D 10231 (Pages : 3) Name.................................

Reg. No.................................

FIFTH SEMESTER B.A./B.Sc. DEGREE EXAMINATION, NOVEMBER 2021

(CUCBCSS–UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. The smallest non abelian group has ——— number of elements.
2. The order of the identity element in any group G is ———.
3. State True or False. "Every abelian group is cyclic".
4. State True or False. "Every group of order 31 is cyclic".
5. Give an example of non-cyclic group with four elements.
6. The total number of subgroups of Z_{12} is ———.
7. What are the orbits of the identity permutation σ of a set A ?
8. How many zero divisors are there for the field Z_7 ?
9. How many units are there for the field Z_7 ?
10. Give an example of integral domain which not a field.
11. State True or False. Z is a sub field of Q.
12. Write the number of generators of the group Z_5 under addition modulo 5.

(12 × 1 = 12 marks)

Part B

Answer any **ten** *questions.*
Each question carries 4 *marks.*

13. Show that left and right cancellation holds in a group G.
14. Let G be a group and suppose that $a * b * c = e \forall a, b, c \in G$. Show that $b * c * a = e$.
15. Prove that a group G has exactly one idempotent element.
16. Can the identity element be a generator of a cyclic group ?
17. Prove that every cyclic group is abelian.

Turn over

18. Consider the group $\mathbb{Z}_{12}$, under the operation addition modulo 12. Find the order of the cyclic subgroup generated by $3 \in \mathbb{Z}_{12}$.
19. Show that the permutation (1, 4, 5, 6) (2, 3, 1, 5) is an even permutation.
20. What is the order of the cycle (1, 4, 5, 7) in S_8 ?
21. Find the partition of the group $\mathbb{Z}_6$, under the operation addition modulo 6, into cosets of the subgroup H = {0, 3}.
22. Consider the rings $\langle \mathbb{Z}, +, \cdot \rangle$ and $\langle 2\mathbb{Z}, +, \cdot \rangle$. Verify whether the map $\phi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\phi(x) = 2x \forall x \in \mathbb{Z}$ is a ring homomorphism or not.
23. Find the number of generators of the cyclic group of order 8.
24. Solve the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$.
25. Consider the following two binary structures :
    (a) $\mathbb{Z}$, the set of integers with ordinary addition ; and
    (b) $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ the set of even integers with ordinary addition.

    Show that the above two binary structures are isomorphic.
26. Let n be a positive integer. Give an example of a group containing n elements.

(10 × 4 = 40 marks)

Part C

Answer any **six** *questions.*
Each question carries 7 marks.

27. Let * be defined by Q^+ by $a * b = \frac{ab}{2}$. Show that $(Q^+, *)$ is an abelian group.
28. Let G be a group. For all $a, b \in G$, prove that $(a * b)^{-1} = b^{-1} * a^{-1}$.
29. Prove that a necessary and sufficient condition that a non-empty subset H of a group G is a subgroup of G is that $a \in H, b \in H \Rightarrow ab^{-1} \in H$.
30. Show that the subgroups of $\mathbb{Z}$ under addition are precisely the groups $n\mathbb{Z}$ under addition for $n \in \mathbb{Z}$.
31. Show that any permutation of a finite set of at least two elements is a product of transpositions.
32. Show that a homomorphism ϕ of a group G is a one-to-one function if and only if Ker $\phi = \{e\}$.
33. Show that cancellation law holds in a ring R if and only if R has no zero divisors.

34. M_2 denotes the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are rational numbers. Is M_2 a field ? Justify your answer.

35. Prove that any integral domain D can be enlarged to (or embedded in) a field F such that every element of F can be expressed as a quotient of two elements of D.

(6 × 7 = 42 marks)

Part D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. Show that subgroup of a cyclic group is cyclic.

37. (a) Define the term orbit, cycle and transposition with respect to a permutation.

(b) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$ as product of disjoint cycles.

(c) Define even and odd permutation. Write (1, 4, 5, 6) (2, 1, 5) as a product of transpositions.

