\cdot|\cdot)$ is positive definite. □

Definition 10.24 Consider n (complex) Hilbert spaces $(\mathsf{H}_i, (\cdot|\cdot)_i)$, $i = 1, 2, \ldots, n$. The **(Hilbert) tensor product** of the H_i, written $\bigotimes_{i=1}^n \mathsf{H}_i$ or $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$, is the completion of $\widetilde{\bigotimes}_{i=1}^n \mathsf{H}_i$ with respect to the inner product $(\cdot|\cdot)$ of Proposition 10.23.

It is immediate to verify that the definition reduces to the elementary (algebraic) one if the spaces H_i are finite-dimensional. Moreover, the next results hold.

Proposition 10.25 *Take n (complex) Hilbert spaces $(\mathsf{H}_i, (\cdot|\cdot)_i)$ with Hilbert bases $N_i \subset \mathsf{H}_i$, $i = 1, 2, \ldots, n$. Then*

$$N := \{z_1 \otimes \cdots \otimes z_n \mid z_i \in N_i \, , \; i = 1, 2, \ldots, n\}$$

is a Hilbert basis for $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$. In particular, $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$ is separable if every H_i is.

Proof By construction N is an orthonormal system (by definition of inner product on the tensor product). We have to prove $\langle N \rangle$ is dense in $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$. As linear combinations of the $v_1 \otimes \cdots \otimes v_n$ are dense in $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$, it is enough to show any $v_1 \otimes \cdots \otimes v_n$ can be approximated arbitrarily well by combinations of $z_1 \otimes \cdots \otimes z_n$ in N. To simplify the notation we will reduce again to $n = 2$, since the general case $n > 2$ goes in the same way. For suitable coefficients γ_z and $\beta_{z'}$, we have $v_1 = \sum_{z \in N_1} \gamma_z z$ and $v_2 = \sum_{z' \in N_2} \beta_{z'} z'$, so (Theorem 3.26 and Definition 3.19):

$$||v_1||^2 = \sup \left\{ \sum_{z \in F_1} |\gamma_z|^2 \;\middle|\; F_1 \text{ finite subset of } N_1 \right\} \tag{10.37}$$

and

$$||v_2||^2 = \sup \left\{ \sum_{z' \in F_2} |\beta_{z'}|^2 \;\middle|\; F_2 \text{ finite subset of } N_2 \right\} . \tag{10.38}$$

If $F \subset N_1 \times N_2$ is finite, a direct computation using the orthonormality of $z \otimes z'$ and the definition of inner product on $\mathsf{H}_1 \otimes \mathsf{H}_2$ gives

$$\left|\left| v_1 \otimes v_2 - \sum_{(z,z') \in F} \gamma_z \beta_{z'} z \otimes z' \right|\right|^2 = ||v_1||^2 ||v_2||^2 - \sum_{(z,z') \in F} |\gamma_z|^2 |\beta_{z'}|^2 .$$

Having (10.37) and (10.38) we can make the right-hand side as small as we like by increasing F. This ends the proof. $\square$

Proposition 10.26 *Let* $(\mathsf{H}_i, (\cdot|\cdot)_i)$ *be (complex) Hilbert spaces,* $D_i \subset \mathsf{H}_i$ *dense subspaces,* $i = 1, 2, \ldots, n$*. The subspace* $D_1 \otimes \cdots \otimes D_n \subset \mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$*, spanned by tensor products* $v_1 \otimes \cdots \otimes v_n$*,* $v_i \in D_i$*,* $i = 1, \ldots, n$*, is dense in* $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_n$*.*

Proof As is by now customary, we prove the claim for $n = 2$. Finite combinations of tensor products $u \otimes v$ are dense in $\mathsf{H}_1 \otimes \mathsf{H}_2$, so it is enough to prove the following: if $\psi := u \otimes v \in \mathsf{H}_1 \otimes \mathsf{H}_2$, there exists a sequence in $D_1 \otimes D_2$ converging to ψ. By construction there exist sequences $\{u_n\}_{n \in \mathbb{N}} \subset D_1$ and $\{v_n\}_{n \in \mathbb{N}} \subset D_2$ respectively tending to u and v. Then

$$||u_n \otimes v_n - u \otimes v|| \le ||u_n \otimes v_n - u_n \otimes v|| + ||u_n \otimes v - u \otimes v|| .$$

But $||u_n \otimes v_n - u_n \otimes v||^2 = ||u_n \otimes (v_n - v)||^2 = ||u_n||_1^2 ||v_n - v||_2^2 \to 0$ as $n \to +\infty$, because if the u_n converge then $\{||u_n||_1\}_{n \in \mathbb{N}}$ is bounded. Similarly $||u_n \otimes v - u \otimes v||^2 = ||(u_n - u) \otimes v||^2 = ||u_n - u||_1^2 ||v||_2^2 \to 0$ as $n \to +\infty$. $\square$

Example 10.27

(1) We will exemplify Hilbert tensor products by showing that for separable L^2 spaces (the spaces of wavefunctions in QM), the Hilbert tensor product may be understood alternatively using product measures.

Consider a pair of *separable* Hilbert spaces $L^2(\mathsf{X}_i, \mu_i)$, $i = 1, 2$, and assume both measures are σ*-finite*, so that the product $\mu_1 \otimes \mu_2$ is well defined on $\mathsf{X}_1 \times \mathsf{X}_2$.

Proposition 10.28 *Let* $L^2(\mathsf{X}_i, \mu_i)$ *be separable Hilbert spaces,* $i = 1, 2$*, with* σ*-finite measures. Then*

$$L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2) \quad \textit{and} \quad L^2(\mathsf{X}_1, \mu_1) \otimes L^2(\mathsf{X}_2, \mu_2)$$

are canonically isomorphic as Hilbert spaces.

The unitary identification is the unique linear bounded extension of:

$$U_0 : L^2(\mathsf{X}_1, \mu_1) \otimes L^2(\mathsf{X}_2, \mu_2) \ni \psi \otimes \phi \mapsto \psi \cdot \phi \in L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$$

where $(\psi \cdot \phi)(x, y) := \psi(x)\phi(y)$, $x \in \mathsf{X}_1$, $y \in \mathsf{X}_2$.

Proof First, if $N_1 := \{\psi_n\}_{n\in\mathbb{N}}$ and $N_2 := \{\phi_n\}_{n\in\mathbb{N}}$ are bases of $L^2(\mathsf{X}_1, \mu_1)$ and $L^2(\mathsf{X}_2, \mu_2)$ respectively, then $N := \{\psi_n \cdot \phi_m\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is a basis in $L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$. It is clearly an orthonormal system by elementary properties of the product measure, and if $f \in L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$ is such that

$$\int_{\mathsf{X}\times\mathsf{X}_2} \overline{f(x, y)}\psi_n(x)\phi_m(y)d\mu_1(x) \otimes d\mu_2(y) = 0$$

for any $\psi_n \cdot \phi_m$, by Fubini–Tonelli we have:

$$\int_{\mathsf{X}_2} \left(\int_{\mathsf{X}_1} \overline{f(x, y)}\psi_n(x)d\mu_1(x) \right) \phi_m(y)d\mu_2(y) = 0 \, .$$

As the ϕ_m form a basis

$$\int_{\mathsf{X}_1} \overline{f(x, y)}\psi_n(x)d\mu_1(x) = 0 \, ,$$

except for a set $S_m \subset \mathsf{X}_2$ of zero μ_2-measure. Then for $y \notin S := \cup_{m\in\mathbb{N}} S_m$ (of zero measure as *countable* union of zero-measure sets):

$$\int_{\mathsf{X}_1} \overline{f(x, y)}\psi_n(x)d\mu_1(x) = 0$$

for any $\psi_n \in N_1$, which implies $f(x, y) = 0$ except for $x \in B$, B having zero μ_1-measure. Overall $f(x, y) = 0$, with the exception of the points in $S \times B$, of zero measure for $\mu_1 \otimes \mu_2$ by elementary properties of product measure. Viewing f as in $L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$, we then have $f = 0$. Consequently N is a basis, being an orthonormal system with trivial orthogonal complement.

Consider the unique bounded linear function U mapping the basis element $\psi_n \otimes \phi_m$ of $L^2(\mathsf{X}_1, \mu_1) \otimes L^2(\mathsf{X}_2, \mu_2)$ to the basis element $\psi_n \cdot \phi_m$ of $L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$. By construction U is unitary. Moreover, U sends $\psi \otimes \phi \in L^2(\mathsf{X}_1, \mu_1) \otimes L^2(\mathsf{X}_2, \mu_2)$ to the corresponding $\psi \cdot \phi \in L^2(\mathsf{X}_1 \times \mathsf{X}_2, \mu_1 \otimes \mu_2)$ (just note $\psi \otimes \phi$ and $\psi \cdot \phi$ have the same components in the respective bases), so U is a linear unitary extension of U_0. Any other linear bounded extension U' of U_0 must reduce to U on bases $\psi_n \otimes \phi_m$, $\psi_n \cdot \phi_m$, and as such it coincides with U by linearity and continuity. $\square$

The result clearly generalises to n-fold products of L^2 spaces with separable σ-finite measures.

(2) Here is another important result about Hilbert tensor products, that deals with the case where all summands H_k of a Hilbert sum (Definition 3.67) coincide.

Proposition 10.29 *If* H *is a Hilbert space and* $0 < n \in \mathbb{N}$ *is fixed, the Hilbert space* $\mathsf{H} \otimes \mathbb{C}^n$ *is naturally isomorphic to* $\bigoplus_{k=1}^n \mathsf{H}$.

The unitary identification is the unique linear bounded extension of

$$V_0 : \mathsf{H} \otimes \mathbb{C}^n \ni \psi \otimes (v_1, \ldots, v_n) \mapsto (v_1\psi, \ldots, v_n\psi) \in \bigoplus_{k=1}^{n} \mathsf{H}\,.$$

Proof The proof is similar to the one in example (1). Fix a Hilbert basis $N \subset \mathsf{H}$, so by construction the vectors

$$(\psi, 0, \ldots, 0), (0, \psi, 0, \ldots, 0), \ldots, (0, \ldots, 0, \psi)$$

form a basis of $\bigoplus_{k=1}^n \mathsf{H}$ as ψ varies in N. Take the unique linear bounded transformation mapping $\psi \otimes e_i$ to $(0, \ldots, \psi, \ldots, 0)$, where: $\psi \in N$, e_i is the ith canonical vector in $\mathbb{C}^n$, and the only non-zero entry in the n-tuple, ψ, is in the ith place. This is easily unitary, and restricts to V_0 on $\psi \otimes (v_1, \ldots, v_n)$. Uniqueness is proved in analogy to example (1). □

(3) The **Fock space** $\mathscr{F}(\mathsf{H})$ generated by H is the infinite Hilbert sum

$$\mathscr{F}(\mathsf{H}) := \bigoplus_{n=0}^{+\infty} \mathsf{H}^{n\otimes}$$

where $\mathsf{H}^{0\otimes} := \mathbb{C}$, $\mathsf{H}^{n\otimes} := \underbrace{\mathsf{H} \otimes \cdots \otimes \mathsf{H}}_{n \text{ times}}$. Notice $\mathscr{F}(\mathsf{H})$ is separable if H is. ■

Remark 10.30 This discussion begs the question whether it makes sense to define a tensor product of *infinitely many* Hilbert spaces. The answer is yes (see [BrRo02, vol.1]). The definition, however, depends on certain choices. Consider a collection $\{\mathsf{H}_\alpha\}_{\alpha\in A}$ of Hilbert spaces (over $\mathbb{C}$) of any cardinality. Fix unit vectors $\psi_\alpha \in \mathsf{H}_\alpha$ and $U := \{\psi_\alpha\}_{\alpha\in A}$. We can construct the Hilbert tensor product $\bigotimes_{\alpha\in A}^{(U)} \mathsf{H}_\alpha$ of as many Hilbert spaces as we like in the following way.

(1) Take the subspace in $\times_{\alpha\in A}\mathsf{H}_\alpha$ of elements $(x_\alpha)_{\alpha\in A}$ for which only finitely many x_α are distinct from the corresponding ψ_α. Define conjugate-linear maps in each argument $\otimes_\alpha\phi_\alpha : \times_{\alpha\in A}\mathsf{H}_\alpha \to \mathbb{C}$ by $\otimes_\beta\phi_\beta((x_\alpha)_\alpha) := \Pi_{\alpha\in A}(x_\alpha|\phi_\alpha)_\alpha$, where, again, only finitely many ϕ_α (depending on $\otimes_\alpha\phi_\alpha$) do not belong in U. Consider the finite span $\widetilde{\otimes^{(U)}}_{\alpha\in A}\mathsf{H}_\alpha$ of the functionals $\otimes_{\alpha\in A}\phi_\alpha$.

(2) Now define $\otimes_{\alpha\in A}^{(U)}\mathsf{H}_\alpha$ to be the completion of $\widetilde{\otimes^{(U)}}_{\alpha\in A}\mathsf{H}_\alpha$ in the norm generated by the only inner product such that $(\otimes_\alpha\phi_\alpha | \otimes_\alpha \phi'_\alpha) := \Pi_\alpha(\phi_\alpha|\phi'_\alpha)_\alpha$.

If A is finite it is not hard to see that the definition reduces to the previous one, and does not depend on the choice of U. This fact ceases to hold, in general, if A is infinite. ■

10.2.2 Tensor Product of Operators

As final mathematical topic we discuss the *Hilbert tensor product of operators*. If A and B are operators with domains $D(A)$ and $D(B)$ in the respective Hilbert spaces H_1 and H_2, we will denote by $D(A) \otimes D(B) \subset \mathsf{H}_1 \otimes \mathsf{H}_2$ the subspace of finite combinations of elements $\psi \otimes \phi$, with $\psi \in D(A)$, $\phi \in D(B)$. Let us try to define an operator:

$$A{\otimes}B : D(A) \otimes D(B) \to \mathsf{H}_1 \otimes \mathsf{H}_2$$

by extending linearly $\psi \otimes \phi \mapsto (A\psi) \otimes (B\phi)$. The question is whether it is well defined. So suppose, for $\Psi \in D(A) \otimes D(B)$, to have two (finite!) decompositions

$$\Psi = \sum_k c_k \psi_k \otimes \phi_k = \sum_j c'_k \psi'_j \otimes \phi'_j \ .$$

We have to check that

$$\sum_k c_k (A\psi_k) \otimes (B\phi_k) = \sum_j c'_j (A\psi'_j) \otimes (B\phi'_j) \ .$$

Take a basis (finite!) of vectors f_r for the joint span of the ψ_k and the ψ'_j, and a similar basis g_s for the span of ϕ_k and ϕ'_j. In particular,

$$\psi_i \otimes \phi_i = \sum_{r,s} \alpha^{(i)}_{rs} f_r \otimes g_s \ , \quad \psi'_j \otimes \phi'_j = \sum_{r,s} \beta^{(j)}_{rs} f_r \otimes g_s \ .$$

Having started with a single Ψ decomposed in different ways, necessarily

$$\sum_i c_i \alpha^{(i)}_{rs} = \sum_j c'_j \beta^{(j)}_{rs} \ .$$

From these identities we obtain

$$A{\otimes}B \left(\sum_i c_i \psi_i \otimes \phi_i \right) = \sum_{rs} (\sum_i c_i \alpha^{(i)}_{rs})((Af_r) \otimes (Bg_s))$$

$$= \sum_{rs} (\sum_j c'_j \beta^{(j)}_{rs})((Af_r) \otimes (Bg_s)) = A \otimes B \left(\sum_j c'_j \psi'_j \otimes \phi'_j \right) \ ,$$

making $A \otimes B$ well defined indeed. The procedure extends trivially to N operators $A_k : D(A_k) \to \mathsf{H}_k$, with domains $D(A_k)$ contained in the H_k.

Definition 10.31 If H_k, $k = 1, 2, \ldots, N$, are Hilbert spaces and $A_k : D(A_k) \to \mathsf{H}_k$ operators with domains $D(A_k) \subset \mathsf{H}_k$, the **(Hilbert) tensor product** of the A_k is the unique operator

$$A_1 \otimes \cdots \otimes A_N : D(A_k) \otimes \cdots \otimes D(A_N) \to \mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_N$$

extending

$$A_1 \otimes \cdots \otimes A_N(v_1 \otimes \cdots \otimes v_N) = (A_1 v_1) \otimes \cdots \otimes (A_N v_N) \quad \text{for } v_k \in D(A_k), k = 1, \ldots, N.$$

In view of the applications the next elementary result is useful.

Proposition 10.32 *Let $A_k : D(A_k) \to \mathsf{H}_k$, $k = 1, \ldots, N$, be operators on Hilbert spaces H_k.*

(a) *If $\overline{D(A_k)} = \mathsf{H}_k$ and A_k is closable for any k, then all operators of the form*

$$B_1 \otimes \cdots \otimes B_n \qquad \text{with } B_k \in \{I_k, A_k\} \text{ for } k = 1, \ldots, N$$

(I_k is the identity operator on H_k) or A_k and their finite combinations, defined on

$$D(A_1) \otimes \cdots \otimes D(A_N)\,,$$

are closable;

(b) *if $D(A_k) = \mathsf{H}_k$ and $A_k \in \mathfrak{B}(\mathsf{H}_k)$ for $k = 1, 2, \ldots, N$, then:*

(i) $A_1 \otimes \cdots \otimes A_N \in \mathfrak{B}(\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_N)$ *and*

(ii) $||A_1 \otimes \cdots \otimes A_N|| = ||A_1|| \cdots ||A_N||$.

Proof (a) Let us study $n = 2$, the rest being completely analogous. Note $D(A_1) \otimes D(A_2)$ is dense by construction (use Proposition 10.26), so the operators in (a) have adjoints. The generic element $\Psi \in D(A_1^*) \otimes D(A_2^*)$, by definition, satisfies $(\Psi | A_1 \otimes A_2 \Phi) = (A_1^* \otimes A_2^* \Psi | \Phi)$ for any $\Phi \in D(A_1) \otimes D(A_2)$. By definition of adjoint

$$D(A_1^*) \otimes D(A_2^*) \subset D((A_1 \otimes A_2)^*)\,.$$

Apply Theorem 5.10(b), for A_1, A_2 densely defined and closable, to the effect that the adjoints are densely defined and so $D((A_1 \otimes A_2)^*)$ is, too. Therefore $A_1 \otimes A_2$ is closable, by Theorem 5.10(b) again. For linear combinations the argument is the same. Claim (b) is proved in the exercise section. □

At this point we wish to consider *polynomials of operators* $A_1 \otimes \cdots \otimes A_N$, when A_k are self-adjoint [ReSi80]. In the ensuing statement, the argument A_k of the polynomial Q should be understood as $I \otimes \cdots \otimes I \otimes A_k \otimes I \otimes \cdots \otimes I$, but we will simplify the otherwise cumbersome notation.

Theorem 10.33 *Let $A_k : D(A_k) \to \mathsf{H}_k$, $D(A_k) \subset \mathsf{H}_k$, $k = 1, 2, \ldots, N$, be self-adjoint operators and $Q(a_1, \ldots, a_n)$ a real polynomial of degree n_k in the kth variable. Let $D_k \subset D(A_k)$ be a domain on which $A_k^{n_k}$ is essentially self-adjoint ($D_k := D(A_k)$, in particular). The following statements are true.*

(a) *$Q(A_1, \ldots, A_1)$ is essentially self-adjoint on $\bigotimes_{k=1}^N D_k$;*
(b) *If every H_k is separable, the spectrum of $\overline{Q(A_1, \ldots, A_N)}$ is:*

$$\sigma\left(\overline{Q(A_1, \ldots A_N)}\right) = \overline{Q(\sigma(A_1), \ldots, \sigma(A_N))}\,.$$

Proof (a) The operator $Q(A_1, \ldots, A_n)$ is well defined on $D := \otimes_{k=1}^N D(A_k^{n_k})$ (in particular by Theorem 9.4(d)) and symmetric, by a direct computation involving the definition of tensor product and the fact that Q has real coefficients and every A_k^m, $m \le n_k$ is symmetric on $D(A_k^{n_k})$. More can be said: $Q(A_1, \ldots, A_n)$ is essentially self-adjoint on D, by Nelson's Theorem 5.47, for we can exhibit a set of analytic vectors for $Q(A_1, \ldots, A_n)$ whose span is dense in the overall Hilbert space. Keeping in mind Example 10.27(1) and the proof of Proposition 9.25(f), a set of analytic vectors is given by tensor products $\psi_{\alpha_L}^{(L,1)} \otimes \cdots \otimes \psi_{\alpha_L}^{(L,N)}$, $L = 1, 2, \ldots$, where $\{\psi_{\alpha_L}^{(L,k)}\}_{\alpha_L \in G_L} \subset D(A_k)$ is a Hilbert basis for the closed subspace $P^{(A_k)}([-L, L) \cap \sigma(A_k))$, and $P^{(A_k)}$ is the spectral measure of A_k. Each time L increases by 1, and $[-L, L)$ gets replaced by $[-L-1, -L) \cup [-L, L) \cup [L, L+1)$, we must care to keep the same basis for the subspace associated to $[-L, L)$. That those vectors are analytic for $Q(A_1, \ldots, A_n)$ is easy, just replicating the proof of Proposition 9.25(f). To prove $Q(A_1, \ldots, A_n)$ is essentially self-adjoint on $D^{(e)} := \otimes_{k=1}^N D_k$ it suffices to prove the inclusion $\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} \supset Q(A_1, \ldots, A_n)\restriction_D$ (by construction, in fact, $Q(A_1, \ldots, A_n)\restriction_{D^{(e)}} \subset Q(A_1, \ldots, A_n)\restriction_D$ so $\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} \subset \overline{Q(A_1, \ldots, A_n)\restriction_D}$; if, further, $\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} \supset Q(A_1, \ldots, A_n)\restriction_D$, then

$$\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} = \overline{Q(A_1, \ldots, A_n)\restriction_D}$$

and the right side is self-adjoint, whence $Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}$ is essentially self-adjoint being symmetric with self-adjoint closure.)

To prove $\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} \supset Q(A_1, \ldots, A_n)\restriction_D$ assume $\otimes_{k=1}^N \phi_k \in D$. Then $\phi_k \in D(A_k^{n_k})$, and D_k being the domain of essential self-adjointness of $A_k^{n_k}$, there exists a sequence $\{\phi_k^l\}_{l \in \mathbb{N}}$ with $\phi_k^l \to \phi_k$ and $A_k^{n_k}\phi_k^l \to A_k^{n_k}\phi_k$, as $l \to +\infty$. A simple estimate involving the spectral decomposition of A_k tells $A_k^m \phi_k^l \to A_k^m \phi_k$, $l \to +\infty$, when $1 \le m \le n_k$. Therefore $\otimes_{k=1}^N \phi_k^l \to \otimes_{k=1}^N \phi_k$ and $Q(A_1, \ldots, A_N)(\otimes_{k=1}^N \phi_k^l) \to Q(A_1, \ldots, A_N)(\otimes_{k=1}^N \phi_k)$ as $l \to +\infty$. The results generalises to finite combinations of $\otimes_{k=1}^N \phi_k$, which implies $\overline{Q(A_1, \ldots, A_n)\restriction_{D^{(e)}}} \supset Q(A_1, \ldots, A_n)\restriction_D$.

(b) Using Theorem 9.18(c) and remembering the separability of each H_k, we represent every A_k via a multiplication operator by F_k on the Hilbert space $L^2(M_k, \mu_k)$ identified with H_k. We remind $\otimes_k^N L^2(M_k, \mu_k)$ is isomorphic to $L^2(\times_{k=1}^N M_k, \mu)$, $\mu = \otimes_{k=1}^N \mu_k$, as we saw in Example 10.27(1). Under that correspondence $Q(A_1, \ldots, A_N)$ on D is mapped to the multiplication by $Q(F_1, \ldots, F_N)$, and D corresponds to the span, inside $L^2(\times_{k=1}^N M_k, \mu)$, of finite products $\phi_1(m_1)\cdots\phi_N(m_N)$ such that $F_k^{n_k} \cdot \phi_k \in L^2(M_k, \mu_k)$.

Suppose $\lambda \in \overline{Q(\sigma(A_1), \ldots, \sigma(A_N))}$. If $I \ni \lambda$ is an open interval, $Q^{-1}(I) \supset \times_{k=1}^N I_k$ for some open interval $I_k \subset \mathbb{R}$, so that $I_k \cap \sigma(A_k) \neq \emptyset$ for any $k = 1, 2, \ldots, N$. Notice $\sigma(A_k) = \text{ess ran}(F_k)$, by Exercise 9.4. Therefore $\mu_k(F_k^{-1}(I_k)) \neq 0$, and so $\mu[Q(F_1, \ldots, F_N)^{-1}(I)] \neq 0$. This means $\lambda \in \text{ess ran} Q(F_1, \ldots, F_N) = \sigma(\overline{Q(A_1, \ldots, P_N)})$ by Exercise 9.4. Conversely, if $\lambda \notin \overline{Q(\sigma(A_1), \ldots, \sigma(A_N))}$ the function

$$(\lambda - Q(F_1, \ldots, F_N)) : \times_{k=1}^N M_k \to \mathbb{R}$$

is bounded, hence $\lambda \in \rho(\overline{Q(A_1, \ldots, A_N)})$, i.e. $\lambda \notin \sigma(\overline{Q(A_1, \ldots, A_N)})$. □

To conclude, consider two von Neumann algebras $\mathfrak{R}_1, \mathfrak{R}_2$ on complex Hilbert spaces $\mathsf{H}_1, \mathsf{H}_2$. There is a corresponding *tensor product of von Neumann algebras* $\mathfrak{R}_1 \otimes \mathfrak{R}_2$.

Definition 10.34 If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are von Neumann algebras on complex Hilbert spaces H_1 and H_2 respectively, their **tensor product** $\mathfrak{R}_1 \otimes \mathfrak{R}_2$ is the von Neumann algebra on $\mathsf{H}_1 \otimes \mathsf{H}_2$ given by the strong completion in $\mathfrak{B}(\mathsf{H}_1 \otimes \mathsf{H}_2)$ of the unital *-algebra of finite combinations of products $A_1 \otimes A_2$, with $A_1 \in \mathfrak{R}_1$, $A_2 \in \mathfrak{R}_2$.

The generalisation to finite products is straightforward, while the tensor product of infinitely many von Neumann algebras is more complicated to define.

The important **theorem on the commutant of tensor products of von Neumann algebras** [KaRi97, vol. II, p. 821] asserts what follows.

Theorem 10.35 *If $\mathfrak{R}_k$, $k = 1, \ldots, N$, are von Neumann algebras on complex Hilbert spaces H_k respectively, then*

$$(\mathfrak{R}_1 \otimes \cdots \otimes \mathfrak{R}_k)' = \mathfrak{R}_1' \otimes \cdots \otimes \mathfrak{R}_k' . \tag{10.39}$$

An elementary but important consequence is the following. Referring to the largest von Neumann algebras on H_1 and H_2 we then have

$$(\mathfrak{B}(\mathsf{H}_1) \otimes \mathfrak{B}(\mathsf{H}_2))' = \{c_1 I_1\}_{c_1 \in \mathbb{C}} \otimes \{c_2 I_2\}_{c_2 \in \mathbb{C}} = \{cI\}_{c \in \mathbb{C}} ,$$

and hence

$$(\mathfrak{B}(\mathsf{H}_1) \otimes \mathfrak{B}(\mathsf{H}_2))'' = \{cI\}_{c \in \mathbb{C}}' = \mathfrak{B}(\mathsf{H}_1 \otimes \mathsf{H}_2) .$$

On the other hand, using (10.39) twice, we also get:

$$(\mathfrak{B}(\mathsf{H}_1) \otimes \mathfrak{B}(\mathsf{H}_2))'' = \mathfrak{B}(\mathsf{H}_1)'' \otimes \mathfrak{B}(\mathsf{H}_2)'' = \mathfrak{B}(\mathsf{H}_1) \otimes \mathfrak{B}(\mathsf{H}_2) .$$

Summing up, we recover the known fact

$$\mathfrak{B}(\mathsf{H}_1) \otimes \mathfrak{B}(\mathsf{H}_2) = \mathfrak{B}(\mathsf{H}_1 \otimes \mathsf{H}_2) .$$

10.2.3 An Example: The Orbital Angular Momentum

The observables corresponding to the three Cartesian components of the orbital angular momentum of a particle, in QM, are the unique self-adjoint extensions of the operators:

$$\mathscr{L}_1 := X_2\restriction_{\mathscr{S}(\mathbb{R}^3)} P_3\restriction_{\mathscr{S}(\mathbb{R}^3)} - X_3\restriction_{\mathscr{S}(\mathbb{R}^3)} P_2\restriction_{\mathscr{S}(\mathbb{R}^3)} ,$$

$$\mathscr{L}_2 := X_3\restriction_{\mathscr{S}(\mathbb{R}^3)} P_1\restriction_{\mathscr{S}(\mathbb{R}^3)} - X_1\restriction_{\mathscr{S}(\mathbb{R}^3)} P_3\restriction_{\mathscr{S}(\mathbb{R}^3)}$$

$$\mathscr{L}_3 := X_1\restriction_{\mathscr{S}(\mathbb{R}^3)} P_2\restriction_{\mathscr{S}(\mathbb{R}^3)} - X_2\restriction_{\mathscr{S}(\mathbb{R}^3)} P_1\restriction_{\mathscr{S}(\mathbb{R}^3)} , \tag{10.40}$$

on the Hilbert space $L^2(\mathbb{R}^3, dx)$ (dx being the Lebesgue measure on $\mathbb{R}^3$). Notationwise, recall X_i and P_i are the operators position and momentum of Sect. 5.3, while $\mathscr{S}(\mathbb{R}^3)$ is the Schwartz space of smooth complex functions that vanish at infinity faster than any inverse power of $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$ together with all their derivatives (cf. Sect. 3.7). In the sequel we will take $D(\mathscr{L}_1) = D(\mathscr{L}_2) = D(\mathscr{L}_3) = \mathscr{S}(\mathbb{R}^3)$ as domain, since $\mathscr{S}(\mathbb{R}^3)$ is invariant under X_i and P_i (hence under $\mathscr{L}_i$). We will show that the **orbital angular momentum operators** $\mathscr{L}_i$ are essentially self-adjoint on the aforementioned domain, and we will find the spectrum and a spectral expression for them. In this section we shall concentrate on the mathematical features, reserving any physical consideration for Chaps. 11 and 12.

We focus on $\mathscr{L}_3$, because the results will apply to the other two by rotating coordinates. Explicitly:

$$\mathscr{L}_3 = -i\hbar \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) ,$$

where x_1, x_2 are viewed as multiplicative operators by the corresponding functions. A fourth operator used in the sequel is the **total angular momentum** (squared):

$$\mathscr{L}^2 := \mathscr{L}_1^2 + \mathscr{L}_2^2 + \mathscr{L}_3^2 ,$$

defined on $\mathscr{S}(\mathbb{R}^3)$. This, too, is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$. We will compute its spectrum and make the spectral expansion of $L^2 := \overline{\mathscr{L}^2}$ explicit.

It is convenient to write the operators in spherical coordinates r, θ, ϕ, where $x_1 = r \sin\theta \cos\phi$, $x_2 = r \sin\theta \sin\phi$, $x_3 = r \cos\theta$, so $r \in (0 + \infty)$, $\theta \in (0, \pi)$, $\phi \in (-\pi, \pi)$. In this manner a simple computation produces

$$\mathscr{L}_3 = -i\hbar \frac{\partial}{\partial \phi} , \quad \mathscr{L}^2 = -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) \right] , \tag{10.41}$$

where operators act on functions in $\mathscr{S}(\mathbb{R}^3)$ whose argument has undergone coordinate change. From (10.41) it is evident that the operators do not depend on the radial coordinate r, a fact of the utmost importance. Keeping that in mind, note that $\mathbb{R}^3 = \mathbb{S}^2 \times [0, +\infty)$, where (up to zero-measure sets) the unit sphere $\mathbb{S}^2$ is the domain of θ, ϕ, whilst $[0, +\infty)$ is where r varies. Furthermore, the Lebesgue measure on $\mathbb{R}^3$ can be seen as the product

$$dx = d\omega(\theta, \phi) \otimes r^2 dr \ ,$$

where

$$d\omega(\theta, \phi) = \sin\theta d\theta d\phi$$

is the standard measure on $\mathbb{S}^2$ identified with the rectangle $(0, \pi) \times (-\pi, \pi)$ by the spherical angles (θ, ϕ) (the complement to $(0, \pi) \times (-\pi, \pi)$ in $\mathbb{S}^2$ has null $d\omega$-measure, so it does not interfere). Thus we obtain the decomposition:

$$L^2(\mathbb{R}^3, dx) = L^2(\mathbb{S}^2 \times [0, +\infty), d\omega(\theta, \phi) \otimes r^2 dr) \ .$$

By Example 10.27(1):

$$L^2(\mathbb{R}, dx) = L^2(\mathbb{S}^2, d\omega) \otimes L^2((0, +\infty), r^2 dr) \ . \tag{10.42}$$

At this point we introduce two operators *on the Hilbert space* $L^2(\mathbb{S}^2, d\omega)$:

$$_{\mathbb{S}^2}\mathscr{L}_3 = -i\hbar \frac{\partial}{\partial \phi} \ , \quad _{\mathbb{S}^2}\mathscr{L}^2 = -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) \right] \ , \tag{10.43}$$

with domain $C^\infty(\mathbb{S}^2)$. As the sphere is a C^∞ manifold,[2] the space $C^\infty(\mathbb{S}^2)$ of smooth maps on $\mathbb{S}^2$ is dense in $L^2(\mathbb{S}^2, d\omega)$ (exercise). These operators are Hermitian, hence symmetric, as a simple computation reveals. Long before the formulation of QM, it was known from the study of the Laplace equation (and classical electrodynamics) that there exists a distinguished basis of $L^2(\mathbb{S}^2, d\omega)$ whose elements are called **spherical harmonics** [NiOu82]:

$$Y_l^m(\theta, \phi) := \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)}{4\pi} \frac{(l+m)!}{(l-m)!}} e^{im\phi} \frac{1}{\sin^m \phi} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (1 - \cos^2\theta)^l \ , \tag{10.44}$$

where:

$$l = 0, 1, 2, \dots \quad m \in \mathbb{N} \ , |m| \leq l \ . \tag{10.45}$$

[2]See Appendix B: the idea is to cover $\mathbb{S}^2$ with local charts in θ,ϕ, by rotating the Cartesian axes. Three local charts suffice to cover $\mathbb{S}^2$. The functions of $C^\infty(\mathbb{S}^2)$, by definition, have domain in $\mathbb{S}^2$ and codomain $\mathbb{C}$, and are C^∞ when restricted to any local chart of the sphere.

The maps $Y^l_m \in C^\infty(\mathbb{S}^2)$ are notoriously eigenfunctions of the differential operators ${}_{\mathbb{S}^2}\mathscr{L}_3$ and ${}_{\mathbb{S}^2}\mathscr{L}^2$ given in (10.43):

$$ {}_{\mathbb{S}^2}\mathscr{L}_3 Y^l_m = \hbar\, m Y^l_m\,, \quad {}_{\mathbb{S}^2}\mathscr{L}^2 Y^l_m = \hbar^2\, l(l+1) Y^l_m\,. \tag{10.46} $$

Note how the first is obvious by definition of Y^l_m. In particular the vectors Y^l_m are analytic for the symmetric operators ${}_{\mathbb{S}^2}\mathscr{L}^2$, ${}_{\mathbb{S}^2}\mathscr{L}_3$ defined on $C^\infty(\mathbb{S}^2)$. As Y^l_m form a basis of $L^2(\mathbb{S}^2, d\omega)$, by Nelson's criterion (Theorem 5.47) they warrant essential self-adjointness to ${}_{\mathbb{S}^2}\mathscr{L}^2$, ${}_{\mathbb{S}^2}\mathscr{L}_3$ on $C^\infty(\mathbb{S}^2)$. Following the recipe of Sect. 9.1.4 concerning the Hamiltonian operator of the one-dimensional harmonic oscillator, we obtain analogue spectral decompositions (in the strong operator topology):

$$ \overline{{}_{\mathbb{S}^2}\mathscr{L}^2} = \sum_{l\in\mathbb{N},\, m\in\mathbb{Z}\,, |m|\leq l} \hbar^2 l(l+1) Y^l_m (Y^l_m|\ ) \quad \text{and} \quad \overline{{}_{\mathbb{S}^2}L_3} = \sum_{l\in\mathbb{N},\, m\in\mathbb{Z}\,, |m|\leq l} \hbar m Y^l_m (Y^l_m|\ )\,. \tag{10.47} $$

In this context the spectra read

$$ \sigma(\overline{{}_{\mathbb{S}^2}\mathscr{L}^2}) = \sigma_p(\overline{{}_{\mathbb{S}^2}\mathscr{L}^2}) = \left\{ \hbar^2\, l(l+1)\ \middle|\, l = 0, 1, 2 \ldots \right\}\,, \tag{10.48} $$

and

$$ \sigma(\overline{{}_{\mathbb{S}^2}\mathscr{L}_3}) = \sigma_p(\overline{{}_{\mathbb{S}^2}\mathscr{L}_3}) = \{ \hbar\, m \ \mid |m| \leq l\,, m \in \mathbb{Z},\ l = 0, 1, 2 \ldots \}\,. \tag{10.49} $$

Now let us go back to $L^2(\mathbb{R}^3, dx)$. As the space $\mathscr{D}(0, +\infty)$ of smooth maps with compact support in $(0, +\infty)$ is dense in the separable Hilbert space $L^2((0 + \infty), r^2 dr)$, by Proposition 3.31(b), there will be a basis $\{\psi_n\}_{n\in\mathbb{N}}$ of maps in $\mathscr{D}(0, +\infty)$. Passing to Cartesian coordinates it is easy to see that

$$ f_{l,m,n}(x) = Y^l_m(\theta, \phi)\psi_n(r) \tag{10.50} $$

belong in $C^\infty(\mathbb{R}^3)$ (the only singularity could be at $x = 0$, but around that point the maps vanish by construction). By definition $f_{l,m,n}$ have compact support, so they live in $\mathscr{S}(\mathbb{R}^3)$. By the definitions and domains given,

$$ {}_{\mathbb{S}^2}\mathscr{L}_3 \otimes I\restriction_{\mathscr{D}(0,+\infty)} \subset \mathscr{L}_3 \quad \text{and} \quad {}_{\mathbb{S}^2}\mathscr{L}^2 \otimes I\restriction_{\mathscr{D}(0,+\infty)} \subset \mathscr{L}^2\,, $$

so $\{Y^l_m \otimes \psi_n \mid n, l \in \mathbb{N}, |m| \leq l\,, m \in \mathbb{Z}\} \subset \mathscr{S}(\mathbb{R}^3)$ is a basis of $L^2(\mathbb{R}^3, dx) = L^2(\mathbb{S}^2, d\omega) \otimes L^2((0, +\infty), r^2 dr)$ by Example 10.27(1). Thinking $\mathscr{L}_3$ and $\mathscr{L}^2$ as acting on $\mathscr{S}(\mathbb{R}^3)$,

$$ \mathscr{L}_3 Y^l_m \otimes \psi_n = \hbar\, m Y^l_m \otimes \psi_n\,, \quad \mathscr{L}^2 Y^l_m \otimes \psi_n = \hbar^2\, l(l+1) Y^l_m \otimes \psi_n\,. \tag{10.51} $$

Again, $\mathscr{L}^2$, $\mathscr{L}_3$ are essentially self-adjoint on that domain, and their unique self-adjoint extensions $L^2 := \overline{\mathscr{L}^2}$, $L_3 := \overline{\mathscr{L}_3}$ decompose spectrally (in the strong operator topology) as

$$L^2 = \sum_{l,n\in\mathbb{N},\ m\in\mathbb{Z}\ ,|m|\leq l} \hbar^2\, l(l+1) Y^l_m \otimes \psi_n (Y^l_m \otimes \psi_n|) \tag{10.52}$$

and

$$L_3 = \sum_{l,n\in\mathbb{N},\ m\in\mathbb{Z}\ ,|m|\leq l} \hbar\, m Y^l_m \otimes \psi_n (Y^l_m \otimes \psi_n|)\ . \tag{10.53}$$

The spectra of L^2, L_3 remain those of (10.48), (10.49). Note how the spectral measures of L^2, L_3 commute.

The same conclusions can be reached using Theorem 10.33 appropriately.

10.3 Polar Decomposition Theorem for Unbounded Operators

Consider a densely-defined closed operator $A : D(A) \to \mathsf{H}$ on the Hilbert space H. Using the fact that A^*A is self-adjoint and positive (as we will see) and by the spectral theorem for unbounded operators, it is possible to define the positive self-adjoint operator $|A| := \sqrt{A^*A}$. Setting $U = A|A|^{-1}$ at least on $Ran(|A|)$, and then extending trivially (as zero) to the complement of $Ran(|A|)$, we immediately find the decomposition

$$A = U|A|\ .$$

Formally, and without caring too much about domains, $U\restriction_{Ran(|A|)}$ is an isometry. Heuristically, this is a generalisation of Theorem 3.82, which we proved for bounded operators defined on the entire Hilbert space. Our hands-on approach is nevertheless flawed, in that is does not say where the polar decomposition should be valid (the domains of A and $|A|$ could be different, a priori) and any attempt to formalise the argument soon becomes punishing. That is why we will follow an indirect route based on a more general theorem.

The generalised polar decomposition we will eventually prove plays a crucial part in rigorous Quantum Field Theory, especially in relationship to the *modular theory* of Tomita and Takesaki, and in defining *KMS thermal states* [BrRo02].

10.3.1 Properties of Operators A^*A, Square Roots of Unbounded Positive Self-adjoint Operators

We proceed in steps, proving first that if A is closed and densely defined, A^*A is self-adjoint and $D(A^*A)$ is a core for A. Then we will show a result that in some sense generalises the polar decomposition, thus specifying properly the domains involved.

At last we will prove the existence and uniqueness of positive self-adjoint square roots of unbounded self-adjoint operators.

Theorem 10.36 (von Neumann) *Consider a closed, densely-defined operator* $A : D(A) \to \mathsf{H}$ *on the Hilbert space* H. *Then:*

(a) A^*A*, defined on the standard domain* $D(A^*A)$*, is self-adjoint;*
(b) *the dense subspace* $D(A^*A)$ *is a core for* A*:*

$$\overline{A\restriction_{D(A^*A)}} = A \ . \tag{10.54}$$

Proof For (a), call $I : \mathsf{H} \to \mathsf{H}$ the identity and introduce $I + A^*A$ on its standard domain (coinciding with $D(A^*A)$, by Definition 5.1). We claim there is a positive operator $P \in \mathfrak{B}(\mathsf{H})$ such that

$$(I + A^*A)P = I \ , \quad P(I + A^*A) = I\restriction_{D(I+A^*A)} \ . \tag{10.55}$$

By Proposition 3.60(f) $P \in \mathfrak{B}(\mathsf{H})$ is self-adjoint as positive. By uniqueness of the inverse the operator $I + A^*A$ coincides with the inverse of P, obtained by spectral decomposition:

$$P^{-1} = \int_{\sigma(P)} \lambda^{-1} dP^{(P)}(\lambda) \ .$$

This is self-adjoint by Theorem 9.4. Therefore $A^*A = (I + A^*A) - I$ is self-adjoint on $D(I + A^*A) = D(A^*A)$, which is consequently dense.

Now we have to exhibit the aforementioned positive $P \in \mathfrak{B}(\mathsf{H})$ satisfying (10.55). If $f \in D(I + A^*A) = D(A^*A)$ then $Af \in D(A^*)$ by definition of $D(A^*A)$. Hence

$$(f|f) + (Af|Af) = (f|f) + (f|A^*Af) = (f|(I + A^*A)f) \ .$$

We proved $(I + A^*A) \geq 0$, and by Cauchy–Schwarz also $||f||^2 \leq ||f||\,||(I + A^*A)f||$, so $I + A^*A : D(A^*A) \to \mathsf{H}$ is injective. Consider the operator A, closed and densely defined. The identity of Theorem 5.10(d) says that for any $h \in \mathsf{H}$ there are unique $Ph \in D(A)$ and $Qh \in D(A^*)$ such that

$$(0, h) = (-APh, Ph) + (Qh, A^*Qh) \tag{10.56}$$

in $\mathsf{H} \oplus \mathsf{H}$. By construction P, Q are defined on all of H, and the two vectors on the right, seen in $\mathsf{H} \oplus \mathsf{H}$, are orthogonal. By definition of norm on $\mathsf{H} \oplus \mathsf{H}$, the identity also tells:

$$||h||^2 \geq ||Ph||^2 + ||Qh||^2 \ ,$$

for any $h \in \mathsf{H}$, so $P, Q \in \mathfrak{B}(\mathsf{H})$ because $||P||, ||Q|| \leq 1$. Considering the single components in (10.56), we have

$$Q = AP \quad \text{and} \quad h = Ph + A^*Qh = Ph + A^*APh = (I + A^*A)Ph \ ,$$

for all $h \in \mathsf{H}$. Hence $(I + A^*A)P = I$ and $P : \mathsf{H} \to D(I + A^*A)$ must be one-to-one, but also onto since we saw $(I + A^*A)$ is injective. The inverse of a bijection is unique, so

$$P(I + A^*A) = I\restriction_{D(I+A^*A)} .$$

Up to now we have proved $P \in \mathfrak{B}(\mathsf{H})$ has range covering $D(I + A^*A)$, and that (10.55) holds. Let us show that $P \geq 0$. If $h \in \mathsf{H}$, then $h = (I + A^*A)f$ for some $f \in D(A^*A)$, so:

$$(Ph|h) = (P(I + A^*A)f|(I + A^*A)f) = (f|(I + A^*A)f) \geq 0 .$$

To finish we prove (b). As A is closed, its graph is closed in $\mathsf{H} \oplus \mathsf{H}$, so a Hilbert space itself. Suppose $(f, Af) \in G(A)$ is orthogonal to $G(A\restriction_{D(A^*A)})$. Then for any $x \in D(A^*A)$:

$$0 = ((f, Af)|(x, Ax)) = (f|x) + (Af|Ax) = (f|x) + (f|A^*Ax) = (f|(I + A^*A)x) .$$

But $Ran(I + A^*A) = \mathsf{H}$, so $f = 0$ and the orthogonal complement to $G(A\restriction_{D(A^*A)})$ in the Hilbert space $G(A)$ is trivial. Therefore $\overline{G(A\restriction_{D(A^*A)})} = G(A)$. $\square$

Together with the uniqueness for positive roots of (unbounded) positive self-adjoint operators, the next theorem contains, as subcase, the polar decomposition theorem for closed and densely-defined operators. Recall that for a pair P, Q with the same domain D, $P \leq Q$ means $(f|Pf) \leq (f|Qf)$ for any $f \in D$.

Theorem 10.37 *Let $A : D(A) \to \mathsf{H}$, $B : D(B) \to \mathsf{H}$ be closed and densely defined on the Hilbert space H.*

(a) *If*

$$D(A^*A) \supset D(B^*B) \quad and \quad A^*A\restriction_{D(B^*B)} \leq B^*B , \tag{10.57}$$

then $D(A) \supset D(B)$ and there exists $C \in \mathfrak{B}(\mathsf{H})$ uniquely determined by:

$$A \supset CB , \quad Ker(C) \supset Ker(B^*) . \tag{10.58}$$

Furthermore, $||C|| \leq 1$ and $C\restriction_{(Ran(B))^\perp} = 0$.

(b) *If*

$$A^*A \supset B^*B \tag{10.59}$$

then $C\restriction_{\overline{Ran(B)}}$ is an isometry and $Ker(C) = Ker(B^*)$.

(c) *If*

$$A^*A = B^*B , \tag{10.60}$$

then $D(A) = D(B)$.

Proof (a) Begin by the uniqueness of C. If $C, C' \in \mathfrak{B}(\mathsf{H})$ and $Ax = CBx$, $Ax = C'Bx$ for $x \in D(B)$, then $C - C'$ is the null operator on $Ran(B)$. By continuity $C\restriction_{\overline{Ran(B)}} = C'\restriction_{\overline{Ran(B)}}$. From the splitting $\mathsf{H} = \overline{Ran(B)} \oplus (\overline{Ran(B)})^\perp$, where $(\overline{Ran(B)})^\perp = Ker(B^*)$, having $KerC \supset Ker(B^*)$, $KerC' \supset Ker(B^*)$ implies $C\restriction_{(\overline{Ran(B)})^\perp} = C'\restriction_{(\overline{Ran(B)})^\perp} = 0$. Hence $C = C'$.

Let us prove there exists $C \in \mathfrak{B}(\mathsf{H})$ such that $A \supset CB$ (hence $D(B) \subset D(A)$), $Ker(C) \supset Ker(B^*)$, $||C|| \leq 1$ and $C\restriction_{(\overline{Ran(B)})^\perp} = 0$.

Call A', B' the restrictions of A, B to $D(A^*A)$, $D(B^*B)$ respectively. By the previous theorem these are cores for A, B, so $\overline{Ran(A')} = \overline{Ran(A)}$ and $\overline{Ran(B')} = \overline{Ran(B)}$ in particular. Note $Ker(A) = Ker(A')$, $Ker(B) = Ker(B')$ for the very definition of $D(A^*A)$ and $D(B^*B)$.

Let us define an operator such that $A' \supset CB'$, to begin with. Define C on $Ran(B')$ such that:

$$A'f = CB'f\ , \quad \text{for every } f \in D(B^*B) \subset D(A^*A)\ .$$

For it to be well defined, we need $B'f = B'g$ to imply $A'f = A'g$, i.e. $B'h = 0 \Rightarrow A'h = 0$ for any $h \in D(B^*B) \subset D(A^*A)$. But the latter is true, for: $B'h = 0$ implies $(B'h|B'h) = 0$, so $0 = (B'h|B'h) = (h|B^*Bh) \geq (h|A^*Ah) = (A'h|A'h) = ||A'h||^2 \geq 0$. Hence $A'h = 0$. The claim is that C is bounded on $Ran(B')$ with $||C|| \leq 1$. Since $A^*A \leq B^*B$, using $D(A^*A) \subset D(A)$ and $D(B^*B) \subset D(B)$, we have

$$\begin{aligned} ||C(B'f)||^2 = (CB'f|CB'f) = (A'f|A'f) = (f|A^*Af) \leq (f|B^*Bf) \\ = (B'f|B'f) = ||B'f||^2\ , \end{aligned} \tag{10.61}$$

if $f \in D(B^*B) \subset D(A^*A)$. Therefore C extends uniquely to $\overline{Ran(B')} = \overline{Ran(B)}$, preserving $||C|| \leq 1$. To fully define $C : \mathsf{H} \to \mathsf{H}$ it suffices to know it on the complement $(\overline{Ran(B)})^\perp = Ker(B^*)$. Let C be null there. Then $C : \mathsf{H} \to \mathsf{H}$ is bounded, $||C|| \leq 1$ and $Ker(C) \supset Ker(B^*)$. By construction:

$$Af = CBf \quad \text{for any } f \in D(B^*B) \subset D(A^*A)\ .$$

Since $D(B^*B)$ is a core for B, if $g \in D(B)$ there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(B^*B) \subset D(A^*A)$ such that $f_n \to g$ and $Bf_n \to Bg$. By continuity of C,

$$\lim_{n\to+\infty} Af_n = \lim_{n\to+\infty} CBf_n = C \lim_{n\to+\infty} Bf_n = CBg\ .$$

But A is closed, so $g \in D(A)$ and $\lim_{n\to+\infty} Af_n = Ag$. Hence $A' = CB'$ actually extends to $A = CB$ on $D(B) \subset D(A)$.

(b) Assuming $A^*A \supset B^*B$, Eq. (10.61) is replaced by:

$$\begin{aligned} ||C(B'f)||^2 = (CB'f|CB'f) = (A'f|A'f) = (f|A^*Af) = (f|B^*Bf) = (B'f|B'f) \\ = ||B'f||^2 \end{aligned}$$

if $f \in D(B^*B) \subset D(A^*A)$. Therefore C is an isometry on $Ran(B)$ and by continuity on $\overline{Ran(B)}$ as well. There remains to prove $Ker(C) \subset Ker(B^*)$, for the other inclusion is valid in the general case (a). If $s \in Ker(C)$, from $\mathsf{H} = \overline{Ran(B)} \oplus Ker(B^*)$ we have $s = r + n$, $r \in \overline{Ran(B)}$, $n \in Ker(B^*)$. Since $Ker(B^*) \subset Ker(C)$, we obtain $0 = Cs = C(r+n) = Cr + Cn = Cr + 0 = Cr$. On the other hand C is isometric on $\overline{Ran(B)}$, so $0 = ||Cr|| = ||r||$ and $r = 0$. Therefore $s \in Ker(C)$ implies $s = n \in Ker(B^*)$, ending the proof of $Ker(C) \subset Ker(B^*)$.

(c) We show that $D(A) = D(B)$ if $A^*A = B^*B$. From the proof of the more general case (a), $D(B) \subset D(A)$. In the present case A and B can be swapped, so $D(A) = D(B)$. □

And now the last ingredient, generalising part of Theorem 3.77.

Theorem 10.38 *Let $A : D(A) \to \mathsf{H}$ be self-adjoint on the Hilbert space H. Then*

(a) $A \geq 0 \Leftrightarrow \sigma(A) \subset [0 + \infty)$*;*

(b) *if $A \geq 0$, there exists a unique self-adjoint operator $B \geq 0$ such that $B^2 = A$, where the left-hand side is defined on its standard domain $D(B^2)$, coinciding with $D(A)$.*

By using the integral in the spectral measure of A it turns out that $B = \sqrt{A}$.

Proof (a) If $\sigma(A) \subset [0 + \infty)$, Theorem 9.4(g), with reference to the PVM $P^{(A)}$ of A, implies $A \geq 0$. Vice versa, suppose $A \geq 0$ and, by contradiction, that there exists λ with $0 > \lambda \in \sigma(A)$. If λ were in $\sigma_p(A)$ there would be a λ-eigenvector $\psi \in \mathsf{H} \setminus \{\mathbf{0}\}$, and $0 \leq (\psi|A\psi) = ||\psi||^2\lambda < 0$, impossible. Instead, if $\lambda \in \sigma_c(A)$, by Theorem 9.13 $P^{(A)}_{(a,b)} \neq 0$ for any open interval $(a, b) \ni \lambda$. So we could choose $(a, b) = (2\lambda, \lambda/2)$, getting, for $\mathbf{0} \neq \psi \in P^{(A)}_{(a,b)}(\mathsf{H})$,

$$0 \leq (\psi|A\psi) = \int_{\mathbb{R}} x d\mu_\psi(x) = \int_{(2\lambda,\lambda/2)} x d\mu_\psi(x) \leq \int_{(2\lambda,\lambda/2)} \frac{\lambda}{2} d\mu_\psi(x) = \frac{\lambda}{2}||\psi||^2 < 0\,,$$

using Theorem 9.4 and that μ_ψ vanishes outside (a, b). This is absurd.

(b) A positive self-adjoint root of A is just

$$B = \sqrt{A} := \int_{\sigma(A)} \sqrt{x} dP^{(A)}(x)\,.$$

The operator is well defined, since $\sigma(A) \subset [0, +\infty)$, it is self-adjoint by Theorem 9.4(b) and $B^2 = A$, where B^2 is defined on its standard domain $D(B^2) = D(A)$ by Theorem 9.4(c, d). At last $B \geq 0$ by Theorem 9.4(g). Let us pass to uniqueness. Assume $B \geq 0$ is self-adjoint and $B = \int_{[0,+\infty)} x dP^{(B)}(x)$. If $B^2 = A$ with $A \geq 0$, by (9.54) we obtain

$$\int_{[0,+\infty)} x dP^{(A)}(x) = \int_{[0,+\infty)} x^2 dP^{(B)}(x) = \int_{[0,+\infty)} x dP^{(B)}(f^{-1}(x))\,,$$

where $f(x) = x^2$, $x \geq 0$, so $f^{-1} : [0, +\infty) \to [0 + \infty)$ is a well-defined map. The spectral measure of A is unique, so in particular $P^{(B)}(f^{-1}(E')) = P^{(A)}(E')$ for any Borel set $E' \subset [0, +\infty)$. If $E \subset [0, +\infty)$ is a Borel set, $f(E) \subset [0, +\infty)$. Setting $E' = f(E)$ gives $P^{(B)}(E) = P^{(A)}(f(E))$ for any Borel set $E \subset [0, +\infty)$ (and $P^{(B)}(E) = 0$ if $E \subset (-\infty, 0)$). Therefore A determines B completely, for it determines its unique PVM. □

10.3.2 Polar Decomposition Theorem for Closed and Densely-Defined Operators

We can finally prove the polar decomposition for closed, densely-defined operators. The idea of the proof is to start, rather than from A, from A^*A. If the polar decomposition is to hold, one expects $A^*A = |A|\,|A|$, with $|A| := \sqrt{A^*A}$ defined spectrally, remembering A^*A is self-adjoint. Now Theorem 10.37(c) yields the required polar decomposition of A. The powerfulness of this approach becomes apparent when considering the properties of the domains involved: usually hard to study by a more direct method, they are now automatic, by Theorem 10.37.

Theorem 10.39 *Let $A : D(A) \to \mathsf{H}$ be closed and defined densely on the Hilbert space H.*

(a) *There exists a unique pair P, U on H such that:*

(1) the **polar decomposition**
$$A = UP \tag{10.62}$$
holds,
(2) P is positive, self-adjoint and $D(P) = D(A)$,
(3) $U \in \mathfrak{B}(\mathsf{H})$ is isometric on $Ran(P)$,
(4) $Ker(U) \supset Ker(P)$.

(b) *Moreover:*

(i) $P = |A| := \sqrt{A^*A}$,
(ii) $Ker(U) = \underline{Ker(P)} = Ker(A) = (Ran(P))^\perp$ and $\overline{Ran(P)} = \overline{Ran(A^*)}$,
(iii) $Ran(U) = \overline{Ran(A)}$.

Remark 10.40 The operator U of (10.62) is a *partial isometry* (Definition 3.72) with *initial space*
$$[Ker(U)]^\perp = \overline{Ran(P)} = [Ker(A)]^\perp = \overline{Ran(A^*)}$$

and *final space*
$$Ran(U) = \overline{Ran(A)}\,.$$

■

Proof (a) We prove uniqueness by finding P and U explicitly, assuming (10.62) plus (2), (3), (4). First we show $D(A^*A) = D(PP)$. By definition of adjoint, as $U \in \mathfrak{B}(\mathsf{H})$, (10.62) implies $A^* = P^*U^* = PU^*$. Hence $f \in D(A^*A)$ if and only if $f \in D(PU^*UP)$. Splitting H into $\overline{Ran(P)} \oplus Ker(P^*) = \overline{Ran(P)} \oplus Ker(P)$, and since U is assumed to be isometric on $\overline{Ran(P)}$ and zero on $Ker(P)$, we get $(U^*U)\restriction_{Ran(P)} = I\restriction_{Ran(P)}$. Hence the claim $f \in D(A^*A) \Leftrightarrow f \in D(PU^*UP)$ is equivalent to $f \in D(A^*A) \Leftrightarrow f \in D(PP)$. So we have proved $D(A^*A) = D(PP)$. Let us use it towards uniqueness. If $g \in D(A^*A) \subset D(A)$ (i.e. $g \in D(PP) \subset D(P)$), recalling U is isometric on $Ran(P)$, then

$$(f|A^*Ag) = (Af|Ag) = (UPf|UPg) = (Pf|Pg) = (f|PPg) \quad \text{for } f \in D(A) = D(P).$$

Being $D(A) = D(P)$ dense, we conclude $A^*A = PP$. Therefore P is a positive self-adjoint root of A^*A, hence unique by Theorem 10.38, and $P = \sqrt{A^*A} =: |A|$. Now we can apply Theorem 10.37 with $B = P$ (closed and densely defined, being self-adjoint) to find that U is uniquely determined and coincides with C satisfying (3) and (4). We have proved that, if they exist, P and U are uniquely determined by (1)–(4). Reversing our reasoning, let A be as in the hypothesis, and define $P := |A|$ and $U := C$ (see Theorem 10.37, with $B = |A|$ so that $A^*A = B^*B$). Then requirements (1), (2) are valid.

(b) We already know that $P = |A|$. That $Ker(U) = Ker(P) = (Ran(P))^\perp$ follows from Theorem 10.37(b), because $B = P = P^* = B^*$ in our case and $(Ran(P^*))^\perp = Ker(P)$. The claim $Ker(A) = Ker(P)$ goes as follows:

$$0 = ||Af||^2 = (Af|Af) = (f|A^*Af) = (f|PPf) = (Pf|Pf) = ||Pf||^2\,,$$

where we used the fact that $Af = 0$ implies $f \in D(A^*A)$ by definition of the latter. Now $\overline{Ran(P)} = \overline{Ran(A^*)}$ follows immediately from the previous properties: $\overline{Ran(P)} = Ker(P)^\perp = Ker(A)^\perp = \overline{Ran(A^*)}$. In the final equality we used Theorem 5.10(c), as A is closed and the domain of A^* is dense by Theorem 5.10(b). Let us prove $Ran(U) = \overline{Ran(A)}$. By the decomposition $UP = A$, taking $Ker(U) = [Ran(P)]^\perp$ into account, we have $Ran(U) = U(\overline{Ran(P)}) = Ran(A)$, so $\overline{Ran(U)} = \overline{U(Ran(P))} = \overline{Ran(A)}$. Then it suffices to show $Ran(U)$ is closed. Let $y \in \overline{Ran(U)} \setminus \{\mathbf{0}\}$. There exists $\{x_n\}_{n\in\mathbb{N}} \subset (KerU)^\perp$ with $Ux_n \to y$, $n \to +\infty$. Since $||U(x_n - x_m)|| = ||x_n - x_m||$, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Define $x = \lim_{n\to+\infty} x_n$, so $Ux = y$, $y \in Ran(U)$ and then $Ran(U)$ contains its limit points, i.e. it is closed. □

Corollary 10.41 *In the hypotheses of Theorem 10.39 the operator $U : \mathsf{H} \to \mathsf{H}$ is unitary precisely when A is injective and $Ran(A)$ is dense in H.*

In particular U is unitary and $U = A|A|^{-1}$ in case A is bijective.

Proof If U is unitary, in particular it is one-to-one and onto, so (ii) and (iii) in (b) imply A is injective and $Ran(A)$ dense in H. Conversely, if A is injective, by (a) and (b) in the theorem above and by continuity of U, we have U isometric on

$\overline{Ran(|A|)} = Ker(|A|)^\perp = Ker(A)^\perp = \mathsf{H}$. As $Ran(U) = \overline{Ran(A)}$ in case $Ran(A)$ is dense, we conclude $U : \mathsf{H} \to \mathsf{H}$ is isometric and onto, hence unitary as claimed. If A is further bijective, it is injective and $Ran(A)$ is trivially dense, so U is unitary as seen before. From $A = U|A|$, then, A and U being bijective, we obtain $|A|$ is bijective, so $U = A|A|^{-1}$, ending the proof. □

10.4 The Theorems of Kato–Rellich and Kato

The last results we will state and prove are those of Kato–Rellich and Kato. They are extremely useful to study self-adjointness and lower boundedness for QM operators (especially the so-called *Hamiltonian operators*), in the framework of *perturbation theory*. The Kato–Rellich theorem provides sufficient conditions for an operator of the form $T + V$, called a *perturbation* of T, to be self-adjoint, and have lower-bounded spectrum when T has. Kato's theorem considers specific situations, where T is the Laplacian on $\mathbb{R}^3$ or $\mathbb{R}^n$. A general treatise, with applications to quantum physics, is [ReSi80], from which several proofs of this section are taken.

10.4.1 The Kato–Rellich Theorem

A preliminary definition is in order.

Definition 10.42 Let $T : D(T) \to \mathsf{H}$ and $V : D(V) \to \mathsf{H}$ be densely-defined operators on the Hilbert space H, with $D(T) \subset D(V)$. If there are $a, b \in [0, +\infty)$ such that

$$||V\varphi|| \leq a||T\varphi|| + b||\varphi|| \quad \text{for any } \varphi \in D(T)\,, \tag{10.63}$$

V is called ***T*-bounded**. The greatest lower bound of the numbers a satisfying (10.63) for some b is called the **relative bound** of V with respect to T. If the relative bound is zero, V is called **infinitesimally small** with respect to T.

Remark 10.43

(1) If T and V are closable, by Definition 5.20 it suffices to verify (10.63) over a core of T.

(2) Equation (10.63) is equivalent to:

$$||V\varphi||^2 \leq a_1^2||T\varphi||^2 + b_1^2||\varphi||^2 \quad \text{for any } \varphi \in D(T)\,, \tag{10.64}$$

In fact, (10.64) implies (10.63) by putting $a = a_1$, $b = b_1$. For the converse, take $a_1^2 = (1+\delta)a^2$, $b_1^2 = (1 + 1/\delta)b^2$ for any $\delta > 0$: then (10.63) implies (10.64). (The greatest lower bound of the numbers a_1 satisfying (10.64) for some b_1 is also called the relative bound of V with respect to T. Due to the arbitrariness of $\delta > 0$, it coincides with the relative bound computed using (10.63).) ■

Let us pass to the *Kato–Rellich theorem*. For a self-adjoint operator $A : D(A) \to \mathsf{H}$, we know $\sigma(A) \subset [M, +\infty) \Leftrightarrow (\psi|A\psi) \geq M(\psi|\psi)$ for any $\psi \in D(A)$, by Theorem 10.38(a). Therefore statement (c) below may be equivalently expressed in terms of lower-bounded quadratic forms.

Theorem 10.44 (Kato–Rellich) *Let $T : D(T) \to \mathsf{H}$ and $V : D(V) \to \mathsf{H}$ be densely-defined operators on the Hilbert space H such that:*

(i) *T is self-adjoint,*
(ii) *V is symmetric,*
(iii) *V is T-bounded with* **relative bound** *$a < 1$.*

Then

(a) *$T + V$ is self-adjoint on $D(T)$.*
(b) *$T + V$ is essentially self-adjoint on every core of T.*
(c) *If $\sigma(T) \subset [M, +\infty)$ then $\sigma(T + V) \subset [M', +\infty)$ where:*

$$M' = M - \max\left\{\frac{b}{(1-a)}, \ a|M| + b\right\}, \quad \textit{with } a, b \textit{ satisfying } (10.63).$$

Proof For (a) we try to apply Theorem 5.18, showing that if we choose $D(T)$ as domain for the symmetric operator $T+V$, we obtain $Ran(T+V \pm iI) = \mathsf{H}$. Actually we will prove there exists $\nu > 0$ such that

$$Ran(T + V \pm i\nu I) = \mathsf{H}\,,$$

giving the previous relation by linearity. If $\varphi \in D(T)$, T self-adjoint implies $Ran(T + i\mu I) = \mathsf{H}$ and

$$||(T + i\mu I)\varphi||^2 = ||T\varphi||^2 + \mu^2||\varphi||^2\,.$$

Setting $\varphi = (T + i\mu I)^{-1}\psi$, gives

$$||T(T + i\mu I)^{-1}|| \leq 1 \quad \text{and} \quad ||(T + i\mu I)^{-1}|| \leq \mu^{-1}\,.$$

Applying (10.63) with $\varphi = (T + i\mu I)^{-1}\psi$ produces

$$||V(T+i\mu I)^{-1}\psi|| \leq a||T(T+i\mu I)^{-1}\psi||+b||(T+i\mu I)^{-1}\psi|| \leq \left(a + \frac{b}{\mu}\right)||\psi||.$$

If $\mu = \nu$ is large enough, the bounded operator

$$U := V(T + i\nu I)^{-1}\,,$$

defined on H, satisfies $||U|| < 1$, as $a < 1$. This implies $-1 \notin \sigma(U)$ by (iii) in Theorem 8.4(c). By Theorem 8.4(a) (U is closed as bounded), we have $Ran(I+U) = \mathsf{H}$. At the same time, since T is self-adjoint, $Ran(T + i\nu I) = \mathsf{H}$ by Theorem 5.18. Hence

$$(I + U)(T + i\nu I)\varphi = (T + V + i\nu I)\varphi \,, \quad \varphi \in D(T)$$

implies, as claimed, $Ran(T + V + i\nu I) = \mathsf{H}$. The proof of $Ran(T + V - i\nu I) = \mathsf{H}$ is completely similar, so (a) is proved.

Let us pass to (b). Equation (10.63) implies, if $\mathscr{D} \subset D(T)$ is a core for T:

$$D(T) = D\left(\overline{T\restriction_{\mathscr{D}}}\right) \subset D\left(\overline{(T+V)\restriction_{\mathscr{D}}}\right) .$$

On the other hand, by construction and because $T + V$ is self-adjoint on $D(T)$ hence closed:

$$D\left(\overline{(T+V)\restriction_{\mathscr{D}}}\right) \subset D\left(\overline{(T+V)}\right) = D\left(T+V\right) = D(T) .$$

Putting all inclusions together produces $D\left(\overline{(T+V)\restriction_{\mathscr{D}}}\right) = D\left(T+V\right)$ so $\overline{(T+V)\restriction_{\mathscr{D}}} = T + V$, as $T + V$ is closed. Then $(T + V)\restriction_{\mathscr{D}}$ is essentially self-adjoint by Proposition 5.21.

Now (c). By assumption, the spectral theorem implies $\sigma(T) \geq M$ (with obvious notation). Choosing $s > -M$ (with $s \in \mathbb{R}$) gives $\sigma(T + sI) > 0$, so $0 \notin \sigma(T + sI)$. But $T + sI$ is self-adjoint, so it is closed, and by Theorem 8.4(a) $Ran(T + sI) = \mathsf{H}$. The same estimates used before prove $||V(T + sI)^{-1}|| < 1$ if

$$-s < M' := M - \max\left\{\frac{b}{(1-a)}, a|M| + b\right\} .$$

Consequently, for these s:

$$Ran(T + V + sI) = \mathsf{H} \quad \text{and} \quad (T + V + sI)^{-1} = (T + sI)^{-1}(I + U)^{-1} ,$$

implying $-s \in \rho(T + V)$, and then $-s \notin \sigma(T + V)$. But $T + V$ self-adjoint has real spectrum, whence $\sigma(T + V) \geq M'$. $\square$

10.4.2 An Example: The Operator $-\Delta + V$ and Kato's Theorem

Condition (10.63) arises naturally in certain contexts, and is of great use, in physical applications, to study the Schrödinger equation, where the *Laplace operator* Δ is perturbed by a potential V. To discuss this application of the Kato–Rellich theorem we begin with a proposition and a lemma.

Proposition 10.45 *Let*

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{10.65}$$

be the Laplace operator on $\mathbb{R}^n$ thought of as operator on (suitable domains of) $L^2(\mathbb{R}^n, dx)$.

(a) *If $\widehat{\mathscr{F}} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dk)$ is the unitary Fourier–Plancherel operator (cf. Sect. 3.7), then Δ is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^n)$, on $\mathscr{D}(\mathbb{R}^n)$ and on $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^n))$ with the same (unique) self-adjoint extension $\overline{\Delta}$.*

(b) *If $k^2 = k_1^2 + k_2^2 + \cdots + k_n^2$ then*

$$\left(\widehat{\mathscr{F}}\overline{\Delta}\widehat{\mathscr{F}}^{-1} f\right)(k) = -k^2 f(k) \tag{10.66}$$

on the standard domain

$$D(\widehat{\mathscr{F}}\overline{\Delta}\widehat{\mathscr{F}}^{-1}) = \left\{ f \in L^2(\mathbb{R}^n, dk) \;\middle|\; \int_{\mathbb{R}^n} k^4 |f(k)|^2 dk < +\infty \right\}.$$

(c) *The operator $\overline{-\Delta} = -\overline{\Delta}$ is bounded from below:*

$$\sigma(\overline{-\Delta}) \subset [0, +\infty)\,, \quad \textit{or equivalently } (\psi|\overline{-\Delta}\psi) \geq 0 \textit{ for any } \psi \in D(\overline{-\Delta}). \tag{10.67}$$

Proof Most of (a) and (b) were proven in Exercises 5.13, 5.14. What we still do not have is that Δ is essentially self-adjoint on $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^n))$ and has a common self-adjoint extension over both $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$. To this end notice $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^n)), \mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, so the three extensions coincide because there is one self-adjoint extension to any essentially self-adjoint operator. That Δ is essentially self-adjoint on $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^n))$, given $\widehat{\mathscr{F}}$ is unitary and (10.66), is equivalent to the essential self-adjointness of the symmetric multiplication by $-k^2$ on $\mathscr{D}(\mathbb{R}^n)$. In turn the latter, in view of Nelson's Theorem 5.47, follows from the observation that every $\varphi = \varphi(k)$ in $\mathscr{D}(\mathbb{R}^n)$ is analytic for the multiplication by $-k^2$, since $||-(k^2)^n\varphi|| \leq ||\varphi||(\sup_{k \in supp\varphi} |k|^2)^n$. Statement (c) descends from (b) and from Theorem 10.38(a). □

Now a fundamental, classical result.

Lemma 10.46 *Fix $n = 1, 2, 3$ and consider $f \in D(\overline{\Delta})$. Then f coincides almost everywhere with a continuous bounded map, and for any $a > 0$ there exists $b > 0$ independent from f such that:*

$$||f||_\infty \leq a||\overline{\Delta} f|| + b||f||\,. \tag{10.68}$$

Proof Let us prove (10.68) for $n = 3$, the other cases being similar. Call $\hat{f} := \widehat{\mathscr{F}} f$. By Proposition 3.105(a) and Plancherel's theorem (Theorem 3.108), the claim is true if we manage to prove $\hat{f} \in L^1(\mathbb{R}^3, dk)$, and for any given $a > 0$ there is $b \in \mathbb{R}$ such that:

$$||\hat{f}||_1 \leq a||k^2 \hat{f}||_2 + b||\hat{f}||_2 \, . \tag{10.69}$$

If $f \in D(\overline{\Delta})$, by Proposition 10.45, $\hat{f} \in D(\widehat{\mathscr{F}\overline{\Delta}\mathscr{F}^{-1}})$, so also $(1 + k^2)\hat{f} \in L^2(\mathbb{R}^3, dk)$. Since $(k_1, k_2, k_3) \mapsto 1/(1 + k^2)$ belongs to that same space, $\hat{f} \in L^1(\mathbb{R}^3, dk)$ by the Hölder inequality. Moreover:

$$||\hat{f}||_1 \leq c||(1 + k^2)\hat{f}||_2 \leq c(||k^2 \hat{f}||_2 + ||\hat{f}||_2) \tag{10.70}$$

where $c := \sqrt{\int (1 + k^2)^{-1} dk}$. If $r > 0$ define $\hat{f}_r(k) := r^3 \hat{f}(rk)$. Then $||\hat{f}_r||_1 = ||\hat{f}||_1$, $||\hat{f}_r||_2 = r^{3/2}||\hat{f}||_2$ and $||k^2 \hat{f}_r||_2 = r^{-1/2}||k^2 \hat{f}||_2$. Using (10.70) for $\hat{f}_r$ the three previous identities give

$$||\hat{f}||_1 \leq cr^{-1/2}||k^2 \hat{f}||_2 + cr^{3/2}||\hat{f}||_2 \quad \text{for any } r > 0.$$

Then (10.69) holds for $a = cr^{-1/2}$. □

Remark 10.47 The lemma can be generalised (see [ReSi80, vol. II]) by this statement based on Young's inequality: *Consider $f \in L^2(\mathbb{R}^n, dx)$ with $f \in D(\overline{\Delta})$. If $n \geq 4$ and $2 \leq q < 2n/(n-4)$ then $f \in L^q(\mathbb{R}^n, dk)$, and for any $a > 0$ there exists $b \in \mathbb{R}$ not depending on f (but on q, n, a) such that $||f||_q \leq a||\overline{\Delta} f|| + b||f||$.* ■

We can eventually apply the Kato–Rellich theorem to a very interesting case for Quantum Mechanics, and prove a result due to Kato. Later we will see a more general statement, known in the literature as Kato's theorem.

Theorem 10.48 (Essential self-adjointness of $-\Delta + V$) *Fix $n = 1, 2, 3$ and take $V = V_2 + V_\infty$, with $V_2 \in L^2(\mathbb{R}^n, dx)$, $V_\infty \in L^\infty(\mathbb{R}^n, dx)$ real functions.*

(a) *$-\Delta + V$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$ and on $\mathscr{S}(\mathbb{R}^n)$.*
(b) *The only self-adjoint extension $\overline{-\Delta + V}$ of the operators of (a) coincides with the (self-adjoint) operator $\overline{-\Delta} + V$ defined on $D(\overline{\Delta})$.*
(c) *$\sigma(\overline{-\Delta + V})$ is bounded from below.*

Proof As V is real it gives a multiplicative operator on the domain

$$D(V) := \{\varphi \in L^2(\mathbb{R}^n, dx) \mid V\varphi \in L^2(\mathbb{R}^n, dx)\} \, .$$

Using the definition it is easy to see the operator is self-adjoint. By construction, moreover,

$$||V\varphi||_2 \leq ||V_2||_2||\varphi||_\infty + ||V_\infty||_\infty||\varphi||_2 < +\infty \tag{10.71}$$

for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ or $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Hence $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset D(V)$. What is more, since $\mathscr{S}(\mathbb{R}^n) \subset D(\overline{\Delta})$ (by Proposition 10.45), using (10.68) in Lemma 10.46 ($n \leq 3$) we find, for any $a > 0$, a number $b > 0$ such that:

$$||V\varphi||_2 \le a||V_2||_2\,||-\Delta\varphi||_2 + (b + ||V_\infty||_\infty)||\varphi||_2 \quad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}^n)\,.$$

That is to say: given $a' > 0$ there is $b' > 0$ with

$$||V\varphi||_2 \le a'||-\Delta\varphi||_2 + b'||\varphi||_2 \quad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}^n) \tag{10.72}$$

so in particular for any $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Consequently

$$||V\varphi - V\varphi'||_2 \le a'\,||(-\Delta\varphi) - (-\Delta\varphi')||_2 + b'||\varphi - \varphi'||_2$$

with φ, φ' in $\mathscr{S}(\mathbb{R}^n)$. Now, V is closed, as self-adjoint, and $\mathscr{S}(\mathbb{R}^n)$ is a core for the self-adjoint (hence closed) operator $\overline{-\Delta}$ (by Proposition 10.45), so the inequality proves $D(V) \supset D(\overline{-\Delta})$. Exploiting the closure of operators we conclude that (10.72) holds on the entire domain of $\overline{-\Delta}$:

$$||V\varphi||_2 \le a'||\overline{-\Delta}\varphi||_2 + b'||\varphi||_2 \quad \text{for any } \varphi \in D(\overline{-\Delta}).$$

If we choose $a' < 1$, $T := \overline{-\Delta}$ satisfies the assumptions of Theorem 10.44, with V as we have it now. By Kato–Rellich, using that $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$ are cores for $\overline{-\Delta}$ by Proposition 10.45, we conclude. □

Remark 10.49 Remembering Remark 10.47, this theorem generalises to $n > 3$ with these modifications: $V = V_p + V_\infty$ with $V_p \in L^p(\mathbb{R}^n, dx)$, $V_\infty \in L^\infty(\mathbb{R}^n, dx)$, where $p > 2$ for $n = 4$, $p = n/2$ for $n \ge 5$. The proof is analogous. ■

For the classical result known as *Kato's theorem*, we shall interpret $f \in L^p(\mathbb{R}^n, dx) + L^q(\mathbb{R}^n, dx)$ to mean f is the sum of a function in $L^p(\mathbb{R}^n, dx)$ and one in $L^q(\mathbb{R}^n, dx)$.

Theorem 10.50 (Kato) *Fix $n = 1, 2, 3$ and denote by $(\mathbf{y}_1, \dots, \mathbf{y}_N)$ the elements in $\mathbb{R}^{nN}$, where $\mathbf{y}_k \in \mathbb{R}^n$ for any $k = 1, \dots, N$. If Δ is the Laplacian (10.65) on $\mathbb{R}^{nN}$, consider the differential operator $-\Delta + V$, V being the multiplicative operator given by:*

$$V(\mathbf{y}_1, \dots, \mathbf{y}_N) := \sum_{k=1}^N V_k(\mathbf{y}_k) + \sum_{i,j=1\, i<j}^N V_{ij}(\mathbf{y}_i - \mathbf{y}_j)\,, \tag{10.73}$$

where

$$\{V_k\}_{k=1,\dots,N} \subset L^2(\mathbb{R}^n, dx) + L^\infty(\mathbb{R}^n, dx)\,,\ \ \{V_{ij}\}_{i<j\, i,j=1,\dots,N} \subset L^2(\mathbb{R}^n, dx) + L^\infty(\mathbb{R}^n, dx)$$

are real functions. Then

(a) *$-\Delta + V$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^{nN})$ and $\mathscr{S}(\mathbb{R}^{nN})$.*

(b) *The only self-adjoint extension $\overline{-\Delta + V}$ of the operators in (a) coincides with the (self-adjoint) operator $\overline{-\Delta} + V$ defined on $D(\overline{-\Delta})$.*

(c) *$\sigma(\overline{-\Delta + V})$ is lower bounded.*

Proof We prove for $n = 3$, for the other cases are identical. Consider the potential $V_{12}(\mathbf{y}_1 - \mathbf{y}_2)$ and call Δ_1 the Laplacian corresponding to the coordinates of $\mathbf{y}_1$. Take $\varphi \in \mathscr{S}(\mathbb{R}^{3N})$. Fix $\mathbf{y}_2, \dots \mathbf{y}_N \in \mathbb{R}^{3(N-1)}$ and define $\mathbb{R}^3 \ni \mathbf{y}_1 \mapsto \varphi'(\mathbf{y}_1) := \varphi(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N)$. Then φ' belongs in $\mathscr{D}(\mathbb{R}^{3N})$ or $\mathscr{S}(\mathbb{R}^{3N})$, according to whether $\varphi \in \mathscr{D}(\mathbb{R}^{3N})$ or $\varphi \in \mathscr{S}(\mathbb{R}^{3N})$ respectively. Similarly, let $\mathbb{R}^3 \ni \mathbf{y}_1 \mapsto V'_{12}(\mathbf{y_1}) := V_{12}(\mathbf{y}_1 - \mathbf{y}_2)$. As in the previous proof, by decomposing $V_{12} = (V_{12})_2 + (V_{12})_\infty$ we arrive at the estimate, for any $a > 0$ and any $\mathbf{y}_2, \dots, \mathbf{y}_N$:

$$||V'_{12}\varphi'||_{L^2(\mathbb{R}^3)} \leq a||(V_{12})_2||_{L^2(\mathbb{R}^3)}\, ||-\Delta_1\varphi'||_{L^2(\mathbb{R}^3)} + (b + ||(V_{12})_\infty||_{L^\infty(\mathbb{R}^3)})||\varphi'||_2$$

where $b > 0$ depends on a, *not* on $\mathbf{y}_2, \dots \mathbf{y}_N \in \mathbb{R}^{3(N-1)}$. Norms are in the spaces over the first copy of $\mathbb{R}^3$ in $\mathbb{R}^{3N}$. It is important to note, due to the invariance of $(\mathbf{y}_1, \mathbf{y}_2) \mapsto V_{12}(\mathbf{y}_1 - \mathbf{y}_2)$ under translations, that the norms $||(V_{12})_k||_{L^k(\mathbb{R}^3)}$ do not depend on the variable $\mathbf{y}_2$. From Remark 10.43 this inequality is the same as

$$||V'_{12}\varphi'||^2_{L^2(\mathbb{R}^3)} \leq a'\, ||-\Delta_1\varphi'||^2_{L^2(\mathbb{R}^3)} + b'||\varphi'||^2_{L^2(\mathbb{R}^3)}$$

for certain $a', b' > 0$ with a' arbitrarily small because of $a||V_{12}||_2$. Integrating the inequality in the variables $\mathbf{y}_2, \dots \mathbf{y}_N \in \mathbb{R}^{3(N-1)}$ produces, for any $a' > 0$, a corresponding $b' > 0$ such that

$$||V_{12}\varphi||^2_{L^2(\mathbb{R}^{3N})} \leq a'\, ||-\Delta_1\varphi||^2_{L^2(\mathbb{R}^{3N})} + b'||\varphi||^2_{L^2(\mathbb{R}^{3N})}\,. \tag{10.74}$$

Transforming with Fourier–Plancherel on $\mathbb{R}^{3N}$, we now have

$$||-\Delta_1\varphi||^2_{L^2(\mathbb{R}^{3N})} = \int_{\mathbb{R}^{3N}} \left|\sum_{r=1}^{3} k_r^2\right|^2 |(\widehat{\mathscr{F}}\varphi)(k_1, \dots, k_{3N})|^2 dk_1 \cdots dk_{3N}$$
$$\leq \int_{\mathbb{R}^{3N}} \left|\sum_{r=1}^{3N} k_r^2\right|^2 |(\widehat{\mathscr{F}}\varphi)(k_1, \dots, k_{3N})|^2 dk_1 \cdots dk_{3N} = ||-\Delta\varphi||^2_{L^2(\mathbb{R}^{3N})}\,.$$

Substituting in (10.74) we conclude that if $\varphi \in \mathscr{D}(\mathbb{R}^{3N})$ or $\mathscr{S}(\mathbb{R}^{3N})$, then for any $a > 0$ there exists $b_{12} > 0$ satisfying

$$||V_{12}\varphi||^2_{L^2(\mathbb{R}^{3N})} \leq a\, ||-\Delta\varphi||^2_{L^2(\mathbb{R}^{3N})} + b_{12}||\varphi||^2_{L^2(\mathbb{R}^{3N})}\,.$$

The same result holds for the other potentials V_{ij}, V_k: the proof goes along the same lines, and is even simpler. If $\varphi \in \mathscr{D}(\mathbb{R}^{3N})$ or $\mathscr{S}(\mathbb{R}^{3N})$, for any $a > 0$ there are corresponding $b_i > 0$ and $b_{ij} > 0$ $(i, j = 1, \dots, N,\ j > i)$ such that:

$$||V_i\varphi||^2_{L^2(\mathbb{R}^{3N})} \leq a\, ||-\Delta\varphi||^2_{L^2(\mathbb{R}^{3N})} + b_i||\varphi||^2_{L^2(\mathbb{R}^{3N})}\,, \tag{10.75}$$

$$||V_{ij}\varphi||^2_{L^2(\mathbb{R}^{3N})} \leq a\, ||-\Delta\varphi||^2_{L^2(\mathbb{R}^{3N})} + b_{ij}||\varphi||^2_{L^2(\mathbb{R}^{3N})}\,. \tag{10.76}$$

On any Hermitian inner product space the Cauchy–Schwartz inequality implies $\left|\left|\sum_{r=1}^{M} \psi_r\right|\right|^2 \leq \left(\sum_{r=1}^{M} ||\psi_r||\right)^2$. There are $N + N(N-1)/2 = N(N+1)/2$ potentials V_k and V_{ij}, so Cauchy–Schwartz and (10.75)–(10.76) force

$$\left|\left|\left(\sum_{k=1}^{N} V_k + \sum_{i,j=1\, i<j}^{N} V_{ij}\right)\varphi\right|\right|^2_{L^2(\mathbb{R}^{3N})}$$

$$\leq \left(\frac{N(N+1)}{2}\right)^2 a\, || - \Delta\varphi||^2_{L^2(\mathbb{R}^{3N})} + \left(\frac{N(N+1)}{2}\right)^2 b||\varphi||^2_{L^2(\mathbb{R}^{3N})}$$

where b is the maximum of the b_k, b_{ij}. From Remark 10.43 the result has an equivalent formulation. For every $a' > 0$ there exists a $b' > 0$ such that

$$||V\varphi|| \leq a'|| - \Delta\varphi|| + b'||\varphi|| \quad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}^{3N}).$$

From this point onwards the proof picks up from Eq. (10.72) in the proof of Theorem 10.48, replacing $\mathbb{R}^n$ with $\mathbb{R}^{3N}$. □

In conclusion we mention, without full proof, another important result of Kato. The demands on V to have $-\Delta + V$ essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$ are different (and weaker than Theorem 10.48 if $n = 3$). Recall that $f : \mathbb{R}^n \to \mathbb{C}$ is called **locally square-integrable** if $f \cdot g$ is in $L^2(\mathbb{R}^n, dx)$ for every $g \in \mathscr{D}(\mathbb{R}^n)$.

Theorem 10.51 *The operator $-\Delta + V_\Delta + V_C$ defined on $L^2(\mathbb{R}^n, dx)$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$, and its unique self-adjoint extension $\overline{-\Delta + V_\Delta + V_C}$ is bounded from below, provided the following conditions hold.*

(i) $V_\Delta : \mathbb{R}^n \to \mathbb{R}$ is measurable and induces a $(-\Delta)$-bounded multiplicative operator with relative bound $a < 1$ (cf. Definition 10.42).
(ii) $V_C : \mathbb{R}^n \to \mathbb{R}$ is locally square-integrable with $V_C \geq C$ almost everywhere, for some $C \in \mathbb{R}$.

Part (i) holds if

$$V_\Delta \in L^p(\mathbb{R}^n, dx) + L^\infty(\mathbb{R}^n, dx)\,,$$

with $p = 2$ when $n \leq 3$, $p > 2$ when $n = 4$ and $p = n/2$ when $n \geq 5$.

Sketch of proof. The final statement was proved with Theorem 10.48 if $n \leq 3$. The argument is the same for $n > 4$ by the remark ensuing Lemma 10.46. If (i) holds $-\Delta + V_\Delta$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$ and $\overline{-\Delta + V_\Delta}$ is lower bounded by the Kato–Rellich theorem. If (ii) holds as well, $-\Delta + V_\Delta + (V_C - C)$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$ by [ReSi80, vol.II, Theorem X.29], for $V_c - C \geq 0$. Therefore $-\Delta + V_\Delta + V_C = (-\Delta + V_\Delta + (V_C - C)) + CI$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^n)$. Since $-\Delta + V_\Delta$ and V_C are both bounded from below on that domain, so are $-\Delta + V_\Delta + V_C$ and $\overline{-\Delta + V_\Delta + V_C}$. □

Example 10.52

(1) A case in $\mathbb{R}^3$ that is interesting to physics is one where the Laplacian perturbation V is the *attractive Coulomb potential*:

$$V(x) = \frac{eQ}{|x|} ,$$

with $e < 0$, $Q > 0$ constants, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$. The hypotheses of Kato's Theorem 10.50 (or 10.48) are valid for the operator:

$$H_0 := -\frac{\hbar^2}{2m}\Delta + V(x)$$

(the constants $m, \hbar > 0$ are irrelevant to the previous theorem, since we may multiply the operator by $2m/\hbar^2$ and then apply it, without losing in generality). So H_0 is essentially self-adjoint if defined on $\mathscr{D}(\mathbb{R}^3)$ or $\mathscr{S}(\mathbb{R}^3)$. The self-adjoint extension $\overline{H_0}$, if $Q = -e$, corresponds to the *Hamiltonian* operator of an electron in the electric field of a proton (neglecting spin effects and viewing the proton as a classical object). This gives the simplest quantum description of the Hamiltonian operator of the hydrogen atom. Here $-e$ is the common absolute value of the charge of electron and proton, m is the electronic mass, $\hbar > 0$ is Planck's constant divided by 2π. The spectrum of the unique self-adjoint extension of this operator determines, in physics, the admissible values of the energy of the system. Despite V is not bounded from below, it is important that the spectrum of the operator considered is always bounded, and therefore also the energy values that are physically admissible have a lower bound. In Chaps. 11, 12 and 13 we will examine better the meaning of the operators here briefly described.

(2) A second case of physical interest, always in $\mathbb{R}^3$, is given by the *Yukawa potential*:

$$V(x) = \frac{-e^{-\mu|x|}}{|x|} ,$$

where $\mu > 0$ is another positive number. Here, too, the operator $H_0 = -\frac{\hbar^2}{2m}\Delta + V(x)$ is essentially self-adjoint if defined on $\mathscr{D}(\mathbb{R}^3)$ or on $\mathscr{S}(\mathbb{R}^3)$, as we know from Kato's Theorem 10.50 (or 10.48). The Yukawa potential describes, roughly, interactions between a *pion* and a source of the *strong force*, the latter thought of, in this manner of speaking, as being caused by a macroscopic source.

(3) The third physically-relevant case is the Hamiltonian of a system of N particles that interact under an external Coulomb potential and the Coulomb potentials of all pairs (not necessarily attractive). Call $\mathbf{x}_i \in \mathbb{R}^3$ the position vectors, $m_i > 0$ the masses and $e_i \in \mathbb{R} \setminus \{0\}$ the charges ($i = 1, \dots, N$). The full operator is

$$H_0 := \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\Delta_i + \sum_{i=1}^{N} \frac{Q_i e_i}{|\mathbf{x}_i|} + \sum_{i<j}^{N} \frac{e_i e_j}{|\mathbf{x}_i - \mathbf{x}_j|},$$

where Δ_i is the Laplacian in the three coordinates of $\mathbf{x}_i$. In order to apply Kato's theorem we must eliminate all factors $\frac{\hbar^2}{2m_i}$ multiplying the Δ_i. For this we can just change coordinates to $\mathbf{y}_i := \frac{\sqrt{2m_i}}{\hbar}\mathbf{x}_i$. Thus the first sum above gives the Laplacian on $\mathbb{R}^{3N}$ in the collective $3N$ components of the $\mathbf{y}_i$. It is not hard to see that the perturbation $V(\mathbf{y}_1, \ldots, \mathbf{y}_N)$ satisfies Kato's Theorem 10.50, so H_0 is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^{3N})$ and its unique self-adjoint extension is bounded from below.

(4) Theorem 10.51 allows to say the following. Adding any real function V', locally integrable and bounded from below, to the Hamiltonian operators H_0 seen in the previous examples gives an essentially self-adjoint operator on the corresponding $\mathscr{D}(\mathbb{R}^n)$. An important instance is the harmonic potential (non-isotropic, in general) $V'(x) = kx_1^2 + k_2x^2 + k_3x_3^2$ with $k_1, k_2, k_3 \geq 0$. ■

Exercises

10.1 Referring to Example 10.16(1), assume $\gamma > 0$. Prove the solution to the Klein–Gordon equation with source $e^{i\omega}\psi$ and dissipative term has the form:

$$u_t = e^{-\gamma t}\left[\cos\left(t\sqrt{A-\gamma I}\right)v + \sin\left(t\sqrt{A-\gamma I}\right)(A-\gamma I)^{-1/2}(v' + \gamma v)\right]$$

$$+ e^{-\gamma t}C_t\psi + e^{i\omega t}(A^2 - \omega^2 I + 2i\gamma\omega I)^{-1}\psi,$$

where $||C_t|| \leq 1$.

Hint. Apply the definition of $\int_a^b L_\tau\psi_\tau d\tau$ given in (10.10). Then pass to the spectral measures of A and use the Fubini–Tonelli theorem, carefully verifying the assumptions.

10.2 If $A_k \in \mathfrak{B}(\mathsf{H}_k)$, $k = 1, \ldots, N$, prove

$$A_1 \otimes \cdots \otimes A_k \in \mathfrak{B}(\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_N).$$

Solution. Consider $N = 2$, the general case being similar. If $\psi = \{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ are bases of H_1 and H_2, take the finite sum $\psi := \sum_{ij} c_{ij} f_i \otimes g_j$. Then $||(A_1 \otimes I)\psi||^2 = \sum_j ||\sum_i c_{ij}A_1 f_i||^2 \leq \sum_j ||A_1||^2 \sum_i |c_{ij}|^2 = ||A_1||^2||\psi||^2$. A density argument allows to conclude $||A_1 \otimes I|| \leq A$, and therefore $||A_1 \otimes A_2|| \leq ||A_1 \otimes I||\ ||I \otimes A_2|| \leq ||A_1||\ ||A_2||$.

10.3 If $A_k \in \mathfrak{B}(\mathsf{H}_k)$, $k = 1, \ldots, N$, prove that

$$||A_1 \otimes \cdots \otimes A_k|| = ||A_1|| \cdots ||A_N|| \, .$$

Solution. We already know that $A_1 \otimes \cdots \otimes A_k \in \mathfrak{B}(\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_N)$ from Exercise 10.2. Take $n = 2$, the generalisation being similar. If $A_1 = 0$ or $A_2 = 0$ the claim is obvious, so assume $||A_1||, ||A_2|| > 0$. When solving (2) we found $||A_1 \otimes A_2|| \leq ||A_1|| \, ||A_2||$, so it is enough to obtain the opposite inequality. By definition of $||A_1||$ and $||A_2||$, for any $\varepsilon > 0$ there are $\psi_1^{(\varepsilon)} \in \mathsf{H}_1$ and $\psi_2^{(\varepsilon)} \in \mathsf{H}_2$, $||\psi_1^{(\varepsilon)}||, ||\psi_2^{(\varepsilon)}|| = 1$, such that $| \, ||A_i \psi_i^{(\varepsilon)}|| - ||A_i|| \, | < \varepsilon$. In particular, $||\psi_i^{(\varepsilon)}|| \geq ||A_i|| - \varepsilon$. With these choices

$$||(A_1 \otimes A_2)(\psi_1^{(\varepsilon)} \otimes \psi_2^{(\varepsilon)})|| = ||A_1 \psi_1^{(\varepsilon)}|| \, ||A_2 \psi_2^{(\varepsilon)}|| \geq ||A_1|| \, ||A_2|| - \varepsilon(||A_1|| + ||A_2||) + \varepsilon^2 .$$

Since $||\psi_1^{(\varepsilon)} \otimes \psi_2^{(\varepsilon)}|| = 1$, and from

$$||A_1 \otimes A_2|| = \sup_{||\psi||=1} ||A_1 \otimes A_2 \psi|| \geq ||(A_1 \otimes A_2)(\psi_1^{(\varepsilon)} \otimes \psi_2^{(\varepsilon)})|| \, ,$$

for any $\varepsilon > 0$ we have $||A_1 \otimes A_2|| \geq ||A_1|| \, ||A_2|| - \varepsilon(||A_1|| + ||A_2||) + \varepsilon^2$, where $-\varepsilon(||A_1|| + ||A_2||) + \varepsilon^2 < 0$. That value tends to 0 as $\varepsilon \to 0^+$. Eventually, $||A_1 \otimes A_2|| \geq ||A_1|| \, ||A_2||$ as required.

10.4 If $A_k \in \mathfrak{B}(\mathsf{H}_k)$, $k = 1, \ldots, N$ prove

$$(A_1 \otimes \cdots \otimes A_k)^* = A_1^* \otimes \cdots \otimes A_N^* \, .$$

Hint. Check $A_1^* \otimes \cdots \otimes A_N^*$ satisfies the properties of the adjoint to a bounded operator (Proposition 3.36).

10.5 If $P_k \in \mathfrak{B}(\mathsf{H}_k)$, $k = 1, \ldots, N$ are orthogonal projectors, show that $P_1 \otimes \cdots \otimes P_k$ is an orthogonal projector.

10.6 Suppose that $A : D(A) \to \mathsf{H}$ is a closed, densely defined, normal operator ($A^*A = AA^*$). Prove that $D(A) = D(A^*)$.

Solution. Since $A^*A = AA^*$ can be written $A^*A = (A^*)^*A^*$, Theorem 10.37(c) implies $D(A) = D(A^*)$.

10.7 Suppose that $A : D(A) \to \mathsf{H}$ is a closed, densely defined, normal operator ($A^*A = AA^*$). Referring to the polar decompositions $A = UP$ and $A^* = VS$, show the following facts hold

(i) $V = U^*$ and $S = P$,
(ii) U is normal,
(iii) $UA \subset AU$, $U^*A \subset AU^*$, $UP \subset PU$, $U^*P \subset PU^*$, $AP \subset PA$, $A^*P \subset PA^*$.

Solution. Start from $UPPU^* = AA^* = A^*A = PU^*UP$. Since $U^*U|_{Ran(P)} = I|_{Ran(P)}$, we have $UP^2U^* = P^2$ and so $P^2U^* = U^*P^2$. Therefore U^* commutes with the PVM of P^2, and hence with every function of it, in particular $P = \sqrt{P^2}$: $U^*P \subset PU^*$. Taking the adjoint we also have $UP \subset PU$. Taking the adjoint of both sides of $A = UP$ we get $A^* = PU^*$, and then $A^*x = U^*Px$ provided $x \in D(P)$. However $D(P) = D(A^*)$ because $D(P) = D(A) = D(A^*)$ (Theorem 10.37), so that $A^* = U^*P$. Now, notice that U^* is an isometry on $Ran(P)$ because U is an isometry on $Ran(U)$ and $(U^*Px|U^*Px) = (x|PUU^*Px) = (x|AA^*x) = (x|A^*Ax) = (x|PU^*UPx) = (x|PPx) = (Px|Px)$. A similar argument proves that $(Px|(U^*U - UU^*)Px) = 0$. Actually, since $Ker(P) \perp Ran(P)$, $Ker(P) \oplus \overline{Ran(P)} = \mathsf{H}$ and finally $U(Ker(P)) \subset Ker(P)$ and $U^*(Ker(P)) \subset Ker(P)$ (for U and U^* commute with P), then $(Px|(U^*U - UU^*)Px) = 0$ can be extended to $(y|(U^*U - UU^*)y) = 0$ for every $y \in \mathsf{H}$. We conclude that $UU^* = U^*U$ as requested, but also $Ker(U^*) = Ker(U)$. The latter contains $Ker(P)$ by hypothesis, so $Ker(U) \supset Ker(P)$. In summary, $A^* = U^*P$ satisfies all requirements defining the polar decomposition of A^*. Uniqueness concludes the proof of the fact that $A^* = U^*P$ is the polar decomposition of A^*. Finally observe that, since U commutes with P, we have $UA = UUP \subset UPU = AU$. A similar argument proves $UA^* \subset A^*U$. Regarding the last inclusions: $AP = UPP \subset PUP = PA$ and $A^*P = U^*PP \subset PU^*P = PA^*$.

10.8 Suppose that $A : D(A) \to \mathsf{H}$ is a closed, densely defined, self-adjoint (or anti-self-adjoint) operator and assume that $Ker(A) = \{\mathbf{0}\}$. Referring to the polar decomposition $A = UP$ prove that $U = U^* = U^{-1}$ (respectively, $U = -U^* = U^{-1}$).

Hint. Use the spectral decomposition theorem.

10.9 Study the polar decomposition $\overline{A} = UP$ for the operator A of Sect. 9.1.4. Prove that U satisfies

$$U\psi_n = \psi_{n-1}$$

if $n \geq 1$ and $\{\psi_n\}_{n\in\mathbb{N}}$ is the basis of $L^2(\mathbb{R}, dx)$ defined in Sect. 9.1.4.

10.10 Study the polar decomposition $\overline{A}^* = VP_1$ for the operator A of Sect. 9.1.4. Prove that V satisfies

$$V\psi_n = \psi_{n+1}$$

if $n \geq 0$ and $\{\psi_n\}_{n\in\mathbb{N}}$ is the basis of $L^2(\mathbb{R}, dx)$ defined in Sect. 9.1.4.

10.11 Referring to Exercises (10.9) and (10.10), prove that

$$P = \sqrt{N} \quad \text{and} \quad P_1 = \sqrt{N + I}$$

where N is the unique self-adjoint extension of the symmetric operator defined on the finite span of the Hilbert basis $\{\psi_n\}_{n\in\mathbb{N}}$, introduced in Sect. 9.1.4, such that $N\psi_n = n\psi$ for $n \in \mathbb{N}$.

10.12 Consider the operator A of Sect. 9.1.4 and prove that $\sigma_p(A^*) = \sigma_c(A^*) = \emptyset$ and $\sigma_r(A^*) = \mathbb{C}$.

Solution. We already know from Exercise 9.8 that $\sigma_p(\overline{A}) = \mathbb{C}$, while Exercises 10.10 and 10.11 tell us that $A^* = V\sqrt{N+1}$. Consider $\psi \in \mathsf{H}$ such that $A^*\psi = \lambda\psi$ for some $\lambda \in \mathbb{C}$. Since $\sqrt{N+1}$ is invertible, this equation is equivalent to $V\phi = \lambda\sqrt{N+1}^{-1}\phi$ where both V and $\sqrt{N+1}^{-1}$ are bounded and everywhere defined. Expanding $\phi = \sum_{n\in\mathbb{N}} c_n\psi_n$, with reference to the Hilbert basis $\{\psi_n\}_{n\in\mathbb{N}}$ introduced in Sect. 9.1.4, we find the identity

$$\sum_n c_n\psi_{n+1} = \lambda\sum_n \frac{c_n}{\sqrt{n+1}}\psi_n$$

and hence

$$\lambda c_n = \sqrt{n+1}c_{n-1} \quad n = 1, 2, \dots .$$

If $\lambda = 0$ the only solution is $c_n = 0$ for every n, and so $\psi = \sqrt{N+1}^{-1}\mathbf{0} = \mathbf{0}$. If $\lambda \neq 0$, we have $c_n = \frac{\sqrt{(n+1)!}}{\lambda^n}c_0$. Unless $c_0 = 0$, giving $\psi = \mathbf{0}$ again, we have $\sum_n |c_n|^2 = +\infty$, and we conclude ψ cannot exist. We have proved that $\sigma_p(A^*) = \emptyset$. Next observe that, as $\overline{A}$ is closed and the domain of $A^* = \overline{A}^*$ is dense, Theorem 5.10(c) implies $Ran(A^* - \lambda I)^\perp = Ker(\overline{A} - \overline{\lambda}I) \neq \{\mathbf{0}\}$ (the last inequality was proved in Exercise 9.8). Hence $Ran(A^* - \lambda I)$ cannot be dense for every $\lambda \in \mathbb{C}$, and therefore $\sigma_r(A^*) = \mathbb{C}$ because $A^* - \lambda I$ is injective for every $\lambda \in \mathbb{C}$, as we proved above.

10.13 Prove the statement in Remark 10.43(2).

10.14 Suppose $V : \mathbb{R}^3 \to \mathbb{R}$ makes the symmetric operator H_1, given by the differential operator $-\Delta_x + V(x)$, essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$, where $\Delta_x := \sum_{k=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian. Prove that the symmetric operator H on $L^2(\mathbb{R}^3 \times \mathbb{R}^3, dx \otimes dy)$ defined by the differential operator $-\Delta_x + V(x) - \Delta_y + V(y)$ is essentially self-adjoint on the span of finite products of a map in x in $\mathscr{S}(\mathbb{R}^3)$ and a map in y in $\mathscr{S}(\mathbb{R}^3)$. Then show

$$\sigma(\overline{H}) = \overline{\sigma(H_1) + \sigma(H_1)} .$$

10.15 Prove that the *attractive Coulomb potential* in $\mathbb{R}^3$:

$$V(x) = \frac{eQ}{|x|} ,$$

with $e < 0$, $Q > 0$ constants, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$, satisfies Kato's Theorem 10.50. What happens if we increase or decrease the dimension of $\mathbb{R}^3$?

Chapter 11
Mathematical Formulation of Non-relativistic Quantum Mechanics

Every science would be redundant if the essence of things and their phenomenic appearance coincided.

Karl Marx

In this chapter we shall mainly enucleate the axioms of QM for the elementary system made by a non-relativistic spinless particle, and discuss a series of important results related to the *canonical commutation relations* (CCRs).

In section one we will revisit, especially in the light of spectral theory, the four axioms of Chap. 7. We will introduce the *von Neumann algebra of observables* and *complete sets of commuting observables*.

Section two further develops the notion of *superselection rules*, and presents a few new technical results in relation with von Neumann algebras.

Section three deals with several technical facts, of physical relevance, about the notion of observable viewed as a self-adjoint operator.

Then, in section four, we will add an axiom for the formalisation of the quantum theory of the spin-zero particle. We shall introduce the CCRs and prove they cannot be satisfied by bounded operators. We will show how *Heisenberg's uncertainty principle* is actually a theorem in the formulation.

The penultimate section is dedicated to the famous *theorem of Stone–von Neumann*, later refined by Mackey, which characterises continuous unitary representations of the CCRs. To prove the theorem we will introduce *Weyl* **-algebras* and discuss their main properties. After proving the theorems of Stone–von Neumann and Mackey, we will use the formalism to extend Heisenberg's relations under rather weak hypotheses on the states involved, and then generalise them to mixtures. We shall reformulate the results of Stone–von Neumann and Mackey in terms of the *Heisenberg group*. A short description of *Dirac's correspondence principle* and its relationship to the procedure called *deformation quantisation* closes the chapter.

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11.1 Round-up and Further Discussion on Axioms A1, A2, A3, A4

In Chap. 7 we saw the general axioms of QM. Let us summarise part of that chapter in the light of the spectral theory developed subsequently. We focus in particular on axiom **A4**, the notion of observable and superselection rules, to which we can add further theoretical material.

11.1.1 Axioms A1, A2, A3

A1. *Given a quantum system S described in an (inertial) frame system $\mathscr{I}$, experimentally testable propositions on S at any given time correspond bijectively to (a sublattice, at least in presence of superselection rules, of) the lattice $\mathscr{L}(\mathsf{H}_S)$ of orthogonal projectors of a complex separable Hilbert space $\mathsf{H}_S \neq \{\mathbf{0}\}$, respectively called the* **logic of elementary propositions** of *S and the* **Hilbert space associated to** *S. Moreover (using the same letter for propositions and corresponding projectors):*

(1) *the compatibility of two propositions (from measuring processes attributing simultaneous truth values to both) corresponds to the commutation of the orthogonal projectors;*
(2) *the logical implication of two compatible propositions $P \Rightarrow Q$ corresponds to the projectors' relation $P \leq Q$;*
(3) *I (identity operator) and 0 (null operator) correspond to the tautology and the contradiction;*
(4) *the negation $\dashv P$* of *P corresponds to the orthogonal projector $\neg P = I - P$;*
(5) *the propositions $P \mathscr{O} Q$ and $P \mathscr{E} Q$ have a physical meaning only when P, Q are compatible, and correspond to the orthogonal projectors $P \vee Q$ and $P \wedge Q$ (respectively projecting onto the closure of the union and the intersection of the projection spaces of P, Q);*
(6) *if $\{Q_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise-compatible propositions, the propositions corresponding to $\vee_{n\in\mathbb{N}} Q_n$ and $\wedge_{n\in\mathbb{N}} Q_n$ are physically meaningful.*

In the sequel we shall assume, loosely speaking, that all the elements in the logic $\mathscr{L}(\mathsf{H}_S)$ describe elementary propositions on S. Different choices, especially in presence of superselection rules but not only, will be discussed separately.

Remarks 11.1 **(1)** The Hilbert space H_S actually depends on the frame system $\mathscr{I}$ as well, as explained in Remark 7.23(4). Another system will give an isomorphic Hilbert space. We will return to this in Chap. 13.
(2) The fact that H_S is *separable* turns out to be useful in several technical constructions. For elementary non-relativistic systems H_S is automatically separable, as we shall see in this chapter. We will also explain that the Hilbert space of a system made of a finite number of elementary systems is, in turn, separable as it is described as

a finite tensor product of separable spaces. As for quantum fields, their the Hilbert space is the *Fock space*, again separable. However, if we stick to the Hilbert space description of quantum systems where states are trace-class operators, there is at least one physical reason to assume that the Hilbert space of a physical system is separable, and the argument is purely thermodynamical. Thermodynamical states of a system in equilibrium with a thermostat, or in equilibrium with other thermodynamical systems, are represented quantistically by mixed states (see axiom **A2** below) of the form $Z_\beta e^{-\beta H}$ where $Z_\beta := tr(e^{-\beta H})$. Here H is the Hamiltonian of the system, $\beta := 1/(k_B T)$ where T is the absolute temperature, and k_B is *Boltzmann's constant* (the case of an open system can be studied similarly by introducing *chemical potentials*). In this situation $e^{-\beta H}$ must obviously be of trace-class so, in particular, $e^{-\beta H}$, and hence H itself, must have purely point spectrum (up to perhaps the value $0 \in \sigma_c(e^{-\beta H})$). It should be clear that if H_S were not separable, the trace-class operator $e^{-\beta H}$ would not exist, because it would have an uncountable set of orthonormal eigenvectors with strictly positive eigenvalue, giving rise to a divergent trace.

When the setup seems to lead inevitably to a non-separable scenario or, more generally, if the description of trace-class states generates insurmountable problems, a more interesting possibility is to abandon Hilbert spaces completely. One radically different line of action describes the system by the so-called *algebraic approach*, which we will introduce in the last chapter. With this approach it does not matter whether the Hilbert space representations are separable or not. When one deals with extended thermodynamical systems, where the notion of state in terms of trace-class operator cannot be used, it is much better to rely on *algebraic states* (which we shall address in Chap. 14) and the celebrated *KMS condition* in order to characterise algebraic thermodynamical states [Haa96]. ■

A2. *A* **state** ρ *at time* t *on a quantum system* S, *with associated Hilbert space* H_S, *is a positive, trace-class operator on* H_S *with trace one.*

The probability that the proposition $P \in \mathscr{L}(\mathsf{H}_S)$ *on* ρ **is true equals** $tr(\rho P)$.

If, as we said, we suppose elementary propositions are described by the whole logic $\mathscr{L}(\mathsf{H}_S)$, states ρ are convex combinations (also infinite, if we consider spectral decompositions of states) of extreme states in the convex set $\mathfrak{S}(\mathsf{H}_S)$ of states. Extreme states are called **pure** and have the form $\rho = \psi(\psi|\,)$ for some unit vector $\psi \in \mathsf{H}_S$. The space of pure states is denoted by $\mathfrak{S}_p(\mathsf{H}_S)$ and is in one-to-one correspondence with the (projective) space of rays of H_S, i.e. the quotient of $\mathsf{H}_S \setminus \{\mathbf{0}\}$ by the equivalence relation $\phi \sim \phi' \;\Leftrightarrow\; \phi = a\phi'$ for some $a \in \mathbb{C} \setminus \{0\}$. States that are *not* pure are called **mixed states** or **mixtures**, and the corresponding trace-class operators are often called **statistical operators** or **density matrices** in the literature. The convex decomposition of a mixed state in terms of pure states, arising for instance from the spectral theorem, is called **incoherent superposition** of pure states. There are, typically, several convex decompositions into pure states for a single mixed state.

An important notion in physics, also historically speaking, is the **transition (or probability) amplitude** $(\psi|\phi)$ of the pure state determined by the unit vector ϕ on the pure state determined by the unit vector ψ. The square modulus of the transition amplitude represents the probability that the system in state ϕ passes to state ψ after a measurement. Note that we may swap states, by the symmetry of Hermitian inner products, without changing the transition probability.

A3. *If the quantum system S is in state $\rho \in \mathfrak{S}(\mathsf{H}_S)$ at time t and proposition $P \in \mathscr{L}(\mathsf{H}_S)$ is validated by a measurement taken at the same t, the system's immediate post-measurement state is*

$$\rho_P := \frac{P\rho P}{tr(\rho P)} ,$$

in accordance with the Lüders-von Neumann collapse postulate. In particular if ρ is pure and given by $\psi \in \mathsf{H}_S$, $||\psi|| = 1$, the post-measurement state is still pure, and given by the vector

$$\psi_P = \frac{P\psi}{||P\psi||} .$$

We emphasise that this axiom refers to ideal *first-kind measurements*, or *non-destructive* or *indirect*, as they are known; a lab's practice adopts several types of testing, that in general do not obey the axiom.

Remark 11.2 States can alternatively be described in terms of *σ-additive probability measures* on $\mathscr{L}(\mathsf{H}_S)$. However only for $\dim \mathsf{H}_S \neq 2$ is the correspondence between states and measures one-to-one, in view of the celebrated Gleason Theorem 7.26. We have two alternative formulations of **A2** and **A3**, respectively denoted by **A2 (measure-theory version)** and **A3 (measure-theory version)** and stated in Sects. 7.4.1 and 7.4.4. With the reformulation axiom **A3** turns out to have a very natural interpretation in terms of conditional probability, as established in Proposition 7.40: if the state before the measurement of P is represented by the probability measure μ over $\mathscr{L}(\mathsf{H}_S)$ and P turns out to be true, the probability measure μ_P after the measurement is the only probability measure over $\mathscr{L}(\mathsf{H}_S)$ satisfying $\mu_P(Q) = \mu(Q)/\mu(P)$ if $\mathscr{L}(\mathsf{H}_S) \ni Q \leq P$. ■

11.1.2 A4 Revisited: von Neumann Algebra of Observables

The subsequent axiom, introduced in Chap. 7, is concerned with the observables of a quantum system.

A4. *Every observable A of the quantum system S is described by a projector-valued measure $P^{(A)}$ on $\mathbb{R}$, on the system's Hilbert space H_S, so that the projector $P^{(A)}(E)$ corresponds to the proposition "the outcome of measuring A falls in E", for any Borel set E in $\mathbb{R}$.*

The spectral theorem for unbounded self-adjoint operators, proved in its maximal generality in Chap. 9 (Theorem 9.13), allows to associate to any observable a *unique* self-adjoint operator on the Hilbert space of the physical system. With this, if H_S is the Hilbert space of some system, the spectrum $\sigma(A) \subset \mathbb{R}$ of an observable A, i.e. of a self-adjoint operator $A : D(A) \to \mathsf{H}_S$, contains all possible outcomes of a measurement of the observable A, theoretically viewed as Borel subsets of $\sigma(A)$, especially intervals $(a_0-\delta, a_0+\delta)$. Mathematically, $\sigma(A)$ coincides with the support of the PVM $P^{(A)}$ associated to the observable. From the spectral theorem we also easily have that

$$||A|| = \sup\{|a| \mid a \in \sigma(A)\} .$$

This subsumes also the case A unbounded, or equivalently $D(A) \subsetneq \mathsf{H}_S$, corresponding to the (formally incorrect) value $||A|| = +\infty$. As a consequence, observables may be *unbounded* self-adjoint operators A, simply because the possible (experimental) values of A belong to an *unbounded set* $\sigma(A)$ of real numbers.

Remark 11.3 If a measurement process regards an observable A with $\sigma(A) = \sigma_p(A)$ and the elements of the spectrum are isolated points, it is always possible to assume the existence of a measuring instrument whose sensitivity $\delta > 0$ is smaller than the distance between (finitely many) consecutive values of A. In this way, even if the measurement is affected by an experimental error represented by $\delta > 0$, we can distinguish between couples of eigenvalues, and the Lüders-von Neumann axiom **A3** takes the more familiar and standard form presented in elementary formulations, i.e. when the associated eigenspaces have dimension 1: after a measurement with outcome a_0 the state is represented by the a_0-eigenvector. More generally, the same result is achieved by measuring simultaneously a finite set of mutually-compatible observables whose common eigenspaces have dimension 1 (see the discussion after Definition 11.11 below). The case of a continuous spectrum, and in particular the precise form of the post-measurement state, is much more problematic, and it was analysed by several authors. Ozawa [Oza84, Oza85], using a natural theoretical description of the measurement procedure, established that the measurements of continuous-spectrum observables are not repeatable. Also for this reason, the postulate by Lüders and von Neumann is viewed with suspicion in presence of continuous spectrum, and a more accurate description of the quantum measurement process might be given in terms of quantum operations, *POVMs* and the related *measuring operators*, see Sect. 13.2.2. ■

Sticking for the moment to bounded observables, a theoretical tool of great relevance is the so-called **von Neumann algebra of observables** of S, henceforth denoted by $\mathfrak{R}_S$. It is the von Neumann algebra *generated* (in the sense of Definition 3.92) by the set all of *bounded* observables. Thus the self-adjoint elements of $\mathfrak{R}_S$ represent all possible bounded observables of the physical system S. A fundamental physical issue is to determine $\mathfrak{R}_S$ for a given physical system S. The problem can be traced back to the analogous issue of selecting the lattice of elementary propositions of the system. The assumption that the elementary propositions on S are described by *all* the projectors in $\mathscr{L}(\mathsf{H}_S)$ is questionable, and anyway incompatible with superselection rules (see below). Generally speaking, one could ask less, like having elementary propositions described by the sublattice $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ of orthogonal projectors of the von Neumann algebra $\mathfrak{R}_S$ (still called the **logic of elementary propositions** of S). By Proposition 7.61

$$\mathfrak{R}_S = \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)'' ,$$

so that, in turn, the elementary propositions of the system completely fix the von Neumann algebra of observables of the system. Rather relevantly, the centre of the logic generates the centre of the algebra of observables $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) \cap \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)'$,

$$(\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) \cap \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)')'' = \mathfrak{R}_S \cap \mathfrak{R}_S'$$

as established in Theorem 7.61.

In absence of superselection rules it is usually assumed, and we are indeed doing this in discussing axioms **A1**–**A4**, that $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) = \mathscr{L}(\mathsf{H}_S)$. Then

$$\mathfrak{R}_S = \mathfrak{B}(\mathsf{H}_S) ,$$

so that $\mathfrak{R}_S' = \{cI\}_{c\in\mathbb{C}}$ and $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) \cap \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)' = \{0, I\}$ (the lattice is irreducible). This, though, is not the general case.

Just like the lattice $\mathscr{L}(\mathsf{H}_S)$ of all orthogonal projectors onto H_S, the logic $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ is an orthomodular (hence bounded and orthocomplemented), complete lattice which is separable if, as we assume, H_S is separable (Proposition 7.61). The remaining properties of $\mathscr{L}(\mathsf{H}_S)$ listed in Theorem 7.56 (atomicity and atomisticity, the covering property, irreducibility) are not valid in general. We will come back to these issues when we address superselection rules.

Remark 11.4 If $\mathfrak{R}_S$ is a proper subalgebra of $\mathfrak{B}(\mathsf{H}_S)$ the notion of state defined as a trace-class operator becomes redundant, since different states of $\mathfrak{S}(\mathsf{H}_S)$ can determine the same probability measure on $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. In this case the notion of state as a probability measure $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}_S)$ is a more faithful representation of the physical realm. However, even in this case, the elements of $\mathfrak{S}(\mathsf{H}_S)$ exhaust the convex body of σ-additive probability measures on $\mathscr{L}_{\mathfrak{R}}(\mathsf{H}_S)$. This is because every such measure can be described by a positive trace-class operator with trace one, provided the splitting of $\mathfrak{R}_S$ into algebras of definite type does not include type-I_2 terms (Remark 7.73). ■

How can the information of unbounded observables be included in $\mathfrak{R}_S$?

We have the following useful technical result regarding unbounded observables A.

Proposition 11.5 *Let $A : D(A) \to \mathsf{H}$ be a (typically unbounded) self-adjoint operator on the Hilbert space H and define the collection of operators*

$$A_n = \int_{[-n,n]} \lambda d P^{(A)}(\lambda) \quad \textit{for } n \in \mathbb{N}. \tag{11.1}$$

Set $id : \mathbb{R} \ni \lambda \mapsto \lambda \in \mathbb{R}$, in part (b) below. The following facts hold:

(a) $A_n^* = A_n \in \mathfrak{B}(\mathsf{H})$;
(b) $P^{(A_n)}(E) = P^{(A)}((\chi_{[-n,n]} \cdot id)^{-1}(E))$ *for every Borel set $E \subset \mathbb{R}$, so that*

$$P^{(A)}(E)\psi = \lim_{n\to+\infty} P^{(A_n)}(E)\psi \quad \textit{for every } E \in \mathscr{B}(\mathbb{R}) \textit{ and } \psi \in \mathsf{H}; \tag{11.2}$$

(c) $\cup_{n\in\mathbb{N}}\sigma(A_n)$ *coincides with* $\sigma(A)$*, possibly up to the value* 0*. More precisely,*
(i) $\sigma_p(A) \subset \cup_{n\in\mathbb{N}}\sigma_p(A_n) \subset \sigma_p(A) \cup \{0\}$*,*
(ii) $\sigma_c(A) \setminus \{0\} \subset \cup_{n\in\mathbb{N}}\sigma_c(A_n) \subset \sigma_c(A)$*;*

(d) *If* $\psi \in D(A)$*, then* $A\psi = \lim_{n+\infty} A_n\psi$*.*

Proof (a) The statement is an immediate consequence of Theorem 9.4(b, f).
(b) The first identity is a trivial application of Theorem 9.4(h). The second identity arises from the former, because either

$$(\chi_{[-n,n]} \cdot id)^{-1}(E) = E \cap [-n, n] \quad \text{if } 0 \notin E$$

or

$$(\chi_{[-n,n]} \cdot id)^{-1}(E) = (E \cap [-n, n]) \cup (\mathbb{R} \setminus [-n, n]) \quad \text{if } 0 \in E.$$

Consequently

$$P^{(A_n)}(E) = P^{(A)}(E \cap [-n, n]) \quad \text{if } 0 \notin E$$

or, respectively,

$$P^{(A_n)}(E) = P^{(A)}(E \cap [-n, n]) + P^{(A)}(\mathbb{R} \setminus [-n, n]) \quad \text{if } 0 \in E,$$

which can be rephrased as

$$P^{(A_n)}(E)\psi = \int_{\mathbb{R}} \chi_E \chi_{[-n,+n]} dP^{(A)}\psi \quad \text{if } 0 \notin E$$

or

$$P^{(A_n)}(E)\psi = \int_{\mathbb{R}} \chi_E \chi_{[-n,+n]} dP^{(A)}\psi + \int_{\mathbb{R}} \chi_{\mathbb{R}\setminus[-n,n]} dP^{(A)}\psi \quad \text{if } 0 \in E.$$

Eventually, Theorem 9.4(f) together with the monotone convergence theorem (used for the measure μ_ψ) implies (11.2).
(c) Let us prove (c)(i). If $\lambda \in \sigma_p(A_n)$, where $\lambda \in [-n, n]$ since $\sigma(A_n) \subset [-n, n]$, then $P^{(A_n)}(\{\lambda\}) \neq 0$ from Theorem 9.13(b)(i). Assuming $\lambda \neq 0$, from the proof of (b) we have that $P^{(A)}(\{\lambda\}) = P^{(A_n)}(\{\lambda\}) \neq 0$. Therefore $\lambda \in \sigma_p(A)$ again by Theorem 9.13(b)(i). We have established that $\cup_n\sigma_p(A_n) \subset \sigma(A) \cup \{0\}$. With the same argument, we see that if $\lambda \in \sigma_p(A)$ and $\lambda \in (-n, n)$, then either $P^{(A_n)}(\{\lambda\}) = P^{(A)}(\{\lambda\}) \neq 0$ or (if $\lambda = 0$) $P^{(A_n)}(\{\lambda\}) = P^{(A)}(\{\lambda\}) + P^{(A)}(\mathbb{R} \setminus [-n, n]) \neq 0$ again. In both cases $\lambda \in \sigma_p(A_n)$ from Theorem 9.13(b)(i). Therefore $\sigma_p(A) \subset \cup_n\sigma_p(A_n)$ concluding the proof of (i).

Let us prove (ii). Suppose that $\lambda \in \sigma_c(A)$. From Theorem 9.13(b)(ii), $P^{(A)}(\{\lambda\}) = 0$ and every open set $J \ni \lambda$ satisfies $P^{(A)}(J) \neq 0$. Fix n and an open set J' such that $\lambda \in J' \subset (-n, n)$. As before $P^{(A_n)}(\{\lambda\}) = P^{(A)}(\{\lambda\}) + P^{(A)}(\mathbb{R} \setminus [-n, n])$, the second term on the right-hand side appearing only if $\lambda = 0$. If $\lambda \neq 0$, $P^{(A)}(\{\lambda\}) = 0$ implies $P^{(A_n)}(\{\lambda\}) = 0$. With the same reasoning we have $P^{(A_n)}(J') = P^{(A)}(J') + P^{(A)}(\mathbb{R} \setminus [-n, n])$, the second term on the right-hand side appearing only if $0 \in J'$.

Since $P^{(A)}(J') \neq 0$ we conclude that $P^{(A_n)}(J') \neq 0$. If $J \supset J'$ is any other open set containing λ, we similarly have $P^{(A_n)}(J) \geq P^{(A_n)}(J') \neq 0$. Hence $P^{(A_n)}(\{\lambda\}) = 0$ and $P^{(A_n)}(J) \neq 0$, Theorem 9.13(b)(ii) yields $\lambda \in \sigma_c(A_n)$ and so $\sigma_c(A) \setminus \{0\} \subset \cup_n \sigma_c(A_n)$. To establish the other inclusion, assume $\lambda \in \sigma_c(A_n)$ for some n. Therefore, using Theorem 9.13(b)(ii) and part (b) above, $0 = P^{(A_n)}(\{\lambda\}) = P^{(A)}(\{\lambda\}) + P^{(A)}(\mathbb{R}\setminus[-n, n])$, the second summand present only if $\lambda = 0$. We conclude that both $P^{(A)}(\{\lambda\})$ and $P^{(A)}(\mathbb{R}\setminus[-n, n])$ vanish. If $J \ni \lambda$ is any open set containing λ with $J \subset (-n, n)$, again from Theorem 9.13(b)(ii) and (b) above, $0 \neq P^{(A_n)}(J) = P^{(A)}(J) + P^{(A)}(\mathbb{R}\setminus[-n, n])$, where the last term may be there only if $0 \in J$. However we know that $P^{(A)}(\mathbb{R} \setminus [-n, n]) = 0$ and so $P^{(A)}(J) \neq 0$. By enlarging J, what we have found remains valid exactly as before. Theorem 9.13(b)(ii) implies that $\lambda \in \sigma_c(A)$ since $P^{(A)}(\{\lambda\}) = 0$ and $P^{(A)}(J) \neq 0$ for every open set $J \ni \lambda$. Summing up, we have established the remaining inclusion $\cup_n \sigma_c(A_n) \subset \sigma_c(A)$.

There is no way to fix the problem with 0, since 0 may belong to $\cup_n \sigma_p(A_n)$ even if $0 \notin \sigma_p(A)$, and in particular also when $0 \in \sigma_c(A)$. In fact, $0 \in \sigma_p(A_n)$ if $\sigma(A) \not\subset [-n, n]$ for some n. In that case, defining $E := \sigma(A) \setminus [-n, n]$ we have $P^{(A)}_E \neq 0$ (otherwise $\sigma(A) \subset [-n, n]$) and every $\psi \in P^{(A)}_E(\mathsf{H})$ with $\psi \neq \mathbf{0}$ therefore satisfies $A_n\psi = \mathbf{0}$, proving that $0 \in \sigma_p(A_n)$. And this happens also when $0 \notin \sigma_p(A)$. (As an example consider the operator X on $L^2(\mathbb{R}, dx)$ such that $X\psi(x) = x\psi(x)$ for $x \in \mathbb{R}$, with domain $D(X) := \{\psi \in L^2(\mathbb{R}, dx) \mid \int |x\psi(x)|^2 dx < +\infty\}$. Here $\sigma(X) = \sigma_c(X) = \mathbb{R}$, but $\sigma_p(X_n) = \{0\}$ for every $n = 1, 2, \ldots$.).

(d) The proof descends again from Theorem 9.4(f) together with the monotone convergence theorem. $\square$

The physical meaning of A_n is not completely obvious. Roughly speaking, we can say that this observable is A itself, but it is measured by an instrument capable of producing outcomes inside $[-n, n]$.[1] Every concrete instrument is of this type, for n sufficiently large. It is therefore natural to assume, if the observable A is not bounded, that every bounded observable A_n as above belongs to $\mathfrak{R}_S$ nevertheless. This is equivalent to saying that the PVMs of the bounded self-adjoint operators A_n belong to $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$, because the elements of the PVM of a bounded normal operator B commute with every bounded operator commuting with B (Theorem 8.56(c), with a characteristic function in place of f). Since the PVM of A can be constructed out of the PVMs of the A_n by exploiting the strong operator topology (Proposition 11.5(b)), we have that $\mathfrak{R}_S$ and therefore $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ contains the PVM of A as well. If, conversely, the PVM of A belongs in $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$, every bounded operator A_n is contained in $\mathfrak{R}_S$ as it is the strong limit of linear combinations of the elements of the PVM of A, by (11.1).

In summary, we may say that although it contains only bounded observables, $\mathfrak{R}_S$ retains however the full information of all observables of the system, including the unbounded ones. This is because (we shall assume that henceforth) $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ also

[1] This characterisation is not 100% consistent with the features of A_n, since A_n more properly describes an observable with value 0 if the outcome of the measurement of A belongs to $\mathbb{R}\setminus[-n, n]$. A real instrument with finite range would not perform any measurement outside its range.

contains the elements of the PVMs of every unbounded observable A. Equivalently, $\mathfrak{R}_S$ contains the physically relevant bounded approximations A_n of A.

Technically speaking the mathematical notion relevant in this context is the following.

Definition 11.6 Given a von Neumann algebra $\mathfrak{R}$ over the Hilbert space H, an operator $A : D(A) \to \mathsf{H}$, with $D(A) \subset \mathsf{H}$, is said to be **affiliated to** $\mathfrak{R}$ if

$$UA \subset AU \quad \text{for every unitary operator } U \in \mathfrak{R}'.$$

In this case we write $A\eta\mathfrak{R}$.

The condition displayed above is equivalent to an apparently stronger requirement.

Proposition 11.7 *Given a von Neumann algebra $\mathfrak{R}$ over the Hilbert space H, an operator $A : D(A) \to \mathsf{H}$ with $D(A) \subset \mathsf{H}$ is affiliated to $\mathfrak{R}$ if and only if*

$$UA = AU \quad \textit{for every unitary operator } U \in \mathfrak{R}'$$

(which is equivalent to

$$UAU^* = A \quad \textit{for every unitary operator } U \in \mathfrak{R}').$$

In particular, it turns out that $U(D(A)) = D(A)$ if $U \in \mathfrak{R}'$ is unitary.

Proof If $UA = AU$ for every unitary operator $U \in \mathfrak{R}'$ then A is trivially affiliated to $\mathfrak{R}$. Conversely, if $A\eta\mathfrak{R}$, left-applying U^* gives $U^*UA \subset U^*AU$, that is $A \subset U^*AU$. Right-applying U^* produces $AU^* \subset U^*A$. To conclude, observe that $U \in \mathfrak{R}' \Leftrightarrow U^* \in \mathfrak{R}'$, so we may swap U^* and U to obtain $AU \subset UA$. Since also $UA \subset AU$ is true, we conclude that $UA = AU$ for $U \in \mathfrak{R}'$. As a consequence $U(D(A)) \subset D(A)$, but since $U^*A = AU^*$ holds too, we also have $U^*(D(A)) \subset D(A)$. Finally, applying U gives $D(A) \subset U(D(A))$, so that $U(D(A)) = D(A)$. $\square$

The following useful result – especially condition (c) – holds.

Proposition 11.8 *Let $\mathfrak{R}$ be a von Neumann algebra over the complex Hilbert space H and $A : D(A) \to \mathsf{H}$ a closed operator with $D(A) \subset \mathsf{H}$ dense. The following facts are equivalent.*

(a) $A\eta\mathfrak{R}$.
(b) *If $A = VP$ is the polar decomposition of A, then*
(i) $V \in \mathfrak{R}$,
(ii) $P\eta\mathfrak{R}$.
If A is self-adjoint, (a) and (b) are equivalent to
(c) *the PVM of A satisfies $P^{(A)}_E \in \mathfrak{R}$ for every Borel set $E \subset \mathbb{R}$.*
If $A \in \mathfrak{B}(\mathsf{H})$, then (a) and (b) are equivalent to
(d) $A \in \mathfrak{R}$.

Proof See Exercise 11.3. □

The conclusion is that given a quantum system described by a certain von Neumann algebra of observables $\mathfrak{R}_S$, the observables, bounded and unbounded, are represented by self-adjoint operators *affiliated to* $\mathfrak{R}_S$. Bounded observables are also elements of $\mathfrak{R}_S$.

11.1.3 Compatible Observables and Complete Sets of Commuting Observables

The notion of *compatible observables* is important in physics:

Definition 11.9 Let S be a quantum system described on the Hilbert space H_S. Two observables A, B of S are **compatible** if the spectral measures $P^{(A)}$, $P^{(B)}$ of the corresponding self-adjoint operators commute, i.e.

$$P^{(A)}(E)P^{(B)}(E) = P^{(B)}(E)P^{(A)}(E) \, , \quad \text{for any Borel set } E \subset \mathbb{R}.$$

Two observables that are not compatible are called **incompatible**.

In physics, compatibility means the observables can be measured at the same time (in agreement with axiom **A1** and with the meaning of the associated spectral measures). If we have a finite set of compatible observables $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$, a *joint spectral measure* $P^{(\mathbf{A})}$ on $\mathbb{R}^n$ can be constructed using the spectral measures of the self-adjoint operators representing the observables, by Theorem 9.19. Retaining those notations, if $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, the self-adjoint operator

$$f(A_1, \ldots, A_n) := \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) dP^{(\mathbf{A})}(x_1, \ldots, x_n) \tag{11.3}$$

– with domain given by vectors $\psi \in \mathsf{H}$ for which $f \in L^2(\mathbb{R}^n; \mu_\psi)$, where $\mu_\psi(E) = (\psi|P^{(\mathbf{A})}(E)\psi)$, $E \in \mathscr{B}(\mathbb{R}^n)$, as usual – has the customary meaning of an *observable* that is *function* of the observables $A_1, \ldots, A_n$.

From the physical point of view, if f is real-valued and the observables A_k are pairwise compatible, the meaning of the observable $f(A_1, \ldots, A_n)$ should be clear: $f(A_1, \ldots, A_n)$ is measured by simultaneously measuring $A_1, \ldots, A_n$, and then evaluating f on the values of the observables found. Obviously, this interpretation also holds for $n = 1$, that is for $f(A)$.

Remark 11.10 A necessary condition to have compatible observables is that the corresponding operators commute, paying attention to domains as prescribed by Theorem 9.41. For unbounded self-adjoint operators this condition is not sufficient, despite what certain physics books might say: Nelson [Nel59] proved that there are pairs of operators that commute on a dense subspace, invariant for both and on which both are essentially self-adjoint, yet the spectral measures of the self-adjoint extensions do not commute. A useful necessary *and sufficient* condition for A and B to be compatible is Theorem 9.41(c):

$$e^{itA}e^{isB} = e^{isB}e^{itA} \quad \text{for any} s, t \in \mathbb{R} .$$

Another necessary *and sufficient* condition is (d) in the same theorem. ■

The following notion, relying on Definition 9.22 and due to Jauch [Jau60] (see also [BeCa81]), plays a relevant role in the theory.

Definition 11.11 Consider a quantum system S described on the Hilbert space H_S and a family $\mathbf{A} = \{A_1, \ldots, A_n\}$ of observables with pairwise-commuting spectral measures. Suppose

$$\mathbf{A}' = \mathbf{A}'' .$$

In other words, every operator $B \in \mathfrak{B}(\mathsf{H}_S)$ that commutes with the spectral measures of the A_k necessarily belongs to $\mathbf{A}''$, which is equivalent to saying (Proposition 9.23) $B = f(A_1, \ldots, A_n)$ for some $f : supp(P^{(\mathbf{A})}) \to \mathbb{C}$ bounded and measurable. Then $\mathbf{A} = \{A_1, \ldots, A_n\}$ is called a **complete set of commuting observables**.

The condition $\mathbf{A}' = \mathbf{A}''$ is equivalent to the apparently weaker one $\mathbf{A}' \subset \mathbf{A}''$, as the inclusion $\mathbf{A}' \supset \mathbf{A}''$ follows from the fact that the bounded measurable functions $f(A_1, \ldots, A_n)$ commute with the spectral measure of each A_k by construction. The requirement $\mathbf{A}' = \mathbf{A}''$ can be proved to be the same as asking the Hilbert space be isomorphic to an L^2 space on the joint spectrum of the A_k [BeCa81], or to the existence of a cyclic vector for the joint spectral measure. Dirac speculated that the set of observables of a quantum system always admits a complete set of commuting observables. Jauch [Jau60] gave the general version of Dirac's postulate in terms of von Neumann algebras, positing the existence of a set of pairwise-commuting observables $\mathbf{A}$ such that $\mathbf{A}' = \mathbf{A}''$. Dirac's original conjecture referred only to observables with point spectrum. Some simple examples of complete sets of commuting observables will be given below.

If $\mathfrak{R}_S = \mathfrak{B}(\mathsf{H}_S)$, we can use any complete set of compatible observables $\mathbf{A} = \{A_1, \ldots, A_n\}$ to *prepare the system in a pure state*, in the sense of Remark 7.39(3), when each A_k admits a point spectrum part. Indeed, if $\lambda_k \in \sigma_p(A_k)$ for $k = 1, \ldots, n$, then the orthogonal projector

$$P_{(\lambda_1,\ldots,\lambda_n)} := P^{(A_1)}_{\{\lambda_1\}} \cdots P^{(A_n)}_{\{\lambda_n\}}$$

projects onto a subspace $\mathsf{H}_{(\lambda_1,\ldots,\lambda_n)} \subset \mathsf{H}_S$ which must be one-dimensional. (If that were not the case, it would be easy to construct an observable commuting with the spectral measures of all of the A_k, which is not a function of them: if $\mathsf{H}_{(\lambda_1,\ldots,\lambda_n)}$ contains two normalised orthogonal vectors ψ_1, ψ_2, the orthogonal projector $B = (\psi_1|\ )\psi_1$ is such an observable.) Therefore, whenever the outcome of the simultaneous (non-destructive, ideal) measurements of $A_1, \ldots, A_n$ is the set of values $(\lambda_1, \ldots, \lambda_n)$, the state after the measurement procedure must be the pure state represented by the unit vector (unique, up to phase) $\psi \in \mathsf{H}_{(\lambda_1,\ldots,\lambda_n)}$, as follows immediately from **A3**.

In the general case where $\mathfrak{R}_S$ does not necessarily coincide with the whole $\mathfrak{B}(\mathsf{H}_S)$, the existence of a complete set of commuting observables implies that the commutant $\mathfrak{R}'_S$ is contained in $\mathfrak{R}_S$, and therefore it coincides with the *centre* of $\mathfrak{R}_S$. In particular, a self-adjoint operator or an orthogonal projector that commutes with the observables is an observable itself. We have, in fact, the following elementary, though important, result by Jauch [Jau60].

Proposition 11.12 *If $\mathfrak{R}_S$ contains a complete set of commuting observables then $\mathfrak{R}'_S$ is Abelian and coincides with the centre $\mathfrak{R}_S \cap \mathfrak{R}'_S$.*

Proof Let $\mathbf{A} \subset \mathfrak{R}_S$ be a complete set of commuting observables. As $\mathbf{A}' = \mathbf{A}''$ we also have $\mathbf{A}'' = \mathbf{A}'''$. On the other hand $\mathfrak{R}_S \supset \mathbf{A}'' \Rightarrow \mathbf{A}''' \supset \mathfrak{R}'_S \Rightarrow \mathfrak{R}_S \supset \mathbf{A}'' = \mathbf{A}''' \supset \mathfrak{R}'_S$. We conclude $\mathfrak{R}_S \supset \mathfrak{R}'_S$, and therefore $\mathfrak{R}'_S$ coincides with the centre of $\mathfrak{R}_S$. In particular $\mathfrak{R}'_S$ is Abelian. □

Example 11.13
(1) Consider a quantum particle without spin, with reference to the rest space $\mathbb{R}^3$ of an inertial frame system (see below, axiom **A5**). In this case $\mathsf{H}_S = L^2(\mathbb{R}^3, dx)$. The three position operators $\mathbf{A} = \{X_1, X_2, X_3\}$ or the momentum operators $\mathbf{B} = \{P_1, P_2, P_3\}$ give complete sets of commuting observables. A further property of these complete sets of commuting observables is that $\mathfrak{R}_S$ is the von Neumann algebra generated by $\mathbf{A} \cup \mathbf{B}$. It is in fact possible to prove that the commutant (which coincides with the centre) of this von Neumann algebra is trivial (for it contains an irreducible unitary representation of the Weyl–Heisenberg group), so $(\mathbf{A} \cup \mathbf{B})'' = \mathfrak{B}(\mathsf{H}) = \mathfrak{R}_S$. (See also Remark 11.46.)
(2) If we add to the picture the spin space (for instance when we consider an electron "without charge"), then $\mathsf{H}_S = L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2$. A complete set of commuting observables is $\mathbf{A} = \{X_1 \otimes I, X_2 \otimes I, X_3 \otimes I, I \otimes S_3\}$, where S_3 is the spin observable (see Sect. 12.3.1) along the z-axis. Another is $\mathbf{B} = \{P_1 \otimes I, P_2 \otimes I, P_3 \otimes I, I \otimes S_1\}$. As before $(\mathbf{A} \cup \mathbf{B})''$ is the von Neumann algebra of observables of the system (note the crucial change of spin component in $\mathbf{A}$ and $\mathbf{B}$). In this case, too, the commutant of the von Neumann algebra of observables is trivial, yielding $\mathfrak{R}_S = \mathfrak{B}(\mathsf{H}_S)$. ■

The von Neumann algebra $\mathfrak{A}$ generated by a complete set of observables is a **maximal Abelian** von Neumann subalgebra of $\mathfrak{R}_S$, namely it satisfies $\mathfrak{A} = \mathfrak{A}'$. Spelt out: (a) it is Abelian, $\mathfrak{A}' \subset \mathfrak{A}$, and (b) it satisfies the maximality requirement that $\mathfrak{A}$ already contains all the elements of $\mathfrak{B}(\mathsf{H}_S)$ commuting with each element of $\mathfrak{A}$. The existence of a maximal Abelian von Neumann subalgebra of $\mathfrak{R}_S$ is equivalent to the commutativity of $\mathfrak{R}'_S$:

Proposition 11.14 *Let $\mathfrak{R}$ be a von Neumann algebra. The following three facts are equivalent.*

(a) *$\mathfrak{R}$ admits a maximal Abelian von Neumann subalgebra $\mathfrak{A} \subset \mathfrak{R}$ with $\mathfrak{A} = \mathfrak{A}'$.*
(b) *$\mathfrak{R}'$ coincides with the centre of $\mathfrak{R}$: $\mathfrak{R}' = \mathfrak{R} \cap \mathfrak{R}'$.*
(c) *$\mathfrak{R}'$ is Abelian: $\mathfrak{R}' \subset \mathfrak{R}''$.*

Proof The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) have already been established in Proposition 11.12: $\mathfrak{A} \subset \mathfrak{R}$ entails $\mathfrak{R}' \subset \mathfrak{A}'$, but $\mathfrak{A}' = \mathfrak{A} \subset \mathfrak{R}$ by (a), so $\mathfrak{R}' \subset \mathfrak{R}$ and hence (b) holds. If (b) holds, $\mathfrak{R}' \subset \mathfrak{R} = \mathfrak{R}''$ so (c) holds. It is evident that (c) $\Rightarrow$ (b): $\mathfrak{R}' \subset \mathfrak{R}(= \mathfrak{R}'')$, so that $\mathfrak{R}' \cap \mathfrak{R}' \subset \mathfrak{R}' \cap \mathfrak{R} \subset \mathfrak{R}'$ and so $\mathfrak{R}' = \mathfrak{R}' \cap \mathfrak{R}$.

Let us finally prove (b) $\Rightarrow$ (a). From Zorn's lemma one easily obtains that every von Neumann algebra $\mathfrak{R}$ always contains an Abelian von Neumann subalgebra $\mathfrak{A}$ that is maximal *with respect to* $\mathfrak{R}$: there is no larger Abelian von Neumann subalgebra of $\mathfrak{R}$. In particular, if $A \in \mathfrak{R}$ commutes with every element of $\mathfrak{A}$, A (and hence A^*) must belongs to $\mathfrak{A}$, for otherwise the algebra generated by A, A^* and $\mathfrak{A}$ would be Abelian and larger than $\mathfrak{A}$. Our algebra $\mathfrak{A}$ therefore satisfies $\mathfrak{A}' \cap \mathfrak{R} \subset \mathfrak{A}$. As $\mathfrak{A} \subset \mathfrak{A}'$ and $\mathfrak{A} \subset \mathfrak{R}$ we also have $\mathfrak{A} \subset \mathfrak{A}' \cap \mathfrak{R}$. Summarising, every von Neumann algebra $\mathfrak{R}$ always contains a von Neumann subalgebra $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R}$. Let us resume the proof of (b) $\Rightarrow$ (a). Take $\mathfrak{A} \subset \mathfrak{R}$ such that $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R}$. Part (b) implies $\mathfrak{R}' = \mathfrak{R} \cap \mathfrak{R}'$, but $\mathfrak{R} \cap \mathfrak{R}' \subset \mathfrak{R} \cap \mathfrak{A}'$, because $\mathfrak{A}' \supset \mathfrak{R}'$ as $\mathfrak{A} \subset \mathfrak{R}$. Finally $\mathfrak{R} \cap \mathfrak{A}' = \mathfrak{A}$ in view of our initial choice for $\mathfrak{A}$. To sum up, we have obtained $\mathfrak{R}' \subset \mathfrak{A}$, and consequently $\mathfrak{R}(= \mathfrak{R}'') \supset \mathfrak{A}'$. The hypothesis $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R}$ entails $\mathfrak{A} = \mathfrak{A}'$, concluding the proof. $\square$

Remark 11.15 A further result established by Jauch in [Jau60] is that a maximal Abelian von Neumann subalgebra $\mathfrak{A} \subset \mathfrak{R}$ can always be written as $\mathfrak{A} = \{A\}'$ for a suitable self-adjoint operator $A \in \mathfrak{A}$. Assuming that $\mathfrak{R} = \mathfrak{R}_S$ is the algebra of observables of a quantum system, since $\{A\}' = \mathfrak{A} = \mathfrak{A}' = \{A\}''$, Propositions 11.14 and 11.12 prove that the existence of a complete set of commuting observables in $\mathfrak{R}_S$ (in particular $\{A\}$) is *equivalent* to the fact that $\mathfrak{R}'_S$ is Abelian. ■

11.2 Superselection Rules

This section is devoted to the extension of the concept of superselection rules that were introduced in Sects. 7.71 and 7.7.2.