38. Show that in the ring $\mathbb{Z}_n$, the divisors of 0 are precisely those elements that are not relatively prime to n also show that $\mathbb{Z}_p$ is a field.

(2 × 13 = 26 marks)

D 90231 (Pages : 3) Name.....................................

Reg. No.....................................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2020

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *questions.*

Each question carries 1 *mark.*

1. Define a Group.
2. Fill in the blanks : The units in the ring of integers $\mathbb{Z}$ are ————.
3. Write the order of the permutation (1, 2) (1 9 8) in S_9.
4. Give an example of a finite group of order 4 which is not cyclic.
5. Calculate the remainder obtained when 45^{72} is divided by 73.
6. Find the inverse of the product (7, 5) (2, 5, 7) in S_7. Is the inverse a cycle ?
7. What is the characteristic of the ring $< \mathbb{Z}_9 , +_9, \times_9 >$.
8. Give an example for an integral domain which is not a field.
9. What is the necessary condition for a homomorphism ϕ from a group G to G' to be injective.
10. Define normal subgroup of a group.
11. What is the index of A_n in S_n.
12. Define a cyclic group and give an example.

(12 × 1 = 12 marks)

Section B

Answer at least **eight** *questions.*

Each question carries 6 *marks.*

All questions can be attended.

Overall Ceiling 48.

13. Write criteria to be checked to determine whether a function $\Phi : S \rightarrow S'$ is an isomorphism of a binary structure $< S,* >$ with $< S',*' >$.

Turn over

14. Is $\mathbb{Z}^+$ a group under usual addition ? Establish your claim.

15. Solve : $x^2 - 1 = 0$ in the field $\mathbb{Z}_p$.

16. Show that every field is an integral domain.

17. Find the multiplicative inverse of 53 in $\mathbb{Z}_{57}$.

18. Construct group table for the Klein group. What is the order of every element in this group ?

19. Define kernel of a group homomorphism. Find the $\ker(\phi)$ for $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = 0$ for all $x \in \mathbb{R}$.

20. Define a ring. Give an example of a non-commutative ring.

21. Define center of a group and show that center of the symmetric group S_3 is the trivial group.

22. In any ring R, show that $a.0 = 0 = 0.a$ and $a.(-b) = -(a.b)$ for all a, b in R.

23. Show that for any group, its identity element and inverse of any element are unique.

24. Evaluate the product of (2, 3) and (3, 5) in $\mathbb{Z}_5 \times \mathbb{Z}_9$.

25. Show that $a^2 - b^2 = (a-b)(a+b)$ in a ring R if and only if R is commutative.

26. Define factor group and give an example.

(8 × 6 = 48 marks)

Section C

Answer at least **five** *questions.*
Each question carries 9 *marks.*
All questions can be attended.
Overall Ceiling 45.

27. Show that the binary structure $< \mathbb{R}, + >$ with operation the usual addition is isomorphic to the structure $< \mathbb{R}^+, . >$ where . is the usual multiplication.

28. (a) State and prove Lagrange's theorem ; and (b) Establish one of its corollary.

29. Show that the subset S of $M_n(\mathbb{R})$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group.

30. Show that every permutation σ of a finite set is a product of disjoint cycles.

31. Let G and G' be groups and let $\phi : G \to G'$ be one to one function such that $\phi(xy) = \phi(x)\Phi(y)$ for all $x, y \in G$. Then prove that $\phi[G]$ is a subgroup of G' and ϕ provides an isomorphism of G with $\Phi[G]$.

32. Show that subgroup a cyclic group is cyclic.

33. Show that M is a maximal normal subgroup of G if and only if G/M is simple.

34. Show that the cancellation law in a ring R holds if and only if it has no zero divisors.

35. Find all solutions of the congruence $12x \equiv 27(\text{mod}\,18)$.

(5 × 9 = 45 marks)

Section D

Answer any **one** *question.*
The question carries 15 *marks.*

36. (a) Show that $|\langle a^s \rangle| = |\langle a^t \rangle|$ if and only if g.c.d (n, s) = g.c.d (n, t) where $n = |\langle a \rangle|$.