11.2.1 Superselection Rules and von Neumann Algebra of Observables

Let us summarise the elementary theory of superselection rules of Sects. 7.7.1, 7.7.2 and add further material and remarks. The focus will be on the interplay between the algebra of observables $\mathfrak{R}_S$ and the associated logic of elementary propositions $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. We shall mainly discuss the mathematical structures, since physical examples were presented in 7.7.1 (see also Sect. 12.3.2). A further instance regarding the Bargmann superselection rule will be presented in Sect. 12.3.4. The discussion will continue in Sect. 14.1.7. As a general reference on the subject we suggest [Ear08].

In presence of superselection rules, axioms **Ss1** and **Ss2** (Sect. 7.7.1) are constraints on **A1–A3**.[2] We will analyse first the von Neumann algebra $\mathfrak{R}_S$, and next the space of states in presence of superselection rules.

Speaking about the von Neumann algebra of observables $\mathfrak{R}_S$ over the separable Hilbert space H_S, axiom **Ss2** is mathematically equivalent to the identification

$$\mathscr{L}(\mathsf{H}_S)_{adm} := \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) \tag{11.4}$$

together with the demand of a non-trivial centre for $\mathfrak{R}_S$, because the centre of the associated lattice $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ is supposed non-trivial and atomic. More precisely (according to Proposition 7.71) the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ contains a family of orthogonal projectors $\{P_k\}_{k\in K}$ with the following properties:

(i) $P_k \neq 0$,
(ii) $P_k \perp P_h$ if $h \neq k$,
(iii) s-$\sum_{k\in K} P_k = I$,
(iv) there is no orthogonal projector $Q \in \mathfrak{R}_{Sk} \cap \mathfrak{R}'_{Sk}$ with $0 < Q < P_k$ for any $k \in K$.

(Proposition 7.70 permits to replace (iii) with
(iii)' the central family $\{P_k\}_{k\in K}$ is maximal with respect to (i) and (ii).)
The structure of $\mathfrak{R}_S$ and H_S according to **Ss2** (taking Proposition 7.70 into account) can now be described as follows. Setting $\mathsf{H}_{Sk} := P_k(\mathsf{H}_S)$:

(a) the family $\{P_k\}_{h\in K}$ is unique up to relabelling.
(b) Each H_{Sk} is invariant under $\mathfrak{R}_S$: $A(\mathsf{H}_{Sk}) \subset \mathsf{H}_{Sk}$ if $A \in \mathfrak{R}_S$.
(c) $\mathfrak{R}_S \ni A \mapsto \pi_k(A) := A\restriction_{\mathsf{H}_{Sk}} : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk}$ is a *-algebra representation of $\mathfrak{R}_S$.
(d) Each $\mathfrak{R}_{Sk} := \pi_k(\mathfrak{R}_S)$ is a *factor* in the Hilbert space H_{Sk}.
(e) $\mathfrak{R}_S$ has a direct decomposition into factors $\mathfrak{R}_{Sk}$:

$$\mathsf{H} = \bigoplus_{k\in K} \mathsf{H}_{Sk} \,, \qquad \mathfrak{R}_S = \bigoplus_{k\in K} \mathfrak{R}_{Sk} \tag{11.5}$$

(f) There is a partition $K = \sqcup_{j\in J} K_j$ such that

$$\mathsf{H}_S^{(j)} = \bigoplus_{k\in K_j} \mathsf{H}_{Sk} \quad \text{and} \quad \mathfrak{R}_S^{(j)} = \bigoplus_{k\in K_j} \mathfrak{R}_{Sk} \,, \tag{11.6}$$

where $\mathsf{H}_S^{(j)}$ and $\mathfrak{R}_S^{(j)}$ are the closed subspaces and definite-type von Neumann algebras of Theorem 7.68. In particular every factor $\mathfrak{R}_{Sk}$, for $k \in K_j$, is of the same type as the corresponding $\mathfrak{R}_S^{(j)}$.

[2] Alternatively, if we wish to describe states in terms of σ-additive measures, superselection rules are encapsulated in axioms **Ss1 (measure-theory formulation)** and **Ss2 (measure theory formulation)** (Sect. 7.7.2). These, too, should be understood as limitations to **A1**, **A2 (measure-theory version)**, **A3 (measure-theory version)**.

The pairwise orthogonal spaces H_{Sk} are the well-known **coherent sectors** or **superselection sectors**. The representations π_k defined in these sectors enjoy a couple of relevant properties.

Proposition 11.16 *In presence of superselection rules on the separable Hilbert space $\mathsf{H}_S \neq \{\mathbf{0}\}$ with coherent sectors H_{Sk}, $k \in K$, the above *-algebra representations π_k of the von Neumann algebra of observables $\mathfrak{R}_S$ are*

(i) not faithful*, i.e. not injective,*

(ii) pairwise unitarily non-equivalent*: there is no unitary operator $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sh}$ such that $U\pi_k(A)U^{-1} = \pi_h(A)$ for every $A \in \mathfrak{R}_S$ when $h \neq k$.*

Proof As $\pi_k(P_k) = \pi_k(I)$ for $I \neq P_k$, π_k cannot be injective. A unitary operator U intertwining π_k and π_h with $k \neq h$ cannot exist because it would satisfy

$$0_h = \pi_h(P_k) = U\pi_k(P_k)U^{-1} = UI_kU^{-1} = I_h \ ,$$

where 0_h is the zero operator and I_h the identity on H_h ($\mathsf{H}_h \neq \{\mathbf{0}\}$ since $P_h \neq 0$). $\square$

Remark 11.17 When superselection rules are present $\mathfrak{R}'_S \cap \mathfrak{R}_S$ is not trivial, since it contains the orthogonal projectors onto coherent sectors. A more difficult question is the converse one: to decide whether superselection rules are present in case $\mathfrak{R}'_S \cap \mathfrak{R}_S$ is non-trivial. The point is to understand whether the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ is atomic. If not, the desired decomposition $\sum_k P_k = I$ in terms of central atoms is not achievable. However, in that case the Hilbert space H_S can be written as a *direct integral*, and one can talk about *continuous superselection rules* [Jau60, Giu00]. These have a different nature, and will not be of our concern. (See also Remark 11.22 below.) ■

Let us address states in presence of superselection rules.

To complete the description of superselection rules, **Ss1** requires that the **admissible states** in presence of superselection rules are those in the convex subset (7.61):

$$\mathfrak{S}(\mathsf{H}_S)_{adm} := \{\rho \in \mathfrak{S}(\mathsf{H}_S) \mid \rho P_k = P_k \rho \quad \text{for any } k \in K\}.$$

The extreme elements of $\mathfrak{S}(\mathsf{H}_S)_{adm}$ identify naturally with elements of the sets $\mathfrak{S}_p(\mathsf{H}_{Sk})$, and hence define **admissible pure states** (7.57) in presence of superselection rules

$$\mathfrak{S}_p(\mathsf{H}_S)_{adm} := \bigsqcup_{k \in K} \mathfrak{S}_p(\mathsf{H}_{Sk}) \ .$$

An alternative description employs **σ-additive probability measures** over $\mathscr{L}(\mathsf{H}_S)_{adm} = \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ (see (7.63)). These are maps

$$\mu : \mathscr{L}(\mathsf{H})_{adm} \to [0, 1] \quad \text{with} \quad \mu(I) = 1$$

such that

$$\mu\left(\text{s-}\sum_{i=0}^{+\infty} Q_i\right) = \sum_{i=0}^{+\infty} \mu(Q_i) \quad \text{for } \{Q_i\}_{i\in\mathbb{N}} \subset \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S) \text{ with } Q_i \perp Q_j \text{ if } i \neq j.$$

It turns out that both the space of admissible states $\mathfrak{S}(\mathsf{H}_S)_{adm}$ and the space of σ-additive probability measures over $\mathscr{L}(\mathsf{H})_{adm}$ are big enough to separate the elements of $\mathscr{L}(\mathsf{H})_{adm}$ (Propositions 7.85 and 7.86). The space of measures is, *vice versa*, separated by admissible propositions $\mathscr{L}(\mathsf{H}_S)_{adm}$ (Proposition 7.86).

Proposition 7.87 establishes that the two descriptions of quantum states, via trace-class operators or measures, are essentially equivalent also in the presence of super-selection rules. Every state $\rho \in \mathfrak{S}(\mathsf{H}_S)_{adm}$ defines a *unique* associated σ-additive probability measure: $\mu(P) := tr(\rho P)$ for every $P \in \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. Conversely, if none of the $\mathfrak{R}_{Sk}$ is a type-I_2 factor, for every σ-additive probability measure over $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ there is *at least one* state $\rho \in \mathfrak{S}(\mathsf{H}_S)_{adm}$ satisfying the condition above.

There remains the problem that two states $\rho, \rho' \in \mathfrak{S}(\mathsf{H}_S)_{adm}$ may result *physically equivalent*, as they determine the same probability measure on $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$, that is $\mu(P) = tr(P\rho) = tr(P\rho')$ for every $P \in \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. In this respect the next result is interesting: it relies on Proposition 7.88 and has an important interplay with Proposition 11.12. It is a refinement of an analogous result established in [Jau60] when the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ is atomic.

Proposition 11.18 *In presence of superselection rules, assume that $\mathfrak{R}'_S$ is Abelian (which happens, in particular, if $\mathfrak{R}_S$ contains a complete set of commuting observables). The following two facts hold.*

(a) *The subalgebra $\mathfrak{R}_{Sk}$ associated to the superselection sector H_{Sk}, as in (11.5), is a maximal factor: $\mathfrak{R}_{Sk} = \mathfrak{B}(\mathsf{H}_{Sk})$ and the representation π_k is irreducible. In particular, $\mathfrak{R}_S$ is of type I.*

(b) *If two states $\rho, \rho' \in \mathfrak{S}(\mathsf{H}_S)_{adm}$ satisfy $tr(P\rho) = tr(P\rho')$ for every $P \in \mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)_{adm}$, they must coincide: $\rho = \rho'$.*
If, conversely, $\mathfrak{R}'_S$ is not Abelian, both (a) and (b) are false (but $\mathfrak{R}_S$ may still be of type I).

Proof (a) Consider an orthogonal projector $P \in \mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk})'$ for some $k \in K$. Since $\mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk})'' = \mathfrak{R}_k$ we furthermore have $P \in \mathfrak{R}'_k$. Extending it to the null operator on $\mathsf{H}_{Sk}^{\perp}$ we find an orthogonal projector in $\mathfrak{R}'_S$, still denoted by P. In our hypotheses $\mathfrak{R}'_S \subset \mathfrak{R}''_S = \mathfrak{R}_S$ so that $\mathfrak{R}'_S = \mathfrak{R}_S \cap \mathfrak{R}_S$ (this is the case in presence of a complete set of commuting observables, by Proposition 11.12). Therefore $P \in \mathfrak{R}'_S \cap \mathfrak{R}_S$ and, in particular, P belongs to the centre of the lattice $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. More precisely, it belongs to the centre of $\mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk})$, which is trivial simply due to the superselection rule. So, either $P = 0$ or $P = P_k$. We have obtained that $\mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk})' = \{0, I_k\}$ where I_k is the identity operator in H_{Sk} viewed as Hilbert space. Hence $\mathfrak{R}_{Sk} = \mathfrak{B}(\mathsf{H}_{Sk})$ (Proposition 7.61(e)) and so $\mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk}) = \mathscr{L}(\mathsf{H}_{Sk})$. Since $\mathfrak{R}_s$ is the direct sum of type-I factors, it has type I as well. Evidently π_k is irreducible, because its image coincides with the whole $\mathfrak{B}(\mathsf{H}_{Sk})$ and by Proposition 3.93.

(b) Immediately descends from Proposition 7.88.
Let us prove the last statement. Suppose that $\mathfrak{R}'_S$ is not Abelian. By decomposing every operator $B \in \mathfrak{R}'_S$ as $B = C + iD$ with $C, D \in \mathfrak{R}'_S$ self-adjoint, we conclude that there must exist an operator $A \in \mathfrak{R}'_S$ that is self-adjoint but not of the form λI (otherwise $\mathfrak{R}'_S$ would be Abelian). Therefore $U = e^{itA} \in \mathfrak{R}'_S$ is not a pure phase but it commutes with every observable, and in particular with every superselection projector P_k. For some $k \in K$ there must exist $\psi \in \mathsf{H}_{Sk}$ with unit norm such that $\psi' = U\psi \neq e^{ia}\psi$ for every $a \in \mathbb{R}$ (otherwise U would be a pure phase). Since U commutes with every observable, we have $tr(P\rho_\psi) = tr(P\rho_{\psi'})$ for every $P \in \mathfrak{B}(\mathsf{H}_S)_{adm}$, and therefore (b) is violated. Notice that U restricted to H_{Sk} defines a non-trivial unitary operator in $\mathfrak{R}'_{Sk}$, so $\mathfrak{R}_{Sk}$ is strictly contained in $\mathfrak{B}(\mathsf{H}_{Sk})$ and (a) is false. $\square$

Definition 11.19 If a given physical system S with algebra of observables $\mathfrak{R}_S$ is subjected to superselection rules and $\mathfrak{R}'_S$ Abelian, then the superselection rules are said to be **Abelian**.

If a quantum system admits *Abelian* superselection rules, then the simplest versions of axioms **A1**–**A4** hold as soon as we restrict to a coherent sector H_{Sk} and the factor $\mathfrak{R}_{Sk}$ (not of type I_2): *all* orthogonal projectors represent elementary observables, *all* self-adjoint operators represent observables and states are correspond *one-to-one* to probability measures on the lattice of elementary propositions.

11.2.2 Abelian Superselection Rules Induced by Central Observables

In the exhaustive survey [Wigh95], Wightman conjectured that superselection rules are associated to a set of *pairwise compatible*, physically meaningful observables $Q_1, \dots, Q_n$, sometimes called **superselection charges**, whose spectral measures (as well as any bounded Q_j, if any) belong to the centre of $\mathfrak{R}_S$. It is also supposed that

$$\sigma(Q_j) = \sigma_p(Q_j) \, .$$

(What follows is however valid also in case $\sigma_c(Q_j) \neq \emptyset$ for some j, provided $P^{(Q_j)}_{\sigma_c(Q_j)} = 0$, where $P^{(Q_j)}$ is the PVM of Q_j.) As every Q_j refers to a different superselection rule, it is implicit that the different superselection rules are compatible with one another, in the obvious sense. This comes from experimental evidence.

From this point of view the Hilbert space splits in an orthogonal sum of closed subspaces H_{S_k}, $k \in K$, and every k is fixed by the values that all charges Q_j *simultaneously* assume on H_{S_k}. In other words, if $P^{(Q_j)}_{q_j}$ is the orthogonal projector of Q_j onto the q_j-eigenspace, the family of central projectors $\{P_k\}_{k \in K}$ is the family of spectral projectors

$$\left\{P_{k(q_1,\ldots,q_n)} := P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}\right\}_{(q_1,\ldots,q_n)\in\sigma(Q_1)\times\cdots\times\sigma(Q_n)} ,$$

where $k \in K$ ($\subset \mathbb{N}$) labels bijectively n-tuples of values $(q_1, \ldots, q_n) \in \sigma(Q_1) \times \cdots \times \sigma(Q_n)$. To obtain a fully fledged superselection rule, in view of **Ss2**, we have to impose that the orthogonal projectors P_k are *atoms* of the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ and that **Ss1** holds: the admissible states are those commuting with the P_k.

Example 11.20 The proton and the neutron can be viewed as two different pure states of the so-called **isotopic spin** τ_3, described by the self-adjoint operator

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11.7}$$

on the Hilbert space $\mathbb{C}^2$. This quantity is related to the electric charge observable Q (here assumed normalised to ± 1) under $Q = \frac{1}{2}(\tau_3 + I)$. Neutron states correspond to eigenvalue -1 whereas proton states correspond to eigenvalue 1. In view of the superselection rule of the electric charge, no coherent superposition of a proton state and a neutron state is possible. In other words, in every admissible state the charge must have a definite value. On the other hand, if we treat these particles as non-relativistic particles, the superselection rule of the mass is present (see Sect. 12.3.4): the mass observable M must have a definite value in every pure state. The Hilbert space of the proton/neutron system must split into coherent sectors where the mass and the charge of the system are defined simultaneously in every possible vector state.

The full isotopic spin consists of further components τ_1 and τ_2 together with τ_3. These bounded, self-adjoint operators are described by the other Pauli matrices σ_1 and σ_2 (which are also used for the *spin* components of a particle with spin $1/2$, see later). The operators τ_i are defined on an internal Hilbert space isomorphic to $\mathbb{C}^2$ but different from the spin space. Within the isospin model of strong interactions, the physical demand is that strong interactions (i.e. the part of the Hamiltonian operator associated with the strong force) are invariant under the unitary group $SU(2)$ spanned by the τ_i: $U(\beta_1, \beta_2, \beta_3) = e^{i\sum_{j=1}^{3}\beta_j\tau_j}$. However, as Q, that is τ_3, is associated to a superselection rule (τ_3 belongs to the centre of $\mathfrak{R}_S$), the remaining self-adjoint operators τ_1 and τ_2 cannot live in $\mathfrak{R}_S$ nor in $\mathfrak{R}'_S$, since they do not commute with $\tau_3 \in \mathfrak{R}_S \cap \mathfrak{R}'_S$. Of the three, in other words, only the self-adjoint operator τ_3 corresponds mathematically to an observable. ■

The requirement that the orthogonal projectors $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ generating the joint spectral measure of the Q_j are *atoms* of the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ is equivalent to the demand that every bounded, central observable must be a function of $Q_1, \ldots, Q_n$ in the sense of Eq. (11.3):

$$\{Q_1, \ldots, Q_n\}'' = \mathfrak{R}_S \cap \mathfrak{R}'_S .$$

(As usual $\{Q_1, \ldots, Q_n\}'$ is defined as the commutant of the spectral measures of all the Q_j if one of them is not bounded, cf. Definition 9.22.) We have indeed the following result.

Proposition 11.21 *Let $\mathfrak{R}_S$ be the von Neumann algebra of a quantum system S described on the separable Hilbert space H_S. Consider n observables $Q_1, \ldots, Q_n$ such that, for $j = 1, \ldots, n$,*

(i) the spectral measure $P^{(Q_j)}$ belongs to the centre of $\mathfrak{R}_S$,

(ii) $\sigma(Q_j) = \sigma_p(Q_j)$.

If $P^{(Q_j)}_{q_j}$ is the orthogonal projector onto the q_j-eigenspace of Q_j, the following facts are equivalent.

(a) *The orthogonal projectors $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ with $(q_1, \ldots, q_n) \in \sigma(Q_1) \times \cdots \times \sigma(Q_n)$ are atoms of the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$.*

(b) *Every observable $A \in \mathfrak{R}_S \cap \mathfrak{R}'_S$ is a function of $Q_1, \ldots, Q_n$ as in (11.3).*

If the above facts hold and A is a (typically unbounded) observable with spectral measure $P^{(A)}$ in $\in \mathfrak{R}_S \cap \mathfrak{R}'_S$, then

$$P^{(A)}_{\sigma_c(A)} = 0\,.$$

In particular, the interior of $\sigma_c(A)$ must be empty.

Proof (a) $\Rightarrow$ (b). Assume that (a) is valid. If A is a self-adjoint operator in $\mathfrak{R}_S$ we must have $AP^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n} = P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n} A$ because $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ belongs to the centre of $\mathfrak{R}_S$. Consequently:

$$A_{(q_1,\ldots,q_n)} := A\restriction_{\mathsf{H}_{(q_1,\ldots,q_n)}} : \mathsf{H}_{(q_1,\ldots,q_n)} \to \mathsf{H}_{(q_1,\ldots,q_n)}$$

is a bounded self-adjoint operator on the common eigenspace $\mathsf{H}_{(q_1,\ldots,q_n)}$ of $Q_1, \ldots Q_n$ with eigenvalues $(q_1, \ldots, q_n)$, viewed as Hilbert space. Suppose by contradiction

$$A_{(q_1,\ldots,q_n)} \neq rI_{(q_1,\ldots,q_n)}$$

for every $r \in \mathbb{R}$ ($I_{(q_1,\ldots,q_n)}$ is the identity operator on $\mathsf{H}_{(q_1,\ldots,q_n)}$). Then the spectral measure of $A_{(q_1,\ldots,q_n)}$ would contain some non-trivial projector $P \neq 0, I_{(q_1,\ldots,q_n)}$. Extending P to the null operator over $\mathsf{H}^{\perp}_{(q_1,\ldots,q_n)}$ would generate a central orthogonal projector P such that $0 < P < P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$, because P commutes with every operator commuting with A and A is in the centre of $\mathfrak{R}_S$. This is incompatible with the hypothesis that $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ is a central atom of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$. Therefore

$$A_{(q_1,\ldots,q_n)} = r(q_1, \ldots, q_n) I_{(q_1,\ldots,q_n)} \quad \text{for some real number } r(q_1, \ldots, q_n).$$

In other words (in strong sense)

$$A = \sum_{(q_1,\ldots,q_n) \in \sigma(Q_1)\times\cdots\times\sigma(Q_n)} r(q_1, \ldots, q_n) P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$$

which is (11.3) specialised to our case. Hence (b) holds. (Since $\sigma_p(Q_1)\times\cdots\times\sigma_p(Q_n)$ is a countable subset of $\mathbb{R}^n$, as H_S is separable, it easy to prove that every map $r : \sigma_p(Q_1) \times \cdots \times \sigma_p(Q_n) \to \mathbb{R}$ extended as zero on the rest of $\mathbb{R}^n$ is Borel-measurable.)

Next, we claim that if (a) fails, then (b) fails. For that we notice preliminarily that any observable in $\mathfrak{R}_S \cap \mathfrak{R}'_S$ that is function of $Q_1, \dots, Q_n$ has the form (in strong sense)

$$f(Q_1, \dots, Q_n) = \sum_{(q_1,\dots,q_n)\in\sigma(Q_1)\times\cdots\times\sigma(Q_n)} f(q_1, \dots, q_n) P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n} , \quad (11.8)$$

for some bounded function $f : \sigma(Q_1) \times \cdots \times \sigma(Q_n) \to \mathbb{R}$. As a consequence, the operator $f(Q_1, \dots, Q_n)$ is the identity up to a constant factor (the eigenvalue $f(q_1, \dots, q_n)$) over each $\mathsf{H}_{(q_1,\dots,q_n)}$. If $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ were not an atom of the centre of $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$, for fixed $(q_1, \dots, q_n)$, there would be a central orthogonal projector Q with $0 < Q < P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$. This would be a central observable different from a constant multiple of the identity on $\mathsf{H}_{(q_1,\dots,q_n)}$, so it could not possibly be a function of $Q_1, \dots, Q_n$. This proves that (a) false $\Rightarrow$ (b) false.

The last statement is easy. Since (Proposition 11.5) A is the strong limit on $D(A)$ of $A_n := AP^{(A)}_{(-n,n]}$ and each observable A_n is bounded, belongs to $\mathfrak{R}_S \cap \mathfrak{R}'_S$, and $\sigma_c(A) = \cup_{n\in\mathbb{N}}\sigma_c(A_n)$ up to the zero value, it is enough to prove the claim for $A \in \mathfrak{R}_S\cap\mathfrak{R}'_S$. From (11.8) every such A (which we know is a function of $Q_1, \dots, Q_n$) admits every common eigenvector of $Q_1, \dots, Q_n$ as eigenvector. Fixing a Hilbert basis (of common eigenvectors) in every $\mathsf{H}_{(q_1,\dots,q_n)}$ we have an overall Hilbert basis of H_K made of eigenvectors of A. This is equivalent to saying that $I = \text{s-}\sum_{a\in\sigma_p(A)} P^{(A)}_{\{a\}}$, where $P^{(A)}_{\{a\}}$ is the orthogonal projector onto the a-eigenspace. From (i) in the spectral Theorem 8.56(b) we know that $P^{(A)}_{\{a\}}$ is also the projector of the PVM of A associated with the Borel set $\{a\}$. Since these sets are pairwise disjoint when $a \in \sigma_p(A)$, we have:

$$P^{(A)}_{\sigma_p(A)} = \text{s-}\sum_{a\in\sigma_p(A)} P^{(A)}_{\{a\}} = I . \quad (11.9)$$

But $\sigma_c(A) \cap \sigma_p(A) = \emptyset \Rightarrow 0 = P^{(A)}_{\sigma_c(A)} P^{(A)}_{\sigma_p(A)}$, so (11.9) forces $0 = P^{(A)}_{\sigma_c(A)}$. If $(a, b) \subset \sigma_c(A)$, by (ii) in the spectral Theorem 8.56(b), $P^{(A)}_{(a,b)} \neq 0$. But that is impossible since $P^{(A)}_{(a,b)} \leq P^{(A)}_{\sigma_c(A)} = 0$. $\square$

The superselection rules induced by the central observables $Q_1, \dots, Q_n$ give rise to the usual factor decomposition (11.5). Each sector H_{Sk} carries a corresponding representation π_k of the algebra $\mathfrak{R}_S$ of observables of the system. The fact that different representations π_k are unitarily inequivalent can now be proved by looking at the superselection charges. For $k \neq h$, there is no Hilbert space isomorphism (i.e., unitary operator) $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sh}$ with

$$U\pi_k(A)U^{-1} = \pi_h(A) \quad \forall A \in \mathfrak{R}_S .$$

This is because $\pi_k(Q_j) = q_{jk} I_k$ and the sets $(q_{1k}, \ldots, q_{jk})$ are different for different k by hypothesis, so that every such U would produce the contradiction:

$$U\pi_k(Q_j)U^{-1} = q_{jk} U I_k U^{-1} = q_{jk} I_h \neq q_{jh} I_h = \pi_h(Q_j) \quad \text{for some } j = 1, 2, \ldots, n.$$

As we already know in the general case, the representations π_k *cannot be faithful* because $\pi_k(Q_j - q_{jk} I) = 0$, albeit $Q_j - q_{jk} I \neq 0$ for some j, otherwise the centre $\mathfrak{R}_S \cap \mathfrak{R}'_S$ would be trivial and no superselection rule would occur. If, finally, $\mathfrak{R}'_S$ is Abelian, the representations π_k are also irreducible, as proved in Proposition 11.18, since their images are the whole spaces $\mathfrak{B}(\mathsf{H}_{Sk})$.

In summary: *if the superselection rules are Abelian*, the π_k are *non-faithful, irreducible and unitarily inequivalent* representations of the type-I von Neumann algebra of observables $\mathfrak{R}_S$, labelled by the set of eigenvalues of the superselection charges Q_j. Notice that the charges Q_j generate the full $\mathfrak{R}'_S$, for this coincides with the centre $\mathfrak{R}_S \cap \mathfrak{R}'_S$ in the case at hand.

We shall return to these issues in Sect. 14.1.7, when we compare this description of the superselection rules with the one arising from the algebraic formulation of quantum theories.

Remarks 11.22 **(1)** It is worth stressing that Wightman's hypothesis on Abelian superselection rules cannot be the general way to select admissible states and admissible observables. In *non-Abelian* gauge theories, like quantum chromodynamics, the commutant of the algebra of observables is not Abelian. This is evident from the fact that every admissible observable must be invariant under a non-Abelian (unitary) gauge (Lie) group. It is not possible to define simultaneously the self-adjoint generators of the corresponding Lie algebra because these do not commute by hypothesis. Hence these self-adjoint operators do *not* represent central observables (it is even disputable whether they all represent observables, if truth be told), nor is the selection of admissible states and observables a consequence of Wightman's superselection rule in this setting. We remark that, in agreement with Proposition 11.12, quantum chromodynamics and other non-Abelian gauge theories cannot admit any complete set of commuting observables, nor is it possible, in these theories, to prepare a state represented by a vector (up to phases) by a sequence of measurements.
(2) It has been conjectured more than once that quantities with classical behaviour are actually quantum observables whose spectral projectors describe superselection rules [BGJKS00], i.e. superselection charges. From this perspective certain superselection rules are dynamical and arise from the interaction between the physical system and the ambient, thereby producing the phenomenon called *decoherence* [BGJKS00].
(3) Theoretically speaking, the picture in which superselection rules are associated to central observables with only point spectrum is completely general, and applies to all quantum systems affected by superselection rules. Indeed, if a physical system admits superselection rules defined by the central family $\{P_k\}_{k\in K}$ satisfying the usual properties (i)–(iv) on the separable Hilbert space H_S, it is always possible to define a central observable Q inducing the family $\{P_k\}_{k\in K}$. It suffices to relabel the family by $N \subset \mathbb{N}$ (the whole $\mathbb{N}$ if K is infinite) and define

$$Q := \text{s-}\sum_{n\in N} nP_n \,.$$

Obviously $\sigma(Q) = \sigma_p(Q) = N$.
(4) In view of (3) and the last statement in Proposition 11.21, if the centre of the von Neumann algebra of observables $\mathfrak{R}_S$ contains at least one observable A with $P^{(A)}_{\sigma_c(A)} \neq 0$ (in particular $\sigma_c(A)$ contains an interval), no superselection rules are allowed. However, it would still be possible to understand the structure of $\mathfrak{R}_S$ (and the convex set of states) in terms of *continuous superselection rules* (Remark 11.17). ■

11.2.3 Non-Abelian Superselection Rules and the Gauge Group

Given a physical system S described on the (separable) Hilbert space H_S with von Neumann algebra of observables $\mathfrak{R}_S$, either in presence of superselection rules or not, $\mathfrak{R}'_S$ may or not be Abelian. It is non-Abelian if and only if $\mathfrak{R}'_S$ is larger that the centre of $\mathfrak{R}_S$. In this case, and if we restrict to one superselection sector H_{Sk}, not every orthogonal projector represents an elementary proposition, and not each self-adjoint operator corresponds mathematically to an observable. Therefore we are led to conclude that there exist *further constraints* on the nature of physical observables, besides the commutation with the central projectors P_k describing superselection rules, since the elements of $\mathfrak{R}_S$ are supposed to commute with *all* elements in $\mathfrak{R}'_S$ and not only the central elements. It may happen that the centre of $\mathfrak{R}_S$ is trivial, so that no superselection (described by central projectors) takes place, yet $\mathfrak{R}'_S$ is non-trivial. In the remaining part of the section we shall consider the generic case, where $\mathfrak{R}_S$ has a non-trivial centre and $\mathfrak{R}'_S$ is not Abelian.

In relation to the noncommutativity of $\mathfrak{R}'_S$, Jauch and Misra [JaMi61] studied the interplay between superselection rules and *gauge symmetries*.[3]

Definition 11.23 Given a physical system S described on the Hilbert space $\mathsf{H}_S \neq \{\mathbf{0}\}$ with von Neumann algebra of observables $\mathfrak{R}_S$, a **gauge symmetry** (also known as a **gauge transformation**) is a unitary operator on H_S that commutes with every element of $\mathfrak{R}_S$. These unitary operators form a unitary group called the **gauge group** which is a subset of $\mathfrak{R}'_S$ called, in turn, **gauge algebra**.

Evidently, the interesting case is that in which the gauge group and the gauge algebra are not Abelian.

Remark 11.24 **(1)** It should be clear that a densely-defined closed operator $A : D(A) \to \mathsf{H}_S$ commutes with every element of the gauge group if and only if it is affiliated to $\mathfrak{R}_S$. In particular, $A \in \mathfrak{B}(\mathsf{H}_S)$ commutes with every gauge symmetry if and only if $A \in \mathfrak{R}_S$.

[3] called *supersymmetries* in their paper.

(2) We stress that we can have a non-Abelian gauge group in absence of superselection rules. ■

If U belongs to the gauge group of $\mathfrak{R}_S$, U is unitary and so it admits a spectral decomposition and an associated PVM $P^{(U)}$ in fulfilment of Theorem 8.56. Every orthogonal projector $P_E^{(U)}$ must commute with every bounded operator commuting with U, therefore $P_E^{(U)} \in \mathfrak{R}'_S$ and the closed subspaces $\mathsf{H}_E^{(U)} := P_E^{(U)}(\mathsf{H}_S)$ are invariant under every element of $\mathfrak{R}_S$. In case U does not belong to the centre of $\mathfrak{R}_S$ we therefore have further $\mathfrak{R}_S$-invariant subspaces, in addition to the possible coherent sectors. If the spectrum of U is a point spectrum, we have a direct Hilbert decomposition of H_S into eigenspaces of U: this decomposition is invariant under $\mathfrak{R}_S$, exactly as the decomposition into coherent sectors, but it does not coincide with it. Finally, if there is another gauge transformation U' that does not commute with U, it gives rise to its own decomposition of H_S, different from that of U. So some of the features of superselection rules linger in the presence of a non-Abelian gauge group, even if the centre of $\mathfrak{R}_S$ is trivial. For this reason one sometimes says that **non-Abelian superselection rules** are present if the gauge group is non-Abelian. Referring to *Non-Abelian Gauge Field Theories*, this is the case for the so-called *global gauge transformations* acting on the physical Hilbert space (after all non-physical states with “non-positive norm"have been removed by the gauge-fixing procedures) [BNS91].

Due to Proposition 11.12, however, the existence of a complete set of commuting observables implies that *the gauge group is Abelian*. Said otherwise, a non-Abelian gauge group prevents the existence of complete sets of commuting observables. This happens for instance in *quantum chromodynamics*, where the gauge group of quarks, called *internal colour gauge symmetry*, is isomorphic to $SU(3)$. Remark 13.45 will address another example of an algebra of observables admitting a non-Abelian gauge group, in relation to quantum systems made of identical subsystems (see also Remark 13.47 and the end of Sect. 13.4.8).

What is the general structure of $\mathfrak{R}_S$? First of all we have to focus on the centre $\mathfrak{R}_S \cap \mathfrak{R}'_S$ of $\mathfrak{R}_S$, and check whether it is trivial or not. In view of Proposition 11.18 there are two instances of the utmost physical interest.

(A) $\mathfrak{R}_S \cap \mathfrak{R}'_S$ is non-trivial (with atomic associated lattice) and $\mathfrak{R}'_S$ is Abelian. Then superselection rules occur and $\mathfrak{R}_S$ splits in a direct sum of factors $\mathfrak{R}_{Sk}$. As $\mathfrak{R}'_S$ is Abelian this is the end of the story, since $\mathfrak{R}_{Sk} = \mathfrak{B}(\mathsf{H}_{Sk})$ and $\mathfrak{R}_S$ is of type I. Consequently, the representations π_k of $\mathfrak{R}_S$ on each sector H_{Sk} are not faithful, irreducible and unitarily inequivalent (Remark 11.22(3)). Bounded observables correspond exactly to self-adjoint operators in $\bigvee_{k\in K} \mathfrak{B}(\mathsf{H}_{Sk})$. This is the same as saying that observables (even unbounded ones) exhaust the self-adjoint operators affiliated to that von Neumann algebra.

In this situation, describing quantum states in terms of trace-class operators in $\mathfrak{S}(\mathsf{H}_S)_{adm}$ is physically sound, because elementary propositions separate these operators. An equivalent description (up to the issues with type-I_2 factors) is the one in terms of probability measures on $\mathfrak{B}_{\mathfrak{R}_S}(\mathsf{H}_S)$.

(B) $\mathfrak{R}_S \cap \mathfrak{R}'_S$ is non-trivial (with atomic and central associated lattice) and $\mathfrak{R}'_S$ is non-Abelian. Additional constraints on the observables are present, so $\mathfrak{R}_{Sk} \neq \mathfrak{B}(\mathsf{H}_{S_k})$ for some $k \in K$. Since $\mathfrak{R}_S \cap \mathfrak{R}'_S = \mathfrak{R}''_S \cap \mathfrak{R}'_S$, the centre of $\mathfrak{R}_S$ coincides with the centre of $\mathfrak{R}'_S$. Having assumed the lattice is atomic, also $\mathfrak{R}'_S$ is a direct sum of factors $(\mathfrak{R}')_{Sk} \subset \mathfrak{B}(\mathsf{H}_{Sk})$. It is easy to see that $(\mathfrak{R}')_{Sk} = (\mathfrak{R}_{Sk})' =: \mathfrak{R}'_{Sk}$ for every $k \in K$. Hence the observables pertinent to each sector are constrained to commute with every element of $\mathfrak{R}'_{Sk}$. Each sector H_{Sk} is therefore associated with a pair of, generally non-trivial, *factors* $\mathfrak{R}_{Sk}, \mathfrak{R}'_{Sk}$ (which satisfy, in particular: (a) $(\mathfrak{R}'_{Sk})' = \mathfrak{R}_{Sk}$, (b) $\mathfrak{R}_{Sk} \wedge \mathfrak{R}'_{Sk} = \{cI\}_{c\in\mathbb{C}}$ and (c) $\mathfrak{R}_{Sk} \vee \mathfrak{R}'_{Sk} = \mathfrak{B}(\mathsf{H}_{Sk})$). Each sector H_{Sk} has its own gauge group and at least one of them is not Abelian. The representations π_k of $\mathfrak{R}_S$ on each sector H_{Sk} are not faithful nor unitarily equivalent, and some representations are reducible. When its centre is not trivial, the factor $\mathfrak{R}_{Sk}$ may or not be of type I, and there is no complete set of commuting observables.

The overall von Neumann algebra of observables is $\bigvee_{k\in K} \mathfrak{R}_{Sk}$, with commutant $\bigvee_{k\in K} \mathfrak{R}'_{Sk}$.

In this situation, a description of quantum states in terms of trace-class operators on $\mathfrak{S}(\mathsf{H}_S)_{adm}$ turns out to be redundant, because elementary propositions do not separate operators of this kind. Probability measures on $\mathfrak{B}_{\mathfrak{R}_S}(\mathsf{H}_S)$ work better. In particular, pure states cannot be faithfully described by unit vectors (up to phase), and furthermore it is not possible to prepare the state of the system in a vector state by measuring a family of compatible observables with pure point spectrum.

There are three main ways, at least, to describe the physical properties of a quantum system in terms of the non-Abelian commutant of the von Neumann algebra of observables.

1. Elementary systems (particles) with non-Abelian internal gauge symmetry. This is the case of quarks and other hadrons. The Hilbert space representing a single sector of the Abelian superselection rules affecting the system (e.g., the electric charge) is a product $\mathsf{H} \otimes \mathsf{H}_G$. While H_G is isomorphic to $\mathbb{C}^n$ for some $n \geq 3$ and carries an irreducible representation of $SU(3)$ (the *colour* group $SU(3)$), H supports an irreducible representation of the (universal covering of the proper orthochronous) Poincaré group with positive energy and mass, and definite spin s. The bounded observables are the elements in the von Neumann algebra generated by the representation of the Poincaré group, hence they are of the form $A \otimes I$ for every self-adjoint operator $A \in \mathfrak{B}(\mathsf{H})$. As H is isomorphic to $L^2(\mathbb{R}^3, dk) \otimes \mathbb{C}^{2s+1}$, the algebra of observables $\mathfrak{R}$ is a type-I_∞ factor. Instead $\mathfrak{R}'$, which is made of operators $I \otimes B$ with $B \in \mathfrak{B}(\mathbb{C}^n)$, is a factor of type I_n.
2. Infinitely extended thermodynamical systems at finite (non-zero) temperature. These systems may be studied as thermodynamical limits of localised systems (e.g. systems defined on increasingly large lattices), or as genuinely infinitely extended systems like quantum fields in Minkowski or curved spacetime (for instance represented by a Weyl C^*-algebra via the GNS construction associated to a *KMS state*, see Chap. 14). The von Neumann algebra of observables is reducible (as required by the very KMS condition), but under suitable hypotheses

– corresponding to a pure phase state – it is a type-*III* factor [Emc72, Haa96, ReSu07].

3. A von Neumann algebra of observables $\mathfrak{R}(O)$ localised on a bounded causally-complete region O in spacetime for a quantum field in Minkowski spacetime within the so-called *Haag–Kastler formulation* [Emc72, Haa96, Rob04, ReSu07]. Here $\mathfrak{R}(O)'$ is made of the observables localised on the causal completion of O and is a highly non-trivial von Neumann algebra. In general $\mathfrak{R}(O)'$ is not a factor, but a type-III von Neumann algebra.

11.3 Miscellanea on the Notion of Observable

This section focuses on some features of either practical or theoretical nature regarding observables.

11.3.1 Mean Value and Standard Deviation

Proposition 7.52 enables us to associate to any pair "observable–state", A, ρ, a probability measure on $\mathbb{R}$, $\mu_\rho^{(A)} : E \mapsto tr(\rho P^{(A)}(E))$ with $E \in \mathscr{B}(\mathbb{R})$ (coinciding with $\mu_\psi^{(A)} = (\psi|P^{(A)}(E)\psi)$, cf. Theorem 8.52(c), if ρ is pure and determined by the unit vector $\psi \in \mathsf{H}$). By construction $supp(\mu_\rho^{(A)}) \subset \sigma(A)$. Since by definition $\mu_\rho^{(A)}(E)$ is the probability that the measurement of A belongs to E, in the state ρ, it makes sense to define the *mean value* and the *standard deviation* of A in the state ρ.

Definition 11.25 Let A be an observable of the physical system S described on the Hilbert space H_S, $\rho \in \mathfrak{S}(\mathsf{H}_S)$ a state of S and $\mu_\rho^{(A)}$ the probability measure associated to ρ and A as above. The **mean value** of A^n, $n = 1, 2, \ldots,$ is

$$\langle A^n \rangle_\rho := \int_{\mathbb{R}} \lambda^n \, d\mu_\rho^{(A)}(\lambda) \,, \quad \text{when } \mathbb{R} \ni \lambda \mapsto \lambda^n \text{is in } L^1(\mathbb{R}, \mu_\rho^{(A)}). \tag{11.10}$$

The **standard deviation** of A **in state** ρ is

$$\Delta A_\rho := \sqrt{\int_{\mathbb{R}} \left(\lambda - \langle A \rangle_\rho\right)^2 \, d\mu_\rho^{(A)}(\lambda)}, \quad \text{when } \mathbb{R} \ni \lambda \mapsto \lambda^2 \text{is in } L^1(\mathbb{R}, \mu_\rho^{(A)}), \tag{11.11}$$

The mean value of A^n (or the standard deviation) does not exists for ρ in case expression (11.10) (or (11.11)) cannot be made sense of.

Remark 11.26 If the map λ^n belongs to $L^1(\mathbb{R}, \mu_\rho^{(A)})$ then also λ^k does, for any $k = 1, 2, \ldots, n-1$, because $\mu_\rho^{(A)}$ is finite. Therefore if $\langle A^n \rangle_\rho$ exists, so does $\langle A^k \rangle_\rho$ (and ΔA_ρ if $n \geq 2$) for any $k = 1, 2, \ldots, n-1$. ■

The properties of Example 7.53 generalise as follows.

Proposition 11.27 *Let A be an observable for a system described on the separable Hilbert space H_S, and take $\rho_\psi := \psi(\psi|\cdot) \in \mathfrak{S}_p(\mathsf{H}_S)$ associated to $\psi \in \mathsf{H}_S$ with $||\psi|| = 1$ and $\rho \in \mathfrak{S}(\mathsf{H}_S)$.*
(a) *(i) $\langle A\rangle_{\rho_\psi}$ exists $\Leftrightarrow$ $\psi \in D(|A|^{1/2})$, and ΔA_{ρ_ψ} exists $\Leftrightarrow$ $\psi \in D(A)$.*
(ii) If $\psi \in D(A)$ then $\langle A\rangle_{\rho_\psi}$ exists, and:

$$\langle A\rangle_{\rho_\psi} = (\psi|A\psi)\,, \tag{11.12}$$

(iii) If $\psi \in D(A^2)$ then $\langle A\rangle_{\rho_\psi}$ and ΔA_{ρ_ψ} exist, Eq. (11.12) holds, and:

$$\Delta A^2_{\rho_\psi} = \left(\psi \left|(A - \langle A\rangle_\psi I)^2 \psi\right.\right) = (\psi|A^2\psi) - (\psi|A\psi)^2\,. \tag{11.13}$$

(b) *(i) $\langle A\rangle_\rho$ exists $\Leftrightarrow$ $Ran(\rho^{1/2}) \subset D(|A|^{1/2})$ and $|A|^{1/2}\rho^{1/2} \in \mathfrak{B}_2(\mathsf{H})$.*
(ii) ΔA_ρ exists (equivalently $\langle A^2\rangle_\rho$ exists) $\Leftrightarrow$ $Ran(\rho^{1/2}) \subset D(A)$ and $A\rho^{1/2} \in \mathfrak{B}_2(\mathsf{H})$.
(iii) If $\langle A^2\rangle_\rho$ exists, then $A\rho \in \mathfrak{B}_1(\mathsf{H}_S)$ and:

$$\langle A\rangle_\rho = tr(A\rho)\,. \tag{11.14}$$

(iv) If $\langle A^4\rangle_\rho$ exists, then $A\rho \in \mathfrak{B}_1(\mathsf{H}_S)$, Eq. (11.14) holds, $(A - \langle A\rangle_\rho I)^2\rho \in \mathfrak{B}_1(\mathsf{H}_S)$ and:

$$\Delta A^2_\rho = tr\left((A - \langle A\rangle_\rho I)^2\rho\right) = tr(A^2\rho) - tr(A\rho)^2\,. \tag{11.15}$$

Proof (a) We have $tr(\rho_\psi P^{(A)}(E)) = (\psi|P^{(A)}(E)\psi) = \mu^{(A)}_\psi(E)$. Therefore asking that $\mathbb{R} \ni \lambda \mapsto \lambda$ and $\mathbb{R} \ni \lambda \mapsto \lambda^2$ belong in $L^1(\mathbb{R}, \mu^{(A)}_{\rho_\psi})$ is respectively equivalent to $\psi \in D(|A|^{1/2})$ and $\psi \in D(A)$, by Definition 9.14. By definition, and using Theorem 9.4(e, f) for the standard deviation:

$$\langle A\rangle_{\rho_\psi} = \int_{\mathbb{R}} \lambda d\mu^{(A)}_\psi(\lambda), \tag{11.16}$$

$$\Delta A^2_{\rho_\psi} = \left(A\psi - \left(\int_{\mathbb{R}} \lambda d\mu^{(A)}_\psi(\lambda)\right)\psi \,\middle|\, A\psi - \left(\int_{\mathbb{R}} \lambda d\mu^{(A)}_\psi(\lambda)\right)\psi\right). \tag{11.17}$$

Using Theorem 9.4(e) these imply (11.12) and (11.13) if $\psi \in D(A)(\subset D(|A|^{1/2}))$ and $\psi \in D(A^2)(\subset D(A))$, respectively.
(b) Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a basis of H_S (separable). Then $\mu^{(A)}_\rho(E) = tr(\rho P^{(A)}(E)) = tr(\rho^{1/2}P^{(A)}(E)\rho^{1/2}) = \sum_{n=0}^{+\infty}(\rho^{1/2}\psi_n|P^{(A)}(E)\rho^{1/2}\psi_n) = \sum_{n=0}^{+\infty}\mu^{(A)}_{\rho^{1/2}\psi}(E)$, for any Borel set $E \in \mathscr{B}(\mathbb{R})$, where we used $\rho^{1/2} \in \mathfrak{B}_2(\mathsf{H}_S)$ (as $\mathfrak{B}_1(\mathsf{H}) \ni \rho \geq 0$) and Proposition 4.38(c). If $f : \mathbb{R} \to \mathbb{C}$ is measurable, then,

$$\int_{\mathbb{R}} |f(\lambda)| d\mu_\rho^{(A)}(\lambda) = \sum_{n=0}^{+\infty} \int_{\mathbb{R}} |f(\lambda)| d\mu^{(A)}_{\rho^{1/2}\psi_n}(\lambda) \leq +\infty\,. \tag{11.18}$$

Moreover, if the right-hand side in (11.18) (so also the left side) is finite, then:

$$\int_{\mathbb{R}} f(\lambda) d\mu_\rho^{(A)}(\lambda) = \sum_{n=0}^{+\infty} \int_{\mathbb{R}} f(\lambda) d\mu^{(A)}_{\rho^{1/2}\psi_n}(\lambda) \in \mathbb{C}\,. \tag{11.19}$$

In fact, $\mu_\rho^{(A)}(E) = \sum_{n=0}^{+\infty} \mu^{(A)}_{\rho^{1/2}\psi}(E)$ implies that (11.18) is trivially true if $|f| = s$ is a simple non-negative map. For any Borel-measurable function $g \geq 0$ there is a simple sequence $0 \leq s_0 \leq s_1 \leq \cdots \leq s_n \to g$ (Proposition 7.49). By monotone convergence on the single integrals and on the counting measure of $\mathbb{N}$ we obtain (11.18), with $|f|$ replaced by an arbitrary $g \geq 0$. If f is real-valued, and in (11.18) we have $< +\infty$, decomposing f in its positive and negative parts $f = f_+ - f_-$, $0 \leq f_+, f_- \leq |f|$, gives (11.19) by linearity. If f is complex-valued the argument is similar, we just work with real and imaginary parts separately. Now, $\langle f(A)\rangle_\rho$ exists precisely when the left-hand side in (11.18) is finite. In turn, this is the same as saying every summand on the right is finite and the sum is finite. The generic term is finite if and only if $\rho^{1/2}\psi_n \in D(|f(A)|^{1/2})$ by definition of $D(g(A))$ (Definition 9.14). Since ψ_n is an arbitrary unit vector in H, $Ran(\rho^{1/2}) \subset D(|f(A)|^{1/2})$. Every integral on the right in (11.18) can be written as (see Theorem 9.4(f)) $|||f(A)|^{1/2}\rho^{1/2}\psi_n||^2$, where $|f(A)|^{1/2}\rho^{1/2} \in \mathfrak{B}(\mathsf{H}_S)$ by Proposition 5.7. By Definition 4.2.4 we conclude that the left-hand side (11.18) is finite iff $|f(A)|^{1/2}\rho^{1/2} \in \mathfrak{B}_2(\mathsf{H}_S)$. Choose $f(\lambda) = \lambda$, so (i) in (b) holds, then choose $f(\lambda) = \lambda^2$ to obtain (ii) in (b), because λ^2 integrable in $\mu_\rho^{(A)}$ implies λ integrable, plus $D(A) = D(|A|)$. To prove (iii) assume $\langle A^2\rangle_\rho$ exists (so also $\langle A\rangle_\rho$ exists), and notice that $Ran(\rho) \subset D(A)$ from $Ran(\rho^{1/2}) \subset D(A)$, since $\rho = \rho^{1/2}\rho^{1/2} \Rightarrow Ran(\rho^{1/2}) \supset ran(\rho)$. Applying (11.19) to $f(\lambda) = \lambda$ and recalling Theorem 9.4(e) we find:

$$\langle A\rangle_\rho = \sum_{n\in\mathbb{N}} (\rho^{1/2}\psi_n|A\rho^{1/2}\psi_n) = \sum_{n\in\mathbb{N}} (\psi_n|\rho^{1/2}A\rho^{1/2}\psi_n) = tr(\rho^{1/2}A\rho^{1/2})\,,$$

where we used $\rho^{1/2}$, $A\rho^{1/2} \in \mathfrak{B}_2(\mathsf{H}_2)$, so their product (in any order) is of trace class. Since the trace is invariant under cyclic permutations (Proposition 4.38(c)) we have $\langle A\rangle_\rho = tr(A\rho^{1/2}\rho^{1/2}) = tr(A\rho)$, concluding (iii). The proof of (iv) is similar: replace A with A^2 and observe that if $\langle A^4\rangle_\rho$ exists, so does $\langle A^2\rangle_\rho$, and (iii) holds. The second identity in (11.15) now follows from the first by obvious algebraic manipulations. $\square$

Remark 11.28 The right-hand sides of (11.14) and (11.15) ((11.12) and (11.13) for pure states) are *not* the definitions of mean value and standard deviation. These are given, in general, by (11.10) and (11.11), independently from Proposition 11.27. ■

11.3.2 An Open Problem: What is the Meaning of $f(A_1, \ldots, A_n)$ if $A_1, \ldots, A_n$ are Not Pairwise Compatible?

The interpretation, both mathematical and physical, of certain formal objects that appear in textbooks has been deliberately swept under the carpet, until now. We have in mind particular expressions of the sort $f(A_1, \ldots, A_n)$, for a real-valued function f and a set of observables $A_1, \ldots, A_n$ that are not pairwise compatible. Expressions of this kind have *great* relevance in physics: think of the Hamiltonian operator of the electron in the hydrogen atom. i.e. $\frac{P^2}{2m} + V(x)$. Here the two summands are incompatible observables. Even von Neumann tackled the issue in his celebrated book on theoretical foundations of QM without reaching a conclusive answer. Here we shall just introduce the physical side of the problem, without finding a solution to it. The mathematics will be discussed briefly in one case of physical relevance in Sect. 11.5.8.

Even the physical meaning of $f(A_1, \ldots, A_n)$ is dubious, since the observables $A_1, \ldots, A_n$ cannot be measured simultaneously. If we were able to measure a bunch of observables simultaneously, $f(A_1, \ldots, A_n)$ would clearly be the observable that is measured by evaluating f on the simultaneous readings of the A_k. This physical interpretation for compatible observables agrees with the mathematical definition of $f(A_1, \ldots, A_n)$ given by (11.3). Not all is lost though, for in certain cases something can be said for incompatible observables, too. Consider:

$$S := f(A_1, \ldots, A_n) = a_1 A_1 + \cdots + a_n A_n$$

for some real coefficients a_k. Suppose that S is (essentially) self-adjoint on some dense domain $\mathscr{D}$ common to all operators A_k. Recalling the definition of *mean value*, we can think of the observable S as *the only observable that satisfies*:

$$\langle S \rangle_{\rho_\psi} = a_1 \langle A_1 \rangle_{\rho_\psi} + \cdots + a_n \langle A_n \rangle_{\rho_\psi} \quad \text{for every unit vector } \psi \in \mathscr{D}.$$

(The interpretation can be extended to include mixed states.) The point is that to check the relation above it is not necessary to measure $A_1, \ldots, A_n$ simultaneously. It is enough to measure them separately on a corresponding statistical ensemble of identical physical systems, when the physical systems of the various statistical ensembles are all prepared in the pure state ρ_ψ. The advantage of this proposal is that it holds also when $A_1, \ldots, A_n$ are pairwise incompatible. Its drawback is that it will not work for more complicated functions f, though it sometimes suggests other remarkable interpretations. First of all, even if $A_1, \ldots, A_n$ are pairwise incompatible, the observable $(a_1 A_1 + \cdots + a_n A_n)^k$ is well defined and has a clear physical meaning whenever $a_1 A_1 + \cdots + a_n A_n$ is defined as above. But more interestingly, we take inspiration from the real function

$$f(a,b) = ab = \frac{1}{2}\left((a+b)^2 - a^2 - b^2\right)$$

where $a, b \in \mathbb{R}$, and define:

$$f(A,B) := \frac{1}{2}\left((A+B)^2 - A^2 - B^2\right) , \tag{11.20}$$

when A and B are incompatible. In principle, the right-hand side has a clear physical interpretation relying upon the iteration of the mean-value interpretation presented above. Life, however, is not that easy. Indeed, if we forget that meaning for the moment, the right-hand side in (11.20) can equivalently be rewritten as:

$$\frac{1}{2}(AB + BA) .$$

Unfortunately, this operator is only Hermitian and not (essentially) self-adjoint in general, when A and B are self-adjoint but not bounded (more or less the standard in Quantum Mechanics). Therefore the initial, simplistic physical interpretation cannot always be supported by the mathematics we have developed. Every case has to be investigated separately, for instance by choosing some essential self-adjoint extension of the Hermitian operator at hand, constructed as a function of incompatible observables.

11.3.3 The Notion of Jordan Algebra

From a completely theoretical point of view, it is worth noticing that (11.20) can be understood as a **Jordan product**, when A and B are elements of the algebra of bounded operators $\mathfrak{B}(\mathsf{H})$:

$$A \circ B := \frac{1}{2}(AB + BA) . \tag{11.21}$$

Domains cause no trouble, and $A \circ B$ is self-adjoint if A and B are self-adjoint. The Jordan product is *commutative*:

$$A \circ B = B \circ A \quad \text{for all } A, B \in \mathfrak{B}(\mathsf{H}) \tag{11.22}$$

but *not* associative. On the contrary it satisfies:

$$(A \circ B) \circ (A \circ A) = A \circ (B \circ (A \circ B)) \quad \text{for all } A, B \in \mathfrak{B}(\mathsf{H}). \tag{11.23}$$

As a consequence of latter two, it can be proved that a weak form of associativity holds, for powers of a fixed element. Hence $A \circ A \circ \cdots \circ A$ is well defined.

Definition 11.29 A vector space $\mathfrak{A}$ (over $\mathbb{C}$ or $\mathbb{R}$) equipped with an operation $\circ : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ that is bilinear, distributive with respect to the product by scalars, and satisfies (11.22) and (11.23), is called a (complex or real) **Jordan algebra**.

A Jordan algebra is said to be **special** if it is constructed out of an associative algebra by defining the Jordan product as in (11.21), making use of the product in the algebra itself. So, in particular, $\mathfrak{B}(\mathsf{H})$ is a complex special Jordan algebra. Similarly, the real algebra of bounded self-adjoint operators in $\mathfrak{B}(\mathsf{H})$ generates a corresponding real special Jordan algebra. More generally, if $\mathfrak{R}_S \subset \mathfrak{B}(\mathsf{H}_S)$ is the von Neumann algebra of observables of a physical system, the class of self-adjoint operators of $\mathfrak{R}_S$ form a real special Jordan algebra when equipped with the natural structure induced by $\mathfrak{B}(\mathsf{H}_S)$. Sometimes it is convenient to develop the quantum theory on S by referring directly to this Jordan algebra instead of the whole $\mathfrak{R}_S$.

The notion of real special Jordan algebra plays an important role in the attempts of several authors, especially I. Segal and G.G. Emch [Emc72, Str05a], to justify physically the *algebraic formulation* of quantum theories that we will introduce in the last chapter.

11.4 Axiom A5: Non-relativistic Elementary Systems

In order to move farther into the mathematical formulation of QM we need to set axioms about special elementary systems. These correspond to the *particles* of the non-relativistic theory. In other terms, the group of transformations under which the theory is invariant is the Galilean group, not the Poincaré group. We will return to this point later. In physics this description is adequate until speeds do not reach the order of the speed of light (about 300.000 km/s). However certain mathematical concepts, like the *Weyl* *-**algebra** introduced in the forthcoming non-relativistic description, are significant in a wider context. They are employed in relativistic regimes as well, in formulations of *Quantum Field Theory* that we will not discuss.

In elementary formulations of quantum theories, complex systems are built by composing elementary systems via the Hilbert tensor product, as we shall see when we study compound systems.

The simplest elementary system in non-relativistic QM consists in a quantum particle of mass $m > 0$ and spin 0. The next axiom holds in this system.

A5. Consider an inertial frame system $\mathscr{I}$ with orthonormal Cartesian coordinates x_1, x_2, x_3 on the rest space of the frame. A non-relativistic particle of mass $m > 0$ and spin 0 is described as follows.

(a) The system's Hilbert space is $\mathsf{H} = L^2(\mathbb{R}^3, dx)$, where $\mathbb{R}^3$ is identified with the rest space of $\mathscr{I}$ under the coordinates x_1, x_2, x_3, and dx is the ordinary Lebesgue measure on $\mathbb{R}^3$.

(b) The three observables associated to x_1, x_2, x_3 are self-adjoint operators, called **position operators**:

$$(X_i\psi)(x_1, x_2, x_3) = x_i\psi(x_1, x_2, x_3)\,, \tag{11.24}$$

$i = 1, 2, 3$, with domains:

$$D(X_i) := \left\{\psi \in L^2(\mathbb{R}^3, dx) \;\middle|\; \int_{\mathbb{R}^3} |x_i\psi(x_1, x_2, x_3)|^2\, dx < +\infty\right\}\,.$$

(c) The three observables associated to the momentum components p_1, p_2, p_3 in $\mathscr{I}$ are self-adjoint operators, called **momentum operators**:

$$P_k = \overline{-i\hbar\frac{\partial}{\partial x_k}}, \tag{11.25}$$

$k = 1, 2, 3$, where the operator on the right is the closure of the essentially self-adjoint differential operator:

$$-i\hbar\frac{\partial}{\partial x_k} : \mathscr{S}(\mathbb{R}^3) \to L^2(\mathbb{R}^3, dx)$$

and $\mathscr{S}(\mathbb{R}^3)$ is the Schwartz space on $\mathbb{R}^3$ (see Sect. 3.7).

Vectors (normalised to 1) of $L^2(\mathbb{R}^3, dx)$ associated to a particle are called its **wavefunctions** in physics' literature. Wavefunctions determine (not uniquely, owing to arbitrary numerical factors) the particle's pure states.

Remarks 11.30 **(1)** The discussion of Sect. 5.3 explains that the same notions can be given if we define $D(X_i)$ using $\psi \in \mathscr{D}(\mathbb{R}^3)$ or $\psi \in \mathscr{S}(\mathbb{R}^3)$, instead of $\psi \in L^2(\mathbb{R}^3, dx)$. In either case one has to take the unique self-adjoint extension of the operator on $\mathscr{D}(\mathbb{R}^3)$ or $\mathscr{S}(\mathbb{R}^3)$.
(2) P_i may be defined equivalently by (see Definition 5.27, Proposition 5.29 and the ensuing discussion):

$$(P_i f)(\mathbf{x}) = -i\hbar \text{w-}\frac{\partial}{\partial x_i} f(\mathbf{x}), \tag{11.26}$$

$$D(P_i) := \left\{ f \in L^2(\mathbb{R}^3, dx) \;\middle|\; \text{w-}\frac{\partial}{\partial x_i} f \in L^2(\mathbb{R}^3, dx)\text{exists} \right\}\,.$$

As usual w-$\frac{\partial}{\partial x_i}$ denotes the weak derivative. The study of Sect. 5.3 also shows that P_i (see Proposition 5.29) can be defined, equivalently, substituting the Schwartz space with $\mathscr{D}(\mathbb{R}^3)$ and taking the unique self-adjoint extension of the operator obtained, which is still essentially self-adjoint.
(3) Let K_i denote the ith position operator on the codomain of the Fourier–Plancherel transform $\widehat{\mathscr{F}} : L^2(\mathbb{R}^3, dx) \to L^2(\mathbb{R}^3, dk)$, see Sect. 3.7. Then Proposition 5.31 gives

$$P_i = \hbar\, \widehat{\mathscr{F}}^{-1} K_i \widehat{\mathscr{F}}\,,$$

an alternative definition of momentum.

(4) From Sect. 9.1.5 we know

$$\sigma(X_i) = \sigma_c(X_i) = \mathbb{R}\,, \quad \sigma(P_i) = \sigma_c(P_i) = \mathbb{R} \quad i = 1, 2, 3. \tag{11.27}$$

■

11.4.1 The Canonical Commutation Relations (CCRs)

The definition of position and momentum is such that there exist *invariant* spaces $\mathsf{H}_0 \subset L^2(\mathbb{R}^3, dx)$ for all six observables, despite the latter's domains are different: $X_i(\mathsf{H}_0) \subset \mathsf{H}_0$ and $P_i(\mathsf{H}_0) \subset \mathsf{H}_0$, $i = 1, 2, 3$. One instance is the Schwartz space $\mathsf{H}_0 = \mathscr{S}(\mathbb{R}^3)$, whose invariance is immediate, by definition. On $\mathscr{S}(\mathbb{R}^3)$ a direct computation that uses (11.24) and (11.25) yields Heisenberg's **canonical commutation relations** (CCRs) :

$$[X_i, P_j] = i\hbar\delta_{ij} I\ ,$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. More precisely

Lemma 11.31 *The operators position X_i and momentum P_j, $i, j = 1, 2, 3$, defined in* **A5**, *obey* **Heisenberg's commutation relations***:*

$$[X_i, P_j]\psi = i\hbar\delta_{ij}\psi \quad \textit{for every } \psi \in D(X_iP_j) \cap D(P_jX_i), i, j = 1, 2, 3\ . \tag{11.28}$$

Equations (11.28) do not change when we replace X_i with $X'_i := X_i + a_iI$ and P_j with $P'_j := P_j + b_jI$, for any constants $a_i, b_j \in \mathbb{R}$.

Proof A straightforward computation shows $D(X'_iP'_j) \cap D(P'_jX'_i) = D(X_iP_j) \cap D(P_jX_i)$. On $\varphi \in \mathscr{D}(\mathbb{R}^3)$, the operator P'_j acts as $-i\hbar\partial/\partial x_j + b_jI$ by construction. Since X'_i multiplies by the shifted coordinate $x_i + a_i$, we obtain $P'_jX'_i\varphi = -i\hbar\delta_{ij}\varphi + X'_iP'_j\varphi$. Therefore

$$\left(P'_jX'_i\varphi - X'_iP'_j\varphi + i\hbar\delta_{ij}\varphi\middle|\,\psi\right) = 0\,, \quad \varphi \in \mathscr{D}(\mathbb{R}^3), \psi \in L^2(\mathbb{R}^3, dx).$$

In turn, if $\psi \in D(X'_iP'_j) \cap D(P'_jX'_i) = D(X_iP_j) \cap D(P_jX_i)$, since P_j and X_i are self-adjoint, the identity reads

$$\left(\varphi\,\middle|X'_iP'_j\psi - P'_jX'_i\psi - i\hbar\delta_{ij}\psi\right) = 0\,.$$

As $\mathscr{D}(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3, dx)$, (11.28) holds with X'_i, P'_j replacing X_i, P_j. Consequently (11.28) holds by taking $a_j = b_j = 0$, $j = 1, 2, 3$. □

Pairs of observables solving Heisenberg's relations (11.28), on some invariant domain such as $\mathscr{S}(\mathbb{R}^3)$, are often called **conjugate observables**. The relations are the simplest manifestation of general CCRs for *bosonic* quantum fields, where position and momentum are defined suitably to befit the theory. From the physical viewpoint it has been noticed over and over, during the history of QM and its evolution, that the canonical commutation relations are much more important than the operators X_i and P_i themselves.

As the definitions make obvious, position and momentum are unbounded operators, and are not defined on the entire Hilbert space. Technically speaking this is a thorn in the side, for it forces to bring into the picture the spectral theory of unbounded operators, which is more involved than the bounded theory. This begs the question whether a substitute definition of X_i, P_i might exists, that preserves Heisenberg's relations and makes the operators bounded. The answer is no, and the reason is dictated by Heisenberg's CCRs.

Proposition 11.32 *There are no self-adjoint operators X and P such that, on a common invariant subspace, $[X, P] = i\hbar I$ and at the same time X, P are bounded.*

Proof Suppose, by contradiction, $[X, P] = i\hbar I$ on a common invariant space D where X, P are bounded. Restrict to D (or its closure $\overline{D}$, by extending X, P to self-adjoint operators defined on $\overline{D}$), and consider it as the Hilbert space. The restrictions will now be self-adjoint and bounded. From $[X, P] = i\hbar I$:

$$PX^n - X^n P = -inX^{n-1} \, .$$

If n is odd, using Proposition 3.38(a) repeatedly (as $X^p = (X^p)^*$ for any natural number p) and the norm's properties:

$$n||X||^{n-1} = n||X^{n-1}|| \leq 2||P||||X^n|| \leq 2||P||||X||||X^{n-1}|| = 2||P||||X||||X||^{n-1} .$$

As $||X|| \neq 0$ (because of (11.28)), we obtain the absurd: $n \leq 2||P||||X|| < +\infty$ for any $n = 1, 3, 5, \ldots$. $\square$

11.4.2 Heisenberg's Uncertainty Principle as a Theorem

A comforting immediate consequence of the CCRs and of the formalism is to turn *Heisenberg's Uncertainty Principle* for the variables position and momentum (cf. Sect. 6.4) into a theorem. We shall prove the principle in its classical form on *pure states*, only to reformulate it later under weaker assumptions on vectors and then *extend it to mixed states*.

Theorem 11.33 ("Heisenberg's Uncertainty Principle") *Let ψ be a unit vector, describing a pure state of a classical spinless particle, such that:*

$$\psi \in D(X_i P_i) \cap D(P_i X_i) \cap D(X_i^2) \cap D(P_i^2)$$

(in particular $\psi \in \mathscr{S}(\mathbb{R}^3)$). Then

$$(\Delta X_i)_\psi (\Delta P_i)_\psi \geq \frac{\hbar}{2} \quad i = 1, 2, 3 \tag{11.29}$$

(where we wrote $_\psi$ instead of $_{\rho_\psi}$ for simplicity):

Proof The hypotheses imply, in particular, $\psi \in D(X_i^2) \cap D(P_i^2)$, so standard deviations are defined and can be found using formula (11.13).

By definition (11.11) we see $(\Delta X_i)_\psi = (\Delta X'_i)_\psi$ and $(\Delta P_i)_\psi = (\Delta P'_i)_\psi$ for $X'_i := X_i + a_i I$, $P'_i := P_i + b_i I$ with a_i, b_i real constants. Hence replacing X_i, P_i by X'_i, P'_i produces an equivalent formula to (11.29). Let us choose $a_i = -\langle X_i \rangle_\psi$, $b_i = -\langle P_i \rangle_\psi$ and prove (11.29) for the operators X'_i. P'_i. From (11.13) the choices force $(\Delta X'_i)_\psi = ||X'_i \psi||$ and $(\Delta P'_i)_\psi = ||P'_i \psi||$. So we need to prove

$$||X'_i \psi|| ||P'_i \psi|| \geq \hbar/2 \,. \tag{11.30}$$

As X'_i, P'_i satisfy (11.28), Schwarz's inequality, the operators' self-adjointness and the properties of the inner product give

$$||X'_i \psi|| ||P'_i \psi|| \geq |(X'_i \psi | P'_i \psi)| \geq |Im(X'_i \psi | P'_i \psi)| = \frac{1}{2} \left| (\psi | X'_i P'_i \psi) - \overline{(\psi | X'_i P'_i \psi)} \right|$$

$$= \frac{1}{2} \left| (\psi | (X'_i P'_i - P'_i X'_i) \psi) \right| = \frac{\hbar}{2} (\psi | \psi) = \frac{\hbar}{2} \,,$$

i.e., (11.30). Lemma 11.31 was used in the penultimate equality. □

Remark 11.34 This proof shows more generally that $\Delta A_\psi \Delta B_\psi \geq \frac{1}{2} |(\psi | [A, B] \psi)|$ for every vector $\psi \in D(AB) \cap D(BA) \cap D(A^2) \cap D(B^2)$ and any Hermitian operators A, B on H. ■

11.5 Weyl's Relations, the Theorems of Stone–von Neumann and Mackey

The CCRs satisfy a remarkable property: in the statement of axiom **A5** it is somehow superfluous that the Hilbert space be $L^2(\mathbb{R}^3, dx)$, as is to ask that the position and momentum operators have the given form. This information is by some means contained in Heisenberg's relations so long as, loosely put, the representation of position and momentum is *irreducible*. This fact is the heart of the famous theorem of Stone–von Neumann, that we will prove in this section. By dropping irreducibility Mackey proved (as a consequence of more general facts in the theory of *imprimitivity systems*) that the Hilbert space is an orthogonal sum of irreducible representations (countably many if the space is separable). We will prove Mackey's theorem after Stone–von Neumann's.

11.5.1 Families of Operators Acting Irreducibly and Schur's Lemma

Before we get going, a few generalities on *families of operators acting irreducibly* are necessary. Tightly linked to this notion is *Schur's lemma*, a very useful result from abstract representation theory of unitary groups that we will encounter in the next chapter. We remind the reader that a subspace M in a vector space X is said to be *invariant* under the operator $A : \mathsf{X} \to \mathsf{X}$ when $A(\mathsf{M}) \subset \mathsf{M}$ (Definition 2.35).

Definition 11.35 Let H be a Hilbert space and $\mathscr{A} := \{A_i\}_{i\in J}$ a family of operators $A_i : \mathsf{H} \to \mathsf{H}$. The space H is called **irreducible** under $\mathscr{A}$, and $\mathscr{A}$ is an **irreducible family of operators** on H, if there is no closed subspace in H different from H and $\{\mathbf{0}\}$ that is invariant under every element in $\mathscr{A}$ simultaneously.

Remark 11.36 Every so often it is necessary to distinguish between **irreducibility** and **topological irreducibility** under a given family of operators. *Irreducibility* refers to the absence of non-trivial invariant subspaces, whereas *topological irreducibility* concerns the absence of *closed* non-trivial invariant subspaces. In this book we shall not make that distinction: for us irreducibility will refer to *closed* subspaces implicitly, and we shall omit the term "topological" everywhere. ■

Here is *Schur's lemma*.

Proposition 11.37 (Schur's lemma) *Let $\mathscr{A} := \{A_i\}_{i\in J} \subset \mathfrak{B}(\mathsf{H})$ be a family of operators on a complex Hilbert space, closed under Hermitian conjugation ($A_i^* \in \mathscr{A}$ if $A_i \in \mathscr{A}$). Then*
(a) *$\mathscr{A}$ is irreducible $\Leftrightarrow$ every operator $V \in \mathfrak{B}(\mathsf{H})$ satisfying*

$$VA_i = A_i V \quad \textit{for every } i \in J,$$

has the form $V = \chi I$ for some complex number $\chi \in \mathbb{C}$.
(b) *Let $\mathscr{A}' := \{A'_i\}_{i\in J} \subset \mathfrak{B}(\mathsf{H}')$ be a family on another Hilbert space H', indexed by the same set J and closed under the Hermitian conjugation. Suppose*

$$A_i^* = A_{j_i} \;\Rightarrow\; A'^*_i = A'_{j_i} \textit{ for every } i \in J \textit{ and some } j_i \in J. \tag{11.31}$$

If H and H' are irreducible, then every bounded linear operator $S : \mathsf{H} \to \mathsf{H}'$ such that

$$SA_i = A'_i S \quad \textit{for every } i \in J,$$

has the form $S = rU$ where $U : \mathsf{H} \to \mathsf{H}'$ is unitary and $r \in \mathbb{R}$.

Proof Let us begin with the more involved part (b), which we will employ for (a).
(b) Taking adjoints of $SA_i = A'_i S$ gives $A_i^* S^* = S^* A'^*_i$ for any $i \in J$. That is to say $A_{j_i} S^* = S^* A'_{j_i}$, $i \in J$. Note how j_i covers J as i varies in J, since for every $A_i \in \mathscr{A}$, $(A_i^*)^* = A_i$, so we may rephrase the identity as $A_i S^* = S^* A'_i$ for every

$i \in J$. Comparing with $SA_i = A'_i S$ gives $A_i S^* S = S^* S A_i$ and $A'_i S S^* = S S^* A'_i$. From the former the bounded self-adjoint operator $V := S^* S$ commutes with every A_i, so by Theorem 8.56(c) the spectral measure $P^{(V)}$ on $\mathbb{R}$ commutes with each A_i. But then, for every $E \in \mathscr{B}(\mathbb{R})$, the closed space $P_E^{(V)}(\mathsf{H})$ is invariant under each A_i. As the space is irreducible, either $P_E^{(V)}(\mathsf{H}) = \mathsf{H}$ i.e. $P_E^{(V)} = I$, or $P_E^{(V)}(\mathsf{H}) = \{\mathbf{0}\}$ i.e. $P_E^{(V)} = 0$, for any Borel set $E \subset \mathbb{R}$. Suppose the spectrum of V contains at least two values $\alpha \neq \alpha'$ and let us use Theorem 9.13(b). Consider two open disjoint real intervals $E \ni \alpha$, $E' \ni \alpha'$. Then $P_E^{(V)} \neq 0$, $P_{E'}^{(V)} \neq 0$ since the intervals intersect the spectrum, and therefore the only possibility is $P_E^{(V)} = P_{E'}^{(V)} = I$. On the other hand $P_E^{(V)} P_{E'}^{(V)} = 0$ since $E \cap E' = \emptyset$. This is absurd because standard properties of PVM imply $P_{E'}^{(V)} + P_E^{(V)} \leq I$. So the spectrum of V (never empty) contains a single isolated point, which is in the point spectrum. Hence $S^* S = V = \lambda I$ for some $\lambda \in [0, +\infty)$, V being clearly positive. In a similar manner we obtain $S S^* = \lambda' I$ for some $\lambda' \in [0, +\infty)$. But then

$$\lambda S^* = S^* S S^* = \lambda' S^* .$$

Consequently either $\lambda = \lambda'$ or $S^* = 0$ and so $S = (S^*)^* = 0$. In the second case the proof ends. In the first instance, let $U := \lambda^{-1/2} S$, so that $UU^* = I'$ and $U^* U = I$ where I, I' are the identity operators of H, H'. Therefore U is unitary. The claim is proved by taking $r = \lambda^{1/2}$.

Let us pass to (a) and assume H is irreducible under $\mathscr{A}$. If $VA_i = A_i V$, then $A_i^* V^* = V^* A_i^*$, meaning $A_i V^* = V^* A_i$ for any $i \in J$, as $\mathscr{A}$ is *-closed. Then the bounded self-adjoint operators $V_+ := \frac{1}{2}(V + V^*)$ and $V_- := \frac{1}{2i}(V - V^*)$ commute with $\mathscr{A}$, implying that their spectral measures commute with $\mathscr{A}$. Arguing as in part (b) we conclude $V_\pm = \lambda_\pm I$ for some real constants $\lambda_\pm$. Then $V = V_+ + iV_- = (\lambda_+ + i\lambda_-)I = \chi I$, $\chi \in \mathbb{C}$. Conversely, suppose that the only operators commuting with $\mathscr{A}$ are multiples χI. If H_0 is invariant under $\mathscr{A}$ and P is the orthogonal projector onto H_0, then $PA_{iP} = A_{iP}$ for any $i \in J$. Take adjoints: $PA_i^* P = PA_i^*$. As $\mathscr{A}$ is *-closed and $i \in J$ arbitrary, the identity reads $PA_i P = PA_i$. Comparing with the initial relation gives $PA_i = A_i P$, $i \in J$. Therefore $P = \chi I$ for some $\chi \in \mathbb{C}$. But $P^* = P \Rightarrow \chi \in \mathbb{R}$, and $PP = P \Rightarrow \chi^2 = \chi$. So there are two possibilities: $P = 0$, and then $\mathsf{H}_0 = \{\mathbf{0}\}$, or $P = I$ so $\mathsf{H}_0 = \mathsf{H}$. This means H is irreducible under $\mathscr{A}$. $\square$

Remark 11.38 Schur's lemma, in cases (a) and (b), will be particularly useful in these situations:

(i) $\mathscr{A}$, $\mathscr{A}'$ are images of two representations $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$, $\pi' : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}')$ of the same *-algebra (or C^*-algebra) $\mathfrak{A}$.

(ii) $\mathscr{A}$, $\mathscr{A}'$ are images of two unitary representations $\mathsf{G} \ni g \mapsto U_g$, $\mathsf{G} \ni g \mapsto U'_g$ of a single group G.

In either case, closure under Hermitian conjugation in case (a), and (11.31) in case (b), are automatic if one takes respectively G and $\mathfrak{A}$ as the indexing set I. ■

11.5.2 Weyl's Relations from the CCRs

In order to illustrate the Stone–von Neumann theorem we proceed step by step. A relevant technical point is that Heisenberg's commutation relations are too hard to use rigorously, for they involve subtleties about domains. To by-pass these issues we can abandon X_i and P_i for the one-parameter unitary groups they generate. Even better, we may take, for $n = 3$, the operators $\sum_{i=k}^{n} t_k X_k + u_k P_k$, $t_k, u_k \in \mathbb{R}$. These are essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$, so we can look at the exponentials of their self-adjoint extensions $\overline{\sum_{i=k}^{n} t_k X_k + u_k P_k}$. A brute-force computation based on Heisenberg's relations (11.28) and the formal Taylor expansion of the exponential (not yet justified) yields the following identity[4]:

$$\exp\left\{i\overline{\sum_{k=1}^{n} t_k X_k + u_k P_k}\right\} \exp\left\{i\overline{\sum_{k=1}^{n} t'_k X_k + u'_k P_k}\right\}$$

$$= \exp\left\{-\frac{i\hbar}{2}\left(\sum_{k=1}^{n} t_k u'_k - t'_k u_k\right)\right\} \exp\left\{i\overline{\sum_{k=1}^{n}(t_k + t'_k)X_k + (u_k + u'_k)P_k}\right\} .$$

The above are called *Weyl relations*, and follow formally from Heisenberg's commutation relations.

The announced proposition proves, completely independently from previous results that involve different techniques, that the operators X_i, P_i are essentially self-adjoint if restricted to $\mathscr{S}(\mathbb{R}^3)$, even in dimension higher than 3. For convenience, we will assume $\hbar = 1$ in the sequel.

Proposition 11.39 *Consider $L^2(\mathbb{R}^n, dx)$, with given $n = 1, 2, \cdots$ and Lebesgue measure dx on $\mathbb{R}^n$. For $k = 1, 2, \cdots, n$ define symmetric operators:*

$$\mathscr{X}_k : \mathscr{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n, dx) \quad and \quad \mathscr{P}_k : \mathscr{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n, dx)$$

$$(\mathscr{X}_k\psi)(\mathbf{x}) = x_k\psi(\mathbf{x}) , \tag{11.32}$$

$$(\mathscr{P}_k\psi)(\mathbf{x}) = -i\frac{\partial\psi}{\partial x_k}(\mathbf{x}) . \tag{11.33}$$

Then:
(a) *the symmetric operators $\sum_{k=1}^{n} t_k\mathscr{X}_k + u_k\mathscr{P}_k$, defined on $\mathscr{S}(\mathbb{R}^n)$, map $\mathscr{S}(\mathbb{R}^n)$ to itself and are essentially self-adjoint for any $(\mathbf{t}, \mathbf{u}) \in \mathbb{R}^{2n}$.*
(b) *$L^2(\mathbb{R}^n, dx)$ is irreducible under the family of bounded operators:*

[4] If the exponentiated operators were $n \times n$ complex matrices the result would follow from the celebrated *Baker-Campbell-Hausdorff formula*: $e^A e^B = e^{[A,B]/2} e^{A+B}$, valid when the matrix $[A, B]$ commutes with both A and B.

$$W((\mathbf{t},\mathbf{u})) := \exp\left\{ i \overline{\sum_{k=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k} \right\} , \quad (\mathbf{t},\mathbf{u}) \in \mathbb{R}^{2n} . \tag{11.34}$$

(c) *The operators W satisfy* **Weyl's relations***:*

$$W((\mathbf{t},\mathbf{u}))W((\mathbf{t}',\mathbf{u}')) = e^{-\frac{i}{2}(\mathbf{t}\cdot\mathbf{u}'-\mathbf{t}'\cdot\mathbf{u})} W((\mathbf{t}+\mathbf{t}',\mathbf{u}+\mathbf{u}')), \quad W((\mathbf{t},\mathbf{u}))^* = W(-(\mathbf{t},\mathbf{u})). \tag{11.35}$$

(d) *For given $(\mathbf{t},\mathbf{u}) \in \mathbb{R}^{2n}$, every mapping $\mathbb{R} \ni s \mapsto W(s(\mathbf{t},\mathbf{u}))$ satisfies:*

$$s\text{-}\lim_{s\to 0} W(s(\mathbf{t},\mathbf{u})) = W(0) . \tag{11.36}$$

Proof Let us begin with $L^2(\mathbb{R}^m, dx)$ where $m = 1$, for the generalisation to finite $m > 1$ is obvious. We will use tools from Sect. 9.1.4. At present we just have the operators $\mathscr{X}$ and $\mathscr{P}$. Both are well defined when restricted to the Schwartz space $\mathscr{S}(\mathbb{R})$, and admit each a self-adjoint extension that coincides with X, P, as previously discussed.

We want to construct a dense subspace of analytic vectors for the symmetric operators $a\mathscr{X} + b\mathscr{P} : \mathscr{S}(\mathbb{R}) \to L^2(\mathbb{R}, dx)$, for any $a, b \in \mathbb{R}$. Define, on the dense domain $\mathscr{S}(\mathbb{R})$, the **annihilation operator**, **creation operator** and **number operator**:

$$A := \frac{1}{\sqrt{2}}\left(\mathscr{X} + \frac{d}{dx}\right) , \quad A^\star := \frac{1}{\sqrt{2}}\left(\mathscr{X} - \frac{d}{dx}\right) , \quad \mathscr{N} := A^\star A . \tag{11.37}$$

By construction $A^* \supset A^\star$, $(A^\star)^* \supset A$, and $\mathscr{N}$ is symmetric. By direct computation the CCRs (or the above definition) force certain commutation relations on $\mathscr{S}(\mathbb{R})$, namely:

$$[A, A^\star]\restriction_{\mathscr{S}(\mathbb{R})} = I\restriction_{\mathscr{S}(\mathbb{R})} . \tag{11.38}$$

It is a well-known fact in the theory of orthogonal polynomials that the complete orthonormal system in $L^2(\mathbb{R}^n, dx)$ of Hermite functions $\{\psi_n\}_{n=0,1,\dots} \subset \mathscr{S}(\mathbb{R})$ (cf. Example 3.32(4)) satisfies $\psi_0 = \pi^{-1/4}e^{-x^2/2}$ and the recursive formula:

$$\psi_{n+1} = (2(n+1))^{-1/2}\left(x - \frac{d}{dx}\right)\psi_n .$$

By definition of $A^\star$, that is the same as saying Hermite functions arise, once ψ_0 is given, from

$$\psi_n = \sqrt{\frac{1}{n!}}(A^\star)^n \psi_0 . \tag{11.39}$$

At the same time a straightforward computation produces

$$A\psi_0 = 0. \tag{11.40}$$

Equations (11.39), (11.40) and (11.38) justify, by induction, the middle relation in the triple (details in Sect. 9.1.4):

$$A^\star\psi_n = \sqrt{n+1}\,\psi_{n+1}\,, \quad A\psi_n = \sqrt{n}\,\psi_{n-1}\,, \quad \mathscr{N}\psi_n = n\psi_n\,, \tag{11.41}$$

The right side in the second one is assumed null if $n = 0$, and as one sees easily the first identity is just the recursive relation introduced a few lines above; the third one follows from the other two.

As the ψ_n are normalised to 1, the first two in (11.41) give the inequality:

$$||A_1 A_2 \cdots A_k \psi_n|| \leq \sqrt{n+1}\sqrt{n+2}\cdots\sqrt{n+k} \leq \sqrt{(n+k)!}\,, \tag{11.42}$$

where every A_i is either A or $A^\star$. Consider a symmetric operator on $\mathscr{S}(\mathbb{R})$ given by an arbitrary real combination $T := a\mathscr{X} + b\mathscr{P}$, $a, b \in \mathbb{R}$. By (11.37), if $z := a + ib$ we have

$$T = \frac{\overline{z}A + zA^\star}{\sqrt{2}}\,. \tag{11.43}$$

This and (11.42) imply, for any Hermite function ψ_n:

$$||T^k\psi_n|| = 2^{-k/2}||(\overline{z}A + zA^\star)^k\psi_n|| \leq 2^{-k/2}2^k|z|^k\sqrt{(n+k)!} = |z|^k\sqrt{2^k(n+k)!}\,.$$

Hence, for $t \geq 0$:

$$\sum_{k=0}^{+\infty}\frac{t^k}{k!}||T^k\psi_n|| \leq \sum_{k=0}^{+\infty}\frac{(\sqrt{2}|z|t)^k\sqrt{(n+k)!}}{k!} \leq \sum_{k=0}^{+\infty}\frac{(\sqrt{2}|z|t)^k\sqrt{(n+k)^n}}{\sqrt{k!}} < +\infty\,.$$

The last series has finite sum by computing the convergence radius r via

$$1/r = \lim_{k\to+\infty}\left(\sqrt{\frac{(n+k)^n}{k!}}\right)^{1/k} = \lim_{k\to+\infty} e^{\frac{n\ln(k+n)}{2k}-\frac{\ln k!}{2k}} = \lim_{k\to+\infty} e^{-\frac{\ln k!}{2k}} = 0$$

(in the end we used Stirling's formula). Therefore any finite combination of Hermite functions is analytic for every $T := a\mathscr{X} + b\mathscr{P}$ on $\mathscr{S}(\mathbb{R})$. As the latter are symmetric, they must be essentially self-adjoint on $\mathscr{S}(\mathbb{R})$ by Nelson's theorem (Theorem 5.47). This ends the proof of case (a) for $m = 1$. If $m > 1$ the argument is similar, keeping in mind that Hermite functions in n variables:

$$\psi_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \psi_{k_1}(x_1)\cdots\psi_{k_m}(x_m)$$

are a complete orthonormal system in $L^2(\mathbb{R}^m, dx)$ (see Example 10.27(1)). What we have seen proves (a), but also (d) and the second identity in (c): in fact

$$W(s(\mathbf{t},\mathbf{u})) = \exp\left\{is\overline{\sum_{k=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k}\right\} = \exp\left\{is\left(\overline{\sum_{k=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k}\right)\right\}$$

by construction, because by definition of closable operator A we have $\overline{sA} = s\overline{A}$ for every $s \in \mathbb{C}$. Now, as $\overline{\sum_{i=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k}$ is self-adjoint, Theorem 9.33(a) ensures the strong continuity of the one-parameter unitary group $\mathbb{R} \ni s \mapsto W(s(\mathbf{t},\mathbf{u}))$, since $W(0) = \exp\left\{i0\overline{\sum_{i=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k}\right\} = I$. The second identity in (c) is obvious since $\mathbb{R} \ni s \mapsto W(s(\mathbf{t},\mathbf{u}))$ is a one-parameter unitary group.

To prove (b) we shall invoke Lemma 11.40, which we will prove after the present theorem but relies only on part (a). Suppose there is a non-null closed space $\mathsf{H}_0 \subseteq L^2(\mathbb{R}^n)$ invariant under $W((\mathbf{t},\mathbf{u}))$, and let $\psi \neq \mathbf{0}$ be an element of H_0. Taking $\phi \in \mathsf{H}_0^{\perp}$, we will prove $\phi = \mathbf{0}$ and so $\mathsf{H}_0 = L^2(\mathbb{R}^n)$. By assumption, H_0 and the orthogonal complement are invariant:

$$(\phi|W((\mathbf{t},0))W((0,\mathbf{u}))\psi) = 0\,, \quad \text{for any } (\mathbf{t},\mathbf{u}) \in \mathbb{R}^{2n}.$$

I.e.

$$\left(\phi\,\middle|\,e^{i\overline{\sum_k t_k \mathscr{X}_k}} e^{i\overline{\sum_k u_k \mathscr{P}_k}}\,\psi\right) = 0\,, \quad \text{for any } (\mathbf{t},\mathbf{u}) \in \mathbb{R}^{2n}.$$

The left side can be computed with (11.50), (11.51) in Lemma 11.40:

$$\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\overline{\phi(\mathbf{x})}\psi(\mathbf{x}+\mathbf{u})\,dx = 0\,, \quad \text{for any } \mathbf{t},\mathbf{u} \in \mathbb{R}^n.$$

Since the map $\mathbf{x} \mapsto h_{\mathbf{u}}(\mathbf{x}) := \overline{\phi(\mathbf{x})}\psi(\mathbf{x}+\mathbf{u})$ is in $L^1(\mathbb{R}^n, dx)$, as product of $L^2(\mathbb{R}^n, dx)$ maps, and given that $\mathbf{t} \in \mathbb{R}^n$ is arbitrary, the identity simply tells that the Fourier transform of $h_{\mathbf{u}} \in L^1(\mathbb{R}^n, dx)$ is zero. By Proposition 3.105(f)$h_{\mathbf{u}}$ is null almost everywhere. In other terms:

$$\overline{\phi(\mathbf{x})}\psi(\mathbf{x}+\mathbf{u}) = 0 \quad \text{almost everywhere for any given } \mathbf{u} \in \mathbb{R}^n. \tag{11.44}$$

Call $E \subset \mathbb{R}^n$ the set on which ψ is not null, and F the set where ϕ never vanishes. (Both are measurable as pre-images of the open set $\mathbb{C} \setminus \{0\}$ under measurable maps.) Denote by m the Lebesgue measure of $\mathbb{R}^n$, so $m(E) > 0$ by assumption. To satisfy (11.44) we must have:

$$m(F\cap(E-\mathbf{u})) = 0 \text{ for any } \mathbf{u} \in \mathbb{R}^n,\ i.e. \quad \int_{\mathbb{R}} \chi_F(\mathbf{x})\chi_E(\mathbf{x}+\mathbf{u})dx = 0 \text{ for any } \mathbf{u} \in \mathbb{R}^n.$$

Integrating in $\mathbf{u}$ gives

$$\int_{\mathbb{R}} du \int_{\mathbb{R}} \chi_F(\mathbf{x})\chi_E(\mathbf{x}+\mathbf{u})dx = 0 \,.$$

As integrands are non-negative and the double integral is finite, we swap integrals by Fubini–Tonelli and use Lebesgue's invariance under translations to obtain:

$$0 = \int_{\mathbb{R}} dx \chi_F(\mathbf{x}) \int_{\mathbb{R}} \chi_E(\mathbf{x}+\mathbf{u})du = \int_{\mathbb{R}} dx \chi_F(\mathbf{x}) \int_{E-\mathbf{x}} 1 du$$
$$= \int_{\mathbb{R}} dx \chi_F(\mathbf{x}) \int_{\mathbb{R}} \chi_E(\mathbf{u})dy = m(F)m(E) \,.$$

As $m(E) > 0$, we have $m(F) = 0$ and so $F = \emptyset$. Therefore ϕ is null almost everywhere, hence the null vector of $L^2(\mathbb{R}^n, dx)$. So $\mathsf{H}_0 = L^2(\mathbb{R}^n, dx)$, proving irreducibility in (b).

There remains to show

$$W((\mathbf{t},\mathbf{u}))W((\mathbf{t}',\mathbf{u}')) = e^{-\frac{i}{2}(\mathbf{t}\cdot\mathbf{u}'-\mathbf{t}'\cdot\mathbf{u})} W((\mathbf{t}+\mathbf{t}',\mathbf{u}+\mathbf{u}')) \,. \tag{11.45}$$

For this we need two steps. Introduce

$$U((\mathbf{t},\mathbf{u})) := e^{\frac{i}{2}(\mathbf{t}\cdot\mathbf{u})} W((\mathbf{t},0))W((0,\mathbf{u})) \,.$$

Step one will prove that

$$U((\mathbf{t},\mathbf{u}))U((\mathbf{t}',\mathbf{u}')) = e^{-\frac{i}{2}(\mathbf{t}\cdot\mathbf{u}'-\mathbf{t}'\cdot\mathbf{u})} U((\mathbf{t}+\mathbf{t}',\mathbf{u}+\mathbf{u}')) \,. \tag{11.46}$$

Step two consists in showing

$$U((\mathbf{t},\mathbf{u})) = W((\mathbf{t},\mathbf{u})) \,, \tag{11.47}$$

which will conclude the overall proof.

Exactly as in part (b), Lemma 11.40 implies:

$$(U((\mathbf{t},\mathbf{u}))\psi)(\mathbf{x}) = e^{\frac{i}{2}\mathbf{t}\cdot\mathbf{u}} e^{i\mathbf{t}\cdot\mathbf{x}} \psi(\mathbf{x}+\mathbf{u}) \,. \tag{11.48}$$

Hence, for given $\psi \in L^2(\mathbb{R}^n, dx)$,

$$U((\mathbf{t},\mathbf{u}))U((\mathbf{t}',\mathbf{u}'))\psi = e^{-\frac{i}{2}(\mathbf{t}\cdot\mathbf{u}'-\mathbf{t}'\cdot\mathbf{u})} U((\mathbf{t}+\mathbf{t}',\mathbf{u}+\mathbf{u}'))\psi \,.$$

This is the same as (11.46), which is eventually justified. Let us pass to (11.47). Consider, for $\mathbf{t}, \mathbf{u}$ fixed, the unitary family $U_s := U(s(\mathbf{t},\mathbf{u}))$, $s \in \mathbb{R}$. Directly from

(11.48) we verify $U_{s+s'} = U_s U_{s'}$ and $U_0 = I$. Therefore $\{U_s\}_{s\in\mathbb{R}}$ is a one-parameter unitary group. The strategy is now to prove that the group is strongly continuous, find its generator and show it coincides with the generator of $\{W(s(\mathbf{t},\mathbf{u}))\}_{s\in\mathbb{R}}$. By Stone's theorem (Theorem 9.33) the two groups will be the same, hence giving (11.47). As for strong continuity, note that for any $\psi, \phi \in L^2(\mathbb{R}^2, dx)$:

$$(\phi|U_s\psi) = e^{\frac{is^2}{2}\mathbf{t}\cdot\mathbf{u}}\left(\phi \left| e^{is\overline{\sum_k t_k \mathscr{X}_k}} e^{is\overline{\sum_k u_k \mathscr{P}_k}}\psi\right.\right)$$

$$= e^{\frac{is^2}{2}\mathbf{t}\cdot\mathbf{u}}\left(e^{is\overline{\sum_k t_k \mathscr{X}_k}}\phi \left| e^{is\overline{\sum_k u_k \mathscr{P}_k}}\psi\right.\right) \to (\phi|\psi) \quad \text{as } s \to 0\,,$$

because the inner product is continuous, and one-parameter groups generated by the self-adjoint operators $\overline{\sum_k u_k \mathscr{P}_k}$ and $\overline{\sum_k t_k \mathscr{X}_k}$ are strongly continuous. Proposition 9.27 guarantees $\{U_s\}_{s\in\mathbb{R}}$ is strongly continuous. Consider $\psi \in \mathscr{S}(\mathbb{R}^n)$, and let us check

$$\lim_{s\to 0}\left|\left|\frac{U_s\psi - \psi}{s} - i\left(\sum_k t_k \mathscr{X}_k + u_k \mathscr{P}_k\right)\psi\right|\right|^2 = 0\,. \tag{11.49}$$

A few passages give

$$\left|\left|\frac{U_s\psi - \psi}{s} - i\left(\sum_k t_k \mathscr{X}_k + u_k \mathscr{P}_k\right)\psi\right|\right|^2$$

$$= \int_{\mathbb{R}^n}\left|\frac{e^{is^2\mathbf{t}\cdot\mathbf{u}/2}e^{is\mathbf{tx}}\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s} - i\mathbf{t}\cdot\mathbf{x}\psi(\mathbf{x}) - \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi\right|^2 dx\,.$$

Hence

$$\left|\left|\frac{U_s\psi - \psi}{s} - i\left(\sum_k t_k \mathscr{X}_k + u_k \mathscr{P}_k\right)\psi\right|\right|^2$$

$$\leq \int_{\mathbb{R}^n}\left|e^{is^2\mathbf{t}\cdot\mathbf{u}/2}e^{is\mathbf{tx}}\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s} - \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi\right|^2 dx$$

$$+2\int_{\mathbb{R}^n}\left|e^{is^2\mathbf{t}\cdot\mathbf{u}/2}e^{is\mathbf{tx}}\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s} - \mathbf{u}\cdot\nabla_{\mathbf{x}}\psi\right|\left|\frac{e^{is^2\mathbf{t}\cdot\mathbf{u}/2}e^{is\mathbf{tx}} - 1}{s\mathbf{t}\cdot\mathbf{x}} - i\right||\mathbf{t}\cdot\mathbf{x}\psi(\mathbf{x})|dx$$

$$+\int_{\mathbb{R}^n}\left|\frac{e^{is^2\mathbf{t}\cdot\mathbf{u}/2}e^{is\mathbf{t}\cdot\mathbf{x}} - 1}{s\mathbf{t}\cdot\mathbf{x}} - i\right|^2|\mathbf{t}\cdot\mathbf{x}\psi(\mathbf{x})|^2 dx\,.$$

Consider the integrals on the right. The middle one, by Schwarz's inequality, tends to zero when the other two do, because its square is less than the product of the other two. By dominated convergence the last integral is infinitesimal as $s \to 0$, because the integrand tends to 0 pointwise and is uniformly bounded by the L^1 map $C|\mathbf{t} \cdot \mathbf{x}\psi(\mathbf{x})|^2$, for some constant $C > 0$. The first integrand also tends to 0 pointwise, as $s \to 0$. We want to use Lebesgue's theorem, so we need an L^1 upper bound, uniform in s around 0 (hence independent of s). Decomposing the integral and recalling $\psi \in \mathscr{S}(\mathbb{R}^n)$, it suffices to find a uniform L^1 bound in $s \in [-\varepsilon, \varepsilon]$ for the expressions

$$\left|\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s}\right| \quad \text{and} \quad \left|\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s}\right|^2$$

in order to obtain a bound of the whole integrand. Assume ψ real (if not, decompose ψ in real and imaginary parts) and invoke the mean value theorem:

$$\left|\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s}\right| = \left|\mathbf{u} \cdot \nabla\psi|_{\mathbf{x}+s_0\mathbf{u}}\right| ,$$

where $s_0 \in [-\varepsilon, \varepsilon]$. Since $\psi \in \mathscr{S}(\mathbb{R}^n)$, for any $p = 1, 2, \ldots$ there is $K_p \geq 0$ with

$$|\mathbf{u} \cdot \nabla\psi|_{\mathbf{x}}| \leq \frac{K_p}{1 + ||\mathbf{x}||^p} .$$

If we fix $\varepsilon > 0$, $\mathbf{u} \in \mathbb{R}^n$ and $p = 2, 3, \ldots,$ there is $C_{p,\varepsilon} > 0$ such that

$$\frac{1}{1 + ||\mathbf{x} + s_0\mathbf{u}||^p} \leq \frac{C_{p,\varepsilon}}{1 + ||\mathbf{x}||^{p-1}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^n,\ s_0 \in [-\varepsilon, \varepsilon].$$

Therefore, for a certain constant $C \geq 0$:

$$\left|\frac{\psi(\mathbf{x}+s\mathbf{u}) - \psi(\mathbf{x})}{s}\right| \leq \frac{C}{1 + ||\mathbf{x}||^{n+1}} , \quad \mathbf{x} \in \mathbb{R}^n,\ s \in [-\varepsilon, \varepsilon].$$

The map on the right and its square are in $L^1(\mathbb{R}^n, dx)$, and this is what we wanted in order to apply Lebesgue's theorem. Hence (11.49) is proved.

Summing up, the self-adjoint generator of the strongly continuous unitary group $\{U(s(\mathbf{t}, \mathbf{u}))\}_{s\in\mathbb{R}}$ coincides with the generator of $\{W(s(\mathbf{t}, \mathbf{u}))\}_{s\in\mathbb{R}}$ on $\mathscr{S}(\mathbb{R}^n)$. Since the second generator is essentially self-adjoint on that space, and as such it admits a unique self-adjoint extension, the generators coincide everywhere. Consequently the groups coincide, for both arise by exponentiating the same self-adjoint generator. $\square$

The proof of parts (b), (c) rely on the following lemma, itself a consequence of (a). We state it aside given its technical usefulness.

Lemma 11.40 *Retaining the assumptions of Proposition 11.39, if $\psi \in L^2(\mathbb{R}^n, dx)$ and $\mathbf{t}, \mathbf{u} \in \mathbb{R}^n$:*

$$\left(e^{i\overline{\sum_k t_k \mathscr{X}_k}}\psi\right)(\mathbf{x}) = e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x}) \,, \tag{11.50}$$

and

$$\left(e^{i\overline{\sum_k u_k \mathscr{P}_k}}\psi\right)(\mathbf{x}) = \psi(\mathbf{x}+\mathbf{u}) \,. \tag{11.51}$$

Proof By direct calculation the group $\{U_s\}_{s\in\mathbb{R}}$,

$$(U_s\psi)(\mathbf{x}) := e^{is\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x}) \,, \quad \forall \psi \in L^2(\mathbb{R}^n, dx)$$

is strongly continuous and satisfies

$$-i \lim_{s\to 0} \frac{1}{s}\left(U_s\psi - \psi\right) = \left(\sum_k t_k \mathscr{X}_k\right)\psi$$

on $\mathscr{S}(\mathbb{R}^n)$. In fact:

$$\left\|\frac{1}{s}\left(U_s\psi - \psi\right) - i\left(\sum_k t_k \mathscr{X}_k\right)\psi\right\|^2 = \int_{\mathbb{R}^3}\left|\frac{e^{is\mathbf{t}\cdot\mathbf{x}}-1}{s} - i\mathbf{t}\cdot\mathbf{x}\right|^2 |\psi(\mathbf{x})|^2 dx$$

$$= \int_{\mathbb{R}^3}\left|\frac{e^{is\mathbf{t}\cdot\mathbf{x}}-1}{s\mathbf{t}\cdot\mathbf{x}} - i\right|^2 |\mathbf{t}\cdot\mathbf{x}|^2|\psi(\mathbf{x})|^2 dx \to 0 \quad \text{as } s \to 0,$$

where we used three ingredients: $\mathbf{x} \mapsto |\mathbf{t}\cdot\mathbf{x}|^2|\psi(\mathbf{x})|^2$ is L^1 as $\psi \in \mathscr{S}(\mathbb{R}^n)$; the map

$$\mathbb{R}\times\mathbb{R}^3 \ni (s,\mathbf{x}) \mapsto \left|\frac{e^{is\mathbf{t}\cdot\mathbf{x}}-1}{s\mathbf{t}\cdot\mathbf{x}} - i\right|^2$$

is bounded and pointwise (in $\mathbf{x}$) tends to 0, $s \to 0$; Lebesgue dominated convergence. By Stone's theorem the generator of $\{U_s\}_{s\in\mathbb{R}}$ is a self-adjoint extension of $\sum_k t_k\mathscr{X}_k$. At the same time, $\sum_k t_k\mathscr{X}_k$ is essentially self-adjoint by (a) in the theorem above, so the unique extension is its closure. Therefore $\{U_s\}_{s\in\mathbb{R}}$ is generated by $\overline{\sum_k t_k\mathscr{X}_k}$, proving (11.50).

Now the second identity. By (3.81)–(3.84), because the Fourier–Plancherel transform $\hat{\mathscr{F}}$ is a Fourier transform $\mathscr{F}$ on the $\mathscr{F}$-invariant space $\mathscr{S}(\mathbb{R}^n)$:

$$\sum_k u_k\mathscr{P}_k = \hat{\mathscr{F}}^{-1}\sum_k u_k\mathscr{K}_k \;\hat{\mathscr{F}}, \tag{11.52}$$

where $\mathscr{K}_k$ is $\mathscr{X}_k$ (the new name just reflects the fact that the variable of the transformed map is $\mathbf{k}$ not $\mathbf{x}$). The Fourier transform is an isomorphism, so

$$\overline{\sum_k u_k \mathscr{P}_k} = \hat{\mathscr{F}}^{-1} \overline{\sum_k u_k \mathscr{K}_k} \; \hat{\mathscr{F}} \, . \tag{11.53}$$

By Corollary 9.37

$$e^{i\overline{\sum_k u_k \mathscr{P}_k}} = \hat{\mathscr{F}}^{-1} \, e^{i\overline{\sum_k u_k \mathscr{K}_k}} \; \hat{\mathscr{F}} \, . \tag{11.54}$$

Reducing to $\psi \in \mathscr{S}(\mathbb{R}^n)$, where $\hat{\mathscr{F}}$ and its inverse are computed by the Fourier integral and reduce to $\mathscr{F}$ and its inverse (cf. Definition 3.103), Eq. (11.54) implies

$$\left(e^{i\overline{\sum_k u_k \mathscr{P}_k}}\psi\right)(\mathbf{x}) = \left(\mathscr{F}^{-1} e^{i\overline{\sum_k u_k \mathscr{K}_k}}\hat{\psi}\right)(\mathbf{x}) = \psi(\mathbf{x}+\mathbf{u}) \, , \quad \text{for any } \psi \in \mathscr{S}(\mathbb{R}^n). \tag{11.55}$$

Recall $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, dx)$, and $\mathscr{S}(\mathbb{R}^n) \ni \psi_n \to \psi$ is in L^2. Then $\psi_n(\cdot + \mathbf{u}) \to \psi(\cdot + \mathbf{u})$ is in L^2, because Lebesgue's measure is translation-invariant and the continuity of $e^{i\overline{\sum_k u_k \mathscr{P}_k}}$ implies (11.51) by (11.55). $\square$

11.5.3 The Theorems of Stone–von Neumann and Mackey

Weyl's relations are valid for bounded operators $W((\mathbf{t}, \mathbf{u}))$, $(\mathbf{t}, \mathbf{u}) \in \mathbb{R}^{2n}$, that form an irreducible set on a complex Hilbert space H, and imply that $s \mapsto W(s(\mathbf{t}, \mathbf{u}))$ are strongly continuous at $s = 0$. We intend to show how they force H to become isomorphic to $L^2(\mathbb{R}^n, dx)$ under the identification sending $W((\mathbf{t}, \mathbf{u}))$ to $e^{i\overline{\sum_k t_k X_k + u_k P_k}}$. In particular, the Hilbert space H turns out to be separable.

The theorem will be stated in a slightly more general form, for which we shall need symplectic geometry.

Let us recall some facts about symplectic vector spaces.

Definition 11.41 A pair (X, σ) is called a (real) **symplectic vector space** if X is a real vector space and the **symplectic form** $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ is a bilinear, skew-symmetric and **weakly non-degenerate** map: $\sigma(u, v) = 0 \; \forall u \in \mathsf{X} \Rightarrow v = 0$.

If (Y, τ) is another symplectic vector space, we call a linear map $f : \mathsf{X} \to \mathsf{Y}$ a **symplectic linear map** if it preserves the symplectic forms: $\tau(f(x), f(y)) = \sigma(x, y)$, $x, y \in \mathsf{X}$.

A **symplectomorphism** is a bijective symplectic linear map.

Note that any symplectic linear map $f : \mathsf{X} \to \mathsf{Y}$ is one-to-one (see Exercise 11.6), so the image $(f(\mathsf{X}), \tau)$ is a symplectic subspace of (Y, τ) isomorphic to (X, σ). If X is a normed space (infinite-dimensional), there exists a stronger concept of *non-degeneracy*: it requires (a) $\sigma(\cdot, v) \in \mathsf{X}'$ for any $v \in \mathsf{X}$, and (b) $\mathsf{X} \ni v \mapsto \sigma(\cdot, v) \in \mathsf{X}'$ is bijective (where X' denotes the topological dual of X). In finite dimension weak non-degeneracy is the same as this strong non-degeneracy.

The next result is due to Darboux (and is related to a more famous theorem on symplectic manifolds, which we shall not be concerned about [FaMa06]).

Theorem 11.42 (Darboux) *If* (X, σ) *is a (real) symplectic vector space with* $\dim \mathsf{X} = 2n$ *finite, there exists a basis (infinitely many, actually), called* **standard symplectic basis***,* $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{f}_1, \cdots, \mathbf{f}_n\} \subset \mathsf{X}$*, in which* σ *assumes the following* **canonical form***:*

$$\sigma(\mathbf{z}, \mathbf{z}') := \left(\sum_{i=1}^{n} t_i u'_i - t'_i u_i \right) \quad \textit{for any } \mathbf{z}, \mathbf{z}' \in \mathsf{X}\,, \tag{11.56}$$

where $\mathbf{z} = \sum_{i=1}^n t_i \mathbf{e}_i + \sum_{i=1}^n u_i \mathbf{f}_i$, $\mathbf{z}' = \sum_{i=1}^n t'_i \mathbf{e}_i + \sum_{i=1}^n u'_i \mathbf{f}_i$.

It is not hard to prove that an automorphism of a symplectic vector space is a symplectomorphism if and only if it preserves Darboux bases.

Now we can state the Stone–von Neumann theorem, whose proof is postponed to after we have introduced Weyl *-algebras. In a dedicated section ensuing the proof we will comment on the mathematics and the physics of the theorem.

Theorem 11.43 (Stone–von Neumann) *Let* H *be a complex non-trivial Hilbert space and* (X, σ) *a real* $2n$*-dimensional symplectic vector space. Suppose* H *admits a family of operators* $\{W(\mathbf{z})\}_{\mathbf{z}\in\mathsf{X}} \subset \mathfrak{B}(\mathsf{H})$ *with the following properties:*
(a) H *is irreducible under* $\{W(\mathbf{z})\}_{\mathbf{z}\in\mathsf{X}}$*;*
(b) *the Weyl relations*

$$W(\mathbf{z})W(\mathbf{z}') = e^{-\frac{i}{2}\sigma(\mathbf{z},\mathbf{z}')}W((\mathbf{z}+\mathbf{z}'))\,, \quad W(\mathbf{z})^* = W(-\mathbf{z})\,, \quad \mathbf{z}, \mathbf{z}' \in \mathsf{X} \tag{11.57}$$

hold;
(c) *for given* $\mathbf{z} \in \mathsf{X}$*, every mapping* $\mathbb{R} \ni s \mapsto W(s\mathbf{z})$ *satisfies*

$$s\text{-}\lim_{s\to 0} W(s\mathbf{z}) = W(0)\,. \tag{11.58}$$

Then, in a given standard symplectic basis of X *for which* $\mathbf{z} \in \mathsf{X}$ *is determined by* $(\mathbf{t}^{(\mathbf{z})}, \mathbf{u}^{(\mathbf{z})}) \in \mathbb{R}^n \times \mathbb{R}^n$*, there exists a Hilbert space isomorphism* $S : \mathsf{H} \to L^2(\mathbb{R}^n, dx)$ *such that:*

$$S\, W(\mathbf{z})\, S^{-1} := \exp\left\{ i \overline{\sum_{k=1}^{n} t_k^{(\mathbf{z})} \mathscr{X}_k + u_k^{(\mathbf{z})} \mathscr{P}_k} \right\}\,, \quad \textit{for any } \mathbf{z} \in \mathsf{X}. \tag{11.59}$$

where the symmetric operators $\mathscr{X}_i$, $\mathscr{P}_i$ *are as of Proposition 11.39.*

Consequently H *must be necessarily separable, as* $L^2(\mathbb{R}^n, dx)$ *is.*

To complement the Stone–von Neumann theorem we state immediately another result, proved by Mackey, that treats reducible representations of the Weyl *-algebra. The notion of Hilbert sum used below is the one found in Definition 3.67.

Theorem 11.44 (Mackey) *Assume the hypotheses of Theorem 11.43, with (a) replaced by one of the following equivalent facts.*
(a1) *Every generator* $W(\mathbf{z})$*,* $\mathbf{z} \in \mathsf{X}$*, has trivial kernel.*

(a2) *Every generator* $W(\mathbf{z})$ *is unitary.*
(a3) $W(0)$ *is the identity operator on* H.
Then the Hilbert space H *is the Hilbert sum of a family (at most countable if* H *is separable) of closed, irreducible and* $W(\mathbf{z})$*-invariant subspaces. On each such component the Stone–von Neumann theorem holds with respect to the restricted operators* $W(\mathbf{z})$.

With the Darboux theorem in mind, an alternative way to formulate the Stone–von Neumann theorem, more often encountered in the literature, goes as follows. Mackey's theorem has a similar reformulation as well, which we omit but the reader can easily reconstruct.

Theorem 11.45 (Alternative version of the Stone–von Neumann theorem) *Let* H *be a complex non-trivial Hilbert space and suppose* $\{U(\mathbf{t})\}_{\mathbf{t}\in\mathbb{R}^n}, \{V(\mathbf{u})\}_{\mathbf{u}\in\mathbb{R}^n} \subset \mathfrak{B}(\mathsf{H})$ *satisfy the following properties.*
(a) H *is irreducible under* $\{U(\mathbf{t})\}_{\mathbf{t}\in\mathbb{R}^n} \cup \{V(\mathbf{u})\}_{\mathbf{u}\in\mathbb{R}^n}$.
(b) *The relations (also called Weyl relations):*

$$U(\mathbf{t})V(\mathbf{u}) = V(\mathbf{u})U(\mathbf{t})e^{i\mathbf{t}\cdot\mathbf{u}} , \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n,$$

$$U(\mathbf{t})U(\mathbf{t}') = U(\mathbf{t}+\mathbf{t}'), \quad V(\mathbf{u})V(\mathbf{u}) = V(\mathbf{u}+\mathbf{u}'), \quad U(\mathbf{t})^* = U(-\mathbf{t}), \quad V(\mathbf{u})^* = V(-\mathbf{u})$$

hold for all $\mathbf{t}, \mathbf{u}, \mathbf{t}'\mathbf{u}' \in \mathbb{R}^n$.
(c) *For any pair* $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^n$*:*

$$s\text{-}\lim_{s\to 0} U(s\mathbf{t}) = U(0) \quad \textit{and} \quad s\text{-}\lim_{s\to 0} V(s\mathbf{u}) = V(0) . \tag{11.60}$$

Then there there exists an isomorphism $S_1 : \mathsf{H} \to L^2(\mathbb{R}^n, dx)$ *such that:*

$$S_1\, U(\mathbf{t})\, S_1^{-1} := \exp\left\{i\overline{\sum_{k=1}^n t_k \mathscr{X}_k}\right\} \quad \textit{and} \quad S_1\, V(\mathbf{u})\, S_1^{-1} := \exp\left\{i\overline{\sum_{k=1}^n u_k \mathscr{P}_k}\right\} .$$

where the symmetric operators $\mathscr{X}_i$*,* $\mathscr{P}_i$ *are defined as in Proposition 11.39.*

Let us explain how the two versions are equivalent. Assume the Hilbert spaces H of the statements are the same. We begin by proving that Theorems 11.43 implies 11.45. From the hypotheses of Theorem 11.45 and its Weyl relations it is immediate to see that the $W((\mathbf{t}, \mathbf{u})) := e^{i\mathbf{t}\cdot\mathbf{u}/2}U(\mathbf{t})V(\mathbf{u})$ fulfil Theorem 11.43 over the symplectic vector space $(\mathbb{R}^n \times \mathbb{R}^n, \sigma_c)$, where σ_c is the symplectic form already in canonical form:

$$\sigma_c((\mathbf{t}, \mathbf{u}), (\mathbf{t}', \mathbf{u}')) = \left(\sum_{i=1}^n t_i u_i' - t_i' u_i\right)$$

in the standard basis of $\mathbb{R}^n \times \mathbb{R}^n$. If we choose the symplectic basis to be the standard one on $\mathbb{R}^n \times \mathbb{R}^n$, then Theorem 11.43 implies Theorem 11.45 by taking $S_1 = S$.

So let us prove Theorems 11.45 implies 11.43. Choose a standard symplectic basis on X and identify elements in X with pairs $(\mathbf{t}, \mathbf{u})$ in $\mathbb{R}^n \times \mathbb{R}^n$. If the $W((\mathbf{t}, \mathbf{u}))$ satisfy Theorem 11.43, then the new operators $V(\mathbf{t}) := W((\mathbf{t}, 0))$ and $U(\mathbf{u}) := W((0, \mathbf{u}))$ fulfil Theorem 11.45. A direct computation shows 11.45 implies Theorem 11.43 for $S = S_1$.

Remarks 11.46 Referring to Definition 11.11, we can give here an elementary example of complete set of commuting observables and von Neumann algebra of observables $\mathfrak{R}_S$ for a quantum particle without spin. As we know $\mathsf{H}_S = L^2(\mathbb{R}^3, d^3x)$. A complete set of commuting observables is the set of the three position operators $\mathbf{A}_1 = \{X_1, X_2, X_3\}$ or the set of the three momenta $\mathbf{A}_2 = \{P_1, P_2, P_3\}$. We leave the elementary proof to the reader. The algebra $\mathfrak{R}_S$ must contain at least the von Neumann algebra generated by $\mathbf{A}_1 \cup \mathbf{A}_2$. We conclude that all self-adjoint operators X_k and P_k are *affiliated* to $\mathfrak{R}_S$ (Proposition 11.8) and therefore $\mathfrak{R}_S$ contains all bounded functions of these operators and linear combinations of these functions. In particular it must contain the operators $U(\mathbf{t})$ and $V(\mathbf{u})$ and so the whole Weyl algebra generated. The commutant of $\mathfrak{R}_S$ is consequently trivial, as it contains a unitary irreducible representation of the Weyl algebra, in view of the first part of Stone–von Neumann theorem. We conclude that $\mathfrak{R}_S = \mathfrak{B}(\mathsf{H}_S)$. ■

*11.5.4 The Weyl *-Algebra*

The statement of the Stone–von Neumann theorem contains an extremely important notion, both for the proof but also in view of developing QM towards Quantum Field Theory. We are talking about *Weyl *-algebras*. Let us spend some time on this.

Definition 11.47 Let X be a real non-trivial vector space of arbitrary dimension (possibly infinite) and $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ a symplectic form on it.

A *-algebra $\mathscr{W}(\mathsf{X}, \sigma)$ is called **Weyl *-algebra of** (X, σ) if there exists a family $\{W(u)\}_{u \in \mathsf{X}}$ of non-zero elements, called **generators** , such that:
(i) Weyl's (commutation) relations:

$$W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)}W(u+v)\,, \quad W(u)^* = W(-u)\,, \quad u, v \in \mathsf{X} \qquad (11.61)$$

hold;
(ii) $\mathscr{W}(\mathsf{X}, \sigma)$ is **generated** by $\{W(u)\}_{u \in \mathsf{X}}$, i.e. $\mathscr{W}(\mathsf{X}, \sigma)$ is the linear span of finite combinations of finite products of $\{W(u)\}_{u \in \mathsf{X}}$.

What we show now, amongst other things, is that a symplectic vector space (X, σ) determines a unique Weyl *-algebra up to *-isomorphisms.

Theorem 11.48 *Let X be a non-trivial real vector space of arbitrary dimension (possibly infinite) and $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ a symplectic form.*

(a) *There exists, always, a Weyl* **-algebra* $\mathscr{W}(\mathsf{X}, \sigma)$ *associated to* (X, σ).
(b) *Any Weyl* **-algebra* $\mathscr{W}(\mathsf{X}, \sigma)$ *has a unit* $\mathbb{I}$, *and:*

$$W(0) = \mathbb{I}\,, \quad W(u)^* = W(-u) = W(u)^{-1}\,,\ u \in \mathsf{X}. \tag{11.62}$$

The generators $\{W(u)\}_{u\in\mathsf{X}}$ *are linearly independent, so in particular* $W(u) \neq W(v)$ *if* $u \neq v$.
(c) *If* $\mathscr{W}(\mathsf{X}, \sigma)$, *generated by* $\{W(u)\}_{u\in\mathsf{X}}$, *and* $\mathscr{W}'(\mathsf{X}, \sigma)$, *generated by* $\{W'(u)\}_{u\in\mathsf{X}}$, *are Weyl* **-algebras of* (X, σ), *there is a unique* **-isomorphism* $\alpha : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}'(\mathsf{X}, \sigma)$, *which is determined by imposing:*

$$\alpha(W(u)) = W'(u)\,, \quad \textit{for any } u \in \mathsf{X}.$$

(d) *Every representation of* $\mathscr{W}(\mathsf{X}, \sigma)$ *on a Hilbert space* $\mathsf{H} \neq \{\mathbf{0}\}$

$$\pi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(\mathsf{H})$$

is faithful.
(e) *Let* $\mathscr{W}(\mathsf{X}', \sigma')$ *be a Weyl* **-algebra of the symplectic vector space* (X', σ').
If $f : \mathsf{X} \to \mathsf{X}'$ *is a symplectic linear map, there exists a* **-homomorphism* (**-isomorphism if* f *is a symplectomorphism*) $\alpha_f : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}(\mathsf{X}', \sigma')$ *that is completely determined by:*

$$\alpha_f(W(u)) = W'(f(u))\,, \quad u \in \mathsf{X} \tag{11.63}$$

(with obvious notation). Furthermore, α_f *is injective.*

Proof (a) Consider the Hilbert space $\mathsf{H} := L^2(\mathsf{X}, \mu)$ where μ is the counting measure of the set X. With $u \in \mathsf{X}$ consider $W(u) \in \mathfrak{B}(L^2(\mathsf{X}, \mu))$ defined by $(W(u)\psi)(v) := e^{i\sigma(u,v)/2}\psi(u + v)$ for any $\psi \in L^2(\mathsf{X}, \mu)$, $v \in \mathsf{X}$. It is immediate that such operators are non-null and satisfy Weyl's commutation relations (11.62), by using Hermitian conjugation as involution. Finite combinations of finite products form a Weyl *-algebra of (X, σ).
(b) From the first equation in (11.61) we have $W(u)W(0) = W(0) = W(0)W(u)$ and $W(u)W(-u) = W(0) = W(-u)W(u)$, because the $W(u)$ do not vanish and generate the whole *-algebra. Hence $W(0) = \mathbb{I}$ and $W(-u) = W(u)^{-1}$. The latter, bearing in mind the second equation in (11.61), implies $W(u)^* = W(u)^{-1}$. Now let us prove the generators' linear independence. Consider a subset of n generators $\{W(u_j)\}_{j=1,\dots,n}$, with $u_1, \dots, u_n$ all distinct, and let us show the $W(u_j)$ are independent. Over arbitrary subsets (and finite combinations) the claim is proved. Consider the null combination $\sum_{j=1}^n a_j W(u_j) = 0$ and let us prove, by induction, $a_j = 0$ for $j = 1, \dots, n$. If $n = 1$ this is true as every $W(u)$ is non-null by definition. Suppose the claim holds for $n - 1$ generators, however chosen, and let us prove the assertion for n. Without loss of generality (relabelling if necessary) we may assume, by contradiction, $a_n \neq 0$. Then $\sum_{j=1}^n a_j W(u_j) = 0$ implies

$$W(u_n) = \sum_{j=1}^{n-1} \frac{-a_j}{a_n} W(u_j) .$$

Consequently

$$\mathbb{I} = W(u_n)^* W(u_n) = \sum_{j=1}^{n-1} \frac{-a_j}{a_n} W(u_n)^* W(u_j) = \sum_{j=1}^{n-1} \frac{-a_j}{a_n} e^{-i\sigma(-u_n,u_j)/2} W(u_j - u_n)$$

$$= \sum_{j=1}^{n-1} b_j W(u_j - u_n) ,$$

where $b_j := \frac{-a_j}{a_n} e^{-i\sigma(-u_n,u_j)/2}$. To prove the claim it suffices to show $b_j = 0$ for every $j = 1, 2, \ldots, n-1$. To do so, let us fix a $u \in \mathsf{X}$, so by the above identity

$$\mathbb{I} = W(u)\mathbb{I}W(-u) = \sum_{j=1}^{n-1} b_j W(u)W(u_j - u_n)W(-u) = \sum_{j=1}^{n-1} b_j e^{-i\sigma(u,u_j-u_n)} W(u_j - u_n) .$$

Comparing the expressions obtained for $\mathbb{I}$ we have

$$\sum_{j=1}^{n-1} b_j W(u_j - u_n) = \sum_{j=1}^{n-1} b_j e^{-i\sigma(u,u_j-u_n)} W(u_j - u_n) .$$

Multiply by $W(u_n)$ and simplify:

$$\sum_{j=1}^{n-1} b_j W(u_j) = \sum_{j=1}^{n-1} b_j e^{-i\sigma(u,u_j-u_n)} W(u_j) .$$

As the generators $W(u_j)$, $j = 1, 2, \ldots, n-1$, are linearly independent, we have $b_j(1 - e^{-i\sigma(u,u_j-u_n)}) = 0$. If $b_j \neq 0$ for some j then we would have $1 = e^{-i\sigma(u,u_j-u_n)}$, and so $\frac{\sigma(u,u_j-u_n)}{2\pi} = k(u) \in \mathbb{Z}$. But the left-hand side is linear in $u \in \mathsf{X}$, so the mapping $\mathsf{X} \ni u \mapsto k(u)$ must be linear. Being $\mathbb{Z}$-valued it is the zero map. Therefore $\sigma(u, u_j - u_n) = 0$ for any $u \in \mathsf{X}$. The non-degeneracy of σ implies $u_j = u_n$ for $j < n$, an absurd.
(c) The Weyl generators are linearly independent, and the product of two is a complex multiple of a generator (by the first Weyl identity), whence generators form a *vector* basis for the Weyl *-algebra. Consider the unique linear map $\alpha : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}'(\mathsf{X}, \sigma)$ defined by $\alpha(W(u)) = W'(u)$ for any $u \in \mathsf{X}$. As $\{W(u)\}_{u\in\mathsf{X}}$ and $\{W'(u)\}_{u\in\mathsf{X}}$ are bases of the corresponding *-algebras, α is a vector-space isomorphism. But products of elements of the two *-algebras are combinations of the generators, by the first set of Weyl relations (the same for both *-algebras), so

α must preserve products. Moreover, $\alpha(W(0)) = W'(0)$ implies that α preserves the multiplicative unit. Eventually $\alpha(W(-u)) = W'(-u)$ and the second Weyl set imply α commutes with involutions as well. The procedure also shows that α is uniquely determined by fixing $\alpha(W(u)) = W'(u)$ for every $u \in \mathsf{X}$.
(d) Consider a representation $\pi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(\mathsf{H})$. By construction the operators $\{\pi(W(u))\}_{u\in\mathsf{X}}$ satisfy Weyl's relations. If every $\pi(W(u))$ is non-null, they define a Weyl *-algebra of (X, σ). By part (c) the representation π, when the codomain restricts to $\pi(\mathscr{W}(\mathsf{X}, \sigma))$, is a *-isomorphism, making π injective. If, on the contrary, $\pi(W(u)) = 0$ for some $u \in \mathsf{X}$, then π is the zero representation. That is because if $z \in \mathsf{X}$, setting $z - u =: v$ implies $\pi(W(z)) = e^{\frac{i}{2}\sigma(u,v)}\pi(W(u))\pi(W(v)) = e^{\frac{i}{2}\sigma(u,v)}0\pi(W(v)) = 0$ by Weyl's relations. Hence π is null as the $W(v)$ form a basis for $\mathscr{W}(\mathsf{X}, \sigma)$. Since the Weyl algebra is a unital *-algebra and $\mathsf{H} \neq \{\mathbf{0}\}$, the zero representation is not admitted Remarks 3.35(3)).
(e) As the generators of the Weyl *-algebra form a basis, as we said in (c), there is one and only one linear map $\alpha_f : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}(\mathsf{X}', \sigma')$, completely determined by (11.63). Using the Weyl relations, and recalling f preserves symplectic forms, we obtain α_f is a *-homomorphism. Its uniqueness is clear, since any *-homomorphism is linear, and (11.63) determine α_f for they fix its values on given bases. Injectivity goes like this: if $\alpha(\sum_i a_i W(u_i)) = 0$ (summing over an arbitrary, finite, set) then $\sum_i a_i\alpha(W(u_i)) = 0$, i.e. $\sum_i a_i W'(f(u_i)) = 0$, where $f(u_i) \neq f(u_j)$ for $i \neq j$ as f is one-to-one (σ' is weakly non-degenerate). Since the $W'(u')$ are linearly independent, $a_i = 0$ for every i and $\alpha(\sum_i a_i W(u_i)) = 0$ implies $\sum_i a_i W(u_i)$, as we wanted. $\square$

Remarks 11.49 **(1)** In the sense of (a), (c) above, the pair (X, σ) and Eq. (11.61) determine the Weyl *-algebra $\mathscr{W}(\mathsf{X}, \sigma)$ of (X, σ) universally (up to *-isomorphisms). Any concrete Weyl *-algebra of (X, σ) made of operators in $\mathfrak{B}(\mathsf{H})$, for a complex Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ and where the involution is the Hermitian conjugation, is sometimes called a **realisation** of the Weyl *-algebra of (X, σ). In other words, in view of Theorem 11.48(c), a realisation of the Weyl *-algebra of (X, σ) over H is an injective linear map $\phi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(\mathsf{H})$ which preserves the product and the involution, and maps $W(0)$ to an operator $\phi(W(0))$ that defines the identity element on the image of ϕ. In formulas, $\phi(W(u))\phi(W(0)) = \phi(W(0))\phi(W(u)) = \phi(W(u))$ for every $u \in \mathsf{X}$. We henceforth use the notation $W_{\mathsf{H}}(u) := \phi(W(u))$.
(2) If $\phi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(\mathsf{H})$ is a realisation of $\mathscr{W}(\mathsf{X}, \sigma)$ over H, *it is in general false that the identity operator $I : \mathsf{H} \to \mathsf{H}$ coincides with the neutral element $W_{\mathsf{H}}(0)$ of the *-algebra $\mathscr{W}_{\mathsf{H}}(\mathsf{X}, \sigma) := \phi(\mathscr{W}(\mathsf{X}, \sigma))$, though this unital *-algebra is isomorphic to $\mathscr{W}(\mathsf{X}, \sigma)$.*

Let us show a simple counterexample. Start from a realisation of $\mathscr{W}(\mathsf{X}, \sigma)$ on a Hilbert space $(\mathsf{H}, (\cdot|\cdot))$ such that $W_{\mathsf{H}}(0) = I$ and consider the Hilbert space $\mathsf{H}' := \mathsf{H} \oplus \mathbb{C}$ with product $\langle(\psi, z)|(\psi', z')\rangle = (\psi|\psi') + \overline{z}z$. A realisation of $\mathscr{W}(\mathsf{X}, \sigma)$ is the unique *-isomorphism ϕ from $\mathscr{W}(\mathsf{X}, \sigma)$ to the *-algebra $\mathscr{W}_{\mathsf{H}'}(\mathsf{X}, \sigma) \subset \mathfrak{B}(\mathsf{H}')$ generated by the operators $W_{\mathsf{H}'}(u) := W_{\mathsf{H}}(u) \oplus 0$, and such that $\phi(W(u)) = W_{\mathsf{H}'}(u)$ for every $u \in \mathsf{X}$. In this case $W_{\mathsf{H}'}(0)$ is not the identity on H', but just the orthogonal projector onto H viewed as closed subspace of H'.

As established in Proposition 3.55, this annoying possibility is just due to the fact that ϕ is *not* a representation of unital *-algebras. If, conversely, ϕ is a *representation*, then $W_{\mathsf{H}}(0)$ is the identity on H just because this is a requirement in the very definition of *-algebra representation (Definition 3.52). An important case are the so-called *GNS representations*, that we will encounter later. They are fundamental in formulations of Quantum Field Theories.

We finally observe that for a Hilbert space realisation, $W_{\mathsf{H}}(0) = I$ *(the realisation is a faithful representation) is equivalent to saying that all operators $W_{\mathsf{H}}(u)$ are unitary*, because $W_{\mathsf{H}}(u)W_{\mathsf{H}}(u)^* = W_{\mathsf{H}}(u)^*W_{\mathsf{H}}(u) = W_{\mathsf{H}}(0)$.

(3) *If $\phi : \mathscr{W}(\mathsf{V}, \sigma) \to \mathfrak{B}(\mathsf{H})$ is a realisation on the Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ of the Weyl *-algebra of (V, σ) and H is irreducible under the generating set $\{W_{\mathsf{H}}(u)\}_{u\in\mathsf{V}}$, then $W_{\mathsf{H}}(0) = I$ and hence the $W_{\mathsf{H}}(u)$ are unitary.*

Here is the proof. Note $W_{\mathsf{H}}(u) \neq 0$, for some $u \in \mathsf{V}$, for otherwise the set of operators $W(u)$ would act reducibly. Since $W_{\mathsf{H}}(u)W_{\mathsf{H}}(0) = W_{\mathsf{H}}(u) = W_{\mathsf{H}}(0)W_{\mathsf{H}}(u)$, $W_{\mathsf{H}}(0)$ is an orthogonal projector that commutes with every $W_{\mathsf{H}}(u)$, so irreducibility forces $W_{\mathsf{H}}(0) = I$ or $W_{\mathsf{H}}(0) = 0$. The latter is impossible because it would imply $W_{\mathsf{H}}(u) = W_{\mathsf{H}}(u + 0) = W_{\mathsf{H}}(u)W_{\mathsf{H}}(0) = 0$ for every $u \in \mathsf{V}$.

(4) *If $\phi : \mathscr{W}(\mathsf{V}, \sigma) \to \mathfrak{B}(\mathsf{H})$ is a realisation on the Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$ of the Weyl *-algebra of (V, σ), then $W_{\mathsf{H}}(0) = I$ and the $W_{\mathsf{H}}(u)$ are unitary precisely when each generator $W_{\mathsf{H}}(u)$ has trivial kernel $\{\mathbf{0}\}$.*

The proof is straightforward. If every $W_{\mathsf{H}}(u)$ has trivial null space, the orthogonal projector $W_{\mathsf{H}}(0)$ has trivial kernel and must coincide with the projector I. Conversely if $W_{\mathsf{H}}(0) = I$ then the $W_{\mathsf{H}}(u)$ are unitary, hence their null spaces are trivial.

(5) The Weyl *-algebra $\mathscr{W}(\mathsf{V}, \sigma)$ of a symplectic vector space (V, σ) admits a norm rendering its Banach completion a unital C^*-algebra: this is the **Weyl C^*-algebra** of (V, σ). Take, for example, the closure of the realisation of (V, σ) on $\mathfrak{B}(L^2(\mathsf{X}, \mu))$ described in the proof of Theorem 11.48(a). The important fact, proved in Chap. 14, is that this C^*-algebra is determined by (V, σ), for one can prove there is a *unique* norm on a Weyl *-algebra satisfying the C^*-identity $||a^*a|| = ||a||^2$. Moreover, the *-isomorphism of Theorem 11.48(c) extends to an (isometric) *-isomorphism of the C^*-algebras. Weyl C^*-algebras are but one starting point to build the quantum theory of bosonic fields [BrRo02]. See [Str05a] for examples of (Weyl) C^*-algebras used in QM. ∎

11.5.5 Proof of the Theorems of Stone–von Neumann and Mackey

In this section we prove the Stone–von Neumann theorem as given by 11.43, and then Mackey's Theorem 11.44. Part of the arguments are a mere reworking of what is found in [Str05a].

Proof of theorem 11.43 *(Stone–von Neumann)* Begin by observing that every operator $W(\mathbf{z}) \in \mathfrak{B}(\mathsf{H})$ is non-zero: for if $W(\mathbf{z}_0) = 0$, for every $\mathbf{z} \in \mathsf{X}$ with $\mathbf{z} - \mathbf{z}_0 =: \mathbf{v}$, we would have $W(\mathbf{z}) = e^{\frac{i}{2}\sigma(\mathbf{z}_0,\mathbf{v})} W(\mathbf{z}_0)W(\mathbf{v}) = e^{\frac{i}{2}\sigma(\mathbf{z}_0,\mathbf{v})} 0 W(\mathbf{v}) = 0$. Then H would not be irreducible under the entire family $W(\mathbf{z}) \in \mathfrak{B}(\mathsf{H})$. By Definition 11.47, the set of $W(\mathbf{z}) \in \mathfrak{B}(\mathsf{H})$ is a generating system for the realisation of the Weyl *-algebra $\mathscr{W}(\mathsf{X}, \sigma)$ of the symplectic vector space (X, σ). This is given by finite combinations of finite products of the $W(\mathbf{z})$, and realised as the image of an irreducible faithful representation $\pi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(\mathsf{H})$ of (X, σ) (notice that irreducibility forces $W(\mathbf{0}) = I$ by Remark 11.49(3)).

Fix a basis in X, so to associate bijectively every $\mathbf{z} \in \mathsf{X}$ to its components $(\mathbf{t}^{(\mathbf{z})}, \mathbf{u}^{(\mathbf{z})}) \in \mathbb{R}^n \times \mathbb{R}^n$. Consider the Hilbert space $L^2(\mathbb{R}^n, dx)$. The family of non-null (unitary) operators $\left\{\exp\left\{i\overline{\sum_{k=1}^n t_k^{(\mathbf{z})} \mathscr{X}_k + u_k^{(\mathbf{z})} \mathscr{P}_k}\right\}\right\}_{\mathbf{z}\in\mathsf{X}}$ defines, by Proposition 11.39, another realisation (irreducible faithful representation) of $\mathscr{W}(\mathsf{X}, \sigma)$ and a corresponding faithful representation $\Pi : \mathscr{W}(\mathsf{X}, \sigma) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$. We denote by $a_{\mathbf{z}} \in \mathscr{W}(\mathsf{X}, \sigma)$ the generators of $\mathscr{W}(\mathsf{X}, \sigma)$, so that $\pi(a_{\mathbf{z}}) = W(\mathbf{z})$ and also

$$\Pi(a_{\mathbf{z}}) = \exp\left\{i\overline{\sum_{k=1}^{n} t_k^{(\mathbf{z})} \mathscr{X}_k + u_k^{(\mathbf{z})} \mathscr{P}_k}\right\} \quad \text{for any } \mathbf{z} \in \mathsf{X}.$$

Suppose now there are two non-zero vectors $\Phi_0 \in \mathsf{H}$, $\Psi_0 \in L^2(\mathbb{R}^n, dx)$ such that: (i) $\mathscr{D} := \pi(\mathscr{W}(\mathsf{X}, \sigma))\Phi_0$ is dense in H, (ii) $\mathscr{D}_1 := \Pi(\mathscr{W}(\mathsf{X}, \sigma))\Psi_0$ is dense in $L^2(\mathbb{R}^n, dx)$, and (iii):

$$(\Phi_0 | \pi(a)\Phi_0) = (\Psi_0 | \Pi(a)\Psi_0) \ , \quad a \in \mathscr{W}(\mathsf{X}, \sigma). \tag{11.64}$$

Let us show that, consequently, there is a linear map $\widetilde{S} : \mathscr{D} \to \mathscr{D}_1$

$$\widetilde{S}\pi(a)\Phi_0 = \Pi(a)\Psi_0 \ , \quad a \in \mathscr{W}(\mathsf{X}, \sigma), \tag{11.65}$$

extending by continuity to a Hilbert isomorphism $\mathsf{H} \to L^2(\mathbb{R}^n, dx)$ satisfying (11.59), and hence proving the theorem.

The mapping is well defined: suppose $\pi(a)\Phi_0 = \pi(b)\Phi_0$. For (11.65) to be well defined we must have $\Pi(a)\Psi_0 = \Pi(b)\Psi_0$. From $\pi(a)\Phi_0 = \pi(b)\Phi_0$ follows, for any $c \in \mathscr{W}(\mathsf{X}, \sigma)$:

$$(\pi(c)\Phi_0 | \pi(a)\Phi_0) = (\pi(c)\Phi_0 | \pi(b)\Phi_0) \ .$$

Since π is a representation of *-algebras, so $\pi(c^*) = \pi(c)^*$ and $\pi(f)\pi(d) = \pi(fd)$, the displayed equation is equivalent to

$$\left(\Phi_0 | \pi(c^* a)\Phi_0\right) = \left(\Phi_0 | \pi(c^* b)\Phi_0\right)$$

and by (11.64) we have $(\Psi_0|\Pi(c^*a)\Psi_0) = (\Psi_0|\Pi(c^*b)\Psi_0)$. Proceeding backwards, for any $c \in \mathscr{W}(\mathsf{X}, \sigma)$:

$$(\Pi(c)\Psi_0|\Pi(a)\Psi_0) = (\Pi(c)\Psi_0|\Pi(b)\Psi_0) \ .$$

As $\Pi(c)\Psi_0$ roams the dense space $\mathscr{D}_1$, necessarily $\Pi(a)\Psi_0 = \Pi(b)\Psi_0$, as required. Therefore $\widetilde{S}$ in (11.65) is well defined. It is immediate to see that $\widetilde{S}$ is linear, for π, Π are representations. By construction $\widetilde{S}$ preserves the inner product, and so is isometric:

$$\left(\widetilde{S}\pi(a)\Phi_0|\widetilde{S}\pi(b)\Phi_0\right) = (\Pi(a)\Psi_0|\Pi(b)\Psi_0) = \left(\Psi_0|\Pi(a)^*\Pi(b)\Psi_0\right)$$

$$= \left(\Psi_0|\Pi(a^*)\Pi(b)\Psi_0\right) = \left(\Psi_0|\Pi(a^*b)\Psi_0\right) = \left(\Phi_0|\pi(a^*b)\Phi_0\right)$$

$$= \left(\Phi_0|\pi(a^*)\pi(b)\Phi_0\right) = \left(\Phi_0|\pi(a)^*\pi(b)\Phi_0\right) = (\pi(a)\Phi_0|\pi(b)\Phi_0) \ .$$

By Proposition 2.47 we can extend, by linearity and continuity, the transformation $\widetilde{S}$ from the dense domain $\mathscr{D}$ to the Hilbert space, obtaining a linear map $S : \mathsf{H} \to L^2(\mathbb{R}^n, dx)$. The extension S stays isometric by inner product's continuity. Similarly, we can construct, first on the dense space $\mathscr{D}_1$, then on $L^2(\mathbb{R}^n, dx)$, a linear isometry $S' : L^2(\mathbb{R}^n, dx) \to \mathsf{H}$ by extending

$$\widetilde{S}'\Pi(a)\Psi_0 = \pi(a)\Phi_0 \quad \text{for any } a \in \mathscr{W}(\mathsf{X}, \sigma). \tag{11.66}$$

Since $\widetilde{S}\widetilde{S}' = I_{\mathscr{D}_1}$, $\widetilde{S}'\widetilde{S} = I_{\mathscr{D}}$ on the dense spaces $\mathscr{D}_1$, $\mathscr{D}$, these are valid by continuity on the extended domains: $SS' = I_{L^2(\mathbb{R}^n,dx)}$, $S'S = I_{\mathsf{H}}$. Overall, $S : \mathsf{H} \to L^2(\mathbb{R}^n, dx)$ is a Hilbert isomorphism satisfying

$$S\pi(a)\Phi_0 = \Pi(a)\Psi_0 \quad \text{for any } a \in \mathscr{W}(\mathsf{X}, \sigma). \tag{11.67}$$

Invert the identity for $b \in \mathscr{W}(\mathsf{X}, \sigma)$ to obtain $\pi(b)\Phi_0 = S^{-1}\Pi(b)\Psi_0$. Substituting in (11.67), and replacing $\pi(a)$ by $\pi(ab) = \pi(a)\pi(b)$ on the left and $\Pi(a)$ by $\Pi(ab) = \Pi(a)\Pi(b)$ on the right, finally produces:

$$S\pi(a)S^{-1}\,\Pi(b)\Psi_0 = \Pi(a)\,\Pi(b)\Psi_0 \ .$$

The vectors $\Pi(b)\Psi_0$ define a dense space in $L^2(\mathbb{R}^n, dx)$, so

$$S\pi(a)S^{-1} = \Pi(a) \quad \text{for any } a \in \mathscr{W}(\mathsf{X}, \sigma) \ .$$

Picking as $a \in \mathscr{W}(\mathsf{X}, \sigma)$ a generic Weyl generator transforms the identity into (11.59).

To end the proof we have to exhibit vectors Φ_0, Ψ_0 satisfying (11.64) and generating, under the respective representations, dense subspaces. Let $\Phi_0 \in \mathsf{H}$ be any non-zero vector. The closed space $\overline{\pi(\mathscr{W}(\mathsf{X}, \sigma))\Phi_0}$ is invariant under any $\pi(a)$, and in particular under any $\pi(W(\mathbf{z}))$, by construction. Since H is irreducible under these vectors, then $\overline{\pi(\mathscr{W}(\mathsf{X},\sigma))\Phi_0} = \mathsf{H}$, i.e. $\mathscr{D} := \pi(\mathscr{W}(\mathsf{X},\sigma))\Phi_0$ is dense in H. A similar argument says $\mathscr{D}_1 := \Pi(\mathscr{W}(\mathsf{X},\sigma))\Psi_0$ is dense in $L^2(\mathbb{R}^n, dx)$ for every non-zero $\Psi_0 \in L^2(\mathbb{R}^n, dx)$. There remains to determine Φ_0, Ψ_0 fulfilling (11.64). Consider in $L^2(\mathbb{R}^n, dx)$ the vector

$$\Psi_0(\mathbf{x}) = \psi_0(x_1)\cdots\psi_0(x_n) = \pi^{-n/4}e^{-|\mathbf{x}|^2/2}$$

where ψ_0 is the first Hermite function. A straightforward calculation based on Lemma 11.31 gives

$$\left(\Psi_0 \left| \exp\left\{ i\overline{\sum_{k=1}^n t_k\mathscr{X}_k + u_k\mathscr{P}_k}\right\} \Psi_0 \right.\right) = \pi^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}e^{-|\mathbf{x}+\mathbf{u}|^2/2}dx = e^{-|\mathbf{t}|^2/4-|\mathbf{u}|^2/4}$$

and so

$$\left(\Psi_0 \left| \exp\left\{ i\overline{\sum_{k=1}^n t_k\mathscr{X}_k + u_k\mathscr{P}_k}\right\} \Psi_0 \right.\right) = e^{-(|\mathbf{t}|^2+|\mathbf{u}|^2)/4}\,, \quad \text{for any } (\mathbf{t},\mathbf{u}) \in \mathbb{R}^n\times\mathbb{R}^n. \tag{11.68}$$

If we manage to find a vector $\Phi_0 \in \mathsf{H}$ such that

$$(\Phi_0\,|W(\mathbf{z})\;\Phi_0) = e^{-(|\mathbf{t}^{(\mathbf{z})}|^2+|\mathbf{u}^{(\mathbf{z})}|^2)/4}\,, \quad \text{for any } \mathbf{z}\in\mathsf{X}, \tag{11.69}$$

then (11.64) holds by linearity, as any $\Pi(a)$ is a combination of elements $\Pi(a_{\mathbf{z}})$ and the corresponding $\pi(a)$ is a combination (same coefficients) of elements $\pi(a_{\mathbf{z}})$. At this point the existence of Φ_0 is warranted by the next proposition.

Proposition 11.50 *Under the assumptions of Theorem 11.43, if a (standard symplectic) basis on* X *has been fixed so to map every* $\mathbf{z}\in\mathsf{X}$ *to its components* $(\mathbf{t}^{(\mathbf{z})},\mathbf{u}^{(\mathbf{z})})\in\mathbb{R}^n\times\mathbb{R}^n$*, there exists* $\Phi_0\in\mathsf{H}$ *satisfying (11.69).*

Proof First, the operators $W(\mathbf{z})$ are unitary with $W(0) = I$, by Remark 11.49(3) and because H is $W(\mathbf{z})$-irreducible. We claim $\mathsf{X}\ni\mathbf{z}\mapsto W(\mathbf{z})$ is continuous in the strong topology (the regularity assumption s-$\lim_{s\to 0} W(s\mathbf{z}) = W(0) = I$ is only apparently weaker than strong continuity at $\mathbf{z}=0$, since the limit might not be uniform along directions tending to the origin). Set $W((\mathbf{t}^{(\mathbf{z})},\mathbf{u}^{(\mathbf{z})})) := W(\mathbf{z})$ in the sequel. Let us begin by proving $\mathbb{R}^n\ni\mathbf{t}\mapsto W((\mathbf{t},0))$ and $\mathbb{R}^n\ni\mathbf{u}\mapsto W((0,\mathbf{t}))$ are strongly continuous. We will prove it for $\mathbb{R}^n\ni\mathbf{t}\mapsto W((\mathbf{t},0))$ only, as the other case is identical. Weyl's relations imply additivity: $W((\mathbf{t},0))W((\mathbf{t}',0)) = W((\mathbf{t}+\mathbf{t}',0))$. If $\mathbf{e}_1,\ldots,\mathbf{e}_n$ are the basis vectors expressing $\mathbf{t}=\sum_{k=1}^n t_k\mathbf{e}_k$ we can write $W((\mathbf{t},0)) = W((t_1\mathbf{e}_1,0))\cdots W((t_n\mathbf{e}_n,0))$. Each map $\mathbb{R}\ni t_k\mapsto W((t_k\mathbf{e}_k,0))$ is strongly continuous by regularity, i.e. s-$\lim_{s\to 0} W(s\mathbf{z}) = W(0) = I$ in Theorem 11.43. Take $\psi\in\mathsf{H}$ and let us show $||W((\mathbf{t},0))\psi-\psi||\to 0$ as $\mathbf{t}\to 0$. We have

$$||W((\mathbf{t},0))\psi - \psi|| = \left|\left|\prod_{k=1}^{n} W((t_k\mathbf{e}_k,0))\psi - \psi\right|\right|$$

$$\leq \left|\left|\prod_{k=1}^{n} W((t_k\mathbf{e}_k,0))\psi - \prod_{k=1}^{n-1} W((t_k\mathbf{e}_k,0))\psi\right|\right|$$

$$+\left|\left|\prod_{k=1}^{n-1} W((t_k\mathbf{e}_k,0))\psi - \prod_{k=1}^{n-2} W((t_k\mathbf{e}_k,0))\psi\right|\right| + \cdots + ||W((t_1\mathbf{e}_1,0))\psi - \psi||$$

$$= ||W((t_n\mathbf{e}_n,0))\psi - \psi|| + \left|\left|W((t_{n-1}\mathbf{e}_{n-1},0))\psi - \psi\right|\right| + \cdots + ||W((t_1\mathbf{e}_1,0))\psi - \psi|| \ .$$

In the last passage we used that $W((t_k\mathbf{e}_k,0))$ is unitary, so it preserves the norm; in particular

$$\left|\left|\prod_{k=1}^{n} W((t_k\mathbf{e}_k,0))\psi - \prod_{k=1}^{n-1} W((t_k\mathbf{e}_k,0))\psi\right|\right| = \left|\left|\prod_{k=1}^{n-1} W((t_k\mathbf{e}_k,0))\left(W((t_n\mathbf{e}_n,0))\psi - \psi\right)\right|\right|$$

$$= ||W((t_n\mathbf{e}_n,0))\psi - \psi|| \ .$$

The inequality

$$||W((\mathbf{t},0))\psi - \psi|| \leq \sum_{k=1}^{n} ||W((t_k\mathbf{e}_k,0))\psi - \psi||$$

and the continuity of $W((t_k\mathbf{e}_k,0))\psi$ for $t_k \to 0$ imply

$$W((\mathbf{t},0))\psi \to \psi \quad \text{as } \mathbf{t} \to 0,$$

working on products of intervals along the Cartesian axes as neighbourhoods of $\mathbf{z} = 0$. Therefore the function $\mathsf{X} \ni \mathbf{z} \mapsto (\phi_1\,|W(\mathbf{z})\,\phi_2) = \big(W((\mathbf{t}^{(\mathbf{z})},0))^*\phi_1\,|\,W((0,\mathbf{u}^{(\mathbf{z})}))\,\phi_2\big)$ is continuous at $\mathbf{z} = 0$ for any ϕ_1, ϕ_2. Hence $\mathsf{X} \ni \mathbf{z} \mapsto W(\mathbf{z})$ is strongly continuous everywhere, in fact

$$||W(\mathbf{z})\phi - W(\mathbf{z}_0)\phi||^2 = ||e^{i\sigma(\mathbf{z}_0,\mathbf{z})/2}W(\mathbf{z}-\mathbf{z}_0)\phi - \phi||^2$$

$$= 2||\phi||^2 - e^{-i\sigma(\mathbf{z}_0,\mathbf{z})/2}\overline{(\phi|W(\mathbf{z}-\mathbf{z}_0)\phi)} - e^{i\sigma(\mathbf{z}_0,\mathbf{z})/2}(\phi|W(\mathbf{z}-\mathbf{z}_0)\phi) \to 0 \text{ as } \mathbf{z} \to \mathbf{z}_0,$$

for any $\phi \in \mathsf{H}$, by the Weyl relations because $W(\mathbf{z})$ are unitary. We can then apply Proposition 9.31 and define

$$P := (2\pi)^{-n}\int_{\mathbb{R}^{2n}} dz e^{-|\mathbf{z}|^2/4}W(\mathbf{z}). \tag{11.70}$$

By construction $P \in \mathfrak{B}(\mathsf{H})$, and Proposition 9.31 implies $P^* = P$:

$$\left(\phi_1 \,\middle|\, P^*\phi_2\right) = \overline{(\phi_2|P\phi_1)} = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-|\mathbf{z}|^2/4}\overline{(\phi_2\,|W(\mathbf{z})\;\phi_1)}\,dz$$

$$= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-|\mathbf{z}|^2/4}\,(\phi_1\,|W(\mathbf{z})\;\phi_2)\;dz = (\phi_1|P\phi_2)\;,$$

where we used

$$\overline{(\phi_2\,|W(\mathbf{z})\;\phi_1)} = (W(\mathbf{z})\phi_1\mid\phi_2) = \left(\phi_1\mid W(\mathbf{z})^*\phi_2\right) = (\phi_1\mid W(-\mathbf{z})\phi_2)\;,$$

and that the measure dz and $\exp -|\mathbf{z}^2/4|$ are unchanged by the reflection $\mathbf{z}\to-\mathbf{z}$. Notice $P\neq 0$, for otherwise

$$0 = \left(\phi_1|W(\mathbf{z}')PW(\mathbf{z}')\phi_2\right) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-|\mathbf{z}|^2/4}\left(\phi_1\,\middle|W(\mathbf{z}')W(\mathbf{z})W(\mathbf{z}')\;\phi_2\right)\;dz$$

i.e., by Weyl's relations:

$$0 = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-|\mathbf{z}|^2/4}e^{i\mathbf{t}^{(\mathbf{z}')}\cdot\mathbf{t}^{(\mathbf{z})}-i\mathbf{u}^{(\mathbf{z}')}\cdot\mathbf{u}^{(\mathbf{z})}}\,(\phi_1\,|W(\mathbf{z})\;\phi_2)\;dz\quad\forall\mathbf{z}'\in\mathbb{R}^{2n}\;,$$

In other terms the Fourier transform of the L^1 function

$$\mathbf{z}\mapsto e^{-|\mathbf{z}|^2/4}\,(\phi_1\,|W(\mathbf{z})\;\phi_2)$$

is null. Then by Proposition 3.105(f) $\mathbf{z}\mapsto(\phi_1\,|W(\mathbf{z})\;\phi_2) = 0$ almost everywhere. Since the map is continuous it must vanish everywhere, so $W(\mathbf{z}) = 0$. As earlier said this cannot be. To finish the proof we need to justify:

$$PW(\mathbf{z})P = e^{-|\mathbf{z}|^2/4}P\;. \tag{11.71}$$

Indeed, choosing $\mathbf{z}=0$ in (11.71) gives $PP = P$, making P a non-null orthogonal projector. If $\Phi_0\in P(\mathsf{H})\setminus\{\mathbf{0}\}$ with $||\Phi_0|| = 1$, as $P\Phi_0 = \Phi_0$, Eq. (11.71) implies, for any $\mathbf{z}\in\mathsf{X}$

$$(\Phi_0|W(\mathbf{z})\Phi_0) = e^{-|\mathbf{z}|^2/4} = e^{-(|\mathbf{t}^{(\mathbf{z})}|^2+|\mathbf{u}^{(\mathbf{z})}|^2)/4}\,.$$

Hence our Φ_0 satisfies (11.69), as requested.

Let us conclude the proof by establishing (11.71). By definition of P, Proposition 9.31(b) and Weyl's relations give

$$(2\pi)^nPW(\mathbf{z})P = \int_{\mathbb{R}^n}dz'e^{-\mathbf{z}'^2/4}PW(\mathbf{z})W(\mathbf{z}') = \int_{\mathbb{R}^{2n}}dz'e^{-\mathbf{z}'^2/4}e^{-i\sigma(\mathbf{z},\mathbf{z}')/2}PW(\mathbf{z}+\mathbf{z}')\,.$$

Recalling (11.70) we can solve for P the integrand. By Proposition 9.31:

$$(\phi_1|PW(\mathbf{z})P\phi_2)$$

$$= \frac{1}{(2\pi)^{2n}} \int dz' dz'' e^{-(\mathbf{z}'^2+\mathbf{z}''^2)/4} e^{-i\sigma(\mathbf{z},\mathbf{z}')/2} e^{-i\sigma(\mathbf{z}'',\mathbf{z}+\mathbf{z}')/2} \left(\phi_1|W(\mathbf{z}+\mathbf{z}'+\mathbf{z}'')\phi_2\right) \tag{11.72}$$

for any $\phi_1, \phi_2 \in \mathsf{H}$. We have passed from an iterated integral to an integral in the product measure using Fubini–Tonelli. This is possible because the integrand vanishes absolutely: it decays exponentially as the product measure's variables go to infinity, due to the exponentials and the estimate $|\left(\phi_1|W(\mathbf{z}+\mathbf{z}'+\mathbf{z}'')\phi_2\right)| \leq ||\phi_1|| \, ||\phi_2||$. Set $\mathbf{z} = (\alpha, \beta)$, $\mathbf{z}' = (\gamma', \delta')$ and $\mathbf{z}'' = (\gamma, \delta)$. The right side of (11.72) reads:

$$\int_{\mathbb{R}^{4n}} \frac{d\gamma d\delta d\gamma' d\delta'}{(2\pi)^{2n}} e^{-(|\gamma|^2-|\delta|^2-|\gamma'|^2-|\delta'|^2)/4} e^{-\frac{i}{2}(\alpha\cdot\delta'-\beta\cdot\gamma'+\gamma\cdot\beta+\gamma\cdot\delta'-\delta\cdot\alpha-\delta\cdot\gamma')}$$

$$\times \left(\phi_1 \left| W((\alpha+\gamma+\gamma', \beta+\delta+\delta')) \, \phi_2\right.\right)$$

Changing variables to $\kappa, \nu, \mu, \lambda \in \mathbb{R}^n$, where $\gamma = (\kappa+\mu-\alpha)/2$, $\gamma' = (\kappa-\mu-\alpha)/2$, $\delta = (\nu+\lambda-\beta)/2$, $\delta' = (\nu-\lambda-\beta)/2$, the integral can be computed explicitly, because the integrals in μ, λ decouple to produce Gaussian integrals. The right-hand side of (11.72) equals, eventually:

$$\frac{e^{-(|\alpha|^2+|\beta|^2)/4}}{(2\pi)^n} \int_{\mathbb{R}^{2n}} d\kappa d\nu e^{-(|\kappa|^2+|\nu|^2)/4} \left(\phi_1 \left| W((\kappa, \nu)) \, \phi_2\right.\right) = e^{-|\mathbf{z}|^2/4} (\phi_1|P\phi_2)$$

which produces (11.71) since $\phi_1, \phi_2 \in \mathsf{H}$ are free. $\square$

This concludes the proof of Theorem 11.43 (Stone–von Neumann). $\square$

Proof of theorem 11.44 *(Mackey).* The hypotheses (a1), (a2), (a3) are equivalent because of Remark 11.49(3), (4). With those assumptions the $W(\mathbf{z})$ are unitary, with $W(0) = I$. Then we can go through the proof of Proposition 11.50 – which only used that the $W(\mathbf{z})$ were unitary with $W(0) = I$, and did not rely on the representation's irreducibility – and build the orthogonal projector $P \neq 0$:

$$P = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} dz e^{-|\mathbf{z}|^2/4} W(\mathbf{z}) \, , \quad \text{for any } \mathbf{z} \in \mathbb{R}^{2n} \, ,$$

so that every $\Phi_0 \in P(\mathsf{H})$ satisfies

$$(\Phi_0|W(\mathbf{z})\Phi_0) = e^{-|\mathbf{z}|^2/4} = e^{-(|\mathbf{t}^{(\mathbf{z})}|^2+|\mathbf{u}^{(\mathbf{z})}|^2)/4} \, ,$$

as we have seen. First consider the closed space $\mathsf{H}_0 := \overline{< \{W(\mathbf{z})P(\mathsf{H})\}_{z\in\mathsf{X}} >}$ proving that $\mathsf{H}_0 = \mathsf{H}$. By construction H_0 is invariant under $W(\mathbf{z})$. Then $\mathsf{H}_0^\perp$ is also invariant. If

$\mathsf{H}_0^\perp \neq \{\mathbf{0}\}$, working in $\mathsf{H}_0^\perp$ as ambient Hilbert space, using the restrictions $W(\mathbf{z})\restriction_{\mathsf{H}_0^\perp}$ (note $W(0)\restriction_{\mathsf{H}_0^\perp} = I\restriction_{\mathsf{H}_0^\perp} \neq 0$ if $\mathsf{H}_0^\perp \neq \{\mathbf{0}\}$), we construct the unique orthogonal projector $P' \neq 0$ such that

$$(\phi_1'|P'\phi_2') = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{-|\mathbf{z}|^2/4}\left(\phi_1'\,|W(\mathbf{z})\;\phi_2'\right)dz, \quad \text{for any } \mathbf{z}\in\mathbb{R}^{2n}, \phi_1,\phi_2\in\mathsf{H}_0^\perp.$$

We know the integral on the right equals $(\phi_1'|P\phi_2')$, i.e. zero, because $\phi_2' \in \mathsf{H}_0^\perp = (P(\mathsf{H}))^\perp$. Hence $P' = 0$, but this contradicts $P' \neq 0$. Hence $\mathsf{H}_0^\perp = \{\mathbf{0}\}$, and so $\mathsf{H}_0 = \mathsf{H}$.

To conclude, take a Hilbert basis $\{\Phi_k\}_{k\in I}$ of $P(\mathsf{H})$ and consider the closed spaces $\mathsf{H}_k := \overline{<\{W(\mathbf{z})\Phi_k\}_{\mathbf{z}\in\mathsf{X}}>}$ invariant under $W(\mathbf{z})$. Notice $\Phi_k \in \mathsf{H}_k$, since $W(0) = I$, so $\mathsf{H}_k \neq \{\mathbf{0}\}$ for any $k \in I$. By (11.71):

$$(\Phi_j|W(\mathbf{z})\Phi_k) = (\Phi_j|PW(\mathbf{z})P\Phi_k) = e^{-|\mathbf{z}|^2/4}(\Phi_j|P\Phi_k) = 0 \quad \text{if } j \neq k\,.$$

We have found closed subspaces $\mathsf{H}_j \neq \{\mathbf{0}\}$ that are mutually orthogonal (in particular j varies in a countable set if H is separable). By construction, as

$$\overline{<\{W(\mathbf{z})P(\mathsf{H})\}_{z\in\mathsf{X}}>} = \mathsf{H}$$

and $\{\Phi_k\}_{k\in I}$ is a basis in $P(\mathsf{H})$, the space of finite combinations of vectors in the mutually orthogonal H_k is dense in H. Therefore H is the Hilbert sum $\oplus_{k\in I}\mathsf{H}_k$ of the closed spaces H_k, $k \in I$ (Definition 3.67). To finish, on every H_k we can replicate the proof of Stone–von Neumann with H replaced by H_k and $\pi : \mathscr{W}(\mathsf{X},\sigma) \to \mathfrak{B}(\mathsf{H})$ replaced by $\pi_k : \mathscr{W}(\mathsf{X},\sigma) \to \mathfrak{B}(\mathsf{H}_k)$, restriction of the image of each operator in $\pi(\mathscr{W}(\mathsf{X},\sigma))$ to H_k. The only difference is that now $\pi_k(\mathscr{W}(\mathsf{X},\sigma))\Phi_k$ is dense in H_k by assumption, whereas in the theorem it descends from the irreducibility of $\pi_k(\mathscr{W}(\mathsf{X},\sigma))$. Therefore the restriction $\pi_k(\mathscr{W}(\mathsf{X},\sigma))$ of $\pi(\mathscr{W}(\mathsf{X},\sigma))$ to H_k is isomorphic – under a suitable Hilbert space isomorphism $S_k : \mathsf{H}_k \to L^2(\mathbb{R}^n, dx)$ – to the standard representation Π of the Weyl algebra on $L^2(\mathbb{R}^n, dx)$ such that

$$S_k\pi_k(a)S_k^{-1} = \Pi(a)\,, \quad \forall a \in \mathscr{W}(\mathsf{X},\sigma)\,.$$

As Π is irreducible, so must be every π_k. This ends the proof. □

11.5.6 More on "Heisenberg's Principle": Weakening the Assumptions and the Extension to Mixed States

The formalism developed to prove the Stone–von Neumann theorem allows to generalise Theorem 11.33, i.e. Heisenberg's principle, by taking weaker assumptions on the set to which ψ belongs (the existence of $(\Delta X_i)_\psi$, $(\Delta P_j)_\psi$ suffices). It also enables to extend it to cover mixed states. Let us begin with a technical lemma.

Lemma 11.51 *Let X_i, P_j be the position and momentum operators of axiom* **A5***, and define $X'_i := X_i + a_i I$, $P'_j := P_j + b_j I$, with $a_i, b_j \in \mathbb{R}$. If $\psi, \phi \in D(X_i) \cap D(P_j)$ then the CCRs*

$$(X'_i\psi|P'_j\varphi) - (P'_j\psi|X'_i\varphi) = i\hbar\delta_{ij}(\psi|\varphi) \tag{11.73}$$

hold, here written using quadratic forms.

Proof Notice $D(X_i) = D(X'_i)$, $D(P_j) = D(P'_j)$. In case $a_i, b_j = 0$, consider (11.35), so

$$(W((-\mathbf{t}, \mathbf{0}))\psi\,|W((\mathbf{0}, \mathbf{u}))\varphi) - (W((\mathbf{0}, -\mathbf{u}))\psi\,|W((\mathbf{t}, \mathbf{0}))\varphi)$$

$$= (1 - e^{-i(\mathbf{t}\cdot\mathbf{u})/2})\,(W((-\mathbf{t}, \mathbf{0}))\psi\,|W((\mathbf{0}, \mathbf{u}))\varphi)\ .$$

Using Stone's theorem $(X'_i\psi|P'_j\varphi) - (P'_j\psi|X'_i\varphi) = i\hbar\delta_{ij}(\psi|\varphi)$. Add $a_i I$ and $b_j I$ to the operators inside the inner products on the left. Since the X_i, P_j are Hermitian, the terms on the right cancel out, yielding (11.73) in the general case. □

Theorem 11.52 *Let X_i and P_j be the position and momentum operators of axiom* **A5***. If the unit vector $\psi \in \mathsf{H}_S$ is such that $(\Delta X_i)_\psi$ and $(\Delta P_i)_\psi$ exist, then Heisenberg's principle holds:*

$$(\Delta X_i)_\psi(\Delta P_i)_\psi \geq \hbar/2\ .$$

Proof By part (i) in Proposition 11.27(a) if $(\Delta X_i)_\psi$ and $(\Delta P_i)_\psi$ are defined then $\psi \in D(X_i) \cap D(P_i)$. Referring to Lemma 11.51 we choose $a_i = -(\psi|X_i\psi)$, $b_j = -(\psi|P_i\psi)$. By definition of standard deviation (11.11) and Theorem 9.4(f) we have $(\Delta X_i)^2_\psi = \int(\lambda - a_i)^2 d\mu^{(A)}_\psi(\lambda) = ||X'_i\psi||^2$. Similarly, $||P'_i\psi||^2 = (\Delta P_i)^2_\psi$. On the other hand (for any a_i, b_i) from (11.73) we infer:

$$||X'_i\psi||||P'_i\psi|| \geq |(X'_i\psi|P'_i\psi)| \geq |Im(X'_i\psi|P'_i\psi)| = \frac{\hbar}{2}\ . \tag{11.74}$$

Since $(\Delta X_i)_\psi(\Delta P_i)_\psi = ||X'_i\psi||||P'_i\psi||$, the claim is proved. □

So now we can extend "Heisenberg's principle" to mixed states as well.

Theorem 11.53 *Let X_i and P_j be the position and momentum operators of axiom* **A5***. If ρ is a mixed state for the spinless particle such that $(\Delta X_i)_\rho$ and $(\Delta P_i)_\rho$ exist, then:*

$$(\Delta X_i)_\rho(\Delta P_i)_\rho \geq \frac{\hbar}{2}\ .$$

Proof Let us notice, preliminarily, that if $(\Delta X_i)_\rho$ and $(\Delta P_i)_\rho$ can be defined, then also $\langle(X_i)^k\rangle_\rho$ and $\langle(P_i)^k\rangle_\rho$, $k = 0, 1, 2$, are defined, as is easy to see using Definition 11.25, because measures are finite. Furthermore, $Ran(\rho) \subset D(X_i) \cap D(P_i)$

by (ii) in Proposition 11.27(b), as $Ran(\rho^{1/2}) \supset Ran(\rho)$. Set $X'_i := X_i + a_i I$, $P'_i := P_i + b_i I$, and choose $a_i := -\langle X_i \rangle_\rho$, $b_i := -\langle P_i \rangle_\rho$. A direct computation relying on Definition 11.25 tells that $(\Delta X_i)^2_\rho = \langle (X'_i)^2 \rangle_\rho$ and $(\Delta P_i)^2_\rho = \langle (P'_i)^2 \rangle_\rho$. Write $\rho = \sum_n p_n \psi_n (\psi_n | \ )$ in a basis of unit eigenvectors. We argue as in Proposition 11.27 when we proved (11.19). As $A = X'_i, P'_i$ and $f(\lambda) = \lambda$, and since $\mu^{(A)}_\rho(E) = tr(P^{(A)}(E)\rho) = \sum_n p_n \mu_{\psi_n}(E)$, using that $p_n \geq 0$ we can prove:

$$\int |f(\lambda)|^2 d\mu^{(A)}_\rho(\lambda) = \sum_{n=0}^{+\infty} p_n \int |f(\lambda)|^2 d\mu^{(A)}_{\psi_n}(\lambda) = \sum_{n=0}^{+\infty} p_n (f(A)\psi_n | f(A)\psi_n) \leq +\infty \,,$$

where $\psi_n \in D(X'_i) \cap D(P'_i) = D(X_i) \cap D(P_i)$, because $\psi_n \in Ran(\rho) \subset D(X_i) \cap D(P_i)$. Therefore:

$$(\Delta X_i)^2_\rho = \langle (X'_i)^2 \rangle_\rho = \sum_n p_n (X'_i \psi_n | X'_i \psi_n) \quad \text{and} \quad (\Delta P_i)^2_\rho = \langle (P'_i)^2 \rangle_\rho = \sum_m p_m (P'_i \psi_m | P'_i \psi_m)\,.$$

Schwarz's inequality plus (11.74) imply the claim, because

$$\langle (X'_i)^2 \rangle^{1/2}_\rho \langle (P'_i)^2 \rangle^{1/2}_\rho \geq \sum_n p_n^{1/2} p_n^{1/2} (X'_i \psi_n | X'_i \psi_n)^{1/2} (P'_i \psi_n | P'_i \psi_n)^{1/2} \geq \sum_n p_n \frac{\hbar}{2} = \frac{\hbar}{2}$$

(note $p_n \geq 0$ and $\sum_n p_n = 1$). $\square$

11.5.7 The Stone–von Neumann Theorem Revisited: Weyl–Heisenberg Group

Our approach to the proof of Stone–von Neumann relies on the structure of (Weyl) *-algebra. There is, however, another point of view, due to Weyl, in which the **Weyl–Heisenberg group** plays the algebra's role. The Weyl–Heisenberg group in $\mathbb{R}^{2n+1}$, which we shall indicate by $H(n)$, is the simply connected Lie group diffeomorphic to $\mathbb{R}^{2n+1}$ with product law

$$(\eta, \mathbf{t}, \mathbf{u}) \circ (\eta', \mathbf{t}', \mathbf{u}') = \left(\eta + \eta' + \frac{1}{2} \sum_{i=1}^{n} u_i t'_i - u'_i t_i \ , \ \mathbf{t} + \mathbf{t}' \ , \ \mathbf{u} + \mathbf{u}' \right)$$

(as usual $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^n$ whilst $\eta \in \mathbb{R}$). A direct computation of its Lie algebra shows there is a basis of $2n + 1$ generators $\mathbf{x}_i$, $\mathbf{p}_i$, $\mathbf{e}$, $i = 1, 2, \ldots, n$ that satisfy:

$$[\mathbf{x}_i, \mathbf{p}_j] = \delta_{ij} \mathbf{e} \,, \quad [\mathbf{x}_i, \mathbf{e}] = [\mathbf{p}_i, \mathbf{e}] = 0 \,, \quad i, j = 1, 2, \ldots, n \,.$$

The linear mapping determined by $\mathbf{e} \mapsto -iI$, $\mathbf{x}_k \mapsto -iX_k$, $\mathbf{p}_k \mapsto -iP_k$ is an isomorphism from the Heisenberg Lie algebra to the Lie algebra of finite real combinations of the conjugate self-adjoint operators $-iI$, $-iX_k$, $-iP_k$, restricted to the common, dense and invariant domain $\mathscr{S}(\mathbb{R}^n)$, with commutator $[\cdot, \cdot]$ as Lie bracket.

This map induces a Lie group isomorphism. By direct inspection, in fact, if the operators $W((\mathbf{t}, \mathbf{u}))$ are defined by Proposition 11.39, the map

$$\mathbb{R}^{2n+1} \ni (\eta, \mathbf{t}, \mathbf{u}) \mapsto e^{i\eta} W((\mathbf{t}, \mathbf{u})) =: H((\eta, \mathbf{t}, \mathbf{u})) \tag{11.75}$$

is an irreducible unitary representation of the $(2n+1)$-dimensional Weyl–Heisenberg group on $L^2(\mathbb{R}^n, dx)$ (not faithful since $H(\eta, \mathbf{t}, \mathbf{u}) = H(\eta + 2k\pi, \mathbf{t}, \mathbf{u})$ if $k \in \mathbb{Z}$). Moreover,

$$\text{s-}\lim_{s\to 0} H(s(\eta, \mathbf{t}, \mathbf{u})) = I \quad \text{for any given } (\eta, \mathbf{t}, \mathbf{u}) \in \mathbb{R}^{2n+1}\,. \tag{11.76}$$

Conversely,

Proposition 11.54 *An irreducible unitary representation of the Weyl–Heisenberg group $H(n) \ni (\eta, \mathbf{t}, \mathbf{u}) \mapsto H((\eta, \mathbf{t}, \mathbf{u}))$ on the non-trivial complex Hilbert space H, satisfying (11.76), has the form (11.75):*

$$H((\eta, \mathbf{t}, \mathbf{u})) = e^{ic\eta} W((\mathbf{t}, \mathbf{u}))\,,$$

where the $W((\mathbf{t}, \mathbf{u}))$ satisfy the Stone–von Neumann Theorem 11.43 with σ replaced by $c\sigma$ and the constant $c \in \mathbb{R} \setminus \{0\}$ is determined by $H((1, \mathbf{0}, \mathbf{0})) = e^{ic} I$.

Representations H, H' over respective Hilbert spaces H, H' with different values of c are unitarily inequivalent: there is no unitary operator $U : \mathsf{H} \to \mathsf{H}'$ such that $UH((\eta, \mathbf{t}, \mathbf{u}))U^{-1} = H'((\eta, \mathbf{t}, \mathbf{u}))$ for every $(\eta, \mathbf{t}, \mathbf{u}) \in H(n)$.

Proof The centre $\mathbb{R}$ of the Weyl–Heisenberg group is represented by a unitary Abelian subgroup. As the elements of $\mathbb{R}$ commute with the Weyl–Heisenberg group, every element $H((\eta, \mathbf{0}, \mathbf{0}))$ commutes with the whole representation. But the latter is irreducible, so Schur's lemma forces $H((\eta, \mathbf{0}, \mathbf{0})) = \chi(\eta) I$, with $\chi(\eta) \in \mathbb{C}$, and $|\chi(\eta)| = 1$ as $H((\eta, \mathbf{0}, \mathbf{0}))$ is unitary. Eventually, since $\eta \to H((\eta, \mathbf{0}, \mathbf{0}))$ is strongly continuous, Stone's theorem implies $\chi(\eta) = e^{ic\eta}$ for every $\eta \in \mathbb{R}$ and some constant c so that $H((1, \mathbf{0}, \mathbf{0})) = e^{ic} I$ in particular. The group's commutation rules require $c \neq 0$, but also ensure that the $W((\mathbf{t}, \mathbf{u})) := e^{-ic\eta} H((\eta, \mathbf{t}, \mathbf{u})) = H((\eta, \mathbf{0}, \mathbf{0}))^{-1} H((\eta, \mathbf{t}, \mathbf{u})) = H((1, \mathbf{0}, \mathbf{0}))^{-1} H((0, \mathbf{t}, \mathbf{u}))$ obey Weyl's relations as in Theorem 11.43 with σ replaced by $c\sigma$, the proof being elementary.

If H and H' are representations associated to constants $c \neq c'$, there is no unitary operator U such that $UH(u)U^{-1} = H'(u)$ for every u, because choosing $u = (\eta, \mathbf{0}, \mathbf{0})$ we would find $e^{ic\eta} = e^{ic'\eta}$ for every $\eta \in \mathbb{R}$, which is impossible. □

The absolute value of the constant c equals Planck's constant $\hbar$, as will become evident in the next theorem. The fact that the sign of c can be reversed and we still have a representation of the Weyl–Heisenberg group is related to the fact that the operation of time reversal ($\mathbf{x}_i \to \mathbf{x}_i$, $\mathbf{p}_i \to -\mathbf{p}_i$) produces a representation of the Weyl–Heisenberg group. In this framework we have an alternative statement of Stone–von Neumann, first proved by Weyl.

Theorem 11.55 *Every irreducible unitary representation of the Weyl–Heisenberg group $H(n)$ satisfying (11.76) is unitarily equivalent to the representation:*

$$\mathbb{R}^{2n+1} \ni (\eta, \mathbf{t}, \mathbf{u}) \mapsto e^{ic\eta} W((\mathbf{t}, \mathbf{u}))$$

on $L^2(\mathbb{R}^n, dx)$, where the $W((\mathbf{t}, \mathbf{u}))$ are the operators of Proposition 11.39 but the $\mathscr{P}_k$ now are:

$$(\mathscr{P}_k \psi)(\mathbf{x}) = -ic \frac{\partial \psi}{\partial x_k}(\mathbf{x})$$

for some constant $c \neq 0$.

Proof It is an immediate consequence of Proposition 11.54 and the first identity in (11.35). □

Remarks 11.56 **(1)** The Stone–von Neumann theorem proves that the non-relativistic elementary particle with spin 0 is described by an *irreducible* representation of a certain Lie group. The same happens for particles with spin, charge, *etc.*, provided one picks the right group. *Elementary systems* are therefore described by *irreducible representations* of a group, which is typically related to the symmetries of the physical system. This point of view has proved – thanks to Wigner in particular – incredibly rewarding for the development of relativistic quantum theories, where irreducible representations of the Poincaré group are employed to define elementary particles, and irreducibility is a characteristic feature of elementary systems.
(2) There exist more or less rigorous formulations of the Stone–von Neumann theorem that rely only on Heisenberg's relations (11.28) and do not need exponentials. To set up these formulations, though, the technical assumptions on domains (spaces of analytic vectors) and on the existence of self-adjoint extensions are neither obvious, nor is their physical meaning straightforward. Beside the foundational work of E. Nelson [Nel59], an important and thorough result is that of J. Dixmier [Dix56], which we shall return to in the next chapter. In a nutshell, the theorem, generalised to an arbitrary finite dimension, states that if P, Q are symmetric on a dense invariant space on which Heisenberg's relations hold, and on that same space $P^2 + Q^2$ is essentially self-adjoint, then P, Q give a strongly continuous representation of the Weyl algebra on the Hilbert space; hence, up to isomorphisms, P, Q have the usual form on a (Hilbert sum of spaces) $L^2(\mathbb{R}, dx)$. ■

11.5.8 Dirac's Correspondence Principle, Weyl's Calculus and Deformation Quantisation

The formulation of QM we have presented leaves open the question of how to pick out operators on H that correspond to observables of physical interest, other than position and momentum. Several respected authors have written much about procedures allowing to pass from relevant classical observables to major quantum observables.

But that is somewhat like fighting a losing battle: from a physical perspective Quantum Mechanics is 'more central' than Classical Mechanics, whence the latter should be seen as a limiting case of the former. Even this fact is by no means easy to prove, apart in a few general cases: one such is *Ehrenfest's theorem*, whose precise mathematical formulation was found only recently [FrKo09]. Therefore one expects there should be quantum entities, observables in particular, without classical counterparts (for instance the "parity" of elementary particles, and in many respects also the spin).

That said, certain quantum observables for the spinless particle will, in principle, be "functions" of the observables X_i, P_i. The common belief is that the quantum quantity corresponding to the classical $F(x, p)$ should look something like $F(X, P)$. But going down this road is a real challenge, more than what mathematics prospects, as already partially discussed (see Sect. 11.3.2). In fact: (1) is it not at all obvious what meaning one should assign to a function of X and P when these operators have non-commuting spectral measures (in the commuting case there are ways out that use *joint measures*, like (11.3)); (2) naïve recipes in this direction do not produce self-adjoint, not even symmetric, operators when the operators do not commute.

For the sake of clarity, let us consider the classical quantity $x \cdot p$. Which observable – i.e. *self-adjoint* operator – should it correspond to? Passing to the spectral measures is ill-advised, because they do not commute. So let us try to use the operators themselves, restricted to an invariant and dense subspace where they are both defined. The hope is to produce an essentially self-adjoint operator, or at least symmetric, and then in some way or another choose among its self-adjoint extensions (if any at all, in case the operator is symmetric). The tentative answer:

"$x \cdot p$ corresponds to $X \cdot P (= \sum_{i=1}^{n} X_i P_i)$"

is totally inadequate, even if we view the operators on the invariant dense space $\mathscr{S}(\mathbb{R}^3)$. That is because $X \cdot P$ is not symmetric on $\mathscr{S}(\mathbb{R}^3)$, for X_i and P_i do not commute (exercise). Nor would it make sense to seek self-adjoint extensions of $X \cdot P$.

Another possibility is to associate to $x \cdot p$ the symmetric operator $(X \cdot P + P \cdot X)/2$ defined on $\mathscr{S}(\mathbb{R}^3)$, and study its self-adjoint extensions. When examining more complicated situations, like $x_k^2 p_k$, this recipe reveals itself very ambiguous, because a priori there are several possibilities: $(X_k^2 P_k + P_k X_k^2)/2$ is symmetric on the domain $\mathscr{S}(\mathbb{R}^3)$, but also $X_k(X_k P_k + P_k X_k)/4 + (X_k P_k + P_k X_k)X_k/4$ is, and there are others. These choices correspond to "symmetrised" products, of sorts, of (non-commuting) operators, that should produce an operator that is at least symmetric.

A criterion, helpful but not decisive to solve the issues raised, was found by Dirac, and goes under the accepted name of "Dirac's correspondence principle". A short but clear technical analysis of Dirac's correspondence principle can be found in [Str12]. Here we shall only consider some elementary issues. To present Dirac's correspondence principle, let us recall that a **Lie algebra** $(\mathsf{V}, [\cdot, \cdot])$ is a vector space (here, over $\mathbb{R}$) equipped with a skew-symmetric bilinear map $[\cdot, \cdot] : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$, called a **Lie bracket**, that satisfies the Jacobi identity:

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 , \quad \text{for any } u, v, w \in \mathsf{V}.$$

In studying the phase space $\mathscr{F}$ of the classical particle (although the setup is fully general), Dirac considered the real vector space $\mathscr{G}(\mathscr{F})$ of sufficiently regular maps $\mathscr{F} \to \mathbb{R}$ with *Poisson bracket*:

$$\{f, g\} := \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}\,, \quad f, g \in \mathscr{G}(\mathscr{F})\,.$$

He observed that $(\mathscr{G}(\mathscr{F}), \{\cdot,\cdot\})$ is a *Lie algebra*. In particular the CCRs:

$$\{x_i, p_j\} = \delta_{ij}$$

hold. These equations are Heisenberg's relations once we substitute $x_i \to X_i$, $p_i \to P_i$ and $\{\cdot,\cdot\} \to -i\hbar^{-1}[\cdot,\cdot]$. The idea behind "Dirac's correspondence principle" is the following.

Let $\hat{f}$ denote the quantum analogue (an operator at least symmetric, and defined on a dense invariant domain, irrespective of the specific quantity) of the generic classical quantity $f \in \mathscr{G}(\mathscr{F})$. Under Dirac's correspondence, if

$$h = \{f, g\}$$

for classical $f, g, h \in \mathscr{G}(\mathscr{F})$, the corresponding $\hat{f}, \hat{g}, \hat{h}$ in the quantum realm satisfy

$$\hat{h} = -i\hbar^{-1}[\hat{f}, \hat{g}]\,.$$

Even though it is not very often declared explicitly, it is also assumed that the map $f \mapsto \hat{f}$ is linear and that the constant function 1 is mapped to the identity operator. Just as an example, consider the usual classical particle. The components of the classical angular momentum

$$l_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} x_j p_k$$

correspond to

$$L_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} X_j P_k\,,$$

which are essentially self-adjoint operators on $\mathscr{S}(\mathbb{R}^3)$. The classical commutation relations

$$\{l_i, l_j\} = \sum_{k=1}^{3} \varepsilon_{ijk} l_k$$

have quantum counterparts on $\mathscr{S}(\mathbb{R}^3)$:

$$[L_i, L_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} L_k \, .$$

Dirac's principle could be explained for observables corresponding to the generators of unitary transformations in a symmetry group of the system; these, though, do not exhaust all possible observables (all this might be more enlightening after reading the book's final three chapters). In that case it is only natural to request that (a) the Lie algebra of the symmetry group, (b) the Lie algebra of the unitary representation of transformations on the quantum system, and (c) the Lie algebra of generators of the group of classical canonical transformations that correspond to symmetries of the classical system, be all isomorphic.

Although we will not push the study any further, we have to mention that serious technical hurdles crop up if one pursues Dirac's idea literally. Suppose, in particular, of working with polynomial functions of arbitrarily large degree in the canonical variables x_i, p_j. Then [Stre07] it is not possible to define a "symmetrised product" of self-adjoint operators corresponding to canonical variables (so to produce operators that are at least symmetric) that does not depend on the degree and that yields the isomorphism $f \mapsto \hat{f}$. More generally, the **Groenewold—van Hove theorem** establishes that the Lie algebra P of real polynomials in the variables $(x_1, \dots, x_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$ equipped with the canonical Poisson bracket $\{\cdot, \cdot\}$ has no *quantisation map*: this would be a linear map $Q : h \mapsto \hat{h}$ from P to a vector space of symmetric operators defined on a common invariant and dense domain in some Hilbert space H, satisfying some natural requirements (for a survey see [Got99]). In particular $Q(1) = I$ and, obviously,

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}) \tag{11.77}$$

as assumed by Dirac's correspondence principle.

In spite of the inherent difficulties of Dirac's original principle, some of the underlying ideas have found a rigorous treatment within quantisation procedures called *Weyl quantisation* or *Weyl calculus* (in particular see [Jef04], [ZFC05], [Gra04] and [DA10]). The following formula, proved by Weyl and based on the Fourier transform, tells how to associate to a function $f = f(x_1, \dots, x_n, p_1, \dots, p_n)$ the operator

$$f(X_1, \dots, X_n, P_1, \dots, P_n)$$

$$:= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \exp\left\{ -i \overline{\sum_{k=1}^{n} t_k \mathscr{X}_k + u_k \mathscr{P}_k} \right\} \widetilde{f}(t_1, \dots, t_n, u_1, \dots, u_n) dt du \, .$$

This expression is a function of the operators $X_1, \dots, X_n, P_1, \dots, P_n$, $\widetilde{f}$ is the Fourier transform of f, and the integral is meant in the sense of Proposition 9.31, assuming f is suitable, e.g. a Schwartz function on $\mathbb{R}^{2n}$. By using duality theorems the definition extends to Schwartz distributions f, for which also polynomial functions can be considered (see Sect. 2 in [Jef04] for a brief and precise technical account of Weyl calculus, and Chap. 1 for other, related procedures). Weyl's procedure does provide operators that can be viewed as functions of the non-commuting X_k, P_k, but still has problems. First of all, it maps real functions to self-adjoint operators, but it does not preserve positivity (positive functions are not mapped, in general, to positive operators). Furthermore, it maps the Poisson bracket of two polynomials to the commutator of the corresponding functions of operators only if the polynomials are at most quadratic.

Deformation quantisation [BFFL78] provides another interesting way out, and of different nature. It keeps the space of classical observables P, but deforms the classical *commutative* product $\cdot$ to a quantum noncommutative product $*$ by means of a (formal) power series in $\hbar$ of the form

$$f * g = f \cdot g + \hbar i\{f, g\} + \hbar^2 G_2(f, g) + \cdots . \tag{11.78}$$

Within this picture $Q(f) = f$, but condition (11.77) is weakened into

$$[Q(f), Q(g)] = f * g - g * f = i\hbar Q(\{f, g\}) + O(\hbar^2) , \tag{11.79}$$

and the Groenewold–van Hove obstruction is avoided.

Remark 11.57 Expression (11.78) suggests that in the classical limit $\hbar \to 0$ the quantum noncommutativity of observables disappears, but a relic of the noncommutative structure survives in the *classical* Poisson structure. ■

In conclusion, despite interesting and remarkable technical attempts, the broad validity of Dirac's correspondence principle is shaky. Its many snags are critical, apparently inescapable and borne by the attempt to provide a serious framework to Dirac's original idea, even in its most rigorous versions such as Weyl's calculus. That said, the extension of Dirac's idea to deformation quantisation seems to be very fruitful also in Quantum Field Theory [BrFr09].

Exercises

11.1 Prove that if $A : D(A) \to \mathsf{H}$ is closable and affiliated to the von Neumann algebra $\mathfrak{R}$, then $\overline{A}$ is affiliated to $\mathfrak{R}$.

Hint. It is sufficient to prove that $U\overline{A}x = \overline{A}Ux$ for every $x \in D(\overline{A})$ and every $U \in \mathfrak{R}'$. To this end, consider $x \in D(\overline{A})$ and a sequence $D(A) \ni x_n \to x$ such that $\overline{A}x_n = Ax_n \to \overline{A}x$, then use the fact that $\overline{A}$ is closed and U continuous.

11.2 Prove that, if $A : D(A) \to \mathsf{H}$ is densely defined and affiliated to the von Neumann algebra $\mathfrak{R}$, then A^* is affiliated to $\mathfrak{R}$.

Solution. As $UA \subset AU$, taking adjoints gives $A^*U^* \supset U^*A^*$, since U^* is a generic unitary element of $\mathfrak{R}'$, and the thesis is proved.

11.3 Prove Proposition 11.8.
Proposition. Let $\mathfrak{R}$ be a von Neumann algebra over the complex Hilbert space H and $A : D(A) \to \mathsf{H}$ a closed operator with $D(A) \subset \mathsf{H}$ dense. The following facts are equivalent.
(a) $A\eta\mathfrak{R}$.
(b) If $A = VP$ is the polar decomposition of A, then
(i) $V \in \mathfrak{R}$,
(ii) $P\eta\mathfrak{R}$.
If A is self-adjoint, (a) and (b) are equivalent to
(c) the PVM of A satisfies $P_E^{(A)} \in \mathfrak{R}$ for every Borel set $E \subset \mathbb{R}$.
If $A \in \mathfrak{B}(\mathsf{H})$, then (a) and (b) are equivalent to
(d) $A \in \mathfrak{R}$.

Solution. (a) $\Leftrightarrow$ (b). If (a) holds, that is $A = VP$ is affiliated to $\mathfrak{R}$, then $UA = AU$ for every $U \in \mathfrak{R}'$ unitary. Taking adjoints gives $A^*U^* = U^*A^*$ which, as $\mathfrak{R}'$ is *-closed, means $UA^* = A^*U$ for every $U \in \mathfrak{R}'$ unitary. Consequently $UA^*A = UA^*U^*UA = A^*AU$. In other words $U \in \mathfrak{R}'$ commutes with the operator A^*A which is self-adjoint (Theorem 10.36). Due to Theorem 9.41(iii), U commutes with the PVM of A^*A and with every function $f(A^*A)$. In particular U commutes with $P = \sqrt{A^*A}$, which is therefore affiliated to $\mathfrak{R}$. Notice that $UP = PU$ implies in particular that $U(Ran(P)) \subset Ran(P)$ and $U(Ker(P)) \subset Ker(P)$. From the polar decomposition $A = VP$ and from $UA = AU$ we also have $UVP = VPU$, so that $UVP = VUP$. This means $UVx - VUx = 0$ if $x \in Ran(P)$ and also $x \in \overline{Ran(P)}$ by continuity. However $\mathsf{H} = Ker(P) \oplus \overline{Ran(P)}$ and $Ker(P) = Ker(V)$ (Theorem 10.39(b) where V is called U). We conclude that $UV - VU = 0$ for every $U \in \mathfrak{R}'$. Let $Q \in \mathfrak{R}'$ be an orthogonal projector. Then $U := iQ - i(I - Q)$ is unitary and belongs to $\mathfrak{R}'$. The condition $UV = VU$ immediately produces $QV = VQ$, so we conclude that V commutes with every orthogonal projector of $\mathfrak{R}'$. Therefore, exploiting Proposition 7.61 and the fact that $S''' = S'$ we find $V \in (\mathscr{L}_{\mathfrak{R}'}(\mathsf{H}))' = (\mathscr{L}_{\mathfrak{R}'}(\mathsf{H}))''' = (\mathfrak{R}')' = \mathfrak{R}$. We have established (a) $\Rightarrow$ (b). If, conversely, (b) is true and $U \in \mathfrak{R}'$, then $UA = UVP = VUP = VPU = AU$, so (b) $\Rightarrow$ (a).
(a) $\Leftrightarrow$ (c). Suppose that $A = A^*$. The operator A is affiliated to $\mathfrak{R}$ iff $UA = AU$ for every $U \in \mathfrak{R}'$. Theorem 9.41(iii) guarantees that this is equivalent to the fact that the PVMs of A and U commute. This concludes the proof that (a) $\Leftrightarrow$ (c).
(d) $\Leftrightarrow$ (a). If $A \in \mathfrak{B}(\mathsf{H})$, then $A \in \mathfrak{R} \Rightarrow A$ is affiliated to $\mathfrak{R}$ immediately. If $A \in \mathfrak{B}(\mathsf{H})$ is affiliated to $\mathfrak{R}$, by decomposing $A = B + iC$ with $B, C \in \mathfrak{B}(\mathsf{H})$ self-adjoint, and using the fact that A^* is also affiliated to $\mathfrak{R}$, both B and C turn out to be affiliated to $\mathfrak{R}$. Now (c) proves that the spectral measures of B, C belong to $\mathfrak{R}$. Since $\mathfrak{R}$ is closed in the strong operator topology, this entails that B, C (which are strong limits of linear combinations of orthogonal projectors in $\mathfrak{R}$) stay in $\mathfrak{R}$. Hence $A = B + iC$ stays in $\mathfrak{R}$ as well.

11.4 Prove that, if $\dim(\mathsf{H}) = n < +\infty$, there are no operators $A, B : \mathsf{H} \to \mathsf{H}$ such that $[A, B] = cI$ for any $c \in \mathbb{C} \setminus \{0\}$. (Do not use Proposition 11.32.)

Solution. If such operators A, B existed, one would have $0 = tr(AB) - tr(AB) = tr(AB) - tr(BA) = tr([A, B]) = tr(cI) = nc$.

11.5 Consider a particle moving on the real line, and suppose the pure state represented by the differentiable function $\psi \in D(X^2) \cap D(P^2) \cap D(XP) \cap D(PX)$, with $||\psi|| = 1$, satisfies $(\Delta X)_\psi (\Delta P)_\psi = \hbar/2$. Prove

$$\psi(x) = (\pi \hbar \gamma)^{-1/4} e^{i \frac{\langle P \rangle_\psi x}{\hbar}} e^{-\frac{(x - \langle X \rangle_\psi)^2}{2\hbar\gamma}}$$

for some $\gamma > 0$.

Hint. We refer to the proof of Theorem 11.33, and note that we can have $(\Delta X)_\psi (\Delta P)_\psi = \hbar/2$ only if $||X'\psi|| ||P'\psi|| = |(X'\psi|P'\psi)|$, plus $Re(X'\psi|P'\psi) = 0$. The first condition implies, by Proposition 3.3(i), that $X'\psi = cP'\psi$ for some $c \in \mathbb{C}$. Since $\sigma_p(X) = \emptyset$ and $\psi \neq \mathbf{0}$, the second condition implies $Re(c) = 0$. Solving the differential equation $X'\psi = i Im(c) P'\psi$, and using $||\psi|| = 1$, leads to the required expression for ψ.

11.6 Prove that a symplectic linear map $f : (\mathsf{X}, \sigma) \to (\mathsf{X}', \sigma')$ is one-to-one.

Solution. Remember that symplectic forms are weakly non-degenerate, and $f(x) = 0 \Rightarrow \sigma(y, x) = \sigma'(f(y), 0) = 0$ for any $y \in \mathsf{X}$ so that $x = 0$.

11.7 Consider the Hilbert space $\mathsf{H} := L^2([a, b], dx)$ and the self-adjoint operator X on H defined by $(X\psi)(x) := x\psi(x)$, for any $\psi \in \mathsf{H}$ such that $X\psi \in \mathsf{H}$. Prove there is no self-adjoint extension P of the symmetric operator $-i\frac{d}{dx}$, defined on the subspace of C^1 maps either vanishing, or periodic, at the boundary of $[a, b]$, so that the one-parameter unitary groups $U(u) := e^{iuX}$, $V(v) := e^{ivP}$ satisfy Weyl's relations: $U(u)V(v) = V(v)U(u)e^{iuv}$ for any $u, v \in \mathbb{R}$.

Hint. First note that, trivially, $V(sv), U(su) \to I$ in strong sense, as $s \to 0$, because one-parameter unitary groups generated by self-adjoint operators are strongly continuous. There are various ways to solve the exercise. For example we can prove $\sigma(X) = [a, b]$. This is impossible if P as above exists, because by Theorem 11.45 there should be a unitary operator S mapping X and P to the operators on $L^2(\mathbb{R}, dx)$ of axiom **A5** (passing from $\mathbb{R}^3$ to $\mathbb{R}^1$ in the obvious way). Another possibility is to split $L^2([a, b], dx)$ in a Hilbert sum of closed, X- and P-invariant spaces, on each of which there is the aforementioned unitary operator S. In either case we can prove $\sigma(X) = \sigma(SXS^{-1}) = \mathbb{R} \neq [a, b]$.

11.8 Refer to the proof of Proposition 11.39 and adapt the definitions of $A, A^\star$ by considering $L^2(\mathbb{R}, dx)$ with Hermite functions $\{\psi_n\}_{n\in\mathbb{N}}$ as basis, and the Bargmann–Hilbert space B_1 (see Example 3.32(6)) with entire functions $\{u_n\}_{n\in\mathbb{N}}$ as basis:

$$u_n(z) := \frac{z^n}{\sqrt{n!}} \quad \text{for any } z \in \mathbb{C}.$$

Call **Segal–Bargmann transformation** the unitary operator

$$U : L^2(\mathbb{R}, dx) \to \mathsf{B}_1$$

determined by $U\psi_n := u_n$, $n = 0, 1, 2, \ldots$. Prove

$$UA^{\star}U^{*} = z \quad \text{and} \quad UAU^{*} = \frac{d}{dz} \tag{11.80}$$

over the dense spans of finite combinations of elements of the two bases.

11.9 On the Bargmann–Hilbert space B_1 (see Example 3.32(6)), consider

$$K_0 := z\frac{d}{dz}\ ,$$

defined on $D(H_0) = \{f \in \mathsf{B}_1 \,|\, z df/dz \in \mathsf{B}_1\}$. Prove that it is essentially self-adjoint and find its spectrum. Does $2\overline{K_0} + I$ have any physical meaning?

Hint. Prove it is symmetric, and show $\{u_n\}_{n\in\mathbb{N}}$ is an eigenvector basis of H_0 (hence of analytic vectors). Up to a factor, $2\overline{K_0} + I$ is the Hamiltonian of the harmonic oscillator.

Chapter 12
Introduction to Quantum Symmetries

Mathematical sciences, in particular, display order, symmetry and clear limits: and these are the uppermost instances of beauty.

Aristotle

This chapter continues in the description of the mathematical structure of Quantum Mechanics, by introducing fundamental notions and tools of great relevance.

Section one is devoted to defining and characterising *quantum symmetries*. We will present examples, discuss what happens in presence of superselection rules, and define *Kadison symmetries* and *Wigner symmetries*. We shall then prove the *theorems of Wigner and Kadison*, which show that the two notions of symmetry actually coincide, and manifest themselves via unitary or anti-unitary operators.

In section two we will address the problem of representing symmetry groups, by introducing *projective representations*, *projective unitary representations* and *central $U(1)$-extensions of a (symmetry) group*. A part of the section will be in particular dedicated to *topological groups* and the study of strongly continuous projective unitary representations. We will examine the special case of the Abelian group $\mathbb{R}$, that has important applications in QM. Next, after recalling the basics on *Lie* groups and Lie algebras, we will discuss key results due to Bargmann, Gårding and Nelson (and a few generalisations thereof) about projective unitary and unitary representations of Lie groups. We will consider the Peter–Weyl theorem on strongly continuous unitary representations of compact Lie groups (or better, compact Hausdorff topological groups).

Section three will present and discuss several physically important examples. As an instance of primary importance in physics, we will study unitary representations of the symmetry group $SO(3)$ in relationship to the *spin*. Eventually we will apply the machinery to the *Galilean group*, and prove *Bargmann's superselection rule of the mass*.

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12.1 Definition and Characterisation of Quantum Symmetries

A truly crucial notion in QM, also in view of the subsequent developments in Quantum Field Theories, is that of *symmetry* of a quantum system. There are two ideas of symmetry in quantum physics: one is dynamic and concerns conserved quantities under temporal evolution, while the other, more elementary, one does not involve temporal evolution. In this first section we will deal with the second kind only, and tackle the dynamic type in the next chapter.

Consider a physical system S described on the Hilbert space H_S (not necessarily separable in this context), and denote by $\mathfrak{S}(\mathsf{H}_S)$ the space of states and by $\mathfrak{S}_p(\mathsf{H}_S)$ the space of pure states. When we act by a transformation g on S we alter its quantum state. To the physical transformation g there corresponds a map $\gamma_g : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$ of states, or $\gamma_g : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S)$ if we restrict to pure states. The relationship between g and γ_g is not relevant at present, and we will take it for granted; at any rate, it will depend upon the description of S. We shall call γ_g a *symmetry* of the system if it obeys certain conditions. Abusing the terminology we will often say g is a symmetry of S. Two are the requisites for γ_g to be a symmetry:
Sym1. γ_g must be bijective,
Sym2. γ_g should preserve some mathematical structure of $\mathfrak{S}(\mathsf{H}_S)$ or $\mathfrak{S}_p(\mathsf{H}_S)$. For the moment we will not specify which structure exactly, although this will have a precise physical interpretation.
In physics, requisite **Sym1** can actually be forced upon the transformation g acting on the system, and corresponds to asking that g be *reversible*, in other words (i) there must exist an inverse transformation g^{-1}, associated to $\gamma_g^{-1} : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$, that takes back to the original state, and (ii) any quantum state should be reachable via γ_g, by choosing the initial state suitably.

The differences between the several symmetry notions known depend on the interpretation of condition **Sym2**, i.e. on the γ_g-invariant structure. There are at least two possible choices. The simplest structure that the map can preserve is the convexity of the space of states, physically corresponding to the fact that a state arises from mixing states with certain statistical weights. Symmetry operations modify the constituent states, but do not change the weights. Quantum symmetries of this sort were studied by Kadison [Kad51], and are nowadays called "Kadison symmetries". Another type, due to Wigner [Wig59], refers to functions on $\mathfrak{S}_p(\mathsf{H}_S)$. For these one requires that the metric structure of the projective space of rays be preserved. We will call them "Wigner symmetries". In the language of physics Wigner symmetries modify pure states but do not change the transition probabilities of pure pairs.

As we shall see later, the symmetries of Kadison and Wigner have a *dual action* on the observables of the theory. In this sense, symmetries can be viewed as bijective transformations of the observables of the theory (preserving some algebraic structures) instead of states. This approach can be adopted right from the start, in particular by defining symmetries as automorphisms of the lattice of the elementary propositions of a quantum system. A more elaborated version of this idea of quantum symmetry, introduced by Segal, enlarges the lattice to a *Jordan algebra* of observables (see Sect. 11.3.3) and defines symmetries as its automorphisms.

In the sequel we will mainly study the first two types of symmetries defined as transformations of states. We will prove that mathematically they reduce to the same concept, and that they are described by unitary or anti-unitary operators (hence Wigner symmetries may be extended to Kadison symmetries defined on the entire space of states). The characterisation in terms of (anti-)unitary operators is hugely important in physics, and is formulated in two results known as *Kadison's theorem* and *Wigner's theorem*. The latter is much more renowned in the physical community, despite the former is equally important. Some results on symmetries as transformations of observables, and Segal's symmetries in particular, appear in Sect. 12.1.7; there we shall state a theorem establishing a bijection between Segal symmetries and Kadison (or Wigner) symmetries, at least in absence of superselection rules. In that section we shall also prove that Kadison symmetries are in one-to-one correspondence with automorphisms of the lattice of elementary propositions (provided the Hilbert space is separable and has dimension $\neq 2$).

Remark 12.1 In quantum theory there is another physically relevant notion of transformation, similar to – and sometimes confused with – a symmetry acting on observables. We are referring to **gauge transformations** (rather regrettably, also known as **gauge symmetries**). A gauge symmetry is a unitary transformation of the Hilbert space of the theory which leaves invariant every (typically unbounded) self-adjoint operator representing an observable. This is equivalent to saying that it fixes every element of the von Neumann algebra of observables (in agreement with Definition 11.23, Exercise 11.3 and Remark 11.24). This is very different from a symmetry: there is always at least one observable that is changed by any non-trivial symmetry, whereas no observable is changed under gauge symmetries. ■

12.1.1 Examples

Before going into mathematical subtleties, let us describe a few examples of physical operations that are (both Wigner and Kadison) symmetries for quantum systems.

Suppose we take an isolated physical system S in a certain inertial frame system $\mathscr{I}$. Transformations known to generate symmetries of S are the rigid translations of S along a given vector, or the rotations about a fixed real axis. Any continuous isometry of the rest space of inertial frames produces a quantum symmetry. Another instance is the transformation of inertial frame system (in relativistic theories as well): the isolated system S in the inertial frame $\mathscr{I}$ is transformed so that the final system appears, in a different inertial system $\mathscr{I}' \neq \mathscr{I}$, as it appeared at the beginning in $\mathscr{I}$. A third transformation giving symmetries, for isolated systems in inertial frames, is time translation (not to be confused with time evolution), which we will see later.

All these transformations are *active*, meaning that they *change* the system S, or better, its quantum state.

It must be crystal clear that the transformations we are talking about do not occur because the system's state evolves in time: they are idealised transformations, pure mathematical notions. Some of them, by the way, could never occur in reality in a system that evolves under its own dynamics, and others could hardly exist. A classical example is the *inversion of parity*. This physical transformation, loosely speaking, substitutes a system S with its mirror image. Sometimes the only way to invert the parity, ideally, is to destroy the system and rebuild its symmetric image from scratch. And sometimes even this abstract operation is physically hollow, owing to the very nature of physical laws. Particles that interact under the weak force, surprisingly, constitute systems whose states do not admit parity transformation as a symmetry, in a rather radical sense: the space of states has no transformation γ representing the ideal physical transformation of parity inversion. This simply means that the alleged symmetry is *not* a true symmetry of the system.

Another type of transformation that shares some features with parity inversion, and that is at times associated to symmetries, is *time reversal*. The examples seen so far have to do with spacetime isometries. Albeit active on states, they are related to *passive* transformations of frame systems (or just of coordinates) by means of passive isometries of spacetime. In this case one expects (not always true, as we saw) active transformations on states to be symmetries, precisely because the various frames or coordinate systems – relative to passive (Galilean or Poincaré) transformations used to describe reality, at least macroscopically – are equivalent. In other words: if we act on the physical system S by an active transformation, we can always revoke the outcome by changing frame system (or just coordinates), knowing the new framing is physically equivalent to the original one.

In contrast to all this, there exist transformations related to symmetries which are neither associated to spacetime isometries, nor reversed by changing frame. A standard example is charge conjugation, which flips the sign of all charges (of the type considered) present in S, and thus changes the superselection sector of the charge. There exist even more abstract transformations relative to internal symmetries and gauge symmetries, on which we will not spend any time.

In conclusion we wish to underline an important physical fact. The lesson that weak interactions teach us is this: deciding whether a transformation acting ideally on a system is indeed a quantum symmetry, is ultimately to be established – after **Sym2** has been specified – experimentally.

After the theorems of Kadison and Wigner we will describe symmetries in terms of (anti-)unitary operators, for the case in which the physical transformations form an abstract, topological or Lie *group* [War75, NaSt82].

In the next chapter we shall treat dynamical symmetries, which emerge when one defines the *time evolution* of the quantum state of a system S. In that context we will recover the tight link between dynamical symmetries and associated conservation laws. It is well known, in the classical setup, that this relationship is encoded into the various formulations of the celebrated *Noether's theorem*.

12.1.2 Symmetries in Presence of Abelian Superselection Rules

As was observed already in Chap. 7, if M is a closed subspace in the Hilbert space H we can identify $\mathfrak{S}(\mathsf{M})$ (or $\mathfrak{S}_p(\mathsf{M})$) with a subset of $\mathfrak{S}(\mathsf{H})$ (resp. $\mathfrak{S}_p(\mathsf{H})$) in a natural manner, i.e. by viewing $\mathfrak{S}(\mathsf{M})$ ($\mathfrak{S}_p(\mathsf{M})$) as the collection of states $\rho \in \mathfrak{S}(\mathsf{H})$ ($\mathfrak{S}_p(\mathsf{H})$) such that $Ran(\rho) \subset \mathsf{M}$. This is the same as extending each $\rho \in \mathfrak{S}(\mathsf{M})$ to an operator on H by declaring it equals zero on $\mathsf{M}^\perp$. In the remaining part of the chapter we will implicitly make this identification, which is useful in the next situation.

In certain circumstances the possible state of a physical system is not an element in $\mathfrak{S}(\mathsf{H}_S)$ ($\mathfrak{S}_p(\mathsf{H}_S)$ if pure), because certain convex combinations are forbidden. This is the case when we have *superselection rules* (see Sects. 7.7.1, 7.7.2 and 11.2.1). Without repeating everything, let us only recall that in presence of superselection rules H_S splits into a Hilbert sum of closed, pairwise orthogonal, non-trivial subspaces called *coherent sectors*:

$$\mathsf{H}_S = \bigoplus_{k\in K} \mathsf{H}_{Sk} \,.$$

Remark 12.2 To guarantee the maximum generality we do not assume, unless explicitly stated, that the spaces H_S and H_{Sk} are separable nor that K is countable, even though in concrete physical cases these are physical requirements. ■

Then we can define the spaces of states $\mathfrak{S}(\mathsf{H}_{Sk})$ and pure states $\mathfrak{S}_p(\mathsf{H}_{Sk})$ of each sector. Note $\mathfrak{S}(\mathsf{H}_{Sk}) \cap \mathfrak{S}(H_{Sj}) = \varnothing$ and $\mathfrak{S}_p(\mathsf{H}_{Sk}) \cap \mathfrak{S}_p(\mathsf{H}_{Sj}) = \varnothing$ if $k \neq j$. Concerning physically-admissible pure states for the superselection rule, these will be precisely the constituents of the disjoint union

$$\bigsqcup_{k\in K} \mathfrak{S}_p(\mathsf{H}_{Sk}) \,.$$

Admissible mixtures by the superselection rule for the system S on H, instead, will be all possible convex combinations (also infinite, in the strong operator topology) in

$$\bigsqcup_{k\in K} \mathfrak{S}(\mathsf{H}_{Sk}) \,.$$

The previous is equivalent to imposing that admissible states are the ρ in $\mathfrak{S}(\mathsf{H}_S)$ (or $\mathfrak{S}_p(\mathsf{H}_S)$) under which every H_{Sk} is invariant, namely $P_k\rho = \rho P_k$ for every $k \in K$ where P_k is the orthogonal projector onto H_{Sk} (see Sect. 7.7.1).

It is worth noting that this picture is not the most general possible, because different elements $\rho, \rho' \in \mathfrak{S}(\mathsf{H}_{Sk})$ can be physically indistinguishable if a *gauge group* is present, as discussed in Sect. 11.2.1. When the gauge group is trivial, one speaks of *Abelian superselection rules*, the only type we shall consider henceforth. We will come back on this issue at the end of Sect. 12.1.4.

In case of Abelian superselection rules the symmetries must respect the coherent decomposition of H, and one allows symmetries that jump sector, i.e. functions $\gamma_{kk'} : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$, $k, k' \in K$, possibly with $k' \neq k$. Every mapping $\gamma_{kk'} : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$ must be invertible and satisfy Wigner's or Kadison's invariance.

12.1.3 Kadison Symmetries

Consider a quantum system S described on the Hilbert space H_S, with space of states $\mathfrak{S}(\mathsf{H}_S)$. There is a (physically weak) demand to have a symmetry that refers to the mixing procedure of quantum states. An operation on S defines a symmetry if the mixing procedure is invariant under it, or more precisely:

if a state is obtainable as mixture of certain states with given statistical weights, then by transforming the system under an operation that generates a symmetry, the same state must be obtainable as mixture of the transformed constituent states with the same *statistical weights.*

Put equivalently, a bijection $\gamma : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$ is a symmetry when it preserves the *convex structure* of $\mathfrak{S}(\mathsf{H}_S)$: if $\rho_i \in \mathfrak{S}(\mathsf{H}_S)$, $0 \leq p_i \leq 1$ and $\sum_{i \in J} p_i = 1$, then

$$\gamma\left(\sum_{i \in J} p_i \rho_i\right) = \sum_{i \in J} p_i \gamma(\rho_i) \,.$$

Henceforth J will be finite. It is obvious that we may take J made of two points, without loss of generality. Now we can present the formal definition in the general case, when coherent superselection sectors are present.

Definition 12.3 (*Kadison symmetry*). Consider a quantum physical system S described on the Hilbert space $\mathsf{H}_S = \oplus_{k \in K} \mathsf{H}_{Sk}$ split in coherent sectors due to Abelian superselection rules.

A **symmetry of** S **according to Kadison** from sector H_{Sk} to sector $\mathsf{H}_{Sk'}$, $k, k' \in K$, is a map $\gamma : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$ such that:
(a) γ is bijective;
(b) γ preserves the convex structures of $\mathfrak{S}(\mathsf{H}_{Sk})$ and $\mathfrak{S}(\mathsf{H}_{Sk'})$. Equivalently:

$$\gamma\,(p_1\rho_1 + p_2\rho_2) = p_1\gamma(\rho_1) + p_2\gamma(\rho_2) \quad \text{for } \rho_1, \rho_2 \in \mathfrak{S}(\mathsf{H}_{Sk})\,,\ p_1 + p_2 = 1,\ p_1, p_2 \in [0, 1]\,. \tag{12.1}$$

If the Hilbert space H does not have coherent sectors, every bijection $\gamma : \mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ preserving convexity is called a **Kadison automorphism** on H.

A symmetry according to Definition 12.3 is induced by an operator $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ that is either unitary or anti-unitary (Definition 5.39):

$$\gamma^{(U)}(\rho) := U\rho U^{-1}\,, \quad \rho \in \mathfrak{S}(\mathsf{H}_{Sk}). \tag{12.2}$$

To prove this we need an elementary lemma.

Lemma 12.4 *Let $U : \mathsf{H} \to \mathsf{H}'$ be an anti-unitary operator from H to H', and $N \subset \mathsf{H}$ a Hilbert basis. Then $U = VC$, where $V : \mathsf{H} \to \mathsf{H}'$ is unitary and $C : \mathsf{H} \to \mathsf{H}$ is the natural conjugation (Definition 5.41) associated to N:*

$$C\psi := \sum_{z\in N} \overline{(z|\psi)}z \, .$$

Proof Define $V\psi := \sum_{z\in N}(z|\psi)Uz$. The proof is immediate, because an anti-unitary operator is anti-isometric and continuous and by elementary properties of bases. Note that $\{Uz\}_{z\in N}$ is basis of H'. $\square$

Proposition 12.5 *Let $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ be unitary (isometric and onto), or anti-unitary, where H_{Sk} and $\mathsf{H}_{Sk'}$ are coherent sectors of the Hilbert space H_S associated to the quantum system S with space of states $\mathfrak{S}(\mathsf{H})$. Then $\gamma^{(U)} : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{B}(\mathsf{H})$ as defined in (12.2) is a symmetry $\mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ according to Kadison.*

Proof Property (12.1) is trivial under either assumption on U (not so, though, if we allowed complex coefficients p_i). Let us prove $\gamma^{(U)}(\rho) \in \mathfrak{S}(\mathsf{H}_{Sk'})$ for $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$. Begin by assuming U unitary. If ρ is of trace class on H_S so must be $U\rho U^{-1}$ as well, since trace-class operators form an ideal in $\mathfrak{B}(\mathsf{H}_S)$ by Theorem 4.34(b) if we view $U\rho U^{-1}$ as composite in $\mathfrak{B}(\mathsf{H}_S)$. For this it suffices to think of ρ in $\mathfrak{S}(\mathsf{H}_S)$ as vanishing on the complement to H_{Sk}, and $\rho(\mathsf{H}_{Sk}) \subset \mathsf{H}_{Sk}$, then extend U and U^{-1} trivially on the orthogonal to H_{Sk} and $\mathsf{H}_{Sk'}$ respectively, hence viewing them as in $\mathfrak{B}(\mathsf{H}_S)$. If $\rho \geq 0$ then $(\psi|U\rho U^{-1}\psi) = (U^*\psi|\rho U^*\psi) \geq 0$, so $\gamma^{(U)}(\rho) \geq 0$. Using the basis formed by merging a basis on H_{Sk} and a basis on $(\mathsf{H}_{Sk})^\perp$ we obtain $tr\left(\gamma^{(U)}(\rho)\right) = tr\left(U\rho U^{-1}\right) = tr(U^{-1}U\rho) = tr(\rho) = 1$. In the last passage we used $U^{-1}U\restriction_{\mathsf{H}_{Sk}} = I\restriction_{\mathsf{H}_{Sk}}$ in computing the trace, and that $\rho = 0$ on $(\mathsf{H}_{Sk})^\perp$. Therefore $\gamma^{(U)}(\rho) \in \mathfrak{S}(\mathsf{H}_{Sk'})$ for $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$. Now assume U anti-unitary. Decompose U as in Lemma 12.4: $U = VC$ with respect to some basis $N \subset \mathsf{H}_S$, specified later. We claim $U\rho U^{-1}$ is positive, of trace class and with trace one. As V is unitary (in which case the claim holds by what we have just seen) and $U\rho U^{-1} = V(C\rho C^{-1})V^{-1}$, it is enough to prove the claim for $U = C$. Choose N to be made of eigenvectors ψ for ρ (Theorem 4.20 ensures its existence). Hence

$$\rho\phi = \sum_{\psi\in N} p_\psi(\psi|\phi)\psi \, ,$$

$\phi \in \mathsf{H}$. Now recall C is continuous and antilinear, $CC = I$, $(f|g) = \overline{(Cf|Cg)}$ by definition of conjugation, every eigenvector p_ψ of ρ is real (positive), and $C\psi = \psi$. Consequently

$$C\rho C^{-1}\phi = \sum_{\psi\in N} p_\psi\overline{(\psi|C\phi)}C\psi = \sum_{\psi\in N} p_\psi\overline{(CC\psi|C\phi)}C\psi =$$

$$= \sum_{\psi\in N} p_\psi(C\psi|\phi)C\psi = \sum_{\psi\in N} p_\psi(\psi|C\phi)\psi = \rho\phi \, .$$

We proved $C\rho C^{-1} = \rho$, so $C\rho C^{-1}$ is of trace class, positive and has trace 1 for $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$. $\square$

Example 12.6 If the superselection rule regards the electrical charge of a system, there will be (infinitely many, in general) sectors H_q, one for each value q of the charge. *Charge conjugation* can be constructed as a collection of symmetries of type $\gamma^{(U_q)}$, where $U_q : H_q \to H_{-q}$ for any q. ■

12.1.4 Wigner Symmetries

Now let us pass to quantum symmetries according to Wigner. Consider the usual quantum system S, described on the Hilbert space H_S and with space of states $\mathfrak{S}(\mathsf{H}_S)$. We focus on pure states $\mathfrak{S}_p(\mathsf{H}_S)$ (the rays of H_S). Let us restrict to transformations

$$\delta : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S) .$$

From the experimental viewpoint we can control the *transition probability*

$$|(\psi|\psi')|^2 = tr(\rho\rho')$$

of two pure states $\rho = \psi(\psi|\ )$, $\rho' = \psi'(\psi'|\ )$. Wigner's condition for a bijection $\delta : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S)$ to be a symmetry is that it preserves transition probabilities: *if two pure states have a certain transition probability, when transforming the system by a physical operation that determines a symmetry the transformed states must maintain the* same *transition probability.*

The next definition takes into account coherent sectors.

Definition 12.7 (*Wigner symmetry*). Consider a quantum system S described on the Hilbert space H_S with space of states $\mathfrak{S}(\mathsf{H}_S)$. Assume H_S splits coherently as $\mathsf{H}_S = \bigoplus_{k\in K} \mathsf{H}_{Sk}$ due to Abelian superselection rules.

A **symmetry of** S **according to Wigner** from H_{Sk} to $\mathsf{H}_{Sk'}$, $k, k' \in K$, is a mapping $\delta : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sk'})$ with the following properties:
(a) δ is bijective;
(b) δ preserves transition probabilities:

$$tr\,(\rho_1\rho_2) = tr\,(\delta(\rho_1)\delta(\rho_2)) \ , \quad \rho_1, \rho_2 \in \mathfrak{S}_p(\mathsf{H}_{Sk}) . \tag{12.3}$$

If H has no coherent sectors every bijection $\delta : \mathfrak{S}_p(\mathsf{H}) \to \mathfrak{S}_p(\mathsf{H})$ that preserves transition probabilities is a **Wigner automorphism** on H.

An example according to Definition 11.25 is, as with Kadison symmetries, the symmetry induced by the (anti-)unitary operator $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ (Definition 5.32), where:

$$\delta^{(U)}(\rho) := U\rho U^{-1} \ , \quad \rho \in \mathfrak{S}_p(\mathsf{H}_{Sk}). \tag{12.4}$$

In contrast to Kadison's symmetries, here the proof is really straightforward.

Remarks 12.8 **(1)** Since pure states have the form $\psi(\psi|\ )$, $||\psi|| = 1$, the action of $\delta^{(U)}$ on pure states can be described, equivalently though sloppily, by saying $\delta^{(U)}$ sends the pure state ψ to the pure state $U\psi$. This is the way in which QM books often describe symmetries induced by (anti-)unitary operators.
(2) Every Kadison symmetry transforms pure states to pure states, so it defines a bijective map on the space of pure states. However, we do not know a priori that this will define a Wigner symmetry, because it is far from evident that it will preserve transition probabilities. On the other hand, a Wigner symmetry does not extend naturally from pure to mixed states. Therefore it is not obvious that the two notions coincide. Yet every unitary or anti-unitary operator determines at the same time a Wigner symmetry and a Kadison symmetry by means of $\rho \mapsto U\rho U^{-1}$. ■

To finish, here is a general notion of Wigner symmetry.

Definition 12.9 (*General Wigner symmetry*). Suppose the Hilbert space H_S of system S decomposes in coherent sectors due to Abelian superselection rules, so that admissible pure states are the elements of

$$\mathfrak{S}_p(\mathsf{H}_S)_{adm} := \bigsqcup_{k\in K} \mathfrak{S}_p(\mathsf{H}_{Sk}) .$$

A **symmetry according to Wigner** (no mention of sectors) is a bijective map δ from $\mathfrak{S}_p(\mathsf{H}_S)_{adm}$ to itself that preserves transition probabilities.

We can recover the above definition using Wigner symmetries between pairs of sectors, as follows.

Proposition 12.10 *Let δ be a Wigner symmetry of S, and suppose the Hilbert space H_S of S splits coherently in such a way that admissible pure states are only those in:*

$$\mathfrak{S}_p(\mathsf{H}_S)_{adm} = \bigsqcup_{k\in K} \mathfrak{S}_p(\mathsf{H}_{Sk}) .$$

There exists a bijection $f : K \to K$ and a family of Wigner symmetries

$$\delta_{f,f(k)} : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sf(k)}) , \quad k \in K,$$

with fixed sectors, such that $\delta\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})} = \delta_{f,f(k)}$ for every k. In this sense δ is just a collection of Wigner symmetries exchanging sectors and not overlapping.

Proof Define on $\mathfrak{S}_p(\mathsf{H}_S)$ the distance

$$d(\rho,\rho') := ||\rho-\rho'||_1 := tr(|\rho-\rho'|) ,$$

where $|| \, ||_1$ is the canonical norm of trace-class operators. Then the sets $\mathfrak{S}_p(\mathsf{H}_{Sk})$ are the connected components of $\mathfrak{S}_p(\mathsf{H}_S)_{adm}$ (Exercise 12.6). The map $\delta : \mathfrak{S}_p(\mathsf{H}_S)_{adm} \to \mathfrak{S}_p(\mathsf{H}_S)_{adm}$ is a surjective isometry for d as follows from Proposition 12.41 (the latter is independent of the present result). In particular δ is a homeomorphism. As such, it preserves maximal connected subsets, and so it must split as a sum of isometric bijections between distinct sectors, i.e. Wigner symmetries between distinct sectors. □

Before we examine the mathematical structure of symmetries we wish to put the spotlight one more time on the remaining physical limitation of the picture we are drawing. As already stressed in Sect. 7.7.2 (especially the comment below Proposition 7.86 and Remark 7.89(2)) and Sect. 11.2.1, distinct density matrices $\rho \neq \rho' \in \mathfrak{S}(\mathsf{H}_{Sk})$ represent different physical situations only when observables separate states. This is equivalent to the demand that the von Neumann algebra of observables pertinent to the sector H_{Sk} is the whole $\mathfrak{B}(\mathsf{H}_{Sk})$ (see Proposition 11.18), and this is the case when Abelian superselection rules occur, the only type we have so far considered. But this is not true in the general situation, in which the commutant $\mathfrak{R}'_S$ of the algebra of observables is larger than the centre $\mathfrak{R}_S \cap \mathfrak{R}'_S$: think of quarks and other hadrons for instance. When observables do not separate trace-class operators of trace one, the physical meaning of the facts we are about to prove becomes dubious (although the statements are valid), because the requirements **Sym1** and **Sym2**, which we have assumed define symmetries, are no longer physically justified when interpreted in terms of density matrices. For instance there is no need for a symmetry to be bijective as a map $\mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$, since copies of distinct operators $\rho, \rho' \in \mathfrak{S}(\mathsf{H}_{Sk})$ can still represent exactly the same measure on $\mathscr{L}_{\mathfrak{R}_{Sk}}(\mathsf{H}_{Sk})$ and such measures are better suited to retain, without redundancies, the experimental information of a physical state. To encompass the most general situation of non-Abelian superselection rules, the definition of symmetry should be reformulated by relying upon axioms **A1 (measure-theory version)** and **A2 (measure-theory version)** regarding states and observables and by adopting the description **Ss1 (measure-theory formulation)** and **Ss2 (measure-theory formulation)** of superselection rules. A very small step towards this more general description will appear in Sect. 12.1.7. Alternatively, if we stick to the description of states in terms of operators in $\mathfrak{S}(\mathsf{H}_{Sk})$, symmetries should be defined as maps between spaces $\mathfrak{S}(\mathsf{H}_{Sk})$ that are bijective *up to* transformations of the *gauge group* of the theory (see Sect. 11.2.3). Then requisite **Sym2** should also be relaxed accordingly.

12.1.5 The Theorems of Wigner and Kadison

We begin by Wigner's theorem. Using that, we will prove Kadison's result along the lines of [Sim76]. The proof of Wigner's theorem is quite direct. Although there are more elegant, but indirect arguments, our approach has the advantage of showing explicitly how to manufacture U with a basis. Let us remark, in passing, that several authors (including Emch, Piron, Bargmann and Varadarajan) proved that a slightly modified version of Wigner's theorem holds within QM formulations based on real and quaternionic Hilbert spaces.

Theorem 12.11 (Wigner). *Consider a quantum system S described on the (not necessarily separable) complex Hilbert space* H_S. *Suppose* H_S *coherently splits*[1] *as* $\mathsf{H}_S = \oplus_{k\in K}\mathsf{H}_{Sk}$ *and* $\dim \mathsf{H}_{Sk} > 1$ *for every* $k \in K$. *Assume*

$$\delta : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sk'})$$

is a symmetry of S according to Wigner from H_{Sk} *to* $\mathsf{H}_{Sk'}$, $k, k' \in K$. *Then*
(a) *there exists an operator* $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$, *unitary or anti-unitary depending on* δ, *such that:*

$$\delta(\rho) = U\rho U^{-1} \quad \textit{for any pure state } \rho \in \mathfrak{S}_p(\mathsf{H}_{Sk}); \tag{12.5}$$

(b) *U is determined up to a phase factor, i.e.,* U_1 *satisfies (12.5) (replacing U) if and only if* $U_1 = \chi U$ *for some* $\chi \in \mathbb{C}$, $|\chi| = 1$.
In case $\dim \mathsf{H}_{Sk} = 1 (= \dim \mathsf{H}_{Sk'})$ *for some k, (a) is still valid with the difference that* δ *(which is unique) can be represented by a unitary or an anti-unitary operator indifferently.*

Proof (a) Let us build an operator U representing δ. Take a Hilbert basis $\{\psi_n\}_{n\in N}$ in H_{Sk} where N is any finite or infinite set of indices, with $\psi_n \neq \psi_m$ when $n \neq m$, and the space is not necessarily separable but has dimension ≥ 2. To each ψ_n associate the pure state $\rho_{\psi_n} := \psi_n(\psi_n| \ )$. Let δ act on these states, obtaining pure states $\delta(\rho_{\psi_n}) = \psi'_n(\psi'_n| \ ) \in \mathfrak{S}_p(\mathsf{H}_{Sk'})$, where the unit vectors $\psi'_n \in \mathsf{H}_{Sk'}$ are determined up to a phase factor. Fix once and for all this phase, arbitrarily. Note $\{\psi'_n\}_{n\in N}$ is a Hilbert basis of $\mathsf{H}_{Sk'}$: the vectors are in fact orthonormal, because $|(\psi'_n|\psi'_m)|^2 = tr(\delta(\rho_n)\delta(\rho_m)) = tr(\rho_{\psi_n}\rho_{\psi_m}) = |(\psi_n|\psi_m)|^2 = \delta_{nm}$. We claim that $\psi' \perp \psi'_n \Rightarrow \psi' = \mathbf{0}$. Let $\psi' \perp \psi'_n$ for every $n \in N$. If $\psi' \neq 0$, without loss of generality we assume $||\psi'|| = 1$ and define $\rho' := \psi'(\psi'| \ ) \in \mathfrak{S}_p(\mathsf{H}_{Sk'})$. Since δ is onto, then $\rho' = \delta(\rho)$ with $\rho = \psi(\psi| \ )$ for some $\psi \in \mathsf{H}_{Sk}$, $||\psi|| = 1$. Therefore:

$$|(\psi'|\psi'_n)|^2 = tr(\delta(\rho)\delta(\rho_{\psi_n})) = tr(\rho\rho_{\psi_n}) = |(\psi|\psi_n)|^2 = 0$$

and then $\psi = \mathbf{0}$, for $\{\psi_n\}_{n\in N}$ is a basis. But this is impossible as $||\psi|| = 1$. Consequently $\psi' = \mathbf{0}$, and $\{\psi'_n\}_{n\in N}$ is a basis.

Using the bases $\{\psi_n\}_{n\in N}$ and $\{\psi'_n\}_{n\in N}$ we will define the operator U in stages. Select a preferred index $o \in N$ and define unit vectors

$$\Psi_k := 2^{-1/2}\,(\psi_o + \psi_k)\ , \quad k \in N \setminus \{o\}$$

and corresponding pure states: $(\Psi_k| \ )\Psi_k$, $k \in N \setminus \{o\}$. The transformed state $\delta(\Psi_k(\Psi_k| \ )) = \Psi'_k(\Psi'_k| \ )$ satisfies, in particular:

[1] If $K = \{1\}$ one should replace $\mathsf{H}_{Sk}, \mathsf{H}_{Sk'}$ by H in the statement.

$$|(\Psi'_k|\psi'_n)|^2 = tr\left(\Psi'_k(\Psi'_k| \ \ )\delta(\rho_n)\right) = tr\left(\delta(\Psi_k(\Psi_k| \ \ ))\delta(\rho_n)\right) = |(\Psi_k|\psi_n)|^2 = \frac{\delta_{on}+\delta_{on}}{2} ,$$

plus $||\Psi'_k|| = 1$. Decomposing $\Psi'_k = \sum_n a_n \psi'_n$, the only possibility is

$$\Psi'_k = \chi'_k 2^{-1/2}(\psi'_o + \chi_k \psi'_k)$$

with $|\chi'_k| = |\chi_k| = 1$. The χ_k are given by δ, while the χ'_k can be chosen as we want. The χ_k carry the information of δ, and we shall employ them soon.

Let us define the action of a map U (for the moment not necessarily linear or antilinear) on all vectors ψ_n and on $(\psi_o + \psi_k)/\sqrt{2}$ by declaring:

$$U(\psi_o) := \psi'_o \,, \quad U(\psi_k) := \chi_k \psi'_k \,, \quad U(2^{-1/2}(\psi_0 + \psi_k)) := 2^{-1/2}(\psi'_o + \chi_k \psi'_k) \,, \tag{12.6}$$

$k \in N \setminus \{o\}$. With this we are sure that if ϕ is one of the above arguments of U and ρ_ϕ its pure state, then $\delta(\rho_\phi)$ is associated to $\rho_{U\phi}$ as requested by the thesis.

Now we extend U to any normalised vector $\psi = \sum_{n\in N} a_n \psi_n \in \mathsf{H}_{Sk}$ in a way that U continues to represent δ. Assume $||\psi|| = 1$ and let $\psi' \in \mathsf{H}_{Sk'}$ with $||\psi'|| = 1$ be such that $\psi'(\psi'| \ \ ) = \delta(\rho_\psi)$. Then

$$\psi' = \sum_{n\in N} a'_n \psi'_n \,. \tag{12.7}$$

The coefficients a'_k are given, up to a global phase factor, by the coefficients a_n and by δ. With our assumptions on δ we have

$$|(\psi'|\psi'_n)|^2 = tr(\delta(\rho_\psi)\delta(\rho_{\psi_n})) = tr(\rho_\psi \rho_{\psi_n}) = |(\psi|\psi_n)|^2 \,.$$

In other words, $|a'_n| = |a_n|$. Using this, together with the first two of (12.6), identity (12.7) can be rephrased as

$$\psi' = \chi\left(a_o U(\psi_o) + \sum_{n\in N\setminus\{o\}} \chi_n^{-1} a'_n U(\psi_n)\right) ,$$

where χ, $|\chi| = 1$, is arbitrary. Now define

$$U(\psi) := a_o U(\psi_o) + \sum_{n\in N\setminus\{o\}} \chi_n^{-1} a'_n U(\psi_n) \,. \tag{12.8}$$

This ensures, by construction, $\rho_{U(\psi)} = \delta(\rho_\psi)$, and one verifies that the definition extends (12.6). However, $U(\psi)$ is not yet fixed, because we still do not know the coefficients a'_n in terms of the components a_n of ψ. Let us find this relation. By construction of U and under the hypotheses on δ, $|(\Psi_k|\psi)| = |(U(\Psi_k)|U(\psi))|$. By (12.8) this means

$$|a_o + a_k|^2 = |a_o + \chi_k^{-1} a'_k|^2 \, .$$

Since $|a_k| = |a'_k|$, the latter implies

$$Re(a_o a_k) = Re(a_o \chi_k^{-1} a'_k) \, .$$

If we assume $a_o \in \mathbb{R} \setminus \{0\}$, recalling that $|a_k| = |a'_k|$, the previous equations occur only in one of these cases

$$a'_k = \chi_k a_k \quad \text{or} \quad a'_k = \chi_k \overline{a_k} \, ,$$

where we henceforth assume $\chi_o := 1$ according to the first identity in (12.6), Hence, for any $\psi = \sum_n a_n \psi_n$ with $a_o \in \mathbb{R} \setminus \{0\}$ we have

$$U(\psi) = \sum_{n \in A_\psi} a_n \chi_n \psi'_n + \sum_{n \in B_\psi} \overline{a_n} \chi_n \psi'_n \, .$$

For the given ψ we can always choose one of A_ψ, B_ψ empty.[2] Suppose the contrary. The components of ψ' satisfy $a'_p = \chi_p a_p$ and $a'_q = \chi_q \overline{a_q}$, for some pair $p \neq q$, where $Ima_p, Ima_q \neq 0$. If $\phi = 2^{-1/2}(\psi_p + \psi_q)$, then by construction:

$$|(\phi'|\psi')|^2 = |(\phi|\psi)|^2 \, ,$$

where $\phi' := U(\phi) = 2^{-1/2}(\chi_p \psi'_p + \chi_q \psi'_q)$. The displayed equation reads

$$\left|a_p + a_q\right|^2 = \left|a_p + \overline{a_q}\right|^2 \, ,$$

i.e.

$$Re(a_p a_q) = Re(a_p \overline{a_q}).$$

This would imply $Ima_q = -Ima_q$, an absurd.

In summary, if $\psi = \sum_n a_n \psi_n \in \mathsf{H}_{Sk}$, $||\psi|| = 1$ and $a_o \in \mathbb{R} \setminus \{0\}$, there are two possible definitions for $U\psi$:

$$U\psi = \sum_{n \in N} a_n \chi_n \psi'_n \quad \text{or} \quad U\psi = \sum_{n \in N} \overline{a_n} \chi_n \psi'_n \, . \tag{12.9}$$

With both choices we are sure that $U\psi$ represents $\delta(\rho_\psi)$ when $||\psi|| = 1$ and $a_o \in \mathbb{R} \setminus \{0\}$. We claim the choice of one definition does *not* depend on ψ. Consider a generic unit vector $\psi = \sum_n a_n \psi_n \in \mathsf{H}_{Sk}$ and $a_o \in \mathbb{R} \setminus \{0\}$. Define $\psi^{(nc)}$ associated to $c \in \mathbb{C}$, $Imc \neq 0$, for every $n = 1, 2, \ldots$:

---

[2]There is a certain ambiguity in defining A_ψ and B_ψ, because the subscripts n of the possible real coefficients a_n can be chosen either in A_n or in B_n indifferently.

$$\psi^{(nc)} := \frac{1}{\sqrt{1+|c|^2}}(\psi_o + c\psi_n)\ .$$

Since $|(\psi|\psi^{(nc)})| = |(U\psi|U\psi^{(nc)})|$ has to hold, necessarily $\psi^{(nc)}$ and ψ are of the same type with respect to the option of (12.9). Therefore all $\psi = \sum_n a_n\psi_n \in \mathsf{H}_{Sk}$, $||\psi|| = 1$, and $a_o \in \mathbb{R}\setminus\{0\}$ are of the same type.

We can finally pass to the whole Hilbert space, dropping the requirement $||\psi|| = 1$ and define the operators $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ by:

$$U : \psi = \sum_{n\in N} a_n\psi_n \mapsto \sum_{n\in N} a_n\chi_n\psi'_n \quad \text{in the linear case,}$$

$$U : \psi = \sum_{n\in N} a_n\psi_n \mapsto \sum_{n\in N} \overline{a_n}\chi_n\psi'_n \quad \text{in the antilinear case.}$$

By construction, since both $\{\psi_n\}_{n\in N}$ and $\{\chi_n\psi'_n\}_{n\in N}$ are Hilbert bases, the former are isometric and onto (unitary), the latter anti-isometric and onto (anti-unitary). Moreover, restricting to unit vectors $\psi = \sum_n a_n\psi_n$ with $a_0 \in \mathbb{R}\setminus\{0\}$, we have

$$\rho_{U\psi} = \delta(\rho_\psi) \tag{12.10}$$

in either case, as required. There remains to prove that (12.10) holds for pure states associated to $\psi = \sum_{n\in N} a_n\psi_n$ with $a_o \notin \mathbb{R}\setminus\{0\}$. First observe that only the case $a_o = 0$ has to be investigated, since $a_o \notin \mathbb{R}$ can be reduced to $a_o \in \mathbb{R}\setminus\{0\}$ by multiplying ψ by a phase χ, because this change does not affect ρ_ψ and $\rho_{U\psi}$. The remaining case $a_o = 0$ can be treated with a continuity argument. From Proposition 12.41 (whose proof is independent from the present result), if $\rho_\psi, \rho_{\psi'} \in \mathfrak{S}_p(\mathsf{H})$ then

$$||\rho_\psi - \rho_{\psi'}||_1 = 2\sqrt{1-|(\psi|\psi')|^2}\ .$$

It is therefore obvious that both maps $\delta : \mathfrak{S}_p(\mathsf{H}_{Sk}) \ni \rho_\psi \mapsto \delta(\rho_\psi) \in \mathfrak{S}_p(\mathsf{H}_{Sk'})$ and $u : \mathfrak{S}_p(\mathsf{H}_{Sk}) \ni \rho_\psi \mapsto \rho_{U\psi} \in \mathfrak{S}_p(\mathsf{H}_{Sk'})$, with $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ unitary or anti-unitary from earlier, are continuous with respect to these distances because they are isometric. A pure state in $\mathfrak{S}_p(\mathsf{H}_{Sk})$ described by a unit vector with $a_0 = 0$ is a limit point of pure states described by unit vectors with $a_o \neq 0$, as is evident if we look at the components. For these states $\delta(\rho) = u(\rho)$ as established in (12.10), hence continuity implies that this identity holds also for pure states associated to unit vectors with $a_o = 0$: (12.10) is valid everywhere on $\mathfrak{S}_p(\mathsf{H}_{Sk})$.

Let us finally prove that the unitary and anti-unitary character of U is decided by δ. The above construction of U is affected by a choice of phases at several places. On the contrary, the next argument is independent from the construction. Here only the requirement $\dim(\mathsf{H}_{Sk}) > 1$ is crucial. Suppose that there exist U unitary and V anti-unitary such that both implement the same δ. As a consequence, if $\rho_\psi = \psi(\psi|\cdot)$ for $||\psi|| = 1$, we immediately have $U\rho_\psi U^{-1} = \delta(\rho_\psi) = V\rho_\psi V^{-1}$. Defining the anti-unitary operator $C := V^{-1}U$ we therefore have $C\rho_\psi = \rho_\psi C$ for every unit vector ψ. Hence $C\psi = \lambda_\psi \psi$ for some $\lambda_\psi \in U(1)$ (this restriction arises from $C^2\psi = |\lambda_\psi|^2\psi$ where C^2 is unitary so that $|\lambda_\psi|^2 \in U(1)$, i.e. $|\lambda_\psi| = 1$). If ϕ is another unit vector normal to ψ, which exists for $\dim(\mathsf{H}) > 1$, then

$$\lambda_\phi \phi + \lambda_\psi \psi = C\phi + C\psi = C(\phi + \psi) = \lambda_{2^{-1/2}(\phi+\psi)}(\phi + \psi)$$

and so

$$(\lambda_\phi - \lambda_{2^{-1/2}(\phi+\psi)})\phi = -(\lambda_\phi - \lambda_{2^{-1/2}(\phi+\psi)})\psi \ .$$

Since ϕ and ψ are linearly independent, the identity found implies

$$\lambda_\phi = \lambda_\psi = \lambda_{2^{-1/2}(\phi+\psi)} \ .$$

Repeating the argument for each pair of vectors of a Hilbert basis $B := \{\psi_n\}_{n\in N}$ (recall that $\dim(\mathsf{H}_{Sk}) > 1$), we get $C\psi_n = \lambda_B \psi_n$ for every $n \in N$ and some common $\lambda_B \in U(1)$. As a consequence, for every vector $\psi = \sum_{n\in N} a_n \psi_n \in \mathsf{H}_{Sk}$,

$$C\psi = \lambda_B \sum_{n\in N} \overline{a_n}\psi_n \ . \tag{12.11}$$

Next, consider another basis $B_1 := \{\phi_n := e^{i\beta_n}\psi_n\}_{n\in\mathbb{N}}$, where the constants $\beta_n \in \mathbb{R}$ are fixed arbitrarily, so that $\psi = \sum_{n\in N} e^{-i\beta_n} a_n \phi_n$. The previous argument applied to B_1 yields, for some other constant $\lambda_{B_1} \in U(1)$,

$$C\psi = \lambda_{B_1} \sum_{n\in N} \overline{e^{-i\beta_n} a_n}\phi_n = \lambda_{B_1} \sum_{n\in N} \overline{a_n} e^{i\beta_n}\phi_n \ . \tag{12.12}$$

On the other hand, (12.11) and $\phi_n = e^{i\beta_n}\psi_n$ furnish

$$C\psi = \lambda_B \sum_{n\in N} \overline{a_n} e^{-i\beta_n}\phi_n \ .$$

Comparing with (12.12) and using the fact that $\{\phi_n\}_{n\in N}$ is a basis, we conclude that, in particular, for a pair of indices $n_1 \neq n_2$,

$$\lambda_B \lambda_{B_1}^{-1} = e^{2i\beta_{n_1}} = e^{2i\beta_{n_1}} \ .$$

Since we are free to define B_1 by fixing $\beta_{n_1} \neq \beta_{n_2} + k\pi$ ($k \in \mathbb{Z}$), the identity above is impossible and therefore U and V cannot exist simultaneously.

(b) Let us prove that if U exists, it is unique up to a phase factor. Clearly if U satisfies the thesis for δ, then $U_1 := \chi U$ will do the same for any $\chi \in \mathbb{C}$, $|\chi| = 1$. We claim this is the only possibility. Suppose there is another U_1 for δ. From the proof of (a) U and U_1 must be both unitary or both anti-unitary. If $\rho = \psi(\psi|\ )$ then, setting $L := U^{-1}U_1$ we have $L\psi(\psi|L^{-1}\phi) = \psi(\psi|\phi)$ for any unit vectors ψ, ϕ. Hence $L\psi(L\psi|\phi) = \psi(\psi|\phi)$, as L is unitary. Since $L\psi(L\psi|\ ) = \psi(\psi|\ )$, $L\psi$ and ψ determine the same pure state, so $L\psi = \chi_\psi \psi$ for every unit vector $\psi \in \mathsf{H}_{Sk}$ and for some unit number $\chi_\psi \in \mathbb{C}$. Choose ψ, ψ' orthogonal (they exist because $\dim \mathsf{H}_{Sk} > 1$). The linearity of L implies

$$\chi_{2^{-1/2}(\psi+\psi')}(\psi + \psi') = L(\psi + \psi') = L\psi + L\psi' = \chi_\psi \psi + \chi_{\psi'}\psi' .$$

Therefore

$$(\chi_{2^{-1/2}(\psi+\psi')} - \chi_\psi)\psi = (\chi_{\psi'} - \chi_{2^{-1/2}(\psi+\psi')})\psi' .$$

As ψ, ψ' are linearly independent, we have $(\chi_{2^{-1/2}(\psi+\psi')} - \chi_\psi) = 0$ and $(\chi_{\psi'} - \chi_{2^{-1/2}(\psi+\psi')}) = 0$, so $\chi_\psi = \chi_{\psi'}$. Hence, decomposing L along a Hilbert basis, for some unit $\chi \in \mathbb{C}$ we have

$$L\psi = \chi\psi \quad \text{for every } \psi \in \mathsf{H}_{Sk} ,$$

and so either $U_1 = \chi U$ or $U_1 = \overline{\chi} U$ according to the unitary or anti-unitary nature of U and U_1.

The last statement is obvious: if H_{Sk} and $\mathsf{H}_{Sk'}$ have dimension 1, for every choice of unit vectors $\psi \in \mathsf{H}_{Sk}$ and $\psi' \in \mathsf{H}_{Sk'}$, necessarily $\delta(\psi(\psi|\ )) = \psi'(\psi'|\ )$, so both $U : a\psi \mapsto a\psi'$ and $U_1 : a\psi \mapsto \overline{a}\psi'$ for every $a \in \mathbb{C}$ implement δ. $\square$

Remark 12.12 It is worth stressing that, in spite of the multiple descriptions of $\delta|_{\mathsf{H}_{Sk}}$ in terms of unitary and anti-unitary operators if $\dim \mathsf{H}_{Sk} = 1$, the map $\delta|_{\mathsf{H}_{Sk}}$ is uniquely fixed, because H_{Sk} (and $\mathsf{H}_{Sk'}$) contains only one equivalence class of unit vectors. ■

Let us move on to Kadison's theorem, which we will reduce to Wigner's theorem following an idea of Roberts and Roepstorff [RoRo69], see [Sim76]. We start by proving the theorem in dimension two.

Proposition 12.13 *Let* H *be a two-dimensional Hilbert space. If* $\gamma : \mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ *is a Kadison automorphism, there exists* $U : \mathsf{H} \to \mathsf{H}$ *unitary, or anti-unitary, such that:*

$$\gamma(\rho) = U\rho U^{-1} , \quad \rho \in \mathfrak{S}(\mathsf{H}).$$

Proof Let us characterise states and pure states on H by means of the *Poincaré sphere*. A state $\rho \in \mathfrak{S}(\mathsf{H})$ is, in the present situation, a positive-definite Hermitian matrix with unit trace. The real space of Hermitian matrices has a basis made by I and the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{12.13}$$

So for some $a, b_n \in \mathbb{R}$

$$\rho = aI + \sum_{n=1}^{3} b_n \sigma_n \, .$$

The condition $tr(\rho) = 1$ fixes $a = 1/2$, since the σ_n are traceless. Positive definiteness, i.e. the demand the two eigenvalues of ρ are positive, is equivalent to $\sqrt{b_1^1 + b_2^2 + b_3^2} \le 1/2$, by direct computation. Overall, the elements ρ of $\mathfrak{S}(\mathsf{H})$ are in one-to-one correspondence to vectors $\mathbf{n} \in \mathbb{R}^3$, $|\mathbf{n}| \le 1$:

$$\rho = \frac{1}{2}(I + \mathbf{n} \cdot \sigma) \, . \tag{12.14}$$

Having ρ pure, i.e. having a unique eigenvalue 1, is equivalent to $|\mathbf{n}| = 1$, as a direct check shows. Altogether $\mathfrak{S}(\mathsf{H})$ is in one-to-one correspondence with the closed unit ball B in $\mathbb{R}^3$ centred at the origin; the subset of pure states $\mathfrak{S}_p(\mathsf{H})$ corresponds one-to-one to the surface ∂B. We call B, viewed in this way, the **Poincaré sphere**. The correspondence just defined:

$$B \ni \mathbf{n} \mapsto \rho_{\mathbf{n}} \in \mathfrak{S}(\mathsf{H})$$

is a true isomorphism: by (12.14), in fact,

$$\rho_{p\mathbf{n}+q\mathbf{m}} = p\rho_{\mathbf{n}} + q\rho_{\mathbf{m}} \quad \text{for any } \mathbf{n}, \mathbf{m} \in B \text{ if } p, q \ge 0,\ p + q = 1 \, .$$

Hence the convex geometry of the spaces is preserved. An important property, for later, is the formula

$$tr\,(\rho_{\mathbf{m}}\rho_{\mathbf{n}}) = \frac{1}{2}(1 + \mathbf{m} \cdot \mathbf{n}) \tag{12.15}$$

that descends directly from $tr(\sigma_j) = 0$, $tr(\sigma_i\sigma_j) = 2\delta_{ij}$ (easy to prove). Now we are ready to characterise Kadison automorphisms. Assigning a Kadison automorphism $\gamma : \mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ is patently the same as defining a bijection $\gamma' : B \to B$ such that

$$\gamma'(p\mathbf{n} + q\mathbf{m}) = p\gamma'(\mathbf{n}) + q\gamma'(\mathbf{m}) \quad \text{for any } \mathbf{n}, \mathbf{m} \in B \text{ if } p, q \ge 0,\ p + q = 1 \, .$$

If the Kadison automorphism $\gamma : \mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ fixes $\gamma' : B \to B$, the map $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\Gamma(\mathbf{0}) := \mathbf{0} \, , \quad \Gamma(\mathbf{v}) := |\mathbf{v}|\gamma'\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) \, , \quad \mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$$

extends γ', is linear and invertible. The proof is straightforward. Kadison automorphisms, being isomorphisms, map extreme elements to extreme elements, so $\Gamma(\mathbf{n}) = \gamma'(\mathbf{n}) = 1$ if $|\mathbf{n}| = 1$, and by linearity:

$$|\Gamma(\mathbf{v})| = |\mathbf{v}|\,, \quad \mathbf{v} \in \mathbb{R}^3\,.$$

In conclusion the linear map $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ associated to the Kadison automorphism γ is an isometry of $\mathbb{R}^3$ with the origin as fixed point, so $\Gamma \in O(3)$, the *three-dimensional orthogonal group*. Conversely, $\Gamma \in O(3)$ implies $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H})}$ is an automorphism according to Wigner. In fact if $\rho_{\mathbf{n}}$ and $\rho_{\mathbf{m}}$ are pure, their transition probability equals $tr\,(\rho_{\mathbf{n}}\rho_{\mathbf{m}})$, which we can express via (12.15). Since Γ is orthogonal:

$$tr\,(\gamma(\rho_{\mathbf{n}})\gamma(\rho_{\mathbf{m}})) = \frac{1}{2}\,(1 + \Gamma(\mathbf{n})\cdot\Gamma(\mathbf{m})) = \frac{1}{2}\,(1 + \mathbf{n}\cdot\mathbf{m}) = tr\,(\rho_{\mathbf{n}}\rho_{\mathbf{m}})\,.$$

Recalling $\gamma'\restriction_{\partial B} = \Gamma\restriction_{\partial B}: \partial B \to \partial B$ is trivially a bijection (this is true for every orthogonal matrix), then $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H})}: \mathfrak{S}_p(\mathsf{H}) \to \mathfrak{S}_p(\mathsf{H})$ is bijective, and hence a Wigner automorphism. Wigner's theorem implies the existence of a unitary or anti-unitary operator $U : \mathsf{H} \to \mathsf{H}$ such that $\gamma(\rho) = U\rho U^{-1}$ for any $\rho \in \mathfrak{S}_p(\mathsf{H})$. If $\rho \in \mathfrak{S}(\mathsf{H})$ we can decompose it as convex combinations of two pure states associated to the eigenvectors of ρ. If $\rho_1, \rho_2 \in \mathfrak{S}_p(\mathsf{H})$ are the states in question for some $p \in [0, 1]$, then $\rho = p\rho_1 + (1-p)\rho_2$, and so

$$\gamma(\rho) = p\gamma(\rho_1) + (1-p)\gamma(\rho_2) = pU\rho_1U^{-1} + (1-p)U\rho_2U^{-1}$$

$$= U\,(p\rho_1 + (1-p)\rho_2)\,U^{-1} = U\rho U^{-1}\,.$$

Therefore the unitary (or anti-unitary) operator U satisfies the theorem's claim. □

Let us state and prove Kadison's theorem in general. (Kadison originally proved the non-trivial statements (a) and (b)).

Theorem 12.14 (Kadison). *Consider a quantum system S described on the (not necessarily separable) complex Hilbert space H_S. Suppose H_S coherently splits*[3] *as $\mathsf{H}_S = \oplus_{k\in K}\mathsf{H}_{Sk}$ and* $\dim \mathsf{H}_{Sk} > 1$ *for every $k \in K$. Suppose the map*

$$\gamma : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$$

is a symmetry of S according to Kadison from H_{Sk} to $\mathsf{H}_{Sk'}$, $k, k' \in K$. Then
(a) *there exists an operator $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$, unitary or anti-unitary depending on γ, such that:*

$$\gamma(\rho) = U\rho U^{-1} \quad \textit{for every state } \rho \in \mathfrak{S}(\mathsf{H}_{Sk}). \tag{12.16}$$

(b) *U is determined up to phase, i.e. U_1 satisfies (12.16) (replacing U) if and only if $U_1 = \chi U$ with $\chi \in \mathbb{C}$, $|\chi| = 1$.*
(c) *The restriction of γ to pure states is a Wigner symmetry.*
(d) *Every Wigner symmetry $\delta : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sk'})$ extends, uniquely, to a Kadison symmetry $\gamma^{(\delta)} : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk'})$.*

[3] If $K = \{1\}$ one should replace H_{Sk}, $\mathsf{H}_{Sk'}$ by H in the statement.

In case $\dim \mathsf{H}_{Sk} = 1 (= \dim \mathsf{H}_{Sk'})$ *for some k, then (a), (c), (d) are still valid, with the difference that* γ *(which is unique) can be represented indifferently by a unitary or an anti-unitary operator.*

Proof (b) and (c). Suppose U in (a) exists. Since γ is bijective and preserves convexity, it preserves extreme and non-extreme sets, and maps pure states to pure states and mixed ones to mixed ones. It is therefore clear that $\delta := \gamma\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})}$ defines a Wigner symmetry associated to the same U and thus the common character (unitary or anti-unitary) of U and any other operator satisfying (12.16) is fixed by δ. By the same argument it is clear that U is determined up to a phase by δ, in view of Wigner's theorem. Part (d) will show that δ completely defines γ and so the character of U is determined by γ, and U itself is determined up to a phase by γ.

(d) If δ is a Wigner symmetry, by Wigner's theorem there is a unitary or anti-unitary operator U such that $\delta(\rho) = U\rho U^{-1}$ for any pure state, which defines the Kadison symmetry $\gamma^{(\delta)}(\rho) = U\rho U^{-1}$ extending δ to the whole space of states. Let us prove the uniqueness of $\gamma^{(\delta)}$. If two Kadison symmetries γ, γ', associated to U, U' (unitary or anti-unitary), coincide on $\mathfrak{S}_p(\mathsf{H}_{Sk})$, then the Wigner symmetries $\delta^{(U)} = U \cdot U^{-1}$, $\delta^{(U')} = U' \cdot U'^{-1}$ are the same. By Wigner's theorem U and U' are both unitary or both anti-unitary, and $U = \chi U'$ with $|\chi| = 1$. Therefore

$$\gamma(\rho) = U\rho U^{-1} = \chi U'\rho U'^{-1}\chi^{-1} = \chi\chi^{-1}U'\rho U'^{-1} = U'\rho U'^{-1} = \gamma'(\rho)$$

for every $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$, so $\gamma = \gamma'$.

Let us pass to (a) and divide the proof in steps. As already remarked γ maps pure states to pure states and mixed ones to mixed ones, bijectively. We claim that if $\mathsf{M} \subset \mathsf{H}_{Sk}$ is a two-dimensional subspace there is a two-dimensional space $\mathsf{M}' \subset \mathsf{H}_{Sk'}$ such that $\gamma(\mathfrak{S}(\mathsf{M})) = \mathfrak{S}(\mathsf{M}')$. If ψ_1, ψ_2 are unit vectors forming a (non-orthogonal, in general) basis of M, the generic element in $\mathfrak{S}(\mathsf{M})$ is $\rho = p\psi_1(\psi_1|\ ) + q\psi_2(\psi_2|\ )$, $p + q = 1$ and $p, q \geq 0$. Hence:

$$\gamma(\rho) = p\gamma(\psi_1(\psi_1|\ )) + q\gamma(\psi_2(\psi_2|\ )) = p\psi_1'(\psi_1'|\ ) + q\psi_2'(\psi_2'|\ )\ ,$$

where the unit vectors ψ_1', ψ_2' arise (up to phase) from the corresponding pure states $\gamma(\psi_1(\psi_1|\ )), \gamma(\psi_1(\psi_1|\ ))$. The latter must be distinct, otherwise the bijection $\gamma^{-1} : \mathfrak{S}(\mathsf{H}_{Sk'}) \to \mathfrak{S}(\mathsf{H}_{Sk})$, that preserves convexity, would map pure to mixed. So ψ_1' and ψ_2', both unit, satisfy $\psi_1' \neq a\psi_2'$ for any $a \in \mathbb{C}$, and hence are linearly independent. The space M' is then generated by ψ_1', ψ_2'.

Now we need two lemmas.

Lemma 12.15 *Under our assumptions on* γ*, there exists a Wigner symmetry* $\delta : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sk'})$ *such that* $\gamma(\rho) = \delta(\rho)$ *for every* $\rho \in \mathfrak{S}_p(\mathsf{H}_{Sk})$.

Proof of Lemma 12.15. Since γ and γ^{-1} preserve extreme and non-extreme sets, $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})} : \mathfrak{S}_p(\mathsf{H}_{Sk}) \to \mathfrak{S}_p(\mathsf{H}_{Sk'})$ is invertible, because the left and right inverse is just $\gamma^{-1}\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk'})} : \mathfrak{S}_p(\mathsf{H}_{Sk'}) \to \mathfrak{S}_p(\mathsf{H}_{Sk})$. The proof ends once we show $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})}$ preserves transition probabilities. Given $\phi, \psi \in \mathsf{H}_{Sk}$ unit and distinct, let M be their span and $\mathsf{M}' \subset \mathsf{H}_{Sk'}$ the two-dimensional space such that $\gamma(\mathfrak{S}(\mathsf{M})) \subset \mathfrak{S}(\mathsf{M}')$, as above. Call $U : \mathsf{M}' \to \mathsf{M}$ an arbitrary unitary operator. Define

$$\gamma'(\rho) := U\gamma(\rho)U^{-1}\,, \quad \rho \in \mathfrak{S}(\mathsf{M}).$$

Immediately, γ' is a Kadison symmetry if we restrict to the 2-dimensional Hilbert space $\mathsf{H} = \mathsf{M}$. As shown in Proposition 12.13, Kadison's theorem holds and there is a unitary, or anti-unitary, map $V : \mathsf{M} \to \mathsf{M}$ such that $\gamma'(\rho) = U\gamma(\rho)U^{-1} = V\rho V^{-1}$. Otherwise said:

$$\gamma(\rho) = UV\rho(UV)^{-1}\,, \quad \rho \in \mathfrak{S}(\mathsf{M}).$$

We stress that even though U and V depend on M and M', their existence is enough to conclude the proof. Consider any two linearly independent unit vectors $\psi, \phi \in \mathsf{H}_{Sk}$, define M as their span and M' as the span of the associated $\psi', \phi' \in \mathsf{H}_{Sk'}$, obtained (up to phase) by applying γ as we said above. Using the result we have found,

$$tr\,(\gamma(\psi(\psi|\ ))\gamma(\phi(\phi|\ ))) = tr\left(UV\psi(\psi|\ )(UV)^{-1}UV\phi(\phi|\ )(UV)^{-1}\right) =$$

$$= tr\left(UV\psi(\psi|\ )\phi(\phi|\ )(UV)^{-1}\right) = tr\,(\psi(\psi|\ )\phi(\phi|\ ))\ .$$

In other words $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})}$ preserves transition probabilities between different pure states. If instead $\psi(\psi|\ ) = \phi(\phi|\ )$, we have

$$tr\,(\gamma(\psi(\psi|\ ))\gamma(\psi(\psi|\ ))) = 1 = tr\,(\psi(\psi|\ )\psi(\psi|\ ))\ ,$$

trivially. So we have proved that $\gamma\restriction_{\mathfrak{S}_p(\mathsf{H}_{Sk})}$ preserves transition probabilities, and hence is a Wigner symmetry. □

By the previous lemma, and invoking Wigner's Theorem 12.11, there exists a unitary, or anti-unitary, operator $U : \mathsf{H}_{Sk} \to \mathsf{H}_{Sk'}$ such that

$$\gamma(\rho) = U\rho U^{-1}\,, \quad \rho \in \mathfrak{S}_p(\mathsf{H}_{Sk}). \tag{12.17}$$

The proof now ends if we prove that the above identity holds also for $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$, and not only for $\mathfrak{S}_p(\mathsf{H}_{Sk})$. For that, note (12.17) is equivalent to:

$$U^{-1}\gamma(\rho)U = \rho\,, \quad \rho \in \mathfrak{S}_p(\mathsf{H}_{Sk}). \tag{12.18}$$

Therefore $\Gamma := U^{-1}\gamma(\cdot)U : \mathfrak{S}(\mathsf{H}_{Sk}) \to \mathfrak{S}(\mathsf{H}_{Sk})$ is a Kadison symmetry (a Kadison automorphism, actually) that reduces to the identity on pure states. Kadison's theorem is eventually proved after we establish the following lemma.

Lemma 12.16 *Let $\mathsf{H} \neq \{\mathbf{0}\}$ be a Hilbert space. If a Kadison automorphism Γ : $\mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ restricts to the identity on pure states, it is the identity.*

Proof of Lemma 12.16. Let $\rho = \sum_{k=0}^{N} p_k\psi_k(\psi_k|\ )$ be a finite convex combination of pure states. Then

$$\Gamma(\rho) = \Gamma\left(\sum_{k=0}^{N} p_k \psi_k (\psi_k|\ )\right) = \sum_{k=0}^{N} p_k \Gamma\left(\psi_k(\psi_k|\ )\right) = \left(\sum_{k=0}^{N} p_k\right) I = I\ .$$

Therefore the claim holds for every $\rho \in \mathfrak{S}(\mathsf{H})$ provided finite (convex) combinations of pure states are dense in $\mathfrak{S}(\mathsf{H})$ in some topology for which Γ is continuous. Let us show this works if we take the topology of trace-class operators induced by the norm $||T||_1 := tr(|T|)$ (see Chap. 4).

If $\rho \in \mathfrak{S}(\mathsf{H})$ we can decompose the operator spectrally:

$$\rho = \sum_{k\in\mathbb{N}} p_k \psi_k (\psi_k|\ )\ ,$$

where $p_k > 0$, $\sum_{k\in\mathbb{N}} p_k = 1$. Convergence is understood in the strong topology, and also in uniform topology (if $p_k \geq p_{k-1}$), as we know from Chap. 4. Let us prove, further, that we may approximate ρ by finite (convex) combinations of pure states $\rho_N \in \mathfrak{S}(\mathsf{H})$, so that $||\rho_N - \rho||_1 \to 0$ as $N \to +\infty$. To this end set:

$$\rho_N := \sum_{k=0}^{N} q_k^{(N)} \psi_k(\psi_k|\ )\ , \quad q_k^{(N)} := \frac{p_k}{\sum_{j=0}^{N} p_j}\ , \quad \text{N=0,1,2,}\ldots\ .$$

Evidently $\rho_N \in \mathfrak{S}(\mathsf{H})$ for any $N \in \mathbb{N}$. Since $q_k^{(N)} > p_k$ and the unit vectors ψ_k (adding a basis of $ker(\rho) \supset ker(\rho_N)$) give a basis of H of eigenvectors of ρ, ρ_N and hence of $\rho - \rho_N$. The trace of $|\rho - \rho_N|$ in that basis satisfies

$$||\rho - \rho_N||_1 = tr\left(|\rho - \rho_N|\right) = \sum_{k=0}^{N} |p_k - q_k^{(N)}| + \sum_{k=N+1}^{+\infty} |p_k|$$

$$= \frac{1 - \sum_{j=0}^{N} p_j}{\sum_{j=0}^{N} p_j} \sum_{k=0}^{N} p_k + \sum_{k=N+1}^{+\infty} p_k \to 0 \quad \text{as } N \to +\infty\ .$$

The limit exists and vanishes because $p_n > 0$ and $\sum_{n=1}^{+\infty} p_n = 1$.

We will show Γ is continuous for $||\ ||_1$, and conclude. First of all extend Γ from $\mathfrak{S}(\mathsf{H})$ to positive trace-class operators on H, by defining:

$$\Gamma_1(A) := tr(A)\Gamma\left(\frac{1}{trA}A\right)\ , \quad \Gamma_1(0) := 0$$

where $A \in \mathfrak{B}_1(\mathsf{H})$, $A \geq 0$ (so $tr(A) > 0$ if $A \neq 0$). It follows that $\Gamma_1(A) \in \mathfrak{B}_1(\mathsf{H})$, $\Gamma_1(A) \geq 0$, and:

$$\Gamma_1(\alpha A) = \alpha\Gamma_1(A) \quad \text{if}\ \alpha \geq 0, \text{ and } \quad tr\left(\Gamma_1\left(A\right)\right) = tr(A)\ .$$

Since Γ preserves convexity, it is not hard to see

$$\Gamma_1\left(A+B\right)=\Gamma_1(A)+\Gamma_1(B)\,.$$

To conclude extend Γ_1 to self-adjoint trace-class operators:

$$\Gamma_2(A):=\Gamma_1(A_+)-\Gamma_1(A_-)\,,$$

where $A_-:=-\int_{(-\infty,0)}x\,dP^{(A)}(x)$ and $A_+:=\int_{[0,+\infty)}x\,dP^{(A)}(x)$. Observe $A_+-A_-=A$ and $|A|=A_++A_-$ by definition, since $P^{(A)}$ is the PVM of A.

If $A\in\mathfrak{B}_1(\mathsf{H})$ is self-adjoint, then $\Gamma_2(A)$ belongs to $\mathfrak{B}_1(\mathsf{H})$ and is self-adjoint. Moreover:

$$||\Gamma_2(A)||_1\leq||\Gamma_1(A_+)||_1+||\Gamma_1(A_-)||_1=tr\,(A_+)+tr\,(A_-)=||A||_1\,.$$

Therefore Γ_2 is continuous for $||\ \ ||_1$, and so also its restriction $\Gamma:\mathfrak{S}(\mathsf{H})\to\mathfrak{S}(\mathsf{H})$ is. □

Altogether we have proved the existence of U unitary, or anti-unitary, satisfying $\gamma(\rho)=U\rho U^{-1}$ for any $\rho\in\mathfrak{S}(\mathsf{H}_{Sk})$. This ends part (a), and the proof of Kadison's theorem is concluded since the proof of the last statement in the thesis is trivial. □

Form the last part of the proof of part (a) we can extract yet another fact, interesting by its own means.

Proposition 12.17 *Let γ be a Wigner (or Kadison) automorphism of the complex Hilbert space H, and denote by $\mathfrak{B}_1(\mathsf{H})_{\mathbb{R}}\subset\mathfrak{B}_1(\mathsf{H})$ the real space of trace-class self-adjoint operators with norm $||\ \ ||_1$.*

There exist a unique continuous linear operator $\gamma_2:\mathfrak{B}_1(\mathsf{H})_{\mathbb{R}}\to\mathfrak{B}_1(\mathsf{H})_{\mathbb{R}}$ that restricts to γ on $\mathfrak{S}_p(\mathsf{H})$ (or on $\mathfrak{S}(\mathsf{H})$, respectively). More precisely

$$||\gamma_2(A)||_1\leq||A||_1\quad\textit{for every } A\in\mathfrak{B}_1(\mathsf{H})_{\mathbb{R}}.$$

Finally, $\gamma_2(A)\geq 0$ if $A\geq 0$.

Proof If γ is a Kadison automorphisms, the proof is contained in Lemma 12.16, where we proved the existence of Γ_2 given Γ (above called γ_2 and γ). Uniqueness holds because linearity implies

$$\gamma_2(A)=tr(A_+)\gamma_2\left(\frac{A_+}{tr(A_+)}\right)-tr(A_-)\gamma_2\left(\frac{A_-}{tr(A_-)}\right)\,,\quad A\in\mathfrak{B}_1(\mathsf{H})_{\mathbb{R}}\,,$$

where the right-hand side depends on γ only (and a summand is omitted if the relevant trace vanishes). For Wigner automorphisms the proof follows from the Kadison case, by statement (d) in Kadison's theorem. □

12.1.6 Dual Action and Inverse Dual Action of Symmetries on Observables

The theorems of Wigner and Kadison enable us to define in a very elementary manner the (dual) *action* of a symmetry *on the observables* of the physical system.

Consider a physical system S described on the Hilbert space H_S. For simplicity we shall consider the case of one sector only, as the generalisation to several coherent sectors is immediate. We know the set $\mathscr{L}(\mathsf{H}_S)$ of elementary propositions on S is described by orthogonal projectors on H. Observables on S are PVMs built with these projectors, i.e. self-adjoint (in general unbounded) operators associated to the PVMs.

Suppose $\gamma : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$ is a symmetry associated with the (anti-)unitary operator U, up to a phase. We define its **dual action** $\gamma^* : \mathscr{L}(\mathsf{H}_S) \to \mathscr{L}(\mathsf{H}_S)$ on the lattice of projectors by:

$$\gamma^*(P) := U^{-1}PU \ , \quad P \in \mathscr{L}(\mathsf{H}_S) \tag{12.19}$$

(the arbitrary phase affecting U being irrelevant). A duality identity holds:

$$tr\left(\rho\gamma^*(P)\right) = tr\left(\gamma(\rho)P\right) \ . \tag{12.20}$$

This follows $\gamma(\rho) = U\rho U^{-1}$ by Kadison's theorem and the fact that (when computing traces if U is anti-unitary) anti-unitary operators preserve bases.

The mapping $\gamma^* : \mathscr{L}(\mathsf{H}_S) \to \mathscr{L}(\mathsf{H}_S)$ not only transforms orthogonal projectors into orthogonal projectors, but also preserves orthocomplemented, σ-complete bounded lattices. It is a *(bounded, orthocomplemented, σ-complete) lattice automorphism* according to Definition 7.13. For example, the orthogonal projectors P, Q of $\mathscr{L}(\mathsf{H}_S)$ commute if and only if $\gamma^*(P)$ commutes with $\gamma^*(Q)$. Moreover $\gamma^*(P \vee Q) = \gamma^*(P) \vee \gamma^*(Q)$, and so on.

If $A : D(A) \to \mathsf{H}$ is self-adjoint on H with spectral measure $P^{(A)} \subset \mathscr{L}(\mathsf{H}_S)$, then $U^{-1}AU : U^{-1}D(A) \to \mathsf{H}_S$ is self-adjoint with spectral measure $\gamma^*\left(P^{(A)}\right)$ (see Exercise 9.1 if unitary, Exercise 12.8 if anti-unitary). This fact allows to extend the action of γ^* to *every observable* in agreement with the spectral decomposition: just define, for a self-adjoint operator $A : D(A) \to \mathsf{H}_S$ representing an observable of S:

$$\gamma^*(A) := U^{-1}AU \ . \tag{12.21}$$

The physical meaning of $\gamma^*(A)$ is the following. When we define a Kadison symmetry γ, we are prescribing an experimental procedure under which *the system S* should be transformed. Mathematically speaking the action on states is described precisely by $\gamma : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$. The action γ^* on observables, instead, represents operative procedures *on measuring instruments* which, intuitively, correspond and generalise *passive transformations of the coordinates*. Better said, *the procedure is such that if we act by γ act on the system, or by γ^* on the instrument, we obtain the* *same result (expectation values, variances, outcome frequencies) as when we take the measurements.*

For instance, $\langle\gamma^*(A)\rangle_\rho$ and $\langle A\rangle_{\gamma(\rho)}$ are equal expectation values:

$$\langle\gamma^*(A)\rangle_\rho = tr\left(\gamma^*(A)\rho\right) = tr\left(U^{-1}AU\rho\right) = tr\left(AU\rho U^{-1}\right) = tr(A\gamma(\rho)) = \langle A\rangle_{\gamma(\rho)} .$$

This is, in practice, the content of the duality Eq. (12.20). The result is equivalent to saying that *the action of* γ *on the system can be neutralised, concerning measurement readings on the system, by the simultaneous action of* $(\gamma^*)^{-1}$ *on the instruments.*

Another action of symmetries on observables of great relevance in physical application is the aforementioned **inverse dual action** on the self-adjoint operator A (in particular an orthogonal projector $P = A$)

$$\gamma^{*-1}(A) := UAU^{-1} . \tag{12.22}$$

It plays a crucial role in Quantum Field Theory when transforming field operators. The meaning of γ^{*-1} is just *the action of* γ *on instruments that neutralises the action of* γ *on the system.*

We shall return to these actions in Sect. 12.2.2 when we deal with groups of symmetries, and in Sect. 14.3.2 when discussing symmetries by the algebraic approach.

In the rest of the book γ^* and γ^{*-1} will be often interpreted as maps $\mathfrak{B}(\mathsf{H}_S) \to \mathfrak{B}(\mathsf{H}_S)$, as a matter of fact automorphisms (or anti-automorphisms if U is anti-unitary) of $\mathfrak{B}(\mathsf{H}_S)$, defined by (12.21) and (12.22) respectively when U is given (up to phase). The extension to many superselection sectors is trivial.

Remark 12.18 From the experimental point of view it is not obvious that a transformation acting on the system can be cancelled by a simultaneous action on the measuring instrument. Symmetries, *à la* Kadison or Wigner, have this property. ■

Example 12.19
(1) Consider a spinless quantum particle described on $\mathbb{R}^3$, thought of as rest space of an inertial frame system with given positively-oriented orthonormal coordinates. From Chap. 11 we know the particle's Hilbert space is $L^2(\mathbb{R}^3, dx)$. Pure states are thus determined, up to arbitrary phases, by *wavefunctions*, i.e. by vectors $\psi \in L^2(\mathbb{R}^3, dx)$ such that $\int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 dx = 1$.

We want to explain how the isometries of $\mathbb{R}^3$ determine Wigner symmetries (hence Kadison symmetries) by the *invariance of the Lebesgue measure*.

The notions of group theory used in the sequel will be summarised at a later stage (elementary facts are present in the book's appendix). Denote by $IO(3)$ the **isometry group of** $\mathbb{R}^3$, which is the *semidirect product* (see the appendix) of $O(3)$ with the *Abelian group of translations* $\mathbb{R}^3$. In practice, every element $\Gamma \in IO(3)$ is a pair $\Gamma = (R, \mathbf{t})$ acting on $\mathbb{R}^3$ as follows: $\Gamma(\mathbf{x}) := \mathbf{t} + R\mathbf{x}$. The composition law of $IO(3)$ is:

$$(\mathbf{t}', R') \circ (\mathbf{t}, R) = (\mathbf{t}' + R'\mathbf{t},\ R'R) \quad \text{hence} \quad (\mathbf{t}, R)^{-1} = (-R^{-1}\mathbf{t},\ R^{-1}) .$$

Let $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ belong to $IO(3)$, so Γ could in particular be: a translation $\Gamma : \mathbb{R}^3 \ni \mathbf{x} \mapsto \mathbf{x}+\mathbf{t}$ along an axis $\mathbf{t}$, a rotation of $O(3)$ about the origin $\mathbb{R}^3 \ni \mathbf{x} \mapsto R\mathbf{x}$ (including rotations with negative determinant), or a combinations of the two. We can define a transformation of square-integrable maps:

$$(U_\Gamma \psi)(\mathbf{x}) := \psi\left(\Gamma^{-1}\mathbf{x}\right), \quad \psi \in L^2(\mathbb{R}^3, dx). \tag{12.23}$$

The operator U is clearly linear, surjective (every isometry Γ of $\mathbb{R}^3$ is bijective) and isometric, as the Jacobian matrix J of an isometry has determinant ± 1:

$$||U_\Gamma \psi||^2 = \int_{\mathbb{R}^3} \left|\psi\left(\Gamma^{-1}\mathbf{x}\right)\right|^2 dx = \int_{\mathbb{R}^3} \left|\psi\left(\mathbf{x}'\right)\right|^2 |\det J| dx' = \int_{\mathbb{R}^3} \left|\psi\left(\mathbf{x}'\right)\right|^2 dx' = ||\psi||^2 \;.$$

The transformation γ_Γ induced by the unitary operator U_Γ on states (pure or mixed) is a symmetry (Wigner or Kadison, respectively), which naturally represents the action of the isometry Γ on S given by the particle examined.

The map $IO(3) \ni \Gamma \mapsto U_\Gamma$ satisfies

$$U_{id} = I\,, \quad U_\Gamma U_{\Gamma'} = U_{\Gamma \circ \Gamma'}\,, \quad \Gamma, \Gamma' \in IO(3)$$

where id is the identity of $IO(3)$, because of (12.23). Hence $IO(3) \ni \Gamma \mapsto U_\Gamma$ preserves the group structure (in particular $U_{\Gamma^{-1}} = (U_\Gamma)^{-1}$); as such it is *a representation* of the group $IO(3)$ by unitary operators. We will discuss these representations in the next section.

Take now a PVM on $\mathbb{R}^3$, denoted $P^{(X)}$, that coincides with the *joint spectral measure* (see Theorem 9.19) of the three position operators:

$$(P^{(X)}_E \psi)(\mathbf{x}) = \chi_E(\mathbf{x})\psi(\mathbf{x})\,, \quad \psi \in L^2(\mathbb{R}^3, dx).$$

It is easy to prove the position operators arise by integrating the corresponding functions in this PVM:

$$X_i = \int_{\mathbb{R}^3} x_i dP^{(X)}(x) \quad i = 1, 2, 3.$$

Directly from Definition (12.23) the so-called *imprimitivity condition* holds:

$$U_\Gamma P^{(X)}_E U_\Gamma^{-1} = P^{(X)}_{\Gamma(E)}\;. \tag{12.24}$$

In fact for a generic map $\psi \in L^2(\mathbb{R}^3, dx)$:

$$\left(U_\Gamma P^{(X)}_E U_\Gamma^{-1} \psi\right)(\mathbf{x}) = \chi_E\left(\Gamma^{-1}(\mathbf{x})\right)\psi\left(\left(\Gamma\left(\Gamma^{-1}(\mathbf{x})\right)\right)\right) = \chi_{\Gamma(E)}(\mathbf{x})\psi(\mathbf{x})$$

$$= \left(P^{(X)}_{\Gamma(E)}\psi\right)(\mathbf{x})\;.$$

Equation (12.24) follows since ψ is arbitrary. Note that the imprimitivity equation can be written equivalently in terms of the inverse dual action of the Kadison symmetry:

$$\gamma_\Gamma^{*-1}\left(P_E^{(X)}\right) = P_{\Gamma(E)}^{(X)} .$$

In general a **system of imprimitivity** on X according to Mackey is given by: (i) a PVM P on the separable complex Hilbert space H for the Borel σ-algebra of the metrisable space X (that admits a metric making it complete and separable), (ii) a second-countable, locally compact group G acting on X so that the action[4] $\mathsf{G} \times X \ni (g, x) \mapsto gx \in X$ is measurable, (iii) a unitary representation $\mathsf{G} \ni g \mapsto V_g \in \mathfrak{B}(\mathsf{H})$ that is strongly continuous and satisfies the **imprimitivity condition** (which is better written using the inverse dual action of the symmetry induced by V_g):

$$V_g P_E V_g^{-1} = P_{g(E)} \quad \text{for any } E \in \mathscr{B}(X),\, g \in \mathsf{G} .$$

The imprimitivity system is said to be **transitive** when the action of G on X is transitive, i.e. such that any two points $x_1, x_2 \in X$ can be transformed into one another, $x_2 = gx_1$, by some $g \in \mathsf{G}$. The unitary representations of G for any imprimitivity system, up to unitary equivalence, are determined by the famous **Imprimitivity theorem of Mackey**, which we shall not be concerned with (see for instance [Jau73, Var07]).

We have verified that $P^{(X)}, IO(3), U$ form a *transitive imprimitivity system on* $\mathbb{R}^3$ (we did not check the topological requests, which hold if we embed $IO(3)$ in the Lie group $GL(4)$). Transitivity is obvious from elementary geometry.

The action of γ_Γ^* on the position operators can be obtained by direct computation, in analogy to the imprimitivity condition, or using the latter to integrate the spectral measure. Let $\mathbf{X} = (X_1, X_2, X_3)$ be the column vector of the X_1, X_2, X_3 restricted to the common invariant Schwartz domain $\mathscr{S}(\mathbb{R}^3)$, where the operators are essentially self-adjoint. Then

$$\gamma_\Gamma^*(\mathbf{X}) = U_\Gamma^{-1}\mathbf{X}U_\Gamma = R\mathbf{X} + \mathbf{t}I , \tag{12.25}$$

and in particular, considering pure translations:

$$\gamma_{(\mathbf{t},I)}^*(\mathbf{X}) = U_{(\mathbf{t},I)}^{-1}\mathbf{X}U_{(\mathbf{t},I)} = \mathbf{X} + \mathbf{t}I , \tag{12.26}$$

and pure rotations:

$$\gamma_{(\mathbf{0},R)}^*(\mathbf{X}) = U_{(\mathbf{0},R)}^{-1}\mathbf{X}U_{(\mathbf{0},R)} = R\mathbf{X} . \tag{12.27}$$

The element $(\mathbf{0}, -I) \in IO(3)$ defines the reflection about the origin. The unitary representation $\mathscr{P} := U_{(\mathbf{0},-I)}$, and the associated Wigner (Kadison) symmetry $\gamma_{\mathscr{P}}$, are called **parity inversion**. Not so precisely, one often calls $(\mathbf{0}, -I)$ parity inversion. Easily, $\mathscr{P}^* = \mathscr{P}$ (so $\mathscr{P}\mathscr{P} = I$ as $\mathscr{P}^{-1} = \mathscr{P}^*$). Therefore the inversion of parity

[4]The map $(g, x) \mapsto gx$ is customarily taken so that $g'(gx) = (g'g)x$ and $ex = x$ for every $g, g' \in \mathsf{G}$, $x \in X$, where $e \in \mathsf{G}$ is the neutral element.

admits an associated observable, called **parity**, with two possible eigenvalues ± 1. We must emphasise that the unitary operator representing $(\mathbf{0}, -I)$ is actually defined, as usual, up to phase, so the observable $\mathscr{P}$ associated to the parity symmetry corresponds to a specific choice of phase. There remain two possibilities for the phase, namely the sign of the observable $\mathscr{P}$, since also $-\mathscr{P}$ is an observable representing the inversion of parity.

(2) Consider the system of the previous example, and let us study it via the *momentum representation*. Using the Fourier–Plancherel transform, in other terms, we identify H and $L^2(\mathbb{R}^3, dk)$, so that the three momentum observables (the components of the momentum in the orthonormal Cartesian coordinates of the inertial frame) are represented by the multiplication operators:

$$\left(P_i\widetilde{\psi}\right)(\mathbf{k}) = \hbar k_i \widetilde{\psi}(\mathbf{k}) \; ,$$

as we saw in Chap. 11. We indicate by $\widetilde{\psi} = \widehat{\mathscr{F}}(\psi)$ the Fourier–Plancherel transform of $\psi \in L^2(\mathbb{R}^3, dx)$. An extremely interesting symmetry in physics is the **time reversal** $\gamma_{\mathscr{T}}$, described by anti-unitary operators (later we will see why). This symmetry corresponds to flipping the sign of time, but also changing sign to particles' velocities and hence to their momentum. The anti-unitary operator $\widetilde{\mathscr{T}}$ describing time reversal can be chosen (uniquely, up to phase) thus:

$$\left(\widetilde{\mathscr{T}}\widetilde{\psi}\right)(\mathbf{k}) := \overline{\widetilde{\psi}(-\mathbf{k})} \; , \quad \widetilde{\psi} \in L^2(\mathbb{R}^3, dk). \tag{12.28}$$

In contrast to $\mathscr{P}$ in the previous example, any chosen phase for $\widetilde{\mathscr{T}}$ maintains $\widetilde{\mathscr{T}}\widetilde{\mathscr{T}} = I$ because $\widetilde{\mathscr{T}}$ is anti-unitary. Nonetheless, $\widetilde{\mathscr{T}}$ is not an observable since the operator is not linear. Reverting to the *position representation* with the chosen phase, it can be proved that the symmetry $\gamma_{\mathscr{T}}$ is associated to an anti-unitary operator

$$\mathscr{T} := \widehat{\mathscr{F}}^{-1}\widetilde{\mathscr{T}}\widehat{\mathscr{F}}$$

($\widehat{\mathscr{F}}$ is the Fourier–Plancherel transform as in Chap. 11), such that

$$(\mathscr{T}\psi)(\mathbf{x}) := \overline{\psi(\mathbf{x})} \; , \quad \psi \in L^2(\mathbb{R}^3, dx). \tag{12.29}$$

We will come back to the time-reversal symmetry in Example 13.22 and determine completely its form.

(3) Consider a particle having electric charge represented by the observable Q with discrete spectrum made by eigenvalues ± 1. Fix an inertial frame $\mathscr{I}$, with orthonormal Cartesian coordinates for which the rest space is $\mathbb{R}^3$. Then the system's Hilbert space is

$$\mathsf{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3, dx) \equiv L^2(\mathbb{R}^3, dx) \oplus (\mathbb{R}^3, dx) \; ,$$

where $\oplus$ denotes orthogonal sum. The canonical isomorphism between the above spaces (cf. Example 10.27(2) as well) descends from the fact that any $\Psi \in \mathbb{C}^2 \otimes$

$L^2(\mathbb{R}^3, dx)$ can be written:

$$\Psi = |+\rangle \otimes \psi_+ \ + \ |-\rangle \otimes \psi_- \ ,$$

where $\{|+\rangle, |-\rangle\}$ is the canonical basis of $\mathbb{C}^2$ made by eigenvectors of the Pauli matrix σ_3 (cf. (12.13)) with eigenvalues $+1$ and -1 respectively. Hence the isomorphism reads:

$$L^2(\mathbb{R}^3, dx) \oplus (\mathbb{R}^3, dx) \ni (\psi_+, \psi_-) \mapsto |+\rangle \otimes \psi_+ + |-\rangle \otimes \psi_- \in \mathbb{C}^2 \otimes L^2(\mathbb{R}^3, dx) .$$

It preserves the Hilbert structure (the inner product) if we view $L^2(\mathbb{R}^3, dx) \oplus (\mathbb{R}^3, dx)$ as an orthogonal sum. The charge observable can be thought of as the Pauli matrix σ_3 in $\mathbb{C}^2$, so on the complete space

$$Q = \sigma_3 \otimes I \ ,$$

where I is the identity on $L^2(\mathbb{R}^3, dx)$. The superselection rule of the charge, in this simple situation, requires that the space split in two coherent sectors $\mathsf{H} = \mathsf{H}_+ \oplus \mathsf{H}_-$, where $\mathsf{H}_\pm$ are the ± 1-eigenspaces of Q. By construction, the coherent decomposition coincides exactly with the natural:

$$\mathsf{H} = L^2(\mathbb{R}^3, dx) \oplus L^2(\mathbb{R}^3, dx) \ .$$

Referring to the latter, admissible pure states are only those determined by vectors $(\psi, \mathbf{0})$ or $(\mathbf{0}, \psi)$, with $\psi \in L^2(\mathbb{R}^3, dx)$. Therefore the symmetry $\gamma_{\mathscr{C}_+}$, called **conjugation of the charge from the sector** H_+ *to the sector* H_- is represented by the unitary operator $\mathscr{C} : \mathsf{H}_+ \to \mathsf{H}_-$:

$$\mathscr{C}_+ : (\psi, \mathbf{0}) \mapsto (\mathbf{0}, \psi) \ , \quad \psi \in L^2(\mathbb{R}^3, dx). \tag{12.30}$$

The symmetry $\gamma_{\mathscr{C}_-}$, called **conjugation of the charge from the sector** H_- **to the sector** H_+ is similar:

$$\mathscr{C}_- : (\mathbf{0}, \phi) \mapsto (\phi, \mathbf{0}) \ , \quad \phi \in L^2(\mathbb{R}^3, dx). \tag{12.31}$$

Notice that $\mathscr{C}_-$ is the inverse of $\mathscr{C}_+$. Eventually, we define the Wigner symmetry called **conjugation of the charge**, that acts on the entire Hilbert space (respecting sectors) and restricts to the two above on the relative coherent spaces:

$$\mathscr{C} := \mathscr{C}_+ \oplus \mathscr{C}_- \ .$$

By construction $\mathscr{C}\mathscr{C} = I$, so $\mathscr{C} = \mathscr{C}^*$ is self-adjoint. Moreover

$$\mathscr{C}^* Q \mathscr{C} = -Q \quad . \tag{12.32}$$

■

12.1.7 Symmetries as Transformations of Observables: Symmetries as Ortho-Automorphisms and Segal Symmetries

In this section we briefly discuss the opposite approach, where quantum symmetries are *defined* from the start as bijective transformations of observables, as opposed to states which preserve some relevant algebraic structures. For the sake of simplicity we shall only consider physical systems which are not affected by superselection rules.

Since a quantum system is mainly defined by the lattice of its elementary propositions $\mathscr{L}(\mathsf{H})$, it is natural to define a symmetry of that quantum system as an *automorphism of an orthocomplemented σ-complete lattice*, according to Definition 7.13.

Remark 12.20 Proposition 7.15 establishes that an automorphism of an orthocomplemented lattice $h : \mathsf{X} \to \mathsf{X}$ is automatically an automorphism of orthocomplemented (σ-)complete lattices if X is (σ-)complete. Therefore a symmetry of a quantum system can be defined as an automorphism of the orthocomplemented lattice $\mathscr{L}(\mathsf{H})$. In the following we shall call **ortho-automorphisms** the automorphisms of an orthomodular lattice. ■

Remarkably, if the Hilbert space of the system is separable and has dimension $\neq 2$, this definition is completely equivalent to the notion of Kadison and Wigner symmetry, as we go to illustrate.

First of all observe that, if H is separable with dimension $\neq 2$, every ortho-automorphism $\alpha : \mathscr{L}(\mathsf{H}) \to \mathscr{L}(\mathsf{H})$ induces a Kadison automorphism. In fact, the property $\alpha(\vee_{j\in J} P_j) = \vee_{j\in J}\alpha(P_j)$ (Proposition 7.15) and (ii) in Theorem 7.22(b) imply

$$\alpha\left(s-\sum_{i=1}^{+\infty} P_i\right) = s-\sum_{i=1}^{+\infty}\alpha_g(P_i) \quad \text{for } \{P_i\}_{i\in\mathbb{N}} \subset \mathscr{L}(\mathsf{H}) \text{ with } P_iP_j = 0 \text{ if } i \neq j\ ,$$

where $\alpha(P_i)\alpha(P_j) = 0$ if $i \neq j$. By the above identities, on quantum systems obeying Gleason's Theorem 7.26 (systems associated to separable complex Hilbert spaces of dimension $\neq 2$) each ortho-automorphism α induces a corresponding Kadison symmetry acting on states by a duality process. In fact, if $\mu : \mathscr{L}(\mathsf{H}) \to [0, 1]$ is a quantum state in the sense of axiom **A2** (measure-formulation form) and $\alpha : \mathscr{L}(\mathsf{H}) \to \mathscr{L}(\mathsf{H})$ satisfies the previous identity, $\mu \circ \alpha : \mathscr{L}(\mathsf{H}) \to [0, 1]$ is still a state for axiom **A2** (measure-formulation form). If the Hilbert space has dimension $\neq 2$, for every mixed state (positive trace-class operator of trace one) $\rho \in \mathfrak{S}(\mathsf{H}_S)$ whose associated measure is μ, there exists a unique mixed state $\gamma_\alpha(\rho)$ associated to $\mu \circ \alpha$ due to Gleason's theorem. Therefore $\gamma_\alpha(\rho)$ is completely determined by the requirement $tr\,(\gamma_\alpha(\rho)P) = tr(\rho\alpha(P))$ for any $P \in \mathscr{L}(\mathsf{H})$. Then $\gamma_\alpha : \mathfrak{S}(\mathsf{H}) \to \mathfrak{S}(\mathsf{H})$ is immediately bijective (α is) and maps convex combinations of states to convex combinations, preserving statistical weights, and the mapping $\alpha \mapsto \gamma_\alpha$ is injective.

Put differently, every ortho-automorphism α of the lattice of elementary propositions faithfully corresponds to a Kadison symmetry γ_α. An immediate consequence of Kadison's Theorem 12.14 is that $\alpha \mapsto \gamma_\alpha$, viewed from the set of symmetries of observables to the set of Kadison symmetries, is surjective as well. In fact, if γ is a Kadison symmetry on H, then there exists a unitary or anti-unitary operator $U : \mathsf{H} \to \mathsf{H}$ such that $\gamma(\rho) = U\rho U^{-1}$ for every $\rho \in \mathfrak{S}(\mathsf{H})$ due to the aforementioned theorem. On the other hand, $\alpha : \mathscr{L}(\mathsf{H}) \ni P \mapsto U^{-1}PU \in \mathscr{L}(\mathsf{H})$ is a ortho-automorphism and $\gamma_\alpha = \gamma$ evidently. Finally observe that the dual action on observables of the Kadison symmetry γ_α associated to the lattice automorphism α coincides with the action of α itself:

$$\gamma_\alpha^*(P) = \alpha(P) \quad \text{for every } P \in \mathscr{L}(\mathsf{H}). \tag{12.33}$$

In summary we have achieved the following result.

Proposition 12.21 *For a quantum system associated to a separable Hilbert space with dimension > 2, in absence of superselection rules, ortho-automorphisms of the lattice of elementary propositions α and Kadison symmetries γ_α are in one-to-one correspondence, and they are physically equivalent under (12.33).*

Remark 12.22 The mathematical byproduct of the discussion above is the most elementary version of **Dye's theorem**.