(b) Show that the only subgroups of $\mathbb{Z}$ are the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$.

37. State Cayley's theorem and give the proof in detail.

38. (a) Show that any two fields of quotients of an integral domain are isomorphic.

(b) Prove or disprove : Factor group of a cyclic group is cyclic.

(1 × 15 = 15 marks)

D 70321 (Pages : 3) Name..................................

Reg. No..................................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2019

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours

Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Fill in the blanks : The total number of commutative binary operations on a set of n elements is ————.

2. Fill in the blanks : The number of elements in the ring $M_2(\mathbb{Z}_3)$ is ————.

3. Fill in the blanks : The least value of n such that a group G of order n is non-abelian is ————.

4. Define a group.

5. Give an example of a finite integral domain.

6. Define skew fields.

7. Calculate the order of the permutation $\mu = (1)(1\ 2)(1\ 3)$ in S_4.

8. Solve : $-3x + 2 = 4$ in the group $< \mathbb{Z}_6, +_6 >$.

9. Show that the identity element in a group is unique.

10. How many left cosets are there for $p\mathbb{Z}$ in $\mathbb{Z}$ if p is a prime.

11. What is a Klein group ?

12. Give a group theoretic definition of greatest common divisor of two positive integers.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 *marks.*

13. If H is a finite non-empty subset of a group G, establish that H will be a subgroup if it is closed under the binary operation in G.

14. Show that a group is a finite group if it has finite number of subgroups.

Turn over

15. Show that every cyclic group is abelian.

16. Find all group homomorphism from $\mathbb{Z}$ into itself.

17. Let G be a group of order pq where p and q are primes. Show that every proper subgroup of G is cyclic.

18. Let S be a set and let f, g and h be functions mapping S into S. Prove that $f*(g*h)=(f*g)*h$ where the binary operation * is the function composition.

19. Is the union of two subgroups a subgroup ? Justify your claim.

20. Show that the coset multiplication given by $(aH)(bH) = abH$ is a well defined operation when H is a normal subgroup of G.

21. Draw the subgroup diagram for $\mathbb{Z}_{18}$.

22. Show that any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

23. Find a group isomorphic to the Klein group other than the Klein group. Establish that it is so.

24. Give any necessary and sufficient condition for a ring R to have no zero divisors. Justify your claim.

25. Is Q, the set of rationals, the field of quotients for integers? Give reasons to establish your assertion or denial.

26. Show that factor group of a cyclic group is always abelian.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Draw the group table for the dihedral group D_4. Is D_4 a cyclic group ? Justify your claim.

28. Define kernal of a group homomorphism and show that it is a normal subgroup of the domain of the homomorphism.

29. Define order of an element in any group G. Show that in a finite group G, order of any element divides order of G.

30. Show that every permutation σ of a finite set is a product of disjoint cycles.

31. Prove or disprove : Every finite integral domain is a field.

32. Define the alternate group A_n. Show that it is a normal subgroup and find the group isomorphic to S_n/A_n.

33. Let G be a finite group in which for each positive integer m, the number of solutions of $x^m = e$ is at most m. Then show that G is cyclic.

34. If $< R, + >$ is an abelian group, show that $< R, +, . >$ is a ring if $a.b$ is defined as 0 for all $a, b \in R$.

35. Prove in some detail that every field L containing an integral domain D contains the field of quotients of D.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 marks.

36. (a) If $\phi : G \to G'$ is a group homomorphism then show that $\phi[H] \leq G'$ whenever $H \leq G$.

(b) Find the index of the subgroup generated by $\sigma = (1, 5, 3, 4)(2, 3)$ in S_5.

37. (a) Prove that the converse of the Lagranges theorem need not be true.

(b) Express $\sigma = (1\ 2\ 3)(1\ 3\ 4)^2 \in S_4$ as a product of disjoint cycles.

38. (a) Define rings and ring homomorphisms. Show that the ring of real numbers and complex numbers are not isomorphic.