Theorem 12.23 (Dye) *If $\mathsf{H} \neq \{\mathbf{0}\}$ is a complex Hilbert space, separable and with dimension $\neq 2$, an ortho-automorphism of $\mathscr{L}(\mathsf{H})$ is of the form*

$$\mathscr{L}(\mathsf{H}) \ni P \mapsto VPV^{-1} \in \mathscr{L}(\mathsf{H})$$

for some (unitary or anti-unitary) $V : \mathsf{H} \to \mathsf{H}$. For $\dim \mathsf{H} > 1$, *V is determined up to a phase by the automorphism itself.*

See the solution of Exercise 12.1 for the detailed proof. ■

As mentioned in the introduction to this chapter, there is a another notion of quantum symmetry introduced by Segal, that relies on the *Jordan algebra* (see Sect. 11.3.3) of operators instead of the lattice of elementary propositions. If a quantum system is described on the complex separable Hilbert space H, in absence of superselection rules the associated (real) **Jordan algebra of observables** $\mathfrak{J}(\mathsf{H}_S)$ is the real vector space of bounded self-adjoint operators in $\mathfrak{B}(\mathsf{H}_S)$ equipped with the commutative, non-associative Jordan product (11.21)

$$A \circ B := \frac{1}{2}(AB + BA) \quad \text{for all } A, B \in \mathfrak{B}(\mathsf{H}_S)\,.$$

In absence of superselection rules, $\mathfrak{J}(\mathsf{H}_S)$ contains all bounded observables of the system S, in particular the elementary propositions of $\mathscr{L}(\mathsf{H}_S)$. The product $\circ$ is the most natural product of bounded observables, as it produces self-adjoint operators when the factors are self-adjoint (see Sect. 11.3.3). It is natural to define a notion of symmetry which involves operators representing (bounded) observables, hence enlarging $\mathscr{L}(\mathsf{H}_S)$ to the whole $\mathfrak{J}(\mathsf{H}_S)$ and preserving the relevant algebraic structure. We state the corresponding definition by focusing only on quantum systems not affected by superselection rules, for the sake of simplicity.

Definition 12.24 (*Segal symmetry*). Consider a quantum physical system S described on the complex separable Hilbert space H_S in absence of superselection rules. A **weak symmetry of S according to Segal** (or **weak Segal automorphism**) is a map $\sigma : \mathfrak{J}(\mathsf{H}_S) \to \mathfrak{J}(\mathsf{H}_S)$ such that:
(a) σ is bijective;
(b) $\sigma(A \circ B) = \sigma(A) \circ \sigma(B)$ for $A, B \in \mathfrak{J}(\mathsf{H}_S)$ with $AB = BA$.
A weak symmetry according to Segal is said to be **strong** (a **strong Segal automorphism**) if (b) is valid regardless of $AB = BA$.

Evidently, if $U : \mathsf{H} \to \mathsf{H}$ is unitary or anti-unitary, the map

$$\mathfrak{J}(\mathsf{H}) \ni A \mapsto UAU^{-1} \in \mathfrak{B}(\mathsf{H})$$

is a strong Segal automorphism. A natural question is whether or not this sort of Segal automorphism exhausts all possible Segal automorphisms. The answer is once again yes: weak and strong symmetries coincide, and also coincide with Kadison and Wigner symmetries. The following remarkable result rephrases a statement in [Sim76], and will not be proved.

Theorem 12.25 *Every weak Segal automorphism $\sigma : \mathfrak{J}(\mathsf{H}) \to \mathfrak{J}(\mathsf{H})$, on a complex separable Hilbert space of $\dim(\mathsf{H}) \geq 2$, is a strong Segal automorphism, and there exists an either unitary or anti-unitary map $U : \mathsf{H} \to \mathsf{H}$, depending on σ and defined up to phase, such that*

$$\sigma(A) = UAU^{-1} \quad \textit{for every } A \in \mathfrak{J}(\mathsf{H}).$$

Therefore, every weak Segal automorphism σ defines a unique Wigner and Kadison automorphism γ, the only one satisfying

$$\gamma^{*-1}(A) = \sigma(A) \quad \textit{for every } A \in \mathfrak{J}(\mathsf{H}).$$

This means that Segal, Kadison and Wigner symmetries are physically equivalent.

12.2 Introduction to Symmetry Groups

This section is devoted to elementary topics from the theory of projective representations, applied to quantum symmetry groups.

12.2.1 Projective and Projective Unitary Representations

Suppose we look at a group G (with product $\cdot$ and neutral element e) as a group of transformations acting on a physical system S, described on the Hilbert space H_S. For simplicity we assume H_S is the only coherent sector. Suppose, further, each transformation $g \in \mathsf{G}$ is associated to a symmetry γ_g, which we can then view as a Kadison (or Wigner) automorphism. We have already met this setup in Example 12.6(1), where G was the isometry group of the three-dimensional rest space of an inertial frame and S was the particle with no charge nor spin. Kadison automorphisms in $\mathfrak{S}(\mathsf{H}_S)$ clearly form a group under composition. Hence the idea is taking shape that there is a *representation of* G *in terms of Kadison automorphisms*: these should describe the action of G on the quantum states of S. In other words we can suppose the map $\mathsf{G} \ni g \mapsto \gamma_g$ is a *group homomorphism* from G to the group of invertible maps on $\mathfrak{S}(\mathsf{H}_S)$:

$$\gamma_{g\cdot g'} = \gamma_g \circ \gamma_{g'} \,, \quad \gamma_e = id \,, \quad \gamma_{g^{-1}} = \gamma_g^{-1} \,, \quad g, g' \in \mathsf{G},$$

where id is the identity automorphism. Actually, the last condition is unnecessary because it follows from the former two by uniqueness of the inverses. We also expect, as it happens in the majority of concrete physical cases, the representation $\mathsf{G} \ni g \mapsto \gamma_g$ to be *faithful*, which means the homomorphism $\mathsf{G} \ni g \mapsto \gamma_g$ is injective. This is very often the case in physics.

Definition 12.26 (*Projective representation*). Consider a quantum system S described on the non-trivial Hilbert space H_S. Let G be a group with an injective homomorphism (a faithful representation) $\mathsf{G} \ni g \mapsto \gamma_g$ defined by Wigner automorphisms $\gamma_g : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S)$. Then G is called a **symmetry group** of S, and $\mathsf{G} \ni g \mapsto \gamma_g$ is its **projective representation on** $\mathfrak{S}_p(\mathsf{H}_S)$.

Remarks 12.27 **(1)** Referring only to Wigner symmetries is not restrictive since Kadison's theorem (in our formulation) warrants every Wigner automorphism γ_g extends, uniquely, to a Kadison automorphism $\gamma'_g : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$. It is straightforward that $\mathsf{G} \ni g \mapsto \gamma'_g$ is an injective homomorphism, i.e. a faithful representation of G by Kadison automorphisms. Conversely, every faithful G-representation by Kadison automorphisms determines a unique faithful G-representation by Wigner automorphisms, by restriction to $\mathfrak{S}_p(\mathsf{H}_S)$. In the sequel, despite mentioning Wigner symmetries most of the times, we will think the representation $\mathsf{G} \ni g \mapsto \gamma_g$ as given by Wigner or Kadison automorphisms, according to what will suit us best.
(2) The name *projective representation* is appropriate because $\mathfrak{S}_p(\mathsf{H}_s)$ is a *projective space*, as we saw in Chap. 7, and the map $\gamma_g : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S)$ is well defined.
(3) Since the homomorphism $\mathsf{G} \ni g \mapsto \gamma_g$ is explicitly required to be injective, we can equivalently take, as group of symmetries, the set of automorphisms γ_g, with $g \in \mathsf{G}$, equipped with the natural group structure coming from composition of maps. Indeed, this group is isomorphic to G by construction. ■

Now here is an interesting issue. Suppose we have a symmetry group and a projective representation $\mathsf{G} \ni g \mapsto \gamma_g$. The map $\mathsf{G} \mapsto \gamma_g$ is certainly a representation, but *not a linear representation*, because the $\gamma_g : \mathfrak{S}_p(\mathsf{H}_S) \to \mathfrak{S}_p(\mathsf{H}_S)$ are not linear maps. Yet since to every automorphism γ_g there corresponds a unitary or anti-unitary operator $U_g : \mathsf{H}_S \to \mathsf{H}_S$ that satisfies $\gamma_g(\rho) = U_g \rho U_g^{-1}$ for any $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$, a natural question arises: can $\mathsf{G} \ni g \mapsto U_g$ be an *(anti)linear representation* of G? Can it be given, in other terms, by *(anti)linear* (unitary and/or anti-unitary) operators in $\mathfrak{B}(\mathsf{H})$? We are equivalently asking whether $\mathsf{G} \ni g \mapsto U_g$ is a *group homomorphism*, i.e. if it preserves the group structure:

$$U_{g\cdot g'} = U_g U_{g'} \ , \quad U_e = I \ , \quad U_{g^{-1}} = U_g^{-1} \quad \text{for any } g, g' \in \mathsf{G}, \tag{12.34}$$

where $I : \mathsf{H}_S \to \mathsf{H}_S$ is the identity operator. The matter is relevant from a technical point to view, too: the profusion of results available on linear representations over (Hilbert) spaces can be used to study symmetry groups of quantum systems. The answer to the preceding questions is typically *negative*, because the condition $U_{g\cdot g'} = U_g U_{g'}$ in general fails. Namely, as $\gamma_g \circ \gamma_{g'} = \gamma_{gg'}$, we have

$$U_g U_{g'} \rho (U_g U_{g'})^{-1} = U_{g\cdot g'} \rho U_{g\cdot g'}^{-1} \quad \text{for any } \rho \in \mathfrak{S}(\mathsf{H}_S).$$

Consequently:

$$(U_{g\cdot g'})^{-1} U_g U_{g'} \rho (U_g U_{g'})^{-1} U_{g\cdot g'} = \rho \ , \quad \rho \in \mathfrak{S}_p(\mathsf{H}_S).$$

This means that $(U_{g\cdot g'})^{-1} U_g U_{g'}$ defines the trivial symmetry given by the identity operator I. Consequently, for $\dim(\mathsf{H}) > 1$: (a) $(U_{g\cdot g'})^{-1} U_g$ is linear (that is, $U_{g\cdot g'}$ and $U_g U_{g'}$ are both unitary or anti-unitary) and (b) $(U_{g\cdot g'})^{-1} U_g U_{g'}$ amounts to a pure phase $\omega(g, g')I$ depending on g, g'. This result is sharp – the best possible – because the U_g themselves are defined up to phase. Overall, if the (unitary or anti-unitary) U_g are associated to a projective representation of a certain symmetry group, the condition $U_{g\cdot g'} = U_g U_{g'}$ weakens, in the general case, to

$$U_g U_{g'} = \omega(g, g') U_{g\cdot g'} \ , \quad g, g' \in \mathsf{G},$$

where $\omega(g, g') \in \mathbb{C}$, $|\omega(g, g')| = 1$, are complex numbers depending on how the U_g are associated to the automorphisms γ_g, but in any case respecting the theorems of Wigner and Kadison.[5] Therefore if, as usual, $U(1)$ denotes the group of unit complex numbers, $\omega(g, g') \in U(1)$. In particular, setting $g = g' = e$, the above implicit definition of $\omega(g, g')$ tells

$$U_e = \omega(e, e) I \ , \tag{12.35}$$

[5]For $\dim(\mathsf{H}) = 1$, every γ_g coincides with the identity map and we are free to choose, for instance, $U_g = I$ so that $\omega(g, g') = 1$ for every $g, g' \in \mathsf{G}$.

(e being the neutral element of G) hence $U_e \rho U_e^{-1} = \rho$ as it should be.

It is not at all obvious that one can redefine phases so to obtain $\omega(g, g') = 1$ for every $g, g' \in \mathsf{G}$.

Remark 12.28 Henceforth we will work with unitary operators, and drop the anti-unitary case, since this is the most common case when dealing with groups of symmetries. The explanation, and a further discussion, is put off until the end of the section. ■

The functions $\mathsf{G} \times \mathsf{G} \ni (g, g') \mapsto \omega(g, g') \in U(1)$ are not totally arbitrary, because associativity holds:

$$(U_g U_{g'}) U_{g''} = U_g (U_{g'} U_{g''}) \,.$$

A computation shows that the above is equivalent to:

$$\omega(g, g')\omega(g \cdot g', g'') = \omega(g, g' \cdot g'')\omega(g', g'') \tag{12.36}$$

In turn, the latter implies:

$$\omega(g, e) = \omega(e, g)\,, \quad \omega(g, e) = \omega(g_1, e)\,, \quad \omega(g, g^{-1}) = \omega(g^{-1}, g)\,, \quad g, g_1 \in \mathsf{G}\,, \tag{12.37}$$

The next definition transcends the physical meaning of the objects involved.

Definition 12.29 (*Projective unitary representation*). Let G be a group and H a (complex) Hilbert space.

(a) A **projective unitary representation of G on H** is a map

$$\mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H}) \tag{12.38}$$

such that: U_g are unitary operators, and the **multiplier** of the representation

$$\Omega(g, g') := U_{g \cdot g'}^{-1} U_g U_{g'}\,, \quad g, g' \in \mathsf{G}, \tag{12.39}$$

has the form

$$\Omega(g, g') = \omega(g, g') I \quad \text{with}\, \omega(g, g') \in U(1)\, \text{for any}\, g, g' \in \mathsf{G} \tag{12.40}$$

(hence (12.36) holds).

The projective representation on $\mathfrak{S}_p(\mathsf{H})$ given by (with obvious notation)

$$\mathsf{G} \ni g \mapsto U_g \cdot U_g^{-1}$$

is **induced** by the projective unitary representation (12.38).

The projective unitary representation (12.38) is called **irreducible** if the family of operators $\{U_g\}_{g \in \mathsf{G}}$ is irreducible on H.

Given Hilbert spaces H, H' (possibly equal), two projective unitary representations

$$\mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H}) \text{ and } \mathsf{G} \ni g \mapsto U'_g \in \mathfrak{B}(\mathsf{H}')$$

are said to be **(unitarily) equivalent** if there exist a unitary operator $S : \mathsf{H} \to \mathsf{H}'$ and a map $\chi : \mathsf{G} \ni g \mapsto \chi(g) \in U(1)$ satisfying:

$$\chi(g) S U_g S^{-1} = U'_g\,, \quad g \in \mathsf{G}. \tag{12.41}$$

(b) A group homomorphism

$$\mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H}) \tag{12.42}$$

mapping elements of G to unitary operators on H is a **(proper) unitary representation of G on H**. (That is to say, a unitary representation is a projective unitary representation whose multipliers equal 1.)

The unitary representation (12.42) is **irreducible** if the family of operators $\{U_g\}_{g\in\mathsf{G}}$ is irreducible on H.

Given Hilbert spaces H, H' (possibly equal), two unitary representations

$$\mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H}) \text{ and } \mathsf{G} \ni g \mapsto U'_g \in \mathfrak{B}(\mathsf{H}')$$

are **(unitarily) equivalent** if there is a unitary operator $S : \mathsf{H} \to \mathsf{H}'$ such that

$$S U_g S^{-1} = U'_g \quad \text{for every } g \in \mathsf{G}. \tag{12.43}$$

Important remark. The reader should now be able to see the difference between *projective* representations, *projective unitary* representations and *unitary* representations. The first type act on $\mathfrak{S}_p(\mathsf{H}_S)$ or $\mathfrak{S}(\mathsf{H}_S)$ representing symmetry groups, and do not involve choices without physical meaning. The other two kinds act on H_S, induce projective representations, but are affected by physically arbitrary choices of the phases of the unitary operators by which they act. ■

Remarks 12.30 **(1)** The notion of *unitary equivalence* of two projective unitary representations is transitive, symmetric and reflexive, so it is an honest *equivalence relation* on the space of projective unitary representations of a given group on a given Hilbert space. If G is a symmetry group for the physical system S, described on the Hilbert space H_S, projective representations of G on $\mathfrak{S}_p(\mathsf{H}_S)$ are patently in one-to-one correspondence with equivalence classes of projective unitary representations of G.

(2) The property that a projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ be *equivalent* to a unitary representation is actually a property of the coset of the projective unitary representation: it means that the equivalence class contains a unitary representative. When talking about symmetry groups of a quantum system, that is a feature of the projective representation on $\mathfrak{S}(\mathsf{H}_S)$ corresponding to the class.

(3) The property that a projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ be *irreducible* is a property of the coset of the projective unitary representation: if one representative in the equivalence class is irreducible, all other elements are irreducible, as is clear from the definitions. Irreducible representations are important because every representation can be constructed as a direct sum, or *direct integral*, of irreducible representations [Var07]. ■

Here is a more concrete way of asking whether a projective representation $\mathsf{G} \ni g \mapsto \gamma_g$ of a symmetry group G can be described, on H_S, by a unitary representation of G. Inside the equivalence class of projective unitary representations associated to $\mathsf{G} \ni g \mapsto \gamma_g$ we fix an arbitrary representative (the ensuing discussion does not depend on this element, by remark (2) above) and consider its multipliers.

Thus we reduce to decide whether there might be a map $\chi : \mathsf{G} \ni g \mapsto \chi(g) \in U(1)$ such that:

$$\omega(g, g') = \frac{\chi(g \cdot g')}{\chi(g)\chi(g')} \quad \text{for any } g, g' \in \mathsf{G} . \tag{12.44}$$

Proof: if said χ exists, inserting it on the left in (12.41) renders the multipliers of $\mathsf{G} \ni g \mapsto U'_g$ trivial by (12.44). Conversely, if the multipliers of $\mathsf{G} \ni g \mapsto U'_g$ are trivial for some choice of the function χ in (12.41), that particular χ solves (12.44).

There are many strategies to tackle and solve the existence problem of χ [Var07], and one can see there exist groups, e.g. the *Lorentz group* and the *Poincaré group*, whose projective representations are described by unitary representations on the Hilbert space of the system. At the same time there exist other groups, like the *Galilean group*, whose (non-trivial) projective representations cannot be given by unitary representations, but only by projective unitary representations: the multipliers cannot be suppressed by smart choices of the phases.

There is a colossal literature on the topic. Irreducible projective unitary representations of the groups of interest in physics, especially Lie groups, have been studied and classified (e.g., see [BaRa86] for a broad, though not completely rigorous nor mathematically complete, treatise on the subject).

12.2.2 Representations of Actions on Observables: Left and Right Representations

Given a symmetry group G with projective representation $\mathsf{G} \ni g \mapsto \gamma_g$, two functions are automatically defined that represent group actions on observables, namely $\mathsf{G} \ni g \mapsto \gamma_g^*$ and $\mathsf{G} \ni g \mapsto \gamma_g^{*-1}$, *the dual action* (12.19) and the *the inverse dual action* (12.22) respectively, discussed in Sect. 12.1.6. The definitions hold beyond the particular choice of projective unitary representation of the theory on the system's Hilbert space: the phases that we have to fix to pass from γ_g to the U_g cancel out when we transfer the action to observables: $\gamma_g^*(A) = U_g^{-1} A U_g$ and $\gamma_g^{*-1}(A) = U_g A U_g^{-1}$.

Note that $\mathsf{G} \ni g \mapsto \gamma_g^*$ *does not define a left* G*-representation*; it is easy to see, from the definition of γ_g^*, that:

$$\gamma_g^* \circ \gamma_{g'}^* = \gamma_{g' \cdot g}^* , \quad g, g' \in \mathsf{G}$$

by construction, and not $\gamma_g^* \gamma_{g'}^* = \gamma_{g \cdot g'}^*$. Furthermore, $\gamma_e^* = id$ and $\gamma_{g^{-1}}^* = (\gamma_g^*)^{-1}$. The function $\mathsf{G} \ni g \mapsto \gamma_g^*$ is a *right representation* of G, provided we endow observables with the structure of a vector space at least – for instance restricting to (self-adjoint) operators in $\mathfrak{B}(\mathsf{H}_S)$.

Definition 12.31 Let G have neutral element e. A (linear) **right representation** of G on a (real or complex) vector space V is a map $\mathsf{G} \ni g \mapsto \alpha_g \in GL(V)$ such that

$$\alpha_a \circ \alpha_b = \alpha_{b \cdot a} , \quad \alpha_e = id , \quad (\alpha_c)^{-1} = \alpha_{c^{-1}}$$

for any $a, b, c \in \mathsf{G}$.

The inverse dual action $\gamma_g^{*-1}(A) = U_g A U_g^{-1}$ defines a standard (left) representation of G as the reader immediately proves:

$$\gamma_g^{*-1} \circ \gamma_{g'}^{*-1} = \gamma_{g \cdot g'}^{*-1} , \quad g, g' \in \mathsf{G} .$$

12.2.3 Projective Representations and Anti-unitary Operators

Let us return to the 'unitary vs. anti-unitary' issue of the operators U_g. Suppose to have a symmetry group with projective representation $\mathsf{G} \ni g \mapsto \gamma_g$. To an automorphism γ_g there corresponds either a unitary operator or an anti-unitary operator $U_g : \mathsf{H}_S \to \mathsf{H}_S$ satisfying $\gamma_g(\rho) = U_g \rho U_g^{-1}$ for every $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$, by Wigner's theorem. Are there criteria to decide whether the U_g are all unitary, all anti-unitary, or maybe both depending on $g \in \mathsf{G}$? If U_g and $U_{g'}$ were anti-unitary, the constraint $U_g U_{g'} = \chi(g, g') U_{g \cdot g'}$ would force $U_{g \cdot g'}$ to be unitary. Therefore representations of groups with more than two elements, all made by anti-unitary operators (identity apart, which is always unitary) cannot exist. The hybrid case when a certain number of anti-unitary operators (more than one) are present is anyway non-trivial, due to constraints such as the aforementioned one. In this respect the next proposition shows that the group G might impose the operators be all unitary.

Proposition 12.32 *Let* H *be a complex Hilbert space with* $\dim \mathsf{H} > 1$ *and* G *a group. Suppose each* $g \in \mathsf{G}$ *is the product of elements* $g_1, g_2, \ldots, g_n \in \mathsf{G}$ *(dependent on* g*, with* n *dependent on* g*) that admit a square root (there exist* $r_k \in \mathsf{G}$ *such that* $g_k = r_k \cdot r_k$ *for every* $k = 1, \ldots, n$*). Then for every projective representation* $\mathsf{G} \ni g \mapsto \gamma_g$*, the elements* γ_g *can only be associated to unitary operators under Wigner's theorem (or Kadison's).*

Proof The proof is obvious, for $U_{r_k}U_{r_k}$ is linear even when U_{r_k} is antilinear. As U_{g_k} and $U_{r_k}U_{r_k}$ represent the same Wigner symmetry associated to $g_k = r_k \cdot r_k$, they are both unitary or anti-unitary and hence they differ by a phase. From $U_{g_k} = \chi(r_k, r_k)U_{r_k}U_{r_k}$ follows U_{g_k} is linear, so also U_g must be linear. □

Remark 12.33 The case $\dim \mathsf{H} = 1$ has no direct physical interest as each symmetry γ_g must be the identity. In particular, $\mathsf{G} \ni g \mapsto \gamma_g$ can always be unitarily represented by the trivial representation $U_g = I$ for all $g \in \mathsf{G}$. Nevertheless, the case $\dim \mathsf{H} = 1$ may result mathematically interesting, and in the following we will also consider the case of (non-trivial) unitary representations on one-dimensional spaces. ■

The following result is important in the applications, especially the case $n = 1$.

Proposition 12.34 *In relationship to Proposition 12.32, the projective representations of the additive group $\mathsf{G} = \mathbb{R}^n$ are associated to unitary operators only.*

Proof If $\mathbf{t} \in \mathbb{R}^n$ then $\mathbf{t} = \mathbf{t}/2 + \mathbf{t}/2$, and the rest is a corollary of Proposition 12.32. □

We shall see later that Proposition 12.32 is automatic when we assume G is a connected Lie group, so anti-unitary operators appear only with discrete groups or when we change connected component. For this reason in the sequel we will deal with the case where the U_g are all unitary.

12.2.4 Central Extensions and Quantum Group Associated to a Symmetry Group

The approach we are about to illustrate allows to study all projective unitary representations of a certain group, by looking at them as restrictions of unitary representations of a larger group, a *central extension* of the starting one. The recipe, albeit apparently overcomplicated, is technically useful (also to detect possible unitary representations of G) in that it lets us use the specific toolbox of the much developed theory of unitary representations (of the extension). Let us briefly explain the basic idea of the procedure, postponing the fundamental example where G is the Galilean group; the reader might skip this section at first and return to it when needed.

Take any group G and a projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ on a Hilbert space H with multipliers ω. Define another group $\widehat{\mathsf{G}}_\omega$ consisting of pairs $(\chi, g) \in U(1) \times \mathsf{G}$ with product:

$$(\chi, g) \circ (\chi', g') = \left(\chi\chi'\omega(g, g')\,,\; g \cdot g'\right)\,, \quad (\chi, g), (\chi', g') \in U(1) \times \mathsf{G}.$$

The reader can check the definition is well posed, owing to the fact ω satisfies (12.36), and that it produces a group with neutral element $(\omega(e, e)^{-1}, e)$, e being the neutral element of G (remember (12.37)), and inverse

$$(\chi, g)^{-1} = (\chi^{-1}\omega(e,e)^{-1}\omega(g, g^{-1})^{-1}, g^{-1}) .$$

The following definition disregards the origin of the function ω, and only requires Eq. (12.36) to be valid.

Definition 12.35 (*Central extension*). Let G be a group and $\omega : \mathsf{G} \times \mathsf{G} \to U(1)$ any function satisfying (12.36). The group $\widehat{\mathsf{G}}_\omega = U(1) \times \mathsf{G}$ with product

$$(\chi, g) \circ (\chi', g) = \left(\chi\chi'\omega(g, g') \,,\; g \cdot g'\right) \,, \quad (\chi, g), (\chi', g') \in U(1) \times \mathsf{G},$$

is a **central extension** of G by $U(1)$ with **multiplier function** ω. The injective map $U(1) \ni \chi \mapsto (\chi, e) \in \widehat{\mathsf{G}}_\omega$ and the surjective homomorphism $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto g \in \mathsf{G}$ are respectively called **canonical injection** and **canonical projection of the central extension**.

The names (see the appendix at the end of the book for a minimal dictionary of group theory) come about as follows: the canonical projection $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto g \in \mathsf{G}$ is a surjective homomorphism, whose null space is the normal subgroup $\mathscr{N}$ (range of the canonical injection and isomorphic to $U(1)$) of pairs (χ, e) with $\chi \in U(1)$. The kernel $\mathscr{N}$ is contained in the *centre* of $\widehat{\mathsf{G}}_\omega$, as its elements commute with the whole $\widehat{\mathsf{G}}_\omega$ (in fact $\omega(e, g) = \omega(g, e)$). In practice the group G has been extended, to $\widehat{\mathsf{G}}_\omega$, by adding the kernel of the surjection $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto g \in \mathsf{G}$), which is central. Notice that G is naturally identified with the quotient group $\widehat{\mathsf{G}}_\omega/\mathscr{N}$.

The procedure for obtaining all projective unitary representations $\mathsf{G} \ni g \mapsto U_g$ of G relies on three important points.

(1) If $\mathsf{G} \ni g \mapsto U_g$ has multiplier function ω, the map

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto V_{(\chi,g)} := \chi U_g \,,$$

is a *unitary* $\widehat{\mathsf{G}}_\omega$-representation on H. In fact the operators $V_{(\chi,g)} : \mathsf{H} \to \mathsf{H}$ are unitary, so $V_{(\omega(e,e)^{-1},e)} = I$ and

$$V_{(\chi,g)}V_{(\chi',g')} = \chi U_g \chi' U_{g'} = \chi\chi'\omega(g, g')U_{g\cdot g'} = V_{(\chi,g)\circ(\chi',g')} \,.$$

(2) The initial representation arises from the unitary representation $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto V_{(\chi,g)}$ *by restriction*: i.e., restricting the domain of V to elements $(1, g)$, $g \in \mathsf{G}$. We will say that the unitary representation V restricts to G in this case.

(3) Given any unitary representation

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto V_{(\chi,g)}$$

of a central extension, the restriction to $\{1\} \times \mathsf{G}$, say $U_g := V_{(1,g)}$, is a projective unitary representation if and only if:

$$V_{(\chi,e)} = \chi\omega(e, e)I \quad \text{for every } \chi \in U(1). \tag{12.45}$$

In fact, $V_{(\chi,g)} = \chi U_g$ implies $V_{(\chi,e)} = \chi U_e = \chi\omega(e,e)I$ (for any projective unitary representation $\omega(e,e) := U_{ee}^{-1}U_eU_e = U_e$). From $(\chi, g) = (\chi\omega(e,e)^{-1}, e)(1, g)$, conversely, if (12.45) holds we can write

$$V(\chi, g) = V(\chi\omega(e,e)^{-1}, e)V(1, g) = \chi V(1, g) =: \chi U_g\,.$$

So we have this proposition.

Proposition 12.36 *Every projective unitary representation of a group* G *is the restriction of a unitary representation of a suitable central extension* $\widehat{\mathsf{G}}_\omega$ *whose multiplier function satisfies (12.45).*

The extension procedure, especially when G is a Lie group, is extremely powerful. Cohomology techniques enable to catalogue *all* projective unitary representations that are continuous in some topology (and all unitary representations of a simply connected Lie group) starting from the Lie algebra of G. We will return here at a later stage.

As a matter of fact we need not know *all* central extensions of G to classify projective unitary representations. It suffices to know central extensions whose multipliers *are non-equivalent*. Two multiplier functions on the same group, $\mathsf{G}\times\mathsf{G} \ni (g, g') \mapsto \omega(g, g') \in U(1)$ and $\mathsf{G}\times\mathsf{G} \ni (g, g') \mapsto \omega'(g, g') \in U(1)$, are called **equivalent** if there is a map $\chi : G \to U(1)$ such that

$$\omega(g, g') = \frac{\chi(g\cdot g')}{\chi(g)\chi(g')}\,\omega'(g, g')\,,\quad g, g' \in G.$$

If two projective unitary representations U, U' of G are equivalent, they are restrictions of unitary representations of central extensions $\widehat{\mathsf{G}}_\omega$, $\widehat{\mathsf{G}}_{\omega'}$ with equivalent multiplier functions ω, ω'. Hence, by knowing central extensions of G whose multipliers *are not equivalent* and their unitary representations, we actually know the equivalence classes of projective unitary representations of G, and so all projective unitary representations of G.

Further, if $\omega(e, e) \neq 1$ for a certain ω, using an equivalence transformation by a constant function χ we can reduce to the case $\omega(e, e) = 1$. Multipliers such that $\omega(e, e) = 1$ (whence $\omega(e, g) = \omega(g, e) = \omega(e, e) = 1$) are **normalised**. *In this case the canonical injection turns out to be a group homomorphism*. The central extension has neutral element $(1, e)$, and (12.45) reads

$$V_{(\chi,e)} = \chi I\,,\quad \chi \in U(1)\,. \tag{12.46}$$

Projective unitary representations arising thus satisfy $U_e = I$.
To finish, we make a few physical considerations on the meaning of $\widehat{\mathsf{G}}_\omega$, when there are no unitary representations of G, only projective unitary representations. Suppose we have a symmetry group $\mathsf{G} \ni g \mapsto \gamma_g$ for the physical system S, hence a projective representation on $\mathfrak{S}(\mathsf{H}_S)$, that is *not* describable by means of a unitary representation. We can anyway choose phases arbitrarily and extend G to $\widehat{\mathsf{G}}_\omega$ using the multipliers

found, and then take $\widehat{\mathsf{G}}_\omega$ as the *true* symmetry group of S. The latter admits in this way two representations. One from G itself:

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto g \in \mathsf{G}\,,$$

that captures the *classical* action of the group. But there is also a *quantum* and *unitary* representation:

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto \chi U_g\,,$$

describing the group action on the states of the system (actually on the system's Hilbert space, and so on states, too).

In this light the group $\widehat{\mathsf{G}}_\omega$ is sometimes called the *quantum group* associated to G. Note, however, that a specific central extension $\widehat{\mathsf{G}}_\omega$ cannot be selected using the construction seen above, for which only projective representations given by Wigner or Kadison automorphisms have a physical meaning. In order to choose among the various central extensions it is necessary to give a physical meaning to the single projective unitary representations of G, or to the unitary representations of the possible extensions $\widehat{\mathsf{G}}_\omega$. This can be done, by enriching G and turning it into a Lie group, as we will see. For the projective unitary representations of the Galilean group, multipliers have a straightforward meaning, for they are related to the *mass* of the physical system. This will be all the more clear after discussing Lie groups of symmetries.

12.2.5 Topological Symmetry Groups

We turn to *topological* symmetry groups and *Lie* groups of symmetries. Lie groups are a subclass of topological groups. The majority of quantum symmetry groups, with the notable exclusion of discrete symmetries (parity inversion and time reversal) in particular, are Lie groups. We will study in depth the additive Lie group $\mathbb{R}$, whose importance should not go amiss, both physically and technically.

Definition 12.37 A **topological group** is a group G and a topological space at the same time, whose operations of product $\mathsf{G} \times \mathsf{G} \ni (f, g) \mapsto f \cdot g \in \mathsf{G}$, and inversion $\mathsf{G} \ni g \mapsto g^{-1}$, are continuous in the product topology of $\mathsf{G} \times \mathsf{G}$ and the topology of G, respectively.

The theory of topological groups and their representations occupies a huge chapter of mathematics [NaSt82], and we shall be just concerned with a few very elementary results that befit our physical models. We present below some examples and properties of topological groups, with an eye to the *Haar measure*.

Example 12.38
(1) The real **general linear group** $GL(n, \mathbb{R})$ and the complex general linear group $GL(n, \mathbb{C})$ of non-singular $n \times n$ real and complex matrices, are (evident) topological groups, if we equip them with the topology induced by $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$.

(2) Using the standard topology any subgroup of the above two is a topological group. For instance:

- the **unitary group**
  $U(n) = \{U \in GL(n, \mathbb{C}) \mid UU^* = I\}$,
- the **special unitary group**
  $SU(n) := \{U \in U(n) \mid \det U = 1\}$,[6]
- the **orthogonal group**
  $O(n) := \{R \in GL(n, \mathbb{R}) \mid RR^t = I\}$
- the **special orthogonal group**
  $SO(n) := \{R \in O(n) \mid \det R = 1\}$,
- the **special linear group**
  $SL(n, \mathbb{R}) := \{A \in GL(n) \mid \det A = 1\}$,
- the **Lorentz group** ($\eta := \mathrm{diag}(-1, 1, 1, 1)$)
  $O(1, 3) := \{\Lambda \in GL(4) \mid \Lambda\eta\Lambda^t = \eta\}$
- the **orthochronous Lorentz group**
  $O(1, 3)\!\Uparrow := \{\Lambda \in O(1, 3) \mid \Lambda_{11} > 0\}$,
- the **special orthochronous Lorentz group**
  $SO(1, 3)\!\Uparrow := \{\Lambda \in O(1, 3)\!\Uparrow \mid \det \Lambda > 0\}$.

The list (including $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$) is made of *closed* subsets in $\mathbb{R}^{n^2}$, or $\mathbb{C}^{n^2}$ if matrices are complex. This comes from the definitions: just notice that by continuity every sequence in one of those groups converges in $\mathbb{R}^{n^2}$, or $\mathbb{C}^{n^2}$, to an element of the group.

The groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ (not the others listed above) are *bounded*, and therefore *compact groups*. Boundedness follows from the definition and the Cauchy–Schwarz inequality. For example, for $U \in U(n)$ we have $\sum_{k=1}^{n} U_{ik}\overline{U_{jk}} = \delta_{ij}$ by definition of $U(n)$. Hence $\sum_{i,k=1}^{n} U_{ik}\overline{U_{ik}} = n$, so $\sum_{i,k=1}^{n} |U_{ik}|^2 = n$ and $U(n)$ is contained in the closed ball of radius $\sqrt{n}$ in $\mathbb{C}^{n^2}$.

(3) Some groups do not look like matrix groups, like the additive group $\mathbb{R}$. But it, too, just like the **isometry group of** $\mathbb{R}^n$, $IO(n)$, built as in Example 12.19(1) replacing $O(3)$ with $O(n)$, can be realised by matrices. For $IO(n)$, one representation is by real $(n + 1) \times (n + 1)$ matrices

$$M((R, \mathbf{t})) := \begin{bmatrix} 1 & \mathbf{0}^t \\ \mathbf{t} & R \end{bmatrix}, \quad \mathbf{t} \in \mathbb{R}^n, R \in O(n) \tag{12.47}$$

(a subgroup of the topological group $GL(n + 1, \mathbb{R})$ with induced topology). The map $IO(n) \ni (R, \mathbf{t}) \mapsto M((R, \mathbf{t}))$ is an isomorphism. The additive group $\mathbb{R}^n$ arises by restriction, via the homeomorphism $\mathbb{R}^n \ni \mathbf{t} \mapsto M((I, \mathbf{t}))$ ($\mathbb{R}^n$ with standard structure).

The *Galilean group* (Sect. 12.3.3) and the *Poincaré* group are topological groups, built analogously via matrices.

[6] The word *special*, for matrix groups, indicates *determinant equal to* 1, and is alwaysdenoted by an S before the group's name.

(4) Yet there exist topological groups (even Lie groups) that cannot be viewed as matrix groups, an example being the *universal covering* (Definition 12.54) of $SL(2, \mathbb{R})$.

(5) *Locally compact Hausdorff* groups, like $\mathbb{R}^n$ (an Abelian group with the sum as composition law), $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ and subgroups thereof, admit a special *regular* Borel measure, called the *Haar measure*. The Haar measure is defined up to a factor and is translation-invariant by group elements.

Its definition is contained in the following classical theorem, proved by Weil in full generality [Hal69], of which we provide no proof. Recall that if G has product $\circ$, the **left** and **right orbits of** $B \subset \mathsf{G}$ under $g \in \mathsf{G}$ are:

$$gB := \{g \circ b \mid b \in B\} \quad \text{and} \quad Bg := \{b \circ g \mid b \in B\}$$

respectively. A positive σ-additive measure μ on the Borel σ-algebra $\mathscr{B}(\mathsf{G})$ of the locally compact Hausdorff group G is called **left-invariant** if

$$\mu(gB) = \mu(B) \quad \text{for any } B \in \mathscr{B}(\mathsf{G}),\, g \in \mathsf{G},$$

and **right-invariant** if

$$\mu(Bg) = \mu(B) \quad \text{for any } B \in \mathscr{B}(\mathsf{G}),\, g \in \mathsf{G}.$$

Note that $\mu(gB)$, $\mu(Bg)$ are well defined. Since the multiplication by $h \in \mathsf{G}$ on the left, $\mathsf{G} \ni b \mapsto f_h(b) := h \circ b$, is an homeomorphism, and since $gB = (f_{g^{-1}})^{-1}(B)$, we have $gB \in \mathscr{B}(\mathsf{G})$ if $B \in \mathsf{G}$. Similarly $Bg \in \mathscr{B}(\mathsf{G})$ if $B \in \mathsf{G}$.

Theorem 12.39 *Let G be a locally compact Hausdorff group. Up to a positive factor, there exists a unique positive σ-additive measure μ_{G} on the Borel σ-algebra $\mathscr{B}(\mathsf{G})$ such that it is regular ($\mu_{\mathsf{G}}(B) = \inf\{\mu_{\mathsf{G}}(U) \mid B \subset U, U\, open\}$ and $\mu_{\mathsf{G}}(B) = \sup\{\mu_{\mathsf{G}}(K) \mid K \subset B, K\, compact\}$) and:*

(i) μ_{G} is left-invariant,

(ii) $\mu_{\mathsf{G}}(B) > 0$ if $B \neq \varnothing$ is open and $\mu_{\mathsf{G}}(K) < +\infty$ if K is compact.[7]

Furthermore, if G is compact, μ_{G} is also right-invariant because $\mu_{\mathsf{G}}(E) = \mu_{\mathsf{G}}(E^{-1})$, where $E^{-1} := \{g^{-1} \mid g \in E\}$ for any $E \in \mathscr{B}(\mathsf{G})$.

μ_{G} is called **left-invariant Haar measure of** G.

A similar result for right-invariant measures defines, up to the usual positive factor, the **right-invariant Haar measure** ν_{G}. This in general is different (factor apart) from the (left-invariant) Haar measure μ_{G}: they coincide in case G is compact, by the theorem, because $\nu(E) := \mu(E^{-1})$ is right-invariant on $\mathscr{B}(\mathsf{G})$ if μ is left-invariant on $\mathscr{B}(\mathsf{G})$. If so, one speaks of the **bi-invariant Haar measure**.

The Abelian group $(\mathbb{R}, +)$ has the Lebesgue measure as Haar measure: the left- and right-invariant Haar measures coincide. The group $GL(n, \mathbb{R})$ (and its subgroups of (2)) has Haar measure:

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[7] Some authors require the condition on compact sets in the definition of regular Borel measure.

$$\mu_{GL(n,\mathbb{R})}(B) := \int_B |\det g(x_{11}, \dots, x_{nn})|^{-n} dx;$$

where $g \in GL(n, \mathbb{R})$ has entries x_{ij} seen as coordinates of $\mathbb{R}^{n^2}$, and dx is the Lebesgue measure on $\mathbb{R}^{n^2}$. ■

At this point we want to specialise the notion of symmetry group to topological groups, which entails imposing topological constraints on the associated projective representation.

Suppose we have a symmetry group $\mathsf{G} \ni g \mapsto \gamma_g$ for the quantum system S described on the Hilbert space H_S. If G is a topological group, we expect the homomorphism $g \mapsto \gamma_g$ to be continuous in some sense. This means choosing a topology on the space of maps γ_g, which we may think of as either Kadison automorphisms or Wigner automorphisms. In the sequel we adopt Wigner's point of view. We give first the definition and then explain it mathematically and physically.

Definition 12.40 Consider a quantum system S described on the Hilbert space H_S. Let G be a topological group with a projective representation $\mathsf{G} \ni g \mapsto \gamma_g$ on H, such that

$$\lim_{g \to g_0} tr\left(\rho_1 \gamma_g(\rho_2)\right) = tr\left(\rho_1 \gamma_{g_0}(\rho_2)\right) , \quad g_0 \in \mathsf{G}, \rho_1, \rho_2 \in \mathfrak{S}_p(\mathsf{H}_S).$$

Then G is a **topological** group of symmetries for S, and $\mathsf{G} \ni g \mapsto \gamma_g$ is a **continuous** projective representation on $\mathfrak{S}_p(\mathsf{H}_S)$.

Physically this is reasonable, for it says the transition probability between two pure states, one of which is the image of the action of the symmetry group, is a *continuous* map for the action. In Wigner's quantum-symmetry setup, this is more than sound.

But the definition is also natural in mathematical terms, as we explain now. Let $\mathfrak{B}_1(\mathsf{H}_S)_{\mathbb{R}}$ be the real vector space of self-adjoint, trace-class operators with norm $|| \ ||_1$. By Proposition 12.17 every Wigner automorphism γ_g is the restriction to $\mathfrak{S}_p(\mathsf{H}_S)$ of a *linear operator* $(\gamma_2)_g : \mathfrak{B}_1(\mathsf{H}_S)_{\mathbb{R}} \to \mathfrak{B}_1(\mathsf{H}_S)_{\mathbb{R}}$, determined by γ_g and continuous for $|| \ ||_1$. Consider then the mapping $\Gamma : \mathsf{G} \ni g \mapsto (\gamma_2)_g$. Putting the strong topology on $\mathfrak{B}_1(\mathsf{H}_S)_{\mathbb{R}}$ and the standard one on the domain, we will say Γ is continuous if for any $\rho \in \mathfrak{B}_1(\mathsf{H}_S)$, $g_0 \in \mathsf{G}$:

$$\lim_{g \to g_0} ||(\gamma_2)_g(\rho) - (\gamma_2)_{g_0}(\rho)||_1 = 0 .$$

Now restrict to $\mathfrak{S}_p(\mathsf{H}_S)$ with the induced topology, thus reverting to the representation $\mathsf{G} \ni g \mapsto \gamma_g$ in terms of Wigner automorphisms. Then $\mathsf{G} \ni g \mapsto \gamma_g$ is continuous if, for any $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$, $g_0 \in \mathsf{G}$:

$$\lim_{g \to g_0} ||\gamma_g(\rho) - \gamma_{g_0}(\rho)||_1 = 0 .$$

This notion of continuity is, apparently, different from that of Definition 12.40. The next proposition tells they are indeed the same.

Proposition 12.41 *Let* H *be a complex Hilbert space and* $||\rho||_1 = tr(|\rho|)$ *the norm on trace-class operators* $\mathfrak{S}(\mathsf{H}_S)$. *Then restricting to pure states:*

$$||\rho - \rho'||_1 = 2\sqrt{1 - (tr(\rho\rho'))^2}\,, \quad \rho, \rho' \in \mathfrak{S}_p(\mathsf{H}). \tag{12.48}$$

Equivalently:

$$||\psi(\psi|\ ) - \psi'(\psi'|\ )||_1 = 2\sqrt{1 - |(\psi|\psi')|^2}\,, \quad \psi, \psi' \in \mathsf{H}, ||\psi|| = ||\psi'|| = 1. \tag{12.49}$$

Therefore $\mathfrak{S}_p(\mathsf{H})$ *is a metric space with distance function:*

$$d(\rho, \rho') := 2\sqrt{1 - (tr(\rho\rho'))^2}\,, \quad \rho, \rho' \in \mathfrak{S}_p(\mathsf{H}).$$

Proof The first assertion is a trivial transcription of the second one, and the third is obvious once the first two are proven, by general properties of norms. To prove the second statement in the non-trivial case $\psi(\psi|\ ) \neq \psi'(\psi'|\ )$, it suffices to observe that $\rho = \psi(\psi|\ ) - \psi'(\psi'|\ )$, viewed as an operator in the span of ψ, ψ', is self-adjoint with zero trace, so its eigenvalues are $\pm\lambda$ for some $\lambda > 0$. Hence

$$2\lambda^2 = tr(\rho^2) = 2 - 2|(\psi|\psi')|^2\,,$$

where we have expanded the trace of ρ^2 in a Hilbert basis to obtain the second identity. We conclude that $\lambda = \sqrt{1 - |(\psi|\psi')|^2}$, so

$$||\psi(\psi|\ ) - \psi'(\psi'|\ )||_1 = \lambda + |-\lambda| = 2\sqrt{1 - |(\psi|\psi')|^2}\,.$$

□

Remarks 12.42 **(1)** The last claim of the proposition is quite interesting, for $\mathfrak{S}_p(\mathsf{H})$ is not a normed space, not even being a vector space. Nonetheless, it is a *metric space* (Definition 2.82) and the distance has a meaning: it is related to the probability amplitude.
(2) By direct inspection one also sees that, referring to the Hilbert–Schmidt norm $||\cdot||_2$ (Definition 4.24),

$$||\psi(\psi|\ ) - \psi'(\psi'|\ )||_2 = \sqrt{2}\sqrt{1 - |(\psi|\psi')|^2} = ||\psi(\psi|\ ) - \psi'(\psi'|\ )||_1/\sqrt{2}\,,$$

for any pair of vectors ψ, ψ' with $||\psi|| = ||\psi'|| = 1$. So, up to a multiplicative constant, the distance on $\mathfrak{S}_p(\mathsf{H}_S)$ constructed out of the Hilbert–Schmidt norm is the same as the distance constructed out of $||\cdot||_1$. ■

Mathematics and physics eventually meet in the next result.

Proposition 12.43 *Consider a quantum system* S *described on the Hilbert space* H_S. *Let* G *be a topological group. A projective representation* $\mathsf{G} \ni g \mapsto \gamma_g$ *on* H *is* *continuous according to Definition 12.40, so* G *is a topological symmetry group for S, if and only if it is continuous with respect to:*

(i) the topology of G *on the domain,*

(ii) the strong topology on the codomain, restricted to $\mathfrak{S}_p(\mathsf{H}_S)$*:*

$$\lim_{g\to g_0} ||\gamma_g(\rho) - \gamma_{g_0}(\rho)||_1 = 0\,, \quad \rho \in \mathfrak{S}_p(\mathsf{H}_S),\, g_0 \in \mathsf{G}. \tag{12.50}$$

Proof Equation (12.48) implies

$$||\gamma_g(\rho) - \gamma_{g_0}(\rho)||_1 = 2\sqrt{1 - tr\left(\gamma_g(\rho)\gamma_{g_0}(\rho)\right)}\,.$$

If $\mathsf{G} \ni g \mapsto \gamma_g$ is continuous for Definition 12.40 then $\lim_{g\to g_0} tr\left(\gamma_g(\rho)\gamma_{g_0}(\rho)\right) = tr\left(\gamma_{g_0}(\rho)\gamma_{g_0}(\rho)\right) = 1$. Substituting above yields (12.50). Conversely, from (12.48), the trace's invariance under cyclic permutations gives

$$tr\left(\gamma_{g_0}(\rho)\gamma_g(\rho)\right) = 1 - \frac{1}{4}||\gamma_g(\rho) - \gamma_{g_0}(\rho)||_1^2\,.$$

Namely:

$$tr\left(\rho\gamma_{g_0^{-1}g}(\rho)\right) = 1 - \frac{1}{4}||\gamma_g(\rho) - \gamma_{g_0}(\rho)||_1^2\,.$$

Changing the names of the elements of the group:

$$tr\left(\rho\gamma_g(\rho)\right) = 1 - \frac{1}{4}||\gamma_{g_0 g}(\rho) - \gamma_{g_0}(\rho)||_1^2\,.$$

All that implies that the map $g \mapsto tr\left(\rho_1\gamma_g(\rho_2)\right)$ is continuous at e if $\rho_1 = \rho_2$. The general case arises immediately from the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product:

$$\left|tr\left(\rho_1\left(\gamma_g\rho_2 - \gamma_{g_0}\rho_2\right)\right)\right|^2 \le tr(\rho_1\rho_1)tr\left(\left(\gamma_g\rho_2 - \gamma_{g_0}\rho_2\right)\left(\gamma_g\rho_2 - \gamma_{g_0}\rho_2\right)\right)\,.$$

The second factor in the right-hand side is nothing but:

$$2tr\left(\rho_2\rho_2\right) - tr\left(\rho_2\gamma_{g^{-1}g_0}\rho_2\right) - tr\left(\rho_2\gamma_{g_0^{-1}g}\rho_2\right)$$

and it tends to 0 as $g \to g_0$. Hence (12.50) implies continuity for Definition 12.40. $\square$

12.2.6 Strongly Continuous Projective Unitary Representations

Consider a physical system S described on the Hilbert space H_S, a topological symmetry group G and a projective representation $\mathsf{G} \ni g \mapsto \gamma_g$. Let us associate to G a projective unitary representation $\mathsf{G} \ni g \mapsto V_g$, in the sense $\gamma_g(\rho) = V_g \rho V_g^{-1}$, for every pure state $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$ of the system and every element $g \in \mathsf{G}$. Clearly if $\mathsf{G} \ni g \mapsto V_g$ is strongly continuous, then $\mathsf{G} \ni g \mapsto \gamma_g$ is a continuous projective representation: Definition 12.40 holds, in fact, since if $\rho_i = \psi_i(\psi_i|\ )$, $i = 1, 2$:

$$tr\left(\rho_1 V_g \rho_2 V_g^*\right) = |(\psi_1|V_g\psi_2)| \to |(\psi_1|V_{g_0}\psi_2)| = tr\left(\rho_1 V_{g_0} \rho_2 V_{g_0}^*\right) \text{ as } g \to g_0\,.$$

Here is an interesting problem: knowing $\mathsf{G} \ni g \mapsto \gamma_g$ is continuous, establish if the phases χ_g of the unitary operators $U_g = \chi_g V_g$ can be fixed so to obtain a projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ (yet associated to the initial $\mathsf{G} \ni g \mapsto \gamma_g$) that is *strongly continuous*. We would like, in other words,

$$U_g\psi \to U_{g_0}\psi \quad \text{as } g \to g_0 \text{ for any } \psi \in \mathsf{H}.$$

In its general form the question is very hard, although Wigner gave a *local* answer. We will show that given a topological symmetry group G and a continuous projective representation $\mathsf{G} \ni g \mapsto \gamma_g$, it is possible to fix the multipliers ω so to make the projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ become strongly continuous on a neighbourhood of the neutral element of G. Moreover, also the multipliers will be continuous on that neighbourhood. This local result is not usually global. We will prove that for $\mathsf{G} = \mathbb{R}$ the result holds everywhere on the group and multipliers can be fixed to 1, so that the representation is simultaneously *unitary* and *strongly continuous*. The consequences in physics reach far: we will be able to justify the postulate of time evolution, and explain the relationship between the existence of symmetries and the presence of preserved quantities as the system S evolves in time: a quantum formulation, in other words, of Noether's theorem. All this later though, because now we shall focus on mathematical aspects.

Proposition 12.44 *Consider a quantum system S described on the Hilbert space H_S, and let G be a topological group with continuous projective representation $\gamma : \mathsf{G} \ni g \mapsto \gamma_g$.*

There exist an open neighbourhood $A \subset \mathsf{G}$ of $e \in \mathsf{G}$ and a projective unitary representation associated to γ, $\mathsf{G} \ni g \mapsto U_g$, that is strongly continuous on A.

The multipliers

$$\omega(g, g')I = \left(U_{g\cdot g'}\right)^{-1} U_g U_{g'}\,, \quad g, g' \in \mathsf{G}$$

define a continuous map on an open neighbourhood A' of e with $A' \cdot A' \subset A$.

Proof Fix $\phi \in \mathsf{H}$, $||\phi|| = 1$. As $\mathsf{G} \ni g \mapsto tr(\phi(\phi|\ )\gamma_g(\phi(\phi|\ )))$ is continuous and equals 1 for $g = e$, there is an open neighbourhood A_0 of e where

$$tr(\phi(\phi|\ )\gamma_g(\phi(\phi|\ ))) \neq 0\,.$$

Represent γ by a projective unitary representation V, arbitrarily chosen, which we have by Wigner's theorem. Around A_0, then:

$$0 \neq tr(\phi(\phi|\ )\gamma_g(\phi(\phi|\ ))) = |(\phi|V_g\phi)|^2\,.$$

Define ($(\phi|V_g\phi) \neq 0$ guarantees it is possible):

$$\chi_g := \frac{\overline{(\phi|V_g\phi)}}{|(\phi|V_g\phi)|}$$

and pass to a new projective unitary representation U:

$$U_g := \chi_g V_g\,, \quad \text{if } g \in A_0 \text{ and} \quad U_g := V_g\,, \quad \text{if } g \notin A_0\,.$$

Then on A_0:

$$0 < \frac{|(\phi|V_g\phi)|^2}{|(\phi|V_g\phi)|} = (\phi|U_g\phi)$$

so

$$0 < (\phi|U_g\phi)\,. \tag{12.51}$$

Equation (12.51) has two consequences on some open neighbourhood A of e, $A \subset A_0$:

$$U_e = 1\,, \quad \text{and} \quad U_{g^{-1}} = U_g^{-1}\,, \quad g \in A\,. \tag{12.52}$$

In fact, $U_e = \chi I$ for some $\chi \in U(1)$, so $(\phi|U_e\phi) = \chi(\phi|\phi) = \chi$. As $(\phi|U_e\phi) > 0$, we can only have $\chi = 1$. As for the second property, $U_{g^{-1}} = \chi'_g U_g^{-1}$ for some $\chi'_g \in U(1)$. Since $g \mapsto g^{-1}$ is continuous and $e^{-1} = e$, there is an open neighbourhood of e, $A \subset A_0$, for which $g^{-1} \in A_0$ if $g \in A$. Working on A,

$$0 < (\phi|U_{g^{-1}}\phi) = \chi'_g(\phi|U_g^{-1}\phi) = \chi'_g(\phi|U_g^*\phi) = \chi'_g(U_g\phi|\phi) = \chi'_g(\phi|U_g\phi)$$

because $(\phi|U_g\phi)$ is real so $(\phi|U_g\phi) = (U_g\phi|\phi)$. Since $(\phi|U_g\phi) > 0$, necessarily $\chi'_g = 1$. This proves (12.52).

Fix a unit vector $\psi \in \mathsf{H}$, possibly distinct from the above ϕ. By continuity of γ, as in Definition 12.40 with $\rho_1 = U_s\psi(U_s\psi|\ )$ and $\rho_2 = \psi(\psi|\ )$, we find

$$\lim_{r\to s} |(U_r\psi|U_s\psi)| = |(U_s\psi|U_s\psi)| = 1\,. \tag{12.53}$$

Choosing $\rho_1 = \phi(\phi|\ )$, $\rho_2 = \psi(\psi|\ )$ gives

$$\lim_{r\to s} |(\phi|U_r\psi)| = |(\phi|U_s\psi)| \ . \tag{12.54}$$

Substituting in the general identities produces

$$||U_s\psi - (U_r\psi|U_s\psi)U_r\psi||^2 = 1 - |(U_r\psi|U_s\psi)|^2 \ , \tag{12.55}$$

so

$$\lim_{r\to s}(U_r\psi|U_s\psi)U_r\psi = U_s\psi \tag{12.56}$$

and in particular, for $\psi = \phi$,

$$\lim_{r\to s}(U_r\phi|U_s\phi)(\phi|U_r\phi) = (\phi|U_s\phi) \ . \tag{12.57}$$

On the other hand, our choice of ϕ and of the phase in U implies

$$\lim_{r\to s}(\phi|U_r\phi) = \lim_{r\to s} |(\phi|U_r\phi)| = |(\phi|U_s\phi)| = (\phi|U_s\phi) \ , \tag{12.58}$$

and so using (12.58) in (12.57), tells

$$\lim_{r\to s}(U_r\phi|U_s\phi) = 1 \ . \tag{12.59}$$

Now, U_t is unitary, and for any $\psi \in \mathsf{H}$ (any $\psi = \phi$) we have

$$||U_r\psi - U_s\psi||^2 = 2 - 2Re(U_r\psi|U_s\psi) \ , \tag{12.60}$$

so (12.59) and (12.60) imply, for $r \in A$, that the map $r \mapsto U_r\phi$ is continuous, with the chosen ϕ. Therefore $r \mapsto (U_r)^{-1}\phi$ is continuous, since (12.57) holds when r is replaced by r^{-1} and s by s^{-1} ($g \mapsto g^{-1}$ is continuous, and $(U_r)^{-1} = U_{r^{-1}}$ by (12.52)). From (12.56) follows

$$\lim_{r\to s}(U_r\psi|U_s\psi)((U_r)^{-1}\phi|\psi) = (\phi|U_s\psi) \ .$$

In other words,

$$\lim_{r\to s}(U_r\psi|U_s\psi)(\phi|U_r\psi) = (\phi|U_s\psi) \ . \tag{12.61}$$

If ψ is a unit vector with $(\phi|U_s\psi) \neq 0$, (12.61) entails

$$\lim_{r\to s}(U_r\psi|U_s\psi) = 1 \ . \tag{12.62}$$

Using this result in (12.60) we establish

$$\lim_{r \to s} ||U_r \psi - U_s \psi|| = 0 \tag{12.63}$$

for the vectors ψ. Notice that (12.63) trivially extends to every $\psi \in \mathsf{H}$ with $(\phi|U_s\psi) \neq 0$ even if its norm is not 1 and, obviously, it is also valid for $\psi = \mathbf{0}$.

In case $(\phi|U_s\psi) = 0$, (12.63) holds however with ψ replaced by $\psi' := \psi + U_s^{-1}\phi$, since it satisfies $(\phi|U_s\psi') \neq 0$ by direct inspection. With this choice we have by construction

$$U_r\psi - U_s\psi = (U_r\psi' - U_s\psi') + (U_r U_s^{-1}\phi - \phi) .$$

The first difference in the right-hand side vanishes when $r \to s$ since (12.63) is valid for ψ', the second difference vanishes as well because

$$||U_r U_s^{-1}\phi - \phi|| = ||U_s^{-1}\phi - U_r^{-1}\phi|| \to 0 \quad \text{as}\, r \to s,$$

as established at the beginning of this proof. We conclude that (12.63) holds for every $\psi \in \mathsf{H}$ and $A \ni g \mapsto U_g$ is strongly continuous.

Now the second claim. From $U(e) = 1$ and $U_{g^{-1}} = U_g^{-1}$, on A we have

$$\omega(g, e) = \omega(e, g) = 1 , \tag{12.64}$$

From

$$(U_r^{-1}\phi|U_s\phi) = \omega(r, s)^{-1}(\phi|U_{r\cdot s}\phi) \tag{12.65}$$

and $(\phi|U_{r\cdot s}\phi) > 0$ if $r \cdot s \in A$, we infer $(r, s) \mapsto \omega(r, s)^{-1}$ is continuous for $r, s, r\cdot s \in A$. Since the product of G is continuous if $e\cdot e = e$, there is a neighbourhood $A' \subset A$ of e where $r, s \in A'$ implies $r \cdot s \in A$. Taking A' small enough renders $A' \times A' \ni (r, s) \mapsto \omega(r, s) = \overline{\omega(r, s)^{-1}}$ continuous. $\square$

12.2.7 A Special Case: The Topological Group $\mathbb{R}$

We prove in this section a very important theorem about continuous representations of the *additive group* $\mathbb{R}$ equipped with the standard topology. The result is crucial in physics, as we will have time to explain.

Theorem 12.45 *Let $\mathbb{R} \ni r \mapsto \gamma_r$ be a continuous projective representation of $\mathbb{R}$ on the Hilbert space H.*

(a) *There exists a strongly continuous one-parameter unitary group (Definition 9.26) $\mathbb{R} \ni r \mapsto W_r$ such that*

$$\gamma_r(\rho) = W_r \rho W_r^{-1} \quad \textit{for any}\, r \in \mathbb{R},\, \rho \in \mathfrak{S}_p(\mathsf{H}). \tag{12.66}$$

(b) *A second strongly continuous one-parameter unitary group* $\mathbb{R} \ni r \mapsto U_r$ *satisfies* (12.66) *(with* U_r *replacing* W_r*) if and only if there exists* $c \in \mathbb{R}$ *such that*

$$U_r = e^{icr} W_r \quad \textit{for any}\, r \in \mathbb{R}.$$

(c) *There exists a self-adjoint operator* $A : D(A) \to \mathsf{H}$ *on* H*, unique up to additive constants, such that:*

$$\gamma_r(\rho) = e^{irA} \rho e^{-irA} \quad \textit{for any}\, r \in \mathbb{R},\, \rho \in \mathfrak{S}_p(\mathsf{H}).$$

Proof (a) Let $[-b, b] \subset A$, $b > 0$, be an interval in the open neighbourhood of 0, say $A \subset \mathbb{R}$, satisfying Proposition 12.44 for $\mathsf{G} = \mathbb{R}$. Decompose $\mathbb{R}$ into the disjoint union of intervals $(na, (n+1)a]$, $n \in \mathbb{Z}$, with $a = b/2$. Any $r \in \mathbb{R}$ belongs to one interval only, so $r = n_r a + t_r$ for unique $t_r \in (0, a]$ and $n_r \in \mathbb{Z}$. Since $\gamma_x \gamma_y = \gamma_{x+y}$:

$$\gamma_r = \gamma_{n_r a + t_r} = (\gamma_a)^{n_r} \gamma_{t_r} \,.$$

Hence if $\mathbb{R} \ni r \mapsto U_r$ is the projective unitary representation of Proposition 12.44:

$$\gamma_r(\rho) = \left((U_a)^{n_r} U_{t_r}\right) \rho \left((U_a)^{n_r} U_{t_r}\right)^{-1} \,,$$

for every $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$. For every $t \in (-a - \varepsilon, a + \varepsilon)$ and some $\varepsilon > 0$ the map $t \mapsto U_t$ is strongly continuous, so

$$\mathbb{R} \ni r \mapsto V_r \quad \text{with}\; V_r := (U_a)^{n_r} U_{t_r},\, n_r \in \mathbb{Z}\, \text{and}\, t_r \in (0, a]\, \text{as above} \tag{12.67}$$

is strongly continuous almost everywhere. The only discontinuities can occur at the endpoints of the intervals $(na, (n+1)a]$. Consider then $r \in (na, (n+1)a]$ and let us verify V_r is continuous at na. With $r_- < na$, $r_+ > na$ we have

$$V_{r_-} \psi = (U_a)^{(n-1)} U_{t_{r_-}} \psi \quad \text{and} \quad V_{r_+} \psi = (U_a)^n U_{t_{r_+}} \psi \,.$$

As $(-a, a] \ni t \mapsto U_t \psi$ is continuous, by definition of V:

$$\lim_{r_- \to na^-} V_{r_-} \psi = V_{na} \psi \,.$$

To prove that the map $\mathbb{R} \ni r \to V_r$ defined in (12.67) is continuous at na we also need to check that the right and left limits coincide, i.e. that the limit of $(U_a)^{(n-1)} U_{t_{r_-}} \psi$, as $t_{r_-} \to a^-$ (computed above), coincides with the limit of $(U_a)^n U_{t_{r_+}} \psi$ as $t_{r_+} \to 0^+$. We have

$$\lim_{t \to a^-} (U_a)^{n-1} U_t \psi = \lim_{t-a \to 0^-} (U_a)^{n-1} \omega(a, t-a)^{-1} U_a U_{t-a} \psi$$

$$= \lim_{t-a\to 0^-} \omega(a, t-a)^{-1}(U_a)^n U_{t-a}\psi = \lim_{\tau\to 0^-} \omega(a,\tau)^{-1}(U_a)^n U_\tau \psi \ .$$

By the previous proof $(0, a] \ni \tau \mapsto \omega(a,\tau)^{-1}$ is continuous, since $a, \tau, a+\tau \in A$ by construction. Moreover, $\chi(a, 0) = 1$ from (12.64). We also know $(0, a] \ni t \mapsto U_t\psi$ is continuous, so:

$$\lim_{t\to a^-} (U_a)^{n-1}U_t\psi = \lim_{\tau\to 0^-} \omega(a,\tau)^{-1}(U_a)^n U_\tau\psi = \lim_{\tau\to 0^+} \omega(a,\tau)^{-1}(U_a)^n U_\tau\psi$$

$$= \lim_{t\to 0^+} (U_a)^n U_t\psi \ .$$

We have proved

$$V_{na}\psi = \lim_{r_-\to na^-} V_{r_-}\psi = \lim_{t_{r_-}\to a^-} (U_a)^{(n-1)}U_{t_{r_-}}\psi = \lim_{t_{r_+}\to 0^+} (U_a)^n U_{t_{r_+}}\psi = \lim_{r_+\to na^+} V_{r_+}\psi \ ,$$

as required. Note $(V_r)^{-1} = (U_{t_r})^{-1}(U_a)^{-n_r} = U_{-t_r}(U_a)^{-n_r}$, where the second identity in (12.52) was used. In analogy to the proof for V_r, also $\mathbb{R} \ni r \mapsto (V_r)^{-1}$ is continuous in the strong topology.

We claim the multipliers of V can be set to 1. For this, first we will show they give a continuous map $\mathbb{R}^2 \ni (r, s) \mapsto \omega(r, s) \in U(1)$, using that $\mathbb{R} \ni t \mapsto V_t\psi$ and $\mathbb{R} \ni t \mapsto (V_t)^{-1}\psi$ are continuous. Then we will prove the latter function is equivalent to the constant map 1. By definition

$$\omega(r, s)V_{r+s} = V_r V_s \ .$$

Fix $(r_0, s_0) \in \mathbb{R}^2$. There must exist $\psi, \phi \in \mathsf{H}\setminus\{\mathbf{0}\}$ so that $(\psi|V_{r_0+s_0}\phi) \neq 0$, otherwise $V_{r_0+s_0} = 0$, which is impossible by hypothesis as V_t is unitary. By continuity there is a neighbourhood B of (r_0, s_0) such that $(r, s) \in B$ implies $(\psi|V_{r+s}\phi) \neq 0$. Then

$$\omega(r, s) = \frac{((V_r)^{-1}\psi|V_s\phi)}{(\psi|V_{r+s}\phi)} \ .$$

Hence $\mathbb{R}^2 \ni (r, s) \mapsto \omega(r, s) \in U(1)$ is continuous around (r_0, s_0), and so continuous on $\mathbb{R}^2$. We may write $\omega(r, s) = e^{-if(r,s)}$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$. The continuous map ω can be thought of as valued in the unit circle $\mathbb{S}^1$, homeomorphic to $U(1)$. The map f can be chosen to be continuous (the fundamental group of $\mathbb{R}^2$ is trivial, so the lifting property of covering spaces holds, cf. [Ser94II, Theorem 18.2]). Equation (12.36) now reads

$$f(s,t) - f(r+s,t) + f(r, s+t) - f(r, s) = 2\pi k_{r,s,t} \quad \text{for} k_{r,s,t} \in \mathbb{Z} \ .$$

Continuous functions map connected sets ($\mathbb{R}^3$) to connected sets (a subset of $2\pi\mathbb{Z}$ with induced standard topology), so the right-hand side is constant. But the left-hand side is zero for $r = s = t = 0$ as a consequence of (12.64), so:

$$f(s,t) - f(r+s,t) + f(r,s+t) - f(r,s) = 0 \quad \text{for every } r, s, t \in \mathbb{R}\,. \tag{12.68}$$

Fix a C^1 map $g : \mathbb{R} \to \mathbb{R}$ with compact support such that:

$$\int_{\mathbb{R}} g(x)dx = 1 \quad \text{and obviously} \quad \int_{\mathbb{R}} \frac{dg}{dx}\, dx = 0\,. \tag{12.69}$$

Define the continuous function:

$$\chi(r) := e^{-ih(r)} \quad \text{where} \quad h(r) := -\int_0^r du \int_{\mathbb{R}} f(u,t)\, \frac{dg}{dt}\, dt - \int_{\mathbb{R}} f(r,t)g(t)dt\,.$$

The new representation $W_r := \chi(r)V_r$ has multiplier $\omega'(r,s) = \omega(r,s)\frac{\chi(r)\chi(s)}{\chi(r+s)}$, so

$$\omega(r,s)' = e^{-if'(r,s)} \quad \text{where} \quad f'(r,s) = f(r,s) - h(r+s) + h(r) + h(s)\,.$$

A moderately involved computation on the right side, using h, (12.68), (12.69), and the easy relation

$$\int_0^{r+s} du\; F(u) - \int_0^r du\; F(u) - \int_0^s du\; F(u) = \int_0^s du\; (F(u+r) - F(u))$$

eventually gives $f'(r,s) = 0$, i.e. $\chi'(r,s) = 1$ for any $(r,s) \in \mathbb{R}^2$. This makes the projective unitary representation $\mathbb{R} \ni r \mapsto W_r$ actually unitary. Since $\mathbb{R} \ni x \mapsto \chi(x)$ is continuous by construction and $\mathbb{R} \ni r \mapsto V_r$ is strongly continuous, also $W = \chi V$ is strongly continuous. That is to say, $\mathbb{R} \ni r \mapsto W_r$ is a strongly continuous one-parameter unitary group satisfying (12.66), thus ending (a).

(b) Suppose there is another strongly continuous one-parameter unitary group U representing γ:

$$U_{-r}W_r\psi = \chi(r)\psi\,, \quad \psi \in \mathsf{H}. \tag{12.70}$$

(We have already proved $\chi(r)$ and ψ are independent in similar situations.) Consequently $W_r = \chi(r)U_r$. Multiply by $W_s = \chi(s)U_s$, and use the additivity of W and U in the parameter:

$$W_{r+s} = \chi(r)\chi(s)U_{r+s} \quad \text{so} \quad U_{-(r+s)}W_{r+s} = \chi(r)\chi(s)I.$$

Comparing with $U_{-(r+s)}W_{r+s} = \chi(r+s)I$, produces

$$\chi(r+s) = \chi(r)\chi(s)\,. \tag{12.71}$$

Equation (12.70) has another corollary:

$$(U_r\phi|W_r\psi) = \chi(r)(\phi|\psi)\,.$$

By Stone's theorem (Theorem 9.33) we can write $U_t = e^{itB}$, $W_t = e^{itA}$ for self-adjoint operators defined on dense domains $D(A)$, $D(B)$. Choose $\phi \in D(B)$, $\psi \in D(A)$ so that $(\phi|\psi) \neq 0$ (always possible by density). By Stone the first derivative of $\mathbb{R} \ni t \mapsto \chi(r)$ has to satisfy

$$\frac{d}{dt}(U_r\phi|W_r\psi) = \left(\frac{d}{dt}U_r\phi\middle| W_r\psi\right) + \left(U_r\phi \middle| \frac{d}{dt}\, W_r\psi\right),$$

hence it exists and equals

$$(iBU_r\phi|W_r\psi) + (U_r\phi|iAW_r\psi)\ .$$

Since the derivative of χ exists, and (12.71) holds:

$$\frac{d}{dx}\chi(x) = \lim_{h\to 0}\frac{1}{h}\left(\chi(x+h) - \chi(x)\right) = \chi(x)\lim_{h\to 0}\frac{1}{h}\left(\chi(h) - \chi(0)\right) = -\chi(x)c\ .$$

Hence $\chi(x) = e^{-icx}$ for some $c \in \mathbb{R}$ and then

$$W_x = e^{-icx}U_x\ .$$

Conversely let W be as in (a) and fix $c \in \mathbb{R}$. A direct computation shows $U_x := e^{icx}W_x$ is a strongly continuous one-parameter unitary group that represents γ.

(c) The strongly continuous one-parameter unitary group $\mathbb{R} \ni r \mapsto W_r$, built in (a), represents γ and has a self-adjoint generator A, by Stone's theorem. Therefore $W_r = e^{irA}$. If $B : D(B) \to \mathsf{H}$ is another self-adjoint operator representing γ, its one-parameter group $U_t = e^{itB}$ fulfils (b). Then there is $c \in \mathbb{R}$ such that $e^{itA} = e^{itc}e^{itB}$. By Stone's theorem the left-hand side admits strong derivative at $t = 0$ on $D(A)$, and the derivative is iA. Similarly, the right-hand side admits strong derivative at $t = 0$, at least on $D(B)$, which equals $icI + iB$. Consequently $D(A) \subset D(B)$ and $A = (cI + B)\restriction_{D(A)}$. Note $cI + B$ is self-adjoint on $D(B)$. Since A is self-adjoint it does not have proper self-adjoint extensions, and then $D(A) = D(B)$ and $A = B + cI$. □

Example 12.46

(1) Consider Example 12.19(1). The physical system is a quantum particle with no spin, described on the Hilbert space $L^2(\mathbb{R}^3, dx)$ if we fix an inertial frame system and identify $\mathbb{R}^3$ with the rest space via orthonormal Cartesian coordinates.

The subgroup *ISO*(3) of isometries of $\mathbb{R}^3$ consists of functions:

$$(\mathbf{t}, R) : \mathbb{R}^3 \ni \mathbf{x} \mapsto \mathbf{t} + R\mathbf{x}\ , \tag{12.72}$$

with $\mathbf{t} \in \mathbb{R}^3$, $R \in SO(3)$. Taking $R \in SO(3)$, as opposed to $R \in O(3)$ explains the '*S*' in *ISO*(3). As said in Example 12.38(3) (about $IO(n)$ there, but the argument is the same), *ISO*(3) is a matrix group. Consider 4×4 real matrices:

$$g_{(\mathbf{t},R)} := \begin{bmatrix} 1 & \mathbf{0}^t \\ \mathbf{t} & R \end{bmatrix}, \quad \mathbf{t} \in \mathbb{R}^n, R \in SO(3). \tag{12.73}$$

The topology is inherited from $GL(4, \mathbb{R})$ i.e. $\mathbb{R}^{16}$. The matrices $g_{(\mathbf{t},R)}$ correspond one-to-one to elements of $ISO(3)$, and $ISO(3) \ni (\mathbf{t}, R) \mapsto g_{(\mathbf{t},R)}$ is an isomorphism, beside a linear representation of $ISO(3)$. In order to make the action of $ISO(3)$ explicit on points in $\mathbb{R}^3$, let us write points as column vectors $(1, x_1, x_2, x_3)^t$ of $\mathbb{R}^4$, where x_1, x_2, x_3 are the Cartesian coordinates of $\mathbf{x} \in \mathbb{R}^3$. In this way we recover the action of $g_{(\mathbf{t},R)}$ on $\mathbb{R}^3$ described by (12.72). We can indifferently see $ISO(3)$ as the group of maps (12.72) or the matrix group (12.73). In either case it will be a topological group from now on. Similarly we may imagine $IO(3)$ as a matrix group, simply allowing R to vary in the whole $O(3)$. With the given topologies, the construction makes $ISO(3)$ a topological subgroup of $IO(3)$ and its connected component at the identity $(\mathbf{0}, I)$.

The linear unitary $ISO(3)$-representation on $L^2(\mathbb{R}^3, dx)$ seen in Example 12.19(1):

$$(U_\Gamma \psi)(\mathbf{x}) := \psi(\Gamma^{-1}\mathbf{x}), \quad \Gamma \in ISO(3), \psi \in L^2(\mathbb{R}^3, dx)$$

is strongly continuous, since

$$||U_\Gamma \psi - U_{\Gamma_0}\psi|| = ||U_{\Gamma_0^{-1}\circ\Gamma}\psi - \psi|| \to 0 \quad \text{as } \Gamma \to \Gamma_0. \tag{12.74}$$

Now look at U_Γ acting on pure states of $\mathsf{H} = L^2(\mathbb{R}^3, dx)$:

$$\gamma_\Gamma(\psi(\psi|\ )) := U_\Gamma\psi\,(\psi|\ )\,U_\Gamma^{-1}.$$

The strongly continuous unitary representation $ISO(3) \ni \Gamma \mapsto U_\Gamma$ renders $ISO(3)$ a *topological group of symmetries* for the spinless quantum particle.
(2) Let P_i be the self-adjoint operator of the momentum observable along the axis x_i, and $\mathbf{P}$ the column vector $(P_1, P_2, P_3)^t$. With an eye on the previous example, let us focus on the subgroup of translations along an axis $\mathbf{t} \in \mathbb{R}^3$. Such subgroup is the strongly continuous one-parameter unitary group $\mathbb{R} \ni r \mapsto U_r^{(\mathbf{t})}$, with

$$\left(U_r^{(\mathbf{t})}\psi\right)(\mathbf{x}) := \psi(\mathbf{x} - r\mathbf{t}), \quad t \in \mathbb{R}, \psi \in L^2(\mathbb{R}^3, dx).$$

It is easy to prove the symmetric operator $\mathbf{t} \cdot \mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)}$ is essentially self-adjoint, so (cf. Lemma 11.40)

$$\left(e^{-i\frac{r}{\hbar}\overline{\mathbf{t}\cdot\mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)}}}\psi\right)(\mathbf{x}) = \psi(\mathbf{x} - r\mathbf{t}), \quad \psi \in L^2(\mathbb{R}^3, dx). \tag{12.75}$$

Therefore
the self-adjoint operator, which exists by Theorem 12.45(c), generating the strongly continuous one-parameter unitary group of translations along $\mathbf{t}$ *is the momentum*

operator along $-\mathbf{t}$*, i.e. the only self-adjoint extension of* $-\frac{1}{\hbar}\mathbf{t}\cdot\mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)}$ *(up to the constant* $\hbar^{-1}$*).*

Observe that the generator can be modified by adding constants. ■

12.2.8 Round-Up on Lie Groups and Algebras

In this last section we assume the reader is familiar with differentiable manifolds, including real-analytic ones (the basic notions are summarised in the appendix with some detail). We recall fundamental results in the theory of Lie groups and provide a few examples, all without proofs [Kir76, NaSt82, Var84].

Definition 12.47 (*Lie group*). A **real Lie group of dimension** n is a real-analytic n-manifold G equipped with two analytic maps:

$$\mathsf{G} \ni g \mapsto g^{-1} \in \mathsf{G} \quad \text{and} \quad \mathsf{G}\times\mathsf{G} \ni (g,h) \mapsto g\cdot h \in \mathsf{G}$$

(where $\mathsf{G}\times\mathsf{G}$ has the analytic product structure), that make G a group with neutral element e.

The **dimension** of the Lie group G is the dimension n the manifold G.

Definition 12.48 (*Lie group morphism*). Consider Lie groups G, G', with respective neutral elements e, e' and operations $\cdot$, $\circ$.

A **Lie group homomorphism** is an analytic map $f : \mathsf{G} \to \mathsf{G}'$ that is also a group homomorphism.

If the homomorphism $f : \mathsf{G} \to \mathsf{G}'$ is invertible and f^{-1} is a homomorphism, f is called **Lie group isomorphism**, and G, G' are **isomorphic** (under f).

A **local homomorphism of Lie groups** is an analytic map $h : \mathscr{O}_e \to \mathsf{G}'$, where $\mathscr{O}_e \subset \mathsf{G}$ is an open neighbourhood of e and $h(g_1\cdot g_2) = h(g_1)\circ h(g_2)$ provided $g_1\cdot g_2 \in \mathscr{O}_e$. (This forces $h(e) = e'$[8] and $h(g^{-1}) = h(g)^{-1}$ for $g, g^{-1} \in \mathscr{O}_e$.)

If the local homomorphism h is an analytic diffeomorphism on its range (given by an open neighbourhood $\mathscr{O}_{e'}$ of e'), and the inverse $f^{-1} : \mathscr{O}_{e'} \to \mathsf{G}$ is a local homomorphism, then h is a **local isomorphism of Lie groups**. The Lie groups G, G' are **locally isomorphic** (under h).

Remark 12.49 Analyticity in Definition 12.47 can be watered down to having G just a topological manifold with continuous operations in the manifold topology (i.e. a topological group that is Hausdorff, paracompact, and locally homeomorphic to $\mathbb{R}^n$). In fact, a famous theorem proved in 1952 by Gleason, Montgomery and Zippin – solving part of *Hilbert's fifth problem* – proves the following.

Theorem 12.50 (Gleason, Montgomery, Zippin) *Every topological group with the structure of a* C^0 *manifold is isomorphic (as a topological group) to a Lie group, and the real-analytic structure is unique up to Lie group isomorphisms.*

---

[8] In fact $h(e) = h(e\cdot e) = h(e)\circ h(e)$, so applying $h(e)^{-1}$ we get $e' = h(e)$.

Notice that the theorem implies that the initial (maximal) C^0 atlas $\mathscr{A}_0$ of the topological group whose operations are continuous contains at least one sub-atlas consisting of a (maximal) real-analytic atlas $\mathscr{A}_\omega$ rendering the group operations analytic, and producing a Lie group structure. Any other analytic structure on the initial topological group compatible with the group operations must be analytically isomorphic to $\mathscr{A}_\omega$ and produces an isomorphic Lie group structure. ■

In the same spirit of the previous remark we may weaken the assumptions of differentiability when defining local (and global) homomorphisms [NaSt82].

Proposition 12.51 *Let* G, G' *be Lie groups,* $\mathscr{O}_e \subset \mathsf{G}$ *an open neighbourhood of the identity* $e \in \mathsf{G}$.

If $h : \mathscr{O}_e \to \mathsf{G}'$ *is continuous and* $h(g_1 \cdot g_2) = h(g_1) \circ h(g_2)$ *provided* $g_1 \cdot g_2 \in \mathscr{O}_e$, *then* h *is analytic on its domain and hence a local homomorphism of Lie groups.*

In particular, a continuous group homomorphism between Lie groups is a Lie group homomorphism.

Remark 12.52 A Lie group can be defined by requiring that the group operations are C^∞, rather than analytic, with respect to some smooth differentiable structure. This approach [War75, HiNe13] preserves all aforementioned results as well as all the results we will state in the rest of the section, simply by replacing the word analytic with smooth everywhere. Further, an analytic Lie group structure is contained in a unique (maximal) C^∞ Lie group structure. Conversely a C^∞ Lie group structure defines an analytic Lie group by direct application of the GMZ theorem. The larger C^∞ structure associated to that analytic Lie group is isomorphic to the initial C^∞ Lie group. This is due to Proposition 12.51 stated for C^∞ groups and where h is the global identity map. ■

Two important concepts for our purposes are *one-parameter subgroups* and *Lie algebras*, which we now recall.

Let G be a Lie group with neutral element e and product $\cdot$. The tangent space at a point $g \in \mathsf{G}$ is denoted $\mathsf{T}_g\mathsf{G}$. Every $g \in \mathsf{G}$ defines an (analytic) map $L_g : \mathsf{G} \ni h \mapsto g \cdot h$, and let us write $dL_g : \mathsf{T}_h\mathsf{G} \to \mathsf{T}_{g \cdot h}\mathsf{G}$ for its differential. Given $A \in \mathsf{T}_e\mathsf{G}$, we consider the first-order *Cauchy problem* on G: find a differentiable map $f : (-\alpha, \beta) \to \mathsf{G}$, $\alpha, \beta > 0$ such that

$$\frac{df}{dt} = dL_{f(t)}A \quad \text{with } f(0) = e.$$

The maximal solution is always complete, i.e. with largest-possible domain $(-\alpha, \beta) = \mathbb{R}$. We will indicate the maximal solution by

$$\mathbb{R} \ni t \mapsto \exp(tA)$$

and we will call it the **one-parameter subgroup generated by** A. If $T \in T_e\mathsf{G}$ it can be proved that

$$\exp(tT)\exp(t'T) = \exp((t+t')T)\,, \quad (\exp(tT))^{-1} = \exp(-tT)\,, \quad \forall\, t, t' \in \mathbb{R}.$$

Consider now a given $T \in \mathsf{T}_e\mathsf{G}$ and the collection of maps:

$$F_{t,T} : \mathsf{G} \ni g \mapsto \exp(tT)\, g\, \exp(-tT)$$

parametrised by $t \in \mathbb{R}$. As $F_{t,T}(e) = e$, the differential $dF_{t,T}|_e$ maps $\mathsf{T}_e\mathsf{G}$ to itself, and is called the **adjoint** of $F_{t,T}$

$$Ad\ F_{t,T} : \mathsf{T}_e\mathsf{G} \to \mathsf{T}_e\mathsf{G}\,.$$

The **commutator** [War75] is the map $\mathsf{T}_e\mathsf{G} \times \mathsf{T}_e\mathsf{G} \to \mathsf{T}_e\mathsf{G}$:

$$[T, Z] := \frac{d}{dt}|_{t=0}(Ad\ F_{t,T})\, Z\,, \quad T, Z \in \mathsf{T}_e\mathsf{G}\,.$$

The commutator has three properties:

$$\begin{array}{rl} \textbf{linearity:} & [aA + bB, C] = a[A, C] + b[B, C]\,, \\ \textbf{skew-symmetry:} & [A, B] = -[B, A]\,, \\ \textbf{Jacobiidentity:} & [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \end{array}$$

holding for any $a, b \in \mathbb{R}$ and $A, B, C \in \mathsf{T}_e\mathsf{G}$. The first two actually imply bilinearity. The third property is a consequence of the associativity of the group law.

Let us fix a local coordinate system $x_1, \dots, x_n$ compatible with the (analytic) structure of G over an open neighbourhood U of e, so that the neutral element becomes the origin. In these coordinates we can expand the group law on $U \times U$ in Taylor series up to the second order

$$\psi(X, X') = X + X' + B(X, X') + O\left(\left(|X|^2 + |X'^2|\right)^{3/2}\right)\,, \tag{12.76}$$

where $X, X' \in \mathbb{R}^n$ are the column vectors of the coordinates of elements $g, g' \in U$ whose product $g \cdot g'$ belongs to U. The mapping $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, and it is easy to see that the commutator, in the coordinate basis of $\mathsf{T}_e\mathsf{G}$, becomes:

$$[T, T'] = B(T, T') - B(T', T)\,, \tag{12.77}$$

where T, T' are (column) vectors in $\mathsf{T}_e\mathsf{G}$.

Definition 12.53 (*Lie algebra*). A vector space V (with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) endowed with a bilinear, skew-symmetric map $\{\ ,\ \} : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ that satisfies the Jacobi identity is called a **Lie algebra**, and $\{\ ,\ \}$ is the **Lie bracket**.

Given Lie algebras $(\mathsf{V}, \{\ ,\ \})$, $(\mathsf{V}', \{\ ,\ \}')$, a linear mapping $\phi : \mathsf{V} \to \mathsf{V}'$ is a **Lie algebra homomorphism** if $\{\phi(A), \phi(B)\}' = \phi(\{A, B\})$ for any $A, B \in \mathsf{V}$. If ϕ is also bijective one calls it a **Lie algebra isomorphism**.

If V' is a $\mathbb{K}$-linear space of linear operators on a given vector space X and $\{\ ,\ \}'$ is the standard commutator of operators on X, the Lie algebra homomorphism ϕ is called **representation** of the Lie algebra $(\mathsf{V}, \{\ ,\ \})$ on X.

Given a Lie group G, the tangent space $\mathsf{T}_e\mathsf{G}$ with the Lie bracket $[\ ,\ ]$ given by the commutator is the **Lie algebra of the Lie group** G.

A **Lie subalgebra** V' in a Lie algebra $(\mathsf{V}, [\ ,\ ])$ is a closed subspace under the Lie bracket, $[A, B] \in \mathsf{V}'$ for $A, B \in \mathsf{V}'$. An **ideal** J in a Lie algebra $(\mathsf{V}, [\ ,\ ])$ is a Lie subalgebra such that

$$[A, B] \in J \quad \text{for any } A \in J, B \in \mathsf{V}.$$

A Lie algebra V is said to be **simple** if it is not Abelian and it contains no non-trivial proper (i.e. different from $\{\mathbf{0}\}$ and V itself) ideals, and **semisimple** if it is a direct sum of simple Lie algebras.

A Lie group whose Lie algebra is semisimple is called a **semisimple Lie group**.

A crucial feature of Lie groups for physics is that *the Lie algebra of a Lie group determines the group almost entirely*, as the classical, and famous, next results shows [NaSt82]. First, though, a topological reminder.

Definition 12.54 (*Covering space*). Let X be a topological space. Another topological space R is a **covering space** of X if there exists a continuous, onto map $\pi : \mathsf{R} \to \mathsf{X}$, called **covering map**, as follows:
(i) for any $x \in \mathsf{X}$ there exists an open set $U \ni x$ such that $\pi^{-1}(U) = \cup_{j \in J} A_j$, with $A_j \subset \mathsf{R}$ open, $A_j \cap A_i = \varnothing$ if $i \neq j, i, j \in J$,
(ii) $\pi\restriction_{A_j} : A_j \to U$ is a homeomorphism for every $j \in J$.

A covering R of X is a **universal covering** if it is simply connected (Definition 1.28).

Two universal coverings R, R' of X are homeomorphic under a map $f : \mathsf{R} \to \mathsf{R}'$ such that $\Pi = f \circ \Pi'$, where $\Pi : \mathsf{R} \to \mathsf{X}$, $\Pi' : \mathsf{R}' \to \mathsf{X}$ are the covering maps. Similarly, if X has a universal covering R and a covering R', with covering maps $\Pi : \mathsf{R} \to \mathsf{X}$, $\pi : \mathsf{R}' \to \mathsf{X}$, then there is a covering map $p : \mathsf{R} \to \mathsf{R}'$ with $\pi \circ p = \Pi$ [Ser94II].

The first result, customarily called *Lie's third theorem* in the literature, is stated below [NaSt82]. A **discrete subgroup** of a topological group G is a subgroup G' such that, for every element of G' there is an open set in G containing only that element of the subgroup. In other words G' is discrete iff every singlet $\{g\} \subset \mathsf{G}'$ is open in the induced topology.

Theorem 12.55 (Lie's third theorem) *Let* V *be a finite-dimensional (real) Lie algebra.*
(a) *There exists a connected and simply connected (real) Lie group* G_V *with Lie algebra* V.
(b) G_V *is, up to isomorphisms, the universal covering of any Lie group* G *having* V *as Lie algebra, and the covering map* $\pi : \mathsf{G}_\mathsf{V} \to \mathsf{G}$ *is a Lie group homomorphism.*

(c) *If a connected Lie group* G *has* V *as Lie algebra, it is isomorphic to a quotient* $\mathsf{G}_{\mathsf{V}}/H_{\mathsf{G}}$, *where* $H_{\mathsf{G}} \subset \mathsf{G}_{\mathsf{V}}$ *is a discrete normal subgroup, contained in the centre of* G_{V},[9] *that coincides with the kernel of* π.

Another important result the following one.

Theorem 12.56 (Lie, Pontrjagin) *Let* G, G' *be (real) Lie groups with Lie algebras* V, V'.
(a) $f : \mathsf{V} \to \mathsf{V}'$ *is a Lie algebra homomorphism if and only if there is a local Lie group homomorphism* $h : \mathsf{G} \to \mathsf{G}'$ *such that* $dh|_e = f$, *where* e *is the neutral element of* G. *Moreover:*
(i) h *is determined completely by* f,
(ii) f *is an isomorphism* $\Leftrightarrow$ h *is a local Lie group isomorphism.*
(b) *If* $h : \mathsf{G} \to \mathsf{G}'$ *is a Lie group homomorphism then:*
(i) $dh|_e : \mathsf{V} \to \mathsf{V}'$ *is surjective* $\Leftrightarrow$ *the connected component of* G *containing* e *is surjectively mapped onto the connected component of* G' *containing the neutral element of* G'.
(ii) $dh|_e : \mathsf{V} \to \mathsf{V}'$ *is injective* $\Leftrightarrow$ *the kernel of* h *is a discrete subgroup of* G.
(c) *If* G, G' *are connected and* G *also simply connected, then* $f : \mathsf{V} \to \mathsf{V}'$ *is a homomorphism if and only if there is a homomorphism* $h : \mathsf{G} \to \mathsf{G}'$ *such that* $dh|_e = f$. *Moreover:*
(i) h *is determined completely by* f,
(ii) f *isomorphism* $\Rightarrow$ h *onto,*
(iii) f *isomorphism and* G' *simply connected* $\Rightarrow$ h *isomorphism.*

Definition 12.57 (*Lie subgroup – simple Lie group*). An embedded (analytic) submanifold $\mathsf{G}' \subset \mathsf{G}$ in a Lie group that is also a subgroup inherits a Lie group structure from G. In such case G' is a **Lie subgroup** of G.

The Lie subgroup G' is said to be **discrete** when the set $\{e\}$ (e being the neutral element of both) is open in the induced topology on G'.

A Lie group G is said to be **simple** if it does not admit non-trivial proper (i.e. different from $\{e\}$ and G) connected Lie subgroups G' that are normal ($\{g \cdot h \cdot g^{-1} \,|\, h \in \mathsf{G}'\} \subset \mathsf{G}'$ for every $g \in \mathsf{G}$).

Remark 12.58
(1) The definition of discrete Lie subgroup we gave is equivalent to the aforementioned standard definition, valid for topological groups (G' is discrete iff every singlet $\{g\} \subset \mathsf{G}'$ is open in the induced topology) because the translation $\mathsf{G}' \to \mathsf{G}' : h \mapsto gh$ is a homeomorphism. Evidently a discrete Lie group is a 0-dimensional submanifold, made of a finite or countable set of points, and its Lie algebra is the trivial vector space.
(2) A connected Lie group is simple iff its Lie algebra is simple.
(3) According to Sect. A.2, a simple Lie group may be not simple as an abstract group. In particular, a simple Lie group (e.g., $SL(2,\mathbb{C})$) can contain a non-trivial discrete normal subgroup. ■

[9] It is also possible to prove that H_{G} is isomorphic to the fundamental group (the first homotopy group) of G which, for Lie groups, is known to be Abelian.

We have a pair of relevant results [NaSt82].

Theorem 12.59 (Cartan) *If* $\mathsf{G}' \subset \mathsf{G}$ *is a closed subgroup of the Lie group* G*, then* G' *is a Lie subgroup of* G *(including the case of a discrete Lie group).*

It is possible to prove the converse fact (see, e.g., [Lee06, Proposition 8.30]).

Proposition 12.60 *An embedded submanifold of a Lie group which is a Lie group with respect to the induced structure is necessarily closed.*

Summing up, closure completely characterises Lie subgroups.

Theorem 12.61 *Let* G *be a Lie group with Lie algebra* V.
(a) *The Lie algebra of a Lie subgroup* G' *is a Lie subalgebra of* V.
(b) *If* $L \subset \mathsf{T}_e\mathsf{G}$ *is a Lie subalgebra of* V*, there exists a unique connected Lie subgroup* $\mathsf{G}' \subset \mathsf{G}$ *whose Lie algebra is* L.

Remarks 12.62 **(1)** In principle an abstract Lie algebra can have infinite dimension as vector space. The dimension of the Lie algebra of a Lie group G, instead, is always finite for it coincides with the dimension of the manifold G.
(2) Theorem 12.59 clearly subsumes discrete subgroups as special cases. Then the manifold underlying the Lie subgroup has dimension zero.
(3) Let G be a Lie group of dimension n and $\{T_1, \dots T_n\}$ a basis of the Lie algebra $\mathsf{T}_e\mathsf{G}$. As the Lie bracket is bilinear it can be written in components

$$[T_i, T_j] = \sum_{k=1}^{\dim \mathsf{T}_e\mathsf{G}} C_{ijk} T_k \ .$$

The coefficients C_{ijk} are the **structure constants** of the Lie group.[10] The Jacobi identity is equivalent to the following equation (of obvious proof):

$$\sum_{s=1}^{n} \left(C_{ijs}C_{skr} + C_{jks}C_{sir} + C_{kis}C_{sjr}\right) = 0\,, \quad r = 1, \dots, n. \tag{12.78}$$

If two Lie groups have the same structure constants with respect to some bases of their Lie algebras, they are *locally isomorphic* in the sense of Theorems 12.55, 12.56. (If the structure constants are equal, the linear map identifying bases is an isomorphism.) Conversely, the structure constants of locally isomorphic Lie groups are the same in two bases related by the pullback of the local isomorphism. ■

Given a Lie group G, the **exponential mapping** is the analytic function

$$\exp : \mathsf{T}_e\mathsf{G} \ni T \mapsto \exp(tT)|_{t=1} \ . \tag{12.79}$$

[10]The structure constants, sometimes denoted by C_{ij}^k, are the components of a *tensor*, called the *structure tensor* of the Lie group.

The exponential mapping has an important property, sanctioned by the next result [NaSt82].

Theorem 12.63 *Let* G *be a Lie group with neutral element* e *and exponential map* exp.
(a) *There exist open neighbourhoods* U *of* $\mathbf{0} \in \mathsf{T}_e\mathsf{G}$ *and* V *of* $e \in \mathsf{G}$ *such that*

$$\exp\restriction_U : U \to \mathsf{V}$$

is an analytic diffeomorphism (bijective, analytic, with analytic inverse).
(b) *If* G *is compact then* $\exp(\mathsf{T}_e\mathsf{G}) = \mathsf{G}$.
(c) *If* G' *is a Lie group with exponential map* $\exp'$ *and* $h : \mathsf{G} \to \mathsf{G}'$ *a Lie group homomorphism:*

$$h \circ \exp = \exp' \circ dh|_e \,.$$

Property (a) has a useful corollary. Fix a basis $T_1, \dots, T_n$ on the Lie algebra of G, Then the inverse to

$$F : (x_1, \dots, x_n) \mapsto \exp\left(\sum_{k=1}^{n} x_n T_n\right)$$

defines a local chart, compatible with the analytic structure, around the neutral element. This is called a **normal coordinate system** or system of **coordinates of first type**. Normal coordinates, in general, do not cover G. In normal coordinates a vector $T \in \mathsf{T}_e\mathsf{G} \equiv \mathbb{R}^n$ determines a point of G only around e. Hence the group multiplication in G becomes a map $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Expanding the latter with Taylor around the origin of $\mathbb{R}^n \times \mathbb{R}^n$ gives

$$\psi(T, T') = T + T' + \frac{1}{2}[T, T'] + O\left(\left(|T|^2 + |T'^2|\right)^{3/2}\right) , \qquad (12.80)$$

where $[T, T'] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the commutator in the basis of $\mathsf{T}_e\mathsf{G} \times \mathsf{T}_e\mathsf{G}$ associated to normal coordinates. The proof is left to the reader. Property (a) has also another consequence (whose proof is an exercise).

Proposition 12.64 *Let* G *be a Lie group with neutral element* e *and product* $\cdot$.
(a) *There exists an open set* $A \ni e$ *in* $\mathsf{T}_e\mathsf{G}$ *such that any* $g \in A$ *can be written as* $g = \exp(tT)$ *for some* $t \in \mathbb{R}$ *and some* $T \in \mathsf{T}_e\mathsf{G}$.
(b) *If* G *is connected and* $g \notin A$*, there are finitely many elements* $g_1, g_2, \dots, g_n \in A$ *such that* $g = g_1 \cdot \cdots \cdot g_n$.

Another useful local chart around the neutral element, called system of **coordinates of second type** is obtained as the inverse of

$$G : (x_1, \dots, x_n) \mapsto \exp(x_1 T_1) \cdots \exp(x_n T_n)$$

where $\cdot$ is the group multiplication and $(x_1, \dots, x_n)$ varies in a sufficiently small open neighbourhood of $(0, \dots, 0)$ in $\mathbb{R}^n$.

The fundamental **Baker–Campbell–Hausdorff formula** [NaSt82]:

$$\exp(X)\exp(Y) = \exp(Z(X,Y)) \tag{12.81}$$

holds on any connected and simply connected Lie group G, with X, Y in the open neighbourhood U of the origin where exp is a local diffeomorphism onto the open neighbourhood $\exp(U) \subset \mathsf{G}$ of the neutral element. In (12.81) the term $Z(X,Y)$ is defined by the series:

$$Z(X,Y) = \sum_{\mathbb{N}\ni n>0} \frac{(-1)^{n-1}}{n} \sum_{r_i+s_i>0\,,1\le i\le n} \frac{\left(\sum_{i=1}^n (r_i+s_i)\right)^{-1}}{r_1!s_1!\dots r_n!s_n!} \left[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\dots X^{r_n}Y^{s_n}\right] \tag{12.82}$$

$$\left[X^{r_1}Y^{s_1}\dots X^{r_n}Y^{s_n}\right] := \underbrace{[X,[X,\dots[X,}_{r_1\text{times}}\underbrace{[Y,[Y,\dots[Y,}_{s_1\text{times}}\dots\underbrace{[X,[X,\dots[X,}_{r_n\text{times}}\underbrace{[Y,[Y,\dots Y}_{s_n\text{times}}]]\cdots]] \tag{12.83}$$

and the right-hand side is taken to be zero if $s_n > 1$ or $s_n = 0$ and $r_n > 1$.

Example 12.65
(1) $M(n,\mathbb{R})$ will denote from now on the set of real $n\times n$ matrices, and $M(n,\mathbb{C})$ complex $n \times n$ matrices.

The group $GL(n,\mathbb{R})$ of invertible real $n\times n$ matrices is a n^2-dimensional Lie group with analytic structure induced by $\mathbb{R}^{n^2}$. Its Lie algebra is the set of real $n\times n$ matrices $M(n,\mathbb{R})$ and the Lie bracket is the usual commutator $[A,B] := AB - BA$, $A, B \in M(n,\mathbb{R})$.

An important feature of $GL(n,\mathbb{R})$ is that its one-parameter subgroups have this form:

$$\mathbb{R} \ni t \mapsto e^{tA} := \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k\,,$$

for any $A \in M(n,\mathbb{R})$, and the convergence is meant in any of the equivalent norms of the Banach space $\mathbb{R}^{n^2}$ (Sect. 2.5).
(2) Any closed subgroup of $GL(n,\mathbb{R})$ we have met as topological group, like $O(n)$, $SO(n)$, $IO(n)$, $ISO(n)$, $SL(n,\mathbb{R})$, the Galilean, Lorentz and Poincaré groups, are therefore Lie groups. As $GL(n,\mathbb{C})$ can be seen as a subgroup in $GL(2n,\mathbb{R})$ (decomposing every matrix element in real and imaginary part), complex matrix groups like $U(n)$ and $SU(n)$, too, are real Lie groups. We must emphasise that working with matrix Lie groups is not a major restriction, since every compact Lie group is isomorphic to a matrix group [War75]. For non-compact Lie groups the story is completely different, a counterexample being the universal covering of $SL(2,\mathbb{R})$.
(3) The exponential of matrices $A, B \in M(n,\mathbb{C})$ has interesting characteristics. First, $e^{A+B} = e^A e^B = e^B e^A$ if $AB = BA$. The proof is similar to the case of numbers, for which one uses Taylor's expansion. There is, though, another useful fact: $A \in M(n,\mathbb{C})$ satisfies, for any $t \in \mathbb{C}$,

$$\det\ e^{tA} = e^{t\operatorname{tr} A}\ , \quad \text{in particular} \quad \det e^{A} = e^{\operatorname{tr} A}\ .$$

Let us prove this identity. We want to differentiate $\mathbb{C} \ni t \mapsto \det e^{tA}$, i.e. find

$$\lim_{h\to 0} \frac{\det e^{(t+h)A} - \det e^{tA}}{h} = \lim_{h\to 0} \frac{\det (e^{tA}e^{hA}) - \det e^{tA}}{h} = \det e^{tA} \lim_{h\to 0} \frac{\det e^{hA} - 1}{h}$$

as long as the last limit exists. Since $e^{hA} = I + hA + ho(h)$, with $o(h) \to 0$ as $h \to 0$ in the standard topology of $\mathbb{C}^{n^2}$, it follows

$$\lim_{h\to 0} \frac{\det e^{(t+h)A} - \det e^{tA}}{h} = \det e^{tA} \lim_{h\to 0} \frac{\det (I + hA + ho(h)) - 1}{h}\ .$$

There are many ways to see that $\det(I + hA + ho(h)) = 1 + h\sum_{i=1}^{n} A_{ii} + ho(h)$, and substituting above we find

$$\frac{d \det e^{tA}}{dt} = \det\ e^{tA}\operatorname{tr}A\ .$$

That also proves the function is smooth. Hence $f_A : \mathbb{C} \ni t \mapsto \det e^{tA}$ solves the differential equation:

$$\frac{df_A(t)}{dt} = (\operatorname{tr} A) f_A(t)\ .$$

Also $g_A : \mathbb{C} \ni t \mapsto e^{t\operatorname{tr} A}$ solves the equation. And both functions satisfy the initial condition $f_A(0) = g_A(0) = 1$, by uniqueness of maximal solutions of first-order equations we obtain $\det e^{tA} = e^{t\operatorname{tr} A}$, any $t \in \mathbb{R}$.

(4) The group of rotations $O(n) := \{R \in M(n, \mathbb{R}) \mid RR^t = I\}$ of $\mathbb{R}^n$ is an important Lie group in physics. That it is a subgroup of $GL(n, \mathbb{R})$ is evident because $\{R \in M(n, \mathbb{R}) \mid RR^t = I\}$ is closed in the Euclidean topology. (Clearly $O(n)$ contains its limit points: $A_k \in O(n)$ and $A_k \to A \in \mathbb{R}^{n^2}$ as $k \to \infty$ imply $A_k^t \to A^t$ and $I = A_k A_k^t \to AA^t$.) The Lie algebra of $O(n)$, denoted $o(n)$, is the vector space of real, skew-symmetric $n \times n$ matrices, and has dimension $n(n-1)/2 = \dim O(n)$. The proof is that Lie algebra vectors are tangent vectors $\dot{R}(0)$ at the identity of the group (the identity matrix) to curves $R = R(u)$ such that $R(u)R(u)^t = I$, $R(0) = I$. By definition, then, they satisfy $\dot{R}(0)R(0)^t + R(0)\dot{R}(0)^t = 0$, i.e. $\dot{R}(0) + \dot{R}(0)^t = 0$. But this defines real skew-symmetric $n\times n$ matrices, a space of dimension $n(n-1)/2$. On the other hand, if A is a real skew-symmetric $n \times n$ matrix, $R(t) = e^{tA} \in O(n)$ as follows from the elementary properties of the exponential function, and $\dot{R}(0) = A$. We conclude that the Lie algebra of $O(n)$ is nothing but the whole collection of real skew-symmetric $n \times n$ matrices.

Eventually note that $O(n)$ is compact since closed and bounded, as we saw earlier. Boundedness is explained in analogy to $U(n)$:

$$||R||^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} R_{ij} R_{ij} \right) = \sum_{i=1}^{n} \delta_{ii} = n \;, \quad \text{for any } R \in O(n) \;.$$

The three-dimensional Lie group $O(3)$ has two connected components: the compact (connected) group $SO(3) := \{R \in O(3) \mid \det R = 1\}$ and the compact set (not a subgroup) $\mathscr{P}SO(3) := \{\mathscr{P}R \in O(3) \mid R \in SO(3)\}$, where $\mathscr{P} := -I$ is the parity transformation.

(5) We will explain how *the exponential map is a covering map for the whole group* $SO(3)$. Define a special basis of $so(3)$ given by matrices $(T_i)_{jk} = -\varepsilon_{ijk}$ where $\varepsilon_{ijk} = 1$ if i, j, k is a cyclic permutation of 1, 2, 3, $\varepsilon_{ijk} = -1$ if i, j, k is a non-cyclic permutation, $\varepsilon_{ijk} = 0$ otherwise. More explicitly

$$T_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad T_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{12.84}$$

All are skew-symmetric so they belong in $so(3)$, and they are clearly linearly independent, hence a basis of $so(3)$. Structure constants are simple in this basis:

$$[T_i, T_j] = \sum_{k=1}^{3} \varepsilon_{ijk} T_k \;, \tag{12.85}$$

The exponential representation of $SO(3)$ is as follows: $R \in SO(3)$ if and only if there exist a unit vector $\mathbf{n} \in \mathbb{R}^3$ and a number $\theta \in \mathbb{R}$ such that

$$R = e^{\theta \mathbf{n} \cdot \mathbf{T}} \;, \quad \text{where} \quad \mathbf{n} \cdot \mathbf{T} := \sum_{i=1}^{3} n_i T_i \;.$$

(6) The Lie algebra of the compact group $SU(2)$, seen as a *real* Lie group, is the real vector space of traceless skew-Hermitian matrices (because the determinant in the group equals 1). Consequently it has a basis formed by $-\frac{i}{2}\sigma_j$, $j = 1, 2, 3$, where σ_k are the Pauli matrices of (12.13). The factor $1/2$ is present so to satisfy the commutation relations:

$$\left[-\frac{i\sigma_i}{2} \;,\; -\frac{i\sigma_j}{2} \right] = \sum_{k=1}^{3} \varepsilon_{ijk} \left(-\frac{i\sigma_k}{2} \right) \;. \tag{12.86}$$

By the remark ensuing Theorem 12.59 the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic. Hence by Theorems 12.55, 12.56 the Lie groups are locally isomorphic. As $SU(2)$ is connected and simply connected (it is homeomorphic to the boundary $\mathbb{S}^3$ of the unit ball in $\mathbb{R}^4$), whereas $SO(3)$ is not simply connected, $SU(2)$ must be the universal covering of $SO(3)$. The Lie algebra isomorphism should arise from differentiating a surjective homomorphism $SU(2) \to SO(3)$. The latter is actually well known (Exercise 12.18), so let us recall it briefly. The exponential map of $SU(2)$ covers the entire group by compactness. In practice every matrix $U \in SU(2)$ can be written

$$U = e^{-i\theta\mathbf{n}\cdot\frac{\sigma}{2}}$$

where $\theta \in \mathbb{R}$ and $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. The aforementioned surjective morphism is the onto map

$$R : SU(2) \ni e^{-i\theta\mathbf{n}\cdot\frac{\sigma}{2}} \mapsto e^{\theta\mathbf{n}\cdot\mathbf{T}} \in SO(3) .$$

Clearly this is not invertible, because the right-hand side is invariant under translations $\theta \to \theta + 2\pi$, while the left-hand side changes sign (take the unit vector $\mathbf{n} = \mathbf{e}_3$ along the axis x_3). In fact it is easy to see that the kernel of h consists of two points $\pm I \in SU(2)$. ■

12.2.9 Continuous Unitary Finite-Dimensional Representations of Connected Non-compact Lie Groups

We discuss here a technical result that is mentioned very often, and used to prove that *there are no non-trivial unitary continuous finite-dimensional representations of the special orthochronous Lorentz group* $SO(1,3)\!\uparrow$. The result extends to its universal covering $SL(2,\mathbb{C})$, as we shall explain after the proof. Unfortunately the proof of many books, and often the statement itself, of this remarkable proposition contains mistakes.

Observe that the theorem does not refer to strong or weak continuity, just because, in an n-dimensional Hilbert space, these notions of continuity are evidently equivalent to the continuity in the standard $\mathbb{C}^n$.

Theorem 12.66 *Let* G *be a connected non-compact Lie group and*

$$U : \mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$$

a continuous unitary representation on a finite-dimensional Hilbert space H. *Then*
(a) U *cannot be faithful;*
(b) *If* G *is a simple group or, more generally, if it does not contain non-trivial proper normal closed subgroups, then* U *is the trivial representation* $U : \mathsf{G} \ni g \mapsto I$.

Proof Let us identify H with $\mathbb{C}^n$ by means of an orthonormal basis. In this way, the representation U can be viewed as an injective continuous group homomorphism $f : \mathsf{G} \to U(n)$.

(a) Our final goals is proving that $f(\mathsf{G})$ is a compact embedded submanifold of $U(n)$ and that the injective homomorphism $f : \mathsf{G} \to f(\mathsf{G})$ is actually a homeomorphism. This will not be possible, because G is not compact by hypothesis.

Due to Theorem 12.56, f is differentiable (analytic) and $df|_e$ is a Lie algebra homomorphism which is injective if f is faithful (because the kernel of f is the discrete subgroup $\{e\}$). Assuming that f is injective (i.e., U is faithful), consider the Lie subalgebra $u_0 := df|_e \mathsf{T}_e\mathsf{G} \subset u(n)$ where $u(n)$ is the Lie algebra of $U(n)$. Since $df|_e$ is injective, u_0 is isomorphic to $\mathsf{T}_e\mathsf{G}$. There is exactly one connected Lie subgroup $\mathsf{U}_0 \subset U(n)$ whose Lie algebra is u_0 in view of Theorem 12.61. By definition of Lie subgroup, U_0 is an embedded submanifold of $U(n)$. It must be clear that $f(\mathsf{G}) \cap \mathsf{U}_0$ contains all one-parameter subgroups of $U(n)$ generated by the elements of u_0, because these subgroups are simultaneously in U_0 and in $f(\mathsf{G})$, as the reader can prove immediately using Theorem 12.63(c). On the other hand, every element $h \in \mathsf{U}_0$ is a finite product of elements belonging to the one-parameter subgroups of U_0 as a consequence of Proposition 12.64, and hence h is also a finite product of elements of $f(\mathsf{G})$. Since f is a group homomorphism, every element $h \in \mathsf{U}_0$ satisfies $h \in f(\mathsf{G})$. We have so far established that $\mathsf{U}_0 = f(\mathsf{G})$. The map $f : \mathsf{G} \to \mathsf{U}_0$ is a bijective differentiable map from the manifold G to the embedded submanifold U_0 of $U(n)$. Since $df|_g = dL_{g^{-1}} \circ df|_e \circ dR_g$ where $R_g : \mathsf{G} \ni h \mapsto hg \in \mathsf{G}$ and $L_k : U(n) \ni r \mapsto kr \in U(n)$ are diffeomorphisms and therefore both $dL_{g^{-1}}$ and dR_g are a bijections, we conclude that $df|_g$ is everywhere injective. As a consequence [Wes78, Proposition 4.2(2)], if $p = \dim \mathsf{G}$ and $q = \dim U(n) \geq p$, then for any chart (S_g, ϕ) around any $g \in \mathsf{G}$ there is some chart (V_g, ψ) in $U(n)$ around $f(g)$ with

$$\psi \circ f \circ \phi^{-1}(x^1, \ldots, x^p) = (x^1, \ldots, x^p, 0, \ldots, 0)$$

where $(x^1, \ldots, x^p)$ belongs to the open set $\phi(V_g) \subset \mathbb{R}^q$. Since $f(\mathsf{G}) = \mathsf{U}_0$ is an embedded submanifold of $U(n)$, we have that $V_g \cap f(\mathsf{G}) = f(S_g)$ possibly restricting V_g around $f(g)$. In other words $f(S_g)$ is open in the induced topology of $f(\mathsf{G}) \subset U(n)$. Since $g \in \mathsf{G}$ is arbitrary and the property is valid by replacing S_g with any smaller open set containing g, the injectivity of f proves that $f : \mathsf{G} \to \mathsf{U}_0 = f(\mathsf{G})$ is open: every open set $A \subset \mathsf{G}$ is the union of open sets $A = \cup_{g \in \mathsf{G}} A \cap S_g$; since f is bijective onto U_0 we also have that $f(A) = \cup_{g \in \mathsf{G}} f(A \cap S_g)$, which is open because union of open sets. The inverse $f^{-1} : \mathsf{U}_0 \to \mathsf{G}$ exists (because f is bijective onto U_0), and is therefore continuous. By Proposition 12.60 U_0 is closed and hence compact ($U(n)$ is compact). This is absurd, because $f^{-1}(\mathsf{U}_0) = \mathsf{G}$ is not compact by hypothesis and f^{-1} is continuous. We conclude that $f : \mathsf{G} \to U(n)$ cannot be injective, that is, $U : \mathsf{G} \to \mathfrak{B}(\mathsf{H})$ cannot be faithful.

(b) If G does not contain non-trivial proper closed normal subgroups, the normal closed subgroup $U^{-1}(I)$ of G must equal either G or $\{e\}$. In the second case U would be faithful, which is not permitted by (a). Summing up, $U^{-1}(I) = \mathsf{G}$ so that $U(\mathsf{G}) = \{I\}$. □

Remarks 12.67 **(1)** The theorem applies to $SO(1, 3)\uparrow$ since this is non-compact, connected and it has no non-trivial closed normal subgroups: its strongly continuous unitary representations are infinite-dimensional or trivial. The same result is valid for $SL(2, \mathbb{C})$, which is non-compact and connected but not simple[11]: $SL(2, \mathbb{C})$ *does not admit non-trivial proper finite-dimensional continuous unitary representations*. Indeed, $\{\pm I\}$ is the unique non-trivial proper normal closed subgroup of $SL(2, \mathbb{C})$. A finite-dimensional continuous unitary representation $U : SL(2, \mathbb{C}) \to \mathfrak{B}(\mathsf{H})$ cannot be faithful by (a). Therefore the closed normal subgroup $U^{-1}(I)$ cannot be trivial and therefore coincides with either $SL(2, \mathbb{C})$, making U trivial, or with $\{\pm I\}$. Let us examine this second possibility, and prove that U has to be trivial also in this case. As is well known, the Lie group $SL(2, \mathbb{C})$ is the universal covering of the Lie group $SO(1, 3)\!\uparrow$ in accordance with Theorem 12.55, and $\{\pm I\}$ is just the kernel of the covering homomorphism, so $SO(1, 3)\!\uparrow$ is diffeomorphic to $SL(2, \mathbb{C})/\{\pm I\}$. It is easy to prove that, consequently, $U : SL(2, \mathbb{C}) \to \mathfrak{B}(\mathsf{H})$ defines a finite-dimensional continuous unitary representation

$$U' : SO(1, 3)\!\uparrow \ni \pm A \ \mapsto U_A \ \in \mathfrak{B}(\mathsf{H}) .$$

The representation U' must be trivial by (b). In turn, U must be trivial as well because $U'(SO(1, 3)\!\uparrow) = U(SL(2, \mathbb{C}))$.
(2) Since $O(n)$ is compact, like $U(n)$, the same argument exploited for the theorem above can be used to prove that a connected non-compact Lie group cannot admit faithful continuous representations on orthogonal matrices and, in particular, all these representations are trivial if the hypotheses in (b) are true. ■

12.2.10 Bargmann's Theorem

We proceed in the study of symmetry groups and deal with connected Lie groups G. Any projective G-representation is representable by unitary operators.

Proposition 12.68 *Let G be a connected Lie group. For any projective representation $\mathsf{G} \ni g \mapsto \gamma_g$ on a Hilbert space H of* $\dim \mathsf{H} > 1$*, the images γ_g can be associated to unitary operators only, according to Wigner's theorem (or Kadison's). If* $\dim \mathsf{H} = 1$ *a unitary representation is always possible.*

Proof If $\dim \mathsf{H} > 1$, by Proposition 12.64, every $g \in \mathsf{G}$ is the product of a finite number of elements $h = \exp(tT)$. Then $h = r \cdot r$ with $r = \exp(tT/2)$. Using Proposition 12.32 the claim follows. If $\dim \mathsf{H} = 1$ every γ_g is the identity and therefore it can be represented unitarily by the trivial representation $\mathsf{G} \ni g \mapsto I$. □

At this point we shall present a number of general results on strongly continuous unitary representations of Lie groups.

It will be useful, in the sequel, to observe that any projective representation of a topological group G may be seen as projective representation of its universal covering

[11] Though it is a *simple Lie group* because it has no non-trivial proper *connected* normal Lie subgroups.

group $\widetilde{\mathsf{G}}$. In fact if $\pi : \widetilde{\mathsf{G}} \to \mathsf{G}$ is the covering map (a continuous homomorphism of topological groups [NaSt82], also guaranteed by Theorem 12.55 for Lie groups), and $\gamma : \mathsf{G} \ni g \mapsto \gamma_g$ is a continuous projective G-representation on the Hilbert space H, then $\gamma \circ \pi : \widetilde{\mathsf{G}} \ni h \mapsto \gamma_{\pi(h)}$ is a continuous projective $\widetilde{\mathsf{G}}$-representation; note that it does not distinguish elements $h, h' \in \widetilde{\mathsf{G}}$ if $\pi(h) = \pi(h')$. Put equivalently, if $h \cdot h'^{-1} \in Ker(\pi)$ then $\gamma \circ \pi(h) = \gamma \circ \pi(h')$ i.e. $(\gamma \circ \pi)(Ker(\pi)) = id$, or $Ker(\pi) \subset Ker(\gamma \circ \pi)$. This proves the following.

Proposition 12.69 *Let G be a topological group and $\pi : \widetilde{\mathsf{G}} \to \mathsf{G}$ its universal covering. Every continuous projective representation $\gamma : \mathsf{G} \ni g \mapsto \gamma_g$ of G on the Hilbert space H arises from the continuous projective representation $\gamma' : \widetilde{\mathsf{G}} \ni g \mapsto \gamma'_g$ of $\mathsf{G} \equiv \widetilde{\mathsf{G}}/Ker(\pi)$ on H such that $Ker(\pi) \subset Ker(\gamma')$.*

Remark 12.70 When needed, henceforth, we will use projective unitary representations of $\widetilde{\mathsf{G}}$ instead of G, because the latter are determined by the former and, in case of Lie groups, $\widetilde{\mathsf{G}}$ is determined by its Lie algebra. ■

We will prepare the ground for an important theorem due to Bargmann [Bar54], that provides sufficient conditions for a continuous projective representation to be given by a unitary representation when the groups in questions are Lie groups. An exhaustive discussion on the mathematical technology necessary to prove the theorem of Bargmann appears in [Var07, Chap. VII] together with several physical examples. The preliminary idea, presented in Sect. 12.2.5, is that a projective unitary representation

$$\mathsf{G} \ni g \mapsto U_g$$

of a group G is the restriction $U_g := U_{(1,g)}$ of a unitary representation

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto U_{(\chi,g)} = \chi U_g$$

of a suitable central extension $\widehat{\mathsf{G}}_\omega$ of G. This is always possible by virtue of Proposition 12.36. Assume G is a topological group, and the projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ induces a continuous projective representation. We can choose the phases of the U_g so that the representation $\mathsf{G} \ni g \mapsto U_g$ is continuous around the identity of G by Proposition 12.44. This cannot be extended to the entire G, in general. The next technical result extends the result to when G is a Lie group.

Theorem 12.71 *Let G be a connected Lie group and $\gamma : \mathsf{G} \ni g \mapsto \gamma_g$ a continuous projective representation on the Hilbert space H. There exist a central extension $\widehat{\mathsf{G}}_\omega$ and a unitary representation*

$$\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto U_{(\chi,g)} \in \mathfrak{B}(\mathsf{H})$$

with[12] $\omega(e,e) = \omega(g,e) = \omega(e,g) = 1$ *and* $U_{(\chi,e)} = \chi I$ *for any* $\chi \in U(1)$*, such that:*

(a) $\widehat{\mathsf{G}}_\omega$ *is a connected Lie group, the canonical injection* $U(1) \to \widehat{\mathsf{G}}_\omega$ *and canonical projection* $\widehat{\mathsf{G}}_\omega \to \mathsf{G}$ *are Lie group homomorphisms;*

(b) *as topological space* $\widehat{\mathsf{G}}_\omega$ *is the product* $U(1) \times A_g$ *around every element* (χ, g)*, where* $A_g \subset \mathsf{G}$ *is an open neighbourhood of* $g \in \mathsf{G}$ *(but* $\widehat{\mathsf{G}}_\omega$ *is not* $U(1) \times \mathsf{G}$ *globally in general);*

(c) *as* C^∞ *manifold* $\widehat{\mathsf{G}}_\omega$ *is the product* $U(1) \times A$ *around the unit element* $(1, e)$*, where* $A \subset \mathsf{G}$ *is an open neighbourhood of* $e \in \mathsf{G}$*;*

(d) *the map* $\mathsf{G} \ni g \mapsto U_{(1,g)} =: U_g$ *is a projective unitary representation that induces* γ*:*

$$\gamma_g(\rho) = U_g \rho U_g^{-1} \quad \text{for any } g \in \mathsf{G},\ \rho \in \mathfrak{S}_p(\mathsf{H}) \tag{12.87}$$

and is strongly continuous on an open neighbourhood of the unit $e \in \mathsf{G}$*;*

(e) *the unitary representation* $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto U_{(\chi,g)}$ *is strongly continuous.*

Outline of the proof. Given γ, consider an initial projective unitary representation $\mathsf{G} \ni g \mapsto V_g$ inducing γ with multiplier ω_0 and the associated unitary representation of the corresponding central extension $\widehat{\mathsf{G}}_{\omega_0}$. Using Proposition 12.44, pass to a different but equivalent projective unitary representation $\mathsf{G} \ni g \mapsto U_g$ such that in a neighbourhood A of $e \in \mathsf{G}$ the map $A \ni g \mapsto U_g$ is strongly continuous. According to Lemma 7.20 in [Var07], it is possible to replace the multiplier of that extension with an equivalent multiplier that is C^∞ on a neighbourhood of e and that we shall indicate by ω again. Notice that this change preserves the strong continuity of $A \ni g \mapsto U_g$. Finally, define the unitary representation $\widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto U_{(\chi,g)} := \chi U_g$ inducing γ so that $\gamma(\rho) = U_{(\chi,g)} \rho U_{(\chi,g)}^{-1} = U_g \rho U_g^{-1}$. According to [Kir76, Exercises 1 and 2, Sect. 14.3], exploiting again the argument of the proof of Proposition 12.44, every $g \in \mathsf{G}$ admits a neighbourhood A_g and a corresponding local homeomorphisms $\widehat{\mathsf{G}}_\omega \to U(1) \times \mathsf{G}$:

$$\widehat{\mathsf{G}}_\omega \supset U(1) \times A_g \ni (\chi, h) \mapsto (\chi_h^{(g)} \chi, h) \in U(1) \times A_g \subset U(1) \times \mathsf{G}\,,$$

where the maps $A_g \ni h \mapsto \chi_h^{(g)}$ are such that $A_g \ni h \mapsto \chi_h^{(g)} U_h$ is strongly continuous. By construction $\chi_g^{(e)} = 1$ for $g \in A_e =: A$. This class of local homeomorphisms defines a C^0 atlas on $\widehat{\mathsf{G}}_\omega$ (made of charts with domain $U(1) \times A_g$) compatible with the group operations of $\widehat{\mathsf{G}}_\omega$. Hence the Gleason–Montgomery–Zippin theorem implies that the C^0 structure of $\widehat{\mathsf{G}}_\omega$ contains an analytic Lie substructure. (a) and (b) hold because the canonical maps $U(1) \to \widehat{\mathsf{G}}_\omega$ and $\widehat{\mathsf{G}}_\omega \to \mathsf{G}$ are topological group homomorphisms (the proof is immediate) and therefore they are Lie group homomorphisms in view of Proposition 12.51. To prove (c) consider the topological group structure of $\widehat{\mathsf{G}}_\omega$ on a neighbourhood $U(1) \times A$ of the

[12] As we know, we can always reduce to this case via an equivalence transformation by a constant map.

unit element. The group operations of $\widehat{\mathsf{G}}_\omega$ constructed out of the smooth function ω are smooth with respect to the product of C^∞ structures of $U(1)$ and A. Applying Theorem 9.4.4 in [HiNe13] to $U(1) \times A$, one sees that there is a unique C^∞ Lie group structure on the topological group $\widehat{\mathsf{G}}_\omega$ such that the inclusion $U(1) \times A \to \widehat{\mathsf{G}}_\omega$ is a C^∞ diffeomorphism onto its image. From Corollary 2.17 in [HiNe13], the said global C^∞ structure must coincide with the C^∞ Lie group structure obtained by enlarging the analytic GMZ structure. In summary, the C^∞ structure around the identity of $\widehat{\mathsf{G}}_\omega$ coincides with that of $U(1) \times \mathsf{G}$. To prove (d), observe that $A \ni g \mapsto U_g$ coincides with the unitary projective representation found in Proposition 12.44, that is strongly continuous in A. Therefore $A \ni g \mapsto U_g$ is strongly continuous. Statement (e) is easy. Since $(\chi, g) \mapsto U_{(\chi,g)} = \chi U_g$ is a unitary representation, it is strongly continuous iff it is strongly continuous at the unit element $(1, e)$. The latter is true because $g \mapsto U_g$ is continuous on a neighbourhood A of e and $\widehat{\mathsf{G}}_\omega$ is homeomorphic to $U(1) \times A$ around $(1, e)$, so that $(\chi, g) \to (1, e)$ in the topology of $\widehat{\mathsf{G}}_\omega$ means $(\chi, g) \to (1, e)$ in the product topology and $||U_{(\chi,g)}\psi - U_{(1,e)}\psi|| = ||\chi U_g \psi - 1U_e\psi|| = ||\chi U_g\psi - \psi|| \to 0$ if $(\chi, g) \to (1, e)$ in the product topology. $\square$

So let us assume, by the above theorem, that any continuous projective representation of a Lie group G is obtainable as strongly continuous projective unitary representation of a central extension of G, itself a Lie group. This allows to introduce Bargmann's theorem.

Let us go through the proof's idea, heuristically. Take a Lie group G (connected and simply connected in the theorem) and its central $U(1)$-extensions $\widehat{\mathsf{G}}_\omega$. Projective unitary representations of G are honest continuous unitary representations of $U(1)$-extensions of G. The question is when are continuous unitary representations of $\widehat{\mathsf{G}}_\omega$ reducible to continuous unitary representations of G. Since the C^∞ structure of $\widehat{\mathsf{G}}_\omega$ is isomorphic to that of $U(1) \times \mathsf{G}$ around the identity, this is enough to guarantee that the Lie algebra of $\widehat{\mathsf{G}}_\omega$ is the vector space $\mathbb{R} \oplus \mathsf{T}_e\mathsf{G}$ with bracket

$$\left[r \oplus T, r' \oplus T'\right] = \alpha(T, T') \oplus [T, T'] ,$$

where $r \oplus T$ is the generic element in $\mathbb{R} \oplus \mathsf{T}_e\mathsf{G}$ and $\alpha : \mathsf{T}_e\mathsf{G} \times \mathsf{T}_e\mathsf{G} \to \mathsf{T}_e\mathsf{G}$ a bilinear skew-symmetric map. A common alternative way to write this is to fix a basis $\{T_k\}_{k=1,\ldots,n}$ of $T_e\mathsf{G}$ and set

$$[I, I] = [I, T_k] = \mathbf{0} , \quad [T_i, T_j] = \alpha_{ij} I + \sum_{k=1}^{n} C_{ijk} T_k , \tag{12.88}$$

where $I := 1 \oplus \mathbf{0}$, T_k actually indicates $0 \oplus T_k$ and $\mathbf{0}$ means $0 \oplus \mathbf{0}$.

The coefficients $\alpha_{ij} := \alpha(T_i, T_j)$ are by construction skew-symmetric, and, in consequence of Jacobi's identity (and corresponding to (12.94) under Bargmann's Theorem) (12.90) holds:

$$\alpha_{ij} = -\alpha_{ji} \,, \tag{12.89}$$

$$0 = \sum_{s=1}^{n} \left(C_{ijs}\alpha_{sk} + C_{jks}\alpha_{si} + C_{kis}\alpha_{sj} \right) \,. \tag{12.90}$$

The numbers α_{ij} are often called **central charges**. The key idea behind Bargmann's theorem is to change basis in the Lie algebra of $\widehat{\mathsf{G}}_\omega$, passing to the new generators

$$I' := I \,, \quad T'_k := \beta_k I + T_k$$

so that

$$[I, I] = [I, T'_k] = \mathbf{0} \,, \quad [T'_i, T'_j] = \alpha_{ij} I + \sum_{k=1}^{n} C_{ijk}(T'_k - \beta_k I) \,, \tag{12.91}$$

and therefore

$$[T'_i, T'_j] = \left(\alpha_{ij} - \sum_{k=1}^{n} C_{ijk}\beta_k \right) I + \sum_{k=1}^{n} C_{ijk} T'_k \,, \tag{12.92}$$

In case we find constants β_k that *absorb the central charges*, i.e.

$$\alpha_{ij} = \sum_{k=1}^{n} C_{ijk}\beta_k \,, \tag{12.93}$$

(note C_{ijk} and α_{ij} are given, once $\widehat{\mathsf{G}}_\omega$ is known) we can write the bracket relations of the Lie algebra of $\widehat{\mathsf{G}}_\omega$ as:

$$[I, I] = [I, T'_k] = \mathbf{0} \,, \quad [T'_i, T'_j] = \sum_{k=1}^{n} C_{ijk} T'_k \,.$$

These are the very commutation relations of the Lie algebra of the direct product of $U(1)$ and G, where $\omega(g, g') = 1$ always. If this is possible, we expect to view a unitary $\widehat{\mathsf{G}}_\omega$-representation as an honest unitary G-representation times $U(1)$, getting rid of the phases.

The hypothesis of Bargmann's Theorem [Bar54], formulated by (12.95), is just condition (12.93), as the proof will explain. The linear function β of the statement, in fact, is completely determined by the coefficients β_k if we set $\beta(T_k) := \beta_k$.

Theorem 12.72 (Bargmann) *Let* G *be a connected, simply connected Lie group. Every continuous projective* G*-representation on the Hilbert space* H *is induced by a strongly continuous unitary representation on* H *provided the following condition holds. For any skew-symmetric bilinear map* $\alpha : \mathsf{T}_e\mathsf{G} \times \mathsf{T}_e\mathsf{G} \to \mathbb{R}$ *satisfying*

$$\alpha\left([T,T'],T''\right)+\alpha\left([T',T''],T\right)+\alpha\left([T'',T],T'\right)=0\,,\quad T,T',T''\in \mathsf{T}_e\mathsf{G}, \tag{12.94}$$

there exists a linear map $\beta : \mathsf{T}_e\mathsf{G} \to \mathbb{R}$ *with*

$$\alpha(T,T') = \beta\left([T,T']\right)\,,\quad T,T'\in \mathsf{T}_e\mathsf{G}. \tag{12.95}$$

Proof Consider a continuous projective representation $\gamma : \mathsf{G} \ni g \mapsto \gamma_g$ on the Hilbert space H. By Theorem 12.71, there is a central $U(1)$-extension $\widehat{\mathsf{G}}_\omega$ of G with the structure of Lie group, and a projective unitary representation $U : \mathsf{G} \ni g \mapsto U_g$ that is strongly continuous around $e \in \mathsf{G}$ and induces γ. The canonical inclusion and projection are Lie homomorphisms. Moreover, around the origin the C^∞ structure of $\widehat{\mathsf{G}}_\omega$ is the product $U(1)\times A$ for some neighbourhood A of $e \in \mathsf{G}$. The multiplier function is *normalised* so that $\omega(e,e)=\omega(e,g)=\omega(g,e)=1$, hence the neutral element of $\widehat{\mathsf{G}}_\omega$ is $(1,e)$. The real vector space underlying the Lie algebra of $\widehat{\mathsf{G}}_\omega$ is $\mathbb{R}\oplus\mathsf{T}_e\mathsf{G}$, where $\oplus$ is the direct sum. We will denote by $r\oplus T$ the elements, where $r\in\mathbb{R}$ and $T\in\mathsf{T}_e\mathsf{G}$. As already said, by the definition of Lie bracket $[\ ,\ ]$ of $\mathsf{T}_e\mathsf{G}$, a few computations involving (12.77) say that the bracket $[\ ,\ ]_\omega$ of $\mathsf{T}_{1\oplus e}\widehat{\mathsf{G}}_\omega$ has the form:

$$\left[r\oplus T, r'\oplus T'\right]_\omega = \alpha(T,T')\oplus[T,T'] \tag{12.96}$$

where $\alpha : \mathsf{T}_e\mathsf{G}\times\mathsf{T}_e\mathsf{G}\to\mathbb{R}$ is a bilinear skew-symmetric map satisfying (12.94), owing to the Jacobi identity of $[\ ,\ ]_\omega$. Now we show that, retaining the hypotheses of the theorem, the universal covering of $\widehat{\mathsf{G}}_\omega$ is the Lie group $\mathbb{R}\otimes\mathsf{G}$, where $\otimes$ is the *direct product* ($\mathbb{R}$ is an additive Lie group). The topological space underlying $\mathbb{R}\otimes\mathsf{G}$ is the product $\mathbb{R}\times\mathsf{G}$, simply connected as the factors are. By Theorem 12.55 $\mathbb{R}\otimes\mathsf{G}$ is the unique simply connected Lie group, up to isomorphisms, having that Lie algebra, and hence is the universal covering of all Lie groups with the Lie algebra of $\mathbb{R}\otimes\mathsf{G}$. We will show $\widehat{\mathsf{G}}_\omega$ is one of those. The Lie algebra of $\mathbb{R}\otimes\mathsf{G}$ is $\mathbb{R}\oplus\mathsf{T}_e\mathsf{G}$ with bracket:

$$\left[r\oplus T, r'\oplus T'\right]_\otimes = 0\oplus[T,T'] \tag{12.97}$$

To prove the claim it suffices to exhibit an isomorphism mapping the Lie algebra of $\mathbb{R}\otimes\mathsf{G}$ to the Lie algebra of $\widehat{\mathsf{G}}_\omega$, when there is $\beta : \mathsf{T}_e\mathsf{G}\to\mathbb{R}$ satisfying (12.95). Let us construct the isomorphism. Fix a basis $T_1,\dots,T_n$ in the Lie algebra of G, and a corresponding basis

$$1\oplus 0,\ \ 0\oplus T_1,\dots,\ \ 0\oplus T_n\ \ \in \mathsf{T}_{(0,e)}\mathbb{R}\otimes\mathsf{G}$$

in the Lie algebra of $\mathbb{R}\otimes\mathsf{G}$. Consider the new basis in the Lie algebra of $\widehat{\mathsf{G}}_\omega$:

$$1\oplus 0,\ \ \beta(T_1)\oplus T_1,\dots,\ \ \beta(T_n)\oplus T_n\ \ \in \mathsf{T}_{(1,e)}\widehat{\mathsf{G}}_\omega\,.$$

This is clearly a basis because the vectors are linearly independent if $T_1,\dots,T_n$ form a basis. Consider the unique linear bijection $f : \mathsf{T}_{(0,e)}\mathbb{R}\otimes\mathsf{G}\to\mathsf{T}_{(1,e)}\widehat{\mathsf{G}}_\omega$ such that:

$$f(1 \oplus 0) := 1 \oplus 0\,, \quad f(0 \oplus T_k) := \beta(T_k) \oplus T_k \quad \text{for } k = 1, 2, \ldots, n.$$

We claim it preserves brackets:

$$[f(r \oplus T)\,,\; f(r' \oplus T')]_\omega = f([r \oplus T\;,\; r' \oplus T]_\otimes)\,,$$

and hence is an isomorphism. As f is linear and brackets are bilinear and skew, it is enough to prove the claim on pairs of distinct basis elements. Evidently $[f(1 \oplus 0)\,,\; f(0 \oplus T_k)]_\omega = 0 = f([1 \oplus 0\,,\; 0 \oplus T_k]_\otimes)$. As for the remaining non-trivial commutators,

$$[f(0 \oplus T_h)\,,\; f(0 \oplus T_k)]_\omega = [\beta(T_h) \oplus T_h\;,\; \beta(T_k) \oplus T_k]_\omega = \alpha(T_h, T_k)[T_h, T_k]$$

$$= \beta([T_h, T_k]) \oplus [T_h, T_k] = \beta\left(\sum_{s=1}^{n} C_{hks} T_s\right) \oplus \sum_{s=1}^{n} C_{hks} T_s = \sum_{s=1}^{n} C_{hks}\,(\beta(T_s) \oplus T_s)$$

$$= \sum_{s=1}^{n} C_{hks} f\,(0 \oplus T_s) = f\left(\sum_{s=1}^{n} C_{hks} 0 \oplus T_s\right) = f\,([0, \oplus T_h\;,\; 0 \oplus T_s]_\otimes)\;.$$

where C_{hks} are the structure constants of G in the basis $T_1, \ldots, T_n$. Therefore the universal covering of $\widehat{\mathsf{G}}_\omega$ is $\mathbb{R} \otimes \mathsf{G}$, and there is a surjective Lie homomorphism

$$\Pi : \mathbb{R} \otimes \mathsf{G} \ni (r, g) \mapsto (\chi(r, g), h(r, g)) \in \widehat{\mathsf{G}}_\omega\,,$$

such that

$$d\Pi|_{(0,g)} = f \tag{12.98}$$

(the latter determines the map uniquely, by Theorem 12.56). Now let us study Π, exploiting the fact that $\widehat{\mathsf{G}}_\omega$ is a central $U(1)$-extension of G. Easily $h(r, e) = e$ for any $r \in \mathbb{R}$. Consider in fact the one-parameter group of $\mathbb{R} \otimes \mathsf{G}$

$$\mathbb{R} \ni r \mapsto (r, e) = \exp\{r(1 \oplus 0)\}\,;$$

Π maps it, by Theorem 12.55(c), to the one-parameter subgroup of $\widehat{\mathsf{G}}_\omega$:

$$\mathbb{R} \ni r \mapsto \exp\{rf(1 \oplus 0)\} = (\chi(r, e), h(r, e)) = \exp\{r(1 \oplus 0)\} = \exp\{(r \oplus 0)\}\,.$$

The Baker–Campbell–Hausdorff formula (12.81) and the relations (12.96) give, for any $r \in \mathbb{R}$ around 0:

$$(\chi(r, e), h(r, e)) = \exp\{(r \oplus 0)\} = (\chi(r, e), e)\,.$$

As $h(r,e)h(s,e) = h(r+s,e)$ by the properties of one-parameter subgroups, the identity found extends to any $r \in \mathbb{R}$, so $h(r,e) = e$ for every $r \in \mathbb{R}$. Define $\chi(r) := \chi(r,e)$. Then

$$\Pi : (r,e) \mapsto (\chi(r), e) \quad \text{and} \quad \chi(r)\chi(r') = \chi(r+r') \quad \text{for any } r, r' \in \mathbb{R}.$$

The second equation follows because $r \mapsto \exp\{rf(1 \oplus 0)\} = (\chi(r,e), h(r,e))$ is a one-parameter subgroup. Setting $h(g) := h(0,g)$ and $\phi(g) := \chi(0,g)$, we can write

$$\Pi : \mathbb{R} \otimes \mathsf{G} \ni (r,g) \mapsto (\chi(r)\phi(g), h(g)) \in \widehat{\mathsf{G}}_\omega\,. \tag{12.99}$$

Let us study the map $h : (0,g) \mapsto g$ and prove it is an isomorphism. As Π is a group homomorphism it maps the product $(r,g)\cdot(r',g')$ to the images' product, so

$$(\chi(r), h(g)) \cdot (\chi(r), h(g')) = (\chi(r+r')\phi(g)\phi(g')\omega(h(g), h(g'))\,,\ h(gg'))\,.$$

This implies $h : \mathsf{G} \ni g \equiv (0,g) \mapsto h(g) \in \mathsf{G}$ is a group homomorphism, the domain G being a Lie subgroup in $\mathbb{R} \otimes \mathsf{G}$. But Π is onto, so h is onto, too. The map $\widehat{\mathsf{G}}_\omega(\chi, s) \mapsto s \in \mathsf{G}$ is a surjective Lie homomorphism by definition of central extension, so we conclude $h : \mathsf{G} \ni g \mapsto h(g) \in \mathsf{G}$ is a surjective Lie homomorphism. By (12.98), it is easy to see $dh : 0 \oplus T_k \to T_k$. Consequently, by (iii) of Theorem 12.56(c) dh is the differential at the identity of a unique Lie isomorphism from G (subgroup of $\mathbb{R} \oplus \mathsf{G}$) to G. By construction it must coincide with h.

To finish take the multiplier function ω and $\phi : \mathsf{G} \to U(1)$. Then $\phi(e) = 1$, because $\Pi : (0,e) \mapsto (1,e)$. Since $\Phi : (0,g) \mapsto (\phi(g), h(g))$ is a Lie homomorphism and the C^∞ structure of $\widehat{\mathsf{G}}_\omega$ is the product around the identity, there ϕ is differentiable. The projection Π maps $(0,g)\cdot(0,g')$ to the product of the images. Therefore

$$(\phi(g)\phi(g')\omega(h(g), h(g'))\,,\ h(gg')) = (\phi(gg'), h(gg'))\,,$$

so

$$\phi(g)\phi(g')\omega(h(g), h(g')) = \phi(gg')\,, \quad g, g' \in \mathsf{G}. \tag{12.100}$$

There remains to find a continuous unitary representation

$$W : \mathsf{G} \ni g \mapsto W_g$$

inducing the projective representation γ. Since $h : \mathsf{G} \to \mathsf{G}$ is an isomorphism, define

$$W_g := \phi(h^{-1}(g))U_g\,, \quad g \in \mathsf{G}.$$

By construction this projective unitary representation induces γ, since $\phi(h^{-1}(g)) \in U(1)$. At the same time, by (12.100):

$$W_g W'_g = \phi(h^{-1}(g))\phi(h^{-1}(g'))U_g U_{g'}$$

$$= \omega(g, g')\phi(h^{-1}(g))\phi(h^{-1}(g'))U_{gg'} = \phi(gg')U_{gg'} = W_{gg'} .$$

Hence W is a proper unitary representation. To finish we show W is continuous. As U is continuous around e, h^{-1} is continuous, $h^{-1}(e) = e$ and ϕ is continuous around e, then $g \mapsto W_g = \phi(h^{-1}(g))U_g$ is certainly continuous on a neighbourhood A of e. That W is a representation of unitary operators implies its continuity (in the strong topology) at every point. In fact, if $\psi \in \mathsf{H}$:

$$||W_g\psi - W_{g_0}\psi|| = ||W_{g_0^{-1}}(W_g\psi - W_{g_0}\psi)|| = ||W_{g_0^{-1}g}\psi - \psi|| \to 0 \quad \text{as } g \to g_0.$$

We used the fact that $g_0^{-1}g \in A$ if g is sufficiently close to g_0, as G is a topological group. □

Remarks 12.73 **(1)** By a previous remark, Bargmann's theorem provides informations also in case the connected Lie group is not simply connected, by looking at its projective representations as representations of the (simply connected) universal covering.
(2) An alternative, and more sophisticated, way to state Bargmann's theorem relies on the *cohomology theory of Lie groups*.

Definition 12.74 If $(\mathsf{V}, \{\ ,\ \})$ is a real Lie algebra, let $Z^2(\mathsf{V}, \mathbb{R})$ indicate the real vector space of real bilinear skew-symmetric maps $\alpha : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ that have the property

$$\alpha(R, \{S, T\}) + \alpha(S, \{T, R\}) + \alpha(T, \{R, S\}) = 0 \quad \text{for any } S, R, T \in \mathsf{V} ,$$

and let $B^2(\mathsf{V}, \mathbb{R})$ be the subspace of $Z^2(\mathsf{V}, \mathbb{R})$ consisting of maps of the form

$$\mathsf{V} \times \mathsf{V} \ni (S, T) \mapsto \beta(\{S, T\}) \quad \text{for every linear } \beta : \mathsf{V} \to \mathbb{R}.$$

The quotient $H^2(\mathsf{V}, \mathbb{R}) := Z^2(\mathsf{V}, \mathbb{R})/B^2(\mathsf{V}, \mathbb{R})$ (viewed as an additive group) is the **real second cohomology group** of V.

It is now obvious that the existence of a linear map β for *every* bilinear skew map α satisfying (12.94) is equivalent to imposing that *the real second cohomology group* $H^2(T_e\mathsf{G}, \mathbb{R})$ of the Lie algebra of G is trivial: namely, that it is made of the zero element 0 only.
(3) It is worth stressing that Bargmann's theorem gives sufficient conditions for *every* continuous projective G-representation to be induced by a strongly continuous unitary representation. These conditions may still fail, but a certain continuous projective G-representation (induced by a unitary representation of some $\widehat{G}_\omega$ according to Theorem 12.71, and therefore associated to some particular $\alpha \in H^2(T_e\mathsf{G}, \mathbb{R})$) is nevertheless induced by a strongly continuous unitary representation. The argument

of Bargmann's proof actually also shows that this happens exactly for the continuous projective G-representation whose α equals zero as an element in $H^2(T_e\mathsf{G}, \mathbb{R})$.

Proposition 12.75 *Let $\mathsf{G} \ni g \mapsto \gamma$ be a continuous projective representation of a connected simply connected Lie group G on the Hilbert space H.*

Suppose that γ is induced by a strongly continuous unitary representation of a central extension $\widehat{\mathsf{G}}_\omega$ as in Theorem 12.71. If ω is represented on $T_e\mathsf{G}$ by the map α such that $[\alpha] = [0] \in H^2(T_e\mathsf{G}, \mathbb{R})$, then there exists a strongly continuous unitary representation $\mathsf{G} \ni g \mapsto U_g \in \mathsf{H}$ such that

$$\gamma_g(\rho) = U_g \rho U_g^{-1} \quad \textit{if } g \in \mathsf{G} \textit{ for any } \rho \in \mathfrak{S}(\mathsf{H}).$$

Proof We established this during the proof of Bargmann's theorem. □

(4) The discussion before the statement Bargmann's theorem implies that $H^2(T_e\mathsf{G}, \mathbb{R})$ retains relevant information about the class of smoothly inequivalent (Lie) central extensions of a connected Lie group G. This is because every such central extension $\widehat{\mathsf{G}}_\omega$ defines a corresponding element $\alpha \in Z^2(T_e\mathsf{G}, \mathbb{R})$. Furthermore, suppose that α, α' are associated to central extensions $\widehat{\mathsf{G}}_\omega$ and $\widehat{\mathsf{G}}'_\omega$ and these extensions are equivalent under an assignment of phases, smooth on a neighbourhood of the identity of G,

$$K : \mathsf{G} \ni g \mapsto \chi_g \in U(1) \quad \text{where } \chi_e = 1, \tag{12.101}$$

(i.e. $\omega(g, g') = \chi_g \chi_{g'} \chi_{gg'}^{-1} \omega'(g, g)$ for any $g, g' \in \mathsf{G}$). Then

$$\phi_K : \widehat{\mathsf{G}}_\omega \ni (\chi, g) \mapsto (\chi_g \chi, g) \in \widehat{\mathsf{G}}_{\omega'} \tag{12.102}$$

is a local Lie group isomorphism, as one proves without effort. It is easy to see that

$$d\phi_K : T_e\widehat{\mathsf{G}}_\omega \ni r \oplus T \to (r + \beta(T)) \oplus T \in \widehat{\mathsf{G}}_{\omega'} ,$$

for some linear map $\beta : T_e\mathsf{G} \to T_e\mathsf{G}$. Since $d\phi_K$ is a Lie algebra isomorphism and the Lie bracket has the form (12.96), then

$$\alpha'(T, S) = \alpha(T, S) + \beta([T, S]) \quad \text{for any } T, S \in T_e\mathsf{G}.$$

In other words, central extensions $\widehat{\mathsf{G}}_\omega$ and $\widehat{\mathsf{G}}'_\omega$ that are equivalent under a locally smooth change of phase define maps α and α' in the same class in $H^2(T_e\mathsf{G}, \mathbb{R})$.
(5) An important result is that Bargmann's theorem holds for connected simply connected Lie groups G whose Lie algebra is simple or semisimple [Bar54].

Proposition 12.76 *If the real Lie algebra $(\mathsf{V}, \{\cdot, \cdot\})$ is simple or semisimple, then $H^2(\mathsf{V}, \mathbb{R}) = \{0\}$.*

The result extends to Lie algebras of affine groups, like the Poincaré group [Bar54].
■

Physically important cases where Bargmann's theorem applies are [Bar54] $SL(2,\mathbb{C})$ (the universal covering of the proper orthochronous Lorentz group) and the universal covering of the connected component at the identity of the Poincaré group. In particular the Lie algebra of the former is semisimple. Therefore, within relativistic quantum theories one can always take advantage of Bargmann's theorem to deal with spacetime symmetries. Conversely, the treatment of spacetime symmetries in Galilean quantum mechanics is much more complicated, as we shall see soon, just because $H^2(\mathsf{T}_e\mathsf{G},\mathbb{R})$ is non-trivial when G is (the universal covering of the connected component of the) Galilean group. Non-trivial elements of the group $H^2(\mathsf{T}_e\mathsf{G},\mathbb{R})$ classify the possible inequivalent central extensions of G: the latter are parametrised by a number corresponding to the mass of the physical system.

Example 12.77

(1) The Abelian Lie group $\mathbb{R}$ is the simplest instance, yet far from trivial, to which Bargmann's theorem applies. The assumptions are automatic, for the Lie algebra is $\mathbb{R}$ with zero bracket, and the only skew functional $\alpha : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the null map. However, the result is not obvious, as confirmed by the fact that we proved it with a certain effort using Theorem 12.45 (only the topological group structure, actually).

The same arguments applies to the Abelian group $U(1)$.

(2) Consider the simply connected Lie group $SU(2)$, and indirectly $SO(3)$, which has $SU(2)$ as universal covering (Examples 12.65(5, 6)). We want to prove all continuous projective unitary $SU(2)$-representations (hence of $SO(3)$ by Proposition 12.69) are induced by corresponding strongly continuous unitary $SU(2)$-representations, because the latter's Lie algebra befits Bargmann's theorem. Actually this follows from the fact that the Lie algebra of $SU(2)$ is semisimple and from Proposition 12.76, but we intend to present an explicit proof.

The Lie algebra $su(2)$ of $SU(2)$ (Example 12.65(6)) has a basis made of $-i\sigma_k/2$, where $\sigma_1,\sigma_2,\sigma_3$ are the *Pauli matrices* seen several times. Identify $su(2)$ with $\mathbb{R}^3$ by the vector space isomorphism that sends the basis of $su(2)$ to the canonical basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ of $\mathbb{R}^3$. Every linear skew functional $\alpha : su(2)\times su(2)\to\mathbb{R}$ is a real skew-symmetric matrix A, in the sense there is a unique real skew-symmetric 3×3 matrix A such that $\alpha(u,v)=\sum_{i,j=1}^3 u_iA_{ij}v_k$, $u,v\in\mathbb{R}^3$ (the proof is left to the reader). By (12.86), condition (12.95) reads, in terms of the A associated to the functional α,

$$\sum_{i,j=1}^{3} u_iA_{ij}v_k = \beta\left(\sum_{r,s,k=1}^{3}\varepsilon_{rsk}u_rv_s\mathbf{e}_k\right),$$

for any $u,v\in\mathbb{R}^3$ (i.e. $su(2)$) and a given linear functional $\beta:\mathbb{R}^3\to\mathbb{R}$ (to be determined). By the latter's linearity, we can rephrase:

$$\sum_{i,j=1}^{3} u_iA_{ij}v_k = \sum_{r,s,k=1}^{3}\varepsilon_{rsk}u_rv_sb_k\ ,$$

for any $u, v \in \mathbb{R}^3$ and some $b \in \mathbb{R}^3$ whose components $b_k = \beta(\mathbf{e}_k)$ determine β. Observe that the vector b, i.e. the functional β satisfying (12.95), exists, since every real skew matrix A acting on $\mathbb{R}^3$ corresponds one-to-one to some $b \in \mathbb{R}^3$: $A_{ij} = \sum_{k=1}^3 \varepsilon_{ijk} b_k$ (inverting $b_k = \frac{1}{2}\sum_{i,j=1}^3 \varepsilon_{ijk} A_{ij}$), as is well known and as one proves with ease.

Therefore condition (12.95) holds for any linear skew functional, and so Bargmann's theorem applies. Note that we did not have to assume (12.94) for $\alpha : su(2) \times su(2) \to \mathbb{R}$, for it is granted: using $\alpha(u, v) = \sum_{i,j,k=1}^3 \varepsilon_{ijk} u_i v_j b_k$, where $b \in \mathbb{R}^3$ determines α, a direct computation shows (12.94) is valid, because of the known formula

$$\sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \,.$$

■

12.2.11 Theorems of Gårding, Nelson, FS³

Now we will discuss the converse problem: construct continuous projective representations that give a Lie group of symmetries. We already know it suffices to build continuous unitary representations of the group's central extensions, so we concentrate on the problem of manufacturing strongly continuous unitary representations of a given Lie group (see [Schm90, Chap. 10] for a quick rigorous review). The idea is to start from a Lie algebra representation in terms of self-adjoint operators, reminiscent of the exponentiation of the generators of a Lie group. Physically, the procedure is appealing because generators have a precise meaning. In the next chapter we will see that the generators (self-adjoint operators) represent preserved quantities during motion, if the time evolution is a subgroup of the symmetry group.

As first step we construct an operator *representation for the Lie algebra*, in presence of a strongly continuous unitary *representation of the Lie group*. Consider a strongly continuous unitary representation of the Lie group G

$$\mathsf{G} \ni g \mapsto U_g$$

on the Hilbert space H. Fix a one-parameter subgroup $\mathbb{R} \ni t \mapsto \exp(tT) \in \mathsf{G}$ associated to the element $T \in \mathsf{T}_e\mathsf{G}$. Stone's Theorem 9.33 ensures

$$U_{\exp(tT)} = e^{-itA_U(T)}\,, \quad \text{for any } t \in \mathbb{R}, \tag{12.103}$$

where $A_U(T)$ is a self-adjoint operator on H, in general unbounded (the sign $-$ is conventional, and we fix it for future convenience) with domain $D(A_U(T))$, and completely determined by $T \in \mathsf{T}_e\mathsf{G}$. We will call the self-adjoint operators $A_U(T)$, $T \in \mathsf{T}_e\mathsf{G}$, the **generators** of the representation U. From Stone's theorem they are defined as

$$A_U(T)\psi := i\frac{d}{dt}|_{t=0}U_{\exp(tT)}\psi \quad \text{iff } \psi \in D(A_U(T)). \tag{12.104}$$

Regarding the fact the $T_e\mathsf{G} \ni T \mapsto -iA_U(T)$ define a representation of the real Lie algebra of G, we can only hope they satisfy:

$$[-iA_U(T), -iA_U(T')]\psi = -iA_U([T, T'])\psi \ , \tag{12.105}$$

$\psi \in \mathscr{D}$, where $\mathscr{D} \subset D(A_U(T))$ is an *invariant* subspace for all the $A_U(T)$, and that the map $T_e\mathsf{G} \ni T \mapsto -iA_U(T)\restriction_{\mathscr{D}}$ is obviously ($\mathbb{R}$-)linear.

As a matter of fact it is well known that such a $\mathscr{D}$ exists and is *dense* in H. A candidate is the **Gårding space** $\mathscr{D}_G$.

Definition 12.78 (*Gårding space*). Let G be a (finite-dimensional real) Lie group and consider a strongly continuous unitary representation $\mathsf{G} \ni g \mapsto U_g$ on the complex Hilbert space H. If $f \in C_0^\infty(\mathsf{G})$ and $x \in \mathsf{H}$, in analogy to Proposition 9.31, define

$$x[f] := \int_{\mathsf{G}} f(g)U_g x \, dg \ , \tag{12.106}$$

where dg denotes the left-invariant Haar measure on G (the normalisation does not matter) and the integral is defined in a weak sense exploiting Riesz' lemma: since the map $\mathsf{H} \ni x \mapsto \int_{\mathsf{G}} f(g)(y|U_g x)dg$ is continuous (the proof being elementary), $x[f]$ is the unique vector in H such that

$$(y|x[f]) = \int_{\mathsf{G}} f(g)(y|U_g x)dg \ , \quad \forall y \in \mathsf{H} \, .$$

The complex span of vectors $x[f] \in \mathsf{H}$ with $f \in C_0^\infty(\mathsf{G}; \mathbb{C})$ and $x \in \mathsf{H}$ is called the **Gårding space** of the representation and is denoted by $\mathscr{D}_G$.

The subspace $\mathscr{D}_G$ enjoys very remarkable properties stated in the next theorem. In the following $L_g : C_0^\infty(\mathsf{G}) \to C_0^\infty(\mathsf{G})$ denotes the standard left action of $g \in \mathsf{G}$ on complex-valued, smooth, compactly supported functions defined on G:

$$(L_g f)(h) := f(g^{-1}h) \quad \forall h \in \mathsf{G} \, , \tag{12.107}$$

and, if $T \in T_e\mathsf{G}$, $X_T : C_0^\infty(\mathsf{G}) \to C_0^\infty(\mathsf{G})$ is the smooth vector field on G (a smooth differential operator) defined by:

$$(X_T(f))\,(g) := \lim_{t\to 0} \frac{f\,(\exp(-tT)g) - f(g)}{t} \quad \forall g \in \mathsf{G} \, . \tag{12.108}$$

Then the map

$$T_e\mathsf{G} \ni T \mapsto X_T \tag{12.109}$$

defines a representation of the Lie algebra $T_e\mathsf{G}$ in terms of vector fields (differential operators) on $C_0^\infty(\mathsf{G})$. In particular,

$$X_T \circ X_{T'} - X_{T'} \circ X_T = X_{[T,T']} \,. \tag{12.110}$$

We have the following theorem, establishing that the Gårding space has all the expected properties.

Theorem 12.79 (Gårding) *Let G be a Lie group and consider a strongly continuous unitary representation $\mathsf{G} \ni g \mapsto U_g$ on the complex Hilbert space H. The Gårding space $\mathscr{D}_G$ satisfies the following properties.*

(a) *$\mathscr{D}_G$ is dense in H.*

(b) *If $g \in \mathsf{G}$, then $U_g(\mathscr{D}_G) \subset \mathscr{D}_G$. In other words the Gårding space is invariant under the action of the unitary representation U. More precisely, if $f \in C_0^\infty(\mathsf{G})$, $x \in \mathsf{H}$, $g \in \mathsf{G}$, then*

$$U_g x[f] = x[L_g f] \,. \tag{12.111}$$

(c) *If $T \in T_e\mathsf{G}$, then $\mathscr{D}_G \subset D(A_U(T))$ and furthermore $A_U(T)(\mathscr{D}_G) \subset \mathscr{D}_G$. More precisely*

$$-\,i A_U(T)x[f] = x[X_T(f)] \,. \tag{12.112}$$

(d) *The map*

$$T_e\mathsf{G} \ni T \mapsto -i A_U(T)\restriction_{\mathscr{D}_G} \tag{12.113}$$

is a representation by anti-symmetric operators on $\mathscr{D}_G$. In other words, the map (12.113) is $\mathbb{R}$-linear and (12.105) is valid if $\psi \in \mathscr{D}_G$.

(e) *$A_U(T)$ with $T \in T_e\mathsf{G}$ is essentially self-adjoint on $\mathscr{D}_G$, namely*

$$A_U(T) = \overline{A_U(T)\restriction_{\mathscr{D}_G}} \,, \quad \forall T \in T_e\mathsf{G} \,. \tag{12.114}$$

Remark 12.80 Item (e) can be strengthened: $\mathscr{D}_G$ is a core for every operator $p(A_U(T))$ where p is a real polynomial of arbitrary degree (see, e.g., Corollaries 10.11.15 and 10.11.16 in [Schm90]). ■

Proof (a) Let $f \in C_0^\infty(\mathsf{G})$ have support K_f and satisfy $f \geq 0$ and $\int_{K_f} f dg = 1$. From the very definition (12.106),

$$x[f] - x = \int_{\mathsf{G}} f(g)(U_g - I)x dg \,.$$

Hence $||x[f]-x|| \leq \max_{g\in K_f} ||U_g x - x||$. Consequently, if we choose a sequence of f_n such that K_{f_n} shrinks to $e \in \mathsf{G}$, we have $x[f_n] \to x$ by continuity of $\mathsf{G} \ni g \mapsto U_g$. As $x \in \mathsf{H}$ is arbitrary, $\mathscr{D}_G$ is dense in H.

(b) This is easy because $U_g x[f]$ can be written as

$$\int_{\mathsf{G}} f(h)U_g U_h x dh = \int_{\mathsf{G}} f(h)U_{gh} x dh = \int_{\mathsf{G}} f(g^{-1}z)U_z x dz = \int_{\mathsf{G}} (L_g f)(z)U_z x dz = x[L_g f] \,.$$

Notice that $L_g f \in C_0^\infty(\mathsf{G})$ if $f \in C_0^\infty(\mathsf{G})$, so that $U_g x[f] \in \mathscr{D}_G$.

(c) Due to the invariance of the Haar measure if $h(t) := \exp(tT)$ for $T \in T_e\mathsf{G}$, we obtain

$$\int_{\mathsf{G}} f(h^{-1}g)U_g x dg = \int_{\mathsf{G}} f(g)U_{h(t)g} x dg = U_{h(t)} \int_{\mathsf{G}} f(g)U_g x dg \,.$$

Hence

$$t^{-1}(U_{h(t)} - I)x[f] = \int_{\mathsf{G}} t^{-1}[f(h(t)^{-1}g) - f(g)]U_g x dg \,.$$

The function $g \mapsto t^{-1}[f(h(t)^{-1}g) - f(g)]$ is integrable on G and the limit as $t \to 0$ is $X_T(f) \in C_0^\infty(\mathsf{G})$. Moreover, the mean value theorem implies that

$$t^{-1}[f(h(t)^{-1}g) - f(g)] = (X_T(f))\,(h^{-1}(\tau(t,g))g)$$

for some point $\tau(t,g) \in [-|t|, |t|]$. As a consequence, it is not so difficult to prove that there is a function $f_0 \in C_0^\infty(\mathsf{G})$ with

$$\left|t^{-1}[f(h(t)^{-1}g) - f(g)] - (X_T(f))(g)\right| \leq f_0(g)$$

when t varies in a neighbourhood of 0 with compact closure I. (If the compact set $K \subset \mathsf{G}$ is the support of $\mathsf{G} \ni g \mapsto (X_T(f))\,(g)$, the set $(h^{-1}(I))(K) = K' \subset \mathsf{G}$ is compact since it is the continuous image of the compact set $I \times K$, and contains all supports of the functions $\mathsf{G} \ni g \mapsto (X_T(f))\,(h^{-1}(\tau)g)$ for $\tau \in I$. If $M < +\infty$ is greater than all values of the continuous compactly-supported function $I \times K \ni (\tau, g) \mapsto |\,(X_T(f))\,(h^{-1}(\tau)g)|$, we have $\left|t^{-1}[f(h(t)^{-1}g) - f(g)] - (X_T(f))(g)\right| \leq 2M$ if $t \in I$ and $g \in \mathsf{G}$. The function f_0 can be defined as a non-negative smooth compactly-supported function on G that equals the value $2M$ on K'.) Lebesgue's dominated convergence therefore implies that

$$\int_{\mathsf{G}} \left|t^{-1}[f(h(t)^{-1}g) - f(g)] - (X_T(f))(g)\right| dg \to 0 \quad \text{if } t \to 0. \tag{12.115}$$

On the other hand, a direct use of the Definition 12.78, the fact that $|(U_{g'}x|U_g x)| \leq ||x||^2$ and Fubini–Tonelli produce

$$\left|\left|t^{-1}(U_{h(t)} - I)x[f] - x[X_t(f)]\right|\right| \leq ||x|| \int_{\mathsf{G}} \left|t^{-1}[f(h(t)^{-1}g) - f(g)] - (X_T(f))(g)\right| dg \,.$$

Therefore (12.115) yields $-iA_U(T)x[f] = x[X_T(f)]$ as wanted, including $x[f] \in D(A_U(T))$ in view of Stone theorem.

(d) The fact that $T_e\mathsf{G} \ni T \mapsto -iA_U(T)\restriction_{\mathscr{D}_G}$ is linear is obvious form (12.112) because $T_e\mathsf{G} \ni T \mapsto X_T$ is linear, as can be proved directly. Making use of (12.112) and (12.110), we have

$$[-iA_U(T), -iA_U(T')]x[f] = x[(X_T \circ X_{T'} - X_{T'} \circ X_T)(f)] = x[X_{[T,T']}(f)]$$

$$= -iA_U([T, T'])x[f] .$$

The fact that $A_U(T)$ is self-adjoint and $\mathscr{D}_G$ is dense and invariant, immediately imply that $A_U(T)\restriction_{\mathscr{D}_G}$ is symmetric, so that $-iA_U(T)\restriction_{\mathscr{D}_G}$ is anti-symmetric.

(e) The proof descends from Corollary 9.34, noting that $\mathscr{D}_G \subset D(A_U(T))$ is dense and invariant under the unitary one-parameter group generated by $A_U(T)$, because it is invariant under the whole representation U. $\square$

From (b) and (c) it follows easily that, if $\psi \in \mathscr{D}_G$, then $\mathsf{G} \ni g \mapsto U_g\psi$ is a *smooth* map (differentiating in the Hilbert topology and with respect to any local coordinate system on G). In other words $\psi \in \mathscr{D}_G$ is a **smooth vector of the representation** U. However an important result proves that also the converse is true (Theorem 3.3 in [DiMa78]; the result remains valid if H is replaced by a Fréchet space).

Theorem 12.81 (Dixmier–Malliavin) *Let G be a Lie group, $\mathsf{G} \ni g \mapsto U_g$ a strongly continuous unitary representation on the Hilbert space H. Then $\psi \in \mathscr{D}_G$ if and only if the map $\mathsf{G} \ni g \mapsto U_g\psi$ is C^∞.*

There is another relevant subspace of H introduced by Nelson [Nel59], which turns out to be more useful than the Gårding space to *recover* the representation U by exponentiating the Lie algebra representation.

Definition 12.82 Let G be a Lie group, $\mathsf{G} \ni g \mapsto U_g$ a strongly continuous unitary representation on the Hilbert space H. We denote by $\mathscr{D}_N$ the space of vectors $\psi \in \mathsf{H}$ such that $\mathsf{G} \ni g \mapsto U_g\psi$ is *analytic* in g, i.e. developable in power series in analytic coordinates around every point of G.

The elements of $\mathscr{D}_N$ are called **analytic vectors of the representation** U and $\mathscr{D}_N$ is the **space of analytic vectors of the representation** U.

Since the composition of elements in G is analytic with respect to the analytic atlas of G, the Dixmier–Malliavin Theorem 12.81 has the following consequence.

Proposition 12.83 *Let G be a Lie group, $\mathsf{G} \ni g \mapsto U_g$ a strongly continuous unitary representation on the Hilbert space H. The subspace $\mathscr{D}_N \subset \mathsf{H}$ satisfies the following properties.*

(a) $U_g(\mathscr{D}_N) \subset \mathscr{D}_N$ *for every* $g \in \mathsf{G}$.

(b) $\mathscr{D}_N \subset \mathscr{D}_G$.

An important relationship exists between analytic vectors in $\mathscr{D}_N$ and analytic vectors of the self-adjoint operators $A_U(T)$ according to Chap. 9.

Proposition 12.84 *Let G be a Lie group, $\mathsf{G} \ni g \mapsto U_g$ a strongly continuous unitary representation on the Hilbert space H. Fix a basis $T_1, \ldots, T_n$ of $T_e\mathsf{G}$. Then $\psi \in \mathscr{D}_N$ if and only if ψ is an analytic vector for every skew-symmetric operator*

$$-i\sum_{j=1}^{n} c_j A_U(T_j)$$

defined on its natural domain for $c_1, \dots, c_n \in \mathbb{R}$.

Proof The proof immediately follows from Lemma 7.1 and the comment at the top of p. 575 in [Nel59]). □

Nelson [Nel59] proved an important result ((b) in the theorem below), which implies that $\mathscr{D}_N$ is not trivial, and is even dense in H. It involves an operator, called *Nelson operator*, that sometimes is related to the *Casimir operators* of the group represented.

Proposition 12.85 *Let* G *be a Lie group and* $\mathsf{G} \ni g \mapsto U_g$ *a strongly continuous unitary representation on the Hilbert space* H. *Take* $T_1, \dots, T_n \in \mathsf{T}_e\mathsf{G}$ *a basis and define* **Nelson's operator** *as the non-negative symmetric operator*

$$\Delta := \sum_{k=1}^{n} A_U(T_k)\restriction^2_{\mathscr{D}_G} \ . \tag{12.116}$$

Then
(a) Δ *is essentially self-adjoint.*
(b) *Every analytic vector of* $\overline{\Delta}$ *belongs to* $\mathscr{D}_N$, *so that* $\mathscr{D}_N$ *is dense.*

Proof It is useful to remember that the analytic vectors of a self-adjoint operator ($\overline{\Delta}$) are dense by Proposition 9.25(f). (b) is proved in in [Nel59, Theorem 3]. The comment after that proof proves (a). □

Remark 12.86 Nelson's operator was originally introduced [Nel59] as the elliptic second-order differential operator $\sum_{k=1}^{n} T_k T_k$, where each T_k is interpreted as *left-invariant vector field* on the Lie group G. Our $-\Delta$ is a representation of that differential operator through the unique extension of the representation $T_e\mathsf{G} \ni T \mapsto -iA_U(T)|_{\mathscr{D}_G}$ to a representation of the *universal enveloping algebra* generated by $T_e\mathsf{G}$ [Nel59]. ■

Corollary 12.87 *Referring to Proposition 12.85, the following statements hold for every fixed basis* $T_1, \dots, T_n \in \mathsf{T}_e\mathsf{G}$.
(a) *If* $c_1, \dots, c_n \in \mathbb{R}$,

$$-\, i\sum_{j=1}^{n} c_j A_U(T_j) \subset -iA_U\left(\sum_{j=1}^{n} c_j T_j\right) \tag{12.117}$$

and

$$\overline{-i\sum_{j=1}^{n} c_j A_U(T_j)\restriction_{\mathscr{D}_N}} = \overline{-i\sum_{j=1}^{n} c_j A_U(T_j)\restriction_{\mathscr{D}_G}} = \overline{-i\sum_{j=1}^{n} c_j A_U(T_j)} = -iA_U\left(\sum_{j=1}^{n} c_j T_j\right) .$$

(b) *If* $T \in T_e\mathsf{G}$, $\mathscr{D}_N$ *is a core for* $A_U(T)$.
(c) *If* $\psi \in \mathscr{D}_N$,

$$U_{\exp(tT)}\psi = \sum_{n=0}^{+\infty} \frac{(-it)^n}{n!} A_U(T)^n \psi \ , \quad \textit{for sufficiently small } |t|. \tag{12.118}$$

Proof Let us prove (a). The symmetric operator in the left-hand side is essentially self-adjoint from Propositions 12.84, 12.85(b) and Nelson's criterion Theorem 5.47, so it admits a unique self-adjoint extension. On the other hand, since $T_e\mathsf{G} \ni T \mapsto A_U(T)\restriction_{\mathscr{D}_N}$ is linear by Theorem 12.79(d), the restriction to $\mathscr{D}_G$ of the same operator satisfies

$$-i\sum_{j=1}^{n} c_j A_U(T_j)\restriction_{\mathscr{D}_G} = -iA_U\left(\sum_{j=1}^{n} c_j T_j\right)\restriction_{\mathscr{D}_G} ,$$

which is essentially self-adjoint by Theorem 12.79, and its unique self-adjoint extension is $-iA_U\left(\sum_{j=1}^{n} c_j T_j\right)$. Summing up, (12.117) holds. The second equation in (a) summarises the above argument to prove (12.117). Parts (b) and (c) are easy. If $T \in T_e\mathsf{G}$, we can complete $T_1 = T$ to a basis by adding $n-1$ suitable vectors. Clearly $\mathscr{D}_N$ is a core for the trivial linear combination $A_U(T) = A_U(T_1)$ since it is essentially self-adjoint on $\mathscr{D}_N$ as we have just seen. Proposition 9.25(d) yields now the final expansion (12.118). □

The validity of expansion (12.118) shows a feature of $\mathscr{D}_N$ which $\mathscr{D}_G$ does not possess: $\mathscr{D}_N$ permits us to reconstruct part of the action of the (representation of) the group from the action of the representatives of the Lie algebra on the Hilbert space. However, it is not evident that the restriction of $A_U(T)$ to $\mathscr{D}_N$ defines a proper representation of $T_e\mathsf{G}$ as T varies. This would be true if we replaced $\mathscr{D}_N$ by $\mathscr{D}_G$, but with this change (12.118) generally fails. The next theorem collects various results due to R. Goodman [Goo69][13] and sorts out all problems.

Theorem 12.88 *Let* G *be a connected Lie group and* $\mathsf{G} \ni g \mapsto U_g$ *a strongly continuous unitary representation on the Hilbert space* H. *The subspace* $\mathscr{D}_N \subset \mathsf{H}$ *has the following properties.*

(a) $\psi \in \mathscr{D}_N$ *if and only if* ψ *is an analytic vector of* $\sqrt{I + \overline{\Delta}}$, *where* Δ *is defined by (12.116) with respect to any fixed basis of* $T_e\mathsf{G}$.

(b) $\mathscr{D}_N$ *is invariant under* $A_U(T)$ *if* $T \in T_e\mathsf{G}$.

(c) $T_e\mathsf{G} \ni T \mapsto -iA_U(T)\restriction_{\mathscr{D}_N}$ *is a representation of* $T_e\mathsf{G}$ *by skew-symmetric operators (with* $A_U(T) = \overline{A_U(T)\restriction_{\mathscr{D}_N}}$ $\forall T \in T_e\mathsf{G}$ *and verifying (12.118)).*

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[13] Here the opposite convention on the sign of Δ is adopted, and the representation of the Lie algebra is constructed on the whole space of smooth vectors, which coincides with $\mathscr{D}_G$ by the Dixmier–Malliavin Theorem 12.81.

We can finally state the famous theorem of Nelson [Nel59] that enables to associate representations of the only simply connected Lie group with a given Lie algebra to representations of that Lie algebra.

Theorem 12.89 (Nelson) *Consider an n-dimensional real Lie algebra V of operators $-iS$ (with each S symmetric on the Hilbert space H, defined on a common subspace $\mathscr{D}$, dense in H and V-invariant) with the usual commutator of operators as Lie bracket.*

Let $-iS_1, \ldots, -iS_n \in \mathsf{V}$ be a basis of V and define Nelson's operator:

$$\Delta := \sum_{k=1}^{n} S_k^2$$

with domain $\mathscr{D}$. If Δ is essentially self-adjoint, there exists a strongly continuous unitary representation

$$\mathsf{G_V} \ni g \mapsto U_g$$

on H of the unique connected simply connected Lie group $\mathsf{G_V}$ with Lie algebra V, that is completely determined by

$$\overline{S} = A_U(-iS) \quad \textit{for every } -iS \in \mathsf{V}.$$

In particular, the symmetric operators S are essentially self-adjoint on $\mathscr{D}$, their closures being self-adjoint.

The above assumptions were weakened by Flato, Simon, Snellman and Sternheimer [FSSS72]:

Theorem 12.90 (FS^3 (Flato, Simon, Snellman, Sternheimer)) *Consider a real n-dimensional Lie algebra V of operators $-iS$ (with each S symmetric on the Hilbert space H, defined on a common subspace $\mathscr{D}$, dense in H and V-invariant) with the usual commutator of operators as Lie bracket.*

Let $-iS_1, \ldots, -iS_n \in \mathsf{V}$ be a basis. If the elements of $\mathscr{D}$ are analytic vectors for every S_k, $k = 1, \ldots, n$, then there is a strongly continuous unitary representation

$$\mathsf{G_V} \ni g \mapsto U_g$$

on H, of the unique simply connected Lie group $\mathsf{G_V}$ with Lie algebra V, that is completely determined by:

$$\overline{S} = A_U(-iS) \quad \textit{for every } -iS \in \mathsf{V}$$

In particular, the symmetric operators S are essentially self-adjoint on $\mathscr{D}$, their closures being self-adjoint.

Example 12.91
(1) Consider two families of operators $\mathscr{P}_k, \mathscr{X}_k, k = 1, 2, \dots, n$, on a dense subspace $\mathscr{D} \subset \mathsf{H}$ in a Hilbert space, and suppose they are symmetric. Assume they satisfy, on the domain, Heisenberg's commutation relations, seen in Chap. 11 (where we set $\hbar = 1$):

$$[-i\mathscr{X}_h, -i\mathscr{P}_k] = -i\delta_{hk} I \quad k, h = 1, \dots, n. \tag{12.119}$$

We may add $-iI$ to the generators. Then $-iI, -i\mathscr{X}_1, \dots, -i\mathscr{X}_n, -i\mathscr{P}_1, \dots, -i\mathscr{P}_n$ form a basis for the Lie algebra of the Heisenberg group $H(n)$ on $\mathbb{R}^{2n+1}$ (see the end of Chap. 11). The Heisenberg group is simply connected. Nelson's theorem guarantees that if, on $\mathscr{D}$, the operator:

$$\Delta - I := \sum_{k=1}^{n} \mathscr{X}_k^2 + \sum_{k=1}^{n} \mathscr{P}_k^2$$

is essentially self-adjoint (we should consider Δ, but it is clear that Δ is essentially self-adjoint if and only if $\Delta - I$ is), then there is a unique unitary and strongly continuous representation $H(n) \ni (\eta, \mathbf{t}, \mathbf{u}) \mapsto H((\eta, \mathbf{t}, \mathbf{u}))$ on H with I, $\overline{\mathscr{X}_h} =: X_h$ and $\overline{\mathscr{P}_h} =: P_h, h = 1, \dots, n$ as (self-adjoint) generators. Therefore if this representation of the *Heisenberg group* is irreducible, by the Stone-von Neumann theorem (Theorem 11.55) there is a unitary transformation $\mathsf{H} \to L^2(\mathbb{R}^n, dx)$ mapping X_h and P_h to the usual position and momentum operators of axiom A.5, Chap. 11 (for $n = 3$ and with the obvious generalisation for $n > 3$).

An elementary example is to take $n = 1$, $\mathsf{H} = L^2(\mathbb{R}, dx)$, the operator $\mathscr{X}$ seen as multiplication by the coordinate x, $\mathscr{P} := -i\frac{\partial}{\partial x}$, and defining $\mathscr{D}$ to be the Schwartz space $\mathscr{S}(\mathbb{R})$. In this case $\overline{\Delta - I}$ coincides with the Hamiltonian of the harmonic oscillator of Chap. 9. The operator $\Delta - I$ has an eigenvector basis made by Hermite functions (belonging in $\mathscr{S}(\mathbb{R})$), which are a basis of $L^2(\mathbb{R}, dx)$ as well. Hence $\Delta - I$ (and so Δ, by Proposition 9.25) admits a set of analytic vectors (Hermite functions) whose finite combinations are dense in the Hilbert space. By Nelson's criterion $\Delta - I$ is essentially self-adjoint, and we may apply the above result.
(2) We have a result about commuting spectral measures.

Theorem 12.92 *Let $A : D(A) \to \mathsf{H}$, $B : D(B) \to \mathsf{H}$ be symmetric operators. If there is a dense space $D \subset D(A^2 + B^2) \cap D(AB) \cap D(BA)$ on which A and B commute, and where $A^2 + B^2$ is essentially self-adjoint, then A and B are essentially self-adjoint on D and the spectral measures of $\overline{A}$ and $\overline{B}$ commute.*

The proof is an easy consequence of Nelson's Theorem 12.89. ∎

12.2.12 A Few Words About Representations of Abelian Groups and the SNAG Theorem

We summarise here the most important theorem concerning continuous unitary representations of Abelian topological groups. Some preliminary notions are necessary.

Definition 12.93 Let G be a commutative, Hausdorff, locally compact group with product $\cdot$ and neutral element e.

A **character of the group** G is a continuous map $\chi : \mathsf{G} \to U(1)$ such that

$$\chi(g)\chi(g_1) = \chi(g \cdot g_1) \quad \text{for all } g, g_1 \in \mathsf{G}.$$

The set $\widehat{\mathsf{G}}$ of characters of G is called the **dual space** of G.

Remarks 12.94 **(1)** It is clear that a character satisfies $\chi(e) = 1$ and $\chi(g^{-1}) = \chi(g)^{-1}$, and that $\widehat{\mathsf{G}}$ is an Abelian group with respect to the pointwise product of functions, with unit element given by the constant function 1.
(2) A classical result by Weil [Wei40, Sect. 2g] says that $\widehat{\mathsf{G}}$ is a locally compact Hausdorff group when equipped with the *compact-open topology*. The **compact-open topology** on the space $C(\mathsf{X}, \mathsf{Y})$ of continuous functions between topological spaces X, Y has a basis formed by $C(\mathsf{X}, \mathsf{Y})$ itself and all finite intersections of sets $\{f \in C(\mathsf{X}, \mathsf{Y}) \mid f(K) \subset A\}$, for all open sets $A \subset \mathsf{Y}$ and compact sets $K \subset \mathsf{X}$.[14] ■

An elementary but important result is the following one.

Theorem 12.95 *According to Definition 12.93, every $\chi \in \widehat{\mathsf{G}}$ is an irreducible, continuous, unitary representation of G on $\mathbb{C}$. Up to unitary equivalence, $\widehat{\mathsf{G}}$ is in* 1-1 *correspondence with strongly continuous irreducible unitary representations of* G.

Proof If $\chi \in \widehat{\mathsf{G}}$, the map $\chi : \mathsf{G} \to U(1)$ is evidently a strongly continuous irreducible unitary representation of G on $\mathbb{C}$. Now consider a strongly continuous unitary representation $U : \mathsf{G} \to \mathsf{H}$. Since G is commutative, every U_g commutes with every other element U_{g_1}. However, since U is irreducible, Schur's lemma requires that $U_g = \chi(g)I$ for a complex number $\chi(g)$ with $|\chi(g)| = 1$. As $U_g U_{g_1} = U_{g \cdot g_1}$, necessarily $\chi(g) \cdot \chi(g_1) = \chi(g \cdot g_1)$. Furthermore, the map $\mathsf{G} \ni g \mapsto \chi(g)$ is continuous by construction, so $\chi \in \widehat{\mathsf{G}}$. Finally notice that, since $U_g = \chi(g)I$, every subspace of H must be invariant. This is not possible when U is irreducible, unless H is one-dimensional. Therefore, there exists an isometric surjective operator $V : \mathsf{H} \to \mathbb{C}$. In particular, $VU_gV^{-1} = V\chi(g)IV^{-1} = \chi(g)$. In other words U is unitarily equivalent to a character. This correspondence between classes of strongly continuous irreducible representations and characters is evidently one-to-one. □

[14] The compact-open topology is Hausdorff if Y is Hausdorff. When Y is a metric space (as $U(1)$, in our case), the compact-open topology on $C(X, Y)$ coincides with the topology of uniform convergence on compact sets of X.

The result characterises continuous irreducible unitary representations of an Abelian topological group in terms of its characters completely. The natural issue is to find how strongly continuous reducible unitary representations are built. The answer is contained in famous theorem by Stone, Najmark, Ambrose and Godement. Traditionally referred to as the *SNAG theorem*, we shall state it without proof because the argument is based on *Bochner's integration* technique [Fol95]. The SNAG theorem establishes that unitary representations are constructed out of irreducible unitary representations by means of a suitable PVM defined on $\widehat{\mathsf{G}}$.

Theorem 12.96 (SNAG theorem) *Let $U : \mathsf{G} \to \mathsf{H}$ be a strongly continuous unitary representation of the Abelian, locally compact Hausdorff group G on the complex Hilbert space H. Then there exist a PVM P, on the Borel sets of $\widehat{G}$ equipped with the compact-open topology, such that*

$$U_g = \int_{\widehat{G}} \chi(g) dP(\chi) .$$

The measure P is uniquely determined by requiring that for every $x \in \mathsf{H}$, varying $E \in \mathscr{B}(\widehat{G})$, $\mu_x(E) := (x|P(E)x)$ is the unique finite regular Borel positive measure on $\widehat{G}$ such that

$$(x|U_g x) = \int_{\widehat{\mathsf{G}}} \chi(g) d\mu_x(\chi) \quad \textit{for every} x \in \mathsf{H} \textit{ and } g \in \mathsf{G}.$$

Example 12.97
(1) Consider the additive group $\mathsf{G} = \mathbb{R}^n$ equipped with the standard topology. Every character $\chi : \mathbb{R} \to U(1)$ has the form

$$\chi_y(x) = e^{i(y_1 x_1 + \cdots + y_n x_n)} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n ,$$

where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is arbitrary and uniquely fixes χ_y. It is not difficult to prove that the compact-open topology on $\widehat{\mathbb{R}^n}$ coincides with the standard topology. Consequently $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$ are isomorphic topological groups under the map

$$\mathbb{R}^n \ni y \mapsto \chi_y \in \widehat{\mathbb{R}^n} ,$$

which is both a group isomorphism and a homeomorphism.

A strongly continuous unitary representation $U : \mathbb{R}^n \ni x \mapsto U_x$ on a Hilbert space H can always be decomposed as

$$U_x = \int_{\mathbb{R}^n} e^{i(y_1 x_1 + \cdots + y_n x_n)} dP(y) \quad x \in \mathbb{R}^n ,$$

by the SNAG theorem. It is not difficult to prove that, if H is *separable*, P is nothing but the *joint measure* defined in Theorem 9.19 for the n self-adjoint operators A_k that generate the one-parameter groups $\mathbb{R} \ni t \mapsto U_{(0,\ldots,0,t,0,\ldots0)}$, where t is in the kth slot. If H is not separable P is a generalisation of a joint measure.

(2) If $\mathsf{G} = U(1)$ then $\widehat{U(1)}$ is made of maps

$$\chi_n(x) = e^{inx} \quad \text{for all } x \in \mathbb{R}\,,$$

where $n \in \mathbb{Z}$ is arbitrary and uniquely fixes χ_n. Hence $\widehat{U(1)}$ is isomorphic to the additive group $\mathbb{Z}$. The compact-open topology on $\widehat{U(1)} \equiv \mathbb{Z}$ coincides with the natural discrete topology on $\mathbb{Z}$. The SNAG decomposition of a strongly continuous unitary representation of $U(1)$ on a Hilbert space H becomes a Hilbert sum of one-dimensional representations. The same result can be obtained as an elementary case of the next section's Peter–Weyl theorem (since $U(1)$ is compact). ■

12.2.13 Continuous Unitary Representations of Compact Hausdorff Groups: The Peter–Weyl Theorem

This section contains a general theorem about strongly continuous unitary representations of compact Hausdorff groups: the celebrated *Peter–Weyl theorem*. Compact Lie groups are covered, of course, due to their structure of differentiable manifolds. The theorem by Peter and Weyl states, in particular, two remarkable facts that we will prove: strongly continuous unitary representations of compact Hausdorff groups can be split in *orthogonal sums* (even with uncountably many summands) of (topologically) *irreducible* representations, and strongly continuous irreducible unitary representations are necessarily *finite-dimensional*. Both are far from obvious. In particular, Theorem 12.66 proves that compactness is crucial, for Lie groups at least. In general, a strongly continuous unitary representation of a topological group might be a *direct integral* of strongly continuous irreducible unitary representations (e.g., unitary representations of the Abelian Lie group $\mathbb{R}$); moreover, there could be infinite-dimensional irreducible representations (like for the Lorentz group).

Let us start with a lemma taking care of the finite-dimensional case.

Lemma 12.98 *Let $\pi : \mathsf{G} \ni g \mapsto U_g$ be a unitary representation (not necessarily continuous) of the group G (even if not topological) on the finite-dimensional Hilbert space H. Then H decomposes in an orthogonal sum $\mathsf{H} = \bigoplus_{k=1}^{n} \mathsf{H}_k$ where for each $k = 1, 2, \ldots, n$:*

(i) $U_g(\mathsf{H}_k) \subset \mathsf{H}_k$ for every $g \in \mathsf{G}$,

(ii) $\pi_k : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_k}$ is an irreducible unitary G-representation on H_k.

If G is a topological group and $\mathsf{G} \ni g \mapsto U_g$ is strongly continuous, so are all maps π_k.

Proof If π is not irreducible it will have a non-trivial invariant subspace $\hat{\mathsf{H}}_1 \subset \mathsf{H}$, with $0 < \dim(\hat{\mathsf{H}}_1) \leq \dim(\mathsf{H}) - 1$. Consider the new unitary representation of $\hat{\pi}_1 : \mathsf{G} \ni g \mapsto U_g\restriction_{\hat{\mathsf{H}}_1}$. If this is not irreducible, as above we can find a non-trivial π-invariant space $\hat{\mathsf{H}}_2 \subset \hat{\mathsf{H}}_1$ with $0 < \dim(\hat{\mathsf{H}}_2) \leq \dim(\mathsf{H}) - 2$. The iteration stops after a finite number of steps, since $\dim(\mathsf{H}) < +\infty$, and produces an invariant subspace $\mathsf{H}_1 \neq \{\mathbf{0}\}$ for which $\pi_1 : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_1}$ is irreducible.

Consider $\mathsf{H}'_2 := \mathsf{H}_1^\perp$. By construction $U_g(\mathsf{H}_1^\perp) \subset \mathsf{H}_1^\perp$, since $z \in \mathsf{H}_1^\perp$ and $x \in \mathsf{H}_1$ imply $(U_g z|x) = (z|U_g^* x) = (z|U_{g^{-1}}x) = 0$ ($U_{g^{-1}}x \in \mathsf{H}_1$ by assumption). Hence $\mathsf{H} = \mathsf{H}_1 \oplus \mathsf{H}'_2$ and $\pi'_2 : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}'_2}$ is a unitary G-representation on H'_2. If π'_2 is irreducible we finish, otherwise we iterate to obtain $\mathsf{H}'_2 = \mathsf{H}_2 \oplus \mathsf{H}'_3$, where $\pi_2 : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_2}$ is irreducible, $\mathsf{H}_2, \mathsf{H}'_3$ are orthogonal to H_1, $\pi(\mathsf{H}'_3) = \pi_2(\mathsf{H}'_3) \subset \mathsf{H}'_3$ and $\pi'_3 : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}'_3}$ is a unitary G-representation on H'_3. By induction the algorithm is finite, and yields $\mathsf{H}'_k = \{\mathbf{0}\}$ if $k = n+1$ for n large enough, because every H_k has dimension at least 1, so $\sum_{k=1}^n \dim(\mathsf{H}_k) \geq n$, but we also have $\sum_{k=1}^n \dim(\mathsf{H}_k) \leq \dim(\mathsf{H}) < +\infty$.

The last claim is immediate, because everything is finite-dimensional. □

Now let us generalise the lemma to infinitely many dimensions for compact Hausdorff groups. The result, part of the more general statement of Peter and Weyl, makes use of the *Haar measure* of Example 12.38(5) in Theorem 12.39.

Remark 12.99 In the rest of the section we refer to Definition 11.35 and Remark 11.36. Therefore *irreducibility* is now understood in *topological sense*: there is no non-trivial *closed* subspace invariant under the representation considered. ■

Theorem 12.100 (Peter–Weyl, part I) *Let* G *be a compact Hausdorff group and* $\pi : \mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ *a strongly continuous unitary representation on* $\mathsf{H} \neq \{\mathbf{0}\}$.
(a) *If* π *is irreducible, then* H *is finite-dimensional:* $\dim(\mathsf{H}) < +\infty$.
(b) *If* π *is reducible, it can be decomposed in a sum of strongly continuous, finite-dimensional irreducible unitary representations of* G. *That is,* H *is the Hilbert sum of mutually orthogonal closed subspaces* $\mathsf{H} = \bigoplus_{k\in K} \mathsf{H}_k$*, where for each* $k \in K$*:*
(i) $\mathsf{H}_k \subset \mathsf{H}$ *is finite-dimensional,*
(ii) $U_g(\mathsf{H}_k) \subset \mathsf{H}_k$ *for every* $g \in \mathsf{G}$*,*
(iii) $\pi_k : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_k}$ *is a strongly continuous and irreducible unitary* G*-representation on* H_k.

Remark 12.101 Every representation π_k constructed out the initial representation π as explained in the theorem will be called a **subrepresentation** of π henceforth. ■

Proof From now on μ_G will be the Haar measure of G, which by Theorem 12.39 is bi-invariant and may be normalised so that $\mu_\mathsf{G}(\mathsf{G}) = 1$ (since G is compact). The final statement in Theorem 12.39 implies that if $f \in L^1(\mathsf{G}, \mu_\mathsf{G})$ and G is compact, then
$$\int_\mathsf{G} f(g) d\mu_\mathsf{G}(g) = \int_\mathsf{G} f(g^{-1}) d\mu_\mathsf{G}(g)\,, \tag{12.120}$$
to be used later.

(a) For $x \in \mathsf{H}$ define the operator $K_x : \mathsf{H} \to \mathsf{H}$ by asking, for any $z, y \in \mathsf{H}$:
$$(z\,|K_x y) = \int_\mathsf{G} \left(z\,\middle|U_g x\right)(U_g x|y)\, d\mu_\mathsf{G}(g)\,. \tag{12.121}$$

As in the proof of Proposition 9.31, using Riesz's representation and the definition of adjoint to bounded operators, K_x is well defined and $K_x \in \mathfrak{B}(\mathsf{H})$. In particular, since U_g is isometric:

$$||K_x y||^2 \leq \int_{\mathsf{G}} |(K_x y|U_g x)|\,|(U_g x|y)| d\mu_{\mathsf{G}}(g) \leq \int_{\mathsf{G}} ||K_x y||\,||U_g x||\,|(U_g x|y)| d\mu_{\mathsf{G}}(g)$$

so

$$||K_x y|| \leq \int_{\mathsf{G}} ||U_g x||\,|(U_g x|y)| d\mu_{\mathsf{G}}(g) \leq \int_{\mathsf{G}} ||U_g x||\,||U_g x||\,||y|| d\mu_{\mathsf{G}}(g) = ||x||^2 ||y|| \int_{\mathsf{G}} d\mu_{\mathsf{G}}(g)$$
$$= ||x||^2 ||y|| \,,$$

and then $||K_x|| \leq ||x||^2$. Moreover

$$U_g K_x = K_x U_g \quad \text{for any } x \in \mathsf{H},\ g \in \mathsf{G}\,. \tag{12.122}$$

In fact, U_g is unitary and $U_g U_{g'} = U_{gg'}$, so

$$\left(z|U_g K_x y\right) = \left(U_g^* z|K_x y\right) = \int_{\mathsf{G}} \left(U_g^* z|U_{g'} x\right) (U_{g'} x|y)\, d\mu_{\mathsf{G}}(g')$$

$$= \int_{\mathsf{G}} \left(z|U_{gg'} x\right) (U_{g'} x|y)\, d\mu_{\mathsf{G}}(g') = \int_{\mathsf{G}} \left(z|U_{gg'} x\right) (U_g U_{g'} x|U_g y)\, d\mu_{\mathsf{G}}(g')\,.$$

Now as $\mu_{\mathsf{G}}(gA) = \mu_{\mathsf{G}}(A)$ (it is the Haar measure), the last integral becomes

$$\int_{\mathsf{G}} \left(z|U_{gg'} x\right) (U_{gg'} x|U_g y)\, d\mu_{\mathsf{G}}(g') = \int_{\mathsf{G}} \left(z|U_{gg'} x\right) (U_{gg'} x|U_g y)\, d\mu_{\mathsf{G}}(gg')$$

$$= \int_{\mathsf{G}} (z|U_s x)\, (U_s x|U_g y)\, d\mu_{\mathsf{G}}(s) = \left(z|K_x U_g y\right)\,.$$

But $z \in \mathsf{H}$ is arbitrary, so (12.122) holds. With this settled, let us begin to prove (a). If the representation $\mathsf{G} \ni g \mapsto U_g \in \mathfrak{B}(\mathsf{H})$ is irreducible, by Proposition 11.37 (Schur's lemma) Eq. (12.122), valid for every $g \in \mathsf{G}$, is valid only if $K_x = \chi(x) I$ for some $\chi(x) \in \mathbb{C}$. Hence

$$\int_{\mathsf{G}} \left(y|U_g x\right) (U_g x|y)\, d\mu_{\mathsf{G}}(g) = (y|K_x y) = \chi(x) ||y||^2\,,$$

$x, y \in \mathsf{H}$, and so

$$\int_{\mathsf{G}} |\left(y|U_g x\right)|^2 \, d\mu_{\mathsf{G}}(g) = \chi(x)||y||^2 \, . \tag{12.123}$$

As $U_g^* = U_{g^{-1}}$, the latter reads

$$\int_{\mathsf{G}} \left|\left(U_{g^{-1}} y\right| x\right)\right|^2 \, d\mu_{\mathsf{G}}(g) = \chi(x)||y||^2$$

or

$$\int_{\mathsf{G}} \left|\left(x\left|U_{g^{-1}} y\right)\right|^2 \, d\mu_{\mathsf{G}}(g) = \chi(x)||y||^2$$

and even, by (12.120),

$$\int_{\mathsf{G}} \left|\left(x|U_{g'} y\right)\right|^2 \, d\mu_{\mathsf{G}}(g') = \chi(x)||y||^2 \, .$$

Using (12.123) with x, y swapped allows to conclude the left-hand side equals $\chi(y)||x||^2$, so that $\chi(x)||y||^2 = \chi(y)||x||^2$ irrespective of $x, y \in \mathsf{H}$. This means $\chi(x) = c||x||^2$ for any $x \in \mathsf{H}$ and some constant $c \geq 0$. Set $x = y$, $||x|| = 1$, so (12.123) gives:

$$\int_{\mathsf{G}} |\left(x|U_g x\right)|^2 \, d\mu_{\mathsf{G}}(g) = \chi(x)||x||^2 = c||x||^4 = c \, ;$$

hence $c > 0$ because the continuous, non-negative map $\mathsf{G} \ni g \mapsto |(x|U_g x)|$ reaches $||x|| = 1$ at $g = e$, and non-empty open sets have non-zero Haar measure (Theorem 12.39(ii)). To finish with (a), consider n orthonormal vectors $\{z_k\}_{k=1,\ldots,n} \subset \mathsf{H}$. Setting $x = e_k$ and $y = e_1$ in (12.123) gives:

$$\int_{\mathsf{G}} |\left(e_1|U_g e_k\right)|^2 \, d\mu_{\mathsf{G}}(g) = \chi(e_k)||e_1||^2 = c > 0 \, , \quad k = 1, 2, \ldots$$

By the orthonormality of the $U_g e_k$ and Bessel's inequality (3.17):

$$nc = \sum_{k=1}^{n} \int_{\mathsf{G}} |(e_1|U_g e_k)|^2 d\mu_{\mathsf{G}}(g) = \int_{\mathsf{G}} \sum_{k=1}^{n} |(e_1|U_g e_k)|^2 d\mu_{\mathsf{G}}(g)$$

$$\leq \int_{\mathsf{G}} ||e_1||^2 d\mu_{\mathsf{G}}(g) = 1 \, .$$

Whichever $c > 0$, the number n cannot be arbitrarily large: it must be finite. Therefore the dimension of H is finite and not bigger than $1/c$. This concludes (a).

(b). For this we need a lemma.

Lemma 12.102 *Suppose that for every non-trivial closed subspace $\mathsf{H}_1 \subseteq \mathsf{H}$ such that $U_g(\mathsf{H}_1) \subset \mathsf{H}_1$, $g \in \mathsf{G}$, there exists a non-trivial finite-dimensional space $\mathsf{H}_0 \subset \mathsf{H}_1$ such that, for any $g \in \mathsf{G}$, $U_g(\mathsf{H}_0) \subset \mathsf{H}_0$ and $\mathsf{G} \ni g \mapsto \pi\restriction_{\mathsf{H}_0}$ is irreducible on H_0. Then H is the Hilbert sum of mutually orthogonal, closed subspaces $\mathsf{H} = \bigoplus_{k\in K} \mathsf{H}_k$, where for each $k \in K$:*

(i) $\mathsf{H}_k \subset \mathsf{H}$ is finite-dimensional,

(ii) $U_g(\mathsf{H}_k) \subset \mathsf{H}_k$ for every $g \in \mathsf{G}$,

(iii) $\pi_k : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_k}$ is a strongly continuous and irreducible unitary G-representation on H_k.

Proof of Lemma 12.102. Consider the family $\mathscr{X} = \{\mathsf{H}_j\}_{j\in J}$, where J has arbitrary cardinality and the sets $\mathsf{H}_j \subset \mathsf{H}$ are finite-dimensional, non-null and such that $\pi(\mathsf{H}_j) \subset \mathsf{H}_j$, $\mathsf{H}_j \perp \mathsf{H}_{j'}$ for $j \neq j'$. Furthermore, we require $\pi_j : \mathsf{G} \ni g \mapsto U_g\restriction_{\mathsf{H}_j}$ be an irreducible G-representation on H_j. (Note that $\mathscr{X}$ is not empty because, defining $\mathsf{H}_1 := \mathsf{H}$, our hypotheses guarantee the existence of a non-trivial irreducible finite-dimensional subspace H_0. The argument applies replacing H with $\mathsf{H}_0^\perp$ and so on.) The π_j are certainly strongly continuous since π is. Endow $\mathscr{X}$ with the order relation given by inclusion. Clearly any ordered subset $\mathscr{E} \subset \mathscr{X}$ is upper bounded by the union of elements in $\mathscr{E}$. Zorn's lemma tells we have a maximal element in $\mathscr{X}$. By construction this is a chain $\{\mathsf{H}'_m\}_{m\in M} \in \mathscr{X}$ not properly contained in any $\{\mathsf{H}_j\}_{j\in J} \in \mathscr{X}$. Now consider the closed Hilbert sum $\mathsf{H}' := \bigoplus_{m\in M} \mathsf{H}'_m$. By construction $U_g(\mathsf{H}') \subset \mathsf{H}'$, because every U_g is continuous. The orthogonal complement $\mathsf{H}'^\perp$ is π-invariant, because $x \in \mathsf{H}'^\perp$ and $y \in \mathsf{H}'$ imply $(U_g x|y) = (x|U_{g^{-1}}y) = 0$ since $U_{g^{-1}}y \in \mathsf{H}'$, $y \in \mathsf{H}'$. Suppose $\mathsf{H}'^\perp \neq \{\mathbf{0}\}$. Then $\mathsf{H}'^\perp$ contains a finite-dimensional subspace $\mathsf{H}_0 \neq \{\mathbf{0}\}$. By construction $\{\mathsf{H}'_m\}_{m\in M} \cup \{\mathsf{H}_0\}$ is in $\mathscr{X}$ and contains the maximal $\{\mathsf{H}'_m\}_{m\in M}$, a contradiction. Therefore $\mathsf{H}'^\perp = \{\mathbf{0}\}$, i.e. $\mathsf{H} = \bigoplus_{m\in M} \mathsf{H}'_m$. □
To finish the proof of part (b) it suffices to prove the next result.

Lemma 12.103 *Let $\mathsf{H}_1 \subseteq \mathsf{H}$ be a closed non-trivial subspace such that $U_g(\mathsf{H}_1) \subset \mathsf{H}_1$, $g \in \mathsf{G}$. There exists a finite-dimensional non-trivial space $\mathsf{H}_0 \subset \mathsf{H}_1$ such that, for any $g \in \mathsf{G}$, $U_g(\mathsf{H}_0) \subset \mathsf{H}_0$ and $\mathsf{G} \ni g \mapsto \pi\restriction_{\mathsf{H}_0}$ is irreducible on H_0.*

Proof of Lemma 12.103. From (12.121) and the inner product's elementary properties $(K_x z|y) = (z|K_x y)$ for any $x, y, z \in \mathsf{H}$. Since $K_x \in \mathfrak{B}(\mathsf{H})$, we have $K_x^* = K_x$, i.e. K_x is self-adjoint. By (12.121):

$$(x\,|K_x x\,) = \int_{\mathsf{G}} |(U_g x|x)|^2 \, d\mu_{\mathsf{G}}(g) \geq 0\,.$$

At the same time $|(U_g x|x)|^2 = 1$ if $g = e$, and by continuity $(x\,|K_x x\,) > 0$ since non-empty open sets have finite measure. Hence $K_x \neq 0$ for any $x \in \mathsf{H}$. Now we claim $K_x \in \mathfrak{B}_2(\mathsf{H})$ (K_x is a Hilbert–Schmidt operator). For this it suffices to show Definition 4.24 applies. If $\{e_k\}_{k\in S}$ indicates a Hilbert basis in H, a few manipulations give

$$\sum_{k\in F}||K_x e_k||^2 = \sum_{k\in F}\int_{\mathsf{G}}\left(\int_{\mathsf{G}}(e_k|U_h x)(U_h x|U_g x)(U_g x|e_k)d\mu_{\mathsf{G}}(h)\right)d\mu_{\mathsf{G}}(g)\,.$$

for every finite $F \subset S$. For a given k, the iterated integral coincides with the integral in the product measure, by Fubini–Tonelli: in fact we are integrating a continuous map on a compact set ($\mathsf{G} \times \mathsf{G}$), so a bounded map, and the integration domain has finite measure ($= 1$). Swapping the integral and the (finite) sum:

$$\sum_{k\in F}||K_x e_k||^2 = \int_{\mathsf{G}\times\mathsf{G}}(U_h x|U_g x)\sum_{k\in F}(U_g x|e_k)(e_k|U_h x)\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g)\,.$$

Using the obvious upper bounds, from $|(U_h x|U_g x)| \leq ||x||^2$ and Schwarz's inequality:

$$\sum_{k\in F}||K_x e_k||^2 \leq \int_{\mathsf{G}\times\mathsf{G}}|(U_h x|U_g x)|\sum_{k\in F}|(U_g x|e_k)|\,|(e_k|U_h x)|\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g)$$

$$\leq ||x||^2\int_{\mathsf{G}\times\mathsf{G}}\sqrt{\sum_{k\in F}|(e_k|U_h x)|^2}\sqrt{\sum_{k\in F}|(U_g x|e_k)|^2}\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g)$$

$$\leq ||x||^2\int_{\mathsf{G}\times\mathsf{G}}\sqrt{\sum_{k\in S}|(e_k|U_h x)|^2}\sqrt{\sum_{k\in S}|(U_g x|e_k)|^2}\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g)$$

$$\leq ||x||^2\int_{\mathsf{G}\times\mathsf{G}}||U_g x||\,||U_h x||\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g) \leq ||x||^4\int_{\mathsf{G}\times\mathsf{G}}1\,d\mu_{\mathsf{G}}(h)\otimes d\mu_{\mathsf{G}}(g)$$

$$= ||x||^4 < +\infty\,.$$

But $F \subset S$ was finite but arbitrary, so

$$\sum_{k\in S}||K_x e_k||^2 \leq ||x||^4 < +\infty$$

and $K_x \in \mathfrak{B}_2(\mathsf{H})$. Since any Hilbert–Schmidt operator, like K_x, is compact, and at present $K_x = K_x^*$, we invoke Hilbert's Theorems 4.19 and 4.20 to decompose H in a Hilbert sum of eigenspaces $\mathsf{H}_\lambda^{(x)}$ of K_x:

$$\mathsf{H} = \bigoplus_{\lambda\in\sigma_p(K_x)}\mathsf{H}_\lambda^{(x)}\,.$$

Each summand, with the possible exclusion of $\mathsf{H}_0^{(x)}$, has finite dimension. Since $K_x \neq 0$, by Theorem 4.20(a) there is an eigenvalue $\lambda_1 \neq 0$. By (12.122) every eigenspace $\mathsf{H}_\lambda^{(x)}$ is π-invariant. Therefore $\mathsf{H}_0 := \mathsf{H}_{\lambda_1}^{(x)}$ satisfies the requests. □

Remark 12.104
Theorem 12.100, actually, applies to a wider class of strongly continuous representations of compact Hausdorff groups. Let $\pi : \mathsf{G} \ni g \mapsto A_g \in \mathfrak{B}(\mathsf{H})$ be a strongly continuous representation of the compact Hausdorff group G, given by bounded (perhaps non-unitary) operators on the Hilbert space H. Using the Haar measure of G, we define the inner product

$$\langle u|v\rangle_{\mathsf{G}} := \int_{\mathsf{G}} (A_g u|A_g v) d\mu_{\mathsf{G}}(g)\ , \quad u, v \in \mathsf{H}$$

on H. This inner product: (1) is well defined, (2) renders $(\mathsf{H}, \langle\ |\ \rangle_{\mathsf{G}})$ a Hilbert space, (3) makes its associated norm $||\ \ ||_{\mathsf{G}}$ *equivalent* (Definition 2.103) to the norm $||\ \ ||$ of $(\ |\ )$. In addition, $\pi : \mathsf{G} \ni g \mapsto A_g \in \mathfrak{B}(\mathsf{H})$ is a strongly continuous *unitary* representation on the Hilbert space $(\mathsf{H}, \langle\ |\ \rangle_{\mathsf{G}})$. ■

We want to study further general features of strongly continuous unitary representations of compact Hausdorff groups. Some of them are encompassed in the second part of the Peter–Weyl theorem, which we shall discuss later.

Notation 12.105 In the rest of this section, given a compact Hausdorff group G, we will denote by $\{T^s\}_{s\in S}$ a family of strongly continuous *irreducible* unitary representations $T^s : \mathsf{G} \ni g \mapsto T^s_g \in \mathfrak{B}(\mathsf{H}^s)$. Later on we will assume the family also exhausts, up to unitary equivalence, the class of strongly continuous irreducible unitary representations of G. We adopt this notation also in Proposition 12.114 although, there, irreducibility will be relaxed. ■

As a first result we have the following proposition.

Proposition 12.106 *Under the assumptions of Theorem 12.100, if T^s and $T^{s'}$ are a pair of strongly continuous unitary irreducible representations of G, then the matrix elements $D^s(g)_{ij} = (\phi_i|T^s(g)\phi_j)$ and $D^{s'}(g)_{ij} = (\psi_i|T^{s'}(g)\psi_j)$, in orthonormal bases for the respective (finite-dimensional) Hilbert spaces, satisfy the following relations:*

$$\int_{\mathsf{G}} \overline{D^{s'}(g)_{mn}} D^s(g)_{ij} d\mu_{\mathsf{G}}(g) = 0 \quad \textit{if } T^s \textit{ and } T^{s'} \textit{ are not equivalent;} \tag{12.124}$$

if T^s and $T^{s'}$ are unitarily equivalent, i.e. $UT^{(s)}(g)U^{-1} = T^{s'}(g)$ for a unitary map $U : \mathsf{H}^s \to \mathsf{H}^{s'}$ and every $g \in \mathsf{G}$, then

$$\int_{\mathsf{G}} \overline{D^{s'}(g)_{mn}} D^s(g)_{ij} d\mu_{\mathsf{G}}(g) = \frac{1}{d_s}\delta_{im}\delta_{jn}\ . \tag{12.125}$$

where $\psi_k = U\phi_k$ and $d_s = \dim \mathsf{H}^s$.

Proof Define operators $E_{ij} : \mathsf{H}^{s'} \to \mathsf{H}^s$ by

$$E_{ij}\psi := \int_{\mathsf{G}} T^s(g)\phi_i(\psi_j|T^{s'}(g^{-1})\psi)d\mu_{\mathsf{G}}(g)$$

for $\psi \in \mathsf{H}^{s'}$. As the Hilbert spaces are finite-dimensional, the above integral is a standard integral of $\mathbb{C}^k$-valued functions. We claim $E_{ij}T^{s'}(h) = T^s(h)E_{ij}$ for every $h \in \mathsf{G}$. Indeed,

$$T^s(h)\int_{\mathsf{G}} T^s(g)\phi_i(\psi_j|T^{s'}(g^{-1})\psi)d\mu_{\mathsf{G}}(g) = \int_{\mathsf{G}} T^s(hg)\phi_i(\psi_j|T^{s'}(g^{-1})\psi)d\mu_{\mathsf{G}}(g)$$

$$= \int_{\mathsf{G}} T^s(g)\phi_i(\psi_j|T^{s'}(g^{-1}h)\psi)d\mu_{\mathsf{G}}(g) = \int_{\mathsf{G}} T^s(g)\phi_i(\psi_j|T^{s'}(g^{-1})\psi)d\mu_{\mathsf{G}}(g)T^{s'}(h)\,.$$

If the two representations are not unitarily equivalent, since $E_{ij}T^{s'}(h) = T^s(h)E_{ij}$, part (b) of Schur's lemma (Proposition 11.37) entails $E_{ij} = 0$. Hence in particular

$$\int_{\mathsf{G}} (\phi_k|T^s(g)\phi_i)(\psi_j|T^{s'}(g^{-1})\psi_l)d\mu_{\mathsf{G}}(g) = 0\,,$$

which can be rephrased as

$$\int_{\mathsf{G}} D^s_{ki}(g)\overline{D^{s'}_{lj}}(g)d\mu_{\mathsf{G}}(g) = 0\,.$$

We have found (12.124). If, instead, the representations are unitarily equivalent, representing everything in H^s with respect to the same Hilbert basis, part (a) of Schur's lemma (Proposition 11.37) implies $E_{ij} = \lambda_{ij}I$, namely

$$\int_{\mathsf{G}} D^s_{ki}(g)\overline{D^s_{lj}}(g)d\mu_{\mathsf{G}}(g) = \lambda_{ij}\delta_{kl}\,.$$

Since the matrices of coefficients $D^s_{ki}(g)$ are unitary, we have that $\sum_k D^s_{ki}(g)\overline{D^s_{kj}}(g) = \delta_{ij}$ so that, assuming $l = k$ and summing over k in the identity above, we find $\delta_{ij} = \lambda_{ij}d_s$. In other words, for some $\lambda \in \mathbb{C}$,

$$\int_{\mathsf{G}} D^s_{ki}(g)\overline{D^s_{lj}}(g)d\mu_{\mathsf{G}}(g) = \lambda\delta_{ij}\delta_{kl}\,.$$

Finally, setting $i = l$ and $k = j$ and summing over i and j,

$$\int_{\mathsf{G}} \sum_{ij} D^s_{ji}(g)\overline{D^s_{ij}}(g)d\mu_{\mathsf{G}}(g) = \lambda\sum_{ij}\delta_{ij}\delta_{ji}\,,$$

that is

$$\int_{\mathsf{G}} \sum_i \delta_{ii} d\mu_{\mathsf{G}}(g) = \lambda \sum_i \delta_{ii} ,$$

and

$$\int_{\mathsf{G}} d_s d\mu_{\mathsf{G}}(g) = \lambda 1 ,$$

which means $\lambda = d_s$, eventually proving (12.125). □

To state the second part of the famous Peter–Weyl theorem we need to use the **right regular representation** of G, i.e., the strongly continuous unitary representation $R : \mathsf{G} \ni g \mapsto R_g$ acting on $L^2(\mathsf{G}, \mu_{\mathsf{G}})$ by

$$(R_g f)(h) := f(hg) , \quad g, h \in \mathsf{G},\ f \in L^2(\mathsf{G}, \mu_{\mathsf{G}}).$$

That R_g is unitary is a consequence of the fact that the Haar measure is bi-invariant, if G is compact. The right regular representation turns out to play a crucial role in the theory of unitary representations of a compact group G. A first important result, arising from Proposition 12.106, is that R subsumes *all* strongly continuous irreducible unitary representations of G.

Proposition 12.107 *Every strongly continuous unitary representation of a compact Hausdorff group G is unitarily equivalent to a subrepresentation of the right regular representation $R : \mathsf{G} \ni g \mapsto R_g$ in accordance with Theorem 12.100.*

Proof Let $T : \mathsf{G} \ni g \mapsto T_g \in \mathfrak{B}(\mathsf{H})$ be a strongly continuous unitary representation of G. Call $D_{ij}(g)$, $i, j = 1, \ldots, \dim \mathsf{H} =: d$ the coefficients of a matrix of T in a Hilbert basis of H. For i arbitrarily fixed, consider the functions $e_j(g) := \sqrt{d} D_{ij}(g)$ for $g \in \mathsf{G}$. These functions are continuous and bounded on the compact set G. Hence they are in $L^2(\mathsf{G}, \mu_{\mathsf{G}})$. Furthermore

$$(R_g e_j)(h) = e_j(hg) = \sqrt{d} D_{ij}(hg) = \sqrt{d} \sum_{k=1}^{d} D_{ik}(h) D_{kj}(g) = \sum_{k=1}^{d} D_{kj}(g) e_k(h) .$$

In other words, the (closed, because finite-dimensional) space H_T spanned by the d vectors e_j is invariant under the action of R, and the subrepresentation of R in this invariant subspace has a matrix with elements

$$\Delta_{rj}(g) = \int_{\mathsf{G}} \overline{e_r(h)} (R_g e_j)(h) d\mu_{\mathsf{G}}(h) = \sum_{k=1}^{d} d \int_{\mathsf{G}} \overline{D_{ir}(h)} D_{ik}(h) D_{kj}(g) d\mu_{\mathsf{G}}(h)$$

$$= \sum_{k=1}^{d} d D_{kj}(g) \int_{\mathsf{G}} \overline{D_{ir}(h)} D_{ik}(h) d\mu_{\mathsf{G}}(h) = \sum_{k=1}^{d} D_{kj}(g) \delta_{rk} = D_{rj}(g) ,$$

where we have exploited Proposition 12.106. This result proves that the restriction of R to H_T is unitarily equivalent to T, since these representations have the same matrix. $\square$

We focus again on the family of strongly continuous irreducible unitary representations of G and consider unitary equivalence classes. In every equivalence class $[T^s]$ of irreducible unitary representations, we select a representative T^s acting on the d_s-dimensional Hilbert space H_s common to the entire class $[T^s]$, thus obtaining a family $\{T^s\}_{s\in\widehat{\mathsf{G}}}$ where T^s is unitarily inequivalent to $T^{s'}$ if $s \neq s'$. The family, moreover, exhausts the whole set of strongly continuous unitary irreducible representations of G up to unitary equivalence. A bit improperly, we call $\{T^s\}_{s\in\widehat{\mathsf{G}}}$ the **family of strongly continuous irreducible representations of** G. The set of indices $\widehat{\mathsf{G}}$ is actually in one-to-one correspondence to *equivalence classes* of strongly continuous irreducible representations of G under unitary equivalence.

We are now in a position to state and prove the second part of the Peter–Weyl theorem, which concerns the nice interplay between irreducible representations of G and the right regular representation R.

Theorem 12.108 (Peter–Weyl, part II) *Let* G *be a compact Hausdorff group.*
(c) *Let* $\{T^s\}_{s\in\widehat{\mathsf{G}}}$ *be the family of strongly continuous irreducible representations of* G*. For every* s*, consider an orthonormal basis* $\{\phi^s_k\}_{k=1,\dots,d_s}$ *of the Hilbert space* H_s *of* T^s*, and the corresponding matrix elements* $D^s(g)_{ij} = (\phi^s_i|T^s(g)\phi^s_j)$*. Then the functions*

$$\mathsf{G} \ni g \mapsto \sqrt{d_s}D^s(g)_{ij} \in \mathbb{C}\,, \quad s \in S, i, j \in \{1, 2, \dots, d_s\}$$

form a basis of $L^2(\mathsf{G}, \mu_\mathsf{G})$*.*
(d) $L^2(\mathsf{G}, \mu_\mathsf{G})$ *decomposes in a Hilbert sum*

$$L^2(\mathsf{G}, \mu_\mathsf{G}) = \bigoplus_{s\in\widehat{\mathsf{G}}} \bigoplus_{k=1}^{d_s} \mathsf{H}^s_k$$

of finite-dimensional subspaces H^s_k *that are invariant and irreducible under the right regular representation* R *of* G*:*

$$(R_g\psi)(h) := \psi(hg) \quad \textit{if}\ \psi \in L^2(\mathsf{G}, \mu_\mathsf{G})\ \textit{and}\ g, h \in \mathsf{G}.$$

The subrepresentation of R *given by the restriction to* H^s_k*, for each* $k = 1, \dots, d_s$ *is unitarily equivalent to* T^s*. Up to unitary equivalence, every element of* $\{T^s\}_{s\in\widehat{\mathsf{G}}}$ *appears with multiplicity* d_s *in the decomposition of* R*.*

Proof (c) Let $\mathsf{H} \subset L^2(\mathsf{G}, \mu_\mathsf{G})$ be the closed subspace generated by the functions D^s_{ij}, for all s and $i, j = 1, \dots, d_s$. (These functions are continuous and belong to $L^2(\mathsf{G}, \mu_\mathsf{G})$, as they are bounded on the compact group G with finite measure.) Our goal is to show $\mathsf{H}^\perp = \{\mathbf{0}\}$. The space H is R-invariant, because R is continuous and the finite span of the functions D^s_{ij} is R-invariant, as immediately follows from the proof of Proposition 12.107. As a consequence $\mathsf{H}^\perp$ is R-invariant as well, because each R_g is unitary.

Assume that there is $v \in \mathsf{H}^\perp \setminus \{\mathbf{0}\}$ and define the function over G

$$u(g) := \int_{\mathsf{G}} v(hg)\overline{v(h)}d\mu_{\mathsf{G}}(h) . \tag{12.126}$$

It is continuous because, from (12.126) and the strong continuity of R,

$$|u(g) - u(g_0)| \leq ||R_g v - R_{g_0} v||\, ||v|| \to 0 \quad \text{for } g \to g_0 .$$

Therefore, as G is compact, u is bounded and therefore it stays in $L^2(\mathsf{G}, \mu_{\mathsf{G}})$ for the measure of G is finite. Furthermore $u \in \mathsf{H}^\perp \setminus \{\mathbf{0}\}$ because, again by (12.126), $u(e) = ||v||^2 \neq 0$ and

$$\int_{\mathsf{G}} u(g)\overline{D^s_{jk}(g)}d\mu_{\mathsf{G}}(g) = \sum_l \int_{\mathsf{G}} v(z)\overline{D^s_{lk}(z)}d\mu_{\mathsf{G}}(z)\overline{\int_{\mathsf{G}} v(h)\overline{D^s_{lj}(h)}d\mu_{\mathsf{G}}(h)} = 0 ,$$

where we have exploited $d\mu_{\mathsf{G}}(g) = \mu_{\mathsf{G}}(hg)$, we defined $z = hg$ and used

$$D^s_{jk}(h^{-1}z) = \sum_l D^s_{jl}(h^{-1})D^s_{lk}(z) = \sum_l \overline{D^s_{lj}(h)}D^s_{lk}(z) .$$

Next we define $w(g) := u(g) + \overline{u(g^{-1})}$, which is again a continuous element of $\mathsf{H}^\perp$: the proof is direct, just take $\mu_{\mathsf{G}}(E^{-1}) = \mu_{\mathsf{G}}(E)$ into account (Theorem 12.39)). Consider the operator

$$(A\psi)(g) := \int_{\mathsf{G}} w(gh^{-1})\psi(h)d\mu_{\mathsf{G}}(h) \quad \psi \in L^2(\mathsf{G}, \mu_{\mathsf{G}}) . \tag{12.127}$$

Since $K(g,h) := w(gh^{-1})$ is continuous and $\mathsf{G} \times \mathsf{G}$ is compact with finite product Haar measure, A is compact (Examples 4.18(4)). As the kernel w is real, $A = A^*$. Since $w \neq 0$, we have $A \neq 0$ and therefore it admits an eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ whose eigenspace H_λ has finite dimension due to the compactness of A. If $\psi \in \mathsf{H}_\lambda \setminus \{\mathbf{0}\}$ we have $\psi \in \mathsf{H}^\perp \setminus \{\mathbf{0}\}$ because, applying the definition of A,

$$\int_{\mathsf{G}} \psi(g)\overline{D^s(g)_{ij}}d\mu_{\mathsf{G}}(g) = \frac{1}{\lambda}\int_{\mathsf{G}} (A\psi)(g)\overline{D^s(g)_{ij}}d\mu_{\mathsf{G}}(g)$$

$$= \frac{1}{\lambda}\sum_k \int_{\mathsf{G}} w(h)\overline{D^s(h)_{ik}}d\mu_{\mathsf{G}}(h)\int_{\mathsf{G}} \psi(g)\overline{D^s(g)_{kj}}d\mu_{\mathsf{G}}(g) = 0 .$$

The operator A commutes with R as the reader can immediately prove just by applying the definition of A and R. Therefore H_λ is invariant under R and can be decomposed in a Hilbert sum of (strongly continuous) unitary irreducible representations of G. The Hilbert spaces of these representations are finite-dimensional by Theorem 12.100. We end up with at least one finite-dimensional subspace $\mathsf{K}_\lambda \subset \mathsf{H}_\lambda \subset \mathsf{H}$ made of eigenvectors of A, and supporting a unitary strongly continuous representation of G. If $e_1, \ldots, e_n$ is a Hilbert basis of K_λ, we must have

$$(R_g e_k)(h) = e_k(hg) = \sum_{l=1}^{n} D(g)_{kl} e_l(h) .$$

In particular, setting $h = e$

$$e_k(g) = \sum_{l=1}^{n} D(g)_{kl} e_l(e) , \quad \forall g \in \mathsf{G} ,$$

which is impossible since it would imply $e_k \in \mathsf{H}$, whereas $\mathbf{0} \neq e_k \in \mathsf{H}^\perp$. We conclude that $\mathsf{H}^\perp = \{\mathbf{0}\}$ as we wanted. Observe that $e_k(e) \in \mathbb{C}$ is well defined since the function $\mathsf{G} \ni g \mapsto e_k(g)$ is continuous because it is an eigenfunction of A. The continuity of the eigenfunctions of A follows easily from (12.127), by noting that w is continuous and then arguing as for the continuity of u (12.126).

(d) The proof is easy. Consider the Hilbert basis made by the D^s_{ij}, where $i, j = 1, \ldots, d_s$ and s labels inequivalent irreducible representations, and let H^s be the closed subspace spanned by the D^s_{ij} with fixed i. This space supports exactly d_s irreducible, strongly continuous representations of G acting on orthogonal subspaces whose sum is H^s itself. In fact, for k fixed in $\{1, \ldots, d_s\}$, the subspace $\mathsf{H}^s_k \subset \mathsf{H}^s$, spanned by the orthonormal functions $e^{(sk)}_j := \sqrt{d_s} D^s_{kj}$, $j = 1, \ldots, d_s$ is R-invariant and R-irreducible because

$$R_g e^{(sk)}_j = \sum_{l=1}^{d_s} D^s_{jl}(g) e^{(sk)}_l ,$$

as established in the proof of Proposition 12.107. For $s \neq s'$ we have $\mathsf{H}^s \perp \mathsf{H}^{s'}$ and, for $k \neq k'$, $\mathsf{H}^s_k \perp \mathsf{H}^s_{k'}$. Finally, from Proposition 12.107, every strongly continuous unitary irreducible representation of G is unitarily equivalent to one of the representations acting on some H^s_k. $\square$

The last, remarkable, result by Peter and Weyl regards the dense span of the $\sqrt{d_s} D^s_{ij}$. We already know that these functions span a dense set in $L^2(\mathsf{G}, \mu_\mathsf{G})$. And we know they are continuous. As a matter of fact, the Hilbert basis of the $\sqrt{d_s} D^s_{ij}$ can be used to approximate every *continuous* function of G in the natural topology of the space of continuous maps over compact sets. We need a pair of lemmas that are interesting in their own right.

Lemma 12.109 *Referring to the statement of Theorem 12.108, we take the Banach space $C(\mathsf{G})$ with norm $|| \cdot ||_\infty$, and an element $\phi \in C(\mathsf{G})$. Define the operator*

$L_\phi : L^2(\mathsf{G}, d\mu_\mathsf{G}) \to L^2(\mathsf{G}, d\mu_\mathsf{G})$

$$(L_\phi f)(h) := \int_\mathsf{G} \phi(g) f(g^{-1}h) d\mu_\mathsf{G}(g) \quad \forall f \in L^2(\mathsf{G}, d\mu_\mathsf{G}) .$$

The following facts hold.
(a) *$L_\phi(L^2(\mathsf{G}, d\mu_\mathsf{G})) \subset C(\mathsf{G})$, and $L_\phi : L^2(\mathsf{G}, d\mu_\mathsf{G}) \to C(\mathsf{G})$ is continuous in the natural norms.*
(b) *If $N \subset \widehat{\mathsf{G}}$ is finite and $\mathsf{H}_N \subset C(\mathsf{G})$ is the span of the functions $\sqrt{d_s} D^s_{ij}$, with $s \in N$, $i, j = 1, \ldots, d_s$, then $L_\phi(\mathsf{H}_N) \subset \mathsf{H}_N$.*

Proof (a) From the Cauchy–Schwarz inequality,

$$|(L_\phi f)(x)|^2 = \left| \int_\mathsf{G} \phi(y) f(y^{-1}x) d\mu_\mathsf{G}(y) \right|^2 \leq \int_\mathsf{G} |\phi(y)|^2 d\mu_\mathsf{G}(y) \int_\mathsf{G} |f(y^{-1}x)|^2 d\mu_\mathsf{G}(y) . \tag{12.128}$$

Similarly

$$|(L_\phi f)(x) - (L_\phi f)(x_0)|^2 \leq \int_\mathsf{G} |\phi(y)|^2 d\mu_\mathsf{G}(y) \int_\mathsf{G} |f(y^{-1}x) - f(y^{-1}x_0)|^2 d\mu_\mathsf{G}(y) ,$$

so that Lebesgue's dominated convergence theorem proves that $\mathsf{G} \ni x \mapsto (L_\phi f)(x)$ is continuous, using the fact that $\mathsf{G} \times \mathsf{G}$ is compact and hence the continuous map $\mathsf{G} \times \mathsf{G} \ni (x, y) \mapsto |f(y^{-1}x) - f(y^{-1}x_0)|$ is bounded and $\mu_\mathsf{G}(\mathsf{G})$ is finite. This proves in particular that L_ϕ is well defined, for $L_\phi f \in C(\mathsf{G}) \subset L^2(\mathsf{G}, \mu_\mathsf{G})$. Inequality (12.128) and $d\mu_\mathsf{G}(y) = d\mu_\mathsf{G}(y^{-1}) = d\mu_\mathsf{G}(y^{-1}x)$ imply that

$$||L_\phi f||_\infty \leq ||\phi||_{L^2} ||f||_{L^2} ,$$

establishing the continuity of $L_\phi : L^2(\mathsf{G}, d\mu_\mathsf{G}) \to C(\mathsf{G})$.

(b) If $f \in \mathsf{H}_N$, we can write $f = \sum_{s \in N} \sum_{j,j=1}^{d_s} c_s^{ij} D^s_{ij}$. Therefore

$$(L_\phi f)(x) = \sum_{s \in N} \sum_{j,k=1}^{d_s} c_s^{ij} \int_\mathsf{G} \phi(y) D^s_{ij}(y^{-1}x) d\mu_\mathsf{G}(y)$$

$$= \sum_{s \in N} \sum_{i,j,k=1}^{d_s} c_s^{ij} \int_\mathsf{G} \phi(y) D^s_{ik}(y^{-1}) d\mu_\mathsf{G}(y) D^s_{kj}(x) = \sum_{s \in N} \sum_{k,i=1}^{d_s} a_s^{ki} D^s_{kj}(x)$$

where

$$a_s^{ki} := \sum_{j=1}^{d_s} c_s^{ij} \int_\mathsf{G} \phi(y) D^s_{ik}(y^{-1}) d\mu_\mathsf{G}(y) ,$$

so that $L_\phi f \in \mathsf{H}_N$. $\square$

Lemma 12.110 *Let* G *be a compact group and* $f : \mathsf{G} \to \mathbb{C}$ *a continuous function. For every* $\varepsilon > 0$ *there exists an open neighbourhood* V_ε *of* $e \in \mathsf{G}$ *such that* $|f(x) - f(y)| < \varepsilon$ *when* $xy^{-1} \in V_\varepsilon$.

Proof Fix $y \in \mathsf{G}$. Since f is continuous, for every $\varepsilon > 0$ there is an open neighbourhood U_ε of y such that $|f(x) - f(y)| < \varepsilon/2$ if $x \in U_\varepsilon$. Defining the open neighbourhood of e (this is well defined as the right translation is a homeomorphism) $V_\varepsilon^y := U_\varepsilon y^{-1}$, we have that $|f(x) - f(y)| < \varepsilon/2$ for $xy^{-1} \in V_\varepsilon^y$. Let W_ε^y be another open neighbourhood of e such that $W_\varepsilon^y \circ W_\varepsilon^y \subset V_\varepsilon^y$. For $\varepsilon > 0$, the collection of the analogous open sets $W_\varepsilon^y y$, with y varying in G, define an open covering of G. Since G is compact, we can extract a finite subcovering $\{W_\varepsilon^{y_k} y_k\}_{k=1,\ldots,n}$ using a finite number of elements $y_1, \ldots, y_n$. Define the open neighbourhood of e

$$V_\varepsilon := \cap_{k=1}^n W_\varepsilon^{y_n} .$$

Since $\cup_{k=1}^n W_\varepsilon^{y_k} y_k = \mathsf{G}$, if $x, y \in \mathsf{G}$ and $xy^{-1} \in V_\varepsilon$, for some $k \in \{1, 2, \ldots, n\}$, then $y \in W_\varepsilon^{y_k} y_k$, so that $|f(y) - f(y_k)| < \varepsilon/2$. Next,

$$xy_k^{-1} = xy^{-1}yy_k^{-1} \subset V_\varepsilon \circ W_\varepsilon^{y_k} \subset W_\varepsilon^{y_k} \circ W_\varepsilon^{y_k} \subset V_\varepsilon^{y_k}$$

and hence $|f(x) - f(y_k)| < \varepsilon/2$. Summing up,

$$|f(x) - f(y)| \le |f(x) - f(y_k)| + |f(y) - f(y_k)| < \varepsilon$$

as required. □

Theorem 12.111 (Peter–Weyl, part III) *Let* G *be a compact Hausdorff group.*
(e) *The finite span of the continuous functions* $\mathsf{G} \ni g \mapsto \sqrt{d_s} D_{ij}^s(g)$, *where* $s \in \widehat{\mathsf{G}}$ *and* $i, j = 1, \ldots, d_s$, *(see Theorem 12.108) is dense in* $C(\mathsf{G})$ *in the uniform norm* $||\cdot||_\infty$.

Proof Take $f \in C(\mathsf{G})$. As G is compact and $f : \mathsf{G} \to \mathbb{C}$ is continuous, Lemma 12.110 says that for every $\varepsilon > 0$, there is an open neighbourhood $V_\varepsilon \subset \mathsf{G}$ of $e \in \mathsf{G}$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \quad \text{if } x_1 x_2^{-1} \in V_\varepsilon. \tag{12.129}$$

Next take $\phi_\varepsilon \in C(\mathsf{G})$, with $\phi_\varepsilon(x) \ge 0$ for $x \in \mathsf{G}$, with support contained in V_ε and such that $\int_\mathsf{G} \phi_\varepsilon d\mu_\mathsf{G} = 1$. Defining L_{ϕ_ε} as in Lemma 12.109, we have

$$||L_{\phi_\varepsilon} f - f||_\infty \le \sup_{x\in\mathsf{G}} \int_\mathsf{G} \phi_\varepsilon(y)|f(y^{-1}x) - f(x)| d\mu_\mathsf{G}(y) < \varepsilon$$

since the only relevant values of x, y in the integrand are those satisfying $x(y^{-1}x)^{-1} = y \in V_\varepsilon$, otherwise $\phi_\varepsilon(y) = 0$, and for these values $|f(y^{-1}x) - f(x)| < \varepsilon$. On the other hand, Theorem 12.100 says that every element $f \in L^2(\mathsf{G}, \mu_\mathsf{G})$ can be arbitrarily approximated in L^2-norm by elements f_N in H_N. In particular,

$$||f - f_{N_\varepsilon}||_{L^2} \leq \frac{\varepsilon}{||\phi_\varepsilon||_{L^2}}$$

for a suitable finite set $N_\varepsilon \subset \widehat{\mathsf{G}}$. (Notice that $||\phi_\varepsilon||_{L^2}=||\phi_\varepsilon||_{L^2}||1||_{L^2} \geq ||\phi_\varepsilon||_{L^1} = 1$.) The inequality, together with the earlier relation $||L_{\phi_\varepsilon} f - f||_\infty < \varepsilon$, gives to

$$|f(x) - (L_{\phi_\varepsilon} f_{N_\varepsilon})(x)| \leq |f(x) - (L_{\phi_\varepsilon} f)(x)| + |L_{\phi_\varepsilon}(f - f_{N_\varepsilon})(x)|$$

$$< \varepsilon + ||f - f_{N_\varepsilon}||_{L_2}||\phi_\varepsilon||_{L^2} < 2\varepsilon \, .$$

Since $\varepsilon > 0$ is arbitrary and $L_{\phi_\varepsilon} f_{N_\varepsilon} \in C(\mathsf{G})$ from Lemma 12.109, we have proved the theorem. □

12.2.14 Characters of Finite-Dimensional Group Representations

A technically useful notion is that of *character of a finite-dimensional representation* of a group.

Definition 12.112 The **character** of a finite-dimensional unitary group representation $\mathsf{G} \ni g \mapsto T_g \in \mathfrak{B}(\mathsf{H})$ is the map

$$\mathsf{G} \ni g \mapsto \chi(g) := tr(T_g) \in \mathbb{C} \, .$$

Remark 12.113 This notion is evidently related to the one of Definition 12.93. If G is Abelian and $\chi : \mathsf{G} \to U(1)$ is a character for Definition 12.93, χ is a character for Definition 12.112: in fact, $tr(\chi(g)) = \chi(g)$ when we think of $\chi(g)$ as an operator on the one-dimensional Hilbert space $\mathbb{C}$. ■

Proposition 12.106 and the definition of character of a representation entail the following elementary but very useful proposition.

Proposition 12.114 *The characters χ of a finite-dimensional unitary representation T of a group G satisfy the following properties.*
(a) $\chi(h^{-1}gh) = \chi(g)$ *if* $h, g \in \mathsf{G}$.
(b) $\chi(g^{-1}) = \overline{\chi(g)}$ *if* $g \in \mathsf{G}$.
Assume that G is Hausdorff and compact, and T is strongly continuous.
(c) *If the representations T and T' are irreducible, the associated characters χ and χ' satisfy*
(i) $\chi = \chi'$ *if T and T' are unitarily equivalent.*
(ii) $\int_{\mathsf{G}} \overline{\chi(g)}\chi'(g)d\mu_{\mathsf{G}}(g) = 1$ *if T and T' are unitarily equivalent.*
(iii) $\int_{\mathsf{G}} \overline{\chi(g)}\chi'(g)d\mu_{\mathsf{G}}(g) = 0$ *if T and T' are unitarily inequivalent.*
(d) *According to Lemma 12.98, assume that the finite-dimensional unitary representation T, with character χ, is decomposed in a direct sum of irreducible unitary*

representations $T^1, T^2, \ldots, T^n$, where unitarily equivalent representations are identified and T^k is unitarily inequivalent to T^h for $h \neq k$. Let us indicate the respective characters by $\chi_1, \chi_2, \ldots, \chi_n$, with multiplicities $m_1, m_2, \ldots, m_n$. Then

(i) $\chi = \sum_i m_i \chi_i$,

(ii) $m_i^2 = \int_{\mathsf{G}} \chi(g)\overline{\chi_i(g)} d\mu_{\mathsf{G}}(g)$,

(iii) $\sum_i m_i^2 = \int_{\mathsf{G}} \chi(g)\overline{\chi(g)} d\mu_{\mathsf{G}}(g)$.

(e) *A finite-dimensional strongly continuous unitary representation T of G is irreducible if and only if its character χ satisfies $\int_{\mathsf{G}} \chi(g)\overline{\chi(g)} d\mu_{\mathsf{G}}(g) = 1$.*

Proof (a) and (b) are immediate by definition of character. (c)(i) follows straightforwardly from the invariance of the trace under change of orthonormal basis. (ii) and (iii) are easy consequences of Proposition 12.106, by computing the traces of the matrices in the integrals of the statement.

(d) According to Lemma 12.98, we choose as Hilbert basis of the Hilbert space H of the representation T a union of Hilbert basis for each invariant irreducible subspace. Then (i) follows from (c)(i) and the definition of trace. Eventually, (c)(i), (c)(ii) and (c)(iii) immediately yield (d)(ii), (d)(iii) and (e). □

Remark 12.115 Items (a) and (b) are valid for finite-dimensional unitary representations of any group G. The remaining items (c), (d), (e) hold also for finite-dimensional unitary representations of a *finite group* G, with N_{G} elements. This can be proved, disregarding any topological issue, by equipping G with the measure that counts the number of elements, normalised by $1/N_{\mathsf{G}}$. This measure is evidently invariant under left and right translations. This is equivalent to replacing $\int_{\mathsf{G}} f(g) d\mu_{\mathsf{G}}(g)$ by $N_{\mathsf{G}}^{-1} \sum_{g \in \mathsf{G}} f(g)$. ■

12.3 Examples

In this section we discuss a few important quantum-mechanical examples of the theory we have developed, with particular regard to the Peter–Weyl theorem applied to $SU(2)$ and the Galilean group.

12.3.1 The Symmetry Group $SO(3)$ and the Spin

We now concentrate on unitary representations of the compact Lie group $SU(2)$, seen as the universal covering of $SO(3)$ (Example 12.38(2)). With the aid of Bargmann's theorem and Proposition 12.69 (see Example 12.77(2) as well), unitary $SU(2)$-representations will be used to define an $SO(3)$-action – by a continuous projective representation – on the physical system made by a particle of spin s.

By Theorem 12.100 unitary $SU(2)$-representations are direct sums of irreducible, finite-dimensional unitary representations. In the sequel we shall describe them.

Until now we discussed the quantum system of a particle on the Hilbert space $L^2(\mathbb{R}^3, dx)$ (fixing an inertial frame $\mathscr{I}$ that identifies $\mathbb{R}^3$ with the rest space, with a right-handed triple of Cartesian axes). Experience shows that this description is not physically adequate: $L^2(\mathbb{R}^3, dx)$ is not always good enough to account for the physical structure of real particles. The latter possess a feature, called *spin*, determined by an associated constant s, just like the mass is attached to the particle; this constant can only be integer or semi-integer $s = 0, 1/2, 1, 3/2, \ldots$.

Having a spin means, physically, that the particle possesses an *intrinsic angular momentum* [Mes99, CCP82], and there are observables, not representable by the fundamental position and momentum, that describe the intrinsic angular momentum. Let us overview the mathematics involved, referring to [Mes99, CCP82] for a sweeping physics' debate on this crucial topic.

If a particle has spin $s = 0$ the description is the usual one for spinless particles. If $s = 1/2$, the particle's Hilbert space is larger, and in fact is the tensor product $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ (seen as Hilbert space) is the **spin space**. The three **spin operators** are the Hermitian matrices (for the moment we introduce the constant value $\hbar$, only to set it to 1 subsequently for simplicity) $S_k := \frac{\hbar}{2}\sigma_k$, $k = 1, 2, 3$ and the σ_k are the *Pauli matrices* seen earlier. The commutation relations:

$$[-iS_i, -iS_j] = \hbar \sum_{k=1}^{3} \varepsilon_{ijk}(-iS_k) \tag{12.130}$$

hold. The associated observables are the components of the particle's intrinsic angular momentum in the given inertial frame system. For $s = 1/2$ the possible values of each component are $-\hbar/2$ and $\hbar/2$, since the eigenvalues of a Pauli matrix are ± 1.

For generic spin s the description is similar, but the spin space is now $\mathbb{C}^{2s+1}$. There the matrices S_k of the spin operators, replacing $\frac{\hbar}{2}\sigma_k$, are Hermitian, satisfy (12.130) and have $2s+1$ eigenvalues $-\hbar s, -\hbar(s-1), \ldots, \hbar(s-1), \hbar s$ of multiplicity 1. For $m, m' = s, s-1, \ldots, -s+1, -s$, here is what they look like, explicitly:

$$(S_1)_{m'm} = \frac{\hbar}{2}\left(\sqrt{(s-m)(s+m+1)}\delta_{m',m+1} + \sqrt{(s+m)(s-m+1)}\delta_{m',m-1}\right),$$

$$(S_2)_{m'm} = \frac{\hbar}{2i}\left(\sqrt{(s-m)(s+m+1)}\delta_{m',m+1} - \sqrt{(s+m)(s-m+1)}\delta_{m',m-1}\right),$$

$$(S_3)_{m'm} = m\hbar\delta_{m',m}\,.$$

For the recipe to construct the S_k and a deeper analysis of the spin we suggest consulting [Mes99, CCP82]. Here we just make three comments.

(a) The operator $S^2 := \sum_{k=1}^{3} S_k^2$ satisfies

$$S^2 = \hbar^2 s(s+1)I$$

where $I : \mathbb{C}^{2s+1} \to \mathbb{C}^{2s+1}$ is the identity matrix.

(b) The space $\mathbb{C}^{2s+1}$ is *irreducible* under the $SU(2)$-representation given by exponentiating $-iS_k$:

$$V^s : SU(2) \ni e^{-i\theta\frac{\hbar}{2}\mathbf{n}\cdot\sigma} \mapsto e^{-i\theta\mathbf{n}\cdot\mathbf{S}} \, . \tag{12.131}$$

For $s = 0, 1/2, 1, 3/2, \dots$ and up to unitary equivalence, the V^s produce *every* irreducible finite-dimensional unitary $SU(2)$-representation.

(c) The matrix S_3 is chosen so to coincide with
$\hbar\,\mathrm{diag}(s, s-1, \dots, -s+1, -s)$.

Typically the eigenvector basis of S_3, i.e. the canonical basis of $\mathbb{C}^{2s+1}$, is denoted $\{|s, s_3\rangle\}_{|s_3|\le s}$. Pure states $\Psi(\Psi| \quad)$ are thus determined by a collection of $2s+1$ wavefunctions ψ_{s_3} in $L^2(\mathbb{R}^3, dx)$ with unit norm, and therefore a pure state is given by a unit vector

$$\Psi = \sum_{|s_3|\le s} \psi_{s_3} \otimes |s, s_3\rangle \, .$$

Because of this $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$ becomes naturally isomorphic to the orthogonal sum of $2s+1$ copies of $L^2(\mathbb{R}^3, dx)$, so Ψ is identified with a column vector

$$\Psi \equiv (\psi_s, \psi_{s-1}, \cdots, \psi_{-s+1}, \psi_{-s})^t$$

of wavefunctions. In QM jargon these are called **spinors** of dimension s.

If s is an integer, the representation $SU(2) \ni e^{-i\theta\frac{1}{2}\mathbf{n}\cdot\sigma} \mapsto e^{-i\frac{\theta\mathbf{n}\cdot\mathbf{S}}{\hbar}}$ on $\mathbb{C}^{2s+1}$, associated to the spin matrices, is a faithful $SO(3)$-representation, since the kernel of the covering map $SU(2) \to SO(3)$ consists of the identity I and of $-I$. If s is half an integer, instead, the above is a faithful $SU(2)$-representation.

One last important remark on the construction of the observables S_k and the relative irreducible $SU(2)$-representations, found in all QM manuals and based on the commutation relations of the S_k only, is the following. The purely algebraic construction works *because we assume the observables S_k are defined on the whole Hilbert space, and have discrete spectrum*. This is theoretically not obvious, and is merely due to the *finite-dimensional* ambient one works in, so the operators S_k are Hermitian matrices. This is guaranteed by the Peter–Weyl theorem, provided one uses *irreducible unitary representations* of a *compact* group like $SU(2)$. The same procedure would not work as well with non-compact groups such as the Lorentz group.

This is the point where we start setting $\hbar = 1$ to simplify notations. We wish to discuss the relationship between the total angular momentum and $SU(2)$, or the rotation group $SO(3)$. For a particle of spin s let

$$\mathscr{J}_k = \mathscr{L}_k \otimes I + I \otimes S_k \tag{12.132}$$

be the **(total) angular momentum operators** on $\mathsf{H} = L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$. The *orbital angular momentum operators* $\mathscr{L}_k$, defined in (10.40) and discussed in Chap. 10, have as closure the observables associated to the components of the orbital angular momentum. Above, the first I denotes the identity operator on $\mathbb{C}^{2s+1}$ and the second the identity on $L^2(\mathbb{R}^3, dx)$. The domain is the invariant linear space $\mathscr{D} := \mathscr{S}(\mathbb{R}^3) \otimes \mathbb{C}^{2s+1}$. By construction these operators satisfy the bracket relations defining the Lie algebra $so(3)$:

$$[-i\mathscr{J}_i, -i\mathscr{J}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}(-i\mathscr{J}_k) \ . \tag{12.133}$$

We wish to apply Nelson's Theorem 12.89 to the Lie algebra spanned by the operators $\mathscr{J}_k$. Consider the symmetric operator

$$\mathscr{J}^2 = \sum_{k=1}^{3} (\mathscr{L}_k \otimes I + I \otimes S_k)^2$$

defined on $\mathscr{D}$. It admits an eigenvector basis

$$|l, m, s_z, n\rangle := Y^l_m \psi_n \otimes |s, s_z\rangle \in \mathscr{D} \subset L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$$

obtained from the basis of H, with $l = 0, 1, 2 \ldots,$, $m = -l, -l+1, \ldots, l-1, l$, $n = 0, 1, 2, \ldots$, $s_z = -s, -s+1, \ldots, s-1, s$, and where the $|s, s_z\rangle \in \mathbb{C}^{2s+1}$ are unit eigenvectors of S_3 relative to s_z. As S_3 is Hermitian, the $2s+1$ vectors $|s, s_z\rangle$ are an orthonormal basis in $\mathbb{C}^{2s+1}$. The space $L^2(\mathbb{R}^3, dx)$ has a basis made by the $Y^l_m \psi_n$ of (10.50), Chap. 10. Proposition 10.25 ensures the $Y^l_m \psi_n \otimes |s, s_z\rangle$ form a basis for the product space. The $|l, m, s_z, n\rangle$ are not eigenvectors of $\mathscr{J}^2$. The purely algebraic *Clebsch–Gordan procedure*[15] [Mes99, CCP82] shows how to build, out of finite combinations of vectors $|l, m, s_z, n\rangle$, an eigenvector basis

$$|j, j_3, l, n\rangle$$

for $\mathscr{J}^2$, $\mathscr{J}_z$, $\mathscr{L}^2$, where $|l+s| \geq j \geq |l-s|$, $l = 0, 1, 2, \ldots$ $j_3 = -j, -j+1, \ldots, j+1, j$, $n = 0, 1, 2, \ldots$ (the j implicitly differ by integers). Then

$$\mathscr{J}^2|j, j_3, l, n\rangle = j(j+1)|j, j_3, n\rangle \ , \quad \mathscr{J}_3|j, j_3, n\rangle = j_z|j, j_3, n\rangle \ ,$$

$$\mathscr{L}^2|j, j_3, n\rangle = l(l+1)|j, j_3, n\rangle \ .$$

The $|j, j_3, l, n\rangle$ belong in $\mathscr{D}$ being finite combinations of $|l, m, s, s_z, n\rangle$. As eigenvectors, they are analytic vectors for $\mathscr{J}^2$. Nelson's criterion tells $\mathscr{J}^2$ is essentially self-adjoint on $\mathscr{D}$. Then there exists a strongly continuous unitary $SU(2)$-representation on H, by Nelson's theorem, the generators of which are the self-adjoint operators

[15] Back when the author was an undergraduate, the procedure was impertinently known among students by the cheeky name of computation of "Flash Gordon coefficients".

$J_k := \overline{\mathscr{J}_k} = \overline{\mathscr{L}_k} \otimes I + I \otimes S_k$. (Notice $\overline{\mathscr{L}_k} \otimes I = \overline{\mathscr{L}_k \otimes I}$ since I is in that case defined on a finite-dimensional space.)

In the exercises we will show that the strongly continuous unitary representation obtained when exponentiating the J_k, if $s = 0$, is an $SO(3)$-representation, and coincides with the known one from Example 12.19(1), where $\Gamma \in IO(3)$ specialises to $\Gamma = R \in SO(3)$. (The representation is strongly continuous owing to Example 12.46(1).) This fact easily implies (see the exercises), when $s \neq 0$, that the $SU(2)$-representation arising by exponentiating the generators J_k as in Nelson's theorem has the form:

$$SU(2) \ni e^{-i\theta\frac{1}{2}\mathbf{n}\cdot\sigma} \mapsto e^{-i\theta\mathbf{n}\cdot\mathbf{J}} = e^{-i\theta\mathbf{n}\cdot\mathbf{L}} \otimes V^s\left(e^{-i\frac{\theta}{2}\mathbf{n}\cdot\sigma}\right) \tag{12.134}$$

where $L_k := \overline{\mathscr{L}_k}$ is the self-adjoint operator associated to the kth component of the orbital angular momentum, as in Chap. 9. Furthermore

$$\left(e^{-i\theta\mathbf{n}\cdot\mathbf{L}}\psi\right)(\mathbf{x}) = \psi\left(e^{-\theta\mathbf{n}\cdot\mathbf{T}}\mathbf{x}\right), \tag{12.135}$$

where

$$SU(2) \ni e^{-i\theta\frac{1}{2}\mathbf{n}\cdot\sigma} \mapsto e^{\theta\mathbf{n}\cdot\mathbf{T}} \in SO(3)$$

is the covering map $SU(2) \to SO(3)$ discussed in Example 12.65(6).

Remark 12.116 Because of Proposition 12.69, the physical assumption is that the projective $SO(3)$-representation induced by the unitary $SU(2)$-representation (12.134) corresponds to the action of $SO(3)$ on the spin-s particle, when we view $SO(3)$ as symmetry group of the system. ■

12.3.2 The Superselection Rule of the Angular Momentum

We consider a generic quantum system admitting a continuous projective representation of the rotation group $SO(3)$ illustrating the physical effect of rotating states. We may view the representation as a strongly continuous unitary $SU(2)$-representation by Bargmann's theorem and Proposition 12.69. Using Peter–Weyl we conclude the system's Hilbert space decomposes in a sum $\mathsf{H} = \bigoplus_{s\in A} \mathsf{H}_s$ of closed orthogonal spaces H_s, on which irreducible, hence finite-dimensional, unitary representations of $SU(2)$ act. Each such is unitarily equivalent to one V^s of the previous section, where now $s(s+1)$ will not correspond to the spin squared of a particle, but rather to the squared eigenvalue of the total angular momentum J^2 on V^s, including orbital and spin components. From the previous section the parameter s can only be integer or semi-integer, $s = 0, 1/2, 1, 3/2, 2, \ldots$, so the index set A cannot be larger than the set of those values. Suppose the set A of our physical system contains either type of values. Let J_3 be the self-adjoint generator of rotations about the z-axis, however fixed, corresponding to the component of the total angular momentum along z by definition. Consider a pure state given by $\Psi = \psi_s + \psi_{s'} \in \mathsf{H}_s + \mathsf{H}_{s'}$, with s integer and s' semi-integer. Irrespective of the axis x_3, and remembering the expression for S_3 of the previous section:

$$e^{-i2\pi J_3}\Psi = e^{-i2\pi S_3^{(s)}}\psi_s + e^{-i2\pi S_3^{(s')}}\psi_{s'} = \psi_s - \psi_{s'} \neq \Psi .$$

This is physically nonsense, for it says that a complete revolution (by 2π) about an axis alters the pure state $\Psi(\Psi|\ )$. Therefore, when A contains both integers and semi-integers, we need to assume a **superselection rule for the angular momentum** that forbids coherent superpositions of pure states with total angular momentum (the s giving the irreducible $SU(2)$-representations) partly integer and partly semi-integer. As remarked in Sect. 7.7, a pure state can have *undefined* angular momentum, when the state's vector is a combination of vectors corresponding to pure states with different angular momenta. The superselection rule, however, forces the values to be all either integer or semi-integer. Another approach leading to the same result is based on the *time reversal symmetry* and will be briefly discussed in Example 13.22.

12.3.3 The Galilean Group and Its Projective Unitary Representations

This subsection analyses the elementary structure of continuous projective unitary representations of the Galilean group, viewed as a Lie group. We shall construct the action of the group on wavefunctions adopting a physical viewpoint, that is, supposing we know the physical meaning of (some of) the self-adjoint generators of the one-parameter subgroups of the Galilean group. We shall find that this action is intrinsically *projective unitary*. This fact leads to a superselection rule of the mass. However, another more mathematically minded and powerful approach is available [CDLL04], whereby the irreducible projective unitary representations are constructed from scratch as consequence of the structure of the second cohomology group of Galilean group, and using the imprimitivity technology of Mackey. This second approach is more general and the physical interpretation of the generators is given *a posteriori*. Nevertheless, the physically meaningful representations arising thus take exactly the form we shall find by our way, and the superselection rule of the mass shows up again.

In classical physics the transformations between the orthonormal Cartesian coordinates of two inertial frames $\mathscr{I}$, $\mathscr{I}'$ are elements of the **Galilean group** $\mathscr{G}$. In this sense Galilean transformations are *passive* transformations. With the obvious notation we can write them as:

$$\begin{cases} t' = t + c\,, \\ x'_i = c_i + tv_i + \sum_{j=1}^{3} R_{ij}x_j\,, \quad \text{i=1,2,3}, \end{cases} \tag{12.136}$$

where $c \in \mathbb{R}$ (*not* the speed of light!), $c_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are any constants, and the numbers R_{ij} define a matrix $R \in O(3)$. Every element of $\mathscr{G}$ is then given by four quantities $(c, \mathbf{c}, \mathbf{v}, R) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times O(3)$. Composing Galilean transformations rephrases as

$$(c_2, \mathbf{c_2}, \mathbf{v_2}, R_2) \cdot (c_1, \mathbf{c_1}, \mathbf{v_1}, R_1) = (c_1 + c_2,\ R_2\mathbf{c_1} + c_1\mathbf{v_2} + \mathbf{c_2},\ R_2\mathbf{v_1} + \mathbf{v_2},\ R_2R_1)\ . \tag{12.137}$$

This composition law turns $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times O(3)$ into a group, the **Galilean group**. In particular, the neutral element is $(0, \mathbf{0}, \mathbf{0}, I)$ and the inverse:

$$(c, \mathbf{c}, \mathbf{v}, R)^{-1} = (-c, R^{-1}(c\mathbf{v} - \mathbf{c}), -R^{-1}\mathbf{v}, R^{-1})\ . \tag{12.138}$$

We may interpret Galilean transformations as *active* transformations, that actively move spacetime events seen as column vectors $(\mathbf{x}, t)^t$ of Cartesian coordinates (orthonormal, right-handed) in an inertial frame system fixed once and for all.

The group $\mathscr{G}$ acts by matrix multiplication if we identify the generic element $(c, \mathbf{c}, \mathbf{v}, R) \in \mathscr{G}$ with the real 5×5 matrix:

$$\begin{bmatrix} R & \mathbf{v} & \mathbf{c} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \tag{12.139}$$

and the columns $(\mathbf{x}, t)^t \in \mathbb{R}^4$ with $(\mathbf{x}, t, 1)^t \in \mathbb{R}^5$. In this way $\mathscr{G}$ becomes a Lie subgroup of $GL(5, \mathbb{R})$ (the analytic structure coincides with the one inherited from $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times O(3)$).

In the sequel we shall reduce to the **restricted Galilean group** $S\mathscr{G}$, the connected Lie subgroup where R has positive determinant, i.e. $R \in SO(3)$. We will not consider the *inversion of parity*, which is known to not always be a symmetry and must be treated separately, at least at a quantum level.

The universal covering $\widetilde{S\mathscr{G}}$, arises by replacing $SO(3)$ with $SU(2)$ (real Lie group of dimension 3 inside $GL(4, \mathbb{R})$). As a matter of fact $\widetilde{S\mathscr{G}}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times SU(2)$ with product

$$(c_2, \mathbf{c_2}, \mathbf{v_2}, U_2) \cdot (c_1, \mathbf{c_1}, \mathbf{v_1}, U_1) = (c_1 + c_2,\ R(U_2)\mathbf{c_1} + c_1\mathbf{v_2} + \mathbf{c_2},\ R(U_2)\mathbf{v_1} + \mathbf{v_2},\ U_2U_1)\ , \tag{12.140}$$

where $SU(2) \ni U \mapsto R(U) \in SO(3)$ is the covering homomorphism of Example 12.65(6) (see also the exercises). This Lie group is the universal covering of $S\mathscr{G}$, being simply connected (as product of simply connected spaces) and having the same Lie algebra as $S\mathscr{G}$.

An interesting basis, in physics, of the Lie algebra of $\widetilde{S\mathscr{G}}$ is given by the 10 vectors

$$-\mathbf{h}\ , \mathbf{p}_i\ , \mathbf{j}_i\ , \mathbf{k}_i \quad \text{i=1,2,3,} \tag{12.141}$$

(note the conventional $-$ sign in the first one), where:

(i) $-\mathbf{h}$ generates the one-parameter subgroup $\mathbb{R} \ni c \mapsto (c, \mathbf{0}, \mathbf{0}, I)$ of **time translations**, called **time displacement symmetry**,
(ii) the three $\mathbf{p}_i$ span the Abelian subgroup $\mathbb{R}^3 \ni \mathbf{c} \mapsto (0, \mathbf{c}, \mathbf{0}, I)$ of **space translations**,
(iii) the three $\mathbf{j}_i$ span the subgroup $SU(2) \ni U \mapsto (0, \mathbf{0}, \mathbf{0}, U)$ of **space rotations**[16],
(iv) the three $\mathbf{k}_i$ generate the Abelian subgroup $\mathbb{R}^3 \ni \mathbf{v} \mapsto (0, \mathbf{0}, \mathbf{v}, I)$ of **pure Galilean transformations**.

The generators obey commutation relations that detect the structure constants:

$$[\mathbf{p}_i, \mathbf{p}_j] = \mathbf{0}\,, \quad [\mathbf{p}_i, -\mathbf{h}] = \mathbf{0}\,, \quad [\mathbf{j}_i, -\mathbf{h}] = \mathbf{0}\,, \quad [\mathbf{k}_i, \mathbf{k}_j] = \mathbf{0}\,, \tag{12.142}$$

$$[\mathbf{j}_i, \mathbf{p}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathbf{p}_k\,, \quad [\mathbf{j}_i, \mathbf{j}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathbf{j}_k\,, \quad [\mathbf{j}_i, \mathbf{k}_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\mathbf{k}_k\,, \tag{12.143}$$

$$[\mathbf{k}_i, -\mathbf{h}] = \mathbf{p}_i\,, \quad [\mathbf{k}_i, \mathbf{p}_j] = \mathbf{0}\,. \tag{12.144}$$

The Galilean group is in all likelihood the most important group in all of classical physics, given that classical laws are invariant under the active action of this group. Galilean invariance is a way to express the equivalence of all inertial frame systems, interpreting passively the group transformations. We expect the restricted Galilean group, *seen as group of active transformations from now on*, to be a symmetry group for any quantum system in low-speed regimes (compared to the speed of light), when relativistic effects are petty.

Projective unitary $S\mathscr{G}$-representations describing the action of the symmetry group $S\mathscr{G}$ on a physical system are well understood (see [Mes99, CCP82], for example). To start discussing them, take a physical system given by a particle of spin s (cf. previous section) and mass $m > 0$, not subject to any forces. Fix an inertial frame system $\mathscr{I}$ with right-handed orthonormal Cartesian coordinates that identify the rest space with $\mathbb{R}^3$. The system's Hilbert space H is $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$. Pure states are wavefunctions with spin:

$$\sum_{|s_3| \leq s} \psi_{s_3} \otimes |s, s_3\rangle$$

The wavefunctions $\widetilde{\psi} \in L^2(\mathbb{R}^3, dk)$ are given in *momentum representation*, and are images under the unitary Fourier–Plancherel transform (cf. Chap. 3)

$$\widehat{\mathscr{F}} : L^2(\mathbb{R}^3, dx) \to L^2(\mathbb{R}^3, dk)$$

of wavefunctions ψ in *position representation*: $\widetilde{\psi} = \widehat{\mathscr{F}}\psi$. In particular (Proposition 5.31), the momentum observable P_j is given on $L^2(\mathbb{R}^3, dk)$ by the operator $\widetilde{P}_j = \widehat{\mathscr{F}} P_j \widehat{\mathscr{F}}^{-1}$, i.e. by the multiplication by $\hbar k_j$ on $L^2(\mathbb{R}^3, dk)$. From now on we set $\hbar = 1$ for simplicity. Assume $s = 0$ for a moment. In this representation of H, the

[16] Properly speaking, the rotations are the associated elements $R(U)$.

action of each element of $S\mathscr{G}$ is induced by a unitary operator $\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)}$:

$$\left(\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)}\widetilde{\psi}\right)(\mathbf{k}) := e^{i(c\mathbf{v}-\mathbf{c})\cdot(\mathbf{k}-m\mathbf{v})}e^{i\frac{c}{2m}(\mathbf{k}-m\mathbf{v})^2}\widetilde{\psi}\left(R(U)^{-1}(\mathbf{k}-m\mathbf{v})\right) \quad (12.145)$$

When $s \neq 0$, the unitary transformations $\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)}$ are replaced by

$$\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)} \otimes V^s(U)\ , \quad (12.146)$$

where V^s was introduced in (12.131).

Back in position representation, i.e. viewing pure states of a spinless particle as unit vectors in $L^2(\mathbb{R}^3, dx)$, the unitary operators $\widetilde{Z^{(m)}}_g$ correspond to unitary operators $Z_g^{(m)} := \widehat{\mathscr{F}}^{-1}\widetilde{Z^{(m)}}_g\widehat{\mathscr{F}}$ under the Fourier–Plancherel transform. In the sequel we will use the two representations without distinction, even though the explicit action of $Z_g^{(m)}$ in position representation will have to wait until the next chapter.

Remarks 12.117 **(1)** Let us evaluate the action on $(c, \mathbf{c}, \mathbf{v}, U)^{-1}$ rather than $(c, \mathbf{c}, \mathbf{v}, U)$, for this is more illuminating

$$\left(\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)^{-1}}\widetilde{\psi}\right)(\mathbf{k}) := e^{i\mathbf{c}\cdot(R(U)\mathbf{k}+m\mathbf{v})}e^{-i\frac{c}{2m}(R(U)\mathbf{k}+m\mathbf{v})^2}\widetilde{\psi}\left(R(U)\mathbf{k}+m\mathbf{v}\right)\ . \quad (12.147)$$

To give a meaning to this, decompose $(c, \mathbf{c}, \mathbf{v}, U)^{-1}$ into

$$(c, \mathbf{c}, \mathbf{v}, U)^{-1} = (0, \mathbf{0}, \mathbf{0}, U)^{-1} \cdot (0, \mathbf{0}, \mathbf{v}, I)^{-1} \cdot (0, \mathbf{c}, \mathbf{0}, I)^{-1} \cdot (c, \mathbf{0}, \mathbf{0}, I)^{-1}\ ,$$

and let us examine the single actions one by one. From the right

$$\left(\widetilde{Z^{(m)}}_{(c,0,\mathbf{0},I)^{-1}}\widetilde{\psi}\right)(\mathbf{k}) = e^{-i\frac{c}{2m}\mathbf{k}^2}\widetilde{\psi}\left(\mathbf{k}\right)\ .$$

In the next chapter we will see that multiplying by the phase $e^{-i\frac{c}{2m}\mathbf{k}^2}$ corresponds to rewinding by a time lapse c (this is the *time evolution*[17] by the same lapse). Taking in also the second one,

$$\left(\widetilde{Z^{(m)}}_{(0,\mathbf{c},\mathbf{0},I)^{-1}\cdot(c,\mathbf{0},\mathbf{0},I)^{-1}}\widetilde{\psi}\right)(\mathbf{k}) = e^{i\mathbf{c}\cdot\mathbf{k}}e^{-i\frac{c}{2m}\mathbf{k}^2}\widetilde{\psi}\left(\mathbf{k}\right)\ .$$

The multiplication by $e^{i\mathbf{c}\cdot\mathbf{k}}$ corresponds (under Fourier–Plancherel) to an active translation by $-\mathbf{c}$ of the wavefunction. Subsuming the third one, we obtain

$$\left(\widetilde{Z^{(m)}}_{(0,\mathbf{0},\mathbf{v},I)^{-1}\cdot(0,\mathbf{c},\mathbf{0},I)^{-1}\cdot(c,\mathbf{0},\mathbf{0},I)^{-1}}\widetilde{\psi}\right)(\mathbf{k}) = e^{i\mathbf{c}\cdot(\mathbf{k}+m\mathbf{v})}e^{-i\frac{c}{2m}(\mathbf{k}+m\mathbf{v})^2}\widetilde{\psi}\left(\mathbf{k}+m\mathbf{v}\right)\ .$$

[17] The inverse transformation of time displacement.

If $\mathbf{k}$ is understood as momentum vector, $\mathbf{k} \to \mathbf{k} + m\mathbf{v}$ is precisely the transformation of the momentum under a Galilean transformation that changes the velocity of the frame of reference, but does not contain space or time translations, nor rotations. The transformation corresponds to an active transformation of the wavefunction under a pure Galilean transformation associated to $-\mathbf{v}$. Eventually, incorporating the rotation $R(U)$, i.e. actively transforming the wavefunction by $R(U)^{-1}$, gives:

$$\left(\widetilde{Z^{(m)}}_{(0,\mathbf{0},\mathbf{0},U)^{-1}\cdot(0,\mathbf{0},\mathbf{v},I)^{-1}\cdot(0,\mathbf{c},\mathbf{0},I)^{-1}\cdot(c,\mathbf{0},\mathbf{0},I)^{-1}}\widetilde{\psi}\right)(\mathbf{k}) =$$

$$e^{i\mathbf{c}\cdot(R(U)\mathbf{k}+m\mathbf{v})}e^{-i\frac{c}{2m}(R(U)\mathbf{k}+m\mathbf{v})^2}\,\widetilde{\psi}\,(R(U)\mathbf{k} + m\mathbf{v})\ .$$

Overall the right-hand side of (12.147) corresponds to the *combined* action (in agreement with the Galilean product) of the subgroups of transformations. Bearing in mind (12.138) our discussion now justifies (12.145).

(2) The operators $\widetilde{Z^{(m)}}_g$ (i.e. the $Z^{(m)}{}_g$, in the position representation) are associated to the universal covering $\widetilde{S\mathscr{G}}$ rather than the group $S\mathscr{G}$ itself. We made this choice in order to apply the theory of previous sections. We know, in fact, that projective representations of a group are obtained from the universal covering's projective representations, and this is particularly convenient because the Galilean group contains a subgroup isomorphic to $SO(3)$. We saw in the previous section that if the spin s is a semi-integer, the projective unitary $SO(3)$-representations of physical interest are unitary $SU(2)$-representations. ■

Using Definition (12.145), the representation $\widetilde{S\mathscr{G}} \ni g \mapsto Z^{(m)}_g$ (equivalently, $\widetilde{S\mathscr{G}} \ni g \mapsto \widetilde{Z^{(m)}}_g$ working in the momentum representation) is *projective* unitary, due to the presence of a multiplier function

$$\omega^{(m)}(g', g) = e^{im\left(-\frac{1}{2}c'\mathbf{v}^2 - c'(R(U')\mathbf{v})\cdot\mathbf{v}' + (R(U')\mathbf{v})\cdot\mathbf{c}'\right)}\ , \quad g = (c, \mathbf{c}, \mathbf{v}, U), g' = (c', \mathbf{c}', \mathbf{v}', U') \tag{12.148}$$

after a boring computation. The result (clearly) remains valid in case the spin s is non-zero, and the unitary operators $\widetilde{Z^{(m)}_g}$ generalise to the unitary operators (12.146), because the representation $U \mapsto V^s(U)$ on the spin space $\mathbb{C}^{2s+1}$ is unitary and does not affect the multiplier function.

It is easy to prove the projective unitary representation $\widetilde{S\mathscr{G}} \ni g \mapsto \widetilde{Z^{(m)}}_g$ (equivalently $\widetilde{S\mathscr{G}} \ni g \mapsto Z^{(m)}{}_g$ in the position representation) is strongly continuous. To that end, as operators are unitary, $\omega^{(m)}$ is continuous and $\omega^{(m)}(e, e) = 1$, it suffices to prove $\widetilde{Z^{(m)}}_g\widetilde{\psi} \to \widetilde{\psi}$ as $g \to e$, for any $\psi \in \mathsf{H}$. This is an easy consequence of the explicit form of $\widetilde{Z^{(m)}}_g$.

We do not know whether the projective unitary representation $\widetilde{S\mathscr{G}} \ni g \mapsto Z^{(m)}{}_g$ is equivalent to a unitary representation, by multiplying $Z^{(m)}{}_g$ by suitable phases $\chi(g)$. The Galilean Lie algebra shows that Bargmann's Theorem 12.72 does *not* hold. But the aforementioned theorem gives sufficient conditions, not necessary ones, so we are not in a position to answer the question. What we will see now is that the representations found are intrinsically projective: they cannot be made unitary by a clever choice of phase.

In order to keep general, we consider *every* possible projective unitary representation $\widetilde{\mathscr{SG}} \ni g \mapsto Z_g^{(m)}$, on any Hilbert space, with multipliers as in (12.148), but irrespective of the fact the $Z_g^{(m)}$ are as in (12.145) or (12.146) on $L^2(\mathbb{R}^3, dk) \otimes \mathbb{C}^{2s+1}$.

Proposition 12.118 *Let $\widetilde{\mathscr{SG}} \ni g \mapsto Z_g^{(m)}$ be projective unitary representations with multipliers (12.148) and $m \in \mathbb{R}$.*
(a) *Given $m \neq 0$, it is not possible to define the phases of $Z_g^{(m)}$ to obtain a unitary $\widetilde{\mathscr{SG}}$-representation (nor strongly continuous).*
(b) *Representations with distinct numbers m cannot belong to the same unitary equivalence class.*

Proof We prove (a) and (b) simultaneously. If two representations with $m_1 > m_2$ belong to the same equivalence class, there exists a map $\chi = \chi(g)$ such that

$$\omega^{(m_1)}(g', g)\left(\omega^{(m_2)}(g', g)\right)^{-1} = \frac{\chi(g' \cdot g)}{\chi(g')\chi(g)}, \quad g, g' \in \widetilde{\mathscr{SG}}. \tag{12.149}$$

Writing $m := m_1 - m_2$, this is the same as

$$\omega^{(m)}(g', g) = \frac{\chi(g' \cdot g)}{\chi(g')\chi(g)}, \quad g, g' \in \widetilde{\mathscr{SG}}. \tag{12.150}$$

We claim that for any given $m \neq 0$ there is no function χ satisfying (12.150), proving the theorem.

By contradiction if such a χ existed, letting $V_g := \chi(g)Z_g^{(m)}$ would force the multipliers of $\widetilde{\mathscr{SG}} \ni g \mapsto V_g$ to be 1, hence the representation would be unitary. Consider the elements in $\widetilde{\mathscr{SG}}$ of the form $f := (0, \mathbf{c}, \mathbf{0}, I)$ and $g := (0, \mathbf{0}, \mathbf{v}, I)$. By (12.137) they commute, so $f^{-1} \cdot g^{-1} \cdot f \cdot g = e$. The corresponding computation for $Z^{(m)}$, keeping (12.145) in account, gives $Z_{f^{-1}}^{(m)} Z_{g^{-1}}^{(m)} Z_f^{(m)} Z_g^{(m)} = e^{-i2m\mathbf{c}\cdot\mathbf{v}} Z_e^{(m)}$. This becomes, with our assumptions:

$$\left(\chi(f^{-1})\chi(g^{-1})\chi(f)\chi(g)\right)^{-1} V_{f^{-1}} V_{g^{-1}} V_f V_g = e^{-i2m\mathbf{c}\cdot\mathbf{v}} \chi(e)^{-1} I \,;$$

as the multipliers of V are trivial because V is unitary by assumption, we have $f \cdot g = g \cdot f$, and permuting the order of the coefficients χ_h:

$$\left(\chi(f^{-1})\chi(f)\chi(g^{-1})\chi(g)\right)^{-1} V_{f^{-1}\cdot f \cdot g^{-1}\cdot g} = \left(\chi(f^{-1})\chi(f)\chi(g^{-1})\chi(g)\right)^{-1} V_e$$
$$= e^{-i2m\mathbf{c}\cdot\mathbf{v}} \chi(e)^{-1} I \,.$$

Therefore

$$\frac{\chi(f^{-1} \cdot f)}{\chi(f^{-1})\chi(f)} \frac{\chi(g^{-1} \cdot g)}{\chi(g^{-1})\chi(g)} = \chi(e) e^{-i2m\mathbf{c}\cdot\mathbf{v}} \,.$$

Using (12.150) this identity becomes $\omega(f, f^{-1})\omega(g, g^{-1}) = \chi(e)e^{-i2m\mathbf{c}\cdot\mathbf{v}}$. Computing the left-hand side explicitly, with the help of (12.148), yields

$$1 = \chi(e)e^{-i2m\mathbf{c}\cdot\mathbf{v}}.$$

This has to be true for any $\mathbf{c}, \mathbf{v} \in \mathbb{R}^3$, hence $m = 0$ and $\chi(e) = 1$. But $m = 0$ was excluded right from the start. The contradiction invalidates the initial assumption, so χ does not exist. □

By virtue of this proposition, and since the quantity m labelling equivalence classes of projective unitary representations has a very explicit physical meaning (for $m > 0$), we might think that the symmetry group of a non-relativistic quantum system of mass m, instead of being the Galilean group, is the central extension $\widehat{\widetilde{S\mathscr{G}}}_m$ given by the multiplier function of the value m of the mass. From the general theory the representation $\widetilde{S\mathscr{G}} \ni g \mapsto Z_g^{(m)}$ arises thus: (a) build the central $U(1)$-extension $\widehat{\widetilde{S\mathscr{G}}}_m$, with multiplier function (12.148) ($\widehat{\widetilde{S\mathscr{G}}}_m$ is a product manifold since $\omega^{(m)}$ is analytic on $\widetilde{S\mathscr{G}} \times \widetilde{S\mathscr{G}}$); (b) restrict to $\widetilde{S\mathscr{G}}$ the strongly continuous unitary representation

$$\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi Z_g^{(m)} .$$

On that account, (intrinsically) projective unitary $\widetilde{S\mathscr{G}}$-representations are substituted by unitary $\widehat{\widetilde{S\mathscr{G}}}_m$-representations. There is a price to pay: the symmetry group changes when the mass varies. Consider the strongly continuous unitary representation

$$\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi Z_g^{(m)} .$$

Restrict to the space $\mathscr{D} \subset L^2(\mathbb{R}^3, dk)$ of smooth complex functions $\widetilde{\psi} = \widetilde{\psi}(\mathbf{k})$ with compact support. By (12.145) every map

$$\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi \widetilde{Z^{(m)}}_g \widetilde{\psi}$$

is smooth whenever $\widetilde{\psi} \in \mathscr{D}$. Hence $\mathscr{D}$ is contained in the Gårding space of $\widehat{\widetilde{S\mathscr{G}}}_m$. With a minor notational misuse we indicate by $\mathscr{D}$ the inverse Fourier–Plancherel image of $\mathscr{D}$ inside $L^2(\mathbb{R}^3, dx)$. Consider the 11 one-parameter Lie subgroups of $\widehat{\widetilde{S\mathscr{G}}}_m$ generated by the Lie algebra basis:

$$1 \oplus \mathbf{0},\ -0 \oplus \mathbf{h},\ 0 \oplus \mathbf{p}_i,\ 0 \oplus \mathbf{j}_i,\ 0 \oplus \mathbf{k}_i\ , \quad i = 1, 2, 3\ .$$

Composing each one with $\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi Z_g^{(m)}$ produces eleven strongly continuous one-parameter unitary groups. Let us find their self-adjoint generators. If we *restrict to* $\mathscr{D}$ when differentiating in the strong topology, the generators are (note the $-$ sign of H):

$$I, \ -H\restriction_{\mathscr{D}}, \ P_i\restriction_{\mathscr{D}}, \ L_i\restriction_{\mathscr{D}}, \ K_i\restriction_{\mathscr{D}}, \quad i = 1, 2, 3.$$

Above, P_k and L_k are the self-adjoint operators representing momentum and orbital angular momentum about the kth axis, which we met already. The self-adjoint operators $H := \widehat{\mathscr{F}}^{-1}\widetilde{H}\widehat{\mathscr{F}}$, called **Hamiltonian operator**, and K_i, called **boost** along the ith axis, are defined as:

$$(\widetilde{H}\widetilde{\psi})(\mathbf{k}) := \frac{\mathbf{k}^2}{2m}\widetilde{\psi}(\mathbf{k}) \quad \text{where} \quad D(\widetilde{H}) := \left\{\widetilde{\psi} \in L^2(\mathbb{R}^3, dk) \,\middle|\, \int_{\mathbb{R}^3} |\mathbf{k}|^4 |\widetilde{\psi}(\mathbf{k})|^2 dk < +\infty\right\} \tag{12.151}$$

and

$$K_j := mX_j \ . \tag{12.152}$$

Since $\mathscr{D}$ is a core for all of the above, the self-adjoint generators of one-parameter group representations of $\widehat{S\mathscr{G}}_m$ associated to:

$$1 \oplus \mathbf{0}, \ -0 \oplus \mathbf{h}, \ 0 \oplus \mathbf{p}_i, \ 0 \oplus \mathbf{j}_i, \ 0 \oplus \mathbf{k}_i \ , \quad i = 1, 2, 3 \tag{12.153}$$

must coincide with the corresponding:

$$I, \ -H, \ P_i, \ L_i, \ K_i \ , \quad i = 1, 2, 3.$$

Each one, as an observable, has a physical meaning. We will talk about the observable H in the next chapter. By considering Lie algebra relations, for instance on $\mathscr{D}$, we realise we are actually working with a central extension of the Galilean group, because one bracket (the last one below) is new: the fault is of a *central charge* that is represented by the mass:

$$[-iP_i, -iP_j] = 0 \ , \quad [-iP_i, iH] = 0 \ , \quad [-iL_i, iH] = 0 \ , \quad [-iK_i, -iK_j] = \mathbf{0} \ ,$$

$$[-iL_i, -iP_j] = \sum_{k=1}^{3} \varepsilon_{ijk}(-iP_k) \ , \quad [-iL_i, -iL_j] = \sum_{k=1}^{3} \varepsilon_{ijk}(-iL_k) \ ,$$

$$[-iL_i, -iK_j] = \sum_{k=1}^{3} \varepsilon_{ijk}(-iK_k), \quad [-iK_i, iH] = -iP_i, \quad [-iK_i, -iP_j] = -m\delta_{ij}(-iI).$$

Referring to (12.88), the central extension we have found is therefore determined by

$$\alpha(\mathbf{k}_i, \mathbf{p}_j) = -\alpha(\mathbf{p}_j, \mathbf{k}_i) = -m\delta_{ij} \ ,$$

and $\alpha(\mathbf{a}, \mathbf{b}) = 0$ in all remaining cases with $\mathbf{a}, \mathbf{b}$ ranging in the basis (12.141). It is possible to prove that this α (with $m \neq 0$) does not comply with Bargmann's Theorem 12.72.

Remarks 12.119 **(1)** We started from a precise central extension of $\widetilde{\mathscr{G}}$ based on physical requirements. Our results, however, are general and Proposition 12.118 holds for every projective unitary representation of $\widetilde{\mathscr{G}}$. By Remark 12.73(2)–(5), in fact, the direct inspection of $H^2(T_e\mathscr{G}, \mathbb{R})$ proves [Bar54] that (a) every equivalence class in $H^2(T_e\mathscr{G}, \mathbb{R})$ contains a representative whose function α has the above form for a precise value of $m \in \mathbb{R}$; (b) representatives with different m are inequivalent, i.e. define different classes of $H^2(T_e\mathscr{G}, \mathbb{R})$. In other words, $m \in \mathbb{R}$ labels the elements of $H^2(T_e\mathscr{G}, \mathbb{R})$ bijectively, and consequently also the different (inequivalent) continuous projective unitary representations of $\widetilde{\mathscr{G}}$. Having said that, physically speaking the values $m \leq 0$ have no meaning. Only $m = 0$ (as of yet, still 'unphysical') gives rise to proper unitary representations.
(2) Since $K_j = mX_j$, the unitary representation $(\chi, g) \mapsto \chi \widetilde{Z^{(m)}}_g$ incorporates operators that obey Weyl's relations on $L^2(\mathbb{R}^3, dk)$. By Proposition 11.39(b) $L^2(\mathbb{R}^3, dk)$ is irreducible for these operators, hence for the representation $\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi \widetilde{Z^{(m)}}_g$. In this sense the non-relativistic spinless quantum particle is an *elementary object* for the Galilean group.
(3) If we take into account the portion of Hilbert space due to the spin, the difference from above is that to have $\widehat{\widetilde{S\mathscr{G}}}_m$ act on states we must replace L_k by $J_k = L_k + S_k$ in every formula. That is to say, the unitary representation reads

$$\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi \widetilde{Z^{(m)}}_g \otimes V^s(U) ,$$

where $g = (c, \mathbf{c}, \mathbf{v}, U)$. The irreducibility seen for the case $s = 0$ extends, so for the particle with spin s the above representation is irreducible on $L^2(\mathbb{R}^3, dk) \otimes \mathbb{C}^{2s+1}$. ■

12.3.4 *Bargmann's Rule of Superselection of the Mass*

Now we shall consider systems more complicated than a free particle. We refer to the next chapter for the general matter, and recall here that when we study an isolated system of N interacting particles of masses $m_1, \ldots, m_N$, the theory's Hilbert space splits:

$$L^2(\mathbb{R}^3, dx) \otimes \mathsf{H}_{int} \otimes \mathbb{C}^{2s_1+1} \otimes \cdots \otimes \mathbb{C}^{2s_N+1} .$$

The Hilbert space H_{int} is relative to the system's internal orbital degrees of freedom (the particles mutual positions, for example in terms of *Jacobi coordinates*, e.g. see [AnMo12] for a more explicit construction). $L^2(\mathbb{R}^3, dx)$ is the Hilbert space of the *centre of mass*. The centre of mass of the system is the unique particle of mass $M := \sum_{n=1}^N m_n$ determined by the observables X_k (the position of the centre of mass) and P_k (total momenta of the system), $k = 1, 2, 3$, of the usual form on $L^2(\mathbb{R}^3, dx)$.

Each factor $\mathbb{C}^{2s_n+1}$ is the spin space of one particle. Via Fourier transform $L^2(\mathbb{R}^3, dx)$ can be seen as $L^2(\mathbb{R}^2, dk)$, which we will assume from now on.

In this context – exactly as in classical mechanics – the symmetry group $S\mathscr{G}$ acts by

$$\widetilde{S\mathscr{G}} \ni (c, \mathbf{c}, \mathbf{v}, U) \;\; \mapsto \;\; Z^{(M)}_{(c,\mathbf{c},\mathbf{v},U)} \otimes V^{(int)}_{R(U)} W^{(int)}_c \otimes V^{S_1}(U) \otimes \cdots \otimes V^{S_N}(U)\,.$$

Above,

$$SO(3) \ni R \mapsto V^{(int)}_R \quad \text{and} \quad \mathbb{R} \ni c \mapsto W^{(int)}_c$$

are representations – both *unitary* and strongly continuous – of the rotation subgroup of $S\mathscr{G}$ (of elements $(0, \mathbf{0}, \mathbf{0}, R)$), and of time translations (of type $(c, \mathbf{0}, \mathbf{0}, I)$) respectively. In addition, $V^{(int)}_R W^{(int)}_c = W^{(int)}_c V^{(int)}_R$ for every $R \in SO(3)$, $c \in \mathbb{R}$. These two representations depend on how we define orbital coordinates and on the kind of inner interactions among the particles. The transformation $Z^{(M)}_{(c,\mathbf{c},\mathbf{v},U)}$ acts only on the freedom degrees of the centre of mass. Since every representation involved is unitary except $Z^{(M)}$, the multiplier function $\omega^{(M)}$ of the overall representation on $L^2(\mathbb{R}^3, dk) \otimes \mathsf{H}_{int} \otimes \mathbb{C}^{2s_1+1} \otimes \cdots \otimes \mathbb{C}^{2s_N+1}$ is the same we had before, using the total mass M as parameter m. Therefore the previous proposition reaches to this much more general instance of quantum system.

Let us look at a physical system S obtained by putting together a finite number, though *not fixed*, of the previous systems. Or even more generally, we may assume that the value of the mass of S, for some reason, is *not fixed*. The mass of S may then range over several values m_i, with $i \in I$ at most countable. It is only natural to associate to the mass a quantum observable, i.e. a self-adjoint operator M whose spectrum is the values of mass (even if all that follows is completely general, explicit models have been constructed in [Giu96, AnMo12]). Likewise, we define a Hilbert space for the system:

$$\mathsf{H}_S = \bigoplus_{m \in \sigma(M)} \mathsf{H}^{(m)}_S\,,$$

where the $\mathsf{H}^{(m)}_S$ are the eigenspaces of the mass operator with distinct eigenvalues $m > 0$. What happens if the Galilean group acts on S? A different projective unitary representation $Z^{(m)}$, depending on m, will act on each $H^{(m)}_S$. The representation of the restricted Galilean group will thus look like:

$$S\mathscr{G} \ni g \mapsto Z_g := \bigoplus_{m \in \sigma(M)} \chi^{(m)}(g) Z^{(m)}_g\,. \tag{12.154}$$

We claim this structure leads to a superselection rule. Since the representation is projective unitary, the multiplier

$$\Omega(g, g') := Z(gg')^{-1} Z(g) Z(g')\,,$$

computed using (12.154), produces

$$\Omega(g, g')I = \bigoplus_{m\in\sigma(M)} \omega^{(m)}(g, g')I_m ,$$

where the $\omega^{(m)}$ account for possible new phases $\chi^{(m)}$ and the I_m on the right are the identities on each $\mathsf{H}_S^{(m)}$. Since

$$I = \bigoplus_{m\in\sigma(M)} I_m ,$$

so

$$\Omega(g, g')I = \bigoplus_{m\in\sigma(M)} \Omega(g, g')I_m ,$$

necessarily we have:

$$\omega^{(m_1)}(g, g') = \omega^{(m_2)}(g, g') = \Omega(g, g') \quad \text{for every } m_1, m_2 \in \sigma(M) .$$

Bu this is not possible, because it implies, solving for $\chi^{(m)}$, the false relation (12.149).

The net result is this: if the Galilean group is to be a symmetry group for our physical system, we are compelled to ban pure states arising from combinations of different subspaces $\mathsf{H}_S^{(m)}$. Therefore we have found a superselection rule related to the mass, known as **Bargmann's superselection rule for the mass**. The coherent sectors of this rule are the summands $\mathsf{H}_S^{(m)}$ with given mass. The result is deeply rooted in the fact that physically-interesting projective representations of the Galilean group do not come from unitary representations, and the mass appears in the multiplier function.

A physically more appropriate situation is that in which one replaces the restricted Galilean group with the (proper orthochronous) Poincaré group: then the superselection rule ceases to hold, because irreducible projective representations of the Poincaré group always arise from irreducible unitary representations [Var07], and states with indefinite (relativistic) mass are allowed.

Remark 12.120 Since m multiplies the exponent in (12.148), we may introduce a central extension $\mathscr{G}_1$ of the Galilean group (of the universal covering to be precise) that does not depend on m. It is enough to redefine the multiplier by setting $m = 1$ in the right-hand side of (12.148). The value of the mass is subsequently fixed by a particular unitary representation (raising the multiplier and the variables $\chi \in U(1)$ to the mth power) when a physical system is chosen to have that mass and to be invariant by the Galilean group. This extension $\mathscr{G}_1$ should be considered as the quantum version of the Galilean group. This approach, adopted in [Giu96], lets the superselection rule of the mass arise dynamically, by enlarging the system with more degrees of freedom, already at the classical level. The mass becomes, a priori, a (classical) variable and defines a self-adjoint operator (the mass operator of the physical system) after quantisation. The approach was improved in [AnMo12], in particular by presenting a physical procedure giving rise to the superselection rule once the mass spectrum is supposed discrete. ■

Exercises

12.1 Prove Theorem 12.23.

Proposition. *If* $\mathsf{H} \neq \{\mathbf{0}\}$ *is a separable complex Hilbert space of dimension* $\neq 2$, *an ortho-automorphisms of* $\mathscr{L}(\mathsf{H})$ *is of the form*

$$\mathscr{L}(\mathsf{H}) \ni P \mapsto VPV^{-1} \in \mathscr{L}(\mathsf{H})$$

for a given unitary or anti-unitary $V : \mathsf{H} \to \mathsf{H}$. *For* $\dim \mathsf{H} > 1$, V *is determined by the automorphism, up to phase.*

Solution. We first assume $\dim \mathsf{H} > 1$. If $h : \mathscr{L}(\mathsf{H}) \to \mathscr{L}(\mathsf{H})$ is an automorphism, the map $h_* : \mathfrak{S}(\mathsf{H}) \ni \rho \mapsto h_*(\rho) \in \mathfrak{B}(\mathsf{H})$, $h_*(\rho)(P) = \rho(h(P))$ for $P \in \mathscr{L}(\mathsf{H})$, is a Kadison symmetry. In fact, $h_*(\rho) \in \mathfrak{S}(\mathsf{H})$ easily follows from the fact that h is an ortho-automorphism of $\mathscr{L}(\mathsf{H})$. Moreover, if $\rho, \rho' \in \mathfrak{S}(\mathsf{H})$ and $p + q = 1$ for $p, q \in [0, 1]$, one has $h_*(p\rho + q\rho')(P) = (p\rho + q\rho')(h(P)) = p\rho(h(P)) + q\rho'(h(P)) = ph_*(\rho)(P) + qh_*(\rho')(P)$, so that h_* is a Kadison automorphism. By Kadison's theorem, there exists $U : \mathsf{H} \to \mathsf{H}$ unitary or anti-unitary depending on h_*, such that $h_*(\rho) = U\rho U^{-1}$ (viewing ρ as a trace-class positive operator with unit trace). Consequently $tr(\rho(h(P) - U^{-1}PU)) = 0$ for every $P \in \mathscr{L}(\mathsf{H})$ and every ρ. In particular, $(\psi|\rho(h(P) - U^{-1}PU)\psi) = 0$ for every $\psi \in \mathsf{H}$. Consequently, $h(P) = U^{-1}PU$, for every $P \in \mathscr{L}(\mathsf{H})$, which is what we wanted (just rename $V := U^{-1}$). Finally, suppose that V and V' (both unitary or anti-unitary) are associated to the same h. Then $V'V^{-1}$ commutes with every one-dimensional orthogonal projector. The standard argument used in the proof of Wigner's theorem shows that $V' = e^{ia}V$ for some $a \in \mathbb{R}$. *Vice versa*, it is obvious that $V' = e^{ia}V$ and V, either unitary or anti-unitary, define the same ortho-automorphism $h_{V'}$ given by $\mathscr{L}(\mathsf{H}) \ni P \mapsto VPV^{-1} =: h_V(P)$.

If $\dim \mathsf{H} = 1$ the identity is the only automorphism. It will be unitary or anti-unitary depending on whether we extend $\psi \mapsto \psi$ linearly or antilinearly.