(b) Find all the units in the ring $< Z_{18}, +_{18}, \times_{18} >$.

(2 × 13 = 26 marks)

D 50600 (Pages : 3) Name.................................

Reg. No..................................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2018

(CUCBCSS—UG)

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Define subgroup of a group.
2. Fill in the blanks : The units in the ring of integers $\mathbb{Z}$ are ————.
3. Write the order of the permutation (1, 2) (1 9 8) in S_9.
4. Give an example of a finite group of order 4 which is not cyclic.
5. Calculate the remainder obtained when 45^{72} is divided by 73.
6. Compute (1, 4) (7, 5) (2, 5, 7) in S_7.
7. What is the characteristic of the ring $< \mathbb{Z}_9, +_9, \times_9 >$.
8. Give an example for an integral domain which is not a field.
9. What is the necessary condition for a homomorphism ϕ from a group G to G' to be injective.
10. Write two equivalent conditions for the subgroup of a group to be a normal subgroup.
11. What is the index of A_n in S_n.
12. Denne a cyclic group and give an example.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 *marks.*

13. Let S be a set and let *f*, *g* and *h* be functions mapping S into S. Prove that $(fog)\,oh = fo\,(goh)$.
14. Show that every group of prime order is abelian.
15. Draw the group table for S_3.

Turn over

16. Define * by $a*b=\frac{ab}{3}$ then show that Q^+, the set of positive rationals, with operation * s a group.

17. Show that A_n is a normal subgroup of S_n and find a group to which S_n/A_n is isomorphic.

18. Define a field. Show that $(a+b)^2 = a^2 + 2ab + b^2$ in any field using the axioms of the field.

19. Give an example of non-commutative finite ring. Establish that it is so.

20. Give any necessary and sufficient condition for a ring R to have no zero divisors. Justify your claim.

21. Find a formula for identifying units in the ring of guassian integers $\{a+ib : a, b \in \mathbb{Z}\}$.

22. Write all the left cosets of $3\mathbb{Z}$ in $\mathbb{Z}$.

23. Show that factor group of a cyclic group is cyclic.

24. Solve : $x^2 = i$ in S_3 where i is the identity.

25. Define group homomorphism and state fundamental theorem of homomorphism.

26. Is Q, the set of rationals, the field of quotients for integers ? Justify your claim.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks.

27. Show that the binary structure $< \mathbb{R}, + >$ with operation the usual addition is isomorphic to the structure $< \mathbb{R}^+, . >$ where is the usual multiplication.

28. Define order of an element in any group G. Show that in a finite group G, order of any element divides order of G.

29. Show that the subset S of $M_n(\mathbb{R})$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group.

30. Show that every permutation σ of a finite set is a product of disjoint cycles.

31. Let G and G′ be groups and let $\Phi : G \longrightarrow G'$ be one to one function such that $\Phi(xy) = \Phi(x)\,\Phi(y)$ for all $x, y \in G$. Then prove that $\Phi[G]$ is a subgroup of G′ and Φ provides an isomorphism of G with $\Phi[G']$.

32. Show that every proper subgroup of a group G, with $o(G) = pq$ where p and q are prime, is cyclic.

33. Solve the equation : $x^2 - 5x + 6 = 0$ completely in $\mathbb{Z}_{12}$.

34. Establish a formula for computing the number of zero divisors in the ring $\mathbb{Z}_n$ by giving its proof.

35. Give a necessary and sufficient condition for union of two subgroups of a group to be a subgroup.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. (a) Find the index of the subgroup generated by $\sigma = (1, 2, 5, 4)(2, 3)$ in S_5.

(b) Write all the subgroups of Z_{10}.

37. State and prove Cayley's theorem in detail.

38. (a) List all the units in the matrix ring $M_2(\mathbb{Z}_2)$.

(b) Show that every finite integral domain is a field.

(2 × 13 = 26 marks)

C 30800 (Pages : 2) Name..............................

Reg. No..............................