12.2 Referring to Example 12.19(1), with $IO(3) \ni \Gamma = (\mathbf{t}, R)$, prove

$$\gamma^*_\Gamma(\mathbf{P}) = U_\Gamma^{-1}\mathbf{P}U_\Gamma = R\mathbf{P}\,, \tag{12.155}$$

where $\mathbf{P}$ is the triple of operators corresponding to the components of momentum, and the relation holds on the Schwartz space $\mathscr{S}(\mathbb{R})$, taken as domain of the momenta.

12.3 Referring to Examples 12.19(1) and (2) and retaining the convention of Exercise 12.2, prove:

$$\gamma^*_{\mathscr{P}}(\mathbf{X}) = \mathscr{P}^{-1}\mathbf{X}\mathscr{P} = -\mathbf{X}\,, \quad \gamma^*_{\mathscr{P}}(\mathbf{P}) = \mathscr{P}^{-1}\mathbf{P}\mathscr{P} = -\mathbf{P} \tag{12.156}$$

while

$$\gamma^*_{\mathscr{T}}(\mathbf{X}) = \mathscr{T}^{-1}\mathbf{X}\mathscr{T} = \mathbf{X}\,, \quad \gamma^*_{\mathscr{T}}(\mathbf{P}) = \mathscr{T}^{-1}\mathbf{P}\mathscr{T} = -\mathbf{P}\,. \tag{12.157}$$

The triple $\mathbf{P}$ corresponds to the momentum's components, and the identities hold on the Schwartz space $\mathscr{S}(\mathbb{R})$, domain of the position and momentum operators.

12.4 Prove that $\mathscr{P}$ and $\mathscr{T}$ are only defined up to a phase. In other words, a unitary operator U_P satisfying

$$U_P^{-1}\mathbf{X}U_P = -\mathbf{X}\,, \quad U_P^{-1}\mathbf{P}U_P = -\mathbf{P}\,, \tag{12.158}$$

will also satisfy $U_P = e^{ia}\mathscr{P}$ for a real constant $a \in \mathbb{R}$ ($a = \pm 1$ if, additionally, $U_P^* = U_P$). Similarly, an anti-unitary operator U_T satisfying

$$U_T^{-1}\mathbf{X}U_T = \mathbf{X}\,, \quad U_T^{-1}\mathbf{P}U_T = -\mathbf{P} \tag{12.159}$$

will also satisfy $U_T = e^{ib}\mathscr{T}$ for some real constant $b \in \mathbb{R}$.

Hint. Prove that $\mathscr{P}U_P$ and $\mathscr{T}U_T$ are unitary operators commuting with the Weyl operators $W((\mathbf{t}, \mathbf{u}))$ of the Weyl algebra associated with the position and momentum operators of our particle, as in Proposition 11.39. Then observe that the family of the $W((\mathbf{t}, \mathbf{u}))$ is irreducible (Proposition 11.39) and use Proposition 11.37.

12.5 Consider the self-adjoint operators L_1, L_2, L_3 representing the components of the *orbital angular momentum* (Chap. 10). If $\mathbf{L}$ indicates their column vector, then

$$\mathbf{L}\restriction_{\mathscr{S}(\mathbb{R}^3)} = \mathbf{X}\restriction_{\mathscr{S}(\mathbb{R}^3)} \wedge \mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)}\ .$$

Restrict domains to $\mathscr{S}(\mathbb{R}^3)$ and prove the following facts. Referring to Example 12.19(1), with $SO(3) \ni \Gamma = (\mathbf{0}, R)$:

$$\gamma^*_{\Gamma}(\mathbf{L}) = U_\Gamma^{-1}\mathbf{L}U_\Gamma = R\mathbf{L}\,, \tag{12.160}$$

$$\gamma^*_{\mathscr{P}}(\mathbf{L}) = \mathscr{P}^{-1}\mathbf{L}\mathscr{P} = \mathbf{L}\,, \tag{12.161}$$

$$\gamma^*_{\mathscr{T}}(\mathbf{L}) = \mathscr{T}^{-1}\mathbf{L}\mathscr{T} = -\mathbf{L}\,. \tag{12.162}$$

Recall $SO(3)$ is the subgroup in $O(3)$ with determinant $+1$, and the wedge product $\wedge$ is defined by the above formal determinant in a right-handed basis.

12.6 Decompose the Hilbert space H_S of a system S in coherent sectors, so that the space of admissible pure states reads:

$$\mathfrak{S}_p(\mathsf{H}_S)_{adm} = \bigsqcup_{k \in K} \mathfrak{S}_p(\mathsf{H}_{Sk}) \,.$$

Equip $\mathfrak{S}_p(\mathsf{H}_S)$ with distance $d(\rho, \rho') := ||\rho - \rho'||_1 := tr(|\rho - \rho'|)$, where $||\ \ ||_1$ is the natural trace-class norm. Prove the sets $\mathfrak{S}_p(\mathsf{H}_{Sk})$ are the connected components of $\mathfrak{S}_p(\mathsf{H}_S)_{adm}$. (It might be useful to recall $\rho = \psi(\psi|\ )$, $\rho' = \psi'(\psi'|\ )$ in $\mathfrak{S}_p(\mathsf{H}_{Sk})$ imply $||\rho - \rho'||_1 = 2\sqrt{1 - |(\psi|\psi')|^2}$, as was proved in the chapter).

Sketch of the solution. Consider pure states $\rho, \rho' \in \mathfrak{S}_p(\mathsf{H}_{Sk})$ with $\rho = \psi(\psi|\ )$ and $\rho' = \psi'(\psi'|\ )$, and ψ not parallel to ψ' (otherwise they give the same state). Define $\psi_t = t\psi + (1-t)\psi'$ and prove the curve $[0,1] \ni t \mapsto \frac{\psi_t}{||\psi_t||^2}(\psi_t|\ )$ is continuous and entirely contained in $\mathfrak{S}_p(\mathsf{H}_{Sk})$. This makes $\mathfrak{S}_p(\mathsf{H}_{Sk})$ path-connected, hence connected. To prove the $\mathfrak{S}_p(\mathsf{H}_{Sk})$ are disjoint, it is sufficient to find $||\rho - \rho'||_1$ for $\rho \in \mathfrak{S}_p(\mathsf{H}_{Sk})$, $\rho' \in \mathfrak{S}_p(\mathsf{H}_{Sk'})$ with $k \neq k'$. In that case the vectors of ρ, ρ' are orthogonal, so $\rho - \rho'$ is actually the sum of the positive and negative parts of the element $\rho - \rho'$ itself. Hence $|\rho - \rho'| = \rho + \rho'$, i.e. $||\rho - \rho'||_1 = 2$. Consider an open set $A_k \supset \mathfrak{S}_p(\mathsf{H}_{Sk})$ union of open balls of radius $1/2$ centred in $\mathfrak{S}_p(\mathsf{H}_{Sk})$, and another open set $A_{k'} \supset \mathfrak{S}_p(\mathsf{H}_{Sk'})$ union of similar balls centred in $\mathfrak{S}_p(\mathsf{H}_{Sk'})$. These two sets cannot intersect by the triangle inequality, so $\mathfrak{S}_p(\mathsf{H}_{Sk})$ and $\mathfrak{S}_p(\mathsf{H}_{Sk'})$ are disjoint.

12.7 Prove the distance $d(\rho, \rho')$ of pure states (Exercise 12.6) satisfies:

$$d\left(\psi(\psi|\ ), \psi'(\psi|\ )\right) = \left|\left|\psi(\psi|\ ) - \psi'(\psi'|\ )\right|\right|_{\mathfrak{B}(\mathsf{H})}$$

for any unit vectors $\psi, \psi' \in \mathsf{H}$, where $||\ \ ||_{\mathfrak{B}(\mathsf{H})}$ is the standard operator norm.

12.8 Let $U : \mathsf{H} \to \mathsf{H}$ be an anti-unitary operator on the Hilbert space H and $A : D(A) \to \mathsf{H}$ a self-adjoint operator on H. Prove:

(a) $U^{-1}AU : U^{-1}(D(A)) \to \mathsf{H}$ is self-adjoint,

(b) $\sigma(U^{-1}AU) = \sigma(A)$,

(c) $\mathscr{B}(\mathbb{R}) \ni E \mapsto U^{-1}P_E^{(A)}U$ is the spectral measure associated to $U^{-1}AU$ by the spectral theorem:

$$U^{-1}\int_{\mathbb{R}} \lambda dP^{(A)}(\lambda)U = \int_{\mathbb{R}} \lambda d(U^{-1}P^{(A)}U)(\lambda)\,,$$

(d) $U^{-1}e^{itA}U = e^{-itU^{-1}AU}$.

Hint. (a) and (b) descend from the definitions of self-adjointness and spectrum. (c) follows from proving that $U^{-1}\int_{\mathbb{R}} f(x)dP^{(A)}(x)U = \int_{\mathbb{R}} f(x)d(U^{-1}P^{(A)}U)(x)$ for bounded maps $f : \mathbb{R} \to \mathbb{C}$. This comes directly from the definition of *integral of a bounded map* in a PVM (Chap. 8). Observing that any self-adjoint operator satisfies $T = \text{s-lim}_{n\to+\infty} \int_{\mathbb{R}} \chi_{[-n,n]}(x)dP^{(T)}(x)$, Stone's theorem and (a) imply (d).

12.9 Prove formula (12.74).

Outline of the solution. The first equality in (12.74) descends from U_Γ unitary, $U_{\Gamma_0}^{-1} = U_{\Gamma_0^{-1}}$ and $U_{\Gamma'}U_\Gamma = U_{\Gamma'\circ\Gamma}$. Hence it is enough to show, for any $\psi \in L^2(\mathbb{R}^3, dx)$:

$$||U_\Gamma\psi - \psi|| \to 0 \quad \text{as } \Gamma \to (\mathbf{0}, I).$$

Let us prove this for compactly-supported continuous maps ϕ. As $ISO(3) \times \mathbb{R}^3 \ni (\Gamma, \mathbf{x}) \mapsto \phi(\Gamma^{-1}\mathbf{x})$ is continuous, if Γ restricts to a relatively compact neighbourhood J of the identity, there is $K \geq 0$ such that $|\phi(\Gamma^{-1}\mathbf{x})| \leq K$ if $(\Gamma, \mathbf{x}) \in J \times \mathbb{R}^3$. Because of Γ there is a compact set $S \subset \mathbb{R}^3$ containing every support of $\phi(\Gamma^{-1}\cdot)$. So there is $\phi_0 \in L^2(\mathbb{R}^3, dx)$ such that $|(U_\Gamma\phi)(\mathbf{x}) - \phi(\mathbf{x})| \leq |\phi_0(\mathbf{x})|$ if $(\Gamma, \mathbf{x}) \in J \times \mathbb{R}^3$: it suffices to choose a continuous map ϕ_0 with absolute value larger than $2K$ at each point of S, and vanishing fast outside S. Since $(U_\Gamma\phi)(\mathbf{x}) \to \phi(\mathbf{x})$ pointwise, by dominated convergence $||U_\Gamma\psi - \psi|| \to 0$ as $\Gamma \to (\mathbf{0}, I)$, in L^2 norm. Let us pass to ψ generic in $L^2(\mathbb{R}^3, dx)$. If $\varepsilon > 0$, take ϕ continuous with compact support and such that $||\psi - \phi|| < \varepsilon/3$. Then

$$||U_\Gamma\psi - \psi|| \leq ||U_\Gamma\psi - U_\Gamma\phi|| + ||U_\Gamma\phi - \phi|| + ||\phi - \psi|| = ||U_\Gamma\phi - \phi|| + 2||\phi - \psi||,$$

since U_Γ is isometric so $||U_\Gamma\psi - U_\Gamma\phi|| = ||\psi - \phi||$. Choose Γ close enough to $(\mathbf{0}, I)$. By the above argument, $||U_\Gamma\phi - \phi|| \leq \varepsilon/3$. Hence for any $\varepsilon > 0$, if Γ is close enough to $(\mathbf{0}, I)$ we have $||U_\Gamma\psi - \psi|| \leq \varepsilon$.

12.10 Using Exercise 12.2, prove $\mathbf{t} \cdot \mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)}$ is essentially self-adjoint.

Hint. If $\mathbf{t} = \mathbf{0}$ the claim is trivial. Otherwise we know $P_1\restriction_{\mathscr{S}(\mathbb{R}^3)}$ is essentially self-adjoint. Consider the unitary operator U_R representing an active rotation moving the axis $\mathbf{t}/|\mathbf{t}|$ onto $\mathbf{e}_3$. Show $U_R\mathbf{t} \cdot \mathbf{P}\restriction_{\mathscr{S}(\mathbb{R}^3)} U_R^{-1} = |\mathbf{t}|P_3\restriction_{\mathscr{S}(\mathbb{R}^3)}$ and conclude.

12.11 Using Exercise 12.2, prove formula (12.75).

Hint. Prove the statement for P_3, passing from wavefunctions in $\mathbf{x}$ to wavefunctions in $\mathbf{k}$ via the Fourier transform. Extend to the general case as in the previous exercise. Note that U unitary and $A : D(A) \to \mathsf{H}$ closable imply that UAU^{-1} (defined on $U(D(A))$) is closable and

$$\overline{UAU^{-1}} = U\overline{A}U^{-1}\,.$$

12.12 Starting from (12.36), show formula (12.37).

Hint. First, substitute the neutral element e appropriately for one among g, g', g'', then write g^{-1} in place of g' and/or g''.

12.13 Let G be a *connected* topological group and $\mathsf{G} \ni g \mapsto \gamma_g$ a strongly continuous projective representation (Proposition 12.43) on the Hilbert space H_S, associated to a physical system. Suppose H_S decomposes in coherent sectors H_{Sk}. Can there be a function γ_g mapping a certain sector to a different sector?

Hint. Decompose $\mathfrak{S}_p(\mathsf{H})$ in a disjoint union of pure states of each sector, and equip sectors with $|| \ ||_1$. Remember that continuous maps preserve connected sets.

12.14 Prove that the Lie algebra of $SU(2)$ is the real vector space of skew-Hermitian matrices. Then show $SU(2)$ is simply connected.

Hint. $SU(2)$ is closed in $GL(4, \mathbb{R})$, hence a Lie group. Therefore one-parameter groups are of type $\mathbb{R} \ni t \mapsto e^{tA}$, with A in the Lie algebra $su(2)$. Impose $e^{tA}(e^{tA})^* = I$ and $tr(e^{tA}) = 1$ for every t, and infer how A has to look like. *Vice versa*, suppose A is skew-Hermitian and check that the above two conditions hold. As for simple connectedness, parametrise the group by 4 real variables so that $SU(2)$ is in one-to-one correspondence with the unit sphere in $\mathbb{R}^4$. Show the parametrisation is a homeomorphism.

12.15 Prove that $U \in SU(2)$ iff there exist a unit vector $\mathbf{n} \in \mathbb{R}^3$ and a real number θ such that:
$$U = e^{-i\theta \mathbf{n}\cdot\frac{\sigma}{2}} .$$

Hint. Use the spectral theorem for the unitary operator $U \in SU(2)$, keeping in account that the Pauli matrices and I form a real basis of 2×2 Hermitian matrices. Conversely, if $U = e^{-i\theta \mathbf{n}\cdot\frac{\sigma}{2}}$, what are U^*U and $\det U$?

12.16 Prove the matrices T in (12.84) satisfy:

$$RT_kR^t = \sum_{i=1}^{3}(R^t)_{ki}T_i \quad \text{for any } R \in SO(3).$$

Hint. Use $(T_i)_{jk} = -\varepsilon_{ijk}$ and write the above equations component-wise. Recall ε_{ijk} are the coefficients of a pseudo-tensor that is invariant under proper rotations.

12.17 Show that $R \in SO(3)$ iff there exist a unit vector $\mathbf{n} \in \mathbb{R}^3$ and a real angle θ such that:
$$R = e^{\theta \mathbf{n}\cdot\mathbf{T}} .$$

Hint. Prove the claim for $\mathbf{n} = \mathbf{e}_3$ by taking, for $R \in SO(3)$, a rotation about $\mathbf{e}_3$. Show every $R \in SO(3)$ admits an eigenvector $\mathbf{n}$. Rotate the axes so to move $\mathbf{n}$ onto $\mathbf{e}_3$, and recall the previous exercise. If, conversely, $R = e^{-i\theta \mathbf{n}\cdot\mathbf{T}}$, what are R^tR and $\det R$?

12.18 Demonstrate that for every $U \in SU(2)$ there exists a unique $R_U \in SO(3)$ such that:
$$U\mathbf{t}\cdot\sigma U^* = (R_U\mathbf{t})\cdot\sigma \quad \text{for any } \mathbf{t} \in \mathbb{R}^3.$$

Then verify
$$SU(2) \ni U \mapsto R_U \in SO(3)$$

is a surjective homomorphism that coincides with:

$$R : SU(2) \ni e^{-i\theta \mathbf{n}\cdot\frac{\sigma}{2}} \mapsto e^{\theta \mathbf{n}\cdot\mathbf{T}} \in SO(3) .$$

Eventually prove the kernel is $\{\pm I\} \subset SU(2)$.

Sketch of the solution. Note $|\mathbf{t}|^2 = \det(\mathbf{t}\cdot\sigma)$, and conclude every $U \in SU(2)$ determines a unique transformation of $\mathbb{R}^3$ mapping $\mathbf{t}$ to some $\mathbf{t}'$, with $|\mathbf{t}| = |\mathbf{t}'|$, defined by $U\mathbf{t}\cdot\sigma U^* = U\mathbf{t}'\cdot\sigma U^*$. The transformation $\mathbf{t} \to \mathbf{t}'$ is then an orthogonal matrix $R(U) \in O(3)$. That $R : SU(2) \ni U \mapsto R(U) \in O(3)$ is a homomorphism is immediate by construction. In the case $U_\theta = e^{-i\theta\frac{\sigma_3}{2}}$ one checks in various ways (e.g. directly, expanding the exponentials) that $R(U_\theta) = e^{\theta T_3}$. The general case relies on Exercise (12.16), rotating $\mathbf{e}_3$ onto an arbitrary unit vector $\mathbf{n}$. Clearly, $R(U_\theta) = e^{\theta\mathbf{n}\cdot\mathbf{T}}$ implies $R(U) \in SO(3)$. Surjectivity is a consequence of the fact that every $SO(3)$ matrix can be written as $e^{\theta\mathbf{n}\cdot\mathbf{T}}$. The kernel is computed by reducing to the one-parameter subgroup generated by σ_3, by rotation of $\mathbf{n}$. The result becomes thus obvious by direct computation.

12.19 Referring to Sect. 12.3.1, prove the strongly continuous unitary $SU(2)$-representation obtained by exponentiating the $\overline{\mathscr{L}_k}$ is the representation $SO(3) \ni R \mapsto U_R$ of Example 12.19 (where $\Gamma \in IO(3)$ is now restricted to $\Gamma = R \in SO(3)$), which is strongly continuous (cf. Example 12.46(1)).

Hint. By Nelson's theorem 12.89 it suffices to check the one-parameter groups $\theta \mapsto U_{e^{\theta\mathbf{n}\cdot\mathbf{T}}\mathbf{x}}$, with $\mathbf{n} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, are generated by the self-adjoint elements L_1, L_2, L_3. It is convenient to work with polar coordinates, using the core of L_1, L_2, L_3 given by spherical harmonics multiplied by a basis of $L^2(\mathbb{R}_+, r^2dr)$.

12.20 Show that the $SU(2)$-representation obtained by exponentiating the generators J_k, by Nelson's theorem, has the form:

$$SU(2) \ni e^{-i\theta\frac{1}{2}\mathbf{n}\cdot\sigma} \mapsto e^{-i\theta\mathbf{n}\cdot\mathbf{J}} = e^{-i\theta\mathbf{n}\cdot\mathbf{L}} \otimes V^s\left(e^{-i\frac{\theta}{2}\mathbf{n}\cdot\sigma}\right) .$$

Hint. Employ the properties of the tensor product of operators to show

$$e^{-i\theta\mathbf{n}\cdot\mathbf{J}} = e^{-i\theta\mathbf{n}\cdot\mathbf{L}} \otimes V^s\left(e^{-i\frac{\theta}{2}\mathbf{n}\cdot\sigma}\right) .$$

Hence we have to prove the representation $SO(3) \ni R \mapsto U_R$ of the previous exercise can be written as $U_{e^{\theta\mathbf{n}\cdot\mathbf{T}}} = e^{-i\theta\mathbf{n}\cdot\mathbf{L}}$. This is certainly true, for instance, for $\mathbf{n} = \mathbf{e}_3$. As for the general case: on one hand we have

$$U_R^* e^{-i\theta\mathbf{n}\cdot\mathbf{L}} U_R = e^{-i\theta\mathbf{n}\cdot\mathbf{U_R^*LU_R}} ,$$

and Exercise 12.5; on the other Exercise 12.16 holds.

12.21 Consider a strongly continuous unitary representation $\mathsf{G} \ni g \mapsto U_g$ of a connected Lie group G on the Hilbert space H. Suppose $E : D(E) \to \mathsf{H}$ is a closable operator for which $\mathscr{D}_N$ is invariant and a core[18].

Prove the following facts.

(1) If E commutes with $A_U(T)$ on $\mathscr{D}_N$ for some $T \in T_e\mathsf{G}$, then $\overline{E}$ commutes with $U_{\exp(tT)}$ for every $t \in \mathbb{R}$.

(2) If furthermore E is essentially self-adjoint on $\mathscr{D}_N$, then the spectral measures of $\overline{E}$ and $A_U(T)$ commute.

Solution. Fix $\psi \in \mathscr{D}_N$. Using the fact that $\mathscr{D}_N$ is E-invariant, and E commutes with $A_U(T)$ on it, we have

$$\overline{E}\sum_{n=0}^{N}\frac{(-it)}{n!}A_U(T)^n\psi = \sum_{n=0}^{N}\frac{(-it)}{n!}A_U(T)^n E\psi \ .$$

Since E is closed, and both ψ, $E\psi$ belong to $\mathscr{D}_N$ and hence are analytic for $A_U(T)$, taking the limit for $N \to +\infty$, we have

$$\overline{E}\sum_{n=0}^{+\infty}\frac{(-it)}{n!}A_U(T)^n\psi = \sum_{n=0}^{+\infty}\frac{(-it)}{n!}A_U(T)^n E\psi \ ,$$

if $|t| < r_{E,\psi}$. In other words

$$\overline{E}e^{-itA_U(T)}\psi = e^{-itA_U(T)}E\psi \ .$$

Suppose that $s_{E,\psi}$ is the supremum of the real numbers t for which the above relation holds. If $s_{E,\psi} < +\infty$, the fact that $\overline{E}$ is closed and that $t \mapsto e^{-itA_U(T)}E\psi$ is continuous at $t = s_{E,\psi}$ immediately implies that

$$\overline{E}e^{-is_{E,\psi}A_U(T)}\psi = e^{-is_{E,\psi}A_U(T)}E\psi \ . \tag{12.163}$$

Defining $\phi := e^{-is_{E,\psi}A_U(T)}\psi$ and using again the same argument, we have that, for some $\tau > 0$,

$$e^{-i\tau A_U(T)}\overline{E}e^{-is_{E,\psi}A_U(T)}\psi = \overline{E}e^{-i\tau A_U(T)}e^{-is_{E,\psi}A_U(T)}\psi = \overline{E}e^{-i(s_{E,\psi}+\tau)A_U(T)}\psi$$

so that, from (12.163),

$$\overline{E}e^{-i(s_{E,\psi}+\tau)A_U(T)}\psi = e^{-i(s_{E,\psi}+\tau)A_U(T)}E\psi \ .$$

This is impossible by definition of $s_{E,\psi}$ unless $s_{E,\psi} = +\infty$. An analogous procedure can be carried out for the infimum of the numbers t such that

[18]The first condition is valid for instance if $E = p(A_U(T_1), \ldots, A_U(T_n))$, where p is a polynomial of finite degree and $T_1, \ldots, T_n$ form a basis of the Lie algebra $T_e\mathsf{G}$.

$$\overline{E}e^{-itA_U(T)}\psi = e^{-itA_U(T)}E\psi \ .$$

Eventually the inf will be $-\infty$. We have found that

$$\overline{E}U_{\exp(tT)}\psi = U_{\exp(tT)}E\psi \quad \forall t \in \mathbb{R} \, .$$

Since $U_g(\mathscr{D}_N) \subset \mathscr{D}_N$, that means

$$U^*_{\exp(tT)}\overline{E}\restriction_{\mathscr{D}_N} U_{\exp(tT)}\psi = E\restriction_{\mathscr{D}_N} \ .$$

But $\mathscr{D}_N$ is a core for E, so taking the closure gives

$$\overline{E}U_{\exp(tT)} = U_{\exp(tT)}\overline{E} \quad \forall t \in \mathbb{R} \, .$$

If E is essentially self-adjoint on $\mathscr{D}_N$, the last point follows easily from Theorem 9.41.

Chapter 13
Selected Advanced Topics in Quantum Mechanics

Give up telling God what to do with his dice.
Niels Bohr, to Einstein

With this chapter we complete the list of axioms for non-relativistic Quantum Mechanics, by defining *time evolution* and *compound systems*. Certain notions, here defined formally, have already been introduced in the final part of the previous chapter when we were talking about symmetry groups. More advanced reference texts, which we have followed here and there, are [Pru81] and [DA10].

In the first section we will state the *axiom of time evolution*, described by a strongly continuous one-parameter unitary group that is generated by the *Hamiltonian operator* of the system. We will define *dynamical symmetries* as a special kind of the symmetries seen earlier. Then we shall discuss the nature of *Schrödinger's equation* and introduce the important concept of *stationary state*. As a classical example of this formalism we will analyse in depth the action of the Galilean group in the position representation (we saw it in the momentum representation in the previous chapter). We will also explain how wavefunctions transform under changes of inertial frame systems. Then we will pass to the basic theory of non-relativistic scattering. We will make a few remarks on the existence of the unitary time-evolution operator in absence of time homogeneity (we will examine the convergence in $\mathfrak{B}(\mathsf{H})$ of the *Dyson series* for a Hamiltonian), and discuss the anti-unitary nature of the time-reversal symmetry.

In the following section we will present a version of *Pauli's theorem*, whose concern is the possibility of defining the "time operator" as self-adjoint conjugate to the Hamiltonian. In this respect we will briefly discuss POVMs, which may be employed to give a weaker meaning to the time observable.

Heisenberg's picture of observables will be introduced in section three, where we shall address the relationship between constants of motion and dynamical symmetries, present the quantum version of *Noether's theorem* and study the case of constants of motion associated to generators of a Lie group, including the one-parameter

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subgroup of time evolution. A digression will give us the chance to present the mathematical problems raised by the *Ehrenfest theorem*. The section will close with the constants of motion associated to the Galilean group.

Section four is devoted to the theory of *compound quantum systems*: systems with an inner structure and multi-particle systems. We will consider, in particular, *entangled states* and discuss some problems related to the *EPR paradox* and the notion of *decoherence*. Eventually, we will pass to the general theory of *systems of identical particles*, and finish with the *spin-statistic* correlation.

13.1 Quantum Dynamics and Its Symmetries

As we quickly recalled in Sect. 7.2.1, physical systems evolve in time according to their dynamics. In the classical Hamiltonian formulation of mechanics the evolution in time of a system's state is described in phase spacetime by the solutions to *Hamilton's equations*. Let us consider the situation in which the *Hamiltonian function H does not depend explicitly on time* in the coordinates of a given inertial frame $\mathscr{I}$. We will talk in this case of time being *homogeneous* with respect to the considered physical system. Hamilton's equations are *autonomous* PDEs, i.e. the time variable does not show up explicitly if the equations are written in those canonical coordinates and the phase spacetime splits naturally in a product $\mathbb{R} \times \mathscr{F}$, where $\mathscr{F}$ is the *phase space*. The solutions to Hamilton's equations determine a one-parameter group of diffeomorphisms $\phi_\tau : \mathscr{F} \to \mathscr{F}$ mapping the initial state $r \in \mathscr{F}$, at time 0 (taken to be sharp for simplicity) to the state $\phi_\tau(r) \in \mathscr{F}$, at time τ. The basic mathematical tool to construct the *time-evolution operator* – the one-parameter group $\{\phi_\tau\}_{\tau\in\mathbb{R}}$ – is the Hamiltonian H of the system, which is identified with the total *mechanical energy* of the frame system $\mathscr{I}$ [GPS01, FaMa06]. In the sequel we will present the quantum analogues of the Hamiltonian function and the evolution operator.

13.1.1 Axiom A6*: Time Evolution*

The quantum setting is not dissimilar to the classical case. The following axiom comprises time evolution in a quantum system S, described on the Hilbert space H_S for given inertial frame $\mathscr{I}$, with homogeneous time. The axiom defines the *Hamiltonian* (operator) of the quantum system as the generator of the one-parameter unitary group capturing the evolution, hence the dynamics, of the quantum state. (We will return to this in Sect. 13.1.6 when looking at a more general situation.) Using the notion of time evolution makes it possible to treat *dynamical symmetries* and, as we will see later, state the quantum *Noether theorem*.

A6. *Let S be a quantum system described on the Hilbert space H_S associated to the inertial frame $\mathscr{I}$. There exists a self-adjoint operator H, called the* **Hamiltonian**

of the system *S in the frame* $\mathscr{I}$ *and corresponding to the observable of the total mechanical energy of S in the frame* $\mathscr{I}$, *such that*

(i) $\sigma(H)$ *is lower bounded,*

(ii) setting $U_\tau := e^{-\frac{i\tau}{\hbar}H}$, *if the system's state at time t is* $\rho_t \in \mathfrak{S}(\mathsf{H}_S)$, *then the state at time* $t+\tau$ *is:*

$$\rho_{t+\tau} = \gamma_\tau^{(H)}(\rho_t) := U_\tau \rho_t U_\tau^{-1} . \tag{13.1}$$

The strongly continuous one-parameter unitary group $\mathbb{R} \ni \tau \mapsto U_\tau$ *is called* **time-evolution operator** *of S in the frame* $\mathscr{I}$, *and the continuous projective representation* $\mathbb{R} \ni \tau \mapsto \gamma_\tau^{(H)}$ of $\mathbb{R}$ *induced by U is called* **dynamical flow** *of S in the frame* $\mathscr{I}$.

Remarks 13.1 **(1)** From now on, unless strictly necessary for better physical clarity, we will omit to write $\hbar$ explicitly in formulas, and set $\hbar = 1$.

(2) The evolution of states is therefore given by a continuous projective representation of the Abelian group $\mathbb{R}$. This fact enables us to phrase differently axiom **A6**, using the results of the preceding chapter. With the intent to weaken the axiom's assumptions as much as possible, and think of the evolution as a function $\rho \mapsto \gamma_\tau(\rho)$ mapping states to states for any $\tau \in \mathbb{R}$, we may require γ_τ to satisfy the following conditions, all rather reasonable from the physical viewpoint:

(i) γ_τ preserves the convexity of the space of states (Kadison symmetry), or equivalently, it preserves transition probabilities (Wigner symmetry);

(ii) γ_τ is additive: $\gamma_\tau \circ \gamma_{\tau'} = \gamma_{\tau+\tau'}$, for $\tau, \tau' \in \mathbb{R}$,

(iii) (viewing symmetries γ_τ *à la* Wigner) γ_τ is continuous for Definition 12.40 or equivalently, continuous in the topology of $\mathfrak{S}_p(\mathsf{H}_S)$ induced by $|| \ ||_1$, as in (12.50). Then Theorem 12.45 proves that the projective representation $\mathbb{R} \ni \tau \mapsto \gamma_\tau$ has the form predicted by axiom **A6**. One of the possible self-adjoint generators – which exist and differ by an additive constant, by Theorem 12.45 – is the system's Hamiltonian, by definition. But we still need to impose the spectrum be bounded from below. In defining the Hamiltonian, the ambiguity coming from the additive constant is actually present in physics, because the energy of a classical system (non-relativistic) is given up to constant. (As classical physics arises as an approximation of relativistic physics [AnMo12], however, the picture is not so obvious because one must account for the superselection of the mass, or the like.)

(3) That the Hamiltonian spectrum of a real physical systems is bounded stems from thermodynamical stability. Unless we consider an ideal system – perfectly isolated from the environment, which in reality does not exist (also by deep theoretical motivations that demand Quantum Field Theory to be explained properly) – the lower limit constraining the spectrum of H (the mechanical energy) is unavoidable. In absence of a threshold there could be transitions to states with decreasingly lower energy. This (infinite!) energy loss towards the outside, in some form or other (particles, electromagnetic waves), would in practice make the system collapse. The lower limit of $\sigma(H)$ has other important theoretical repercussions we will see later.

(4) The *inverse* symmetry to time evolution is called **time displacement**. We met this symmetry when we were studying the Galilean group. Physically it is an

active transformation of S. In other words, for given τ, it is a Kadison symmetry $\gamma_\tau^{(-H)} : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$ that transforms the state ρ at a generic given time t_0 into the state $\tau_\tau(\rho)$ *at the same time* t_0, so that $\gamma_\tau^{(H)}\left(\gamma_\tau^{(-H)}(\rho)\right)$ coincides with ρ. By construction $\gamma_\tau^{(-H)} = \left(\gamma_\tau^{(H)}\right)^{-1}$. Evidently the unitary generator of $\gamma_\tau^{(-H)}$ is $-H$, as the name already suggests. This explains the "−" sign used for the self-adjoint generator of the time displacement symmetry when we were discussing the subgroups of the Galilean group in Sect. 12.3.3 interpreted as *active transformations* of the physical system.
(5) The use of an *inertial* reference frame is not strictly necessary, and a reference frame with temporal homogeneity is in fact sufficient for the physical validity of all statements up to Sect. 13.1.6. In that section we shall address what happens in absence of time homogeneity. ■

Now let us suppose H_S is decomposed in coherent sectors H_{Sk}, $k \in K$, because Abelian superselection rules occur. Then the space of admissible pure states splits in the disjoint union of sets $\mathfrak{S}_p(\mathsf{H}_{Sk})$, and mixed states are convex combinations of elements in the various $\mathfrak{S}(\mathsf{H}_{Sk})$. The next result shows that the dynamical flow preserves this splitting, as expected.

Proposition 13.2 *Let S be a quantum system described on the Hilbert space H_S associated to the inertial frame $\mathscr{I}$, with dynamical flow $\gamma^{(H)}$. Suppose H_S splits in coherent sectors H_{Sk}, $k \in K$. Then the dynamical flow preserves both pure and mixed states. More precisely:*
(a) *if $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$ for some $k \in K$, then $\gamma_t^{(H)}(\rho) \in \mathfrak{S}(\mathsf{H}_{Sk})$ for every $t \in \mathbb{R}$;*
(b) *if $\rho \in \mathfrak{S}_p(\mathsf{H}_{Sk})$ for some $k \in K$, then $\gamma_t^{(H)}(\rho) \in \mathfrak{S}_p(\mathsf{H}_{Sk})$ for every $t \in \mathbb{R}$.*

Proof Since

$$\gamma_t^{(H)}(\psi(\psi|\ )) = e^{-itH}\psi\left(e^{-itH}\psi\right|\ ) \ ,$$

clearly the representation $\gamma^{(H)}$ maps pure states to pure states, so mixed to mixed ones. Restrict $\gamma^{(H)}$ to pure states. Fix $\rho \in \mathfrak{S}_p(\mathsf{H}_{Sk})$ and consider the path $\mathbb{R} \ni t \mapsto \gamma_t^{(H)}(\rho)$. By Proposition 12.43 it is continuous for $||\cdot||_1$. We know $\mathfrak{S}_p(\mathsf{H}_{Sk})$ are the connected components of $\mathfrak{S}_p(\mathsf{H}_S)_{adm}$ for the topology induced by the aforementioned norm (Exercise 12.6), so the curve is confined to live in one component only. The latter is $\mathfrak{S}_p(\mathsf{H}_{Sk})$, since the path passes through there at $t = 0$. If $U_t = e^{-itH}$, then, for any unit vector $\psi \in \mathsf{H}_{Sk}$ we have $U_t\psi \in \mathsf{H}_{Sk}$ for all t. Consider now $\rho \in \mathfrak{S}(\mathsf{H}_{Sk})$ and its spectral decomposition $\rho = \sum_{j\in J} p_j\psi_j(\psi_j|\ )$. The series converges strongly and by construction $\psi_j \in \mathsf{H}_{Sk}$, $j \in J$, is a unit vector. Therefore for any $t \in \mathbb{R}$:

$$\gamma_t^{(H)}(\rho) = U_t\sum_{j\in J} p_j\psi_j(\psi_j|\ )U_t^{-1} = \sum_{j\in J} p_jU_t\psi_j(\psi_j|U_t^*\ ) = \sum_{j\in J} p_jU_t\psi_j(U_t\psi_j|\ ) \in \mathfrak{S}(\mathsf{H}_{Sk}) \, ,$$

ending the proof. □

Remark 13.3 From now on the system's Hilbert space H_S will not contain coherent sectors, apart from a few cases we will comment upon. We leave it to the reader to generalise the ensuing definitions and results to the multi-sector case. ■

13.1.2 Dynamical Symmetries

Time evolution allows to refine the notion of symmetry seen in the previous chapter, and define *dynamical symmetries*.

Consider a quantum system S with dynamical flow $\gamma^{(H)}$. Let us assume, as we said, the Hilbert space consists of a single coherent sector. Take a symmetry σ (Kadison or Wigner) acting on states, paying attention that now states evolve in time following the dynamics of the flow $\gamma^{(H)}$. If we apply σ to the evolved state $\gamma_t^{(H)}(\rho)$ and obtain $\rho'_t := \sigma(\gamma_t^{(H)}(\rho))$, nothing guarantees that the function $\mathbb{R} \ni t \mapsto \rho'_t$ will describe the possible evolution under $\gamma^{(H)}$ of a certain state (necessarily $\rho'_0 = \sigma(\gamma_0^{(H)}(\rho)) = \sigma(\rho)$). But if this does happen (for any choice of initial state ρ), σ is called a *dynamical symmetry*, because its action *is compatible* with the system's dynamics.

A variant to having $\mathbb{R} \ni t \mapsto \sigma(\gamma_t^{(H)}(\rho))$ describing the evolution of a state in S, is to take a whole family of symmetries $\sigma^{(t)}$ parametrised by time $t \in \mathbb{R}$. To have a *time-dependent* dynamical symmetry we require $\mathbb{R} \ni t \mapsto \sigma^{(t)}(\gamma_t^{(H)}(\rho))$ still be an evolution under $\gamma^{(H)}$ for some state of S.
More formally:

Definition 13.4 Let S be a quantum system described on the Hilbert space H_S (made of one sector) and associated to the inertial frame $\mathscr{I}$, with Hamiltonian H and dynamical flow $\gamma^{(H)}$. A symmetry $\sigma : \mathfrak{S}(\mathsf{H}_S) \to \mathfrak{S}(\mathsf{H}_S)$ is called a **dynamical symmetry** of S if

$$\gamma_t^{(H)} \circ \sigma = \sigma \circ \gamma_t^{(H)} \quad \text{for every } t \in \mathbb{R}. \tag{13.2}$$

A family of symmetries parametrised by time $\{\sigma^{(t)}\}_{t\in\mathbb{R}}$ is a **time-dependent dynamical symmetry** when:

$$\gamma_t^{(H)} \circ \sigma^{(0)} = \sigma^{(t)} \circ \gamma_t^{(H)} \quad \text{for every } t \in \mathbb{R}. \tag{13.3}$$

The first result we prove characterises dynamical symmetries. Part (c) is a consequence of the spectral lower bound of H and characterises dynamical symmetries when $\sigma(H)$ is unbounded, as for the majority of concrete physical systems.

Theorem 13.5 *Let S be a quantum system described on the Hilbert space H_S associated to the inertial frame $\mathscr{I}$ with Hamiltonian H (hence, with lower-bounded spectrum) and dynamical flow $\gamma^{(H)}$.*

(a) *Consider a family of symmetries labelled by time $\{\sigma^{(t)}\}_{t\in\mathbb{R}}$ and induced by unitary (or anti-unitary) operators $V^{(\sigma^{(t)})} : \mathsf{H}_S \to \mathsf{H}_S$. Then $\{\sigma^{(t)}\}_{t\in\mathbb{R}}$ is a time-dependent dynamical symmetry of S if and only if*

$$\chi_t V^{(\sigma^{(t)})} e^{-itH} = e^{-itH} V^{(\sigma^{(0)})} \quad \textit{for every } t \in \mathbb{R} \textit{ and some unit number } \chi_t \in \mathbb{C}$$

(b) *Consider a symmetry σ induced by a unitary (or anti-unitary) $V^{(\sigma)} : \mathsf{H}_S \to \mathsf{H}_S$. Then σ is a dynamical symmetry of S if and only if*

$$e^{-iat} V^{(\sigma)} e^{-itH} = e^{-itH} V^{(\sigma)} \quad \textit{for every } t \in \mathbb{R} \textit{ and some } a \in \mathbb{R}$$

where $a = 0$ in the unitary case.

(c) *Consider a symmetry σ induced by a unitary (or anti-unitary) operator $V^{(\sigma)} : \mathsf{H}_S \to \mathsf{H}_S$ and suppose $\sigma(H)$ is not bounded above (but is bounded below). Then σ is a dynamical symmetry of S if and only if*

$$V^{(\sigma)} e^{-itH} = e^{-itH} V^{(\sigma)} \quad \textit{for every } t \in \mathbb{R},$$

or equivalently, if and only if the following hold:
(i) $V^{(\sigma)}$ is unitary and
(ii) $V^{(\sigma)} H = H V^{(\sigma)}$.

Proof (a) and (b). For $S : \mathsf{H}_S \to \mathsf{H}_S$ unitary (or anti-unitary), $S\psi(\psi|S^{-1}\cdot) = S\psi(S\psi|\cdot)$. Set $U_t := e^{-itH}$, $V^{(t)} := V^{(\sigma^{(t)})}$ and use the identity with the unitary operator $S := (V^{(t)}U_t)^{-1}U_t V^{(0)}$. Then (13.3) implies, for any pure $\rho = \psi(\psi|\ )$:

$$(V^{(t)}U_t)^{-1}U_t V^{(0)}\psi\left((V^{(t)}U_t)^{-1}U_t V^{(0)}\psi|\ \right) = \psi(\psi|\ ),$$

hence for some unit number $\chi_t \in \mathbb{C}$:

$$(V^{(t)}U_t)^{-1}U_t V^{(0)}\psi = \chi_t\psi \quad \text{for all } \psi \in \mathsf{H}.$$

The same proof of the analogous fact in Theorem 12.11 says that χ_t does not depend on ψ. Therefore if $\sigma^{(t)}$ is a time-dependent dynamical symmetry:

$$\chi_t V^{(\sigma^{(t)})} U_t = U_t V^{(\sigma^{(0)})} \quad \text{for all } t \in \mathbb{R} \text{ and some } \chi_t \in \mathbb{C}, |\chi_t| = 1.$$

Conversely, if the condition holds, trivially $\sigma^{(t)}$ is a time-dependent dynamical symmetry. Statement (b) is a subcase, except for the proof that $\chi_t = e^{ict}$ for some $c \in \mathbb{R}$, which we will settle at the end.
(c) We claim that if σ is a dynamical symmetry then (i), (ii) hold. By (a), if σ is a dynamical symmetry:

$$\chi_t V^{(\sigma)} U_t = U_t V^{(\sigma)} \quad \text{for some unit } \chi_t \in \mathbb{C} \tag{13.4}$$

Hence $\chi_t I = (V^{(\sigma)} U_t)^{-1} U_t V^{(\sigma)}$ and $\chi_t (\phi|\psi) = \left(V^{(\sigma)} U_t \phi \middle| U_t V^{(\sigma)} \psi\right)$ if $V^{(\sigma)}$ is unitary, or $\overline{\chi_t}(\psi|\phi) = \left(V^{(\sigma)} U_t \phi \middle| U_t V^{(\sigma)} \psi\right)$ if $V^{(\sigma)}$ is anti-unitary. Choose $\phi \in D(H)$ not orthogonal to $\psi \in V^{(\sigma)-1}(D(H))$ (since $D(H)$ is dense), apply Stone's theorem and conclude $t \mapsto \chi_t$ is smooth everywhere. We rewrite (13.4) as:

$$\chi_t U_t = e^{\pm it V^{(\sigma)-1} H V^{\sigma}} , \tag{13.5}$$

with '$-$' sign if $V^{(\sigma)}$ is unitary, '$+$' if anti-unitary (in the latter case the final χ_t coincides with the initial $\overline{\chi_t}$). Using Stone's theorem in (13.5) we obtain $D(V^{(\sigma)-1} H V^{(\sigma)}) \subset D(H) = D(cI + H)$ and

$$\mp V^{(\sigma)-1} H V^{(\sigma)} \restriction_{D(H)} = cI + H \quad \text{where } c := i \frac{d\chi_t}{dt}|_{t=0}. \tag{13.6}$$

Note c must be real since $\mp V^{(\sigma)-1} H V^{(\sigma)} - H$ is symmetric on $D(H)$. Actually (13.6) holds on the entire domain of $V^{(\sigma)-1} H V^{(\sigma)}$ because the latter is self-adjoint and does not have self-adjoint extensions $(cI + H)$ other than $\mp V^{(\sigma)-1} H V^{(\sigma)}$ itself. Therefore

$$V^{(\sigma)-1} H V^{(\sigma)} = \mp cI \mp H \, . \tag{13.7}$$

In particular (cf. Exercise 12.8 in the anti-unitary case):

$$\sigma(H) = \sigma(V^{(\sigma)-1} H V^{(\sigma)}) = \sigma\left(\mp cI \mp H\right) = \mp c \mp \sigma(H) \, .$$

If $\sigma(H)$ is bounded below but not above, the identity cannot be valid if on the right side we have the minus sign, irrespective of the constant c. Hence $V^{(\sigma)}$ must be unitary. Therefore $\inf \sigma(H) = \inf(c + \sigma(H)) = c + \inf \sigma(H)$ and $c = 0$, for $\inf \sigma(H)$ is finite by hypothesis ($\sigma(H) \neq \varnothing$ is bounded below). We obtained that a dynamical symmetry σ fulfils (i) and (ii): $V^{(\sigma)}$ is unitary and $V^{(\sigma)} H = H V^{(\sigma)}$. If so, $H = V^{(\sigma)-1} H V^{(\sigma)}$. Exponentiating,

$$V^{(\sigma)} e^{-itH} = e^{-itH} V^{(\sigma)} \quad \text{for every } t \in \mathbb{R},$$

eventually showing that σ is a dynamical symmetry, and ending part (c).

We still have to finish part (b). If σ is a symmetry, using the proof of (c) we arrive at (13.7). Yet we cannot say $c = 0$, unless $V^{(\sigma)}$ is unitary, for in that case $\sigma(H) = c + \sigma(H)$ and the reasoning still works. Exponentiating (13.7) gives:

$$e^{-ict} U_t = V^{(\sigma)-1} U_t V^{(\sigma)} ,$$

whence

$$e^{-iat} V^{(\sigma)} e^{-itH} = e^{-itH} V^{(\sigma)}$$

where $a = -c$ if $V^{(\sigma)}$ anti-unitary. This ends part (b) and the proof. □

13.1.3 Schrödinger's Equation and Stationary States

Consider a pure initial state $\rho \in \mathfrak{S}_p(\mathsf{H}_S)$. As already noticed, the evolution is such that any evolved state ρ_t is pure. Theoretical physicists refer to this property[1] by saying that the evolution of quantum states is *unitary*. If $t \mapsto \rho_t \in \mathfrak{S}_p(\mathsf{H}_S)$ denotes the evolution of a pure state, we can determine any ρ_t, up to phase, by a vector ψ_t normalised to 1. Choosing the simplest phases for the pure states involved, the equation governing the evolution of pure states becomes (reintroducing the constant $\hbar$):

$$\psi_{t'} = e^{-\frac{i(t'-t)}{\hbar}H}\psi_t \, . \tag{13.8}$$

We can manipulate this relation to obtain an equation of great historical relevance. For this we observe that $\psi_t \in D(H)$ implies $\psi_{t'} \in D(H)$ for any other time $t' \in \mathbb{R}$. In fact, $\psi_t \in D(H)$ means $\int_{\mathbb{R}} \lambda^2 d\mu_{\psi_t}^{(H)} < +\infty$, where $\mu_{\psi_t}^{(H)}(E) = (\psi_t|P^{(H)}(E)\psi_t) = (\psi_{t'}|e^{+\frac{i\tau}{\hbar}H}P^{(H)}(E)e^{-\frac{i\tau}{\hbar}H}\psi_{t'})$, for $t - t' = \tau$. On the other hand, trivially,

$$e^{+\frac{i\tau}{\hbar}H}P^{(H)}(E)e^{-\frac{i\tau}{\hbar}H} = P^{(H)}(E) \, ,$$

since $P^{(H)}(E)$ is a projector of the PVM of H. Hence $\int_{\mathbb{R}} \lambda^2 d\mu_{\psi_t}^{(H)} < +\infty$ is equivalent to $\int_{\mathbb{R}} \lambda^2 d\mu_{\psi_{t'}}^{(H)} < +\infty$, i.e. $\psi_{t'} \in D(H)$. Let us suppose $\psi_t \in D(H)$ for some t, from which $\psi_{t'} \in D(H)$ for every t'. Applying Stone's theorem to (13.8) and interpreting the resulting derivative in strong sense, we obtain

$$i\hbar\frac{d}{dt}\psi_t = H\psi_t \, . \tag{13.9}$$

This is the fundamental **time-dependent Schrödinger equation**. We have to notice that (13.9) only holds if $\psi_t \in D(H)$, whereas the evolution Eq. (13.1) has a general reach.

Let us make a few comments on Schrödinger's equation and then pass to more general matters.

Consider a system formed by one particle of mass m (without spin for simplicity) subjected to a force with sufficiently regular potential energy $V = V(\mathbf{x})$, in the inertial frame $\mathscr{I}$ with right-handed orthonormal coordinates. Following the discussion about *Dirac's correspondence principle*, at the end of Chap. 11, one expects the Hamiltonian of this system to correspond, quantum-wise, to a certain self-adjoint extension H of the symmetric operator

$$H_0 := \frac{1}{2m}\sum_{i=1}^{3} P_i^2 + V(\mathbf{X}) \, ,$$

[1]Especially in relationship to the evolution of states of quantum fields in spacetimes comprising dynamical black holes, where the unitary evolution is rather problematic.

initially defined on some invariant dense subspace where P_i and X_i are well defined. This choice formally fulfils Dirac's correspondence, at least with reference to the commutation relations of H_0 and X_k, P_k on domains where everything is well defined. This expectation turns out to be correct, and the Hamiltonian observables do have the mentioned form in the physical world. Systems formed by atoms and molecules, for instance, behave like that [Mes99, CCP82].

We shall identify the particle's Hilbert space with $L^2(\mathbb{R}^3, dx)$ so that position operators are multiplicative. If we work with functions that are regular enough, the starting expression for H is

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{X}) , \tag{13.10}$$

where Δ is the familiar Laplacian on $\mathbb{R}^3$ and $V(\mathbf{X})$ is the multiplication by the original function $V = V(\mathbf{x})$. Schrödinger's equation then reads:

$$\left[-\frac{\hbar^2}{2m}\Delta + V(X)\right]\psi_t(x) = i\hbar\frac{\partial}{\partial t}\psi_t(x) ,$$

which is precisely how Schrödinger wrote it in his astounding 1926 papers. Beware, however, that the equation should not be taken literally, as a usual PDE, because: (1) the t-derivative is not meant pointwise, but in Hilbert sense[2]; (2) the equation is valid up to zero-measure sets for x, since wavefunctions belong to $L^2(\mathbb{R}^3, dx)$. If we were to find "naïve" solutions (functions $f(t, x)$ in t and x), we would then have to prove they solve (13.9) in the unknown $\psi_t = f(t, \cdot) \in L^2(\mathbb{R}^3, dx)$.

Let us return to how to define the Hamiltonian operator from the symmetric differential operator (13.10) defined on a dense domain. We have to verify, case by case, if the operator admits self-adjoint extensions or if it is essentially self-adjoint. In this respect the symmetric operator H_0 commutes with the operator $C : L^2(\mathbb{R}^3, dx) \to L^2(\mathbb{R}^3, dx)$ representing the complex conjugation of L^2 functions. By von Neumann's theorem 5.43, then, there are self-adjoint extensions. The general theory of self-adjoint extensions of operators like H_0 was developed and harvested by T. Kato [Kat66]. For several potentials of interest, like the attractive Coulomb potential and the harmonic oscillator, one can prove H_0 is essentially self-adjoint. We saw these results in Examples 10.52, Sect. 10.4, as consequences of general theorems. There is a whole branch of functional analysis in Hilbert spaces devoted to this sort of problems. We mention just one easy corollary of Theorem 10.50.

Theorem 13.6 (Kato) *Consider the differential operator on* $\mathbb{R}^3$*:*

$$H_0 := -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}) , \tag{13.11}$$

[2]Observe, nevertheless, that if the derivative exists both in the ordinary and in the L^2 sense, the two coincide by Proposition 2.32 for $p = 2$.

defined on some dense domain $D(H_0) \supset \mathscr{S}(\mathbb{R}^3)$. *Suppose*

$$V(\mathbf{x}) = \sum_{j=1}^{N} \frac{g_j}{|\mathbf{x} - \mathbf{x}_j|} + U(\mathbf{x}) , \tag{13.12}$$

where g_j *are constants,* $\mathbf{x}_j \in \mathbb{R}^3$ *are given points and* $U : \mathbb{R}^3 \to \mathbb{R}$ *is measurable and (essentially) bounded. Then*
(a) H_0 *is essentially self-adjoint on* $D(H_0)$, $\mathscr{D}(\mathbb{R}^3)$ *and* $\mathscr{S}(\mathbb{R}^3)$.
(b) *The common self-adjoint extension* $\overline{H_0}$ *of the operators in (a) coincides with the self-adjoint operator* $\overline{-\Delta} + V$ *defined on* $D(\overline{-\Delta})$.
(c) $\sigma(\overline{H_0})$ *is bounded from below.*

In general, if the Hamiltonian H of a certain system has point spectrum $\sigma_p(H)$, every eigenvector ψ_E of H, for $E \in \sigma_p(H)$, has a trivial evolution:

$$U_t\psi_E = e^{-i\frac{tE}{\hbar}}\psi_E .$$

This says the pure state $\rho_E := \psi_E(\psi_E|\ )$ associated to ψ_E ($||\psi_E|| = 1$) *does not evolve in time*. These very special states are called **stationary states** of the system. When one studies the macroscopic system of an atom or a molecule, the starting point is describe the heavier parts – nuclei – as classical systems, that act by electric Coulomb forces on peripheral electrons viewed as quantum particles. The electrons' quantum states are stationary for their Hamiltonian. More on this in Example 13.8(3).

Remarks 13.7 **(1)** Referring to Theorem 13.6 it can be proved (cf. [CCP82, Sect. VI] and especially [Hel64, Sect. 11]) that if some g_j vanish and the remaining are strictly negative then $\sigma_p(\overline{H_0}) \neq \varnothing$.
(2) By virtue of Theorem 10.51, H_0 continues to be essentially self-adjoint on $\mathscr{D}(\mathbb{R}^3)$ and the only self-adjoint extension is bounded from below provided U is non-negative and lower bounded. In that case [CCP82, Hel64], if $g_j = 0$ for every j and U is regular enough and tends to infinity as $|\mathbf{x}| \to +\infty$, then $\sigma(\overline{H_0}) = \sigma_p(\overline{H_0}) \neq \varnothing$.
(3) One of the highest mountain tops the inexperienced student has to conquer when taking on QM is to understand the motivations behind the regularity constraints imposed on the eigenvalues of the theory's Hamiltonian. The characteristic equation

$$H_0\psi_E = E\psi_E , \quad \text{with } E \in \mathbb{R} , \quad \psi_E \in L^2(\mathbb{R}^3, dx) ,$$

should give, roughly speaking, the stationary states of the system whose Hamiltonian is determined by H_0. Consider, as often in physics, an operator of the form (13.11) where the function $U : \mathbb{R}^3 \to \mathbb{R}$ of (13.12) has finite discontinuities on some regular surfaces σ_k, $k = 1, 2, \ldots, N$ (disjoint from one another and from other isolated singularities of V) and is continuous everywhere else. We also want U to be bounded (by remark (2) we could just require lower boundedness). QM manuals typically require the functions ψ_E further satisfy the following conditions:

(1) away from the singularities of V the ψ_E are C^2 (actually C^∞),

(2) the ψ_E solve $H_0\psi_E = E\psi_E$ for some $E \in \mathbb{R}$, i.e. interpreting the operator as if it were a differential operator, away from the singularities of V,

(3) on singular surfaces σ_k the maps ψ_E and the normal derivatives are continuous,

(4) at isolated singularities ψ_E admits finite limits.

The constraints are sometimes justified in a sort-of-whimsical way in textbooks (this happens in particular for the analogous statements for $\mathbb{R}^1$).

What we can say is, first, that H_0 is *not* the operator representing the Hamiltonian observable, because H_0 is not self-adjoint! The operator in question is a self-adjoint extension of H_0. Theorem 13.6 warrants, under the assumptions made, H_0 is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^3)$, so there is one self-adjoint extension that coincides with the closure of H_0 and with its adjoint as well: $\overline{H_0} = H_0^*$. Stationary states are given by the spectrum of H_0^*, i.e. by solutions to

$$H_0^*\psi_E = E\psi_E\,, \quad E \in \mathbb{R}\,, \quad \psi_E \in D(H_0^*)\,.$$

This equation, since $\mathscr{D}(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3, dx)$, may be written:

$$(\varphi|H_0^*\psi_E) = E(\varphi|\psi_E), \quad E \in \mathbb{R}, \quad \text{for every } \varphi \in \mathscr{D}(\mathbb{R}^3) \text{ and a given } \psi_E \in D(H_0^*).$$

Using the definition of adjoint, the equation reads

$$(H_0\varphi|\psi_E) = E(\varphi|\psi_E)\,, \quad E \in \mathbb{R}\,, \quad \text{for any } \varphi \in \mathscr{D}(\mathbb{R}^3) \text{ and a given } \psi_E \in D(H_0^*).$$

Put differently, we seek functions $\psi_E \in L^2(\mathbb{R}^3, dx)$ such that, for any $\varphi \in \mathscr{D}(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} \left(-\frac{\hbar^2}{2m}\Delta\overline{\varphi(\mathbf{x})} + V(\mathbf{x})\overline{\varphi(\mathbf{x})} - E\overline{\varphi(\mathbf{x})}\right)\psi_E(\mathbf{x})\,dx = 0\,. \tag{13.13}$$

Hence the ψ_E do not necessarily solve $H_0^*\psi_E = E\psi_E$, for it is enough they solve it *weakly*: they must satisfy (13.13) for any $\varphi \in \mathscr{D}(\mathbb{R}^3)$. Issues of this kind [ReSi80] are dealt with by the general theory of *elliptic regularity*, which proves [CCP82, Hel64] $\psi_E \in L^2(\mathbb{R}^3, dx)$ satisfies (13.13), with the aforementioned assumptions on the potential V, if and only if ψ_E satisfies conditions (1)–(4). ■

Examples 13.8

(1) The simplest example is the free spinless particle of mass $m > 0$, described on the Hilbert space $L^2(\mathbb{R}^3, dx)$ associated to the axes of an inertial system $\mathscr{I}$. Pure states are represented by *wavefunctions*, i.e. unit elements $\psi \in L^2(\mathbb{R}^3, dx)$. The Hamiltonian is simply:

$$H := \frac{1}{2m}\overline{\sum_{k=1}^{3} P_k\restriction^2_{\mathscr{S}(\mathbb{R}^3)}} = -\frac{\hbar^2}{2m}\overline{\Delta\restriction_{\mathscr{S}(\mathbb{R}^3)}}\,. \tag{13.14}$$

Let us briefly discuss its self-adjointness. Although everything should be clear from Proposition 10.45, we think it might be interesting to go over a few facts. The left-hand side of (13.14) is self-adjoint since

$$H_0 := \frac{1}{2m}\sum_{k=1}^{3} P_k\restriction^2_{\mathscr{S}(\mathbb{R}^3)} = -\frac{\hbar^2}{2m}\Delta\restriction_{\mathscr{S}(\mathbb{R}^3)}$$

is essentially self-adjoint. The proof is direct via the unitary Fourier-Plancherel operator $\widehat{\mathscr{F}}$, noting that in the space $L^2(\mathbb{R}^3, dk)$ of transformed maps $\widetilde{\psi} := \widehat{\mathscr{F}}(\psi)$, the above operator multiplies by:

$$\mathbf{k} \mapsto \frac{\hbar^2}{2m}\mathbf{k}^2 ,$$

and has dense domain $D(\widehat{\mathscr{F}}^{-1} H_0 \widehat{\mathscr{F}}) := \mathscr{S}(\mathbb{R}^3)$. By construction $\widehat{\mathscr{F}}^{-1} H_0 \widehat{\mathscr{F}}$ is symmetric, and it is easy to prove its essential self-adjointness by showing

$$Ker((\widehat{\mathscr{F}}^{-1} H_0 \widehat{\mathscr{F}})^* \pm iI) = \{\mathbf{0}\} ,$$

or by proving that each vector of $\mathscr{D}(\mathbb{R}^3) \subset \mathscr{S}(\mathbb{R}^3)$ is analytic for $\widehat{\mathscr{F}}^{-1} H_0 \widehat{\mathscr{F}}$. The same holds for H_0, since $\widehat{\mathscr{F}}$ is unitary.

By construction if $H := \overline{H_0}$, then $\widetilde{H} := \widehat{\mathscr{F}}^{-1} H \widehat{\mathscr{F}}$ acts as multiplicative operator:

$$\left(\widetilde{H}\widetilde{\psi}\right)(\mathbf{k}) = \frac{\hbar^2}{2m}\mathbf{k}^2\widetilde{\psi}(\mathbf{k}) ,$$

where

$$D(\widetilde{H}) = \left\{ \widetilde{\psi} \in L^2(\mathbb{R}^3, dk) \,\middle|\, \int_{\mathbb{R}^3} |\mathbf{k}|^4 |\widetilde{\psi}(\mathbf{k})|^2 dk < +\infty \right\} .$$

An alternative definition for H comes from taking the unique self-adjoint extension of H_0 defined on $\mathscr{D}(\mathbb{R}^3)$ instead of $\mathscr{S}(\mathbb{R}^3)$:

$$H_0 := \frac{1}{2m}\sum_{k=1}^{3} P_k\restriction^2_{\mathscr{D}(\mathbb{R}^3)} = -\frac{\hbar^2}{2m}\Delta\restriction_{\mathscr{D}(\mathbb{R}^3)} .$$

However, H_0 is still essentially self-adjoint and its self-adjoint extension is the previous H. Or, we could define H_0 on $\widehat{\mathscr{F}}(\mathscr{D}(\mathbb{R}^3))$, and find the same result. All this descends from Proposition 10.45.

(2) An interesting case in $\mathbb{R}^3$ is where the free Hamiltonian is modified by the potential energy of the *attractive Coulomb potential*:

$$V(\mathbf{x}) = \frac{eQ}{|\mathbf{x}|} ,$$

where $e < 0$, $Q > 0$ are constants expressing the electric charges of the particle and the centre of attraction respectively. The assumptions of Kato's theorem 10.50 (or 10.48) hold ($m, \hbar > 0$ are constants that play no role, since we can multiply the operator by $2m/\hbar^2$ without loss of generality). Therefore:

$$H_0 := -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})$$

is essentially self-adjoint, whether defined on $\mathscr{D}(\mathbb{R}^3)$ or $\mathscr{S}(\mathbb{R}^3)$. If $-Q = e$ is the charge of the electron ($-1.60 \cdot 10^{-19}$ C), and $m = m_e$ its mass ($9.11 \cdot 10^{-31}$ Kg), the only self-adjoint extension $\overline{H_0}$ corresponds to the Hamiltonian of an electron inside the electric field of a proton (neglecting spin effects and envisaging the proton as a classical object of infinite mass). This gives the simplest quantum description of the Hamiltonian operator of the hydrogen atom. Although V is not bounded from below, it is important to note the spectrum of the operator is always bounded, so also the admissible values of energy are constrained. This implies the hydrogen atom is an energetically stable system: it cannot collapse under an infinite energy loss caused by the interaction with the electromagnetic field (i.e. losing the energy of photons emitted by the atom: this way will not be treated in this physically-very-elementary book). The analogous classical model, for which the electron and the attractive centre are dimensionless points, would not have total energy bounded from below.[3] Studying the spectrum of $\overline{H_0}$ [Mes99, CCP82] shows $\sigma_c(\overline{H_0}) = [0, +\infty)$, while $\sigma_p(\overline{H_0}) = \{E_n\}_{n=1,2,\ldots}$, where

$$E_n = -\frac{2\pi R\hbar c}{n^2} \quad n = 1, 2, 3, \ldots \tag{13.15}$$

$R = me^4/(4\pi c\hbar^3)$ is the *Rydberg constant* and c the speed of light. Eigenvectors have a complicated expression [Mes99, CCP82]. For each of the values E_n, $n = 1, 2, 3$, the corresponding eigenspace has a finite basis in spherical coordinates:

$$\psi_{nlm}(r,\theta,\phi) = -\sqrt{\left(\frac{2}{na_0}\right)^2 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-\frac{r}{na_0}} \left(\frac{2r}{na_0}\right)^l L^{2l+1}_{n+l}\left(\frac{2r}{na_0}\right) Y^m_l(\theta,\phi)\,, \tag{13.16}$$

where $l = 0, 1, \ldots, n-1$ and $m = -l, -l+1, \ldots, l-1, l$. The maps Y^m_l are the spherical harmonics (10.44), $a_0 = \hbar^2/e^2 m_e = 0,529$ Å is the radius of Bohr's first orbit and $L^\alpha_n(x)$, for $x \geq 0$, is the **Laguerre polynomial**:

$$L^\alpha_n(x) := \frac{d^\alpha}{dx^\alpha}\left[e^x \frac{d^n}{dx^n}(x^n e^{-x})\right], \quad n \in \mathbb{N}, \alpha = 0, 1, \ldots, n.$$

[3] Such a classical model would not, anyway, be consistent because of the *Bremsstrahlung* of the accelerated electron; as is well known, this fact produces mathematical inconsistencies when the electronic radius tends to zero.

By examining the interaction between photons and the hydrogen atom [Mes99, CCP82] we know that the electron, initially in a stationary state determined by an eigenvector of $\overline{H_0}$ with eigenvalue E_n, can change state and pass to a new stationary state of energy $E_m < E_n$ transferring the excess of energy to a photon. The reverse process may occur, whereby the electron acquires energy from a photon and passes from a state of energy E_m to a state of energy E_n. Due to the interactions with photons, it can be proved that only the state of minimum energy $E_1 = 2\pi R\hbar c$, the so-called **ground state**, is stable, while the others are all unstable. The electron decays to the ground state after a certain mean lifetime to be determined. (Therefore the name *stationary state* is not completely appropriate for the system formed by an atom and the electromagnetic field described by photons. One should rather just speak of eigenvalues of the Hamiltonian for the hydrogen atom.) The collection of energy differences $E_n - E_m$ determine all possible photonic frequencies (light frequencies) that a gas of hydrogen atoms can emit or absorb, by Einstein's formula $E_n - E_m = h\nu_{n,m}$. The latter relates the frequency $\nu_{n,m}$ of photons emitted by the atom to the energy needed by photons that switch from energy E_n to E_m (see Chap. 6). XIX century spectroscopists, though puzzled by the values $\nu_{n,m}$, knew them long before QM was formulated [Mes99, CCP82]. Finding the same values and being able to *explain them* in a completely theoretical manner is certainly one of the pinnacles of physics in the past century.

(3) A second interesting situation, in $\mathbb{R}^3$, is that in which to the Hamiltonian of the free particle of example (1) we add the *Yukawa potential*:

$$V(\mathbf{x}) = \frac{-e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|},$$

where $\mu > 0$ is a positive constant. Here, too, $H_0 = -\frac{\hbar^2}{2m}\Delta + V(x)$ is essentially self-adjoint if either defined on $\mathscr{D}(\mathbb{R}^3)$ or on $\mathscr{S}(\mathbb{R}^3)$, because of Kato's theorem 10.50 (or 10.48). The Yukawa potential describes, roughly speaking, the interaction processes between a *pion* and the *strong force* originating from a macroscopic source.

(4) Still referring to example (1), the action of the evolution operator is evident using the Fourier representation:

$$(\widetilde{U}_t\widetilde{\psi})(\mathbf{k}) = \left(e^{-\frac{it}{\hbar}\widetilde{H}}\widetilde{\psi}\right)(\mathbf{k}) = e^{-\frac{it\hbar}{2m}\mathbf{k}^2}\widetilde{\psi}(\mathbf{k}). \tag{13.17}$$

The proof is immediate from the spectral decompositions of $\widetilde{H}$ and the commutation of the spectral measures of P_1, P_2, P_3:

$$e^{-\frac{it}{\hbar}\widetilde{H}} = e^{-\frac{it}{2\hbar m}\widetilde{P}_1^2}e^{-\frac{it}{2\hbar m}\widetilde{P}_2^2}e^{-\frac{it}{2\hbar m}\widetilde{P}_3^2},$$

where each $\widetilde{P}_j = \widehat{\mathscr{F}}^{-1}P_j\widehat{\mathscr{F}}$ multiplies by

$$\left(\widetilde{P}_j\widetilde{\psi}\right)(\mathbf{k}) = \hbar k_j\widetilde{\psi}(\mathbf{k}).$$

Back in position representation we look at the evolution of a wavefunction determining the state $U_t \rho U_t^*$ when $\rho = \psi(\psi|\ \ )$. This is

$$\psi(t, \mathbf{x}) := \left(e^{-i\frac{t}{\hbar}H}\psi\right)(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \widetilde{\psi}(\mathbf{k}) e^{-i\frac{\hbar t}{2m}\mathbf{k}^2}\, dk \tag{13.18}$$

where

$$\psi(\mathbf{x}) = \psi(0, \mathbf{x}) := \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \widetilde{\psi}(\mathbf{k})\, dk\ , \tag{13.19}$$

for $\psi \in \mathscr{S}(\mathbb{R}^3)$. In general the integrals should be understood in the sense of the Fourier-Plancherel transform.

(5) In the previous example Eq. (13.18) can be written without Fourier-transforming the initial datum ψ, as this proposition establishes.

Proposition 13.9 *Take $\psi \in \mathscr{S}(\mathbb{R}^3)$ and $H = \overline{-\frac{\hbar}{2m}\Delta}$, $\hbar, m > 0$ (the Laplacian Δ is initially defined on $\mathscr{S}(\mathbb{R}^3)$ or equivalently $\mathscr{D}(\mathbb{R}^3)$).*

(a) *For any given $t \in \mathbb{R}$, the map $\psi(t, \mathbf{x}) := \left(e^{-i\frac{t}{\hbar}H}\psi\right)(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, belongs to $\mathscr{S}(\mathbb{R}^3)$.*

(b) *If $t \neq 0$ and $\mathbf{x} \in \mathbb{R}^3$:*

$$\psi(t, \mathbf{x}) = \left(\frac{m\hbar}{2\pi i t}\right)^{3/2} \int_{\mathbb{R}^3} e^{im\hbar|\mathbf{x}-\mathbf{y}|^2/(2t)} \psi(\mathbf{y}) dy \tag{13.20}$$

where the multi-valued square root is computed by branching the complex plane along the negative real axis.

(c) *Let $C_\psi := \left(\frac{m\hbar}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} |\psi(\mathbf{x})| dx$. Then*

$$||\psi(t, \cdot)||_\infty \leq C_\psi |t|^{-3/2} \quad \textit{for every } t \neq 0\ . \tag{13.21}$$

Proof (a) The Fourier transform $\widetilde{\psi}$ of $\psi \in \mathscr{S}(\mathbb{R}^3)$ is in $\mathscr{S}(\mathbb{R}^3)$. Multiplying by the exponential $e^{-i\hbar\mathbf{k}^2/(2m)}$ produces a map of $\mathscr{S}(\mathbb{R}^3)$. Since $\mathscr{S}(\mathbb{R}^3)$ is Fourier-invariant, Eq. (13.18) implies $\psi(t, \cdot) \in \mathscr{S}(\mathbb{R}^3)$.
(b) Equation (13.18) can be rewritten using the Fourier transform and Lebesgue's dominated convergence:

$$\psi(t, \mathbf{x}) = \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} e^{-i\frac{\hbar t}{2m}\mathbf{k}^2} \left(\int_{\mathbb{R}^3} \frac{e^{-i\mathbf{k}\cdot\mathbf{y}}}{(2\pi)^3} \psi(\mathbf{y})\, dy\right) dk$$

$$= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} e^{-i\frac{\hbar(t-i\epsilon)}{2m}\mathbf{k}^2} \left(\int_{\mathbb{R}^3} \frac{e^{-i\mathbf{k}\cdot\mathbf{y}}}{(2\pi)^3} \psi(\mathbf{y})\, dy\right) dk\ .$$

If $\epsilon > 0$, Fubini-Tonelli allows to write

$$\psi(t,\mathbf{x}) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) - \hbar(t-i\epsilon)\mathbf{k}^2/(2m))} dk \right) \psi(\mathbf{y})\, dy\,.$$

The inner Gaussian integral can be computed explicitly (e.g., with residue techniques):

$$\psi(t,\mathbf{x}) = \lim_{\epsilon\to 0^+} \left(\frac{m\hbar}{2\pi i(t-i\epsilon)} \right)^{3/2} \int_{\mathbb{R}^3} e^{im\hbar|\mathbf{x}-\mathbf{y}|^2/(2(t-i\epsilon))} \psi(\mathbf{y}) dy\,.$$

For $t \neq 0$ and $\mathbf{x} \in \mathbb{R}^3$ fixed, we can take the limit inside the integral due to dominated convergence: the integrand, in fact, is in absolute value smaller than $|\psi| \in L^1(\mathbb{R}^3, dx)$, uniformly in $\epsilon > 0$. This gives (13.20).
(c) Follows from (b) directly. □

Equation (13.20) holds on $\mathbb{R}^d$ if we replace the exponent $3/2$ with $d/2$. (The wavefunctions $\psi(t,\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, evolve under the evolution operator generated by the self-adjoint closure of $-\frac{1}{2m}\Delta$, where $\Delta : \mathscr{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d, dx)$ is the Laplacian in d dimensions.)

Since the integral of $|\psi(t,\mathbf{x})|^2$ over $\mathbb{R}^3$ is constant in time, and $|\psi(t,\mathbf{x})|^2$ at any point $\mathbf{x} \in \mathbb{R}^3$ is infinitesimal by (13.21), a wavefunction that is initially non-zero on a small region in space must increase its support as time goes by, and "spread out" over increasingly larger regions. ■

13.1.4 The Action of the Galilean Group in Position Representation

Example 13.8(4) allows to make explicit, in position representation, the Galilean group's action, which we saw at the end of Chap. 12 in momentum representation for the free particle of spin s. If $(\tau, \mathbf{c}, \mathbf{v}, U)$ is the generic element of the universal covering $\widetilde{S\mathscr{G}}$ of the restricted Galilean group, the aforementioned representation is induced by the unitary operators $Z^{(m)}_{(\tau,\mathbf{c},\mathbf{v},U)}$ that act, in momentum representation, as (12.145):

$$\left(\widetilde{Z^{(m)}}_{(\tau,\mathbf{c},\mathbf{v},U)}\widetilde{\psi}\right)(\mathbf{k}) := e^{i(\tau\mathbf{v}-\mathbf{c})\cdot(\mathbf{k}-m\mathbf{v})} e^{i\frac{\tau}{2m}(\mathbf{k}-m\mathbf{v})^2} \widetilde{\psi}\left(R(U)^{-1}(\mathbf{k}-m\mathbf{v})\right)\,.$$

In position representation, anti-transforming with Fourier-Plancherel $\psi = \widehat{\mathscr{F}}^{-1}\widetilde{\psi}$ easily gives

$$\left(U_t Z^{(m)}_{(\tau,\mathbf{c},\mathbf{v},U)}\psi\right)(\mathbf{x}) = e^{im(\mathbf{v}\cdot\mathbf{x}-\mathbf{v}^2t/2)} \psi\left(t-\tau,\, R(U)^{-1}(\mathbf{x}-\mathbf{c}) - (t-\tau)R(U)^{-1}\mathbf{v}\right)$$

for $\psi \in L^2(\mathbb{R}^2, dx)$. Put otherwise, if $\psi'(t, \mathbf{x}) := \left(U_t Z^{(m)}_{(\tau,\mathbf{c},\mathbf{v},U)}\psi\right)(\mathbf{x})$ is the wavefunction acted upon by the element $(\tau, \mathbf{c}, \mathbf{v}, U)$ of the (universal covering of the) Galilean group at $t = 0$, which evolves to time t, we have:

$$\psi'(t, \mathbf{x}) = e^{im(\mathbf{v}\cdot\mathbf{x}-\mathbf{v}^2 t/2)}\psi\left((\tau, \mathbf{c}, \mathbf{v}, U)^{-1}(t, \mathbf{x})\right) \tag{13.22}$$

by (12.138). For particles with spin s, as we saw in the previous chapter, for fixed inertial frame the Hilbert space is $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$ and wavefunctions are unit vectors

$$\Psi = \sum_{s_z=-s}^{s} \psi_{s_z} \otimes |s, s_z\rangle\ ,$$

where $|s, s_z\rangle$ form the canonical basis of $\mathbb{C}^{2s+1}$ in which the spin operator S_z is diagonal with eigenvalues s_z.

By this decomposition $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$ becomes naturally isomorphic to the orthogonal sum of $2s + 1$ copies of $L^2(\mathbb{R}^3, dx)$. Consequently, the vectors Ψ define *spinors* of order s, that is, column vectors of wavefunctions for particles without spin:

$$\Psi \equiv (\psi_s, \psi_{s-1}, \cdots, \psi_{-s+1}, \psi_{-s})^t\ .$$

Similarly, let $\Psi'_t := \left(U_t Z^{(m)}_{(\tau,\mathbf{c},\mathbf{v},U)} \otimes V^{(s)}(U)\Psi\right)$, where $V^{(s)}(U)$ is the action of $U \in SU(2)$ on spinors for particles of spin s (cf. Sect. 12.3.1). Then the active Galilean action, in terms of spinors, reads:

$$\psi'_{s'_z}(t, \mathbf{x}) = e^{im(\mathbf{v}\cdot\mathbf{x}-\mathbf{v}^2 t/2)} \sum_{s_z=-s}^{s} V^{(s)}(U)_{s'_z s_z}\psi_{s_z}\left((\tau, \mathbf{c}, \mathbf{v}, U)^{-1}(t, \mathbf{x})\right)\ , \tag{13.23}$$

where $V^{(s)}(U)_{ij}$ is the matrix entry of $V^{(s)}(U)$ in the canonical basis of $\mathbb{C}^{2s+1}$.

Now think of Galilean transformations *passively*, hence view the $Z^{(m)}_{(\tau,\mathbf{c},\mathbf{v},U)}$ as unitary operators between *distinct Hilbert spaces associated to different frame systems* that describe the *same* physical system. We can thus describe the transformations of quantum states between different frame systems. The basic idea is that when one acts on a state by an active Galilean transformation, and then changes to the transformed reference system by the *same* active map, in the *new* frame the transformed state must look like the original, pre-transformation, one. Therefore the law of passive transformations of states (coordinate change) corresponds to the inverse active transformation seen above, meaning that we replace $(\tau, \mathbf{c}, \mathbf{v}, U)$ with $(\tau, \mathbf{c}, \mathbf{v}, U)^{-1}$ in (13.23). Let us see this recipe implemented. Take two inertial frames $\mathscr{I}, \mathscr{I}'$ with right-handed Cartesian coordinates x_1, x_2, x_3 and x'_1, x'_2, x'_3 and time coordinate t, t' respectively. Suppose the coordinate change is the Galilean transformation:

$$\begin{cases} t' = t + \tau \ , \\ x'_i = c_i + tv_i + \sum_{j=1}^{3} R_{ij} x_j \ , \quad \text{i = 1,2,3.} \end{cases} \tag{13.24}$$

where $\tau \in \mathbb{R}, c_i \in \mathbb{R}, v_i \in \mathbb{R}, R \in SO(3)$. Consider a particle of spin s, so the theory's Hilbert space is $\mathsf{H} := L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^{2s+1}$ for $\mathscr{I}$, and $\mathsf{H}' := L^2(\mathbb{R}'^3, dx') \otimes \mathbb{C}^{2s+1'}$ for $\mathscr{I}'$. The spaces $\mathbb{R}^3$ and $\mathbb{R}'^3$ are identified with the rest spaces of their frame systems by the coordinates. The canonical bases of $\mathbb{C}^{2s+1}$, $\mathbb{C}^{2s+1'}$ are the eigenvector bases of the spin operators along the third axes, S_3 and $S_{3'}$. Choose a matrix $U \in SU(2)$ whose image under the covering of $SO(3)$ is R. (Note the parameters $\mathbf{v}$, U also show up in the phase factor, and are given up to sign, as seen in the previous chapter: this sign may change the vectors representing a pure state, but does not alter the state itself.) Consider a pure state described in $\mathscr{I}$ by the unit vector Ψ and its evolution in $\mathscr{I}$. The state Ψ corresponds to a state Ψ' in $\mathscr{I}'$, together with its evolution. The relationship between the spinors Ψ and Ψ' evolves according to

$$\psi'_{s'_z}(t', \mathbf{x}') = e^{-im(\mathbf{v}\cdot R(U)\mathbf{x} + \mathbf{v}^2 t/2)} \sum_{s_z=-s}^{s} V^{(s)}(U)_{s'_z s_z} \psi_{s_z} \left(t + \tau, R(U)\mathbf{x} + \tau \mathbf{v} + \mathbf{c}\right) \ , \tag{13.25}$$

obtained replacing $(\tau, \mathbf{c}, \mathbf{v}, U)$ by $(\tau, \mathbf{c}, \mathbf{v}, U)^{-1}$ in (13.23) (the parameters $\mathbf{v}$, U also appear in the phase, and the ones of the inverse Galilean transformation must be used). For spin $s = 0$, in particular:

$$\psi'(t', \mathbf{x}') = e^{-im(\mathbf{v}\cdot R(U)\mathbf{x} + \mathbf{v}^2 t'/2)} \psi \left(t + \tau, R(U)\mathbf{x} + \tau \mathbf{v} + \mathbf{c}\right) \ , \tag{13.26}$$

where the coordinates $(t, \mathbf{x})$ and $(t', \mathbf{x}')$ are related by (13.24).

Remark 13.10 Notice how the term $e^{-im(\mathbf{v}\cdot R(U)\mathbf{x} + \mathbf{v}^2 t/2)}$ *cannot* be removed by taking another representative for the projective ray, since the phase depends on the variable $\mathbf{x}$. The resulting equation, therefore, is not the transformation we would expect intuitively, if we imagine that the spinless wavefunction and each component of the wavefunction with spin $s \neq 0$ are *scalar fields* on the spacetime of classical physics. The scalar-field interpretation of wavefunctions in position representation is *a priori* not automatic, and totally false (not just for one choice of phase) in relativistic theories, where wavefunctions in position representation (within the so-called *Newton-Wigner formalism* [Var07]) are highly nonlocal objects.[4] ∎

[4]One should not confuse a wavefunction in position representation with the field of second quantisation, which is a local object instead.

13.1.5 Basic Notions of Scattering Processes

Consider a quantum system, for instance a single quantum particle, described on the Hilbert space H (after an inertial system has been fixed) and whose evolution is given by a Hamiltonian operator $H = H_0 + V$. The term H_0 is the Hamiltonian of the "non-interacting theory" that we may think, to fix ideas, as described by the purely kinetic Hamiltonian of the particle, even if we could consider more involved multi-particle quantum systems. The other term V represents therefore the interaction with an external field or the self-interaction, and is often unknown or partially known. In certain circumstances, in the distant past or future a state described by ψ behaves "as if it evolved" under the non-interacting Hamiltonian H_0. This happens typically in *scattering* processes.

Consider for example one particle: initially free, it interacts briefly with a scattering centre – a system we can treat as semi-classical – and then becomes again free. Experimentally speaking, we can say the system is prepared at $t \to -\infty$ in an approximatively free state, and after the interaction, as $t \to +\infty$, it manifests itself in a state that can still be seen as free. Examining the difference between prepared state and observed state gives informations on the structure of the scattering centre, and more generally on the type of interaction described by V. In more complicated situations there is no scattering centre, and one has to deal with two or more particles, or even systems with an unknown number of particles that (self-)interact very briefly and return swiftly to a non-interacting setup.

We will introduce the basic mathematical ideas to formalise all that, referring the reader to advanced texts [ReSi80, Pru81, Mes99, CCP82] for details and generalisations to several particles (or relativistic processes with unknown number of particles).

The fact that *for certain state vectors in the system*, generically indicated by ϕ, the evolution in time is approximated by the non-interacting evolution in the far future, is expressed by

$$\lim_{t\to+\infty} ||e^{-itH}\phi - e^{-itH_0}\psi|| = 0 . \tag{13.27}$$

for some state ψ distinct from, but determined by, ϕ. Equivalently, since e^{itH} is unitary:

$$\lim_{t\to+\infty} ||\phi - e^{itH}e^{-itH_0}\psi|| = 0 . \tag{13.28}$$

The argument can be clearly replicated for $t \to -\infty$, to describe what happens long before the interaction takes place, when the evolution is taken to be free. For several reasons, both theoretical and experimental, it is convenient to describe scattering using vectors like ψ, *that evolve by the Hamiltonian of the non-interacting theory*, rather than ϕ, *which evolves under the interacting Hamiltonian*. This motivates the introduction of **wave operators** $\Omega_\pm$, also known as **Møller operators**:

$$\Omega_\pm\psi := \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\psi , \quad \psi \in \mathsf{H}, \tag{13.29}$$

assuming the limit exists.

If the operators $\Omega_\pm : \mathsf{H} \to \mathsf{H}$ exist they must be *isometries*, since they are strong limits of unitary operators. More precisely $\Omega_\pm$ are *partial isometries* (Definition 3.72) with initial space the whole H. Consequently the final spaces

$$\mathsf{H}_\pm := Ran(\Omega_\pm) \tag{13.30}$$

are closed in H by Proposition 3.73. By construction if $\phi_\pm \in \mathsf{H}_\pm$, so $\phi_\pm = \Omega_\pm \psi$ for some $\psi \in \mathsf{H}$, it follows[5]

$$||e^{-itH}\phi_\pm - e^{-itH_0}\psi|| \to 0 \quad \text{as } t \to \pm\infty. \tag{13.31}$$

Hence $\mathsf{H}_\pm$ determine the class of states whose long-time future evolution, or long-time past evolution, can be approximated by the free evolution of the states obtained by swapping the $\Omega_\pm$. Equation (13.31), exactly as we wanted, tells that the state *of the interacting system* $\phi_\pm$, evolving under the full Hamiltonian $H = H_0 + V$, has the asymptotic behaviour (as $t \to \pm\infty$, respectively) of the state ψ *in the non-interacting system*, which evolves under the free Hamiltonian H_0.
In detail:

Proposition 13.11 *If the surjective isometry $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ of (13.29) is defined, then*

$$e^{-itH}\Omega_\pm = \Omega_\pm e^{-itH_0}\,. \tag{13.32}$$

Consequently

$$e^{-itH}\mathsf{H}_\pm \subset \mathsf{H}_\pm, \quad e^{-itH}\restriction_{\mathsf{H}_\pm} = \Omega_\pm e^{-itH_0}\Omega_\pm^{-1}, \quad H\restriction_{\mathsf{H}_\pm \cap D(H)} = \Omega_\pm H_0 \Omega_\pm^{-1}, \tag{13.33}$$

and in particular:

$$\sigma(H\restriction_{\mathsf{H}_\pm \cap D(H)}) = \sigma(H_0)\,. \tag{13.34}$$

Proof As for the first statement

$$e^{-itH}\Omega_\pm\psi = \lim_{s\to\pm\infty} e^{i(s-t)H}e^{-isH_0}\psi = \lim_{z\to\pm\infty} e^{izH}e^{-izH_0}e^{-itH_0}\psi = \Omega_\pm e^{-itH_0}\psi\,,$$

whence $e^{-itH}\mathsf{H}_\pm \subset \mathsf{H}_\pm$ and $e^{-itH}\restriction_{\mathsf{H}_\pm} = \Omega_\pm e^{-itH_0}\Omega_\pm^{-1}$. Stone's theorem easily implies the other relation. Eventually, $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ being isometric proves (13.34) by the last identity in (13.33) (Exercise 8.9). $\square$

For the usual non-relativistic particle the non-interacting Hamiltonian H_0, that accounts for the kinetic energy only, has spectrum $\sigma(H_0) = \sigma_c(H_0) = [0, +\infty)$. Under the above proposition's assumptions, then, $\sigma(H\restriction_{\mathsf{H}_\pm \cap D(H)}) = \sigma_c(H\restriction_{\mathsf{H}_\pm \cap D(H)}) = [0 + \infty)$.

[5] In fact $||e^{-itH}\Omega_\pm\psi - e^{-itH_0}\psi|| = ||\Omega_\pm\psi - e^{itH}e^{-itH_0}\psi|| \to ||\Omega_\pm\psi - \Omega_\pm\psi|| = 0$ as $t \to \pm\infty$.

So let us assume the wave operators $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ do exist on some physical system, and suppose the system has been prepared in a state that, as $t \to -\infty$, tends to be described by the non-interacting evolution of $e^{-itH_0}\psi_{in}$. Hence the system's real state will be described, at $t = 0$, by the state $\phi_- := \Omega_-\psi_{in}$. After the interaction, as $t \to +\infty$, the state will be described approximatively by a non-interacting vector $e^{-itH_0}\psi_{out}$. The real state, at $t = 0$, is described by $\phi_+ := \Omega_+\psi_{out}$. The probability of the process is thus:

$$|(\phi_+|\phi_-)|^2 = |(\Omega_+\psi_{out}|\Omega_-\psi_{in})|^2 = |(\psi_{out}|\Omega_+^*\Omega_-\psi_{in})|^2 .$$

Define the **scattering operator**, also called **S matrix**:

$$S := \Omega_+^*\Omega_- : \mathsf{H} \to \mathsf{H} . \tag{13.35}$$

The transition amplitude from a state that behaves as a non-interacting state $e^{-itH_0}\psi_{in}$, as $t \to -\infty$, to the state that behaves like a non-interacting state $e^{-itH_0}\psi_{out}$ as $t \to +\infty$, equals:

$$(\psi_{out}|S\psi_{in}) . \tag{13.36}$$

In this picture, the interaction V is completely "withheld" by S, while we can consider the states $\psi_{in/out}$ as being indeed free. Overall we have, as we were saying at the beginning, a recipe to describe the scattering in terms of states in a non-interacting system. To conclude, we have a proposition.

Proposition 13.12 *If the surjective isometries $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ of (13.29) exist, and $\mathsf{H}_+ = \mathsf{H}_-$ (in particular, under* asymptotic completeness*, see Remark 13.13(1)), the scattering operator (13.35) is unitary.*

Proof It is enough to prove $S^*S = SS^* = I$. Since $\Omega_\pm$ is a partial isometry with initial space H and final space $\mathsf{H}_\pm$, by Proposition 3.74

$$\Omega_\pm^*\Omega_\pm = I , \quad \Omega_\pm\Omega_\pm^* = P_{\mathsf{H}_\pm} .$$

where $P_{\mathsf{H}_\pm} : \mathsf{H} \to \mathsf{H}$ is the orthogonal projector onto $\mathsf{H}_\pm$. Therefore

$$S^*S = \Omega_-^*\Omega_+\Omega_+^*\Omega_- = \Omega_-^*P_{\mathsf{H}_+}\Omega_- = \Omega_-^*P_{\mathsf{H}_-}\Omega_- = \Omega_-^*\Omega_- = I .$$

Similarly,

$$SS^* = \Omega_+^*\Omega_-\Omega_-^*\Omega_+ = \Omega_+^*P_{\mathsf{H}_-}\Omega_+ = \Omega_+^*P_{\mathsf{H}_+}\Omega_+ = \Omega_+^*\Omega_+ = I ,$$

ending the proof. □

Remarks 13.13 **(1)** Next to $\mathsf{H}_\pm$ it is useful to introduce the **space of stationary states** H_p, given by the closure of the span of eigenvectors of H, which describe stationary states (see Remark 9.15(2)). Physically, one expects elements $\phi \in \mathsf{H}_p$ to represent precisely states whose evolution *cannot* be approximated, at large times, by non-interacting states. That is because the evolution of such a state $\phi(\phi|\ )$ (with $||\phi|| = 1$) is trivial, for ϕ is an eigenvector of H. What we expect, said more accurately, is to have an orthogonal sum

$$\mathsf{H} = \mathsf{H}_\pm \oplus \mathsf{H}_p \,. \tag{13.37}$$

If this happens one speaks about **asymptotic completeness**. Note that (13.37) implies:

$$\mathsf{H}_+ = \mathsf{H}_- = \mathsf{H}_p^\perp \,, \tag{13.38}$$

and by (13.34), also

$$\sigma_{pc}(H) = \sigma(H_0) \tag{13.39}$$

(Remark 9.15(2)). At last, asymptotic completeness and (13.38) make the operator S unitary by Proposition 13.12.

(2) The next easy result relates the orthogonality of $\mathsf{H}_\pm$ and H_p with the properties of the evolution operator generated by H_0.

Proposition 13.14 *If the surjective isometry $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ of (13.29) exists, and $(\psi|e^{-itH_0}\psi') \to 0$ as $t \to \pm\infty$ for any $\psi, \psi' \in \mathsf{H}$, then $\mathsf{H}_\pm \perp \mathsf{H}_p$.*

Proof Define $\phi_\pm := \Omega_\pm\psi$ and suppose $H\phi_E = E\phi_E$. Then

$$(\phi_\pm|\phi_E) = \lim_{t\to\pm\infty}(e^{itH}e^{-itH_0}\psi|\phi_E) = \lim_{t\to\pm\infty}(e^{-itH_0}\psi|e^{itH}\phi_E)$$

$= \lim_{t\to\pm\infty} e^{-iEt}(e^{-itH_0}\psi|\phi_E) = 0.$ □

(3) The short and compressed description of scattering theory we have presented does not work in Quantum Field Theory: defining the unitary operators $\Omega_\pm$ is *not possible* under simple, physically plausible hypotheses on spatial homogeneity (the theory's invariance under the group of space translations). The obstruction is exquisitely theoretical and goes under the name of *Haag theorem* [Haa96]. In order to overcome the problem we can turn to the *LSZ formalism* [Haa96], in which scattering descriptions employ the weak topology. However, these issues assume an ivory-tower flavour, so to speak, when compared to the much more serious problem of *renormalisation*. ■

Example 13.15 Take a free spinless particle (in the sequel $\hbar = 1$) of mass $m > 0$, subject to a square-integrable potential V on $\mathbb{R}^3$ in a given inertial system. Then $\mathsf{H} = L^2(\mathbb{R}^3, dx)$, $H_0 = \overline{-\frac{1}{2m}\Delta}$ (the Laplacian Δ is as usual initially defined on $\mathscr{D}(\mathbb{R}^3)$ or $\mathscr{S}(\mathbb{R}^3)$), and $V \in L^2(\mathbb{R}^3)$. Theorem 10.48 (redefining the coordinates of $\mathbb{R}^3$ so to comprise the factor $(2m)^{-1}$) guarantees $H = H_0 + V$ is self-adjoint on $D(H_0)$, so $D(H) = D(H_0)$.

We wish to show the wave operators $\Omega_\pm$ are well defined, and that $\mathsf{H}_\pm \perp \mathsf{H}_p$. First, a technical lemma.

Lemma 13.16 *If* $\mathsf{H} = L^2(\mathbb{R}^3, dx)$, $H_0 = \overline{-\frac{1}{2m}\Delta}$ *and* $V \in L^2(\mathbb{R}^3)$, *let* $H = H_0 + V$ *and* $U_0(t) := e^{-itH_0}$, $U(t) := e^{-itH}$. *Then*

$$\frac{d}{dt}U(-t)U_0(t)\psi = U(-t)iVU_0(t)\psi \ , \quad \psi \in D(H_0) = D(H). \tag{13.40}$$

Hence, for $T > t$:

$$||(U(-T)U_0(T) - U(-t)U_0(t))\psi)|| \leq \int_t^T ||VU_0(s)\psi||ds \ , \tag{13.41}$$

Proof Set $\Omega_t := U(-t)U_0(t)$. Then

$$\frac{d}{dt}\Omega_t\psi = \lim_{h\to 0}\frac{U(-(t+h))U_0(t+h) - U(-t)U_0(t)}{h}\psi \ .$$

Decompose the derivative

$$\frac{d}{dt}\Omega_t\psi = \lim_{h\to 0}\frac{U(-(t+h))(U_0(t+h) - U_0(t))}{h}\psi + \lim_{h\to 0}\frac{(U(-(t+h)) - U(-t))U_0(t)}{h}\psi \ .$$

Since $U_0(t)D(H_0) \subset D(H_0) = D(H)$, Stone's theorem shows the second limit equals

$$U(-t)iHU_0(t)\psi \ .$$

As for the first limit, we compute the norm squared of the difference between $-iU(-t)H_0U_0(t)\psi$ and $\frac{U(-(t+h))(U_0(t+h)-U_0(t))}{h}\psi$. By Stone's theorem and the unitarity of $U(-(t+h))$:

$$\lim_{h\to 0}\frac{U(-(t+h))(U_0(t+h) - U_0(t))}{h}\psi = -iU(-t)H_0U_0(t)\psi \ .$$

As $H - H_0 = V$, we have:

$$\frac{d}{dt}\Omega_t\psi = U(-t)iVU_0(t)\psi \ ,$$

for $\psi \in D(H_0)$, so for any $\phi \in \mathsf{H}$

$$\frac{d}{dt}(\phi|\Omega_t\psi) = (\phi|U(-t)iVU_0(t)\psi) \ .$$

The right-hand side is continuous, so the fundamental theorem of calculus gives

$$(\phi|\Omega_T\psi) - (\phi|\Omega_t\psi) = \int_t^T (\phi|U(-s)iVU_0(s)\psi)ds \ .$$

But $U(s)$ is unitary,

$$|(\phi|(\Omega_T - \Omega_t)\psi)| \leq \int_t^T ||\phi||||VU_0(s)\psi||ds\,,$$

and choosing $\phi = (\Omega_T - \Omega_t)\psi$ we recover (13.41):

$$||(\Omega_T - \Omega_t)\psi)|| \leq \int_t^T ||VU_0(s)\psi||ds\,,$$

ending the proof. □

This lemma allows to prove the existence of wave operators on the system considered.

Proposition 13.17 *Take* $\mathsf{H} = L^2(\mathbb{R}^3, dx)$, $H_0 = \overline{-\frac{1}{2m}\Delta}$, $V \in L^2(\mathbb{R}^3)$, *and consider the self-adjoint operator* $H = H_0 + V$.
(a) *The wave operators* $\Omega_\pm : \mathsf{H} \to \mathsf{H}_\pm$ *in (13.29) are well defined.*
(b) $\mathsf{H}_\pm \perp \mathsf{H}_p$.

Remark 13.18 The theorem applies to the special case where V is a Yukawa potential (Example 13.8(3)). One can reach a stronger conclusion: by assuming $V \in L^1(\mathbb{R}^3, dx) \cap L^2(\mathbb{R}^3, dx)$, as for the Yukawa potential, asymptotic completeness holds, hence the scattering operator is unitary [ReSi80]. ■

Proof (a) Let us begin with the existence of Ω_+, for Ω_- is similar. If $\psi \in \mathscr{S}(\mathbb{R}^3) \subset D(H_0) = D(H)$, estimate (13.41) implies immediately:

$$||(U(-T)U_0(T) - U(-t)U_0(t))\psi)|| \leq \int_t^T ||V||_2||U_0(s)\psi||_\infty ds$$

because if $\psi \in \mathscr{S}(\mathbb{R}^3)$ then $U_0(t)\psi \in \mathscr{S}(\mathbb{R}^3)$ (cf. Example 13.8(5)). Using (13.21) we find:

$$||(U(-T)U_0(T)\psi - U(-t)U_0(t)\psi)|| \leq 2C_{V,\psi}\left(\frac{1}{\sqrt{|t|}} - \frac{1}{\sqrt{|T|}}\right)\,. \tag{13.42}$$

This shows every sequence of vectors $\psi_n := U(-t_n)U_0(t_n)\psi$ is a Cauchy sequence when $t_n \to +\infty$ for $n \to +\infty$, so it converges to $\phi \in \mathsf{H}$. On the other hand Eq. (13.42) proves such limit does not depend on the sequence chosen. Hence if $\psi \in \mathscr{S}(\mathbb{R}^3)$ there exist a (unique) $\phi \in \mathsf{H}$ so that

$$\lim_{t\to+\infty} U(-t)U_0(t)\psi = \phi\,. \tag{13.43}$$

This extends easily to $\psi \in \mathsf{H}$, because $\mathscr{S}(\mathbb{R}^3)$ is dense in H. Let us prove the latter assertion. Set $\Omega_t := U(-t)U_0(t)$. By the above considerations $\Omega'\psi := \lim_{t\to+\infty} \Omega_t\psi$ is well defined provided $\psi \in \mathscr{S}(\mathbb{R}^3)$. Since this space is dense in

H and every Ω_t is isometric, the operator Ω' extends to a linear isometry on H. To conclude it suffices to prove $\Omega_t\psi \to \Omega'\psi$, $t \to +\infty$, for any $\psi \in \mathsf{H}$. If $\psi \in \mathsf{H}$ consider a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^3)$ with $\psi_n \to \psi$, $n \to +\infty$ in H. Then

$$||\Omega_t\psi - \Omega'\psi|| \leq ||\Omega_t\psi - \Omega_t\psi_n|| + ||\Omega_t\psi_n - \Omega'\psi_n|| + ||\Omega'\psi_n - \Omega'\psi|| \,.$$

Since Ω_t and Ω' are isometric, we can rewrite it as

$$||\Omega_t\psi - \Omega'\psi|| \leq ||\psi - \psi_n|| + ||\Omega_t\psi_n - \Omega'\psi_n|| + ||\psi_n - \psi|| \,.$$

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ large enough so that $||\psi - \psi_n|| < 2\epsilon/3$. For that n, by the first part of the proof, we can pick $T \in \mathbb{R}$ so that $||\Omega_t\psi_n - \Omega'\psi_n|| < \epsilon/3$ for $t > T$. Hence we can determine, for every $\epsilon > 0$, $T \in \mathbb{R}$ such that $t > T$ gives $||\Omega_t\psi - \Omega'\psi|| \leq \epsilon$. And this holds for any $\psi \in \mathsf{H}$, ending (a).
(b) It is enough to prove, in our hypotheses, that Proposition 13.14 holds. We will show that this descends from (13.21). Fix $\psi, \phi \in \mathsf{H}$ and consider corresponding sequences $\psi_n, \phi_n \in \mathscr{S}(\mathbb{R}^3)$ with $\psi_n \to \psi$, $\phi_n \to \phi$ in H as $n \to +\infty$. If $\delta_n := \psi - \psi_n$ and $\delta'_n := \psi' - \psi'_n$, we have

$$|(\psi|U_0(t)\psi')| \leq |(\delta_n|U_0(t)\delta'_n)| + |(\delta_n|U_0(t)\psi'_n)| + |(\psi_n|U_0(t)\delta'_n)| + |(\psi_n|U_0(t)\psi'_n)|.$$

By using Schwarz's inequality and the fact $U_0(t)$ is isometric we find

$$|(\psi|U_0(t)\psi')| \leq ||\delta_n||||\delta'_n|| + ||\delta_n||||\psi'_n|| + ||\psi_n||||\delta'_n|| + |(\psi_n|U_0(t)\psi'_n)| \,.$$

But the norm is obviously continuous in H, and $\delta_n \to 0$ and $\delta'_n \to 0$ as $n \to +\infty$. Hence for any given $\epsilon > 0$, $||\delta_n||||\delta'_n||$, $||\delta_n||||\psi'_n||$ and $||\psi_n||||\delta'_n||$ are all smaller than $\epsilon/4$ for some large $n \in \mathbb{N}$. Therefore

$$|(\psi|U_0(t)\psi')| \leq 3\epsilon/4 + |(\psi_n|U_0(t)\psi'_n)| \,.$$

Computing the inner product on $L^2(\mathbb{R}^3, dx)$ explicitly, and since $\psi_n, \psi'_n, U_0(t)\psi'_n \in \mathscr{S}(\mathbb{R}^3)$, we obtain

$$|(\psi_n|U_0(t)\psi'_n)| \leq ||U_0(t)\psi'_n||_\infty \int_{\mathbb{R}^3} |\psi_n(\mathbf{x})|dx \,.$$

By (13.21) there exists $T > 0$ for which the right-hand side above is bounded by $\epsilon/4$ when $t > T$. Altogether, for any pair $\psi, \psi' \in \mathsf{H}$, if $\epsilon > 0$ there is $T > 0$ such that $|(\psi|U_0(t)\psi')| \leq \epsilon$ whenever $t > T$. □

■

13.1.6 The Evolution Operator in Absence of Time Homogeneity and Dyson's Series

We return to the evolution operator to discuss a generalisation that has to do with Schrödinger's equation. An important remark, made in axiom **A6**, is that the evolution operator U_τ is actually *independent of the initial instant*. If we fix the state ρ at initial time t, $U_\tau \rho U_\tau^*$ will be the state at time $t+\tau$. Had we fixed the same state ρ at initial time $t' \neq t$, the state at time $t'+\tau$ would have been $U_\tau \rho U_\tau^*$ again. So the system's laws of dynamics are unaffected in the time interval $[t, t']$. In other terms axiom **A6** adopts, for the system S in the frame $\mathscr{I}$, the *homogeneity of time*. Classically, this situation corresponds to having the Hamiltonian not explicitly dependent on time in the coordinates of a certain frame. This is not the case in more general dynamical situations, like when S interacts with an evolving external world. If, on the contrary, S is isolated (though this is not the only possibility) and we describe it in an inertial system, then time is homogeneous, as in classical mechanics.

But if time is not homogeneous, time evolution is axiomatised as follows.

A6'. *Let the quantum system S be described in an (inertial) reference frame $\mathscr{I}$, with space of states H_S. There exists a family $\{U(t_2, t_1)\}_{t_2, t_1 \in \mathbb{R}}$ of unitary operators on H_S, called* **evolution operators** *from t_1* to *t_2, satisfying, for $t, t', t'' \in \mathbb{R}$:*

(i) $U(t, t) = I$,

(ii) $U(t'', t')U(t', t) = U(t'', t)$,

(iii) $U(t', t) = U(t, t')^* = U(t', t)^{-1}$

and such that the function $\mathbb{R}^2 \ni (t, t') \mapsto U(t, t')$ b is strongly continuous.

Furthermore, if ρ is the state at time t_0, the evolved state at time t_1 (which may precede t_0) is $U(t_1, t_0)\rho U(t_1, t_0)^*$.

The main difference with axiom **A6** is that now we cannot associate a self-adjoint generator to the family $\{U(t_2, t_1)\}_{t_2, t_1 \in \mathbb{R}}$. What is more, speaking of Hamiltonian of the system makes no longer sense, in general. We may still retain such a notion nonetheless (in the sense of a *time-dependent* Hamiltonian) by generalising Schrödinger's equation and defining the $U(t', t)$ as its solutions. Formally, the operator U_τ of **A6** satisfies Schrödinger's equation (with $\hbar = 1$):

$$s\text{-}\frac{d}{d\tau} U_\tau = -iHU_\tau \ .$$

For the generalised evolution operator $U(t', t)$, we can assume an analogous equation

$$s\text{-}\frac{d}{d\tau} U(\tau, t) = -iH(\tau)U(\tau, t) \ , \tag{13.44}$$

whenever to each instant τ an observable is assigned, called **Hamiltonian at time** τ, that expresses the system's energy (in the given frame) at time τ. This energy is, in general, not a preserved quantity. In order to treat Eq. (13.44) rigorously we must address a few delicate technical problems concerning the distinct domains of the various $H(\tau)$. Despite that, the equation remains extremely useful in a number of practical applications. The *Dyson series*, pivotal in Quantum Electrodynamics and Quantum Field Theory, is a formal solution to (13.44). To this end let us prove a result that illustrates the simplified situation where each Hamiltonian $H(\tau)$ is bounded and defined on the entire Hilbert space. In that case the collection of the $H(\tau)$ determines, via (13.44), a *continuous* family of evolution operators $U(t', t)$ given by the Dyson series.

Proposition 13.19 *Let* H *be a Hilbert space and* $\mathbb{R} \ni t \mapsto H(t) = H(t)^* \in \mathfrak{B}(\mathsf{H})$ *strongly continuous. Consider the* **Dyson series** *of the operators* $U(t, s)$*:*

$$U(t,s) := I + \sum_{n=1}^{\infty}(-i)^n \int_s^t dt_1 H(t_1) \int_s^{t_1} dt_2 H(t_2) \cdots \int_s^{t_{n-1}} dt_{n-1} H(t_n) \quad (13.45)$$

where iterated integrals are defined as in Proposition 9.31. Then the series converges uniformly. Moreover:
(a) *the* $U(t, s)$ *are unitary and satisfy (i), (ii), (iii) in* **A6'***;*
(b) *the map* $\mathbb{R} \ni (t, s) \mapsto U(t, s)$ *is continuous in the uniform topology;*
(c) *the* **generalised Schrödinger equation** *holds:*

$$s-\frac{d}{dt}U(t,s) = -iH(t)U(t,s) \quad \textit{for every } t, s \in \mathbb{R}; \quad (13.46)$$

(d) *expression (13.45) may be written:*

$$U(t,s) = \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int_s^t \int_s^t \cdots \int_s^t T\left[H(t_1)H(t_2)\cdots H(t_n)\right] dt_1 dt_2 \cdots dt_n \,. \quad (13.47)$$

Above,

$$T[H(t_1)H(t_2)\cdots H(t_n)] := H(\tau_n)H(\tau_{n-1})\cdots H(\tau_1)$$

is the **chronological reordering operator** *of the product:* τ_n *is the largest among* $t_1, \ldots, t_n$*, then comes* $\tau_{n-1} \leq \tau_n$ *as second-largest and so on for every* $t_1, \ldots, t_n$*, up to the smallest value* τ_1*.*

Proof First of all every term in Dyson's expansion

$$U_n(t,s) = (-i)^n \int_s^t dt_1 H(t_1) \int_s^{t_1} dt_2 H(t_2) \cdots \int_s^{t_{n-1}} dt_{n-1} H(t_n)$$

makes sense, since by Proposition 9.31(c) each integral on the right, starting from the right-most $(t_{n-1}, s) \mapsto \int_s^{t_{n-1}} dt_{n-1} H(t_n)$, is an operator-valued map ranging in $\mathfrak{B}(\mathsf{H})$ and jointly strongly continuous in the integration limits (hence in the upper limit alone, too). The product (as pointwise operation) of two such maps is still strongly continuous and valued in $\mathfrak{B}(\mathsf{H})$, so it can be integrated. Using Proposition 9.31, where now the L^1 map is the characteristic function of the interval $[s, t_k]$, the nth term $U_n(t,s)$ in Dyson's series, $t, s \in [T, S]$, satisfies

$$||U_n(t,s)|| \leq A_{a,b} := \frac{|b-a|^n}{n!}\left(\sup_{\tau\in[a,b]} ||H(\tau)||\right)^n , \quad (t,s) \in [a,b]^2 . \quad (13.48)$$

As we observed in the proof of Proposition 9.31, since $\tau \mapsto H(\tau)$ is strongly continuous, $\sup_{\tau\in[a,b]} ||H(\tau)|| < +\infty$ by Banach-Steinhaus. Hence $0 \leq A_{a,b} < +\infty$. Since the series of positive terms $A_{a,b}$ converges, the Dyson series converges in the uniform topology, uniformly in (s,t) on every compact set. Therefore if every Dyson term is uniformly continuous (proved next) then $(t,s) \mapsto U(t,s)$ is uniformly continuous. To show that the Dyson terms are uniformly continuous, we must resort to their recurrence relation:

$$U_n(t,s) = -i \int_s^t H(\tau)U_{n-1}(\tau,s)d\tau . \quad (13.49)$$

It implies, working on the compact set $[a,b] \times [a,b]$,

$$||U_n(t,s) - U_n(t',s')|| \leq \left|\left|\int_{t'}^t H(\tau)U_{n-1}(\tau,s)d\tau\right|\right|$$

$$+\left|\left|\int_s^{t'} H(\tau)(U_{n-1}(\tau,s) - U_{n-1}(\tau,s'))d\tau\right|\right| + \left|\left|\int_s^{s'} H(\tau)U_{n-1}(\tau,s')d\tau\right|\right| ,$$

so by Proposition 9.31(a):

$$||U_n(t,s) - U_n(t',s')|| \leq |t-t'| \sup_{(\tau,\sigma)\in[a,b]^2} ||H(\tau)|| ||U_{n-1}(\tau,\sigma)||$$

$$+(b-a) \sup_{\tau\in[a,b]} ||H(\tau)|| ||U_{n-1}(\tau,s) - U_{n-1}(\tau,s')||$$

$$+|s-s'| \sup_{(\tau,\sigma)\in[a,b]^2} ||H(\tau)|| ||U_{n-1}(\tau,\sigma)|| .$$

Hence if $(t,s) \mapsto U_{n-1}(t,s)$ is uniformly continuous, so is $(t,s) \mapsto U_n(t,s)$; in particular

$$\sup_{\tau\in[a,b]} ||H(\tau)|| ||U_{n-1}(\tau,s) - U_{n-1}(\tau,s')|| \to 0 \quad \text{as } s \to s'$$

because the continuity of $(t,s) \mapsto U_{n-1}(t,s)$ on $[a,b]^2$ implies uniform continuity (besides, $\sup_{\tau\in[a,b]} ||H(\tau)|| < +\infty$ exists). The induction principle tells we can just prove that

$$U_1(t, s) := -i \int_s^t dt_1 H(t_1)$$

is continuous. But this is true by (i) in Proposition 9.31(c). With this we proved (b) and part of (a). To finish (a) we will use (c), so let us prove that first. Applying Proposition 9.31(b, c) to the terms of the Dyson series computed on ψ, differentiating term by term and using (13.49), we arrive at

$$\frac{d}{dt}U(t, s)\psi = -iH(t)U(t, s)\psi, \quad \text{i.e.} \quad s-\frac{d}{dt}U(t, s) = -iH(t)U(t, s), \quad (13.50)$$

provided we can exchange sum and derivative. Using (13.48) together with

$$\sup_{t\in[a,b]} ||H(t)|| < +\infty$$

tells the derivatives' series converges uniformly on compact sets in the uniform topology, hence uniformly in the strong topology. Hence the Dyson series can be differentiated in t (strongly) term by term, which proves (13.50) and thus (c). Now we can finish claim (a). With a similar procedure, in particular employing Proposition 9.31(ii), we obtain

$$\frac{d}{ds}(\phi|U(t, s)\psi) = i(\phi|U(t, s)H(s)\psi) .$$

From this and (13.50) follows

$$\frac{d}{ds}(\phi|U(t, s)U(s, t)\psi) = i(\phi|U(t, s)(H(s) - H(s))U(s, t)\psi) = 0 ,$$

so in particular $(\phi|U(t, s)U(s, t)\psi) = (\phi|U(t, t)U(t, t)\psi)$. But $U(t, t) = I$, so $U(s, t) = U(t, s)^{-1}$. From (13.50) we have

$$\frac{d}{dt}||U(t, s)\psi||^2 = \frac{d}{dt}(U(t, s)\psi|U(t, s)\psi) .$$

The right-hand side is easy, and equals

$$(-iH(t)U(t, s)\psi|U(t, s)\psi) + (U(t, s)\psi| - iH(t)U(t, s)\psi) = 0$$

by linearity in the right entry, antilinearity in the left, and because $H(t) = H(t)^*$. In other words $||U(t, s)\psi|| = ||U(s, s)\psi|| = ||\psi||$. Consequently every $U(t, s)$ is unitary, being isometric and onto. So we proved

$$U(t, s)^* = U(s, t) = U(t, s)^{-1} .$$

There remains to check (iii) of **A6'**. The operator $V(t,s) := U(t,s) - U(t,u)U(u,s)$ clearly satisfies $\frac{d}{dt}V(t,s)\psi = -iH(t)V(t,s)\psi$. Exactly as before

$$\frac{d}{dt}||V(t,s)\psi||^2 = \frac{d}{dt}(V(t,s)\psi|V(t,s)\psi) = 0\,,$$

by the inner product's linearity and by $H(t) = H(t)^*$. Hence $||V(t,s)\psi|| = ||V(s,s)\psi||$. But this is null, for $U(s,s) = I$ and $U(s,u)U(u,s) = I$. Eventually, then,

$$U(t,s)\psi = U(t,u)U(u,s)\psi \quad \text{for every } \psi \in \mathsf{H}.$$

To show (13.47) it suffices, starting from the last relation, to express the iterated integrals of each series using suitable maps θ, and change names to variables, to get (13.45). For instance

$$T[H(t_1)H(t_2)] = \theta(t_1 - t_2)H(t_1)H(t_2) + \theta(t_2 - t_1)H(t_2)H(t_1)\,,$$

where $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ otherwise. Integrating the sum in $dt_1 dt_2$ over $[t,s]^2$, and swapping t_1, t_2 in the term with $\theta(t_2 - t_1)$, produces the second summand on the right in (13.45), apart from the constant $(-i)^2/2!$. □

Remarks 13.20 **(1)** Dyson's series, written as in (13.47), resembles the expansion of the time-ordered exponential. For that reason the series is often encountered, with $\hbar$ back in, in the integral form:

$$U(t,s) = T\left[e^{-\frac{i}{\hbar}\int_s^t H(\tau)d\tau}\right]. \tag{13.51}$$

If H does not depend on time, the right-hand side reduces precisely to $e^{-i\frac{(t-s)}{\hbar}H}$ as expected.

(2) We have already noted that the Dyson series is central in Quantum Field Theory. It is even more fundamental in perturbation theory, where the Hamiltonian decomposes as $H = H_0 + V$, and V is a correcting term to H_0 and to the dynamics it generates. In such cases one proceeds by the so-called *Dirac's interaction picture* [Mes99, CCP82], in which the Dyson series plays a key part. In general concrete applications the Dyson series is used also when H is not bounded. For that reason the above theorem does not apply and the series should be understood in a weak sense of sorts [ReSi80]. ■

13.1.7 Anti-unitary Time Reversal

Let us return to general matters in relation to the time-evolution axiom **A6**, i.e. *under time homogeneity*, and show two more important corollaries to the lower boundedness of the spectrum of the Hamiltonian H.

In the previous chapter we saw that if a system admits a symmetry (whether Kadison or Wigner is irrelevant to Theorem 12.14), the latter is a unitary or anti-unitary transformation. If a system S with Hamiltonian H possesses the *time reversal* symmetry $\gamma_{\mathscr{T}}$ (cf. Example 12.19(2)), the unitary or anti-unitary map $\mathscr{T} : \mathsf{H}_S \to \mathsf{H}_S$ it determines (suppose the Hilbert space has one coherent sector) must satisfy

$$\gamma_{\mathscr{T}}\left(\gamma_t^{(H)}(\rho)\right) = \gamma_{-t}^{(H)}\left(\gamma_{\mathscr{T}}(\rho)\right) .$$

(We set $\hbar = 1$ henceforth). Equivalently,

$$e^{-itH}\mathscr{T}\rho\mathscr{T}^{-1}e^{+itH} = \mathscr{T}e^{+itH}\rho e^{-itH}\mathscr{T}^{-1} \quad \text{for every } \rho \in \mathfrak{S}(\mathsf{H}_S). \tag{13.52}$$

Therefore time reversal, when present, is *not* a dynamical symmetry in the sense of Definition 13.4, owing to the sign flip of time in the dynamical flow. The following important result rephrases, partially, Proposition 13.2.

Theorem 13.21 *Consider a system S with Hamiltonian H (of lower-bounded spectrum) on the Hilbert space H_S. If the spectrum of H is unbounded above, every operator $\mathscr{T} : \mathsf{H}_S \to \mathsf{H}_S$ satisfying (13.52) is anti-unitary and such that*

$$\mathscr{T}^{-1}H\mathscr{T} = H .$$

This applies in particular to the time-reversal symmetry (if it exists).

Proof If $V : \mathsf{H}_S \to \mathsf{H}_S$ is unitary (or anti-unitary), then $V\psi(\psi|V^{-1}\cdot) = V\psi(V\psi|\cdot)$. Setting $U_t := e^{-itH}$ and taking the unitary operator $V := (\mathscr{T}U_{-t})^{-1}U_t\mathscr{T}$, for any pure state $\rho = \psi(\psi|\ )$ we have

$$(\mathscr{T}U_{-t})^{-1}U_t\mathscr{T}\psi\left((\mathscr{T}U_{-t})^{-1}U_t\mathscr{T}\psi\right|\ \Big) = \psi(\psi|\ ) .$$

Hence for some $\chi_t \in \mathbb{C}$ with $|\chi_t| = 1$:

$$(\mathscr{T}U_{-t})^{-1}U_t\mathscr{T}\psi = \chi_t\psi , \quad \psi \in H.$$

Replicating the argument of Theorem 12.11 shows χ_t does not depend on ψ. What is more, the map $\mathbb{R} \ni t \mapsto \chi_t$ is differentiable: take $\phi \in D(H)$, $\psi \in \mathscr{T}^{-1}D(H)$ with $(\phi|\psi) \neq 0$ ($D(H)$ is dense) and differentiate the identity $(\mathscr{T}U_{-t}\phi|\mathscr{T}^{-1}U_t\mathscr{T}\psi) = \chi_t(\phi|\psi)$, if $\mathscr{T}$ is unitary, or $\overline{(\mathscr{T}U_{-t}\phi|\mathscr{T}^{-1}U_t\mathscr{T}\psi)} = \chi_t(\phi|\psi)$ if $\mathscr{T}$ is anti-unitary. Stone's theorem guarantees derivatives exist. Hence there is a differentiable map $\mathbb{R} \ni t \mapsto \chi_t$ such that $e^{-itH}\mathscr{T} = \mathscr{T}\chi_t e^{itH}$, so $\mathscr{T}^{-1}e^{-itH}\mathscr{T} = \chi_t e^{itH}$. Therefore

$$e^{\mp it\mathscr{T}^{-1}H\mathscr{T}} = \chi_t e^{itH}$$

with '$-$' if $\mathscr{T}$ is unitary and '$+$' if anti-unitary (cf. Exercise 12.8 for the latter). Note $\mathscr{T}^{-1}H\mathscr{T}$ is self-adjoint, so the left-hand side is a strongly continuous unitary group parametrised by $t \in \mathbb{R}$. Applying Stone's theorem tells $D(\mathscr{T}^{-1}H\mathscr{T}) \subset D(H) = D(cI + H)$ and

$$\mp \mathscr{T}^{-1}H\mathscr{T}\restriction_{D(H)} = cI + H \quad \text{where } c := -i\frac{d\chi_t}{dt}|_{t=0} \, . \tag{13.53}$$

The constant c is real, for $\mp\mathscr{T}^{-1}H\mathscr{T} - H$ is symmetric on $D(H)$. As a matter of fact (13.53) is valid everywhere on the domain of the self-adjoint operator $\mp\mathscr{T}^{-1}H\mathscr{T}$, which has no self-adjoint extensions $(cI + H)$ other than itself. Therefore

$$\mathscr{T}^{-1}H\mathscr{T} = \mp cI \mp H \, .$$

In particular (cf. Exercise 12.8 for the anti-unitary case):

$$\sigma(H) = \sigma(\mathscr{T}^{-1}H\mathscr{T}) = \sigma\,(\mp cI \mp H) = \mp c \mp \sigma(H) \, .$$

Suppose $\sigma(H)$ is bounded below but not above: the last identity cannot hold if on the right there is a $-$ sign, whichever the constant c. Then $\mathscr{T}$ must be anti-unitary, and $\inf\sigma(H) = \inf(c + \sigma(H)) = c + \inf\sigma(H)$. Hence $c = 0$, as $\inf\sigma(H)$ is finite by assumption ($\sigma(H) \neq \varnothing$ is bounded below). □

Example 13.22

(1) Let us consider a spinless non-relativistic particle described on $L^2(\mathbb{R}^3, dx)$, as discussed in Sect. 11.4. Assume that $\mathscr{T}$ is anti-unitary, as when the Hamiltonian is bounded from below but unbounded above. Since the time-reversal symmetry leaves unchanged the positions of the particles but reverts their velocities, it is expected that the action of $\mathscr{T}$ on the position and momentum operators will look like this

$$\mathscr{T}^{-1}X_k\mathscr{T} = X_k \, , \quad \mathscr{T}^{-1}P_k\mathscr{T} = -P_k \quad k = 1, 2, 3 \, . \tag{13.54}$$

By direct inspection one sees that, if $K : L^2(\mathbb{R}^3, dx) \to L^2(\mathbb{R}^3, dx)$ is the standard complex conjugation of wavefunctions ($(K\psi)(x) := \overline{\psi(x)}$ for every $x \in \mathbb{R}^3$ and $\psi \in L^2(\mathbb{R}^3, dx)$), then

$$K^{-1}X_kK = X_k \, , \quad K^{-1}P_kK = -P_k \quad k = 1, 2, 3 \, , \tag{13.55}$$

so that K is a candidate to represent $\mathscr{T}$. Let us prove that it is essentially unique. Suppose $\mathscr{T}$ is an anti-unitary operator (representing the time-reversal symmetry). Then $U := \mathscr{T}K^{-1}$ is unitary, and (13.54) and (13.55) entail

$$UX_k = X_kU \, , \quad UP_k = P_kU \, , \quad k = 1, 2, 3 \, . \tag{13.56}$$

We leave to the reader the proof of the fact that these relations imply that U commutes with all the operators $W((\mathbf{t}, \mathbf{u}))$ of the Weyl algebra associated with the position and momentum operators of our particle, as discussed in Proposition 11.39. Alternatively one may assume from scratch the validity of

$$\mathscr{T}^{-1}W((\mathbf{t},\mathbf{u}))\mathscr{T} = W((\mathbf{t},-\mathbf{u}))\ , \quad \text{for all } (\mathbf{t},\mathbf{u}) \in \mathbb{R}^3 \times \mathbb{R}^3\ , \tag{13.57}$$

instead of the physically equivalent identities (13.54), (13.55). Now (13.57) immediately implies that $U = \mathscr{T}K^{-1}$ commutes with all the operators $W((\mathbf{t},\mathbf{u}))$.

Since the family of operators $W((\mathbf{t},\mathbf{u}))$ is irreducible (Proposition 11.39), Proposition 11.37 implies that $U = cI$ for some complex number c with $|c| = 1$ because $U^*U = I$. Summing up, every operator representing the time-reversal symmetry of a spin-0 particle must have the form

$$\mathscr{T}_0 = e^{ia}K\ , \quad a \in \mathbb{R}\ .$$

And every such operator satisfies (13.54), (13.55) and (13.57), so it may be used to represent the time-reversal symmetry of a spinless particle. Note that this anti-unitary time-reversal operator leaves fixed Hamiltonian operators of the form (13.14) and, at least formally, also the self-adjoint extensions of

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x})\ .$$

(2) What happens by introducing the spin? We confine ourselves to spin-1/2 particles. In this case the Hilbert space of the particle is enlarged to the tensor product $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2$, where the second factor is the spin-1/2 Hilbert space (cf. Sect. 12.3.1). Here the three spin operators are represented by ($\hbar = 1$)

$$S_k = \frac{1}{2}\sigma_k\ ,$$

using the standard *Pauli matrices*. Let us again assume that the time-reversal operator $\mathscr{T} : L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2$ is anti-unitary, as when the Hamiltonian is bounded below but not above. In addition to (13.54) and (13.55) – obviously interpreting the operators X_k and P_k as $X_k \otimes I$ and $P_k \otimes I$ respectively – the time-reversal symmetry is also supposed to satisfy

$$\mathscr{T}^{-1}I \otimes S_k\mathscr{T}^{-1} = -I \otimes S_k \quad k = 1, 2, 3\ .$$

The reason is that the relations above must be valid for the orbital angular momentum constructed out of the X_k and P_k, and it is natural to extend these to the spin which, in some respects, may be considered as a sort of angular momentum. By direct inspection one sees that an anti-unitary operator which satisfies (13.54), (13.55) and the constraint above is

$$(I \otimes i\sigma_2)C\ .$$

where now C is the natural complex conjugation in $L^2(\mathbb{R}^3, dx) \otimes \mathbb{C}^2$: the unique antilinear extension of $C(\psi \otimes \chi) := (K\psi) \otimes \overline{\chi}$ for every $\chi \in \mathbb{C}^2$ and $\psi \in L^2(\mathbb{R}^3, dx)$. Exactly as for the spinless case $U = \mathscr{T} C^{-1}$ commutes with all operators $W((\mathbf{t}, \mathbf{u})) \otimes I$ and $I \otimes \sigma_k$. We leave to the reader the proof of the fact that the family of operators $W((\mathbf{t}, \mathbf{u})) \otimes I$ and $I \otimes \sigma_k$ where $(\mathbf{t}, \mathbf{u}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $k = 1, 2, 3$ is irreducible. Hence, U is again a pure phase. We conclude that the operator describing the time-reversal symmetry of a spin-1/2 particle has necessarily the form

$$\mathscr{T}_{1/2} = e^{ia} I \otimes i\sigma_2 C \,, \quad a \in \mathbb{R} \,.$$

Comparing $\mathscr{T}_0$ and $\mathscr{T}_{1/2}$ we discover an important difference:

$$\mathscr{T}_0^2 = I \quad \text{whereas} \quad \mathscr{T}_{1/2}^2 = -I \,.$$

This result is general: if $\mathscr{T}_s$ denotes the time-reversal operator for a particle with spin $s = 0, 1/2, 1, 3/2, \ldots$, we find

$$\mathscr{T}_s^2 = (-1)^{2s} I \,.$$

This result can be used to justify the *superselection rule of the angular momentum* introduced in Example 7.80. ■

13.2 From the Time Observable and Pauli's Theorem to POVMs

There is yet another consequence of the spectral lower bound of H. It addresses the existence of a quantum observable corresponding to the classical quantity of time, which satisfies canonical commutation relations with the Hamiltonian. The existence of such an operator may be suggested by Heisenberg's 'time-energy' uncertainty relationship, mentioned in Chap. 6. In Chap. 11 we deduced Heisenberg's uncertainty principle for position and momentum as a theorem, following the CCR

$$[X, P] = i\hbar I \,.$$

We might expect a self-adjoint operator T to correspond to the *observable time* (*the instant* at which a phenomenon occurs, or its *duration* in a given quantum system); it should moreover satisfy a similar commutation relation with the Hamiltonian, on some domain:

$$[T, H] = i\hbar I \,,$$

and therefore there should be an analogue time-energy uncertainty

$$(\Delta H)_\psi (\Delta T)_\psi \geq \hbar/2 \,,$$

exactly as for the pair position-momentum. We saw in Chap. 11 that by interpreting in strong sense the position-momentum CCR, i.e. passing from operators to the exponential algebra, the commutation of the exponentials determined the operators up to unitary transformations, by virtue of the Stone-von Neumann theorem. These alleged relations would read $e^{-i\frac{h}{\hbar}T}e^{-i\frac{t}{\hbar}H} = e^{i\frac{ht}{\hbar}}\, e^{-i\frac{t}{\hbar}H}e^{-i\frac{h}{\hbar}T}$. But in the case at stake that is not possible. There no way to define properly the operator time, and make sense of the time-energy relations: a no-go result that bears the name of *Pauli's theorem*. It is however possible to try to define the observable time, case by case, by invoking *generalised observables*, which are useful in other contexts like the theory of Quantum Information.

13.2.1 Pauli's Theorem

Putting together a series of results collected from previous chapters we will prove our version of a result known as Pauli's theorem.

Theorem 13.23 *Consider a system S with Hamiltonian H (with lower-bounded spectrum) on the Hilbert space H_S. Suppose there exist a self-adjoint operator $T : D(T) \to \mathsf{H}_S$ and a subspace $\mathscr{D} \subset D(H) \cap D(T)$ in H_S on which TH and HT are well defined and the CCR ($\hbar = 1$)*

$$[T, H] = iI$$

holds. Then none of the following facts can occur.
(a) *$\mathscr{D}$ is dense and invariant under T and H; the symmetric operator $T^2 + H^2$ is essentially self-adjoint on $\mathscr{D}$.*
(b) *$\mathscr{D}$ is dense, invariant under T and H, and made of analytic vectors for both T and H.*
(c) *The exponential operators satisfy CCRs:*

$$e^{ihT}e^{itH} = e^{iht}\, e^{itH}e^{ihT}\,, \quad t, h \in \mathbb{R}.$$

Proof If (a) were true, by Nelson's theorem 12.89 $H\restriction_{\mathscr{D}}$ and $T\restriction_{\mathscr{D}}$ would be essentially self-adjoint (making $\mathscr{D}$ a core for both self-adjoint operators H, T) and there would be a strongly continuous unitary representation of the unique simply connected Lie group whose Lie algebra is generated by I, H, T under the CCRs and the trivial brackets $[T, I] = [H, I] = 0$. But that defines the Heisenberg group $H(2)$, as seen in the previous chapter, and we would have proven (c). The same conclusion follows from assuming (b) because of Theorem 12.90. So let us suppose (c) holds.

Going through the argument after Theorem 11.45, we could prove that the $W(t,h) := e^{iht/2}e^{itH}e^{ihT}$ satisfy Weyl's relations and the hypotheses of Mackey's theorem 11.44. Then the Hilbert space H_S would split in an orthogonal sum $\mathsf{H}_S = \oplus_k \mathsf{H}_k$ of closed invariant spaces under e^{itH} and e^{ihT} for any t, h; and for any k there would be a unitary map $S_k : \mathsf{H}_k \to L^2(\mathbb{R}, dx)$, so $S_k e^{itH} \restriction_{\mathsf{H}_k} S_k^{-1} = e^{itX}$ in particular, with X denoting the standard position operator on $\mathbb{R}$. Applying Stone's theorem to $e^{itH}\mathsf{H}_s \subset \mathsf{H}_s$ we would obtain these consequences: firstly $H(\mathsf{H}_k \cap D(H)) \subset \mathsf{H}_k$, secondly $H\restriction_{\mathsf{H}_k \cap D(H)}$ is self-adjoint on H_k, and then $e^{itH}\restriction_{\mathsf{H}_k} = e^{itH\restriction_{\mathsf{H}_k\cap D(H)}}$. At this point the condition satisfied by S_k would read $e^{itH\restriction_{\mathsf{H}_k\cap D(H)}} = S_k^{-1}e^{itX}S_k$. Reapplying Stone's theorem would produce $H\restriction_{\mathsf{H}_k\cap D(H)} = S_k^{-1}XS_k$, hence

$$\sigma(H) \supset \sigma(H\restriction_{\mathsf{H}_k\cap D(H)}) = \sigma(S_k^{-1}XS_k) = \sigma(X) = \mathbb{R}\,.$$

(For the first inclusion it suffices to use the definition of spectrum.) But that is impossible because $\sigma(H)$ is bounded from below. $\square$

13.2.2 Generalised Observables as POVMs

The problem raised by Pauli's theorem about the definition of time is hard, and not yet completely construed. One attempt, that weakens the notions of observable and PVM, has found several other uses in QM, especially in *Quantum Information* [NiCh07].

Let us look into the proof of Proposition 7.52, which associates probability measures to observables seen as PVMs on $\mathbb{R}$: given a state $\rho \in \mathfrak{S}(\mathsf{H})$, we did *not* employ requirement (b) in Proposition 7.44; and concerning property (d), we only made use of *weak convergence* (implied by strong convergence). So we may rephrase Proposition 7.52 like this.

Proposition 13.24 *Let H be a Hilbert space and $\{P(E)\}_{E\in\mathscr{B}(\mathbb{R})}$ a collection of operators in $\mathfrak{B}(\mathsf{H})$ satisfying:*
(a)' *$P(E) \geq 0$ for every $E \in \mathscr{B}(\mathbb{R})$;*
(b)' *$P(\mathbb{R}) = I$;*
(c)' *for any countable set $\{E_n\}_{n\in\mathbb{N}}$ of pairwise-disjoint Borel sets in $\mathbb{R}$,*

$$w\text{-}\sum_{n=0}^{+\infty} P(E_n) = P\left(\cup_{n\in\mathbb{N}} E_n\right)\,.$$

If $\rho \in \mathfrak{S}(\mathsf{H})$, the mapping $\mu_\rho : E \mapsto tr(P(E)\rho)$ is a probability measure on $\mathbb{R}$.

The numbers $\mu_\rho(E)$ are the probabilities the experimental readings of the observable $\{P(E)\}_{E\in\mathscr{B}(\mathbb{R})}$ fall in the Borel set E. Sometimes it is convenient to adopt *generalised observables*, assuming they are given by maps $E \mapsto P(E)$ satisfying conditions (a)', (b)', (c)': these are weaker than the ones for PVMs, but still guarantee μ_ρ is a probability measure. In particular, the $P(E)$ are no longer orthogonal projectors, but mere bounded positive operators. All this leads to the following definition. We refer to [Ber66] for a broad mathematical treatise and [BGL95] for an extensive discussion on the applications to QM.

Definition 13.25 Let H be a Hilbert space and $(\mathsf{X}, \Sigma(\mathsf{X}))$ a measurable space. A mapping $A : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ is called **positive-operator valued measure** (POVM) on X if:
(a)' $A(E) \geq 0$ for any $E \in \Sigma(\mathsf{X})$;
(b)' $A(\mathsf{X}) = I$;
(c)' for any countable set $\{E_n\}_{n\in\mathbb{N}}$ of disjoint measurable subsets in X:

$$\text{w-}\sum_{n=0}^{+\infty} A(E_n) = A(\cup_{n\in\mathbb{N}} E_n) .$$

A **generalised observable** on H is a collection of operators $\{A(E)\}_{E\in\Sigma(\mathsf{X})}$ such that $\Sigma(\mathsf{X}) \ni E \mapsto A(E)$ is a positive-operator valued measure.

If A is a POVM on H, since $\mathfrak{B}(\mathsf{H}) \ni A(E) \geq 0$ for every $E \in \Sigma(\mathsf{X})$, we have $A(E) = A(E)^*$ by Proposition 3.60(f). Moreover, by Definition 13.25(c, d) $0 \leq A(E) \leq I$, so $||A(E)|| \leq 1$ from Proposition 3.60(a).

On a Hilbert space H the convex set $\mathfrak{E}(\mathsf{H})$ of elements $A \in \mathfrak{B}(\mathsf{H})$ with $0 \leq A \leq I$ is called the **space of effects**, and the **effects** are the operators A.

Remark 13.26 Differently from PVMs, the elements of a POVM are not assumed to commute. This feature has the nice consequence that a convex linear combination of a pair of POVMs over the same $\Sigma(\mathsf{X})$ is still POVM. ■

The effects on H are the operators used to build every POVM on H, and their space is the analogue of $\mathscr{L}(\mathsf{H})$ in defining observables via PVMs. The space $\mathfrak{E}(\mathsf{H})$ contains $\mathscr{L}(\mathsf{H})$ and is partially ordered by the usual relation $\geq$, though it is not a lattice.This prevents a generalised interpretation of orthogonal projectors as propositions on the system.

Extending axiom **A3** from post-measurement states to generalised observables is problematic. It is not possible to establish, in practice, in which state the system is after a measurement whose reading is $E \in \mathscr{B}(\mathbb{R})$ if the observable is represented by a POVM $\{A(E)\}_{E\in\mathscr{B}(\mathbb{R})}$, and without further information. The extra data is assigned by decomposing each $A(E) = B(E)^*B(E)$ in the POVM, where the operators $B(E) \in \mathfrak{B}(\mathsf{H})$ are called **measuring operators**. If so, the post-measurement state, if E is the outcome, is supposed to look like

$$\rho' = \frac{B(E)\rho B(E)^*}{tr(A(E)\rho)} .$$

For PVMs, clearly, $A(E) = B(E)$ are orthogonal projectors. In the general case, since the $B(E)$ are not required to be positive, there are an infinite number of solutions $B(E)$ to the equation $B(E)^*B(E) = A(E)$ when $A(E)$ is given. From the physical viewpoint, this implies that there are infinitely many different experimental apparatuses that give the same probabilities for the outcomes.

Another remarkable difference from standard measurements is that a POVM is not repeatable in general, even if the possible values of the measured observable form a discrete set. Indeed, if the state ρ' is subjected to the same measurement associated with the same Borel set E, the new state turns out to be:

$$\rho'' = \frac{B(E)\rho' B(E)^*}{tr(B(E)\rho' B(E)^*)} = \frac{B(E)B(E)\rho B(E)^* B(E)^*}{tr(B(E)B(E)\rho B(E)^* B(E)^*)}$$

which, in general, coincides to ρ' only if $B(E)^2 = B(E)$. However this identity is false for a generic element of a POVM. See [BGL95] for further details and many examples.

Here is an interesting application of generalised observables to the definition of time. Suppose time is defined as the observable associated to the lapse it takes a particle to hit a detector. By Pauli's theorem 13.23 such observable is unlikely going to be defined via projectors if we impose that the observable is somehow "conjugated" to the Hamiltonian.

The attempts to define the *time observable* in terms of POVMs are very promising. Candidates for a *generalised time observable* T, e.g. the arrival time of a free particle, arise from a suitable POVM $T := \{A(E)\}_{E\in\mathscr{B}(\mathbb{R})}$ dependent on the system [Gia97, BrFr02]. Introducing measures $\mu^{(T)}_{\psi,\phi}(E) := (\psi|A(E)\phi)$, $E \subset \mathscr{B}(\mathbb{R})$, and setting $\mu^{(T)}_{\psi} := \mu^{(T)}_{\psi,\psi}$, we can define $\langle T\rangle_\psi$ and $(\Delta T)_\psi$ as we did for PVMs. If T is built appropriately, on suitable domains, then $(\Delta T)_\psi(\Delta H)_\psi \geq \hbar/2$ and analogues hold [Gia97, BrFr02], where H is the system's Hamiltonian. In analogy to PVMs, we may associate to the POVM T an operator, denoted by T, characterised by being the unique operator such that:

$$(\psi|T\phi) := \int_{\mathbb{R}} \lambda d\mu^{(T)}_{\psi,\phi}(\lambda) ,$$

where $\psi \in \mathsf{H}$ and ϕ belongs to a dense and suitable domain $D(T)$. This T turns out to be symmetric, but non self-adjoint. For a particle of mass $m > 0$, free to move along the real axis, the operator T of [Gia97] has the obvious form $T = \frac{m}{2}(XP^{-1}+P^{-1}X)$, on a suitable dense subspace of $L^2(\mathbb{R}, dx)$.

Remarks 13.27 **(1)** Gleason's theorem 7.26 has an important extension (yet much easier to prove) to generalised observables due to Busch [Bus03].

Theorem 13.28 (Busch) *Let* H *be a complex Hilbert space of finite dimension* ≥ 2 *or separable. For any map* $\mu : \mathfrak{E}(\mathsf{H}) \to [0, 1]$ *such that* $\mu(I) = 1$ *and* $\mu\left(w-\sum_{n=0}^{+\infty} A_n\right) = \sum_{n=0}^{+\infty} \mu(A_n)$ *for every sequence* $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{E}(\mathsf{H})$ *satisfying* $w\text{-}\sum_{n=0}^{+\infty} A_n \leq I$*, there exists* $\rho \in \mathfrak{S}(\mathsf{H})$ *such that* $\mu(A) = tr(A\rho)$*,* $A \in \mathfrak{E}(\mathsf{H})$*.*

(2) An important theorem shows the tight relationship between PVMs and POVMs.

Theorem 13.29 (Neumark) *Let* $(\mathsf{X}, \Sigma(\mathsf{X}))$ *be a measurable space and* H *a Hilbert space. If* $A : \Sigma(\mathsf{X}) \to \mathfrak{B}(\mathsf{H})$ *is a POVM, there exist a Hilbert space* H'*, an operator* $U : \mathsf{H} \to \mathsf{H}'$ *and a PVM* $P : \Sigma(\mathsf{X}) \to \mathscr{L}(\mathsf{H}')$ *such that* $A(E) = U^* P(E) U$ *for every* $E \in \Sigma(\mathsf{X})$.

Condition (b)' in Definition 13.25 and the analogue for PVMs imply $U^*U = I_{\mathsf{H}}$, so U is an isometry (not surjective, otherwise A would be a PVM). Hence H is isomorphic to a (proper) closed subspace of H'. Yet $P(E)$ does not, in general, have a direct physical meaning, because H' is *not* the system's Hilbert space. ■

13.3 Dynamical Symmetries and Constants of Motion

This section is devoted to extending to QM the various versions of *Noether's theorem*. In classical theories Noether's theorem relates dynamical symmetries to constants of motion. In QM this relationship is as straightforward as it can get. To state the relative theorem we need to introduce the so-called *Heisenberg picture* of observables.

13.3.1 Heisenberg's Picture and Constants of Motion

Take a quantum system S described in the inertial frame $\mathscr{I}$ with evolution operator $\mathbb{R} \ni \tau \mapsto e^{-i\tau H}$. Fix once for all the instant $t = 0$ for the initial conditions. Then consider the associated continuous projective representation of $\mathbb{R}$, $\mathbb{R} \ni t \mapsto \gamma_t^{(H)} := e^{-itH} \cdot e^{itH}$, and the *dual action* (cf. Sect. 12.1.6) on observables. If A is an observable (possibly an orthogonal projector representing an elementary property of S) we call

$$A_H(t) := \gamma_t^{(H)*}(A) = e^{itH} A e^{-itH}$$

the **Heisenberg picture** of A at time τ. By construction $\sigma(A_H(\tau)) = \sigma(A)$ and the observables's spectral measures satisfy $P^{(A_H(t))}(E) = \gamma_t^{(H)*}(P^{(A)}(E))$ for any $E \in \mathscr{B}(\mathbb{R})$.

In Heisenberg's picture, coherently with the symmetries' dual action of Chap. 12, quantum states do *not* evolve in time and the dynamics acts on observables. In particular, the expectation value of A on the state ρ_t (the evolution at time t of the initial state ρ) can be computed either as $\langle A \rangle_{\rho_t}$ or equivalently as $\langle A_H(t) \rangle_\rho$, because

$$\langle A \rangle_{\rho_t} = tr\left(AU_t\rho U_t^{-1}\right) = tr\left(U_\tau^{-1} A U_t \rho\right) = \langle A_H(\tau) \rangle_\rho$$

if we put ourselves in the hypotheses of Proposition 11.27 (using the measure $\mu_\rho^{(A)}$ directly shows that the result holds generally). And the same happens for the probability that the reading of A at time τ falls within the Borel set E, if ρ was the state at time 0:

$$tr(P_E^{(A)}\rho_t) = tr(P_E^{(A_H(t))}\rho)\ .$$

Remarks 13.30 **(1)** According to Sect. 12.2.2, Heisenberg's evolution of observables coincides with the *inverse dual action* (Sect. 12.22) on observables of the *time displacement* symmetry (Remark 13.1(4)). This is an immediate consequence of the fact that time displacement is generated by $-H$, which entails $\gamma_t^{(-H)*} = (\gamma_t^{(H)*})^{-1}$. This fact plays a crucial role in Quantum Field Theory, where the action of symmetries on quantum-field operators is pictured in terms of inverse dual actions (so that these actions give rise to standard left representations of the symmetry group instead of *right* representations). Heisenberg's evolution is then the natural representation of the time displacement symmetry.
(2) To distinguish Heisenberg's picture from the ordinary picture in which states – not observables – evolve, the latter is often called **Schrödinger picture**, a convention we will adopt.
(3) It must be noted that an observable may depend on time in Schrödinger's picture as well. Better said, it is convenient to use a self-adjoint family $\{A_t\}_{t\in\mathbb{R}}$ parametrised by time t, and view it as a single observable denoted A_t. If so, we say the observable **depends on time explicitly**. In Heisenberg's picture time dependency takes care of both (implicit and explicit) dependencies:

$$A_{Ht}(t) := \gamma_t^{(H)*}(A_t) = e^{itH}A_t e^{-itH}\ . \tag{13.58}$$

Now that we have seen the evolution of observables in Heisenberg's picture, we can introduce *constants of motion* by mimicking the classical definition. ■

Definition 13.31 Let S be a quantum system described on the Hilbert space H_S associated to the inertial frame $\mathscr{I}$ with Hamiltonian H. An observable A is a **constant of motion** or a **first integral** if its Heisenberg picture is does not depend on time:

$$A_H(t) = A_H(0) \quad \text{for any } t \in \mathbb{R}. \tag{13.59}$$

An explicitly time-dependent observable A_t is called constant of motion or first integral provided

$$A_{Ht}(t) = A_{H0}(0) \quad \text{for any } t \in \mathbb{R}. \tag{13.60}$$

Remarks 13.32 **(1)** An observable that does not depend on time explicitly is a constant of motion if and only if its Heisenberg and Schrödinger pictures coincide.
(2) The notions of Heisenberg's picture and constants of motion extend to situations where time is not homogeneous and with evolution operators different from $U(t_2, t_1)$. We will not worry about this.
(3) Identity (13.60) is oftentimes found in books written as

$$\frac{\partial A_{Ht}}{\partial t} + i[H, A_{Ht}(t)] = 0\ , \tag{13.61}$$

where the partial derivative refers to the explicit time variable only, i.e. the subscript $_{Ht}$. In practice if we do not care about domain issues, that equation is a trivial consequence of (13.60), and implies (13.60) if we also assume (13.58). The equivalence, in general false, is however troublesome to prove. At any rate, the concept of *constant of motion* is perfectly formalised, physically, by (13.60), with no need to differentiate in time and incur in spurious technical problems. ■

Notation 13.33 Lest we overburden notations for (explicitly) time-dependent observables, we will simply write $A_H(t)$ instead of $A_{Ht}(t)$ from now on, if no confusion arises. ■

We are ready to exhibit the relationship between constants of motion and dynamical symmetries. In classical physics one-parameter symmetry groups are known to correspond, in the various formulations of Noether's theorem, to constants of motion. We wish to extend that to QM. Let us start with an easy case.

Proposition 13.34 *Let $\sigma(\cdot) := V^{(\sigma)} \cdot V^{(\sigma)-1}$ be a dynamical symmetry with $V^{(\sigma)}$ simultaneously unitary and self-adjoint. Then the observable $V^{(\sigma)}$ is a constant of motion.*

Proof If U_t is the evolution operator, by Theorem 13.5(b) $U_t V^{(\sigma)} U_t^{-1} = V^{(\sigma)}$. □

It is not rare that an interesting operator is together unitary and self-adjoint (and thus represents a symmetry and an observable). An example is the *parity inversion*, which we discussed in Examples 12.19. The situation is completely different from that of classical mechanics, where a system invariant under parity inversion (or any discrete symmetry) does not gain an associated constant of motion.

Let us deal with one-parameter groups of continuous symmetries, for which the link dynamical symmetries–constants of motion is forthright.

To begin with we consider a time-dependent observable $\{A_t\}_{t\in\mathbb{R}}$, in a certain system S with Hamiltonian H. If A_t is a constant of motion, then, by the previous definitions

$$e^{itH} A_t e^{-itH} = A_0 \,.$$

If we exponentiate the self-adjoint operators in the equation we obtain

$$e^{iae^{itH} A_t e^{-itH}} = e^{iaA_0} \,,$$

a relation that known properties transform into

$$e^{itH} e^{iaA_t} e^{-itH} = e^{iaA_0} \,,$$

i.e.

$$e^{iaA_t} e^{-itH} = e^{-itH} e^{iaA_0} \,, \quad a \in \mathbb{R}, t \in \mathbb{R}.$$

This equation's interpretation in terms of dynamical symmetries is quite relevant. It says that for any given $a \in \mathbb{R}$ the symmetries $\{\sigma_a^{(A_t)}\}_{t\in\mathbb{R}}$, with

$$\sigma_a^{(A_t)}(\rho) := e^{iaA_t}\rho e^{-iaA_t} \quad \text{where } \rho \in \mathfrak{S}(\mathsf{H}_S)$$

form, by Theorem 13.5, a *time-dependent dynamical symmetry* of S. If we restrict to $A_t = A$ time-independent, the same argument proves that

$$\sigma_a^{(A)}(\rho) := e^{iaA}\rho e^{-iaA} \quad \text{where } \rho \in \mathfrak{S}(\mathsf{H}_S),$$

defines a *dynamical symmetry* for every $a \in \mathbb{R}$.

All this shows that *constants of motion determine dynamical symmetries* but also *continuous* projective representations $\mathbb{R} \ni a \mapsto \sigma_a^{(A_t)}(\cdot)$ of $\mathbb{R}$, since $\mathbb{R} \ni a \mapsto e^{iaA_t}$ is strongly continuous by Definition 12.40 (cf. Sect. 12.2.6).

Now we ask about *the converse*.

Given a family of time-dependent dynamical symmetries $\{\sigma_a^{(t)}\}_{t\in\mathbb{R}}$ where $\mathbb{R} \ni a \mapsto \sigma_a^{(t)}$ is a continuous projective representation of the group $\mathbb{R}$ for every $t \in \mathbb{R}$, is it possible to write each one of them as $\sigma_a^{(A_t)}(\cdot) := e^{iaA_t} \cdot e^{-iaA_t}$, so that the self-adjoint operators A_t give an (explicitly time-dependent) observable that is a constant of motion?

According to Theorem 12.45 we can always find self-adjoint operators A_t such that $\sigma_a^{(A_t)}(\cdot) := e^{iaA_t} \cdot e^{-iaA_t}$ for every $a \in \mathbb{R}$. But these are determined up to a real constant $A_t \to A'_t := A_t - c(t)I$, so the point is whether one can fix the maps $c(t)$ so that

$$e^{itH}A'_t e^{-itH} = A'_0\,.$$

The answer of the next theorem, the quantum version of Noether's theorem, is yes.

Theorem 13.35 ("Quantum Noether theorem") *Let S be a quantum system, described on the Hilbert space H_S associated to the inertial frame $\mathscr{I}$, with Hamiltonian H and dynamical flow $\gamma^{(H)}$. If constants of motion and dynamical symmetries refer to $\gamma^{(H)}$, the following facts holds.*
(a) *If A is a constant of motion:*

$$\sigma_a^{(A)}(\rho) := e^{iaA}\rho e^{-iaA}\,, \quad \forall \rho \in \mathfrak{S}(\mathsf{H}_S)$$

defines a dynamical symmetry for every $a \in \mathbb{R}$, and $\mathbb{R} \ni a \mapsto \sigma_a^{(A)}(\cdot)$ is continuous.
(b) *Let $\{A_t\}_{t\in\mathbb{R}}$ be a time-dependent observable and a constant of motion. As $t \in \mathbb{R}$ varies,*

$$\sigma_a^{(A_t)}(\rho) := e^{iaA_t}\rho e^{-iaA_t}\,, \quad \forall \rho \in \mathfrak{S}(\mathsf{H}_S)$$

defines a time-dependent dynamical symmetry for every $a \in \mathbb{R}$, and $\mathbb{R} \ni a \mapsto \sigma_a^{(A_t)}(\cdot)$ is continuous $\forall t \in \mathbb{R}$.
(c) *Let σ_a be a dynamical symmetry and $\mathbb{R} \ni a \mapsto \sigma_a$ a continuous projective representation, $\forall a \in \mathbb{R}$. Then there exists a constant of motion A such that*

$$\sigma_a(\cdot) := e^{iaA} \cdot e^{-iaA}\,, \quad a \in \mathbb{R}. \tag{13.62}$$

A constant of motion A' satisfies (13.62) if and only if

$$A' = A + cI \quad \textit{for some constant } c \in \mathbb{R}.$$

(d) *Let $\{\sigma_a^{(t)}\}_{t\in\mathbb{R}}$ be a time-dependent dynamical symmetry $\forall a \in \mathbb{R}$, and such that the map $\mathbb{R} \ni a \mapsto \sigma_a^{(t)}$ is a continuous projective representation $\forall t \in \mathbb{R}$. Then there exists a time-dependent observable $\{A_t\}_{t\in\mathbb{R}}$ that is a constant of motion plus*

$$\sigma_a^{(t)}(\cdot) := e^{iaA_t} \cdot e^{-iaA_t}\,, \quad a \in \mathbb{R}, t \in \mathbb{R}. \tag{13.63}$$

A time-dependent observable $\{A'_t\}_{t\in\mathbb{R}}$ is a constant of motion satisfying (13.63) if and only if

$$A'_t = A_t + cI \quad \textit{for every } t \in \mathbb{R} \textit{ and some constant } c \in \mathbb{R}.$$

Proof Claims (a), (b) were proved above, while (c) is evidently a subcase of (d) if we set $\sigma_a^{(t)} = \sigma_a$ and $A_t = A$ for any $t \in \mathbb{R}$. So there remains to prove (d). By Theorem 12.45, for any $t \in \mathbb{R}$ we can write $\sigma_a^{(t)}(\rho) := e^{iaA'_t}\rho e^{-iaA'_t}$, $a \in \mathbb{R}$, $\rho \in \mathfrak{S}(\mathsf{H}_S)$, where the self-adjoint operators A'_t are given by the group $\mathbb{R} \ni\mapsto \sigma_a^{(t)}$ and can be redefined to $A'_t + c(t)I = A_t$ by adding real constants $c(t)$. We want to show that it is possible to fix $c = c(t)$ in order that $\{A_t\}_{t\in\mathbb{R}}$ be a time-dependent constant of motion. Let us imagine we have made a choice for those operators. By Theorem 13.5(a), for suitable unit complex numbers $\chi(t, a)$ we have

$$\chi(t, a) = e^{iaA'_t} e^{-itH} e^{-iaA'_0} e^{itH}\,, \tag{13.64}$$

whence $\chi(t, 0) = 1$ for every $t \in \mathbb{R}$. Furthermore

$$\chi(t, a)(\psi|\phi) = \left(e^{itH} e^{-iaA'_t}\psi \,\middle|\, e^{-iaA'_0} e^{itH}\phi\right).$$

Choosing, for given $t \in \mathbb{R}$, $\psi \in (D(A'_t))$ and $\phi \in e^{itH}(D(A'_0))$ not orthogonal (the domains are dense because A_t is self-adjoint and e^{itH} unitary), and applying Stone's theorem on the right for the variable a, we obtain that the derivative in a of the left-hand side exists for every $a \in \mathbb{R}$. At the same time (13.64) implies, for given $t \in \mathbb{R}$:

$$\chi(t, a + a') = e^{i(a+a')A'_t} e^{-itH} e^{-i(a+a')A'_0} e^{itH}$$

$$= e^{iaA'_t}\left(e^{ia'A'_t} e^{-itH} e^{-ia'A'_0} e^{itH}\right) e^{-itH} e^{-iaA'_0} e^{itH}$$

$$= e^{iaA'_t}\chi(t, a') e^{-itH} e^{-iaA'_0} e^{itH} = \chi(t, a')\chi(t, a)\,.$$

For $t \in \mathbb{R}$ given, the map $\mathbb{R} \ni a \mapsto \chi(t, a)$ is differentiable and satisfies $\chi(t, a + a') = \chi(t, a)\chi(t, a')$, so $\frac{\partial\chi(t,a)}{\partial a} = \frac{\partial\chi(t,a)}{\partial a}|_{a=0}\chi(t, a)$. Since $|\chi(t, a)| = 1$, $\chi(t, 0) =$

1 for all $t \in \mathbb{R}$, the differential equation is solved by $\chi(t,a) = e^{-ic(t)a}$ with $c(t) = i\frac{\partial \chi(t,a)}{\partial a}|_{a=0} \in \mathbb{R}$. So we have

$$e^{-ic(t)a} = e^{iaA'_t}e^{-itH}e^{-iaA'_0}e^{itH} ,$$

and hence

$$e^{ia(A'_t+c(t)I)}e^{-itH} = e^{-itH}e^{iaA'_0} .$$

By (13.64) $e^{-ic(0)a} = 1$ for any $a \in \mathbb{R}$, so necessarily $c(0) = 0$. Then the above identity reads

$$e^{ia(A'_t+c(t)I)}e^{-itH} = e^{-itH}e^{ia(A'_0+c(0)I)} .$$

As we said earlier we are free to modify the A'_t by adding constants, so with $A_t := A'_t + c(t)I$ we obtain

$$e^{iaA_t}e^{-itH} = e^{-itH}e^{iaA_0} . \tag{13.65}$$

The identity implies $e^{itH}A_te^{-itH} = A_0$ so that $\{A_t\}_{t\in\mathbb{R}}$ defines a time-dependent constant of motion. To conclude observe that we are still free to modify A_t by adding a real constant $d(t)$ for every given $t \in \mathbb{R}$. But from (13.65) we immediately have that $(A_1)_t := A_t + d(t)$ satisfies

$$e^{ia(A_1)_t}e^{-itH} = e^{-itH}e^{ia(A_1)_0} , \quad \forall t \in \mathbb{R} ,$$

only if $d(t)$ is constant: $d(t) = c \in \mathbb{R}$ for every $t \in \mathbb{R}$. □

Remark 13.36 Suppose the system's Hilbert space splits in coherent sectors under a superselection rule, and assume this rule corresponds to a certain observable Q being defined and taking a precise value in every sector, on each pure state. This is the case of the electric charge, for example. The self-adjoint operator representing Q is a constant of motion, since the evolution prevents the (pure) state to escape the sector where it initially lives. This observation unveils a deep relationship, between superselection rules and constants of motion, that proved extremely relevant in the algebraic formulation of quantum theories [Haa96]. We will talk about it in Chap. 14. ■

13.3.2 A Short Detour on Ehrenfest's Theorem and Related Mathematical Issues

Before we go on to examine the constants of motion of the Galilean group, we would like to spend some time on a topic related to the evolution of observables. In QM manuals there is a statement of acclaimed heuristic importance, especially for relating QM to its classical limit, known as *Ehrenfest theorem*. The heart of Ehrenfest's theorem is, formally, quite straightforward. Take a quantum system S

described on the Hilbert space H_S and an observable or self-adjoint operator A (for simplicity time-independent). Fix a pure state/unit vector ψ and consider its evolution under the operator e^{-itH}. In formal terms (overlooking domains),

$$\frac{d}{dt}\langle A\rangle_{\psi_t} = \frac{d}{dt}\left(e^{-itH}\psi\middle| Ae^{-itH}\psi\right) = i\,(H\psi_t|A\psi_t) - i\,(\psi_t|AH\psi_t)$$

for $\psi_t := e^{-itH}\psi$. This implies the general **Ehrenfest relation**:

$$\frac{d}{dt}\langle A\rangle_{\psi_t} = \langle i[H, A]\rangle_{\psi_t}\ . \tag{13.66}$$

Although to obtain (13.66) we ignored important mathematical details, it is easy to prove (exercise) that the relation is implied by the following three conditions: (i) $A \in \mathfrak{B}(H)$; (ii) $\psi_\tau \in D(H)$ around t, or equivalently $\psi \in D(H)$, since $D(H)$ is evolution-invariant; (iii) $\psi_\tau \in D(HA)$ around t. It is far from easy to make assumptions of some help to physical applications that only concern H, A, ψ and are valid on a neighbourhood of some t. We can nevertheless weaken (i), (ii), (iii): beside $A \in \mathfrak{B}(H)$, assume only $\psi \in D(H)$, and interpret $\langle i[H, A]\rangle_{\psi_t}$ in (13.66) as a *quadratic form*:

$$\langle i[H, A]\rangle_{\psi_t} := i(H\psi_t|A\psi_t) - i(A\psi_t|H\psi_t)\ .$$

This yields a weaker version of Ehrenfest's theorem:

$$\frac{d}{dt}\langle A\rangle_{\psi_t} = i(H\psi_t|A\psi_t) - i(A\psi_t|H\psi_t)\ . \tag{13.67}$$

Even in this reading the statement is still too abstract, because practically every observable A of interest in QM is not a bounded operator. In fact, the importance of Ehrenfest's theorem becomes evident precisely when applied to the unbounded operators position and momentum.

Consider, to that end, a system made by a spin-zero particle of mass m, subject to a potential V, in an inertial frame. The Hamiltonian is a self-adjoint extension of the differential operator $H_0 := -\frac{\hbar^2}{2m}\Delta + V$. Suppose we work with $\tau \mapsto \psi_\tau$, which around t belongs to some subdomain of $D(X_i\overline{H_0}) \cap D(\overline{H_0}X_i)$ on which the Hamiltonian $\overline{H_0}$ is differentiable. Then (reintroducing $\hbar$ everywhere):

$$[\overline{H_0}, X_i]\psi = -\frac{\hbar}{2m}\sum_{j=1}^{3}\left[\frac{\partial^2}{\partial x_j^2}, x_i\right]\psi = -\frac{\hbar}{m}\frac{\partial\psi}{\partial x_i}\ ,$$

whence (13.66) gives

$$m\frac{d}{dt}\langle X_i\rangle_{\psi_t} = \langle P_i\rangle_{\psi_t}\ . \tag{13.68}$$

Similarly, working around t with $\tau \mapsto \psi_\tau$ in some domain inside $D(P_i\overline{H_0}) \cap D(\overline{H_0}P_i)$ where $\overline{H_0}$ is differentiable, we obtain

$$[\overline{H_0}, P_i]\psi = -i\left[-V, \frac{\partial}{\partial x_i}\right]\psi = -i\frac{\partial V}{\partial x_i}\psi \ ,$$

so from (13.66):

$$\frac{d}{dt}\langle P_i\rangle_{\psi_t} = -\left\langle \frac{\partial V}{\partial x_i}\right\rangle_{\psi_t} \ . \tag{13.69}$$

The classical statement of Ehrenfest's theorem consists of the pair of Eqs. (13.68)–(13.69), from which the mean values of position and momentum have a classical-like behaviour. Precisely, assume the gradient of V does not vary much on the spatial image of the wavefunction $\psi_t(\mathbf{x})$. Then we can estimate the right-hand side of (13.69) by

$$\left\langle \frac{\partial V}{\partial x_i}\right\rangle_{\psi_t} \simeq \int_{\mathbb{R}^3} \overline{\psi_t(\mathbf{x})}\left.\frac{\partial V}{\partial x_i}\right|_{\langle\mathbf{X}\rangle_{\psi_t}}\psi_t(\mathbf{x})dx = \left(\int_{\mathbb{R}^3}\overline{\psi_t(\mathbf{x})}\psi_t(\mathbf{x})dx\right)\left.\frac{\partial V}{\partial x_i}\right|_{\langle\mathbf{X}\rangle_{\psi_t}} = \left.\frac{\partial V}{\partial x_i}\right|_{\langle\mathbf{X}\rangle_{\psi_t}} .$$

Substituting in (13.69) we get the classical equation:

$$\frac{d}{dt}\langle P_i\rangle_{\psi_t} \simeq -\left.\frac{\partial V}{\partial x_i}\right|_{\langle\mathbf{X}\rangle_{\psi_t}} \ . \tag{13.70}$$

The punchline is that *under Ehrenfest's equations* (13.68)–(13.69), *the more wave packets cluster around their mean value – under a potential whose force varies slowly on the packet's spatial range – the better the momentum and position mean values obey the evolution laws of classical mechanics.*

Alas, the entire discussion is rather academic, because establishing mathematical conditions on H_0 that are *physically sound* and can justify in full the argument leading to (13.68)–(13.69), is a largely unsolved problem.

Remarks 13.37 **(1)** Recently, conditions on H and A have been found that realise (13.67) when A is *neither bounded* nor self-adjoint, including the case where A is the position or the momentum. The result we are talking about is the following [FrKo09].

Theorem 13.38 *Let $H : D(H) \to \mathsf{H}$, $A : D(A) \to \mathsf{H}$ be densely-defined operators on the Hilbert space H such that:*
(H1) *H is self-adjoint and A Hermitian (hence symmetric);*
(H2) *$D(A) \cap D(H)$ is invariant under $\mathbb{R} \ni t \mapsto e^{-itH}$, for every $t \in \mathbb{R}$;*
(H3) *if $\psi \in D(A) \cap D(H)$ then $\sup_I ||Ae^{-itH}\psi|| < +\infty$ for any bounded interval $I \subset \mathbb{R}$.*
Let $\psi_t := e^{-itH}\psi$. Then for any $\psi \in D(A) \cap D(H)$ the map $t \mapsto \langle A\rangle_{\psi_t}$ is C^1 and

$$\frac{d}{dt}\langle A\rangle_{\psi_t} = i(H\psi_t|A\psi_t) - i(A\psi_t|H\psi_t) \ .$$

As earlier claimed, the above hypotheses subsume the case where A is the position or the momentum on $\mathsf{H} = L^2(\mathbb{R}^n, dx)$, even though proving it is highly non-trivial (*op. cit.*, corollary 1.2). For it to happen it is enough that H is the only self-adjoint extension of $H_0 = -\Delta + V$ on $\mathscr{D}(\mathbb{R}^n)$ with V real and $(-\Delta)$-bounded with relative bound $a < 1$, in the sense of Definition 10.42.

(2) From the point of view of physics it is impossible to build an experimental device capable of measuring all possible values of an observable described by an unbounded self-adjoint operator. For the position observable, for instance, it would mean filling the universe with detectors! So we expect any observable represented by the *unbounded* self-adjoint operator A to be – physically speaking – indistinguishable from the observable of the self-adjoint operator $A_N := \int_{\sigma(A)\cap[-N,N]} \lambda dP^{(A)}(\lambda) \in \mathfrak{B}(H)$, with $N > 0$ large but finite. The general form of Ehrenfest's theorem (13.66) applies to such class of observables, if we assume (ii) and (iii), or only $\psi \in D(H)$ to have (13.67). In this case, though, it is not easy to use the formal commutation of position and momentum with a Hamiltonian like $-\frac{\hbar^2}{2m}\Delta + V$, which would bring to (13.68), (13.69). ■

13.3.3 *Constants of Motion Associated to Symmetry Lie Groups and the Case of the Galilean Group*

Consider a quantum system S with Hilbert space H_S, Hamiltonian H and inertial frame $\mathscr{I}$. Suppose there is a Lie group G with a strongly continuous unitary representation $\mathsf{G} \ni g \mapsto U_g$ on H_S, and assume the evolution operator $\mathbb{R} \ni t \mapsto e^{-itH}$ coincides with the representation of a one-parameter subgroup of G (clearly G is a symmetry group for S, since the representation U induces a projective representation of the same group). What we want to prove is that every $T \in \mathsf{T}_e\mathsf{G}$ determines a dynamical symmetry and a constant of motion (explicitly time-dependent, in general). In fact,

Theorem 13.39 *Let S be a quantum system on the Hilbert space H_S, with Hamiltonian H (in some inertial frame). Let $\mathsf{G} \ni g \mapsto U_g$ be a strongly continuous unitary representation on H_S of the n-dimensional Lie group G, and define the evolution operator $\mathbb{R} \ni t \mapsto e^{-itH}$ as the representation of a given one-parameter subgroup generated by $\mathbf{h} \in \mathsf{T}_e\mathsf{G}$:*

$$e^{-itH} = U_{\exp(t\mathbf{h})}, \quad t \in \mathbb{R}.$$

(a) *To each $\mathbf{b} \in \mathsf{T}_e\mathsf{G}$ there correspond a constant of motion $\{B_t\}_{t\in\mathbb{R}}$, in general time-dependent, and an associated dynamical symmetry according to Theorem 13.35.*

(b) *The operator $-iB_t$ (restricted to the Gårding space) for $t \in \mathbb{R}$ is the image under the Lie-algebra isomorphism (12.113) of some element $\mathbf{b}_t$ of the Lie algebra of G*

such that $\mathbf{b}_0 = \mathbf{b}$.

(c) *If* $[\mathbf{h}, \mathbf{b}] = 0$ *the constant of motion* $\{B_t\}_{t\in\mathbb{R}}$ *is time-independent.*

Proof (a)–(b). Consider the map $\mathbb{R} \to \mathsf{G}$:

$$\mathbb{R} \ni a \mapsto \exp(t\mathbf{h})\exp(a\mathbf{b})\exp(-t\mathbf{h}) \ .$$

It is certainly a one-parameter subgroup for any given $\mathbf{b} \in \mathsf{T}_e\mathsf{G}$ and every $t \in \mathbb{R}$. So if $T_1, \ldots T_n$ is a basis of $\mathsf{T}_e\mathsf{G}$, for suitable real functions $c_j = c_j(t)$ we can write

$$\exp(t\mathbf{h})\exp(a\mathbf{b})\exp(-t\mathbf{h}) = \exp(a\sum_{j=1}^{n} c_j(t)T_j) =: \exp(a\mathbf{b}_t) \ .$$

Apply U and pass to the Lie algebra representation $\mathsf{T}_e\mathsf{G} \ni \mathbf{b} \mapsto A_U[\mathbf{b}] := A_U(\mathbf{b})\restriction_{\mathscr{D}_G}$, where the Gårding space $\mathscr{D}_G$ is invariant and a core for the self-adjoint operators $A_U(\mathbf{b})$ (Chap. 12, in particular Theorem 12.79 and Corollary 12.87). Then

$$e^{-itH}e^{-ia\overline{A_U[\mathbf{b}]}}e^{itH} = e^{-ia\overline{\sum_{j=1}^{n} c_j(t)A_U[T_j]}} = e^{-iaA_U(\mathbf{b}_t)} \ . \tag{13.71}$$

Define self-adjoint operators parametrised by time

$$B_t := \overline{\sum_{j=1}^{n} c_j(t)A_U[T_j]} = A_U(\mathbf{b}_t) \ .$$

Then (13.71) shows B_t is a constant of motion that depends explicitly on time, for (13.71) implies:

$$e^{itH}B_te^{-itH} = A_U(\mathbf{b}) = B_0 \ , \quad t \in \mathbb{R} \ .$$

Again (13.71) shows that the family of symmetries $\sigma_a^{(t)} := e^{-iaB_t} \cdot e^{iaB_t}$, for any $a \in \mathbb{R}$, is a time-dependent dynamical symmetry. In fact (13.71) forces

$$e^{-iaB_t}e^{-itH} = e^{-itH}e^{-iaB_0} \ , \quad t \in \mathbb{R},$$

and then Theorem 13.5 proves the claim.
(c) Assuming $[\mathbf{b}, \mathbf{h}] = 0$, and using the Baker–Campbell–Hausdorff formula (12.81), (12.82), (12.83), we obtain

$$\exp(\tau\mathbf{h})\exp(a\mathbf{b}) = \exp(a\mathbf{b})\exp(\tau\mathbf{h}) \tag{13.72}$$

so long as $|a|, |\tau| < \epsilon$ with $\epsilon > 0$ small enough. Those formulas actually hold for any value of $a, \tau \in \mathbb{R}$. To see that, it suffices to observe, irrespective of a and τ, that we can write $a = \sum_{r=1}^{N} a_r$ and $\tau = \sum_{r=1}^{N} \tau_r$ so that $|a_r|, |\tau_r| < \epsilon$ for any r. For example,

$$\begin{aligned}\exp(\tau\mathbf{h})\exp(a\mathbf{b}) &= \exp(\tau_N\mathbf{h})\cdots\exp(\tau_2\mathbf{h})\exp(\tau_1\mathbf{h})\exp(a_1\mathbf{b})\exp(a_2\mathbf{b})\cdots\exp(a_N\mathbf{b})\\ &= \exp(\tau_N\mathbf{h})\cdots\exp(\tau_2\mathbf{h})\exp(a_1\mathbf{b})\exp(\tau_1\mathbf{h})\exp(a_2\mathbf{b})\cdots\exp(a_N\mathbf{b})\\ &\cdots\\ &= \exp(a_1\mathbf{b})\exp(\tau_N\mathbf{h})\cdots\exp(\tau_2\mathbf{h})\exp(a_2\mathbf{b})\cdots\exp(a_N\mathbf{b})\exp(\tau_1\mathbf{h})\\ &\cdots\\ &= \exp(a_1\mathbf{b})\exp(a_2\mathbf{b})\cdots\exp(a_N\mathbf{b})\exp(\tau_N\mathbf{h})\cdots\exp(\tau_2\mathbf{h})\exp(\tau_1\mathbf{h}) = \exp(a\mathbf{b})\exp(\tau\mathbf{h})\,,\end{aligned}$$

so

$$e^{\tau\mathbf{h}}e^{a\mathbf{b}} = e^{a\mathbf{b}}e^{\tau\mathbf{h}}\,.$$

Consequently, defining $t = -\tau$ and using U we get

$$e^{itH}e^{-iaA_U(\mathbf{b})}e^{-itH} = e^{-iaA_U(\mathbf{b})}\,,$$

whence the claim. □

To exemplify the general result found above, we revert to the Galilean group and its projective unitary representations seen at the end of the previous chapter. We will show there are 10 first integrals for a system having the restricted Galilean group $S\mathscr{G}$ as symmetry group (described by a unitary representation of a central extension of the universal covering $\widetilde{S\mathscr{G}}$). We consider in particular the spinless particle of mass m, and refer to the unitary representation of the central extension $\widehat{\widetilde{S\mathscr{G}}}_m$ of Chap. 12. The Lie algebra is the extension of the Lie algebra of $\widetilde{S\mathscr{G}}$, which has 10 generators $-\mathbf{h}$, $\mathbf{p}_i$, $\mathbf{j}_i$, $\mathbf{k}_i$, i = 1,2,3, such that:
(i) $-\mathbf{h}$ generates the subgroup $\mathbb{R} \ni c \mapsto (c, \mathbf{0}, \mathbf{0}, I)$ of **time displacements**,
(ii) the $\mathbf{p}_i$ generate the Abelian subgroup $\mathbb{R}^3 \ni \mathbf{c} \mapsto (0, \mathbf{c}, \mathbf{0}, I)$ of **space translations**,
(iii) the $\mathbf{j}_i$ generate the subgroup $SO(3) \ni R \mapsto (0, \mathbf{0}, \mathbf{0}, R)$ of **space rotations**,
(iv) the $\mathbf{k}_i$ generate the Abelian subgroup $\mathbb{R}^3 \ni \mathbf{v} \mapsto (0, \mathbf{0}, \mathbf{v}, I)$ of **pure Galilean transformations**.

Remark 13.40 We have already remarked that *time displacement* and *time evolution* are one the inverse of the other. Hence $\mathbf{h}$ is the generator of time evolution in accordance with the notation used in Theorem 13.39, whereas $-\mathbf{h}$ is the generator of time displacement. This observation should explain the conventional sign of the generator of time translations: one considers time evolution to be physically more important than time displacement. ■

These elements obey the commutation relations (12.144). To pass from the Lie algebra of $\widetilde{S\mathscr{G}}$ to that of $\widehat{\widetilde{S\mathscr{G}}}_m$ we add a generator commuting with the above ones, plus central charges for the commutation relations between $\mathbf{k}_i$ and $\mathbf{p}_j$ equal to the mass m

(cf. (12.153) and ensuing discussion). The strongly continuous unitary representation of $\widehat{\widetilde{S\mathscr{G}}}_m$ of our concern is the following one:

$$\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi \widetilde{Z^{(m)}}_g \ ,$$

induced by unitary operators $\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)}$ (12.145):

$$\left(\widetilde{Z^{(m)}}_{(c,\mathbf{c},\mathbf{v},U)}\widetilde{\psi}\right)(\mathbf{k}) := e^{i(c\mathbf{v}-\mathbf{c})\cdot(\mathbf{k}-m\mathbf{v})} e^{i\frac{c}{2m}(\mathbf{k}-m\mathbf{v})^2}\widetilde{\psi}\left(R(U)^{-1}(\mathbf{k}-m\mathbf{v})\right) \ .$$

Notice the Lie group $\widehat{\widetilde{S\mathscr{G}}}_m$ contains the subgroup spanned by $\mathbf{h}$, corresponding to the evolution operator on the system's Hilbert space H_S. Among the commutation relations (12.144) defining the Lie algebra of $\widetilde{S\mathscr{G}}$ (valid on the central extension $\widehat{\widetilde{S\mathscr{G}}}_m$), we are interested in the ones directly involving $\mathbf{h}$:

$$[\mathbf{p}_i, -\mathbf{h}] = \mathbf{0}\ , \quad [\mathbf{j}_i, -\mathbf{h}] = \mathbf{0}\ , \quad [\mathbf{k}_i, -\mathbf{h}] = \mathbf{p}_i \quad i = 1, 2, 3. \tag{13.73}$$

Adapting the proof of Theorem 13.39 to the representation $\widehat{\widetilde{S\mathscr{G}}}_m \ni (\chi, g) \mapsto \chi Z_g^{(m)}$, the first two brackets give

$$e^{-i\tau H} e^{-iaP_i} = e^{-iaP_i} e^{-i\tau H} \tag{13.74}$$

and

$$e^{-i\tau H} e^{-iaL_i} = e^{-iaL_i} e^{-i\tau H} \ . \tag{13.75}$$

Using Theorem 13.5 and Definition 13.31, these tell, in agreement with Theorem 13.39:

(a) the three momentum components and the three orbital angular momentum components are (time-independent) constants of motion;

(b) the symmetries generated by these integrals of motion, i.e. the *translations along the axes* and the *rotations about the axes* are (time-independent) dynamical symmetries (see Examples 12.46 and (12.135) for the explicit action on wavefunctions).

Let us address the last bracket in (13.73). A direct use of the Baker–Campbell–Hausdorff formula is not trivial, even if technically possible with a bit of work, also in the general case. To understand what this third identity corresponds to in terms of the associated one-parameter subgroups, let us study the matter in the Galilean group. The subgroup generated by $-\mathbf{h}$ is the time displacement:

$$\exp(\tau\mathbf{h}) = (-\tau, \mathbf{0}, \mathbf{0}, I) \quad \tau \in \mathbb{R}\ .$$

The subgroup generated by $\mathbf{k}_j$ is a pure Galilean transformation along the jth axis with unit vector $\mathbf{e}_j$:

$$\exp(a\mathbf{k}_j) = (0, \mathbf{0}, a\mathbf{e}_j, I) \quad a \in \mathbb{R} : .$$

Immediately, then, the group law (12.137) gives

$$\exp(\tau\mathbf{h})\exp(a\mathbf{k}_j)\exp(-\tau\mathbf{h}) = \exp(a(\tau\mathbf{p}_j + \mathbf{k}_j))\ .$$

Applying the unitary representation these become

$$e^{-i\tau H}e^{-aK_j}e^{i\tau H} = e^{-ia\left(\overline{\tau P_j\restriction_{\mathscr{D}_G} + K_j\restriction_{\mathscr{D}_G}}\right)}\ .$$

Therefore, if we define self-adjoint operators

$$K_{jt} := \overline{tP_j\restriction_{\mathscr{D}_G} + K_j\restriction_{\mathscr{D}_G}} \quad j = 1, 2, 3,$$

each observable is a constant of motion explicitly dependent on time, and each one defines a dynamical symmetry for every $a \in \mathbb{R}$:

$$e^{-iaK_{jt}}e^{-itH} = e^{-itH}e^{-iaK_{j0}}$$

The dynamical symmetry $e^{-iaK_{jt}}$ thus defines a *pure Galilean transformation along* $\mathbf{e}_j$ at time t.

Remarks 13.41 **(1)** It can be interesting to question about the meaning of the conservation law of K_{jt}, which is not at all obvious. We remind that the boost is defined (see (12.152)) as $K_j = -mX_j$. Choosing $\psi \in \mathscr{D}_G$ and letting it evolve under the evolution operator, $\psi_t := e^{-itH}\psi$, the conservation law for K_{jt} implies:

$$t(\psi_t|P_j\psi_t) - m(\psi_t|X_j\psi_t) = const,$$

i.e.

$$\langle P_j\rangle_{\psi_t} = m\frac{d}{dt}\langle X_j\rangle_{\psi_t}\ . \tag{13.76}$$

Hence the average momentum of the particle is, in some sense, the product of the mass times the velocity, the latter indicating the average position of the particle. The result is *a priori* not evident, since in QM the momentum is *not* the product of mass and velocity.
(2) Suppose we work with a multi-particle system, admitting the Galilean group as symmetry group described by a unitary representation of a central extension associated to the total mass M (see Chap. 12). Identity (13.76) is proved in the same way, and hence holds true. But now P_j is the component along $\mathbf{e}_j$ of the *total* momentum, and X_j is the $\mathbf{e}_j$-component of the *position vector of the centre of mass*. A similar relationship holds for systems invariant under the Poincaré group, and follows from the invariance under pure Lorentz transformations. The term corresponding to the total mass accounts for the energy contributions of the single components (like the kinetic energies of the isolated points making the system), in conformity to equation $M = E/c^2$. ■

This accounts for 9 first integrals, but we said there are 10 in total.

The attentive reader will have noticed there is still a dynamical symmetry around, and a corresponding conservation law: yes, energy! Namely, the obvious commutation relation $[\mathbf{h}, \mathbf{h}] = \mathbf{0}$ holds on the Lie algebra, or $[H, H] = 0$ at the level of self-adjoint generators, or $[e^{-i\tau H}, e^{-i\tau' H}] = 0$ for the exponentials. By Theorem 13.5 and Definition 13.6, the last identity, in agreement with Theorem 13.39, says that

(a) the Hamiltonian is a constant of motion,

(b) the symmetry generated by $-H$ (the time displacement) is a dynamical symmetry.

The result is completely general and does not depend on having the Galilean group as symmetry group; it suffices that the Hamiltonian exists.

13.4 Compound Systems and Their Properties

We met in Chap. 12 systems composed by subsystems, and we saw that the overall Hilbert space is the tensor product of the Hilbert spaces relative to the subsystems. But this is actually an axiom of the theory. Compound systems bear a host of fascinating non-classical features, which we will review in this section.

13.4.1 Axiom A7: Compound Systems

We are ready to state the seventh axiom of QM, the one about compound quantum systems. For the mathematical contents we refer to the definitions and results of Sect. 10.2.1.

A7. *When a quantum system consists of a finite number N of subsystems, each described on a Hilbert space $\mathsf{H}_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, N$, the comprehensive system is described on the Hilbert space $\bigotimes_{i=1}^{N} \mathsf{H}_i$.*

Any observable $A_i : D(A_i) \to \mathsf{H}_i$ on the ith subsystem (including elementary observables defined by orthogonal projectors) is identified in the larger system with the observable $\overline{I \otimes \cdots \otimes I \otimes A_i \otimes I \otimes \cdots \otimes I}$.

Two are the types of compound systems we have already met: those made of *elementary particles with internal structure*, and *multi-particle* systems (elementary particles with or without internal structure). In the first case the Hilbert space is $L^2(\mathbb{R}^3, dx) \otimes \mathsf{H}_0$, where H_0 is *finite-dimensional* and describes the particle's internal degrees of freedom: spin and charges of various sort (cf. Chap. 11). By elementary particle with internal structure we mean that the internal space is finite-dimensional. The literature, when referring to systems of elementary particles with space H_0, calls $L^2(\mathbb{R}^3, dx)$ the **orbital space** or **space of orbital degrees of freedom**, and H_0 the

internal space or **space of internal degrees of freedom**. In case the space of internal freedom degrees describes a (certain type of) charge, we should also keep possible superselection rules into account.

We would like to make a few remarks on the Hamiltonian operator of multi-particle systems, when the single Hilbert spaces are $L^2(\mathbb{R}^3, dx)$ with a fixed inertial frame, and $\mathbb{R}^3$ is the rest space (using orthonormal Cartesian coordinates). The Hilbert space of a system on N particles with masses $m_1, \ldots, m_N$ is the N-fold tensor product of $L^2(\mathbb{R}^3, dx)$. From Example 10.27(1) this product is naturally isomorphic to $L^2(\mathbb{R}^{3N}, dx)$. Indicate by $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ the generic point in $\mathbb{R}^{3N}$, where $\mathbf{x}_k = ((x_k)_1, (x_k)_2, (x_k)_3)$ is the triple of orthonormal Cartesian coordinates of the kth factor of $\mathbb{R}^{3N} = \mathbb{R}^3 \times \cdots \times \mathbb{R}^3$. The natural isomorphism turns (prove it as exercise) the position operator of the kth particle into the multiplication by the corresponding $\mathbf{x}_k = ((x_k)_1, (x_k)_2, (x_k)_3)$, and the momentum into the unique self-adjoint extension of the $\mathbf{x}_k$-derivatives (times $-i\hbar$), for instance on $\mathscr{D}(\mathbb{R}^{3N})$. The Hamiltonians of each particle, assumed free, coincide with the self-adjoint extension, say on $\mathscr{D}(\mathbb{R}^{3N})$, of the corresponding Laplacian $-\Delta_k = \sum_{i=1}^{3} \frac{\partial^2}{\partial (x_k)_i^2}$ times $-\hbar^2/(2m_k)$. Relying on Sect. 11.5.8, if the particles undergo interactions described classically by a potential $V = V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, the Hamiltonian is expected to be some self-adjoint extension of

$$H_0 := \sum_{k=1}^{N} -\frac{\hbar^2}{2m_k}\Delta_k + V(\mathbf{x}_1, \ldots, \mathbf{x}_N)\,.$$

For instance, particles with charges e_k interacting with one another under Coulomb forces and also with external charges Q_k are expected to have as Hamiltonian a self-adjoint extension of

$$H_0 := \sum_{k=1}^{N} -\frac{\hbar^2}{2m_k}\Delta_k + \sum_{k=1}^{N} \frac{Q_k e_k}{|\mathbf{x}_k|} + \sum_{i<j}^{N} \frac{e_i e_j}{|\mathbf{x}_i - \mathbf{x}_j|}\,.$$

As we explained in Sect. 10.4 and Examples 10.52, important results mainly due to Kato imply, under natural assumptions on V, that not only H_0 is essentially self-adjoint on standard domains like $\mathscr{D}(\mathbb{R}^{3N})$ or $\mathscr{S}(\mathbb{R}^{3N})$, but the only self-adjoint extension is *bounded below*, therefore *making the system energetically stable*. This happens in particular for the operator with the Coulomb interaction presented above (cf. Examples 10.52).

If the N particles have an internal structure, with internal Hilbert space H_{0k}, the overall Hilbert space will be isomorphic to $L^2(\mathbb{R}^{3N}, dx) \otimes_{k=1}^{N} \mathsf{H}_{0k}$, and the possible Hamiltonians are more complicated, usually. We encourage the reader to consult the specialised texts [Mes99, CCP82, Pru81, ReSi80] for examples of this kind.

13.4.2 *Independent Subsystems: The Delicate Viewpoint of von Neumann Algebra Theory*

Apropos compounds of independent subsystems, we shall make a few superficial observations from a more abstract point of view (see [ReSu07, ReSu10]). Suppose that a quantum system S described by the von Neumann algebra $\mathfrak{R}$ on the Hilbert space H contains mutually independent subsystem S_k, $k = 1, \ldots, N$ with corresponding von Neumann algebras of observables $\mathfrak{R}_k$ on H. We expect that

$$\mathfrak{R}_k \subset \mathfrak{R}'_h \quad \text{if}\, k \neq h, \tag{13.77}$$

because the subsystems are independent and any two observables of different subsystems are therefore always compatible. For the same reason

$$\mathfrak{R}_k \cap \mathfrak{R}_h = \{cI \mid c \in \mathbb{C}\} \quad \text{when}\, k \neq h \tag{13.78}$$

because $\mathfrak{R}_k$ contains non-trivial observables of S_k but not of $\mathfrak{R}_h$. If S is consists of S_k only (no other independent subsystem exists) then we may assume that

$$\mathfrak{R} = \left(\bigcup_{k=1}^{N} \mathfrak{R}_k \right)'' . \tag{13.79}$$

Further possible requirements for describing independent subsystems will be about states. Consider a system made of N subsystems, so that its von Neumann algebra $\mathfrak{R}$ is the product $\mathfrak{R}_1 \otimes \cdots \otimes \mathfrak{R}_N$ of N von Neumann algebras on the same Hilbert space, whose pairwise intersections are trivial. Given N *normal states* $\rho_1, \ldots, \rho_N$ (i.e. positive trace-class operators of unit trace, Remark 7.73(2)) on the respective von Neumann algebras $\mathfrak{R}_k$, it is reasonable to assume that there is an extension ρ defining a (normal) state on $\mathfrak{R}$ with the *separability property*

$$tr(\rho(A_1 \cdots A_n)) = tr(\rho_1 A_1) \cdots tr(\rho_N A_N) \quad \forall A_k \in \mathfrak{R}_k, k = 1, \ldots, N. \tag{13.80}$$

(Another condition is the injectivity of the *GNS representation* associated to ρ when the GNS representations of the ρ_k are injective. We will not pursue this now, as it is related to Chap. 14). We can assume an even more abstract viewpoint. Suppose we start from N von Neumann algebras $\mathfrak{M}_k$ over *different* Hilbert spaces H_k, forming a compound system described on H by the von Neumann algebra $\mathfrak{R}$. The previous discussion is still valid if there exist injective *-homomorphisms of von Neumann algebras $h_k : \mathfrak{M}_k \to \mathfrak{R}$ (in particular, h_k maps the identity to the identity: $h_k(I_k) = I$), so that each $\mathfrak{M}_k$ is identified with a von Neumann subalgebra $\mathfrak{R}_k \subset \mathfrak{R}$. This identification is also topological, by Remark 3.95(2). A stronger version would require that the isomorphisms h_k be implemented unitarily. Conditions (13.77)–(13.80) are supposed to be valid for the h_k-images $\mathfrak{R}_k$ in place of the algebras $\mathfrak{M}_k$

themselves. We leave it to the reader to discover the related picture involving elementary propositions (orthogonal projectors) of $\mathscr{L}_{\mathfrak{M}_k}(\mathsf{H}_k)$, $\mathscr{L}_{\mathfrak{R}}(\mathsf{H})$ and isomorphisms of (σ-)complete orthocomplemented lattices.

What is the relationship between axiom **A7** and this abstract and apparently more general approach to composite systems? Let us prove that they are in agreement axiom **A7**, as a special case of the more general notion of composite system.

When $\mathfrak{R}_k \subset \mathfrak{B}(\mathsf{H}_k)$, axiom **A7** gives a natural procedure to build H and $\mathfrak{R}$ under (13.77)–(13.79): beside $\mathsf{H} = \otimes_{k=1}^{n} \mathsf{H}_k$ we also have $\mathfrak{R} = \otimes_{k=1}^{N} \mathfrak{R}_k$, the *tensor product of von Neumann algebras* as of Definition 10.34 (in particular, Theorem 10.35 holds). Each map

$$h_k \ : \ \mathfrak{R}_k \ \ni A \mapsto \ I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I \otimes \ \in \ \otimes_{k=1}^{N} \mathfrak{R}_k \ ,$$

with A in the kth entry in the right-hand side, is a *-isomorphism onto its image, the von Neumann subalgebra $\mathbb{C} I_{\mathsf{H}_1} \otimes \cdots \otimes \mathfrak{R}_k \otimes \cdots \otimes \mathbb{C} I_{\mathsf{H}_N}$ of $\otimes_{k=1}^{N} \mathfrak{R}_k$. Moreover (13.80) is also valid, simply by setting $\rho := \rho_1 \otimes \cdots \otimes \rho_N$, where each ρ_k is a positive trace-class operator of trace one on H_k, defining a normal state on $\mathfrak{R}_k$.

Yet no one says that tensor products of Hilbert spaces and von Neumann algebras is the only possibility to describe compound system in terms of the algebras of observables of the subsystems. Indeed, there exist compound systems – in the abstract sense of (13.77)–(13.79) – for which $\mathfrak{R} \subset \mathfrak{B}(\mathsf{H})$ *cannot* be interpreted as the tensor product of the $\mathfrak{R}_k \subset \mathfrak{B}(\mathsf{H})$ of its subsystems, if *at the same time* we wish to identify each $\mathfrak{R}_k$ with a von Neumann algebra $\hat{\mathfrak{R}}_k \subset \mathfrak{B}(\mathsf{H}_k)$ over a suitable Hilbert space H_k, so that H is isomorphic to $\otimes_{k=1}^{N} \mathsf{H}_k$.

The identifications simultaneously regard Hilbert spaces and von Neumann algebras, so they should be implemented by a unitary operator. As a matter of fact, we must look for a surjective, norm-preserving operator $U : \mathsf{H} \to \otimes_{k=1}^{N} \mathsf{H}_k$ such that

$$U \mathfrak{R}_k U^{-1} = \mathbb{C} I_{\mathsf{H}_1} \otimes \cdots \otimes \hat{\mathfrak{R}}_k \otimes \cdots \otimes \mathbb{C} I_{\mathsf{H}_N} \quad \text{for } k = 1, 2, \ldots, N$$

and

$$\mathfrak{R} = U^{-1} \hat{\mathfrak{R}}_1 \otimes \cdots \otimes \hat{\mathfrak{R}}_N U$$

hold.

Here is an elementary example to make the point. Suppose that $\mathfrak{R}_1$ is a factor on the Hilbert space H, and define $\mathfrak{R}_2 := \mathfrak{R}_1'$. Then $\mathfrak{R}_1 \cap \mathfrak{R}_2 = \{cI \mid c \in \mathbb{C}\}$ and $(\mathfrak{R}_2 \cup \mathfrak{R}_2)'' = \mathfrak{B}(\mathsf{H})$, and hence (13.77)–(13.79) are satisfied. We interpret[6] the picture as that of a bipartite system with total algebra of observables $\mathfrak{R} := \mathfrak{B}(\mathsf{H})$ and subsystems' algebras $\mathfrak{R}_1, \mathfrak{R}_2$. It will not be possible to interpret the tensor product with axiom **A7**. In fact, when $\mathfrak{R}_1$ is *not* of *type I*, we cannot find Hilbert spaces H_1, H_2 with von Neumann algebras $\hat{\mathfrak{R}}_1 \subset \mathfrak{B}(\mathsf{H}_1)$, $\hat{\mathfrak{R}}_2 \subset \mathfrak{B}(\mathsf{H}_2)$, and a unitary operator

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[6] According to Sect. 11.2.3, another natural interpretation would be that $\mathfrak{R}_2$ is the algebra generated by the *gauge group* of a system with von Neumann algebra of observables $\mathfrak{R}_1$, in absence of Abelian superselection rules.

$U : \mathsf{H} \to \mathsf{H}_1 \otimes \mathsf{H}_2$, satisfying $U\mathfrak{R}_1 U^{-1} = \hat{\mathfrak{R}}_1 \otimes \mathbb{C}I_{\mathsf{H}_2}$ and $U\mathfrak{R}_2 U^{-1} = \mathbb{C}I_{\mathsf{H}_1} \otimes \hat{\mathfrak{R}}_2$ [KaRi97, vol. II, p. 816]. On the other hand, if $\mathfrak{R}_1$ has type I_n, so $\mathfrak{R}_2$ has type I_m ($n, m = 1, 2, \ldots, +\infty$), then U does exist, and moreover $\hat{\mathfrak{R}}_1 = \mathfrak{B}(\mathsf{H}_1)$, $\hat{\mathfrak{R}}_2 = \mathfrak{B}(\mathsf{H}_2)$. Hence we also have (13.80). In standard Quantum Mechanics, fortunately, all operator algebras are of type I (or direct sums of such), so no problems arise. For extended thermodynamical systems, or in Quantum Field Theory, the picture is much more delicate: factors of other type appear (type-III factors, see [Haa96, Rob04, ReSu07] for instance) and tensor products may no longer be appropriate to describe compound systems. In Chap. 14 we shall take the next step in the abstraction and describe observables using C^*-algebras. When one plays that game, though, there is no unique definition of tensor product [KaRi97, Sect. 11.3].

We will not address these advanced issues in this book and just stick to axiom **A7** as it stands. For a detailed study of independent subsystems from the viewpoint of von Neumann algebras (and C^*-algebras) the reader is advised to look in [Red98, ch. 10–12]. Those three chapters also analyse the implications in Quantum Field Theory, especially with regard to the EPR paradox. We will cover the latter in the next section, albeit from a much more elementary point of view.

13.4.3 Entangled States and the So-Called "EPR Paradox"

A measuring device is not necessarily located at a point in space. On the contrary, if we want to measure quantities defined in space, first and foremost the position of a quantum particle, we must fill space with instruments: particle detectors that measure the position. The process of reduction of the state described by axiom **A3** is "instantaneous". This means that once a device has detected the particle at the point p and at time t, from that instant onwards no other device, as remote in space as we want from the first detector, will be able to detect the particle. The reduction of state therefore seems to be a *nonlocal* process: apparently it implies a "simultaneous" transmission of information between faraway places. This appears to violate the principles of the theory of relativity. In 1935 Einstein, Podolsky and Rosen [Des99, Bon97, Ghi07, Alb94], whilst considering systems of two particles, showed that the question can be phrased in physically-operative terms by which the violation seems to materialise effectively [EPR35].

Axiom **A7** describes the possible states of a compound quantum system. Let S be a system made of two subsystems A, B. The Hilbert space of S is $\mathsf{H}_S = \mathsf{H}_A \otimes \mathsf{H}_B$, in the obvious notations. The vectors of $\mathsf{H}_A \otimes \mathsf{H}_B$ are not just of the *factorised* sort $\psi_A \otimes \psi_B$, with one tensor product, for there are linear combinations of these products, too, like

$$\Psi = \frac{\psi_A \otimes \psi_B - \psi'_A \otimes \psi'_B}{\sqrt{2}} . \tag{13.81}$$

Pure states corresponding to unit vectors of the above form are called **entangled pure states**.[7]

Consider the entangled state associated to the vector Ψ of (13.81), and let us suppose ψ_A and ψ'_A are eigenstates normalised to 1 of some observable G_A with discrete spectrum on system A, respectively corresponding to distinct eigenvalues a and a'. Assume the same for ψ_B, ψ'_B: they are unit eigenstates of an observable G_B with discrete spectrum on system B, with eigenvalues $b \neq b'$.

The discrete-spectrum observables G_A, G_B are, for instance, relative to internal freedom degrees of the systems A and B. They can typically be components of the spin or the polarisation of the particles. In that case the spaces H_A, H_B are also factorised into orbital space and internal space.

Until we measure it, the quantity G_A is not defined on the system, if the latter is in the state given by Ψ; there are two possible values a, a' with probability $1/2$ each. The same pattern is valid for G_B. The minute we measure G_A, reading – say – a (*a priori* unpredictable, at least in principle), the state of the *total system* changes, in allegiance to axiom **A3**, becoming the pure state of the unit vector

$$\psi_A \otimes \psi_B \ .$$

The crucial point is the following: if the initial state is the entangled state Ψ, the measurement of G_A determines *a measurement of* G_B *as well*: in the pure state associated to $\psi_A \otimes \psi_B$ the value of G_B is well defined, and equals b in our conventions. Any measurement of G_B can only give b.

Following the famous study of Einstein, Podolsky and Rosen, consider now compound systems of two particles A, B, prepared in the entangled pure state of the vector Ψ of (13.81), that *move away from each other* at great speed (i.e., the state's orbital part is the product of two very concentrated packets that separate rapidly from each other). In principle we can measure G_A and G_B on the respective particles in distant places and at lapses so short that no physical signal, travelling below the speed of light, can be transmitted from one experiment to the other in good time.

If axiom **A3** is to be valid, there should be a correlation between the outcomes: every time the reading of G_A is a (or a'), G_B will give b (respectively, b').

How can system A communicate to system B the outcome of the measurement of G_A in time to produce the aforementioned correlations, without breaching the cornerstones of relativity?

This is a common situation in classical systems too, and in that case the explanation is very easy: there is no superluminal communication between the systems, for the correlations *pre-exist* the measurements. We call *local realism* this viewpoint, though, philosophically speaking, local realism is a much more articulate position. For example, let the observed quantities G_A, G_B be some particle "charge" or the like, and suppose the overall system S has charge 0 in the state in which it has been prepared, while the particles could have charge ± 1 corresponding to the aforementioned

[7] Analogously, for mixed states: $\rho \in \mathfrak{S}(\mathsf{H}_A \otimes \mathsf{H}_B)$ is **entangled** if it is *not* a convex combination of states of the form $\rho_A \otimes \rho_B$, $\rho_A \in \mathfrak{S}(\mathsf{H}_A)$, $\rho_B \in \mathfrak{S}(\mathsf{H}_B)$.

values a, a' and b', b. *If we reason with classical particles*, we have to conclude one particle has charge 1, the other -1. If the values of charge are pre-existent, i.e. if they exist before and independently of the fact that we take a measurement to observe the charge, we can rest assured that if a particle has charge 1 when measured, the second one will give -1 when observed, irrespective of where and when charges are measured, because the values are fixed *beforehand*.

The picture described by QM is, however, different: even if the state associated to Ψ has total charge $G = G_A + G_B$ equal to 0, the subsystems's charges *are not defined* on the state of Ψ, and become fixed *at the time of the measurement* (of either state). Therefore the situation pre-existing the measurement cannot be held responsible for the correlations predicted by QM, if we accept the standard interpretation of QM.

The idea of Einstein, Podolsky and Rosen was that if said correlations were really observed (as required by QM), and since defying the assumption of relativistic locality was out of the question, the reason for the correlations was due to a *pre-existing* state to the measurements. As this cannot be described in the framework of the standard formulation of QM, it would have proved that the standard formulation of QM was, by nature, *incomplete*. (The probabilities used in QM, moreover, would reduce to mere epistemic probabilities).

J. Bell, in a brilliant paper published in 1964 [Bel64, Bon97, Ghi07], while measuring at least three types of "charges" producing correlations (in reality one measures three spin components for massive particles or polarisation states of photons), proved it is possible to *distinguish experimentally* between two situations, where the charges are either:
(i) fixed *before* the measurements,
or
(ii) fixed *at the same time* of the measurements.

Bell proved that case (i) occurs only if the outcomes obey a series of inequalities: the celebrated *Bell inequalities*.

It is important to remark that a potential experimental infringement of Bell's inequalities does not automatically validate the standard formulation of QM. Non-local correlations, if observed experimentally, could in principle be justified without QM. What is true is that Quantum Mechanics, in contrast to Classical Mechanics, *forecasts* the presence of the correlations and at the same time the *violation* of Bell's inequalities, as we will see in short.

13.4.4 Bell's Inequalities and Their Experimental Violation

We will discuss briefly the simplified version of Bell's inequalities proposed by Wigner and d'Espagnat. The *Wigner-d'Espagnat inequality* has never, up to now, been used in practical experiments. We essentially follow the presentation by Sakurai [Sak94].

Take two particles A, B of spin $1/2$ produced together, in a region O, in the "singlet state", i.e. in the unique pure state of *zero total spin*. Fix an inertial system where the phenomenon is described. The entangled pure spin state is representable by Ψ_{sing} in the spin space $\mathsf{H}_{Aspin} \otimes \mathsf{H}_{Bspin}$:

$$\Psi_{sing} = \frac{\psi_+^{(n)} \otimes \psi_-^{(n)} - \psi_-^{(n)} \otimes \psi_+^{(n)}}{\sqrt{2}}\,, \tag{13.82}$$

where each H_{Aspin}, H_{Bspin} is isomorphic to $\mathbb{C}^2$, since each particle has spin $s = 1/2$. Moreover, $\psi_+^{(n)}$ and $\psi_-^{(n)}$ are unit eigenvectors with respective eigenvalues $1/2, -1/2$ for the spin operator $S_n := \mathbf{n} \cdot \mathbf{S}$ along $\mathbf{n}$ (unit three-dimensional vector), for the single particle (as usual, $\hbar = 1$). The decomposition (13.82) holds for the singlet state Ψ_{sing}, *irrespective of where the axis* $\mathbf{n}$ *is*.

We suppose the particles part from each other. In other words the state's orbital part will, for example, be a product of wavefunctions, one in the orbital variables of A and one in the orbital variables of B, given by packets concentrated around their centres. The packets move away quickly from O in the chosen frame, so that the packets never overlap when the spin measurements are taken on A and B (we will not discuss the case of identical particles, which is practically the same anyway). To study the correlation of spin measurements that violate locality, actually, it is not even necessary to assume the orbital part has the form we said. It suffices to place the devices measuring spin in faraway regions O_A, O_B (and far from O), and make sure the axis of the spin analyser of A in O_A can be re-oriented during consecutive measurements (see below) fast enough to prevent signals from propagating subluminally from O_A and reach O_B during measurements on the spin of B. This setup was concretely put into practice by Aspect's experiments.

To fix ideas imagine O_B is on the right of O and O_A on the left. The spin measurements in A and B (even two or more consecutive readings along distinct axes for each particle) can be taken, independent of one another, along given independent directions $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, not necessarily orthogonal. Assume at last that N pairs AB in spin singlet state are generated in O, and that each pair is then analysed by spin measurements on A, B in O_A, O_B along three given independent unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Suppose that on the N pairs the values of the spin components are fixed *before* measuring in O_A and O_B. This is the contrary of what the standard formulation of QM predicts. *In order to have zero total spin*, for each pair AB the spin triples $(S_u, S_v, S_w)_A$ for A and $(S_u, S_v, S_w)_B$ for B must have opposite corresponding components. For instance, $(+, -, +)_A$ and $(-, +, -)_B$ are admissible, whereas $(+, +, +)_A$ and $(-, +, -)_B$ are not (from now on we shall abbreviate $+1/2$ by $+$ and $-1/2$ by $-$). There are 8 possible combinations altogether, tabled below.

Among the N pairs there will be N_1 pairs $(+, +, +)$ for A and $(-, -, -)$ for B *irrespective of whether measured or not*, N_2 pairs $(+, +, -)$ for A and $(-, -, +)$ for B *irrespective of whether measured or not*, and so on. At any rate we will have $N = \sum_{n=1}^{8} N_k$.

With our "classical" hypotheses, every pair among the N examined must belong, after it has been created, in one of the sets, independently of what sort of spin measurements is taken. So let us suppose, for a certain pair, we measure S_u on A

	part. A	part. B
N_1	$(+,+,+)$	$(-,-,-)$
N_2	$(+,+,-)$	$(-,-,+)$
N_3	$(+,-,+)$	$(-,+,-)$
N_4	$(+,-,-)$	$(-,+,+)$
N_5	$(-,+,+)$	$(+,-,-)$
N_6	$(-,+,-)$	$(+,-,+)$
N_7	$(-,-,+)$	$(+,+,-)$
N_8	$(-,-,-)$	$(+,+,+)$

finding $+$, and S_v on B finding $+$. Then the pair can only belong to class 3 or 4, and there are $N_3 + N_4$ possibilities out of N that this happens. If we call $p(u+, v+)$ the probability of finding $+$ measuring S_u on A and $+$ measuring S_v on B, we have

$$p(u+, v+) = \frac{N_3 + N_4}{N} . \tag{13.83}$$

Similarly

$$p(u+, w+) = \frac{N_2 + N_4}{N} , \quad p(w+, v+) = \frac{N_3 + N_7}{N} . \tag{13.84}$$

Since $N_2, N_7 \geq 0$:

$$p(u+, v+) = \frac{N_3 + N_4}{N} \leq \frac{N_2 + N_4}{N} + \frac{N_3 + N_7}{N} = p(u+, w+) + p(w+, v+) ,$$

i.e. **Bell's inequalities** hold:

$$p(u+, v+) \leq p(u+, w+) + p(w+, v+) . \tag{13.85}$$

These inequalities hold whatever basis of unit vectors (not necessarily orthogonal) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ we pick, provided the values of the spin components along them are defined *before* we take the spin measurements, and the total spin of each pair is null. QM's prediction leads to a *violation* of the inequalities if we choose the axes suitably. Compute first $p(u+, v+)$ with the quantum recipe. Suppose the measure on A of S_u is $+$. Measuring S_u on B will give (or has already given) $-$, by (13.82). Anyway, particle B will have spin state represented by $\psi_-^{(u)}$ in the eigenvector basis of S_u. So we can evaluate $p(u+, v+)$ as:

$$p(u+, v+) = \frac{1}{2} \left| \left(\psi_-^{(u)} \middle| \psi_+^{(v)} \right) \right|^2 , \tag{13.86}$$

where $1/2$ is the initial probability of having $+$ on A when measuring S_u in state Ψ_{sing}. It is an easy exercise to compute the right-hand side of (13.86) in terms of the angle θ_{uv} between $\mathbf{u}$ and $\mathbf{v}$:

$$p(u+, v+) = \frac{1}{2}\sin^2\left(\frac{\theta_{uv}}{2}\right) . \tag{13.87}$$

The other terms in (13.85) are similar, so Bell's inequality (13.85) is equivalent to:

$$\sin^2\left(\frac{\theta_{uv}}{2}\right) \leq \sin^2\left(\frac{\theta_{uw}}{2}\right) + \sin^2\left(\frac{\theta_{wv}}{2}\right) . \tag{13.88}$$

It is not hard to see that a smart choice of angles invalidates the inequality. For example $\theta_{uv} = \pi/2$ implies $\sin^2\left(\frac{\theta_{uv}}{2}\right) = 1/2$. Setting $\theta_{uw} = \theta_{wv} = 2\phi$, the inequality becomes

$$\frac{1}{4} \leq \sin^2\phi ,$$

clearly contradicted by independent axes $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ with $\theta_{uv} = \pi/2$ and $\theta_{uw} = \theta_{wv} = 2\phi \in (\pi/4, \pi/3)$.

Remark 13.42 Despite the whole theory has unfolded within the non-relativistic formalism, one could already at this juncture raise an important question, unabated when passing to the relativistic formalism. In the quantum commutation of $p(u+, v+)$ we assumed we had measured *first* the spin of A, and *then* the spin of B. If measurements are taken in *causally disjoint* spacetime events – events that cannot be connected by future-directed timelike or spacelike paths – then the chronological order of the events is conventional, and depends on the choice of (inertial) frame, as is well known in special relativity. Thus we can find a frame where B is measured before A. So the question is whether computing $p(u+, v+)$ in this situation – which by the principle of relativity is physically equivalent to the previous one – gives the same result found earlier. Leaving behind the issue of a relativistic formalisation, the probability $p(u+, v+)$ is easily seen not to change, since the particles' spin observables commute. We will return to this kind of problem later. ■

Since 1972 several experiments have been conducted to test the existence of the aforementioned correlations, and the truth or falsity of Bell's inequalities (an important experiment was made in 1982 by A. Aspect, J. Dalibard and G. Roger [Bon97, Ghi07]). As a byproduct of the large number of experimental tests over the years, various common deficiencies in the testing of Bell's theorem have been found, in particular the *detection loophole* and *communication loophole*. The experiments have been gradually improved to better deal with these loopholes. In 2015, for the first time, R. Hanson and collaborators[8] corroborated the violation of Bell's inequalities during an experimental test of Bell's theorem. The reported results are free of any additional assumptions or loophole. Within the acceptability range of experimental errors, we can conclude that (a) the nonlocal correlations predicted by QM *do exist*, (b) Bell's inequalities are *violated*. Unless we deny the validity of the above

[8] Hanson, R. et al. : *Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres*. Nature. 526: 682–686 (2015); arXiv:1508.05949.

experiments, therefore, and *independent of whether we accept or not the standard formulation of Quantum Mechanics*, we must agree that the correlations anticipated by Quantum Mechanics exist, and the outcomes are fixed at the moment of the measurements. Most physicists nowadays think that local realism is untenable. From a philosophical viewpoint, however, local realism cannot be excluded completely if we adopt extreme (unfalsifiable) assumptions such as the hypothesis of *superdeterminism*, where everything, including all experiments and outcomes, is predetermined.

13.4.5 EPR Correlations Cannot Transfer Information

Although we developed QM in its non-relativistic version, the problems posed by the EPR analysis do not substantially change in the relativistic framework. But one question remains unanswered (we retain the conventions and notations of Sect. 13.4.3): *how does system A communicate to system B the outcome of the measurement of* G_A, *in time to produce the correlations we know of, and without destroying the cornerstones of relativity?*

The answer is quite intricate, and by no means conclusive. First of all, one has to say that the question is ill posed, because it understates that the outcome of measuring G_A *causes* the outcome of G_B. In spacetime regions where the two measurements are taken (or can be taken) the latter are, in relativistic language, *causally disjoint*: there is no future-directed "spacelike" or "timelike" path in spacetime joining them. It is well known from relativity that there exists an inertial system in which A is measured *before* B, and another one where the situation is opposite: the measurement of B *precedes in time* the measurement of A. So it makes no sense to say that the outcome of the experiment on A is the consequence, or the cause, of the outcome on B. One could, notwithstanding, resort to the partial conventionality of Einstein's synchronisation procedure in order to dismiss that problem. But despite the conventional choices underpinning special relativity, it is known that the correlations between causally disjoint events are "dangerous" in relativistic theories, for they can spawn causal paradoxes: using a chain of causally disjoint events we can put two events in the history of a given system in any chronological order whatsoever. If it were possible to use the correlations of causally disjoint events to transfer information either way, we would be able to communicate with the past (inside the light cone) and thereby obtain causal paradoxes.

It can be proved (see [Bon97] and references for a detailed study) that *by accepting the standard formulation of QM* for systems made by entangled states like (13.81) (but also general entangled mixed states), no piece of information can be transmitted from (event) X, where part of the system is measured, to (event) Y, where the other part is measured, by measuring arbitrary pairs of quantities and exploiting the quantum correlations between the readings. Not only that, but observing the outcomes on one part of the system we cannot establish *whether* on the other part any measurements have been taken, if they are being taken as we speak, nor if they will be taken in the future.

Let us examine two ways of transferring information from X to Y via EPR correlations.

(a) Consider the single pairs of measurements on A, B of the observables G_A, G_B, which we know have correlated outcome. We cannot pass information from X to Y using the correlation, because the outcome, albeit correlated, is completely accidental. It is like having two coins A, B with the remarkable property that each time one shows "heads", the other one gives "tails", independent of the fact they are tossed far away, rapidly, and that A is tossed before or after B in some frame. The coins, though, have a quantum character and it is physically impossible to force one to give a certain result: the outcome of the toss is determined in a probabilistic way and whatever our wish is. Thus the two coins, i.e. our quantum system made by parts A and B, cannot be used as a Morse telegraph of sorts to transmit information between X and Y.

(b) The second possibility is to consider not the single measurements of G_A and G_B, but a large number thereof, and study the statistical features of the outcome distributions. The statistics of the measurements of G_A might be different according to whether we measure G_B as well, or whether we measure a new quantity G'_B. In this way, by measuring or not measuring G_B (and measuring G'_B or measuring nothing at all) in Y, we can send an elementary signal to X, of the type "yes" or "no", that we recover by checking experimentally the statistics of A. We claim that neither this procedure allows to transfer information, since the statistics relative to G_A is exactly the same in case we also measure G_B (or any other G'_B) or we do not measure G_B. Consider the state $\rho \in \mathfrak{S}(\mathsf{H}_A \otimes \mathsf{H}_B)$ of the system composed by A, B. Suppose $G_A = G^{(A)} \otimes I_B$, with $G^{(A)}$ self-adjoint on H_A, has discrete and finite spectrum $\{g_1^{(A)}, g_2^{(A)}, \ldots, g_n^{(A)}\}$, with eigenspaces $H_{g_k^{(A)}} \subset \mathsf{H}_A \otimes \mathsf{H}_B$ as ranges of the orthogonal projectors $P_k^{(G_A)} := P_k^{G^{(A)}} \otimes I_B$. Similarly, $G^{(B)}$ is self-adjoint on H_B, $G_B = I_A \otimes G^{(B)}$ has spectrum $\{g_1^{(B)}, g_2^{(B)}, \ldots, g_m^{(B)}\}$ discrete and finite, the eigenspaces $H_{g_k^{(B)}} \subset \mathsf{H}_A \otimes \mathsf{H}_B$ are targets of orthogonal projectors $P_k^{(G_B)} := I_A \otimes P_k^{G^{(B)}}$. If we measure G_B on state ρ reading $g_k^{(B)}$, the post-measurement state is

$$\frac{1}{tr\left(P_k^{(G_B)} \rho P_k^{(G_B)}\right)} P_k^{(G_B)} \rho P_k^{(G_B)} .$$

Considering all possible readings of B, if we measure first B and then A (in some frame), the system we want to test on A is the mixture

$$\rho' = \sum_{k=1}^{m} \frac{p_k}{tr\left(P_k^{(G_B)} \rho P_k^{(G_B)}\right)} P_k^{(G_B)} \rho P_k^{(G_B)}$$

where $p_k = tr(P_k^{(G_B)} \rho P_k^{(G_B)})$ is the probability of reading $g_k^{(B)}$ for B. Altogether

$$\rho' = \sum_{k=1}^{m} P_k^{(G_B)} \rho P_k^{(G_B)} .$$

Hence the probability of getting $g_h^{(A)}$ for A, when B has been measured (irrespective of the latter's outcome), is:

$$\mathscr{P}(g_h^{(A)}|B) = tr(\rho' P_h^{(G_A)}) = tr\left(\sum_{k=1}^{m} P_k^{(G_B)} \rho P_k^{(G_B)} P_h^{(G_A)}\right)$$

The trace is linear and invariant under cyclic permutations, so

$$\mathscr{P}(g_h^{(A)}|B) = \sum_{k=1}^{m} tr(P_k^{(G_B)} \rho P_k^{(G_B)} P_h^{(G_A)}) = \sum_{k=1}^{m} tr(\rho P_k^{(G_B)} P_h^{(G_A)} P_k^{(G_B)})$$

$$= \sum_{k=1}^{m} tr(\rho P_k^{(G_B)} P_k^{(G_B)} P_h^{(G_A)}) .$$

In the last passage we used $P_k^{(G_B)} P_h^{(G_A)} = P_h^{(G_A)} P_k^{(G_B)}$, coming from the structure of the projectors. On the other hand $P_k^{(G_B)} P_k^{(G_B)} = P_k^{(G_B)}$ and $\sum_k P_k^{(G_B)} = I$ by the spectral theorem. Therefore

$$\mathscr{P}(g_h^{(A)}|B) = \sum_{k=1}^{m} tr\left(\rho P_k^{(G_B)} P_h^{(G_A)}\right) = tr\left(\rho \sum_{k=1}^{m} P_k^{(G_B)} P_h^{(G_A)}\right) = tr\left(\rho P_h^{(G_A)}\right) = \mathscr{P}(g_h^{(A)}) .$$

The final result is: *the probability of obtaining $g_h^{(A)}$ from A when the quantity B has been measured (with any possible outcome) coincides with the probability of obtaining $g_h^{(A)}$ from A without measuring B.*

So even by considering the statistics of outcomes of A, there is no way to transmit information by EPR correlations: when measuring part B of the system, the presence or the absence of the correlations is completely irrelevant if we observe only part A.

Therefore, Quantum Mechanics and Special Relativity seem to coexist peacefully. In reality the above discussion turns a blind eye on whether spacetime is classical or relativistic. Apparently, the lesson learned is that the processes of compound quantum systems are not describable in spacetime. Only the *outcomes* of measurements, interpreted as states of macroscopic systems (detectors, meters, *etc*…) can be described in spacetime using events. Spacetime allegedly resembles an "*a posteriori*" structure in which macroscopic phenomena are recorded, sometimes in relationship to microscopic phenomena. But this is not the only possible way to look at things. The apparent violation of locality due to the "collapse of the state" might in fact be a purely speculative construction, related to an all-too-simplistic model of the measuring procedures. Furthermore, a careful analysis might reveal that spacetime categories carry on being fundamental at the quantum level as well. In this respect see the recent study [Dop09].

13.4.6 *The Phenomenon of Decoherence as a Manifestation of the Macroscopic World*

It must be clear that the point of view outlined in the previous section has to be considered as a starting point and not the end of the journey, at least until we understand, experimentally, what a macroscopic/classical system is, what a microscopic/quantum system is, and which are the reasons for switching from one regime to the other.

An interesting perspective for recovering the classical world from the quantum one is based on the notion of *decoherence* [BGJKS00]. We present the main idea quite rapidly (see in particular [Kup00], [Zeh00]). Consider a quantum system S interacting with another quantum system E, the latter seen as the *ambient* where the evolution takes place (E may include measurement instruments and any other object that interacts with S). The evolution is described on the Hilbert space $\mathsf{H} = \mathsf{H}_S \otimes \mathsf{H}_E$ by an operator (unitary and strongly continuous) $\mathbb{R} \ni t \mapsto U_t$. If $\rho(0)$ is a state (mixed in general) of the total system at time $t = 0$, measurements of observables on S at time t are taken using an effective statistical operator $\rho_S(t)$ of the form:

$$\rho_S(t) = tr_E \left(U_t \rho(0) U_t^{-1} \right) , \tag{13.89}$$

where $tr_E(W)$ denotes the **partial trace** with respect to E of the self-adjoint operator $W \in \mathfrak{B}_1(\mathsf{H}_S \otimes \mathsf{H}_E)$ (we used it tacitly in the previous section as well). Then $tr_E(W)$ is the unique self-adjoint operator in $\mathfrak{B}_1(\mathsf{H}_S)$ for which

$$\sum_{n \in \mathbb{N}} (\phi \otimes \psi_n | W \phi \otimes \psi_n) = (\phi | tr_E(W) \phi) , \quad \phi \in \mathsf{H}_S$$

in any basis $\{\psi_n\}_{n \in \mathbb{N}} \subset \mathsf{H}_E$. The role of the partial trace (for the subsystem E) is to assign, in a natural way, a state ($tr_E(W)$) to the subsystem (S) with respect to which the trace is not taken, whenever we have a state (W) for the total system ($S + E$). If an observable $A \otimes I_{\mathsf{H}_E}$ is bounded on S, as expected $tr(A\, tr_E(W)) = tr(A \otimes I_{\mathsf{H}_E} W)$.

The evolution given by (13.89), in general, cannot be expressed canonically by a unitary evolution operator acting directly on $\rho_S(0)$. This approach seems to account well for the experimental behaviour of many systems that interact intensely with the ambient (like macromolecules). In certain cases the interaction with the ambient determines a collection $\{P_k\}_{k \in K} \subset \mathfrak{B}(\mathsf{H}_S)$ of pairwise-orthogonal projectors, not dependent on the overall state, for which *almost instantaneously* the state ρ_t satisfies:

$$\rho_S(t) = \sum_{k \in K} P_k \rho_S(t) P_k .$$

Any mechanism due to the interaction of S and E that produces this situation is called a **decoherence** process. In practice decoherence corresponds to a *dynamical procedure that generates a superselection rule* for S, whose coherent sectors are the projection spaces of the P_k which give propositions about quantities that are typically considered completely classical. A mechanism of this sort (see [Kup00] and the models therein) could shed light on the reasons why large molecules, for example, have geometric features that vary with continuity and can be described in classical terms. What is more, it could elucidate why certain macroscopic objects behave classically. Perhaps it could explain, alternatively, what in the common interpretation of the formalism goes under the name of collapse of the state (which would never occur in reality), even though it is not clear how to justify the apparent violation of locality [BGJKS00]. An elementary physical process leading to the superselection of the mass, once we assume that the spectrum mass is a discrete set of positive reals, was presented in [AnMo12].

It is worth remarking that the decoherence process is especially used to try to describe *quantum measurement procedures* [BGJKS00], assuming that, even during the interaction system-apparatus, an overall Schrödinger evolution holds. Actually, in this case the actors are three: the system S, the environment E and the experimental apparatus A. The latter is described separately from the environment, and is devoted to measuring some observable of S. Here the decoherence phenomenon should concern the interaction A-E. It is however disputable whether these approaches really permit to describe the notion of collapse of the state completely, hence removing it from quantum theories.

13.4.7 Axiom **A8**: *Compounds of Identical Systems*

In QM elementary particles are *identical*. The fact that they cannot be distinguished is formalised in QM in a precise way by keeping axiom **A7** in account, as we will see in a moment.

First we need a few technical results.

Definition 13.43 The **permutation group** $\mathscr{P}_n$ on n elements is the set of bijective maps $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ (called **permutations of** n **objects**) equipped with the composition product.

In particular, a **permutation of two objects** is a map $\sigma \in \mathscr{P}_n$ that restricts to the identity on a subset of $n - 2$ elements of $\{1, 2, \ldots, n\}$.

To any $\sigma \in \mathscr{P}_n$ we associate a number $(-1)^\sigma \in \{-1, +1\}$ called its **parity**. If σ is the product of an even number of permutations of two objects then $(-1)^\sigma := 1$, while if the number of permutations is odd, $(-1)^\sigma := -1$. Despite the number of permutations of two objects appearing in σ is not uniquely determined, the parity is, as one can show.

Consider a Hilbert space H and its n-fold tensor product $\mathsf{H}^{\otimes n} := \bigotimes_{i=1}^n \mathsf{H}$. Any $\sigma \in \mathscr{P}_n$ induces a unitary operator $U_\sigma : \mathsf{H}^{\otimes n} \to H^{\otimes n}$ defined as follows. Pick a basis N

for H. By Proposition 10.25 the vectors $\psi_1 \otimes \cdots \otimes \psi_n$, with $\psi_k \in N, k = 1, 2, \ldots, n$, form a basis of $\mathsf{H}^{\otimes n}$. If σ is an arbitrary permutation also $\psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \psi_{\sigma^{-1}(n)}$ will give a basis for $\mathsf{H}^{\otimes n}$. This basis, actually, is precisely the same one we had before acting by σ, up to rearrangements. Define $U_\sigma : \mathsf{H}^{\otimes n} \to \mathsf{H}^{\otimes n}$ as the unique bounded operator satisfying

$$U_\sigma(\psi_1 \otimes \cdots \otimes \psi_n) := \psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \psi_{\sigma^{-1}(n)} , \quad \psi_k \in N, k = 1, 2, \ldots, n.$$

Then U_σ is unitary, for it preserves bases. Moreover if $\phi_1, \ldots \phi_n \in \mathsf{H}$ are arbitrary (even not in N), decomposing over the ψ_i and exploiting linearity and continuity gives

$$U_\sigma(\phi_1 \otimes \cdots \otimes \phi_n) := \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)} .$$

This proves half of the following proposition.

Proposition 13.44 *Consider $\mathsf{H}^{\otimes n} := \bigotimes_{i=1}^n \mathsf{H}$, where H is a Hilbert space, and let $\mathscr{P}_n$ denote the permutation group on n elements.*
(a) *If $\sigma \in \mathscr{P}_n$ there exists a unique unitary operator $U_\sigma : \mathsf{H}^{\otimes n} \to \mathsf{H}^{\otimes n}$ such that:*

$$U_\sigma(\phi_1 \otimes \cdots \otimes \phi_n) := \phi_{\sigma^{-1}(1)} \otimes \cdots \otimes \phi_{\sigma^{-1}(n)} , \tag{13.90}$$

for any $\phi_1, \ldots, \phi_n \in \mathsf{H}$.
(b) *$U : \mathscr{P}_n \ni \sigma \mapsto U_\sigma$ is a faithful unitary representation of $\mathscr{P}_n$.*

Proof (a) The claim descends from the arguments preceding the proposition: just define U_σ via a basis and check (13.90) holds for *any* $\phi_1, \ldots, \phi_n \in \mathsf{H}$. Two bounded operators satisfying (13.90) coincide on a basis, hence everywhere (being bounded).
(b) By direct inspection (using the fact that σ^{-1} appears in the right-hand side of (13.90)) $(U_\sigma U_{\sigma'})(\phi_1 \otimes \cdots \otimes \phi_n) = U_{\sigma\circ\sigma'}(\phi_1 \otimes \cdots \otimes \phi_n)$. Linearity and continuity imply $U_\sigma U_{\sigma'} = U_{\sigma\circ\sigma'}$, making $\sigma \mapsto U_\sigma$ a (unitary) representation of $\mathscr{P}_n$. Faithfulness is granted by the injectivity of U, since $U_\sigma = I$ implies $\phi_{\sigma^{-1}(1)} \otimes \cdots \otimes \phi_{\sigma^{-1}(n)} = \phi_1 \otimes \cdots \otimes \phi_n$ for any orthonormal vectors $\phi_1, \ldots, \phi_n \in \mathsf{H}$, hence $\sigma^{-1} = id = \sigma$. □

Physically, if $\Psi \in \mathsf{H}^{\otimes n}$ is a pure state of a system made of n identical subsystems, each described on its own Hilbert space H, the pure state of $U_\sigma \Psi$ is naturally interpreted as the state in which the n subsystems have been permuted under σ. The action of U_σ extends to all states $\rho \in \mathfrak{S}(\mathsf{H}^{\otimes n})$ by the transformation that maps ρ to $U_\sigma \rho U_\sigma^{-1}$. As U_σ is unitary, the transformation preserves the positivity and trace of ρ ($U_\sigma \rho U_\sigma^{-1}$ is of trace class if ρ is, because trace-class operators form an ideal), so $U_\sigma \rho U_\sigma^{-1} \in \mathfrak{S}(\mathsf{H}^{\otimes n})$ if $\rho \in \mathfrak{S}(\mathsf{H}^{\otimes n})$.

The permutation group's action on states dualises to an action on propositions $P \in \mathscr{L}(\mathsf{H}^{\otimes n})$ on the system. The *inverse dual action*, as usual, is given by the transformation mapping P to $U_\sigma P U_\sigma^{-1}$. Since U_σ is unitary, $U_\sigma P U_\sigma^{-1}$ is an orthogonal projector if P is.

By the properties of the trace (Proposition 4.38(c))

$$tr\left(U_\sigma P U_\sigma^{-1}\;\; U_\sigma \rho U_\sigma^{-1}\right) = tr\left(P \;\; \rho\right) .$$

Then, this action of permutations on propositions cancels the action of permutations on states. The natural interpretation of the transformation associating P to $U_\sigma P U_\sigma^{-1}$ is an *active* action on physical instruments of the permutation σ. The action of U_σ on propositions induces an action on each PVM $\{P^{(A)}(E)\}_{E\in\mathscr{B}(\mathbb{R})}$ (associated to the observable A) that maps it to a PVM $\{U_\sigma^{-1}P^{(A)}(E)U_\sigma\}_{E\in\mathscr{B}(\mathbb{R})}$. From the spectral theorem we know the latter action corresponds to transforming the observable A into $U_\sigma A U_\sigma^{-1}$. The physical meaning is obvious in the light of previous considerations. Now we are ready to state the axiom for compounds of identical systems.

A8. *If a physical system S consists of $n < +\infty$ identical subsystems, each described on one copy of the Hilbert space* H, *physically-admissible propositions correspond to the subset in $\mathscr{L}(\mathsf{H}^{\otimes n})$ of orthogonal projectors that are invariant under the permutation group (cf. Proposition* 13.44).

Equivalently: $P \in \mathscr{L}(\mathsf{H}^{\otimes n})$ makes physical sense on S only if

$$U_\sigma P U_\sigma^{-1} = P\,, \quad \text{for every } \sigma \in \mathscr{P}_n.$$

Therefore physically-admissible observables A on S are those whose spectral measures satisfy the above condition, i.e.

$$U_\sigma A U_\sigma^{-1} = A\,, \quad \text{for every } \sigma \in \mathscr{P}_n.$$

Just for example, if we work with a compound of two identical particles of mass m, with coordinates $x_i^{(1)}$ and $x_i^{(2)}$, an admissible observable is the ith component of the average position $(X_i^{(1)} + X_i^{(1)})/2$. Without going into details, using the spectral measures of $X_i^{(1)}$ and $X_i^{(2)}$ we can construct an admissible proposition (an orthogonal projector commuting with every U_σ) corresponding to the statement: "one of the particles has ith coordinate falling within the Borel set E". Conversely, propositions like " particle 1 has ith coordinate falling in the Borel set E" are not admissible.

Remark 13.45 Referring to the notion of *gauge group* presented in Definition 11.23, we conclude that the von Neumann algebra $\mathfrak{R}$ of a system of n identical systems is a subalgebra of $\mathfrak{B}(\mathsf{H}^{\times n})$ admitting a non-Abelian commutant $\mathfrak{R}'$ whose gauge group contains the representation $\{U_\sigma\}_{\sigma\in\mathscr{P}_n}$. ■

13.4.8 Bosons and Fermions

At last, we would like to show one consequence of axiom **A8** that is worthy of mention. Consider the usual system S made of n identical subsystems. Take $\sigma \in \mathscr{P}_n$ and the λ-eigenspace of U_σ inside $\mathsf{H}^{\otimes n}$:

$$(\mathsf{H}^{\otimes n})_\lambda^{(\sigma)} := \{\Psi \in \mathsf{H}^{\otimes n} \;\mid\; U_\sigma \Psi = \lambda \Psi\} .$$

Note U_σ is unitary, so $|\lambda| = 1$.

Every meaningful proposition must commute with U_σ, so if the system's state $\Psi \in (\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ is initially pure, following a measurement by the admissible (true) proposition P the state will be described by $P\Psi/||P\Psi||$; this is in $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ since $U_\sigma \frac{P\Psi}{||P\Psi||} = \frac{U_\sigma P\Psi}{||P\Psi||} = \frac{PU_\sigma \Psi}{||P\Psi||} = \lambda \frac{P\Psi}{||P\Psi||}$. By taking measurements, therefore, we cannot "make the system leave" the space $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ if it was in a pure state described by a vector in $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ immediately prior to the measurement. Not even time evolution, at least under time homogeneity, "allows the system to leave" the space $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ if it was, at the initial time, in a pure state in $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$. In fact, the Hamiltonian observable H (being admissible) will have spectral measure that commutes with U_σ. Consequently

$$e^{-itH}U_\sigma = \int_{\sigma(H)} e^{-ih}\, dP^{(H)}(h) U_\sigma = U_\sigma \int_{\sigma(H)} e^{-ih}\, dP^{(H)}(h) = U_\sigma e^{-itH} .$$

If $U_\sigma \Psi = \lambda\Psi$, then $U_\sigma \Psi_t = U_\sigma e^{-itH}\Psi = e^{-itH}U_\sigma\Psi = e^{-itH}\lambda\Psi = \lambda\Psi_t$, so $\Psi_t \in (\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$, for any time $t \in \mathbb{R}$. In case the evolution operator is not the exponential of the Hamiltonian (lack of time homogeneity), under suitable assumptions one can still prove the same result. This happens if, for instance, the evolution operator is given by the Dyson series (see Proposition 13.19) for a special class of time-dependent Hamiltonian observables. We have the following result.

Proposition 13.46 *Suppose a compound system S is made of $n < +\infty$ identical subsystems, each described on the same Hilbert space H, and at some time t_0 the system is in a pure state represented by a vector in $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$ for some $\sigma \in \mathscr{P}_n$. Then the evolution (in regime of time homogeneity), or a measurement, leaves the system in a pure state represented by a vector in $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$.*

The experimental evidence not only confirms this fact, but shows that pure states of a compound of identical particles in 4 dimensions (three for space plus time) are determined by vectors in *two subspaces only*, built intersecting the $(\mathsf{H}^{\otimes n})^{(\sigma)}_\lambda$. To explain that fact we need a few comments.

Consider a permutation $\delta \in \mathscr{P}_n$ of two elements, so $\delta \circ \delta = id$ and $U_\delta U_\delta = I$. As U_δ is unitary, U_δ is self-adjoint. Hence U_δ is an observable, actually a constant of motion (exercise). Not just that: $\sigma(U_\delta) \subset \{-1, 1\}$ because $\sigma(U_\delta)$ is contained in $\mathbb{R}$ (U_δ is self-adjoint) and also in the unit circle in $\mathbb{C}$ (U_δ is unitary). Therefore $\sigma(U_\delta) = \sigma_p(U_\delta)$ because the spectrum consists of one or two isolated points. It is easy to prove $\sigma_p(U_\delta) = \{-1, 1\}$. In fact, if δ swaps the kth and jth elements, every vector of $\mathsf{H}^{\otimes n}$ of the form

$$(\psi_1 \otimes \cdots \otimes \psi_k \otimes \cdots \otimes \psi_j \otimes \cdots \otimes \psi_n) \pm (\psi_1 \otimes \cdots \otimes \psi_j \otimes \cdots \otimes \psi_k \otimes \cdots \otimes \psi_n)$$

is an eigenvector of U_δ with eigenvalue ± 1. From this follows, for any $\sigma \in \mathscr{P}_n$, that U_σ admits the eigenvalues (possibly coinciding) 1 and $(-1)^\sigma$. It is enough to write $U_\sigma = U_{\delta_1} \cdots U_{\delta_m}$, where the σ_i are permutations of two elements. The intersections

$$(\mathsf{H}^{\otimes n})_{+}^{(\sigma)} := \cap_{i=1}^{m} (\mathsf{H}^{\otimes n})_{+1}^{(\delta_i)} \quad \text{and} \quad (\mathsf{H}^{\otimes n})_{-}^{(\sigma)} := \cap_{i=1}^{m} (\mathsf{H}^{\otimes n})_{-1}^{(\delta_i)}$$

are eigenspaces for U_σ with respective eigenvalues $+1$ and $(-1)^\sigma$, since $U_\sigma = U_{\delta_1} \cdots U_{\delta_m}$.
The space $\mathsf{H}^{\otimes n}$ has two physically interesting and mutually-orthogonal closed subspaces, obtained from the intersections of all spaces of type $(\mathsf{H}^{\otimes n})_{+}^{(\sigma)}$ and $(\mathsf{H}^{\otimes n})_{-}^{(\sigma)}$, respectively, as $\sigma \in \mathscr{P}_n$ varies. These are the **totally symmetric product**

$$(\mathsf{H}^{\otimes n})_{+} := \{\Psi \in \mathsf{H}^{\otimes n} \mid U_\sigma \Psi = \Psi \ , \forall \sigma \in \mathscr{P}_n\}$$

and the **totally skew-symmetric product**

$$(\mathsf{H}^{\otimes n})_{-} := \{\Psi \in \mathsf{H}^{\otimes n} \mid U_\sigma \Psi = (-1)^\sigma \Psi \ , \forall \sigma \in \mathscr{P}_n\} \ .$$

The physical relevance of these subspaces lies in that every known compound of identical particles has pure states described by vectors either in $(\mathsf{H}^{\otimes n})_{+}$ or in $(\mathsf{H}^{\otimes n})_{-}$. Particles of the first type, called **bosons** (whose mixtures are incoherent superpositions of pure states in $(\mathsf{H}^{\otimes n})_{+}$), have integer spin; particles of the second type (whose mixtures are incoherent superpositions of pure states in $(\mathsf{H}^{\otimes n})_{-}$), called **fermions**, have semi-integer spin. This phenomenon is often referred to as the *spin statistical correlation*.

Remark 13.47 Notice that the restrictions of the representation $\mathscr{P}_n \ni \sigma \mapsto U_\sigma : \mathsf{H}^{\otimes n} \to \mathsf{H}^{\otimes n}$ to these invariant subspaces are Abelian representations. So if we deal with systems of identical subsystems whose Hilbert space is either $(\mathsf{H}^{\otimes n})_{+}^{(\sigma)}$ or $(\mathsf{H}^{\otimes n})_{-}^{(\sigma)}$, i.e. either bosons or fermions, the gauge group of the algebra of observables is still Abelian. ■

Within the non-relativistic formulation of QM there is no proof for the relationship to the value of spin. We can only show, using Proposition 13.46, that if a system of particles has a fermionic or a bosonic behaviour, at time t_0, it will maintain the behaviour so long it is described by pure states.

In the non-relativistic formulation there are states represented by vectors that, in principle, are neither symmetric nor skew-symmetric and belong to a subspace H' of $\mathsf{H}^{\otimes n} = (\mathsf{H}^{\otimes n})_{+} \oplus (\mathsf{H}^{\otimes n})_{-} \oplus \mathsf{H}'$ orthogonal to both $(\mathsf{H}^{\otimes n})_{\pm}$. This subspace is again invariant under measurement procedures and temporal evolution and, obviously, under the unitary representation of the permutation group $\mathscr{P}_n$. Actually H' can be decomposed into a direct sum of subspaces of dimension ≥ 2 which are separately invariant under the algebra of observables, time evolution, and the action of the permutation group. What we are discussing refers to each of these subspaces. In the jargon, one says that elementary physical systems (particles) admitting such vector states obey a *parastatistics*. Particles of this sort have never been observed. It is however theoretically interesting to notice that a physical system made of identical subsystems described on H' cannot admit complete sets of commuting observables (Definition 11.11). (This is because the existence of a maximal observable would imply that the commutant of the algebra of observables is commutative by Proposition 11.12, while the commutant of the von Neumann algebra of observables relative to the invariant carrier space H' contains a unitary representation of $\mathscr{P}_n$ which *cannot be commutative*. This representation is the restriction to H' of the overall unitary faithful representation $\mathscr{P}_n \ni \sigma \mapsto U_\sigma : \mathsf{H}^{\otimes n} \to \mathsf{H}^{\otimes n}$. The restriction to the invariant subspaces $(\mathsf{H}^{\otimes n})_\pm$ is trivially commutative. If it were commutative also when restricted to the remaining invariant subspace H', it would be commutative on the whole $\mathsf{H}^{\otimes n}$, but we know that it is not the case since the overall representation is faithful and $\mathscr{P}_n$ is not Abelian.) A consequence of the fact that the commutant of the algebra of observables restricted to H' is not trivial is that the one-to-one correspondence between pure states and rays no longer holds. Indeed, if V is a unitary non-trivial element of the commutant of the algebra of observables in H', it is impossible to distinguish Ψ and $\Psi' = V\Psi$ by means of measurement procedures. (This fact does not apply to $(\mathsf{H}^{\otimes n})_\pm$ because the representation of $\mathscr{P}_n$ there is just given in terms of global phases.)

Passing to relativistic formulations of QM – more precisely the *Relativistic Quantum Field Theory* on 4-dimensional Minkowski spacetime – many authors (mainly W. Pauli) obtained a famous theorem, aptly called *spin statistical correlation theorem* [StWi00]. It proves that the restriction on pure states and the spin statistical correlation observed experimentally are consequences of the theory's invariance under the Poincaré group, rather than the Galilean group. In three-dimensional spacetime models there are compounds of identical particles that do not abide by Fermi's statistics, nor Bose's.

Remark 13.48 Axiom **A8** imposes constraints on the observables of a system of identical particles. One may adopt a physically much more rigid, quite unmotivated, but very popular, point of view by focusing on vectors instead observables. The requirement is that
a normalized vector $\Psi \in \mathsf{H}^{\otimes n}$ represents a state if and only if the associated ray is fixed under the action of permutations and the arbitrary phase does not depend on the representative vector: $U_\sigma \Psi = \lambda_\sigma \Psi$ for every $\sigma \in \mathscr{P}_n$ and some $\lambda_\sigma \in \mathbb{C}$ *with* $|\lambda_\sigma| = 1$.

This stronger version of **A8** implies that only vectors in $(\mathsf{H}^{\otimes n})_+$ and $(\mathsf{H}^{\otimes n})_+$ can be admitted to represent states, ruling out any parastatistics (Exercise 13.6). ■

In conclusion we mention that when we deal with compounds of *infinitely many* identical subsystems described on H, the natural Hilbert spaces to develop the theory are the subspaces of the Fock space (Example 10.27(3)):

$$\mathscr{F}_+(\mathsf{H}) := \bigoplus_{n=0}^{+\infty} (\mathsf{H}^{n\otimes})_+ \quad \text{and} \quad \mathscr{F}_-(\mathsf{H}) := \bigoplus_{n=0}^{+\infty} (\mathsf{H}^{n\otimes})_- \,,$$

called the **bosonic Fock space** and **fermionic Fock space** generated by H. As usual we assumed $(\mathsf{H}^{0\otimes})_\pm := \mathbb{C}$, and that the unique pure state determined by $(\mathsf{H}^{0\otimes})_\pm$ is the *vacuum state* of the system. This is the framework of *Quantum Field Theory*, in which fields are "replaced" by systems of infinitely many identical bosonic or fermionic particles.

Exercises

13.1 Consider a mixed state $\rho \in \mathfrak{S}(\mathsf{H})$ and an orthogonal sum $\mathsf{H} = \oplus_{k\in K}\mathsf{H}_k$ associated to orthogonal projectors $\{P_k\}_{k\in K}$, with K finite or countable.

Using the strong topology define

$$\rho' := \text{s-}\sum_k P_k \rho P_k \ .$$

Prove ρ' is well defined and $\rho' \in \mathfrak{S}(\mathsf{H})$.

Hint. $P_k P_h = 0$ if $k \neq h$, s-$\sum_k P_k = I$, and $||\rho^2|| \leq 1$. This allows to prove the series converges strongly, using known properties of series of orthogonal vectors.

That ρ' is positive and $||\rho'|| \leq 1$ follows by the construction and the similar properties of ρ. Using the basis N of H made by the union of bases for each summand H_k, with Proposition 4.31 one proves ρ' is of trace class, plus $tr\rho' = tr\rho = 1$.

13.2 In Sect. 13.4.5 it was shown that the probability of measuring $g_k^{(A)}$ for G_A on part A of a quantum system is independent of the fact that G_B is measured on part $B \neq A$. Prove that the result is valid for arbitrary observables (even with continuous and unbounded spectrum). Assume that the device measuring G_B gives as possible readings a countable disjoint family of Borel sets $E_k^{(G_B)}$ whose union is $\sigma(G_B)$.

13.3 Referring to (13.90) prove that $U_\sigma U_{\sigma'} = U_{\sigma\circ\sigma'}$.

Solution. By linearity, and exploiting the fact that the operators are bounded and defined everywhere, it is sufficient proving that

$$U_\sigma(U_{\sigma'}(\phi_1 \otimes \ldots \otimes \phi_n)) = U_{\sigma\circ\sigma'}(\phi_1 \otimes \ldots \otimes \phi_n) \ .$$

Let us establish that identity. If $\sigma, \sigma' \in \mathscr{P}_n$ then:

$$U_\sigma(U_{\sigma'}(\phi_1 \otimes \ldots \otimes \phi_n)) = U_\sigma(\phi_{\sigma'^{-1}(1)} \otimes \ldots \otimes \phi_{\sigma'^{-1}(n)}) \ .$$

Redefining $u_i := \phi_{\sigma'^{-1}(i)}$ so that $u_{\sigma^{-1}(j)} := \phi_{\sigma'^{-1}(\sigma^{-1}(j))}$, one finds

$$U_\sigma(U_{\sigma'}(\phi_1 \otimes \ldots \otimes \phi_n)) = u_{\sigma^{-1}(1)} \otimes \ldots \otimes u_{\sigma^{-1}(n)}$$

$$= \phi_{\sigma'^{-1}\circ\sigma^{-1}(1)} \otimes \ldots \otimes \phi_{\sigma'^{-1}\circ\sigma^{-1}(n)} = \phi_{(\sigma\circ\sigma')^{-1}(1)} \otimes \ldots \otimes \phi_{(\sigma\circ\sigma')^{-1}(n)}$$

$= U_{\sigma\circ\sigma'}(\phi_1 \otimes \ldots \otimes \phi_n)$ as wanted.

13.4 Consider a compound of n identical particles in $\mathsf{H}^{\otimes n}$. Prove that under axiom **A8**, if $\delta \in \mathscr{P}_n$ is a permutation on two elements, then U_δ is a constant of motion.

13.5 Prove that $(\mathsf{H}^{\otimes n})_+$ and $(\mathsf{H}^{\otimes n})_-$ are orthogonal, that $\mathsf{H}^{\otimes 2} = (\mathsf{H}^{\otimes 2})_+ \oplus (\mathsf{H}^{\otimes 2})_-$ if $n = 2$, and that the previous fact is false already for $n = 3$.

13.6 Prove that if $\Psi \in \mathsf{H}^{\otimes n}$ represents a pure state and $U_\sigma \Psi = \lambda_\sigma \Psi$, with $|\lambda_\sigma| = 1$, for every $\sigma \in \mathscr{P}_n$, then either $\Psi \in (\mathsf{H}^{\otimes n})_-$ or $\Psi \in (\mathsf{H}^{\otimes n})_+$.

Solution. Consider a permutation of two elements j, k in $\{1, \dots, n\}$, indicated by $(j, k) \in \mathscr{P}_n$. Hence

$$(j, k) = (2, k) \circ (1, j) \circ (1, 2) \circ (2, k) \circ (1, j) \ .$$

On the other hand $U_{(p,q)}\Psi = \lambda_{(p,q)}\Psi$ with $\lambda_{(p,q)} = \pm 1$, since $U_{(p,q)}U_{(p,q)} = I$ because $(p, q)^2 = id$, for every choice of p and q. Therefore, assuming $\Psi \neq 0$,

$$\lambda_{(j,k)} = \lambda_{(2,k)}\lambda_{(1,j)}\lambda_{(1,2)}\lambda_{(2,k)}\lambda_{(1,j)} = \lambda_{(1,2)} \ .$$

We conclude that, for every permutation of two elements $\delta \in \mathscr{P}_n$,

$$U_\delta \Psi = \lambda_{(1,2)}\Psi \ .$$

Hence, by the definition of $(-1)^\sigma$ stated below Definition 13.43, and setting $1^\sigma := 1$,

$$U_\sigma \Psi = \lambda^\sigma_{(1,2)}\Psi \ , \quad \sigma \in \mathscr{P}_n \ .$$

Therefore if $\lambda_{(1,2)} = 1$ then $\Psi \in (\mathsf{H}^{\otimes n})_+$; if $\lambda_{(1,2)} = -1$, then $\Psi \in (\mathsf{H}^{\otimes n})_-$.

13.7 Assume that, for a system of n identical particles, the vectors representing states define a closed subspace $\mathsf{K} \subset \mathsf{H}^{\otimes n}$ and satisfy $U_\sigma \Psi = \lambda_{\Psi,\sigma}\Psi$, where now $\lambda_{\Psi,\sigma}$ may also depend on Ψ. Prove that K is a subspace of either $(\mathsf{H}^{\otimes n})_-$ or $(\mathsf{H}^{\otimes n})_+$ provided its dimension is ≥ 2.

Outline of the solution. If $\Psi, \Phi \in \mathsf{K}$ are orthogonal and have unit norm, the linearity of U_σ entails $\lambda_{(\Psi+\Phi)/\sqrt{2},\sigma} = \lambda_{\Psi,\sigma} = \lambda_{\Phi,\sigma} =: \lambda_\sigma$. Therefore U_σ is represented by λ_σ along the vectors of an orthonormal basis of K, that is: $U_\sigma|_{\mathsf{K}} = \lambda_\sigma I$. By applying Exercise 13.6 one concludes the proof.

Chapter 14
Introduction to the Algebraic Formulation of Quantum Theories

> *I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert spaces any more.*
> von Neumann, letter to Birkhoff about the mathematical formulation of QM (1935)

In the last chapter of the book we offer a short presentation of the algebraic formulation of quantum theories, and we will state and prove a central theorem about the so-called *GNS construction*. We will discuss how to treat the notion of quantum symmetry in this framework, by showing that an algebraic quantum symmetry can be implemented (anti-)unitarily in GNS representations of states invariant under the symmetry.

As general references, mostly concerned with the algebraic formulation of Quantum Field Theories, we recall [Emc72, Haa96, Ara09, Rob04, BDFY15] and the more recent [Str05a, Str12] on the algebraic formulation of QM. On the mathematical side, detailed and critical studies on the present material are [BrRo02, KaRi97].

We will routinely resort to Definition 3.52 throughout the chapter.

14.1 Introduction to the Algebraic Formulation of Quantum Theories

The fundamental Theorem 11.43 of Stone-von Neumann is stated in the jargon of theoretical physics as follows:

"*all irreducible representations of the CCRs with a finite, and fixed, number of freedom degrees are unitarily equivalent*,".

The expression *unitarily equivalent* refers to the existence of a Hilbert-space isomorphism S, and the finite number of degrees of freedom is the dimension of the symplectic space X on which the Weyl algebra is built.

V. Moretti, *Spectral Theory and Quantum Mechanics*, UNITEXT - La Matematica per il 3+2 110, https://doi.org/10.1007/978-3-319-70706-8_14

What happens then in infinite dimensions? Let us keep irreducibility, and suppose we pass from X finite-dimensional – parametrising, e.g., the coordinates of a point-particle in phase space – to X infinite-dimensional – describing a suitable solution space to free bosonic field equations, say. Then the Stone–von Neumann theorem no longer holds. Theoretical physicists would say that

"*there exist non-equivalent irreducible CCR representations with an infinite number of freedom degrees*".

What happens in this situation, in practice, is that one can find strongly continuous irreducible representations π_1, π_2, on (separable) Hilbert spaces $\mathsf{H}_1, \mathsf{H}_2$, of the Weyl *-algebra $\mathfrak{A} := \mathscr{W}(\mathsf{X}, \sigma)$ (here thought of as C^*-algebra, with no change in the results) associated to the physical system under exam (a quantised bosonic field, typically), that admit *no* isomorphism $S : \mathsf{H}_1 \to \mathsf{H}_2$ satisfying:

$$S\pi_1(a)\, S^{-1} = \pi_2(a)\,, \quad \text{for any } a \in \mathfrak{A}.$$

Pairs of this kind are called *(unitarily) non-equivalent*. Jumping from X finite-dimensional to infinite-dimensional corresponds to passing from Quantum Mechanics to Quantum Field Theory (relativistic QFT, possibly, and on curved spacetime [Wal94, KhMo15, FeVe15]). In these situations (but not only), the existence of non-equivalent representations has often to do with *spontaneous symmetry breaking*. The presence of non-equivalent representations of a single physical system (X, σ) shows that a formulation in a fixed Hilbert space is completely inadequate, and we must free ourselves of the Hilbert structure in order to lay the foundations of quantum theories in broader generality (an interesting and detailed technical analysis of several problems with the Hilbert-space formulation, based on concrete models of quantum fields and statistical mechanics' systems, can be found in [Emc72, Sect. 1, Chap. 1]).

This programme has been developed by and large, starting from the pioneering work of von Neumann himself, and is nowadays called *algebraic formulation of Quantum (Field) Theories*. Within this framework it was possible to formalise, for example, field theories on curved spacetime in relationship to the quantum phenomenology of black-hole thermodynamics.

14.1.1 Algebraic Formulation

The algebraic formulation prescinds, anyway, from the nature of the quantum system, and may be stated for systems with finitely many freedom degrees as well [Str05a]. The viewpoint falls back on two assumptions [Haa96, Ara09, Str05a, Str12, BDFY15] (which somehow generalise the results of Sect. 7.6.5).

AA1. *A physical system S is described by its* **observables**, *viewed now as self-adjoint elements in a certain C^*-algebra $\mathfrak{A}_S$ with unit $\mathbb{I}$ associated to S.*

AA2. *An* **algebraic state** *on* $\mathfrak{A}_S$ *is a linear functional* $\omega : \mathfrak{A}_S \to \mathbb{C}$ *such that*:

$$\omega(a^*a) \geq 0 \quad \forall\, a \in \mathfrak{A}_S, \qquad \omega(\mathbb{I}) = 1 \ ,$$

that is, *positive and normalised to 1.*

We have to stress that $\mathfrak{A}_S$ is not seen as a concrete C^*-algebra of operators on a given Hilbert space, but remains an abstract C^*-algebra. Physically, $\omega(a)$ is the *expectation value* of the observable $a \in \mathfrak{A}$ in state ω.

Remark 14.1 **(1)** $\mathfrak{A}_S$ is usually called *the algebra of observables of S* though, properly speaking, the observables are the self-adjoint elements of $\mathfrak{A}_S$ only.
(2) We can assign $\mathfrak{A}_S$ to S irrespective of any reference frame, provided we assume that the (active) transformations of the various frames are given by automorphisms of $\mathfrak{A}_S$. If the algebra depends on the frame, the time evolution *with respect to the given frame* is described by a one-parameter group of *-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$, where $\alpha_t : \mathfrak{A}_S \to \mathfrak{A}_S$ is a *-homomorphism for any $t \in \mathbb{R}$, α_0 is the identity and $\alpha_t \circ \alpha_{t'} = \alpha_{t+t'}$ for $t, t' \in \mathbb{R}$. It is natural to demand weak continuity in t: for every state ω on $\mathfrak{A}_s$, the map $\mathbb{R} \ni t \mapsto \omega(\alpha_t(a))$ is continuous for every $a \in \mathfrak{A}_S$.

In case the algebra of observables is independent of the reference frame, $\mathfrak{A}_S$ is actually thought of as a *net of algebras*: it will be described by a function $\mathscr{O} \mapsto \mathfrak{A}_S(\mathscr{O})$ mapping to an algebra $\mathfrak{A}_S(\mathscr{O})$ any regular bounded region $\mathscr{O}$ in spacetime (extending both in the space and time directions). From this point of view time evolution is replaced by causal relations between algebras localised at spacetime regions that are causally related, in particular when one region belongs to another's future. The above approach, thoroughly discussed for the first time in the crucial paper of Haag and Kastler [HaKa64], is the modern stepping stone to develop algebraic field theory in the local covariant formulation.
(3) It is important to emphasise that, differently form the Hilbert space formulation, the algebraic approach can be adopted to describe *both classical and quantum systems*. The two cases are distinguished on the base of the commutativity of the algebra of observables $\mathfrak{A}_S$. A commutative algebra is assumed to describe a classical system, whereas a noncommutative one is supposed to be associated with a quantum system. ■

14.1.2 Motivations and Relevance of Lie-Jordan Algebras

Before going on with the mathematical technology, a brief discussion about the motivations underlying the algebraic formulation may be useful. In the rest of this subsection, whose content is quite heuristic, we shall denote the observables with capital letters $A, B, \ldots$ and use corresponding small letters $a, b, \ldots$ to denote the values attained by the observables.

The most evident, *a posteriori*, justification of the algebraic approach lies in its powerfulness [Haa96]. There have been a host of attempts to account for assumptions **AA1** and **AA2** and their physical meaning, in full generality (see the study of [Emc72, Ara09, Str05a, Str12] and especially the work of I. E. Segal [Seg47] based on *Jordan algebras*), yet none seems to be definitive [Stre07].

There are at least two interrelated basic issues to consider when attempting to justify assumptions **AA1** and **AA2** physically.

(a) *The identification of the notions of state and expectation value.*

This identification would be natural within the Hilbert space formulation, where the class of observables includes elementary observables, which are represented by orthogonal projectors and correspond to "yes-no" statements. The expectation value of one such observable coincides with the probability that the outcome of the measurement is "yes". The set of all those probabilities defines, in fact, a quantum state of the system, as we know. The analogues of these elementary propositions, however, generally do not belong to the C^*-algebra of observables in the algebraic formulation (see Sect. 14.1.6). Even so, this obstruction is not insurmountable. Following [Ara09], in a completely general physical system the most general notion of state ω would be the assignment of all probabilities $w_\omega^{(A)}(a)$ that the outcome of measuring the observable A is a, for all observables A and all values a. On the other hand, it is known [Str05a] that all experimental information on the measurement of A in state ω – the probabilities $w_\omega^{(A)}(a)$ in particular – is recorded in the expectation values of the polynomials of A. Here, we should think of $p(A)$ as the observable whose values are given by the $p(a)$, for all values a of A. This characterisation of an observable is theoretically supported by the various solutions to the *momentum problem* in probability theory. To adopt this paradigm, therefore, we have to assume that the set of observables must include all real polynomials $p(A)$ whenever it contains the observable A. This is in agreement with the much stronger requirement **AA1**.

We stress that, within the algebraic formulation, it is natural to suppose that the full set of observables and the whole set of states together encompass the maximum amount of information available on the physical system we are studying. As a byproduct of this assumption and of the identification of the notion of state with that of expectation value, we are committed to conclude that, in our formalism:

(i) **states separate observables**: two observables A, B coincide if and only the corresponding expectation values coincide: $A = B \Leftrightarrow \omega(A) = \omega(B)$ for all states ω;

(ii) **observables separate states**: two states ω, ω' coincide if and only if the corresponding expectation values coincide: $\omega = \omega' \Leftrightarrow \omega(A) = \omega'(A)$ for all observables A.

As a matter of fact, **AA1** and **AA2** theoretically support these two statements. Indeed, (ii) follows immediately from the fact that a state is a linear functional on the C^*-algebra $\mathfrak{A}_S$ and that every element (also not self-adjoint) of $\mathfrak{A}_S$ is a linear combination of self-adjoint elements representing observables. Assertion (i) is a straightforward consequence of the Gelfand-Najmark theorem, as we shall see in Corollary 14.30.

(b) *The incarnation of observables as a C^*-algebra.*

Whilst the nature and the physical meaning of the associative product of the algebra, in particular, is difficult to justify *a priori*, the assumption on the boundedness of observables in a suitable norm does not seem so hard to clarify from an operational point of view [Str05a]. An observable A is defined in terms of a concrete experimental apparatus, which yields the numerical results of measurements in any state ω. Since each concrete experimental apparatus has inevitable limitations that imply a scale bound independent of the state on which measurements are performed, the result of measurements of A on the various states is a bounded set of real numbers, whose bound is related to the scale bound of the associated experimental apparatus. It is then natural to associate to each observable A a finite bound (in perfect agreement with the theoretical result of Corollary 14.30):

$$||A|| := \sup_{\omega \in \Omega} |\omega(A)| < +\infty \tag{14.1}$$

where Ω is the set of all possible states. As the notation suggests, this bound can be interpreted as a norm, once we have built the algebraic structure of the observables (see [Str05a] for details). Let us next focus on the purely algebraic features of the space of observables, the associative algebra of product in particular. In this respect we can observe that, actually, a weaker form of **AA1** must certainly be true. As already noticed, if p is a real polynomial and A is an observable (so that the possible values a are real numbers), $p(A)$ is a well-defined object and it is an observable as well. As we said $p(A)$ is nothing but the observable whose values are $p(a)$, for all values a of A. If p is complex, $p(A)$ deserves the same operative interpretation since its real and imaginary parts are observables. All this indicates the existence, for every observable A, of a natural structure of a commutative *-algebra on complex polynomials $p(A)$. The involution here is the obvious one $p(A)^* := \overline{p}(A)$. It is not difficult to prove that (14.1) turns out to be a norm for this *-algebra.

Following the same route of Sect. 11.3.2, objects like real linear combinations $\alpha A + \beta B$ of observables can be made physically meaningful even if A and B cannot measured simultaneously: $\alpha A + \beta B$ is the observable whose expectation values verify $\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B)$ for every state ω. Notice that, from the physical point of view, we are enlarging the class of observables, for we are assuming that new observables verifying the previous constraint exist (and thus they are uniquely determined, since their expectation values are known). The extension to complex combinations is now straightforward. All that on the one hand justifies the appearance of a complex vector space structure, equipped with an antilinear bijective and involutive operation (extending the previous one), $(\alpha A + \beta B)^* := \overline{\alpha} A + \overline{\beta} B$, and with a norm that extends (14.1), as is not difficult to prove. This structure becomes a fully fledged normed *-algebra if the construction is supplemented by an associative product, that extends the one defined in each algebra of polynomials $p(A)$ generated by any fixed element A. On the other hand, the procedure gives rise to a generally *non-distributive* and *non-associative Jordan product* (see Sect. 11.3.2 and Definition 11.29):

$$A \circ B := \frac{1}{2}\left((A+B)^2 - A^2 - B^2\right) , \tag{14.2}$$

where $(A+B)^2$, A^2 and B^2 are well defined as indicated above.

However, similarly to what happens in the Hilbert space formulation, if p denotes a general polynomial with two entries, and A and B are observables that *cannot be measured simultaneously* (see Sect. 11.3.2), the interpretation of $p(A, B)$ turns out to be very difficult. Certainly, the product of the C^*-algebra of **AA1** provides a sound *mathematical* meaning to $p(A, B)$, thought it does not elucidate the *physical* meaning of $p(A, B)$ in terms of A and B. That said, it is difficult to justify the existence of a C^*-algebra product in the complex vector space of extended observables, even assuming the existence of a meaningful Jordan product and norm [Str05a]. Necessary and sufficient conditions have been found (e.g., see [AlSc03]) for the existence of a C^*-algebra structure that completes the algebraic and topological structures outlined above. However the physical interpretation of these conditions does not seem transparent. (Sect. 2, Chap. 1 of [Emc72] contains a deep study on the Jordan-algebra induced C^*-structure, and the many steps involved are analysed thoroughly.)

Another, much stronger, possibility is to assume – without any true physical justification – that the real linear space of observables is also equipped with a Lie-algebra structure of pure quantum nature (the Lie bracket $[A, B]$ is proportional to the Planck constant), and that the two products enjoy the properties below.

Definition 14.2 A (real) Jordan algebra $\mathfrak{A}$) (Definition 11.29) equipped with a Lie bracket $\{\ ,\ \}$ is called (real) **Lie-Jordan algebra** when, for a constant $c \in \mathbb{R}$, it satisfies:

$$(A \circ B) \circ C - A \circ (B \circ C) = \frac{c}{4}\{\{A, C\}, B\} \quad \text{for all } A, B, C \in \mathfrak{A}. \tag{14.3}$$

and furthermore the **Leibniz rule** is valid:

$$\{A, B \circ C\} = \{A, B\} \circ C + \{A, C\} \circ B \quad \text{for all } A, B, C \in \mathfrak{A}. \tag{14.4}$$

From the physical side, in our context it is natural to assume $c = \hbar^2$. Moreover, if we look at the Hamiltonian formulations of classical physics for instance, $\{\ ,\ \}$ could be interpreted as the quantum analogue of the Poisson bracket. In other words the Lie bracket should be understood as the standard commutator of (self-adjoint) observables multiplied by $i\hbar$, in accordance with a version of Dirac's correspondence principle (see Sect. 11.5.8).

A posteriori, the Lie-Jordan structure arises quite naturally when one knows that (complexified) observables form a *-algebra. In fact, given a complex *-algebra, by defining the Lie bracket in terms of the standard commutator as:

$$\{\ ,\ \} := \frac{i}{\hbar}[\ ,\ ] \tag{14.5}$$

and setting

$$A \circ B = \frac{1}{2}(AB + BA) \ , \tag{14.6}$$

one finds a natural real Lie-Jordan algebra when restricting to the real space of self-adjoint elements. In particular:

$$AB = A \circ B + \frac{i\hbar}{2}\{A, B\} \quad \text{for all } A, B, C \in \mathfrak{A}. \tag{14.7}$$

Taking (14.6) into account, that is the same as:

$$AB = A \circ B + \frac{1}{2}[A, B] \quad \text{for all } A, B, C \in \mathfrak{A} \,. \tag{14.8}$$

The procedure we have outlined can be reversed if we assume that observables possess a Lie-Jordan structure. Given a real Lie-Jordan algebra $\mathfrak{A}$ with $c = \hbar^2$, requiring (14.5) produces an associative complex *-algebra. We can perform this on $\mathfrak{A}_1 := \mathbb{C} \otimes \mathfrak{A}$: the involution * is $(a \otimes A)^* := \overline{a} \otimes A$ for every $a \in \mathbb{C}$ and $A \in \mathfrak{A}$, and the associative product is given by extending (14.8) $\mathbb{C}$-linearly.

It is important to notice that the Lie-Jordan algebra structure can be equipped with a natural norm. After completion, this will generate an associated C^*-algebra [AlSc03].

Coming back to physics, it is finally worth stressing that with this formulation both the non-commutativity and non-associativity underlying the algebra of observables appear to have a quantum nature, and seem to realise quantum *deformations* of classical structures, in the sense that commutativity and associativity are restored as soon as $\hbar \to 0$. However, a relic of quantum non-commutativity remains in the Poisson bracket even at classical level. As a matter of fact, the picture we have described can be developed further. The (algebraic) *deformation quantisation* procedure, introduced in Sect. 11.5.8, starts by assuming that (14.7) is just the first-order expansion in $\hbar$ of the quantum product. This approach has been exploited in algebraic Quantum Field Theory in a modern and powerful fashion to reformulate the perturbative interacting theory, including the renormalisation procedure [BrFr09].

14.1.3 The GNS Reconstruction Theorem

The set of algebraic states on $\mathfrak{A}_S$ is a convex subset in the dual $\mathfrak{A}'_S$ of $\mathfrak{A}_S$: if ω_1 and ω_2 are positive and normalised linear functionals, $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ is clearly still positive and normalised, for any $\lambda \in [0, 1]$.

Hence, just as we saw for the standard formulation, we can define *pure algebraic states* as extreme elements of the convex body.

Definition 14.3 An algebraic state $\omega : \mathfrak{A} \to \mathbb{C}$ on the unital C^*-algebra $\mathfrak{A}$ is called a **pure algebraic state** if it is extreme in the set of algebraic states. An algebraic state that is not pure is called **mixed**.

Later we will show that the space of states is non-empty and compact in the *-weak topology. Consequently, pure states exist.

Surprisingly, most of the abstract apparatus given by a C^*-algebra and a set of states admits elementary Hilbert space representations once a reference algebraic state has been fixed. This is by virtue of a famous procedure Gelfand, Najmark and Segal came up with, that we present below [Haa96, Ara09, Str05a].

Theorem 14.4 (GNS theorem) *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{I}$ and $\omega : \mathfrak{A} \to \mathbb{C}$ a positive linear functional with $\omega(\mathbb{I}) = 1$. Then*
(a) *there exist a triple $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$, where H_ω is a Hilbert space, $\pi_\omega : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_\omega)$ a $\mathfrak{A}$-representation over H_ω and $\Psi_\omega \in \mathsf{H}_\omega$, such that:*
(i) Ψ_ω is cyclic for π_ω: the invariant subspace $\mathscr{D}_\omega := \pi_\omega(\mathfrak{A})\Psi_\omega$ is dense in H_ω;
(ii) $(\Psi_\omega|\pi_\omega(a)\Psi_\omega) = \omega(a)$ for every $a \in \mathfrak{A}$.
(b) *If (H, π, Ψ) satisfies (i) and (ii), there exists a unitary operator $U : \mathsf{H}_\omega \to \mathsf{H}$ such that $\Psi = U\Psi_\omega$ and $\pi(a) = U\pi_\omega(a)U^{-1}$ for any $a \in \mathfrak{A}$.*

Proof (a) For a start we will build the Hilbert space. We will refer to the elementary theory of Hilbert spaces of Chap. 3. Let us define the quadratic form $\langle x, y\rangle_\omega := \omega(x^*y)$, $x, y \in \mathfrak{A}$. This is a Hermitian semi-inner product by the requests made on ω, so for the seminorm $p_\omega(x) := \sqrt{\langle x, x\rangle_\omega}$ the Schwarz inequality

$$\omega(x^*y) \leq \sqrt{\omega(x^*x)}\,\sqrt{\omega(y^*y)} \tag{14.9}$$

holds. Consider now the set $\mathscr{I}_\omega := \{x \in \mathfrak{A} \mid p_\omega(x) = 0\}$. Since p_ω is a seminorm, by (14.9) $\mathscr{I}_\omega$ is a subspace of $\mathfrak{A}$. As the symbol suggests, $\mathscr{I}_\omega$ is a left ideal in $\mathfrak{A}$, i.e. a subspace such that $yx \in \mathscr{I}_\omega$ for any $x \in \mathscr{I}_\omega$, $y \in \mathfrak{A}$. In fact by (14.9):

$$0 \leq p_\omega(yx)^4 = \omega(x^*y^*yx)^2 \leq \omega(y^*yxx^*y^*y)\omega(x^*x) = 0\,.$$

Hence we may define the vector space $\mathscr{D}_\omega := \mathfrak{A}/\mathscr{I}_\omega$, quotient of $\mathfrak{A}$ by the ideal $\mathscr{I}_\omega$. The elements of $\mathscr{D}_\omega$ are thus cosets $[x]$ for the equivalence relation $\mathfrak{A}$: $x \sim y \Leftrightarrow x - y \in \mathscr{I}_\omega$. The vector space structure is naturally inherited by $\mathfrak{A}$, and makes $\alpha[x] + \beta[y] := [\alpha x + \beta y]$ meaningful,[1] for any $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathfrak{A}$. That $\mathscr{I}_\omega$ is a subspace guarantees the structure is well defined. Since $\mathscr{I}_\omega$ is also the left ideal of zeroes of the seminorm associated to the semi-inner product $\langle\ ,\ \rangle_\omega$, as we just showed,

$$([x]\,|[y])_\omega := \langle x, y\rangle_\omega \quad \forall x, y \in \mathfrak{A} \tag{14.10}$$

[1]Notice that there is no identity like $[ab] = [a][b]$ in this construction. Only the linear structure, not the algebra, is constructed on the quotient.

is a well-defined Hermitian inner product on $\mathscr{D}_\omega$. Introduce the Hilbert completion H_ω of $\mathscr{D}_\omega$ for the inner product, which we continue to indicate with (14.10) on the entire H_ω. The representation π_ω is defined in the natural way on the dense subspace $\mathscr{D}_\omega = \mathfrak{A}/\mathscr{I}_\omega \subset \mathsf{H}_\omega$ as:

$$(\pi_\omega(a))([b]) := [ab] \,. \tag{14.11}$$

Then $\mathscr{D}_\omega$ is by construction invariant under every $\pi_\omega(a)$. At last, let $\Psi_\omega := [\mathbb{I}]$. As $a \in \mathfrak{A}$ varies, the sets $\pi_\omega(a)\Psi_\omega = [a]$ fill the dense space $\mathscr{D}_\omega$. Therefore Ψ_ω is cyclic, as needed. It is easy to see that, by construction, $\mathfrak{A} \ni a \mapsto \pi_\omega(a)$ is linear on the dense domain $\mathscr{D}_\omega$ (hence it has an adjoint) and satisfies, for any $a, b, c \in \mathfrak{A}$, $\mu \in \mathbb{C}$:

(i) $\pi_\omega(a)\pi_\omega(b) = \pi_\omega(ab)$,
(ii) $\pi_\omega(a) + \pi_\omega(b) = \pi_\omega(a+b)$,
(iii) $\mu\pi_\omega(a) = \pi_\omega(\mu a)$,
(iv) $(\pi_\omega(b)\Psi_\omega\,|\pi_\omega(a)\pi_\omega(c)\Psi_\omega)_\omega = (\pi_\omega(a^*)\pi_\omega(b)\Psi_\omega\,|\pi_\omega(c)\Psi_\omega)_\omega$.

The last fact, equivalent to

$$\pi_\omega(a)^*\restriction_{\mathscr{D}_\omega} = \pi_\omega(a^*) \,,$$

follows by

$$(\pi_\omega(b)\Psi_\omega\,|\pi_\omega(a)\pi_\omega(c)\Psi_\omega)_\omega = ([b]\,|[ac])_\omega = \omega(b^*ac) = \omega\left((a^*b)^*c\right)$$

$$= \left([a^*b]\,|[c]\right)_\omega = (\pi_\omega(a^*)\pi_\omega(b)\Psi_\omega|\pi_\omega(c)\Psi_\omega)_\omega \,.$$

By construction, for $a \in \mathfrak{A}$:

$$\omega(a) = \omega(\mathbb{I}^*a\mathbb{I}) = ([\mathbb{I}]|[a][\mathbb{I}])_\omega = (\Psi_\omega\,|\pi_\omega(a)\Psi_\omega)_\omega \,. \tag{14.12}$$

To finish (a) it is enough to prove that every operator $\pi_\omega(a) : \mathscr{D}_\omega \to \mathsf{H}_\omega$ is bounded, so it extends uniquely to a bounded operator on H_ω, because $\mathscr{D}_\omega \subset \mathsf{H}_\omega$ is dense. We will call the extended operators with the same names $\pi_\omega(a)$. Therefore properties (i)–(iv) will still be valid, by continuity. In particular, the operators being bounded, (iv) implies $\pi_\omega(a)^* = \pi_\omega(a^*)$. It is obvious that $\pi_\omega(\mathbb{I}) = I$ and so the map $\pi_\omega : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_\omega)$ will be a *-representation.

To prove the boundedness of the $\pi_\omega(a)$, we begin by showing ω is continuous. We will only assume that the linear functional ω is positive, without using $\omega(\mathbb{I}) = 1$. If $h \in \mathfrak{A}$ is normal, since $|\sigma(h)| \leq ||h||$ by the features of the spectral radius, Theorem 8.36(c) gives $\sigma(h \pm ||h||\mathbb{I}) \geq 0$. By Theorem 8.25, $h \pm ||h||\mathbb{I} = c^*c$, so the positivity and linearity of ω allows to say $\omega(h) \pm ||h||\omega(\mathbb{I}) \geq 0$, meaning $|\omega(h)| \leq \omega(\mathbb{I})||h||$. In turn this implies ω is a bounded linear functional. In fact, if $y \in \mathfrak{A}$ is any element, y^*y is self-adjoint and so normal. Using the above result gives immediately $|\omega(y^*y)| \leq \omega(\mathbb{I})||y^*y||$. Finally, (14.9) with $x = \mathbb{I}$ says

$$|\omega(y)|^4 \leq \omega(\mathbb{I})^2\, ||y^*y||^2 = (\omega(\mathbb{I})||y||)^2 \,,$$

hence $||\omega|| \leq \omega(\mathbb{I})$. On the other hand, from $|\omega(\mathbb{I})| = \omega(\mathbb{I})$ and $||\mathbb{I}|| = 1$ we have

$$||\omega|| = \omega(\mathbb{I}) .$$

In our case, as $\omega(\mathbb{I}) = 1$, we obtain $||\omega|| = 1$.

If $\omega(x^*x) > 0$ we can repeat the argument for the linear functional

$$\mathfrak{A} \ni z \mapsto \rho(z) := \frac{\omega(x^*zx)}{\omega(x^*x)} ,$$

by construction linear, positive and such that $\rho(\mathbb{I}) = 1$; therefore $||\rho|| = \rho(\mathbb{I}) = 1$. We conclude that the state ω satisfies

$$\omega(x^*y^*yx) \leq ||y^*y||\omega(x^*x) ,$$

holding also when $\omega(x^*x) = 0$ because the Cauchy-Schwarz inequality forces $0 \leq \omega(x^*y^*yx) = \omega((x^*y^*y)x) \leq \sqrt{\omega((x^*y^*y)^*(x^*y^*y))}\sqrt{\omega(x^*x)}$. Consequently

$$||(\pi_\omega(y))([x])||_\omega = ||[yx]||_\omega = \sqrt{\omega(x^*y^*yx)} \leq \sqrt{||y^*y||}\sqrt{\omega(x^*x)} \leq ||y||\;||[x]||_\omega ,$$

and so $||\pi_\omega(y)|| \leq ||y||$. This ends part (a).

(b) Just asking $U\pi_\omega(a)\Psi_\omega := \pi(a)\Psi$ for any $a \in \mathfrak{A}$ determines a densely-defined isometric operator, which we call U. This is well defined because $\pi_\omega(a)\Psi_\omega = \pi_\omega(a')\Psi_\omega$ implies $\pi(a)\Psi = \pi(a')\Psi$, as is evident from

$$||\pi(a - a')\Psi||^2 = \omega((a - a')^*(a - a')) = ||\pi_\omega(a - a')\Psi_\omega||^2_\omega .$$

For the same reason U is isometric:

$$||U\pi_\omega(a)\Psi_\omega||^2 = ||\pi(a)\Psi||^2 = \omega(a^*a) = ||\pi_\omega(a)\Psi_\omega||^2_\omega .$$

Hence we can extend U to a continuous isometric operator on H with the same name. Similarly, let us construct an isometric operator $V : \mathsf{H} \to \mathsf{H}_\omega$ as the unique continuous extension of $V\pi(a)\Psi = \pi_\omega(a)\Psi_\omega$. By continuity, and using the density of $\pi(\mathfrak{A})\Psi$, we have $UV\Phi = \Phi$ for every $\Phi \in \mathsf{H}$. Therefore U is onto, beside isometric, and so unitary. That $\Psi = U\Psi_\omega$ and $\pi(a) = U\pi_\omega(a)U^{-1}$, $a \in \mathfrak{A}$, are obvious by construction. $\square$

The GNS theorem shows that given an algebraic state, the observables of $\mathfrak{A}$ are still represented by (bounded) self-adjoint operators on a Hilbert space H_ω, where the expectation value of ω takes the usual form $(\Psi_\omega|\pi(a)\Psi_\omega)$ with respect to a reference vector Ψ_ω. The latter vector allows to recover the whole Hilbert space by means of the representation π_ω itself, as we said in the GNS theorem (a), part (i). However the reader should notice that not all algebraic states on $\mathfrak{A}$ are represented by positive trace-class operators on H_ω, as we shall discuss shortly.

The representation π_ω need not be injective, i.e. faithful. From the proof and (14.11) with $b = \mathbb{I}$ in particular, we immediately have

Proposition 14.5 *The GNS representation $\pi_\omega : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_\omega)$ of the algebraic state ω on the unital C^*-algebra $\mathfrak{A}$ is faithful if the* **Gelfand left ideal** *of ω,*

$$\mathscr{I}_\omega := \{a \in \mathfrak{A} \mid \omega(a^*a) = 0\} ,$$

is trivial, i.e., $\mathscr{I}_\omega = \{0\}$.

Algebraic states with trivial Gelfand left ideal are called **faithful**. Notice that the kernel $\mathrm{Ker}(\pi_\omega)$ of the GNS representations π_ω of a C^*-algebra is a closed *two-sided* *-ideal: it only depends on ω, not on the particular GNS triple, and generally it is smaller than $\mathscr{I}_\omega$. In fact, form (14.11) and the construction of the GNS representation, one easily sees that the elements of $\mathscr{I}_\omega$ are the elements $a \in \mathscr{A}$ such that $\pi_\omega(a)\Psi_\omega = 0$. Conversely, $a \in \mathrm{Ker}(\pi_\omega)$ satisfies $\pi_\omega(a)\pi_\omega(b)\Psi_\omega = 0$ for all $b \in \mathfrak{A}$, which is a much stronger requirement. Therefore, in principle ω can have faithful GNS representations even if $\mathscr{I}_\omega \neq \{0\}$ but just because $\mathrm{Ker}(\pi_\omega)$ is trivial.

A technical result that was proved in passing, during the previous proof, and that is useful in itself, is the following.

Theorem 14.6 (Continuity of positive functionals) *If ω is a positive functional on the C^*-algebra $\mathfrak{A}$ with unit $\mathbb{I}$, then ω is continuous and $||\omega|| = \omega(\mathbb{I})$.*

There is a useful technical corollary to the GNS theorem that deserves being stated and proved.

Corollary 14.7 *Let ω be an algebraic state on the unital C^*-algebra $\mathfrak{A}$ with associated GNS triple $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$.*
(a) *If $\psi : \mathfrak{A} \to \mathbb{C}$ is linear, positive and $\psi \leq \omega$ ($\omega - \psi$ is positive), there exists a unique $T \in \mathfrak{B}(\mathsf{H}_\omega)$ such that*

$$\psi(b^*a) = (\pi_\omega(b)\Psi_\omega|T\pi_\omega(a)\Psi_\omega)_\omega \quad \forall a, b \in \mathfrak{A} .$$

Moreover $0 \leq T \leq I$ and $T \in \pi_\omega(\mathfrak{A})'$ (T commutes with each $\pi_\omega(a)$, $a \in \mathfrak{A}$).
(b) *Conversely, if $0 \leq T \leq I$ and $T \in \pi_\omega(\mathfrak{A})'$, then $\psi(a) := (\Psi_\omega|T\pi_\omega(a)\Psi_\omega)_\omega$, for every $a \in \mathfrak{A}$, is a positive functional with $\psi \leq \omega$.*

Proof (a) Take ψ as in the assumptions. Since

$$|\psi(b^*a)|^2 \leq \psi(b^*b)\psi(a^*a) \leq \omega(b^*b)\omega(a^*a) = ||[b]||_\omega \, ||[a]||_\omega ,$$

setting $\psi'([b],[a]) := \psi(b^*a)$, Riesz's theorem warrants the existence of $T \in \mathfrak{B}(\mathsf{H}_\omega)$ with $\psi'([b],[a]) = ([b]|T[a])_\omega$. In other terms $\psi(b^*a) = (\pi_\omega[b]\Psi_\omega| T\pi_\omega(a)\Psi_\omega)$. Furthermore, by construction:

$$([b]\,|(T\pi_\omega(a) - \pi_\omega(a)T)[c]\,)_\omega = \psi(b^*ac) - \psi((a^*b)^*c) = \psi(b^*ac) - \psi(b^*ac) = 0 .$$

(b) is immediate.

Remark 14.8 **(1)** The cyclic vector Ψ_ω is a unit vector, by (a) (ii) in the GNS theorem, since $a = \mathbb{I}$ and $\omega(\mathbb{I}) = 1$.
(2) Irrespective of the way one proves the GNS theorem, the *-representation of unital C^*-algebras π_ω must be continuous, because of Theorem 8.22, and must also satisfy $||\pi_\omega(a)|| \leq ||a||$ for any $a \in \mathfrak{A}$. In addition, the same theorem implies π_ω is isometric ($||\pi_\omega(a)|| = ||a||$ for any $a \in \mathfrak{A}$) precisely when it is faithful (one-to-one).
(3) If we deal with C^*-algebras *without unit*, algebraic states can be defined anyway. They are positive, bounded linear functionals, but $\omega(\mathbb{I}) = 1$ is replaced by $||\omega|| = 1$. These two conditions are equivalent on unital C^*-algebras, by Theorem 14.6. We saw in Sect. 7.6.5 that if we restrict to the C^*-algebra $\mathfrak{B}_\infty(\mathsf{H})$ of compact operators on a Hilbert space H (which has no unit, because in infinite dimensions the identity operator is never compact), algebraic states are exactly the positive operators of trace class with unit trace. ∎

If ω is an algebraic state on $\mathfrak{A}$, every statistical operator on the Hilbert space of a GNS representation of ω, i.e. every positive, trace-class operator with unit trace $T \in \mathfrak{B}_1(\mathsf{H}_\omega)$ (a *statistical operator*), determines an algebraic state

$$\mathfrak{A} \ni a \mapsto tr\,(T\pi_\omega(a)) \ ,$$

evidently. This is true, in particular, for $\Phi \in \mathsf{H}_\omega$ with $||\Phi||_\omega = 1$, in which case the above definition reduces to

$$\mathfrak{A} \ni a \mapsto (\Phi|\pi_\omega(a)\Phi)_\omega \ .$$

To this end we have

Definition 14.9 If ω is an algebraic state on the unital C^*-algebra $\mathfrak{A}$, every algebraic state on $\mathfrak{A}$ obtained either from a statistical operator or a unit vector, in a GNS representation of ω, is called **normal state** of ω. The set $Fol(\omega)$ of normal states is the **folium** of the algebraic state ω.

Note that in order to determine $Fol(\omega)$ one can use a fixed GNS representation of ω. In fact, as the GNS representation of ω varies, normal states do not change, as implied by part (b) of the GNS theorem.

The folium of a state ω of the algebra of observables $\mathfrak{A}$ can be naïvely thought of as *the set of algebraic states arising from the action of observables of* $\mathfrak{A}$ on ω, *possibly through a limiting process*.

By the GNS theorem, namely, every unit vector $\Phi \in \mathsf{H}_\omega$ is a limit of $\pi_\omega(b_n)\Psi_\omega$ as $n \to +\infty$, provided we choose $b_n \in \mathfrak{A}$ suitably. Hence, the GNS theorem implies that the algebraic state associated to Φ, which is an element of $Fol(\omega)$, can be always computed as

$$\omega_\Phi(a) = (\Phi|\pi_\omega(a)\Phi)_\omega = \lim_{n\to+\infty} \omega(b_n^* a b_n) \ .$$

The other algebraic states in the folium of ω are determined by positive, trace-class operators $T \in \mathfrak{B}(\mathsf{H}_\omega)$ with unit trace. Decomposing T spectrally as an infinite convex combination $T = \text{s-}\sum_i p_i(\Phi_i|\ )\Phi_i$, we can eventually write $\omega_T(a) = \sum_i p_i \omega_{\Phi_i}(a)$, and fall back into the previous case.

In case $\mathfrak{A}$ is a von Neumann algebra of operators on H, normal states are defined as follows (recall that there already is a natural representation of $\mathfrak{A}$, the one over $\mathfrak{A}$ itself).

Definition 14.10 Looking at a von Neumann algebra $\mathfrak{R} \subset \mathfrak{B}(\mathsf{H})$ on the Hilbert space H as a C^*-algebra, a **normal state** of $\mathfrak{R}$ is an algebraic state ω satisfying $\omega(A) = tr(\rho_\omega A)$ for some positive $\rho_\omega \in \mathfrak{B}_1(\mathsf{H})$ with unit trace determined by ω, and for every $A \in \mathfrak{R}$.

An important characterisation holds [BrRo02].

Proposition 14.11 *Consider an algebraic state $\omega : \mathfrak{R} \to \mathbb{C}$ on the von Neumann algebra $\mathfrak{R} \subset \mathfrak{B}(\mathsf{H})$ for a given Hilbert space H. The following facts are equivalent:*

(a) *ω is normal;*
(b) *ω is continuous in the σ-weak topology (cf. Definition 3.94);*
(c) *ω is* **completely additive***:*

$$\omega\left(\sum_{j \in J} P_j\right) = \sum_{j \in J} \omega(P_j)$$

for every family $\{P_j\}_{j \in J} \subset \mathfrak{R}$ of pairwise-orthogonal, orthogonal projectors (notation as in Theorem 7.72).

Remark 14.12 If $\mathfrak{R} = \mathfrak{B}(\mathbb{C}^2)$, every state is normal, by direct inspection. Instead, if we consider σ-additive measures on $\mathscr{L}(\mathbb{C}^2)$, Remark 7.28(4) proves that there exist measures which are not normal states. This is the reason why Theorem 7.72 needs the further hypothesis that the direct sum of $\mathfrak{R}$ does not contain type-I_2 algebras (Sect. 7.6.3). In this regard, the notion of state on a von Neumann algebra seems to be more rigid than the notion of measure over the projector lattice of that von Neumann algebra. ■

We will prove during Lemma 14.28 that any unital C^*-algebra always admits states (hence a convex set of states). We can ask whether pure states exist, i.e. if the set of states of a unital C^*-algebra contains extreme elements. The answer is yes, and one shows that every algebraic state can be obtained as a limit of a sequence of a convex combination of pure states, in the *-weak topology.

Theorem 14.13 *The set $S(\mathfrak{A})$ of algebraic states of a C^*-algebra $\mathfrak{A}$ with unit is a bounded and convex compact subset of $\mathfrak{A}'$ in the *-weak topology. Moreover, $S(\mathfrak{A})$ coincides with the *-weak closure of the convex hull of pure states (which is therefore non-empty).*

Proof By Theorem 14.6 the convex set $S(\mathfrak{A})$ is contained in the closed unit ball inside the dual of $\mathfrak{A}$. The latter is *-weakly compact, by Theorem 2.80 of Banach-Alaoglu. As the set of states is closed in that topology (the proof is straightforward), it is also compact in the dual of $\mathfrak{A}$ and convex. The Krein-Milman theorem 2.81 guarantees the set of extreme algebraic states is not empty, and the closure of its convex hull is $S(\mathfrak{A})$. $\square$

14.1.4 Pure States and Irreducible Representations

We devote this section to an important relationship between pure algebraic states and irreducible representations of the algebra of observables: *ω is pure if and only if the representation π_ω is irreducible*. To prove it we need the following lemma.

Lemma 14.14 *An algebraic state ϕ on a unital C^*-algebra $\mathfrak{A}$ is pure if and only if $\phi = \psi_1 + \psi_2$ for positive functionals $\psi_i : \mathfrak{A} \to \mathbb{C}$, $i = 1, 2$, implies $\psi_i = \lambda_i \phi$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$.*

Proof If ϕ is not pure, it is not extreme in the set of algebraic states, so $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ for $\phi_1 \neq \phi \neq \phi_2$. Defining $\psi_i := \frac{1}{2}\phi_i$, we see that $\phi = \psi_1 + \psi_2$, where $\psi_1 \neq \lambda_1 \phi$ irrespective of λ_1. Let us assume ϕ is pure, conversely. First, if $\lambda \in (0, 1)$ and $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ for some states ϕ_i, then $\phi = \phi_1 = \phi_2$. So assume such ϕ satisfies $\phi = \psi_1 + \psi_2$, for some positive functionals ψ_1, ψ_2. We claim $\psi_i = \lambda_i \phi$ for some numbers λ_i.

If $\psi_i(\mathbb{I}) = 0$ for $i = 1$ or $i = 2$, then $\psi_i = 0$ by Theorem 14.6, and the conclusion follows trivially. So suppose $\psi_i(\mathbb{I}) \neq 0, i = 1, 2$. Define $\phi_i(a) := \psi_i(\mathbb{I})^{-1}\psi_i(a)$. Then ϕ_i is a state and $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$, with $\lambda = \psi_1(\mathbb{I})$ and $1 - \lambda = \phi(\mathbb{I}) - \psi_1(\mathbb{I}) = \psi_2(\mathbb{I})$. Since ϕ is extremal, $\phi_1 = \phi_2 = \phi$, hence $\psi_i = \psi_i(\mathbb{I})\phi$, $i = 1, 2$. $\square$

Now the announced result can be stated.

Theorem 14.15 (Characterisation of pure algebraic states) *Let ω be an algebraic state on the unital C^*-algebra $\mathfrak{A}$ and $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$ a corresponding GNS triple. Then ω is pure if and only if π_ω is irreducible.*

Proof By Schur's lemma (see esp. Remark 11.38), π_ω is irreducible iff $\pi_\omega(\mathfrak{A})' = \{cI\}_{c\in\mathbb{C}}$. A direct consequence of Corollary 14.7 is that $\pi_\omega(\mathfrak{A})' = \{cI\}_{c\in\mathbb{C}}$ iff $0 \leq \psi \leq \omega$ implies $\psi = c\omega$ for some $c \in \mathbb{C}$. But $0 \leq \psi \leq \omega$ iff $\omega = \psi + (\omega - \psi)$, together with $\psi \geq 0$ and $\omega - \psi \geq 0$. In summary, π_ω is irreducible iff $\omega = \psi_1 + \psi_2$, $\psi_i \geq 0$ imply $\psi_i = \lambda_i\omega$ for some choice of λ_i. The previous lemma tells π_ω is irreducible iff ω is pure. $\square$

Now we have two important consequences that relate pure states to irreducible representations of a C^*-algebra with unit.

Corollary 14.16 *Let ω be a pure state on the unital C^*-algebra $\mathfrak{A}$ and $\Phi \in \mathsf{H}_\omega$ a unit vector. Then*

(a) *the functional*

$$\mathfrak{A} \ni a \mapsto (\Phi|\pi_\omega(a)\Phi)_\omega ,$$

defines a pure algebraic state and $(\mathsf{H}_\omega, \pi_\omega, \Phi)$ *is a GNS triple for it. In that case, GNS representations of algebraic states given by non-zero vectors in* H_ω *are all unitarily equivalent.*
(b) *Unit vectors* $\Phi, \Phi' \in \mathsf{H}_\omega$ *give the same (pure) algebraic state if and only if* $\Phi = c\Phi'$ *for some* $c \in \mathbb{C}$, $|c| = 1$, *i.e. if and only if* Φ *and* Φ' *belong to the same ray of* H_ω.