FIFTH SEMESTER B.Sc. DEGREE (SUPPLEMENTARY) EXAMINATION, NOVEMBER 2017

(UG–CCSS)

MM 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries ¼ weightage.

1. The operation $*$ defined on z by letting $a * b = ab + 1$. Determine whether $*$ is associative.
2. Define an isomorphism of an algebric structure to another.
3. Give an example of an abelian group with 100 elements.
4. Find the order of the element $-i$ in the group $u_4 = \{1, -1, i, -i\}$ with usual multiplication.
5. Symmetric group of n letters has ———— elements.
6. What is the order of the cycle (1, 4, 5, 7) in S_8.
7. Define index of a subgroup H in a group G.
8. Compute the product (12) (6) in Z_{24}.
9. Define a linear combination of the set $S = \{u_1, u_2, \dots u_n\}$.
10. Is the set of pure imaginary numbers with usual addition and multiplication a ring ?
11. Define a ring homomorphism.
12. Describe all the units in the ring z_5.

(12 × ¼ = 3 weightage)

Part B

Answer **all** *questions.*
Each question carries 1 weightage.

13. Show that set of real numbers excluding -1 is a group under the operation $*$ which is given by $a * b = a + b + ab$.
14. Prove that inverse of any element of a group is unique.
15. Show that Klein 4 group is not cyclic.
16. Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{pmatrix}$ as a product of disjoint cycles.
17. Find the partition of z_6 into cosets of the subgroup $H = \{0, 3\}$.

Turn over

18. Show that every subgroup of an abelian group is normal.

19. Prove that the set z_4 is a ring w.r.to addition modulo 4 and multiplication modulo 4.

20. Find all the units in z_{14}.

21. $S = \{P(x) \in P_n(x) / P(1) + P(3) = 0\}$ where $P_n(x)$ is the vector space of all polynomials of degree $\leq n$. Show that S is a subspace of $P_n(x)$.

(9 × 1 = 9 weightage)

Part C

Answer any **five** *questions.*
Each question carries 2 weightage.

22. Prove that residue classes modulo 5 form a finite abelian group with respect to addition of residue classes.

23. Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ as a product of disjoint cycles and as product of transpositions.

24. Find all the subgroups of z_{12} and draw the lattice diagram.

25. Show that product of two left cosets need not be a coset.

26. Let G be the group of non-zero complex numbers under multiplication and let n be a positive integer. Show that the mapping $\phi : G \to G$ defined by $\phi(z) = z^n$ is a homomorphism. What is the kernal of this homomorphism.

27. In the ring z_n, divisors of 0 are precisely those elements that are not relatively prime to n. Prove this statement.

28. Check whether the set $S = \{x - 1, 2x^2 + x, x^3 + 1, 2x^3 - 3x\}$ of P_3 is linearly independent.

(5 × 2 = 10 weightage)

Part D

Answer any **two** *questions.*
Each question carries 4 weightage.

29. Let G be a cyclic group with generator a. If order of G is infinite then prove that G is isomorphic to $(z, +)$. Also prove that if G has a finite order n, then G is isomorphic to (z_n, t_n).

30. For $n \geq 2$, prove that the no. of even permutations in S_n is same as no. of add permutations in it.

31. Prove the following theorems :

    (a) Every field is an integral domain.

    (b) Every finite integral domain is a field.

(2 × 4 = 8 weightage)

C 30306 (Pages : 4) Name..............................

Reg. No..............................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2017

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer **all** *the twelve questions.*
Each question carries 1 *mark.*

1. True *or* False : Every binary operation on a set consisting of a single element is both commutative and associative.,
2. Describe the isomorphism from $<\mathbb{Z},+>$ into $<n\mathbb{Z},+>$.
3. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.
4. Determine whether the set of all invertible $n \times n$ real matrices with determinant -1 is a subgroup of $GL(n, \mathbb{R})$.
5. Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is a permutation of $\mathbb{R}$.
6. Write the orbits of the identity permutation, i on a set A
7. True *or* false : Every finite group contains an element of every order that divides the order of the group.
8. Find all orbits of the permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$.
9. Find all units in the ring $\mathbb{Z}_4$.
10. Find the characteristics of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$.
11. How many solutions, does the equation $x^2 - 5x + 6 - 0$ have $\mathbb{Z}_7$?
12. Find the number of elements in the set $\{\sigma \in S_5 / \sigma(2) = 5\}$?

(12 × 1 = 12 marks)

Turn over

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 marks.

13. Define isomorphism of algebraic structures. Determine whether the function $\phi : <\mathbb{Q}, +> \to <\mathbb{Q}, +>$ where $\phi(x) = \frac{x}{2}, x \in \mathbb{Q}$ is an isomorphism.

14. Define a group. Is the set $\mathbb{Q}^+$ of positive rational numbers under multiplication a group. Justify your answer ?

15. Let G be a group and a be a fixed element of G. Show that $H_a = \{x \in G : xa = ax\}$ is a subgroup of G.

16. Prove that an infinite cyclic group has exactly two generators.

17. Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$ as a product of disjoint cycles and then as a product of transpositions.

18. Find all generators of $\mathbb{Z}_6, \mathbb{Z}_8$ and $\mathbb{Z}_{20}$.

19. If A is any set and σ is a permutation of A, show that the relation '$\sim$' defined on A by $a \sim b$ if and only if $b = \sigma^n(a)$, for some $n \in \mathbb{Z}, a, b \in A$ is an equivalence relation.

20. Let H be a subgroup of a group G, and let $a \in G$. Define the left and right cosets of H containing a. Exhibit all left and right cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

21. Prove that a group homomorphism $\phi : G \to G'$ is a one-one map if and only if $\text{Ker}(\phi) = \{e\}$.

22. Define Ring. Give an example.

23. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in $\mathbb{Z}_{12}$.

24. Define characteristic of a ring. Find the characteristic of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$ and $\mathbb{Z}_3 \times \mathbb{Z}_4$.

25. Show that the intersection of two normal subgroups of a group is a normal subgroup.

26. If $\mathbb{R}$ is a ring such that $a^2 = a \in \mathbb{R}$. Prove that $\mathbb{R}$ is a commutative ring.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 *marks.*

27. Let G be a set consisting of ordered pairs (a, b) such that a, b are real and $a \neq 0$. A binary operation $*$ is defined on G by $(a, b) * (c, d) = (ac, bc + d)$. Prove that G is a group under *. Is G abelian ? Justify.

28. Show that a nonempty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H, \forall\, a, b \in H$.

29. Let G and G' be groups and let $\phi : G \to G'$ be a one-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Prove that $\phi[G]$ is a subgroup of G' and ϕ is an isomorphism of G with $\phi[G]$.

30. Define abelian group. Prove that a group G is abelian if every element except the identity e is of order 2.

31. Show that the set of all permutations on three symbols forms a finite non-abelian group S_3 of order 6 with respect to permutation multiplication.

32. Let $\phi : G \to G'$ be a group homomorphism and let $H = \text{Ker}(\phi)$. For $a \in G$, prove that the set $\{x \in G \,/\, \phi(x) = \phi(a)\}$ is the left coset aH of H.

33. Show that $a^2 - b^2 = (a + b)(a - b)$ for all a, b in a ring $\mathbb{R}$ if and only if $\mathbb{R}$ is commutative.

34. Show that the characteristics of an integral domain D must be either 0 or a prime p.

35. Describe the field of quotients of an integral domain.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. (a) Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G. (6 marks)

(b) Prove that a subgroup H of a group G is normal in G if and only if each left coset of H in G is a right coset of H in G. (7 marks)

Turn over

37. (a) Prove that no permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions. (7 marks)

(b) Prove that a subgroup of a cycle group is cyclic. (6 marks)

38. (a) Show that the order of an element of a finite group divides the order of the group.

(7 marks)

(b) Prove that every finite integral domain is a field.

(6 marks)

[2 × 13 = 26 marks]