Proof (a) Consider the closed space $\mathsf{M}_\Phi := \overline{\pi_\omega(\mathfrak{A})\Phi}$; we will show it coincides with H_ω. By construction $\pi(a)\mathsf{M}_\Phi \subset \mathsf{M}_\Phi$ for $a \in \mathfrak{A}$, so M_Φ is closed and π_ω-invariant. As the representation is irreducible, necessarily $\mathsf{M}_\Phi = \mathsf{H}_\omega$ or $\mathsf{M}_\Phi = \{\mathbf{0}\}$. The latter case is impossible because $\pi_\omega(\mathbb{I})\Phi = \Phi \neq \mathbf{0}$. Now the claim is clear by construction, because $(\mathsf{H}_\omega, \pi_\omega, \Phi)$ satisfies the GNS assumptions for a triple of an algebraic state given by Φ as above, which is pure because the GNS representation is irreducible. The last statement is obvious since all GNS representations can be constructed as above. The unitary transformation between two such is always the identity operator.
(b) If $\Phi = c\Phi'$ the two vectors give the same pure algebraic state. If, conversely, two unit vectors determine the same pure algebraic state, i.e. $(\Phi|\pi_\omega(a)\Phi)_\omega = (\Phi'|\pi_\omega(a)\Phi')_\omega$ for every $a \in \mathfrak{A}$, then we decompose $\Phi = c\Phi' + \Psi$ with Ψ orthogonal to Φ'. In this way

$$(\Phi|\pi_\omega(a)\Phi)_\omega = |c|^2(\Phi'|\pi_\omega(a)\Phi')_\omega + c(\Psi|\pi_\omega(a)\Phi')_\omega + \overline{c}(\Phi'|\pi_\omega(a)\Psi)_\omega ,$$

whence

$$(1 - |c|^2)(\Phi'|\pi_\omega(a)\Phi')_\omega = c(\Psi|\pi_\omega(a)\Phi')_\omega + \overline{c}(\Phi'|\pi_\omega(a)\Psi)_\omega .$$

Choose $a = \mathbb{I}$, so that the right-hand side vanishes and then the left-hand side does, too. This is possible only if $|c| = 1$. Back to $\Phi = c\Phi' + \Psi$, we obtain $\Psi = \mathbf{0}$ because $1 = ||\Phi'||^2 = |c|^2 + ||\Psi||^2$. □

Corollary 14.17 *If* $\mathfrak{A}$ *is a* C^**-algebra with unit, every irreducible representation* $\pi : \mathfrak{A} \to \mathsf{H}$ *is the GNS representation of a state pure.*

Proof Let $\Psi \in \mathsf{H}$ be a unit vector. As the representation is irreducible, $\pi(\mathfrak{A})\Psi$ is dense in H. It is easy to see that (H, π, Ψ) is a GNS triple for $\omega(\cdot) = (\Psi|\pi(\cdot)\Psi)$. The latter state is pure because of irreducibility. □

Remark 14.18 Consider, in the standard (not algebraic) formulation, a physical system S described on the Hilbert space H_S, and a mixed state $\rho \in \mathfrak{S}(\mathsf{H})$. The map $\omega_\rho : \mathfrak{B}(\mathsf{H}) \ni A \mapsto tr(\rho A)$ defines an algebraic state on the C^*-algebra $\mathfrak{B}(\mathsf{H}_S)$. By the GNS theorem, there exist another Hilbert space H_ρ, a representation $\pi_\rho : \mathfrak{B}(\mathsf{H}_S) \to \mathfrak{B}(\mathsf{H}_\rho)$ and a unit vector $\Psi_\rho \in \mathsf{H}_\rho$ such that

$$tr(\rho A) = (\Psi_\rho|\pi_\rho(A)\Psi_\rho)$$

for $A \in \mathfrak{B}(\mathsf{H}_S)$. Therefore it seems that the initial mixed state has been transformed into a pure state! How is this possible?

The answer follows from Theorem 14.15: Ψ_ρ cannot correspond to any vector $U^{-1}\Psi_\rho$ in H_S, under any unitary transformation $U : \mathsf{H}_S \to \mathsf{H}_\rho$ with $UAU^{-1} = \pi_\rho(A)$. In fact the representation $\mathfrak{B}(\mathsf{H}_S) \ni A \mapsto A \in \mathfrak{B}(\mathsf{H}_S)$ is irreducible, whereas π_ρ cannot be irreducible because the state of ρ is not an extreme point in the space of non-algebraic states, and so it cannot be extreme in the larger space of algebraic states. (The precise form – up to unitary transformations – of the representation π_ρ will be discussed in Example 14.19(4).)

This remark should clarify that the correspondence pure (algebraic) states vs. state vectors, automatic in the standard formulation (in absence of superselection rules), holds in Hilbert spaces of GNS representations of pure algebraic states, but not in Hilbert spaces of GNS representations of mixed algebraic states. ■

Example 14.19 **(1)** Let us focus on the standard theory described on a complex separable Hilbert space H in order to discuss the simplest possible example. Assume that the algebra of observables is the whole $\mathfrak{B}(\mathsf{H})$ and fix a Hilbert-space pure state represented, up to phase, by a unit vector ψ. The map

$$\omega_\psi : \mathfrak{B}(\mathsf{H}) \ni A \mapsto (\psi|A\psi)$$

defines a normal algebraic state. Let us construct a GNS representation of ω_ψ, and remember that all other GNS representations will be unitarily equivalent to it, in view of Theorem 14.4(b). If we set $\imath(A) := A$ for every $A \in \mathfrak{B}(\mathsf{H})$, then $(\mathsf{H}, \imath, \psi)$ is easily a GNS triple for ω_ψ. The only thing to check is that $A\psi$ ranges in a dense subspace as A varies in $\mathfrak{B}(\mathsf{H})$. To this end, define $\mathsf{M} := \overline{\{A\psi \mid A \in \mathfrak{B}(\mathsf{H})\}}$. Evidently, M is a non-trivial closed subspace of H invariant under the action of $\mathfrak{B}(\mathsf{H})$. As a consequence of Proposition 3.93, $\mathsf{M} = \mathsf{H}$. We have proved that $(\mathsf{H}, \imath, \psi)$ is a GNS triple for ω_ψ.

Notice that the representation $\imath$ is clearly irreducible. Using $\mathfrak{B}(\mathsf{H})$ as algebra of observables, this result implies that pure Hilbert-space states, as of Definition 7.36, are pure states also in the algebraic sense, for Theorem 14.15. This is not an evident result because a pure state of Definition 7.36 could be, in principle, a non-trivial convex combination of algebraic states which are not normal.

(2) For commutative C^*-algebras with unit the following characterisation of pure states holds.

Proposition 14.20 *If $\mathfrak{A}$ is a commutative C^*-algebra with unit, a state $\omega : \mathfrak{A} \to \mathbb{C}$ is pure if and only if it is multiplicative: $\omega(ab) = \omega(a)\omega(b)$ for any $a, b \in \mathfrak{A}$.*

Proof That ω is pure implies π_ω is irreducible, but $\pi_\omega(a)$ commutes with every other $\pi_\omega(b)$ since $\mathfrak{A}$ is commutative. By Schur's lemma $\pi_\omega(\mathfrak{A}) = \{cI \mid c \in \mathbb{C}\}$. Using the GNS representation gives $\omega(ab) = \omega(a)\omega(b)$. Conversely if ω is multiplicative, by the GNS theorem we can write

$$(\pi_\omega(a^*)\Psi_\omega|\pi_\omega(b)\Psi_\omega)_\omega = (\pi_\omega(a^*)\Psi_\omega|\Psi_\omega)_\omega(\Psi_\omega|\pi_\omega(b)\Psi_\omega)_\omega \ ,$$

so Ψ_ω, alone, is a basis of H_ω, because $\pi_\omega(\mathfrak{A})\Psi_\omega$ is dense in H_ω. Therefore H_ω has dimension 1, and all its operators are numbers. In particular, $\pi_\omega(\mathfrak{A})' = \{cI \mid c \in \mathbb{C}\}$, which means π_ω is irreducible by Schur's lemma. □

(3) The next example does not originate in QM. Take a compact Hausdorff space X and the commutative C^*-algebra with unit $C(\mathsf{X})$ of $\mathbb{C}$-valued continuous maps on X, equipped with the usual pointwise algebraic operations, involution given by complex conjugation and norm $|| \;||_\infty$. If μ denotes a Borel probability measure on X, then

$$\omega_\mu : C(\mathsf{X}) \ni f \mapsto \int_{\mathsf{X}} f d\mu$$

defines an algebraic state on $C(\mathsf{X})$. The GNS theorem then gives a triple $(\mathsf{H}_\mu, \pi_\mu, \Psi_\mu)$ where: $\mathsf{H}_\mu = L^2(\mathsf{X}, d\mu)$, $(\pi_\omega(f)\psi)(x) := f(x)\psi(x)$ for every $x \in \mathsf{X}$, $\psi \in \mathsf{H}_\mu$ and $f \in C(\mathsf{X})$. The cyclic vector Ψ_ω coincides with the constant map 1 on X.

It can be checked that pure states are the Dirac measures δ_x concentrated at points $x \in \mathsf{X}$. In this sense probability measures can be understood as "thick" points.

(4) Let us again return to the standard theory described on a complex separable Hilbert space H, and suppose that the algebra of observables is the whole $\mathfrak{B}(\mathsf{H})$. Fix a mixed state represented by positive trace-class operator T with unit trace. The map

$$\omega_T : \mathfrak{B}(\mathsf{H}) \ni A \mapsto tr(TA)$$

defines a normal algebraic state. Let us construct a GNS representation of ω_T, remembering that all other GNS representations are unitarily equivalent to this one by Theorem 14.4(b). First of all decompose T spectrally,

$$T = \sum_{k\in N} p_k(\psi_k| \;)\psi_k \; .$$

Above, N may be uncountable but the non-zero p_k form a finite or countable set, $1 \geq p_k \geq p_{k+1} \geq 0$, $\sum_{k\in N} p_k = 1$ and $\{\psi_k\}_{k\in N}$ is a Hilbert basis of H. The series of T converges in the uniform topology. With these hypotheses, consider the Hilbert space H_{ω_T} defined by the Hilbert orthogonal sum

$$\mathsf{H}_{\omega_T} := \oplus_{k\in N'}\mathsf{H}_k \quad \text{with } \mathsf{H}_k = \mathsf{H} \text{ for every } k \in N' \tag{14.13}$$

where $N' \subset N$ contains only the indices k such that $p_k > 0$. Next define the vector

$$\Psi_{\omega_T} := \oplus_{k\in N'}\sqrt{p_k}\psi_k \; . \tag{14.14}$$

It is evident that $\Psi_{\omega_T} \in \mathsf{H}_{\omega_T}$ and it is easy to prove that the map

$$\pi_{\omega_T} : \mathfrak{B}(\mathsf{H}) \ni A \mapsto \oplus_{k\in N'}A_k \quad \text{with } A_k = A \text{ for every } k \in N' \tag{14.15}$$

is a representation of $\mathfrak{B}(\mathsf{H})$ on $\mathfrak{B}(\mathsf{H}_{\omega_T})$. By construction

$$\left(\Psi_{\omega_T}\middle|\,\pi_{\omega_T}(A)\Psi_{\omega_T}\right) = \sum_{k\in N'} p_k(\psi_k|A\psi_k) = \omega_T(A) \quad \text{for every } A \in \mathfrak{B}(\mathsf{H}).$$

To conclude that $(\mathsf{H}_{\omega_T}, \pi_{\omega_T}, \Psi_{\omega_T})$ is a GNS triple of ω_T it is sufficient to establish that Ψ_{ω_T} is cyclic for π_{ω_T}. Let us show it. If $\Phi = \oplus_{k\in N'}\phi_k \in \mathsf{H}_{\omega_T}$, assume that $\left(\Phi|\pi_{\omega_T}(A)\Psi_{\omega_T}\right) = 0$ for every $A \in \mathfrak{B}(\mathsf{H})$, that is

$$\sum_{k\in N'} \sqrt{p_k}(\phi_k|B\psi_k) = 0\,, \quad \forall B \in \mathfrak{B}(\mathsf{H})\,. \tag{14.16}$$

To prove our claim it is enough to obtain $\Phi = \mathbf{0}$. Fix $k \in N'$ and consider the subset of $\mathfrak{B}(\mathsf{H})$ made of elements $B := AP_k$, where $A \in \mathfrak{B}(\mathsf{H})$ and $P_k := (\psi_k|\ )\psi_k$. Specialising (14.16) to these elements, we have

$$(\phi_k|A\psi_k) = 0\,, \quad \forall A \in \mathfrak{B}(\mathsf{H})\,.$$

The set of vectors $A\psi_k$ is dense in $\mathsf{H} = \mathsf{H}_k$ when A varies in $\mathfrak{B}(\mathsf{H})$, as a trivial consequence of Corollary 14.16(a), so that $\phi_k = \mathbf{0}$. Since k is arbitrary, we may conclude that $\Phi = \mathbf{0}$ and thus Ψ_{ω_T} is cyclic for π_{ω_T} as wanted.

We have constructed a GNS triple for ω_T. It is evident from the construction that (14.15) is reducible if N' contains more than one element, as every closed subspace $\mathsf{H}_k \subset \mathsf{H}_{\omega_T}$ is invariant under π_{ω_T}. All this agrees with Theorem 14.15, since a mixed Hilbert-space state T, defined in accordance with Definition 7.36, must be a non-pure algebraic state *a fortiori* when viewed as an algebraic normal state ω_T.

Moreover the representation we have found is evidently injective. This last result is clearly invariant under the action of unitary operators, and hence is valid for every GNS representation of ω_T. In other words the following proposition holds.

Proposition 14.21 *Let* H *be a separable Hilbert space. Normal states of* $\mathfrak{B}(\mathsf{H})$ *necessarily have faithful GNS representations. Therefore if the GNS representations of a state* ω *of* $\mathfrak{B}(\mathsf{H})$ *is not faithful,* ω *cannot be normal.*

The kernel $\mathfrak{K}_\omega$ of a non-faithful GNS representation π_ω of a state ω of $\mathfrak{B}(\mathsf{H})$ is a two-sided *-ideal of $\mathfrak{B}(\mathsf{H})$, as one proves immediately. Since π_ω is continuous, $\mathfrak{K}_\omega$ is also closed in the uniform topology of $\mathfrak{B}(\mathsf{H})$. If H is separable, there is only one non-trivial two-sided *-ideal of $\mathfrak{B}(\mathsf{H})$ that is closed in the uniform topology, namely the ideal of compact operators $\mathfrak{B}_\infty(\mathsf{H})$ (Theorem 4.17). So, in particular, $\omega(A) = 0$ for every compact operator whenever ω admits non-faithful GNS representations and hence is not normal. This suggests that non-normal states of $\mathfrak{B}(\mathsf{H})$ may display quite pathological features. ■

14.1.5 Further Comments on the Algebraic Approach and the GNS Construction

In a general setup, especially when we deal with algebras arising from Quantum Field Theories, it is sometimes reasonable to suppose that the C^*-algebra $\mathfrak{A}$ whose self-adjoint elements represent observables be *simple*.

Definition 14.22 A C^*-algebra $\mathfrak{A}$ is **simple** if its only closed two-sided ideals that are invariant under the involution are $\mathfrak{A}$ and $\{0\}$.

The reason for finding simple algebras appealing is that every non-trivial representation (whether GNS or not), on whichever Hilbert space, is faithful (injective) hence isometric, as the next proposition proves.

Proposition 14.23 *If $\mathfrak{A}$ is a simple C^*-algebra with unit and $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ a representation on the Hilbert space $\mathsf{H} \neq \{\mathbf{0}\}$, then π is faithful (one-to-one) and isometric.*

Proof The null space of $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ is a two-sided ideal in $\mathfrak{A}$ that is closed (π is continuous by Theorem 8.22) and invariant under the involution, as is immediate to see. But $\mathrm{Ker}(\pi) = \mathfrak{A}$ is impossible since $\pi(\mathbb{I}) = I \neq 0$. Hence our representation must be injective because $\mathrm{Ker}(\pi) = \{0\}$, and so isometric by Theorem 8.22. □

This means that every operator representation of a simple, unital C^*-algebra faithfully represents the algebra, quite literally. However, as we saw at the end of Example 14.19(4), there are cases of interest in physics where the relevant algebra of observables is not simple.

Sometimes the C^*-structure is too rigid, whereas a *-algebra with unit is better tailored to described observables. This is the case when one studies bosonic quantum fields without using Weyl C^*-algebras. The key part of the GNS theorem is still valid. In fact, we have the following version of the GNS theorem, whose proof is similar to (actually much simpler than) the previous one.

Theorem 14.24 (GNS theorem for *-algebras with unit) *Let $\mathfrak{A}$ be a *-algebra with unit $\mathbb{I}$ and $\omega : \mathfrak{A} \to \mathbb{C}$ a positive linear functional with $\omega(\mathbb{I}) = 1$. Then*
(a) *there exists a quadruple $(\mathsf{H}_\omega, \mathscr{D}_\omega, \pi_\omega, \Psi_\omega)$ made of a Hilbert space H_ω, a subspace $\mathscr{D}_\omega \subset \mathsf{H}_\omega$, a linear map $\pi_\omega : \mathfrak{A} \to \mathfrak{L}(\mathscr{D}_\omega, \mathsf{H}_\omega)$ and an element $\Psi_\omega \in \mathscr{D}_\omega$, such that:*
(i) $\mathscr{D}_\omega$ is $\pi_\omega(a)$-invariant for every $a \in \mathfrak{A}$, since $\mathscr{D}_\omega = \pi_\omega(\mathfrak{A})\Psi_\omega$;
(ii) Ψ_ω is cyclic for π_ω, that is, $\mathscr{D}_\omega$ is dense in H_ω;
(iii) $\pi_\omega : \mathfrak{A} \to \pi_\omega(\mathfrak{A})$ is an algebra homomorphism satisfying: $\pi_\omega(\mathbb{I}) = I$ and $\pi_\omega(a^*) = \pi_\omega(a)^*\restriction_{\mathscr{D}_\omega}$, $a \in \mathfrak{A}$;
(iv) $(\Psi_\omega|\pi(a)\Psi_\omega) = \omega(a)$, $a \in \mathfrak{A}$.
(b) *If $(\mathsf{H}, \mathscr{D}, \pi, \Psi)$ fulfils (i)–(iv), there exists a unitary operator $U : \mathsf{H}_\omega \to \mathsf{H}$ such that $\Psi = U\Psi_\omega$, $\mathscr{D} = U\mathscr{D}_\omega$ and $\pi(a) = U\pi_\omega(a)U^{-1}$ for any $a \in \mathfrak{A}$.*

Now the function π_ω is not continuous (there is no preferred topology on $\mathfrak{A}$). The operators $\pi_\omega(a)$ do not belong in $\mathfrak{B}(\mathsf{H}_\omega)$, in general. Every operator $\pi_\omega(a)$ is *closable* in $\mathscr{D}_\omega$ by Theorem 5.10(b), since the domain contains the dense subspace $\mathscr{D}_\omega$, making $\pi_\omega(a)^*$ densely defined, and moreover $\overline{\pi_\omega(a^*)} \subset \pi_\omega(a)^*$.

Still referring to Theorem 14.24, it is worth remarking that elements properly corresponding to observables – i.e. self-adjoint elements $a \in \mathfrak{A}$ – are mapped to symmetric operators $\pi_\omega(a)$ by the last requirement in (iii). In general, however, even if $a = a^*$ the associated operators will not be self-adjoint: $\pi_\omega(a)^* = \pi_\omega(a)$. The weaker version $\pi_\omega(a)^{**} = \pi_\omega(a)^*$ (essential self-adjointness) should be valid for self-adjoint elements a under some condition on either the algebra and/or the state. Alas, precise technical conditions and their physical significance are poorly explored in the literature, and deserve further investigation.

A general quantum theory, formulated algebraically, seeks to find, among the immense collection of algebraic states on a C^*-algebra of observables in a given physical system, those states that bear some meaning. We refer to the aforementioned suggestions for a deep study of such a wide-ranging topic. We shall return to this point later, although usually it is the physics that suggests the choice of some privileged state ω. For instance, the reference state of Quantum Field Theories without gravity (Minkowski's flat spacetime) and without interactions is the so-called *vacuum*. The vacuum state corresponds to the absence of particles associated to the field in question, and is invariant under the Poincaré group. The picture changes abruptly when "gravity is turned on", i.e. when one introduces curvature on the spacetime: the absence of the Poincaré symmetry, in general, does not allow to select one's favourite (algebraic) state uniquely, but rather an entire class of states. Most of the times these are known as *Hadamard states* [Wal94, KhMo15]. These enable to make sense of renormalisation, and also define important observables such as the energy-momentum tensor (cf. [Mor03], for example).

14.1.6 Hilbert-Space Formulation Versus Algebraic Formulation

Withholding the point of view adopted up to Chap. 13 included, in which one starts from a given Hilbert space H_S, the C^*-algebra $\mathfrak{A}_S$ of observables associated to a system S can, in the limit situation, be the whole space of bounded operators $\mathfrak{B}(\mathsf{H}_S)$. A choice that makes more physical sense is to define the algebra of observables as a unital C^*-subalgebra in $\mathfrak{B}(\mathsf{H}_S)$: this typically has the structure of a *von Neumann algebra* (see Sect. 3.3.2) of type I when the possible superselection rules are Abelian (Proposition 11.18(a)), and is generated by the PVMs of the system's observables in accordance with Definition 3.92. However, type I arises also in the presence of a non-Abelian superselection rules for elementary particles, as briefly discussed at the end of Sect. 11.2.3.

Since $\mathfrak{A}_S$ is a von Neumann algebra and hence is strongly closed, we can integrate spectral measures, at least in bounded measurable functions, and still obtain elements of the algebra. This von Neumann algebra is actually completely determined by the associated lattice of its orthogonal projections, by Proposition 7.61(b). Taking *bounded* operators is no major restriction from the point of view of physics. Any observable A represented by an unbounded self-adjoint operator, namely, is physically the same as the sequence of observables represented by bounded self-adjoint operators $A_n := \int_{(-n,n]} \lambda dP^{(A)}(\lambda)$, $n = 1, 2, \ldots$ as discussed in Sect. 11.1.2. For the time being we will assume $\mathfrak{A}_S = \mathfrak{R}_S = \mathfrak{B}(\mathsf{H}_S)$, and later return to the general case where superselection rules are turned on.

Clearly every state $\rho \in \mathfrak{S}(\mathsf{H}_S)$ determines a (normal) algebraic state on the C^*-algebra $\mathfrak{B}(\mathsf{H}_S)$ by setting $\omega_\rho(A) := tr(\rho A)$, $A \in \mathfrak{B}(\mathsf{H}_S)$. From what we said, a state in $\mathfrak{S}(\mathsf{H}_S)$ is pure precisely when it is algebraically pure in the C^*-algebra $\mathfrak{B}(\mathsf{H}_S)$. The set of algebraic states on $\mathfrak{B}(\mathsf{H}_S)$ coming from positive trace-class operators with unit trace does not exhaust all algebraic states on $\mathfrak{B}(\mathsf{H}_S)$, but only a small part of them.

Nevertheless, viewing the C^*-algebra of observables as a specific C^*-algebra of operators on a Hilbert space (possibly the entire algebra of bounded operators) in the general framework of the algebraic formulation would be like sliding back in the theory, for it would lead to assume the theoretical existence of a privileged Hilbert space where states are described. This would rule out, for systems with infinitely many degrees of freedom, a host of states corresponding to non-unitarily equivalent representations, which do exist and have a meaning.

In the general case observables are therefore taken to form an abstract C^*-algebra $\mathfrak{A}$; the Hilbert space representation is fixed only *after* a state ω has been given, and is the Hilbert space H_ω of the GNS construction. At this point, in the Hilbert space the C^*-algebra may be enlarged to a von Neumann algebra (still C^*), simply by taking the double commutant $\pi_\omega(\mathfrak{A})''$ generated by $\pi_\omega(\mathfrak{A})$. It is here that type-III von Neumann algebras naturally arise, and may be used to represent physical systems in the peculiar physical states discussed in the second and third case at the end of Sect. 11.2.3. Notice however that as $\pi_\omega(\mathfrak{A})''$ is closed in the weak, strong and uniform topologies, there are elements in $\pi_\omega(\mathfrak{A})''$ that are *not* limits in $\pi(\mathfrak{A})$ in the uniform topology (coinciding with the topology of $\mathfrak{A}$ under π_ω). These elements do not correspond to elements of $\mathfrak{A}$, and cannot be considered, in this sense, "true observables of the system", independent of the choice of state. In particular, elementary propositions like: "the reading of a falls in the Borel set E" are not usually thinkable as elements of $\mathfrak{A}$, i.e. observables. These should correspond to maps $\chi_E(a)$, where the function of the self-adjoint element a is defined via continuous functional calculus under the representation $\Phi_a : C(\sigma(a)) \to \mathfrak{A}$ of Theorem 8.36. But $\chi_E \notin C(\sigma(a))$ in general. Strictly speaking, we can make sense of these observables only *after* having fixed a state, working in its GNS representation.

It would, actually, be possible to narrow down this gap between the two formulations in the following, a bit artificial, manner [Stre07]. From Chap. 8 we know that the integral of a bounded, measurable map in a PVM on the Hilbert space H is defined using the uniform topology, which is the natural topology of the C^*-algebra $\mathfrak{B}(\mathsf{H})$. Hence one could always ask the C^*-algebra $\mathfrak{A}$ of observables of a physical system be generated by the $p \in \mathfrak{A}$ that have the same features of orthogonal projectors in Hilbert spaces: $p = pp$ and $p^* = p$. These elements correspond to orthogonal projectors in the Hilbert space of any GNS representation of $\mathfrak{A}$. Therefore one could choose the elements p, using GNS representations of physically meaningful states, so to obtain the PVMs of the relevant observables, whence also the (bounded measurable) maps of those observables.

However, regardless any *a priori* constraint on the algebra of observables or *a posteriori* additions – a general comprehensive technical discussion on all these possibilities appears in [Emc72, Sect. 2.1.g] – something about elementary propositions can be said in the general case, as we go to illustrate (essentially following a remark in [Wal94]). Let $\mathscr{B}_s(\mathbb{R})$ be the algebra of subsets of $\mathbb{R}$ including $\emptyset$ and all finite disjoint unions of intervals $(a, b]$ and $(c, +\infty)$ with $-\infty \leq a \leq b < +\infty$ and $c \in \mathbb{R}$. One can easily prove that for every $E \in \mathscr{B}_s(\mathbb{R})$ there is a sequence of real continuous functions f_n and a constant $K < +\infty$ such that $|f_n(x)| \leq K$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and $f_n(x) \to \chi_E(x)$ for every $x \in \mathbb{R}$. The elements $f_n(a)$ are well-defined in $\mathfrak{A}$, but the sequence $\{f_n(a)\}_{n\in\mathbb{N}}$ generally does not converge in $\mathfrak{A}$. However, its GNS representation $\{\pi_\omega(f_n(a))\}_{n\in\mathbb{N}}$, referred to any fixed state ω, strongly converges to the correct element of the PVM of $\pi_\omega(a)$:

$$P_E^{(\pi_\omega(a))} = \text{s-}\lim_{n\to+\infty} \pi_\omega(f_n(a)) \ ,$$

as one easily verify (taking (i) in Theorems 8.39(b) and 8.54(c, d) into account, for $T^* = T = \pi_\omega(a)$). Even the probability that, in the state ω, the reading of a falls in the Borel set $E \in \mathscr{B}_s(\mathbb{R})$, can be computed using only that sequence of observables:

$$(\Psi_\omega | P_E^{(\pi_\omega(a))} \Psi_\omega) = \lim_{n\to+\infty} \omega(f_n(a)) \ .$$

In this sense, the sequence of abstract observables $\{f_n(a)\}_{n\in\mathbb{N}} \subset \mathfrak{A}$ embodies the entire information of the elementary proposition "the reading of a falls in the Borel set $E \in \mathscr{B}_s(\mathbb{R})$", though that proposition cannot be represented in terms of an element of $\mathfrak{A}$. Notice that two bounded sequences of continuous functions f_n and g_n, pointwise converging to the same characteristic function χ_E, for $E \in \mathscr{B}_s(\mathbb{R})$, give rise to the same $P_E^{(\pi_\omega(a))}$ for every fixed state ω. Therefore, more properly, elementary propositions are univocally represented by *equivalence classes* of sequences of observables of $\mathfrak{A}$.

Actually, the restriction $E \in \mathscr{B}_s(\mathbb{R})$ is not as strong as it could seem at first glance, since $\{P_E^{(\pi_\omega(a))}\}_{E\in\mathscr{B}_s(\mathbb{R})}$ can uniquely be extended to a PVM that, in turn, coincides with the whole PVM $\{P_F^{(\pi_\omega(a))}\}_{F\in\mathscr{B}(\mathbb{R})}$. This follows straightforwardly from Theorem 1.41, since the σ-algebra generated by $\mathscr{B}_s(\mathbb{R})$ is nothing but the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ itself.

At this juncture the abstract formulation appears to part rather evidently from the elementary formulation, which is based on a pre-existing Hilbert space and on the fundamental nature of elementary propositions about observable readings.

The process of reduction of the state, that follows the outcome of a measurement, should be treated likewise. Take an observable $a \in \mathfrak{A}$, suppose the system is in the pre-measurement state ω, and let the (ideal) reading of a fall in the Borel set E. After the measurement the state is

$$\omega_E : \mathfrak{A} \ni b \mapsto \frac{(\Psi_\omega | P_E \pi_\omega(b) P_E \Psi_\omega)}{(P_E \Psi_\omega | P_E \Psi_\omega)} ,$$

where, to simplify the notation, P_E is the PVM element of the self-adjoint operator $\pi_\omega(a)$ corresponding to the Borel set E.

An interesting theoretical question concerns the possibility to introduce the notion of *coherent superposition* of (algebraic) pure states without explicitly referring to their vector-space representation via the GNS construction, but dealing with the algebraic framework only. The answer is that this is indeed possible, though technically involved [Zan91].

14.1.7 Algebraic Abelian Superselection Rules

The algebraic formulation permits to handle situations – necessary on physical grounds, as we said – in which *non-unitarily equivalent* representations of the same algebra $\mathfrak{A}_S$ of observables of a given system S coexist. Such representations are associated to pairs of distinct algebraic states giving inequivalent GNS representations. If $\simeq$ denotes the unitary equivalence of GNS representations, we may decompose the set of pure states, i.e. of irreducible representations, in equivalence classes under the relation:

$$\omega_1 \sim \omega_2 \quad \text{if and only if} \quad \pi_{\omega_1} \simeq \pi_{\omega_2} \,. \tag{14.17}$$

These classes have a meaning in relationship to *superselection rules* (see Sects. 7.7.1, 7.7.2, 11.2.1, 11.2.2), as Haag and other mathematical physicists noticed.

To get into the matter we need to take a step back. Let us return to the standard formulation in the Hilbert space, though with algebraic focus on observables rather than the logic of admissible propositions. Consider a quantum theory that admits superselection rules. At least in some cases (Sect. 7.7.1) these require an observable Q (like the electric charge) to be always defined, with arbitrary value q, on pure normal states. We will assume for a moment that the possible values are countable (we assume $\sigma(Q) = \sigma_p(Q)$), so to have closed, pairwise orthogonal coherent sectors H_{Sq} in the separable Hilbert space H_S. The H_{Sq} are the q-eigenspaces of Q. The algebra of (bounded) observables is the von Neumann algebra $\mathfrak{A}_S := \mathfrak{R}_S$ generated by the orthogonal projectors in $\mathscr{L}(\mathsf{H}_S)$ (hence all bounded operators) that commute with the projectors P_q onto the H_{Sq}. In other words $\mathfrak{R}_S := (\{P_q \mid q \in \sigma_p(Q)\}')'' = \{P_q \mid q \in \sigma_p(Q)\}'$. Clearly $\mathfrak{R}_S$ has a non-trivial centre that contains P_q. Therefore each coherent sector is invariant under every physically admissible observable, and on every sector there will be a representation of $\mathfrak{R}_{Sq}$ obtained by restricting observables to the closed invariant space. This algebra evidently has the form

$$\mathfrak{R}_{Sq} = \{A\restriction_{\mathsf{H}_{Sq}} \mid A \in \mathfrak{R}_S\} \tag{14.18}$$

and is a von Neumann algebra on H_{Sq} as the reader can easily prove. We simultaneously have a Hilbert decomposition into pairwise orthogonal closed subspaces and a corresponding direct decomposition of von Neumann algebras

$$\mathsf{H}_S = \bigoplus_{q\in\sigma_p(Q)} \mathsf{H}_{Sq}\,, \quad \mathfrak{R}_S = \bigoplus_{q\in\sigma_p(Q)} \mathfrak{R}_{Sq}\,.$$

Every value q that Q can take gives a coherent sector H_{Sq} equipped with its own algebra $\mathfrak{R}_{Sq}$, which by (14.18) is a representation

$$\pi_q : \mathfrak{R}_S \ni A \mapsto A\restriction_{\mathsf{H}_{Sq}} \in \mathfrak{R}_{Sq}$$

of $\mathfrak{R}_S$ itself. Distinct choices of q produce *unitarily inequivalent* and *non-faithful* representations. In fact, if $q_1 \neq q_2$ on H_{Sq_1} and H_{Sq_2}, Q is represented by different multiples of the identity $q_1 I$ and $q_2 I$. Hence $Uq_1IU^{-1} = q_1 I \neq q_2 I$ whichever unitary map $U : \mathsf{H}_{Sq_1} \to \mathsf{H}_{Sq_2}$ we take. Non-faithfulness arises form the fact that if Q assumes the value q on H_{Sq}, then $Q - qI \neq 0$ but $\pi_q(Q - qI) = 0$.

If more than one superselection rule is activated, Wightman [Wigh95] conjectured that the rules are associated to a finite set of pairwise compatible observables $Q_1, \ldots, Q_n$ in the centre of $\mathfrak{R}_S$ (we again assume $\sigma(Q_j) = \sigma_p(Q_j)$ and that the charges are bounded operators. The latter requirement can easily be relaxed by assuming that the charges are simply affiliated to $\mathfrak{R}_S$). Supposing Wightman is right (see however Remark 11.22 and Sect. 11.2.3), the Hilbert space splits in an orthogonal sum of coherent sectors H_{S_k}, $k \in K$, common to all superselection rules. Every k is completely fixed by the values that all charges Q_j assume on H_{S_k}, and for every set of these values there is a $k \in K$ defining a superselection sector H_{S_k} (K is a countable set, or even finite if the spectrum of each Q_j is finite). On each sector we have a representation π_k of the algebra $\mathfrak{R}_S$ of observables of the system. These representations are mutually inequivalent, as we have seen.

If we assume, more restrictively, that the family $\{Q_1, \ldots, Q_n\}$ completely describes all superselection rules of the system – in this case we say that the superselection rules are *Abelian* (see Sect. 11.2.2), then we are committed to suppose that every other central observable must be a function of them. In other words

$$\{Q_1, \ldots, Q_n\}'' = \mathfrak{R}_S \cap \mathfrak{R}'_S\,.$$

It is possible to prove that this hypothesis is equivalent to requiring that the orthogonal projectors $P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}$ generating the joint spectral measure of the Q_j are *atoms* of the centre of the lattice $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ of projectors of $\mathfrak{R}_S$ (Proposition 11.21). As we said before, the family of central projectors $\{P_k\}_{k\in K}$ is nothing but the family of spectral projectors

$$\left\{P^{(Q_1)}_{q_1} \cdots P^{(Q_n)}_{q_n}\right\}_{(q_1,\ldots,q_n)\in\sigma(Q_1)\times\cdots\times\sigma(Q_n)} \ ,$$

and $k \in K$ (a subset of $\mathbb{N}$) simply labels the values $(q_1, \ldots, q_n) \in \sigma(Q_1) \times \cdots \times \sigma(Q_n)$. In this way we recover the description of superselection rules of Sects. 7.7.1, 7.7.2 and 11.2.1. We therefore have decompositions

$$\mathsf{H}_S = \bigoplus_{k\in K} \mathsf{H}_{Sk} \ , \quad \mathfrak{R}_S = \bigoplus_{k\in K} \mathfrak{R}_{Sk} \ , \tag{14.19}$$

where the algebras $\mathfrak{R}_{Sk}$ are now *factors*, as we know from Sect. 7.7.1.

If we further assume that $\mathfrak{R}_S$ contains a *complete set of commuting observables*, we eventually have that $\mathfrak{R}_{Sk} = \mathfrak{B}(\mathsf{H}_{Sk})$ and $\mathfrak{R}_S$ is of type I, by Proposition 11.18. Each representation $\pi_k : \mathfrak{R}_S \ni A \mapsto A\restriction_{\mathsf{H}_{Sk}} \in \mathfrak{R}_{Sk}$ of $\mathfrak{R}_S$ on each superselection sector is also *irreducible*. The representations π_k are therefore non-faithful, *unitarily inequivalent irreducible representations* of $\mathfrak{R}_S$ labelled by the values of the superselection charges Q_j.

States over $\mathfrak{R}_S$, in the sense of the non-algebraic formulation, can be described in terms of σ-additive probability measures over $\mathscr{L}_{\mathfrak{R}_S}(\mathsf{H}_S)$ as discussed in Sects. 7.7.1, 7.7.2. In turn, these measures can be identified with positive trace-class operators of trace one using the generalisation of Gleason's theorem, as discussed in Sect. 7.7.2. (All that happens in accordance with Propositions 14.11, 7.70, 7.72 and Remark 7.73, when we assume H_S separable with $\dim \mathsf{H}_{Sk} \neq 2$ for every $k \in K$). These are *normal states* of $\mathfrak{R}_S$ from the algebraic viewpoint, and form a convex set $\mathfrak{S}(\mathsf{H}_S)_{adm}$ whose subset $\mathfrak{S}_p(\mathsf{H}_S)_{adm}$ of extreme elements still contains normal states, represented by the unit vectors of the sectors H_{Sk}.

Let us pass to the algebraic formulation, based on a C^*-algebra of observables and the general notion of algebraic state – thus freeing ourselves from Hilbert spaces and von Neumann algebras as the characterising structures of a physical system. The picture now is suddenly more straightforward, for the use of C^*-algebras eschews convoluted arguments and technical complications. Extending Wightman's assumption, in the algebraic formalism, superselection rules are accounted for by observables Q in the **centre** of the C^*-algebra $\mathfrak{A}_S$ of observables, i.e. the subalgebra of elements commuting with all of $\mathfrak{A}_S$. Every pure algebraic state ω, corresponding to an irreducible representation of the algebra of observables, must inevitably select a value of Q in the GNS representation by Schur's lemma, as $\pi_\omega(Q)$ commutes with all elements. That is to say, $\pi_\omega(Q) = qI$ for some $q \in \mathbb{R}$ (now the values may be uncountable, since the separable Hilbert space is not unique). Exactly as before, two pure algebraic states ω, ω' with distinct $q \neq q'$ produce inequivalent GNS representations, so there is no unitary operator $U : \mathsf{H}_\omega \to \mathsf{H}_{\omega'}$ such that $U\pi_\omega(a)U^{-1} = \pi_{\omega'}(a)$ for each $a \in \mathfrak{A}_S$ (this identity is false for $a = Q$).

In general we expect that families of non-equivalent pure states (i.e. of inequivalent irreducible representations) can be labelled by distinct values of a charge of sorts, corresponding to some central observable. Eventually, at least for physically important theories, the existence of superselection charges could be the reason for the existence of inequivalent irreducible representations of the C^*-algebra of observables $\mathfrak{A}_S$ [Rob04], though no proof of this conjecture exists.

Remark 14.25 **(1)** The aforementioned interesting conjecture posits that irreducible inequivalent representations of a given C^*-algebra of observables are due to superselection rules; these, in turn, arise from superselection charges. This is untenable for elementary, and hence maybe un-physical, cases. In fact, the representations π_ω of $\mathfrak{A}_S$ associated to different values of Q, if any, are non-faithful since $\pi_\omega(Q - q\mathbb{I}) = 0$ if q is the value of Q in the GNS representation of the state ω. If the algebra $\mathfrak{A}_S$ is *simple*, all representations are faithful (Proposition 14.23) hence no superselection charge Q may exist, *even if unitarily inequivalent irreducible representations do exist*. This is the case for the *Weyl C^*-algebra* we will discuss shortly (see Lemma 14.37). However one expects that adding further elements to the algebra of observables which are natural in the von Neumann algebras of the GNS representations (as the so-called *number of particles* for the Weyl C^*-algebra) will allow to restore a description of the superselection rules in terms of superselection charges.
(2) In principle, further superselection rules can anyway show up in a specific GNS representation of $\mathfrak{A}_S$ associated to a state ω, in case we think of the algebra, in such representation, as the von Neumann algebra $\pi_\omega(\mathfrak{A}_S)''$. This is larger than $\pi_\omega(\mathfrak{A}_S)$, so in general it has a non-trivial centre even if $\mathfrak{A}_S$ does not. (See [Pri00] for this point, in particular concerning the interpretation of central observables of $\pi_\omega(\mathfrak{A}_S)''$ as classical observables.) ■

Even in the absence of superselection charges with clear physical meaning, it is natural to think of the GNS representations of the algebra $\mathfrak{A}_S$ associated with *unitarily inequivalent pure states ω* as *algebraic superselection sectors*. For this, let us look at the equivalence relation (14.17), denote by Ω the space of equivalence classes, and pick a representative ω in every equivalence class $[\omega]$. Then the Hilbert sum of the unitarily inequivalent irreducible GNS representation spaces H_ω,

$$\mathsf{H}_\Omega = \bigoplus_{[\omega]\in\Omega} \mathsf{H}_\omega \,,$$

can be viewed as the Hilbert space of the system in the presence of superselection rules, each H_ω playing the role of a superselection sector. Indeed, the unit-vector states in all sectors exhaust the set of all possible algebraic pure states on $\mathfrak{A}$ by construction. Moreover, coherent superpositions

$$\alpha\Psi + \beta\Psi'$$

of unit vectors $\Psi \in \mathsf{H}_\omega$, $\Psi' \in \mathsf{H}_{\omega'}$ in *different* sectors (with $|\alpha|^2 + |\beta|^2 = 1$) and incoherent superpositions

$$|\alpha|^2\Psi(\Psi|\cdot) + |\beta|^2\Psi'(\Psi'|\cdot)$$

define the same algebraic state. This is because they are indistinguishable when acting on elements $a \in \mathfrak{A}_S$ (represented on H_Ω through the GNS representation of the case):

$$\left(\alpha\Psi + \beta\Psi' \left| \pi_\omega(a) \oplus \pi_{\omega'}(a)(\alpha\Psi + \beta\Psi')\right.\right) = |\alpha|^2(\Psi|\pi_\omega(a)\Psi) + |\beta|^2(\Psi'|\pi_{\omega'}(a)\Psi') .$$

The algebraic mixed states on $\mathfrak{A}_S$ can be obtained as weak limits of incoherent superpositions of unit vectors belonging to the spaces H_ω, due to Theorem 14.13.

There are, however, crucial differences in comparison with the genuine Hilbert-space description of superselection rules previously summarised.

(1) In general, H_Ω is not separable and Ω is uncountable.

(2) Not all (algebraic) mixed states are represented by trace-class operators on H_Ω, though they are elements of the *-weak closure of the convex body generated by the vector states of the sectors H_Ω.

(3) $\mathfrak{A}_S$ is a C^*-algebra, rather than a von Neumann algebra.

(4) In general, there are no elements in $\mathfrak{A}_S$ corresponding to orthogonal projectors onto coherent sectors H_ω. This contrasts with the standard Hilbert-space formulation, where such elements are the central atomic projectors $P_k \in \mathfrak{R}_S$.

(5) A consequence of (4) is that an identity such as

$$\mathfrak{A}_S = \bigoplus_{[\omega]\in\Omega} \pi_\omega(\mathfrak{A}_S) \quad \text{(false)}$$

is generally false, whereas the analogue

$$\mathfrak{R}_S = \bigoplus_{k\in K} \pi_k(\mathfrak{R}_S) = \bigoplus_{k\in K} \mathfrak{R}_{Sk} ,$$

holds for the standard Hilbert-space formulation of superselection rules.

(6) As a second consequence of (4), the representations π_ω of $\mathfrak{A}_S$ may be faithful, again differently form the analogous representations π_k of $\mathfrak{R}_S$ in the standard Hilbert-space formulation, which are never faithful.

Remark 14.26 **(1)** If the set Ω has the cardinality of $\mathbb{R}$ at most, it would be also possible to enlarge $\mathfrak{A}$ by adding an observable Q to its centre, which is represented by qI in each sector, with different q for different inequivalent sectors. Such a Q would however merely be an artificial construction of dubious physical meaning.
(2) We stress that there is no real need for reformulating all the algebraic machinery backwards in the Hilbert space H_Ω. Its introduction was only meant to point out the analogies between the algebraic formulation and the Hilbert-space formulation in relationship to the notion of coherent sectors. On the other hand, the crucial differences listed above would definitely make a full Hilbert-space reformulation, in particular without artificially extending $\mathfrak{A}_S$, impossible. ■

The algebraic interpretation of superselection sectors that we have discussed is appropriate for elementary systems, i.e., *elementary particles* in absence of non-Abelian internal gauge symmetry. Here, pure states, vacuum states and ground states in particular play a crucial role. The picture is the natural algebraic extension of the case of *Abelian superselection rules*, introduced in Sects. 11.2.1 and 11.2.2 within the Hilbert space formulation. For other physically important systems different from elementary particles, like finite-temperature systems in the so-called *thermodynamical limit*, a more appropriate algebraic description of superselection sectors exists. It is provided by unitary equivalence classes of GNS representations of states ω (satisfying the so-called KMS condition) where the irreducibility condition $\pi_\omega(\mathfrak{A})' = \{cI\}_{c\in\mathbb{C}}$ is relaxed to the *factorial* condition $\pi_\omega(\mathfrak{A})' \cap \pi_\omega(\mathfrak{A})'' = \{cI\}_{c\in\mathbb{C}}$. This is the algebraic analogue of *non-Abelian superselection rules* in the Hilbert-space formulation, where the von Neumann algebra of observables in each sector is a factor (usually of type III) with non-trivial commutant. The aforementioned algebraic states states ω still describe some particular physical situation, as opposed to generic mixtures. For instance, in case of thermodynamical equilibrium, some such factorial states – with $\pi_\omega(\mathfrak{A})' \neq \{cI\}_{c\in\mathbb{C}}$ – describe *pure phases* of a quantum system in the thermodynamical limit.

14.1.8 Fell's Theorem

We suggest to consult [Haa96, Rob04, Ear08] to find lucid reviews on the algebraic formalism and superselection sectors, especially for local theories based on *nets* of observable algebras arising from Quantum Field Theory. Let us just make here one general comment. We saw how the space of pure states decomposes in disjoint families of states giving inequivalent representations, and the states of a same family can be viewed as state vectors on one Hilbert space. So we would like to know if it is possible, experimentally speaking, to say to which family a given pure state ω belongs to. The answer is not simple, as shown by a theorem proved by Fell. For pure states on a C^*-algebra with unit, Fell's theorem says that pure states in a given family are dense in the set of all pure states for the *-weak topology. Let us explain why this abstract fact is relevant. In the real world we can conduct only a finite (arbitrarily large) number of experiments. Suppose we can measure N observables $a_1, a_2, \ldots, a_N$. The accuracy is finite, so the true value α_i of the reading $\omega(a_i)$ of a_i is only given up to $\varepsilon_i > 0$:

$$|\omega(a_i) - \alpha_i| < \varepsilon_i \ , \quad i = 1, 2, \ldots, N \ .$$

Now observe that the numbers α_i and ε_i determine a neighbourhood in the space of states with respect to the *-weak topology. Fell's result implies that it is not possible to establish to which family a given pure state ω belongs by using an arbitrarily large, but finite, number of measurements with however small, yet finite, errors.

One way to simplify the problem [Haa96] is to choose *a priori* a family of pure states with some *ad hoc* criterion. Supposing, for instance, that the algebra is spatially localisable, we may assume that outside a certain region the physical system is absent. Then all states of interest are those that outside a given and arbitrarily large region resembling the vacuum state, when we measure on it observables localised outside the region.

14.1.9 Proof of the Gelfand-Najmark Theorem, Universal Representations and Quasi-equivalent Representations

The GNS construction has a purely mathematical consequence known as the *Gelfand-Najmark theorem* (stated in Chap. 8). It says that every abstract C^*-algebra with unit can be realised as a C^*-algebra of operators on a Hilbert space, albeit not uniquely. To prove the result we need a few technical lemmas.

Lemma 14.27 *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{I}$. Any bounded linear functional $\phi : \mathfrak{A} \to \mathbb{C}$ with $\phi(\mathbb{I}) = ||\phi||$ is positive.*

Proof We will make use of Theorem 8.25, and, a usual, $r(c)$ will denote the spectral radius of c. Without loss of generality we assume $\phi(\mathbb{I}) = 1$. Let $a \in \mathfrak{A}$ be positive and set $\phi(a) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$. We have to show $\alpha \geq 0$ and $\beta = 0$. For small $s \geq 0$ we have $\sigma(\mathbb{I} - sa) = \{1 - st \mid t \in \sigma(a)\} \subset [0, 1]$, since $\sigma(a) \subset [0, +\infty)$. Hence $||\mathbb{I} - sa|| = r(\mathbb{I} - sa) \leq 1$. Therefore $1 - s\alpha \leq |1 - s(\alpha + i\beta)| = |\phi(\mathbb{I} - sa)| \leq 1$, so $\alpha \geq 0$. Now define $\beta_n := a - \alpha\mathbb{I} + in\beta\mathbb{I}$, $n = 1, 2, \ldots$. Then

$$||b_n||^2 = ||b_n^* b_n|| = ||(a - \alpha\mathbb{I})^2 + n^2\beta^2\mathbb{I}|| \leq ||a - \alpha\mathbb{I}||^2 + n^2\beta^2 .$$

Consequently

$$(n^2 + 2n + 1)\beta^2 = |\phi(b_n)|^2 \leq ||a - \alpha\mathbb{I}||^2 + n^2\beta^2 \quad n = 1, 2, \ldots$$

and then $\beta = 0$. □

Lemma 14.28 *Let $\mathfrak{A}$ be a C^*-algebra with unit and $a \in \mathfrak{A}$.*
(a) *If $\alpha \in \sigma(a)$ there exists a state $\phi : \mathfrak{A} \to \mathbb{C}$ such that $\phi(a) = \alpha$.*
(b) *If $a \neq 0$, there exists a state $\phi : \mathfrak{A} \to \mathbb{C}$ with $\phi(a) \neq 0$.*
(c) *If $a = a^*$, there exists a state $\phi : \mathfrak{A} \to \mathbb{C}$ such that $|\phi(a)| = ||a||$.*

Proof (a) For any complex numbers β, γ we have $\alpha\beta + \gamma \in \sigma(\beta a + \gamma\mathbb{I})$, so $|\alpha\beta + \gamma| \leq ||\beta a + \gamma\mathbb{I}||$. Hence asking $\phi(\beta a + \gamma\mathbb{I}) := \alpha\beta + \gamma$ defines (unambiguously) a linear functional on the subspace $\{\beta a + \gamma\mathbb{I} \mid \beta, \gamma \in \mathbb{C}\}$ such that $\phi(a) = \alpha, \phi(\mathbb{I}) = 1$ and $||\phi|| = 1$. By a corollary to the Hahn-Banach theorem, we can extend ϕ to a

continuous linear functional on $\mathfrak{A}$ satisfying $||\phi|| = \phi(\mathbb{I}) = 1$. The previous lemma guarantees that the functional is a state on $\mathfrak{A}$ with $\phi(a) = \alpha$.
(b) If $a = a^*$ and $a \neq 0$, then $\sigma(a) \neq \{0\}$, for otherwise the properties of the spectral radius of self-adjoint elements would imply $||a|| = r(a) = 0$. Then the state ϕ of part (a) satisfies $\phi(a) \neq 0$ for $\alpha \in \sigma(a) \setminus \{0\}$. Consider when $a \neq a^*, a \neq 0$. Then we can decompose $a = b + ic$ with $b = b^*$, $c = c^*$. If $\phi(a) = 0$ for any state $\phi : \mathfrak{A} \to \mathbb{C}$, we would have $0 = \phi(a) = \phi(b) + i\phi(c)$ for any ϕ. But the GNS theorem implies $\phi(d) = \overline{\phi(d)}$ for $d = d^*$. Hence $\phi(b) = \phi(c) = 0$ for any ϕ. Since c and d are self-adjoint, the proof's initial argument forces $b = c = 0$ so $a = 0$. As this was excluded, there must exist a state with $\phi(a) \neq 0$.
(c) Since $||a|| = \sup\{|\lambda| \mid \lambda \in \sigma(a)\}$ and $\sigma(a)$ is compact in $\mathbb{R}$, there must be an element $\Lambda \in \sigma(a)$ with $|\Lambda| = ||a||$. Using part (a) with $\alpha = \Lambda$ proves the claim. □

Now we are ready to state and prove the Gelfand-Najmark theorem.

Theorem 14.29 (Gelfand-Najmark) *For any unital C^*-algebra $\mathfrak{A}$ there exist a Hilbert space H and an (isometric) *-isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{B} \subset \mathfrak{B}(\mathsf{H})$ is a C^*-subalgebra of $\mathfrak{B}(\mathsf{H})$.*

Proof For every $x \in \mathfrak{A} \setminus \{0\}$ let us fix a state $\phi_x : \mathfrak{A} \to \mathbb{C}$ with $\phi_x(x) \neq 0$. This state exists by part (b) of the above lemma. Consider the collection of GNS triples $(\mathsf{H}_x, \pi_x, \Psi_x)$ associated to each ϕ_x, and the Hilbert sum

$$\mathsf{H} := \bigoplus_{x \in \mathfrak{A}\setminus\{0\}} \mathsf{H}_x \,.$$

In this way the elements of H are of the form $\psi = \oplus_{x\in\mathfrak{A}\setminus\{0\}} \psi_x := \{\psi_x\}_{x\in\mathfrak{A}\setminus\{0\}}$ with:

$$\sum_{x\in\mathfrak{A}\setminus\{0\}} ||\psi_x||_x^2 < +\infty \,. \tag{14.20}$$

On H we have an inner product making it a Hilbert space:

$$(\psi|\psi') = \sum_{x\in\mathfrak{A}\setminus\{0\}} (\psi_x|\psi'_x)_x \,.$$

Define the map $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ by imposing:

$$\pi(0) := 0 \quad \text{and} \quad (\pi(a)\psi)_x := \pi_x(a)\psi_x \quad \text{for } \psi \in \mathsf{H}, a \in \mathfrak{A} \setminus \{0\} \,.$$

It is not hard to see π is a *-homomorphism of unital C^*-algebras mapping $\mathfrak{A}$ to $\mathfrak{B}(\mathsf{H})$. By Theorem 8.22(c) $\pi(\mathfrak{A})$ is a unital C^*-algebra subalgebra of $\mathfrak{B}(\mathsf{H})$. In particular, $||\pi(a)|| \leq ||a||$, as prescribed by Theorem 8.22. In fact if (14.20) holds, since Theorem 8.22 gives $||\pi_x(a)|| \leq ||a||$, we obtain

$$||\pi(a)\psi||^2 = \sum_{x\in\mathfrak{A}\setminus\{0\}} ||\pi_x(a)\psi_x||_x^2 \le ||a||^2 \sum_{x\in\mathfrak{A}\setminus\{0\}} ||\psi_x||_x^2 = ||a||^2||\psi||^2 < +\infty\,.$$

To end the proof it suffices to show π is isometric. By Theorem 8.22(a) that is equivalent to injectivity. Suppose $\pi(a) = 0$, so $\pi_x(a)\psi_x = 0$ for any $x \in \mathfrak{A} \setminus \{0\}$, $\psi_x \in \mathsf{H}_x$. In particular $\phi_x(a) = (\Psi_x|\pi_x(a)\Psi_x) = 0$, so choosing $x = a$ gives $\phi_a(a) = 0$. But this is not possible if $a \neq 0$. Therefore $a = 0$ and π is one-to-one, hence isometric. □

One elementary corollary of the Gelfand-Najmark theorem, in particular, has physical interest: it proves that the norm of a unital C^*-algebra is determined by the whole collection of states.

Corollary 14.30 *Let $\mathfrak{A}$ be a C^*-algebra with unit $\mathbb{I}$ and denote by $S(\mathfrak{A})$ the collection of algebraic states on $\mathfrak{A}$. Then:*

$$||a|| = \sup_{\omega\in S(\mathfrak{A})} |\omega(a)| \quad \textit{if } a^* = a \in \mathfrak{A}. \tag{14.21}$$

Consequently:

$$||a||^2 = \sup_{\omega\in S(\mathfrak{A})} |\omega(a^*a)| \quad \textit{if } a \in \mathfrak{A}. \tag{14.22}$$

Finally, if $a, b \in \mathfrak{A}$:

$$a = b \quad \textit{if and only if} \quad \omega(a) = \omega(b) \quad \textit{for every } \omega \in S(\mathfrak{A}).$$

Proof Let us start from the first couple of statements. The second is an immediate consequence of the first since $||a||^2 = ||a^*a||$ and $(a^*a)^* = a^*a$, so we have to prove the former only. To this end, in view of the fact that states are positive hence continuous, and that $||\omega|| = \omega(\mathbb{I}) = 1$ due to Theorem 14.6, we have:

$$\sup_{\omega\in S(\mathfrak{A})} |\omega(a)| \le \sup_{\omega\in S(\mathfrak{A})} ||\omega||\,||a|| \le \sup_{\omega\in S(\mathfrak{A})} 1\,||a||\,.$$

We conclude that:

$$\sup_{\omega\in S(\mathfrak{A})} |\omega(a)| \le ||a||\,.$$

To obtain the opposite inequality, we pass to the C^*-algebra $\mathfrak{B} \subset \mathfrak{B}(\mathsf{H})$ representing $\mathfrak{A}$ isometrically under the action of the *-isomorphism ϕ (Gelfand-Najmark theorem). Since $\phi(a) \in \mathfrak{B}(\mathsf{H})$ is self-adjoint, Proposition 3.60(a) yields:

$$||a|| = ||\phi(a)|| = \sup\{|(\psi|\phi(a)\psi)| \mid \psi \in \mathsf{H}\,, ||\psi|| = 1\} = \sup_{||\psi||=1} |\omega_\psi(a)|$$

$$\le \sup_{\omega\in S(\mathfrak{A})} |\omega(a)|\,,$$

where we have exploited the fact that, for $||\psi|| = 1$, the linear map

$$\mathfrak{A} \ni b \mapsto \omega_\psi(b) := (\psi|\phi(b)\psi)$$

is an algebraic state. The proof of the first (and second) statement in the thesis ends here, since we have established that:

$$\sup_{\omega \in S(\mathfrak{A})} |\omega(a)| \le ||a|| \le \sup_{\omega \in S(\mathfrak{A})} |\omega(a)| \,.$$

Concerning the last item in the thesis the only implication to prove is that $a = b$ if $\omega(a) = \omega(b)$ for every $\omega \in S(\mathfrak{A})$. Let us prove it. If $\omega(a) = \omega(b)$ then $\overline{\omega(a)} = \overline{\omega(b)}$, so that $\omega(a^*) = \omega(b^*)$ by the GNS construction. Therefore: $\omega(a + a^* - (b + b^*)) = 0$ and $\omega(i(a - a^*) - i(b - b^*)) = 0$. Since the arguments of ω are self-adjoint in both cases and ω is arbitrary, the first statement of the corollary implies $a - b + (a^* - b^*) = 0$ and $a - b - (a^* - b^*) = 0$, which entail $a - b = 0$. $\square$

The Gelfand-Najmark theorem enables us to introduce an extremely useful technical tool, the *universal representation* of a unital C^*-algebra.

Let $\mathfrak{A}$ be a C^*-algebra with unit and denote by $S(\mathfrak{A}) \subset \mathfrak{A}'$ the convex set of its algebraic states. Take the GNS representation $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$ of state $\omega \in S(\mathfrak{A})$ and consider the Hilbert sum $\bigoplus_{\omega \in S(\mathfrak{A})} \mathsf{H}_\omega$. Its elements $\oplus_{\omega \in S(\mathfrak{A})} \psi_\omega := \{\psi_\omega\}_{\omega \in S(\mathfrak{A})}$ satisfy

$$\sum_{\omega \in S(\mathfrak{A})} ||\psi_\omega||^2_\omega < +\infty \,. \tag{14.23}$$

The space $\bigoplus_{\omega \in S(\mathfrak{A})} \mathsf{H}_\omega$ is a Hilbert space for the inner product

$$(\psi|\psi') = \sum_{\omega \in S(\mathfrak{A})} (\psi_\omega|\psi'_\omega)_\omega \,.$$

The **universal representation** of $\mathfrak{A}$ is the representation:

$$\Pi : \mathfrak{A} \to \mathfrak{B}\left(\bigoplus_{\omega \in S(\mathfrak{A})} \mathsf{H}_\omega\right) \quad \text{given by} \quad \Pi\left(\bigoplus_{\omega \in S(\mathfrak{A})} \psi_\omega\right) := \bigoplus_{\omega \in S(\mathfrak{A})} \pi_\omega(a)\psi_\omega \,.$$

Definition 14.31 Let $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ be a representation of the *-algebra $\mathfrak{A}$ on the Hilbert space H. A **subrepresentation** of π is a representation of the form $\pi\restriction_{\mathsf{H}_0}: \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_0)$, where the subspace $\mathsf{H}_0 \subset \mathsf{H}$ is closed and π-invariant.

Clearly any GNS representation of a C^*-algebra with unit is a subrepresentation of the universal representation. Then the next easy, but useful, fact holds.

Proposition 14.32 *The universal representation of any given C^*-algebra with unit is faithful and isometric.*

Proof That a representation is faithful implies, by Theorem 8.22, that it is isometric. Faithfulness descends immediately from the fact that Π, as a subrepresentation, contains the representation π used in the proof of the Gelfand-Najmark theorem (the latter is injective). $\square$

Eventually we mention a result on the structure of the folium of an algebraic state. First, a notation and an important definition.

Notation 14.33 Let $\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H})$ be a representation of the *-algebra $\mathfrak{A}$, and n a cardinal number. We denote by $n\pi$ the representation on $\bigoplus_{i=1}^n \mathsf{H}_i$, $\mathsf{H}_i := \mathsf{H}$ defined by

$$n\pi(a)(\oplus_{i=1}^n \psi_i) := \oplus_{i=1}^n \pi(a)\psi_i \quad \text{for any } a \in \mathfrak{A},\ \psi_i \in \mathsf{H}\,.$$

■

Definition 14.34 Two representations $\pi_1 : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_1)$, $\pi_2 : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}_2)$ of the same *-algebra $\mathfrak{A}$ are called **quasi-equivalent**, written

$$\pi_1 \approx \pi_2\,,$$

if they are unitarily equivalent up to multiplicities. Equivalently, there exists cardinals n_1, n_2 such that $n_1\pi_1 \simeq n_2\pi_2$.

For example

$$\pi : \mathfrak{A} \to \mathfrak{B}(\mathsf{H}) \quad \text{and} \quad \pi_1 : \mathfrak{A} \ni a \mapsto \pi(a) \oplus U\pi(a)U^{-1} \in \mathfrak{B}(\mathsf{H} \oplus \mathsf{H}')$$

are quasi-equivalent if $U : \mathsf{H} \to \mathsf{H}'$ is a unitary operator. Unitarily equivalent representations are obviously quasi-equivalent. And quasi-equivalence is an equivalence relation. About this (see [Haa96] and [BrRo02, vol. 1]) we have

Proposition 14.35 *Let $\mathfrak{A}$ be a C^*-algebra with unit and $\omega : \mathfrak{A} \to \mathbb{C}$ an algebraic state with GNS representation π_ω.*
(a) *If π_1 and π_2 are representations of $\mathfrak{A}$, $\pi_1 \approx \pi_2$ if and only if the von Neumann algebras $\pi_1(\mathfrak{A})''$, $\pi_2(\mathfrak{A})''$ are isomorphic as *-algebras, and the *-isomorphism restricts to a *-isomorphism $\pi_1(\mathfrak{A}) \to \pi_2(\mathfrak{A})$.*
(b) *The GNS representations of $\mathfrak{A}$ generated by states in $Fol(\omega)$ are quasi-equivalent. In particular, if $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, with $\lambda \in (0,1)$ and $\omega_1 \neq \omega$, the GNS representation of ω_1 is unitarily equivalent to a GNS subrepresentation of ω.*
(c) *If π is a representation of $\mathfrak{A}$ and $\pi \approx \pi_\omega$, then π is a GNS representation of a state in $Fol(\omega)$.*

14.2 Example of a C^*-Algebra of Observables: The Weyl C^*-Algebra

This section is devoted to the simplest non-trivial C^*-algebra of observables used in physics. We are talking about the Weyl C^*-algebra involved in the description of several systems, among which are non-interacting bosonic quantum systems. Almost all systems that are describable using a Weyl C^*-algebra can also be described by weakening the observables' structure to a *-algebra. Yet Weyl C^*-algebras are mathematically attractive, which motivates our interest.

*14.2.1 Further Properties of Weyl *-Algebras $\mathscr{W}(\mathsf{X}, \sigma)$*

Keeping in mind Sect. 11.5.4, let (X, σ) be a symplectic space: a pair consisting of a real vector space X of even dimension (possibly infinite), henceforth non-trivial, and a weakly non-degenerate symplectic form $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$. Let $\mathscr{W}(\mathsf{X}, \sigma)$ denote the Weyl *-algebra of (X, σ) introduced in Definition 11.47. We know (Theorem 11.48) $\mathscr{W}(\mathsf{X}, \sigma)$ is defined up to *-isomorphisms. We wish to explain that it is possible, and in a unique way, to enlarge $\mathscr{W}(\mathsf{X}, \sigma)$ to a C^*-algebra called the *Weyl C^*-algebra associated to* (X, σ). More precisely, we will define on $\mathscr{W}(\mathsf{X}, \sigma)$ a unique norm satisfying the C^* property $||a^*a|| = ||a||^2$. The Weyl C^*-algebra will be the completion of $\mathscr{W}(\mathsf{X}, \sigma)$ for that norm. In order to prove all this we need a few preliminary facts that form the contents of the section. There are various procedures, and distinct (equivalent) formulations, that prove the ensuing properties (see [BrRo02] in particular). We will essentially follow the approach of [BGP07].

Lemma 14.36 *Let X be a non-trivial real vector space, $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ a weakly non-degenerate symplectic form, and consider a Weyl *-algebra $\mathscr{W}(\mathsf{X}, \sigma)$ associated to the system.*
(a) *There exists a norm $||\ ||$ on $\mathscr{W}(\mathsf{X}, \sigma)$ satisfying the C^* property: $||a^*a|| = ||a||^2$ for any $a \in \mathscr{W}(\mathsf{X}, \sigma)$.*
(b) *If $\psi \in \mathsf{X}$, the generator $W(\psi)$ is unitary, so for the above norm $||W(\psi)|| = 1$.*
(c) *If $\psi, \phi \in \mathsf{X}$, $\psi \neq \phi$, in the above norm*

$$||W(\psi) - W(\phi)|| = 2\,,$$

so $\mathscr{W}(\mathsf{X}, \sigma)$ is not separable.
(d) *If we set, for any $a \in \mathscr{W}(\mathsf{X}, \sigma)$:*

$$||a||_c := \sup\{p(a) \mid p : \mathscr{W}(\mathsf{X}, \sigma) \to [0, +\infty) \ \textit{is a } C^*\textit{norm}\}\,,$$

then $||\ ||_c$ is a C^* norm.

Proof (a) Let us focus again on the construction of $\mathscr{W}(\mathsf{X},\sigma)$ of Theorem 11.48(a). Consider the complex Hilbert space $\mathsf{H} := L^2(\mathsf{X},\mu)$ where μ is the counting measure on X. For $u \in \mathsf{X}$ the operators $W(u) \in \mathfrak{B}(L^2(\mathsf{X},\mu))$, $(W(u)\psi)(v) := e^{i\sigma(u,v)/2}$ $\psi(u+v)$ for $\psi \in L^2(\mathsf{X},\mu)$, $v \in \mathsf{X}$, define a Weyl *-algebra associated to (X,σ): $\mathscr{W}(\mathsf{X},\sigma) \subset \mathfrak{B}(\mathsf{H})$. The norm $||\ ||$ of $\mathfrak{B}(\mathsf{H})$ satisfies the C^* property. Starting from a different representation $\mathscr{W}'(\mathsf{X},\sigma)$, $||\ ||$ induces a C^* norm on $\mathscr{W}'(\mathsf{X},\sigma)$ by means of the *-isomorphism $\alpha : \mathscr{W}(\mathsf{X},\sigma) \to \mathscr{W}'(\mathsf{X},\sigma)$ of Theorem 11.48(c).
(b) From the Weyl relations we know $W(\psi)W(\psi)^* = W^*(\psi)W(\psi) = \mathbb{I}$, so $W(\psi)$ is unitary. The C^* property implies $||W(\psi)|| = 1$.
(c) Let us complete $\mathscr{W}(\mathsf{X},\sigma)$ with respect to the norm $||\ ||$ of (a), so to obtain a C^*-algebra. By Weyl's relations we have $W(\chi)W(\phi-\psi)W(\chi)^{-1} = e^{-i\sigma(\chi,\phi-\psi)}W(\phi-\psi)$. Since $W(\phi-\psi)$ is unitary, $\sigma(W(\phi-\psi)) \subset \{z \in \mathbb{C} \mid |z| = 1\}$. By definition of spectrum

$$\sigma(W(\chi)W(\phi-\psi)W(\chi)^{-1}) = \sigma(W(\phi-\psi)) = e^{-i\sigma(\chi,\phi-\psi)}\sigma(W(\phi-\psi))\,.$$

Since $\psi \neq \phi$, $\sigma(\chi,\phi-\psi)$ covers the whole $\mathbb{R}$ as χ varies in X. Hence $\sigma(W(\phi-\psi)) = \{z \in \mathbb{C} \mid |z| = 1\}$. Therefore $\sigma(e^{i\sigma(\psi,\phi)}W(\phi-\psi)-\mathbb{I})$ is the unit circle in $\mathbb{C}$ centred at -1, so if r is the spectral radius, then $r(e^{i\sigma(\psi,\phi)}W(\phi-\psi)) = 2$. But $e^{i\sigma(\psi,\phi)}W(\phi-\psi)$ is normal: $2 = r(e^{i\sigma(\psi,\phi)}W(\phi-\psi)) = ||e^{i\sigma(\psi,\phi)}W(\phi-\psi)-\mathbb{I}||$. Using the norm's C^* property and the generators' unitarity, the Weyl identities imply that $||W(\phi)-W(\psi)||^2$ equals

$$||(W(\phi)^*-W(\psi)^*)(W(\phi)-W(\psi))|| = ||e^{i\sigma(\psi,\phi)}W(\phi-\psi)-\mathbb{I}|| = 4\,.$$

There are uncountably many elements $\psi \in \mathsf{X}$ ($\mathsf{X} \neq \{\mathbf{0}\}$ by assumption), so $\mathscr{W}(\mathsf{X},\sigma)$ is not separable: if $S \subset \mathsf{X}$ were dense, there would be an element of S inside the ball of radius $1/2$ centred at each $W(\psi)$. As said balls do not intersect, S cannot be countable.
(d) Every property of a norm, plus the C^* property $||a^*a||_c = ||a||_c^2$, hold by direct inspection. The only thing left is to show that the supremum defining $||a||_c$ is finite. To this end, on $\mathscr{W}(\mathsf{X},\sigma)$ we have a norm (not C^* in general): $||\sum_i a_i W(\psi_i)||_0 := \sum_i |a_i|$. As every $W(\psi)$ has unit norm in any C^* norm p, as seen in (b), we have $p(a) \leq ||a||_0 < +\infty$. Therefore the least upper bound in $||a||_c$ is smaller than $||a||_0$, hence is finite. □

Lemma 14.37 *Let (X,σ) be a non-trivial, weakly non-degenerate real symplectic space, $\mathscr{W}(\mathsf{X},\sigma)$ a Weyl *-algebra associated to (X,σ). Denote by $C\mathscr{W}(\mathsf{X},\sigma)$ the C^* completion of $\mathscr{W}(\mathsf{X},\sigma)$ in the norm $||\ ||_c$ of Lemma 14.36(d).*

Then $C\mathscr{W}(\mathsf{X},\sigma)$ is simple: it does not admit two-sided closed ideals that are invariant under the involution, other than $\{0\}$ and $C\mathscr{W}(\mathsf{X},\sigma)$ itself.

Proof Write $\mathfrak{A}$ for the C^*-algebra with unit obtained by completion of $\mathscr{W}(\mathsf{X},\sigma)$ under $||\ ||_c$. Suppose $I \subset \mathfrak{A}$ is a closed, two-sided ideal that is *-invariant. Then

$I_0 := I \cap \{cW(0) \mid c \in \mathbb{C}\}$ is a complex subspace of $\{cW(0) \mid c \in \mathbb{C}\}$ identified with $\mathbb{C}$. Hence $I_0 = \{0\}$ or $I_0 = \{cW(0) \mid c \in \mathbb{C}\}$. In the latter case I would then contain $\mathbb{I}$, so it would coincide with $\mathfrak{A}$. So assume $I_0 = \{0\}$ and consider the map:

$$P : \mathscr{W}(\mathsf{X}, \sigma) \to \{cW(0) \mid c \in \mathbb{C}\}\,, \text{ with } P\left(\sum_{\phi \in F \subset \mathsf{X}} W(\phi)\right) = a_0 W(0) \text{ in case } F \subset \mathsf{X} \text{ is finite}\,.$$

We claim P is bounded, and that it extends continuously to an operator, P, defined on $\mathfrak{A}$. To do so let us realise $\mathscr{W}(\mathsf{X}, \sigma)$ in the C^*-algebra of operators $\mathfrak{B}(L^2(\mathsf{X}, \mu))$, as in the proof of Lemma 14.36(a). Call $\delta_0 \in L^2(\mathsf{X}, \mu)$ the map $\delta_0(0) = 1$, $\delta_0(\phi) = 0$ for $\phi \neq 0$. For $a = \sum_{\phi \in F \subset \mathsf{X}} a_\phi W(\phi)$ and $\psi \in \mathsf{X}$ we have

$$(a\delta_0)(\psi) = \left(\sum_{\phi \in F \subset \mathsf{X}} a_\phi W(\phi)\delta_0\right)(\psi) = \sum_{\phi \in F \subset \mathsf{X}} a_\phi e^{i\sigma(\phi,\psi)/2}\delta_0(\phi + \psi)$$
$$= a_{-\psi} e^{i\sigma(-\psi,\psi)/2} = a_{-\psi}\,.$$

Consequently

$$(\delta_0|a\delta_0)_{L^2(\mathsf{X},\mu)} = \sum_{\psi \in \mathsf{X}} \overline{\delta_0(\psi)}(a\delta_0)(\psi) = (a\delta_0)(0) = a_0\,.$$

In addition, $||\delta_0|| = 1$, so

$$||P(a)||_c = ||a_0 W(0)||_c = |a_0| = |(\delta_0|a\delta_0)_{L^2}| \leq ||a||_{op} \leq ||a||_c\,,$$

proving P extends to a bounded operator on $\mathfrak{A}$.

Take now $a \in I \subset \mathfrak{A}$ and fix $\varepsilon > 0$. Write

$$a = a_0 W(0) + \sum_{j=1}^{n} a_j W(\phi_j) + r,$$

where the ϕ_j are all distinct and $||r||_c < \varepsilon$. For $\psi \in \mathsf{X}$ we have

$$I \ni W(\psi)aW(-\psi) = a_0 W(0) + \sum_{j=1}^{n} a_j e^{-i\sigma(\psi,\phi_j)/2} W(\phi_j) + r(\psi)\,,$$

since

$$||r(\phi)||_c = ||W(\psi)rW(-\psi)||_c \leq ||r||_c < \varepsilon\,.$$

Choosing ψ_1 and ψ_2 so that $e^{-i\sigma(\psi_1,\phi_n)} = -e^{-i\sigma(\psi_2,\phi_n)}$, then adding two elements

$$a_0 W(0) + \sum_{j=1}^{n} a_j e^{-i\sigma(\psi_1,\phi_j)/2} W(\phi_j) + r(\psi_1) \in I$$

and

$$a_0 W(0) + \sum_{j=1}^{n} a_j e^{-i\sigma(\psi_2,\phi_j)/2} W(\phi_j) + r(\psi_2) \in I$$

gives

$$a_0 W(0) + \sum_{j=1}^{n-1} a'_j W(\phi_j) + r_1 \in I\ ,$$

where $||r_1||_c = \frac{1}{2}||r(\psi_1) + r(\psi_2)||_c < (\varepsilon + \varepsilon)/2 = \varepsilon$. We can repeat the argument, and eventually obtain, for some r_n with $||r_n||_c < \varepsilon$:

$$a_0 W(0) + r_n \in I\ .$$

As $\varepsilon > 0$ is arbitrary and I closed, we conclude $P(a) = a_0 W(0) \in I_0$, so $a_0 = 0$. With $\psi \in \mathsf{X}$ and $a = \sum_\phi a_\phi W(\phi) \in I$ arbitrary, we similarly have $W(\psi)a \in I$, whence $P(W(\psi)a) = 0$. This means $a_{-\psi} = 0$ for any $\psi \in \mathsf{X}$, so $a = 0$. Therefore $I = \{0\}$, ending the proof. $\square$

Now to the key theorem on Weyl C^*-algebras of a given symplectic space.

Theorem 14.38 *Let (X, σ) be a non-trivial weakly non-degenerate real symplectic space, and consider a Weyl *-algebra $\mathscr{W}(\mathsf{X}, \sigma)$ associated to (X, σ).*
(a) *There exist a unique norm on $\mathscr{W}(\mathsf{X}, \sigma)$ satisfying the C^* property:*

$$||a^* a|| = ||a||^2 \quad \textit{for any } a \in \mathscr{W}(\mathsf{X}, \sigma).$$

(b) *Let $C\mathscr{W}(\mathsf{X}, \sigma)$ be the C^*-algebra completion of $\mathscr{W}(\mathsf{X}, \sigma)$ for the C^* norm of (a). If $\mathscr{W}'(\mathsf{X}, \sigma)$ is another Weyl *-algebra associated to the same space (X, σ) and $||\ \ ||'$ the unique C^* norm, call $CW'(\mathsf{X}, \sigma)$ the corresponding C^*-algebra with unit.*

Then there is a unique isometric *-isomorphism $\gamma : C\mathscr{W}(\mathsf{X}, \sigma) \to C\mathscr{W}'(\mathsf{X}, \sigma)$ such that:

$$\gamma(W(\psi)) = W'(\psi) \quad \textit{for any } \psi \in \mathsf{X},$$

where $W(\psi)$, $W'(\psi)$ are generators of the Weyl *-algebras $\mathscr{W}(\mathsf{X}, \sigma)$, $\mathscr{W}'(\mathsf{X}, \sigma)$.

Proof (a) By Theorem 11.48(c) it is known that two Weyl *-algebras $\mathscr{W}(\mathsf{X}, \sigma)$, $\mathscr{W}'(\mathsf{X}, \sigma)$ on the same symplectic space are *-isomorphic under some $\alpha : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}'(\mathsf{X}, \sigma)$ that is totally determined by $\alpha(W(\psi)) = W'(\psi)$, $\psi \in \mathsf{X}$. Equip $\mathscr{W}(\mathsf{X}, \sigma)$, $\mathscr{W}'(\mathsf{X}, \sigma)$ with C^* norms $||\ \ ||, ||\ \ ||'$. Then $||a||_1 = ||\alpha(a)||'$ is a C^* norm on $\mathscr{W}(\mathsf{X}, \sigma)$, other than $||\ ||$ in general. By definition of $||\ ||_c$ we have $||\alpha(a)||' \leq ||a||_c$, so α extends to a *-homomorphism of C^*-algebras:

$$\widetilde{\alpha} : \overline{\mathscr{W}(\mathsf{X}, \sigma)}_{||\ ||_c} \to \overline{\mathscr{W}(\mathsf{X}', \sigma')}_{||\ ||'} .$$

The kernel of $\widetilde{\alpha}$ is a closed *-invariant two-sided ideal, hence trivial by the previous lemma. In conclusion $\widetilde{\alpha}$ is one-to-one and an isometry by Theorem 8.22(a). Suppose now $\mathscr{W}(\mathsf{X}, \sigma) = \mathscr{W}'(\mathsf{X}, \sigma)$, so $||\ || = ||\ ||'$ too. Then α has to be the identity, extending to the identity $\widetilde{\alpha}$, and also isometric by the above argument. So, $||\ ||_c = ||\ ||' = ||\ ||$ is the only C^* norm on $\mathscr{W}(\mathsf{X}, \sigma)$.
(b) We have to prove that the *-isomorphism $\alpha : \mathscr{W}(\mathsf{X}, \sigma) \to \mathscr{W}'(\mathsf{X}, \sigma)$, determined by $\alpha(W(\psi)) = W'(\psi)$, $\psi \in \mathsf{X}$, extends to a *-isomorphism between the C^*-algebras $C\mathscr{W}(\mathsf{X}, \sigma)$ and $C\mathscr{W}'(\mathsf{X}, \sigma)$. The same argument used above (now we do know $||\ || = ||\ ||_c$) shows that α extends to an injective *-homomorphism $\gamma : C\mathscr{W}(\mathsf{X}, \sigma) \to C\mathscr{W}'(\mathsf{X}, \sigma)$. On the other hand we can swap $\mathscr{W}(\mathsf{X}, \sigma)$ and $\mathscr{W}'(\mathsf{X}, \sigma)$, and extend $\alpha' : \mathscr{W}'(\mathsf{X}, \sigma) \to \mathscr{W}(\mathsf{X}, \sigma)$, determined by $\alpha'(W'(\psi)) = W(\psi)$, $\psi \in \mathsf{X}$, to $\gamma' : C\mathscr{W}'(\mathsf{X}, \sigma) \to C\mathscr{W}(\mathsf{X}, \sigma)$. By construction $\alpha'\alpha = id_{\mathscr{W}(\mathsf{X},\sigma)}$, $\alpha\alpha' = id_{\mathscr{W}'(\mathsf{X},\sigma)}$. These relations continue to hold, by continuity, when extended to $\gamma'\gamma = id_{C\mathscr{W}(\mathsf{X},\sigma)}$, $\gamma\gamma' = id_{C\mathscr{W}'(\mathsf{X},\sigma)}$. Therefore γ is onto, as well, and thus a *-isomorphism. □

14.2.2 The Weyl C-Algebra $C\mathscr{W}(\mathsf{X}, \sigma)$*

By keeping Theorem 14.38 into account, we can define Weyl C^*-algebras.

Definition 14.39 Let X be a non-trivial real vector space equipped with a weakly non-degenerate symplectic form $\sigma : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$. The **Weyl C^*-algebra** $C\mathscr{W}(\mathsf{X}, \sigma)$ associated to (X, σ) is a C^*-algebra with unit generated by non-zero elements $W(\psi)$, $\psi \in \mathsf{X}$, satisfying the Weyl relations:

$$W(\psi)W(\psi') = e^{-\frac{i}{2}\sigma(\psi,\psi')}W(\psi + \psi') , \quad W(\psi)^* = W(-\psi) , \quad \psi, \psi \in \mathsf{X} .$$

This notion is well defined, and as consequence of Theorem 14.38 we obtain the following result. It shows that the Weyl C^*-algebra is unique up to *-isomorphisms.

Theorem 14.40 *Let (X, σ) be a non-trivial weakly non-degenerate real symplectic space, $C\mathscr{W}(\mathsf{X}, \sigma)$ a Weyl C^*-algebra associated to it.*
(a) *If $C\mathscr{W}'(\mathsf{X}, \sigma)$ is a second Weyl C^*-algebra associated to (X, σ), there exists a unique (isometric) *-isomorphism $\gamma : C\mathscr{W}(\mathsf{X}, \sigma) \to C\mathscr{W}'(\mathsf{X}, \sigma)$ such that*

$$\gamma(W(\psi)) = W'(\psi) \quad \textit{for any } \psi \in \mathsf{X},$$

where $W(\psi)$, $W'(\psi)$ generate the Weyl **-algebras $\mathscr{W}(\mathsf{X},\sigma)$, $\mathscr{W}'(\mathsf{X},\sigma)$ respectively.*
(b) *$C\mathscr{W}(\mathsf{X},\sigma)$ is simple: there are no non-trivial closed,* **-invariant two-sided ideals.*
(c) *If $C\mathscr{W}(\mathsf{X}',\sigma')$ is the Weyl C^*-algebra associated to the weakly non-degenerate symplectic space (X',σ') and $f:\mathsf{X}\to\mathsf{X}'$ a symplectic linear map, there is a unique injective and isometric* **-homomorphism $\gamma_f : C\mathscr{W}(\mathsf{X},\sigma)\to C\mathscr{W}(\mathsf{X}',\sigma')$ such that:*

$$\gamma_f(W(\psi)) = W'(f(\psi)) \quad \textit{for any} \quad \psi\in\mathsf{X}.$$

Furthermore, $\gamma_f(C\mathscr{W}(\mathsf{X},\sigma))$ is a unital C^*-subalgebra of $C\mathscr{W}(\mathsf{X}',\sigma')$.

Proof Items (a) and (b) were proven with Theorem 14.38. Let us see to (c). By Theorem 11.48(f) there is one injective *-homomorphism $\alpha_f : \mathscr{W}(\mathsf{X},\sigma)\to\mathscr{W}(\mathsf{X}',\sigma')$ such that $\alpha_f(W(\psi)) = W'(f(\psi))$, $\psi\in\mathsf{X}$. The C^* norm $||\ ||'$ on $C\mathscr{W}(\mathsf{X}',\sigma')$ induces a C^* norm on $\mathscr{W}(\mathsf{X},\sigma)$, $||a|| = ||\alpha_f(a)||'$. By uniqueness of the C^* norm on a Weyl *-algebra, the latter coincides with the original norm of $C\mathscr{W}(\mathsf{X},\sigma)$. Hence α_f is isometric and continuous, and extends continuously to an isometric (so injective) *-homomorphism $\gamma_f : C\mathscr{W}(\mathsf{X},\sigma)\to C\mathscr{W}(\mathsf{X}',\sigma')$. That $\gamma_f(C\mathscr{W}(\mathsf{X},\sigma))$ is a C^*-subalgebra with unit in $C\mathscr{W}(\mathsf{X}',\sigma')$ follows from Theorem 8.22(b). □

Remark 14.41 If $\mu:\mathsf{X}\times\mathsf{X}\to\mathbb{R}$ is a real inner product fulfilling

$$\frac{1}{4}|\sigma(\psi,\phi)|^2 \leq \mu(\psi,\psi)\mu(\phi,\phi)\,, \quad \text{for any } \psi,\phi\in\mathsf{X},$$

it can be proved there exists a unique algebraic state ω_μ on $C\mathscr{W}(\mathsf{X},\sigma)$ such that:

$$\omega_\mu(W(\psi)) = e^{-\frac{1}{2}\mu(\psi,\psi)}\,.$$

States of this type are called *Gaussian* or *quasi-free*, and play a big role in physical theories. The GNS representations of a quasi-free state generates Hilbert spaces with totally symmetric Fock structure (bosonic Fock spaces). ■

Example 14.42 Take Minkowski's spacetime, with coordinates $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^3$, and consider on it the *Klein-Gordon equation*:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \Delta_{\mathbf{x}}\phi - \frac{m^2c^2}{\hbar^2}\phi = 0\,,$$

where c is the speed of light and $m>0$ the mass of the particles associated to the bosonic field ϕ. Indicate with X the vector space of real smooth solutions ϕ such that $\mathbb{R}^3\ni\mathbf{x}\mapsto\phi(t,\mathbf{x})$ have compact support for every $t\in\mathbb{R}$. This space admits a weakly non-degenerate symplectic form:

$$\sigma(\phi,\phi') := \int_{\mathbb{R}^3}\left(\phi(t,\mathbf{x})\frac{\partial}{\partial t}\phi'(t,\mathbf{x}) - \phi'(t,\mathbf{x})\frac{\partial}{\partial t}\phi(t,\mathbf{x})\right)dx\,.$$

For given solutions X, one can prove that the symplectic form does not depend on the choice of $t \in \mathbb{R}$ by the nature of the Klein-Gordon equation itself. The C^*-algebra $C\mathscr{W}(\mathsf{X}, \sigma)$ is the algebra of observables of the Klein-Gordon quantum field ϕ, and can be taken as the starting point for the procedure of "second quantisation" of bosonic fields. In this case an algebraic state of paramount importance is the so-called *Minkowski vacuum*, i.e. the *Gaussian state* (see Remark 14.41) determined by a special μ that takes spatial Fourier transforms of solutions at time $t = 0$. This particular state represents the absence of particles, and is invariant under the Poincaré group. In the GNS representation of the state, the Weyl generators have the form $\pi_{\omega_\mu}(W(\phi)) = e^{i\Phi(\phi)}$. The self-adjoint operator $\Phi(\phi)$ is called *operator of the field of second quantisation*. There exists a well-known generalisation of all these notions to Quantum Field Theory in curved spacetime (see [Wal94, KhMo15] for an introductory review). ■

14.3 Introduction to Quantum Symmetries Within the Algebraic Formulation

In this section we briefly discuss how quantum symmetries are dealt with in the algebraic formulation [Haa96, Str05b]. After recalling the basic notions, we will prove two theorems about the (anti-)unitary representation of symmetries on the GNS space of an invariant algebraic state. The strategy allows to describe precisely, in mathematical terms, the concept known as the *spontaneous breaking of the symmetry*.

14.3.1 The Algebraic Formulation's Viewpoint on Quantum Symmetries

Consider a quantum system S described by the unital C^*-algebra $\mathfrak{A}_S$ of observables. Said better, the observables are the self-adjoint elements of $\mathfrak{A}_S$. A quantum symmetry α should be seen either as a *-automorphism $\alpha : \mathfrak{A}_S \to \mathfrak{A}_S$, i.e. as a bijective *-homomorphism (hence isometric), or as a *-*anti-automorphism*.

Definition 14.43 If $\mathfrak{A}$ is a C^*-algebra with unit $\mathbb{I}$, a ***-anti-automorphism** is a bijective, antilinear isometry $\alpha : \mathfrak{A} \to \mathfrak{A}$ such that $\alpha(\mathbb{I}) = \mathbb{I}$, $\alpha(a^*) = \alpha(a)^*$ and $\alpha(ab) = \alpha(a)\alpha(b)$, for any $a, b \in \mathfrak{A}$.

The above definition of algebraic symmetry has a "psychological explanation" based on the notion of quantum symmetry when the theory is formulated on a Hilbert space (and recalling the theorems of Wigner and Kadison). Fix a GNS triple $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$, suppose the GNS representation $\pi_\omega : \mathfrak{A}_S \to \mathfrak{B}(\mathsf{H})$ is injective (always the case if $\mathfrak{A}_S$ is simple, as we said in Sect. 14.1.6), and represent the symmetry on H_ω by the operator U, which is unitary or anti-unitary. Then we can set

$$\alpha(a) := \pi_\omega^{-1}\left(\gamma^{*-1}(\pi_\omega(a))\right) , \quad a \in \mathfrak{A}_S ,$$

where, mimicking the previous section's definition, the action γ of the symmetry U on observables is the *inverse dual action* (12.22)

$$\gamma^{*-1}(A) := UAU^{-1} .$$

The map α is well defined, and gives a *-automorphism or *-anti-automorphism provided $U \cdot U^{-1}$ maps observables (seen as operators) to observables, as is only natural to suppose. The reason for preferring the inverse dual action to the dual action (12.21) will be clear when we pass to representations of groups of symmetries.

Here is the formal definition.

Definition 14.44 Let S be a physical system described by the unital C^*-algebra of observables $\mathfrak{A}_S$. An **(algebraic) quantum symmetry** of S is a *-automorphism or a *-anti-automorphism $\alpha : \mathfrak{A}_s \to \mathfrak{A}_s$.

This naturally begs a question: given a symmetry α and an algebraic state ω, under which assumptions is α representable by a unitary, or anti-unitary, operator on the Hilbert space H_ω of the GNS representation of ω? The next theorem is a big step forward in this direction.

Theorem 14.45 *Let α be an algebraic quantum symmetry of system S, described by the unital C^*-algebra $\mathfrak{A}_S$ of observables. Suppose ω is an α-**invariant** algebraic state on $\mathfrak{A}_S$:*

$$\omega(\alpha(a)) = \omega(a) \quad \textit{for } a \in \mathfrak{A}_S \textit{ with } a = a^* . \tag{14.24}$$

If $(\mathsf{H}_\omega, \pi_\omega, \Psi_\omega)$ is the GNS triple of ω, there exists only one operator $U_\alpha : \mathsf{H}_\omega \to \mathsf{H}_\omega$, unitary or anti-unitary according to whether α is linear or antilinear, such that:

$$U_\alpha \Psi_\omega = \Psi_\omega \quad \textit{and} \quad U_\alpha \pi_\omega(a) U_\alpha^{-1} = \pi_\omega(\alpha(a)) , \quad a \in \mathfrak{A}_S . \tag{14.25}$$

Remark 14.46 Since any $a \in \mathfrak{A}_S$ can be written as $a = a_1 + ia_2^*$ with a_1, a_2 self-adjoint, and $\omega(a^*) = \overline{\omega(a)}$ for any state ω (straightforward from the GNS theorem), ω being α-invariant is the same as imposing $\omega(\alpha(a)) = \omega(a)$ if α is linear, or $\omega(\alpha(a)) = \overline{\omega(a)}$, if α is antilinear, for any $a \in \mathfrak{A}_S$.

Proof of theorem 14.45. The idea is to define U_α first on the dense space $\pi(\mathfrak{A}_S)\Psi_\omega$ by

$$U\pi_\omega(a)\Psi_\omega := \pi_\omega(\alpha(a))\Psi_\omega , \tag{14.26}$$

and then extend it continuously to H_ω, interpreting the result as U_α. The definition of U is unambiguous if $\pi_\omega(a)\Psi_\omega = \pi_\omega(a')\Psi_\omega$ implies $\pi_\omega(\alpha(a))\Psi_\omega = \pi_\omega(\alpha(a'))\Psi_\omega$, i.e. if $\pi_\omega(b)\Psi_\omega = 0$ implies $\pi_\omega(\alpha(b))\Psi_\omega = 0$. But this is true by the GNS theorem and the invariance of ω:

$$||\pi_\omega(\alpha(b))\Psi_\omega||^2 = (\pi_\omega(\alpha(b))\Psi_\omega|\pi_\omega(\alpha(b))\Psi_\omega)$$

$$= (\Psi_\omega|\pi_\omega(\alpha(b^*))\pi_\omega(\alpha(b))\Psi_\omega)$$

$$= (\Psi_\omega|\pi_\omega(\alpha(b^*b))\Psi_\omega) = \omega(\alpha(b^*b)) = \omega(b^*b) = (\Psi_\omega|\pi_\omega(b^*b)\Psi_\omega)$$

$$= ||\pi_\omega(b)\Psi_\omega||^2 .$$

By construction U, as in (14.26), is linear or antilinear depending on how α is. Moreover, it is isometric/anti-isometric, if α is a *-isomorphism/anti-isomorphism respectively. Hence it is continuous, for above computations show that

$$||U\pi_\omega(b)\Psi_\omega||^2 = ||\pi_\omega(b)\Psi_\omega||^2 .$$

We extend U by continuity to H_ω, since $\pi(\mathfrak{A}_S)\Psi_\omega$ is dense in H, and obtain a linear/antilinear operator $U_\alpha : \mathsf{H}_\omega \to \mathsf{H}_\omega$ preserving the norms. Our U_α is onto, as inverse of the analogous uniquely-defined extension of

$$U_\alpha^{-1}\pi_\omega(a)\Psi_\omega = \pi_\omega(\alpha^{-1}(a))\Psi_\omega . \tag{14.27}$$

Therefore $U_\alpha : \mathsf{H}_\omega \to \mathsf{H}_\omega$ is well defined, unitary/anti-unitary if α is linear/antilinear, and (14.25) are true. The first condition is trivially true if we set $b = \mathbb{I}$ in (14.26). As for the second one, put $a = bc$ in (14.27):

$$U_\alpha\pi_\omega(b)\pi_\omega(c)\Psi_\omega = \pi_\omega(\alpha(b))\pi_\omega(\alpha(c))\Psi_\omega .$$

Using (14.26):

$$U_\alpha\pi_\omega(b)U_\alpha^{-1}\pi_\omega(\alpha(c))\Psi_\omega = \pi_\omega(\alpha(b))\pi_\omega(\alpha(c))\Psi_\omega .$$

That is to say, if $\Phi \in \pi_\omega(\alpha(\mathfrak{A}_S))\Psi_\omega = \pi_\omega(\mathfrak{A}_S)\Psi_\omega$:

$$U_\alpha\pi_\omega(b)U_\alpha^{-1}\Phi = \pi_\omega(\alpha(b))U_\alpha^{-1}\Phi .$$

Since $\pi_\omega(\mathfrak{A}_S)\Psi_\omega$ is dense in H_ω, the second identity in (14.25) holds. Uniqueness is patent by construction, because if U' satisfies (14.25) (U' replacing U_α) it must satisfy (14.27) as well (U' replacing U_α). This fact determines it completely. □

Remark 14.47 **(1)** A system S may admit an algebraic symmetry α that is *not* representable unitarily (or anti-unitarily) on the Hilbert space of the theory (e.g., a GNS representation of a reference algebraic state ω). If so, the symmetry α is said to have been **broken spontaneously** by the representation employed. The phenomenon of **spontaneous symmetry breaking** is hugely important in particle physics and statistical mechanics [Str05b].

(2) An alternate definition for a *-anti-automorphism [BrRo02, Ara09] is a *linear* map $\alpha : \mathfrak{A} \to \mathfrak{A}$ such that $\alpha(ab) = \alpha(b)\alpha(a)$ for every $a, b \in \mathfrak{A}$. With this definition, if ω is fixed under the action of α, the corresponding part of Theorem 14.45 holds anyway, with the same hypothesis, and this change: there is an anti-unitary operator $U_\alpha : \mathsf{H}_\omega \to \mathsf{H}_\omega$ such that

$$U_\alpha \Psi_\omega = \Psi_\omega \quad \text{and} \quad U_\alpha \pi_\omega(a^*) U_\alpha^{-1} = \pi_\omega(\alpha(a)) \,, \quad a \in \mathfrak{A}_S, \tag{14.28}$$

where we stress the presence of a^* in place of a in the second identity when comparing to (14.25). The proof is evident if we observe that $\alpha'(a) := \alpha(a)^*$, for a ranging in $\mathfrak{A}$, defines a *-automorphism in accordance with Definition 14.43, and Theorem 14.45 is valid if α leaves ω invariant. This alternative but essentially equivalent definition of *-anti-automorphism can be exploited when one deals with real *-algebras or other algebraic structures defined over the field of real numbers. ■

14.3.2 (Topological) Symmetry Groups in the Algebraic Formalism

We want to show, concisely, how Theorem 14.45, proven in Sect. 14.3.1 in the algebraic formalism, generalises naturally to the situation where the algebraic symmetry α is replaced by an algebraic symmetry group.

So take a quantum system S described, in the algebraic formalism, by the unital C^*-algebra $\mathfrak{A}_S$, whose self-adjoint elements are the system's observables. Suppose there is a representation $\alpha : \mathsf{G} \ni g \mapsto \alpha_g$ of the group G in terms of *-automorphisms α_g of $\mathfrak{A}_S$. If ω is an invariant algebraic state, Theorem 14.45 guarantees every α_g is representable by a unitary operator U_{α_g} on the Hilbert space H_ω of the GNS representation of ω. We will show that this correspondence produces automatically a unitary representation of G, without the need to redefine the phases of the unitary maps U_g. This representation is also strongly continuous under a certain hypothesis.

Theorem 14.48 *Let S be a quantum system described, in the algebraic formalism, by the unital C^*-algebra $\mathfrak{A}_S$, and let G be a group with a representation*

$$\alpha : \mathsf{G} \ni g \mapsto \alpha_g$$

by *-automorphisms α_g of $\mathfrak{A}_S$. Suppose ω is a G-invariant algebraic state on S represented by α:

$$\omega(\alpha_g(a)) = \omega(a) \quad \textit{for any } g \in \mathsf{G}, a \in \mathfrak{A}_S \, with \, a = a^*. \tag{14.29}$$

(a) *If $U_{\alpha_g} : \mathsf{H}_\omega \to \mathsf{H}_\omega$ is the unitary operator associated to the *-automorphism α_g by Theorem 14.45 (as the unique unitary extension of the operator in (14.26) with $\alpha = \alpha_g$), for any $g \in \mathsf{G}$ the map*

$$\mathsf{G} \ni g \mapsto U_{\alpha_g} \tag{14.30}$$

is a unitary representation of G *on* H_ω.
(b) *If* G *is a topological group,* $\mathsf{G} \ni g \mapsto \omega(a^*\alpha_g(a))$ *is continuous for any given* $a \in \mathfrak{A}_S$ *if and only if the representation (14.30) is strongly continuous.*

Proof Consider the operators U_{α_g} defined by Theorem 14.45. By assumption, since $U_{\alpha_h}\Psi_\omega = \Psi_\omega$ so $\Psi_\omega = U_{\alpha_h}^{-1}\Psi_\omega$, we have:

$$U_{\alpha_g}U_{\alpha_h}\pi_\omega(a)\Psi_\omega = U_{\alpha_g}U_{\alpha_h}\pi_\omega(a)U_{\alpha_h}^{-1}\Psi_\omega = U_{\alpha_g}\pi_\omega(\alpha_h(a))\Psi_\omega$$

$$= U_{\alpha_g}\pi_\omega(\alpha_h(a))U_{\alpha_g}^{-1}\Psi_\omega = \pi_\omega(\alpha_g(\alpha_h(a)))\Psi_\omega = \pi_\omega((\alpha_g \circ \alpha_h)(a))\Psi_\omega =$$

$$= \pi_\omega(\alpha_{g\cdot h}(a))\Psi_\omega = U_{\alpha_{g\cdot h}}\pi_\omega(a)U_{\alpha_{g\cdot h}}^{-1}\Psi_\omega = U_{\alpha_{g\cdot h}}\pi_\omega(a)\Psi_\omega\ .$$

As $\pi_\omega(\mathfrak{A}_S)\Psi_\omega$ is dense in H_S, then $U_{\alpha_g}U_{\alpha_h} = U_{\alpha_{gh}}$. Similarly we can prove $U_{\alpha_g}^{-1} = U_{\alpha_{g^{-1}}}$ and $U_{\alpha_e} = I$. In other terms (14.30) is a unitary representation of G.

Now assume G is a topological group and $\mathsf{G} \ni g \mapsto \omega(a^*\alpha_g(a))$ is continuous for every $a \in \mathfrak{A}_S$. By the GNS theorem, and the fact that $U_{\alpha_g}\Psi_\omega = \Psi_\omega$, this implies $\mathsf{G} \ni g \mapsto \omega(a^*\alpha_g(a)) = (\pi_\omega(a)\Psi_\omega|U_{\alpha_g}\pi_\omega(a)\Psi_\omega)$ is a continuous function. But $a \in \mathfrak{A}_S$ is generic, so we have proved that for every Φ in the dense space $\pi_\omega(\mathfrak{A}_S)\Psi_\omega$, $\mathsf{G} \ni g \mapsto (\Phi|U_{\alpha_g}\Phi)$ is continuous. Using that U_{α_g} is unitary, it is easy to see that, consequently:

$$||U_{\alpha_g}\Phi - U_{\alpha_{g'}}\Phi||^2 = ||U_{\alpha_{g\cdot g'^{-1}}}\Phi - \Phi||^2 \to 0$$

as $g \to g'$, for any Φ in the dense subspace $\pi_\omega(\mathfrak{A}_S)\Psi_\omega$. Therefore $\mathsf{G} \ni g \mapsto U_{\alpha_g}$ is strongly continuous on the dense space $\pi_\omega(\mathfrak{A}_S)\Psi_\omega$. This generalises to the generic vector $\Psi \in \mathsf{H}_S$ as follows. For every $\varepsilon > 0$ we can find $\Phi \in \pi_\omega(\mathfrak{A}_S)\Psi_\omega$ so that $||\Psi - \Phi|| < 2\varepsilon/3$. For such Φ, there is a neighbourhood $I_{g'}$ of g' in G such that $||U_{\alpha_g}\Phi - U_{\alpha_{g'}}\Phi|| < \varepsilon/3$ if $g \in I_{g'}$. Hence for $g' \in \mathsf{G}$ and any $\varepsilon > 0$ there is a neighbourhood $I_{g'}$ of g' such that:

$$||U_{\alpha_g}\Psi - U_{\alpha_{g'}}\Psi|| \le ||U_{\alpha_g}\Psi - U_{\alpha_g}\Phi|| + ||U_{\alpha_g}\Phi - U_{\alpha_{g'}}\Phi|| + ||U_{\alpha_{g'}}\Phi - U_{\alpha_{g'}}\Psi||$$

$$= ||\Psi - \Phi|| + ||U_{\alpha_g}\Phi - U_{\alpha_{g'}}\Phi|| + ||\Phi - \Psi|| < \varepsilon$$

for $g \in I_{g'}$. To conclude, observe that if $\mathsf{G} \ni g \mapsto U_{\alpha_g}$ is strongly continuous, then $\mathsf{G} \ni g \mapsto \omega(a^*\alpha_g(a)) = (\pi_\omega(a)\Psi_\omega|U_{\alpha_g}\pi_\omega(a)\Psi_\omega)$ is continuous for every $a \in \mathfrak{A}_S$. This ends the proof. □

Remark 14.49 **(1)** We could also decide to make the algebraic symmetry α in Theorem 14.45 act in the GNS representation by means of the *dual action* (12.21) of U_α:

$$\pi_\omega(\alpha(a)) = U_\alpha^{-1}\pi_\omega(a)U_\alpha\ ,$$

simply redefining as U_α^{-1} the previous U_α. With this choice, Theorem 14.48 remains valid provided we assume that the representation $\mathsf{G} \ni g \mapsto \alpha_g$ of *-automorphisms of $\mathfrak{A}$ is a *right* representation (Sect. 12.2.2) instead of a standard left representation: $\alpha_e = id$, $\alpha_{g^{-1}} = \alpha_g^{-1}$, but

$$\alpha_g \circ \alpha_h = \alpha_{h\cdot g} \quad g, h \in \mathsf{G} .$$

In the algebraic formulation, where the focus is more on observables than on states, one normally adopts the viewpoint based on inverse dual actions to deal with standard representations in terms of automorphisms instead of the more exotic right representations.

This happens especially for the quantum field operators introduced in Example 14.42 (e.g., see [KhMo15] Proposition 5.1.17 and Sect. 5.2.7):

$$\alpha_g(W(\phi)) = W(\phi \circ g^{-1})$$

where $g \in \mathsf{G}$, G is the isometry group (Poincaré group) of Minkowski spacetime M and $W(\phi)$ is the Weyl element associated to the solution ϕ of the Klein–Gordon equation. In the GNS representation of an isometry-invariant state ω_M, the Gaussian Poincaré vacuum, we therefore have

$$\pi_{\omega_M}(W(\phi \circ g^{-1})) = \pi_{\omega_M}(\alpha_g(W(\phi))) = U_g \pi_{\omega_M}(W(\phi)) U_g^{-1} . \tag{14.31}$$

Passing to the exponential description $\pi_{\omega_M}(W(\phi)) = e^{i\Phi(\phi)}$ in the GNS Hilbert space of ω_M, we easily obtain

$$\Phi(\phi \circ g^{-1}) = U_g \Phi(\phi) U_g^{-1} .$$

Within this formalism, Heisenberg's evolution is nothing but the inverse dual action on observables of the *time displacement symmetry* (Remark 13.1(4)) contained in G.

(2) In case G is the topological group $\mathbb{R}$, right and standard representations coincide. If $\{\alpha_t\}_{t\in\mathbb{R}}$ satisfies part (b) for the invariant state ω, Stone's theorem warrants that the one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ representing $\mathbb{R}$ ($\pi_\omega(\alpha_t(a)) = U_t \pi_\omega(a) U_t^{-1}$) admits a self-adjoint generator $H : D(H) \to \mathsf{H}_\omega$ defined on the dense domain $D(H) \subset \mathsf{H}_\omega$, for which $U_t = e^{itH}$. Then we may think of $\{\alpha_t\}_{t\in\mathbb{R}}$ as a one-parameter group of *-automorphisms that describes the time evolution of the observables of the system with the parameter t as time. This is the algebraic correspondent of *Heisenberg's time evolution* of observables, namely, the algebraic analogue of the *inverse dual action* of the *time displacement symmetry* $\mathbb{R} \ni t \mapsto e^{+itH}$ acting on states, as remarked above.

By Stone's theorem, the first condition in (14.25) implies $\Psi_\omega \in D(H)$ and $H\Psi_\omega = 0$, so $0 \in \sigma_p(H)$. If $\sigma(H) \subset [0, +\infty)$ (and sometimes one further requires $\dim(\mathrm{Ker}(H)) = 1$), ω is called a **ground state** for the time evolution $\{\alpha_t\}_{t\in\mathbb{R}}$.

(3) One may try to prove an analogous theorem for representations $\mathsf{G} \ni g \mapsto \alpha_g$, where each α_g is either a *-automorphism or a *-anti-automorphism. We shall not address this issue here. We only observe that if every element $g \in \mathsf{G}$ can be written as product of squares $g = g_1^2 \cdots g_n^2$, as happens for $\mathbb{R}$ and *connected* Lie groups, then every α_g must be linear since $\alpha_g = \alpha_{g_1}^2 \circ \cdots \circ \alpha_{g_n}^2$ is linear, no matter whether every α_{g_k} is linear or antilinear. So, in a representation of a Lie group, *-anti-automorphisms (and the corresponding anti-unitary operators) may show up only if the group is not connected. ■

Condition (b) in Theorem 14.48 seems non-physical, as it does not concern physically accessible quantities in an evident way. A continuity condition should be formulated in terms of physical quantities like expectation values. As a matter of fact it is possible to reformulate the theorem and replace condition (b) by a more meaningful request, physically. To this end we note that given a state ω on a unital C^*-algebra $\mathfrak{A}_S$ representing a physical quantum system, and having chosen $b \in \mathfrak{A}_S$ with $\omega(b^*b) \neq 0$ (which is equivalent to $\pi_\omega(b)\Psi_\omega \neq 0$), the functional

$$\mathfrak{A}_S \ni a \mapsto \omega_b(a) := \frac{\omega(b^*ab)}{\omega(b^*b)} \tag{14.32}$$

is still a state. Its meaning is clearer when we pass to the GNS representation of ω where, by construction, we immediately find

$$\omega_b(a) = \frac{(\pi_\omega(b)\Psi_\omega|\pi_\omega(a)\pi_\omega(b)\Psi_\omega)}{||\pi_\omega(b)\Psi_\omega||^2} .$$

We see in this way that ω_b is a state in the folium of ω. The finite span $\mathscr{D}_\omega$ of the vectors associated to these states is the one used to construct the Hilbert H_ω space of the GNS Theorem 14.4 as $\mathsf{H}_\omega = \overline{\mathscr{D}_\omega}$.

If we keep all other hypotheses in Theorem 14.48, condition (b) can be replaced by the equivalent demand that the expectation values, with respect to those states, of every observable $a = a^* \in \mathfrak{A}_S$ subjected to the action α_g of the group are continuous functions of $g \in \mathsf{G}$. We have the following physically-enhanced version of Theorem 14.48.

Proposition 14.50 *Let S be a quantum system described, in the algebraic formalism, by the unital C^*-algebra $\mathfrak{A}_S$, and let G be a topological group with a representation $\alpha : \mathsf{G} \ni g \mapsto \alpha_g$ by *-automorphisms α_g of $\mathfrak{A}_S$. Suppose ω is a G-invariant algebraic state on S represented by α as in (14.29), and that $U : \mathsf{G} \ni g \mapsto U_{\alpha_g}$ is the unitary representation of G as in (14.30).*

The following facts are equivalent.

(a) *U is strongly continuous.*

(b) *The map*

$$\mathsf{G} \ni g \mapsto \omega(a^*\alpha_g(a)) \in \mathbb{C}$$

is continuous for any given $a \in \mathfrak{A}_S$.

(c) *The map*

$$\mathsf{G} \ni g \mapsto \omega_b(\alpha_g(a)) \in \mathbb{C}$$

is continuous for any given $a = a^* \in \mathfrak{A}_S$ and any $b \in \mathfrak{A}_S$ with $\omega(b^*b) \neq 0$ as in (14.32) inducing the state ω_b.

Proof We already know that (a) and (b) are equivalent. Moreover, (b) implies (c) because

$$\omega_b(\alpha_g(a)) = \frac{(\pi_\omega(b)\Psi_\omega|U_g\pi_\omega(a)U_g^{-1}\pi_\omega(b)\Psi_\omega)}{||\pi_\omega(b)\Psi_\omega||^2} \tag{14.33}$$

and $\mathsf{G} \ni g \mapsto U_g$ is strongly continuous. Suppose (c) holds so that all maps (14.32) are continuous. From (14.33), we conclude, for every vector Φ in the dense linear space $\mathscr{D}_\omega$ spanned by all $\pi_\omega(b)\Psi_\omega$, that the function

$$\mathsf{G} \ni g \mapsto (\Phi|U_{\alpha_g}\pi_\omega(a)U_{\alpha_g}^{-1}\Phi) \tag{14.34}$$

is continuous for every $a = a^* \in \mathfrak{A}_S$. Observing in particular that $\pi_\omega(a) = \pi_\omega(a)^*$, it is easy to prove that the form

$$\langle \Phi, \Phi' \rangle_{a,g} := (\Phi|U_{\alpha_g}\pi_\omega(a)U_g^{-1}\Phi')$$

is linear in the right entry, antilinear in the left one and $\overline{\langle \Phi, \Phi' \rangle_{a,g}} = \langle \Phi', \Phi \rangle_{a,g}$. By polarisation, we conclude that also the map

$$\mathsf{G} \ni g \mapsto (\Phi|U_{\alpha_g}\pi_\omega(a)U_{\alpha_g}^{-1}\Phi')$$

is continuous, if $\Phi, \Phi' \in \mathscr{D}_\omega$. Taking $\Phi' = \Psi_\omega$ (corresponding to $b = \mathbb{I}$), since $U_{\alpha_g}\Psi_\omega = \Psi_\omega$, we have that

$$\mathsf{G} \ni g \mapsto (\Phi|U_{\alpha_g}\pi_\omega(a)\Psi_\omega) \tag{14.35}$$

is continuous. By linearity the self-adjoint element a can be replaced with any element of $\mathfrak{A}_S$. Since Φ is a finite linear combination of vectors $\pi_\omega(a)\Psi_\omega$ with $a \in \mathfrak{A}_S$, we can replace $\pi_\omega(a)\Psi_\omega$ in (14.35) with Φ itself, and preserve continuity. Hence

$$\mathsf{G} \ni g \mapsto (\Phi|U_{\alpha_g}\Phi)$$

is continuous for every Φ ranging in a dense space. Here the argument used in the proof of Theorem 14.48 also proves that $\mathsf{G} \ni g \mapsto U_{\alpha_g}$ is continuous, and therefore (a) is true. $\square$

Appendix A
Order Relations and Groups

A.1 Order Relations, Posets, Zorn's Lemma

A relation $\geq$ on an arbitrary set X is called a **partial order (relation)** if it is *reflexive* ($x \geq x, \forall x \in \mathsf{X}$), *transitive* ($x \geq y \geq z \Rightarrow x \geq z, \forall x, y, z \in \mathsf{X}$) and *skew-symmetric* ($x \geq y \geq x \Rightarrow x = y, \forall x, y \in \mathsf{X}$). The pair $(\mathsf{X}, \geq)$ is then said a **partially ordered set** (shortened to **poset**).

An equivalent writing of $a \geq b$ is $b \leq a$.

The partial order $\geq$ is a **total order** if, further, either $x \geq y$ or $y \geq x$ for any $x, y \in \mathsf{X}$.

If $(\mathsf{X}, \geq)$ is a partially ordered set:
(i) $Y \subset \mathsf{X}$ is **upper bounded** (resp. **lower bounded**) if it admits an **upper bound** (**lower bound**), i.e. an element $x \in \mathsf{X}$ such that $x \geq y$ ($y \geq x$) for any $y \in Y$;
(ii) an element $x_0 \in \mathsf{X}$ for which there is *no* element $x \neq x_0$ in X such that $x \geq x_0$ is **maximal** in X. (Note that for us a maximal element in X may not be an upper bound in X).
If $(\mathsf{X}, \geq)$ is a poset, a subset $Y \subset \mathsf{X}$ is (**totally**) **ordered** if the relation $\geq$, restricted to $Y \times Y$, is a total order.

Recall that Zorn's lemma is an equivalent statement to the Axiom of Choice (or to the well-ordering axiom).

Theorem A.1 ("Zorn's lemma") *If any ordered subset in a poset* $(\mathsf{X}, \geq)$ *is upper bounded,* X *admits a maximal element.*

Among the various notions related to posets $(\mathsf{X}, \geq)$, those of supremum and infimum are useful:
(i) a is called a **least upper bound** (or **supremum**, or just **sup**) of the set $A \subset \mathsf{X}$, written $a = \sup A$, if a is an upper bound of A and any other upper bound a' of A satisfies $a \leq a'$;
(ii) a is called a **greatest lower bound**, (or **infimum** or **inf**) of $A \subset \mathsf{X}$, written $a = \inf A$, if a is a lower bound for A and any other lower bound a' satisfies $a' \leq a$;
It is immediate to see that any subset $A \subset \mathsf{X}$ has at most one least upper bound and one greatest lower bound.

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A.2 Round-up on Group Theory

A group is an algebraic structure $(\mathsf{G}, \circ)$ consisting in a set G and an operation $\circ : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ (the composition law, often called **product**) satisfying three properties:

(1) $\circ$ is *associative*

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \ , \quad \text{for any} \quad g_1, g_2, g_3 \in \mathsf{G} \ ;$$

(2) there exists an element $e \in \mathsf{G}$, variously called **identity**, **neutral** element or **unit**, such that

$$e \circ g = g \circ e = g \ , \quad \text{for any} \quad g \in \mathsf{G} \ ;$$

(3) each element $g \in \mathsf{G}$ admits an **inverse**, i.e.

$$\text{for any} \quad g \in \mathsf{G} \quad \text{there exists} \quad g^{-1} \in \mathsf{G} \quad \text{such that} \quad g \circ g^{-1} = g^{-1} \circ g = e \ .$$

The identity and the inverse to a given element are easily seen to be unique.
A group $(\mathsf{G}, \circ)$ is **commutative** or **Abelian** if $g \circ g' = g' \circ g$ for any $g, g' \in \mathsf{G}$; otherwise it is **noncommutative** or **non-Abelian**.

A subset $\mathsf{G}' \subset \mathsf{G}$ in a group is a **subgroup** if it becomes a group with the product of G restricted to $\mathsf{G}' \times \mathsf{G}'$. A subgroup N in a group G is **normal** if it is invariant under **conjugation**, i.e. for any $n \in N$ and $g \in \mathsf{G}$ the conjugate element $g \circ n \circ g^{-1}$ belongs to N. A group is said to be **simple** if it does not admit normal subgroups different from $\{e\}$ and itself.

If N is a normal subgroup in G, then G/N denotes the quotient, i.e. the collection of cosets in G defined by the equivalence relation $g \sim g' \ \Leftrightarrow \ g = ng'$ for some $n \in N$. It is easy to prove that G/N inherits a natural group structure from G.

The **centre** Z of G is the commutative subgroup of G made by elements z that commute with every group element: $z \in Z \ \Leftrightarrow \ z \circ g = g \circ z$ for any $g \in \mathsf{G}$.

If $(\mathsf{G}_1, \circ_1)$ and $(\mathsf{G}_2, \circ_2)$ are two groups, a **(group) homomorphism** from G_1 to G_2 is a map $h : \mathsf{G}_1 \to \mathsf{G}_2$ that *preserves the group structure*, i.e.:

$$h(g \circ_1 g') = h(g) \circ_2 h(g') \quad \text{for any} \quad g, g' \in \mathsf{G}_1 \ .$$

Using obvious notation, it is clear that $h(e_1) = e_2$ and $h(g^{-1_1}) = (h(g))^{-1_2}$ for any $g \in \mathsf{G}_1$.

The **kernel** $Ker(h) \subset \mathsf{G}$ of a homomorphism $h : \mathsf{G} \to \mathsf{G}'$ is the pre-image under h of the identity e' of G', i.e. the set of elements g such that $h(g) = e'$. Notice $Ker(h)$ is a normal subgroup. Clearly h is one-to-one if and only if its kernel contains the identity of G only. Moreover, the image $h(\mathsf{G})$ of a homomorphism $h : \mathsf{G} \to \mathsf{G}'$ is a subgroup of G' isomorphic to $\mathsf{G}/Ker(h)$.

A **group isomorphism** is a *bijective* group homomorphism. An isomorphism $h : \mathsf{G} \to \mathsf{G}$ is an **automorphism** of G. The set $Aut(\mathsf{G})$ of automorphisms of G is itself a group under the composition of maps.

If $(\mathsf{G}_1, \circ_1)$, $(\mathsf{G}_2, \circ_2)$ are groups, the **direct product** $\mathsf{G}_1 \otimes \mathsf{G}_2$ is a group with the following structure. Its elements are the pairs (g_1, g_2) of the *Cartesian product* of the *sets* $\mathsf{G}_1, \mathsf{G}_2$. The composition law is

$$(g_1, g_2) \circ (f_1, f_2) := (g_1 \circ_1 f_1, g_2 \circ_2 f_2) \quad \forall (g_1, g_2), (f_1, f_2) \in \mathsf{G}_1 \times \mathsf{G}_2 .$$

The neutral element is obviously (e_1, e_2), where e_1, e_2 are the identities of $\mathsf{G}_1, \mathsf{G}_2$. Moreover, G_1 and G_2 can be identified with normal subgroups of $\mathsf{G}_1 \otimes \mathsf{G}_2$.

The ensuing generalisation of the direct product plays a big role in physical applications. Let $(\mathsf{G}_1, \circ_1)$, $(\mathsf{G}_2, \circ_2)$ be groups and suppose that for any $g_1 \in \mathsf{G}_1$ there is a group isomorphism $\psi_{g_1} : \mathsf{G}_2 \to \mathsf{G}_2$ such that:

(i) $\psi_{g_1} \circ \psi_{g_1'} = \psi_{g_1 \circ_1 g_1'}$,
(ii) $\psi_{e_1} = id_{\mathsf{G}_2}$,

where $\circ$ is the composition of functions and e_1 the neutral element in G_1. (Equivalently, $\psi_g \in Aut(\mathsf{G}_2)$ for any $g \in \mathsf{G}_1$, and the map $\mathsf{G}_1 \ni g \mapsto \psi_g$ is a group homomorphism $\mathsf{G}_1 \to Aut(\mathsf{G}_2)$.) We can endow the Cartesian product $\mathsf{G}_1 \times \mathsf{G}_2$ with a group structure simply by defining the composition law on $(g_1, g_2), (f_1, f_2) \in \mathsf{G}_1 \times \mathsf{G}_2$ as

$$(g_1, g_2) \circ_\psi (f_1, f_2) := (g_1 \circ_1 f_1,\ g_2 \circ_2 \psi_{g_1}(f_2)) .$$

The operation is well defined, so $(\mathsf{G}_1 \otimes_\psi \mathsf{G}_2, \circ_\psi)$ is a group called the **semidirect product** of G_1 and G_2 by ψ. The order of the factors in the product is clearly relevant.

One can prove N is a normal subgroup of $\mathsf{G} \otimes_\psi \mathsf{N}$, and

$$\psi_g(n) = g \circ_\psi n \circ_\psi g^{-1} \quad \text{for any } g \in \mathsf{G}, n \in \mathsf{N}.$$

There is also a converse of sorts. Consider a group $(\mathsf{H}, \circ)$, let G be a subgroup of H and N a normal subgroup. Assume $N \cap G = \{e\}$, e being the identity of H. Suppose also $\mathsf{H} = GN$, meaning that any $h \in \mathsf{H}$ is the product $h = gn$ of an element $g \in G$ and some $n \in N$. Then one can prove that the pair (g, n) is uniquely determined by h, and H is isomorphic to the semidirect product $G \otimes_\psi N$ with

$$\psi_g(n) := g \circ h \circ g^{-1} \quad \text{for any } g \in G, n \in N.$$

Now take a vector space V (real or complex). The group of bijective linear maps $f : \mathsf{V} \to \mathsf{V}$ with the usual composition law is indicated by $GL(\mathsf{V})$, and is called the **(general) linear group** of V.

If $\mathsf{V} := \mathbb{R}^n$ or $\mathbb{C}^n$ then $GL(\mathsf{V})$ is denoted by $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, respectively.

Let us define linear representations of a group. Take a group $(\mathsf{G}, \circ)$ and a vector space V. A **(linear, left) representation** of G on V is a homomorphism $\rho : \mathsf{G} \to GL(\mathsf{V})$.

A representation $\rho : \mathsf{G} \to GL(\mathsf{V})$ is called:
(1) faithful if it is injective;
(2) free if the subgroup made of elements h_v such that $\rho(h_v)v = v$ is trivial for any $v \in \mathsf{V} \setminus \{0\}$, i.e. it contains only the neutral element of G;
(3) transitive if, for any $v, v' \in \mathsf{V} \setminus \{0\}$ there exists $g \in \mathsf{G}$ with $v' = \rho(g)v$;
(4) irreducible if there exists no proper subspace $S \subset \mathsf{V}$ that is **invariant** under the action of $\rho(\mathsf{G})$, i.e. $\rho(g)S \subset S$ for any $g \in \mathsf{G}$.

In case V is a Hilbert or Banach space and $\rho(\mathsf{G})$ are *bounded operators on the entire space* V, the representation is said **topologically irreducible** if there are no *closed* $\rho(\mathsf{G})$-invariant subspaces in V.

Appendix B
Elements of Differential Geometry

Let $n, m = 1, 2, \ldots, k = 0, 1, \ldots$ be fixed integers and $\Omega \subset \mathbb{R}^n$ an open, non-empty set. A map $f : \Omega \to \mathbb{R}^m$ is **of class** C^k (or simply C^k), written $f \in C^k(\Omega; \mathbb{R}^n)$, if all partial derivatives of the components of f are continuous up to order k included. Conventionally, $C^k(\Omega) := C^k(\Omega; \mathbb{R})$.

A function $f : \Omega \to \mathbb{R}^m$ is (of class) C^∞, or **smooth**, if it is C^k for any $k = 0, 1, \ldots$, so one defines

$$C^\infty(\Omega; \mathbb{R}^n) := \bigcap_{k=0,1,\ldots} C^k(\Omega; \mathbb{R}^n)\ .$$

Again, $C^\infty(\Omega) := C^\infty(\Omega; \mathbb{R})$. Eventually, $f : \Omega \to \mathbb{R}^m$ is C^ω or **real-analytic** if it is C^∞ and it admits a Taylor expansion (in several real variables) at any point $p \in \Omega$, on some finite open ball around p contained in Ω. Usually, when the order k of differentiability is not mentioned explicitly it means that $k = \infty$.

Notation B.1 In this section upper indices denote coordinates of $\mathbb{R}^n$ and components of (contravariant) vectors. Hence the standard coordinates on $\mathbb{R}^n$ will be denoted by $x^1, \ldots, x^n$, instead of $x_1, \ldots, x_n$. ■

B.1 Smooth Manifolds, Product Manifolds, Smooth Functions

The most general and powerful tool for describing the features of spacetime, the three-dimensional physical space and the abstract space of physical systems in classical theories, is the notion of *smooth manifold*. In practice a smooth manifold is a collection of objects, generally called *points*, that admits local coordinates identifying points with n-tuples of $\mathbb{R}^n$.

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Definition B.2 Let $n = 0, 1, 2, 3, \dots$ and $k = 1, 2, \dots, \infty, \omega$ be fixed. A C^k **manifold of dimension** n is a set M, whose elements are called **points**, equipped with the geometric structure defined below.
(1) M has a **differentiable structure** $\mathscr{A} = \{(U_i, \phi_i)\}_{i\in I}$ **of class** C^k, that is a collection of pairs (U_i, ϕ_i), called **local charts**, where U_i is a subset in M and ϕ_i a map $U_i \to \mathbb{R}^n$ (the **local coordinate system** or **local frame**) such that:
(i) $\cup_{i\in I} U_i = M$, every mapping ϕ_i is injective and $\phi_i(U_i) \subset \mathbb{R}^n$ is open (so M is called an n-dimensional manifold, or just n-manifold);
(ii) local charts in $\mathscr{A}$ must be pairwise C^k-*compatible*. Two injective maps $\phi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$ with $U, V \subset M$ are C^k-**compatible** if either $U \cap V = \emptyset$, or $U \cap V \neq \emptyset$ and the maps $\phi \circ \psi^{-1} : \psi(U\cap V) \to \phi(U\cap V)$, $\psi \circ \phi^{-1} : \phi(U\cap V) \to \psi(U \cap V)$ are both C^k;
(iii) $\mathscr{A}$ is **maximal**: if $U \subset M$ is open and $\phi : U \to \mathbb{R}^n$ is compatible with every local chart of $\mathscr{A}$, then $(U, \phi) \in \mathscr{A}$.
(2) The topological requirements are:
(i) M is a second-countable Hausdorff space;
(ii) M is, by way of $\mathscr{A}$, *locally homeomorphic to* $\mathbb{R}^n$. In other terms, if $(U, \phi) \in \mathscr{A}$ then U is open and $\phi : U \to \phi(U)$ is a homeomorphism.
A C^ω manifold is more often called **real-analytic manifold**.

Remark B.3
(1) We include in the definition of manifold the case $n = 0$: $\mathbb{R}^0 = \{0\}$ is the typical 0-dimensional manifold. Also $\mathbb{N}$ equipped with the topology induced by $\mathbb{R}$ is a 0-dimensional manifold. Second-countability forces a 0-dimensional manifold to be a finite or countable collection of discrete points, and the value of k becomes irrelevant.
(2) Every local chart (U, ϕ) enables us to assign n real numbers $(x^1_p, \dots, x^n_p) = \phi(p)$ bijectively to every point p of U. The entries of the n-tuple are the **coordinates** of p in the local chart (U, ϕ). Points in U are thus in one-to-one correspondence with n-tuples of $\phi(U) \subset \mathbb{R}^n$.
(3) If $U \cap V \neq \emptyset$, the compatibility of local charts (U, ϕ), (V, ψ) implies that the Jacobian matrix of $\phi\circ\psi^{-1}$ is invertible and so has everywhere non-zero determinant. Conversely, if $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is bijective, of class C^k, and with non-vanishing Jacobian determinant on $\psi(U\cap V)$, then also $\psi\circ\phi^{-1} : \phi(U\cap V) \to \psi(U \cap V)$ is C^k and the local charts are compatible. The proof can be found in the renowned [CoFr98II].

Theorem B.4 (Implicit function theorem) *Let $D \subset \mathbb{R}^n$ be open, non-empty, and $f : D \to \mathbb{R}^n$ a C^k function, for some $k = 1, 2, \dots, \infty$. If the Jacobian of f at $p \in D$ has non-zero determinant, there exist open neighbourhoods $U \subset D$ of p and V of $f(p)$ such that: (i) $f\restriction_U : U \to V$ is bijective, (ii) the inverse $f\restriction_U^{-1} : V \to U$ is C^k.*

(3) The topological requirements in (2)(i) (valid for the standard topology of $\mathbb{R}^n$) are technical and guarantee unique solutions to differential equations on M (necessary in physics when the equations describe the evolution of physical systems) and the existence of integrals on M. Condition (2)(ii) intuitively says that around any of its

points M is undistinguishable from a continuous image of $\mathbb{R}^n$. Standard counterexamples show that the Hausdorff property of $\mathbb{R}^n$ is not carried over to M by local homeomorphisms, so it must be imposed explicitly.
(4) Let M be a second-countable Hausdorff space. A collection of local charts $\mathscr{A}$ on M satisfying (i) and (ii) in (1), but not necessarily (iii), plus (ii) in (2) is called a C^k **atlas** on the n-manifold M. It is not hard to see that any atlas $\mathscr{A}$ on M is contained in some maximal atlas. Two atlases on M such that every chart of one is compatible with any chart of the other induce the same differentiable structure on M. Therefore to assign a differentiable structure it suffices to prescribe a non-maximal atlas, one of the many that determine it. The unique differentiable structure associated to a given atlas is said to be **induced** by the atlas.
(5) If $1 \leq k < \infty$ there might be superfluous charts in the differentiable structure (only a finite number!); by eliminating them we obtain a C^∞ atlas. ■

Examples B.5
1. The simplest examples of differentiable manifolds, of class C^∞ and dimension n, are non-empty open subsets of $\mathbb{R}^n$ (including $\mathbb{R}^n$ itself) with the standard differentiable structure determined by the identity map (the inclusion, alone, defines an atlas).
2. Consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ (with topology inherited from $\mathbb{R}^3$) centred at the origin:

$$\mathbb{S}^2 := \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \;\middle|\; (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\right\}$$

in canonical coordinates x^1, x^2, x^3 of $\mathbb{R}^3$. It has dimension 2 and a smooth structure induced by $\mathbb{R}^3$ by defining an atlas with 6 local charts $(\mathbb{S}^2_{(i)\pm}, \phi^{(i)}_\pm)$ $(i = 1, 2, 3)$ as follows. Take the axis x^i $(i = 1, 2, 3)$ and the pair of open hemispheres $\mathbb{S}^2_{(i)\pm}$ with 'south-to-north' direction given by x^i, and consider local charts $\phi^{(i)}_\pm : \mathbb{S}^2_{(i)\pm} \to \mathbb{R}^2$ that map $p \in \mathbb{S}^2_{(i)\pm}$ to its coordinates on the plane $x^i = 0$. It can be proved (see below) that $\mathbb{S}^2$ cannot be covered by a single (global) chart, in contrast to $\mathbb{R}^3$ (or any open subspace). This shows that the class of smooth manifolds does not reduce to open non-empty subsets of $\mathbb{R}^n$, and hence is quite interesting. A similar example is the unit circle in $\mathbb{R}^2$. ■

Given C^k manifolds M and N of respective dimensions m, n, we can construct a third C^k manifold of dimension $m+n$ over the topological product $M \times N$. (The resulting space will be Hausdorff and second-countable.) This is called *product manifold* of M and N, and denoted simply by $M \times N$. The structure described below is called *product structure*. Given local charts (U, ϕ) on M and (V, ψ) on N it is immediate to see

$$U \times V \ni (p, q) \mapsto (\phi(p), \psi(q)) =: \phi \oplus \psi(p, q) \in \mathbb{R}^{m+n} \tag{B.1}$$

is a local homeomorphism. If (U', ϕ') and (V', ψ') are other charts, compatible with the previous ones, the charts $(U \times V, \phi \oplus \psi)$ and $(U' \times V', \phi' \oplus \psi')$ are obviously compatible. As (U, ϕ) and (V, ψ) vary on M and N the charts $(U \times V, \phi \oplus \psi)$ define an atlas on $M \times N$. The structure this atlas generates is, by definition, the product structure.

Definition B.6 Given C^k manifolds M, N of dimension m, n, the **product manifold** is the set $M \times N$ equipped the with product topology and the C^k structure induced by the local charts $(U \times V, \phi \oplus \psi)$ in (B.1), when (U, ϕ), (V, ψ) vary on M, N.

Since a manifold is locally indistinguishable from $\mathbb{R}^n$, the differentiable structure allows to make sense of *differentiable functions* defined on a manifold other than $\mathbb{R}^n$ or subsets. The idea is simple: reduce locally to the standard notion on $\mathbb{R}^n$ using the local charts that cover the manifold.

Definition B.7 Let M, N be manifolds of dimensions m, n and class C^p, C^q respectively $(p, q \geq 1)$. A continuous map $f : M \to N$ is said C^k $(0 \leq k \leq p, q$, possibly $k = \infty$ or $\omega)$ if $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is a C^k map, for any choice of local charts (U, ϕ) on N and (V, ψ) on M.

The collection of C^k functions from M to N, $k = 0, 1, 2, \ldots, \infty, \omega$ is denoted by $C^k(M; N)$; if $N = \mathbb{R}$ one just writes $C^k(M)$.

A C^k **diffeomorphism** $f : M \to N$ is a bijective C^k map with C^k inverse. If there is a C^k diffeomorphism f mapping M to N, the two manifolds are called **diffeomorphic** (under f).

Remark B.8
(1) Notice how we allowed for differentiable maps of class C^0, which are simply continuous maps (just as C^0 diffeomorphisms are homeomorphisms). Every C^k diffeomorphism is clearly a homeomorphism, which explains why there cannot exist any diffeomorphism between $\mathbb{S}^2$ and (a subset of) $\mathbb{R}^2$, for the former is compact, the latter not. Consequently, the sphere $\mathbb{S}^2$ does not admit global charts.
(2) For $f : M \to N$ to be C^p it is enough that $\psi \circ f \circ \phi^{-1}$ is C^p for any local charts (U, ϕ), (V, ψ) in the given atlases, without having to check the condition for *every possible* local chart on the manifolds. ■

A useful notion is that of *embedded submanifold*. The space $\mathbb{R}^n$ is an embedded submanifold in $\mathbb{R}^m$ if $m > n$. In the canonical coordinates $x^1, \ldots, x^m$ on $\mathbb{R}^m$, $\mathbb{R}^n$ is identified with the subspace given by equations $x^{n+1} = \cdots = x^m = 0$, while the first n coordinates of $\mathbb{R}^m$, $x^1, \ldots, x^n$, are identified with the standard coordinates on $\mathbb{R}^n$. Now the idea is to replace $\mathbb{R}^n$, $\mathbb{R}^m$ using local frames, and generalise to manifolds N, M.

Definition B.9 Let M be a C^k $(k \geq 1)$ manifold of dimension $m > n$. An **embedded C^k submanifold of M of dimension** n is an n-manifold N of class C^k such that
(a) N is a subset in M with induced topology;
(b) the differentiable structure di N is given by the atlas $\{(U_i, \phi_i)\}_{i \in I}$ where:

(i) $U_i = V_i \cap N$, $\phi_i = \psi\restriction_{V_i \cap N}$ for a suitable local chart (V_i, ϕ_i) on M;

(ii) in the frame $x^1, \ldots, x^m$ associated to (V_i, ϕ_i), the set $V_i \cap N$ is determined by $x^{n+1} = \cdots = x^m = 0$, and the remaining coordinates $x^1, \ldots, x^n$ are the local frame associated to ϕ_i.

To finish we state an important result (see [doC92, Wes78] for example) to decide when a subset in a manifold is an embedded submanifold. The proof is straightforward from Dini's theorem [CoFr98II].

Theorem B.10 (On regular values) *Let M be a C^k manifold of dimension m. Consider the set*

$$N := \{p \in M \mid f_j(p) = v_j \,, \quad j = 1, \ldots, c\}$$

determined by $c(< m)$ constants v_j and c functions $f_j : M \to \mathbb{R}$ of class C^k. Suppose that around each point $p \in N$ there exists a local chart (U, ϕ) on M such that the Jacobian matrix $\partial(f_j \circ \phi^{-1})/\partial x^i|_{\phi(p)}$ ($j = 1, \ldots, c$, $i = 1, \ldots, m$) has rank r. Then N is an embedded C^k submanifold in M of dimension $n := m - c$.

In particular, if the square $c \times c$ matrix

$$\frac{\partial f_j \circ \phi^{-1}}{\partial x^k} \,, \quad j = 1, \ldots, c, \quad k = m - c + 1, m - c + 2, \ldots, m$$

is non-singular at $\phi(p)$, $p \in N$, then the first n coordinates $x^1, \ldots, x^n$ define a frame system around p in N.

B.2 Tangent and Cotangent Spaces. Covariant and Contravariant Vector Fields

Let M be C^k manifold of dimension n ($k \geq 1$). Consider the space $C^k(M)$ as an $\mathbb{R}$-vector space under linear combinations

$$(af + bg)(p) := af(p) + bg(p) \,, \quad \text{for any } p \in M$$

where $a, b \in \mathbb{R}$, $f, g \in C^k(M)$. Given a point $p \in M$, a **derivation** at p is an $\mathbb{R}$-linear map $L_p : C^k(M) \to \mathbb{R}$ satisfying the *Leibniz rule*:

$$L_p(fg) = f(p)L_p(g) + g(p)L_p(f) \,, \quad f, g \in C^k(M). \tag{B.2}$$

A linear combination $aL_p + bL'_p$ of derivations at p ($a, b \in \mathbb{R}$),

$$(aL_p + bL'_p)(f) := aL_p(f) + bL'_p(f) \,, \quad f, g \in C^k(M),$$

is still a derivation. Hence derivations at p form a vector space over $\mathbb{R}$, which we denote $\mathscr{D}^k_p$. Every local chart (U, ϕ) with $U \ni p$ automatically gives n derivations at p, as follows. If $x^1, \ldots, x^n$ are coordinates associated to ϕ, define the kth derivation to be

$$\left.\frac{\partial}{\partial x^k}\right|_p : f \mapsto \left.\frac{\partial f \circ \phi^{-1}}{\partial x^k}\right|_{\phi(p)} \,, \quad f, g \in C^1(M). \tag{B.3}$$

If 0 is the null derivation and $c^1, c^2, \ldots, c^n \in \mathbb{R}$ satisfy $\sum_{k=1}^n c^k \frac{\partial}{\partial x^k}\big|_p = 0$, we choose a differentiable function coinciding with the coordinate map x^l on an open neighbourhood of p (whose closure is in U) and vanishing outside. Then the n derivations $\frac{\partial}{\partial x^k}\big|_p$ at p are *linearly independent*: $\sum_{k=1}^n c^k \frac{\partial}{\partial x^k}\big|_p f = 0$ implies $c^l = 0$. Since we are free to choose l arbitrarily, every coefficient c^r is zero for $r = 1, 2, \ldots, n$. Hence the n derivations $\frac{\partial}{\partial x^k}\big|_p$ form a basis for an n-dimensional subspace of $\mathscr{D}_p^k$ (actually if $k = \infty$ the subspace coincides with $\mathscr{D}_p^\infty$). Changing chart to (V, ψ), $V \ni p$, with frame $y^1, \ldots, y^n$, the new derivations are related to the old ones by:

$$\frac{\partial}{\partial y^i}\bigg|_p = \sum_{k=1}^n \frac{\partial x^k}{\partial y^i}\bigg|_{\psi(p)} \frac{\partial}{\partial x^k}\bigg|_p . \tag{B.4}$$

The proof is direct from the definitions. Because the Jacobian $\frac{\partial x^k}{\partial y^i}\Big|_{\psi(p)}$ is invertible by definition of chart, the subspace of $\mathscr{D}_p^k$ spanned by the $\frac{\partial}{\partial y^i}\Big|_p$ coincides with the span of the $\frac{\partial}{\partial x^k}\big|_p$. The subspace is thus *intrinsically defined.*

Definition B.11 Let M be an n-dimensional C^k manifold ($k \geq 1$), and fix a point $p \in M$.

The vector subspace of derivations at $p \in M$ generated by the n derivations $\frac{\partial}{\partial x^k}\big|_p$, $k = 1, 2, \ldots, n$, in any local coordinate system (U, ϕ) with $U \ni p$, is called **tangent space of** M **at** p and is written T_pM. The elements of the tangent space at p are the **tangent vectors at** p to M. Tangent vectors are examples of **contravariant vectors**.

We recall that the space V^* of linear maps from a real vector space V to $\mathbb{R}$ is called **dual space** to V. If the dimension of V is finite, so is the dimension of V^*, for they coincide. In particular, if $\{e_i\}_{i=1,\ldots,n}$ is a basis of V, the **dual basis** in V^* is the basis $\{e^{*j}\}_{j=1,\ldots,n}$ defined via $e^{*j}(e_i) = \delta_i^j$, $i, j = 1, \ldots, n$, by linearity. With $f \in V^*$, $v \in V$, one uses the notation $\langle v, f\rangle := f(v)$.

Definition B.12 Let M be an n-dimensional C^k manifold ($k \geq 1$), $p \in M$ a given point.

The dual space to T_pM is called **cotangent space of** M **at** p, written T_p^*M. Points of the cotangent space at p are called **cotangent vectors at** p or 1**-forms** at p, and are instances of **covariant vectors** (covectors). For any basis $\frac{\partial}{\partial x^k}\big|_p$ of T_pM, the n elements of the dual basis are indicated by $dx^i|_p$. By definition

$$\left\langle \frac{\partial}{\partial x^k}|_p, dx^i|_p \right\rangle = \delta_k^i .$$

Let us move on to *vector fields* on a manifold M.

Suppose M is an n-dimensional C^k manifold (including $k = \infty$ and $k = \omega$). A **contravariant** C^r **vector field**, $r = 0, 1, \ldots, k$, is a map assigning a vector

$v(p) \in T_pM$ to any $p \in M$, so that for any local chart (U, ϕ) with coordinates $x^1, \ldots, x^n$ where

$$v(q) = \sum_{i=1}^{n} v^i(x_q^1, \ldots, x_q^n) \left.\frac{\partial}{\partial x^i}\right|_q ,$$

the n functions $v^i = v^i(x^1, \ldots, x^n)$ are C^r on $\phi(U)$. Similarly, a **covariant** C^r **vector field**, $r = 0, 1, \ldots, k$ is a map sending $p \in M$ to a covector $\omega(p) \in T_p^*M$, so that for any local chart (U, ϕ) with coordinates $x^1, \ldots, x^n$ where

$$\omega(q) = \sum_{i=1}^{n} v_i(x_q^1, \ldots, x_q^n)\, dx^i\big|_q ,$$

the n functions $\omega_i = \omega_i(x^1, \ldots, x^n)$ are C^r on $\phi(U)$.

Remark B.13 Take $v \in T_pM$ and two local charts (U, ϕ), (V, ψ) with $U \cap V \ni p$ and respective coordinates $x^1, \ldots, x^n$, $x'^1, \ldots, x'^n$. Then $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n v'^j \left.\frac{\partial}{\partial x'^j}\right|_p$. Hence $\sum_i^n v^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j,i=1}^n v'^j \left.\frac{\partial x^i}{\partial x'^j}\right|_{\psi(p)} \left.\frac{\partial}{\partial x^i}\right|_p$, so $\sum_{i=1}^n \left(v^i - \sum_{j=1}^n \left.\frac{\partial x^i}{\partial x'^j}\right|_{\psi(p)} v'^j\right) \left.\frac{\partial}{\partial x^i}\right|_p = 0$. Since the derivations $\left.\frac{\partial}{\partial x^i}\right|_p$ are linearly independent, we conclude that the components of a tangent vector in T_pM transform, under coordinate change, as

$$v^i = \sum_{j=1}^{n} \left.\frac{\partial x^i}{\partial x'^j}\right|_{\psi(p)} v'^j , \tag{B.5}$$

The same argument gives the formula for covariant vectors $\omega = \sum_{i=1}^n \omega_i\, dx^i\big|_p = \sum_{j=1}^n \omega'_j\, dx'^j\big|_p$, namely

$$\omega_i = \sum_{j=1}^{n} \left.\frac{\partial x'^j}{\partial x^i}\right|_{\psi(p)} \omega'_j . \tag{B.6}$$

B.3 Differentials, Curves and Tangent Vectors

Let $f : M \to \mathbb{R}$ be a C^r scalar field on the C^k manifold M of dimension n, and assume $k \geq r > 1$. The **differential** df of f is the covariant vector field of class C^{r-1}

$$df|_p = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_{\psi(p)} dx^i|_p$$

in any local chart (U, ψ).

Consider a C^r curve inside the C^k manifold M $(r = 0, 1, \ldots, k)$, i.e. a C^r function $\gamma : I \to M$ where $I \subset \mathbb{R}$ is an open interval thought of as a submanifold in $\mathbb{R}$. Assume explicitly that $r > 1$. We can define the *tangent vector* to γ at $p \in \gamma(I)$ by

$$\dot{\gamma}(p) := \sum_{i=1}^{n} \left.\frac{dx^i}{dt}\right|_{t_p} \left.\frac{\partial}{\partial x^i}\right|_p ,$$

where $\gamma(t_p) = p$, in any local chart around p. The definition does *not* depend on the chart. Had we defined

$$\dot{\gamma}'(p) := \sum_{j=1}^{n} \left.\frac{dx'^j}{dt}\right|_{t_p} \left.\frac{\partial}{\partial x'^j}\right|_p$$

in another frame system around p, using (B.5) would have given

$$\dot{\gamma}(p) = \dot{\gamma}'(p) .$$

So we have this definition.

Definition B.14 A C^r **curve**, $r = 0, 1, \ldots, k$, in the n-dimensional C^k manifold M is a C^r map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval (embedded in $\mathbb{R}$). When $r > 1$, the **tangent vector** to γ at $p = \gamma(t_p)$, $t_p \in I$, is the vector $\dot{\gamma}(p) \in T_pM$ given by

$$\dot{\gamma}(p) := \sum_{i=1}^{n} \left.\frac{dx^i}{dt}\right|_{t_p} \left.\frac{\partial}{\partial x^i}\right|_p , \tag{B.7}$$

in any local frame around p.

B.4 Pushforward and Pullback

Let M and N be manifolds of dimensions m and n, and $f : N \to M$ a function (all at least C^1). Given a point $p \in N$ consider local charts (U, ϕ) around p in N and (V, ψ) around $f(p)$ in M. Indicate by $(y^1, \ldots, y^n)$ the coordinates on U, by $(x^1, \ldots, x^m)$ those on V and introduce maps $f^k(y^1, \ldots, y^n) = x^k(f \circ \phi^{-1})$, $k = 1, \ldots, m$. Now define:

(i) the **pushforward** $df_p : T_pN \to T_{f(p)}M$, in coordinates:

$$df_p : T_pN \ni \sum_{i=1}^{n} u^i \left.\frac{\partial}{\partial y^i}\right|_p \mapsto \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \left.\frac{\partial f^j}{\partial y^i}\right|_{\phi(p)} u^i \right) \left.\frac{\partial}{\partial x^j}\right|_p ; \tag{B.8}$$

(ii) the **pullback** $f_p^* : T^*_{f(p)}M \to T^*_pN$, in coordinates:

$$f_p^* : T^*_{f(p)}M \ni \sum_{j=1}^{m} \omega_j dx^j|_{f(p)} \mapsto \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \frac{\partial f^j}{\partial y^i}\bigg|_{\phi(p)} \omega_j \right) dy^i|_p \, . \tag{B.9}$$

It is not hard to see they *do not depend on local frame systems.* The pushforward is also written $f_{p*} : T_p N \to T_{f(p)} M$.

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Symbols

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